Out: February 17, 2020
Due: March 1, 2020

NTNU – Trondheim
Norwegian University of
Science and Technology

TTK4130 Modeling and Simulation
Assignment 6

Introduction

The objective of this assignment are:

- To understand what differential algebraic equations (DAEs) are, and what makes them different from ordinary differential equations (ODEs).
- To be able to calculate the differential index of a DAE, and perform index reductions.
- To understand and apply Tikhonov's theorem for dynamical systems.

Problem 1 (Differential index)
Consider the DAE:

$$\dot{x}_1 = x_1 + x_2 + z \tag{1a}$$
$$\dot{x}_2 = z + u \tag{1b}$$
$$0 = \frac{1}{2}\left(x_1^2 + x_2^2 - 1\right). \tag{1c}$$

(a) Why is (1) actually a DAE?

(b) What is the differential index of (1)?

(c) Perform an index reduction of (1).

Problem 2 (Tikhonov's theorem)
Consider the differential equation:

$$\dot{\boldsymbol{x}} = -\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \boldsymbol{x} - \boldsymbol{z} \tag{3a}$$
$$\epsilon\dot{\boldsymbol{z}} = \frac{1}{10}\boldsymbol{x} - A\boldsymbol{z}, \tag{3b}$$

where

$$A = \begin{bmatrix} x_1^2 & x_2 \\ 0 & x_2^2 \end{bmatrix} + \alpha I, \tag{4}$$

with ϵ, $\alpha \geq 0$, and where I is the 2-by-2 identity matrix.

(a) Is (3) a DAE or an ODE? Explain.

(b) Simulate (3) numerically for small values of α (e.g. $\alpha = 10^{-3}$) and for $\epsilon \to 0$ (e.g. ϵ in the range $10^{-3} - 10^{-6}$). Compare the results to the ones from the DAE approximation resulting from $\epsilon = 0$. Use the initial conditions

$$\boldsymbol{x}(0) = \boldsymbol{z}(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{5}$$

and a final simulation time of at least 10.

Hint: Use the ode solver `ode15s` in order to reduce the simulation time. This is a "stiff" solver. We will discuss what that means later in the course.

(c) Repeat the previous part, but now with $\alpha = 0$.

(d) Add plots of the simulation results to your answer.
Report what you observe, and explain it from a theoretical point of view.
Hint: The conditions of Tikhonov's theorem.

Problem 3 (ODE or DAE?)

For the following differential equations, determine if they are ODE or DAEs. If they are DAEs, specify (if possible) what are the algebraic and differential states.

(a)

$$\dot{x}_1 + u + x_1 + x_2 = 0 \tag{6a}$$
$$u + x_2 + \dot{x}_2\dot{x}_1 + \dot{x}_2 u + \dot{x}_2 x_1 + \dot{x}_2 x_2 + u^2 = 0. \tag{6b}$$

(b)

$$u + \dot{x}_1 x_1 + \dot{x}_2 x_2 = 0 \tag{8a}$$
$$u\dot{x}_1 x_1 + \dot{x}_2 u x_2 = 0. \tag{8b}$$

Problem 4 (Implicit DAE)

Consider the fully-implicit DAE:

$$\dot{x} + u + \tanh(u\dot{x}) + xz = 0 \tag{10a}$$
$$\tanh(2u - z) = 0\,, \tag{10b}$$

where $x, z, u \in \mathbb{R}$ and $\tanh(\cdot)$ is the tangent hyperbolic function.

(a) Can you put (10) in the form of a semi-explicit DAE?

(b) Does (10) always provide a well-defined trajectory?
Hint: Use the Implicit Function Theorem.

(c) What is the differential index of (10)?