TTK4130 Assignment 1 Eirik Falck

$$\begin{cases} \dot{x} = \alpha x - \beta x y \\ \dot{y} = -\gamma y + \delta x y \end{cases}$$

x: prey

y: predators

Problem 1

1. $$\dot{x}_1 = a u_1 - b\sqrt{x_1} \qquad u_i > 0 \quad i = 1,2,3$$
$$\dot{x}_2 = \frac{a}{x_1}\left(u_1(u_2 - x_2) + c(u_3 - x_2)\right) \qquad a,b,c > 0$$

2. $$\ddot{x} + c\dot{x} + g\left(1 - \left(\frac{x_d}{x}\right)^{\kappa}\right) = 0 \qquad c>0,\ g>0,\ \kappa>1$$

3. $$\dot{x} = \begin{cases} y - \dfrac{x}{\ln\sqrt{x^2+y^2}} & [x,y] \neq [0,0] \\ 0 & [x,y] = [0,0] \end{cases}$$

$$\dot{y} = \begin{cases} -x - \dfrac{y}{\ln\sqrt{x^2+y^2}} & [x,y] \neq [0,0] \\ 0 & [x,y] = [0,0] \end{cases}$$

a) 1. Already in state-space form

3. Already in state-space form

2. $x_1 = x$, $x_2 = \dot{x}$ $\rightarrow$ $$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = \ddot{x} = -c x_2 - g\left(1 - \left(\frac{x_d}{x_1}\right)^{\kappa}\right) \end{cases}$$

b) 1. $\bar{u} = [\bar{u}_1 \ \ \bar{u}_2 \ \ \bar{u}_3]^T = \text{const.}$; $\bar{x} = [\bar{x}_1 \ \ \bar{x}_2]^T$

$$0 = a\bar{u}_1 - b\sqrt{\bar{x}_1} \rightarrow \sqrt{\bar{x}_1} = \frac{a}{b}\bar{u}_1 \rightarrow \boxed{\bar{x}_1 = \left(\frac{a}{b}\right)^2 \bar{u}_1^{\,2}}$$

$$0 = \cancel{\frac{g}{\bar{x}_1}}\left(\bar{u}_1\bar{u}_2 - \bar{u}_1\bar{x}_2 + c\bar{u}_3 - c\bar{x}_2\right)$$

$$= \bar{u}_1\bar{u}_2 - (c+\bar{u}_1)\bar{x}_2 + c\bar{u}_3$$

$$\rightarrow \boxed{\bar{x}_2 = \frac{\bar{u}_1\bar{u}_2 + c\bar{u}_3}{c+\bar{u}_1}}$$

2. Equlibrium when $\ddot{x} = \dot{x} = 0$

$$0 + c0 + g\left(1 - \left(\frac{x_d}{\bar{x}}\right)^{\pi}\right) = 0$$

$$\left(\frac{x_d}{\bar{x}}\right)^{\pi} = 1 \rightarrow \boxed{\bar{x} = x_d}$$

3. Equlibrium when $[\dot{x}, \dot{y}] = [0, 0]$

When $[x, y] \neq [0, 0]$: $\quad x^2 + y^2 = r$

$$\dot{x} = y - \frac{x}{\ln\sqrt{x^2+y^2}} \qquad \dot{y} = -x - \frac{y}{\ln\sqrt{x^2+y^2}}$$

$$y\dot{x} = y^2 - \frac{xy}{\ln\sqrt{x^2+y^2}} \qquad x\dot{y} = -x^2 - \frac{xy}{\ln\sqrt{x^2+y^2}}$$

$$y\dot{x} - x\dot{y} = y^2 - \cancel{\frac{xy}{\ln\sqrt{r}}} - \left(-x^2 - \cancel{\frac{xy}{\ln\sqrt{r}}}\right)$$

$$= y^2 + x^2$$

$$[\dot{x}, \dot{y}] = [0, 0] \rightarrow \bar{x}^2 + \bar{y}^2 = 0$$

$$\Longrightarrow \boxed{[\bar{x}, \bar{y}] = [0, 0]}$$

c) Find A, B in $\dot{x} = Ax + Bu$

1. $$A = \frac{\partial f}{\partial x} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} -\frac{b}{2\sqrt{\bar{x}_1}} & 0 \\ -\frac{a}{\bar{x}_1^2}(\bar{u}_1(\bar{u}_2 - \bar{x}_2) + c(\bar{u}_3 - \bar{x}_2)) & -\frac{a}{\bar{x}_1}(c + \bar{u}_1) \end{bmatrix}$$

$$B = \frac{\partial f}{\partial u} = \begin{bmatrix} \frac{\partial f_1}{\partial u_1} & \frac{\partial f_1}{\partial u_2} & \frac{\partial f_1}{\partial u_3} \\ \frac{\partial f_2}{\partial u_1} & \frac{\partial f_2}{\partial u_2} & \frac{\partial f_2}{\partial u_3} \end{bmatrix}$$

$$= \begin{bmatrix} a & 0 & 0 \\ \frac{a}{\bar{x}_1}(\bar{u}_2 - \bar{x}_2) & \frac{a}{\bar{x}_1}\bar{u}_1 & \frac{ac}{\bar{x}_1} \end{bmatrix}$$

2. $$A = \frac{\partial f}{\partial x} = \begin{bmatrix} 0 & 1 \\ -g x_d^{\kappa} \kappa \bar{x}_1^{-\kappa-1} & -c \end{bmatrix}$$

$\bar{x}_1 = x_d$,

$-g \cancel{x_d^{\kappa}} \kappa \cancel{x_d^{-\kappa}} x_d^{-1}$

$= -\frac{g\kappa}{x_d}$

$$= \begin{bmatrix} 0 & 1 \\ -\frac{g\kappa}{x_d} & -c \end{bmatrix}, \quad B = 0$$

↳ homogeneous system.

3. $\frac{\partial f_1}{\partial x} = \frac{\partial}{\partial x}\left(-2\frac{x}{\ln(x^2+y^2)}\right) = -2\frac{\ln(x^2+y^2) - x\frac{2x}{x^2+y^2}}{\ln^2(x^2+y^2)}$

$= -2\frac{\ln(x^2+y^2) - 2\frac{x^2}{x^2+y^2}}{\ln^2(x^2+y^2)} = -2\left(\frac{1}{\ln(x^2+y^2)} - \frac{2}{\ln^2(x^2+y^2)}\frac{x^2}{x^2+y^2}\right)$

$\xrightarrow{(x,y)\to(0,0)} \underline{0}$

$\frac{\partial f_1}{\partial y} = 1 - \frac{\partial}{\partial y}\left(\frac{2x}{\ln(x^2+y^2)}\right) = 1 - 2x\cdot\frac{2y}{x^2+y^2} \xrightarrow{(x,y)\to(0,0)} \underline{1}$

$\frac{\partial f_2}{\partial x} = \underline{-1}$, $\frac{\partial f_2}{\partial y} = \underline{0}$ (by the same logic as above).

$\Rightarrow \frac{\partial f}{\partial x} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = A$, $\underline{\underline{B = 0}}$

d) 3. $\lambda = \pm i \rightarrow$ marginally stable

1. Lower triangular, so $\lambda_1 = -\frac{b}{2\sqrt{\bar{x}_1}} < 0$, $\lambda_2 = -\frac{a}{\bar{x}_1}(c + \bar{u}_1) < 0$

$\rightarrow$ Both poles in LHP $\rightarrow$ asymptotically stable

2. $\lambda(\lambda + c) + \frac{gK}{xd} = \lambda^2 + \lambda c + \frac{gK}{xd} = 0$

$\lambda = \frac{-c \pm \sqrt{c^2 - 4\frac{gK}{xd}}}{2}$, $\frac{gK}{xd} > 0 \rightarrow \sqrt{c^2 - 4\frac{gK}{xd}} < c^2$

So, $Re\{\lambda\} < 0$ and the system is asymptotically stable.

Problem 2

a) For $\dot{x} = x^2$, $x(0) = 1$

ode45 fails at $t=1$ and the output explodes to infinity.

For $\dot{x} = \sqrt{|x|}$, $x(0) = 0$

ode45 gives a flat response, which does make sense since $\dot{x}(0) = 0$ and no change will occur.

b) We know from the lectures that the solution to $\dot{x} = x^2$, $x(0) = 1$ is

$$x(t) = \frac{1}{1-t}$$

which does explode at $t=1$.

For $\dot{x} = \sqrt{|x|}$, $x(0) = 0$, we know there exists infinitely many solution. In Matlabs case, the solution ended up with "choosing" $t_0 > 5$.

Problem 3

H: healthy people
I: infected, but not zombie
Z: zombies
D: dead (people and zombies)

Modeled by Lokta-Volterra equations.

a) Rules in simulation:

- healthy people birth rate: $\dot{H} = bH - b_d{}^2 H$
- healthy people natural death / interaction with zombies: $\dot{H} = -dH - iZH$, $\dot{D} = dH$
- healthy people becoming infected: $\dot{I} = iZH$
- infected slowly becoming zombie: $\dot{I} = -aI$, $\dot{Z} = aI$
- infected dying of natural causes: $\dot{I} = -dI$, $\dot{D} = dI$
- zombie rise from dead: $\dot{Z} = rD$, $\dot{D} = -rD$
- zombies neutralized: $\dot{Z} = -nHZ$, $\dot{D} = nHZ$

Due to the principle of superposition, we can easily combine the partial equations above to get the dynamics:

$$\begin{cases} \dot{H} = bH - b_d H^2 - dH - iZH \\ \dot{I} = iZH - aI - dI = iZH - (a+d)I \\ \dot{Z} = rD + aI - nHZ \\ \dot{D} = dH + nHZ + dI - rD = (d+n)H + dI - rD \end{cases}$$

b) Q: quarantined people

Additional rules:

- infected and zombies becoming quarantined:

  $\dot{Q} = q_i I + q_z Z$, $\dot{I} = -q_i I$, $\dot{Z} = -q_z Z$

- escapees getting killed: $\dot{Q} = -d_q Q$, $\dot{D} = d_q Q$

New complete model:

$$\begin{cases} \dot{H} = bH - b_d H^2 - dH - iZH = (b-d)H - b_d H^2 - iZH \\ \dot{I} = iZH - aI - dI - q_i I = iZH - (a+d+q_i)I \\ \dot{Z} = rD + aI - nHZ - q_z Z \\ \dot{D} = dH + nHZ + dI - rD + d_q Q \\ \dot{Q} = q_i I + q_z Z - d_q Q \end{cases}$$

c) In both cases, $H \to 0$. However, with quarantine, the H population survives for longer ($t=3$ w/o, $t=4,2$ w/).

Furthermore, without quarantine, the Z population dominates at the end. With quarantine, the number of zombies are contained and more people just end up dead.