Out: March 16, 2020
Due: April 19, 2020

NTNU – Trondheim
Norwegian University of
Science and Technology

TTK4130 Modeling and Simulation

Assignment 10

Introduction

Balance laws are used to describe the change in mass, movement and energy of substances (solids, liquids or gasses), as well as related quantities such as volume, concentration and temperature.

By using balance laws, one can for example model the level of a liquid in a tank, the velocity of a fluid in a pipe, the concentrations of substances in a chemical reaction, the temperature of substances in contact, among many other processes.

Because of these many options, balance laws have applications in many sectors from the chemical and process industries to geophysics and biology.

Chapter 11 in the book treats balance laws. The discussion given here is very general, and correctly models the quantities of interest as functions of both time and space. This general approach leads to partial differential equations (PDEs), whose systematic study and simulation are outside of the scope of this course.

In this course, we will generally assume that the quantities of interest are constant over a volume at a given time. This approach will lead to an ODE or DAE, as we will see in this assignment.

Common assumptions are for example to assume that the temperature of a liquid in a tank is the same overall, or that the velocity of a liquid is the same overall. Note that assuming that some quantities are constant over a volume cannot be reasonable assumptions. An example of this is the pressure in a tank.

It will also be advantageous to read Section 4.2. about valve models.

Problem 1 (Tank System, DAE)

Let us consider the cascade of three tanks shown in Figure 1. The cross-sections of the tanks are A_1, A_2, and A_3, respectively. The valves are denoted by R_1, R_2. The mass inflow ω_0 and its derivatives are known inputs. The mass flow between the tanks and the outflow of tank 3 are given by ω_1, ω_2 and ω_3, respectively. Moreover, we assume that the liquid density ρ is a known constant.

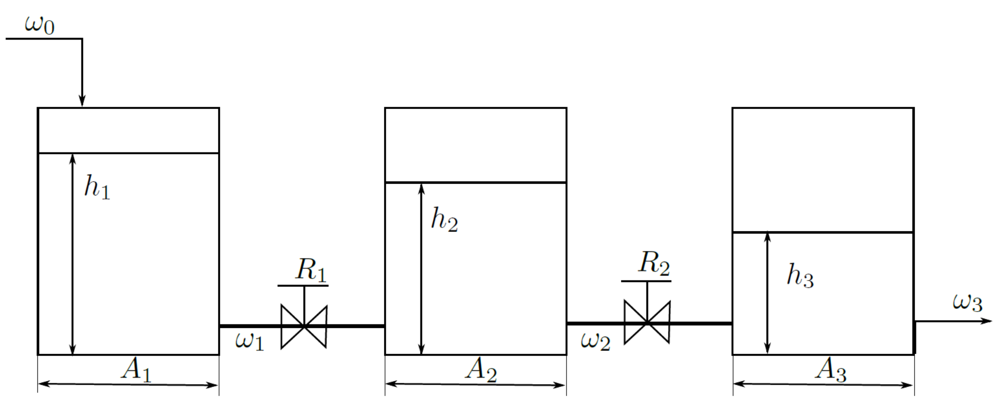

Figure 1: Mixing cascade of three tanks.

(a) Show that the mass of the fluid in tank i is given by $m_i = \rho A_i h_i$, where h_i is the level of tank i. Finally, apply a mass balance on each fluid mass m_i to show that

$$\frac{dh_i}{dt} = \frac{1}{\rho A_i}\left(\omega_{i-1} - \omega_i\right), \quad i = 1, 2, 3. \tag{1}$$

Solution: $m_i = \rho A_i h_i$ because density is constant and defined as mass per volume. Since ρ and A_i are constant, the mass balance for each tank gives

$$\begin{aligned}\frac{dm_i}{dt} &= \omega_{i-1} - \omega_i \\ \rho A_i \frac{dh_i}{dt} &= \omega_{i-1} - \omega_i\,, \quad i = 1, 2, 3.\end{aligned}$$

Let us first assume that the valve equations and the outflow are given by

$$g_1(\omega_1, h_1, h_2) = 0 \tag{2a}$$

$$g_2(\omega_2, h_2, h_3) = 0 \tag{2b}$$

$$g_3(\omega_3, h_3) = 0 \tag{2c}$$

with

$$\frac{\partial g_i}{\partial \omega_i} \neq 0. \tag{3}$$

(b) Use the results from part a. to derive the DAE that models both the fluid heights, $h_{1,2,3}$, and the mass flows $\omega_{1,2,3}$.
What is the differential index of this DAE?

Solution: By putting together the differential equations for the tank levels and the valve equations, we get the following DAE:

$$\begin{aligned}\frac{dh_1}{dt} &= \frac{1}{\rho A_1}(\omega_0 - \omega_1) \\ \frac{dh_2}{dt} &= \frac{1}{\rho A_2}(\omega_1 - \omega_2) \\ \frac{dh_3}{dt} &= \frac{1}{\rho A_3}(\omega_2 - \omega_3) \\ 0 &= g_1(\omega_1, h_1, h_2) \\ 0 &= g_2(\omega_2, h_2, h_3) \\ 0 &= g_3(\omega_3, h_3).\end{aligned}$$

The differential variables are

$$\boldsymbol{x} = [h_1, h_2, h_3]^T\,,$$

and the algebraic variables are

$$\boldsymbol{y} = [\omega_1, \omega_2, \omega_3]^T.$$

The algebraic equation $\boldsymbol{g}(\boldsymbol{x}, \boldsymbol{y}) = 0$ is

$$\begin{aligned}0 &= g_1(\omega_1, h_1, h_2) \\ 0 &= g_2(\omega_2, h_2, h_3) \\ 0 &= g_3(\omega_3, h_3).\end{aligned}$$

The Jacobian of $\boldsymbol{g}(\boldsymbol{x}, \boldsymbol{y})$ with respect to the algebraic variables, $\boldsymbol{y}$, is

$$\frac{\partial \boldsymbol{g}(\boldsymbol{x}, \boldsymbol{y})}{\partial \boldsymbol{y}} = \begin{bmatrix} \frac{\partial g_1}{\partial \omega_1} & 0 & 0 \\ 0 & \frac{\partial g_2}{\partial \omega_2} & 0 \\ 0 & 0 & \frac{\partial g_3}{\partial \omega_3} \end{bmatrix},$$

which is invertible. Therefore, the differential index is 1.

Let us now assume that the tank cross-sections are identical, i.e. $A_i = A$ for $i = 1, 2, 3$, and that the valves equations are given by

$$h_1 = h_2 \tag{4a}$$
$$h_2 = h_3 \tag{4b}$$
$$\omega_3 = A\rho\sqrt{2gh_3}, \tag{4c}$$

where g is the acceleration of gravity.

(c) Derive the new DAE that models the fluid heights and mass flows, and find its differential index.

Solution:

The new DAE-system is given by

$$\begin{aligned} \frac{dh_1}{dt} &= \frac{1}{\rho A}(\omega_0 - \omega_1) \\ \frac{dh_2}{dt} &= \frac{1}{\rho A}(\omega_1 - \omega_2) \\ \frac{dh_3}{dt} &= \frac{1}{\rho A}(\omega_2 - \omega_3) \\ 0 &= h_1 - h_2, \\ 0 &= h_2 - h_3, \\ 0 &= \omega_3 - A\rho\sqrt{2gh_3}. \end{aligned}$$

The first differentiation of the algebraic equations results in

$$\begin{aligned} 0 &= \frac{dh_1}{dt} - \frac{dh_2}{dt} \\ 0 &= \frac{dh_2}{dt} - \frac{dh_3}{dt} \\ 0 &= \frac{d\omega_3}{dt} - A\rho\sqrt{\frac{g}{2h_3}}\frac{dh_3}{dt}. \end{aligned}$$

By using the differential equations of the DAE, we get

$$\begin{aligned} 0 &= \frac{\omega_0 - \omega_1}{\rho A} - \frac{\omega_1 - \omega_2}{\rho A} \\ 0 &= \frac{\omega_1 - \omega_2}{\rho A} - \frac{\omega_2 - \omega_3}{\rho A} \\ 0 &= \frac{d\omega_3}{dt} - \sqrt{\frac{g}{2h_3}}(\omega_2 - \omega_3). \end{aligned}$$

We now have a differential equation for ω_3, but we still need differential equations for ω_1 and ω_2. Before we differentiate one more time, we solve the 2 algebraic equations above for ω_1 and ω_2. This gives

$$\omega_1 = \frac{2\omega_0 + \omega_3}{3}$$
$$\omega_2 = \frac{\omega_0 + 2\omega_3}{3}.$$

The second differentiation will transfer the system into a ODE. Therefore, the differential index is two.

This second differentiation gives the following differential equations for ω_1 and ω_2

$$\frac{d\omega_1}{dt} = \frac{2}{3}\frac{d\omega_0}{dt} + \frac{1}{9}\sqrt{\frac{g}{2h_3}}(\omega_0 - \omega_3)$$
$$\frac{d\omega_2}{dt} = \frac{1}{3}\frac{d\omega_0}{dt} + \frac{1}{9}\sqrt{\frac{2g}{h_3}}(\omega_0 - \omega_3).$$

However, one does not need to calculate these differential equations in order to find the differential index.

Problem 2 (Wind turbine in open wind channel)

Let us consider a wind turbine of diameter d placed in an open cylindrical wind channel of diameter D, as depicted in Figure 2.

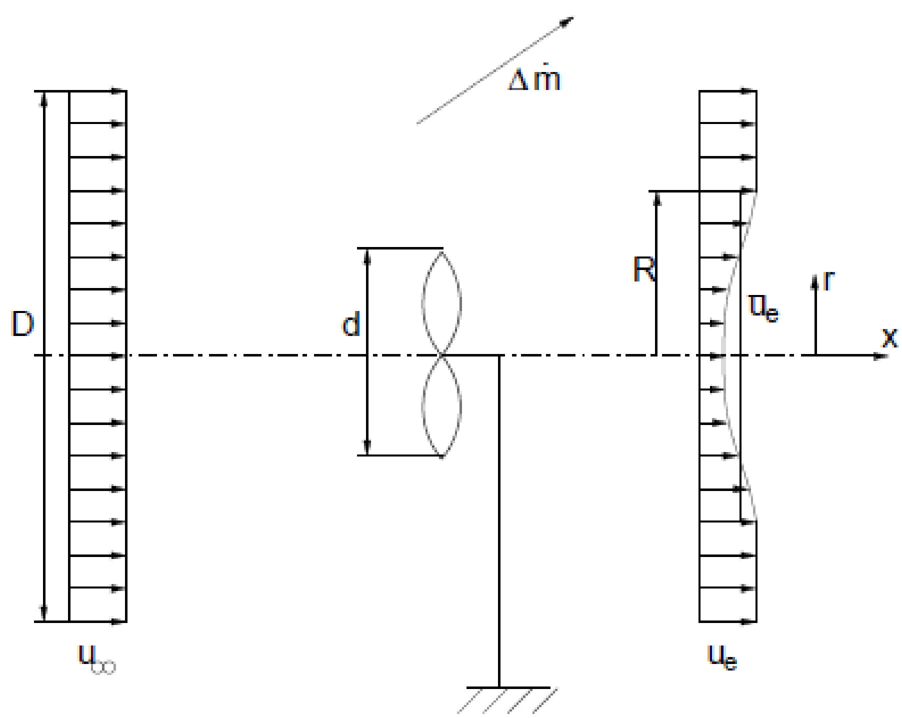

Figure 2: Longitudinal section of the open cylindrical wind channel with wind turbine.

The upstream wind velocity that enters the wind channel, u_∞, is assumed constant. This gives a downstream wind that exits the channel, u_e, which is measured to be

$$u_e(r,\phi) = \begin{cases} \bar{u}_e, & 0 \leq r \leq R \\ u_\infty, & R < r \leq D/2. \end{cases} \tag{5}$$

Note that the downstream wind is defined in polar coordinates.

Since the wind channel is open, there is a displaced mass flow of air, $\Delta\dot{m}$ [$\mathrm{kg\,s^{-1}}$], that exists the channel opening.

Assume that the air flow has a constant density ρ, that the wind channel is at steady-state, and that friction forces can be neglected. Moreover, $d \ll D$ and $d/2 < R < D/2$.

In this problem, the chosen control volume will be the cylinder with the same axis and length as the wind channel, but with a radius of R instead of a radius of $\frac{D}{2}$.

(a) Use the mass balance

$$\frac{d}{dt}\iiint_{V_c} \rho dV = -\iint_{\partial V_c} \rho\vec{v}\cdot\vec{n} dA \tag{6}$$

and the consequence of the steady-state assumption

$$0 = \frac{d}{dt}\iiint_{V_c} \rho dV \tag{7}$$

to show that the displaced mass flow $\Delta\dot{m}$ is

$$\Delta\dot{m} = \rho\pi R^2 \left(u_\infty - \bar{u}_e\right). \tag{8}$$

Solution: The control volume used in this problem, is shown in Figure 3.

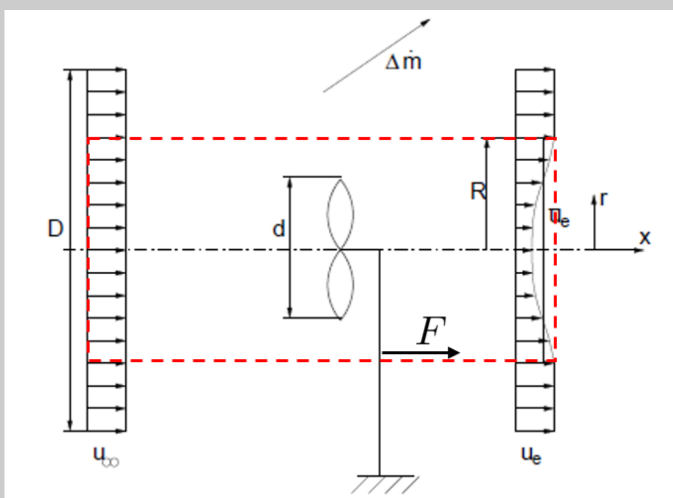

Figure 3: The control volume.

The mass balance and the steady-state assumption give

$$\begin{aligned} \frac{d}{dt}\iiint_{V_c} \rho dV &= -\iint_{\partial V_c} \rho\vec{v}\cdot\vec{n} dA \\ 0 &= \rho u_\infty \pi R^2 - \rho\bar{u}_e\pi R^2 - \Delta\dot{m}. \end{aligned}$$

Hence,

$$\Delta \dot{m} = \rho \pi R^2 \left(u_\infty - \bar{u}_e \right).$$

(b) There are infinite suitable control volumes for this problem.
Why is the chosen control volume better than a cylinder with the same axis and length, but larger radius?
Moreover, why does the steady-state assumption imply (7)?

Solution: A cylinder with a larger radius will also work. However, since the upstream and downstream wind velocities are equal for $r > R$, their contributions to the surface integrals in this region will just cancel each other out. In other words, the use of a larger cylinder as the control volume will just imply more calculations.

The steady-state assumption implies that the mass in the wind-channel does not change with time. This is exactly what (7) expresses.

(c) Use the momentum balance

$$\frac{d}{dt} \iiint_{V_c} \rho \vec{v} dV = - \iint_{\partial V_c} \rho \vec{v} \vec{v} \cdot \vec{n} dA - \begin{bmatrix} F \\ 0 \\ 0 \end{bmatrix} \tag{9}$$

and the consequence of the steady-state assumption

$$0 = \frac{d}{dt} \iiint_{V_c} \rho \vec{v} dV \tag{10}$$

to show that the force F that acts on the wind turbine is

$$F = \rho \pi R^2 (u_\infty - \bar{u}_e) \bar{u}_e. \tag{11}$$

Solution: The momentum balance and the consequence of the steady-state are vector equations. In our case, we are only interested in what happens with the first coordinate of these equations. Therefore, we multiply them with $\boldsymbol{e}_1$, and obtain:

$$\frac{d}{dt} \iiint_{V_c} \rho \vec{v} dV = - \iint_{\partial V_c} \rho \vec{v} \vec{v} \cdot \vec{n} dA - \begin{bmatrix} F \\ 0 \\ 0 \end{bmatrix}$$
$$\Rightarrow \quad 0 = \rho u_\infty^2 \pi R^2 - \rho u_c^2 \pi R^2 - \Delta \dot{m} u_\infty - F,$$

where the contribution to the surface integral over the sides of the cylindrical control volume, S_c, has been calculated using that the x coordinate of the velocity is u_∞ and that

$$\iint_{S_c} \rho \vec{v} \cdot \vec{n} dA = \Delta \dot{m}$$

as proved in part a. This gives

$$\left(\iint_{S_c} \rho \vec{v} \vec{v} \cdot \vec{n} dA \right) \cdot \boldsymbol{e}_1 = \iint_{S_c} \rho \vec{v} \cdot \boldsymbol{e}_1 \vec{v} \cdot \vec{n}$$
$$= u_\infty \iint_{S_c} \rho \vec{v} \cdot \vec{n} dA = \Delta \dot{m} u_\infty.$$

Hence,

$$F = \rho \pi R^2 (u_\infty - \bar{u}_e) \bar{u}_e.$$

Problem 3 (Countercurrent heat exchange, transport equation)

Countercurrent heat exchange is a mechanism for saving energy, where fluids circulate in parallel pipes in contact with each other, but in opposite directions. By doing so, if one fluid travels from a cold to a warm region, and the other fluid does conversely, then the energy loss due to dissipation is minimized.

Countercurrent heat exchange occurs in nature, and has been mimicked in engineering. For example, heat recovery ventilators, which are found in most modern houses, are based on this mechanism.

We will now model the countercurrent heat exchange between to coaxial cylinders as shown in Figure 4.

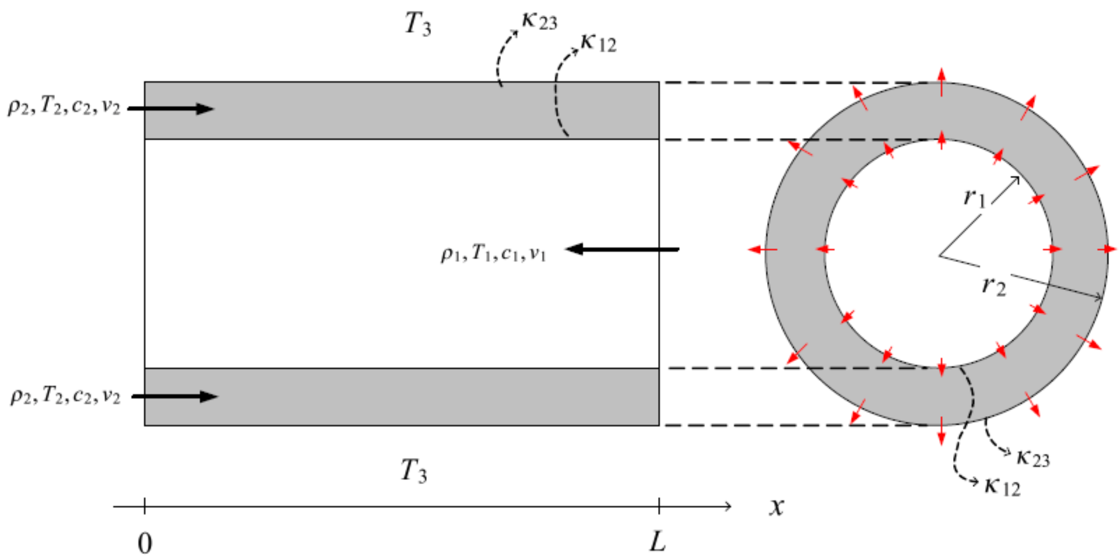

Figure 4: Countercurrent heat exchange between to coaxial cylinders.

The system consists of an inner volume and an outer volume. Liquids flow into both volumes. The variables specified in Figure 4 have the following meaning:

Internal volume:

- ρ_1: Mass density [kg/m^3].
- v_1: The absolute value of velocity [m s^{-1}].
- T_1: Temperature [K]
- c_1: Specific heat capacity [J/(kgK)], i.e., $u_1 = c_1 T_1$.
- r_1: Radius of inner pipe [m]

Outer volume:

- ρ_2: Mass density [kg/m^3].
- v_2: The absolute value of velocity [m s^{-1}].
- T_2: Temperature [K]
- c_2: Specific heat capacity [J/(kgK)], i.e., $u_2 = c_2 T_2$.
- r_2: Radius of outer pipe [m]

Environment:

- T_3: Temperature of environment [K]
- L: Length of pipe [m]
- κ_{12}: Heat transfer coefficient between the inner and outer volumes [W/(m^{2}K)]
- κ_{23}: Heat transfer between the external volume and the surroundings [W/(m^{2}K)]

Assumptions:

- $T_1 > T_2 > T_3$.
- T_1 and T_2 are constant radially, i.e., T_1 and T_2 are only functions of t and x.
- c_1, c_2, v_1, v_2, and the pressure in the inner and outer volumes are constant.
- Conduction occurs radially in the pipes (depicted by red arrows in figure).
- Potential and kinetic energy in the system can be neglected.

First, we will derive the partial differential equation (PDE) for the temperature in the inner pipe, T_1.

This will be done by considering an arbitrary small section of the cylindrical inner pipe, i.e. we will not consider the full length of the inner cylinder from 0 to L, but rather an arbitrary section with length from x to $x + \Delta x$, but still of radius r_1.

Moreover, we will use the general energy balance

$$\frac{d}{dt} \iiint_{V_c} \rho \left(u + \frac{1}{2}|v|^2 + \phi \right) dV = - \iint_{\partial V_c} \rho \left(u + \frac{1}{2}|v|^2 + \phi \right) \vec{v} \cdot \vec{n} dA - \iint_{\partial V_c} \vec{j_Q} \cdot \vec{n} dA. \tag{12}$$

(a) Explain why the energy balance (12) can be reduced to

$$\frac{d}{dt} \iiint_{V_c} \rho u \, dV = - \iint_{\partial V_c} \rho u \vec{v} \cdot \vec{n} dA - \iint_{\partial V_c} \vec{j_Q} \cdot \vec{n} dA \tag{13}$$

in this case.

Solution: Because the assumptions say that the potential and kinetic energies can be neglected.

(b) Show that the left-hand side of (13) for $u = u_1$ is equal to

$$\rho_1 c_1 \pi r_1^2 \int_x^{x+\Delta x} \frac{\partial T_1}{\partial t}(x,t) \, dx. \tag{14}$$

Solution: Since c_1 is constant and T_1 is constant radially, we have that

$$\begin{aligned}\frac{d}{dt} \iiint_{V_c} \rho u_1 \, dV &= \frac{d}{dt} \iiint_{V_c} \rho c_1 T_1 \, dV \\ &= \frac{d}{dt} \left(\rho c_1 \pi r_1^2 \int_x^{x+\Delta x} T_1(x,t) \, dx \right) = \rho_1 c_1 \pi r_1^2 \int_x^{x+\Delta x} \frac{\partial T_1}{\partial t}(x,t) \, dx.\end{aligned}$$

The right-hand side of (13) accounts for several phenomena that vary the energy contained in the control volume. The first term accounts for both the energy increase due to the fluid that arrives into the control volume from the right and the energy decrease due to the fluid that leaves the control volume from the left, while the second term describes the energy decrease due to the heat transfer between the pipes.

(c) Show that the energy increase due to the fluid that arrives into the control volume from the right and the energy decrease due to the fluid that leaves the control volume from the left are respectively

$$\rho_1 c_1 v_1 \pi r_1^2 T_1(x + \Delta x, t) \tag{15a}$$
$$-\rho_1 c_1 v_1 \pi r_1^2 T_1(x, t). \tag{15b}$$

Solution: The energy increase is given by the surface integral of $\rho u_1 v_1$ over the disc that constitutes the right end of the control volume. This integral is equal to

$$\begin{aligned}\iint_{D_c} \rho u_1 v_1 \, dA &= \iint_{D_c} \rho c_1 T_1 v_1 \, dA \\ &= \rho_1 c_1 v_1 \pi r_1^2 T_1(x + \Delta x, t)\end{aligned}$$

since the disc is located at the length $x + \Delta x$ and v_1 is constant.

The energy decrease is solved analogously. In this case, the direction of the fluid flow is outwards with respect to the control volume, and the corresponding disc is located at the length x.

(d) Show that the energy decrease due to the heat transfer between the pipes is equal to

$$-2\pi r_1 \kappa_{12} \int_x^{x+\Delta x} T_1(x, t) - T_2(x, t) \, dx. \tag{16}$$

Hint: What is the power loss per squared meter? What is its direction with respect to the normal vector of the contact surface?

Solution: The heat transfer occurs along the sides of the cylindrical control volume, S_c. Since the power loss per squared meter is

$$\kappa_{12}(T_1 - T_2)$$

and the heat conduction occurs radially, we conclude that the energy decrease is given by

$$\begin{aligned}\iint_{S_c} -\kappa_{12}(T_1 - T_2)\vec{n} \cdot \vec{n} dA &= \iint_{S_c} -\kappa_{12}(T_1 - T_2) \, dA \\ &= -2\pi r_1 \kappa_{12} \int_x^{x+\Delta x} T_1(x, t) - T_2(x, t) \, dx \, ,\end{aligned}$$

where we have used that the temperatures are constant along each cross-section of S_c, which have perimeter $2\pi r_1$.

The results of the previous parts give the following integral equation:

$$\begin{aligned}\rho_1 c_1 \pi r_1^2 \int_x^{x+\Delta x} \frac{\partial T_1}{\partial t}(x, t) \, dx = \rho_1 c_1 v_1 \pi r_1^2 \left(T_1(x + \Delta x, t) - T_1(x, t)\right) \\ - 2\pi r_1 \kappa_{12} \int_x^{x+\Delta x} T_1(x, t) - T_2(x, t) \, dx.\end{aligned} \tag{17}$$

(e) Use (17) to derive the partial differential equation for $T_1(x, t)$:

$$\frac{\partial T_1}{\partial t} - v_1 \frac{\partial T_1}{\partial x} = -\frac{2\kappa_{12}}{\rho_1 c_1 r_1}(T_1 - T_2). \tag{18}$$

Hint: Take $\Delta x \to 0$, and use the definition of derivative and L'Hôpital's rule.

Solution: The partial differential equation follows from diving (17) by $\rho_1 c_1 \pi r_1^2 \Delta x$, and taking the limit $\Delta x \to 0$.

(f) **(Optional)** Derive the partial differential equation for $T_2(x,t)$:

$$\frac{\partial T_2}{\partial t} + v_2 \frac{\partial T_2}{\partial x} = -\frac{2 r_1 \kappa_{12}}{\rho_2 c_2 (r_2^2 - r_1^2)} (T_2 - T_1) - \frac{2 r_2 \kappa_{23}}{\rho_2 c_2 (r_2^2 - r_1^2)} (T_2 - T_3). \tag{19}$$

Solution: The derivation of this PDE is analogous to (18). In this case, the control volume is an arbitrary small section of the difference of the outer and inner pipe. Therefore, the area and perimeter of the cross-sections of the control volume are different. In addition, we have to account for heat transfer from the inner pipe and heat transfer to the environment.