Out: March 9, 2020
Due: April 5, 2020

NTNU – Trondheim
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Science and Technology

TTK4130 Modeling and Simulation

Assignment 9

Introduction

This assignment represents the culmination of the work started in assignment 7, and has as goals:

- To implement and test an arbitrary IRK scheme.
- To study the phenomenon of numerical energy loss.
- To solve numerically implicit DAEs.

Problem 1 (IRK schemes)

(a) Code a Matlab function that implements an arbitrary IRK method based on its Butcher table.
Explain what implicit equations have to be solved at each iteration.
Add the implemented code to your answer.
Hints:

- *A template for such a function can be found on Blackboard.*
- *Use Newton's method to solve the implicit equations. You may want to modify your code from assignment 8 so that it only returns the final root estimate.*
- *It is arguably best to declare the vectors* $\boldsymbol{k}_1, \cdots, \boldsymbol{k}_s$ *as a matrix* $K = [\boldsymbol{k}_1 \cdots \boldsymbol{k}_s]$. *However, when solving the implicit equations, it is probably more advantageous to concatenate this vectors as the column vector* $\boldsymbol{k} = [\boldsymbol{k}_1^\top \cdots \boldsymbol{k}_s^\top]^\top$. *In any case, one can go back and forth these representations using the Matlab command* `reshape`.

(b) The Gauss-Legendre collocation method is an IRK scheme with $s = 2$ stages and of order $2s = 4$, which has the Butcher table :

Table 1: Gauss-Legendre Collocation (IRK4)

$\frac{1}{2} - \frac{\sqrt{3}}{6}$	$\frac{1}{4}$	$\frac{1}{4} - \frac{\sqrt{3}}{6}$
$\frac{1}{2} + \frac{\sqrt{3}}{6}$	$\frac{1}{4} + \frac{\sqrt{3}}{6}$	$\frac{1}{4}$
	$\frac{1}{2}$	$\frac{1}{2}$

Test the code developed in part a. by implementing the IRK4.
Simulate the test system

$$\dot{x} = \lambda x \tag{1}$$

with $\lambda = -2$ and initial condition $x(0) = 1$, using the IRK4 and the RK4 method (assignment 7).
Simulate for a final time $t_f = 2$ and a time-step $\Delta t_k = 0.4$.
Add a plot of the results to your answer, and comment on them.

(c) For what value of $\lambda < 0$ will the IRK4 scheme become unstable?

Problem 2 (Numerical energy loss)

Consider the pneumatic spring without damping:

$$\ddot{x} + g\left(1 - \left(\frac{x_d}{x}\right)^{\kappa}\right) = 0\,, \tag{2}$$

where $x, x_d, g > 0 = 1$ and $\kappa \geq 1$.

Since there is no damping, the physical solution will oscillate around its equilibrium position $x = x_d$.

Moreover, the energy for the system (2) is given by

$$E = \frac{mg}{\kappa - 1}\frac{x_d^{\kappa}}{x^{\kappa-1}} + mgx + \frac{1}{2}m\dot{x}^2\,, \tag{3}$$

where m is the mass.

(a) Use (2) to show that the energy for the physical solution is constant over time, i.e., $\dot{E}(t) = 0$.

(b) Simulate the pneumatic spring without damping (2) using the explicit Euler method, the implicit Euler method and the implicit midpoint rule (Gauss method of order 2).
Use the initial condition $[x(0),\, \dot{x}(0)] = [2\,\mathrm{m},\, 0\,\mathrm{m\,s^{-1}}]$, a step-size of $\Delta t_k = 0.01\,\mathrm{s}$ and a final time $t_f = 10\,\mathrm{s}$ for each simulation.
Moreover, use the model parameter values $x_d = 1.32\,\mathrm{m}$, $\kappa = 2.40$, $g = 9.81\,\mathrm{m\,s^{-2}}$ and $m = 200\,\mathrm{kg}$.
Add a plot with the simulated positions, x, as well as a plot with the corresponding energies, E, to your answer.
Comment on the results, and give a formal explanation.
Hint: Keywords here are stability region, A- and L-stability.

Problem 3 (DAE integration)

In this task, we will use the IRK4 integrator from problem 1 to solve 2 fully-implicit DAEs of the form

$$\boldsymbol{F}(\dot{\boldsymbol{x}}, \boldsymbol{x}, \boldsymbol{z}, t) = 0 \tag{4}$$

Note that for the sake of simplicity we have assumed that the dynamics have no input.

In order to simulate the DAEs, the function `RKDAE.m` has been delivered. This function integrates a DAE of the form (4) using the selected RK method.

We recommend you to use this solver. However, if you prefer, you are free to implement your own DAE integrator.

(a) Test your code on the 3D pendulum with the constraint

$$C(\boldsymbol{q}) = \frac{1}{2}\left(\boldsymbol{p}^{\top}\boldsymbol{p} - L^2\right) = 0\,, \tag{5}$$

where $L = 1$. In this case, the index-reduced model equations read as

$$\dot{\boldsymbol{p}} = \boldsymbol{v} \tag{6a}$$

$$m\dot{\boldsymbol{v}} = -mg\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} - z\boldsymbol{p} \tag{6b}$$

$$0 = \boldsymbol{p}^{\top}\dot{\boldsymbol{v}} + \boldsymbol{v}^{\top}\boldsymbol{v} \tag{6c}$$

Put the dynamics in a state-space form first, using e.g.

$$\boldsymbol{x} = \begin{bmatrix} \boldsymbol{p} \\ \boldsymbol{v} \end{bmatrix}, \tag{7}$$

and simulate the DAE using the IRK4. Try e.g. the consistent initial conditions:

$$\boldsymbol{x}(0) = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \in \mathbb{R}^6, \tag{8}$$

Imporve:
- different IRK?
- variable step length?

and e.g. the final time $t_f = 30$. Try different time-steps Δt.
Plot the simulated states and the corresponding contraint values (5). What do you observe? Explain the reason behind this. What would you do to improve the results?

(b) Test your code on the 3D pendulum model one obtains directly from the Lagrange approach, i.e.

$$\begin{aligned} \dot{\boldsymbol{p}} &= \boldsymbol{v} \\ m\dot{\boldsymbol{v}} &= -mg \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} - z\boldsymbol{p} \\ 0 &= \frac{1}{2}\left(\boldsymbol{p}^\top \boldsymbol{p} - L^2\right). \end{aligned}$$

What happens when you try to simulate? Why is that?

Singular to working precision.
This is because a value for z can't be found since the DAE is of higher order.