TTK4130 Assignment 7 Eirik Falch

Problem 1

$$\dot{x} = f(x, u, t)$$

$$k_1 = f(x(t_k), u(t_k), t_k)$$

$$k_2 = f(x(t_k) + a\Delta t \cdot k_1, u(t_k + c\Delta t), t_k + c\Delta t)$$

$$x_{k+1} = x(t_k) + \Delta t \sum_{j=1}^{2} b_j k_j$$

$$\begin{array}{c|cc} 0 & & \\ c & a & \\ \hline & b_1 & b_2 \end{array}$$

a) $e_k = x_{k+1} - x(t_{k+1})$

$$x(t_{k+1}) = x(t_k) + \Delta t \dot{x}(t_k) + \frac{\Delta t^2}{2} \ddot{x}(t_k) + O(\Delta t^3)$$

$$= x(t_k) + \Delta t f(x(t_k), u_k) + \frac{\Delta t^2}{2} \dot{f}(x(t_k), u_k) + O(\Delta t^3)$$

$$= x(t_k) + \Delta t f(x(t_k), u_k) + \frac{\Delta t^2}{2}\left(\frac{\partial f}{\partial x}\dot{x}\right) + O(\Delta t^3)$$

$$x_{k+1} = x(t_k) + \Delta t \cdot b_1 k_1 + \Delta t b_2 k_2$$

$$= x(t_k) + \Delta t b_1 f(x(t_k), u_k) + \Delta t b_2 f(x(t_k) + a\Delta t f(x(t_k), u_k), u_k) = (*)$$

$$f(x(t_k) + a\Delta t f(x(t_k), u_k), u_k)$$
$$= f(x(t_k), u_k) + a\Delta t \left.\frac{\partial f}{\partial x}\right|_{x(t_k)} f(x(t_k), u_k) + O(\Delta t^2)$$

$$(*) = x(t_k) + b_1 \Delta t\, f(x(t_k), u_k)$$
$$+ b_2 \Delta t \left[f(x(t_k), u_k) + a \Delta t \frac{\partial f}{\partial x} \dot{x} + O(\Delta t^2) \right]$$

$$= x(t_k) + (b_1 + b_2) \Delta t\, f(x(t_k), u_k)$$
$$+ a b_2 \Delta t^2 \frac{\partial f}{\partial x} \dot{x} + b_2 O(\Delta t^3)$$

Conditions

$0 \leq c \leq 1$ $\quad b_1 + b_2 = 1 \quad ab_2 = \frac{1}{2}$

b) The global error reduces the power of the one-step error to $O(\Delta t^2)$ which corresponds to the order of the RK2 method.

c) $x(t) = x(t_k) + \sum_{i=1}^{n} \frac{\alpha_i}{i!} (t - t_k)^i \qquad t \in [t_k, t_{k+1}] \qquad (4)$

$$= x(t_k) + \alpha_1 (t - t_k) + \frac{\alpha_2}{2} (t - t_k)^2 + \sum_{i=3}^{n} \frac{\alpha_i}{i!} (t - t_k)^i$$

$$x(t_{k+1}) = x(t_k) + \sum_{i=1}^{n} \frac{\alpha_i}{i!} (t_{k+1} - t_k)^i$$
$$= x(t_k) + \sum_{i=1}^{n} \frac{\alpha_i}{i!} \Delta t^i$$
$$n=2 \rightarrow = x(t_k) + \alpha_1 \Delta t + \alpha_2 \frac{\Delta t^2}{2}$$

$\underline{\underline{n=2}} \rightarrow x$ is quadratic and $f(x) = \dot{x}$ is linear, as long as the conditions from a) is met, the RK2 is exact.

Problem 2

a) We can see a clear difference between the true solution and the simulations. The simulations are clearly discrete and are built up of line segments.

As expected, RK4 is closest to the true solution. RK2 is closer throughout than RK1, but there's a larger gap at the end. This might just be a coincident, though.

b) Theoretical:

RK1: $\|x_N - x(T)\| \leq c\,O(\Delta t) \rightarrow$ order $= 1$

RK2: $\|x_N - x(T)\| \leq c\,O(\Delta t^2) \rightarrow$ order $= 2$

RK4: $\|x_N - x(T)\| \leq c\,O(\Delta t^4) \rightarrow$ order $= 4$

Notice the plot is log-log. The linear plots corresponds to polynomial functions with linear axis. The order is the slope of the linear plots. Since the linearity of the plots break down for $\Delta t > 0{,}1$, we will only consider for $\Delta t \leq 0{,}1$. The slope is found by

$$\frac{\log_{10}\{e(\Delta t = 10^{-1})\} - \log_{10}\{e(\Delta t = 10^{-3})\}}{\log_{10}(10^{-1}) - \log_{10}(10^{-3})}$$

RK1 : order = 0,98

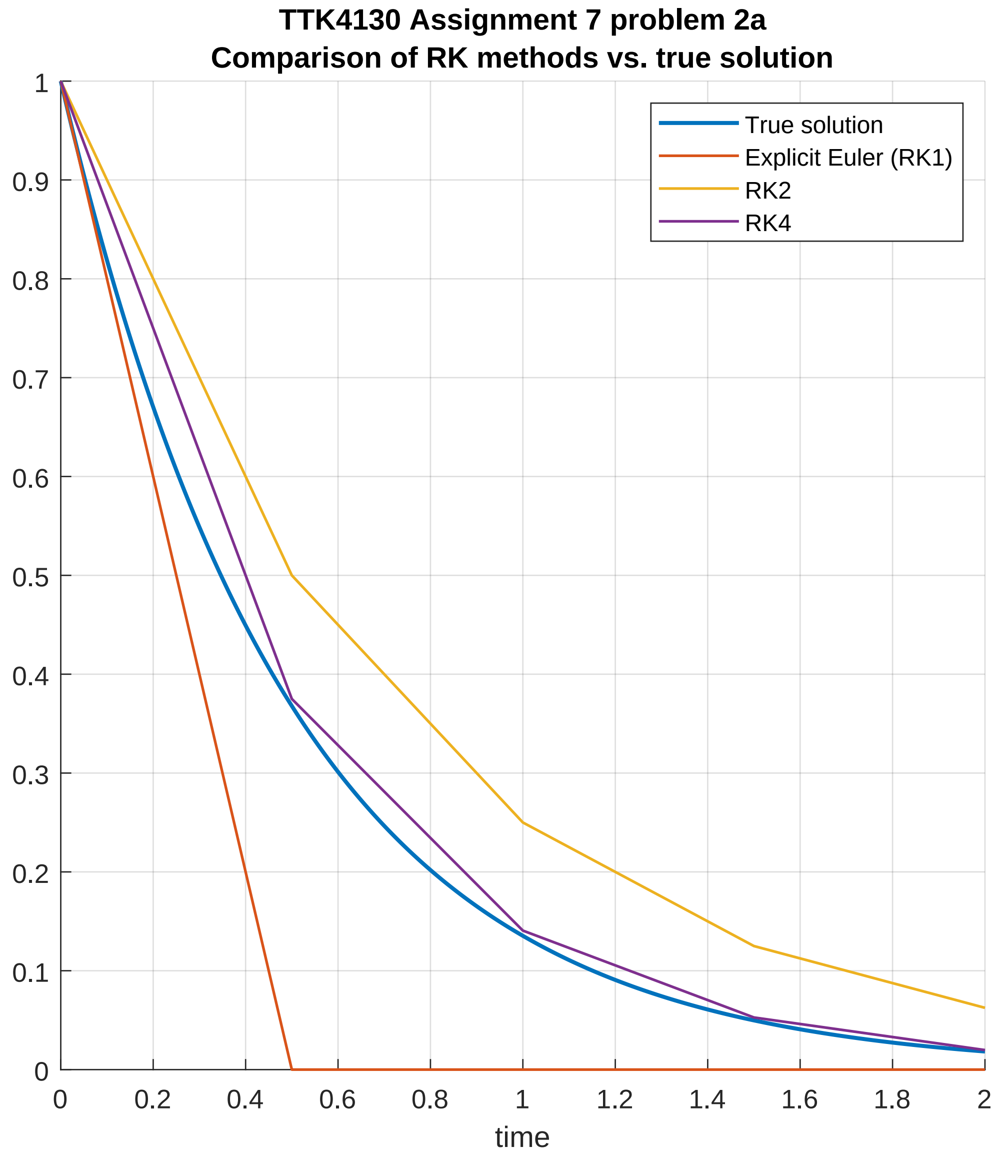
RK2 : order = 2,04

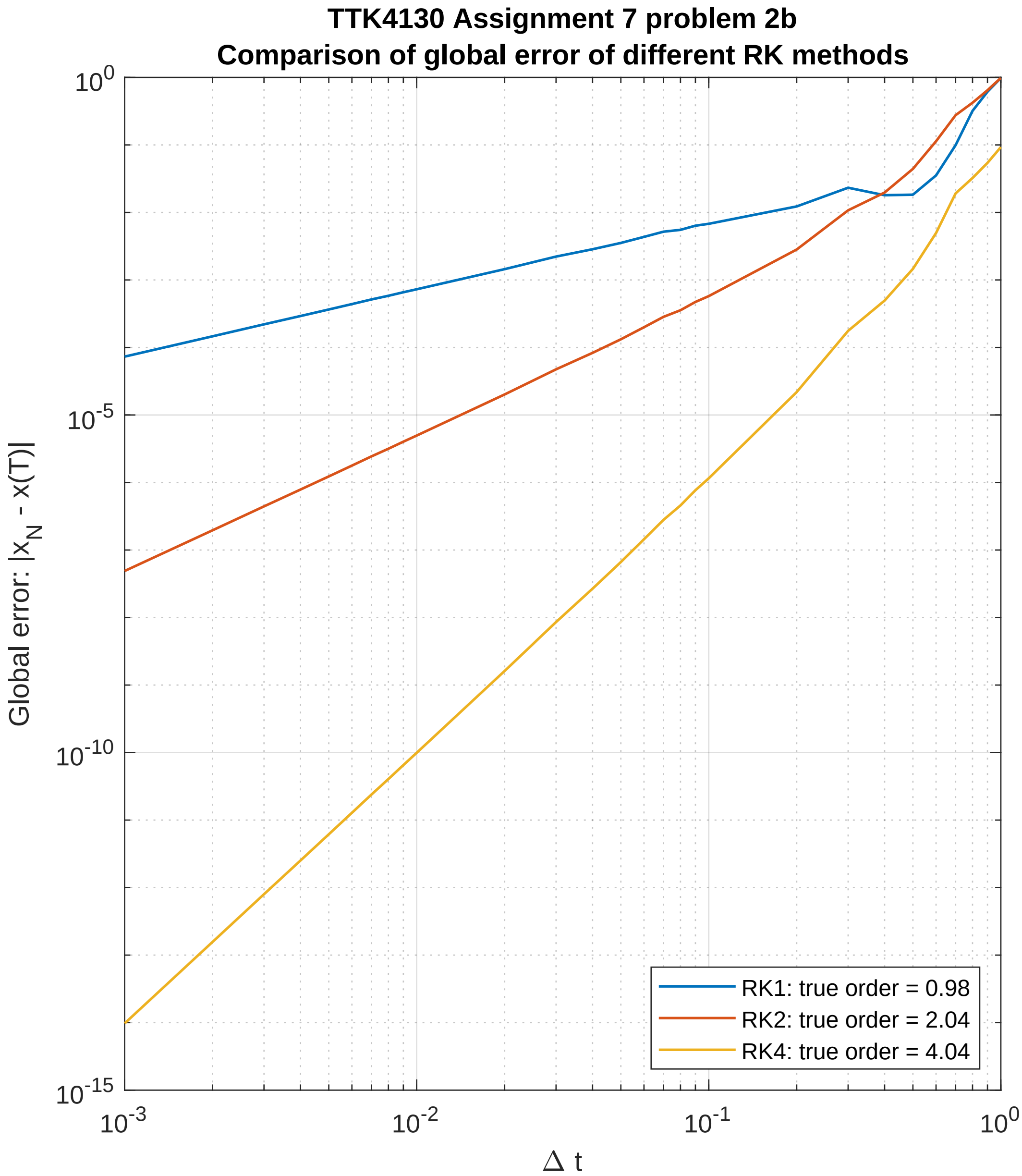
RK4 : order = 4,04

The true order corresponds very well with the theoretical order when $10^{-3} \leq \Delta t \leq 10^{-1}$. Above $\Delta t > 10^{-1}$ it starts to break down

c)

```matlab
function x = ERKTemplate(ButcherArray, f, T, x0)
    % Returns the iterations of an ERK method
    % ButcherArray: Struct with the ERK's Butcher array
    %   A: Nstage x Nstage
    %   b: Nstage x 1
    %   c: Nstage x 1
    %       (NB: both b and c must be standing vectors)
    % f: Function handle
    %   Vector field of ODE, i.e., x_dot = f(t,x)
    % T: Vector of time points, 1 x Nt
    % x0: Initial state, Nx x 1
    % x: ERK iterations, Nx x Nt
    %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
    % Define variables
    % Allocate space for iterations (x) and k1,k2,...,kNstage
    % It is recommended to allocate a matrix K for all kj, i.e.
    % K = [k1 k2 ... kNstage]
    A = ButcherArray.A;
    b = ButcherArray.b;
    c = ButcherArray.c;

    nx = size(x0,1);
    Nt = size(T,2);
    Nstage = size(A,1);

    K = zeros(nx,Nstage);
    x = zeros(nx,Nt);
    x(:,1) = x0;

    %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
    xt = x0; % initial iteration
    % Loop over time points
    for nt=2:Nt
        %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
        % Update variables
        dt = T(nt) - T(nt-1);

        %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
        % Calculation of the K vector relies on the A matrix having zeros
        % on and above the diagonal such that it's explicit RK.
        K(:,1) = f(nt, xt);

        % Loop that calculates k2,...,kNstage
        for nstage=2:Nstage
            K(:,nstage) = f(nt, xt + dt * sum(K .* A(nstage,:),2) );
        end % for
        %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
        % Calculate and save next iteration value x_t
        xt = xt + dt * sum(K .* (b.'),2);
        x(:,nt) = xt;

        %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
    end % for
end % function
```


TTK4130 Assignment 7 problem 2a
Comparison of RK methods vs. true solution
True solution
Explicit Euler (RK1)
RK2
RK4
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2
time



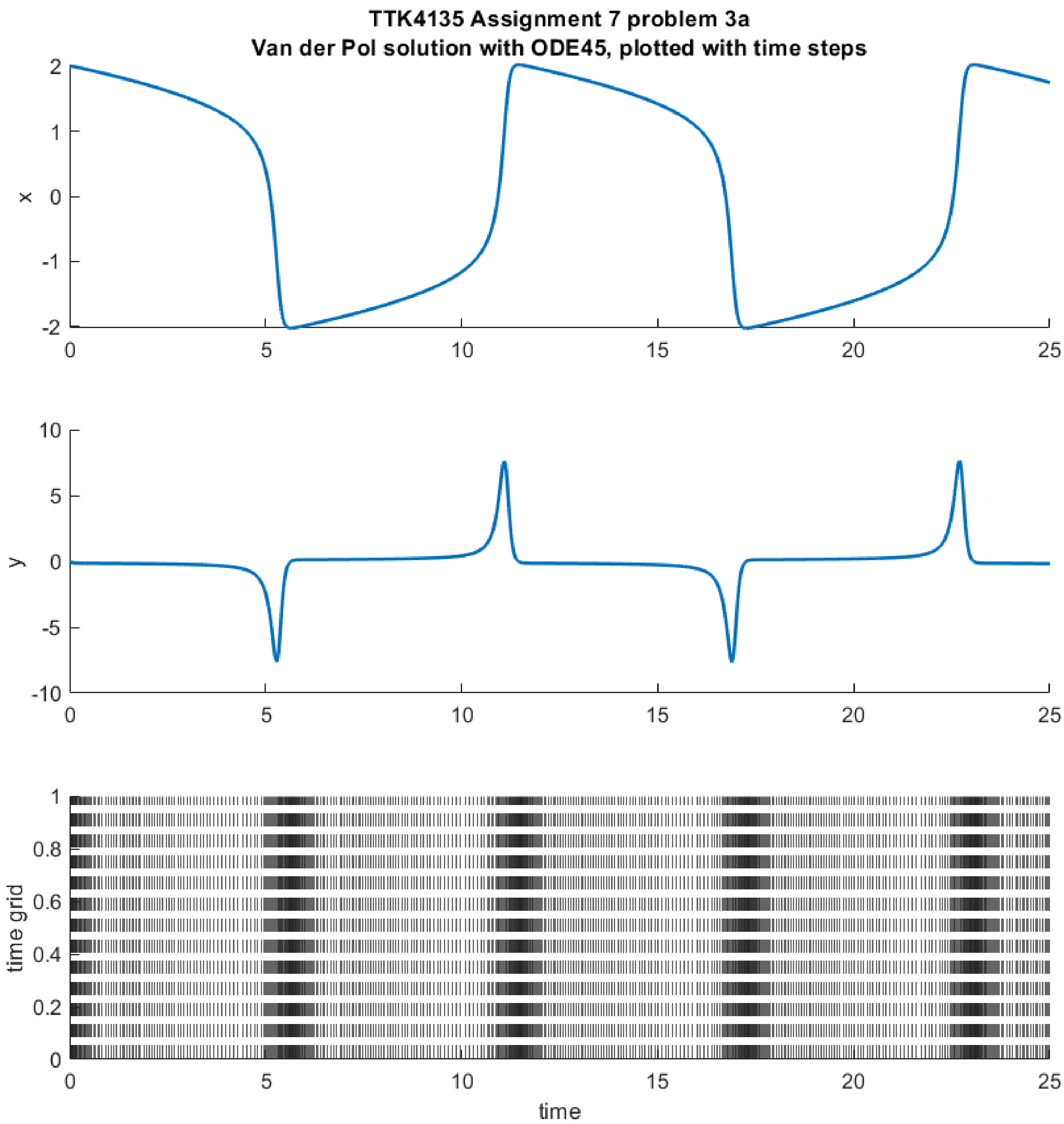
TTK4130 Assignment 7 problem 2b
Comparison of global error of different RK methods
10^{0}
10^{-5}
10^{-10}
10^{-15}
Global error: $|x_N - x(T)|$
10^{-3}
10^{-2}
10^{-1}
10^{0}
Δ t
RK1: true order = 0.98
RK2: true order = 2.04
RK4: true order = 4.04

Problem 3

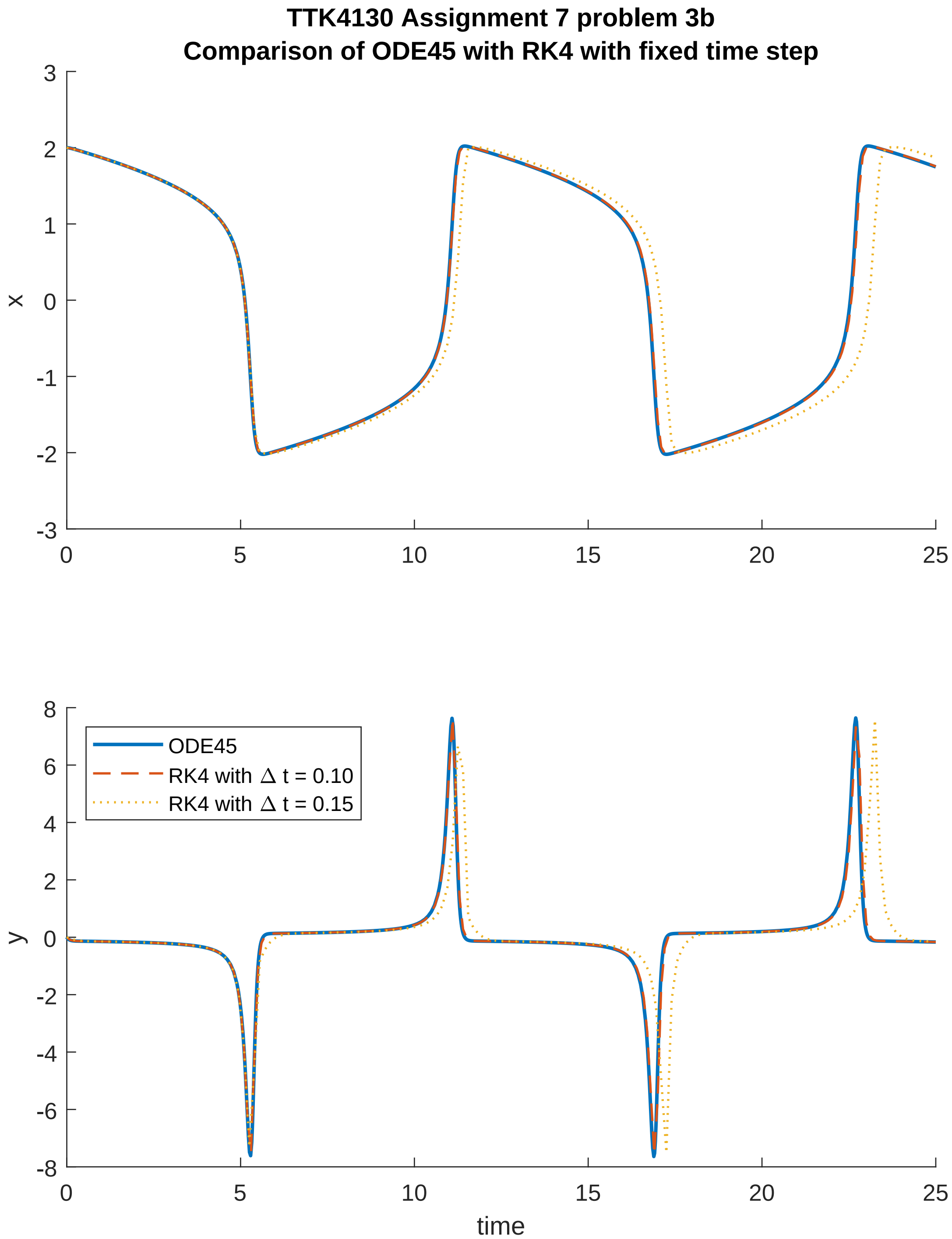
a) As seen, shorter time-steps are taken close to the fast dynamics. This is as expected.

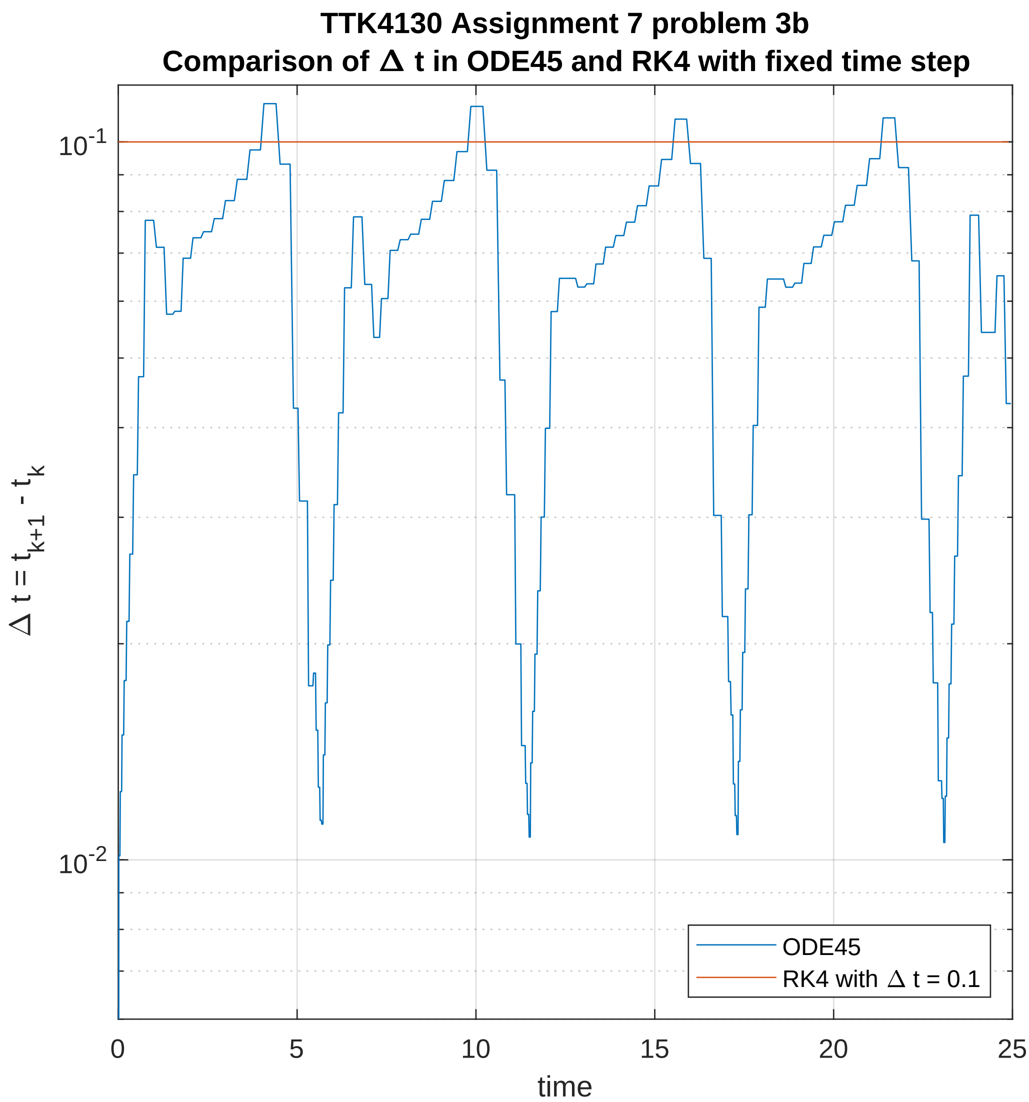
b) With $\Delta t = 0{,}1$, the RK4 solution follows the ODE45 solution to a high degree of accuracy. Increasing the time step slightly to $\Delta t = 0{,}15$ gives undesirable effects such as the solution starts lagging towards the end.

The maximal step size in ODE45 time grid is slightly above 0,1 while the minimal step size is close to 0,01. The step size is periodic in line with the dynamics in Van der Pol.



TTK4135 Assignment 7 problem 3a
Van der Pol solution with ODE45, plotted with time steps
x
y
time grid
time


TTK4130 Assignment 7 problem 3b
Comparison of ODE45 with RK4 with fixed time step
x
y
time
ODE45
RK4 with Δ t = 0.10
RK4 with Δ t = 0.15
3
2
1
0
-1
-2
-3
8
6
4
2
-2
-4
-6
-8
5
10
15
20
25


TTK4130 Assignment 7 problem 3b
Comparison of Δ t in ODE45 and RK4 with fixed time step
10^-1
10^-2
Δ t = t_{k+1} - t_k
0
5
10
15
20
25
time
ODE45
RK4 with Δ t = 0.1