TTK4130 Assignment 2 Eirik Falck

Problem 1

a) $SO(3) = \{R \mid R \in \mathbb{R}^{3\times3},\ R^TR = I,\ \det(R) = 1\}$

$$R_1 = \begin{bmatrix} * & 1 & * \\ 1 & * & * \\ * & * & * \end{bmatrix} \qquad R_2 = \begin{bmatrix} 5/13 & * & * \\ * & 1 & * \\ 12/13 & * & * \end{bmatrix}$$

$$R_1^TR_1 = \begin{bmatrix} *' & 1 & *'' \\ 1 & \bar{*} & \bar{\bar{*}} \\ \underline{*} & \underline{\underline{*}} & \underline{\underline{\underline{*}}} \end{bmatrix}\begin{bmatrix} *' & 1 & \underline{*} \\ 1 & \bar{*} & \underline{\underline{*}} \\ *'' & \bar{\bar{*}} & \underline{\underline{\underline{*}}} \end{bmatrix} = \begin{bmatrix} *'^2 + *''^2 + 1 & 0 & 0 \\ 0 & 1 + \bar{*}^2 + \bar{\bar{*}}^2 & 0 \\ 0 & 0 & \underline{*}^2 + \underline{\underline{*}}^2 + \underline{\underline{\underline{*}}}^2 \end{bmatrix}$$

$$\Rightarrow R_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad \det(R_1) = -1\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = 1$$

$$R_2^TR_2 = \begin{bmatrix} 5/13 & r_{21} & 12/13 \\ r_{12} & 1 & r_{32} \\ r_{13} & r_{23} & r_{33} \end{bmatrix}\begin{bmatrix} 5/13 & \overset{0}{\overset{\shortparallel}{r_{12}}} & r_{13} \\ r_{21} = 0 & 1 & 0 = r_{23} \\ 12/13 & \underset{0}{\underset{\shortparallel}{r_{32}}} & r_{33} \end{bmatrix} \rightarrow R_2 = \begin{bmatrix} 5/13 & 0 & -\frac{12}{13} \\ 0 & 1 & 0 \\ 12/13 & 0 & \frac{5}{13} \end{bmatrix}$$

$$= \begin{bmatrix} \left(\frac{5}{13}\right)^2 + \underset{0}{\cancel{r_{21}^2}} + \left(\frac{12}{13}\right)^2 & & \\ \cancel{r_{12}\cdot\frac{5}{13}} + \cancel{r_{21}} + \cancel{\frac{12}{13} r_{32}} & \cancel{r_{12}^2} + \cancel{r_{32}^2} + 1 & \\ \frac{5}{13} r_{13} + \cancel{r_{23} r_{21}} + \frac{12}{13} r_{33} & \cancel{r_{13} r_{12}} + \cancel{r_{23}} + \cancel{r_{33} r_{32}} & r_{13}^2 + \cancel{r_{23}^2} + r_{33}^2 \end{bmatrix} = I_{3\times3}$$

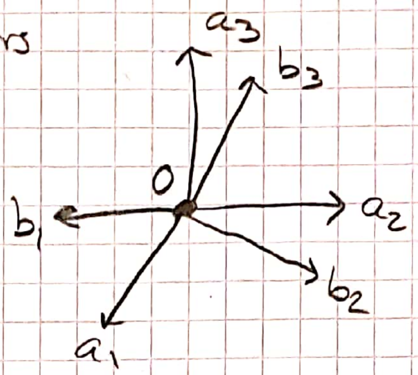
$$\det(R_2) = \frac{5}{13}\begin{vmatrix} 1 & \cancel{r_{23}} \\ \cancel{r_{32}} & r_{33} \end{vmatrix} - \cancel{r_{21}\begin{vmatrix} r_{12} & r_{13} \\ r_{32} & r_{33} \end{vmatrix}} + \frac{12}{13}\begin{vmatrix} \cancel{r_{12}} & r_{13} \\ 1 & \cancel{r_{23}} \end{vmatrix}$$

$$= \frac{5}{13} r_{33} - \frac{12}{13} r_{13} = 1, \quad r_{13}^2 + r_{33}^2 = 1$$

$$\rightarrow r_{13} = -\frac{12}{13} \quad r_{33} = \frac{5}{13}$$

b) $a = \{O, \vec{a}_1, \vec{a}_2, \vec{a}_3\}$ $\quad b = \{O, \vec{b}_1, \vec{b}_2, \vec{b}_3\}$

a_i, b_i : orthonormal vectors



$$R_b^a = [\vec{a}_i \cdot \vec{b}_j]$$

$$= \begin{bmatrix} \vec{a}_1 \\ \vec{a}_2 \\ \vec{a}_3 \end{bmatrix} \begin{bmatrix} \vec{b}_1 & \vec{b}_2 & \vec{b}_3 \end{bmatrix}$$

because a is orthonormal

$$= \begin{bmatrix} a_1 \cdot b_1 & a_1 \cdot b_2 & a_1 \cdot b_3 \\ a_2 \cdot b_1 & a_2 \cdot b_2 & a_2 \cdot b_3 \\ a_3 \cdot b_1 & a_3 \cdot b_2 & a_3 \cdot b_3 \end{bmatrix} = \begin{bmatrix} b_1^a & b_2^a & b_3^a \end{bmatrix}$$

The columns of R_b^a are the basis b represented in frame a coordinates.

c) $v^a = R_b^a v^b$

$$(u^a)^T v^a = (R_b^a u^b)^T R_b^a v^b = (u^b)^T \underbrace{(R_b^a)^T R_b^a}_{I} v^b = \underline{\underline{(u^b)^T v^b}} \quad \square$$

The geometrical interpretation of this is that the scalar product is invariant of the frame it is calculated in.

d) $(Ru)^\times = R u^\times R^T$, $v^a = R^a_b v^b$

$$u^a \times v^a = (u^a)^\times v^a = (R^a_b u^b)^\times R^a_b v^b$$

$$= R^a_b (u^b)^\times \underbrace{R^{a\,T}_b R^a_b}_{I} v^b = R^a_b (u^b)^\times v^b = \underline{\underline{R^a_b (u^b \times v^b)}} \quad \square$$

The interpretation is that the cross-product is dependent on the frame, but can be rotated from frame a to frame b by the rotation matrix R^a_b.

Problem 2

a) $R^a_b = R_1(\rho)\, R_2(\theta)\, R_3(\psi)$

$$\dot{R}^a_b = \frac{\partial R^a_b}{\partial \rho}\dot{\rho} + \frac{\partial R^a_b}{\partial \theta}\dot{\theta} + \frac{\partial R^a_b}{\partial \psi}\dot{\psi}$$

$$\dot{R}^a_b = (\omega^a_{ab})^\times R^a_b \rightarrow (\omega^a_{ab})^\times = \dot{R}^a_b (R^a_b)^T$$

$$(\omega^a_{ab})^\times = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}, \quad \omega^a_{ab} = \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} = M \begin{bmatrix} \dot{\rho} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix}$$

The simulations give sensible results. We can see $\vec{\omega}_{ab}$ is stationary while the red frame b is spinning around it. Larger $\vec{\omega}_{ab}$ also gives faster rotations, as expected.

b) The results of the simulations are also here sensible. The same observations as in a) were observed.

Problem 3

$\|k\|^2 = 1$

Axis-angle representation: angle θ around axis k

$$R = R_{k,\theta} = \cos\theta\, I + \sin\theta\, k^\times + (1-\cos\theta) k k^T$$

a) $$Rk = (\cos\theta\, I + \sin\theta\, k^\times + (1-\cos\theta) k k^T) k$$

$$= \cos\theta\, k + \sin\theta \underbrace{k^\times k}_{0} + (1-\cos\theta) k \underbrace{k^T k}_{\|k\|^2 = 1}$$

$$= \cos\theta\, k + k - \cos\theta\, k = \underline{\underline{k}} \quad \square$$

This implies that a rotation of the axis doesn't change the vector, i.e. the axis k is an eigenvector of the rotation matrix.

b) shepperds(R1) $\Rightarrow$ $\vec{k}_1 = \begin{bmatrix} 0,7071 \\ -0,7071 \\ 0 \end{bmatrix}$, $\theta_k = 3,1416$ rad

shepperds(R2) $\Rightarrow$ $\vec{k}_2 = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$, $\theta_2 = 1,1760$ rad

Comparing to compositions of rotation matrices corresponding to the above results, we can conclude that they are reasonable.