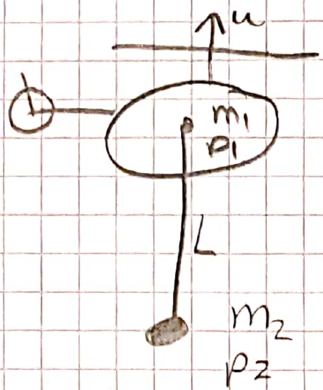
TTK4130 Assignment 5 Eirik Falck

Problem 1



$$q = \begin{bmatrix} p_1 \\ \theta \\ \phi \end{bmatrix} \in \mathbb{R}^5$$

$$u \in \mathbb{R}^3$$

$$L = \| p_1(0) - p_2(0) \|$$

a) $p_2 = p_1 + \begin{bmatrix} L \sin\theta \cos\varphi \\ L \sin\theta \sin\varphi \\ L \cos\theta \end{bmatrix}$ ← spherical to cartesian coordinates.

$$Q = \begin{bmatrix} u \\ 0 \\ 0 \end{bmatrix} \qquad T_1 = \frac{1}{2} m_1 \dot{p}_1^T \dot{p}_1$$

$$T_2 = \frac{1}{2} m_2 \dot{p}_2^T \dot{p}_2$$

$$\rightarrow T = T_1 + T_2 = \frac{1}{2} m_1 \dot{p}_1^T \dot{p}_1 + \frac{1}{2} m_2 \dot{p}_2^T \dot{p}_2$$

$$V = m_1 g \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} p_1 + m_2 g \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} p_2$$

$$\mathcal{L} = T - V \quad , \quad \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}} - \frac{\partial \mathcal{L}}{\partial q} = Q$$

$$M(q) = \frac{\partial^2 \mathcal{L}}{\partial \dot{q}^2} \qquad b(q, \dot{q}, u) = Q + \frac{\partial \mathcal{L}}{\partial q} - \frac{\partial^2 \mathcal{L}}{\partial \dot{q} \partial q} \dot{q}$$

b) $q = \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} \in \mathbb{R}^6$ $\mathcal{L} = T - V$

$$c(q) = \frac{1}{2}\left(e^T e - L^2\right), \quad e = p_1 - p_2$$

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}} - \frac{\partial \mathcal{L}}{\partial q} - z\nabla c(q) = Q$$

$$c(q) = 0$$

$$M(q) = \frac{\partial^2 \mathcal{L}}{\partial \dot{q}^2} \qquad b(q, z, u) = Q + \frac{\partial \mathcal{L}}{\partial q} - \frac{\partial^2 \mathcal{L}}{\partial \dot{q}\partial q}\dot{q} - z\nabla c$$

The M and b matrices are considerably less complicated now.

In this case, using constrained Lagrange result in a much easier model.

Problem 2

a)

$$\left.\begin{aligned} \dot{q} &= v \\ M(q)\,\dot{v} &= b(q,z,u) \end{aligned}\right\} M(q)\,\ddot{q} = b(q,z,u)$$

$$0 = c(q)$$

$$M\ddot{q} = Q + \frac{\partial \mathcal{L}}{\partial q} - \frac{\partial^2 \mathcal{L}}{\partial \dot{q}\,\partial q}\dot{q} - z\nabla c$$

$$(*) \quad M\ddot{q} + z\underbrace{\nabla c}_{= \frac{\partial c}{\partial q}^T} = Q + \frac{\partial \mathcal{L}}{\partial q} - \frac{\partial^2 \mathcal{L}}{\partial \dot{q}\,\partial q}\dot{q}$$

$$\frac{d}{dt}c = \frac{\partial c}{\partial q}\dot{q} = \dot{c} = 0$$

$$\frac{d}{dt}\dot{c} = \frac{d^2}{dt^2}c = \frac{\partial}{\partial q}\left(\frac{\partial c}{\partial q}\dot{q}\right)\dot{q} + \frac{\partial c}{\partial q}\ddot{q} = \ddot{c} = 0$$

$$(**) \quad \frac{\partial c}{\partial q}\ddot{q} = -\frac{\partial}{\partial q}\left(\frac{\partial c}{\partial q}\dot{q}\right)\dot{q}$$

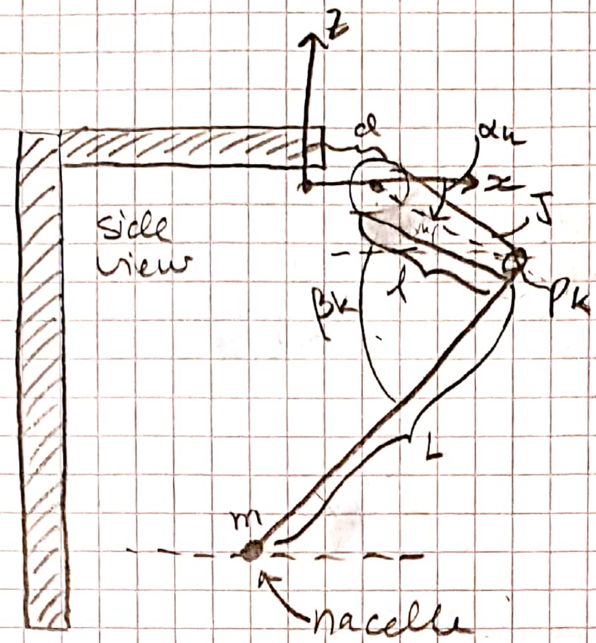
$$\Rightarrow \underbrace{\begin{bmatrix} M & \frac{\partial c}{\partial q}^T \\ \frac{\partial c}{\partial q} & 0 \end{bmatrix}}_{H} \begin{bmatrix} \ddot{q} \\ z \end{bmatrix} = \underbrace{\begin{bmatrix} Q + \frac{\partial \mathcal{L}}{\partial q} - \frac{\partial^2 \mathcal{L}}{\partial \dot{q}\,\partial q}\dot{q} \\ -\frac{\partial}{\partial q}\left(\frac{\partial c}{\partial q}\dot{q}\right)\dot{q} \end{bmatrix}}_{c(q,\dot{q},u)}$$

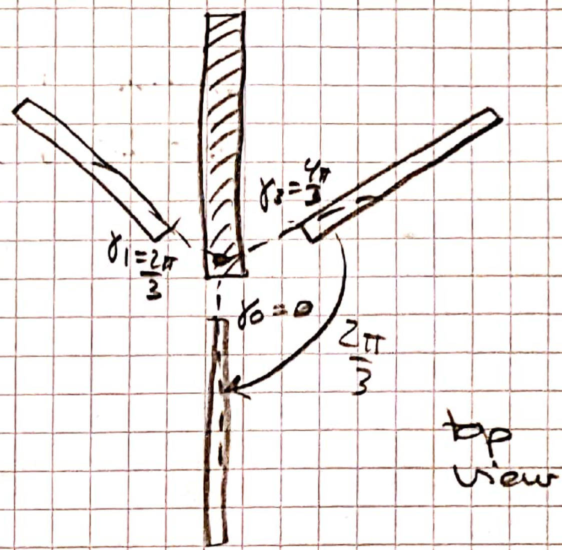
$$a(q) = \nabla c = \frac{\partial c}{\partial q}^T$$

b) The implicit form is considerably simpler. The explicit form is really long and complicated.

For simulation, the implicit form is preferable, and then the inverse can be found numerically.

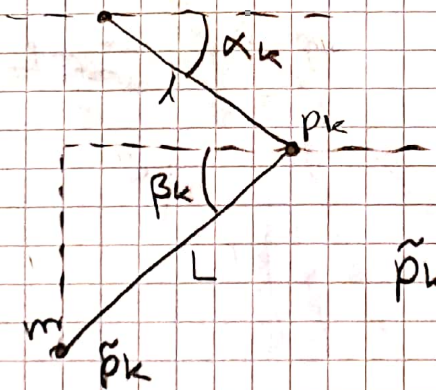
Problem 3





$$p_k = R_k \begin{bmatrix} d + l\cos\alpha_k \\ 0 \\ -l\sin\alpha_k \end{bmatrix} \qquad R_k = \begin{bmatrix} \cos\gamma_k & -\sin\gamma_k & 0 \\ \sin\gamma_k & \cos\gamma_k & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

a) $q = [\alpha_0 \quad \beta_0 \quad \alpha_1 \quad \beta_1 \quad \alpha_2 \quad \beta_2]^T$



$$\tilde{p}_k = R_k \begin{bmatrix} d + l\cos\alpha_k - L\cos\beta_k \\ 0 \\ -l\sin\alpha_k - L\sin\beta_k \end{bmatrix}$$

$\rightarrow \tilde{p}_0 = \tilde{p}_1 = \tilde{p}_2 = \tilde{p}$ since nacelle is modeled as single point.

b) $T_{k,rot} = \frac{1}{2} J \dot{\alpha}_k^2$

$$T_{k,trans} = \frac{1}{2} m \dot{\tilde{p}}_k^T \dot{\tilde{p}}_k = \frac{1}{2} m \dot{\tilde{p}}^T \dot{\tilde{p}}$$

$$\rightarrow T = \frac{1}{2} m \dot{\tilde{p}}^T \dot{\tilde{p}} + \sum_{k=0}^{2} \frac{1}{2} J \dot{\alpha}_k^2$$

$$V = mg\,[0 \;\; 0 \;\; 1]\,\tilde{p}$$

$$\underline{c} = \begin{bmatrix} p_0^T p_0 - \ell^2 \\ p_1^T p_1 - \ell^2 \\ p_2^T p_2 - \ell^2 \\ \tilde{p}^T \tilde{p} - L^2 \end{bmatrix} = 0 \in \mathbb{R}^4$$

$$Q = [\tau_0 \;\; 0 \;\; \tau_1 \;\; 0 \;\; \tau_2]^T$$

τ_k = torque from motor on yellow arm k.

$z \in \mathbb{R}^4$

$$\Rightarrow \quad \mathcal{L} = T - V - z^T c$$

$$= \frac{1}{2} m \dot{\tilde{p}}^T \dot{\tilde{p}} + \frac{1}{2} J (\dot{\alpha}_0^2 + \dot{\alpha}_1^2 + \dot{\alpha}_2^2) - mg[0,0,1]\tilde{p} - z^T \begin{bmatrix} p_0^T p_0 - \ell^2 \\ p_1^T p_1 - \ell^2 \\ p_2^T p_2 - \ell^2 \\ \tilde{p}^T \tilde{p} - L^2 \end{bmatrix}$$

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}} - \frac{\partial \mathcal{L}}{\partial q} = Q \quad , \quad c(q) = 0$$

c) The differential index of DAEs stemming from constrained Lagrange is always index 3.

d) $c(q(0), \dot{q}(0)) = 0$

$\dot{c}(q(0), \dot{q}(0)) = 0$

$\ddot{c}(q, \dot{q}, \ddot{q}) = 0$