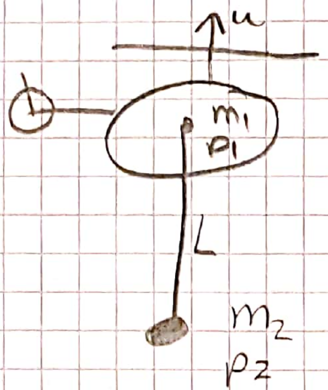
TTK4130 Assignment 5 Eirik Falck

Problem 1



$q = \begin{bmatrix} p_1 \\ \theta \\ \phi \end{bmatrix} \in \mathbb{R}^5$

$u \in \mathbb{R}^3$

$L = \|p_1(0) - p_2(0)\|$

a) $p_2 = p_1 + \begin{bmatrix} L\sin\theta\cos\varphi \\ L\sin\theta\sin\varphi \\ L\cos\theta \end{bmatrix}$ ← spherical to cartesian coordinates.

$Q = \begin{bmatrix} u \\ 0 \\ 0 \end{bmatrix}$ $\quad T_1 = \frac{1}{2} m_1 \dot{p}_1^T \dot{p}_1$

$T_2 = \frac{1}{2} m_2 \dot{p}_2^T \dot{p}_2$

$\rightarrow T = T_1 + T_2 = \frac{1}{2} m_1 \dot{p}_1^T \dot{p}_1 + \frac{1}{2} m_2 \dot{p}_2^T \dot{p}_2$

$V = m_1 g [0 \ 0 \ 1] p_1 + m_2 g [0 \ 0 \ 1] p_2$

$\mathcal{L} = T - V \quad , \quad \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}} - \frac{\partial \mathcal{L}}{\partial q} = Q$

$M(q) = \frac{\partial^2 \mathcal{L}}{\partial \dot{q}^2} \qquad b(q, \dot{q}, u) = Q + \frac{\partial \mathcal{L}}{\partial q} - \frac{\partial^2 \mathcal{L}}{\partial \dot{q} \partial q} \dot{q}$

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clear all
clc

% Parameters
syms m1 m2 L g real
% Force
u = sym('u',[3,1]);

% Position point mass 1
pm1 = sym('p1',[3,1]);
dpm1 = sym('dp1',[3,1]);
ddpm1 = sym('d2p1',[3,1]);
% Angles for point mass 2
a  = sym('a',[2,1]);
da = sym('da',[2,1]);
dda = sym('d2a',[2,1]);
% Generalized coordinates
q  = [pm1;a];
dq = [dpm1;da];
ddq = [ddpm1;dda];

% Position of point mass 2
pm2  = pm1 + [L*sin(a(1))*cos(a(2)); L*sin(a(1))*sin(a(2)); L*cos(a(1))];
% Velocity of point mass 2
dpm2 = jacobian(pm2,q)*dq;
% Generalized forces
Q = [u; 0; 0];
% Kinetic energy
T = 1/2 * m1 * (dpm1.') * dpm1 + 1/2 * m2 * (dpm2.') * dpm2;
T = simplify(T);
% Potential energy
V = m1 * g * pm1(3) + m2 * g * pm2(3);
V = simplify(V);
% Lagrangian
Lag = T - V;

% Derivatives of the Lagrangian
Lag_q = simplify(jacobian(Lag,q)).';
Lag_qdq = simplify(jacobian(Lag_q.',dq));
Lag_dq = simplify(jacobian(Lag,dq)).';
Lag_dqdq = simplify(jacobian(Lag_dq.',dq)); % W

% Matrices for problem 1
M = Lag_dqdq;
b = Q + simplify(Lag_q - Lag_qdq*dq);
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b) $q = \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} \in \mathbb{R}^6$ $\quad \mathcal{L} = T - V$

$$c(q) = \frac{1}{2}\left(e^T e - L^2\right), \quad e = p_1 - p_2$$

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}} - \frac{\partial \mathcal{L}}{\partial q} - z\nabla c(q) = Q$$

$$c(q) = 0$$

$$M(q) = \frac{\partial^2 \mathcal{L}}{\partial \dot{q}^2} \qquad b(q, z, u) = Q + \frac{\partial \mathcal{L}}{\partial q} - \frac{\partial^2 \mathcal{L}}{\partial \dot{q}\partial q}\dot{q} - z\nabla c$$

The M and b matrices are considerably less complicated now.

In this case, using constrained Lagrange result in a much easier model.

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clear all
clc

% Parameters
syms m1 m2 L g real
% Force
u = sym('u',[3,1]);

% Positions of point masses
pm1 = sym('pm1',[3,1]);
pm2 = sym('pm2',[3,1]);
dpm1 = sym('dpm1',[3,1]);
dpm2 = sym('dpm2',[3,1]);
ddpm1 = sym('d2pm1',[3,1]);
ddpm2 = sym('d2pm2',[3,1]);
% Generalized coordinates
q  = [pm1;pm2];
dq = [dpm1;dpm2];
ddq = [ddpm1;ddpm2];
% Algebraic variable
z = sym('z');

% Generalized forces
Q = [u; zeros(3,1)];
% Kinetic energy (function of q and dq)
T = 1/2 * m1 * (dpm1.') * dpm1 + 1/2 * m2 * (dpm2.') * dpm2;
% Potential energy
V = m1 * g * pm1(3) + m2 * g * pm2(3);
% Lagrangian (function of q and dq)
Lag = T - V;
% Constraint
dpm  = pm1 - pm2; % difference of positions
C = 1/2 * ((dpm.') * dpm - L^2);

% Derivatives of constrained Lagrangian
Lag_q = simplify(jacobian(Lag,q)).';
Lag_qdq = simplify(jacobian(Lag_q.',dq));
Lag_dq = simplify(jacobian(Lag,dq)).';
Lag_dqdq = simplify(jacobian(Lag_dq.',dq)); % W
C_q = simplify(jacobian(C,q)).';

% Matrices for problem 1b
M = Lag_dqdq;
b = Q + simplify(Lag_q - Lag_qdq*dq - z*C_q);

% Matrices for problem 2
Himplicit = [M, C_q; C_q.', 0];
c = [Q + simplify(Lag_q - Lag_qdq*dq); -jacobian((C_q.')*dq,q)*dq];

Hexplicit = simplify(inv(Himplicit));
rhs = simplify(Hexplicit*c);

```

Problem 2

a)

$$\left.\begin{aligned} \dot q &= v \\ M(q)\,\dot v &= b(q,z,u) \end{aligned}\right\} M(q)\,\ddot q = b(q,z,u)$$

$$0 = c(q)$$

$$M\ddot q = Q + \frac{\partial \mathcal{L}}{\partial q} - \frac{\partial^2 \mathcal{L}}{\partial \dot q\,\partial q}\dot q - z\nabla c$$

$$(*) \quad M\ddot q + z\underbrace{\nabla c}_{= \frac{\partial c}{\partial q}^T} = Q + \frac{\partial \mathcal{L}}{\partial q} - \frac{\partial^2 \mathcal{L}}{\partial \dot q\,\partial q}\dot q$$

$$\frac{d}{dt}c = \frac{\partial c}{\partial q}\dot q = \dot c = 0$$

$$\frac{d}{dt}\dot c = \frac{d^2}{dt^2}c = \frac{\partial}{\partial q}\left(\frac{\partial c}{\partial q}\dot q\right)\dot q + \frac{\partial c}{\partial q}\ddot q = \ddot c = 0$$

$$(**) \quad \frac{\partial c}{\partial q}\ddot q = -\frac{\partial}{\partial q}\left(\frac{\partial c}{\partial q}\dot q\right)\dot q$$

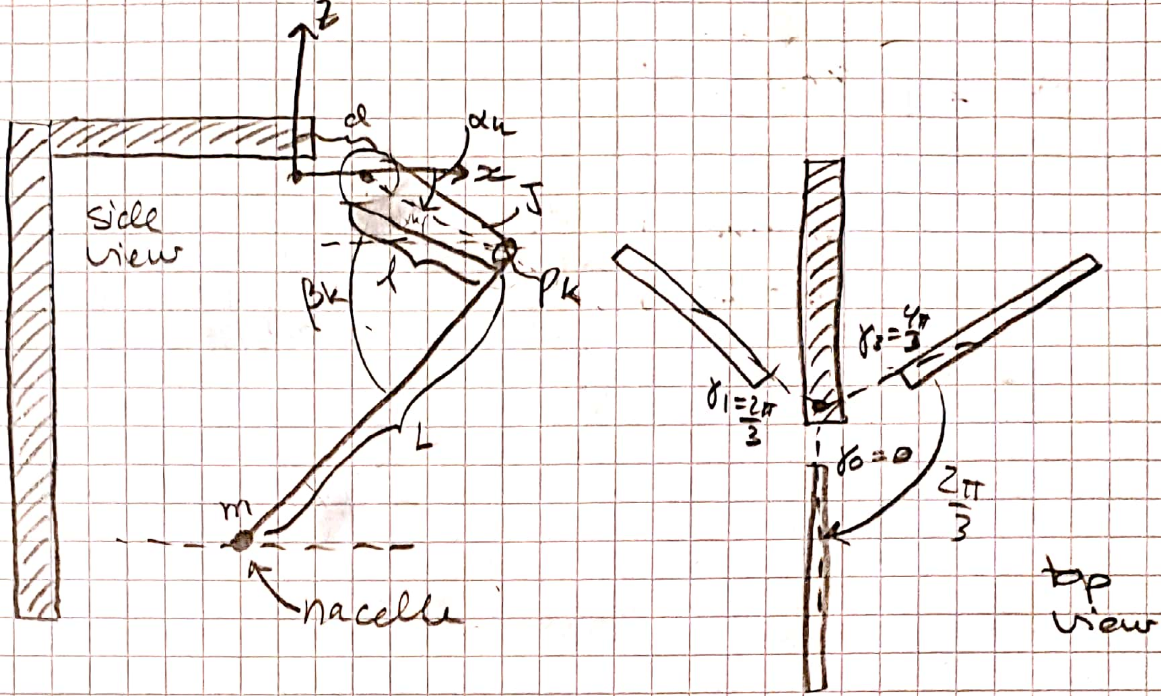
$$\Rightarrow \quad \underbrace{\begin{bmatrix} M & \frac{\partial c}{\partial q}^T \\ \frac{\partial c}{\partial q} & 0 \end{bmatrix}}_{H} \begin{bmatrix} \ddot q \\ z \end{bmatrix} = \underbrace{\begin{bmatrix} Q + \frac{\partial \mathcal{L}}{\partial q} - \frac{\partial^2 \mathcal{L}}{\partial \dot q\,\partial q}\dot q \\ -\frac{\partial}{\partial q}\left(\frac{\partial c}{\partial q}\dot q\right)\dot q \end{bmatrix}}_{c(q,\dot q,u)}$$

$$a(q) = \nabla c = \frac{\partial c}{\partial q}^T$$

b) The implicit form is considerably simpler. The explicit form is really long and complicated.

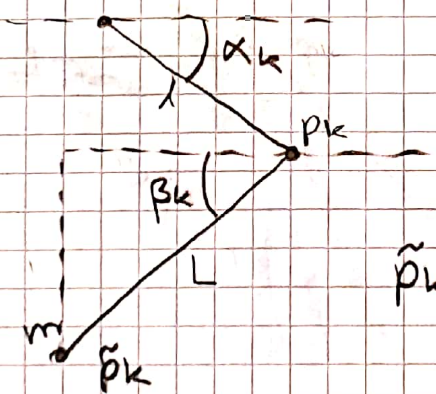
For simulation, the implicit form is preferable, and then the inverse can be found numerically.

Problem 3



$$p_k = R_k \begin{bmatrix} d + l\cos\alpha_k \\ 0 \\ -l\sin\alpha_k \end{bmatrix} \qquad R_k = \begin{bmatrix} \cos\gamma_k & -\sin\gamma_k & 0 \\ \sin\gamma_k & \cos\gamma_k & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

a) $q = [\alpha_0 \quad \beta_0 \quad \alpha_1 \quad \beta_1 \quad \alpha_2 \quad \beta_2]^T$



$$\tilde{p}_k = R_k \begin{bmatrix} d + l\cos\alpha_k - L\cos\beta_k \\ 0 \\ -l\sin\alpha_k - L\sin\beta_k \end{bmatrix}$$

$\rightarrow \tilde{p}_0 = \tilde{p}_1 = \tilde{p}_2 = \tilde{p}$ since nacelle is modeled as single point.

b) $T_{k,rot} = \frac{1}{2} J \dot{\alpha}_k^2$

$$T_{k,trans} = \frac{1}{2} m \dot{\tilde{p}}_k^T \dot{\tilde{p}}_k = \frac{1}{2} m \dot{\tilde{p}}^T \dot{\tilde{p}}$$

$$\rightarrow T = \frac{1}{2} m \dot{\tilde{p}}^T \dot{\tilde{p}} + \sum_{k=0}^{2} \frac{1}{2} J \dot{\alpha}_k^2$$

$$V = mg \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \tilde{p}$$

$$\underline{c} = \begin{bmatrix} p_0^T p_0 - \ell^2 \\ p_1^T p_1 - \ell^2 \\ p_2^T p_2 - \ell^2 \\ \tilde{p}^T \tilde{p} - L^2 \end{bmatrix} = 0 \in \mathbb{R}^4$$

$$Q = \begin{bmatrix} \tau_0 & 0 & \tau_1 & 0 & \tau_2 \end{bmatrix}^T$$

τ_k = torque from motor on yellow arm k.

$z \in \mathbb{R}^4$

$$\Rightarrow \mathcal{L} = T - V - z^T c$$

$$= \frac{1}{2} m \dot{\tilde{p}}^T \dot{\tilde{p}} + \frac{1}{2} J (\dot{\alpha}_0^2 + \dot{\alpha}_1^2 + \dot{\alpha}_2^2) - mg[0,0,1]\tilde{p} - z^T \begin{bmatrix} p_0^T p_0 - \ell^2 \\ p_1^T p_1 - \ell^2 \\ p_2^T p_2 - \ell^2 \\ \tilde{p}^T \tilde{p} - L^2 \end{bmatrix}$$

$$\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}} - \frac{\partial \mathcal{L}}{\partial q} = Q \quad , \quad c(q) = 0$$

c) The differential index of DAEs stemming from constrained Lagrange is always <u>index 3</u>.

d) $c(q(0), \dot{q}(0)) = 0$

$\dot{c}(q(0), \dot{q}(0)) = 0$

$\ddot{c}(q, \dot{q}, \ddot{q}) = 0$