Modsim Assignment 10 Eirik Falck

Problem 1

a) Volume of tank i: $V_i = A_i h_i$

mass: $m_i = \rho V_i = \rho A_i h_i$

$\dot{m}_i = \rho A_i \dot{h}_i = \dot{m}_{in} - \dot{m}_{out}$

$$\rho A_i \dot{h}_i = \omega_{i-1} - \omega_i$$

$$\frac{dh_i}{dt} = \frac{1}{\rho A_i}(\omega_{i-1} - \omega_i)$$

b) $x = [h_1 \quad h_2 \quad h_3 \quad \omega_1 \quad \omega_2 \quad \omega_3]^T$

$$\dot{h}_1 = \frac{1}{\rho A_1}(\omega_0 - \omega_1)$$

$$\dot{h}_2 = \frac{1}{\rho A_2}(\omega_1 - \omega_2)$$

$$\dot{h}_3 = \frac{1}{\rho A_3}(\omega_2 - \omega_3)$$

$$\dot{g}_1 = \frac{\partial g_1}{\partial \omega_1}\dot{\omega}_1 + \frac{\partial g_1}{\partial h_1}\dot{h}_1 + \frac{\partial g_1}{\partial h_2}\dot{h}_2 = 0$$

$$\rightarrow \dot{\omega}_1 = -\frac{\partial g_1}{\partial \omega_1}^{-1}\left(\frac{\partial g_1}{\partial h_1}\dot{h}_1 + \frac{\partial g_1}{\partial h_2}\dot{h}_2\right)$$

$$\dot{g}_2 = \frac{\partial g_2}{\partial \omega_2}\dot{\omega}_2 + \frac{\partial g_2}{\partial h_2}\dot{h}_2 + \frac{\partial g_2}{\partial h_3}\dot{h}_3 = 0$$

$$\rightarrow \dot{\omega}_2 = -\frac{\partial g_2}{\partial \omega_2}^{-1}\left(\frac{\partial g_2}{\partial h_2}\dot{h}_2 + \frac{\partial g_2}{\partial h_3}\dot{h}_3\right)$$

$$\dot{g}_3 = \frac{\partial g_3}{\partial \omega_3}\dot{\omega}_3 + \frac{\partial g_3}{\partial h_3}\dot{h}_3$$

$$\rightarrow \dot{\omega}_3 = -\frac{\partial g_3}{\partial \omega_3}^{-1}\frac{\partial g_3}{\partial h_3}\dot{h}_3$$

If we assume $\frac{\partial g_i}{\partial w_j} << 1$, the equations constitute a DAE.

c) $g_1 = h_1 - h_2 = 0$

$g_2 = h_2 - h_3 = 0$

$g_3 = \omega_3 - A\rho\sqrt{2gh_3} = 0$

Problem 2

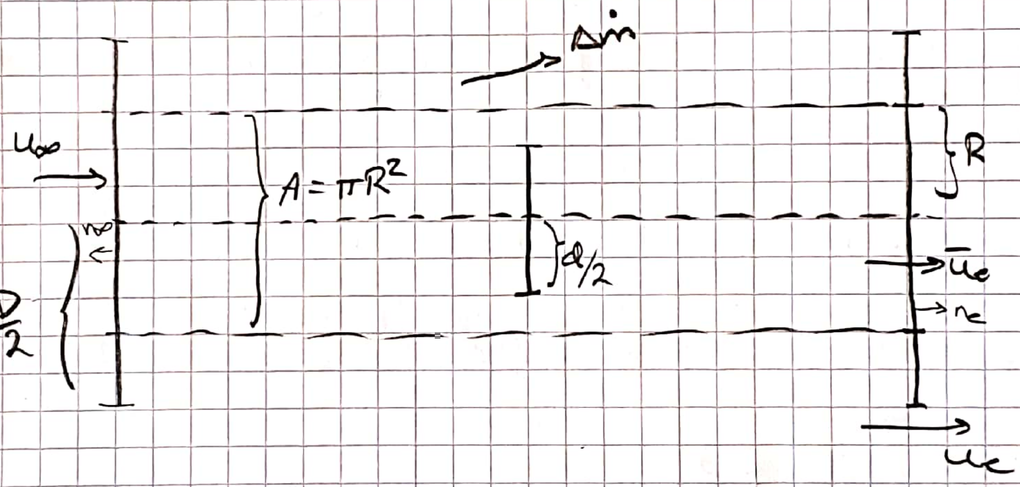
$u_\infty = \text{const.}$ $\qquad u_c = \begin{cases} \bar{u}_c & 0 \leq r \leq R \\ u_\infty & R < r \leq D/2 \end{cases}$

Displaced air: $\Delta\dot{m}$ $\qquad d << D$

Density of air: $\rho = \text{const.}$ $\qquad d/2 < R < D/2$

a) $$\frac{d}{dt}\iiint_{V_c} \rho \, dV = -\iint_{\partial V_c} \rho \vec{v} \cdot \vec{n} \, dA \qquad \text{(mass balance)}$$

$$0 = \frac{d}{dt}\iiint_{V_c} \rho \, dV \qquad \text{(steady state assumption)}$$



$$-\iint_{\partial V_c} \rho \vec{v} \cdot \vec{n} \, dA = -\left[-u_\infty \rho A + \bar{u}_c \rho A + \Delta\dot{m} \right] = 0$$

$$\underline{\underline{\Delta\dot{m} = \rho A (u_\infty - \bar{u}_c) = \rho \pi R^2 (u_\infty - \bar{u}_c)}}$$

b) As we see in the figure in the assignment, only the wind in the control volume with radius R is affected by the turbine, so including the whole wind tunnel is redundant.

Since the density is constant and the tunnel is in steady state, the number of particles entering the tunnel is the same as the number leaving, so the net change in total mass in the tunnel is zero.

c) $$\underbrace{\frac{d}{dt}\iiint_{V_c} \rho\vec{v}\,dV}_{=0} = -\iint_{\partial V_c} \rho\vec{v}\,\vec{v}\cdot\vec{n}\,dA - \begin{bmatrix} F \\ 0 \\ 0 \end{bmatrix}$$

$$-\iint_{\partial V_c} \rho\vec{v}\,\vec{v}\cdot\vec{n}\,dA - \begin{bmatrix} F \\ 0 \\ 0 \end{bmatrix} = -\left[\rho(-u_\infty)A u_\infty + \rho\bar{u}_e A\bar{u}_e + \Delta\dot{m}(u_\infty - \bar{u}_e)\right] - F = 0$$

$$F = u_\infty^2\rho A - \bar{u}_e^2\rho A - \rho A(u_\infty - \bar{u}_e)^2$$

$$= \cancel{u_\infty^2}\rho A - \bar{u}_e^2\rho A - \rho A(\cancel{u_\infty^2} - 2u_\infty\bar{u}_e + \bar{u}_e^2)$$

$$= -2\bar{u}_e^2\rho A + 2\rho A u_\infty\bar{u}_e$$

$$= \underline{\underline{2\rho\pi R^2(u_\infty - \bar{u}_e)\bar{u}_e}}$$

I don't know how to do this without getting the factor 2.

Problem 3

a) $\frac{1}{2}|v|^2$: represent kinetic energy

ϕ : represent potential energy

The last assumption is that kinetic and potential energy can be neglected. Thus, $\frac{1}{2}|v|^2 + \phi \approx 0$ and the energy balance reads

$$\frac{d}{dt}\iiint_{V_c} \rho u \, dV = -\iint_{\partial V_c} \rho u \vec{v}\cdot\vec{n}\, dA - \iint_{\partial V_c} \vec{J_Q}\cdot\vec{n}\, dA$$

b) $u_1 = c_1 T_1$

$$\frac{d}{dt}\iiint_{V_c} \rho_1 u_1 \, dV = \frac{d}{dt}\iiint_{V_c} \rho_1 c_1 T_1 \, dV$$

$$= \rho_1 c_1 \pi r_1^2 \cdot \frac{d}{dt}\int_x^{x+\Delta x} T_1 \, dx = \rho_1 c_1 \pi r_1^2 \int_x^{x+\Delta x} \frac{dT_1}{dt} dx$$

The last equality uses Leibniz integral rule.
The second to last equality uses the fact that ρ and c are constant, and that the cross-sectional area is πr_1^2 and is constant for all x.

c) Enters at $x+\Delta x$:

$$-\iint_{\partial V_c} \rho_1 u_1 \vec{v} \cdot \vec{n} \, dA = -\left(-\rho_1 c_1 T_1(x+\Delta x, t) v_1 \cdot \pi r_1^2\right)$$

$$= \underline{\underline{\rho_1 c_1 v_1 \pi r_1^2 T(x+\Delta x, t)}}$$

Leaves at x:

$$-\iint_{\partial V_c} \rho_1 u_1 \vec{v} \cdot \vec{n} \, dA = -\left(\rho_1 c_1 T_1(x, t) v_1 \pi r_1^2\right)$$

$$= \underline{\underline{-\rho_1 c_1 v_1 \pi r_1^2 T(x, t)}}$$

d) $j_Q = K_{12}(T_1 - T_2)$, direction radially outwards

$$-\iint_{\partial V_c} \vec{j_Q} \cdot \vec{n} \, dA = -\left(K_{12} \int_x^{x+\Delta x} T_1 - T_2 \, dx \cdot 2\pi r_1\right)$$

$$= \underline{\underline{-2\pi r_1 K_{12} \int_x^{x+\Delta x} T_1 - T_2 \, dx}}$$

Amount of heat across the boundary between 1 and 2 with circumference $2\pi r_1$ and x-width Δx.

e) Divide both sides by Δx:

$$\rho_1 c_1 \cancel{\pi} r_1^2 \frac{1}{\Delta x}\int_x^{x+\Delta x} \frac{\partial T_1}{\partial t}\, dx = \rho_1 c_1 v_1 \cancel{\pi} r_1^2 \frac{T_1(x+\Delta x, t) - T_1(x,t)}{\Delta x}$$

$$- 2\cancel{\pi}\cancel{r_1} K_{12} \frac{1}{\Delta x}\int_x^{x+\Delta x} T_1 - T_2 \, dx$$

Let $\tilde{T}_1 = \int \frac{\partial T_1}{\partial t}\, dx$, $\tilde{D} = \int T_1 - T_2 \, dx$ be their respective anti-derivatives. Then, in the limit $\Delta x \to 0$ we get

- $$\lim_{\Delta x \to 0} \frac{1}{\Delta x}\int_x^{x+\Delta x} \frac{\partial T_1}{\partial t}\, dx = \lim_{\Delta x \to 0} \frac{\tilde{T}_1(x+\Delta x, t) - \tilde{T}_1(x,t)}{\Delta x}$$
$$= \frac{\partial \tilde{T}_1}{\partial x} = \frac{\partial}{\partial x}\int \frac{\partial T_1}{\partial t}\, dx = \underline{\frac{\partial T_1}{\partial t}}$$

- $$\lim_{\Delta x \to 0} \frac{1}{\Delta x}\int_x^{x+\Delta x} T_1 - T_2 \, dx = \lim_{\Delta x \to 0} \frac{\tilde{D}(x+\Delta x) - \tilde{D}(x)}{\Delta x}$$
$$= \frac{\partial \tilde{D}}{\partial x} = \frac{\partial}{\partial x}\int T_1 - T_2 \, dx = \underline{T_1 - T_2}$$

- $$\lim_{\Delta x \to 0} \frac{T_1(x+\Delta x, t) - T_1(x,t)}{\Delta x} = \underline{\frac{\partial T_1}{\partial x}}$$

$$\Rightarrow \frac{\partial T_1}{\partial t} = v_1 \frac{\partial T_1}{\partial x} - \frac{2K_{12}}{\rho_1 c_1 r_1}(T_1 - T_2)$$

$$\Leftrightarrow \underline{\underline{\frac{\partial T_1}{\partial t} - v_1 \frac{\partial T_1}{\partial x} = -\frac{2K_{12}}{\rho_1 c_1 r_1}(T_1 - T_2)}}$$