Modsim Assignment 9 Eirik Falck

Problem 1

a) See code below.

b) We can see the simulated solution following the true curve very well and has (almost) the exact same function value at the sample points.

c) $b = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \end{bmatrix}^T$

$$K_1 = f(x_k + \Delta t[a_{11} K_1 + a_{12} K_2])$$
$$= \lambda(x_k + \Delta t a_{11} K_1 + \Delta t a_{12} K_2)$$
$$= \lambda x_k + \lambda \Delta t a_{11} K_1 + \lambda \Delta t a_{12} K_2$$

$$(1 - \lambda \Delta t a_{11}) K_1 = \lambda x_k + \lambda \Delta t a_{12} K_2$$

$$K_1 = \frac{\lambda}{1 - \lambda \Delta t a_{11}} (x_k + \Delta t a_{12} K_2)$$

$$K_2 = \frac{\lambda}{1 - \lambda \Delta t a_{22}} (x_k + \Delta t a_{21} K_1)$$

$$K_1 = \frac{\lambda}{1 - \lambda \Delta t a_{11}} \left(x_k + \Delta t a_{12} \frac{\lambda}{1 - \lambda \Delta t a_{22}} (x_k + \Delta t a_{21} K_1) \right)$$
$$= \frac{\lambda}{1 - \lambda \Delta t a_{11}} \left(x_k + \frac{\Delta t a_{12} \lambda}{1 - \lambda \Delta t a_{22}} x_k + \frac{\Delta t^2 a_{12} a_{21} K_1 \lambda}{1 - \lambda \Delta t a_{22}} \right)$$
$$= \frac{\lambda}{1 - \lambda \Delta t a_{11}} \left(\frac{(1 - \lambda \Delta t a_{22}) x_k + \lambda \Delta t a_{12} x_k}{1 - \lambda \Delta t a_{22}} + \frac{\lambda \Delta t^2 a_{12} a_{21}}{1 - \lambda \Delta t a_{22}} K_1 \right)$$

$$\frac{(1-\lambda\Delta t a_{11})(1-\lambda\Delta t a_{22}) - \lambda^2\Delta t^2 a_{12}a_{21}}{\cancel{(1-\lambda\Delta t a_{22})}} K_1 = \frac{\lambda(1-\lambda\Delta t a_{22}) + \lambda^2 \Delta t a_{12}}{\cancel{(1-\lambda\Delta t a_{22})}} x_k$$

$$K_1 = \frac{\lambda(1-\lambda\Delta t a_{22}) + \lambda^2 \Delta t a_{12}}{\underbrace{(1-\lambda\Delta t a_{11})(1-\lambda\Delta t a_{22}) - \lambda^2\Delta t^2 a_{12}a_{21}}_{\Delta_1}} x_k$$

$$K_2 = \frac{\lambda}{1-\lambda\Delta t a_{22}}\left(x_k + \Delta t a_{21} K_1\right)$$

$$= \left[\frac{\lambda}{1-\lambda\Delta t a_{22}} + \frac{\lambda\Delta t a_{21}}{1-\lambda\Delta t a_{22}} \cdot \frac{\lambda(1-\lambda\Delta t a_{22}) + \lambda^2\Delta t a_{12}}{(1-\lambda\Delta t a_{11})(1-\lambda\Delta t a_{22}) - \lambda^2\Delta t^2 a_{12}a_{21}}\right] x_k$$

$$= \underbrace{\frac{\lambda(1-\lambda\Delta t a_{11})(1-\lambda\Delta t a_{22}) - \lambda^3\Delta t^2 a_{12}a_{21} + \lambda^2\Delta t a_{21}\left([1-\lambda\Delta t a_{22}] + \lambda\Delta t a_{12}\right)}{(1-\lambda\Delta t a_{22})\left([1-\lambda\Delta t a_{11}][1-\lambda\Delta t a_{22}] - \lambda^2\Delta t^2 a_{12}a_{21}\right)}}_{\Delta_2} x_k$$

$$x_{k+1} = x_k + \Delta t\, b_1 K_1 + \Delta t\, b_2 K_2$$

$$= \underbrace{\left(1 + \Delta t\, b_1 \Delta_1 + \Delta t\, b_2 \Delta_2\right)}_{\text{stable if } |\cdot| \le 1} x_k$$

Problem 2

(2) $\ddot{x} + g\left(1 - \left(\frac{x_d}{x}\right)^{\kappa}\right) = 0 \qquad x, x_d, g > 0 \qquad \kappa \geq 1$

(3) $E = \frac{mg}{\kappa - 1} \frac{x_d^{\kappa}}{x^{\kappa-1}} + mgx + \frac{1}{2} m \dot{x}^2$

a) $\dot{E}(t) = \frac{mg\, x_d^{\kappa}}{\kappa - 1} \cdot \frac{d}{dt} x^{1-\kappa} + mg\dot{x} + m\dot{x}\ddot{x}$

$= \frac{-mg\, x_d^{\kappa}}{\cancel{\kappa-1}} \cancel{(1-\kappa)} x^{-\kappa} \cdot \dot{x} + mg\dot{x} + m\dot{x}\ddot{x}$

$= -mg\, x_d^{\kappa} x^{-\kappa} \cdot \dot{x} + \cancel{mg\dot{x}} + mg\dot{x}\left[\left(\frac{x_d}{x}\right)^{\kappa} \cancel{-1}\right]$

$= \cancel{-mg\left(\frac{x_d}{x}\right)^{\kappa}\dot{x}} + 0 + \cancel{mg\left(\frac{x_d}{x}\right)^{\kappa}\dot{x}}$

$= 0$.

b) $\dot{x}_1 = x_2$

$\dot{x}_2 = -g\left(1 - \left(\frac{x_d}{x_1}\right)^{\kappa}\right)$

$\Rightarrow \dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -g\left(1 - \left(\frac{x_d}{x_1}\right)^{\kappa}\right) \end{bmatrix} = f(x)$

$x = [x_1 \;\; x_2]^T = [x \;\; \dot{x}]^T$

With Explicit Euler, the simulation becomes unstable and actually gains energy.

With Implicit Euler, the simulation is stable, but looses energy although from a) we know the energy is constant.

With the midpoint rule, the solution is marginally stable with standing oscillations. the energy fluctuate slightly, but at such a small amplitude we can approximate it as constant (fluctuations of $\pm 0{,}00005$).

We know all IRK methods are A-stable. This means they can handle fast dynamics without becoming unstable. This is observed in the two implicit method plots.

Explicit methods are not A-stable and only stable within a specific region. For Explicit Euler, this is the unit circle around -1. Since we observe the method becoming unstable, we must be outside this region.

Problem 3

a) $c(q) = \frac{1}{2}(p^T p - L^2) = 0$, $L = 1$

$$\dot{p} = v$$

$$\dot{v} = -g\begin{bmatrix}0\\0\\1\end{bmatrix} - \overbrace{\frac{1}{m}\tilde{z}}^{z}p$$

$$0 = p^T\dot{v} + v^T v$$

$$x = \begin{bmatrix}p\\v\end{bmatrix} \qquad x(0) = [1 \;\; 0 \;\; 0 \;\vdots\; 0 \;\; 1 \;\; 0]^T \in \mathbb{R}^6$$

$$\dot{x} = \begin{bmatrix}0 & I\\ -zI & -g\begin{bmatrix}0 & \\ & 0 & \\ & & 1\end{bmatrix}\end{bmatrix}x$$

When $\Delta t = 0{,}5$, the method becomes unstable, and no meaningful path is found.

When $\Delta t = 0{,}1$, the result is a nice graph, although a little choppy. This is due to the "long" time step.

When $\Delta t = 0{,}01$, the result is much smoother.

As we see, when $\Delta t = 0{,}1$ or $\Delta t = 0{,}01$, the constraint value oscillates a bit, but has generally low amplitude (10^{-5} and 10^{-9} respectively).

The reason for the oscillations may be the fixed time step introduces aliasing effects.

To improve the results, variable step length can be used. Alternatively, a different IRK method can be used. IRK6 (Butcher Tableau can be found on Wikipedia) was tried, and the constraint didn't oscillate and had slightly smaller amplitude.

b) When using the model obtained directly from Lagrange, the Jacobian passed to the Newton's method is "singular to working precision". This is because z doesn't enter as a variable in the constraint and thus a value cannot be found for z.