Out: February 10, 2020
Due: February 23, 2020

NTNU – Trondheim
Norwegian University of
Science and Technology

TTK4130 Modeling and Simulation

Assignment 5

Introduction

In this assignment we will study the modeling of complex mechanical systems using the constrained Lagrange equations. Since the calculations of the partial derivatives of the Lagrangian can be involved, we will once more outsource this task by using the Matlab Symbolic Math Toolbox™.

Problem 1 (Hovering mass)

We consider a helicopter hovering a mass. We model this system as two point masses coupled by a rigid link. The masses are m_1 and m_2, and their positions in space are given by $\boldsymbol{p}_1 \in \mathbb{R}^3$ and $\boldsymbol{p}_2 \in \mathbb{R}^3$. We assume that an external force $\boldsymbol{u} \in \mathbb{R}^3$ is applied to m_1 (helicopter).

(a) We will first model this system by using the classical Lagrange approach, where the number of coordinates is equal to the number of degrees of freedom. In this setting, the position of the helicopter is described by $\boldsymbol{p}_1 \in \mathbb{R}^3$, and the position of the hovering mass is described by the two angles θ, ϕ, which give the orientation of the rigid link (cable). Hence, the generalized coordinates are

$$\boldsymbol{q} = \begin{bmatrix} \boldsymbol{p}_1 \\ \theta \\ \phi \end{bmatrix} \in \mathbb{R}^5. \tag{1}$$

As seen in the course and in the previous assignment, the fully assembled model has the form:

$$\dot{\boldsymbol{q}} = \boldsymbol{v} \tag{2a}$$
$$M(\boldsymbol{q})\dot{\boldsymbol{v}} = \boldsymbol{b}(\boldsymbol{q}, \dot{\boldsymbol{q}}, \boldsymbol{u}). \tag{2b}$$

Complete the delivered template `HoveringMassUnconstraintTemplate.m` by doing the following tasks:

1. Write the expression for the position of mass 2.
2. Write the expression for the generalized forces.
3. Write the expression for the kinetic energy.
4. Write the expression for the potential energy.
5. Write the expression for the Lagrangian.
6. Run the routine in order to obtain the expressions for the matrices M and $\boldsymbol{b}$.

Add the implemented code to your answer.

Solution: The expression for $\boldsymbol{p}_2$ depends on the spherical coordinate representation one chooses. In our case, we have defined the angles θ, ψ such that

$$\boldsymbol{p}_2 = \boldsymbol{p}_1 + L \begin{bmatrix} \sin(\theta)\cos(\psi) \\ \sin(\theta)\sin(\psi) \\ -\cos(\theta) \end{bmatrix}.$$

Hence, an implementation of the symbolic calculations is:

```
clear all
clc
% parameters
syms m1 m2 L g real
```

```
% force
u = sym('u',[3,1]);
% point mass 1
pm1 = sym('p1',[3,1]);
dpm1 = sym('dp1',[3,1]);
ddpm1 = sym('d2p1',[3,1]);
% angles for point mass 2
a  = sym('a',[2,1]);
da = sym('da',[2,1]);
dda = sym('d2a',[2,1]);
% generalized coordinates
q  = [pm1;a];
dq = [dpm1;da];
ddq = [ddpm1;dda];
% position point mass 2
pm2  = pm1 + L*[sin(a(1))*cos(a(2));
                sin(a(1))*sin(a(2));
               -cos(a(1))];
dpm2 = jacobian(pm2,q)*dq;
% Lagrangian
T = simplify(0.5*m1*(dpm1.'*dpm1) + 0.5*m2*(dpm2.'*dpm2));
V = g*(m1*pm1(3) + m2*pm2(3));
Lag = T - V;
% derivatives of Lagrangian
Lag_q = simplify(jacobian(Lag,q)).';
Lag_qdq = simplify(jacobian(Lag_q.',dq));
Lag_dq = simplify(jacobian(Lag,dq)).';
Lag_dqdq = simplify(jacobian(Lag_dq.',dq)); % W
% matrices
M = Lag_dqdq;
b = [u;zeros(2,1)]+ simplify(Lag_q - Lag_qdq*dq);
```

Note that the expression for `M` and `b` are fairly complex.

(b) We will now use the constrained Lagrange approach to model the dynamics of the system. The generalized coordinates in this case are

$$\boldsymbol{q} = \begin{bmatrix} \boldsymbol{p}_1 \\ \boldsymbol{p}_2 \end{bmatrix} \in \mathbb{R}^6, \tag{3}$$

and the scalar constraint is

$$C = \frac{1}{2}\left(\boldsymbol{e}^\top \boldsymbol{e} - L^2\right), \quad \text{where} \quad \boldsymbol{e} = \boldsymbol{p}_1 - \boldsymbol{p}_2. \tag{4}$$

The dynamics are then given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{q}}}(\boldsymbol{q}, \dot{\boldsymbol{q}}) - \frac{\partial \mathcal{L}}{\partial \boldsymbol{q}}(\boldsymbol{q}, \dot{\boldsymbol{q}}) - z\nabla C(\boldsymbol{q}) = \boldsymbol{Q} \tag{5a}$$

$$C(\boldsymbol{q}) = 0. \tag{5b}$$

In this case, the fully assembled model should have the form:

$$\dot{\boldsymbol{q}} = v \tag{6a}$$
$$M(\boldsymbol{q})\dot{v} = \boldsymbol{b}(\boldsymbol{q}, z, \boldsymbol{u}) \tag{6b}$$
$$0 = C(\boldsymbol{q}). \tag{6c}$$

Perform the following tasks:

1. Find the model matrices $M(\boldsymbol{q})$ and $\boldsymbol{b}(\boldsymbol{q}, \dot{\boldsymbol{q}}, \boldsymbol{u})$.
   Note that these matrices are not the same as in the previous part.
   In this case, it is fairly easy to derive the model equations by hand.
   However, if you prefer, you can use Matlab's symbolic toolbox to calculate them by completing the delivered template `HoveringMassConstraintTemplate.m`.
2. Compare the complexity of the models from part a. and b., i.e. the complexity of the symbolic expressions for M and $\boldsymbol{b}$. What do you conclude?

Solution: In this case, the Lagrangian is

$$\mathcal{L} = \frac{1}{2} m_1 \dot{\boldsymbol{p}}_1^\top \dot{\boldsymbol{p}}_1 + \frac{1}{2} m_2 \dot{\boldsymbol{p}}_2^\top \dot{\boldsymbol{p}}_2 - m_1 g \boldsymbol{p}_1^\top \boldsymbol{e}_3 - m_2 g \boldsymbol{p}_2^\top \boldsymbol{e}_3\,,$$

and the generalized force is

$$\boldsymbol{Q} = \begin{bmatrix} \boldsymbol{u} \\ \boldsymbol{0} \end{bmatrix}.$$

Note that there is only one multiplier (z) since there is only one constraint.
By applying (5), we obtain the following model:

$$\begin{aligned} m_1 \ddot{\boldsymbol{p}}_1 &= -m_1 g \boldsymbol{e}_3 - z\left(\boldsymbol{p}_1 - \boldsymbol{p}_2\right) + \boldsymbol{u} \\ m_2 \ddot{\boldsymbol{p}}_2 &= -m_2 g \boldsymbol{e}_3 + z\left(\boldsymbol{p}_1 - \boldsymbol{p}_2\right) \\ 0 &= \frac{1}{2}\left(\left(\boldsymbol{p}_1 - \boldsymbol{p}_2\right)^\top \left(\boldsymbol{p}_1 - \boldsymbol{p}_2\right) - L^2\right). \end{aligned}$$

In other words,

$$M = \begin{bmatrix} m_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & m_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & m_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & m_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & m_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & m_2 \end{bmatrix} = \begin{bmatrix} m_1 \boldsymbol{I}_3 & \boldsymbol{0} \\ \boldsymbol{0} & m_2 \boldsymbol{I}_3 \end{bmatrix},$$

$$\boldsymbol{b}(\boldsymbol{q}, z, \boldsymbol{u}) = \begin{bmatrix} -m_1 g \boldsymbol{e}_3 - z\left(\boldsymbol{p}_1 - \boldsymbol{p}_2\right) + \boldsymbol{u} \\ -m_2 g \boldsymbol{e}_3 + z\left(\boldsymbol{p}_1 - \boldsymbol{p}_2\right) \end{bmatrix}.$$

It is evident that the expressions for this model are a lot simpler than the expressions for the previous model. However, in this case we end up with an DAE, and not an ODE.

Problem 2 (Explicit vs. Implicit model)

(a) Write the hovering mass model of part 1.b in the form:

$$\begin{bmatrix} M & \boldsymbol{a}(\boldsymbol{q}) \\ \boldsymbol{a}(\boldsymbol{q})^\top & 0 \end{bmatrix} \begin{bmatrix} \ddot{\boldsymbol{q}} \\ z \end{bmatrix} = \boldsymbol{c}\left(\boldsymbol{q}, \dot{\boldsymbol{q}}, \boldsymbol{u}\right). \tag{7}$$

Specify what the functions $\boldsymbol{a}$ and $\boldsymbol{c}$ are.

Solution: As shown in the lecture notes $\boldsymbol{a}$ is the gradient of the constraints.
In this case, we have that

$$\boldsymbol{a} = \nabla_{\boldsymbol{q}} C(\boldsymbol{q}) = \begin{bmatrix} \boldsymbol{p}_1 - \boldsymbol{p}_2 \\ \boldsymbol{p}_2 - \boldsymbol{p}_1 \end{bmatrix}.$$

Moreover, the vector $\boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}}, \boldsymbol{u})$ is given by

$$\boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}}, \boldsymbol{u}) = \begin{bmatrix} \boldsymbol{b} + z\nabla_{\boldsymbol{q}} C \\ -\frac{\partial}{\partial \boldsymbol{q}}\left(\frac{\partial C}{\partial \boldsymbol{q}}\dot{\boldsymbol{q}}\right)\dot{\boldsymbol{q}} \end{bmatrix} = \begin{bmatrix} -m_1 g \boldsymbol{e}_3 + \boldsymbol{u} \\ -m_2 g \boldsymbol{e}_3 \\ -(\dot{\boldsymbol{p}}_1 - \dot{\boldsymbol{p}}_2)^\top (\dot{\boldsymbol{p}}_1 - \dot{\boldsymbol{p}}_2) \end{bmatrix}.$$

(b) Compare the model in the form (7) to its explicit counterpart:

$$\begin{bmatrix} \ddot{\boldsymbol{q}} \\ z \end{bmatrix} = \begin{bmatrix} M & \boldsymbol{a}(\boldsymbol{q}) \\ \boldsymbol{a}(\boldsymbol{q})^\top & 0 \end{bmatrix}^{-1} \boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}}, \boldsymbol{u}). \tag{8}$$

Make sure to use Matlab's symbolic toolbox for finding the right-hand side of (8).
The last part of `HoveringMassConstraintTemplate.m` provides a template for this.
What do you observe?
Which form of the ODE would you rather evaluate when solving it numerically?

Solution: Since

$$\begin{bmatrix} M & \boldsymbol{a}(\boldsymbol{q}) \\ \boldsymbol{a}(\boldsymbol{q})^\top & 0 \end{bmatrix} = \begin{bmatrix} m_1 \boldsymbol{I}_3 & \boldsymbol{0} & \boldsymbol{p}_1 - \boldsymbol{p}_2 \\ \boldsymbol{0} & m_2 \boldsymbol{I}_3 & \boldsymbol{p}_2 - \boldsymbol{p}_1 \\ \boldsymbol{p}_1^\top - \boldsymbol{p}_2^\top & \boldsymbol{p}_2^\top - \boldsymbol{p}_1^\top & 0 \end{bmatrix},$$

$$\boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}}, \boldsymbol{u}) = \begin{bmatrix} -m_1 g \boldsymbol{e}_3 + \boldsymbol{u} \\ -m_2 g \boldsymbol{e}_3 \\ -(\dot{\boldsymbol{p}}_1 - \dot{\boldsymbol{p}}_2)^\top (\dot{\boldsymbol{p}}_1 - \dot{\boldsymbol{p}}_2) \end{bmatrix},$$

the expressions in (7) are relatively easy to evaluate.
However, as the symbolic calculations show, the right-hand side expression in (8) is very complicated and difficult to grasp. An evaluation of this expression will be more computationally costly than evaluating the expressions in (7) and then solving the corresponding linear system numerically.

Problem 3 (Δ-robot)

Δ-robots are common in ultra-fast packaging applications. Figure 1 illustrates a Δ-robot. The three yellow arms of length l are actuated and can pivot in their vertical planes. These arms drive the three double thin rods of length L (typically made of ultra-light carbon fiber), connected to the nacelle (triangular shape at the bottom). The geometry imposes that the nacelle always remains horizontal. Moreover, the pivots on the nacelle are forced to remain at a distance L from the pivot at the extremities of the yellow arms.

In order to express the Lagrange function and the constraints, the following parameters and variables are defined: The cartesian frame has its origin at the center of the upper platform, with the x-axis aligned with the frontal robot arm, and the z axis pointing up. The position of the pivots of the yellow arms in this cartesian frame, $\boldsymbol{p}_{1,2,3}$, are given by

$$\boldsymbol{p}_k = R_k \begin{bmatrix} d + l\cos\alpha_k \\ 0 \\ -l\sin\alpha_k \end{bmatrix}, \qquad R_k = \begin{bmatrix} \cos\gamma_k & -\sin\gamma_k & 0 \\ \sin\gamma_k & \cos\gamma_k & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{9}$$

where $\gamma_{1,2,3} = \left\{0, \frac{2\pi}{3}, \frac{4\pi}{3}\right\}$, d is the constant distance from the center of the upper platform to the axis of the motors, and α_k are the angles of the yellow arms with respect to the horizontal plane. The yellow arms together with the motors have an inertia J, i.e. their kinetic energy is $T_k = \frac{1}{2}J\dot{\alpha}_k^2$. The nacelle has a mass m. For simplicity, we will assume that the nacelle is just a point where the long arms are all connected.

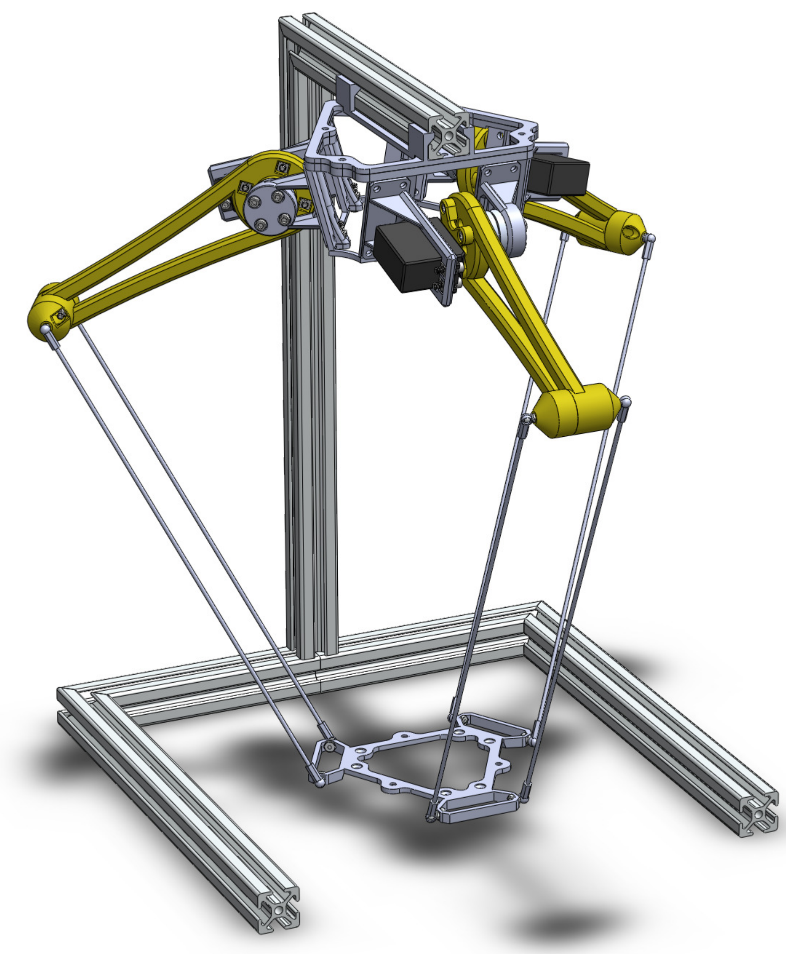

Figure 1: Illustration of the Δ-robot. The yellow arms (length l) are actuated by the motors on the upper platform (black boxes).

(a) Assume that we use the classical unconstraint Lagrange approach with the angles $\alpha_{1,2,3}$ as the generalized coordinates.
Explain what is the challenge with this approach in this particular case.
Hint: How would you find the position of the nacelle as a function of the generalized coordinates?

Solution: The fix length of the thin rods gives the following equations for the nacelle position

$$\|\boldsymbol{p} - \boldsymbol{p}_k\|^2 = L^2$$

for $k = 1, 2, 3$, where $\boldsymbol{p}$ is the position of the nacelle.
Note that $\boldsymbol{p}_k$ is a function of α_k as explained above.
These equations can only be solved numerically since no close-form expressions exist.

We choose the following generalized coordinates for the Δ-robot:

$$\boldsymbol{q} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \boldsymbol{p} \end{bmatrix},$$

where $\boldsymbol{p} \in \mathbb{R}^3$ is the position of the nacelle.

(b) Write down the Lagrange function $\mathcal{L}$ of the Δ-robot, as well as the associated constraints $\boldsymbol{c}$.

Solution: Let us define the constraint function first. It must capture the knowledge that the nacelle position $\boldsymbol{p}$ is at a distance L of the end-points of the arms, $\boldsymbol{p}_k$. This can be written as:

$$\boldsymbol{c}(\boldsymbol{q}) = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \quad \text{, where} \quad c_k = \|\boldsymbol{p} - \boldsymbol{p}_k\|^2 - L^2.$$

The kinetic and potential energy of the robot are given by:

$$T(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \frac{1}{2} J \sum_{k=1}^{3} \dot{\alpha}_k^2 + \frac{1}{2} m \dot{\boldsymbol{p}}^\top \dot{\boldsymbol{p}},$$
$$V(\boldsymbol{q}) = mg\boldsymbol{p}^\top \boldsymbol{e}_3.$$

Hence, the Lagrange function reads as:

$$\mathcal{L}(\boldsymbol{q}, \dot{\boldsymbol{q}}, \boldsymbol{z}) = T - V - \boldsymbol{z}^\top \boldsymbol{c}.$$

(c) What is the differential index of the DAE that results from $\mathcal{L}$ and $\boldsymbol{c}$?

Solution: Since the DAE results from a constrained Lagrange formulation, its differential index is 3. This is due to the structure of the equations. See the lecture notes for the details.

(d) What are the consistency conditions of the Δ-robot?

Solution: The consistency conditions are:

$$C(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \begin{bmatrix} \boldsymbol{c} \\ \dot{\boldsymbol{c}} \end{bmatrix} = \boldsymbol{0}.$$