TTK4130 Assignment 7 Eirik Falch

Problem 1

$$\dot{x} = f(x, u, t)$$

$$k_1 = f(x(t_k), u(t_k), t_k)$$

$$k_2 = f(x(t_k) + a\Delta t \cdot k_1, u(t_k + c\Delta t), t_k + c\Delta t)$$

$$x_{k+1} = x(t_k) + \Delta t \sum_{j=1}^{2} b_j k_j$$

$$\begin{array}{c|cc} 0 & & \\ c & a & \\ \hline & b_1 & b_2 \end{array}$$

a) $e_k = x_{k+1} - x(t_{k+1})$

$$x(t_{k+1}) = x(t_k) + \Delta t \dot{x}(t_k) + \frac{\Delta t^2}{2} \ddot{x}(t_k) + O(\Delta t^3)$$

$$= x(t_k) + \Delta t f(x(t_k), u_k) + \frac{\Delta t^2}{2} \dot{f}(x(t_k), u_k) + O(\Delta t^3)$$

$$= x(t_k) + \Delta t f(x(t_k), u_k) + \frac{\Delta t^2}{2}\left(\frac{\partial f}{\partial x} \dot{x}\right) + O(\Delta t^3)$$

$$x_{k+1} = x(t_k) + \Delta t \cdot b_1 k_1 + \Delta t b_2 k_2$$

$$= x(t_k) + \Delta t b_1 f(x(t_k), u_k) + \Delta t b_2 f(x(t_k) + a\Delta t f(x(t_k), u_k), u_k) = (*)$$

$$f(x(t_k) + a\Delta t f(x(t_k), u_k), u_k)$$

$$= f(x(t_k), u_k) + a\Delta t \left.\frac{\partial f}{\partial x}\right|_{x(t_k)} f(x(t_k), u_k) + O(\Delta t^2)$$

$$(*) = x(t_k) + b_1 \Delta t\, f(x(t_k), u_k)$$
$$+ b_2 \Delta t \left[f(x(t_k), u_k) + a \Delta t \frac{\partial f}{\partial x} \dot{x} + O(\Delta t^2) \right]$$

$$= x(t_k) + (b_1 + b_2) \Delta t\, f(x(t_k), u_k)$$
$$+ a b_2 \Delta t^2 \frac{\partial f}{\partial x} \dot{x} + b_2 O(\Delta t^3)$$

Conditions

$0 \leq c \leq 1 \qquad b_1 + b_2 = 1 \qquad ab_2 = 1/2$

b) The global error reduces the power of the one-step error to $O(\Delta t^2)$ which corresponds to the order of the RK2 method.

c) $x(t) = x(t_k) + \sum_{i=1}^{n} \frac{\alpha_i}{i!} (t - t_k)^i \qquad t \in [t_k, t_{k+1}] \qquad (4)$

$$= x(t_k) + \alpha_1 (t - t_k) + \frac{\alpha_2}{2} (t - t_k)^2 + \sum_{i=3}^{n} \frac{\alpha_i}{i!} (t - t_k)^i$$

$$x(t_{k+1}) = x(t_k) + \sum_{i=1}^{n} \frac{\alpha_i}{i!} (t_{k+1} - t_k)^i$$
$$= x(t_k) + \sum_{i=1}^{n} \frac{\alpha_i}{i!} \Delta t^i$$
$$n=2 \rightarrow\; = x(t_k) + \alpha_1 \Delta t + \alpha_2 \frac{\Delta t^2}{2}$$

$\underline{\underline{n=2}} \rightarrow x$ is quadratic and $f(x) = \dot{x}$ is linear, as long as the conditions from a) is met, the RK2 is exact.

Problem 2

a) We can see a clear difference between the true solution and the simulations. The simulations are clearly discrete and are built up of line segments.

As expected, RK4 is closest to the true solution. RK2 is closer throughout than RK1, but there's a larger gap at the end. This might just be a coincident, though.

b) Theoretical:

RK1: $\|x_N - x(T)\| \leq c\,O(\Delta t) \rightarrow$ order $= 1$

RK2: $\|x_N - x(T)\| \leq c\,O(\Delta t^2) \rightarrow$ order $= 2$

RK4: $\|x_N - x(T)\| \leq c\,O(\Delta t^4) \rightarrow$ order $= 4$

Notice the plot is log-log. The linear plots corresponds to polynomial functions with linear axis. The order is the slope of the linear plots. Since the linearity of the plots break down for $\Delta t > 0{,}1$, we will only consider for $\Delta t \leq 0{,}1$. The slope is found by

$$\frac{\log_{10}\{e(\Delta t = 10^{-1})\} - \log_{10}\{e(\Delta t = 10^{-3})\}}{\log_{10}(10^{-1}) - \log_{10}(10^{-3})}$$

RK1 : order = 0,98

RK2 : order = 2,04

RK4 : order = 4,04

The true order corresponds very well with the theoretical order when $10^{-3} \leq \Delta t \leq 10^{-1}$. Above $\Delta t > 10^{-1}$ it starts to break down

c)

Problem 3

a) As seen, shorter time-steps are taken close to the fast dynamics. This is as expected.

b) With $\Delta t = 0{,}1$, the RK4 solution follows the ODE45 solution to a high degree of accuracy. Increasing the time step slightly to $\Delta t = 0{,}15$ gives undesirable effects such as the solution starts lagging towards the end.

The maximal step size in ODE45 time grid is slightly above 0,1 while the minimal step size is close to 0,01. The step size is periodic in line with the dynamics in Van der Pol.