Out: January 13, 2020
Due: January 26, 2020

NTNU – Trondheim
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TTK4130 Modeling and Simulation
Assignment 1

Introduction

The conditions for the existence of a solution to an ODE, as well as the linearization and study of the stability of the dynamic system that this ODE represents, constitute fundamental knowledge and techniques for modeling and simulation.

Regarding modeling, it is mostly performed by using physical principles, such as Newton's second law, the conservation of mass, momentum, energy, etc. However, one can also model dynamic systems using statistical relations or heuristic assumptions.

An example of the later approach are the classical Lokta-Volterra equations:

$$\begin{cases} \dot{x} &= \alpha x - \beta xy \\ \dot{y} &= -\gamma y + \delta xy \,, \end{cases} \tag{1}$$

which model the dynamics of two isolated animal populations: The prey and the predators.

Here, x is the number of prey (e.g. tapirs) and y is the number predators (e.g. jaguars). In this model, $\alpha > 0$ is the birth-rate of the prey and $\gamma > 0$ is the death-rate of the predators. The interactions between these populations are modeled by the product xy, which is a measure of the number of encounters. Some encounters result in the death of prey, and enough eaten prey enable the predators to have offspring. How much the populations vary due to this interactions is described by the parameters $\beta > 0$ and $\delta > 0$.

Problem 1 (Linearization and stability)

Consider the following 3 dynamic systems:

1. $$\begin{cases} \dot{x}_1 &= au_1 - b\sqrt{x_1} \\ \dot{x}_2 &= \frac{a}{x_1}\left(u_1(u_2 - x_2) + c(u_3 - x_2)\right) \,, \end{cases} \tag{2}$$

   where $u_1, u_2, u_3 > 0$ are inputs and $a, b, c > 0$ are parameters.

2. $$\ddot{x} + c\dot{x} + g\left(1 - \left(\frac{x_d}{x}\right)^{\kappa}\right) = 0 \,, \tag{3}$$

   where $c > 0$, $g > 0$, $\kappa > 1$ and x_d are parameters.

3. $$\dot{x} = \begin{cases} y - \frac{x}{\ln\sqrt{x^2+y^2}} & , [x, y] \neq [0, 0] \\ 0 & , [x, y] = [0, 0] \end{cases} \qquad \dot{y} = \begin{cases} -x - \frac{y}{\ln\sqrt{x^2+y^2}} & , [x, y] \neq [0, 0] \\ 0 & , [x, y] = [0, 0]. \end{cases} \tag{4}$$

(a) Write the systems in state-space form if there are not in this form already.

(b) Find the equilibrium points of each system.
For system 1, find the equilibrium points given a constant input $\mathbf{u} = [u_1,\, u_2,\, u_3]^T$.
Hint: For system 3, $\dot{x} = 0$ and $\dot{y} = 0$ implies $y\dot{x} - x\dot{y} = 0$.

(c) Linearize each system around each of its equilibrium points.
Hint: For system 3, one can avoid cumbersome calculations using that $\frac{r}{\log|r|} = o(r)$.

(d) Are the linearized systems stable, asymptotically stable or unstable? Justify your answers.

Problem 2 (Existence of solution)
Consider the following ODEs:

$$\dot{x} = x^2, \quad x(0) = 1 \tag{5}$$

$$\dot{x} = \sqrt{|x|}, \quad x(0) = 0. \tag{6}$$

For both ODEs, do the following:

(a) Simulate using the Matlab function `ode45` for a final time $t = 5$. What do you observe?

(b) Provide a formal explanation of the obtained results.

Problem 3 (Zombie apocalypse)
The doomsday is upon us!

Zombies have begun to rise from the dead, and violently and indiscriminately kill and infest the living. In order to save humanity, you have to model and simulate the zombie infestation using the little information available on these abominations.

There are 4 well-defined and non-overlapping populations:

- The healthy (H): Healthy people.
- The infected (I): People that have survived a zombie encounter, but have been infected.
- The zombies (Z).
- The dead (D): Recently deceased people and neutralized zombies.

The dynamics and interactions between these populations can be modeled like the classical predator-prey equations, i.e. the Lokta-Volterra equations, and they are given by the following laws:

- The birth-rate of the healthy is given by the parameter $b > 0$. In addition, the birth-rate is damped by a quadratic term with parameter b_d.
- The healthy can either become dead by "natural" causes (non-zombie interaction) with death-rate $d > 0$, or can become infected due to interactions with zombies with parameter i.
- The healthy that become infected, remain infected for some time and then become zombies. This is model as a first order system with rate $a > 0$.
- Infected individuals can still die by "natural" causes before becoming a zombie with the same death-rate as healthy individuals. In such case, they become dead; otherwise, they become a zombie.
- Zombies rise from the dead with rate $r > 0$.
- Some zombies that interact with healthy individuals are neutralized with parameter $n > 0$. This zombies become dead, and may rise again in the future.

(a) Model the dynamics of the populations H, I, Z and D.
Hint: The expressions for $\dot{H}$, $\dot{I}$, $\dot{Z}$ and $\dot{D}$ are given by polynomials in H, I, Z and D. Furthermore, $\dot{H} + \dot{I} + \dot{Z} + \dot{D}$ is equal to the total birth-rate of the healthy, $bH - b_d H^2$.

The situation looks grim: All simulations confirm that the outbreak of zombies will lead to $H \to 0$, i.e. the collapse of civilization.

In order to contain the outbreak, the authorities ask you to model the effects of a partial quarantine of zombies and infected individuals. Quarantined individuals are removed from their original populations and are placed on an special area, where they no longer can infect healthy individuals.

Consider the new population of quarantined individuals (Q), and add the following changes to the previous model:

- Infect individuals and zombies are quarantined with rates q_i and q_z, respectively.
- Some quarantined individuals will try to escape, and are immediately killed or neutralized, i.e. they are moved to the dead population. This happens with rate d_q.
- The quarantined individuals that are moved to the dead population, may still rise as "free roaming" zombies.

(b) Model the dynamics of the populations H, I, Z, D and Q.

(c) Simulate the models with and without quarantine for 100 days. Use the Matlab function `ode45` or another in-built solver of your choosing. Moreover, use the following parameter values and initial conditions:

Parameter	Value [s^{-1}]	Population	Initial value
a	$1.4 \cdot 10^{-6}$	H	$\frac{b-d}{b_d}$
b	$3.1 \cdot 10^{-8}$	I	0
b_d	$5.6 \cdot 10^{-16}$	Z	0
d	$2.8 \cdot 10^{-8}$	D	0
i	$2.6 \cdot 10^{-6}$	Q	0
n	$1.4 \cdot 10^{-6}$		
r	$2.8 \cdot 10^{-7}$		
q_i	$2.7 \cdot 10^{-6}$		
q_z	$2.7 \cdot 10^{-6}$		
d_q	$2.8 \cdot 10^{-5}$		

Table 1: Parameter and initial values.

Add a plot with the obtained results for all populations to your answer.

Comment on the results.

Hint: The time frame of these simulation is very large. Therefore, using the solver directly will not be feasible due to large computational time and lack of memory. Read the documentation of the solver in order to find a way to circumvent this issue.