Out: February 17, 2020
Due: March 1, 2020

NTNU – Trondheim
Norwegian University of
Science and Technology

TTK4130 Modeling and Simulation

Assignment 6

Introduction

The objective of this assignment are:

- To understand what differential algebraic equations (DAEs) are, and what makes them different from ordinary differential equations (ODEs).
- To be able to calculate the differential index of a DAE, and perform index reductions.
- To understand and apply Tikhonov's theorem for dynamical systems.

Problem 1 (Differential index)

Consider the DAE:

$$\dot{x}_1 = x_1 + x_2 + z \tag{1a}$$
$$\dot{x}_2 = z + u \tag{1b}$$
$$0 = \frac{1}{2}\left(x_1^2 + x_2^2 - 1\right). \tag{1c}$$

Moreover, assume that $x_1 + x_2 \neq 0$.

(a) Why is (1) actually a DAE?

Solution: We can answer in a simple or formal way:

- The simple answer: We observe that the variable z does not enter as time-differentiated in (1). Hence, this variable is an algebraic variable, and (1) is therefore a DAE.
- The formal answer: We observe that (1) is given by the fully implicit differential equation:

$$F(\dot{\boldsymbol{s}}, \boldsymbol{s}, u) = \begin{bmatrix} \dot{x}_1 - x_1 - x_2 - z \\ \dot{x}_2 - z - u \\ \frac{1}{2}\left(x_1^2 + x_2^2 - 1\right) \end{bmatrix}$$

where $\boldsymbol{s} = \begin{bmatrix} x_1 & x_2 & z \end{bmatrix}^\top$.
Since the Jacobian matrix

$$\frac{\partial F}{\partial \dot{\boldsymbol{s}}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

is rank-deficient, (1) is a DAE.

(b) What is the differential index of (1)?

Solution: We observe that (1) is a semi-explicit DAE with

$$g(\boldsymbol{x}, z, u) = \frac{1}{2}\left(x_1^2 + x_2^2 - 1\right)$$

and $\frac{\partial g}{\partial z} = 0$. Hence, it is of index larger than 1. In order to find the differential index, we need to perform time-differentiations on (1) until it is transformed into an ODE. Because (1a)-(1b) are already in state-space form, we can leave them alone, and focus on (1c). We then observe that:

$$\frac{\mathrm{d}}{\mathrm{d}t} g\left(\boldsymbol{x}, z, u\right) = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\left(x_1^2 + x_2^2 - 1\right)\right) = x_1\dot{x}_1 + x_2\dot{x}_2.$$

Replacing $\dot{x}_1$, $\dot{x}_2$ by there expressions from (1a)-(1b), we obtain:

$$\frac{\mathrm{d}}{\mathrm{d}t} g\left(\boldsymbol{x}, z, u\right) = x_1\left(x_1 + x_2 + z\right) + x_2\left(z + u\right),$$

which is not yet a differential equation. A second time-derivative gives:

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} g\left(\boldsymbol{x}, z, u\right) = \dot{\boldsymbol{x}}_1\left(x_1 + x_2 + z\right) + \dot{\boldsymbol{x}}_2\left(z + u\right) + x_1\left(\dot{x}_1 + \dot{x}_2 + \dot{z}\right) + x_2\left(\dot{z} + \dot{u}\right)$$

We can then solve $\frac{\mathrm{d}^2}{\mathrm{d}t^2} g\left(\boldsymbol{x}, z, u\right) = 0$ for $\dot{z}$:

$$\dot{z} = \frac{-\dot{\boldsymbol{x}}_1\left(x_1 + x_2 + z\right) - \dot{\boldsymbol{x}}_2\left(z + u\right) - x_1\dot{x}_1 - x_1\dot{x}_2 - x_2\dot{u}}{x_1 + x_2},$$

which is an ODE as long as $x_1 + x_2 \neq 0$.
Since we had to differentiate 2 times in order to obtain a differential equation for z, the differential index of the DAE is 2.

(c) Perform an index reduction of (1).

Solution: We have already performed this task in the previous question. The index-reduced DAE is the one occurring ”one step before getting an ODE”, i.e.

$$\begin{aligned}
\dot{x}_1 &= x_1 + x_2 + z \\
\dot{x}_2 &= z + u \\
0 &= x_1\left(x_1 + x_2 + z\right) + x_2\left(z + u\right) \\
&= x_1^2 + x_1x_2 + x_2u + (x_1 + x_2)z.
\end{aligned}$$

Here as well, we need $x_1 + x_2 \neq 0$ to be able to solve the index-reduced DAE.

Problem 2 (Tikhonov's theorem)

Consider the differential equation:

$$\dot{\boldsymbol{x}} = -\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\boldsymbol{x} - \boldsymbol{z} \tag{3a}$$

$$\epsilon\dot{\boldsymbol{z}} = \frac{1}{10}\boldsymbol{x} - A\boldsymbol{z}, \tag{3b}$$

where

$$A = \begin{bmatrix} x_1^2 & x_2 \\ 0 & x_2^2 \end{bmatrix} + \alpha I, \tag{4}$$

with ϵ, $\alpha \geq 0$, and where I is the 2-by-2 identity matrix.

(a) Is (3) a DAE or an ODE? Explain.

Solution: For $\epsilon > 0$, (3) is an ODE, while for $\epsilon = 0$, it is a DAE.

(b) Simulate (3) numerically for small values of α (e.g. $\alpha = 10^{-3}$) and for $\epsilon \to 0$ (e.g. ϵ in the range $10^{-3} - 10^{-6}$). Compare the results to the ones from the DAE approximation resulting from $\epsilon = 0$. Use the initial conditions

$$x(0) = z(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{5}$$

and a final simulation time of at least 10.

Hint: Use the ode solver `ode15s` *in order to reduce the simulation time. This is a "stiff" solver. We will discuss what that means later in the course.*

Solution:

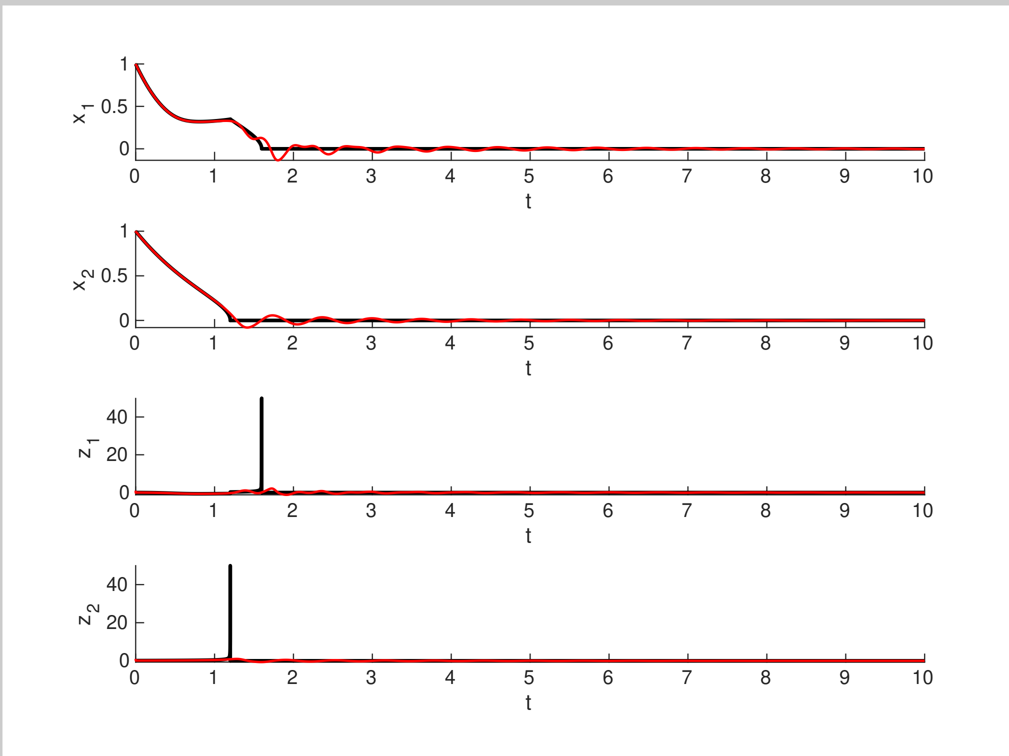

Figure 1: Results for $\epsilon = 10^{-6}$ (red) and results for $\epsilon = 0$ (black). $\alpha = 10^{-3}$.

(c) Repeat the previous part, but now with $\alpha = 0$.

Solution:

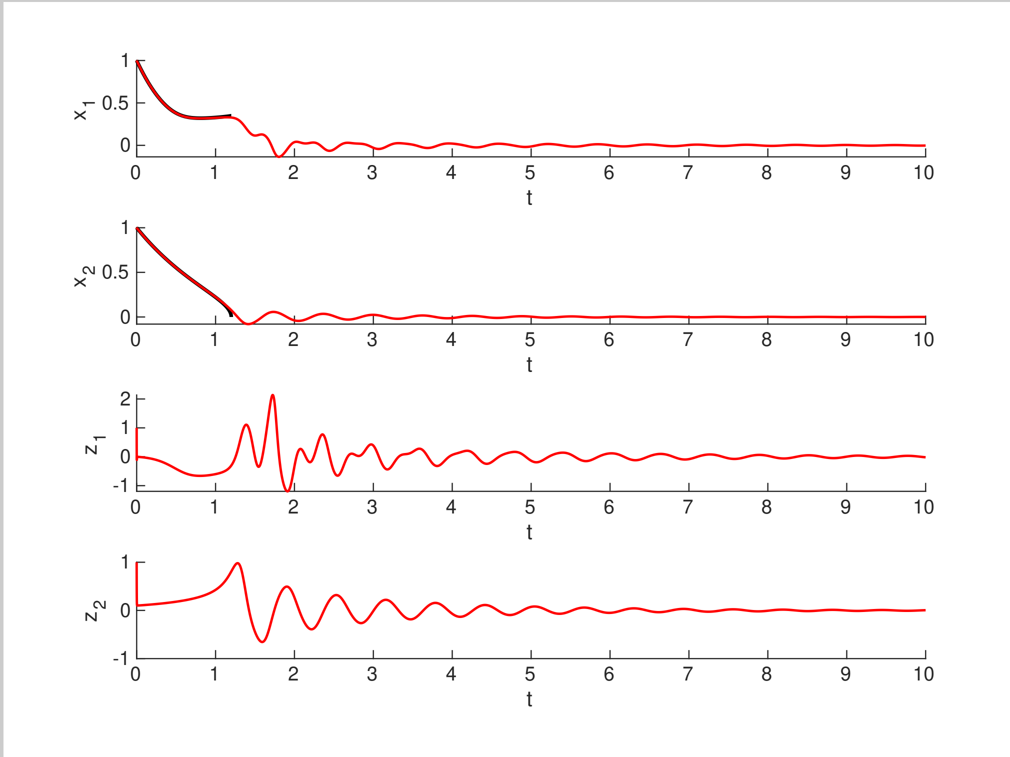

Figure 2: Results for $\epsilon = 10^{-6}$ (red) and results for $\epsilon = 0$ (black). The simulation for $\epsilon = 0$ failed. The corresponding z values are not presented because they diverge.

(d) Add plots of the simulation results to your answer.
Report what you observe, and explain it from a theoretical point of view.
Hint: The conditions of Tikhonov's theorem.

Solution: For $\alpha > 0$, the solution for $\epsilon > 0$ converges to the solution for $\epsilon = 0$ as $\epsilon \to 0$.
However, for $\alpha = 0$, this is not case. Actually z diverges as x_1 or x_2 approaches 0.
This is in accordance to Tikhonov's theorem.
Let $g(\boldsymbol{x}, \boldsymbol{z}) = \frac{1}{10}\boldsymbol{x} - A\boldsymbol{z}$.
One of the conditions of Tikhonov's theorem requires that the Jacobian $\frac{\partial g}{\partial z}$ is invertible for all $\boldsymbol{x} \in \mathbb{R}^2$.
This Jacobian is

$$\frac{\partial g}{\partial \boldsymbol{z}} = A = \begin{bmatrix} x_1^2 & x_2 \\ 0 & x_2^2 \end{bmatrix} + \alpha I \,,$$

and its determinant is

$$\det\left(\frac{\partial g}{\partial z}\right) = (x_1^2 + \alpha)(x_2^2 + \alpha).$$

Hence, we note that if $\alpha > 0$, then $\frac{\partial g}{\partial z}$ is always invertible. However, if $\alpha = 0$, then $\frac{\partial g}{\partial z}$ can be singular if $x_1 = 0$ or $x_2 = 0$.
Therefore, for $\alpha = 0$, Tikhonov's theorem cannot guarantee the convergence of the solutions for $\epsilon > 0$ to the solution for $\epsilon = 0$. In this particular case, we have that the solutions in fact diverge.

Problem 3 (ODE or DAE?)

For the following differential equations, determine if they are ODE or DAEs. If they are DAEs, specify (if possible) what are the algebraic and differential states.

(a)

$$\dot{x}_1 + u + x_1 + x_2 = 0 \tag{6a}$$

$$u + x_2 + \dot{x}_2\dot{x}_1 + \dot{x}_2 u + \dot{x}_2 x_1 + \dot{x}_2 x_2 + u^2 = 0. \tag{6b}$$

Solution: The differential equation (6) can be put in the fully implicit form:

$$\boldsymbol{F}(\dot{\boldsymbol{x}}, \boldsymbol{x}, u) = \begin{bmatrix} \dot{x}_1 + u + x_1 + x_2 \\ u + x_2 + \dot{x}_2(\dot{x}_1 + u + x_1 + x_2) - u^2 \end{bmatrix} = 0\,,$$

and we observe that:

$$\frac{\partial \boldsymbol{F}(\dot{\boldsymbol{x}}, \boldsymbol{x}, u)}{\partial \dot{\boldsymbol{x}}} = \begin{bmatrix} 1 & 0 \\ \dot{x}_2 & \dot{x}_1 + u + x_1 + x_2 \end{bmatrix}.$$

We should note here that the lower-right element of $\frac{\partial \boldsymbol{F}}{\partial \dot{\boldsymbol{x}}}$ is zero as it is the first equation in (6). Hence, $\frac{\partial \boldsymbol{F}}{\partial \dot{\boldsymbol{x}}}$ is rank-deficient as it has a column of zeros. Therefore, this equation system is not an ODE.

Moreover, we observe that (6) can be written as:

$$\dot{x}_1 + u + x_1 + x_2 = 0$$

$$x_2 + u + u^2 = 0.$$

Hence, this system is a DAE, and x_2 plays here the role of an algebraic variable. It is customary to rename algebraic variables as z or similar to stress this fact.

(b)

$$u + \dot{x}_1 x_1 + \dot{x}_2 x_2 = 0 \tag{8a}$$

$$u\dot{x}_1 x_1 + \dot{x}_2 u x_2 = 0. \tag{8b}$$

Solution: We perform the same tasks as before, i.e.:

$$\boldsymbol{F}(\dot{\boldsymbol{x}}, \boldsymbol{x}, u) = \begin{bmatrix} u + \dot{x}_1 x_1 + \dot{x}_2 x_2 \\ u\dot{x}_1 x_1 + \dot{x}_2 u x_2 \end{bmatrix} = 0\,,$$

and

$$\frac{\partial \boldsymbol{F}(\dot{\boldsymbol{x}}, \boldsymbol{x}, u)}{\partial \dot{\boldsymbol{x}}} = \begin{bmatrix} x_1 & x_2 \\ u x_1 & u x_2 \end{bmatrix}.$$

We observe that $\frac{\partial \boldsymbol{F}}{\partial \dot{\boldsymbol{x}}}$ is rank-deficient as the second row is the first one multiplied by u. In case of doubt, one can also verify that $\det\left(\frac{\partial \boldsymbol{F}}{\partial \dot{\boldsymbol{x}}}\right) = 0$. Hence, (8) is not an ODE.

Moreover, we observe that (8) can be rewritten as:

$$u + \dot{x}_1 x_1 + \dot{x}_2 x_2 = 0$$

$$u(\dot{x}_1 x_1 + \dot{x}_2 x_2) = 0\,,$$

such that $u = -\dot{x}_1 x_1 + \dot{x}_2 x_2$ and

$$-u^2 = 0$$

should hold. Then, the system reduces to

$$\dot{x}_1 x_1 + \dot{x}_2 x_2 = 0.$$

Since this system has less equations than variables, it is neither an ODE nor a DAE.

Problem 4 (Implicit DAE)

Consider the fully-implicit DAE:

$$\dot{x} + u + \tanh(\dot{x}) + xz = 0 \tag{10a}$$

$$\tanh(2u - z) = 0\,, \tag{10b}$$

where $x, z, u \in \mathbb{R}$ and $\tanh(\cdot)$ is the tangent hyperbolic function.

(a) Can you put (10) in the form of a semi-explicit DAE?

Solution: As written in the additional lecture note, a fully implicit can be trivially written as a semi-explicit DAE by introducing some "helper variables" (labelled ν here). In our case, this gives

$$\begin{aligned} \dot{x} &= \nu \\ 0 &= \nu + u + \tanh(\nu) + xz \\ 0 &= \tanh(2u - z)\,. \end{aligned}$$

Moreover, we will prove in the next part that this DAE can be written in the form:

$$\begin{aligned} \dot{x} &= f(x, u) \\ z &= g(x, u)\,, \end{aligned}$$

which is even simpler than the default semi-explicit form. However, we cannot provide explicit expressions for f and g. Therefore, we would need to solve for $\dot{x}$ and z numerically.

(b) Does (10) always provide a well-defined trajectory?

Hint: Use the Implicit Function Theorem.

Solution: We use the Implicit Function Theorem here to decide if (10) can be solved for $\dot{x}$ and z. We write:

$$F(\dot{x}, x, z, u) = \begin{bmatrix} \dot{x} + u + \tanh(\dot{x}) + xz \\ \tanh(2u - z) \end{bmatrix} = 0$$

and compute:

$$M = \begin{bmatrix} \frac{\partial F}{\partial \dot{x}} & \frac{\partial F}{\partial z} \end{bmatrix} = \begin{bmatrix} 2 - \tanh(\dot{x})^2 & x \\ 0 & \tanh(2u - z)^2 - 1 \end{bmatrix}$$

and

$$\det(M) = -(\tanh(2u - z)^2 - 1)(\tanh(\dot{x})^2 - 2)$$

Since $\tanh(x) \in (-1, 1)$ for all $x \in \mathbb{R}$, we have that $\det(M) \neq 0$. Therefore, the Implicit Function Theorem guarantees that we can solve for $\dot{x}$ and z, i.e., there are smooth functions f and g such that

$$\begin{aligned} \dot{x} &= f(x, u) \\ z &= g(x, u). \end{aligned}$$

Therefore, the trajectory of x is well-defined, and so is the one for z.

(c) What is the differential index of (10)?

Solution: The function that gives the algebraic equation in (10) is $g(x, z, u) = \tanh(2u - z)$. Since

$$\frac{\partial g}{\partial z} = \tanh(2u - z)^2 - 1 \neq 0 \,,$$

the differential index of (10) is 1.

Another quicker way of finding the differential index, is to realize that equation $\tanh(2u - z) = 0$ is equivalent to the equation $z = 2u$. In particular, the differential index is 1.