TTK4130 Assignment 6 Eirik Falck

Problem 1

$$(1)\quad \begin{cases} \dot{x}_1 = x_1 + x_2 + z \\ \dot{x}_2 = z + u \\ 0 = \frac{1}{2}(x_1^2 + x_2^2 - 1) = g(x, z, u) \end{cases}$$

a) (1) is a DAE because it contains states that doesn't apper time differentiated, namely the z state.

c) $$\frac{d}{dt} g = x_1 \dot{x}_1 + x_2 \dot{x}_2 = x_1(x_1 + x_2 + z) + x_2(z + u) = 0$$

$$= x_1^2 + x_1 x_2 + x_1 z + x_2 z + x_2 u$$

$$= x_1^2 + x_2(x_1 + u) + z(x_1 + x_2) = 0$$

$$\frac{\partial}{\partial z}(\cdot) = x_1 + x_2 = 0 \text{ for some } x$$

$\Rightarrow$ not full rank for all x

$$\frac{d}{dt}(\cdot) = 2x_1\dot{x}_1 + x_2(\dot{x}_1 + \dot{u}) + \dot{x}_2(x_1 + u) + z(\dot{x}_1 + \dot{x}_2) + \dot{z}(x_1 + x_2) = 0$$

$$= 2x_1(x_1 + x_2 + z) + x_2(x_1 + x_2 + z + \dot{u}) + (z + u)(x_1 + u)$$

$$+ z(x_1 + x_2 + z + z + u) + \dot{z}(x_1 + x_2)$$

$$\dot{z} = \frac{1}{x_1 + x_2}(\ldots)$$

b) **Index 2**

Problem 2

$$\dot{x} = -\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x - z$$

$$\epsilon \dot{z} = \frac{1}{10} x - A z \quad , \quad A = \begin{bmatrix} x_1^2 & x_2 \\ 0 & x_2^2 \end{bmatrix} + \alpha I$$

$\epsilon, \alpha \geq 0$

a) (3) is not a DAE as long as $\epsilon > 0$.
If $\epsilon = 0$, z doesn't appear time differentiated and it does become a DAE.

b) As we can see in the figures, the DAE follows well in the start but misses the oscillations at the end when $\epsilon = 1e-3$.

When $\epsilon = 1e-5$, the DAE is almost a perfect fit with the ODE.

c) An interesting observation is that the simulation of DAE breaks down after a while. Up untill that point, however, it follows the ODE well.

d) $\alpha \geq 0$:

$$g(x,z) = \frac{1}{10}x - \begin{bmatrix} \alpha + x_1^2 & x_2 \\ 0 & \alpha + x_2^2 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

$$= \begin{bmatrix} x_1/10 - z_1(\alpha + x_1^2) - z_2 x_2 \\ x_2/10 - z_2(\alpha + x_2^2) \end{bmatrix} = \epsilon \dot{z} \xrightarrow{\epsilon \to 0} 0$$

$$\frac{\partial g}{\partial z} = \begin{bmatrix} -\alpha - x_1^2 & -x_2 \\ 0 & -\alpha - x_2^2 \end{bmatrix} \overset{\alpha = 0}{=} \begin{bmatrix} -x_1^2 & -x_2 \\ 0 & -x_2^2 \end{bmatrix}$$

The dynamics of $\epsilon \dot{z} = g(x,z)$ are always stable because $\lambda \leq 0$.

When $\alpha > 0$, $\frac{\partial g}{\partial z}$ is always full rank, so Tikhonovs theorem holds for all z.

When $\alpha = 0$ $\frac{\partial g}{\partial z}$ is singular when $x_2 = 0$ or $x_1 = 0$ or both. This happens at $t \approx 1,2$ and explains our observations in c) that the DAE breaks down.

Problem 3

a) $$\underbrace{\dot{x}_1 + u + x_1 + x_2}_{} = 0 \qquad (1a)$$

$$u + x_2 + \dot{x}_2(\dot{x}_1 + u + x_1 + x_2) + u^2 = 0 \qquad (1b)$$

$$(1b) \rightarrow u + x_2 + \cancel{\dot{x}_2 \cdot 0} + u^2 = 0$$

$\Rightarrow$ This is a DAE.

- x_1 is differential state
- x_2 is algebraic state.

b) $$u + \dot{x}_1 x_1 + \dot{x}_2 x_2 = 0 \qquad (2a)$$

$$u\dot{x}_1 x_1 + u\dot{x}_2 x_2 = 0 \qquad (2b)$$

$$(2b) \rightarrow u\,(\underbrace{\dot{x}_1 x_1 + \dot{x}_2 x_2}_{=-u}) = 0 \Rightarrow -u^2 = 0$$

$$u = 0$$

$$(2a) \rightarrow \dot{x}_1 x_1 + \dot{x}_2 x_2 = 0$$

Cannot be specified, so it's neither DAE nor ODE.

Problem 4

$\dot{x} + u + \tanh(u\dot{x}) + xz = 0$ (10a) $= f$

$\tanh(2u - z) = 0$ (10b) $= g$

a) (10a) $f(\dot{x}, x, z, u) = 0$

$$\frac{\partial f}{\partial \dot{x}} = 1 + (1 - \tanh^2(u\dot{x}))u$$

$$= 1 + u - u\tanh^2(u\dot{x}) = 0 \text{ for } (x, u) = (0, -1)$$

Since $\frac{\partial f}{\partial \dot{x}}$ is not full rank for every (x, u), by the IFT, (10) cannot be written as a semi-explicit DAE.

b) $\begin{bmatrix} \frac{\partial F}{\partial \dot{x}} & \frac{\partial F}{\partial z} \end{bmatrix} = \begin{bmatrix} 1 + u - u\tanh^2(u\dot{x}) & x \\ 0 & \tanh(2u - z) - 1 \end{bmatrix}$

$\Rightarrow$ Not always full rank

$\rightarrow$ (10) does not always provide a well-defined trajectory.

c) $\frac{\partial g}{\partial z} = \tanh(2u - z) - 1 = 0$

$\hookrightarrow 2u - z \rightarrow \infty$

$\frac{\partial g}{\partial z}$ full rank for all x, z, u and is therefore index $\underline{\underline{1}}$.