Out: February 3, 2020
Due: February 16, 2020

NTNU – Trondheim
Norwegian University of Science and Technology

TTK4130 Modeling and Simulation

Assignment 4

Introduction

In this assignment we will study the modeling of complex mechanical systems based on the Lagrange approach. Since the calculations of the partial derivatives of the Lagrangian can be involved, we will outsource this task by using the Matlab Symbolic Math Toolbox™.

Problem 1 (Double pendulum on a cart)

In this task, we consider the double pendulum depicted in Figure 1. We assume that the system is frictionless. The cart (grey cube in the figure) is of mass $m = 1\,\text{kg}$ and the two black spheres are of mass $M = 1\,\text{kg}$. We assume the rest of the system as massless. The position of the cart on the rail is x and the position of the pendulums are described by the angles $\theta_1, \theta_2 \in \mathbb{R}$, which give the deviation from the vertical. The length of the pendulum arms are L=1 m. A force F is applied laterally on the cart to move it along the rail.

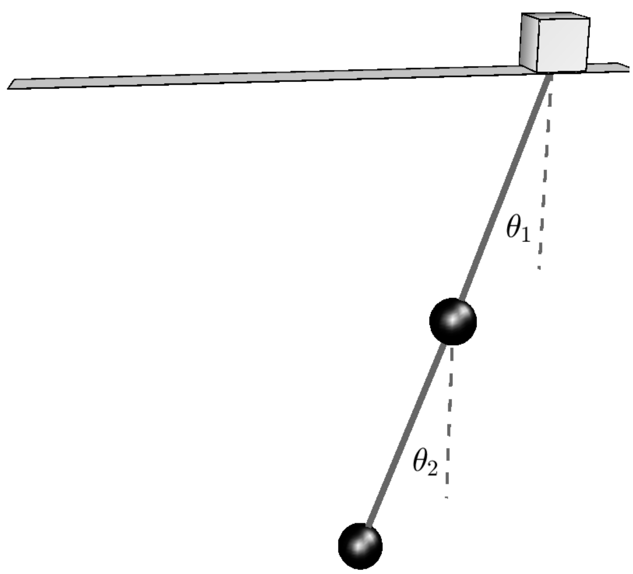

Figure 1: Schematic of the double pendulum.

(a) The Lagrange equations will be build by using the Matlab's symbolic toolbox.
Complete the Matlab routine `PendulumSymbolicTemplate.m` by doing the following tasks:

1. Define the generalized coordinates $\boldsymbol{q} = [x,\, \theta_1,\, \theta_2]^\top$ as a symbolic variable.
2. Define a symbolic variable for the derivative of the generalized coordinates.
3. Write the expressions for the positions of mass 1 and 2.
4. Complete the expressions for the kinetic and potential energies.
5. Write the expression for the Lagrangian function.
6. Run the routine.

Add the implemented code to your answer.

As shown in the course, the Lagrange equations can be written in state-space form as

$$\boldsymbol{x} = \begin{bmatrix} \boldsymbol{q} \\ \dot{\boldsymbol{q}} \end{bmatrix}, \qquad \dot{\boldsymbol{x}} = \begin{bmatrix} \dot{\boldsymbol{q}} \\ \frac{\partial^2 \mathcal{L}}{\partial \dot{\boldsymbol{q}}^2}^{-1} \left(\boldsymbol{Q} + \frac{\partial \mathcal{L}}{\partial \boldsymbol{q}} - \frac{\partial^2 \mathcal{L}}{\partial \dot{\boldsymbol{q}} \partial \boldsymbol{q}} \dot{\boldsymbol{q}} \right) \end{bmatrix}. \tag{1}$$

The routine implemented in part a. exports 2 Matlab functions. One that gives the positions of the masses, while the other returns the terms $\frac{\partial^2 \mathcal{L}}{\partial \dot{\boldsymbol{q}}^2}$ and $\boldsymbol{Q} + \frac{\partial \mathcal{L}}{\partial \boldsymbol{q}} - \frac{\partial^2 \mathcal{L}}{\partial \dot{\boldsymbol{q}} \partial \boldsymbol{q}} \dot{\boldsymbol{q}}$.

Note that these functions return the corresponding numerical values, and not the original symbolic expressions.

(b) Assume that the external force F is given by the PD control law:

$$F = -10x - \dot{x} \tag{2}$$

Write a Matlab function that gives the dynamics of the state-space model (1) by using the exported function `PendulumODEMatrices.m` and the PD control law (2).
In other words, this Matlab function should return the value of the right-hand side of (1) as a function of the states and the parameters.
Call this function `PendulumDynamics.m` and add it to your answer.

(c) Complete the delivered Matlab routine `PendulumSimulation.m` in order to make an animation of the simulation results.
Simulate your system using e.g. $x(0) = 0$, $\theta_1(0) = \frac{\pi}{4}$, $\theta_2(0) = \frac{\pi}{2}$ as initial conditions with initial velocities at rest.
Run the animation.
What do you observe? Are these results reasonable? Explain.
Hint: The function that returns the positions of the masses can come in handy here.

Solution: The generalized coordinates of the system are given by $\boldsymbol{q} = [x, \theta_1, \theta_2]^\top$. The position of the first mass is given by

$$\boldsymbol{p}_1(\boldsymbol{q}) = \begin{bmatrix} x + L\sin\theta_1 \\ -L\cos\theta_1 \end{bmatrix}, \qquad \boldsymbol{p}_2(\boldsymbol{q}) = \boldsymbol{p}_1(\boldsymbol{q}) + \begin{bmatrix} L\sin\theta_2 \\ -L\cos\theta_2 \end{bmatrix}.$$

The kinetic energy of the system is given by

$$T(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \frac{1}{2} m\dot{x}^2 + \frac{1}{2} \dot{\boldsymbol{q}}^\top M_{\mathrm{p}}(\boldsymbol{q})\, \dot{\boldsymbol{q}},$$

where

$$M_{\mathrm{p}}(\boldsymbol{q}) = M \left(\frac{\partial \boldsymbol{p}_1}{\partial \boldsymbol{q}}^\top \frac{\partial \boldsymbol{p}_1}{\partial \boldsymbol{q}} + \frac{\partial \boldsymbol{p}_2}{\partial \boldsymbol{q}}^\top \frac{\partial \boldsymbol{p}_2}{\partial \boldsymbol{q}} \right).$$

The potential energy of the system reads as

$$V(\boldsymbol{q}) = Mg\,(\boldsymbol{p}_1 + \boldsymbol{p}_2)^\top \begin{bmatrix} 0 \\ 1 \end{bmatrix} = -Mg\,(2\cos\theta_1 + \cos\theta_2).$$

The generalized force $\boldsymbol{Q}$ pertains to the cart only, as the only external force in the system is applied on the cart, and the only virtual work that it produces is via displacements of the cart. As a result:

$$\boldsymbol{Q} = \begin{bmatrix} F \\ 0 \\ 0 \end{bmatrix}.$$

We can then blindly apply the Lagrange equations:

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{q}}} - \frac{\partial \mathcal{L}}{\partial \boldsymbol{q}} = \boldsymbol{Q},$$

which take the form:

$$\frac{\partial^2 \mathcal{L}}{\partial \dot{\boldsymbol{q}}^2}\ddot{\boldsymbol{q}} + \frac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{q}}\partial \boldsymbol{q}}\dot{\boldsymbol{q}} - \frac{\partial \mathcal{L}}{\partial \boldsymbol{q}} = \boldsymbol{Q}.$$

Hence, the ODE describing the system reads as (1).

Problem 2 (Ball on a beam)

We consider here a solid ball on a beam system as depicted in Figure 2. The ball rolls without slipping (pure rotation) on a beam that is articulated without friction in the middle. A torque T acts on the beam joint. We will use the following numerical values: The rotational inertia of the rail around its joint is $J = 1\,\mathrm{kg\,m}^2$, the mass of the ball is $M = 10\,\mathrm{kg}$ and its radius is $R = 0.25\,\mathrm{m}$. The position of the ball respect to the joint of the rail will be labelled by x.

A rotation with $\dot{\theta} > 0$ obeys the right-hand rule for reference frames $\vec{a}$ and $\vec{b}$, while a displacement with $\dot{x} > 0$ moves the ball in the direction $\boldsymbol{b}_1$ (see Figure 2).

Here, we will use the generalized coordinates $\boldsymbol{q} = [x,\, \theta]^T$.

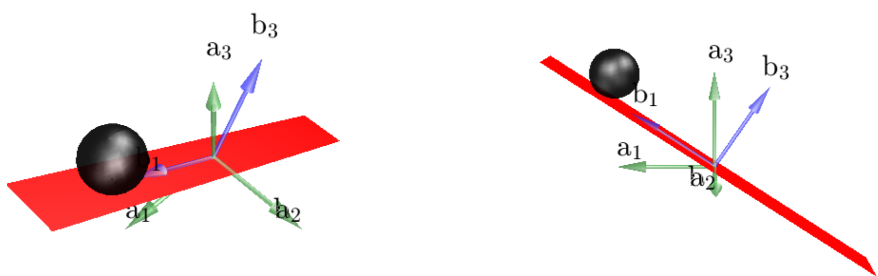

Figure 2: Schematic of the ball on a beam.

(a) What is the position of the ball's center as a function of the generalized coordinates?

(b) The total movement of the ball is the result of its movement with respect to the beam, in addition to the movement of the beam.

What is then the angular velocity of the ball as a function of the generalized coordinates?

(c) The ball is a rigid body that both experiences translation and rotation.

What is the expression for the kinetic energy of the ball?

(d) The beam is also a rigid body, but it only rotates.

What is the expression for the kinetic energy of the beam?

(e) Complete the Matlab routine `BallAndBeamSymbolicTemplate.m` by doing the following tasks:

1. Define the generalized coordinates $\boldsymbol{q} = [x,\, \theta]^\top$ as a symbolic variable.
2. Define a symbolic variable for the derivative of the generalized coordinates.
3. Write the expression for the position of the ball's center.
4. Complete the expressions for the kinetic and potential energies.
5. Write the expression for the Lagrangian function.

6. Run the routine.

Add the implemented code to your answer.

Also in this case, the Lagrange equations can be written in state-space form as given by (1).
The routine implemented in part e. exports 2 Matlab functions. One that gives the position of the ball's center, while the other returns the terms $\frac{\partial^2 \mathcal{L}}{\partial \dot{q}^2}$ and $\boldsymbol{Q} + \frac{\partial \mathcal{L}}{\partial q} - \frac{\partial^2 \mathcal{L}}{\partial \dot{q} \partial q} \dot{\boldsymbol{q}}$.

(f) Assume that the external torque T is given by the PD control law:

$$T = 200(x - \theta) + 70(\dot{x} - \dot{\theta}). \tag{3}$$

Write a Matlab function that gives the dynamics of the state-space model (1) by using the exported function `BallAndBeamODEMatrices.m` and the PD control law (3).
In other words, this Matlab function should return the value of the right-hand side of (1) as a function of the states and the parameters.

(g) Complete the delivered Matlab routine `BallAndBeamSimulation.m` in order to make an animation of the simulation results.
Simulate your system using e.g. $x(0) = 1$, $\theta(0) = 0$ as initial conditions with initial velocities at rest.
Run the animation.
What do you observe? Are these results reasonable? Explain.
Hint: The function that returns the position of the ball's center can come in handy here.

Solution: The generalized coordinates are

$$\boldsymbol{q} = \begin{bmatrix} x \\ \theta \end{bmatrix}.$$

The position of the center of mass of the ball is

$$\boldsymbol{p} = x \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} + R \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}.$$

Its kinetic energy due to the translation of the centre of mass is given by

$$T_{\mathrm{T}}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \frac{1}{2} \dot{\boldsymbol{q}}^\top M_{\mathrm{T}}(\boldsymbol{q})\, \dot{\boldsymbol{q}}, \qquad M_{\mathrm{T}}(\boldsymbol{q}) = M \frac{\partial \boldsymbol{p}}{\partial \boldsymbol{q}}^\top \frac{\partial \boldsymbol{p}}{\partial \boldsymbol{q}}.$$

Its kinetic energy due to its rotation is given by

$$T_{\mathrm{R}}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \frac{1}{2} I_{\mathrm{ball}} \omega^2, \qquad I_{\mathrm{ball}} = \frac{2}{5} M R^2,$$

where is angular velocity ω is given by

$$\omega = \dot{\theta} + \frac{\dot{x}}{R}.$$

The kinetic energy of the rail is given by

$$T_{\mathrm{r}}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \frac{1}{2} J \dot{\theta}^2.$$

The overall kinetic energy then reads as

$$T(\boldsymbol{q}, \dot{\boldsymbol{q}}) = T_{\mathrm{T}} + T_{\mathrm{R}} + T_{\mathrm{r}}.$$

On the other side, the potential energy is

$$V(\boldsymbol{q}) = Mg\boldsymbol{p}(2) = Mg\left(x \sin\theta + R\cos\theta\right).$$

Finally, the generalized forces in this case are

$$\boldsymbol{Q} = \begin{bmatrix} 0 \\ T \end{bmatrix}.$$