TTK4135 Assignment 3 Eirik Falck

Problem 1

$$\min_x \; c^T x \quad \text{s.t.} \quad Ax = b, \; x \geq 0$$

$c, x \in \mathbb{R}^n \qquad b \in \mathbb{R}^m$

KKT conditions: $\mathcal{L}(x, \lambda, s) = c^T x - \lambda^T (Ax - b) - s^T x$

- $\nabla_x \mathcal{L}(x^*, \lambda^*, s^*) = c - A^T \lambda^* - s^* = 0$
  $\hookrightarrow A^T \lambda^* + s^* = c$
- $Ax^* = b$
- $x^* \geq 0$
- $s^* \geq 0$
- $s^{*T} x^* = 0$

a) The Newton direction may not be defined because $\nabla^2 (c^T x) = 0 \not> 0$, i.e. $\nabla^2 (c^T x)$ is not positive definite.

b) Definition of convex function

$$f(\alpha x + (1-\alpha) y) \leq \alpha f(x) + (1-\alpha) f(y) \quad \forall \alpha \in [0,1]$$

In LP: $f(x) = c^T x$

- Objective function convex: ✓

$$c^T(\alpha x + (1-\alpha) y) = c^T(\alpha x) + c^T([1-\alpha] y)$$
$$= \alpha c^T x + (1-\alpha) c^T y$$

To have a convex optimization problem, we need

- convex objective function ✓
- linear equality constraints: $Ax = b$ ✓
- concave inequality constraints: ✓

$$x \geq 0 \; : \; f(\alpha x + (1-\alpha)y) \geq \alpha f(x) + (1-\alpha) f(y)$$

$$(\alpha x + (1-\alpha)y) \geq \alpha x + (1-\alpha)y$$

$\Rightarrow$ The linear program is convex.

c) Dual problem: $\max\limits_{\lambda} b^T\lambda \quad \text{s.t.} \quad A^T\lambda \leq c$

Make dual to standard form:

$$\min_{\lambda} -b^T\lambda \quad \text{s.t.} \quad c - A^T\lambda \geq 0$$

$$\mathcal{L}(\lambda, x) = -b^T\lambda - x^T(c - A^T\lambda)$$

KKT conditions:

- $\nabla_\lambda \mathcal{L}(\lambda^*, x^*) = -b + Ax^* = 0$
  $\rightarrow Ax^* = b$
- $x^* \geq 0$
- $s^* \triangleq (c - A^T\lambda^*) \Rightarrow s^* \geq 0$ $\quad \rightarrow A^T\lambda^* + s^* = c$
- $s^{*T} x^* = 0$

d) $c^T x^* = (A^T\lambda^* + s^*)^T x^* = \lambda^{*T} \underbrace{Ax^*}_{b} + \underbrace{s^{*T} x^*}_{=0}$

$= \lambda^{*T} b = b^T \lambda^*$

$\Rightarrow \quad c^T x^* = b^T \lambda^*$

e) Basic feasible point:

- The possible point that can be optimal points.

- In LP, this is the verteceies of the feasible region.

- $\mathcal{B} \subseteq \{1, \dots, n\}$ with exactly m indices s.t. if $i \notin \mathcal{B} \Rightarrow x_i = 0$ and $B = [A_i]_{i\in\mathcal{B}}$ is a $m \times m$ matrix with columns from A.

f) LICQ: All ∇c_i, $i \in \mathcal{A}(x)$ are linearly independent.

A is full rank $\Leftrightarrow$ all rows of A linearly independent.

This means all constraints are linearly independent, hence ∇c_i are also linearly independent.

Problem 2

a) R_I : $2A + 1B \leq 8$

A, B: loading product.

R_{II} : $1A + 3B \leq 15$

$\rightarrow$ $2A + 1B + x_3 = 8$ $\quad$ $x_3, x_4 \geq 0$

$1A + 3B + x_4 = 15$ $\quad$ ↳ slack variables

$$x = \begin{bmatrix} A \\ B \\ x_3 \\ x_4 \end{bmatrix} ; \quad A = \begin{bmatrix} 2 & 1 & 1 & 0 \\ 1 & 3 & 0 & 1 \end{bmatrix} \quad b = \begin{bmatrix} 8 \\ 15 \end{bmatrix}$$

$c^T = [3/2 \ -1]$ if price of $1B$: 1.

$\Rightarrow$ LP: $\min_x c^T x$ s.t. $Ax = b$, $x \geq 0$

c) Yes, x^* is at the intersection of both constraints, making them both active.

b, d) See figure below.

e) The plot only shows A and B, but we see the iterations move along the constraints $Ax = b$ and $x = 0$. This fits the theory well.

Problem 3

$$\min_x \; q(x) = \frac{1}{2} x^T G x + x^T c \qquad G: n \times n, \text{ sym.}$$

$$c, x, a_i \in \mathbb{R}^n$$

$$\text{s.t} \quad a_i^T x = b_i \quad i \in \mathcal{E}$$

$$a_i^T x \geq b_i \quad i \in \mathcal{I}$$

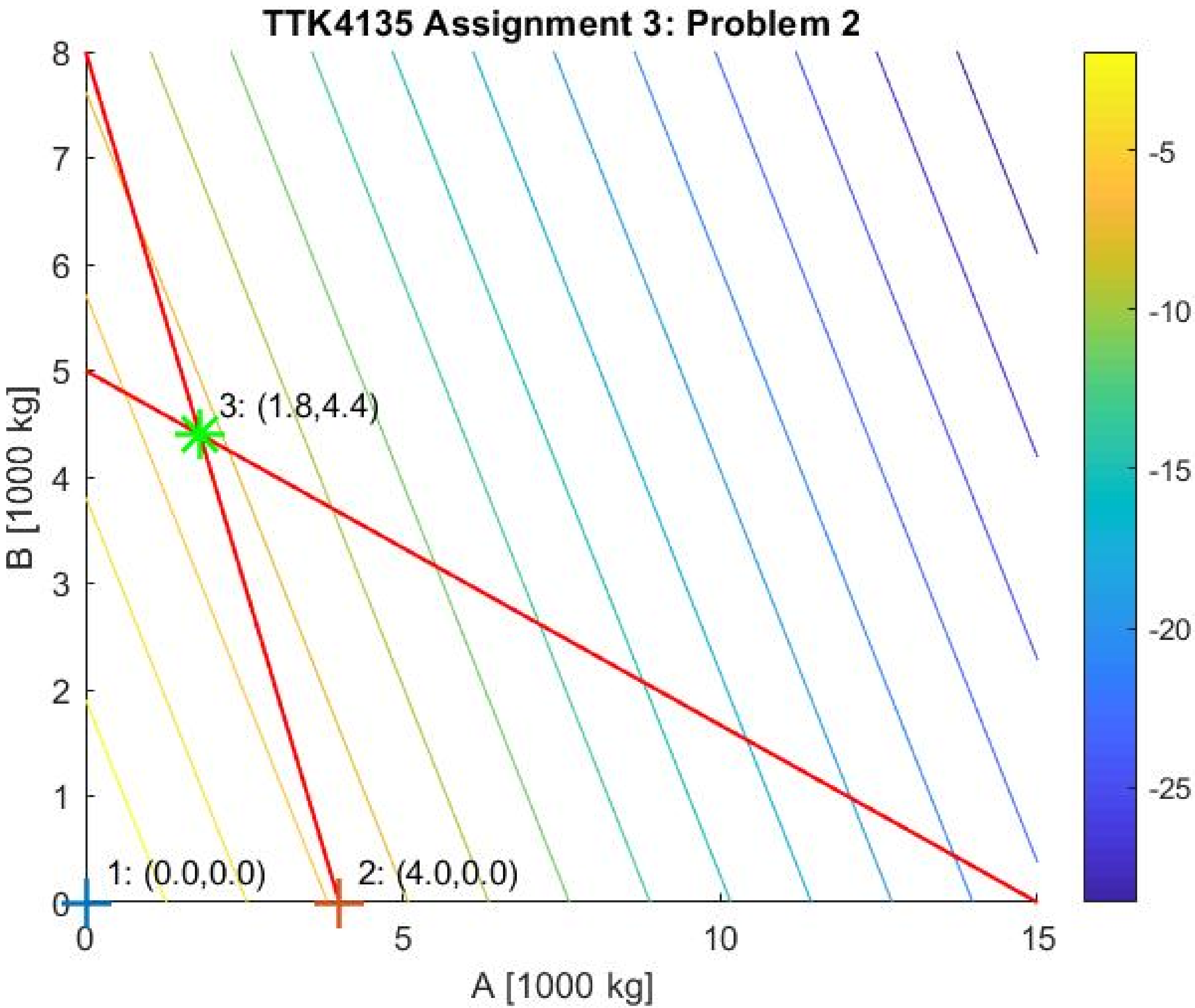
a) $A(x^*) = \mathcal{E} \cup \{ i \in \mathcal{I} / a_i^T x^* = b_i \}$

where x^* is the optimal point.

b) $\mathcal{L}(x^*, \lambda^*) = \frac{1}{2} x^{*T} G x^* + x^{*T} c - \sum_{i \in A(x^*)} \lambda_i (a_i^T x^* - b_i)$

- $\nabla_x \mathcal{L}(x^*, \lambda^*) = G x^* + c - \sum_{i \in A(x^*)} a_i \lambda_i = 0$

- $a_i^T x^* = b_i \quad i \in A(x^*)$
- $a_i^T x^* \geq b_i \quad i \in \mathcal{I} \setminus A(x^*)$
- $\lambda_i^* \geq 0 \quad i \in \mathcal{I} \cap A(x^*)$


TTK4135 Assignment 3: Problem 2
B [1000 kg]
A [1000 kg]
0
1
2
3
4
5
6
7
8
0
5
10
15
1: (0.0,0.0)
2: (4.0,0.0)
3: (1.8,4.4)
-5
-10
-15
-20
-25

```
%% LP definition
% x  = [A; B; x3; x4];
% x3,x4: slack variables
Aeq = [2 1 1 0;
       1 3 0 1];
beq = [8; 15];
c = [-3/2; -1; 0; 0];

f = @(a,b) c'*[a; b; 0; 0];


%% Meshgrid and contour
[X,Y] = meshgrid(0:1:15,0:1:8);
Z = zeros(size(X));
% TODO: Is there an easier way to do this?
for i = 1:size(X,1)
    for j = 1:size(X,2)
        Z(i,j) = f(X(i,j),Y(i,j));
    end
end

figure(1);
hold on;
title('TTK4135 Assignment 3: Problem 2');
xlabel('A [1000 kg]');
ylabel('B [1000 kg]');

contour(X,Y,Z,15);
colorbar;


%% Constraints
% see separate file for function: ploteqconstraints
ploteqconstraints(Aeq, beq, 'r', 'LineWidth', 1.2);


%% Simplex method
x0 = [0 0 beq']';
[xopt, fval, iterations] = simplex(c, Aeq, beq, x0, 'report');


%% Plot x* and iterations
for i = 1:size(iterations, 2)
    p = [iterations(1,i), iterations(2,i)];
    ptxt = sprintf('%d: (%.1f,%.1f)', i, p(1), p(2));
    xshift = 0.3;
    yshift = 0.3;
    text(p(1) + xshift, p(2) + yshift, ptxt);

    if i == size(iterations, 2)
        plot(p(1), p(2), 'g*', 'MarkerSize', 15, 'LineWidth', 1.5);
    else
        plot(p(1), p(2), '+', 'MarkerSize', 15, 'LineWidth', 1.5);
    end
end
```

```
function ploteqconstraints(Aeq, beq, varargin)
% Plot equality contstraints for a 2D linear optimization problem
% in standard form
%
%   min c'*x    subject to:   A*x = b, x >= 0
%    x
%
%
% ploteqconstraints(Aeq, beq, varargin)
%
%   Aeq: Nx2 matrix
%   beq: Nx1 vector
%   varargin: Name,Value used for specs for plotting

N = size(Aeq,1);

for i = 1:N
    temp = num2cell(Aeq(i,:));
    [ai1, ai2] = temp{:};
    bi = beq(i);

    p1 = [bi / ai1; 0];
    p2 = [0; bi / ai2];

    x = linspace(p1(1), p2(1));
    y = linspace(p1(2), p2(2));

    plot(x, y, varargin{:});
end  % for

end  % function
```