TTK4135 Assignment 2 Eirik Falch

Problem 1

a) $x = [0 \quad 0]^T$, $p = [2 \quad 1]^T \Rightarrow \exists\ \alpha \in (0,1)$

Mean value theorem:

$$f(x+p) = f(x) + \nabla f(x+\alpha p)^T p$$

for some $\alpha \in (0,1)$

$$f(x) = x_1^3 + 3x_1 x_2^2 \qquad x = \begin{bmatrix}0\\0\end{bmatrix} \quad p = \begin{bmatrix}2\\1\end{bmatrix}$$

$$\nabla f(\underbrace{x+\alpha p}_{\begin{bmatrix}2\alpha\\ \alpha\end{bmatrix}}) = \begin{bmatrix}3x_1^2 + 3x_2^2 \\ 6x_1 x_2\end{bmatrix} = \begin{bmatrix}3\cdot 4\alpha^2 + 3\cdot\alpha^2 \\ 6\cdot 2\alpha\cdot\alpha\end{bmatrix} = \begin{bmatrix}15\alpha^2 \\ 12\alpha^2\end{bmatrix}$$

$$f(x) = 0, \quad f(x+p) = f(p) = 2^3 + 3\cdot 1\cdot 2^2 = \underline{20}$$

$$20 = \alpha^2 [15 \quad 12]\begin{bmatrix}2\\1\end{bmatrix} = \alpha^2(30+12) = \underline{42\alpha^2}$$

$$\alpha^2 = \frac{20}{42} = \frac{10}{21} \rightarrow \alpha = \sqrt{\frac{10}{21}} \approx \underline{\underline{0{,}69 \in (0,1)}} \quad \square$$

b) $f(x) = x^{1/2} = \sqrt{x}$, $f'(x) = \frac{1}{2\sqrt{x}}$

Lipschitz continuous: $f: D \subset \mathbb{R}^n \to \mathbb{R}^m$

$$\| f(x_1) - f(x_0) \| \leq L \| x_1 - x_0 \| \quad \forall\ x_0, x_1 \in \mathcal{N} \subset D$$

Let $x_0 = x$ and $x_1 = x + h$ s.t.

$$\frac{\| f(x+h) - f(x) \|}{\| h \|} = \frac{f(x+h) - f(x)}{h} \leq L$$

$$\lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = f'(0) \longrightarrow \infty = L$$

L doesn't exist, thus $\underline{\underline{f \text{ is not Lipschitz continuous.}}}$ $\square$

Problem 2

$$\min_x \; c^T x \quad \text{s.t.} \quad Ax = b, \quad x \geq 0$$

$c, x \in \mathbb{R}^n \qquad b \in \mathbb{R}^m$

$$\mathcal{L}(x, \lambda, s) = c^T x - \lambda^T (Ax - b) - s^T x$$

- $\nabla_x \mathcal{L}(x^*, \lambda^*, s^*) = c^* - A^T \lambda^* - s^* = 0$
- $Ax^* = b$
- $x^* \geq 0$
- $s^* \geq 0$
- $s_i^* x_i^* = 0$ for $i = 1, 2, \dots, n$

↳ Alternatively: $x^T s = 0$

Problem 3

R, S, T: tonnes of product

a) Profit: $100R + 75S + 55T = \underbrace{\begin{bmatrix} 100 & 75 & 55 \end{bmatrix}}_{c^T} \underbrace{\begin{bmatrix} R \\ S \\ T \end{bmatrix}}_{x}$

$R, S, T \geq 0 \rightarrow x \geq 0$

Time constraints:

A: $3R + 2S + T = 7200$

B: $2R + 2S + 3T = 6000$

$\rightarrow \underbrace{\begin{bmatrix} 3 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} R \\ S \\ T \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} 7200 \\ 6000 \end{bmatrix}}_{b}$

$$\rightarrow \max_x \; c^T x \quad \text{s.t.} \quad Ax = b, \; x \geq 0$$

$$\boxed{\rightarrow \min_x \; \underbrace{-\begin{bmatrix} 100 & 75 & 55 \end{bmatrix} \begin{bmatrix} R \\ S \\ T \end{bmatrix}}_{-c^T x} \quad \text{s.t.} \quad \underbrace{\begin{bmatrix} 3 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix} \begin{bmatrix} R \\ S \\ T \end{bmatrix} = \begin{bmatrix} 7200 \\ 6000 \end{bmatrix}}_{Ax = b}, \; \underbrace{x \geq 0}_{x \geq 0}}$$

b) $x \in \mathbb{R}^3 \qquad b \in \mathbb{R}^2 \qquad A \in \mathbb{R}^{2\times 3}$

$n = 3 \geq m = 2$

Index set: $\{1, 2, 3\}$, $\begin{array}{cccc} & R & S & T \\ x = & [x_1 & x_2 & x_3]^T \end{array}$

$$A = \begin{bmatrix} 3 \\ 2 \end{bmatrix} x_1 + \begin{bmatrix} 2 \\ 2 \end{bmatrix} x_2 + \begin{bmatrix} 1 \\ 3 \end{bmatrix} x_3$$

$$\mathcal{B} = \{1, 3\} \to B = \begin{bmatrix} 3 & 1 \\ 2 & 3 \end{bmatrix}, \quad B^{-1} = \frac{1}{7}\begin{bmatrix} 3 & -1 \\ -2 & 3 \end{bmatrix}$$

$$x_B = \begin{bmatrix} x_1 \\ x_3 \end{bmatrix}$$

$$Bx_B = b \to x_B = B^{-1}b = \frac{1}{7}\begin{bmatrix} 3 & -1 \\ -2 & 3 \end{bmatrix}\begin{bmatrix} 7200 \\ 6000 \end{bmatrix} = \underline{\frac{1}{7}\begin{bmatrix} 15\,600 \\ 3\,600 \end{bmatrix}}$$

$$\mathcal{B} = \{1, 2\} \to B = \begin{bmatrix} 3 & 2 \\ 2 & 2 \end{bmatrix}, \quad B^{-1} = \frac{1}{2}\begin{bmatrix} 2 & -2 \\ -2 & 3 \end{bmatrix}$$

$$x_B = B^{-1}b = \frac{1}{2}\begin{bmatrix} 2 & -2 \\ -2 & 3 \end{bmatrix}\begin{bmatrix} 7200 \\ 6000 \end{bmatrix} = \underline{\begin{bmatrix} 1\,200 \\ 1\,800 \end{bmatrix}}$$

$$\mathcal{B} = \{2, 3\} \to B = \begin{bmatrix} 2 & 1 \\ 2 & 3 \end{bmatrix}, \quad B^{-1} = \frac{1}{4}\begin{bmatrix} 3 & -1 \\ -2 & 2 \end{bmatrix}$$

$$x_B = B^{-1}b = \frac{1}{4}\begin{bmatrix} 3 & -1 \\ -2 & 2 \end{bmatrix}\begin{bmatrix} 7\,200 \\ 6\,000 \end{bmatrix} = \underline{\begin{bmatrix} 3\,900 \\ -\,600 \end{bmatrix}}$$

The basic feasible points satisfying $Ax = b$ and $x \geq 0$ are

$$\underline{x' = \frac{1}{7}\begin{bmatrix} 15\,600 \\ 0 \\ 3600 \end{bmatrix}} \quad \text{and} \quad \underline{x'' = \begin{bmatrix} 1\,200 \\ 1\,800 \\ 0 \end{bmatrix}}$$

c) $\mathcal{L}(x, \lambda, s) = c^T x - \lambda^T(Ax - b) - s^T x$

$A = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}, \quad c = \begin{bmatrix} -100 \\ -75 \\ -55 \end{bmatrix}$

$\nabla_x \mathcal{L}(x, \lambda, s) = c - A^T\lambda - s = 0$

$A^T\lambda + s = c$

Both BFP from b) solve $Ax = b$.

Check for x': $\mathcal{B} = \{1, 3\}$, $B = \begin{bmatrix} 3 & 1 \\ 2 & 3 \end{bmatrix}$, $c_B = \begin{bmatrix} -100 \\ -55 \end{bmatrix}$

$x'^T s' = \underbrace{x_B'^T}_{x_B' \neq 0} s_B' + \cancel{\overset{0}{x_N'^T}} s_N' = 0 \Rightarrow \underline{s_B' = 0}$

$\Rightarrow$ KKT condition $x^T s$ is fulfilled.

$A^T\lambda' + s' = \begin{bmatrix} B^T \\ N^T \end{bmatrix} \lambda' + \begin{bmatrix} s_B' \\ s_N' \end{bmatrix} = \begin{bmatrix} B^T \\ N^T \end{bmatrix} \lambda' + \begin{bmatrix} 0 \\ s_N' \end{bmatrix} = \begin{bmatrix} c_B \\ c_N \end{bmatrix}$

$B^T \lambda' = c_B' \Rightarrow \lambda' = (B^T)^{-1} c_B = (B^{-1})^T c_B = \frac{1}{7}\begin{bmatrix} 3 & -2 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} -100 \\ -55 \end{bmatrix} = \underline{\frac{1}{7}\begin{bmatrix} -190 \\ -65 \end{bmatrix}}$

$N^T \lambda' + s_N' = c_N'$

$\rightarrow s_N' = c_N' - N^T\lambda' = -75 - \frac{1}{7}\begin{bmatrix} 2 & 2 \end{bmatrix}\begin{bmatrix} -190 \\ -65 \end{bmatrix} = -\frac{15}{7} < 0 \quad X$

$\Rightarrow x' = \frac{1}{7}\begin{bmatrix} 15\,600 \\ 0 \\ 3\,600 \end{bmatrix}$ is <u>not</u> the optimal point.

Check for x'': $\mathcal{B}=\{1,2\}$: $B=\begin{bmatrix}3&2\\2&2\end{bmatrix}$ $c_B=\begin{bmatrix}-100\\-75\end{bmatrix}$

· $Ax''=\begin{bmatrix}7200\\6000\end{bmatrix}$ ✓ $B^{-1}=\frac{1}{2}\begin{bmatrix}2&-2\\-2&3\end{bmatrix}$

· $x''\geqslant 0$ ✓

$x_B''\neq 0$

· $x''^T s'' = x_B''^T s_B'' + \overset{0}{\cancel{x_N''^T}} s_N'' = 0 \Rightarrow \underline{s_B''=0}$

$A^T\lambda''+s''=c$:

$$\begin{bmatrix}B^T\\N^T\end{bmatrix}\lambda''+\begin{bmatrix}\cancel{s_B''}\\s_N''\end{bmatrix}=\begin{bmatrix}B^T\\N^T\end{bmatrix}\lambda''+\begin{bmatrix}0\\s_N''\end{bmatrix}=\begin{bmatrix}c_B''\\c_N''\end{bmatrix}$$

$$B^T\lambda''=c_B'' \Rightarrow \lambda''=(B^{-1})^T c_B''=\frac{1}{2}\begin{bmatrix}2&-2\\-2&3\end{bmatrix}\begin{bmatrix}-100\\-75\end{bmatrix}=\underline{\begin{bmatrix}-25\\-25/2\end{bmatrix}}$$

$$N^T\lambda''+s_N''=c_N''$$

$$\rightarrow s_N''=c_N''-N^T\lambda''=-55-\begin{bmatrix}1&3\end{bmatrix}\begin{bmatrix}-25\\-25/2\end{bmatrix}=\frac{15}{2}>0 \checkmark$$

$\Rightarrow x^*=x''=\begin{bmatrix}1200\\1800\\0\end{bmatrix}$ is <u>the optimal point</u>.

d) Dual problem:

$$\max_{\lambda}\ b^T\lambda \quad \text{s.t.} \quad A^T\lambda+s=c$$
$$s\geqslant 0$$

$\Leftrightarrow$

$$\max_{\lambda}\ \begin{bmatrix}7600&6000\end{bmatrix}\begin{bmatrix}\lambda_1\\\lambda_2\end{bmatrix} \quad \text{s.t.} \quad \begin{bmatrix}3&2\\2&2\\1&3\end{bmatrix}\begin{bmatrix}\lambda_1\\\lambda_2\end{bmatrix}+s=\begin{bmatrix}-100\\-75\\-55\end{bmatrix}$$
$$s\geqslant 0.$$

e) Relevant equations: $Ax = b$

$A^T\lambda + s = c$, $s \geq 0$, $x_i s_i = 0 \Leftrightarrow x^T s = s^T x = 0$

$$c^T x = (A^T\lambda + s)^T x = (A^T\lambda)^T x + \overbrace{s^T x}^{0}$$
$$= \lambda^T A x = (Ax)^T \lambda = b^T \lambda \quad \square.$$

f) $\lambda^* = \begin{bmatrix} -25 \\ -12,5 \end{bmatrix} \begin{pmatrix} A \\ B \end{pmatrix}$

The result from the sensitivity analysis from Ch. 13 is $c^T \Delta x = \varepsilon \lambda_j$, i.e. the change in the objective function value is proportional to the j'th Lagrange multiplier.

Since $|\lambda_1^*| > |\lambda_2^*|$, it's more beneficial to increase the availability of stage $\underline{\underline{A}}$.

Using matlab, we find that it increases profit by 25 NOK as opposed to 12,5 NOK for increase to stage B.