Norwegian University of Science and Technology
Department of Engineering Cybernetics

Exercise 6
MPC and LQR

Problem 1 (30 %) Finite-Horizon LQR (Similar to Exam August 2000)

Newton's second law of motion is $ma = F$.

a Let $m = 1$, $F = u$, and let x_1 represent position and x_2 represent velocity. Write Newton's second law on state-space form $\dot{x} = A_c x + b_c u$ (c for "continuous time").

b Using a sampling interval of $T = 0.5$,

$$A = e^{A_c T}, \text{ and } b = \left(\int_0^T e^{A_c \tau} d\tau \right) b_c \tag{1}$$

show that Newton's second law can be written in discrete time as

$$x_{t+1} = Ax_t + bu_t, \quad A = \begin{bmatrix} 1 & 0.5 \\ 0 & 1 \end{bmatrix}, \; b = \begin{bmatrix} 0.125 \\ 0.5 \end{bmatrix} \tag{2}$$

without using MATLAB.

Hint: The matrix exponential e^A can be written as an infinite series.

c Let the cost function for optimal control be given by

$$f(z) = \frac{1}{2} \sum_{t=0}^{N-1} \left\{ x_{t+1}^\top Q x_{t+1} + u_t^\top R u_t \right\} \tag{3a}$$

with

$$Q = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \quad \text{and} \quad R = 2 \tag{3b}$$

and where $z = [x_1^\top, \ldots, x_N^\top, u_0^\top, \ldots, u_{N-1}^\top]^\top$. Formulate the Riccati equation for this problem and show how the solution of the Riccati equation can be used to find a state-feedback controller that minimizes $f(z)$. Illustrate the solution with a sketch of the system with the controller.

d Let $N \to \infty$. Solve the stationary Riccati equation (use the MATLAB function `dlqr`) and find the optimal feedback law $u_t = -Kx_t$ (use MATLAB to carry out the matrix multiplications). Attach a printout of your code to your homework. Is the resulting closed-loop system stable?

Hint: The upper left element of P is 4.0350; use this to verify your solution to the stationary Riccati equation.

e In general, does the type of controller found in the previous problem give a stable feedback system? Elaborate.

Problem 2 (20 %) Infinite-Horizon Linear-Quadratic Control

Consider the discrete-time system

$$x_{t+1} = 3x_t + 2u_t, \qquad x_t \in \mathbb{R}^1, \quad u_t \in \mathbb{R}^1 \tag{4}$$

and the cost function

$$f^\infty(z) = \frac{1}{2}\sum_{t=0}^{\infty} \left\{ q x_{t+1}^2 + u_t^2 \right\}, \qquad q > 0 \tag{5}$$

a Show that the stationary Riccati equation is

$$p = q + \frac{a^2 p r}{r + b^2 p} \tag{6}$$

Set $q = 2$ and find the solution.

b What is the optimal feedback $u_t = -kx_t$?

c An LQ controller with infinite horizon gives an optimal constant feedback matrix K. Under which conditions does the optimal feedback law give an asymptotically stable closed loop system?

Problem 3 (50 %) MPC and input blocking

MATLAB will be used in this task. If you are using MATLAB 2015a or earlier, you can still access the Active-set method. In 2019, MATLAB 2015a was available at farm.ntnu.no., if you cannot find it, please post a question on BlackBoard and we will check if it is made unavailable. If you are using a newer version of MATLAB, the discussion will be a bit different. Considering that the Active-set method is taught in class, I encourage you to use this method. State in your answer which algorithm(s) you have used.

We now revisit the optimal control problem from Problem 1 f), Assignment 5. The plant model is given by

$$x_{t+1} = \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0.1 & -0.79 & 1.78 \end{bmatrix}}_{A} x_t + \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0.1 \end{bmatrix}}_{B} u_t \tag{7a}$$

$$y_t = \underbrace{\begin{bmatrix} 0 & 0 & 1 \end{bmatrix}}_{C} x_t \tag{7b}$$

where y_t is a measurement. The process has been at the at the origin $x_t = 0$, $u_t = 0$ for a while, but at $t = -1$ a disturbance moved the process so that $x_0 = [0,\ 0,\ 1]^\top$. We wish to solve a finite horizon ($N < \infty$) optimal control problem with the cost (or objective) function

$$f(y_1, \dots, y_N, u_0, \dots, u_{N-1}) = \sum_{t=0}^{N-1} \left\{ y_{t+1}^2 + r u_t^2 \right\}, \quad r > 0 \tag{8}$$

Use $r = 1$ unless otherwise noted. We use $N = 30$ for the entire exercise. The input constraint is

$$-1 \leq u_t \leq 1 \quad t \in [0, N-1] \tag{9}$$

We assume that (7) is a perfect model of the plant and that full state information is available for control.

a Revisit your code from Problem 1 f), Exercise 5, and solve the open-loop optimization problem. If you don't have the code from the previous assignment: The solution of Assignment 5 is available on BlackBoard.

b We will now reduce the number of control variables u_t by dividing the time horizon into 6 blocks of equal length (5 time steps each) and require u to be constant on each of these blocks. We will do this by modifying A_{eq} in equation

$$A_{\text{eq}} z = b_{\text{eq}} \tag{10}$$

which represents the system equations (7a) for all time instants on the horizon. That is, the right part of the matrix A_{eq} (the part containing the B matrices) will have fewer columns. Give the structure of the modified A_{eq} and state how many columns the right part (containing B matrices) now has. G in the objective will also look slightly different. Modify your code from 1) and solve this optimal control problem with the input bound constraints (9) using `quadprog`. Plot y_t and u_t and compare your results with those obtained in 1). Verify that the input blocks work as intended. How many iterations does `quadprog` use to find the solution for this problem?

Hint: Constructing the A_{eq} matrix is now slightly more involved;

```
kron(eye(6),ones(5,1))
```

is a good start. G will also be slightly modified.

c We now use a better input parametrization: we use the same number of blocks but the input blocks are now of increasing lengths: 1, 1, 2, 4, 8, and 14 time steps, in that order. Modify your code to achieve this. Plot and compare with the results obtained in 1) and 2). Does the parametrization change the number of iterations `quadprog` uses to find the solution?

d Revisit your code from Problem 2 b), Assignment 5, and solve the MPC problem (no input blocking).

e Modify your code to solve the same MPC problem as in 4), but with the input blocking scheme from problem 3) (blocks of increasing lengths). Plot and compare the two MPCs (the one in 4) and the one from this sub problem).

f What is the effect of using input blocking with MPC? Why do we choose blocks of increasing length? Discuss based on the results obtained above.