TTK4135 Assignment 4 Eirik Falch

Problem 1

QP:

$$\min_x \frac{1}{2}x^T G x + c^T x$$

$$\text{s.t.} \quad a_i^T x = b_i \quad i \in \mathcal{E}$$

$$a_i^T x \geq b_i \quad i \in \mathcal{I}$$

a) For the QP to be convex the matrix G must be positive semi-definite, $G \geq 0$

Convexity is important because it guarantees that local solutions are infact global.

b) If $Z^T G Z \geq 0$, we would have $q(x) \geq q(x^*)$ meaning x^* would not be a strict local minimizer, only a local minimizer and x^* is only one of possibly many global solutions.

c) $\min_{x} \quad (x_1-1)^2 + (x_2-2,5)^2$

$= x_1^2 - 2x_1 + 1 + x_2^2 - 5x_2 + 2,5^2$

$G = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$

$c^T = [-2 \quad -5]$

s.t.

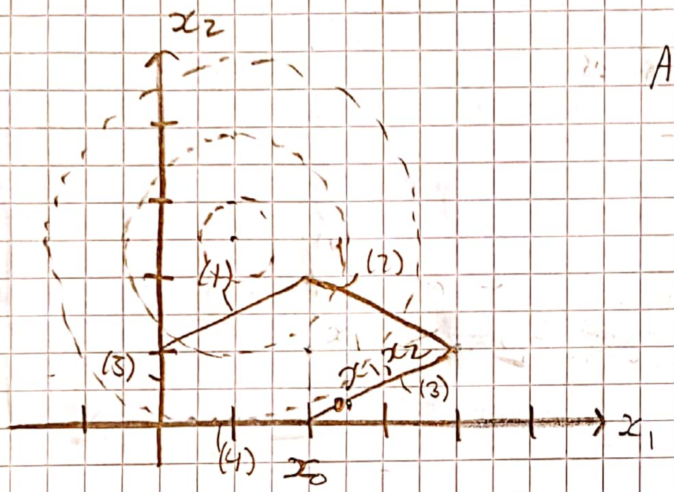
$$x_1 - 2x_2 + 2 \geq 0 \qquad (1)$$

$$-x_1 - 2x_2 + 6 \geq 0 \qquad (2)$$

$$-x_1 + 2x_2 + 2 \geq 0 \qquad (3)$$

$$x_1 \geq 0 \qquad (4)$$

$$x_2 \geq 0 \qquad (5)$$



$$A = \begin{bmatrix} 1 & -2 \\ -1 & -2 \\ -1 & 2 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} x \geq b = \begin{bmatrix} -2 \\ -6 \\ -2 \\ 0 \\ 0 \end{bmatrix}$$

(0) $x_0 = [2 \quad 0]^T, \quad \mathcal{W}_0 = \{3\}$

$$\min_{p} \ \frac{1}{2} p_0^T G p_0 + g_0^T p_0 \qquad \text{s.t.} \quad [-1 \quad 2]\, p = 0$$

$$\rightarrow p_0 = [0{,}2 \quad 0{,}1]^T \neq 0$$

$$\rightarrow \alpha_0 = \min\left(1, \frac{b_i - a_i^T x_0}{a_i^T p_0} \quad i \notin \mathcal{W}_0\right) = 1$$

$$\Rightarrow x_1 = x_0 + \alpha_0 p_0 = [2{,}2 \quad 0{,}1]^T$$

(1) $x_1 = [2.2 \;\; 0.1]^T \quad \mathcal{W}_1 = \{3\}$

$\rightarrow p_1 \approx 0$

$\rightarrow [-1 \;\; 2]^T \hat{\lambda} = g_1 = Gx_1 + c = \begin{bmatrix} 4 \\ 1 \end{bmatrix} + \begin{bmatrix} -2 \\ -5 \end{bmatrix} = \begin{bmatrix} 2 \\ -4 \end{bmatrix}$

$\Leftrightarrow \begin{bmatrix} -1 \\ 2 \end{bmatrix} \hat{\lambda} = \begin{bmatrix} 2 \\ -4 \end{bmatrix} \rightarrow \hat{\lambda} = -2,4 \not\geq 0$

$\Rightarrow x_1$ not optimal point

(2) $x_2 = [2.2 \;\; 0.1]^T \quad \mathcal{W}_2 = \emptyset$

$\rightarrow p_2 = [-1,2 \;\; 2,4]^T \neq 0$

$\rightarrow \alpha_2 = \min(1, \ldots) = 0,607, \quad i = 1$

(3) $x_3 = [1,4 \;\; 1,7]^T \quad \mathcal{W}_3 = \{1\}$

$\rightarrow p_3 \approx 0$

$\rightarrow \hat{\lambda}_1 = 0,8 \geq 0$

$$\Rightarrow \quad \underline{\underline{x^* = x_3 = \begin{bmatrix} 1,4 \\ 1,7 \end{bmatrix}}}$$

d)

$$q(x) = \frac{1}{2}x^T G x + c^T x \quad \text{s.t.} \quad Ax \geq b$$

$$\mathcal{L}(x,\lambda) = \frac{1}{2}x^T G x + c^T x - \lambda^T (Ax - b)$$

$$\nabla_x \mathcal{L}(x,\lambda) = Gx + c - A^T\lambda = 0$$

$$\rightarrow x = G^{-1}(A^T\lambda - c)$$

$\Rightarrow$ Dual objective function:

$$f(\lambda) = \inf_x \mathcal{L}(x,\lambda)$$

$$= \frac{1}{2}(A^T\lambda - c)^T G^{-1}(A^T\lambda - c) + c^T G^{-1}(A^T\lambda - c)$$
$$- \lambda^T(AG^{-1}(A^T\lambda - c) - b)$$

$$= -\frac{1}{2}(A^T\lambda - c)^T G^{-1}(A^T\lambda - c) + \lambda^T b$$

$\Rightarrow$ Dual problem

$$\max_\lambda -\frac{1}{2}(A^T\lambda - c)^T G^{-1}(A^T\lambda - c) + \lambda^T b \quad \text{s.t.} \quad \lambda \geq 0.$$

e) theorem 12.11: $f(\bar{\lambda}) \leq q(\bar{x})$, $\bar{x}$ feasible, $\bar{\lambda} \geq 0$

$$q(x^*) - q(\bar{x}) \leq q(x^*) - f(\bar{\lambda})$$

As we can see, $f(\bar{\lambda})$ will be an upper bound for the error $q(x^*) - q(\bar{x})$ when $q(x^*)$ is unknown.

Problem 2

a)

$$\left.\begin{aligned} R_I &: \ 2A + 1B \le 8 \\ R_{II} &: \ 1A + 3B \le 15 \\ &\quad A, B \ge 0 \end{aligned}\right\} \quad \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} x \le \begin{bmatrix} 8 \\ 15 \end{bmatrix}$$

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} A \\ B \end{bmatrix}$$

Profit: $(3 - 0{,}4x_1)x_1 + (2 - 0{,}2x_2)x_2$

$$= 3x_1 - 0{,}4x_1^2 + 2x_2 - 0{,}2x_2^2$$

$$= x^T \begin{bmatrix} -0{,}4 & 0 \\ 0 & -0{,}2 \end{bmatrix} x + \begin{bmatrix} 3 & 2 \end{bmatrix} x$$

$$= \frac{1}{2} x^T \begin{bmatrix} -0{,}8 & 0 \\ 0 & -0{,}4 \end{bmatrix} x + \begin{bmatrix} 3 & 2 \end{bmatrix} x$$

$\Rightarrow$

$$\min_x \ \frac{1}{2} x^T \underbrace{\begin{bmatrix} +0{,}8 & 0 \\ 0 & +0{,}4 \end{bmatrix}}_{G} x + \underbrace{\begin{bmatrix} -3 & -2 \end{bmatrix}}_{c^T} x$$

$$\text{s.t} \quad \underbrace{\begin{bmatrix} -2 & -1 \\ -1 & -3 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}}_{A} x \ge \underbrace{\begin{bmatrix} -8 \\ -15 \\ 0 \\ 0 \end{bmatrix}}_{b} \quad \begin{matrix} \} \text{ from } R_I, R_{II} \\ \\ \} \text{ from } A, B \ge 0 \end{matrix}$$

b) See below

c) $x^* = \begin{bmatrix} 2,3 \\ 3,5 \end{bmatrix}$ $\qquad x^*_{LP} = \begin{bmatrix} 1,8 \\ 4,4 \end{bmatrix}$

The optimal point of the QP problem x^* is not at the intersection point of the constraints. It's instead on one of the constraints, making only one constraint active.

d) From x_0 the algorithm "jumps" onto one constraint. There, it follows the constraint until it reaches the optimal point.

e) As we can see, the QP solution can be anywhere in the feasible area as opposed to LP where it must be at the intersection of two or more constraints.