Problem 1 (30 %) The Mean-Value Theorem

a The Mean Value Theorem states that for a function $f : \mathbb{R}^n \to \mathbb{R}$ and any vector p,

$$f(x+p) = f(x) + \nabla f(x+\alpha p)^\top p \tag{1}$$

for some $\alpha \in (0,1)$.

We have $f(x) = x_1^3 + 3x_1x_2^2$, $x = [0,\ 0]^\top$ and $p = [2,\ 1]^\top$. This gives

$$f(x) = 0 \quad \text{and} \quad f(x+p) = 14 \tag{2}$$

Furthermore,

$$\nabla f(x+\alpha p) = \begin{bmatrix} 3(x_1+\alpha p_1)^2 + 3(x_2+\alpha p_2)^2 \\ 6(x_1+\alpha p_1)(x_2+\alpha p_2) \end{bmatrix} = \begin{bmatrix} 3\alpha^2 \cdot 4 + 3\alpha^2 \\ 6\alpha \cdot 2 \cdot \alpha \end{bmatrix} = \begin{bmatrix} 15\alpha^2 \\ 12\alpha^2 \end{bmatrix} \tag{3}$$

which gives $\nabla f(x+\alpha p)^\top p = 42\alpha^2$. With these values, the Mean Value Theorem gives

$$14 = 0 + 42\alpha^2 \tag{4a}$$

$$\Rightarrow \alpha = +\sqrt{\frac{14}{42}} = \frac{\sqrt{3}}{3} \in (0,1) \tag{4b}$$

b $f(x) = x^{\frac{1}{2}}$ is a continuous function. Its derivative is

$$f'(x) = \frac{1}{2\sqrt{x}} \tag{5}$$

As x goes to 0, $\|f'(x)\|$ goes to infinity. Hence, in a small neighborhood $\mathcal{N}$ of $x = 0$, there is no Lipschitz constant L so that

$$\frac{\|f(x_1) - f(x_0)\|}{\|x_1 - x_0\|} \le L, \quad \forall x_0, x_1 \in \mathcal{N} \tag{6}$$

This means that $f(x)$ is not Lipschitz continuous (see page 624 in the textbook).

Problem 2 (25 %) LP and KKT-conditions (Exam August 2000)

The Lagrangian for the linear program

$$\min_x c^\top x \qquad \text{s.t.} \qquad Ax = b, \quad x \geq 0 \tag{7}$$

with $c \in \mathbb{R}^n$, $x \in \mathbb{R}^n$, and $b \in \mathbb{R}^m$ is

$$\mathcal{L}(x, \lambda) = c^\top x - \lambda^\top (Ax - b) - s^\top x \tag{8}$$

where $\lambda \in \mathbb{R}^m$ is the Lagrange multipliers for the equality constraints and $s \in \mathbb{R}^n$ is the Lagrange multipliers for the inequality constraints. (Note that both λ and s are vectors!) Using Theorem 12.1 in the textbook, we find the first KKT condition by differentiating the Lagrangian wrt. x at the solution,

$$\nabla_x \mathcal{L}(x^*, \lambda^*) = c - (\lambda^{*\top} A)^\top - s^* \tag{9}$$

Since this gradient must be zero, the first KKT condition can be written

$$A^\top \lambda^* + s^* = c \tag{10a}$$

The remaining four KKT conditions can be written

$$Ax^* = b \tag{10b}$$
$$x^* \geq 0 \tag{10c}$$
$$s^* \geq 0 \tag{10d}$$
$$s_i^* x_i^* = 0, \quad i = 1, \dots, n \tag{10e}$$

Problem 3 (45 %) Linear Programming

In a plant three products R, S and T are made in two process stages A and B. To make a product the following time in each process stage is required:

- 1 tonne of R: 3 hours in stage A plus 2 hours in stage B.
- 1 tonne of S: 2 hours in stage A and 2 hours in stage B.
- 1 tonne of T: 1 hour in stage A and 3 hours in stage B.

During one year, stage A has 7200 hours and stage B has 6000 hours available production time. It is required that *the available production time should be fully utilized in both stages.*

The profit from the sale of the products are:

- R: 100 NOK per tonne.
- S: 75 NOK per tonne.
- T: 55 NOK per tonne.

We wish to maximize the yearly profit.

a We wish to formulate the problem as an LP on standard form. Hence, we minimize the negative of profit. Let the decision variables be the number of tonnes manufactured, i.e., x_1 is the number of tonnes of R produced, x_2 is the number of tonnes of S produced, and x_3 is the number of tonnes of T produced. Let c_i be the negative of the corresponding profits per tonne of these products, so that $c_1 = -100$, $c_2 = -75$, and $c_3 = -55$ ($c^\top = -[100,\ 75,\ 55]$). The profit can then be written

$$-c^\top x = -c_1x_1 - c_2x_2 - c_3x_3 = 100x_1 + 75x_2 + 55x_3 \tag{11}$$

and the objective function becomes $c^\top x$. In order to utilize all 7200 hours of available time in process stage A, the following equation must hold:

$$3x_1 + 2x_2 + 1x_3 = 7200 \tag{12}$$

Similarly, for process stage B:

$$2x_1 + 2x_2 + 3x_3 = 6000 \tag{13}$$

Defining a matrix A and a vector b as

$$A = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}, \quad b = \begin{bmatrix} 7200 \\ 6000 \end{bmatrix} \tag{14}$$

the equations (12) and (13) can be written on the form $Ax = b$. Finally, since negative production is impossible, we have the constraint $x \geq 0$ (all components of x of nonnegative).

With x, c, A and b defined as above, the problem can be written as an LP on standard form:

$$\min c^\top x \qquad \text{s.t.} \qquad Ax = b,\ x \geq 0 \tag{15}$$

b The LP has $m = 2$ constraints and $n = 3$ variables. The index set for the problem is $\{1, 2, 3\}$, and the basis $\mathcal{B}$ contains $m = 2$ indices. Hence, there are three possible bases ("basises") for the problem. Each of these will correspond to a basic feasible point. (See Chapter 13.2 for theory.) We will go through each of these three cases one by one. In short, the basic feasible points can then be found by setting each combination of the $(n - m)$ variables to zero and solve the resulting equation set.

- $\mathcal{B} = \{1, 2\} \Leftrightarrow B = \begin{bmatrix} 3 & 2 \\ 2 & 2 \end{bmatrix} \Leftrightarrow x_3 = 0$

  In this case, we get the equation set

$$\underbrace{\begin{bmatrix} 3 & 2 \\ 2 & 2 \end{bmatrix}}_{B} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 7200 \\ 6000 \end{bmatrix}}_{b} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1200 \\ 1800 \end{bmatrix} \tag{16}$$

  This gives the basic feasible point $x = [1200,\ 1800,\ 0]^\top$

- $\mathcal{B} = \{1,3\} \Leftrightarrow B = \begin{bmatrix} 3 & 1 \\ 2 & 3 \end{bmatrix} \Leftrightarrow x_2 = 0$
  This gives the equation set

$$\underbrace{\begin{bmatrix} 3 & 1 \\ 2 & 3 \end{bmatrix}}_{B} \begin{bmatrix} x_1 \\ x_3 \end{bmatrix} = \underbrace{\begin{bmatrix} 7200 \\ 6000 \end{bmatrix}}_{b} \Rightarrow \begin{bmatrix} x_1 \\ x_3 \end{bmatrix} = \frac{1}{7} \begin{bmatrix} 15600 \\ 3600 \end{bmatrix} \tag{17}$$

  The corresponding basic feasible point $x = \frac{1}{7}[15600,\ 0,\ 3600]^\top$

- $\mathcal{B} = \{2,3\} \Leftrightarrow B = \begin{bmatrix} 2 & 1 \\ 2 & 3 \end{bmatrix} \Leftrightarrow x_1 = 0$
  This results in the equation set

$$\underbrace{\begin{bmatrix} 2 & 1 \\ 2 & 3 \end{bmatrix}}_{B} \begin{bmatrix} x_2 \\ x_3 \end{bmatrix} = \underbrace{\begin{bmatrix} 7200 \\ 6000 \end{bmatrix}}_{b} \Rightarrow \begin{bmatrix} x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3900 \\ -600 \end{bmatrix} \tag{18}$$

  This gives the point $x = [0,\ 3900,\ -600]^\top$, which is infeasible.

c Due to the convexity of linear programs, the KKT conditions are both necessary and sufficient. Hence, one can find the solution to an LP by checking the KKT conditions at all basic feasible points. This is a straightforward task for this small problem.

The last point found in **b)** does obviously not satisfy the KKT conditions, since it is infeasible ($x_3 < 0$). We can hence rule this point out in our search for the solution to the LP.

For the second point, $x = \frac{1}{7}[15600,\ 0,\ 3600]^\top$, we must find s to check the KKT conditions. From the condition $x_i s_i = 0,\ i = 1,2,3$, we can set $s = [0,\ s_2,\ 0]^\top$. From the first KKT condition, $A^\top \lambda^* + s^* = c$, we get

$$\begin{bmatrix} 3 & 2 \\ 2 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} + \begin{bmatrix} 0 \\ s_2 \\ 0 \end{bmatrix} = - \begin{bmatrix} 100 \\ 75 \\ 55 \end{bmatrix} \tag{19}$$

The solution to this system is $\lambda = -\frac{1}{7}[190,\ 65]^\top$ and $s_2 = -\frac{15}{7} < 0$. Since s_2 is negative, the KKT conditions do not hold and $x = \frac{1}{7}[15600,\ 0,\ 3600]^\top$ can not be the solution to the LP.

Similarly, the first point, $x = [1200,\ 1800,\ 0]^\top$, gives $s = [0,\ 0,\ s_3]^\top$. The first KKT condition becomes

$$\begin{bmatrix} 3 & 2 \\ 2 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ s_3 \end{bmatrix} = - \begin{bmatrix} 100 \\ 75 \\ 55 \end{bmatrix} \tag{20}$$

The multipliers then become $\lambda = -[25,\ 12.5]^\top$ and $s_3 = 7.5 > 0$. Hence, all KKT conditions are satisfied and the solution to the LP is $x^* = [1200,\ 1800,\ 0]^\top$. The optimal objective function value (the negative of maximal profit) is $c^\top x^* = -255,000$.

d Since the LP is on standard form, the dual problem can be stated as

$$\max b^\top \lambda \qquad \text{s.t.} \qquad A^\top \lambda \leq c \tag{21}$$

We know that the solution to this problem is $\lambda^* = -[25,\ 12.5]^\top$, and that the optimal objective function value is $b^\top \lambda^* = -255,000$.

e To show that the optimal objective function value for the LP in **a)** equals the optimal objective function value for the dual problem in **d)**, we will use the first KKT condition,

$$A^\top \lambda^* + s^* = c \tag{22a}$$

the equality constraint for the LP at x^*,

$$Ax^* = b \tag{22b}$$

and the last KKT condition

$$\begin{aligned} &s_i^* x_i^* = 0, \quad i = 1, \dots, n \\ \Rightarrow &s^{*\top} x^* = 0 \end{aligned} \tag{22c}$$

(The implication follows from the fact that no element in x^* and s^* can be negative.) We start the derivation from the optimal objective function value of the LP:

$$\begin{aligned} c^\top x^* &= (A^\top \lambda^* + s^*)^\top x^* && \text{(from equation (22a))} \\ &= \lambda^{*\top} A x^* + s^{*\top} x^* && \text{(expanding the parenthesis)} \\ &= \lambda^{*\top} b + s^{*\top} x^* && \text{(from equation (22b))} \\ &= \lambda^{*\top} b + 0 && \text{(from equation (22c))} \\ &= b^\top \lambda^* && \text{(by equivalent forms of the scalar product)} \end{aligned}$$

f Constraint 1 (7200 h in stage A) is associated with the multiplier $\lambda_1^* = -25$, while constraint 2 (6000 h in stage B) is associated with the multiplier $\lambda_2^* = -12.5$. We would then expect that it would be most profitable to increase the capacity of stage A, since $|\lambda_1^*| > |\lambda_2^*|$ (see the discussion of sensitivity on pages 361–362).

Increasing the capacity of A by 1 hour ($b = [7201,\ 6000]^\top$) increases the maximal profit to 255,025 (with $x^* = [1201,\ 1799,\ 0]^\top$).

Increasing the capacity of B by 1 hour ($b = [7200,\ 6001]^\top$) increases the maximal profit to 255,012.5 (with $x^* = [1199,\ 1801.5,\ 0]^\top$).

Observe how the increase in the profit matches the multipliers. Will this nice relationship always hold? And what about $\|\nabla c_i(x^*)\|$?

The MATLAB code on the next page can be modified to find the above results.

```
% Updated Spring 2019 - Joakim R. Andersen
c = -[100; 75; 55];
Aeq = [3, 2, 1; 2, 2, 3];
beq = [7200; 6000]; % Change this vector
lb = zeros(3,1);
options = optimset('Algorithm', 'dual-simplex');
[x,fval,exitflag,output,lambda] = ...
linprog(c,[],[],Aeq,beq,lb,[],[],options);
disp(x);
disp(fval);
```