



**Problem 1 (35 %) LP and KKT conditions (Exam August 2000)**

We consider the following LP on standard form:

$$\min_x c^\top x \quad \text{s.t.} \quad Ax = b, \quad x \geq 0 \quad (1)$$

with  $c, x \in \mathbb{R}^n$  and  $b \in \mathbb{R}^m$ . We know that the KKT conditions for this problem are

$$A^\top \lambda^* + s^* = c \quad (2a)$$

$$Ax^* = b \quad (2b)$$

$$x^* \geq 0 \quad (2c)$$

$$s^* \geq 0 \quad (2d)$$

$$s_i^* x_i^* = 0, \quad i = 1, \dots, n \quad (2e)$$

**a** The Newton direction is  $p_k^N = (-\nabla^2 f_k)^{-1} \nabla f_k$  (see equation (2.15) in the textbook). Here,  $f(x) = c^\top x$ , so that  $\nabla f_k = c$  and  $\nabla^2 f_k = 0$ . Hence,  $(-\nabla^2 f_k)^{-1}$  does not exist, and  $p_k^N = (-\nabla^2 f_k)^{-1} \nabla f_k$  is therefore not defined.

**b** For an optimization problem to be convex, the following must hold:

- the objective function is convex,
- the equality constraint functions  $c_i(\cdot)$ ,  $i \in \mathcal{E}$  are linear, and
- the inequality constraint functions  $c_i(\cdot)$ ,  $i \in \mathcal{I}$  are concave.

(See page 8 in the textbook.)

Using the definition of convexity (equation (1.4) in the textbook) on the objective function, we have

$$\begin{aligned} \alpha f(x_1) + (1 - \alpha)f(x_2) &= \alpha c^\top x_1 + (1 - \alpha)c^\top x_2 \\ &= c^\top (\alpha x_1 + (1 - \alpha)x_2) \\ &= f(\alpha x_1 + (1 - \alpha)x_2) \end{aligned} \quad (3)$$

which shows the convexity of the objective function. The equality constraint  $Ax = b$  is linear, which means the second requirement is satisfied. The inequality constraint is  $x \geq 0$ , so  $c(x) = x$  must be a concave function if the problem is to be convex. Using the definition of concavity (the opposite of convexity), we have that

$$c(\alpha x + (1 - \alpha)y) = \alpha x + (1 - \alpha)y \quad (4a)$$

and that

$$\alpha c(x) + (1 - \alpha)c(y) = \alpha x + (1 - \alpha)y \quad (4b)$$

That is, the function is both convex and concave at the same time (this is true for all linear functions). Hence, problem (1) is a convex optimization problem since all three requirements are satisfied.

**c** The dual problem for (1) is defined as

$$\max_{\lambda} b^{\top} \lambda \quad \text{s.t.} \quad A^{\top} \lambda \leq c \quad (5)$$

We rewrite the dual problem as

$$\min_{\lambda} -b^{\top} \lambda \quad \text{s.t.} \quad c - A^{\top} \lambda \geq 0 \quad (6)$$

and define the Lagrangean for the problem as

$$\bar{\mathcal{L}}(\lambda, x) = -b^{\top} \lambda - x^{\top} (c - A^{\top} \lambda) \quad (7)$$

where  $x \in \mathbb{R}^n$  are multipliers for the constraints. Differentiating with respect to  $\lambda$  and requiring the derivative to be zero gives

$$\nabla_{\lambda} \bar{\mathcal{L}} = -b + (x^{\top} A^{\top})^{\top} = Ax - b = 0 \quad (8)$$

This, together with the general KKT conditions, gives the following KKT conditions for the dual problem:

$$Ax^* = b \quad (9a)$$

$$A^{\top} \lambda^* \leq c \quad (9b)$$

$$x^* \geq 0 \quad (9c)$$

$$x_i^* (c - A^{\top} \lambda^*)_i = 0, \quad i = 1, \dots, n \quad (9d)$$

Defining  $s = c - A^{\top} \lambda$  shows that the KKT-conditions for the dual problem (5) equals the KKT-conditions for problem (1).

**d** The optimal objective  $c^{\top} x^*$  of problem (1) and the optimal objective  $b^{\top} \lambda^*$  of problem (5) have identical values.

**e** A *basic feasible point*  $x$  for problem (1) is defined by the following:

- A subset  $\mathcal{B} \subseteq \{1, \dots, n\}$  can be defined as containing exactly  $m$  indices,
- $i \notin \mathcal{B} \Rightarrow x_i = 0$ ,
- the  $m \times m$  matrix  $B$  defined by  $B = [A_i]_{i \in \mathcal{B}}$ , where  $A_i$  is the  $i$ th column of  $A$ , is nonsingular.

An equivalent definition is that a basic feasible point  $x$  has  $n - m$  elements set to zero, and nonzero elements given by the constraint  $Ax = b$ .

**f** The equality constraints  $Ax = b$  can be written

$$a_i^\top x = b_i, \quad i \in \mathcal{E} \quad (10a)$$

or

$$c_i(x) = a_i^\top x - b_i = 0, \quad i \in \mathcal{E} \quad (10b)$$

where  $a_i^\top$  is the  $i$ th row of the matrix  $A$ . We then have that the gradients of the equality constraints are

$$\nabla c_i(x) = a_i, \quad i \in \mathcal{E} \quad (11)$$

Since  $A$  has full (row) rank, all rows  $a_i^\top$  are linearly independent. This means that all equality constraint gradients  $\nabla c_i(x) = a_i$  are linearly independent.

## Problem 2 (40 %) LP

Two reactors,  $R_I$  and  $R_{II}$ , produce two products  $A$  and  $B$ . To make 1000 kg of  $A$ , 2 hours of  $R_I$  and 1 hour of  $R_{II}$  are required. To make 1000 kg of  $B$ , 1 hour of  $R_I$  and 3 hours of  $R_{II}$  are required. The order of  $R_I$  and  $R_{II}$  does not matter.  $R_I$  and  $R_{II}$  are available for 8 and 15 hours, respectively. The selling price of  $A$  is  $\frac{3}{2}$  of the selling price of  $B$  (i.e., 50 % higher). We want to maximize the total selling price of the two products.

**a** This problem can be formulated as the following standard-form LP:

$$\begin{aligned} \min \quad & [-3 \quad -2 \quad 0 \quad 0] x \\ \text{s.t.} \quad & \begin{bmatrix} 2 & 1 & 1 & 0 \\ 1 & 3 & 0 & 1 \end{bmatrix} x = \begin{bmatrix} 8 \\ 15 \end{bmatrix} \\ & x \geq 0 \end{aligned} \quad (12)$$

**b** Figure 1 shows a contour plot with the constraints indicated in the space of  $x_1$  and  $x_2$ .

**c** By modifying the example file provided to fit this problem, we find the following iteration sequence:

	Iteration number		
	1	2	3
$x$	$\begin{bmatrix} 0 \\ 0 \\ 8 \\ 15 \end{bmatrix}$	$\begin{bmatrix} 4 \\ 0 \\ 0 \\ 11 \end{bmatrix}$	$\begin{bmatrix} 1.8 \\ 4.4 \\ 0 \\ 0 \end{bmatrix}$

From Figure 1 we see that the the solution is at the intersection between the availability constraints  $2x_1 + x_2 \leq 8$  and  $x_1 + 3x_2 \leq 15$ . The non-negativity constraints are not active.

**d** The iterations along with iteration numbers are indicated in Figure 1.

**e** Looking at the algorithm output (the report given at each iteration) shows that the iteration sequence agrees with the theory in Chapter 13.3.

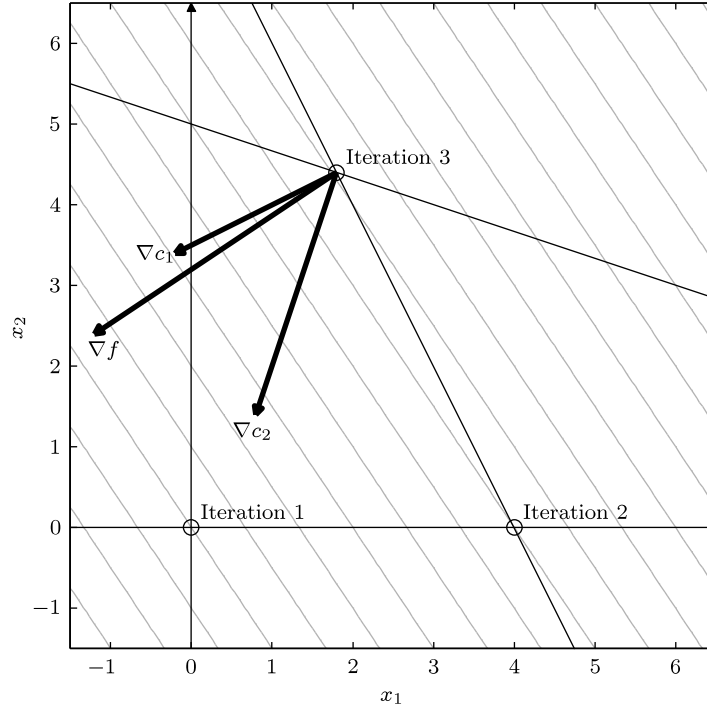


Figure 1: Contour plot with constraints for LP problem (12).

### Problem 3 (25 %) QP and KKT Conditions (Exam May 2000)

A quadratic program (QP) can be formulated as

$$\min_x q(x) = \frac{1}{2}x^\top Gx + x^\top c \quad (13a)$$

$$\text{s.t. } a_i^\top x = b_i, \quad i \in \mathcal{E} \quad (13b)$$

$$a_i^\top x \geq b_i, \quad i \in \mathcal{I} \quad (13c)$$

where  $G$  is a symmetric  $n \times n$  matrix,  $\mathcal{E}$  and  $\mathcal{I}$  are finite sets of indices and  $c$ ,  $x$  and  $\{a_i\}, i \in \mathcal{E} \cup \mathcal{I}$ , are vectors in  $\mathbb{R}^n$ .

- a** The active set  $\mathcal{A}(x^*)$  for problem (13) is defined as the set of indices of the constraints for which equality holds at  $x^*$ . That is,

$$\mathcal{A}(x^*) = \{i \in \mathcal{E} \cup \mathcal{I} | a_i^\top x^* = b_i\} \quad (14)$$

- b** Defining the Lagrangian for problem (13) as

$$\begin{aligned} \mathcal{L}(x, \lambda) &= q(x) - \sum_{i \in \mathcal{E} \cup \mathcal{I}} \lambda_i c_i(x) \\ &= \frac{1}{2}x^\top Gx + x^\top c - \sum_{i \in \mathcal{E} \cup \mathcal{I}} \lambda_i (a_i^\top x - b_i) \end{aligned} \quad (15)$$

From the complementarity condition  $\lambda_i^* c_i(x^*) = 0$ ,  $i \in \mathcal{E} \cup \mathcal{I}$ , we know that for constraints that are not active at the solution, the corresponding multipliers are

zero. Hence, the derivative with respect  $x$  of the Lagrangian at the solution can be written as

$$\begin{aligned}\nabla_x \mathcal{L}(x^*, \lambda^*) &= \frac{1}{2}(Gx^* + G^\top x^*) + c - \sum_{i \in \mathcal{A}(x^*)} \lambda_i^* a_i, \quad G^\top = G \\ &= Gx^* + c - \sum_{i \in \mathcal{A}(x^*)} \lambda_i^* a_i\end{aligned}\tag{16}$$

For all constraints  $i$  in the active set ( $i \in \mathcal{A}(x^*)$ ),

$$a_i^\top x^* = b_i\tag{17}$$

must hold at the solution. For inequality constraints  $i$  that are not active at the solution ( $i \in \mathcal{I} \setminus \mathcal{A}(x^*)$ ),

$$a_i^\top x^* \geq b_i\tag{18}$$

must hold. For active inequality constraints  $i$  ( $i \in \mathcal{I} \cap \mathcal{A}(x^*)$ ), the multipliers must be non-negative. That is,

$$\lambda_i^* \geq 0\tag{19}$$

The KKT conditions for problem (13) can now be summarized as

$$Gx^* + c - \sum_{i \in \mathcal{A}(x^*)} \lambda_i^* a_i = 0\tag{20a}$$

$$a_i^\top x^* = b_i, \quad i \in \mathcal{A}(x^*)\tag{20b}$$

$$a_i^\top x^* \geq b_i, \quad i \in \mathcal{I} \setminus \mathcal{A}(x^*)\tag{20c}$$

$$\lambda_i^* \geq 0, \quad i \in \mathcal{I} \cap \mathcal{A}(x^*)\tag{20d}$$