



### Problem 1 (25 %) Definitions

- a What is the definition of the gradient of a function?
- b What is the definition of the Jacobian of a function?
- c Let  $f(\mathbf{x})$  be a scalar and  $\mathbf{x} \in \mathbb{R}^n$ . What will the size of the gradient be?
- d Let  $\mathbf{f}(\mathbf{x})$  be a column vector of length  $m$  and  $\mathbf{x} \in \mathbb{R}^n$ . What will the size of the Jacobian be?

### Problem 2 (25 %) Linear

Let  $\mathbf{f}(\mathbf{x}) = \mathbf{Ax}$ , where

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

- a Use the definition and calculate  $\frac{\partial \mathbf{f}(\mathbf{x})}{\partial \mathbf{x}}$ . After calculating it, simplify it into a matrix form. Is this the Jacobian or the gradient of  $\mathbf{f}(\mathbf{x})$ ?
- b Can you, without doing any calculations, find  $\frac{\partial \mathbf{Ax}}{\partial \mathbf{x}}$  when  $\mathbf{x}$  is a column vector of length  $n$ , and  $\mathbf{A}$  is a matrix of dimension  $m \times n$  (i.e.,  $m$  rows and  $n$  columns)?

### Problem 3 (25 %) Nonlinear

Let  $f(\mathbf{x}, \mathbf{y}) = \mathbf{x}^T \mathbf{G} \mathbf{y}$ , where

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad \mathbf{G} = \begin{bmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \end{bmatrix}, \quad \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}.$$

- a What is the dimension of  $f(\mathbf{x}, \mathbf{y})$ ? Is  $\nabla_{\mathbf{x}} f(\mathbf{x}, \mathbf{y})$  equal to  $\frac{\partial f(\mathbf{x}, \mathbf{y})}{\partial \mathbf{x}}$  (no calculations are needed)?
- b Use the definition and calculate  $\nabla_{\mathbf{x}} f(\mathbf{x}, \mathbf{y})$ . Then write the answer in matrix form.
- c Use the definition and calculate  $\nabla_{\mathbf{y}} f(\mathbf{x}, \mathbf{y})$ . Then write the answer in matrix form.
- d Let  $f(\mathbf{x}) = \mathbf{x}^T \mathbf{H} \mathbf{x}$ , where  $\mathbf{x} \in \mathbb{R}^n$  and  $\mathbf{H} \in \mathbb{R}^{n \times n}$ . Find  $\nabla f(\mathbf{x})$  using the results from the previous exercises. What will the answer be if  $\mathbf{H}$  is symmetric?

#### Problem 4 (25 %) Common case

This problem is supposed to be solved using matrices only (i.e., the elements of the matrices are irrelevant). During this course you will encounter (1), **many** times.  $\mathcal{L}(\dots)$  is a scalar,  $\mathbf{G}$  is symmetric and  $\mathbf{x}$  is of length  $n$ . The remaining vectors and matrices are of appropriate sizes such that the output of each term is a scalar. If you feel there is a need of specifying the sizes to solve the problem, justify your reasoning and set up some variable sizes (i.e., use variables (m,n,p,q, ...) instead of fixed numbers).

I would recommend you to try to utilize what you have learnt in the previous exercises before looking at the “Matrix Calculus”-note. Use this as a test to see if you have understood the rules.

$$\mathcal{L}(\mathbf{x}, \boldsymbol{\lambda}, \boldsymbol{\mu}) = \mathbf{x}^T \mathbf{G} \mathbf{x} + \boldsymbol{\lambda}^T (\mathbf{C} \mathbf{x} - \mathbf{d}) + \boldsymbol{\mu}^T (\mathbf{E} \mathbf{x} - \mathbf{h}) \quad (1)$$

**a** Find  $\nabla_{\mathbf{x}} \mathcal{L}(\mathbf{x}, \boldsymbol{\lambda}, \boldsymbol{\mu})$ .

**b** Find  $\nabla_{\boldsymbol{\mu}} \mathcal{L}(\mathbf{x}, \boldsymbol{\lambda}, \boldsymbol{\mu})$ .

**c** Find  $\nabla_{\boldsymbol{\lambda}} \mathcal{L}(\mathbf{x}, \boldsymbol{\lambda}, \boldsymbol{\mu})$ .