TTK4135 Assignment 3 Eirik Falck

Problem 1

$$\min_x \; c^T x \quad \text{s.t.} \quad Ax = b, \; x \geq 0$$

$$c, x \in \mathbb{R}^n \qquad b \in \mathbb{R}^m$$

KKT conditions: $\mathcal{L}(x, \lambda, s) = c^T x - \lambda^T (Ax - b) - s^T x$

- $\nabla_x \mathcal{L}(x^*, \lambda^*, s^*) = c - A^T \lambda^* - s^* = 0$
  $\hookrightarrow A^T \lambda^* + s^* = c$
- $Ax^* = b$
- $x^* \geq 0$
- $s^* \geq 0$
- $s^{*T} x^* = 0$

a) The Newton direction may not be defined because $\nabla^2 (c^T x) = 0 \not> 0$, i.e. $\nabla^2 (c^T x)$ is not positive definite.

b) Definition of convex function

$$f(\alpha x + (1-\alpha) y) \leq \alpha f(x) + (1-\alpha) f(y) \quad \forall \; \alpha \in [0,1]$$

In LP: $f(x) = c^T x$

- Objective function convex: ✓
  $$c^T(\alpha x + (1-\alpha) y) = c^T(\alpha x) + c^T([1-\alpha] y)$$
  $$= \alpha c^T x + (1-\alpha) c^T y$$

To have a convex optimization problem, we need

- convex objective function ✓
- linear equality constraint: $Ax = b$ ✓
- concave inequality constraints: ✓

  $x \geq 0$: $f(\alpha x + (1-\alpha) y) \geq \alpha f(x) + (1-\alpha) f(y)$

  $(\alpha x + (1-\alpha) y) \geq \alpha x + (1-\alpha) y$

$\Rightarrow$ The linear program is convex.

c) Dual problem: $\max_{\lambda} b^T \lambda$ s.t. $A^T \lambda \leq c$

Make dual to standard form:

$$\min_{\lambda} -b^T \lambda \quad \text{s.t.} \quad c - A^T \lambda \geq 0$$

$$\mathcal{L}(\lambda, x) = -b^T \lambda - x^T (c - A^T \lambda)$$

KKT conditions:

- $\nabla_\lambda \mathcal{L}(\lambda^*, x^*) = -b + Ax^* = 0$

  $\rightarrow Ax^* = b$

- $x^* \geq 0$ $\rightarrow A^T \lambda^* + s^* = c$
- $s^* \triangleq (c - A^T \lambda^*) \Rightarrow s^* \geq 0$
- $s^{*T} x^* = 0$

d) $c^T x^* = (A^T \lambda^* + s^*)^T x^* = \lambda^{*T} \underbrace{A x^*}_{b} + \underbrace{s^{*T} x^*}_{=0}$

$= \lambda^{*T} b = b^T \lambda^*$

$\Rightarrow \quad \underline{\underline{c^T x^* = b^T \lambda^*}}$

e) Basic feasible point:
- The possible point that can be optimal points.
- In LP, this is the vertecies of the feasible region.
- $\mathcal{B} \subseteq \{1, \dots, n\}$ with exactly m indices s.t. if $i \notin \mathcal{B} \Rightarrow x_i = 0$ and $B = [A_i]_{i \in \mathcal{B}}$ is a $m \times m$ matrix with columns from A.

f) LICQ: All ∇c_i, $i \in \mathcal{A}(x)$ are linearly independent.

A is full rank $\Leftrightarrow$ all rows of A linearly independent.

This means all constraints are linearly independent, hence ∇c_i are also linearly independent.

Problem 2

a) $R_I: \quad 2A + 1B \leq 8$ $\qquad$ A,B: loading product.

$R_{II}: \quad 1A + 3B \leq 15$

$\rightarrow \quad 2A + 1B + x_3 = 8$ $\qquad x_3, x_4 \geq 0$

$\quad 1A + 3B + x_4 = 15$ $\qquad$ ↳ slack variables

$$x = \begin{bmatrix} A \\ B \\ x_3 \\ x_4 \end{bmatrix}; \quad A = \begin{bmatrix} 2 & 1 & 1 & 0 \\ 1 & 3 & 0 & 1 \end{bmatrix} \quad b = \begin{bmatrix} 8 \\ 15 \end{bmatrix}$$

$c^T = [3/2 \;\; -1]$ if price of 1B : 1.

$\Rightarrow$ LP: $\min_x c^T x$ s.t. $Ax = b$, $x \geq 0$

c) Yes, x^* is at the intersection of both constraints, making them both active.

b, d) See figure below.

e) The plot only shows A and B, but we see the iterations move along the constraints $Ax = b$ and $x = 0$. This fits the theory well.

Problem 3

$$\min_x \; q(x) = \frac{1}{2}x^T G x + x^T c$$ G: $n \times n$, sym.

s.t. $a_i^T x = b_i \quad i \in \mathcal{E}$ $\qquad c, x, a_i \in \mathbb{R}^n$

$a_i^T x \geq b_i \quad i \in \mathcal{I}$

a) $\mathcal{A}(x^*) = \mathcal{E} \cup \{ i \in \mathcal{I} / a_i^T x^* = b_i \}$

where x^* is the optimal point.

b) $\mathcal{L}(x^*, \lambda^*) = \frac{1}{2} x^{*T} G x^* + x^{*T} c - \sum_{i \in \mathcal{A}(x^*)} \lambda_i (a_i^T x^* - b_i)$

- $\nabla_x \mathcal{L}(x^*, \lambda^*) = G x^* + c - \sum_{i \in \mathcal{A}(x^*)} a_i \lambda_i = 0$

- $a_i^T x^* = b_i \qquad i \in \mathcal{A}(x^*)$
- $a_i^T x^* \geq b_i \qquad i \in \mathcal{I} \setminus \mathcal{A}(x^*)$
- $\lambda_i^* \geq 0 \qquad i \in \mathcal{I} \cap \mathcal{A}(x^*)$