TMA4180 Optimization Project
BFGS, Penalties and Stress - Optimizing our way to Stable Structures

Eirik Fagerbakke, Synne Gilbu, Henrik Grenersen
and Petter Røe Sundhaug

April 2023

Contents

1 Abstract

Tensegrity structures have long been a topic of interest in arts. However, recently they have been more widely used in engineering because of of their ability to change shape by adjusting the length of the cables. We present in this paper an optimization-based approach for form-finding by minimizing the energies in the structures for different configurations of bars and cables. We have looked at BFGS and a quadratic penalty method to solve these optimization problems numerically. For our numerical experiments, we seemed successful in minimizing the energy and finding the stable tensegrity structures.

2 Introduction

Tensegrity structures are structures consisting of cables and bars connected at joints. The reason that the structures can hold together is because of the tension in the cables. It is also possible for the bars to be compressed, but this does not meaningfully contribute to a structure holding together. We mainly use the principle of internal potential energy together with constraints to give way to optimization problems. We then use known numerical methods to approximate the shape of the structures that has the least internal energy.

In this paper we first explain the mathematical model used and the constraints. We then have sections 4, 5 and 6 that covers the configuration of a model with only cables and fixed nodes, a model with the inclusion of bars, and a free standing model respectively. For each configuration we will present different mathematical properties that will give rise to strategies of how to compute a solution with the help of known optimization algorithms.

3 Choice of Model and Existence of Solutions

3.1 Model

The tensegrity structure is modeled as a directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ with vertex set $\mathcal{V} = \{1, ..., N\}$ and edge set $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$, where the vertices represent the joints in the structure. Edges $e_{ij} = (i, j)$ with $i < j$ indicates that the joints i, j is connected either through a cable or a bar. The set of cables is denoted by $\mathcal{C}$, and $\mathcal{B}$ describes the corresponding set for bars.

The total energy for the system is given by:

$$E(X) = \sum_{e_{ij} \in \mathcal{B}} \left(E_{\text{elast}}^{\text{bar}}(e_{ij}) + E_{\text{grav}}^{\text{bar}}(e_{ij}) \right) + \sum_{e_{ij} \in \mathcal{C}} E_{\text{elast}}^{\text{cable}}(e_{ij}) + E_{\text{ext}}(X) \tag{1}$$

Where:

$$E_{\text{elast}}^{\text{bar}}(e_{ij}) = \frac{c}{2\ell_{ij}^2} \left(\left\| x^{(i)} - x^{(j)} \right\| - \ell_{ij} \right)^2 \tag{2}$$

$$E_{\text{grav}}^{\text{bar}}(e_{ij}) = \frac{\rho g \ell_{ij}}{2} \left(x_3^{(i)} + x_3^{(j)} \right) \tag{3}$$

$$E_{\text{elast}}^{\text{cable}}(e_{ij}) = \begin{cases} \frac{k}{2\ell_{ij}^2} \left(\left\| x^{(i)} - x^{(j)} \right\| - \ell_{ij} \right)^2 & \text{if } \left\| x^{(i)} - x^{(j)} \right\| > \ell_{ij}, \\ 0 & \text{if } \left\| x^{(i)} - x^{(j)} \right\| \leq \ell_{ij} \end{cases} \tag{4}$$

$$E_{\text{ext}}(X) = \sum_{i=1}^{N} m_i g x_3^{(i)} \tag{5}$$

3.2 Constraints

We will look at two types of constraints. The first idea is to fix M of the points, i.e.

$$x^{(i)} = p^{(i)} \qquad i = 1, \ldots, M \tag{6}$$

and then solve the unconstrained optimization problem in $\mathbb{R}^{3(N-M)}$.

The second idea is to use the constraint that the points all have to be above the ground:

$$x_3^{(i)} \geq 0 \qquad i = 1, \ldots, N \tag{7}$$

3.3 Existence of Solutions

We first want to investigate whether our problem (1) admits a global solution or not. We know that the problem admits a global solution if the objective function is lower semi-continuous and coercive, and in addition that our feasible set is closed.

That the function is lower semi-continuous is easily seen from the fact that we only have polynomial terms in our expression, as well as terms involving norms. Since norms and polynomials are continuous, the function is then continuous, and therefore also lower semi-continuous.

We first investigate constraint (6), where we have fixed points. As some of the nodes are fixed, the cables and/or bars will thereby be stretched to infinity if we let the norm of X go to infinity. We must also note that the norm can go to infinity by letting one or more $x_3^{(i)}$ go to negative infinity. This is however counteracted by the elastic energy of the cables/bars, which goes to positive infinity faster because of the squared term. We also need that $\mathcal{G}$ is connected. Otherwise, we could simply place a disconnected structure infinitely far away from the rest of the structure, and let the norm of X go to infinity without letting the energy go to infinity.

For the second constraint, (7), where all nodes need to be above the ground, the external load term goes to infinity if one or more $x_3^{(i)}$ go to infinity. However, here we also have the possibility of moving the entire structure infinitely far away from the origin horizontally, which would lead to an increasing norm, but not to an increase in the energy function. This can be solved by making one of the nodes act as a "reference node" that is constant in the horizontal components. Provided that $\mathcal{G}$ is connected, we can then argue that the energy function must go to infinity when the norm of X goes to infinity, as we did with the previous constraint (6).

Finally, we also need our feasible sets to be closed. For the fixed points, 6, we have a free optimization problem over $\mathbb{R}^{3(N-M)}$, so the set is closed.

For the other case, 7, we want to minimize over $\mathbb{R}^{3N}$, but with the constraint that all the nodes are above the ground. This is also closed, as we have $x_3^{(i)} \geq 0$, rather than $x_3^{(i)} > 0$.

Since we have a lower semi-continuous, coercive function, and closed feasible sets in both (6) and (7), the problem admits a global solution in both cases.

4 Cable-nets

We now restrict the problem to the case where we only have contributions to the energy from the elasticity in the cables (4) and the external loads (5), for which we want to minimize the structure's energy given a set of fixed nodes (6).

4.1 On the Differentiability of our Objective Function

Now, we want to investigate if our objective function is defined in C^1 and C^2.

We know that the last term, $E_{\text{ext}}(X)$, is infinitely differentiable, as we only have linear terms in $x_3^{(i)}$. Computing the gradient with respect to one node yields

$$\nabla_{x^{(i)}} E_{\text{ext}}(X) = \begin{pmatrix} 0 & 0 & m_i g \end{pmatrix}^\top . \tag{8}$$

The only term we need to investigate is therefore $E_{\text{elast}}^{\text{cable}}(e_{ij})$. We begin by considering the gradient of our objective function with respect to one of the nodes in the structure.

Because of our model for the elastic energy in the cables, the cables cannot be compressed, and thus both the function and the gradient is equal to zero for when the distance between the points is less than the resting length, and also when the points approach each other to eventually have a distance between them equal to the resting length. We observe that the gradient also approaches zero when the nodes approach each other in the opposite scenario, i.e. when $\|x^i - x^j\| \to l_{ij}^+$. We will later show that exactly this property is important for differentiability, compared to the case of compressible bars.

Because of symmetry in the elasticity term of a cable 4, we see that computing the gradient with respect to $x^{(i)}$ or $x^{(j)}$ will yield the same result. We have therefore introduced the notation e_{ij}^* to mean a cable that is connected between nodes i and j, without taking into consideration the ordering of the points. The same holds for ℓ_{ij}^*.

The gradient of the elasticity term of a cable, that includes the node $x^{(i)}$, is then

$$\nabla_{x^{(i)}} E_{\text{elast}}^{\text{cable}}(e_{ij}^*) = \begin{cases} \frac{k}{(\ell_{ij}^*)^2} \left(x^{(i)} - x^{(j)}\right) \left[1 - \frac{\ell_{ij}^*}{\|x^{(i)} - x^{(j)}\|}\right] & \|x^{(i)} - x^{(j)}\| > \ell_{ij}^* \\ 0 & \|x^{(i)} - x^{(j)}\| \le \ell_{ij}^* \end{cases} \tag{9}$$

To evaluate whether $E_{\text{elast}}^{\text{cable}}$ is C^2, we need to look at the Hessian. To simplify calculations, we have looked at the first component of $\nabla_{x^{(i)}} E_{\text{elast}}^{\text{cable}}(e_{ij})$, and differentiated with respect to $x_2^{(i)}$:

$$\partial_{x_2^{(i)}} \left(\frac{k}{l_{ij}^2} (x_1^{(i)} - x_1^{(j)}) \left[1 - \frac{\ell_{ij}}{\|x^{(i)} - x^{(j)}\|} \right] \right) = \frac{k}{\ell_{ij}} (x_1^{(i)} - x_1^{(j)}) \|x^i - x^j\|^{-3} (x_2^{(i)} - x_2^{(j)})$$

If we now look at the limit $\|x^{(i)} - x^{(j)}\| \to \ell_{ij}^+$, we get

$$k \ell_{ij}^{-4} (x_1^{(i)} - x_1^{(j)})(x_2^{(i)} - x_2^{(j)})$$

which is not equal to 0 in general. We therefore have a discontinuity in the hessian for $\|x^{(i)} - x^{(j)}\| \to \ell_{ij}^{+}$. In general, our objective function is therefore C^1, but not C^2.

4.2 Convexity

We now look at the properties of convexity of the restricted cable net problem. If all terms in the energy function are convex, then so is $E(X)$ itself. We first look at the internal energy from the elasticity, and use expression (4).

We first note that if we have cables with zero internal energy, then it is trivial that we would have convexity given that the other functions are convex. In the general case, we can consider a function f of the distance between two nodes, which is proportional to the elastic energy in the bar between them. We then consider the derivative of this function:

$$f'(\|x^{(i)} - x^{(j)}\|) = ((\|x^{(i)} - x^{(j)}\| - \ell_{ij})^2)' = 2\|x^{(i)} - x^{(j)}\| - 2\ell_{ij}. \tag{10}$$

As long as $\|x^{(i)} - x^{(j)}\| > l_{ij}$ holds, we can say that f is an increasing function, and therefore also convex. Because of equation (4), we also know that given that the distance between these nodes is equal to the resting length of the cables or less, we remember that the energy is zero, so we need not consider this case.

We now define a function $g : \mathbb{R}^{3\times 3} \to \mathbb{R}$, such that $g(x^{(i)}, x^{(j)}) = f(\|x^{(i)} - x^{(j)}\|)$. We can see that it satisfies the definition of convexity,

$$\begin{aligned}
g\left(\lambda x^{(i)} + (1-\lambda)y^{(i)}, \lambda x^{(j)} + (1-\lambda)y^{(j)}\right) &= f\left(\|\lambda(x^{(i)} - x^{(j)}) + (1-\lambda)(y^{(i)} - y^{(j)})\|\right) \\
&\leq f\left(\|\lambda(x^{(i)} - x^{(j)})\|\right) + f\left(\|(1-\lambda)(y^{(i)} - y^{(j)})\|\right) \\
&= f\left(\lambda\|x^{(i)} - x^{(j)}\|\right) + f\left((1-\lambda)\|y^{(i)} - y^{(j)}\|\right) \\
&\leq \lambda f\left(\|x^{(i)} - x^{(j)}\|\right) + (1-\lambda) f\left(\|y^{(i)} - y^{(j)}\|\right) \\
&= \lambda g\left(x^{(i)}, x^{(j)}\right) + (1-\lambda) g\left(y^{(i)}, y^{(j)}\right),
\end{aligned} \tag{11}$$

where we have used that function f and the norms also satisfies the definition of convexity. Thus, we know that g is also convex. We can then translate this to our problem by having $g(x^{(i)}, x^{(j)}) = E_{\text{elast}}^{\text{cable}}(e_{ij})$, and therefore it is convex.

As for $E_{\text{ext}}(X)$, it will always be convex since it is clearly continuous and increasing. Its second derivative is also non-negative. This means that every part of the sum is convex and therefore our objective function is also convex. Therefore, every local minimum is also a global minimum, but since the function is not strictly convex, we cannot be sure that the function only has one global minima.

4.3 Optimality Conditions

Since our objective function $E(X)$ is convex and we have an unconstrained problem, the necessary and sufficient condition is that the gradient of our objective function is zero. Below we have presented the gradient of the energy function with respect to one node, $x^{(i)}$:

$$\nabla_{x^{(i)}} E(X) = \sum_{j: e_{ij}^* \in \mathcal{C}} \nabla_{x^{(i)}} E_{\text{elast}}^{\text{cable}}(e_{ij}^*) + \nabla_{x^{(i)}} E_{\text{ext}}(X) \qquad i > M. \tag{12}$$

This constitutes a vector of three elements, which can be placed in succession to yield the full gradient, $\nabla_X E(X) \in \mathbb{R}^{3(N-M)}$, by considering each of the free nodes.

We note that in the special case that our structure consists of two nodes connected by a cable, the gradient of the elastic term becomes zero when the cable length is less than or equal to the resting length. This is in line with our physical intuition as well.

4.4 Numerical Approach

We have implemented the BFGS algorithm to find stable structures, given an initialization. In our implementation of BFGS we have followed the details presented in the lectures to initialize our approximation of the Hessian, and we have used strong Wolfe conditions, inspired by the approach presented in the exercises, in order to choose our step lengths.

5 Inclusion of Bars

Although some of the structures presented earlier are quite interesting, they have all been limited in the way that the stable configurations for the free nodes have all lain below our fixed nodes. Perhaps more interesting is however configurations where our free nodes can stand above our fixed nodes, and this is where the bars make their appearance. Before we get to these structures, we will focus on some theoretical results for this new problem.

5.1 Lack of Differentiability

The first of these results concerns our objective function, which now also includes terms that represent the energy of the bars. Because we have already considered the terms concerning the cables and external loads, we only need to consider the new terms when it comes to the gradient.

We see that $E_{\text{grav}}^{\text{bar}}(e_{ij})$ only contains linear terms in $x_3^{(i)}$, so it is infinitely differentiable. For the other terms, we have a similar situation as with $E_{\text{elast}}^{\text{cable}}(e_{ij})$, but we now have the possibility of compressing the cable. By evaluating the gradient of $E_{\text{elast}}^{\text{bar}}(e_{ij})$ as we did in (9), we get

$$\nabla_{x^{(i)}} E_{\text{elast}}^{\text{bar}}(e_{ij}^*) = \frac{c}{(\ell_{ij}^*)^2}\left(x^{(i)} - x^{(j)}\right)\left[1 - \frac{\ell_{ij}^*}{\|x^{(i)} - x^{(j)}\|}\right]. \tag{13}$$

If we now consider the points $x^{(j)}$ and $x^{(i)} = x^{(j)} + t(x_1^*, x_2^*, x_3^*)$ connected by a bar e_{ij}, we see that our expression simplifies to the following:

$$\frac{c}{l_{ij}^2}\,(x_1^*, x_2^*, x_3^*)\left[t - \frac{t l_{ij}}{|t|\sqrt{(x_1^*)^2 + (x_2^*)^2 + (x_3^*)^2}}\right].$$

Now, in order for $x^{(i)}$ to approach $x^{(j)}$, we need to evaluate the limit $t \to 0$. We see that the second term in the brackets has a discontinuity at zero, where it approaches -1 from the left and 1 from the right. Thus, the gradient is discontinuous whenever two points that are connected are pushed towards together, and our objective function is in general not differentiable.

However, this lack of differentiability should in practical situations pose no problems. This would entail that our bars could be compressed to a single point, in which case it would make sense to reevaluate our model, as the energy should then in reality grow to infinity. By using a large enough value for the material parameter c, that properly reflects the rigidity of the bars, we can prevent this from happening.

5.2 Optimality Conditions

Since the problems we just presented concerning differentiability should pose no problems in practical situations, the first order necessary optimality condition still states that the gradient of our objective function must be zero at a point for it to be a local minimizer.

To compute the gradient, we need to add terms involving the gravitational and elastic energy of the bars to the expression in (12). For the gravitational energy, we have the gradient

$$\nabla_{x^{(i)}} E_{\text{grav}}^{\text{bar}}(e_{ij}^*) = \begin{pmatrix} 0 & 0 & \frac{\rho g \ell_{ij}^*}{2} \end{pmatrix}^\top \tag{14}$$

.

We then get that the gradient of our objective function with respect to a node $x^{(i)}$ is

$$\begin{aligned} \nabla_{x^{(i)}} E(X) = \sum_{j:e_{ij}^* \in \mathcal{C}} \nabla_{x^{(i)}} E_{\text{elast}}^{\text{cable}}(e_{ij}^*) + \nabla_{x^{(i)}} E_{\text{ext}}(X) + \qquad & i > M \\ \sum_{j:e_{ij}^* \in \mathcal{B}} \left(\nabla_{x^{(i)}} E_{\text{elast}}^{\text{bar}}(e_{ij}^*) + \nabla_{x^{(i)}} E_{\text{grav}}^{\text{bar}}(e_{ij}^*) \right) & \end{aligned} \tag{15}$$

5.3 Convexity

Since we have not discussed convexity for this new problem, we cannot be certain that the above presented condition is also sufficient. Therefore, we now aim to decide whether our problem is convex. In order to do this we again investigate the $E_{\text{elast}}^{\text{bar}}$ term. This is mainly because we have shown that the rest of the function is convex, and the gravity terms for the bar should also pose no problems for convexity, as they are linear. In order to show that this function is not convex, we need that

$$E_{\text{elast}}^{\text{bar}}(\lambda X + (1-\lambda)Y) \geq \lambda E_{\text{elast}}^{\text{bar}}(X) + (1-\lambda) E_{\text{elast}}^{\text{bar}}(Y)$$

for some $\lambda \in [0, 1]$ and a pair of structures X and Y.

First we assume that our structure has one fixed node, $p^{(1)}$, and that X consists of only one node, $x^{(2)}$. This node is connected to the fixed node by a bar, with distance equal to the resting length of the bar. Then we set $Y = -X$, and since there is no elastic energy in the bars, we get that the convex combination of the elastic energy in the bars is zero.

Now, we only have to show that $E_{\text{elast}}^{\text{bar}}(\lambda X + (1-\lambda)Y) \geq 0$. To show this we will use an example, with a fixed node $p^{(1)} = (0, 0, 0)$ for both structures.

We set $X = (1, 0, 0)$, so that $Y = (-1, 0, 0)$, and let the resting length of the cable in both structures be 1. We then set $\lambda = 0.8$, and get that $\lambda X + (1-\lambda)Y = (0.6, 0, 0)$ which means that the distance between the free node and the fixed node in origin is smaller than the resting length, and we get an increase in energy.

This shows that our function is non-convex if $\mathcal{B} \neq \emptyset$, meaning that our optimality condition is not sufficient. We also provide an example of a situation where our problem admits a (non-global) local minima in section (7.2).

5.4 Numerical Approach

We have here made use of our previous implementation for BFGS, with updates to the objective function and gradient, in accordance with the relations derived in section 4.

6 Free-standing Structures

Now we direct our focus to free standing structures, which are structures for which the only constraint is that all nodes are positioned at or above ground level. This constraint can be formulated mathematically for each node as $c_i(X) = x_3^{(i)} \geq 0$, and we therefore have a constrained optimization problem in $\mathbb{R}^{3N}$.

6.1 Optimality Conditions

We now again aim to derive optimality conditions for our problem, and to decide whether they are sufficient or necessary. Since we have a constrained problem with inequality constraints, the KKT conditions provide us with first order optimality conditions. Had our objective function also been convex, these conditions would actually be sufficient, but, alas, the case is no longer that simple. For a local solution X^* we have that as long as LICQ holds, there exists a Lagrange multiplier $\lambda^* \in \mathbb{R}^N$ such that

$$\nabla E(X^*) - \sum_{i \in \mathcal{I}} \lambda_i^* \nabla c_i(X^*) = 0, \qquad c_i(X^*) \geq 0 \quad \text{if } i \in \mathcal{I}, \tag{16}$$

$$\lambda_i^* \geq 0 \quad \text{if } i \in \mathcal{I}, \qquad \lambda_i^* c_i(X^*) = 0 \quad \text{if } i \in \mathcal{I}. \tag{17}$$

Although these conditions are not sufficient, they are in fact necessary, given that LICQ holds. This means that we need the gradient of the active constraints to be linearly independent. We observe that the gradient of each of these constraints with respect to the node they constrain is the unit vector in the z-direction. Thus, the gradient of $c_i(X)$ with respect to a node i is a vector containing zeros at all indices, except at index $3i$ where we have 1. Therefore we also have that all the gradients of the constraints are linearly independent, and LICQ holds, meaning that the KKT conditions are in fact necessary optimality conditions.

6.2 Invariance under Simultaneous Horizontal Shifts

Since we no longer have any fixed nodes, one could argue that there exists infinitely many structures with the same, low energy. We want to investigate this, i.e. whether the total energy of the structure is invariant to simultaneous horizontal shifts, as it might pose problems for a later numerical approach to solving the problem. This is shown by checking each term in (1) . Since the horizontal positions are not included in $E_{\text{grav}}^{\text{bar}}(e_{ij})$ and $E_{\text{ext}}(X)$, these functions are invariant. Now we only need to consider the elastic energies. In both of these expressions, the only part that is subject to the horizontal coordinates are the norms, which are identical for both functions.

We denote the shifts as a in the x_1-direction and b in the x_2-direction, which gives the following expression for the distance between the nodes

$$||[x_1^{(i)} + a, x_2^{(i)} + b, x_3^{(i)}] - [x_1^{(j)} + a, x_2^{(j)} + b, x_3^{(j)}]||= \|x^{(i)} - x^{(j)}\|.$$

So every norm that is included will be invariant to horizontal shifts. We can then say that $E(X)$ is invariant to simultaneous horizontal shifts in the free standing problem.

A possible solution to address the issue of horizontal shifts is to fix the coordinates x_1 and x_2 of the first node. The optimization problem will then have a point of reference and so we limit the number of possible solutions to the optimization problem.

6.3 Numerical Approach

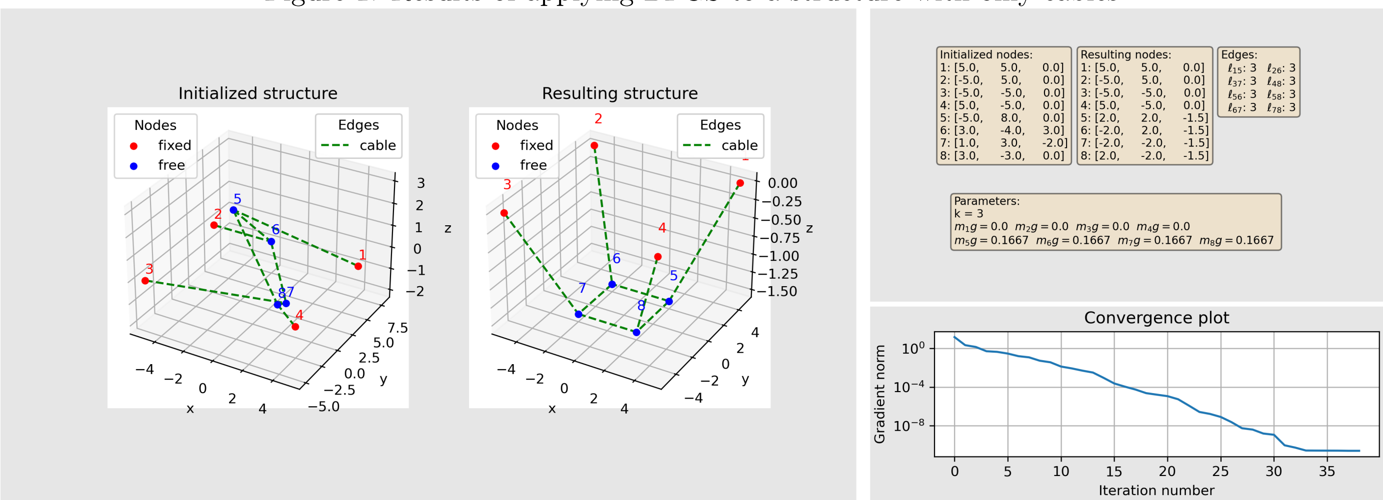
We have here implemented a version of the Quadratic penalty algorithm, and used the solution presented above where two coordinates are fixed, so we effectively solve an optimization problem in $3N - 2$ variables. BFGS is used to minimize the objective function given by the sum of the energy function and the sum of squared constraints, multiplied by a penalty parameter.

7 Numerical Results

7.1 Cable-nets

In the case where we only have contributions to the energy from the elasticity in the cable and the external loads, we get a plot that corresponds quite well with our intuition.

Figure 1: Results of applying BFGS to a structure with only cables

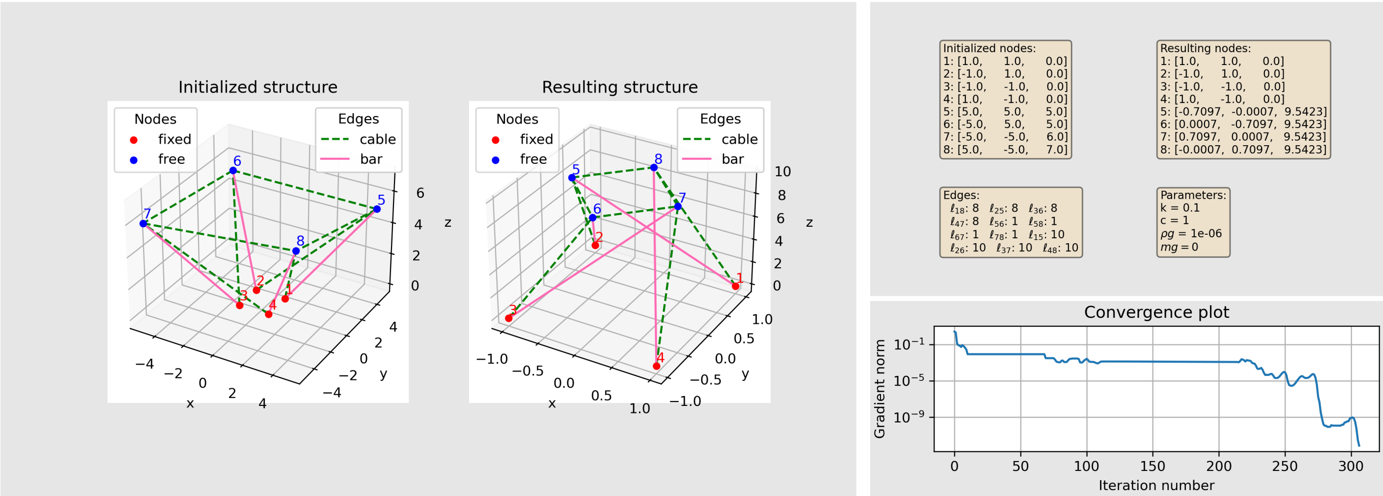


The convergence plot shows us that we get superlinear convergence, which means that the error decreases more rapidly for each iteration.

7.2 Inclusion of Bars

Now that we have included bars in our structures, we also have the possibility that structures might rise above our fixed nodes, given a good initialization. An example of this is presented below, where we start out with a rather large square at the top where the cables are stretched out, and a plot of this is presented below.

Figure 2: Results of applying BFGS to a structure with bars and cables



Then, after a significantly higher number of steps than in subsection (7.1), we also achieve convergence towards the stable structure presented here. The convergence plot shows that we no longer achieve superlinear convergence, and a possible reason for this might the lack of convexity in our objective function. Another possibility, which is more specific for BFGS, is that we have shown that our objective function is no longer twice differentiable, although one could say that this algorithm is designed to work especially well on functions that are indeed twice differentiable.

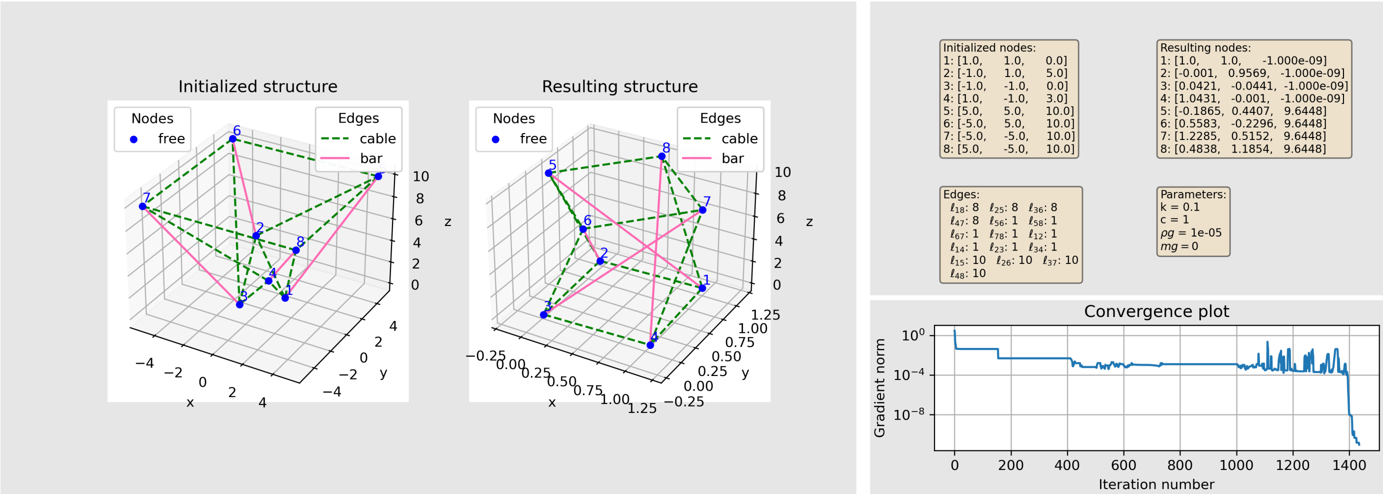
Had we instead initialized the set of free nodes beneath the fixed nodes, we would have attained a minima with lower energy than the structure presented below, which seems rather intuitive. This is done as an additional experiment in the notebook.

7.3 Free-standing Structures

Ultimately, we have also tested our approach for free-standing structures with an initialization that is very similar to the first initialization in the last section, but with cables between the nodes on the ground. This initialization is presented below, along with the final structure and a convergence plot.

Firstly we note that the final structure resembles the structure from subsection (7.2), albeit with some slight modifications. Furthermore, most points of the structure are in fact above ground, except for the first four nodes, which are placed marginally below ground. This indicates that the penalties used seem to fulfil their purpose. We also observe that the convergence rate fluctuates a bit more for this experiment, especially when the number of iterations increases above 1000. Because we allow BFGS to use maximally 500 iterations for each penalty, we can be certain that these rates concern our highest penalty, which is 10^5, as the penalty increases by 10, starting at 100. From class we also know that these problems tend to be ill-posed as the penalty increases, and this convergence plot might be an example of this.

Figure 3: Results of applying quadratic penalty method to a free-standing structure



8 Conclusion

In this project we have investigated different types of tensegrity structures. At first we had a look at the structure where we only had cable nest and external loads. We found this problem to be convex and therefore every local minimum was also a global minimum. For the numerical approach we could observe what our intuition would say, and we got a superlinear convergence.

Then we included bars to our structure. In this case we do not have convexity, and our optimality condition is not sufficient. We get that we can have non-global local minimum. The lack of superlinear convergence might come from the fact that we do not have convexity.

At last, we have our free-standing structures with all nodes positioned over ground level. The optimality conditions for this problem will be necessary, but not sufficient. In the numerical approach we get a fluctuating convergence rate.

To summarize, we would say that throughout the project, we have learned a lot about tensegrity structures, but perhaps even more important about the underlying structures of many optimization problems. Especially about how more fascinating the structures can become as the problems' complexity increases.