Emergency Center Queues, Climate Models and GameStop Stocks - What a Ride!

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Problem 1

a)

To have an M/M/1 queue, we need to have three conditions fulfilled:

1. interarrival times are iid exponentially distributed (memoryless)
2. service times are iid exponentially distributed (memoryless)
3. there is only one server, and the service times are independent of the arrival process.

This is satisfied in problem 1, as the arrival of patients follow a Poisson process with rate λ, meaning that the interarrival times must be iid $Exp(\lambda)$, with mean $1/\lambda$. The treatment times of the patients are also iid exponentially distributed, with mean $1/\mu$. Finally, we only treat one patient at a time. The UUC is therefore an M/M/1 queue.

$X(t)$ determines the amount of patients in the system at time t, which can only increase or decrease by one in a short interval, and the change in $X(t)$ is proportional to t.

We have the following rates:

$$\text{Birth rate:} \quad \lambda_i = \lambda, \qquad i = 0, 1, \ldots$$
$$\text{Death rate:} \quad \mu_0 = 0 \qquad \mu_i = \mu, \qquad i = 1, 2, \ldots$$

since we only have one server.

To find the average time a patient will spend in the UCC, we can use Little's law. In order to do this however, we would first need to determine the expected number of patients, L. In order to do this we can first determine the limiting distribution of the number of patients in the UCC. We firstly make use of the following relation for the limiting distribution

$$\pi_k = \frac{\theta_k}{\sum_{i=0}^{\infty} \theta_i}, \quad k = 0, 1, \ldots$$

$$\theta_0 = 1, \quad \theta_l = \prod_{i=1}^{l} \frac{\lambda_{i-1}}{\mu_i}, \quad l = 1, 2, \ldots$$

In our case, since all birth rates are equal, and all death rates are equal, we then get that

$$\theta_0 = 1, \quad \theta_l = \frac{\lambda^l}{\mu^l}$$

Now we can evaluate the sum of the θs as

$$\sum_{k=0}^{\infty} \left(\frac{\lambda}{\mu}\right)^l = \begin{cases} \frac{1}{1-\lambda/\mu}, & \lambda < \mu \\ \infty, & \text{else} \end{cases}$$

Since we assume that $\lambda < \mu$, we can then determine our limiting probabilities as

$$\pi_k = \left(\frac{\lambda}{\mu}\right)^k \left(1 - \frac{\lambda}{\mu}\right), \quad k = 0, 1, 2, \dots$$

We recognize this as the pdf of a geometric distribution, except for the fact that the indices begin at 0 instead of 1. We can work our way around this small problem by instead considering

$$X(t) + 1 \sim \text{Geom}\left(1 - \frac{\lambda}{\mu}\right)$$

Meaning that we can determine the expected number of patients in the UCC as

$$\underline{L} = E[X(t)] = E[X(t) + 1 - 1] = E[X(t) + 1] - 1 = \frac{1}{1 - \frac{\lambda}{\mu}} - 1 = \underline{\frac{\lambda}{\mu - \lambda}}$$

We then get that the average time spent in the UCC is

$$\underline{W} = \frac{L}{\lambda} = \underline{\frac{1}{\mu - \lambda}}$$

b) Now we wish to simulate the queue for $\lambda = 5$ per hour and $\mu = 0.1$ per minute, and wish to estimate the expected time a patient will spend in the UCC. Below, one realization for the queue is plotted, and a table which contains a CI for the expected time is presented.

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## Warning: Using 'size' aesthetic for lines was deprecated in ggplot2 3.4.0.
## i Please use 'linewidth' instead.
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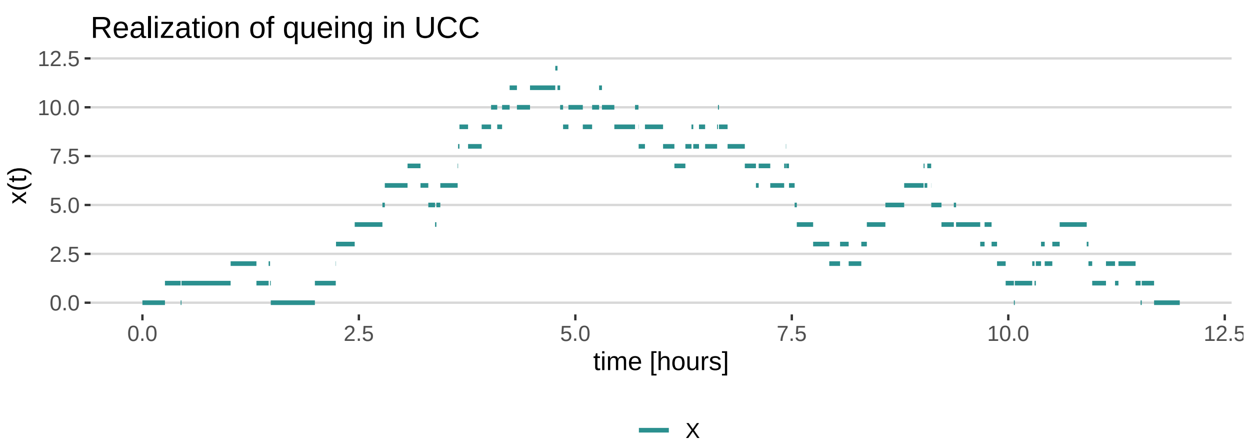


From the plot above we see that there's essentially no idle time. This is a good sign as a higher idle time would indicate a worse and worse use of resources. We also observe that there are some fluctuations between 0 and 1, but that we also have peaks as large as 12, where the time in the queue increases as well.

	Lower bound	Upper bound	Estimate	Theoretical
W	0.9745	1.09	1.032	1

Regarding our C.I., we have chosen to use the function `t.test(x, y)`, because we do not know Var[W] and want a C.I. for $E[W]$. We also note that our theoretical value lies within this interval, which it should do in 95% of the realizations.

c)

We now want to consider a somewhat more realistic model for the UCC, in which patients will now be classified as "urgent" or "non-urgent". The probability that a patient that arrives is urgent is $0 < p < 1$, and we denote the number of urgent and normal patients in the UCC at time t by $U(t)$ and $N(t)$ respectively. This will affect our system in the following way: urgent patients that arrive are immediately treated, while non-urgent patients are pushed behind this patient in the queue. This illustrates an interesting property of the process $\{U(t) : t \geq 0\}$, as the urgent patients will form a queue among themselves, ignoring the non-urgent patients.

Now, since we still only have 1 "server", and the treatments are exponentially distributed as before, this would be a M/M/1 queue, if we could verify that the arrival times of urgent patients are exponentially distributed as well. Since we know that the arrival of patients is in general $\sim \text{Exp}(\lambda)$, we only need to account for the probability of a patient being urgent. Therefore, the arrival of urgent patients $\sim \text{Exp}(p\lambda)$, and the process is a M/M/1 queue.

Analogous to our situation to 1a) we can now actually determine the long-run mean number of urgent patients in the UCC, because we here also have that $p\lambda < \mu$, as $p < 1$,

$$\underline{L_U = \frac{p\lambda}{\mu - p\lambda}}$$

d) For the process $\{N(t) : t \geq 0\}$ we still have that two of the demands for a stochastic process to be an M/M/1 queue are met. Demand 1, regarding the arrival times of patients, and demand 3, the number of servers is still 1, are both met. However, the treatment times are now distributed differently, as a normal patient might receive treatment, and then have it cancelled, because of the arrival of an urgent patient. Say, for instance that $U(t) = 1, N(t) = 0$ at some time t, and one normal patient arrives, increasing $N(t)$ to 1. In a M/M/1 queue this normal patient would have been treated immediately, but now the service time depends on the number of urgent patients, as well as the time needed to treat these patients.

In order to determine the long-run mean number of patients in the UCC, we can now make use of the fact that

$$X(t) = U(t) + N(t)$$

This means that

$$E[N(t)] = E[X(t)] - E[U(t)] = L_X - L_U = \frac{\lambda}{\mu - \lambda} - \frac{p\lambda}{\mu - p\lambda}$$

$$\underline{L_N} = \frac{\lambda(\mu - p\lambda) - p\lambda(\mu - \lambda)}{(\mu - \lambda)(\mu - p\lambda)} = \underline{\frac{\mu\lambda(1-p)}{(\mu - \lambda)(\mu - p\lambda)}}$$

e) Since we have already found expressions for L_U and L_N, we can use Little's law to decide the total time spent in the system for patients of both types. We first need to determine the arrival rate of normal patients, but since

$$\lambda_X = \lambda_U + \lambda_N$$

We get that

$$\lambda_N = \lambda(1 - p)$$

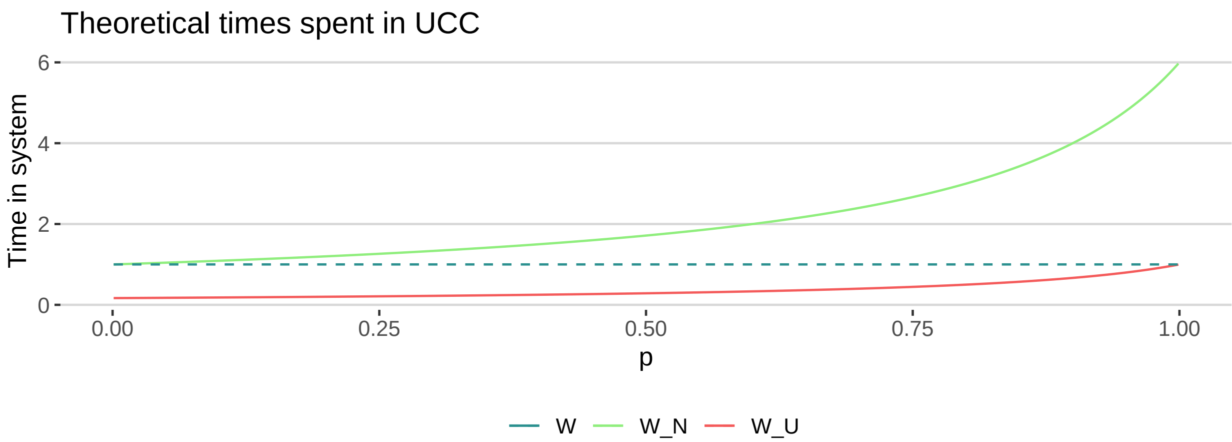
This could of course also be seen from the fact that on average, a proportion $1-p$ of the arriving patients will be normal, analogous to the case for λ_U. We then get the following expression for the times spent in the system

$$W_U = \frac{1}{\mu - p\lambda}$$

$$W_N = \frac{\mu}{(\mu-\lambda)(\mu-p\lambda)}$$

f)

Using the same parameters as earlier, we now wish to plot the above expressions as functions of p, and also compare them to the case with only normal patients, shown as the dashed line W.



When $p \approx 0$ we see that $W_N \to W$, and this makes sense, as the proportion of urgent patients is almost zero, meaning that we have a queue similar to our initial case. For $p \approx 1$ we see that $W_U \to W$, which is again logical, as almost every patient that arrives is urgent, and they will have to wait as patients would in our situation described in a). In the unlikely event that a normal patient arrives in this case, they would have to wait for a longer amount of time, as there are probably many urgent patients in the UCC.

Now we want to consider in more detail the expected time spent at the UCC for a normal patient in the extreme cases $p \approx 0$ and $p \approx 1$, which by inserting these values gives us the expressions

$$W_N(p=0) = \frac{1}{\mu-\lambda}, \quad W_N(p=1) = \frac{\mu}{(\mu-\lambda)^2}$$

We have already covered the case where $p \approx 0$, but for $p \approx 1$, we see that the expected time spent in the UCC is finite. We are also interested in finding the p for which $W_N(p) = 2$. That is, solve the equation

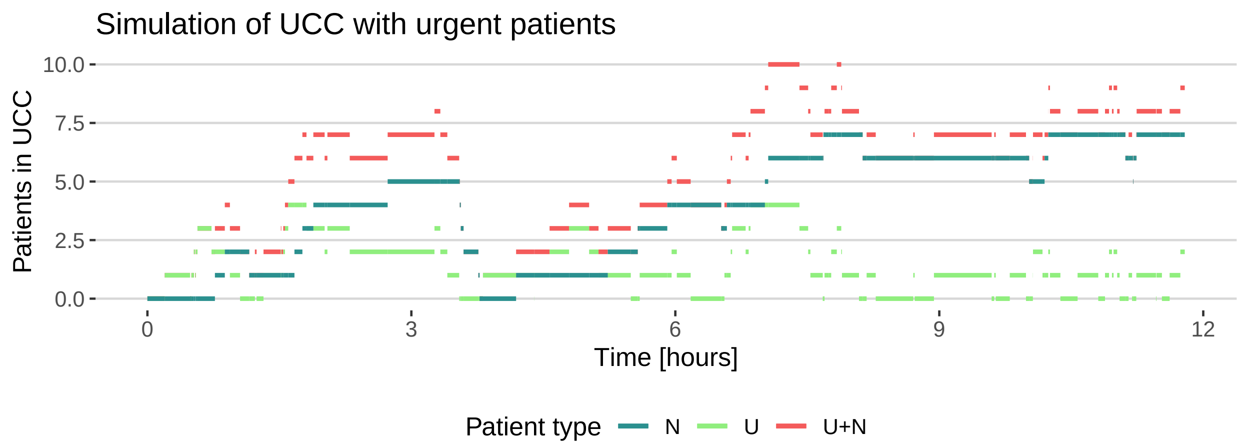
$$\frac{\mu}{(\mu-\lambda)(\mu-p\lambda)} = 2$$

$$\mu - p\lambda = \frac{\mu}{2(\mu-\lambda)}$$

$$\underline{p} = \frac{2\mu(\mu-\lambda)-\mu}{2\lambda(\mu-\lambda)} = \frac{2*6(6-5)-6}{2*3*(6-5)} = \underline{\frac{3}{5}}$$

g)

Now we want to simulate the queues we have worked with analytically until now.



From the plot above, we see that normal patients generally spend a longer time in the UCC than the urgent patients, as expected. We also see that the values for N only decrease when $U = 0$, as it should according to our model. Now we want to look at the mean times in the system, and compare them to the theoretical values discussed earlier.

	Lower bound	Upper bound	Estimate	Theoretical
W_U	0.5032	0.5284	0.5158	0.5
W_N	2.89	3.2	3.045	3

We see that the expected times from the simulation are very close to the theoretical values, but that our numerical values for the CI for W_U is a bit off, as the theoretical value does not lie within the CI, but this is indeed the case for W_N.

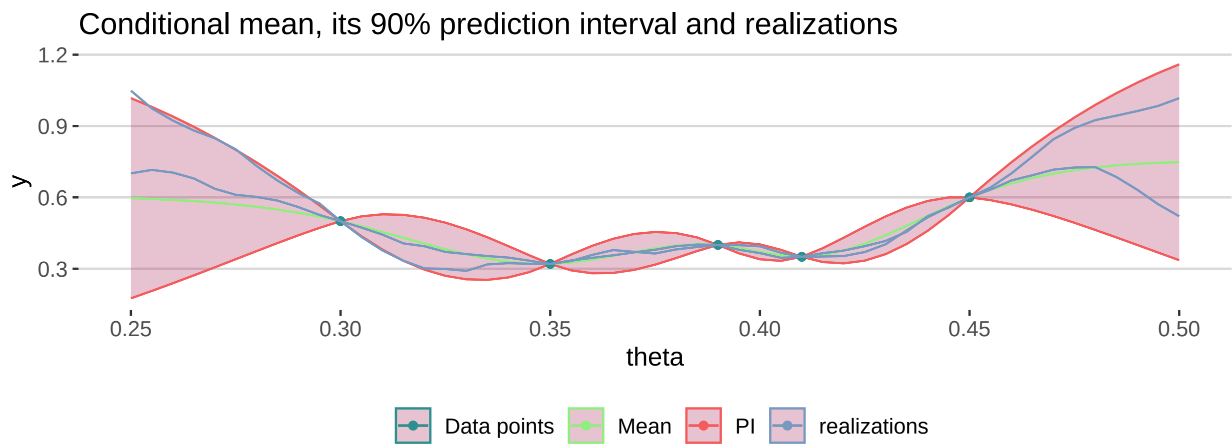
Problem 2

In this problem we will investigate the properties of a parameter of a climate model which is very costly to simulate. Given five, and then six, observation points from researchers that have simulated the model, we will use a Gaussian process model to model the unknown relation between θ and the score $y(\theta)$. In our chosen model we have a Matern type correlation function with decay parameter $\phi_M = 15$, which means that the correlation is given as

$$\text{Corr}[Y(\theta_1, \theta_2)] = (1 + 15|\theta_1 - \theta_2|)\exp(-15|\theta_1 - \theta_2|), \quad \theta_1, \theta_2 \in [0, 1]$$

We will also use that $\text{E[Y}(\theta)] \equiv 0.5$ and that $\text{Var[Y}(\theta)] = 0.5^2$ Initially the evaluation points are $\{(0.30, 0.5), (0.35, 0.32), (0.39, 0.40), (0.41, 0.35), (0.45, 0.60)\}$.

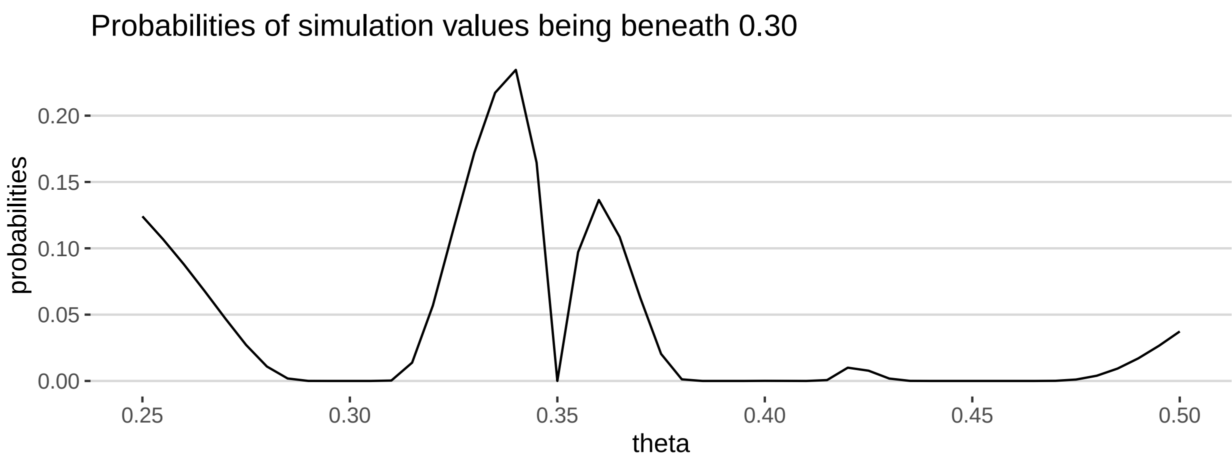
a) We now want to use Algorithm 3 from the GP note in order to find the conditional mean and covariance matrix of the process, given the five evaluation points. We also find the 90% prediction interval for Y by using equation (19) in the GP note.



From the figure above, we see that all plots coincide in our data points, as they also should since the simulation is conditioned on these values. We have also tried to simulate the process, using algorithm 2 from the GP note. Here we ran into a problem with the Cholesky factorization of the covariance matrix, but we were able to work our way around it by adding 10^{-8} to all elements on the diagonal. We have also seen that this has been done in code that has been presented in lectures, but we are still unsure as to why exactly this must be done; is it only due to numerical inaccuracies or is there something inherent in our method that always makes this necessary to do?

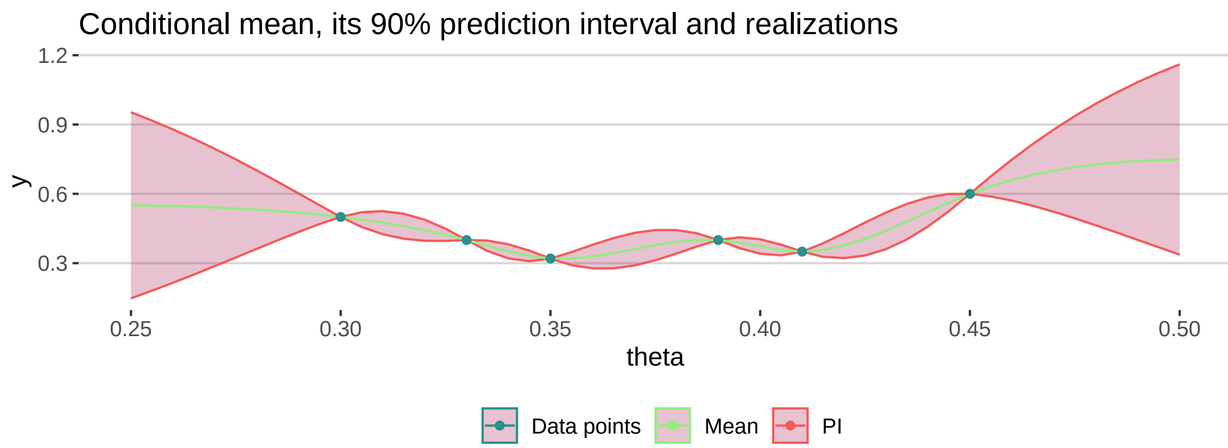
b)

Since the scientists' simulation is so time-costly, they might want to make an informed choice when choosing which θs to test for. Since the scientists' goal is to find a θ such that $y(\theta) < 0.30$, we can consider the conditional probability that $Y(\theta) < 0.30$ given our data. A plot for this probability as a function of θ is presented below.

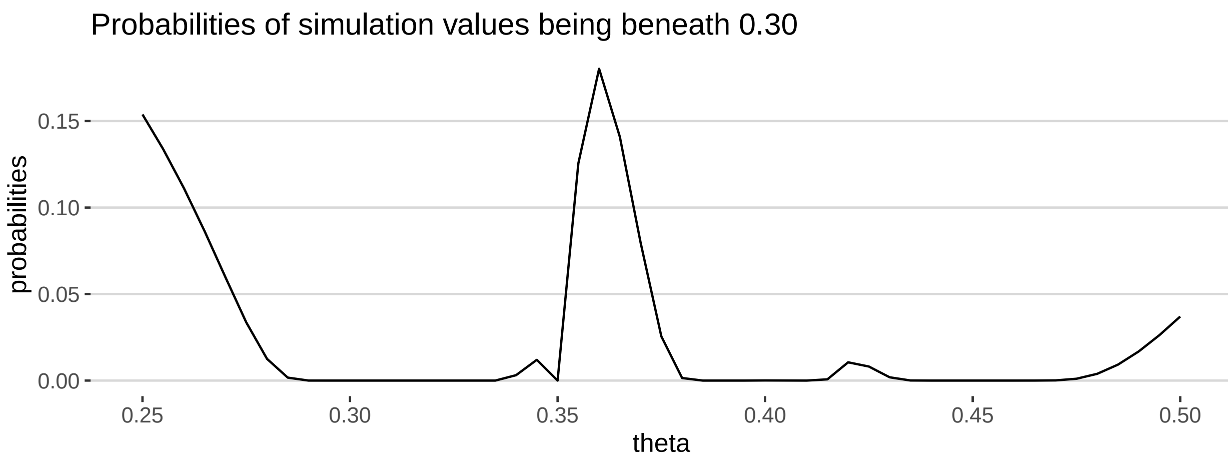


From the plot above we see that $\theta = 0.34$ yields the greatest probability of our desired result.

c) Now the scientists have checked $\theta = 0.33$, which gave the result $y = 0.40$. We now want to expand our data set to contain this new value, and then follow the same procedure in order to determine the next value of θ that should be evaluated. First we present a similar plot to the one presented in 2a).



Here we see the same general trends as we did for our five datapoints. Now we want to determine which θs are most likely to yield our desired result, and present a plot that determines this below.



From the plot above we see that $\theta = 0.36$ is the most probable alternative that gives our wanted result. We would therefore advise the scientists to choose exactly this value as input for their next simulation.

Problem 3

In this problem we will act as Per Ivar's stock advisors in his purchase of GameStop shares. Here we assume that the share prices,Y(t), follow geometric Brownian motion, with

$$\mathrm{Y(t)} = \exp\{\mu + \sigma B(t)\} \quad t \geq 0$$

Where B(t) is standard Brownian motion with unit variance, and $\sigma > 0$ is a scale parameter.We also have the following initial conditions $\mathrm{B}(0) = 0, \mathrm{Y}(0) = e^{\mu}, \mu = \log(50)$ and $\sigma^2 = 4$.

Per Ivar has also instructed us that he will only sell his shares if the price of the stock rises to $75, -$, or in the unfortunate case that it decreases to $25, -$.

a) Our first task as advisors is to provide Per Ivar with the probability that he will profit from selling the shares. This is equivalent to finding the probability that the price of the stock reaches $75, -$ before it reaches $25, -$. If we then rewrite our stochastic variable such that we could instead consider an expression for $X(t) = \sigma B(t)$, we then get that

$$X(t) = \ln Y(t) - \mu$$

Now, we could translate our problem into one that is very similar to the problem described in the project text. Because now we are interested in

$$P(Y(\tau^*_{25,75}) = 75|Y(0) = 50)$$

Where $\tau_{25,75} = \min\{t \geq 0 : Y(t) \in \{25, 75\}$. We can translate these hitting times to hitting times in X, and we then get

$$X(T_{25}) = \ln(25) - \ln(50) = -\ln 2 \qquad X(T_{75}) = \ln(75) - \ln(50) = \ln \frac{3}{2}$$

We then have

$$\tau_{\ln 3/2, \ln 2} = \min\left\{t \geq 0 : Y(t) \in \left\{-\ln 2, \ln \frac{3}{2}\right\}\right\}$$

We also know that

$$P(X(\tau_{a,b}) = a|X(0) = 0) = \frac{b}{a+b}$$

which yields

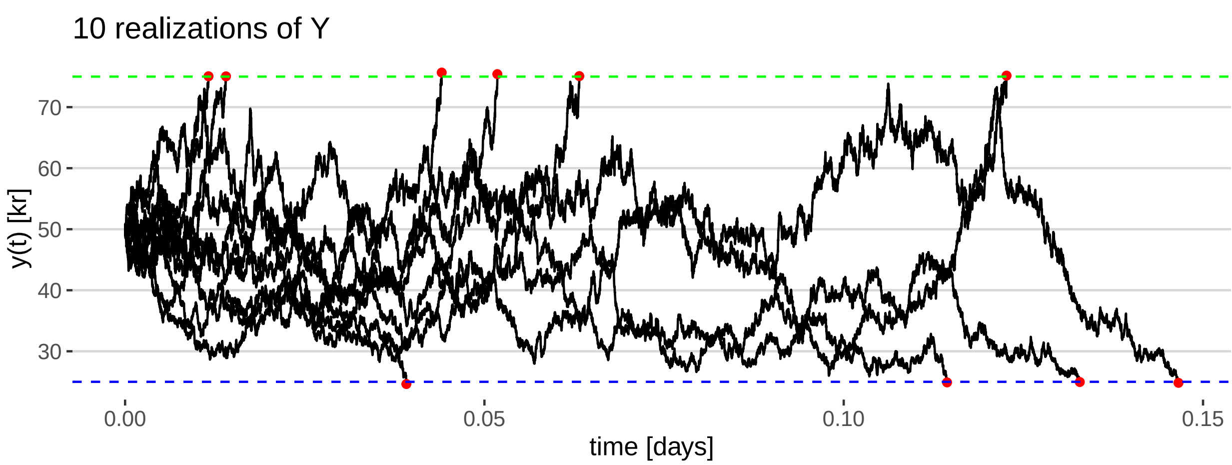
$$\underline{P(Y(\tau^*_{25,75}) = 75|Y(0) = 50)} = P\left(X(\tau_{-\ln 2, \ln 3/2}) = \ln \frac{3}{2}\middle| X(0) = 0\right)$$

$$= \frac{\ln 2}{\ln 3/2 + \ln 2} \approx \underline{0.631}$$

b) Now we can calculate the expected time until Per Ivar will have to sell his shares. From the project text we know that the expected time to reach threshold a or b is given as

$$E[\tau_{a,b}] = \frac{ab}{\sigma^2} \qquad \Longrightarrow \qquad \underline{E} = \frac{\ln 2 \ln \frac{3}{2}}{4} \approx \underline{0.070}$$

Now we also want to simulate this, and see if we can get results that are close to this theoretical calculation. Beneath we present a plot of 10 realizations of the process, which terminate when the stock price reaches a threshold.



Here we see that in some simulations there can be rather great fluctuations, in price, but that the majority of the cases end up in a profitable sale. We have also estimated the following quantities

	Simulated	Analytic
Expected hitting time	0.074033	0.070
Probability of profit	0.6	0.631

Here we see that the expected hitting time and the probability of profit is approximated quite well by our simulations, but that there is some deviation from the theoretical values. This could be because we have not simulated for a very large number of realizations, but only ten.

c) As our last task as Per Ivar's dutiful advisors, we wish to simulate the expected time that Per Ivar owned the shares before selling them with a profit. That is, we want to find

$$E\left[\tau_{\ln 3/2, \ln 2} \middle| X(\tau_{\ln 3/2, \ln 2}) = \ln \frac{3}{2}\right]$$

We have only done this by simulating 100 realizations of the process, because of time limitations.

From the simulations we have found that the expected hitting time, given that we have profit is

$E\left[\tau_{\ln 3/2, \ln 2} \middle| X(\tau_{\ln 3/2, \ln 2}) = \ln \frac{3}{2}\right] = 0.0640543.$

We can also find the expected time by solving an integral. We let T_a denote the first time that $X(t) = a$, and similarly we let T_b be the first time that $X(t) = -b$. We can then find

$$S(t) = P(T_a \geq t | T_b > t)$$

and integrate this to find the expected value.

We look at

$$\begin{aligned} P(T_a \leq t | T_b > t) &= \frac{P(X(\tau_{a,b}) = a)}{P(T_b > t)} \\ &= \frac{b}{a+b} \frac{1}{P(T_b > t)} \end{aligned}$$

From the GP note, we have

$$P(T_a > t) = 1 - 2(1 - \Phi(b/\sqrt{t\sigma^2}))$$

We then get

$$E\left[\tau_{\ln 3/2, \ln 2} \middle| X(\tau_{\ln 3/2, \ln 2}) = \ln \frac{3}{2}\right] = \int_0^\infty \left(1 - \frac{\ln 2}{\ln 3/2 + \ln 2} \frac{1}{1 - 2(1 - \Phi(\ln 2/(2\sqrt{t}))}\right) dt$$

However, computing this in R, we get a value that tends to infinity, so this is not correct.

$E\left[\tau_{\ln 3/2, \ln 2} \middle| X(\tau_{\ln 3/2, \ln 2}) = \ln \frac{3}{2}\right] = \text{NA}$