assignment09

April 24, 2021

0.1 Task 10.1

Implement the system

$$y_k + a y_{k-1} = b u_{k-1} + e_k$$

where:

- e_k is white Gaussian zero-mean noise with variance λ^2
- the input is computed through a state-feedback law $u_k = -K y_k + r_k$ with r_k a reference signal
- K is so that the closed loop system in the absence of the reference signal is asymptotically stable, and the mode of the system is non-oscillatory
- r_k, for the sake of this assignment, is another white Gaussian zero-mean noise with variance σ^2

[2]:
```
# importing the right packages
import numpy as np
import matplotlib.pyplot as plt
import scipy.optimize as optimize

np.random.seed(1)
```

[3]:
```
# main function to simulate the system
def simulate( a, b, K, lambda2, sigma2, y0, N, reference_frequency = 0 ):

    # storage allocation
    y = np.zeros(N)
    u = np.zeros(N)

    # saving the initial condition
    y[0] = y0

    # system noises
    e = np.random.normal(0, np.sqrt(lambda2),  N)
    r = np.random.normal(0, np.sqrt(sigma2), N) + np.sin( reference_frequency *␣
 ↪np.arange(N) )
```

```
    # cycle on the steps
    for t in range(1, N):
        # on last iteration, when t=N-1, we only set u[N-2] so u[N-1] is always␣
↪0.
        u[t-1] = -K*y[t-1] + r[t-1]
        y[t] = -a*y[t-1] + b*u[t-1] + e[t]

    return [y, u]
```

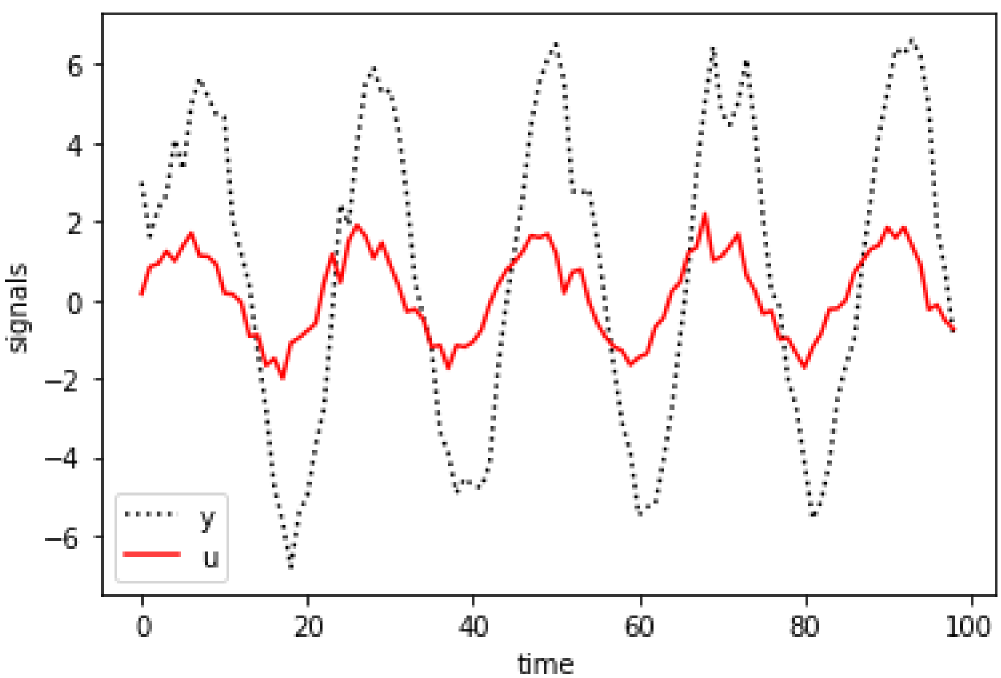
[4]:
```
# define also a function for doing poles allocation, considering
# that eventually if the reference is absent then the ODE is
#
# y_k + ( a + b K ) y_{k-1} = e_k
#


#
# y/e = 1/(1 + (a + b K)*z^-1)
# -> pole = -(a + b K)
#
# -a - b K = pole
# -b K = pole + a
# K = (pole + a) / (-b)
#
# Gain K = -(pole + a) / b
#
def compute_gain( a, b, desired_pole_location ):
    return -(desired_pole_location + a) / b
```

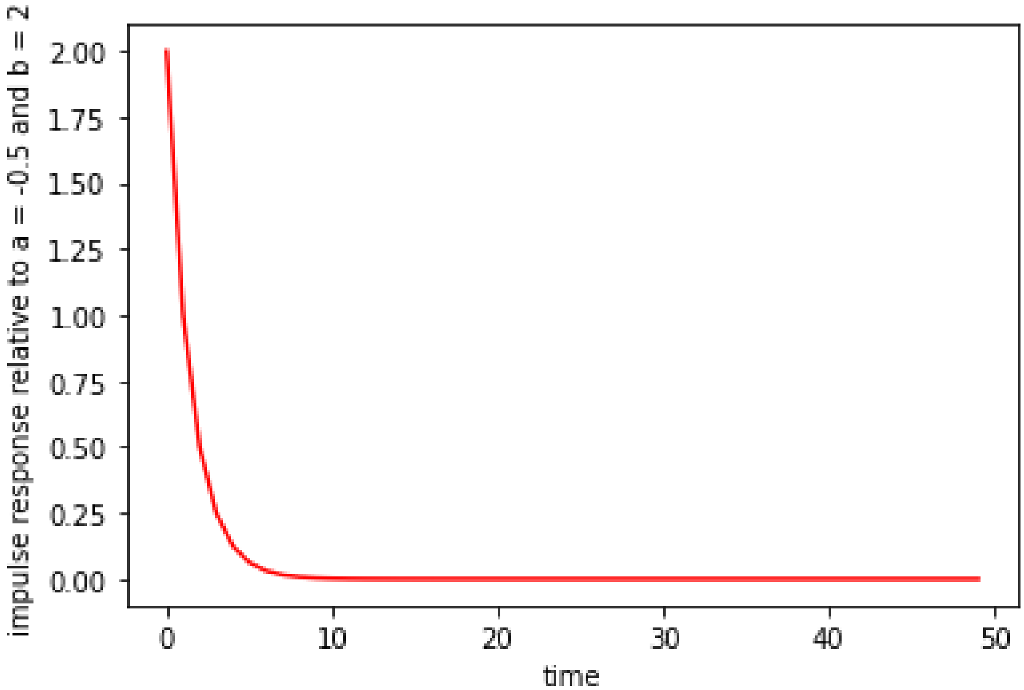
[5]:
```
# plotting of the impulse response
def plot_impulse_response( a, b, figure_number = 1000 ):

    # ancillary quantities
    k = range(0,50)
    y = b * np.power( -a, k )

    # plotting the various things
    plt.figure( figure_number )
    plt.plot(y, 'r-', label = 'u')
    plt.xlabel('time')
    plt.ylabel('impulse response relative to a = {} and b = {}'.format(a, b))
```

[6]:
```
# define the system parameters
a = -0.5
b = 2
K = compute_gain( a, b, 0.7 )

# noises
```

```
lambda2 = 0.1 # on e
sigma2  = 0.1 # on r

# initial condition
y0 = 3

# number of steps
N = 100
```

[7]:
```
# DEBUG - check that things work as expected

# run the system
[y, u] = simulate( a, b, K, lambda2, sigma2, y0, N, 0.3 )

# plotting the various things
plt.figure()
plt.plot(y[:-1], 'k:', label = 'y')
plt.plot(u[:-1], 'r-', label = 'u')
plt.xlabel('time')
plt.ylabel('signals')
plt.legend()

plot_impulse_response( a, b )
```





0.2 Task 10.2

Implement a PEM-based approach to the estimation of the system, assuming to know the correct model structure but not knowing about the existence of the feedback loop given by K.

[8]:
```
# important: the system is an ARX one, and e_k is Gaussian so PEM = ???
# And given this, how can we simplify things?
```

[9]:
```
# define the function solving the PEM problem asked in the assignment

"""
https://www.it.uu.se/edu/course/homepage/systemid/vt12/ch6.pdf

theta = [a1, ..., aN, b1, ..., bN]
y_hat[k] = -a1*y[k-1] - ... - aN*y[k-N] + b1*u[k-1] + ... + bN*u[k-N] = phi.T @␣
 ↪theta
phi = [ -y[k-1], ..., -y[k-N], u[k-1], ..., u[k-N] ]

theta_hat = arg min ( y[k] - y_hat[k](theta) )^2
"""
def PEM_solver( u, y ):
    # y = [ y[0], y[1], y[2], ..., y[t-1], y[t] ]
    # u = [ u[0], u[1], ..., u[t-2], u[t-1], u[t]=0 ] ,  u[t] is never set and␣
 ↪must be ignored
```

```
    phi = np.array([ -y[-2], u[-2] ]).reshape((-1, 1))

    # compute the PEM estimate by directly solving the normal equations
    theta_hat = np.linalg.solve( phi.T @ phi, phi.T * y[-2] ).ravel()

    # explicit the results
    a_hat = theta_hat[0]
    b_hat = theta_hat[1]

    return [a_hat, b_hat]
```

[10]:
```
# compute the solution
[a_hat, b_hat] = PEM_solver( u, y )

# assess the performance
MSE = np.linalg.norm([a - a_hat, b - b_hat])**2

# print debug info
print('MSE: {}'.format(MSE))
print('a, b = {}, {}   --    ahat, bhat = {}, {}'.format(a, b, a_hat, b_hat))
```

```
MSE: 2.293863894155823
a, b = -0.5, 2   --    ahat, bhat = -0.6041427912209901, 0.48903402646104427
```

0.3 Task 10.3

Show from a numerical perspective that for $\lambda^2 = 0.1$ (i.e., a constant variance on the process noise) the estimates are consistent.

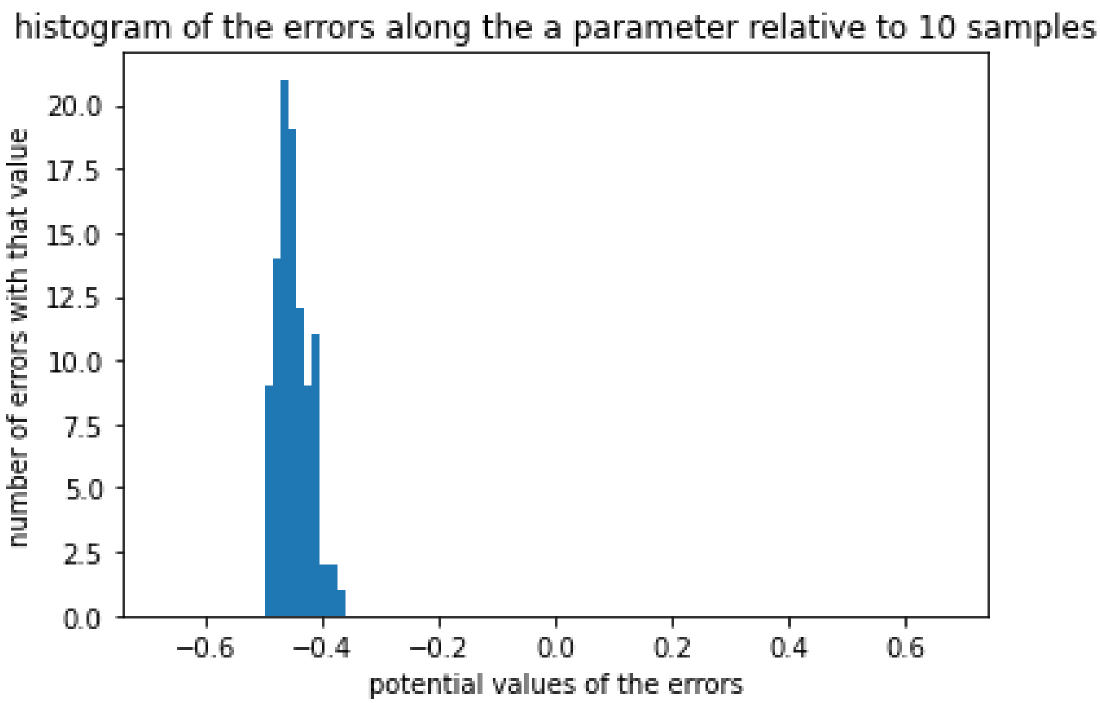
[11]:
```
# the best way to show this is to do a Monte Carlo approach:
# - for each N, compute the distribution of the estimates
#   around the true parameters
# - increase N and show that this distribution tends to
#   converge to the true parameters

# defining the MC simulation
N_MC_runs       = 100
min_order_for_N = 1
max_order_for_N = 4
num_of_N_orders = max_order_for_N - min_order_for_N + 1;

# noises and initial condition
lambda2 = 0.1 # on e
sigma2  = 0.1 # on r
y0      = 0

# storage allocation
MSEs = np.zeros( (num_of_N_orders, N_MC_runs) )
```

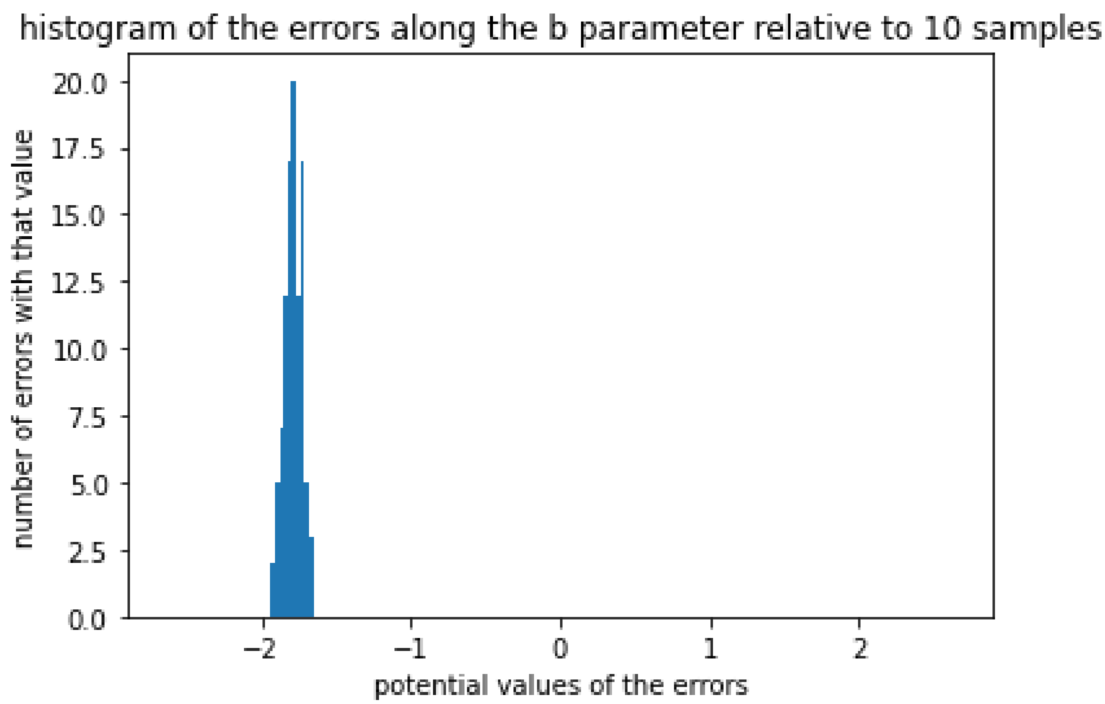
```
theta_hats = np.zeros( (num_of_N_orders, N_MC_runs, 2) )

# cycle on the number of samples
for j, N in enumerate( np.logspace( min_order_for_N, max_order_for_N,␣
↪num_of_N_orders ) ):
    N = int(N)

    # debug
    print('starting computing order {} of {}'.format(j+1, num_of_N_orders))

    # MC cycles
    for m in range(N_MC_runs):

        # simulate the system
        [y, u] = simulate( a, b, K, lambda2, sigma2, y0, N, 0.3 )

        # compute the solution
        [a_hat, b_hat] = PEM_solver(u, y)

        # assess the performance
        MSEs[j, m] = np.linalg.norm([a - a_hat, b - b_hat])**2

        # save the results
        theta_hats[j, m, 0] = a_hat
        theta_hats[j, m, 1] = b_hat
```

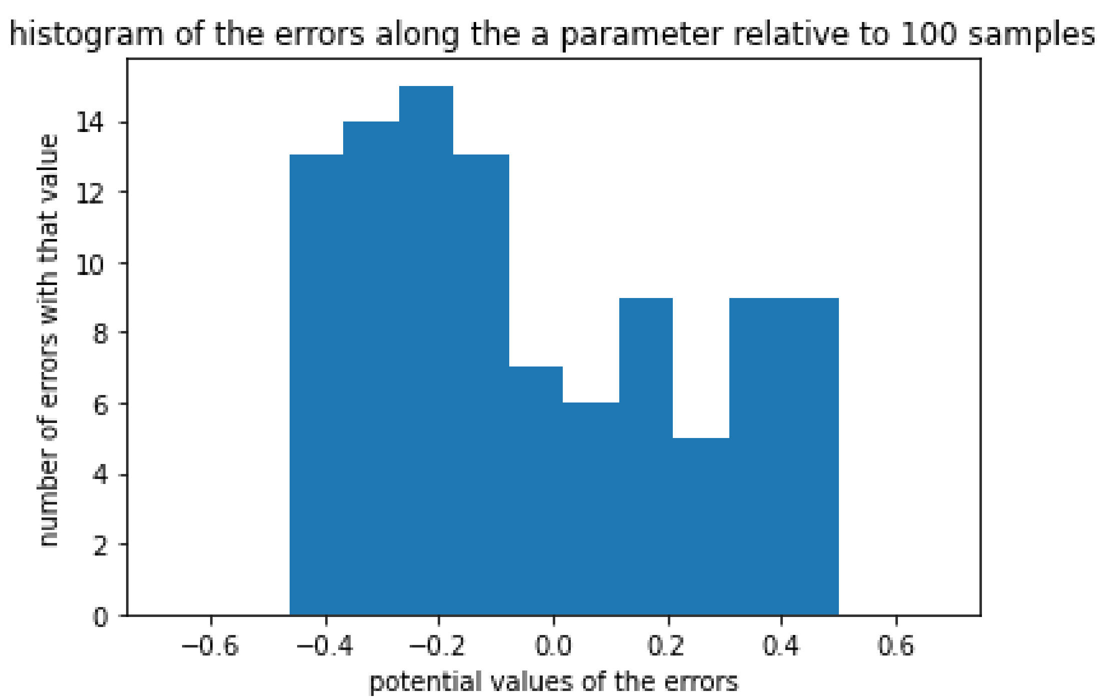
```
starting computing order 1 of 4
starting computing order 2 of 4
starting computing order 3 of 4
starting computing order 4 of 4
```

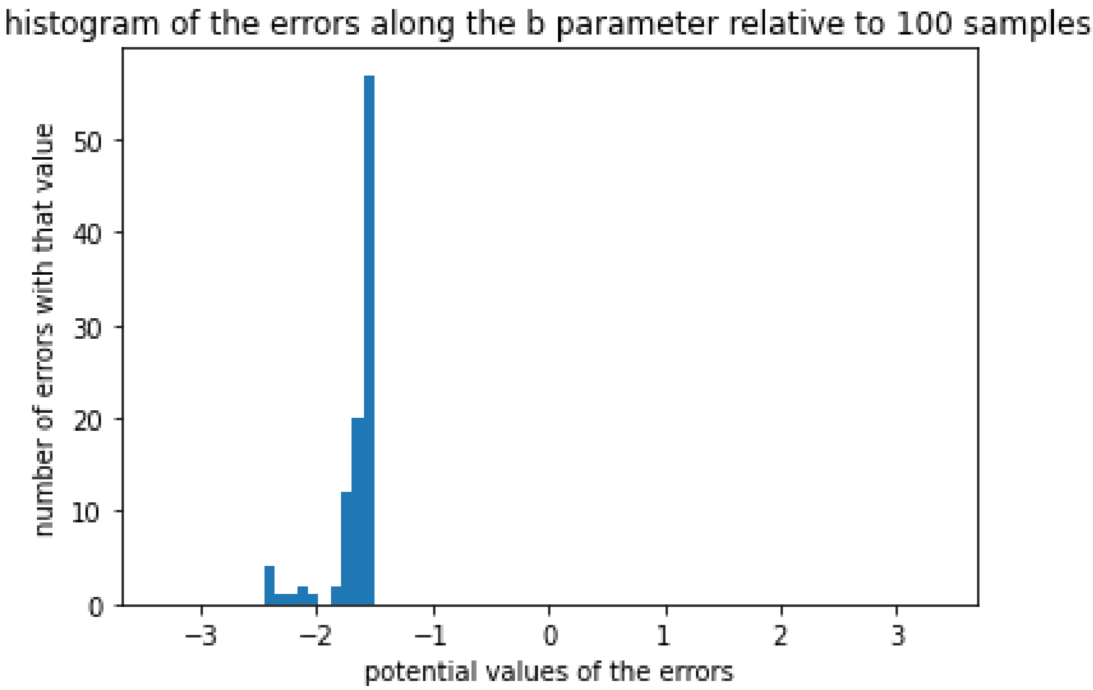
[12]:
```
# cycle on the number of samples
for j, N in enumerate( np.logspace( min_order_for_N, max_order_for_N,␣
↪num_of_N_orders ) ):
    N = int(N)

    # plot the histogram of the MSEs relative to this number of samples
    # plt.figure(j)
    # plt.hist(MSEs[j,:])
    # plt.xlim(0, 1.5 * np.max(MSEs[j,:]))
    # plt.title('histogram of the MSEs relative to {} samples'.format(N))
    # plt.xlabel('potential values of the MSE')
    # plt.ylabel('number of MSEs with that value')

    # plot the histogram of the errors along the a parameter
    plt.figure(j + 100)
    x_lim = np.max(np.abs(theta_hats[j,:,0] - a))
```

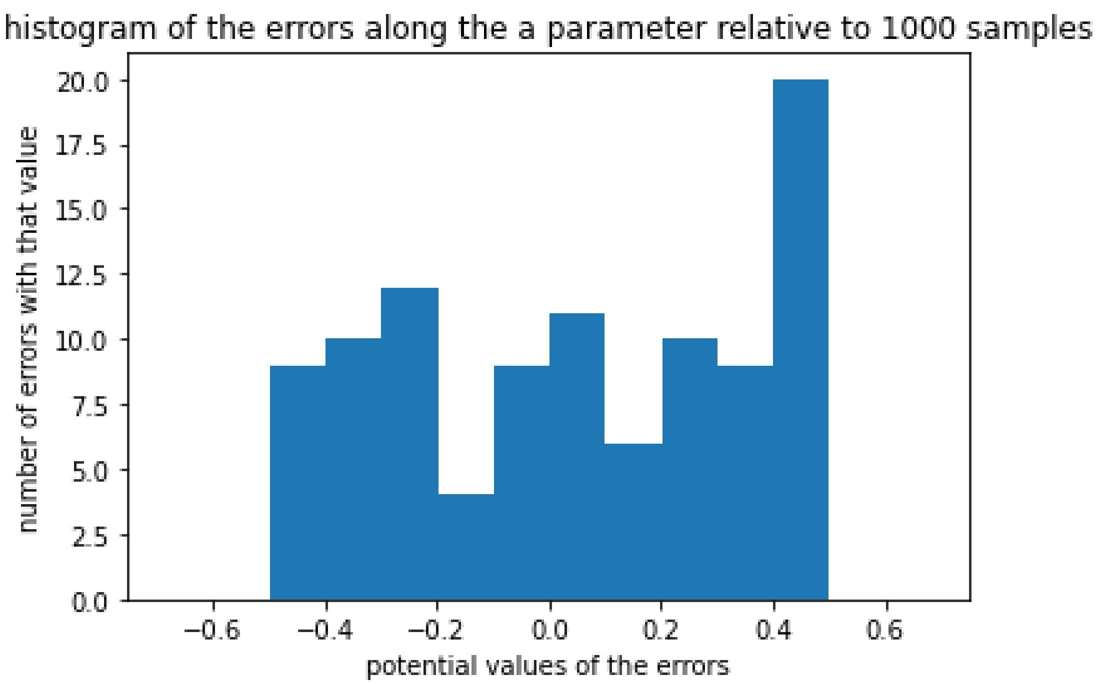
```
    plt.hist(theta_hats[j,:,0] - a)
    plt.xlim(-1.5 * x_lim, 1.5 * x_lim)
    plt.title('histogram of the errors along the a parameter relative to {}␣
↪samples'.format(N))
    plt.xlabel('potential values of the errors')
    plt.ylabel('number of errors with that value')

    # plot the histogram of the errors along the b parameter
    plt.figure(j + 200)
    x_lim = np.max(np.abs(theta_hats[j,:,1] - b))
    plt.hist(theta_hats[j,:,1] - b)
    plt.xlim(-1.5 * x_lim, 1.5 * x_lim)
    plt.title('histogram of the errors along the b parameter relative to {}␣
↪samples'.format(N))
    plt.xlabel('potential values of the errors')
    plt.ylabel('number of errors with that value')
```

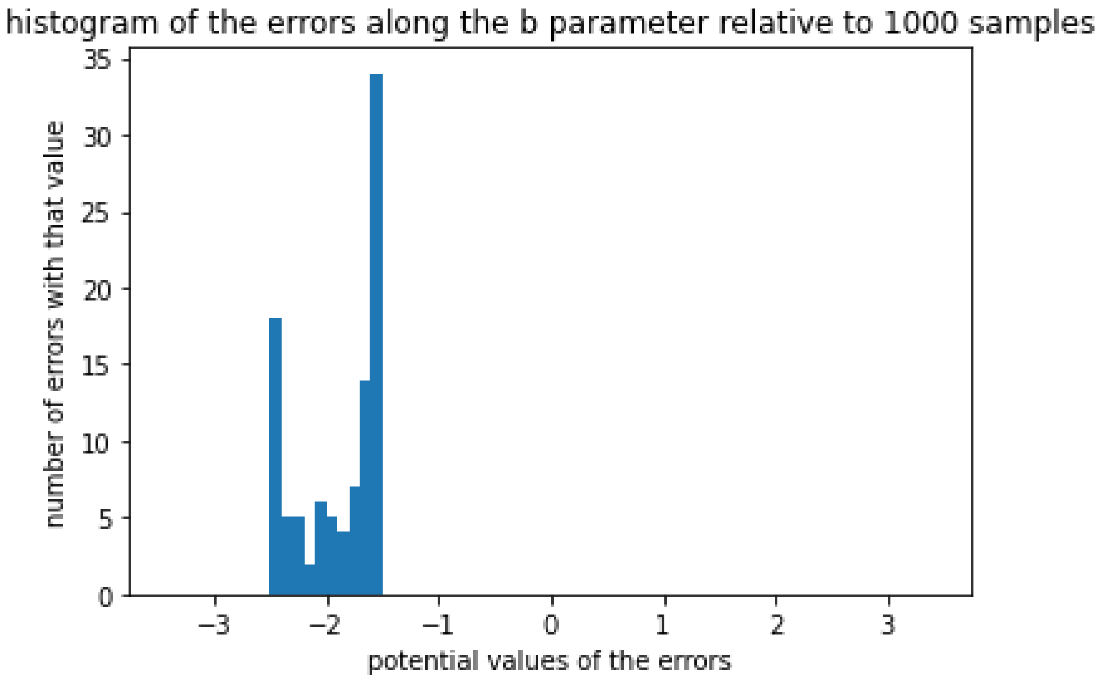


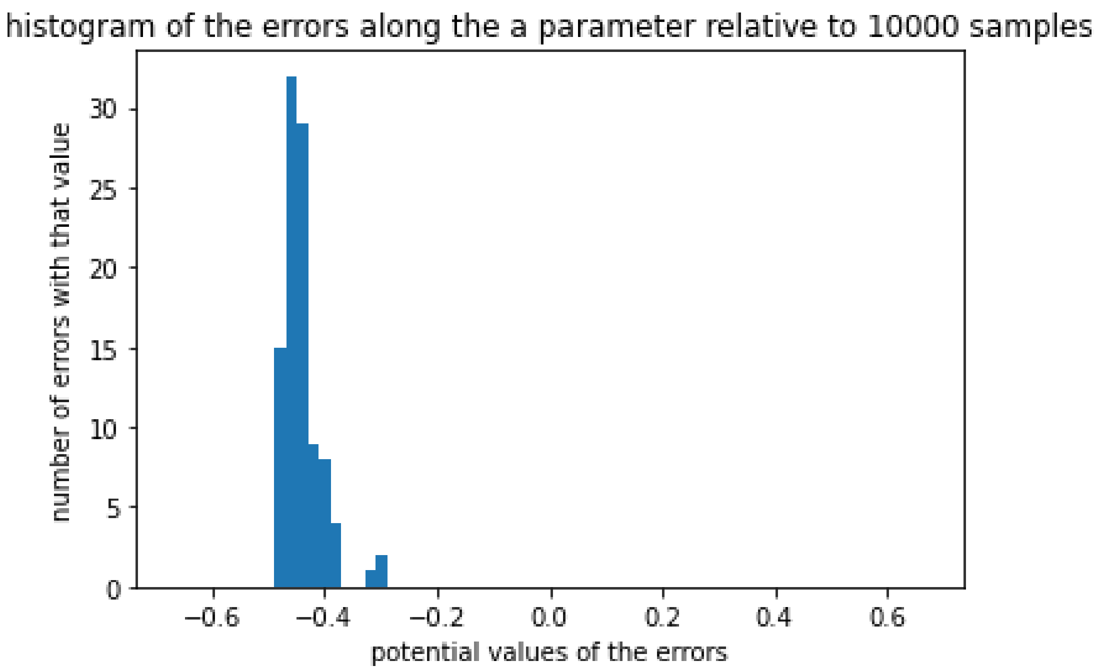

histogram of the errors along the b parameter relative to 10 samples
number of errors with that value
20.0
17.5
15.0
12.5
10.0
7.5
5.0
2.5
0.0
−2
−1
0
1
2
potential values of the errors

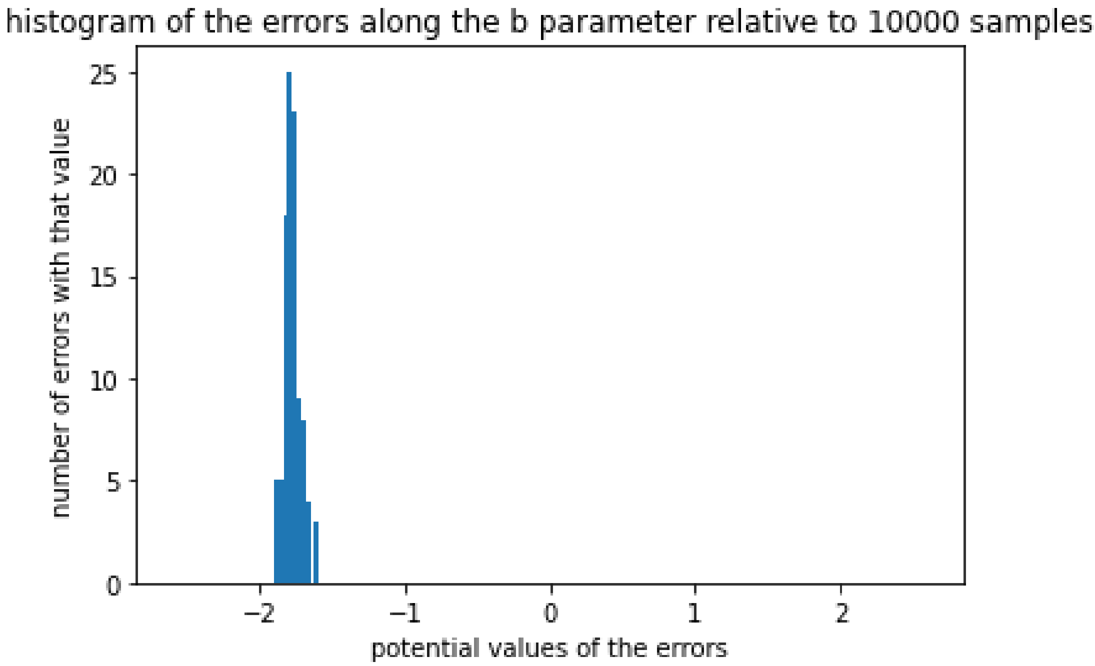

histogram of the errors along the a parameter relative to 100 samples
number of errors with that value
14
12
10
8
6
4
2
0
−0.6
−0.4
−0.2
0.0
0.2
0.4
0.6
potential values of the errors


histogram of the errors along the b parameter relative to 100 samples
number of errors with that value
0
10
20
30
40
50
−3
−2
−1
0
1
2
3
potential values of the errors


histogram of the errors along the a parameter relative to 1000 samples
number of errors with that value
0.0
2.5
5.0
7.5
10.0
12.5
15.0
17.5
20.0
−0.6
−0.4
−0.2
0.0
0.2
0.4
0.6
potential values of the errors


histogram of the errors along the b parameter relative to 1000 samples
number of errors with that value
0
5
10
15
20
25
30
35
−3
−2
−1
0
1
2
3
potential values of the errors


histogram of the errors along the a parameter relative to 10000 samples
number of errors with that value
0
5
10
15
20
25
30
−0.6
−0.4
−0.2
0.0
0.2
0.4
0.6
potential values of the errors



We see as N increases, the distribution becomes smaller, thus showing that the estimates are consistent.

PEM tries to solve for a and b assuming this model:

```
y[k] = -a y[k-1] + b u[k-1]
```

Our (closed-loop) model is actually:

```
y[k] = -(a + b K) y[k-1] + 0 u[k-1]
```

Hence, the estimates should converge towards

```
a_hat --> a + b K = -0.5 + 2 * -0.1 = -0.7
b_hat --> 0
```

The plot showing a error converges to between -0.4 and -0.5, while the plot for b error converges to around -1.8. Since a = -0.5, b = 2, and K = -0.1, the estimates are close to what we would expect them to converge to.

0.4 Task 10.3

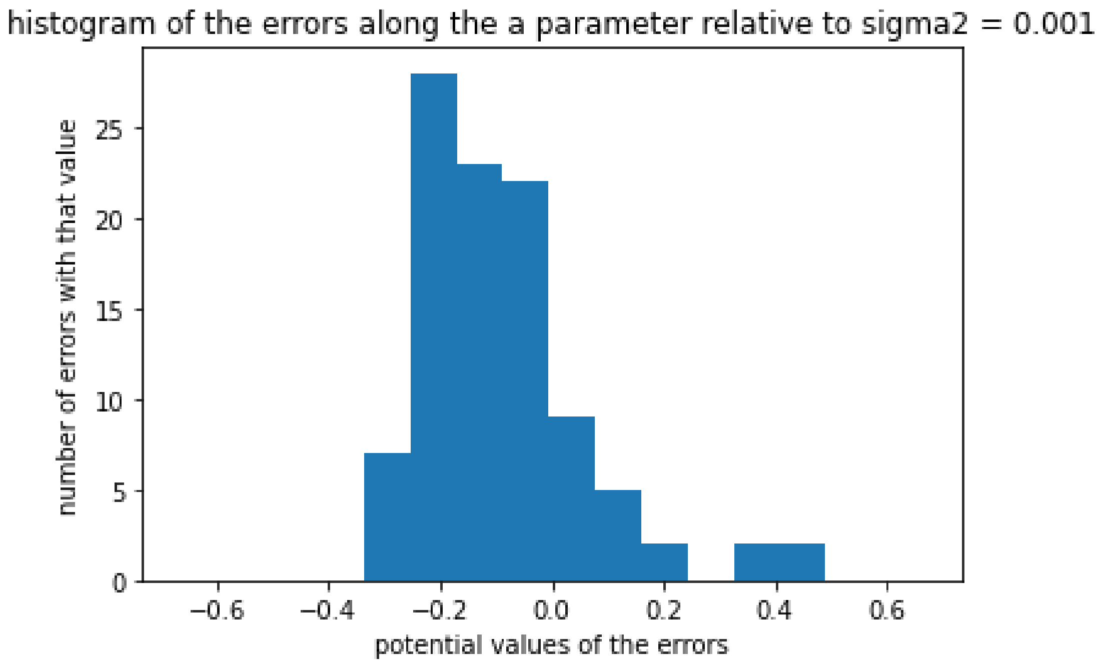
Show that the variances of the estimates though will tend to infinity as $\sigma^2 \to 0$, i.e., the reference becomes a deterministic known signal.

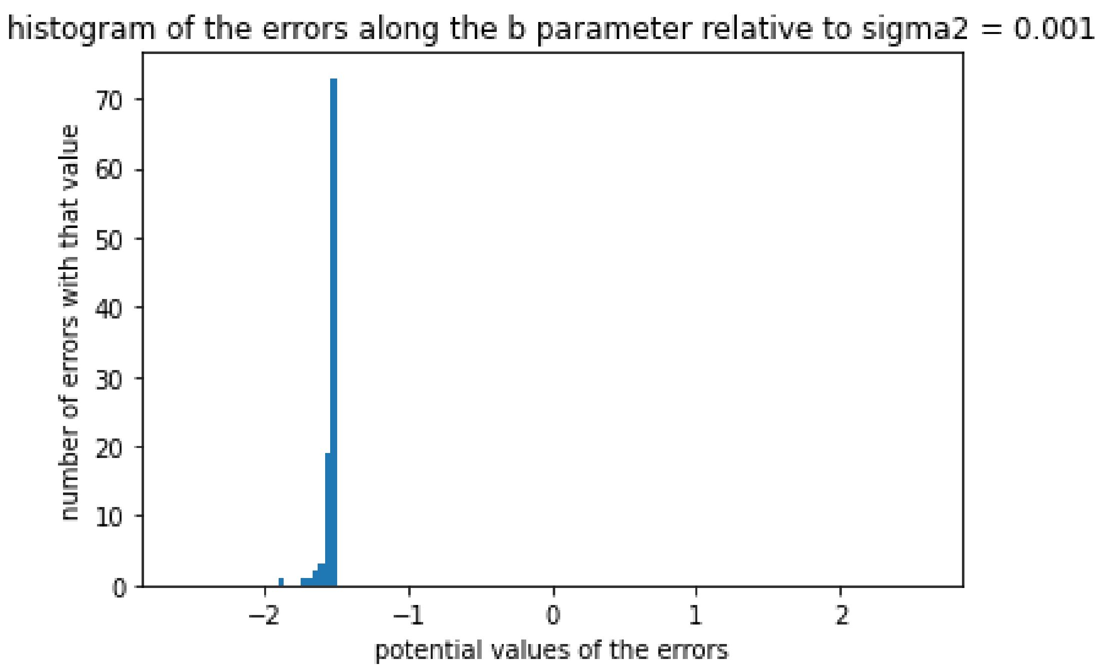
[13]:
```
# again the best way to show this is to do a Monte Carlo approach:
# - for each sigma2, compute the distribution of the estimates
#   around the true parameters
# - diminish sigma2 and show that this distribution tends to
```

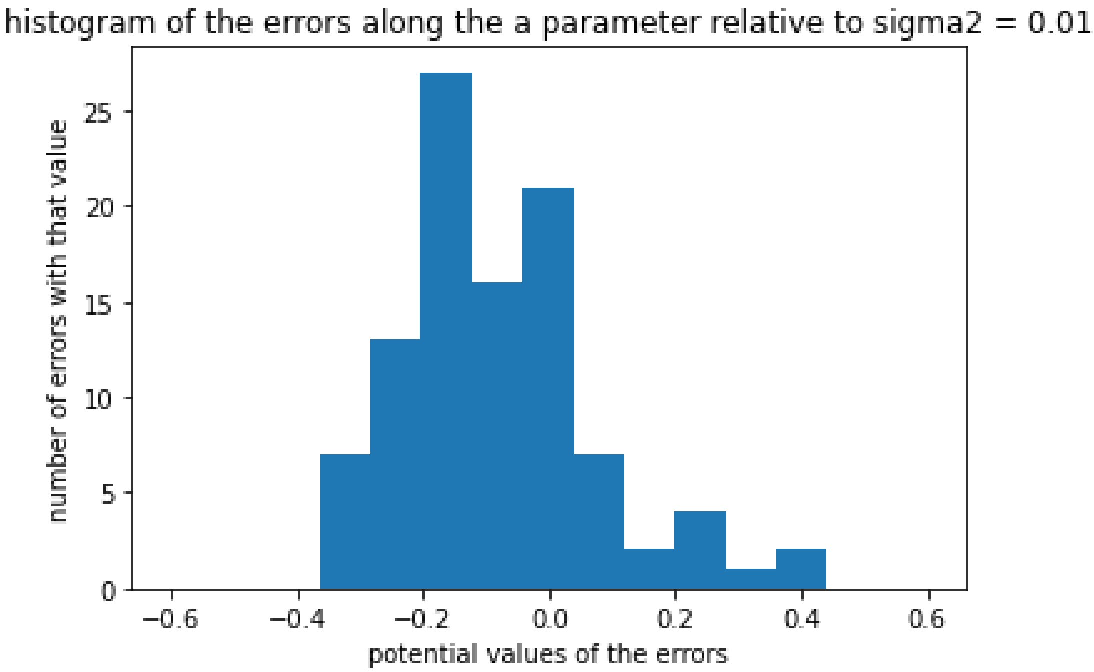
```python
#   diverge

# defining the MC simulation
N                     = 100
N_MC_runs             = 100
min_order_for_sigma2 = -3
max_order_for_sigma2 = 1
num_of_sigma2_orders = max_order_for_sigma2 - min_order_for_sigma2 + 1;

# noises and initial condition
lambda2 = 0.1 # on e
y0      = 0

# storage allocation
MSEs = np.zeros( (num_of_sigma2_orders, N_MC_runs) )
theta_hats = np.zeros( (num_of_sigma2_orders, N_MC_runs, 2) )

# cycle on the variance of the measurement noise
for j, sigma2 in enumerate( np.logspace( min_order_for_sigma2,␣
 ↪max_order_for_sigma2, num_of_sigma2_orders ) ):

    # debug
    print('starting computing order {} of {}'.format(j+1, num_of_sigma2_orders))

    # MC cycles
    for m in range(N_MC_runs):

        # simulate the system
        [y, u] = simulate( a, b, K, lambda2, sigma2, y0, N, 0.3 )

        # compute the solution
        [a_hat, b_hat] = PEM_solver(u, y)

        # assess the performance
        MSEs[j,m] = np.linalg.norm([a - a_hat, b - b_hat])**2

        # save the results
        theta_hats[j,m,0] = a_hat
        theta_hats[j,m,1] = b_hat
```

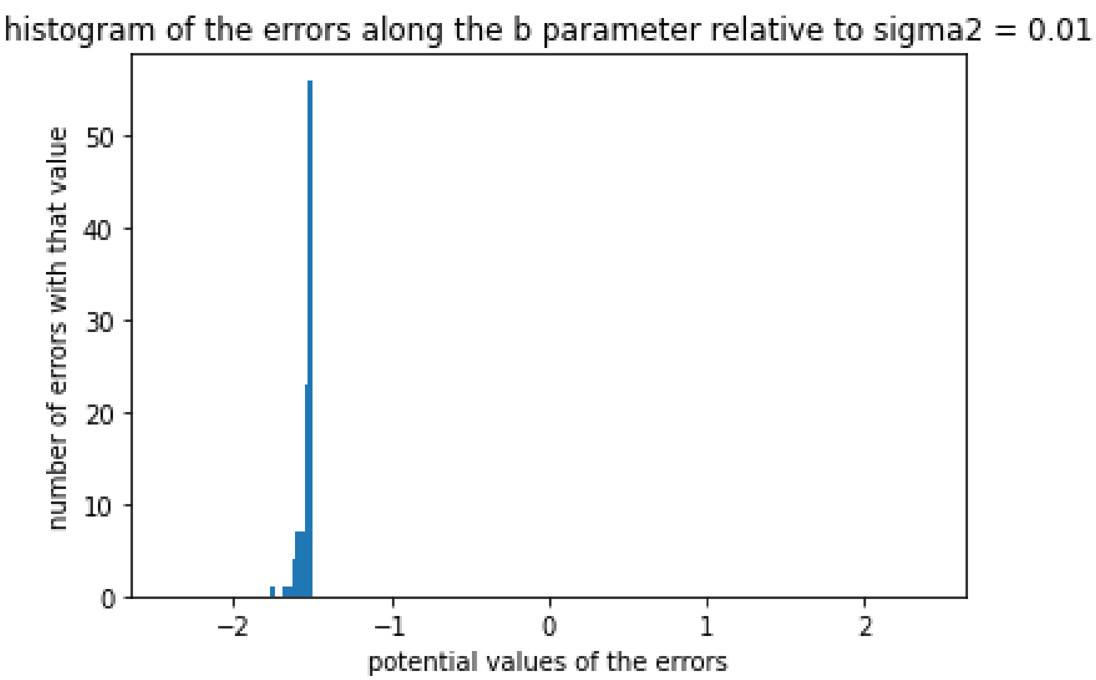
```
starting computing order 1 of 5
starting computing order 2 of 5
starting computing order 3 of 5
starting computing order 4 of 5
starting computing order 5 of 5
```

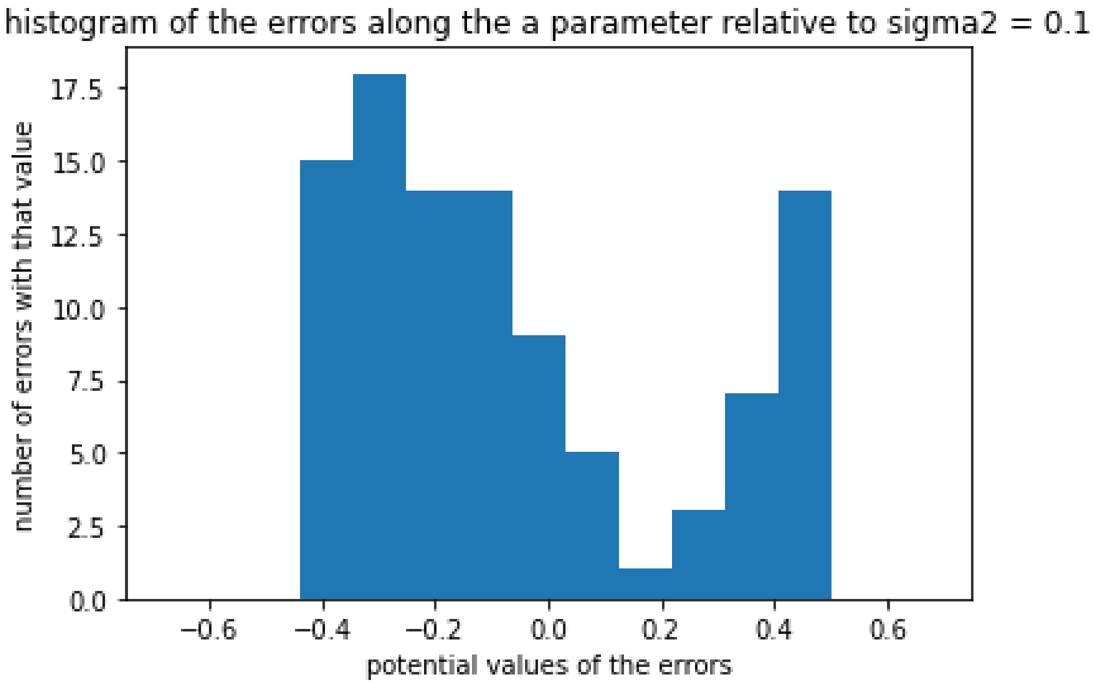
[14]:
```
# cycle on the variance of the measurement noise
for j, sigma2 in enumerate( np.logspace( min_order_for_sigma2,␣
↪max_order_for_sigma2, num_of_sigma2_orders ) ):

    # plot the histogram of the MSEs relative to this number of samples
    # plt.figure(j)
    # plt.hist(MSEs[j,:])
    # plt.xlim(0, 1.5 * np.max(MSEs[j,:]))
    # plt.title('histogram of the MSEs relative to sigma2 = {}'.format(sigma2))
    # plt.xlabel('potential values of the MSE')
    # plt.ylabel('number of MSEs with that value')

    # plot the histogram of the errors along the a parameter
    plt.figure(j + 100)
    x_lim = np.max(np.abs(theta_hats[j,:,0] - a))
    plt.hist(theta_hats[j,:,0] - a)
    plt.xlim(-1.5 * x_lim, 1.5 * x_lim)
    plt.title('histogram of the errors along the a parameter relative to sigma2␣
↪= {}'.format(sigma2))
    plt.xlabel('potential values of the errors')
    plt.ylabel('number of errors with that value')

    # plot the histogram of the errors along the b parameter
    plt.figure(j + 200)
    x_lim = np.max(np.abs(theta_hats[j,:,1] - b))
    plt.hist(theta_hats[j,:,1] - b)
    plt.xlim(-1.5 * x_lim, 1.5 * x_lim)
    plt.title('histogram of the errors along the b parameter relative to sigma2␣
↪= {}'.format(sigma2))
    plt.xlabel('potential values of the errors')
    plt.ylabel('number of errors with that value')
```

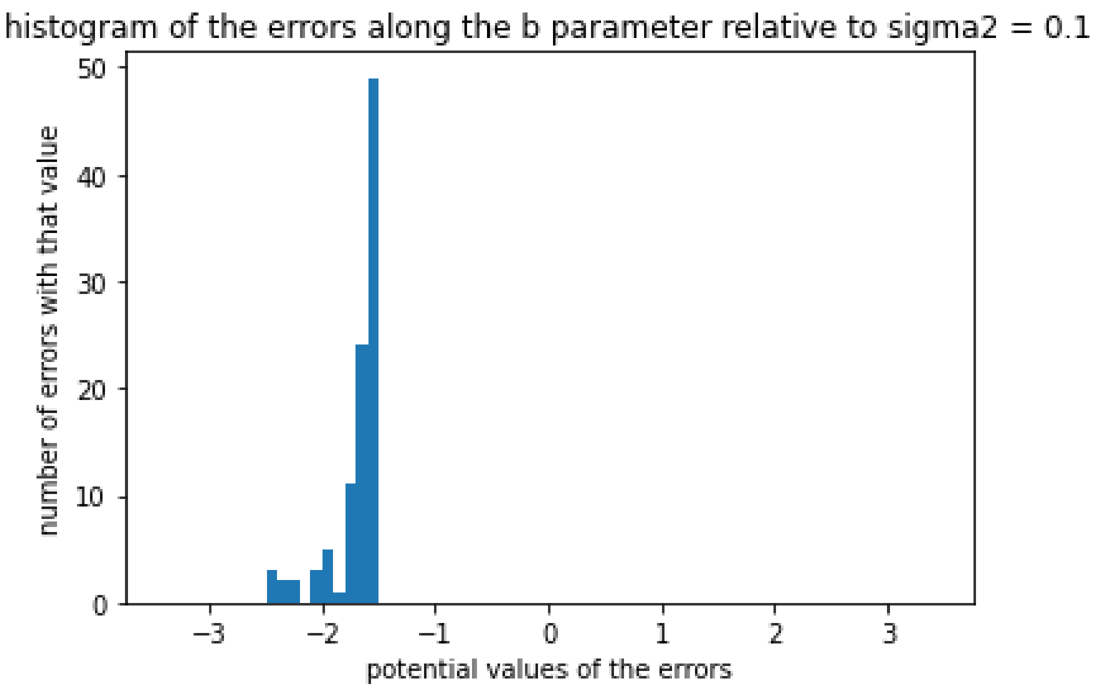

histogram of the errors along the a parameter relative to sigma2 = 0.001
number of errors with that value
potential values of the errors

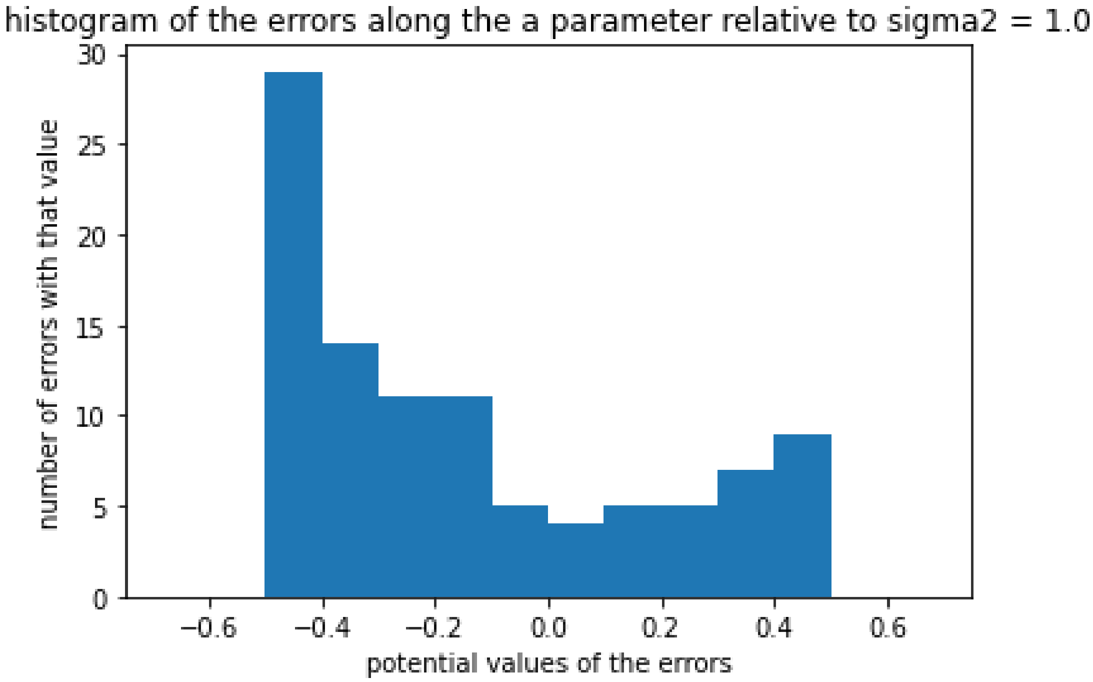

histogram of the errors along the b parameter relative to sigma2 = 0.001
number of errors with that value
potential values of the errors

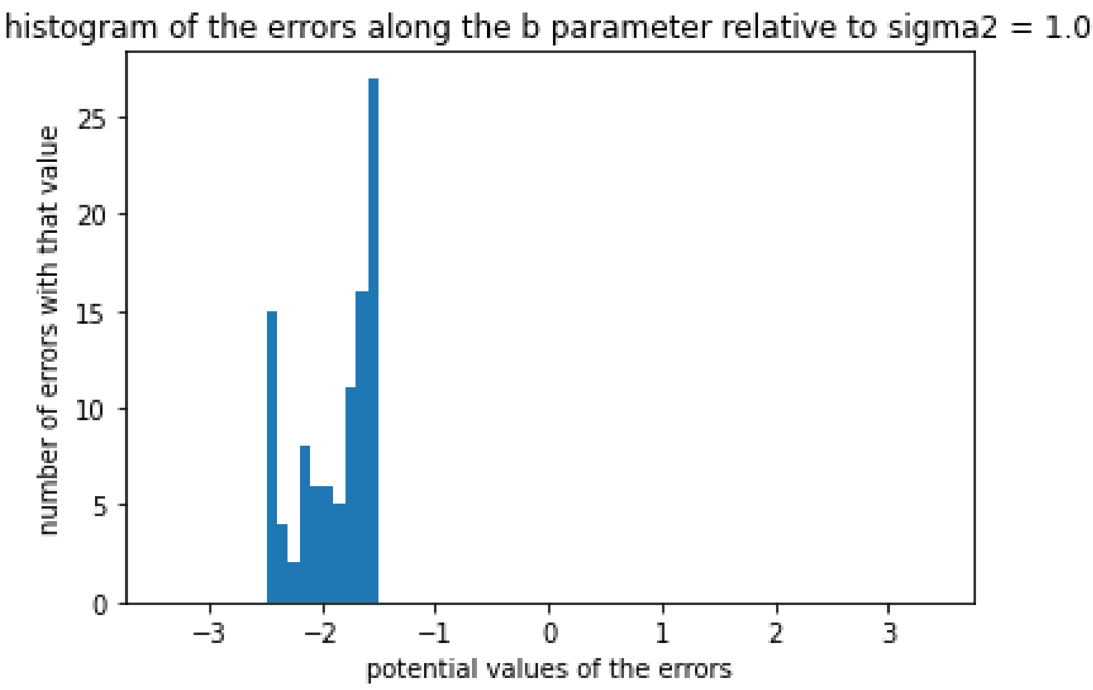

histogram of the errors along the a parameter relative to sigma2 = 0.01
number of errors with that value
potential values of the errors
0
5
10
15
20
25
−0.6
−0.4
−0.2
0.0
0.2
0.4
0.6

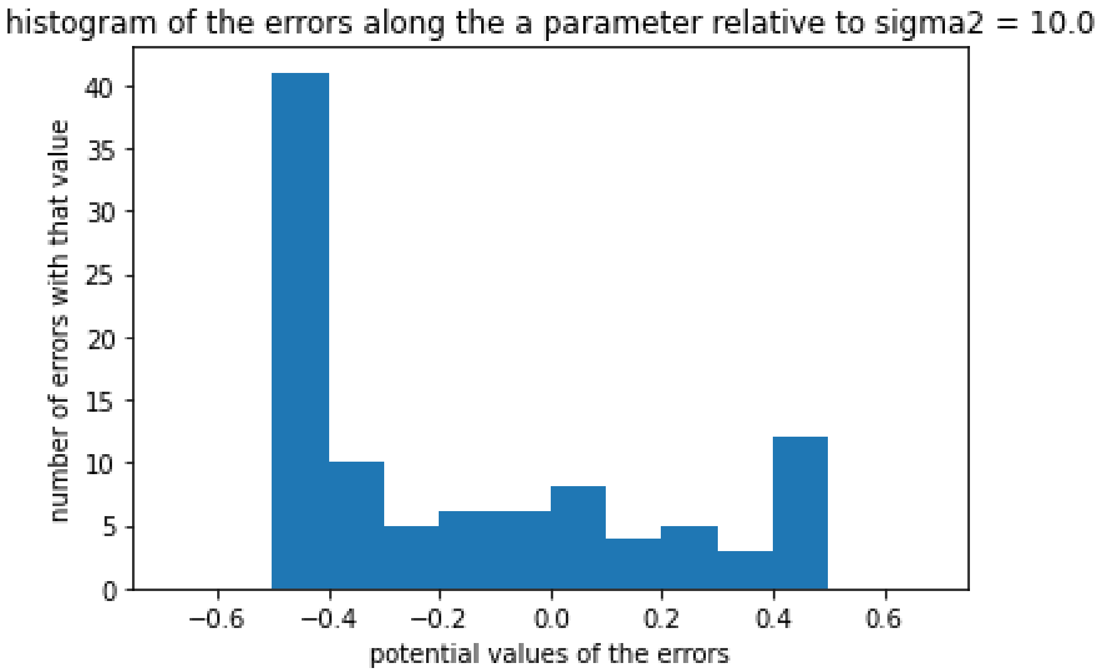

histogram of the errors along the b parameter relative to sigma2 = 0.01
number of errors with that value
potential values of the errors
0
10
20
30
40
50
−2
−1
0
1
2

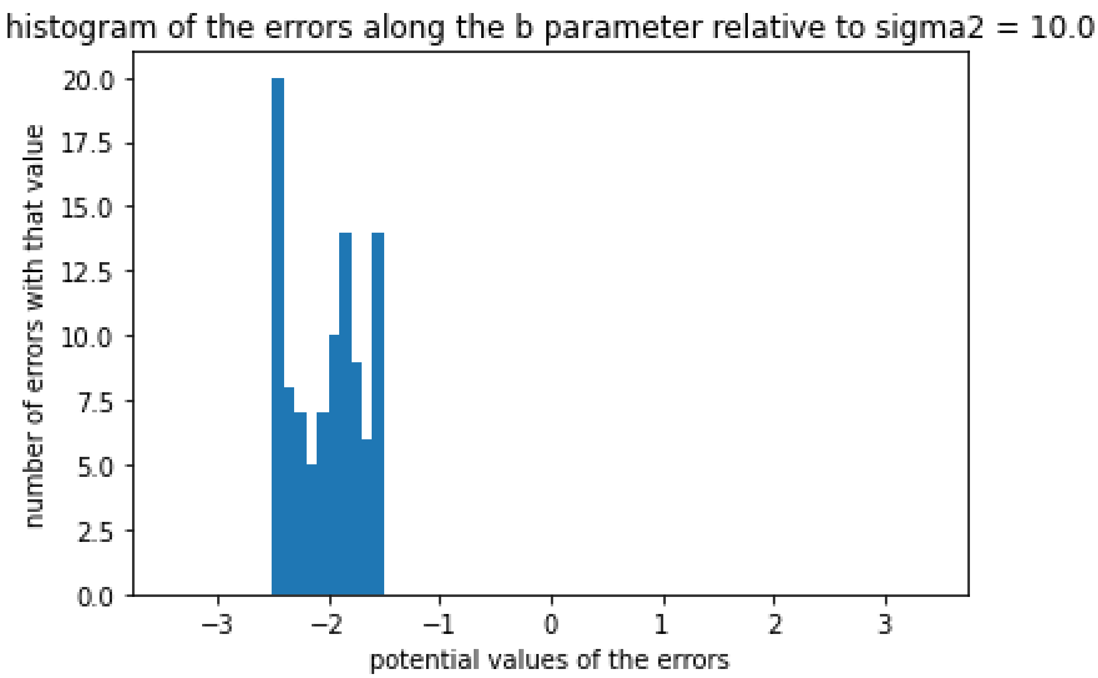

histogram of the errors along the a parameter relative to sigma2 = 0.1
number of errors with that value
17.5
15.0
12.5
10.0
7.5
5.0
2.5
0.0
−0.6
−0.4
−0.2
0.0
0.2
0.4
0.6
potential values of the errors


histogram of the errors along the b parameter relative to sigma2 = 0.1
number of errors with that value
50
40
30
20
10
0
−3
−2
−1
0
1
2
3
potential values of the errors


histogram of the errors along the a parameter relative to sigma2 = 1.0
number of errors with that value
potential values of the errors
0
5
10
15
20
25
30
−0.6
−0.4
−0.2
0.0
0.2
0.4
0.6


histogram of the errors along the b parameter relative to sigma2 = 1.0
number of errors with that value
potential values of the errors
0
5
10
15
20
25
−3
−2
−1
0
1
2
3





0.5 Task 10.4

It is weird that the expected behavior with increasing variance is not seen...