# Reasoning

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**ARTICLE HISTORY**
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**ABSTRACT**
This articles is about reasoning.

**KEYWORDS**
Reasoning; mathematics; Euclid

## 1. Introduction

**TODO Stasinos:** later

## 2. Background

The most widely known mathematician in geometry is Euclid form Ancient Greece and Euclid's most renowned work are the *Elements* (Heath, 1956). The Elements consist of thirteen books, the first six being about geometry of the plane and are based on the ruler-and-compass approach. The first book contains five postulates and nine axioms along with twenty three definitions and forty eight propositions. Postulates and axioms may only be found in the first book. We based this work on the first book because it contains the core elements that are used through out the Elements, thus sets a solid base for implementing the rest as well.

Euclid's propositions have a consistent structure; a set of statements, a construction and a proof. The given statements are used to construct a diagram. Then the proof is obtained using a combination of the given statements, observations made on the diagram, definitions, postulates, axioms and previously defined propositions. A basic characteristic of Euclid's Elements is that each proposition once it is proven, it is considered as given knowledge and so it is used inside other later propositions.

Although the Elements where considered flawless for many years, Euclid's work was not as clear concerning the purity of logic, instead there where many times in which he would use non-logically proven assumptions (Harrison, 2009, Sections 1.1 and 2).

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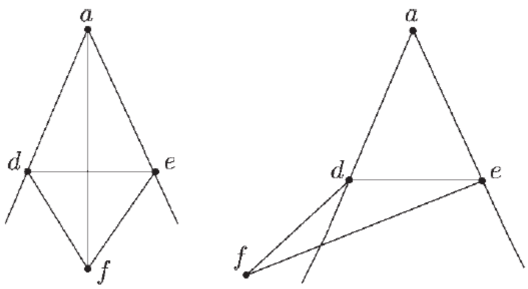



**Figure 1.** Two cases for Proposition I.9 considered by Avigad et al. (2009).

### *2.1. Automated Deduction for Geometry Theorems*

Whether that was because something came from observation or from assumption. This is where Hilbert**?** firstly stepped in and axiomatized Euclid's geometry by redefining the basis of Euclid's definitions, replacing and removing some axioms concerning the plane. There are also other known axiomatizations of the Elements such as Tarski's **?** and Birkhoff's **?**.

Avigad, Dean, and Mumma (2009) present a formal system that models Books I to IV from Euclid's Elements. In order to address the matter of facts that Euclid takes as observations from the diagram, they set a list of axioms to replace the conclusions derived from the diagram Euclid constructs. These axioms are in their core rules that depict the diagram's purpose and part in the proof. They state that in some cases where Euclid reads geometric relations directly from the diagram, in their system this has to be met with one of the system's rules. These axioms are divided into categories according to the impact they depict from the diagram.

First are *construction rules* these are the base of their rule system and are described as the 'built-in theorems that are available from the start'. The construction rules include rules about points, lines, circles and intersections of the above.

Then they define the *diagrammatic inference* axioms, which replace the diagram's purpose. While traditional diagrammatic inference acts on manipulating a diagram, this approach uses a set of axioms, construction and inference rules to conclude on the 'direct consequences' of those. They attempt to define this notion and describe why these attempts may not be adequate and in which cases they work.

Then are the rules about *transfer inferences* on the diagrams, their intention is to depict how the changes on a diagram, as Euclid describes them, apply on their facts.

The last rules they define are about *superposition inferences*, these rules are about how one diagram relates to another and the rules that apply to the former apply to the latter as well.

They conclude this section by explaining the *direct consequence* notion, that they based their approach on. However, while Euclid takes some parts of the construction as granted, in their system they claim that more specificity is needed in order to make use of their rules. In one of the examples they give is with Proposition I.9., where Euclid generates a triangle $dfe$ (as seen in Figure 1), they claim that the $f$ point needs more clarification for its placing in the diagram because it may fall out of the $dae$ angle, which is not true. If one follows Euclid's direction for the construction the $f$ point does cut the angle in half and it's impossible to be placed in the position they claim. Euclid's directions state that to place the $f$ point one must construct an equilateral triangle using his proposition I.1 with $de$ as a base. Creating an equilateral triangle would place point $f$ exactly in a positions precisely cutting the $dae$ angle in half when one connects $f$ and $a$. Another example the use is the one of proposition I.35 (as seen in Figure 2), in this case there is a lot more to be discussed. While all three cases that they present as figures can be generated solely from the proposition's statement, the construction steps of the proposition eliminate the second where there is one point less.

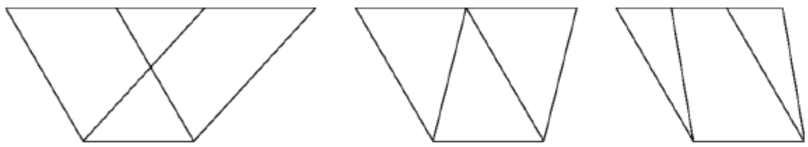


**Figure 2.** Three cases for Proposition I.35 considered by Avigad et al. Avigad et al. (2009).

However, Euclid's aim is to prove the proposition, and while the proof might not be exactly the same for all the cases considered in Figure 2, the proposition is true for all which is of most importance. Euclid is vague enough where he should be, since his intend is to reuse the propositions through out his work, should he have been more precise other proofs that use this one might not be provable. It appears that Euclid is making use of the open world assumption, which makes his work all the more attractive to potential representation.

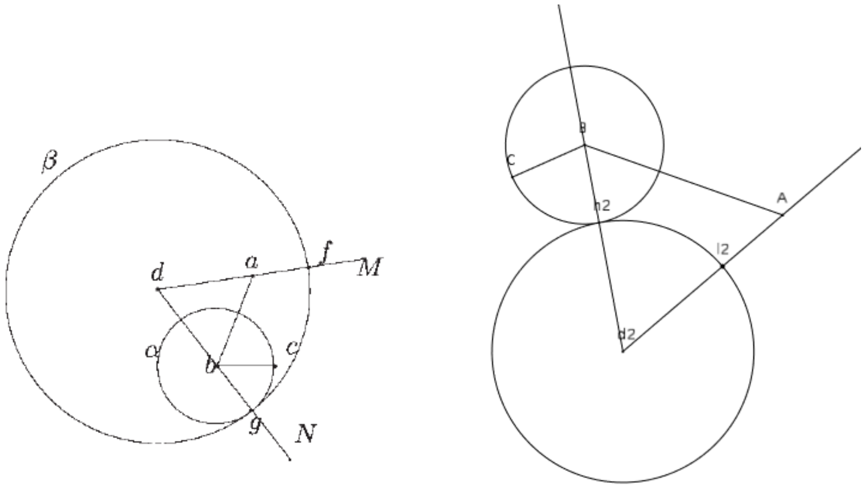
Last but not least, I will refer to their statement of Euclid not stating enough information in his constructions. They take as an example proposition I.2, claiming that point $a$ may not be inside circle $\beta$ and a clarification of $da < dg$ should be made. This is again misinterpreted since if one follows Euclid's steps to the construction, even if $a$ fell out of circle the entire proof still stands true. To prove this, one can consider the case where $ab > bc$ giving Figure 3 (right). In this case the circle with center $B$ (correspinding to circle $\alpha$ on the left-hand side figure), would intersect with line segment $Bd2$ (right) or $db$ (left) in two points. When one takes the point that is between $d2$ and $B$ ($h2$) and forms circle with center $d2$ and radius of $d2h2$ (right) or $\beta$ (left). Then line segment $Ad2$ intersects with the previous circle at point $l2$ and Euclid's proof stands exactly the same, even in this case, since again the proposition's aim was to create a line segment from $A$ equal to $BC$ and that is achieved.

The latest approach on Euclid by Beeson, Narboux, and Wiedijk (2019) approaches the matter of 'proof-checking Euclid'. In their approach they used a custom-designed representation of Euclid's Elements.

In their approach they do not follow Euclid's ways of proving and also do not follow the same order as his. They seem to approach superposition as equality between figures and define equality using Hartshorne (2000) definitions redefined in first-order axioms. What is more, they had to adapt to the tool and reused theorems as implicit assumptions for each lemma.

They claim that their approach is more complete than previous approaches to formalizing Euclid's work. and that they are the first to do a non-paper-and-pencil formalization. They also claim that their work to add to Euclid's work and prove correct and valid Book I of Euclid, using their additions, including 'corrected proofs of those propositions, that are close to Euclid's ideas'.

However, they too do not interpret superposition in the way Euclid does and try to find other ways around this in the construction of propositions and their proofs. For example, when they say that proposition I.9 cannot be proved using I.1 and they try




(a) Proposition I.2 as formed by Avigad et al.

(b) The counter-example that Avigad et al. offer to argue that

**Figure 3.** Different diagrams that both support the proof of Proposition I.2

to work around this.

### *2.2. Diagrammatic Inference*

On the other hand, there are systems that act on diagrams like the one of Miller (2001). In his approach Miller creates a collection of rules that are divided into categories according to the implication of the rule on the diagram, similar to the ones of Avigad et al. First are the construction rules, inference rules follow and then transformation rules. He treats superposition as *lemma incorporation*, for which he defines and proves a theorem. Miller's *Lemma Incorporation Theorem* is used in his system to identify the forms that a diagram may take supposing an ancestor diagram.

This becomes useful in the sense of using this lemma to get the superposition effect by isolating the diagram requiring superposition from the parent diagram, applying is to a smaller 'environment' and using the result in the parent diagram. Miller concludes this section with the presentation of the system CDEG that makes use of all the above. CDEG diagrams are not as conservative and not easy to read, an example diagram is the one of Figure 4 which is the result of CDEG on proposition I.1 of Euclid.

The main issue with this approach is that it does not fully take advantage of what Euclid essentially aims at, which is also the major issue that most publications have difficulty interpreting.

### *2.3. Remarks*

Euclid does indeed have some more vague points, on the contrary to what these publications suggest, work best through his work, since the vague points are not important to the application of a proposition, rather opportunities to apply them even more times. The vague point need no more clarification, if ones does specify in more detail the context of each proposition might become more logically complete, but it will lose the point of versatility.

One approach acts directly on diagrams while others use solely logic to tackle the problem. However, Euclid relies both on diagrams and logic in his proofs. To encode so much information may be challenging, which means that, isolating one or the other requires even more effort in designing and applying such a method.

The notion of Functional Programming (FP) one would say is to transform a problem into a set of steps one should execute, with the intent to focus on the computation of the problem. Thus, instead of creating multiple paths for execution at a time, functional programming focuses on one main goal and each function is a step towards this goal.

We aim that in this process, every step is a function, which itself is a part of the solution to a greater goal. A *function* in FP takes a number of arguments, that it will use to compute the respective output.

## 3. We Need A Name

**TODO Eirini:** later

## 4. Implementation and Validation

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## 5. Conclusions

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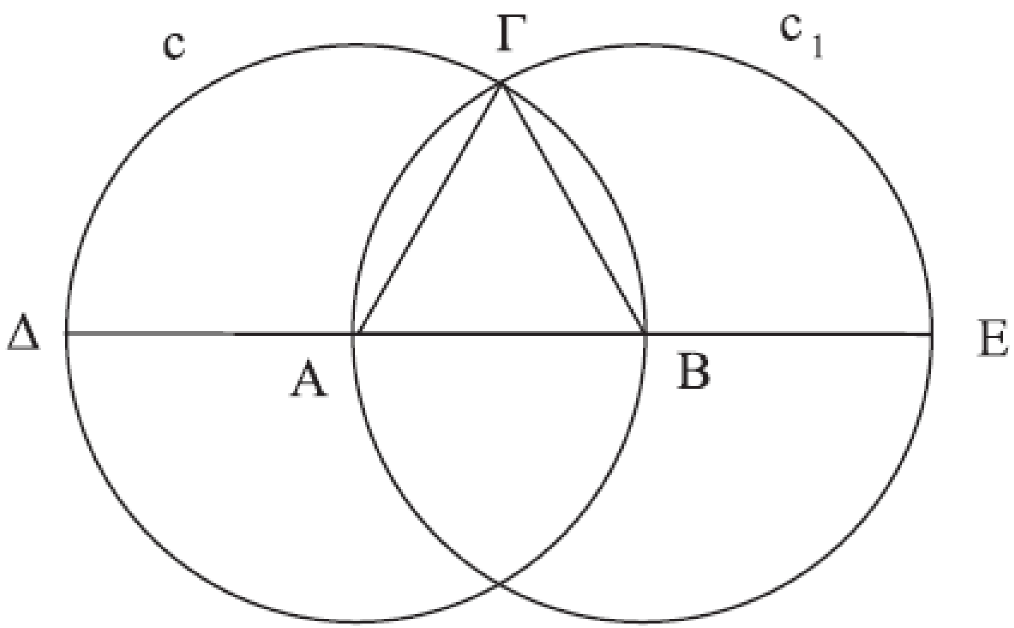
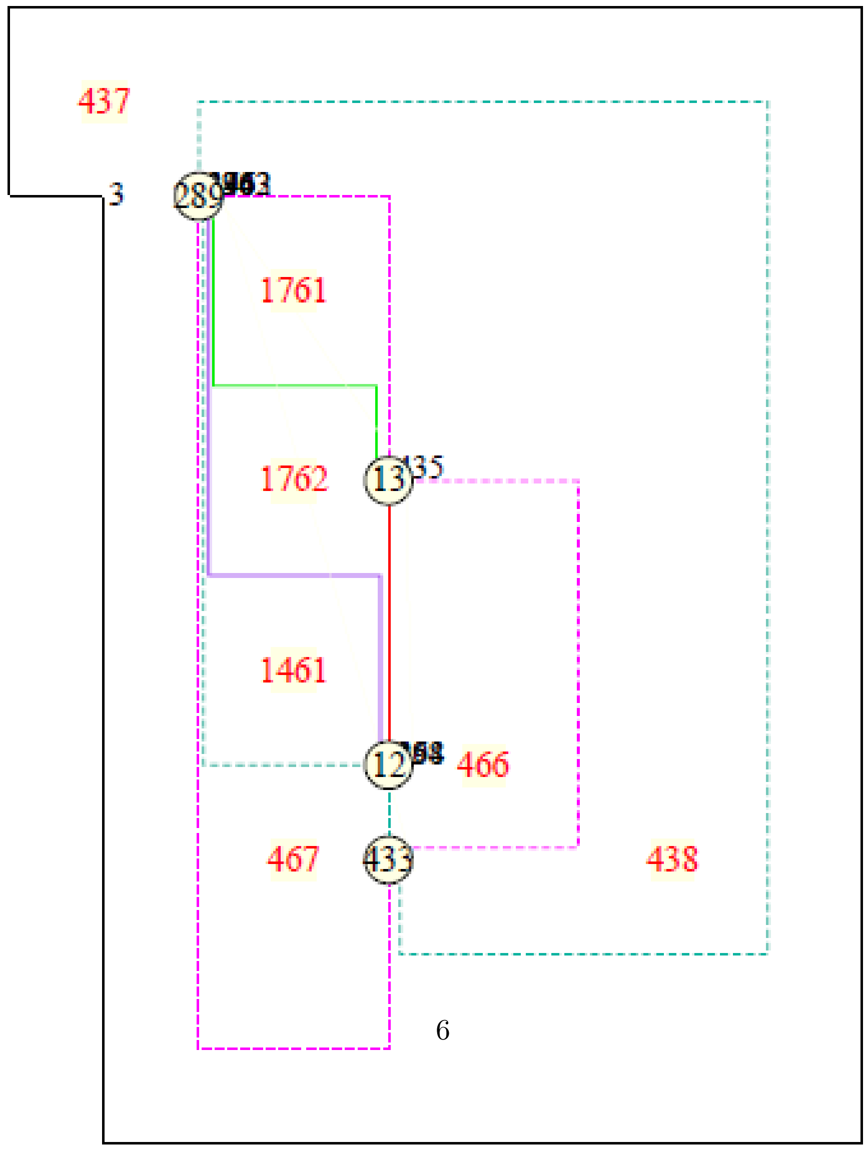



(a) General representation of diagram from proposition I.1




(b) Resulting image of Proposition I.1 from Miller (2001).