INDIAN INSTITUTE OF TECHNOLOGY KANPUR

MTH392A - UGP Report

EFX for Ternary Valuations

Submitted by : Prajjwal Kumar
Roll number : 210736

Supervisor : Dr. Soumyarup Sadhukhan
Signature : Soumyarup Sadhukhan

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*Annexure-II

DECLARATION

I hereby declare that the work presented in the project report entitled **EFX for Ternary Valuations** contains my own ideas in my own words. At places, where ideas and words are borrowed from other sources, proper references, as applicable, have been cited. To the best of our knowledge this work does not emanate from or resemble other work created by person(s) other than mentioned herein.

Name : Prajjwal Kumar

Date : Wednesday 3rd April, 2024

Signature : 

ABSTRACT

We are examining the problem of fairly dividing m indivisible goods among n individuals. Envy-freeness up to any item (EFX) is a highly sought-after concept in discrete fair division. Although there has been extensive research on this topic, the existence of EFX allocations remains a major open problem in fair division literature. Recently, it has been shown that EFX allocations always exist for binary valuations, i.e., $v(B \cup \{g\}) - v(B) \in \{0, 1\}$. In our study, we focus on ternary valuations where the marginal value gained from receiving an extra item is either $0, \frac{1}{2}$, or 1. We intend to develop an algorithm for allocating goods among three agents under ternary valuations maintaining EFX.

Keywords : Fair Division
EFX
Ternary Valuation
3 agents

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1. Introduction

We consider the problem of allocating m indivisible goods among n agents in a fair manner. This area, known as fair division, is an extremely active field of work at the interface of mathematical economics and computer science. The quintessential and classic notion of fairness in fair division is that of envy freeness, which requires that each agent prefers the bundle assigned to her over that of any other agent. However, in the case of indivisible goods, an envy-free allocation is not guaranteed to exist; Consider the example of two agents and one good where both agents valuing this single good at 1. Given that envy-free allocation may not exist, researchers considered other notions of fairness. One such important notion is EFx (envy free upto any good) requiring for any pair of agents, each agent prefers their bundle over the other agent's bundle after removing any single good from that bundle. Although there has been extensive research on this topic, the existence of EFX allocations remains a major open problem in fair division literature. In this project, we intend to examine the existence of EFX allocation for a special class of valuation functions, known as ternary valuations in the literature.

1.1 Related literature

(i) It has been proved that under binary valuation setup, i.e. if for any i $v_i(A) - v_i(A \setminus g) \in \{0,1\}$ where $g \in A$, an EFX allocation always exists for any finite number of goods and finite number of agents and an algorithm has been provided to find the such an EFX allocation Bu et al. (2023).

(ii) Despite a significant amount of effort, the existence of EFX allocation is still one of the most fundamental open problems in the fair division literature. The existence of an EFX allocation is only known for much restricted setups such as only for 2 agents Plaut and Roughgarden (2020)

(iii) For three agents and additive valuations, EFX-allocation always exits (see Chaudhury et al. (2024)).

(iv) In a four agent setting with additive valuations, an almost EFX exists in the sense that it leaves atmost a single item unallocated. Berger et al. (2022)

(v) An EFX allocation always exists when one can leave at most $n-2$ items unallocated Mahara (2023)

(vi) An partial EFX allocation $(X_1,\dots,X_n,P)$, where P is the set of unallocated goods, always exists such that no agent envies P more than his bundle and $|P| < n$ Chaudhury et al. (2021)

In this paper we aim to prove that EFX allocation exists for 3 agents and valuations with ternary marginals, i.e., $v_i(A) - v_i(A \setminus g) \in \{0, \frac{1}{2}, 1\}$ where $g \in A$, algorithmically.

2. Notation and Preliminaries

We study the problem of allocating m indivisible goods among n agents in a fair manner. The cardinal preference of each agent $i \in [n]$, over the subsets of goods, is specified via valuation $v_i : 2^{[m]} \to \mathbb{R}_+$. Here, $v_i(S) \in \mathbb{R}_+$ denotes the valuation that agent i has for a subset of goods $S \subseteq [m]$. We will throughout represent a discrete fair division instance via the tuple $\langle [n], [m], \{v_i\}_{i=1}^n \rangle$. Our algorithms work in the standard value-oracle model. That is, for each valuation v_i, we only require an oracle that—when queried with a subset $S \subseteq [m]$—provides the value $v_i(S)$.

This work focuses on valuations that may have three possible marginals, i.e., for each agent $i \in [n]$, the marginal value of including any good $g \in [m]$ in any subset $S \subseteq [m]$ is either zero or half or one: $v_i(S \cup \{g\}) - v_i(S) \in \{0, \frac{1}{2}, 1\}$. We will refer to such set functions v_is as **ternary** valuations. Note that a ternary valuation v_i is monotone: $v_i(A) \leq v_i(B)$ for all subsets $A \subseteq B \subseteq [m]$. Also, we will assume that the agents' valuations satisfy $v_i(\emptyset) = 0$.

An allocation $\mathscr{A} = (A_1, A_2, \ldots, A_n)$ is an ordered collection of n pairwise disjoint subsets, $A_1, \ldots, A_n \subseteq [m]$, wherein the subset of goods A_i is assigned to agent $i \in [n]$. We refer to each such subset A_i as a bundle. Note that an allocation can be partial in the sense that $\cup_{i=1}^n A_i \neq [m]$. For disambiguation, we will use the term complete allocation to denote allocations wherein all the goods have been assigned and, otherwise, use the term partial allocation. The social (utilitarian) welfare of an allocation $\mathscr{A} = (A_1, \ldots, A_n)$ is the sum of the values that it generates among the agents, $\sum_{i=1}^{n} v_i(A_i)$.

The quintessential notion of fairness is that of envy-freeness Foley (1966). This notion is defined next for allocations of indivisible goods.

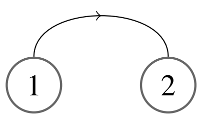
Definition 2.1. An allocation $\mathscr{A} = (A_1, \ldots, A_n)$ is said to be **envy-free (EF)** iff $v_i(A_i) \geq v_i(A_j)$ for all agents $i, j \in [n]$ Varian (1974).

Definition 2.2. An allocation $\mathscr{A} = (A_1, \ldots, A_n)$ is said to be **envy-free upto any-item (EFX)** if $v_i(A_i) \geq v_i(A_j \setminus g)$ for any agents i, j and for any $g \in A_j$ Caragiannis et al. (2019)..

Definition 2.3. Given a partial allocation $\mathscr{A} = (A_1, \ldots, A_n; P)$, where P is the set of allocated items, i is said to **strongly envy** j if $v_i(A_i) < v_i(A_j \setminus g)$ for some $g \in A_j$ Caragiannis et al. (2019).

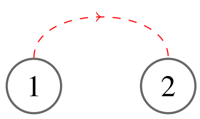
Definition 2.4. An agent is called **source agent** if no agent envies his bundle.

Notation 1.1



denotes $v_1(A_1) \leq v_1(A_2)$. i.e 1 envies 2's bundle or 1 values his bundle same as 2's bundle.

Notation 1.2



denotes $v_1(A_1) > v_1(A_2)$. i.e 1 strictly prefers his bundle over agent 2's bundle.

We call such a graph as an envy graph.

3. Some useful results

1. If there is a cycle between a pair of agents, resolving this cycle leads to EF allocation between the pair.

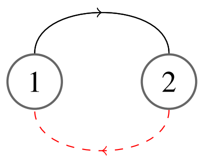
   Proof - Suppose there is a cycle between agent i and j. This implies that $v_i(A_i) \leq v_i(A_j)$ and $v_j(A_j) \leq v_j(A_i)$. After resolving the cycle agent i is allotted A_j and agent j is allotted A_i. The above inequalities show that this allocation is EF between agent i and j.

2. The value of social welfare function in 3 agents scenario is bounded by $3 * m$ where m is the number of items.

   Proof - As $v_i(A_i) - v_i(A_i \setminus g) \in \{0, \frac{1}{2}, 1\}$, $v_i(A_i) \leq m$. Thus $v_1(A_1) + v_2(A_2) + v_3(A_3) \leq 3 * m$.

3. In our problem, if an EFX allocation including the new good h with strict increase in social welfare function is found, we can find a EFX allocation using all the items. This is because the social welfare function is bounded. Since the social welfare function has to increase atleast by half in each iteration, at max $6 * m$ iterations can be made.

4. Consider the 2 agent envy graph :

   Since 1 is the source agent we add h to 1. Suppose after adding h to 1, agent 2 starts strongly envying 1's bundle. i.e $\exists g \in A_1 \cup h$ such that $v_2(A_2) < v_2(A_1 \cup h \setminus g)$. Remove this g

   If $v_1(A_2) \geq v_1(A_1 \cup h \setminus g)$:

   > Exchange the bundles.
   >
   > $\Delta SWF = (v_1(A_2) - v_1(A_1)) + (v_2(A_1 \cup h \setminus g) - v_2(A_2)$. Therefore we are done. The allocation is EFX using result stated in 1.

   Else $v_1(A_2) < v_1(A_1 \cup h \setminus g)$

   > We continue removing such $g's$ until either the If condition is satisfied or 2 no longer strongly envies 1's bundle.
   >
   > If after removing some $g's$ 2 no longer strongly envies 1's bundle. Then the allocation is :
   >
   > $1 \longrightarrow A_1 \cup h \setminus g$
   >
   > $2 \longrightarrow A_2$
   >
   > . Since 2 doesn't strongly envy 1's bundle the allocation is EFX.
   >
   > Also since $v_1(A_1) \leq v_1(A_2)$ and $v_1(A_2) < v_1(A_1 \cup h \setminus g)$ we have $v_1(A_1) < v_1(A_1 \cup h \setminus g)$. Thus the social welfare function strictly increases.

   Thus if in 2 agent case some agent envies the other agent, We have a EFX allocation.

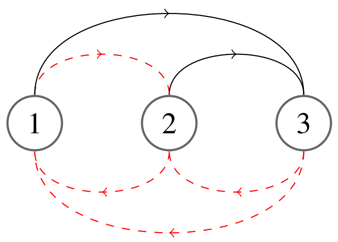
4. Solved Cases

Notice that in our envy-graph if there exists a cycle, we eliminate the cycle as it results in an EF allocation between the agents in the cycle. So we will only consider the envy graphs where there is no cycle formation.

1. If there exists a source agent such that adding h to his bundle causes no one to strongly envy him we add h to his bundle.

   In Particular if $\exists i \in \{1,2,3\}$ such that $v_j(A_j) - v_j(A_i) \geq 1$ $\forall j \in \{1,2,3\} \setminus i$, add h to A_i.

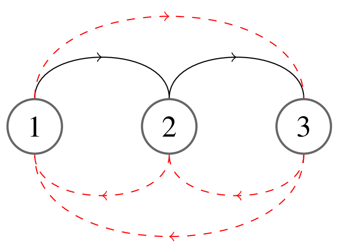
2. Consider the case as shown in the envy graph :

   with $v_2(A_2) - v_2(A_1) \geq 1$.

   $\because v_2(A_2) - v_2(A_1) \geq 1$, if we add h to A_1, $v_2(A_2) - v_2(A_1 \cup h) \geq 0$. Thus 2 can't envy 1's new bundle. Thus it is just a 2 agent scenario with agent 1 envying agent 3's bundle initially and h is added to his bundle. The solution of this case is given in the section useful results point 4.

3. Consider the case as shown in the envy graph :

   with $v_3(A_3) - v_3(A_1) \geq 1$.

   $\because v_3(A_3) - v_3(A_1) \geq 1$, if we add h to A_1, $v_3(A_3) - v_3(A_1 \cup h) \geq 0$. Thus 3 can't envy 1's new bundle. Thus it is just a 2 agent scenario with agent 1 envying agent 2's bundle initially and h is added to his bundle. The solution of this case is given in the section useful results point 4.

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   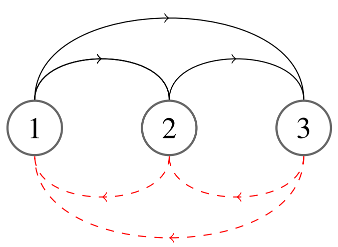

   $\longrightarrow (I)$

   Since 1 is the only source agent, let us add the arbitrary good h to 1. Suppose that after adding h to A_1 both agent 2 and 3 start strongly envying agent 1's bundle.

   $\because$ 2 strongly envies 1's bundle

$\Rightarrow \exists g \in A_1 \cup h$ such that $v_2(A_2) < v_2(A_1 \cup h \setminus g)$

We remove this g from 1's bundle and consider the 2 exhaustive cases:

(i) $v_1(A_2) \geq v_1(A_1 \cup h \setminus g)$ and

(ii) $v_1(A_2) < v_1(A_1 \cup h \setminus g)$

Case (i) :

In this case we exchange the bundles between agent 1 and 2. So now agent 1 has the bundle A_2, agent 2 has the bundle $A_1 \cup h \setminus g$ and agent 3 has the bundle A_3. After this exchange we see that between 1 and 2 we have an EF allocation. There is EFX between agent 1 and 3 because the initial allocation was EFX and agent 1 gets weakly better bundle. Also 2 can't strongly envy 3's bundle because the initial allocation was EFX. Therefore only possible strong envy is 3 envying 2's bundle.

If 3 does not strongly envy 2's bundle we are done as the allocation is EFX and the social welfare function strictly increases.

$\Delta SWF = (v_1(A_2) - v_1(A_1)) + (v_2(A_1 \cup h \setminus g) - v_2(A_2)) + (v_3(A_3) - v_3(A_3))$ and

$v_1(A_2) \geq v_1(A_1), v_2(A_1 \cup h \setminus g) > v_2(A_2)$

If 3 does strongly envy 2's bundle:

$\exists g' \in A_1 \cup h \setminus g$ such that $v_3(A_3) < v_3(A_1 \cup h \setminus (g \cup g'))$

We remove this g' from the bundle and again categorise the problem into 2 exhaustive sub-cases.

(i') $v_2(A_3) \geq v_2(A_1 \cup h \setminus (g \cup g'))$

(ii') $v_2(A_3) < v_2(A_1 \cup h \setminus (g \cup g'))$

Sub-Case (i'):

We now exchange bundles between agents 2 and 3. So now agent 1 has the bundle A_2, agent 2 has the bundle A_3 and agent 3 has the bundle $A_1 \cup h \setminus (g \cup g')$. This resulting allocation is an EFX allocation as agent 3 is getting a better bundle than A_3 and thus doesn't strongly envy agent 1 or 2's bundle.

This allocation also leads to strict increase of social welfare function as :

$\Delta SWF = (v_1(A_2) - v_(A_1)) + (v_2(A_3) - v_2(A_2)) + (v_3(A_1 \cup h \setminus (g \cup g')) - v_3(A_3))$

which is positive.

Sub-Case (ii'):

If this is the case we continue and see whether still agent 3 strongly envies 2. If he does $\exists g'' \in A_1 \cup h \setminus (g \cup g')$ such that $v_3(A_3) < v_3(A_1 \cup h \setminus (g \cup g' \cup g''))$. We remove this g'' and again make similar sub-cases as above and recursively go on until either sub-case (i') is satisfied or agent 3 no longer envies 2's bundle.

Note that if the recursion stops because sub-case (i') or its recursive case is satisfied, we see that the allocation remains EFX due to the same reasons stated in sub-case (i').Also the social welfare increase is guaranteed as all the recursive versions of sub-case (i')

require that 3 strongly envies 2.

But if 3 no longer envies 2 after removing a few such $g's$, the allocation is trivially and EFX. Also the Social Welfare Function is strictly increasing as : $\Delta SWF = (v_1(A_2) - v_(A_1)) + (v_2(A_1 \cup h \setminus (g \cup g')) - v_2(A_2)) + (v_3(A_3) - v_3(A_3))$. The sub-case condition gives us $v_2(A_3) < v_2(A_1 \cup h \setminus (g \cup g'))$ and since $v_2(A_2) \leq v_2(A_3)$, therefore $v_2(A_2) < v_2(A_1 \cup h \setminus (g \cup g'))$ which guarantees SWF increase.

Case (ii):

Again we go on removing such $g's$ tills either case (i) is satisfied or 2 no longer strongly envies 1's bundle.
If after removing some $g's$ 2 no longer strongly envies 1.

Suppose that agent 3 does not strongly envy agent 1's new bundle.
Agent 1 has the bundle $A_1 \cup h \setminus g$. The case (ii) condition is $v_1(A_2) < v_1(A_1 \cup h \setminus g)$. We also have $v_1(A_1) \leq v_1(A_2)$. This implies $v_1(A_1) < v_1(A_1 \cup h \setminus g)$. So the allocation is EFX. Also the Social welfare function strictly increases as $v_1(A_1) < v_1(A_1 \cup h \setminus g)$.

Suppose agent 3 does strongly envy agent 1's new bundle. But this is similar to the two agent case where initially 1 envies 3 and then some h is added to 1's bundle.

Hence we have found an EFX allocation if the initial allocation follows envy graph (I).

In all the cases when in each pair of agents a agent envies the other and there is one source agent can be reduced to the case analogous to the case with envy-graph (I).

5. Problematic cases

1. $v_i(A_i) - v_j(A_j) = \frac{1}{2}$ and $v_j(A_j) - v_i(A_i) = \frac{1}{2}$ for any i, j

2. $v_1(A_1) \leq v_1(A_3)$, $v_3(A_3) > v_3(A_1)$

   $v_2(A_2) \leq v_2(A_3)$, $v_2(A_2) > v_2(A_3)$

   $v_2(A_2) - v_2(A_1) = \frac{1}{2}$, $v_1(A_1) - v_2(A_2) \geq 1$ This is a case where 2 pair of agents satisfy the condition that one weakly envies the other.

   And if after adding h to A_1, 2 and 3 both start strongly envying 1. We can resolve strong envy between 1 and 3. But after resolving 2 can still strongly envy 1.

3. $v_1(A_1) - v_1(A_2) = \frac{1}{2}$

   $v_2(A_2) - v_2(A_1) \geq 1$

   $v_3(A_3) - v_3(A_1) = \frac{1}{2}$

   $v_1(A_1) - v_1(A_3) \geq 1$

   $v_2(A_2) - v_2(A_3) = \frac{1}{2}$

   $v_3(A_3) - v_3(A_2) \geq 1$

   This is a case where no pair of agents satisfy the condition that one weakly envies the other.

   In this case adding h to anyone is not feasible.

   We cannot add h to A_1 as 3 will strongly envy him.

   We cannot add h to A_2 as 1 will strongly envy him.

   We cannot add h to A_3 as 2 will strongly envy him.

4. Suppose only 1 pair satisfies the condition that one weakly envies the other (say this pair is 2 and 3, i.e $v_2(A_2) \leq v_2(A_3)$, $v_3(A_3) > v_3(A_2)$). Then -

   $v_2(A_2) - v_1(A_1) \geq 1$ and $v_1(A_1) - v_1(A_2) = \frac{1}{2}$ (So b/w 1 and 2 we should try to add good to 1 and not 2)

   If $v_3(A_3) - v_3(A_1) = \frac{1}{2}$. Then we cannot add to 1 as 3 can strongly envy 1.

   So we can not add h to either of 1, 2 and 3.

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