HOMOLOGY OF LOCAL RINGS

by

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PREFACE

This monograph is a survey of some of the "classical" results and techniques as well as more recent developments in the theory of homology of local rings.

We wish to express our appreciation to Professor P. Ribenboim for suggesting the project.

Thanks are also due to Mrs. E. M. Wight for her excellent typing of the manuscript.

T.G. and G.L.

Kingston, June 1969.

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INTRODUCTION

The homological approach to commutative ring theory officially begins in 1890 with Hilbert's theory of syzygies, [Hi] and is extended through the work of Koszul [Ko] and Cartan [Ca] in the early 1950's. These pioneering efforts opened the way for the theory of homological dimension.

Let R be a commutative ring and M an R-module. M is said to have finite homological dimension over R if there exists an exact sequence

$$0 \to P_n \to P_{n-1} \to \cdots \to P_1 \to P_o \to M \to 0$$

where the P_i are projective R-modules. The length n of the shortest such sequence is called the homological dimension of M. Equivalently M has homological dimension $\leq n$ if $\mathrm{Ext}_R^{n+1}(M,C) = 0$ for all R-modules C. The supremum of the homological dimensions of all R-modules M is called the global dimension of R and is denoted $g\ell.\dim R$.

Various data concerning a commutative, noetherian ring R can be obtained by looking at all the localizations $R_{\underline{m}}$ where $\underline{m}$ runs through the set of maximal ideals in R. As an example we have

$$g\ell.\dim R = \sup_{\underline{m}} g\ell.\dim R_{\underline{m}}$$

A significant aspect of global dimension is shown by the following characterization of regular local rings; i.e. local rings R for which the Krull dimension equals the embedding dimension $n = \dim_{R/\underline{m}} \underline{m}/\underline{m}^2$.

<u>Theorem (Auslander-Buchsbaum-Serre)</u>: A local ring R is regular if and only if $g\ell\text{-}\dim R < \infty$.

One of the most important applications of the above characterization is the following:

<u>Theorem</u> If $\underline{p}$ is a prime ideal in a regular ring R then the localization $R_{\underline{p}}$ is regular.

For a local (noetherian) ring with residue class field $k = R/\underline{m}$ the global dimension is - at least in principle -

simple to compute. We have $g\ell$-dim $R < n$ if and only if $\mathrm{Ext}_R^n(k,k) = 0$ or equivalently $\mathrm{Tor}_n^R(k,k) = 0$. In these notes we will be interested in the nature of $\mathrm{Tor}^R(k,k)$ and $\mathrm{Ext}_R(k,k)$ for any local ring R. Of particular interest is the Poincaré series of R, i.e. the formal power series

$$P(R) = \sum_{p=0}^{\infty} b_p z^p$$

where $b_p = \dim_k \mathrm{Tor}_p^R(k,k)$ are called the Betti numbers of R.

The regular local rings were the first rings for which $P(R)$ was explicitly computed. It has been shown by Serre that if R is regular then $P(R) = (1+z)^n$ where n is the embedding dimension of R.

Introducing resolutions of the R-module $k = R/\underline{m}$ having a structure of an augmented differential graded algebra over R, Tate showed in [Ta] 1957 that if R is a complete intersection then there exist non-negative integers m,n such that

$$P(R) = \frac{(1+z)^n}{(1-z^2)^m}$$

This result was independently found by Zariski.

In [Go] 1962 Golod gave a far reaching contribution to the theory by showing that if certain homology operations on the Koszul complex vanish, then there exist non-negative integers $n, c_1, \ldots, c_n$ such that

$$P(R) = \frac{(1+z)^n}{1-\sum_{i=1}^{n} c_i z^{i+1}}$$

It has been conjectured by Kaplansky and Serre that $P(R)$ always represents a rational function.

In [Sc] 1964 Scheja established the rationality of $P(R)$ in the case where $\underline{m}$ is generated by two elements, and in [Sh] Shamash established the rationality in the case where R is a factor ring of a regular ring $\tilde{R}$ modulo an ideal generated by two elements. Assuming in the latter case that R is not a complete intersection it was shown in [Gu"] 1968 that

$$P(R) = \frac{(1+z)^n}{1-2z^2-z^3}$$

n being the embedding dimension of R.

Not much is known about the rationality of $P(R)$ in

general, although an inequality due to Serre shows that $P(R)$ is always bounded (coefficientwise) by a rational function.

One of the few results of general nature concerning $P(R)$ is the existence of non-negative integers $\epsilon_q (q \geq 0)$ such that

$$P(R) = \prod_{i=0}^{\infty} \frac{(1 + z^{2i+1})^{\epsilon_{2i}}}{(1 - z^{2i+2})^{\epsilon_{2i+1}}}$$

The ϵ_q are invariants of R. They are called the deviations of R and are denoted by $\epsilon_q(R)$. This formula was established by Assmus in [As'] 1958 in the case where $R/\underline{m}$ has characteristic zero, and in the general case by Levin in [Le] 1965.

The disposition of the present notes is as follows:

In the first Chapter we develop the machinery of differential graded algebras with divided powers, and R-algebra resolutions. We include a discussion of the Koszul complex, R-sequences, regular rings and local complete intersections.

Chapter two exhibits the Hopf algebra structure of $\mathrm{Tor}^R(k,k)$ and shows that the dual Hopf algebra is isomorphic, as an algebra, to $\mathrm{Ext}_R(k,k)$ with the Yoneda product. Examples of this structure are given for some different types of local rings.

The techniques developed in the first Chapter are refined in Chapter three to yield results about the Poincaré series and the deviations of local (noetherian) rings. Some "change of ring"-theorems are given, including the recent result that - under certain conditions - we have

$$\epsilon_q(R_{\underline{p}}) \leq \epsilon_q(R)$$

where $R_{\underline{p}}$ is the localization of R with respect to a prime ideal $\underline{p}$. A section containing homological characterizations of local complete intersections is included.

Chapter four is devoted to proving theorems about the rationality of the Poincaré series, and the development of certain homological techniques.

NOTATIONS AND CONVENTIONS

R will always denote a commutative ring with unit element 1. All modules will be unitary. Ring homomorphisms will be required to preserve the unit element.

If R, $\tilde{R}$ and R' denote local rings, then $\underline{m}$, $\tilde{\underline{m}}$ and $\underline{m}'$ denote their maximal ideal respectively.

$\mathbb{Z}$ denotes the ring of integers. $\mathbb{N}_0$ resp. $\mathbb{N}_1$ denotes the set of non-negative resp. positive integers.

$\amalg$ and $\oplus$ denotes direct sum.

1_X denotes the identity map of an object X.

If X is a graded object then X_q denotes the homogeneous component of degree q. If X is a differential object, then the differential is usually denoted d.

$N \subset N'$ means N is a (not necessarily proper) subset of N'.

$\hookrightarrow$ is sometimes used to denote an inclusion map.

CHAPTER ONE

DIFFERENTIAL GRADED ALGEBRAS

§1. Basic definitions. Augmented R-algebras

Definition 1.1.1 An R-module X with a family of submodules $\{X_q\}_{q\in\mathbb{Z}}$ is called a graded R-module if $X = \coprod_q X_q$. If nothing else is said, we will always assume that $X_q = 0$ for $q < 0$. X is said to be of finite type if X_q is a finitely generated R-module for each q. The modules $X_q (q \geq 0)$ are called the homogeneous components of X. If $x \in X_q$ we say that x is homogeneous and has degree q; we write $\deg x = q$.

An R-linear map $f : X \to Y$ between graded modules is said to be homogeneous of degree p if $f(X_q) \subset Y_{q+p}$ for all q; we write $\deg f = p$. In this case the map $X_q \to Y_{q+p}$ induced by f is denoted by f_q. By a graded algebra over R we mean an associative algebra X over R and a family of R-submodules $\{X_q\}_{q\in\mathbb{Z}}$ such that (i) $X = \coprod_{q\in\mathbb{Z}} X_q$, $X_q = 0$ for $q < 0$.

(ii) $X_p X_q \subset X_{p+q}$.

Unless otherwise stated we will always assume that X has a unit element denoted by 1.

A graded algebra is called <u>strictly skew-commutative</u> if

(iii) $xy = (-1)^{pq} yx$ for $x \in X_p$, $y \in X_q$.

(iv) $x^2 = 0$ if $\deg x$ is odd.

By a <u>homomorphism</u> of graded algebras we mean a homogeneous algebra homomorphism of degree 0, preserving the unit element whenever it is meaningful.

If X, Y are strictly skew-commutative graded algebras over R then the tensor product $X \underset{R}{\otimes} Y$ of the underlying graded modules can be furnished in a canonical way with a structure of a graded strictly skew-commutative algebra. Define the homogeneous components by

$$(X \otimes Y)_n = \coprod_{p+q=n} X_p \otimes Y_q \quad \text{for } n \in \mathbb{Z} .$$

and the multiplication is uniquely determined by the requirement

$$(x \otimes y)(x' \otimes y') = (-1)^{pq}(xx' \otimes yy')$$

$$\text{for } y \in Y_p , \; x' \in X_q .$$

Definition 1.1.2. If S is a symbol given a degree $n > 0$, then $R\langle S\rangle$ will denote the following commutative graded algebra over R: If n is odd, then $R\langle S\rangle$ denotes the exterior algebra generated by S. If n is even, then we take $R\langle S\rangle$ to be the polynomial algebra in a countable number of variables $S = S^{(1)}, S^{(2)}, S^{(3)}, \ldots$ with relations $S^{(i)}S^{(j)} = \frac{(i+j)!}{i!j!} S^{(i+j)}$ for $i,j \geq 1$, and grading defined by giving $S^{(k)}$ the degree kn for $k > 0$. By convention we put $S^{(o)} = 1$. The elements $S^{(k)}$ for $k > 0$ will be called the natural divided powers of S. If $\deg S$ is odd, then by the natural divided powers of S we mean the element S only. In both cases $\deg S$ even and $\deg S$ odd $R\langle S\rangle$ will be called the divided polynomial algebra generated over R by the variable S of degree n.

Definition 1.1.3. A graded algebra X with an R-linear homogeneous map $d: X \to X$ of degree -1 will briefly be called an R-algebra provided that

(i) X is strictly skew-commutative.

(ii) X_o is generated as an R-module by the unit element (connectedness).

(iii) $d^2 = 0$, and for $x \in X_q$, $y \in X$ we have

$$d(xy) = (dx)y + (-1)^q x\, dy .$$

d is called the differential on X . A factor ring $\bar{R}$ of R will be considered as an R-algebra with trivial grading and differential. A map $f: X \to Y$ between R-algebras is called an R-<u>homomorphism</u> if f is a homomorphism of graded algebras satisfying $d_Y f = f d_X$, where d_X and d_Y are the differentials on X and Y respectively. A subset I of an R-algebra X is called an <u>ideal</u> in X if I is an ideal in the underlying algebra satisfying

(i) $I = \coprod_q I_q$ where $I_q = I \cap X_q$

(ii) $dI \subset I$

One checks easily that if I is an ideal in X $(1 \notin I)$ then there is a unique R-algebra structure on the residue class algebra X/I such that the canonical map $X \to X/I$ is an R-homomorphism.

By the <u>tensor product</u> $X \otimes_R Y$ of two R-algebras X,Y we mean the R-algebra whose underlying graded algebra is

the tensor product of the graded algebras X and Y, and whose differential $d_{X\otimes Y}$ is given by

$$d_{X\otimes Y}(x\otimes y) = d_X(x)\otimes y + (-1)^q x \otimes d_Y(y)$$

for $x \in X_q$, $y \in Y$, d_X and d_Y denoting the differentials on X and Y respectively.

An R-algebra X is called an R-<u>subalgebra</u> of an R-algebra Y if

(i) $X \subset Y$

(ii) The inclusion map is an R-homomorphism.

If X is an R-algebra with differential d we will let $Z(X)$ and $B(X)$ denote the set of cycles and boundaries respectively, i.e. $Z(X) = \ker d$, $B(X) = \operatorname{im} d$. We define $Z_q(X) = Z(X) \cap X_q$ and $B_q(X) = B(X) \cap X_q$ for $q \in \mathbb{Z}$. One checks easily that

$$Z(X) = \coprod_q Z_q(X) \ , \quad B(X) = \coprod_q B_q(X)$$

moreover that $Z(X)$ is an R-subalgebra of X and that $B(X)$ is an ideal in $Z(X)$. Thus $H(X) = Z(X)/B(X)$ is an R-algebra. The homogeneous components are $H_q(X) = Z_q(X)/B_q(X)$. Note that $H_o(X)$ is a homomorphic image of R and that $H(X)$ is in fact a $H_o(X)$-algebra. An

R-homomorphism $f: X \to Y$ induces an R-homomorphism $H(X) \to H(Y)$ which we denote by f_*.

An <u>augmented</u> R-<u>algebra</u> is an R-algebra X furnished with an R-homomorphism $\varepsilon: X \to \overline{R}$, where $\overline{R}$ is some factor ring of R. ε will be called the augmentation of X and $\ker \varepsilon$ will be called the augmentation ideal.

Let X and Y be augmented R-algebras with augmentations ε_X and ε_Y respectively. An R-homomorphism $f: X \to Y$ will be called a <u>homomorphism of augmented</u> R-<u>algebras</u> if $\varepsilon_Y f = \varepsilon_X$.

If X is an augmented R-algebra with augmentation $\varepsilon: X \to \overline{R}$ then ε induces R-homomorphisms

$$\varepsilon_Z: Z(X) \to Z(\overline{R}) = \overline{R}$$

$$\varepsilon_*: H(X) \to H(\overline{R}) = R$$

The kernel of ε_Z and ε_* will be denoted by $\widetilde{Z}(X)$ and $\widetilde{H}(X)$ respectively. They are graded algebras (without unit element) with homogeneous components $\widetilde{Z}_q(X) = Z(X) \cap X_q$, $\widetilde{H}_q(X) = \widetilde{H}(X) \cap H_q(X)$. Note that $\widetilde{H}_q(X) = H_q(X)$ for $q > 0$. $\widetilde{H}(X)$ will be called the <u>reduced homology</u> of X.

X will be called <u>acyclic</u> if $\widetilde{H}(X) = 0$. A homomorphism $f: X \to Y$ of augmented R-algebras will be called acyclic if Y is acyclic. Note that if X is acyclic and projective as an R-module, then X may be considered as a projective resolution of the R-module $H_o(X)$.

If $f: X \to Y$ is a homomorphism of augmented R-algebras then f induces a homomorphism of graded algebras (without unit element)

$$\widetilde{f}_*: \widetilde{H}(X) \to \widetilde{H}(Y)$$

The following diagram has exact rows and is commutative

$$\begin{tikzcd}
0 \arrow[r] & \widetilde{H}(X) \arrow[r, hook] \arrow[d, "\widetilde{f}_*"] & H(X) \arrow[r, "(\varepsilon_X)_*"] \arrow[d, "f_*"] & \overline{R} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \widetilde{H}(Y) \arrow[r, hook] & H(Y) \arrow[r, "(\varepsilon_Y)_*"'] & \overline{R} \arrow[r] & 0
\end{tikzcd}$$

This shows that $\widetilde{f}_*$ is an isomorphism if and only if f_* is an isomorphism.

§2. Construction of R-algebra resolutions. The process of adjoining variables to kill cycles.

In this section we will describe a method, due to Tate, of extending an augmented R-algebra to an acyclic one.

Let X be an R-algebra and let s be a cycle in X of degree $n-1 \geq 0$. Noting that X_o is a factor ring of R we let $\overline{R}\langle S\rangle$ be the divided polynomial algebra generated over the ring $\overline{R} = X_o$ by a variable S of degree n ; see (1.1.2).

The canonical homomorphisms of graded algebras

$$X \xrightarrow{f} X \underset{R}{\otimes} \overline{R}\langle S\rangle \xleftarrow{g} \overline{R}\langle S\rangle$$

defined by $f(x) = x \otimes 1$, $g(y) = 1 \otimes y$ for $x \in X$, $y \in \overline{R}\langle S\rangle$ are injective and in the following we will always assume that X and $\overline{R}\langle S\rangle$ are graded subalgebras of $X \underset{R}{\otimes} \overline{R}\langle S\rangle$.

One easily checks that there is a unique differential d on $X \underset{R}{\otimes} \overline{R}\langle S\rangle$ converting it to an R-algebra, satisfying

(1) the inclusion map $X \to X \otimes \bar{R}\langle S\rangle$ is an R-homomorphism

(2) $dS = s$ and if $\deg S = n$ is even, then $dS^{(k)} = sS^{(k-1)}$ for $k > 0$.

This R-algebra will be denoted by

$$X\langle S; dS = s\rangle$$

and is called the R-algebra obtained from X by the <u>adjunction of a variable</u> S <u>which kills the cycle</u> s. If X has an augmentation $\epsilon_X : X \to \bar{\bar{R}}$ and if $\epsilon_X(s) = 0$ then we will always assume that $Y = X\langle S; dS = s\rangle$ is furnished with the unique augmentation $\epsilon_Y : Y \to \bar{\bar{R}}$ extending ϵ_X (trivially).

<u>Lemma 1.2.1</u>. Let X be an augmented R-algebra. Let $\sigma \in \tilde{H}_{n-1}(X)$ be a homology class represented by a cycle $s \in \tilde{Z}_{n-1}(X)$. Let $Y = X\langle S; dS = s\rangle$. We have

(i) $\tilde{H}_q(Y) = \tilde{H}_q(X)$ for $q < n-1$

(ii) $\tilde{H}_{n-1}(Y) = \tilde{H}_{n-1}(X)/\sigma R$.

<u>Proof</u>. (i) follows from the fact that $Y_q = X_q$ for $q < n$. (ii) follows since $\tilde{Z}_{n-1}(Y) = \tilde{Z}_{n-1}(X)$ and

$B_{n-1}(Y) = B_{n-1}(X) + sR$.

We will now fix notation concerning the adjunction of more than one variable.

Given a well ordered set I . Let I^{∞} denote the set of elements in I without an immediate predecessor. Let I^{o} be the set of symbols $\{[i-1] | i \in I^{\infty}\}$. Let $\underline{I}$ be the disjoint union $I \cup I^{o}$. Now the well ordering on I has a unique extension to a well ordering on $\underline{I}$ such that $[i-1] < i$, and $j < i \Rightarrow j < [i-1]$ for $i \in I^{\infty}$ and $j \in I$.

The first element in $\underline{I}$ will be denoted by 0 . If $j \in \underline{I}$ has an immediate predecessor it will be denoted $j-1$, thus $[i-1] = i-1$ for $i \in I$.

Given augmented R-algebras X,Y and a set of variables $\{S_i\}_{i \in I}$. We call Y a <u>free extension of</u> X <u>with basis</u> $\{S_i\}_{i \in I}$ provided that there exists a well ordering of I and a direct system of augmented R-algebras X^i over the well ordered set $\underline{I}$ satisfying

(i) $X^o = X$

(ii) $X^i = X^{i-1}\langle S_i ; dS_i = s_i \rangle$ for $i \in I$

(iii) $X^i = \varinjlim_{h<i} X^h$ if i is a limit element in $\underline{I}$

(iv) $Y = \varinjlim_{i \in \underline{I}} X^i$

If Y is a free extension of X with basis $\{S_i\}_{i \in I}$ we write

$$Y = X\langle \ldots S_i \ldots ;\ dS_i = s_i \rangle$$

or briefly $Y = X\langle \ldots S_i \ldots \rangle$. The maps $X^i \to Y$ are injections and we will identify X^i with its image in Y . Thus we have

$$Y = \bigcup_{i \in \underline{I}} X^i$$

We will now define an increasing filtration $\{F_q Y\}_{q \geq 0}$ on Y . Put $F_o Y = X$ and for $q \geq 1$ let $F_q Y$ be the graded subalgebra of Y generated by X , the adjoined variables of degree $\leq q$ and all the natural divided powers of these variables. We obviously have $Y = \bigcup_q F_q Y$. Note that $F_q Y$ is in fact an R-subalgebra of Y . This filtration will be called the <u>filtration associated with the extension</u> $X \subset Y$.

$X \langle \ldots S_i \ldots \rangle$ will also denote the underlying graded algebra of the R-algebra Y above. We have an isomorphism

of graded algebras

$$X\langle\ldots S_i\ldots\rangle \approx X \underset{R}{\otimes} \left(\underset{i\in I}{\otimes} R\langle S_i\rangle\right)$$

(infinite tensor product of graded algebras over R).

Observe that if there exists another well ordering of I and another direct system of augmented R-algebras $'X^i$ over the corresponding well ordered set $'\underline{I}$ satisfying (i) to (iii) above, then $\varinjlim_{i\in 'I} 'X^i$ and $\varinjlim_{i\in I} X^i$ are canonically isomorphic. In particular, if we are given an augmented R-algebra X and a set of cycles $\{s_i\}_{i\in I}$ in $\widetilde{Z}(X)$ then it makes sense to talk about the augmented R-algebra $X\langle\ldots S_i\ldots; dS_i = s_i\rangle$, without reference to a well ordering on I .

<u>Lemma 1.2.2</u>. Let X be an augmented R-algebra and $\{s_i\}_{i\in I}$ a set of homogeneous cycles in $\widetilde{Z}(X)$ of common degree $n-1$. Put $Y = X\langle\ldots S_i\ldots; dS_i = s_i\rangle$. Then the inclusion map $X \to Y$ induces isomorphisms $\widetilde{H}_q(X) \approx \widetilde{H}_q(Y)$ for $q < n-1$. The map $\widetilde{H}_{n-1}(X) \to \widetilde{H}_{n-1}(Y)$ is surjective with kernel generated by the homology classes σ_i of the cycles s_i .

Proof. By well ordering the set I one easily shows by transfinite induction that $Y_q = X_q$ for $q < n$ and that $Y_n = X_n \oplus \coprod_{i \in I} X_o S_i$. Hence $\tilde{Z}_{n-1}(Y) = \tilde{Z}_{n-1}(X)$ and $B_{n-1}(Y) = B_{n-1}(X) + \sum_{i \in I} Rs_i$. It follows that

$$\tilde{H}_{n-1}(Y) = \tilde{H}_{n-1}(X) / \sum_{i \in I} R\sigma_i$$

Theorem 1.2.3. (Tate, Northcott) Let X be an augmented R-algebra. There exists an acyclic augmented R-algebra of the form

$$Y = X\langle \ldots S_i \ldots ; \; dS_i = s_i \rangle .$$

Moreover if R is noetherian and X is of finite type, then Y can be taken to be of finite type.

Proof. We shall obtain Y as a direct limit of a direct system over $\mathbb{N}_o$ of augmented R-algebras X^n satisfying

(3) $\tilde{H}_q(X^n) = 0$ for $q < n$

The maps of the system will be inclusion maps. We define X^n inductively. Let $X^o = X$. Let $n \geq 1$ and suppose that X^{n-1} has been constructed. If $\tilde{H}_{n-1}(X^{n-1}) = 0$ put $X^n = X^{n-1}$. Otherwise choose a set of homogeneous cycles $\{s_{n,\alpha}\}_{\alpha \in I_n} \subset \tilde{Z}_{n-1}(X^{n-1})$ representing a set of generators for $\tilde{H}_{n-1}(X^{n-1}) = 0$. Put $X^n = X^{n-1} < \ldots S_{n,\alpha} \ldots ; dS_{n,\alpha} = s_{n,\alpha} >$. By (1.2.2) X^n satisfies (3). Let $Y = \varinjlim_{n \in \mathbb{N}_o} X^n$. Y is clearly acyclic. Moreover Y is a free extension of X with basis $\{S_i\}_{i \in I}$ where $I = \{(n,\alpha) | n \in \mathbb{N}_1, \alpha \in I_n\}$. Indeed, well order each of the sets I_n and well order I lexicographically with highest significance on the first coordinate. Consider the following direct system over $\underline{I}$ of augmented R-algebras (all maps being inclusion maps)

$$X^o = X$$

$$X^i = X^{i-1} < S_{n,\alpha} ; dS_{n,\alpha} = s_{n,\alpha} > \quad \text{for} \quad i = (n,\alpha) \in I$$

$$X^i = \bigcup_{h<i} X^h \quad \text{for limit elements} \quad i \in \underline{I} .$$

We have $Y = \bigcup_{i \in \underline{I}} X^i$. Hence Y is a free extension with basis $\{S_i\}_{i \in I}$.

Now suppose that R is noetherian and that X is of finite type. It suffices to show that X^n $(n \in \mathbb{N}_o)$ can be taken to be of finite type. This can easily be shown by induction on n. If $n > 0$ and X^{n-1} is of finite type then $\tilde{H}_{n-1}(X^{n-1})$ is a finitely generated R-module. We may therefore take the set I_n to be finite. It follows that X^n is of finite type.

<u>Corollary 1.2.4</u>. Let $R \to \bar{R}$ be a surjective homomorphism of rings. Then there exists a free resolution of the R-module $\bar{R}$ of the form $R\langle \ldots S_i \ldots ; dS_i = s_i \rangle$.

<u>Proof</u>. This follows from the above theorem considering $R \to \bar{R}$ as an augmented R-algebra.

<u>Remark</u>. The technique for constructing an R-algebra resolution of a factor ring $\bar{R}$ of R was given by Tate [Ta] in the case where R is noetherian. Using transfinite methods Northcott [No"] extended this technique to the non-noetherian case.

§3. Derivations and the exact sequence associated with the adjunction of a variable.

Definition 1.3.1. Let X be an R-algebra with differential d. A derivation j on X is an R-linear homogeneous map $j: X \to X$ satisfying

i) $dj = jd$

ii) $j(xy) = (-1)^{w \cdot q} j(x)y + xj(y)$

where $w = \deg j$ and $y \in X_q$.

Lemma 1.3.2. Let j be a derivation on an R-algebra X, and s a cycle in X. Put $X' = X\langle S; dS = s\rangle$. Then j can be extended to a derivation j' on X' if and only if

(1) $j(s) \in B(X')$.

Proof. If j can be extended, (1) is satisfied because $j(s) = j(dS) = dj'(S)$. On the other hand if (1) is satisfied, choose an element $G \in X'$ with the property

$$dG = j(s)$$

We treat the cases $\deg S$ odd and $\deg S$ even separately.

If $\deg S$ is odd, we have

$$X' = X \oplus XS$$

For $x_o, x_1 \in X$ define

$$(2) \qquad j'(x_o + x_1 S) = j(x_o) + (-1)^{\deg j} j(x_1)S + x_1 G$$

If $\deg S$ is even, we have

$$X' = \coprod_{i=0}^{\infty} XS^{(i)}$$

For $x_o, \ldots, x_m \in X$ define

$$(3) \qquad j' \sum_{i=o}^{m} x_i S^{(i)} = \sum_{i=o}^{m} j(x_i) S^{(i)} + \sum_{i=1}^{m} x_i S^{(i-1)} G$$

We omit the straightforward but tedious verification that in both cases j' becomes a derivation on X'.

<u>Definition 1.3.2'</u>. Let j', S and G be as in the above proof. j' will be called <u>the canonical extension of</u> j <u>satisfying</u> $j'S = G$.

Each time we are given an R-algebra of the form

Proof. Let $\eta \in H_q(X) = H_q(X'')$ be represented by a cycle z. Since $j(zS) = z$, $\partial(\eta)$ is the homology class of $d(zS) = (-1)^q zs$. Hence $\partial(\eta) = (-1)^q \eta\sigma$.

Lemma 1.3.4. Let $j: X \to X$ be a derivation on the R-algebra X. Let $\{s_\alpha\}_{\alpha \in I}$ be a set of cycles in $Z(X)$. Suppose that there exist elements $G_\alpha \in X$ such that $dG_\alpha = js_\alpha$ for $\alpha \in I$. Put $X' = X\langle \ldots S_\alpha \ldots ; dS_\alpha = s_\alpha \rangle$. Then j has an extension to a derivation $j': X' \to X'$ satisfying

$$j'(S_\alpha) = G_\alpha \quad \text{for } \alpha \in I$$

$$j'(S_\alpha^{(i)}) = S_\alpha^{(i-1)} G_\alpha \quad \text{for } i \geq 1 \text{ if } \deg S_\alpha \text{ is even.}$$

Such an extension j' is unique.

Proof. The uniqueness of j' is clear since X' is generated as an algebra over R by X and the natural divided powers of the adjoined variables S_α. The existence of j' follows from the proof of (1.3.2), well ordering I and using transfinite induction. We leave the details to the reader.

Proposition 1.3.5. Let $f:X \to Y$ be a homomorphism of R-algebras inducing an isomorphism $f_*: H(X) \cong H(Y)$. Let s be a cycle in X. Put $X' = X\langle S; dS = s\rangle$ and $Y' = Y\langle T; dT = f(s)\rangle$. f has a canonical extension to an R-homomorphism $f':X' \to Y'$ inducing an isomorphism $f'_*: H(X') \to H(Y')$.

Proof. If $\deg S$ is odd, then for $x_o, x_1 \in X$ define

$$f'(x_o + x_1 S) = f(x_o) + f(x_1)T$$

and if $\deg S$ is even, then for $x_o, \ldots, x_m \in X$ define

$$f'\left(\sum_{i=o}^{m} x_i S^{(i)}\right) = \sum_{i=o}^{m} f(x_i) T^{(i)}$$

Straightforward verifications show that in both cases f' becomes an R-homomorphism. We restrict ourselves to showing that f'_* is an isomorphism.

Let $w = \deg S$. To simplify the notation let $f'': X'' \to Y''$ denote the map $f: X \to Y$ if w is odd and the map $f': X' \to Y'$ if w is even. We have a commutative diagram of complexes

$$
\text{(I)} \qquad \begin{tikzcd}
0 \arrow[r] & X \arrow[r] \arrow[d, "f"] & X' \arrow[r] \arrow[d, "f'"] & X'' \arrow[r] \arrow[d, "f''"] & 0 \\
0 \arrow[r] & Y \arrow[r] & Y' \arrow[r] & Y'' \arrow[r] & 0
\end{tikzcd}
$$

where the horizontal rows are the exact sequences associated with the adjunction of S and T. From (I) we obtain the commutative diagram

$$
\text{(II)} \qquad \begin{tikzcd}
H_{q-w+1}(X'') \arrow[r] \arrow[d] & H_q(X) \arrow[r] \arrow[d] & H_q(X') \arrow[r] \arrow[d] & H_{q-w}(X'') \arrow[r] \arrow[d] & H_{q-1}(X) \arrow[d] \\
H_{q-w+1}(Y'') \arrow[r] & H_q(Y) \arrow[r] & H_q(Y') \arrow[r] & H_{q-w}(Y'') \arrow[r] & H_{q-1}(Y)
\end{tikzcd}
$$

If w is odd then $f''_* = f_*$ and it follows from (II) and the five-lemma that f'_{*q} is an isomorphism for all q. Now suppose that w is even. We will show by induction on q that f'_{*q} is an isomorphism for all q. For $q = 0$ $f'_{*q} = f_{*o}$ is an isomorphism. Let $q \geq 1$ and suppose that f'_{*p} is an isomorphism for all $p < q$. Since $w \geq 2$ we have $q-w+1 < q$. Hence all the four extreme vertical maps in (II) are isomorphisms. Again by the five-lemma f'_{*q} is an isomorphism.

§4. The Koszul complex, R-sequences, Codimension and regular rings.

In this section R will always denote a noetherian ring. We shall study R-algebras obtained from the R-algebra R by the adjunction of variables of degree 1. Note that the elements of R may be considered as cycles.

$\underline{a}$ will always denote an ideal in the Jacobson radical $\underline{m}$ of R.

Definition 1.4.1. Let $a_1,\ldots,a_n$ be a set of generators for $\underline{a}$. The R-algebra

$$E = R\langle T_1,\ldots,T_n; dT_i = a_i\rangle$$

will be called the Koszul complex generated over R by $a_1,\ldots,a_n$. We will also use the notation(1)

$$E = R_{a:1,\ldots,n}$$

---

(1) In [A.B] the symbol combination $R_{a:1,\ldots,n}$ is used to denote the underlying complex of E.

For $n = 0$ $R_{a:1,\ldots,n}$ shall mean the R-algebra R. Note that the underlying graded algebra of E is the exterior algebra generated by the free R-module $E_1 = \coprod_{i=1}^{n} T_i R$. In particular E_n is the free R-module generated by the product $T_1 T_2 \ldots T_n$, and $E_q = 0$ for $q > n$.

<u>Lemma 1.4.2</u>. Let $\underline{a}$ be generated by the elements $a_1,\ldots,a_n$ and put $E = R_{a:1,\ldots,n}$. We have

(i) $H_o(E) = R/\underline{a}$

(ii) $H_n(E) \approx 0:\underline{a}$

(iii) $H_q(E) = 0$ for $q > n$

(iv) $\underline{a}H(E) = 0$

<u>Proof</u>. (i) and (iii) are obvious. (ii) follows from the fact that

$$d(T_1 T_2 \ldots T_n) = \sum_{i=1}^{n} (-1)^{i+1} a_i T_1 \ldots \hat{T}_i \ldots T_n$$

where $\wedge$ over T_i means that T_i is to be omitted from the product. Thus if $r \in R$ then $d(rT_1 T_2 \ldots T_n) = 0$ if and only if $ra_i = 0$ for $i = 1,\ldots,n$. We have therefore

$H_n(E) = Z_n(E) \approx 0:\underline{a}$.

(iv) follows since $\underline{a} \subset B(E)$ and $B(E)Z(E) \subset B(E)$.

<u>Definition 1.4.3</u>. A sequence of elements $x_1,\ldots,x_s$ in $\underline{a}$ is called an R-<u>sequence</u> in $\underline{a}$ if x_1 is a non-zero-divisor in R , and for $2 \leq i \leq s$ x_i is a non-zerodivisor for the ring $R/(x_1,\ldots,x_{i-1})$. By convention we will consider the empty sequence as an R-sequence generating the zero-ideal. If $s = 0$ then $x_1,\ldots,x_s$ shall be interpreted as the empty sequence. An R-sequence $x_1,\ldots,x_s$ is called a maximal R-sequence in $\underline{a}$ if $x_1,\ldots,x_s,y$ is not an R-sequence for any $y \in \underline{a}$. Note that since R is noetherian any R-sequence in $\underline{a}$ can be extended to a maximal one.

<u>Proposition 1.4.4</u>.(*) Let $E = R_{a:1,\ldots,n}$ and let q be the largest integer such that $H_q(E) \neq 0$. Let s be the number of elements in a maximal R-sequence in $\underline{a} = (a_1,\ldots,a_n)$.

(*) This is a special case of a more general theorem due to Auslander and Buchsbaum [A.B, Theorem 1.7].

Then $s+q = n$.

<u>Proof</u>. We prove the proposition by induction on s . If $s = 0$ then every element of $\underline{a}$ is a zerodivisor. Hence $\underline{a}$ is contained in a prime ideal associated to (0) . So by (1.4.2) $H_n(E) \approx 0:\underline{a} \neq 0$ and $q = n$.

Now let $k \geq 1$. Suppose that the proposition is valid for $s = k-1$. Now let $s = k$. Let $x_1,\ldots,x_s$ be a maximal R-sequence in $\underline{a}$. Put $\overline{R} = R/x_1R$, $\overline{E} = E/x_1E$ and let $\bar{a}_i,\bar{x}_j$ and $\bar{\underline{a}}$ be the respective images of a_i,x_j and $\underline{a}$ in $\overline{R}$. Since E is free as an R-module and x_1 is a non-zerodivisor we have an exact sequence of complexes

$$0 \longrightarrow E \xrightarrow{x_1} E \longrightarrow \overline{E} \longrightarrow 0$$

inducing exact sequences

(1)
$$\begin{array}{ccccccc} H_{q+1}(E) & \longrightarrow & H_{q+1}(\overline{E}) & \longrightarrow & H_q(E) & \xrightarrow{x_1} & H_q(E) \\ \| & & & & \not\| & & \\ 0 & & & & 0 & & \end{array}$$

$$(2) \qquad \begin{array}{ccccc} H_p(E) & \longrightarrow & H_p(\bar{E}) & \longrightarrow & H_{p-1}(E) \\ \| & & & & \| \\ 0 & & & & 0 \end{array} \qquad \text{for} \quad p > q+1$$

Since $x_1 \in \underline{a}$ it follows from (1.4.2) that $x_1 H_q(E) = 0$. Hence we obtain from (1) $H_{q+1}(\bar{E}) \neq 0$ and from (2) $H_p(\bar{E}) = 0$ for $p > q+1$. $\bar{E}$ is isomorphic as a complex to $\bar{R}_{\bar{a}:1,\ldots,n}$ and $\bar{x}_2,\ldots,\bar{x}_s$ is obviously a maximal $\bar{R}$-sequence in $\bar{\underline{a}}$. Hence by the induction hypothesis we have $(s-1) + (q+1) = n$ so $s+q = n$.

<u>Corollary 1.4.5</u>. Maximal R-sequences in $\underline{a}$ have the same number of elements.

<u>Definition 1.4.6</u>. The common number of elements in maximal R-sequences in $\underline{m}$ will be called the <u>codimension</u> of R and will be denoted codim R.

<u>Proposition 1.4.7</u>. Let x be a non-zerodivisor in the Jacobson radical $\underline{m}$ of R. Put $\bar{R} = R/xR$. Then codim $\bar{R}$ = codim $R - 1$.

Proof. Let $\underline{a} = \underline{m}$, $x_1 = x$ and let E, $\bar{\underline{a}}$, $\bar{E}$, n and q be as in the proof of (1.4.4). The proposition now follows from (1.4.4) and the exact sequences (1) and (2), since $\bar{\underline{a}}$ is the Jacobson radical of $\bar{R}$. Indeed $\operatorname{codim} \bar{R} = n - (q+1) = (n-q)-1 = \operatorname{codim} R - 1$.

Proposition 1.4.8. Let $a_1,\ldots,a_n$ be elements in the Jacobson radical $\underline{m}$ in R and let $E = R_{a:1,\ldots,n}$. The following statements are equivalent:

(i) $a_1,\ldots,a_n$ is an R-sequence.

(ii) $H_i(E) = 0$ for $i \geq 1$.

(iii) $H_1(E) = 0$.

Moreover if $a_1,\ldots,a_n$ is an R-sequence, then so is any other minimal set of generators for the ideal $(a_1,\ldots,a_n)$.

Proof. We will first give the equivalence of (i), (ii) and (iii). (i) $\Rightarrow$ (ii) follows from (1.4.4), and (ii) $\Rightarrow$ (iii) is trivial. We will now show the implication (iii) $\Rightarrow$ (i) by induction on n . If $n = 0$ the implication is trivially true. Let $k \geq 1$. Suppose that the implication

is true for $n = k-1$. Now let $n = k$. Put $E' = R_{a:1,\ldots,n-1}$. We have $E = E'\langle T; dT = a_n\rangle$. The exact sequence associated with the adjunction of T , see (1.3.2'):

$$0 \longrightarrow E' \longrightarrow E \longrightarrow E' \longrightarrow 0$$

induces an exact sequence

$$H_1(E') \xrightarrow{\partial_1} H_1(E') \longrightarrow H_1(E) \longrightarrow H_o(E') \xrightarrow{\partial_o} H_o(E')$$

According to (1.3.3) the connecting homomorphisms ∂_1 and ∂_o are multiplication by a_n (up to sign). Now suppose that $H_1(E) = 0$. It follows that ∂_1 is surjective and ∂_o is injective. In particular $H_1(E') = a_n H_1(E')$ but $a_n \in \underline{m}$ and $H_1(E')$ is a finitely generated R-module, hence by Nakayama's lemma $H_1(E') = 0$. By the induction hypothesis it follows that $a_1,\ldots,a_{n-1}$ is an R-sequence. Since ∂_o is injective and since $H_o(E') = R/(a_1,\ldots,a_{n-1})$ it follows that a_n is a non-zerodivisor for $R/(a_1,\ldots,a_{n-1})$. Therefore $a_1,\ldots,a_n$ is an R-sequence.

The last statement of the proposition follows from the equivalence (i) $\Leftrightarrow$ (ii) using (1.4.4).

In the rest of this section R will always denote a local, noetherian ring with maximal ideal $\underline{m}$ and residue class field $k = R/\underline{m}$. We shall give two more characterizations of R-sequences.

<u>Proposition 1.4.9</u> (Gulliksen) Let $\underline{a} = (a_1,\ldots,a_\varepsilon)$ be a non-zero ideal of finite homological dimension in a local ring R. Let E be the Koszul complex

$$E = R\langle T_1,\ldots,T_\varepsilon; dT_i = a_i\rangle$$

Then the following statements are equivalent

(i) $\underline{a}$ is generated by an R-sequence

(ii) $H_1(E)$ is an $R/\underline{a}$-free module.

<u>Proof</u>. We will first reduce the problem to the case where $\underline{a}$ is minimally generated by $a_1,\ldots,a_\varepsilon$. We may assume without loss of generality that $\underline{a}$ is minimally generated by $a_1,\ldots,a_{\varepsilon'}$ where $1 \leq \varepsilon' \leq \varepsilon$. For $\varepsilon' \leq \beta \leq \varepsilon$ define

$$E^\beta = R\langle T_1,\ldots,T_\beta; dT_i = a_i\rangle$$

Suppose now that $\epsilon' \leq \beta < \epsilon$. Since $a_{\beta+1}$ is a boundary in E^{β} , the connecting homomorphism in the homology sequence associated with the extension $E^{\beta} \subset E^{\beta+1}$ is the zero-map, see (1.3.3). In particular we obtain the exact sequence

$$0 \longrightarrow H_1(E^{\beta}) \longrightarrow H_1(E^{\beta+1}) \longrightarrow H_o(E^{\beta}) \longrightarrow 0$$

Since $H_o(E^{\beta}) \approx R/\underline{a}$ and since a direct summand of a free $R/\underline{a}$-module is free, cf. [No', 9 Theorem 12], we have that $H_1(E^{\beta})$ is $R/\underline{a}$-free if and only if $H_1(E^{\beta+1})$ is $R/\underline{a}$-free. It follows that $H_1(E^{\epsilon'})$ is $R/\underline{a}$-free if and only if $H_1(E^{\epsilon})$ is $R/\underline{a}$-free.

We may therefore suppose that $a_1,\ldots,a_{\epsilon}$ is a minimal set of generators for $\underline{a}$. In this case the implication (i) $\Rightarrow$ (ii) follows from (1.4.4).

Conversely, suppose that $H_1(E)$ is $R/\underline{a}$-free. It suffices by (1.4.8) to show that $H_1(E) = 0$. Let us assume that $H_1(E) \neq 0$ and let $s_1,\ldots,s_m$ represent a free basis for $H_1(E)$. Let

$$X' = E\langle S_1,\ldots,S_{m-1}; dS_i = s_i\rangle$$

and

$$X = X'\langle S_m; dS_m = s_m\rangle$$

Let $J: X \to X$ be the derivation associated with the extension $X' \subset X$ see (1.3.2'). Let X be furnished with the obvious augmentation $X \to R/\underline{a}$. By (1.2.3) we can find an acyclic R-algebra X^* of the form $X^* = X\langle \ldots U_i \ldots; dU_i = u_i\rangle$, which is of finite type. We will show that J can be extended to a derivation on X^*, which we also denote by J.

Since $\tilde{H}_q(X) = 0$ for $q < 2$ we have $F_2X^* = F_1X^* = X$ and only variables of degree ≥ 3 are adjoined to obtain X^*.

Let $u = e + \sum_{i=1}^{m} r_i S_i$, where $e \in E$, be a typical cycle in $Z_2(X)$. Differentiation yields

$$\sum_i r_i s_i = de \in B_1(E)$$

Since $s_1, \ldots, s_m$ represents a free $R/\underline{a}$-basis for $H_1(E)$ it follows that $r_i \in \underline{a}$. Now, $J(u) = r_m$ hence

$$J(Z_2(X)) \subset \underline{a}X_o = B_o(X)$$

By (1.3.4) J can be extended to a derivation $J: F_3X^* \to$ $\to F_3X^*$.

Let $q \geq 3$ and suppose that J has been extended to a derivation

$$J: F_q X^* \to F_q X^*$$

Since $\deg J = -2$ and $H_p(F_q X^*) = 0$ for $0 < p < q$, we have

$$J(Z_q(F_q X^*)) \subset B(F_q X^*)$$

Again by (1.3.4) J can be extended to a derivation

$$J: F_{q+1} X^* \to F_{q+1} X^*$$

The existence of the required derivation $J: X^* \to X^*$ is now clear.

Let $k = R/\underline{m}$. Since $a_1, \ldots, a_\varepsilon$ is a minimal set of generators for $\underline{a}$ we have $Z_1(E) \subset \underline{m}E_1$. In particular $S_m \otimes 1$ is a cycle in $X^* \otimes k$. In fact $S_m^{(i)} \otimes 1 \in Z(X^* \otimes k)$ for all $i \geq 0$. X^* is a free resolution of the R-module $R/\underline{a}$. Since $\underline{a}$ has finite homological dimension, there exists an integer j such that

$$S_m^{(j)} \otimes 1 \in B(X^* \otimes k)$$

However, since $J^j(S_m^{(j)}) = 1$, and $J^j \otimes 1_k$ commutes with the differential in $X^* \otimes k$ we have $1 \otimes 1 = J^j \otimes 1_k\ (S_m^{(j)} \otimes 1) \in B_o(X^* \otimes k) = 0$ which is <u>absurd</u>.

The following characterization somewhat similar to (1.4.9) was given independently by Ferrand [Fe] and Vasconcelos [Va, Corollary 1]. We shall not have occasion to use the theorem in the following and state the result without proof:

<u>Proposition 1.4.10</u>. (Ferrand, Vasconcelos) Let $\underline{a}$ be an ideal of finite homological dimension in a local ring R . Then $\underline{a}$ is generated by an R-sequence if and only if $\underline{a}/\underline{a}^2$ is a free $R/\underline{a}$-module.

<u>Remark</u>. The following example shows that in (1.4.9) as well as in (1.4.10) the hypothesis that $\underline{a}$ be of finite homological dimension is essential.

Let R, $\underline{m}$ be any non-regular local ring, see definition (1.4.9) below, and let $\underline{a} = \underline{m}$. Let E be the Koszul-complex generated over R by a set of generators for $\underline{a}$

(1.4.13) shows that $\underline{a}$ is not generated by an R-sequence, however $H_1(E)$ as well as $\underline{a}/\underline{a}^2$ are free $R/\underline{a}$-modules since $R/\underline{a}$ is a field.

In a later section, §1 of Chapter three, we shall associate to any local noetherian ring a sequence of non-negative integers $\varepsilon_q(R)$ $(q = 0,1,2,\ldots)$ called the deviations of R. These are the exponents in the product formula for the Poincaré series of R, mentioned in the Introduction. We give already here the definition of the two first deviations.

<u>Definition 1.4.11</u>. We define

$$\varepsilon_o(R) = \dim_k \underline{m}/\underline{m}^2$$

Note that $\varepsilon_o(R)$ is the number of elements in any minimal set of generators for $\underline{m}$, cf. [No', Chapter 9, Proposition 3].

We remark that if $t_1,\ldots,t_n$ and $t_1',\ldots,t_n'$ are minimal sets of generators for $\underline{m}$ and $E = R_{t:1,\ldots,n}$ and $E' = R_{t':1,\ldots,n}$ then there exists a (non-canonical) R-

isomorphism $E \approx E'$ inducing an R-isomorphism $H(E) \approx H(E')$ which in fact is a k-isomorphism since $\underline{m}H(E) = 0$. This will follow from the more general result (1.9.6') which will be proved later.

In view of the above remark we define

$$\epsilon_1(R) = \dim_k H_1(E)$$

where E is the Koszulcomplex generated over R by a minimal set of generators for $\underline{m}$.

The least number of generators for any ideal in R having $\underline{m}$ as radical will be called the <u>dimension</u> of R and will be denoted $\operatorname{Dim} R$. Obviously $\operatorname{Dim} R \leq \epsilon_o(R)$. Whenever $\operatorname{Dim} R = \epsilon_o(R)$ R is called a <u>regular</u> ring.

Standard examples of regular local rings are $k[[x_1,\ldots,x_n]]$, the ring of formal powerseries in n variables over a field k; and localizations of the polynomial ring $k[x_1,\ldots,x_n]$ with respect to prime ideals.

<u>Theorem 1.4.12</u>. (Auslander, Buchsbaum, Serre) A local ring is regular if and only if it has finite global dimension.

Proof. See [No', Chapter 9, Theorem 24].

Theorem 1.4.13. (Auslander, Buchsbaum, Eilenberg) Let R, $\underline{m}$ be a local ring and let E be the Koszul complex generated over R by a minimal set of generators for $\underline{m}$. Then the following statements are equivalent:

(i) R is regular

(ii) $\underline{m}$ is generated by an R-sequence

(iii) $H_i(E) = 0$ for $i > 0$

(iv) $\varepsilon_1(R) = 0$.

Proof. The implication (i) $\Rightarrow$ (ii) follows from Theorem 6 in [No, Section 4.6]. Conversely if $\underline{m}$ is generated by an R-sequence $x_1,\ldots,x_n$, then by (1.4.7) we have $\operatorname{codim} R = n + \operatorname{codim} R/\underline{m} = n$. By [No', Section 9.5] we have $\operatorname{Dim} R \geq \operatorname{codim} R$. Hence $\operatorname{Dim} R \geq n = \varepsilon_0(R)$ so R is regular. The equivalences (ii) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (iv) follows from (1.4.8).

Theorem 1.4.14 (I.S. Cohen). If $R,\underline{m}$ is a local ring which is complete in the $\underline{m}$-adic topology, then there exists a ring isomorphism $R \approx \tilde{R}/\underline{a}$ where $\tilde{R}$ is a regular local

ring and $\underline{a}$ is an ideal contained in the square of the maximal ideal of R.

Proof. Suppose that $R,\underline{m}$ is complete. The existence of a regular local ring R^* and a surjective ring homomorphism $f^*:R^* \to R$ follows from Cohen's Structure Theorem, see [Na, Theorem 31.1]. Put $\underline{a} = \ker f^*$ and let $\underline{m}^*$ be the maximal ideal in R^*. If $\underline{a} \subset (\underline{m}^*)^2$ then we put $R = R^*$, otherwise let $a_1,\ldots,a_s$ be elements in $\underline{a}$ representing a basis for the $R^*/\underline{m}^*$-vectorspace $\underline{a}/\underline{a} \cap (\underline{m}^*)^2$ and put $\tilde{R} = R^*/(a_1,\ldots,a_s)$. s times application of lemma (1.4.16) below shows that $\tilde{R}$ is regular. f^* induces a surjective ring homomorphism $f: \tilde{R} \to R$ whose kernel obviously is contained in $\tilde{\underline{m}}^2$.

Lemma 1.4.15. Let $\tilde{R}$, $\tilde{\underline{m}}$ be a regular local ring. Let $\underline{a}$ be an ideal contained in $\tilde{\underline{m}}^2$. Put $k = \tilde{R}/\tilde{\underline{m}}$ and $R = \tilde{R}/\underline{a}$. We have

$$\varepsilon_1(R) = \dim_k \underline{a}/\tilde{\underline{m}}\,\underline{a}$$

Proof. Let $\tilde{E}$ be the koszul complex generated over $\tilde{R}$ by a minimal set of generators for $\tilde{\underline{m}}$. Since $\underline{a} \subset \tilde{\underline{m}}^2$

the complex $E = \widetilde{E} \underset{R}{\otimes} R$ may be considered as the Koszul complex generated over R by a <u>minimal</u> set of generators for $\underline{m} \subset R$. Tensorizing $\widetilde{E}$ with the exact sequence of $\widetilde{R}$-modules

$$0 \longrightarrow \underline{a} \longrightarrow \widetilde{R} \longrightarrow R \longrightarrow 0$$

we obtain an exact sequence of complexes

$$0 \longrightarrow \underline{a}\,\widetilde{E} \longrightarrow \widetilde{E} \longrightarrow E \longrightarrow 0$$

inducing an exact sequence

$$(3) \quad \begin{array}{ccccccccc} H_1(\widetilde{E}) & \longrightarrow & H_1(E) & \xrightarrow{\partial_1} & H_o(\underline{a}\widetilde{E}) & \longrightarrow & H_o(\widetilde{E}) & \longrightarrow & H_o(E) \longrightarrow 0 \\ & & & & & & \wr\wr & & \wr\wr \\ & & & & & & k & & k \end{array}$$

Since $\widetilde{R}$ is regular, $H_1(\widetilde{E}) = 0$ by (1.4.13), hence ∂_1 is an isomorphism. Moreover $H_o(\underline{a}\,\widetilde{E}) = \underline{a}/\widetilde{\underline{m}}\,\underline{a}$. Hence we have an isomorphism

$$\partial_1 : H_1(E) \longrightarrow \underline{a}/\widetilde{\underline{m}}\,\underline{a}$$

The proof is now complete since $\varepsilon_1(R) = \dim_k H_1(E)$.

<u>Lemma 1.4.16</u>. Let R, $\underline{m}$ be a regular ring and let $x \in \underline{m} \setminus \underline{m}^2$. Then the factor ring R/xR is regular. This follows from Theorem 26 in [Z.S, Ch.VIII, §11].

§5. Local complete intersections and the Tate-Zariski resolution.

In this section R, $\underline{m}$ is a local, noetherian ring. E denotes the Koszul complex generated over R by a minimal set of generators $t_1,\ldots,t_n$ for $\underline{m}$. We suppose that E is furnished with the canonical augmentation $E \to R/\underline{m}$. Put $F = E\langle S_1,\ldots,S_m; dS_i = s_i\rangle$ where $s_1,\ldots,s_m$ represents a minimal set of generators for $H_1(E)$. (If $H_1(E) = 0$ we put $F = E$). As will be shown later (1.9.6') the R-algebra F, as well as the R-algebra E, is an invariant of R. By (1.4.13) in the previous section, E is acyclic if and only if R is regular. It was shown by Assmus that acyclicity of F characterizes another important type of local rings, the local complete intersections.

Definition 1.5.1. A local ring R will be called a local complete intersection if

$$\varepsilon_o(R) - \varepsilon_1(R) = \mathrm{Dim}\ R$$

Proposition 1.5.2. Let $\hat{R}$ denote the $\underline{m}$-adic completion

of R. Then R is a complete intersection if and only if $\hat{R}$ is.

<u>Proof</u>. Consider R as a subring of $\hat{R}$. Let $t_1,\ldots,t_n$ be a minimal set of generators for $\underline{m} \subset R$. $t_1,\ldots,t_n$ is also a minimal set of generators for the maximal ideal $\hat{\underline{m}}$ in $\hat{R}$. Hence $\varepsilon_o(R) = \varepsilon_o(\hat{R})$. Moreover $\mathrm{Dim}\, R = \mathrm{Dim}\, \hat{R}$ and since $\hat{R}$ is R-flat and vector-spaces over $R/\underline{m}$ are complete, we have

$$\begin{aligned} H_1(R_{t:1,\ldots,n}) &\approx H_1(R_{t:1,\ldots,n}) \otimes \hat{R} \approx \\ &\approx H_1(R_{t:1,\ldots,n} \otimes \hat{R}) \\ &\approx H_1(\hat{R}_{t:1,\ldots,n}) \end{aligned}$$

Hence $\varepsilon_1(R) = \varepsilon_1(\hat{R})$.

<u>Proposition 1.5.3</u>. Let $\underline{a}$ be an ideal in a regular local ring $\tilde{R}$. Put $R = \tilde{R}/\underline{a}$. Then R is a local complete intersection if and only if $\underline{a}$ is generated by an $\tilde{R}$-sequence.

<u>Proof</u>. We shall first see that we may assume that $\underline{a}$ is contained in the square of the maximal ideal $\tilde{\underline{m}}$ of

$\tilde{R}$. Indeed, if $\underline{a} \not\subset \tilde{\underline{m}}^2$, let $x \in \underline{a} \setminus \tilde{\underline{m}}^2$. Put $R_1 = \tilde{R}/x\tilde{R}$. By (1.4.16) R_1 is regular. Let $\underline{a}_1$ be the image of $\underline{a}$ in R_1 . We have a canonical map $R_1 \to R$ with kernel $\underline{a}_1$, and $\underline{a}_1$ is generated by an R_1 sequence if and only if $\underline{a}$ is generated by an $\tilde{R}$-sequence. If $\underline{a}_1$ is not contained in the square of the maximal ideal $\underline{m}_1$ of R_1 , we continue the process by reducing R_1 modulo an element $x_1 \in \underline{a}_1 \setminus \underline{m}_1^2$. Since $\tilde{R}$ is noetherian the process stops after a finite number of steps.

We suppose now that $a \subset \tilde{m}^2$ and let $a_1,\ldots,a_\epsilon$ be a minimal set of generators for a . Hence by (1.4.15) we have $\epsilon_1(R) = \epsilon$. Moreover $\epsilon_o(R) = \epsilon_o(\tilde{R}) = \mathrm{Dim}\ \tilde{R}$. Therefore R is a complete intersection if and only if $\mathrm{Dim}\ \tilde{R} - \mathrm{Dim}\ R = \epsilon$. Since $\tilde{R}$ is regular, it is a Macaulay ring; hence the last equation is equivalent to saying that $a_1,\ldots,a_\epsilon$ is an $\tilde{R}$-sequence. Cf. [Z.S, Appendix 6, Theorem 2].

We will prove the already mentioned Assmus characterization and various other homological characterizations of complete intersections later in Chapter three, §5.

For the moment we will only give a description of an R-algebra resolution of $R/\underline{m}$, due to Tate [Ta] and Zariski, in the case where R is a complete intersection which is a factor ring of a regular ring.

<u>Proposition 1.5.4.</u> <u>(Tate,Zariski)</u> Let $\tilde{R}$, $\tilde{\underline{m}}$ be a regular local ring. Let $\tilde{\underline{m}}$ be minimally generated by $\tilde{t}_1,\ldots,\tilde{t}_n$. Let $\underline{a}$ be an ideal contained in $\tilde{\underline{m}}^2$ and minimally generated by $a_i = \sum_{j=1}^{n} \tilde{r}_{ij}\tilde{t}_j$, $i = 1,\ldots,\varepsilon$ where $\tilde{r}_{ij} \in \tilde{R}$. Put $R = \tilde{R}/\underline{a}$. Let r_{ij} and t_j be the image of $\tilde{r}_{ij}$ and $\tilde{t}_j$ in R. Put

$$E = R\langle T_1,\ldots,T_n \; ; \; dT_j = t_j\rangle$$

given the canonical augmentation $E \to R/\underline{m}$. Then the elements $s_i = \sum_{j=1}^{n} r_{ij}T_j$ $(1 \leq i \leq \varepsilon)$ form a minimal set of generators for $H_1(E)$. If R is a local complete intersection then the R-algebra

$$F = E\langle S_1,\ldots,S_\varepsilon ; dS_i = s_i\rangle$$

is acyclic, hence yields a free resolution of the R-module $R/\underline{m}$.

Proof. Since $\underline{a} \subset \tilde{\underline{m}}^2$, $t_1,\ldots,t_n$ is a minimal set of generators for $\underline{m} \subset R$. That $s_1,\ldots,s_\epsilon$, as defined above, is a minimal set of generators can be seen by examining the isomorphism ∂_1 in the proof of (1.4.15). The fact that F is acyclic will follow from (3.5.2).

Remark. (1.5.4) above is essentially a specialization of the more general Theorem 4 in [Ta].

§6. Minimal R-algebra resolutions.

In this section R, $\underline{m}$ is a local ring with residue class field $k = R/\underline{m}$. Unless otherwise stated we make no assumption that R be noetherian.

Definition 1.6.1. A set of generators $\{a_\alpha\}$ for an R-module A will be called minimal if the residue classes of a_α are linearly independent in the $R/\underline{m}$-vectorspace $A/\underline{m}A$. Note that if A is finitely generated then it

always has a minimal set of generators.

<u>Theorem 1.6.2</u>. (Gulliksen) Let X be an augmented R-algebra with augmentation ideal M. Suppose that X/M is a field and that $\tilde{H}_o(X)$ has a minimal set of generators as an R-module. Then there exists an acyclic augmented R-algebra of the form $Y = X\langle \ldots S_i \ldots ; dS_i = s_i \rangle$ satisfying $B(Y) \subset MY$.

<u>Proof</u>. We will use the same construction of Y as in the proof of (1.2.3) and will use the same notation. With the above hypothesis $\tilde{H}_{n-1}(X^{n-1})$ has a minimal set of generators for $n \in \mathbb{N}_1$. Indeed, if $n > 1$ then $\tilde{H}_o(X^{n-1}) = 0$ hence $H_o(X^{n-1}) \approx R/\underline{m}$. Therefore $\tilde{H}_{n-1}(X^{n-1}) = H_{n-1}(X^{n-1})$ is annihilated by $\underline{m}$ and hence is a vectorspace over $R/\underline{m}$. We may therefore assume without loss of generality that the chosen set of cycles $\{s_{n,\alpha}\}_{\alpha \in I_n}$ represents a minimal set of generators for $\tilde{H}_{n-1}(X^{n-1})$ for $n > 0$.

For all $(n,\alpha) \in I$ we define $j^{n,\alpha}$ to be the unique derivation on X^n satisfying

$$j^{n,\alpha}(X^{n-1}) = 0 \ , \ j^{n,\alpha}(S_{n,\beta}) = \delta_{\alpha,\beta} \text{ (Kronecker delta)}$$

and

$$j^{n,\alpha}(S^{(j)}_{n,\beta}) = \delta_{\alpha,\beta} S^{(j-1)}_{n,\beta} \quad \text{for} \quad j \geq 1$$

$$\text{if } \deg S_{n,\beta} \text{ is even.}$$

Such a derivation exists by (1.3.4) since $j^{n,\alpha}(dS_{n,\beta}) = 0 = d\delta_{\alpha,\beta}$. We will show that each of the derivations $j^{n,\alpha}$ can be extended to a derivation on Y which will be denoted by $J^{n,\alpha}$. Using a limit argument it obviously suffices to show the following: Let $\ell \geq n$ and suppose that $j^{n,\alpha}$ has been extended to a derivation j on X^{ℓ}. Then j can be extended further to a derivation j' on $X^{\ell+1}$.

We will now prove this last statement. If $\widetilde{H}_{\ell}(X^{\ell}) = 0$ then $X^{\ell+1} = X^{\ell}$ so we put $j' = j$. Otherwise let $\{s_{\ell+1,\beta}\}_{\beta \in I_{\ell+1}}$ be the chosen set of cycles representing a minimal set of generators for $\widetilde{H}_{\ell}(X^{\ell})$. The existence of the extension j' will follow from (1.3.4) if we can show that

(1) $$j(s_{\ell+1,\beta}) \in B(X^{\ell}) \quad \text{for} \quad \beta \in I_{\ell+1} .$$

Since the degree of j equals $-\deg S_{n,\alpha} = -n < 0$ and

deg $s_{\ell+1,\beta} = \ell$ we have

$$j(s_{\ell+1,\beta}) \in Z_{\ell-n}(X^{\ell})$$

Thus if $\ell > n$ then (1) is satisfied since $H_q(X^{\ell}) = \tilde{H}_q(X^{\ell}) = 0$ for $0 < q < \ell$. Hence we may assume that $\ell = n$. In this case the chosen cycles can be expressed as follows

$$(2) \qquad s_{\ell+1,\beta} = \sum_{\gamma} r_{\beta,\gamma} S_{n,\gamma} + x_{\beta}$$

where $r_{\beta,\gamma} \in R$, $r_{\beta,\gamma} \neq 0$ for only finitely many indices γ, and $x_{\beta} \in X^{n-1}$. Differentiation of (2) yields $0 = \sum_{\gamma} r_{\beta,\alpha} s_{n,\gamma} + dx_{\beta}$. Since $\{s_{n,\gamma}\}_{\gamma \in I_n}$ represents a minimal set of generators for $\tilde{H}_{n-1}(X^{n-1})$ it follows that $r_{\beta,\gamma} \in \underline{m}$ for all $\gamma \in I_n$. We obtain from (2)

$$j(s_{\ell+1,\beta}) = r_{\beta,\alpha}.1 \in \underline{m}\, X_o^n = B_o(X^{\ell})$$

The last equality follows since $\ell = n > 0$.

Still following the notation in the proof of (1.2.3) we consider the direct system over $\underline{I}$ of augmented R-subalgebras of Y $\{X^i\}_{i \in \underline{I}}$. For $i = (n,\alpha) \in I$ we consider the exact sequence

$$0 \longrightarrow X^{i-1} \hookrightarrow X^i \longrightarrow X^i$$

the map $X^i \to X^i$ being the derivation associated with the extension $X^{i-1} \hookrightarrow X^i$, see (1.3.2'). This sequence splits as a sequence of (left) X^{i-1}-modules. Since $X \subset X^{i-1}$ it also splits as a sequence of (left) X-modules. The derivation $X^i \longrightarrow X^i$ is easily seen to be the restriction of $j^{n,\alpha}$. Hence we have a commutative diagram of (left) X-modules

$$\begin{array}{ccccccc} 0 & \longrightarrow & X^{i-1} & \hookrightarrow & X^i & \longrightarrow & X^i \\ & & & & \cap & & \cap \\ & & & & Y & \xrightarrow{J^{n,\alpha}} & Y \end{array} \qquad \text{for } i = (n,\alpha) \in I \tag{3}$$

Consider the functor of (left) X-modules $Y \longmapsto \overline{Y}$ where $\overline{Y} = Y/MY$. X^i is a direct summand (as an X-module) of Y hence the inclusion map $X^i \subset Y$ induces an injection $\overline{X}^i \to \overline{Y}$ and we will identify the elements of $\overline{X}^i$ with their image in $\overline{Y}$. (3) induces a commutative diagram with exact horizontal row

$$\begin{array}{ccccccc} 0 & \longrightarrow & \overline{X}^{i-1} & \longrightarrow & \overline{X}^i & \longrightarrow & X^i \\ & & & & \cap & & \cap \\ & & & & \overline{Y} & \xrightarrow{\overline{J}^{n,\alpha}} & \overline{Y} \end{array} \qquad \text{for } i = (n,\alpha) \in I \tag{4}$$

Let $K = \bigcap_{(n,\alpha)\in I} \ker \overline{J}^{n,\alpha}$. We will now show that $K \subset \overline{X}$. Recall that $Y = \bigcup_{i\in \underline{I}} X^i$ where $\underline{I}$ is well-ordered. With the identifications we have made we have $\overline{Y} = \bigcup_{i\in \underline{I}} \overline{X}^i$. Let $y \in K$ and let i be the least element in $\underline{I}$ such that $y \in \overline{X}^i$. Obviously i is not a limit element in $\underline{I}$. Hence either $i = 0$ or $i \in I$. If $i \in I$ then it follows from (4) and the fact that $\overline{J}^{n,\alpha}(y) = 0$ that $y \in \overline{X}^{i-1}$, which contradicts the minimality of i . Hence $y \in \overline{X}^o = \overline{X}$.

By induction on q we are going to show that

$$B_q(\overline{Y}) = 0 \tag{5}$$

For $q = 0$ this is clear since $B_o(Y) = M_o \subset MY$. Let $r \geq 1$ and assume that (5) has been established for $q < r$. For every $(n,\alpha) \in I$, $\overline{J}^{n,\alpha}$ is of negative degree and commutes with the differential on $\overline{Y}$ induced from Y . Hence

$$\overline{J}^{n,\alpha}(B_r(\overline{Y})) \subset \coprod_{q<r} B_q(\overline{Y}) = 0$$

Hence $B_r(\overline{Y}) \subset B_r(\overline{Y}) \cap K \subset \overline{X}_r = 0$.

Since $B(\overline{Y}) = 0$ it follows that $B(Y) \subset MY$.

Definition 1.6.3. A resolution Y of an R-module A will be called minimal if $B(Y) \subset \underline{m}Y$.

Corollary 1.6.4. Let R be a local ring. Suppose that there exists a minimal set of generators for the maximal ideal $\underline{m}$. Then there exists an augmented R-algebra of the form

$$Y = R\langle \ldots S_i \ldots ; dS_i = s_i \rangle$$

which is a minimal free resolution of the R-module $R/\underline{m}$.

Proof. Consider the augmented R-algebra $R \to R/\underline{m}$ which has augmentation ideal $\underline{m}$. By the previous theorem there exists an acyclic free extension Y

$$Y = R\langle \ldots S_i \ldots ; dS_i = s_i \rangle$$

with the property $B(Y) \subset \underline{m}Y$.

§7. Divided power algebras.

In this section R is a general commutative ring (with unit element).

Definition 1.7.1. A graded algebra X (always strictly skew-commutative with unit element 1) is said to have a system of divided powers if to every element $x \in X$ of even positive degree there is associated a sequence of elements $x^{(k)} \in X$ $(k = 0,1,2,\ldots)$ satisfying (1) to (5) below:

(1) $x^{(0)} = 1$, $x^{(1)} = x$, $\deg x^{(k)} = k.\deg x$

(2) $x^{(h)}x^{(k)} = (h,k)x^{(h+k)}$ where $(h,k) = \dfrac{(h+k)!}{h!k!}$

(3) $(x+y)^{(k)} = \sum\limits_{i+j=k} x^{(i)}y^{(j)}$

(4) for $k \geq 2$

$$(xy)^{(k)} = \begin{cases} = 0 & \text{if } \deg x \text{ and } \deg y \text{ are odd} \\ = x^k y^{(k)} & \text{if } \deg x \text{ is even and} \\ & \deg y \text{ is even and positive.} \end{cases}$$

(5) $(x^{(h)})^{(k)} = [h,k]x^{(hk)}$ for $k \geq 0$, $h \geq 1$

where $[h,k] = \dfrac{(hk)!}{k!(h!)^k}$.

Note that $[h,k]$ is an integer since $[h,0] = 1$ and $[h,k] = [h,k-1]((k-1)h,h-1)$ for $k \geq 1$.

A graded algebra furnished with a system of divided powers will be called a <u>divided power algebra</u>. If X is a divided power algebra, $C(X)$ will denote the submodule generated by all elements xx' where x and x' are elements of X of positive degree, and all elements $y^{(k)}$ where $y \in X$ is of even positive degree and $k > 1$. $C(X)$ will be called the <u>module of decomposable elements in</u> X.

<u>Lemma 1.7.2</u>. Let R be a ring containing the field of rational numbers. Let X be a graded algebra over R. Then X has a unique system of divided powers.

<u>Proof</u>. The uniqueness of such a system is clear since it follows from (1) and (2) that $x^k = k!x^{(k)}$ for $k \geq 0$ and x of even positive degree, hence

$$x^{(k)} = \frac{1}{k!} x^k \quad . \tag{6}$$

On the other hand, taking (6) as a definition, one easily

shows that (1) to (5) are satisfied.

Definition 1.7.3. A homomorphism of divided power algebras $f: X \to Y$ is a homomorphism of the underlying graded algebras which is compatible with the divided powers, i.e.

$$(7) \qquad f(x^{(k)}) = f(x)^{(k)} \quad \text{for} \quad k \geq 0 \quad \text{and} \quad x \in X$$

of even positive degree.

A set N of homogeneous elements in a divided power algebra X will be called a set of generators for X if every element in X is a linear combination of products of elements which are in N or are divided powers of elements in N.

An ideal I in a divided power algebra X is an ideal in the underlying algebra which is closed under divided powers, i.e.

$$x \in I,\ k \geq 1\ ,\ \deg x \text{ even positive} \Rightarrow x^{(k)} \in I$$

Lemma 1.7.4. If I is an ideal in a divided power algebra X, then there is a unique divided power algebra structure

on the residue class algebra X/I such that the canonical map $X \to X/I$ is a homomorphism.

<u>Proof</u>. The uniqueness is clear. Since $X \to X/I$ is an algebra homomorphism we need only define divided powers in X/I. Define $\bar{x}^{(k)} = \overline{x^{(k)}}$ for $k \geq 0$ and $\bar{x} \in X/I$ of even positive degree. To show that this is well defined let $\bar{x} = \bar{y}$, thus $x = y+a$ for some $a \in I$. It suffices to assume that $k \geq 1$. By (3) we have

$$x^{(k)} = y^{(k)} + y^{(k-1)}a + \ldots + a^{(k)}$$

hence $x^{(k)} - y^{(k)} \in I$.

Since $X \to X/I$ is an algebra homomorphism, the identities (1) to (5) are also valid in X/I.

<u>Lemma 1.7.5</u>. Let X and Y be divided power algebras and let N be a set of generators for X. Let $f:X \to Y$ be a homomorphism of the underlying algebras. Then f is compatible with the divided powers if for $k \geq 0$, $x \in N$ and $\deg x$ even positive, we have

$$(7') \qquad f(x^{(k)}) = f(x)^{(k)}$$

Proof. Let X^o and X^1 temporarily denote the set of elements of X of even positive and odd degree respectively. Let N^o be the set of divided powers $y^{(k)}$ $(k \geq 0)$ of elements $y \in N \cap X^o$. Let N' be the set consisting of the unit element and all products of elements in $N \cap X^1$ involving an even number of factors. Then an element $x \in X^o$ can be written

$$x = \sum_i r_i a_i b_{i1} \cdots b_{in_i}$$

where $r_i \in R$, $a_i \in N^1$ and $b_{ij} \in N^o$ for $1 \leq j \leq n_i$. Now suppose that (7') is satisfied by the elements in $N \cap X^o$. It follows from (4) respectively (5) that $r_i a_i$ respectively b_{ij} satisfies (7') whenever they are of positive degree. It follows from (3) and (4) that x also satisfies (7').

Proposition 1.7.6. Let $\{S_i\}_{i \in I}$ be a set of variables and let $X = \bigotimes_{i \in I} R\langle S_i \rangle$ be the (possibly infinite) tensor product of the divided polynomial algebras $R\langle S_i \rangle$. Then X has a unique system of divided powers in which the divided powers of S_i are the natural ones.

Proof. If there exists such a system of divided powers on X it is clearly unique. Indeed, it follows from the axioms that the divided powers of an element $x \in X$ can be expressed algebraically in terms of the natural divided powers of the variables S_i.

Concerning the existence, consider the polynomial ring $\tilde{R} = \mathbb{Z}[x]$ where $x = (x_r)_{r \in R}$. Let $\bar{\mathbb{Z}}$ be the prime ring of R. The canonical map $\mathbb{Z} \longrightarrow \bar{\mathbb{Z}}$ extends to a surjective ring homomorphism $f_o : \tilde{R} \to R$ uniquely determined by $f_o(x_r) = r$. For each $i \in I$ we choose a new variable $\tilde{S}_i$ such that $\deg \tilde{S}_i = \deg S_i$, and define $\tilde{X} = \bigotimes_{i \in I} \tilde{R}\langle \tilde{S}_i \rangle$. f_o has an extension to a surjective homomorphism of graded algebras, $f : \tilde{X} \to X$ uniquely determined by $f(\tilde{S}_i) = S_i$ for $i \in I$ and $f(\tilde{S}_i^{(j)}) = S_i^{(j)}$ for $j \geq 2$ if $\deg S_i$ is even.

Let $\underline{a} = \ker f_o$. Then $\underline{a}\tilde{X} = \ker f$. It follows from (4) that $\ker f$ is closed under divided powers. Hence by (1.7.4) it suffices to show the existence of divided powers on $\tilde{X}$.

Let K be the field of quotients of the integral domain $\tilde{R}$. Since $\tilde{X}$ is an $\tilde{R}$-free module the embedding $\tilde{R} \to K$ induces an embedding $\tilde{X} \to \tilde{X} \otimes_{\tilde{R}} K$ of graded algebras

over $\tilde{R}$. We will consider $\tilde{X}$ as a subalgebra of $\tilde{X} \underset{\tilde{R}}{\otimes} K$ by identifying elements $y \in \tilde{X}$ with their image $y \otimes 1$.

Since $\tilde{R}$ has characteristic zero, K contains the rationals. $\tilde{X} \underset{\tilde{R}}{\otimes} K$ is a graded algebra over K, hence by (1.7.2) it has a unique system of divided powers. It therefore suffices to show that $\tilde{X}$ is closed under divided powers. In fact it suffices to show that

(8) there exist $y_* \in \tilde{X}$ such that $y^k = k!y_*$

for $k \geq 0$ and $y \in \tilde{X}$ of even positive degree.

If y_1, y_2 are elements of common even positive degree in $\tilde{X}$ satisfying (8), then it follows from (3) that y_1+y_2 satisfies (8). If y_1 is of even degree, and if y_2 is of even positive degree and satisfies (8), then so does y_1y_2, by (4). If y is a product of elements of odd degree then y obviously satisfies (8). Hence since $\tilde{X}$ is generated as an algebra by the natural divided powers of the variables $\tilde{S}_i$ $(i \in I)$, it suffices to show that (8) is satisfied by the elements $S_i^{(h)}$ for $h \geq 1$, deg S_i even.

It follows from the definition of natural divided powers and induction on k that

$$(S_i^{(h)})^k = \frac{(hk)!}{(h!)^k} S_i^{(hk)}$$

Since $\frac{(hk)!}{(h!)^k} = k!\,[h,k]$ and $[h,k]$ is an integer by the note in (1.7.1), the proof is now complete.

<u>Definition 1.7.7</u>. A divided power algebra X over R will be called <u>free</u> if it is the tensorproduct of divided polynomial algebras $R\langle S_i\rangle$ $(i \in I)$ and the divided powers of the variables S_i are the natural ones. The set $\{S_i\}_{i\in I}$ will be called a basis for X.

<u>Lemma 1.7.8</u>. Let $\{x_i\}_{i\in I}$ be a set of homogeneous elements in a divided power algebra X. Let $X' = \bigotimes_{i\in I} R\langle S_i\rangle$ where the S_i are variables satisfying $\deg S_i = \deg x_i$ for $i \in I$. Then there exists a unique homomorphism $f: X' \to X$ satisfying $f(S_i) = x_i$.

<u>Proof</u>. The uniqueness is clear since $\{S_i\}_{i\in I}$ is a set of generators for X'. Concerning the existence, define maps $f_i: R\langle S_i\rangle \to X$ by $f_i(S_i) = x_i$, and $f_i(S_i^{(h)}) = x_i^{(h)}$ for $h \geq 1$ if $\deg S_i$ is even. The maps f_i are algebra homomorphisms inducing an algebra

homomorphism $f:X' \to X$. Since $\{S_i\}_{i\in I}$ is a set of generators for the divided power algebra X' , it follows from (1.7.5) that f is compatible with the divided powers.

<u>Corollary 1.7.9</u>. Any divided power algebra is a homomorphic image of a free divided power algebra.

<u>Proposition 1.7.10</u>. Let X,Y be divided power algebras over R . Then the graded algebra $X \underset{R}{\otimes} Y$ has a system of divided powers such that the canonical maps

$$X \longrightarrow X \underset{R}{\otimes} Y \longleftarrow Y$$

becomes homomorphisms of divided power algebras. Such a system is unique.

<u>Proof</u>. It follows from (3) and (4) that the divided powers on $X \otimes Y$ can be expressed algebraically in terms of the divided powers of the algebra generators $x \otimes 1$, $1 \otimes y$ for $X \otimes Y$. The uniqueness of the divided powers on $X \otimes Y$ follows therefore if the above maps are compatible with the divided powers.

We will now prove the existence.

By (1.7.9) we can find free divided power algebras X', Y' and surjective homomorphisms $f: X' \to X$, $g: Y' \to X$. We have a canonical commutative diagram of graded algebras

$$(9) \qquad \begin{tikzcd} X' \arrow[r] \arrow[d, "f"] & X' \otimes Y' \arrow[d, "f\otimes g"] & Y' \arrow[l] \arrow[d, "g"] \\ X \arrow[r] & X \otimes Y & Y \arrow[l] \end{tikzcd}$$

If X' and Y' have basis $\{S_i\}_{i\in I}$ and $\{T_j\}_{j\in J}$ respectively, then $X' \otimes Y'$ is a free divided power algebra with basis $\{S_i\}_{i\in I} \cup \{T_j\}_{j\in J}$ (disjoint union). We have identified the elements $x \in X'$, $y \in Y'$ with $x \otimes 1$ and $1 \otimes y$ respectively. $\ker f$ and $\ker g$ are closed under divided powers.

The elements of $\ker f \otimes g$ are finite sums $\sum_i x_i y_i$ where for each i either $x_i \in \ker f$ or $y_i \in \ker g$. It follows, using (3) and (4) that $\ker f \otimes g$ is closed under divided powers. It follows from (1.7.4) that the divided powers on $X' \otimes Y'$ pass to the algebra $X \otimes Y$. One easily checks that the upper horizontal maps in (9) are compatible with the divided powers. Hence so are the

lower horizontal maps. The proof is now complete.

§8. Augmented (R)-algebras.

If nothing else is said, R will be a general commutative ring (with unit element).

Definition 1.8.1. Let X' be a graded sub algebra of an R-algebra X. A system of divided powers on X' is said to be compatible with the differential d on X if for all x of even positive degree in X we have

$$(1) \qquad d(x^{(k)}) = (dx)x^{(k-1)} \quad \text{for } k \geq 1$$

Note that (1) is automatically satisfied for $k = 1$. An R-algebra X is called an (R)-algebra if it is furnished with a system of divided powers which is compatible with the differential. Note that if $\overline{R}$ is a factor ring of R, then the R-algebra $\overline{R}$ is also an (R)-algebra

with trivial divided powers.

If X,Y are (R)-algebras and $f: X \to Y$ is a homomorphism of the underlying R-algebras which preserves divided powers, then f is briefly called an (R)-<u>homomorphism</u>.

If an augmented R-algebra is also an (R)-algebra it will be called an <u>augmented</u> (R)-<u>algebra</u>. Note that the augmentation map is an (R)-homomorphism.

Whenever we are given factor rings $\overline{R}$, $\overline{\overline{R}}$ of R and a surjective R-linear ring homomorphism $\overline{R} \to \overline{\overline{R}}$ we may interpret $\overline{R} \to \overline{\overline{R}}$ as an augmented (R)-algebra with underlying (R)-algebra $\overline{R}$ and augmentation $\overline{R} \to \overline{\overline{R}}$.

If X and Y are (R)-algebras and $X \subset Y$, then X is said to be an (R)-<u>subalgebra</u> of Y if the inclusion map is an (R)-homomorphism.

A subset N of an (R)-algebra X is called a <u>set of generators</u> for X if it is a set of generators for the underlying divided power algebra. See (1.7.3).

Let $f:X \to Y$ be an (R)-homomorphism. For $q \in \mathbb{N}_o$ let F_qY be the (R)-subalgebra of Y generated by $f(X)$ and all the elements of Y of degreee $\leq q$. $\{F_qY\}_{q\geq 0}$

is called the <u>filtration associated with the extension</u> $f:X \to Y$.

<u>Lemma 1.8.2</u>. Let X,Y be divided power algebras over R . Suppose there is a differential d on $X \otimes_R Y$ which makes it to an R-algebra. Let $\overline{X}$ and $\overline{Y}$ be the images of the homomorphisms $X \to X \otimes Y$ and $Y \to X \otimes Y$ respectively. If the systems of divided powers on $\overline{X}$ and $\overline{Y}$ are compatible with the differential d , then so is the system of divided powers on $X \otimes Y$.

<u>Proof</u>. Let $x \in \overline{X}$ and $y \in \overline{Y}$. If x and y are of odd degree then xy satisfies (1) since, by §7 (4), $(xy)^{(k)} = 0$ for $k \geq 2$, and $(d(xy))xy = (dx)yxy \pm$ $\pm\, x(dy)xy = 0$ since $x^2 = 0 = y^2$. If x and y are of even degree and $\deg y > 0$ then it follows from §7 (4), using the relation $yy^{(k-1)} = ky^{(k)}$, that

$$d((xy)^{(k)}) = d(xy)(xy)^{(k-1)}$$

Finally if $x_1,\ldots,x_n$ are elements in $X \otimes Y$ of common even degree and satisfying (1), then it follows from §7 (3) that $\sum_{i=1}^{n} x_i$ satisfies (1) .

Corollary 1.8.3. Let X and Y be (R)-algebras. The algebra $X \underset{R}{\otimes} Y$ has a unique (R)-algebra structure for which the canonical maps

$$X \longrightarrow X \otimes Y \longleftarrow Y$$

are (R)-homomorphisms.

Proof. This follows from (1.1.3), (1.7.10) and the previous lemma.

Proposition 1.8.3'. Let X be an (R)-algebra and $R \to R'$ a ring homomorphism. Then the R'-algebra $X \underset{R}{\otimes} R'$ has a unique system of divided powers such that the canonical map $X \to X \underset{R}{\otimes} R'$ preserves divided powers. This yields an (R')-algebra structure on $X \underset{R}{\otimes} R'$.

Proof. The uniqueness is almost clear. We restrict ourselves to showing the existence of an (R')-algebra structure on $X \underset{R}{\otimes} R'$.

By (1.7.8) X is a homomorphic image of a free divided power algebra Y. Obviously $Y \underset{R}{\otimes} R'$ is a free divided

is called the <u>filtration associated with the extension</u> $f:X \to Y$.

<u>Lemma 1.8.2</u>. Let X,Y be divided power algebras over R . Suppose there is a differential d on $X \otimes_R Y$ which makes it to an R-algebra. Let $\overline{X}$ and $\overline{Y}$ be the images of the homomorphisms $X \to X \otimes Y$ and $Y \to X \otimes Y$ respectively. If the systems of divided powers on $\overline{X}$ and $\overline{Y}$ are compatible with the differential d , then so is the system of divided powers on $X \otimes Y$.

<u>Proof</u>. Let $x \in \overline{X}$ and $y \in \overline{Y}$. If x and y are of odd degree then xy satisfies (1) since, by §7 (4), $(xy)^{(k)} = 0$ for $k \geq 2$, and $(d(xy))xy = (dx)yxy \pm$ $\pm\, x(dy)xy = 0$ since $x^2 = 0 = y^2$. If x and y are of even degree and $\deg y > 0$ then it follows from §7 (4), using the relation $yy^{(k-1)} = ky^{(k)}$, that

$$d((xy)^{(k)}) = d(xy)(xy)^{(k-1)}$$

Finally if $x_1,\ldots,x_n$ are elements in $X \otimes Y$ of common even degree and satisfying (1), then it follows from §7 (3) that $\sum_{i=1}^{n} x_i$ satisfies (1) .

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Proposition 1.8.3'. Let X be an (R)-algebra and $R \to R'$ a ring homomorphism. Then the R'-algebra $X \underset{R}{\otimes} R'$ has a unique system of divided powers such that the canonical map $X \to X \underset{R}{\otimes} R'$ preserves divided powers. This yields an (R')-algebra structure on $X \underset{R}{\otimes} R'$.

Proof. The uniqueness is almost clear. We restrict ourselves to showing the existence of an (R')-algebra structure on $X \underset{R}{\otimes} R'$.

By (1.7.8) X is a homomorphic image of a free divided power algebra Y. Obviously $Y \underset{R}{\otimes} R'$ is a free divided

power algebra over R'. The existence of divided powers on $X \otimes_R R'$ follows since it is a homomorphic image of $Y \otimes_R R'$, see proof of (1.7.10). Using the arguments in the proof of (1.8.2) one easily shows that the differential on $X \otimes_R R'$ is compatible with the divided powers.

<u>Theorem 1.8.4</u>. Let X be an augmented (R)-algebra and let Y be a free extension of the underlying augmented R-algebra of X with basis $\{S_i\}_{i \in N}$; then Y has an augmented (R)-algebra structure respecting the natural divided powers of S_i $(i \in N)$ such that the inclusion map $X \to Y$ is a homomorphism of augmented (R)-algebras.

<u>Proof</u>. Using an induction argument, we may assume without loss of generality that Y is a free extension of X with a single basis element S. Thus $Y = X\langle S; dS = s\rangle$ where s is a cycle in the augmentation ideal of X. We need only extend the system of divided powers on X to a system on Y which is compatible with the differential on Y. We have an equality of graded algebras $Y = X \otimes_R R\langle S\rangle$. Since X and Y are in fact X_o-algebras we may assume without loss of generality that

$X_o = R$, so that $R\langle S\rangle$ is embedded in the obvious way as a graded sub algebra of $X \underset{R}{\otimes} R\langle S\rangle$. By (1.7.6) $R\langle S\rangle$ is a divided power algebra respecting the natural divided powers of S , and by (1.7.10) there is a unique system of divided powers on $X \underset{R}{\otimes} R\langle S\rangle$ compatible with the inclusion maps

$$X \longrightarrow X \otimes R\langle S\rangle \longleftarrow R\langle S\rangle$$

It remains to show that it is also compatible with the differential on $X \otimes R\langle S\rangle$. By (1.8.2) it suffices to show that (1) is satisfied for elements $x \in R\langle S\rangle$ of even positive degree.

Let $x = rS^{(h)}$, $r \in R$ and $h \geq 1$. Let $k \geq 1$. We have

$$d((rS^{(h)})^{(k)}) = r^k[h,k]dS.S^{(hk-1)}$$

$$= r^k dS[h,k-1]((k-1)h,h-1)S^{(hk-1)}$$

$$= r^k dS.S^{(h-1)}[h,k-1]S^{(h(k-1))}$$

$$= d(rS^{(h)})(rS^{(h)})^{(k-1)}$$

The proof is now complete.

Definition 1.8.5. Let $f: X \to X'$ be a homomorphism of augmented (R)-algebras, and let $\{x_i\}_{i \in I}$ be a set of homogeneous elements in X'. For $i \in I$ consider the divided polynomial algebra $R\langle S_i \rangle$ where $\deg S_i = \deg x_i$ and let $f^i = R\langle S_i \rangle \to X'$ be the homomorphism of divided power algebras sending S_i to x_i. f and the maps f^i $(i \in I)$ induce a homomorphism of divided power algebras

$$u: X \otimes \left(\bigotimes_{i \in I} R\langle S_i \rangle \right) \to X'$$

We say that f is a free extension with basis $\{x_i\}_{i \in I}$ if u is an isomorphism.

It is convenient to introduce the following notation: If $f: X \to X'$ is a free extension with basis $\{x_i\}_{i \in I}$, then for $q > 0$ put

$$N_q(X'|X) = \{x_i | i \in I \text{ and } \deg x_i = q\}$$

$$n_q(X'|X) = \{dx_i | x_i \in N_q(Y|X)\}$$

The map f induces an increasing filtration $\{F_q X'\}_{q \geq 0}$ on X' which is independent of the basis $\{x_i\}_{i \in I}$. Put

$F_oX' = f(X)$ and for $q > 0$ let F_qX' be the (R)-subalgebra of X' generated by $f(X)$ and all elements of X' of degree $\leq q$. It is easy to see that this filtration is the filtration associated with the extension $f(X) \subset X'$ as defined in §2.

Note that while the sets $N_q(X'|X)$ depend on the chosen basis, their cardinality $\# N_q(X'|X)$ is independent of such a choice. In fact for $q \geq 1$ $(F_qX'/F_{q-1}X')_q$ is a free X_o-module of rank $\# N_q(X'|X)$.

<u>Lemma 1.8.6</u>. Given homomorphisms of augmented (R)-algebras

$$\begin{tikzcd} X' & Y' \\ X \arrow[u, "g"] \arrow[r, "f"] & Y \arrow[u, "h"'] \end{tikzcd}$$

such that Y' is acyclic and g is a free extension. Then there exists a homomorphism $f': X' \to Y'$ of augmented (R)-algebras making the above diagram commutative.

<u>Proof</u>. By using an induction argument we see that it suffices to show the lemma in the case where g is

a free extension with a single basis element S. Thus, identifying X with its image in X', X' has the form $X' = X\langle S; dS = s\rangle$.

Since Y' is acyclic we can choose an element $y \in Y'$ such that $dy = f(s)$. If $\deg S$ is even X' is the a free X-module with basis $\{S^{(k)}\}_{k\geq 0}$. Define an X-linear map $f': X' \to Y'$ by $f'(S^{(k)}) = y^{(k)}$ for $k \geq 0$. One easily checks that f' is an R-homomorphism. f' is obviously compatible with the augmentations, and by (1.7.5) it is compatible with the divided powers.

The treatment in the case where $\deg S$ is odd is similar and will be omitted.

<u>Lemma 1.8.7</u>. Let R, $\underline{m}$ be a local ring. Let

$$\begin{tikzcd} X' \arrow[r, "f'"] & Y' \\ X \arrow[u, "g"] \arrow[r, "f"] & Y \arrow[u, "h"'] \end{tikzcd}$$

be a commutative diagram of (R)-algebras of finite type, where g and h are free extensions and f is an isomorphism. Then f' is an isomorphism if and only if the induced map

$$\bar{f}: X'/(C(X')+\underline{m}X'+X) \to Y'/(C(Y')+\underline{m}Y'+Y)$$

is an isomorphism.

<u>Proof</u>. If f' is an isomorphism then clearly $\bar{f}$ is an isomorphism. We suppose now that $\bar{f}$ is an isomorphism. We observe that for $q \geq 1$ we have

$$\dim_{R/\underline{m}}(X'/C(X')+\underline{m}X'+X)_q = \# N_q(X'|X)$$

and similarly for the extension $Y \to Y'$. Hence

$$\# N_q(X'|X) = \# N_q(Y'|Y) .$$

Since X_p and Y_p are isomorphic as R-modules for all $p \geq 0$ we conclude that also X'_q and Y'_q are isomorphic. Moreover X'_q is finitely generated as an R-module. Since R is noetherian it follows that $f'_q: X'_q \to Y'_q$ is an isomorphism if it is surjective.

We show now by induction on q that f'_q is surjective. For $q = 0$ f'_q is obviously surjective. Now let $q > 0$ assume that f'_p is surjective for $p < q$. Since $\bar{f}$ is surjective we have

$$Y'_q \subset f'(X'_q) + C(Y')_q + \underline{m}Y'_q + Y_q$$

Since f is surjective, we have $Y_q \subset f'(X'_q)$, and since f preserves divided powers it follows from the induction hypothesis that $C(Y')_q \subset f'(X'_q)$. Hence $Y'_q \subset f'(X'_q)$ follows by Nakayama's lemma.

<u>Corollary 1.8.8</u>. Let $X \subset X'$ be a free extension of augmented (R)-algebras over a local ring R, $\underline{m}$. Assume that X' is of finite type. Let $\{x_i\}_{i \in I}$ be a set of homogeneous elements of X'. The following are equivalent:

(i) $\{x_i\}_{i \in I}$ is a basis for the extension $X \subset X'$

(ii) $\{x_i\}_{i \in I}$ represents a basis for the graded vectorspace $X'/(C(X') + \underline{m}X' + X)$.

<u>Proof</u>. The implication (i) $\Rightarrow$ (ii) is almost obvious. We restrict ourselves to showing (ii) $\Rightarrow$ (i). Let $\{S_j\}_{j \in J}$ be a basis for the extension $X \subset X'$. Because of the first implication there is no loss of generality in assuming that $I = J$ and that $\deg S_i = \deg x_i$ for $i \in I$. The identity map on X now extends to an algebra homomorphism $u: X' \to X'$ which is compatible with the divided powers and given by $u(S_i) = x_i$. Changing the differentials on X

and X' to the trivial differentials, we obtain a commutative diagram of augmented (R)-algebras

$$\begin{array}{ccc} X' & \xrightarrow{u} & X' \\ \cup & & \cup \\ X & \xrightarrow{1_X} & X \end{array}$$

Assuming now that (ii) is true, it follows from the previous lemma that u is an isomorphism, hence that $\{x_i\}_{i\in I}$ is a basis for $X \subset X'$.

§9. Acyclic closure of augmented (R)-algebras.

In this section R will always denote a local noetherian ring with maximal ideal $\underline{m}$. All (R)-algebras will be assumed to be of finite type, i.e. their homogeneous components are finitely generated R-modules.

Definition 1.9.1. Let X be an augmented (R)-algebra. (X^*,h), or briefly X^*, will be called an acyclic

closure of X if

(i) $h: X \to X^*$ is a free acyclic extension, see (1.8.5)

(ii) $B(X^*) \subset C(X^*) + \underline{m}X^* + X$.

If X is an augmented (R)-algebra, X^* will in the Chapters one and three, always denote an acyclic closure of X and $\{F_q X^*\}_{q \geq 0}$ will denote the filtration associated with the extension $X \to X^*$. See (1.8.1). (In Chapter two A^* denotes the dual of a vectorspace or a Hopf algebra A).

Proposition 1.9.2. Let X be an augmented (R)-algebra, let $h: X \to Y$ be a free acyclic extension with basis $\{S_i\}_{i \in I}$ and let $\{F_n Y\}_{n \geq 0}$ be the filtration associated with the extension. Then (Y,h) is an acyclic closure of X if and only if for all $q > 0$ $n_q(Y|X)$ represents a minimal set of generators for $\widetilde{H}_{q-1}(F_{q-1}Y)$.

Proof. For $q > 0$ put $N_q = N_q(Y|X)$ and $n_q = n_q(Y|X)$. Since Y is acyclic it suffices to show that (a) and (b) below are equivalent:

(a) $Z_q(Y) \subset C(Y)_q + \underline{m}Y_q + X_q$

(b) $n_q(Y|X)$ represents a minimal set of generators for $\tilde{H}_{q-1}(F_{q-1}Y)$.

Since $F_qY = F_{q-1}Y\langle\ldots S\ldots\rangle$ where S runs through N_q , (a) is equivalent to

(a') $Z_q(Y) \subset \sum_{S\in N_q} \underline{m}S + (F_{q-1}Y)_q$ (direct sum).

Proof of the implication (b) $\Rightarrow$ (a'): Let $z = \sum_S r_S S + c$ be a cycle in $Z_q(Y)$ where $c \in F_{q-1}Y$, $S \in N_q$ and $r_S \in R$. Differentiation yields

$$\sum_S r_S dS = -dc \in B_{q-1}(F_{q-1}Y)$$

Now (b) implies that $r_S \in \underline{m}$ for all $S \in N_q$.

Proof of the implication (a') $\Rightarrow$ (b): For $S \in N_q$ let $\overline{dS}$ denote the homology class of dS in $\tilde{H}_{q-1}(F_{q-1}Y)$. Suppose that $\sum_S r_S\overline{dS} = 0$ where $r_S \in R$. Then there exists an element $c \in (F_{q-1}Y)_q$ such that

$$\sum_S r_S dS = dc .$$

Hence

$$\sum_S r_S S - c \in Z_q(Y) .$$

It follows from (a') that $r_S \in \underline{m}$ for $S \in N_q$.

<u>Proposition 1.9.3</u>. Any augmented (R)-algebra X has an acyclic closure.

<u>Proof</u>. We will obtain an acyclic closure X^* as a direct limit of a direct system of augmented (R)-algebras $\{F^q\}_{q=0,1,2,\ldots}$ of finite type and inclusion maps. Put $F^o = X$. Let $q \geq 0$. Suppose that a free extension $X \subset F^q$ has been defined. If $\tilde{H}_q(F^q) = 0$ then put $F^{q+1} = F^q$. Otherwise let $\{s_i\}_{i=1,\ldots,m}$ represent a minimal set of generators for $\tilde{H}_q(F^q)$ and put

$$F^{q+1} = F^q\langle S_1,\ldots,S_m; dS_i = s_i\rangle$$

Put $X^* = \lim\limits_{\substack{\rightarrow \\ q}} F^q$ and let $h: X \to X^*$ be induced by the inclusion maps $X \subset F^q$. One checks easily that X^* is of finite type.

$h: X \to X^*$ is a free extension which is acyclic by (1.2.2).

Let $\{F_q X^*\}_{q\geq o}$ be the filtration associated with

the extension $h: X \to X^*$. We have

$$F^q = F_q X^* \quad \text{for} \quad q \geq 0 .$$

Hence for $q \geq 0$ the killed cycles of degree q represent a minimal set of generators for $\tilde{H}_q(F_q X^*)$. It follows from (1.9.2) that (X^*, h) is an acyclic closure of X.

<u>Examples</u>.

(a) Let X be the canonical augmented (R)-algebra $R \to R/\underline{m}$. Let $X \to Y$ be a free acyclic extension. Then Y is an acyclic closure of X if and only if Y is a minimal resolution. Indeed, if Y is a minimal resolution, i.e. $B(Y) \subset mY$ then obviously (1.9.1 (ii)) is satisfied. Conversely, if Y is an acyclic closure then by (1.9.2) Y is as in (1.6.4), hence it is a minimal resolution.

(b) Let $T_1, \ldots, T_n$ be variables of degree 1. Put $X = R\langle T_1, \ldots, T_n; dT_i = 0\rangle$. We have $X_o = R$. Give X the augmentation $X \to R$ extending the identity map on R. Then $X^* = X\langle S_1, \ldots, S_n; dS_i = T_i\rangle$ is an acyclic closure of

X . Clearly $B(X^*) \subset C(X^*) + X$ so we need only show that X^* is acyclic. For $1 \leq i \leq n$ consider the (R)-subalgebra $X^i = R\langle T_i, S_i; dS_i = T_i\rangle$. The cycles of positive degree in X^i have the form $rTS^{(k)}$ where $r \in R$, $k \geq 0$. We have $rTS^{(k)} = d(rS^{(k+1)})$. Therefore X^i is acyclic. The inclusion maps $X^i \to X$ induce an isomorphism of complexes $\bigotimes_{i=1}^{n} X^i \approx X$. Since $B_o(X^i) = 0$ and X^i is a free R-module we conclude that the tensor product $\bigotimes_{i=1}^{n} X^i$ and hence X is acyclic, see (3.2.2).

<u>Lemma 1.9.4</u>. Let X^* be an acyclic closure of X . Then, for all $q > 0$, the differential d on X^* induces an isomorphism of $R/\underline{m}$-modules

$$(F_qX^*/(C(F_qX^*) + \underline{m}F_qX^* + F_{q-1}X^*))_q \to \widetilde{H}_{q-1}(F_{q-1}X^*)\otimes R/\underline{m}$$

<u>Proof</u>. The above map is well defined since

$$d((F_qX^*)_q) \subset \widetilde{Z}_{q-1}(F_{q-1}X^*) \quad \text{and} \quad C(F_qX^*)_q \subset F_{q-1}X^* .$$

Choosing a basis for the extension $X \subset X^*$, we may write $F_qX^* = F_{q-1}X^*\langle S_1,\ldots,S_m; dS_i = s_i\rangle$ where $\{S_1,\ldots,S_m\} = N_q(X^*|X)$. Hence the domain of the map is the free

$R/\underline{m}$-module generated by the residue classes of $S_1,\ldots,S_m$. Since, by (1.9.2), $s_1,\ldots,s_m$ represent a minimal set of generators for $\tilde{H}_{q-1}(F_{q-1}X^*)$ it follows that the above map induced by d is an isomorphism.

<u>Theorem 1.9.5</u> (Gulliksen) ("Uniqueness" of acyclic closure). Let $f: X \to Y$ be an isomorphism of augmented (R)-algebras. Let X^* and Y^* be acyclic closures of X and Y respectively. Then f can be extended, in a non-canonical way, to an (R)-homomorphism $f^*: X^* \to Y^*$. Moreover any (R)-homomorphism $X^* \to Y^*$ extending f is an isomorphism.

<u>Proof</u>. The existence of f^* follows from (1.8.6). Suppose now that f^* is such an extension of f. Let U_qX and W_qX denote the factor complexes

$$U_qX = F_qX^*/(C(F_qX^*) + \underline{m}F_qX^* + X) \quad \text{for} \quad q \geq 0$$

$$W_qX = F_qX^*/(C(F_qX^*) + \underline{m}F_qX^* + F_{q-1}X^*) \quad \text{for} \quad q > 0$$

U_qY and W_qY are similarly defined. Note that $(W_qX)_p = (W_qY)_p = 0$ for $p \neq q$. For $q \geq 0$ f^* induces an (R)-algebra homomorphism $f^q: F_qX^* \to F_qY^*$. We will

show by induction on q that f^q is an isomorphism. For $q = 0$ we have $f^q = f$, hence an isomorphism. Let $q \geq 1$. Consider the commutative diagrams

$$\text{(I)} \qquad \begin{array}{ccccccccc} 0 & \longrightarrow & U_{q-1}X & \longrightarrow & U_qX & \longrightarrow & W_qX & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \\ 0 & \longrightarrow & U_{q-1}Y & \longrightarrow & U_qY & \longrightarrow & W_qY & \longrightarrow & 0 \end{array}$$

where the vertical maps are induced by f^* and the left horizontal maps are induced by the inclusions $F_{q-1}X^* \subset F_qX^*$ and $F_{q-1}Y^* \subset F_qY^*$. (I) has exact rows.

$$\text{(II)} \qquad \begin{array}{ccc} W_qX = (W_qX)_q & \text{———} & \widetilde{H}_{q-1}(F_{q-1}X^*) \otimes R/\underline{m} \\ \downarrow{\scriptstyle\gamma} & & \downarrow{\scriptstyle\eta} \\ W_qY = (W_qY)_q & \text{———} & \widetilde{H}_{q-1}(F_{q-1}Y^*) \otimes R/\underline{m} \end{array}$$

The horizontal maps are the isomorphisms induced by the differential, see (1.9.4), and η is induced by f^{q-1}.

Now assume that f^{q-1} is an isomorphism. It follows that α and η hence also γ are isomorphisms. From (I) we see that β is an isomorphism, and by (1.8.7) f^q is an isomorphism.

Corollary 1.9.6. Let X be an augmented (R)-algebra and $f: X \to Y$ a free, acyclic extension. Then f has the factorization

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "h"'] & & Y \\ & X^* \arrow[ur, "g"'] & \end{tikzcd}$$

where (X^*, h) is an acyclic closure of X and g is a free extension. In particular

$$\# N_q(Y|X) = \# N_q(Y|X^*) + \# N_q(X^*|X) \quad \text{for} \quad q > 0 .$$

Proof. Let $X^* = X\langle \ldots S_i \ldots ; dS_i = s_i \rangle$ be an acyclic closure of X. By (1.8.6) we have a commutative diagram

$$\begin{tikzcd} Y \arrow[r, dashed, shift left, "f'"] & X^* \arrow[l, dashed, shift left] \\ X \arrow[u, "f"] \arrow[r, equal] & X \arrow[u, "h"'] \end{tikzcd}$$

inducing maps

$$Y/(C(Y) + \underline{m}Y + X) \underset{\bar{g}}{\overset{\bar{f}'}{\rightleftarrows}} X^*/(C(X^*) + \underline{m}X^* + X)$$

$f'g$ extends the identity map on X hence by (1.9.5) it is an isomorphism. It follows that $\bar{f}'\bar{g}$ is an isomorphism, hence $\bar{g}$ is injective. g is injective and will be used to identify X^* with $g(X^*)$. Since $\{S_i\}_i$ represents a basis for $X^*/(C(X^*) + \underline{m}X^* + X)$ it can be extended to a set, say $\{S_i\}_i \cup \{T_j\}_j$ (disjoint union), representing a basis for $Y/(C(Y) + \underline{m}Y + X)$. It follows from (1.8.8) tha Y is a free extension of X^*. We have $Y = X\langle\ldots S_i\ldots\rangle\langle\ldots T_j\ldots\rangle$. The cardinality formula above is obviously valid.

<u>Corollary 1.9.6'</u>. Let X be the augmented (R)-algebra $R \to k$ and let X^* be an acyclic closure of X. Then the (R)-algebras $F_q X^*$ $(q \geq 0)$ are invariants of R.

<u>Proof</u>. Let f be the identity map on X and let X^* and Y^* be acyclic closures of X. By the previous theorem f can be extended to an (R)-isomorphism $f^*: X^* \to Y^*$ inducing (R)-isomorphisms $f^q: F_q X^* \to F_q Y^*$ for $q = 0,1,2,\ldots$

Proposition 1.9.7. Let $f:X \to Y$ be a homomorphism of augmented (R)-algebras inducing an isomorphism $\tilde{f}_*:\tilde{H}(X) \to \tilde{H}(Y)$. Let $X^* = X\langle\ldots S_i\ldots\rangle$ be an acyclic closure of X. Then there exists an acyclic closure $Y^* = Y\langle\ldots T_i\ldots\rangle$ of Y and an (R)-algebra homomorphism $f^*: X^* \to Y^*$ extending f such that $f^*(S_i) = T_i$ for all i, and inducing isomorphisms

(i) $H(F_qX^*) \xrightarrow{\approx} H(F_qY^*)$ for $q \geq 0$.

Proof. We will construct an inductive system of augmented (R)-algebras $\{Y^q\}_{q\geq 0}$ and inclusion maps and a compatible family of homomorphisms of augmented (R)-algebras

$$f^q: F_qX^* \to Y^q$$

inducing an isomorphism

(i') $H(F_qX^*) \xrightarrow{\approx} H(Y^q)$

This will be done by induction on q. Put $Y^o = Y$ and $f^o = f$. Now let $q \geq 0$ and suppose that Y^q and f^q have been constructed. If $F_{q+1}X^* = F_qX^*$ put $Y^{q+1} = Y^q$

and $f^{q+1} = f^q$. Otherwise we may write

$$F_{q+1}X^* = F_qX^*\langle S_1,\ldots,S_m; dS_i = s_i\rangle$$

where $S_1,\ldots,S_m$ are the variables of degree $q+1$. Define $Y^{q+1} = Y^q\langle T_1,\ldots,T_m; dT_i = f^q(s_i)\rangle$. f^q can now be extended to an (R)-homomorphism $f^{q+1}:F_{q+1}X^* \to Y^{q+1}$ defined by $f^{q+1}(S_i) = T_i$ for $1 \leq i \leq m$. Repeated application of (1.3.5) shows that f^{q+1} induces an isomorphism $\widetilde{H}(F_{q+1}X^*) \to \widetilde{H}(Y^{q+1})$.

Put $Y^* = \lim\limits_{\substack{\to \\ q\geq 0}} Y^q$. Y^* is obviously a free acyclic extension of Y . We have $Y^q = F_qY^*$ for $q \geq 0$. Let $q \geq 0$. Since X^* is an acyclic closure of X it follows by (1.9.2) that $s_1,\ldots,s_m$ represent a minimal set of generators for $\widetilde{H}_q(F_qX^*)$. By (i') $f(s_1),\ldots,f(s_m)$ represent a minimal set of generators for $\widetilde{H}_q(Y^q) = \widetilde{H}_q(F_qY^*)$. It follows by (1.9.2) that Y^* is an acyclic closure of Y .

The family $\{f^q\}_{q\geq 0}$ induces an (R)-algebra homomorphism $f^*:X^* \to Y^*$ extending f and satisfying $f^*(S_i) = T_i$ for all i . The map (i) is the isomorphism (i').

<u>Proposition (1.9.8)</u> Let X be an augmented (R)-algebra and let (X^*, h) be an acyclic closure. Suppose that X is free as an R-module. Let $f : R \to R'$ be a homomorphism of local rings and suppose that R' is R-flat and $f(\underline{m}) \subset \underline{m}'$. Then $(X^* \otimes_R R', h \otimes 1_{R'})$ is an acyclic closure of the augmented (R)-algebra $X \otimes_R R'$. We have a canonical isomorphism of (R')-algebras

(i) $H(F_q(X^* \otimes R')) \approx H(F_q X^*) \otimes R' \quad (q \geq 0)$

In particular if $R' = \hat{R}$ is the $\underline{m}$-adic completion of R and $H_o(X^*) = R/\underline{m}$ then (i) yields an isomorphism of $R/\underline{m}$-algebras

(ii) $H(F_q X^* \otimes \hat{R})) \approx H(F_q X^*) \qquad (q \geq 1)$.

<u>Proof</u>. h induces an injection $h \otimes 1_{R'}$ which is a homomorphism of augmented (R')-algebras. We identify $X \otimes R'$ with its image in $X^* \otimes R'$. Choosing a basis for the extension $X \to X^*$ we may write

$$X^* = X < \dots S_i \dots ; dS_i = s_i >$$

We have

$$X^* \otimes R' = X \otimes R' < \dots S_i \otimes 1 \dots ;$$

$$d(S_i \otimes 1) = s_i \otimes 1 >$$

Hence since R' is R-flat, $h \otimes 1_{R'}$ is an acyclic free extension. Moreover (after appropriate identifications)

$$B(X^* \otimes R') = B(X^*) \otimes R' \subset (C(X^*) + \underline{m}X^* + X) \otimes R'$$

$$\subset C(X^* \otimes R') + \underline{m}'(X^* \otimes R') + X \otimes R'$$

since $C(X^*) \otimes R' \subset C(X^* \otimes R')$ by (2.4'), and $f(\underline{m}) \subset \underline{m}'$. Hence $(X^* \otimes R', h \otimes 1_{R'})$ is an acyclic closure of $X \otimes R'$.

(i) follows from the equation

$$(F_q X^*) \otimes R' = F_q(X^* \otimes R')$$

and the canonical R'-algebra isomorphism

$$H(F_q X^*) \underset{R}{\otimes} R' \approx H((F_q X^*) \underset{R}{\otimes} R')$$

Concerning (ii) we remark that $H(F_q X^*)$ is a module over the ring $H_o(F_q X^*)$. Suppose that $H_o(X^*) = R/\underline{m}$. For $q \geq 1$ we have

$$H_o(F_q X^*) = H_o(X^*) = R/\underline{m} \quad ,$$

so $H(F_q X^*)$ is annihilated by $\underline{m}$, hence the map

$$\varphi : H(F_q X^*) \to H(F_q X^*) \otimes_R \hat{R}$$

defined by $\varphi(\eta) = \eta \otimes 1$ for $\eta \in H(F_q X^*)$ is an isomorphism of $R/\underline{m}$ -algebras. Now (ii) follows from (i) .

CHAPTER TWO

THE STRUCTURE AND DUALITY OF $\mathrm{Tor}^R(k,k)$ AND $\mathrm{Ext}_R(k,k)$

§1. Algebras, co-algebras and Hopf algebras

Let k be a field. An algebra over k is a graded vector space A over k together with morphisms of graded k-spaces

$$\varphi : A \otimes A \to A \quad \text{and} \quad \eta : k \to A$$

(in this section, all tensor products are over k) such that the diagrams

a)

$$\begin{tikzcd} A \otimes A \otimes A \arrow[r, "1\otimes\varphi"] \arrow[d, "1\otimes\varphi"'] & A \otimes A \arrow[d, "\varphi"] \\ A\otimes A \arrow[r] & A \end{tikzcd}$$

and b)

$$\begin{tikzcd}
k \otimes A \arrow[rr, "\eta\otimes 1"] \arrow[dr, "\approx"'] & & A \otimes A \arrow[dl, "\varphi"] \\
& A & \\
A \otimes k \arrow[ur, "\approx"] \arrow[rr, "1\otimes\eta"] & & A \otimes A \arrow[ul, "\varphi"']
\end{tikzcd}$$

are commutative. The morphism φ is called the multiplication of the algebra A and η is called the unit of A. Diagram a) says that the multiplication is associative and diagram b) says that A has a unit.

If A and B are graded k-spaces, the twisting morphism $T : A \otimes B \to A \otimes B$ is defined by

$$T(a \otimes b) = (-1)^{pq} b \otimes a$$

for a in A_p and b in B_q. The algebra A is commutative if the diagram

$$\begin{tikzcd}
A \otimes A \arrow[dr, "\varphi"] \arrow[dd, "T"] & \\
& A \\
A \otimes A \arrow[ur, "\varphi"'] &
\end{tikzcd}$$

commutes. (Note that A is actually skew-commutative.)

Reversing all the arrows in the above definitions, we get the notion of a co-algebra. Namely, <u>co-algebra</u> over k is a graded k-space A together with morphisms of graded k-spaces

$$\Delta : A \to A \otimes A \quad \text{and} \quad \epsilon : A \to k$$

such that the diagrams

a)

$$\begin{tikzcd}
A \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & A \otimes A \arrow[d, "\Delta\otimes 1"] \\
A \otimes A \arrow[r, "1\otimes\Delta"] & A \otimes A \otimes A
\end{tikzcd}$$

and b)

$$\begin{tikzcd}
A \otimes A \arrow[rr, "\epsilon\otimes 1"] & & k \otimes A \\
& A \arrow[ul, "\Delta"] \arrow[ur, "\approx"'] \arrow[dl, "\Delta"'] \arrow[dr, "\approx"] & \\
A \otimes A \arrow[rr, "1\otimes\epsilon"] & & A \otimes k
\end{tikzcd}$$

are commutative. The morphism Δ is called the <u>comultiplication</u> of the co-algebra and ε is called the <u>co-unit</u> of A.

Diagram a) says the co-algebra A is associative and diagram b) says it has a co-unit. A co-algebra is <u>commutative</u> if the diagram

$$\begin{tikzcd} & A \otimes A \arrow[d, "T"] \\ A \arrow[ur, "\Delta"] \arrow[r, "\Delta"'] & A \otimes A \end{tikzcd}$$

commutes.

An algebra A is <u>connected</u> if the unit $\eta : k \to A_o$ is an isomorphism. Similarly, a co-algebra is connected if the co-unit $\varepsilon : A_o \to k$ is an isomorphism. From now on, we assume that all algebras and co-algebras are connected and we identify k with A_o.

Let A be an algebra and $I(A)$ the elements of positive degree. The <u>indecomposable elements</u> of A are defined to be the elements of the graded vector space $Q(A) = I(A)/(I(A))^2$. If A is a co-algebra, the <u>primi-</u>

tive elements of A are those x of positive degree such that $\Delta(x) = x \otimes 1 + 1 \otimes x$. The set of primitive elements in A is denoted by $P(A)$.

A morphism of algebras $f : A \to B$ is a morphism of graded k-spaces such that the diagrams

$$\begin{tikzcd} A \otimes A \arrow[r, "\varphi_A"] \arrow[d, "f \otimes f"'] & A \arrow[d, "f"] \\ B \otimes B \arrow[r, "\varphi_B"] & B \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} k \arrow[r, "\eta_A"] \arrow[d, "\approx"'] & A \arrow[d, "f"] \\ k \arrow[r, "\eta_B"] & B \end{tikzcd}$$

are commutative. When the arrows are reversed and φ_A, φ_B, η_A, η_B are replaced by Δ_A, Δ_B, ε_A, ε_B one gets the definition for a morphism of co-algebras.

Finally, a Hopf algebra over k is a graded k-space A together with morphisms of graded k-spaces

$$\varphi : A \otimes A \to A \quad , \quad \eta : k \to A$$

$$\Delta : A \to A \otimes A \quad , \quad \varepsilon : A \to k$$

such that

1) A with φ and η is an algebra over k .

2) A with Δ and ε is a co-algebra over k .

3) $\varepsilon\eta$ = identity on k

4) The diagram

$$\begin{tikzcd}
A \otimes A \arrow[r, "\varphi"] \arrow[d, "\Delta\otimes\Delta"] & A \arrow[r, "\Delta"] & A \otimes A \\
A \otimes A \otimes A \otimes A \arrow[rr, "A\otimes T\otimes A"] & & A \otimes A \otimes A \otimes A \arrow[u, "\varphi\otimes\varphi"']
\end{tikzcd}$$

is commutative.

Condition 3), together with the fact that A is connected, gives that for $x \in A_n$

$$\Delta(x) = x \otimes 1 + \ldots + 1 \otimes x \quad .$$

$A \otimes A$ is, in a natural way, both an algebra and a co-algebra. Its multiplication is defined by

$$(A \otimes A) \otimes (A \otimes A) \xrightarrow{1\otimes T\otimes 1} (A \otimes A) \otimes (A \otimes A) \xrightarrow{\varphi\otimes\varphi} A \otimes A \ .$$

and it's comultiplication by reversing the arrows and replacing φ by Δ . Thus condition 4) says that Δ is a morphism of algebras $A \to A \otimes A$ and φ is a

morphism of co-algebras $A \otimes A \to A$.

We will need a number of results on Hopf algebras. These will be stated without proof. For their proofs, the reader is referred to [M.M.] . The proposition numbers refer to the numbering in [M.M.] .

A morphism of algebras $f : A \to B$ induces in a natural way a morphism $Q(f) : Q(A) \to Q(B)$, and a morphism of co-algebras induces a morphism $P(f) : P(A) \to P(B)$.

<u>Proposition 3.8.</u> If $f : A \to B$ is a morphism of algebras, then f is an epimorphism if and only if $Q(f) : Q(A) \to Q(B)$ is an epimorphism,

<u>Proposition 3.9.</u> If $f : A \to B$ is a morphism of co-algebras, then f is a monomorphism if and only if $P(f) : P(A) \to P(B)$ is a monomorphism.

Call a Hopf algebra commutative if it is commutative as an algebra.

<u>Proposition 4.17.</u> If A is a commutative Hopf algebra

over a field k of characteristic zero, then the natural morphism $P(A) \to Q(A)$ is a monomorphism.

<u>Proposition 4.20.</u> If A is a commutative Hopf algebra over a field k of characteristic $p \neq 0$, and $x^p = 0$ for all x in A, then the natural morphism $P(A) \to Q(A)$ is a monomorphism.

If A is a graded k-space, let A^* denote the dual space. k is a field so if A and B are two graded k-spaces of finite type, then

$$(A \otimes B)^* = A^* \otimes B^* \quad .$$

<u>Proposition 4.8.</u> If A together with φ and Δ is a Hopf algebra whose underlying k-space is of finite type, then A^* together with Δ^* and φ^* is a Hopf algebra.

A^* is called the dual Hopf algebra to A. We also have

<u>Proposition 3.10.</u> $P(A^*) = (Q(A))^*$, i.e. the primitive elements of A^* are precisely those elements of A^* which vanish on $(I(A))^2$.

If A is a Hopf algebra, let $G(A, A)$ denote the vector space endomorphisms of A which are the identity morphism on $k = A_0$. Two elements f and g in $G(A, A)$ may be composed as follows:

$$A \xrightarrow{\Delta} A \otimes A \xrightarrow{f \otimes g} A \otimes A \xrightarrow{\varpi} A$$

<u>Proposition 8.2.</u> $G(A, A)$ is a group under this composition.

The identity of this group is the map $\eta\varepsilon : A \to A$. The inverse of the identity morphism is called the <u>conjugation</u> of A and is denoted by C . C may be defined inductively as follows:

Assume that C has been defined in degrees less than n , and let x be in A_n . Write

$$\Delta x = x \otimes 1 + \Sigma\, x_i' \otimes x_i'' + 1 \otimes x \quad .$$

Then

$$C(x) = -x - \Sigma\, x_i'\, C(x_i'')$$

<u>Propositions 8.6, 8.7, 8.8.</u> C is an anti-isomorphism of Hopf algebras and $C^2 = 1$, i.e. the following diagrams commute:

$$\begin{tikzcd}
A \arrow[r, "C"] \arrow[d, "\Delta"'] & A \arrow[r, "\Delta"] & A \otimes A \\
A \otimes A \arrow[rr, "T"] & & A \otimes A \arrow[u, "C \otimes C"']
\end{tikzcd}
\qquad
\begin{tikzcd}
A \otimes A \arrow[r, "\varphi"] \arrow[d, "C \otimes C"] & A \arrow[r, "C"] & A \\
A \otimes A \arrow[rr, "T"] & & A \otimes A \arrow[u, "\varphi"']
\end{tikzcd}$$

§2. $\mathrm{Tor}^R(k, k)$ <u>as a Hopf algebra</u>.

Let R be a local ring, $\underline{m}$ its maximal ideal, and k the residue field $R/\underline{m}$. The multiplication on $\mathrm{Tor}^R(k, k)$ is the standard external product [Ma, p. 221] and is defined as follows. Let X be a free resolution of k. Then $X \otimes X$ is a free complex over k, so by the comparison theorem [Ma, p. 87], there is a chain

transformation $\varphi : X \otimes X \to X$ lifting the identity on k and any two such transformations are homotopic. Then

$$\varphi \otimes 1 : X \otimes k \otimes X \otimes k \approx X \otimes X \otimes k \to X \otimes k$$

induces

$$(\varphi \otimes 1)_* : H(X \otimes k \otimes X \otimes k) \to H(X \otimes k)$$

If u, v are cycles in $X \otimes k$, then $u \otimes v$ is a cycle in $X \otimes k \otimes X \otimes k$. This defines

$$\alpha : H(X \otimes k) \otimes H(X \otimes k) \to H(X \otimes k \otimes X \otimes k) .$$

Since $\mathrm{Tor}^R(k, k) = H(X \otimes k)$, the product in $\mathrm{Tor}^R(k, k)$ is now defined to be $(\varphi \otimes 1)_*\alpha$. By the comparison theorem, the product is independent of the choice of X and φ.

We already know (1.6.4) and (1.8.4) that k has a free resolution which is an (R)-algebra. Thus $\mathrm{Tor}^R(k, k)$ inherits the properties of this algebra; i.e. $\mathrm{Tor}^R(k, k)$ is a skew-commutative graded algebra over k whose unit is the unit of k. Also, every element of even degree has a sequence of divided powers.

Now using any free resolution X of k, the augmentation $\epsilon : X \to k$ induces isomorphisms

$$(1 \otimes \epsilon)_* : H(X \otimes X) \to H(X \otimes k) = \mathrm{Tor}^R(k, k) \quad .$$

and

$$(\epsilon \otimes 1)_* : H(X \otimes X) \to H(X \otimes k)$$

The twisting morphism $T : X \otimes X \to X \otimes X$ then induces

$$C : \mathrm{Tor}^R(k, k) \xrightarrow{(1\otimes\epsilon)_*^{-1}} H(X \otimes X) \xrightarrow{T_*} H(X \otimes X)$$

$$\xrightarrow{(1\otimes\epsilon)_*} \mathrm{Tor}^R(k, k)$$

Anticipating its role in the Hopf algebra, C is called the <u>conjugation</u> on $\mathrm{Tor}^R(k, k)$. (To see what C means explicitly, let X be a minimal resolution of k . Since $dX_n \subset \underline{m}\, X_{n-1}$, $d \otimes 1 : X_n \otimes k \to X_{n-1} \otimes k$ is zero so $\mathrm{Tor}^R(k, k) = X \otimes k$. Now if x is an element of X_n , denote its image in $X_n \otimes k$ by $\bar{x}$. Then $x \otimes 1$ "begins" a cycle in $X \otimes X$. At the end of this cycle is an element $1 \otimes y$ with y in X_n . The image of y in $X_n \otimes k$ is $C(\bar{x})$.)

Clearly, C^2 = identity on $\mathrm{Tor}^R(k, k)$.

The comultiplication on $\mathrm{Tor}^R(k, k)$ was first introduced by Assmus [As] and may be described as follows.

First, we note that since k is a field, the Künneth formula [Ma, p. 166] yields that

$$\alpha : H(X \otimes k) \otimes H(X \otimes k) \longrightarrow H(X \otimes k \otimes X \otimes k)$$

is an isomorphism.

Hence, the most natural candidate for a comultiplication is the map

$$h : \mathrm{Tor}^R(k, k) = H(X \otimes k) \xrightarrow{(1\otimes\varepsilon)_*^{-1}} H(X \otimes X) \longrightarrow$$

$$H(X \otimes k \otimes X \otimes k) \xrightarrow{\alpha^{-1}} H(X \otimes k) \otimes H(X \otimes k)$$

$$= \mathrm{Tor}^R(k, k) \otimes \mathrm{Tor}^R(k, k) \quad .$$

However, for $u \in \mathrm{Tor}_n^R(k, k)$

$$h(u) = u \otimes 1 + \ldots + 1 \otimes C(u)$$

and according to axiom 3 for Hopf algebras, we are supposed to get $1 \otimes u$ at the end. This is easily corrected. We define $\Delta = (1 \otimes C)h$. We now have to show that $\mathrm{Tor}^R(k, k)$ together with φ and Δ is a Hopf algebra whose conjugation is C .

<u>Lemma 2.2.1.</u> C is an anti-isomorphism of algebras and co-algebras and Δ is a morphism of algebras.

<u>Proof.</u> Since $\mathrm{Tor}^R(k, k)$ with φ is skew-commutative we need to show

1) $C\,\varphi = (\varphi \otimes \varphi)\, C$

2) $\Delta\, C = T(C \otimes C)\, \Delta$

3) $\Delta\, \varphi = (\varphi \otimes \varphi)\, (1 \otimes T \otimes 1)\, (\Delta \otimes \Delta)$

Let X be a minimal algebra resolution of k and let x in X_n, y in X_m represent $\bar{x}$ and $\bar{y}$ in $X_n \otimes k$ and $X_m \otimes k$ respectively. Then there exist cycles $z_1 = x \otimes 1 + \dots + 1 \otimes x'$, $z_2 = y \otimes 1 + \dots$ $\dots + 1 \otimes y'$ in $X \otimes X$ with x' in X_n, y' in X_m. Since the differential on X is a derivation, $z_1\, z_2$ is a cycle in $X \otimes X$. This shows both that $C(\bar{x}\,\bar{y}) =$ $= C(\bar{x})\, C(\bar{y})$ and that $h(\bar{x}\,\bar{y}) = h(\bar{x})\, h(\bar{y})$. Then

$$\Delta\,(\bar{x}\,\bar{y}) = (1 \otimes C)\, h\,(\bar{x}\,\bar{y}) = (1 \otimes C)\,(h(\bar{x})\, h\,(\bar{y}))$$

$$= (1 \otimes C)\, h\,(\bar{x}) \cdot (1 \otimes C)\, h\,(\bar{y}) = \Delta\,(\bar{x})\, \Delta\,(\bar{y})$$

1) and 3) follow. To prove 2), we observe that

$T(C \otimes 1) = (1 \otimes C)\, T$ and $hC = Th$.

Then $\Delta\, C = (1 \otimes C)\, hC = (1 \otimes C)\, Th = T(C \otimes 1)\, h$

$$= T(C \otimes C)\,(1 \otimes C)\, h = T(C \otimes C)\, \Delta$$

This proves the lemma.

Lemma 2.2.2. C is the conjugation of $\mathrm{Tor}^R(k, k)$

Proof. Let X be a minimal algebra resolution of k, and let $x \otimes 1 + \Sigma x_i \otimes y_i + 1 \otimes y$ be a cycle in $X \otimes X$ (the x_i are all assumed to have positive degree with $x \in X_n$, $y \in X_n$). Then since the differential on X is a derivation, $x + \Sigma x_i y_i + y$ is a cycle in X_n and therefore represents the zero class in $\mathrm{Tor}_n^R(k, k)$. Hence $C(\bar{x}) = -\bar{x} - \Sigma \bar{x}_i \bar{y}_i$. Since $\Delta(\bar{x}) = \bar{x} \otimes 1 + \Sigma \bar{x}_i \otimes C(\bar{y}_i) + 1 \otimes \bar{x}$, this agrees with the inductive definition of C given in section 1.

§3. The Duality of $\mathrm{Ext}_R(k, k)$ and $\mathrm{Tor}^R(k, k)$

The Hopf algebra conditions on the unit and co-unit are obviously satisfied so the only thing left is to prove that Δ is associative. This is achieved by studying the dual algebra.

For simplicity, we use the notation $A = \mathrm{Tor}^R(k, k)$. The dual space $A_n^* = \mathrm{Hom}_k(\mathrm{Tor}_n^R(k, k), k) = \mathrm{Hom}_k(X_n \otimes k, k) \approx \mathrm{Hom}_R(X_n, k) = \mathrm{Ext}_R^n(k, k)$. The comultiplication then

induces

$$\Delta_* : \mathrm{Ext}_R(k, k) \otimes \mathrm{Ext}_R(k, k) \to \mathrm{Ext}_R(k, k) \quad .$$

We prove that Δ is associative by showing that Δ_* is associative.

We will need Yoneda's characterization of $\mathrm{Ext}_R(k, k)$ as congruence classes of long exact sequences beginning and ending with k . For details see [Ma, Ch. III] .

Consider all exact sequences of R-modules

$$0 \to k \to B_{n-1} \to \cdots \to B_1 \to B_0 \to k \to 0 \quad .$$

Such a sequence is called an n-<u>fold extension</u> of k by k . Two n-fold extensions E and E' are called <u>congruent</u> if there exists a finite set of n-fold extensions $E_1, \ldots, E_r$ and chain transformations $f_1, \ldots, f_{r-1}$ with

$$E = E_1 \xrightarrow{f_1} E_2 \xleftarrow{f_2} E_3 \cdots \xleftarrow{f_{r-1}} E_r = E'$$

where each f_i is the identity map on both k and k and the f_i go in alternate directions. $\mathrm{Ext}_R^n(k, k)$ is then identified with the set of congruence classes of n-fold extensions of k by k in the following manner.

Let X be any free resolution of k. Then $\mathrm{Ext}^n(k, k) = H^n(\mathrm{Hom}(X, k))$. Setting $L_n = \mathrm{Ker}\,(X_n \to X_{n-1})$, an element in $\mathrm{Ext}^n(k, k)$ may be represented by an R-linear map $f : X_n \to k$ which vanishes on L_n, and thus defines a map $L_{n-1} \to k$, also called f. We then get the following commutative diagram.

$$\begin{tikzcd}
0 \arrow[r] & L_{n-1} \arrow[r] \arrow[d, "f"'] & X_{n-1} \arrow[r] \arrow[d, "g"'] & X_{n-2} \arrow[r] \arrow[d, "1"'] & \cdots \arrow[r] & X_1 \arrow[r] \arrow[d, "1"'] & R \arrow[r] \arrow[d, "1"'] & k \arrow[r] \arrow[d, "1"'] & 0 \\
 & k \arrow[r] & G \arrow[r] & X_{n-2} \arrow[r] & \cdots \arrow[r] & X_1 \arrow[r] & R \arrow[r] & k \arrow[r] & 0
\end{tikzcd}$$

where $G = (k \oplus X_{n-1})/N$, N being the submodule of $k \oplus X_{n-1}$ generated by all elements $(-f(a), a)$ for a in L_{n-1}. $g : X_{n-1} \to G$ is defined by $g(b) = (0, b) + N$; k is included in G by sending c in k into $(c, 0) + N$, and G maps into X_{n-2} by sending $(c, b) + N$ into db. (d is the differential on X). It can be shown that this makes

the bottom sequence exact and thus this sequence is an n-fold extension of k by k. The class of f in $\mathrm{Ext}_R^n(k, k)$ then corresponds to the congruence class of the bottom sequence.

An n-fold extension E of k by k and an m-fold extension E' of k by k may be composed to give an $(n + m)$-fold extension EE' of k by k by splicing E' onto the right end of E and suppressing the k in the middle. The congruence class of the composite depends only on the congruence classes of E and E'. This is called the <u>Yoneda composite</u> and defines an associative product on $\mathrm{Ext}_R(k, k)$. We will show that Δ_* is the Yoneda opposite.

If $E : 0 \to A \to B \to C \to 0$ is a short exact sequence of R-modules and G is any R-module, E defines a map $\underline{E} : \mathrm{Tor}_m^R(G, C) \to \mathrm{Tor}_{m-1}^R(G, A)$ given by

$$\underline{E}(x) = (-1)^m \partial_E(x)$$

for x in $\mathrm{Tor}_m^R(G, C)$. Here ∂_E is the connecting homomorphism associated with the sequence E. $\underline{E}$ depends only on the congruence class of E.

Now, an n-fold extension

$$E : 0 \to k \to B_{n-1} \xrightarrow{d_{n-1}} \cdots \to B_1 \xrightarrow{d_1} B_0 \to k$$

is the Yoneda composite of n short exact sequences

$E_n \quad : 0 \to k \to B_{n-1} \to C_{n-1} \to 0$

$E_{n-1} : \quad 0 \to C_{n-1} \to B_{n-2} \to C_{n-2} \to 0$

$E_2 \quad : 0 \to C_2 \to B_1 \to C_1 \to 0$

$E_1 \quad : 0 \to C_1 \to B_0 \to k \to 0$

where C_i is the image of d_i , $i = 1, \ldots, n-1$.

So E defines $\underline{E} = \underline{E}_n \, \underline{E}_{n-1} \cdots \underline{E}_1 : \mathrm{Tor}_m^R(k, k) \to \mathrm{Tor}_{m-n}^R(k, k)$. Again $\underline{E}$ depends only on the congruence class of E and $\underline{EE'} = \underline{E}\,\underline{E}'$

<u>Lemma 2.3.1.</u> Let f in $\mathrm{Ext}_R^n(k,k)$ correspond to an n-fold extension E . Then $\underline{E} = (1 \otimes f)\, h$.

Proof. Let X be a minimal resolution of k , and identify f with the map it induces $L_{n-1} \to k$. Then we have a commutative diagram

$$\begin{array}{ccccccccccccccc}
E' : & 0 & \to & L_{n-1} & \to & X_{n-1} & \to & X_{n-2} & \to & \cdots & \to & X_1 & \to & R & \to & k & \to & 0 \\
 & & & \downarrow f & & \downarrow g & & \downarrow 1 & & & & \downarrow 1 & & \downarrow 1 & & \downarrow 1 & & \\
E : & 0 & \to & k & \to & A & \to & X_{n-2} & \to & \cdots & \to & X_1 & \to & R & \to & k & \to & 0
\end{array}$$

with both rows exact and the bottom row represents the congruence class associated with f.

Now, let x in X_m represent $\bar{x}$ in $X_m \otimes k = \mathrm{Tor}_m^R(k,k)$, and choose a cycle

$$z = x \otimes 1 + x_{m-1,1} + \dots + x_{1,m-1} + 1 \otimes y$$

in $X \otimes X$ with $X_{m-1,i}$ in $X_{m-1} \otimes X_i$ and y in X_m. Since z is a cycle in $X \otimes X$,

$$dx \otimes 1 + (-1)^{m-1}(1 \otimes d)x_{m-1,1} = 0$$

$$(d \otimes 1)x_{m-1,1} + (-1)^{m-2}(1 \otimes d)\, x_{m-2,2} = 0$$

$$(d \otimes 1)x_{1,m-1} + (-1)^{o}(1 \otimes dy) = 0$$

Decompose E and E' into short exact sequences $E = E_n E_{n-1} \dots E_1$, $E' = E_n' E_{n-1} \dots E_1$. It is now evident that $\underline{E}_1(\bar{x})$ = class of $(1 \otimes d)\, X_{m-1,1}$ in $H_{m-1}(X \otimes \underline{m}) = \mathrm{Tor}_{m-1}^R(k,\underline{m})$. Similarly, $\underline{E}_2 : \mathrm{Tor}_{m-1}^R(k,\underline{m}) \to \mathrm{Tor}_{m-2}^R(k,L_1)$ sends the class of $(1 \otimes d)\,(X_{m-1,1})$ into the class of $(1 \otimes d)\, X_{m-2,2}$. Finally, we get that $\underline{E}'(\bar{x})$ = class of $(1 \otimes d)$ $(X_{m-n,n})$ in $\mathrm{Tor}_{m-n}^R(k,L_{n-1})$. Then the commutativity of the diagram shows that $\underline{E} = f_* \underline{E}' = (1 \otimes f)h$.

<u>Corollary 2·3·2</u> <u>For</u> f <u>in</u> $\mathrm{Ext}_R^n(k, k)$ <u>and</u> x in $\mathrm{Tor}_n^R(k, k)$, <u>if</u> E <u>is the</u> n-<u>fold extension corresponding to</u> f , <u>then</u> $\underline{E}(x) = fC(x)$

The following theorem is due to Levin [Le, Ch. 3]

<u>Theorem 2·3·3·</u> Let R be a local ring with residue class field k . Then the dual of the comultiplication on $\mathrm{Tor}^R(k, k)$ is the opposite of the Yoneda product on $\mathrm{Ext}_R(k, k)$.

<u>Proof·</u> Let f be in $\mathrm{Ext}_R^n(k, k)$ and g in $\mathrm{Ext}_R^m(k, k)$. We have to show that if f corresponds to E and g to E' , then EE' corresponds to $(f \otimes g)\, T\, \Delta$, or i·e· that $\underline{EE}'\, C = (f \otimes g)\, T\, \Delta$ on $\mathrm{Tor}^R_{n+m}(k, k)$.

But
$$\begin{aligned} \underline{EE}'\, C &= \underline{E}\,(1 \otimes g)\, h\, C = \underline{E}\,(1 \otimes g)\, T\, h \\ &= (fC \otimes g)\, T\, h = (f \otimes g)\,(C \otimes 1)\, T\, h \\ &= (f \otimes g)\, T\,(1 \otimes C)\, h = (f \otimes g)\, T\, \Delta \quad . \end{aligned}$$

We have now proved

Theorem 2.3.4. $\mathrm{Tor}^R(k, k)$ with φ and Δ is a commutative Hopf algebra.

Theorem 2.3.5. $\mathrm{Tor}^R(k,k)$ is a free divided power algebra.

Proof. Since the minimal algebra resolution X is a free divided power algebra (Definition 1.7.7), the same is true for $X \otimes k = \mathrm{Tor}^R(k, k)$.

This structure theorem on $\mathrm{Tor}^R(k, k)$ was first proved by Assmus [As] in the case where k has characteristic 0 , by applying the structure theorem for commutative Hopf algebras [M.M, Theorem 7.11.] . The theorem was then proved in general by Levin [Le], using the methods of this section.

§4. Examples

In this section, we illustrate the structure of the Hopf algebra $\mathrm{Tor}^R(k, k)$ in some simple cases. More interesting examples will be found in Chapter 5.

The following lemmas will prove useful.

Let f be in A_n^* and define $\underline{f} : A_m \to A_{m-n}$ by $\underline{f} = (1 \otimes f)\, h$.

<u>Lemma 2.4.1.</u> If f is in A_n^* and g is in A_m^* , then $\underline{fg} = \underline{f}\,\underline{g}$ where $f\,g$ is the Yoneda product of f and g .

<u>Proof.</u> We use the associativity of Δ .

$$\begin{aligned}
\underline{f}\,\underline{g} &= (1 \otimes f)\, h\, (1 \otimes g)\, h = (1 \otimes fC)\, \Delta\, (1 \otimes gC)\, \Delta \\
&= (1 \otimes f \otimes g)\, (1 \otimes C \otimes C)\, (\Delta \otimes 1)\, \Delta \\
&= (1 \otimes f \otimes g)\, (1 \otimes C \otimes C)\, (1 \otimes \Delta)\, \Delta \\
&= (1 \otimes f \otimes g)\, (1 \otimes T)\, (1 \otimes \Delta C)\, \Delta \\
&= (1 \otimes fg\, C)\, \Delta = (1 \otimes fg)\, h = \underline{fg}
\end{aligned}$$

<u>Lemma 2.4.2.</u> If f is a primitive element in A_n^* , then $\underline{f}$ is a derivation on A in the following sense: for $x \in A_i$, $y \in A_j$,

$$f(xy) = (-1)^{nj} f(x)y + x\, f(y)$$

Proof. Since f is a primitive element in A^* it must vanish on $(I(A))^2$. But h is an algebra morphism so

$$\underline{f}(xy) = (1 \otimes f)\, h(x)\, h(y)$$
$$= (1 \otimes f)((x \otimes 1)\, h(y) + h(x)\, (y \otimes 1))$$
$$= x\, \underline{f}(y) + (-1)^{nj} \underline{f}(x)\, y$$

Regular local rings

Let R be a regular local ring with maximal ideal $\underline{m}$ and residul field k. Choose a minimal generating set $t_1, \ldots, t_n$ for $\underline{m}$. Then the exterior algebra $R < T_1, \ldots, T_n ; d\, T_i = t_i >$ is a minimal algebra resolution of k. Hence, as an algebra, A is simply the exterior algebra on A_1. (We use the same symbol to denote the image of T_i mod $\underline{m}$).

Elements of degree one are primitive in any connected Hopf algebra. In A, we also have that

$Q(A) \approx A_1$, so since $P(A) \to Q(A)$ is a monomorphism, $P(A) = A_1$. But $Q(A^*) = (P(A))^* = A_1^*$ so A^* is generated by its elements of degree 1. $T_1, \ldots, T_n$ are a basis for A_1. Take a dual basis $T_1^*, \ldots, T_n^*$ for A_1^*. Since A is generated by primitive elements

it is commutative as a co-algebra, hence A^* is commutative as an algebra. Then from the derivation formula (2.4.2), we deduce that for $i_1 < \dots < i_k$; $j_1 < \dots < j_k$

$$(T_{i_1}^* \dots T_{i_k}^*)\,(T_{i_1} \dots T_{j_k}) = \delta^{i_1 \dots i_k}_{j_1 \dots j_k}$$

(Kronecker delta) It follows that A^* is the exterior algebra on $T_1^*, \dots, T_n^*$.

Local Complete Intersections

First we need to discuss the behavior of Δ with respect to divided powers. Let X be a minimal algebra resolution of k . X has divided powers and it is known (1.8.2) that $X \otimes X$ has a system of divided powers such that for x in X of even degree,

$$(x \otimes 1)^{(i)} = x^{(i)} \otimes 1 \quad \text{and} \quad (1 \otimes x)^{(i)} = 1 \otimes x^{(i)} \quad .$$

Hence we have divided powers induced on A and $A \otimes A$.

<u>Proposition 2.4.3.</u> For $\bar{x}$ in A of even degree,

$$\Delta\ (\bar{x}^{(i)}) = (\Delta\ (\bar{x}))^{(i)}$$

<u>Proof.</u> Let x in X_{2m} represent $\bar{x}$ in A_{2m}, and choose a cycle

$$z = x \otimes 1 + \cdots + 1 \otimes y \quad \text{in} \quad X \otimes X \ .$$

Then since $dz^{(i)} = (dz)\ z^{(i-1)}$, $z^{(i)}$ is a cycle in $X \otimes X$. But

$$z^{(i)} = x^{(i)} \otimes 1 + \cdots + 1 \otimes y^{(i)} \ .$$

This shows that $h(\bar{x}^{(i)}) = (h(\bar{x}))^{(i)}$ and that $C(\bar{x}^{(i)}) = (C(\bar{x}))^{(i)}$. It follows that $\Delta\ (\bar{x}^{(i)}) = (\Delta\ (\bar{x}))^{(i)}$.

Now let $R = \tilde{R}/\underline{a}$ where $\tilde{R}$ is a regular local ring of dimension n and $\underline{a}$ is generated by an $\tilde{R}$-sequence $x_1, \cdots, x_m$ with all x_i in $\tilde{m}^2$. Suppose $\tilde{t}_1, \cdots, \tilde{t}_n$ are a minimal generating set for $\tilde{\underline{m}}$. Then their images $t_1, \cdots, t_n$ in R generate $\underline{m}$ minimally. Furthermore if $x_i = \sum_{j=1}^{n} \tilde{b}_{ij}\ \tilde{t}_j$ in $\tilde{R}$, then

$$X = R < T_1, \cdots, T_n\ ;\ S_1, \cdots, S_m\ ;\ d\ T_i = t_i \ ,$$

$d\, S_i = \sum_j b_{ij}\, T_j >$ is a minimal algebra resolution of k over R, see (1.5.4).

If k has characteristic 0, then since $A = X \otimes k$, we have $Q(A) = A_1 \oplus (A_2/A_1{}^2)$. To extend this result to characteristic p,

let $Q'(A) = I(A)/D$ where D consists of the decomposable elements of $I(A)$, i.e. all sums of products of divided powers. (see definition 1.7.1). Then, in general, $Q'(A) = A_1 \oplus (A_2/A_1^2)$. By (2.4.3), a primitive element of A cannot be decomposable, hence we have a monomorphism $P(A) \to Q'(A)$. Thus, in general $P(A) \subset A_1 \oplus A_2$.

The following formulas are easily verified.

a) $\Delta(S_i) = S_i \otimes 1 - \sum_{j,k=1}^{n} \bar{c}_{ijk}\, T_k \otimes T_j + 1 \otimes S_i$ where $b_{ij} = \sum_{k=1}^{n} c_{ijk} t_k$ and $^-$ denotes passage mod $\underline{m}$.

b) $\Delta(S_i{}^{(j)}) = (\Delta(S_i))^{(j)} = \sum_{r=o}^{j} S_i{}^{(r)} \otimes S_i{}^{(j-r)}$

+ terms involving the T_i's.

From a) it follows that for x in X_2, $\bar{x}$ in

A_2 is primitive if and only if $dx \in \underline{m}^2 X_1$. Let $T_1^*, \dots, T_n^*$ be a dual basis to $T_1, \dots, T_n$, and let $S_1^*, \dots, S_m^*$ be a dual basis to $S_1, \dots, S_m$. Then formula b) shows that $\underline{S_i^*}(S_i^{(j)}) = S_i^{(j-1)}$.

<u>Theorem 2.4.4.</u> If $\underline{a} \subset \tilde{\underline{m}}^3$, then

$$A_* \approx k < T_1^*, \dots, T_n^* > \otimes k [S_1^*, \dots, S_m^*] .$$

(Here $< >$ denotes the exterior algebra and $[\ ]$ denotes the polynomial algebra (no divided powers)).

<u>Proof.</u> Since $a \subset \tilde{\underline{m}}^3$ and $\tilde{R}$ is a regular local ring, we must have all the $b_{ij} \in \underline{m}^2$. So $P(A) = A_1 \oplus A_2$ and A is generated by primitive elements. This makes A^* commutative. The independence of the generators is then guaranteed by the equation

$$T_{i_1}^* \dots T_{i_r}^* (S_1^*)^{j_1} \dots (S_m^*)^{j_m} (T_{i_1} \dots T_{i_r} S_1^{(j_1)} \dots \dots S_m^{(j_m)}) = 1 .$$

The theorem follows.

If $\underline{a} \notin \tilde{\underline{m}}^3$, the structure of A will be more complicated. For example, suppose

$R = k[[x, y]] / (x^2, y^3)$. Then

$X = R < T_1, T_2, S_1, S_2 \; ; dT_1 = x \; , dT_2 = y \; ,$

$dS_1 = xT_1 \; , dS_2 = y^2 T_2 >$ and $\Delta (S_1) = S_1 \otimes 1 -$

$- T_1 \otimes T_1 + 1 \otimes S_1$

$\Delta (S_2) = S_2 \otimes 1 + 1 \otimes S_2$. Hence, $T_1{}^* \, T_2{}^* =$

$= -T_2{}^* \, T_1{}^*$, $(T_2{}^*)^2 = 0$ as before but in this case, $(T_1{}^*)^2 = S_1{}^*$, i.e.

$A^* \approx k< T_2{}^* > \otimes k \, [T_1{}^* \; , \; S_2{}^*]$. Here $k[T_1{}^* \; , \; S_2{}^*]$ is a strictly commutative polynomial algebra with generators $T_1{}^*$ of degree one and $S_2{}^*$ of degree two.

<u>The case when</u> $\underline{m}^2 = 0$

In this case $\underline{m}$ is a k-space, so

$\mathrm{Tor}_i^R(k, k) \approx \mathrm{Tor}_{i-1}^R(k, m) \approx \mathrm{Tor}_{i-1}^R(k, k) \otimes \mathrm{Tor}_1^R(k, k)$.

So $b_i = n \, b_{i-1}$, $P(A) = A_1$ and $Q(A^*) = A_1{}^*$.

This forces A_* to be the free associative algebra on $T_1{}^* , \ldots T_n{}^*$.

<u>Conjecture</u> (Levin) . $\mathrm{Ext}_R(k, k)$ with the Yoneda product is finitely generated as an algebra. (Equivalently, the vector space of all primitive elements in $\mathrm{Tor}^R(k, k)$ is finite dimensional.)

CHAPTER THREE

THE POINCARÉ SERIES OF A LOCAL RING.

Throughout this chapter R , $\underline{m}$ denotes a local noetherian ring with residue class field $k = R/\underline{m}$.

§1. The Poincaré series $P(R)$ and the deviations $\varepsilon_q(R)$.

It was shown in (2.3.5) that $\mathrm{Tor}^R(k, k)$ is a free divided power algebra over k , in the sense of (1.7.7), so this algebra is completely known if we know the number of basis elements ε_q of each degree $q > 0$. The non-negative integers ε_q are invariants of R coinciding with the deviations $\varepsilon_q(R)$ defined below. The invariants ε_q are in turn determined by the Betti numbers of R : $b_p = b_p(R) = \dim_k \mathrm{Tor}^R(k, k)$ $(p \geq 0)$

<u>Definition 3.1.1</u> The formal powerseries

$$\sum_{p=0}^{\infty} b_p(R)\, z^p$$

is called the Poincaré series of R (also called the Betti series of R) and will be denoted by $P(R)$.

If X is an (R)-algebra, $D(X)$ will denote the subcomplex $D(X) = C(X) + \underline{m}\, X + X_o$ where $C(X)$ is the module of decomposable elements in X, see (1.7.1).

For any factor ring $\bar{R}$ of R we will in view of the uniqueness of acyclic closure (1.9.5) let

$$V(R, \bar{R}) = \coprod_{q > 0} V_q(R, \bar{R})$$

denote a representative of the isomorphism class of graded $R/\underline{m}$-modules $X^*/D(X^*)$ where X^* is an acyclic closure of the augmented (R)-algebra $R \to \bar{R}$. Note that by the definition of acyclic closure (1.9.1) we have $B(X^*) \subset D(X^*)$ hence

$V(R, \bar{R}) \approx X^*/D(X^*) = H(X^*/D(X^*))$. Put $\nu_q(R, \bar{R}) = \dim_k \nu_q(R, \bar{R})$ for $q > 0$ and $\varepsilon_q(R) = \nu_{q+1}(R, k)$ for $q \geq 0$. The non-negative integers $\varepsilon_q(R)$ will be called the <u>deviations</u> of R. For $q = 0, 1$ this definition agrees with definition (1.4.11). For $1 \leq q \leq 3$ $\varepsilon_q(R)$ coincide with the "abweichungen" introduced by Scheja [Sc].

Remark We do not know whether $\varepsilon_q(R)$ coincide with the deviations introduced by André [An, p. 117] . See also [An'] .

Lemma 3.1.2 Let X^* be an acyclic closure of an augmented (R) -algebra of the form $R \to \bar{R}$. Then for $q > 0$ we have

(i) $\nu_q(R, \bar{R}) = \# N_q(X^*|R) = \dim_{R/\underline{m}} \tilde{H}_{q-1}(F_{q-1}X^*) \otimes R/\underline{m}$

In particular $\nu_1(R, \bar{R})$ is the number of elements in a minimal set of generators for $\ker (R \to \bar{R})$

(ii) $\varepsilon_o(R) = \dim_{R/\underline{m}} \underline{m}/\underline{m}^2$ and $\varepsilon_q(R) = \dim_{R/\underline{m}} H_q(F_q X^*)$

for $q \geq 1$.

If $\hat{R}$ is the $\underline{m}$ - adic completion of R then

(iii) $\varepsilon_q(\hat{R}) = \varepsilon_q(R)$ for $q \geq 0$.

Proof Choose a basis $\{S_i\}_{i \in I}$ for the extension $R \subset X^*$, and write $X^* = R < \ldots S_i \ldots ; dS_i = s_i >$ Let $\bar{S}_i$ be the residue classes of S_i in $X^*/D(X^*)$.

Then $\{\bar{S}_i\}_{i\in I}$ is a basis for the $R/\underline{m}$ -module $X^*/D(X^*)$. Hence $\nu_q(R, \bar{R}) = \# N_q(X^*|R)$.

(i) now follows since by (1.9.2) $n_q(X^*|R)$ represents a minimal set of generators in $\tilde{H}_{q-1}(F_{q-1}X^*)$, and $\# n_q(X^*|R) = \# N_q(X^*|R)$. Let $\underline{a} = \ker(R \to \bar{R})$. We have $\nu_1(R, \bar{R}) = \dim_{R/\underline{m}} \tilde{H}_0(F_0X^*) \otimes R/\underline{m} = \dim \underline{a}/\underline{ma}$.

(ii) follows from (i), and (iii) follows from (ii) and (1.9.8) .

<u>Corollary 3.1.3</u> Let $P(R)$ be the Poincaré series of R . Then

$$P(R) = \prod_{i=0}^{\infty} \frac{(1+z^{2i+1})^{\epsilon_{2i}}}{(1-z^{2i+2})^{\epsilon_{2i+1}}}$$

where $\epsilon_q = \epsilon_q(R)$ for $q \geq 0$.

<u>Proof</u> Let X denote the augmented (R) -algebra $R \to k$, and let X^* denote an acyclic closure of X . By Example (a) after (1.9.3) X^* is a minimal resolution, i.e. $B(X^*) \subset \underline{m}\, X^*$. Hence for $p \geq 0$ we have $\mathrm{Tor}^R(k, k) \approx H(X^* \otimes k) = X^* \otimes k$. Choose a basis $N(X^*|X)$ for the extension $X \subset X^*$. We have

an isomorphism of graded algebras

$$X^* \otimes k \approx \bigotimes_{S \in N(X^*|X)} k < S >$$

hence considering Poincaré series of graded vector-spaces we have $P(R) = \chi(X^* \otimes k) = \prod_{s \in N(X^*|X)} \chi(k < S >)$

Now the formula for $P(R)$ follows since

$N(X^*|X) = \bigcup_{q>0} N_q(X^*|X)$ (disjoint union) and $\varepsilon_q(R) =$

$= \# N_q(X^*|X)$. Note that if S is a variable of odd degree $2i+1$ then $\chi(k < S >) = 1 + z^{2i+1}$ and if $\deg S = 2i + 2$ then $\chi(k < S >) = \dfrac{1}{1-z^{2i+2}}$

<u>Remark</u> The product representation in (3.1.3) is unique. Precisely, if we are given sequences of non-negative integers $\{\varepsilon_q\}_{q \geq 0}$, $\{\eta_q\}_{q \geq 0}$ such that

$$\prod_{i=0}^{\infty} \frac{(1 + z^{2i+1})^{\varepsilon_{2i}}}{(1 - z^{2i+2})^{\varepsilon_{2i+1}}} = \prod_{i=0}^{\infty} \frac{(1 + z^{2i+1})^{\eta_{2i}}}{(1 - z^{2i+2})^{\eta_{2i+1}}}$$

(coefficientwise equality when developed in power-series with respect to z) then $\epsilon_q = \eta_q$ for all $q \geq 0$.

§2. $V_q(R, \bar{R})$ as <u>homology groups.</u>

In this section we shall use the following notation

$$\pi(R) = \begin{cases} 2p & \text{if } R/\underline{m} \text{ has characteristic } p \neq 0 \\ \infty & \text{otherwise} \end{cases} .$$

If there is no danger of confusion $\pi(R)$ will be denoted by π . $D(X)$ and $V(R, \bar{R})$ are defined in (3.1.1) . k will denote the residue class field $R/\underline{m}$.

<u>Lemma 3.2.1</u> Let Y be an (R) -algebra which is free in the sense of (1.7.7) and admitting an augmentation. Then there exists a free (R) - algebra X satisfying (i) $B(X) \subset C(X) + \underline{m}\, X$ and a surjective (R) -algebra homomorphism $f : Y \to X$ inducing isomorphisms (ii) $H(Y/D(Y) \approx H(X/D(X)) = X/D(X)$

(iii) $H_q(Y) \approx H_q(X)$ for $q < \pi$. If moreover $B_{2r}(Y) \subset C(Y)_{2r} + \underline{m}\, Y_{2r}$ for all r then the induced map $H_q(Y) \to H_q(X)$ is an isomorphism for all q .

<u>Proof</u> Let $\{b_i\}_{i \in I}$ be a set of homogeneous elements in $B(Y)$ representing a basis for the k-vectorspace $B(Y) / (B(Y) \cap (C(Y) + \underline{m}\, Y))$. Choose elements $a_i \in y$ such that $da_i = b_i$. $\{a_i\}_{i \in I} \cup \{b_i\}_{i \in I}$ represents a linearly independent set of vectors in $Y/(C(Y) + \underline{m}Y + Y_o)$. Extend this to a homogeneous basis and let the new basis vectors be represented by homogeneous elements $c_j \in Y$ $j \in J$. By (1.8.8) $\{a_i\}_{i \in I} \cup \{b_i\}_{i \in I} \cup \{c_j\}_{j \in J}$ is a basis for the extension $R \subset Y$. The (R) -algebra $Y' = R < \cdots b_i \cdots , \cdots a_i \cdots ; da_i = b_i >$ is an (R) -subalgebra of Y . Let Y'_+ be the subcomplex of Y' generated by the elements of positive degree. Note that $B_o(Y') = 0$. Y'_+Y is an ideal in the (R)-algebra Y . Consider the canonical (R) -algebra

$$X = Y/Y'_+Y$$

and let $f : Y \to X$ be the canonical map. Let $\bar{c}_j = f(c_j)$. Y is a free extension of Y' with basis $\{c_j\}_{j \in J}$. It follows that X is a free (R)-algebra, in fact a free extension of R with basis $\{\bar{c}_j\}_{j \in J}$. So $X = R < \dots \bar{c}_j \dots ; d\bar{c}_j = f(dc_j) >$. X satisfies (i) since f preserves divided powers and we have $B(Y) \subset \sum_i Rb_i + C(Y) + \underline{m}Y$. Moreover, $C(X) + \underline{m}X \subset D(X)$ hence $H(X/D(X)) = X/D(X)$. f gives rise to the exact sequence of complexes

$$0 \to (D(Y) + Y_+^!Y)/D(Y) \to Y/D(Y) \to X/D(X) \to 0$$

$(D(Y) + Y_+^!Y)/D(Y) = Y_+^!Y/(Y_+^!Y \cap D(Y))$ has as a vector space a homogeneous basis consisting of the residue classes of a_i and b_i, $i \in I$. Since $da_i = b_i$, we have

$$H(Y_+^!Y/Y_+^!Y \cap D(Y)) = 0$$

and the map (ii) is an isomorphism.

We will now show that the induced map $H_q(Y) \to H_q(X)$ is an isomorphism for $q < \pi$. It suffices to show

$$H_q(Y_+^!Y) = 0 \qquad \text{for } q < \pi$$

We will assume that $J = \mathbb{N}_0$ so that the elements c_j are indexed by natural numbers. Only trivial modifications must be carried out if J is finite.

Let $Y^0 = Y'$ and for $\alpha \geq 1$ let $Y^\alpha = Y' < c_1, \ldots$ $\ldots c_\alpha ; dc_j = dc_j >$. Since $Y = \bigcup_{a=0}^{\infty} Y^\alpha$ it suffices to show that

(1) $H_q(Y'_+ Y^\alpha) = 0$ for $q < \pi$, $\alpha \geq 0$.

(1) will be shown by induction on α . For $\alpha = 0$ we have $Y'_+ Y^\alpha = Y'_+$. For all $i \in I$ consider the (R) -subalgebra of Y'

$$U^i = R < b_i , a_i ; da_i = b_i >$$

We have an isomorphism of (R) -algebras

(2) $$Y' \approx \lim_{\substack{\rightarrow \\ n}} \bigotimes_{i=1}^{n} U^i$$

If $\deg b_i$ is odd ($\deg a_i$ is even) the cycles of positive degree in U^i are generated by $b_i a_i^{(k)} = da_i^{(k+1)}$ for $k \geq 0$. Hence $H_q(U^i) = 0$ for $q > 0$. If $\deg b_i$ is even ($\deg a_i$ is odd) the cycles of positive degree less than $\frac{\pi}{2} \deg b_i$ are

generated by $b_i^{(k)} = d(\frac{1}{k} a_i b_i^{(k-1)})$. Indeed k is invertible in R since $k < \frac{\pi}{2}$. Hence $H_q(U^i) = 0$ for $0 < q < \frac{\pi}{2} \deg b_i$. Thus for all $i \in I$ we have

$$H_q(U^i) = 0 \quad \text{for} \quad 0 < q < \pi$$

For all $i \in I$ we have $B_o U^i = 0$ hence it follows from (2) and (3.2.2) below that

$$H_q(Y') = 0 \quad \text{for} \quad 0 < q < \pi \quad .$$

This shows that (1) is valid for $\alpha = 0$. Let $\beta \geq 1$ and suppose that (1) is valid for $\alpha = \beta-1$. Consider the exact sequence associated with the extension $Y^{\beta-1} \to Y^{\beta}$

$$(3) \qquad 0 \to Y^{\beta-1} \to Y^{\beta} \to Y^{(\beta)} \to 0$$

where $Y^{(\beta)} = Y^{\beta}$ if $\deg c_{\beta}$ is even, $Y^{(\beta)} = Y^{\beta-1}$ otherwise, see (1.3.2') (3) splits as a sequence of left Y' -modules hence we have an exact sequence of complexes

$$0 \to Y'_+ Y^{\beta-1} \to Y'_+ Y^{\beta} \xrightarrow{j} Y'_+ Y^{(\beta)} \to 0$$

By the induction hypothesis $H_q(Y_+^! Y^{\beta-1}) = 0$ for $q < \pi$. Let $\rho = \deg c_\beta$. j has degree $-\rho$. Hence for $q < \pi$ j induces an isomorphism

$$j_{*q} : H_q(Y_+^! Y^\beta) \xrightarrow{\sim} H_{q-\rho}(Y_+^! Y^{(\beta)})$$

Since j_* is of negative degree it follows that $H_q(Y_+^! Y^\beta) = 0$.

We will now prove the last statement of the Lemma. Suppose that $B_{2r}(Y) \subset C(Y)_{2r} + \underline{m}Y_{2r}$ for all r. Then the elements b_j are all of odd degree. It follows that $H_q(U^i) = 0$ for $q > 0$. The same argument as used above shows that $H(Y_+^! Y) = 0$. Hence $f_* : H(Y) \to H(X)$ is an isomorphism.

<u>Lemma 3.2.2</u> Let U and V be R-free complexes satisfying $B_o U = 0 = B_o V$

$$H_q(U) = 0 = H_q(V) \quad \text{for} \quad 0 < q < \pi .$$

Then $H_q(U \otimes_R V) = 0$ for $0 < q < \pi$.

<u>Proof.</u> For $0 < q < \pi$ we have

$$H_q(U \underset{R}{\otimes} V) = \mathrm{Tor}_q^R(U_o , V_o) = 0 \quad .$$

<u>Theorem 3.2.3</u> (Gulliksen) Let $\bar{R}$ be a factor ring of R and Y be a free acyclic extension of the augmented (R) -algebra $R \to \bar{R}$. Then there is an injective homomorphism of graded vector spaces over k

$$V(R , \bar{R}) \to H(Y/D(Y))$$

For $q < \pi(R)$ this gives an isomorphism

$$V_q(R , \bar{R}) \approx H_q(Y/D(Y))$$

If moreover $B_{2r}(Y) \subset C(Y)_{2r} + \underline{m}Y_{2r}$ for all r , then we have an isomorphism for all q .

<u>Proof.</u> We apply (3.2.1) to the free (R) -algebra Y , and let $f : Y \to X$ and $\{\bar{c}_j\}_{j \in J}$ be as in the proof of (3.2.1) .

Let us first assume that $B_{2r}(Y) \subset C(Y)_{2r} + \underline{m}Y_{2r}$. Since Y is acyclic, it follows from (3.2.1 (iii)) that

also X is acyclic. Hence X is an acyclic closure of $R \to \bar{R}$. Thus $V(R, \bar{R}) \approx X/D(X) \approx H(Y/D(Y))$ by (3.2.1).

In the general case consider the (R)-subalgebra of X

$$X' = R < \bar{c}_1 , \dots , \bar{c}_j ; d\bar{c}_i = f(dc_i) >$$

where $\bar{c}_1, \dots . \bar{c}_\gamma$ are all the variables of degree less than π. Give X' the obvious augmentation $X' \to \bar{R}$ and let P be an acyclic closure of the augmented (R)-algebra X'.

We shall first see that P is also an acyclic closure of $R \to \bar{R}$. Since P is a free acyclic extension of $R \to \bar{R}$, it suffices to show

(4) $B_q(P) \subset C(P)_q + \underline{m}P_q$ for $q \geq 0$.

Since $H_q(X') = H_q(X) = 0$ for $0 < q < \pi-1$, P is obtained from X' by the adjunction of a (possibly empty) set of variables of degree greater than or equal to π. Suppose first that $0 < q < \pi$. We have

$$B_q(P) = Z_q(X') = Z_q(X)$$

By (3.2.1 (i) and (ii)) and the fact that $H_q(Y) = 0$ we have

$$Z_q(X) = B_q(X) \subset C(X)_q + \underline{m}X_q = C(P)_q + \underline{m}P_q$$

Hence in this case (4) follows. Suppose now that $q \geq \pi$. Since P is an acyclic closure of X', we have

$$B_q(P) \subset C(P)_q + \underline{m}P_q + X'_q \quad .$$

But $X'_q = C(X')_q \subset C(P)_q$. Again (4) follows.

Since P and Y are free acyclic extensions of $R \to \bar{R}$, there exist by (1.8.6) homomorphisms of augmented (R)-algebras

$$P \underset{g}{\overset{h}{\rightleftarrows}} Y$$

extending the identity map on the augmented (R)-algebra $R \to \bar{R}$, and inducing homomorphisms of graded vector-spaces over k.

$$H(P/D(P)) \underset{g'}{\overset{h'}{\rightleftarrows}} H(Y/D(Y))$$

By (1.9.5) hg is an isomorphism, hence $h'g'$ is an isomorphism and g' is injective. Since $B(P) \subset D(P)$, we have $V(R, \bar{R}) \approx P/D(P) = H(P/D(P))$, and the existence of an injective homomorphism $V(R, \bar{R}) \to H(Y/D(Y))$ is now clear.

Concerning the last part of the theorem, let $q < \pi$. Using (3.2.1 (ii) and (i)) we obtain

$$H_q(Y/D(Y)) \approx H_q(X/D(X)) = (X/D(X))_q = (P/D(P))_q \approx V_q(R, \bar{R}) .$$

<u>Theorem 3.2.4</u> (Gulliksen) Let $R \to \bar{R} \to \bar{\bar{R}}$ be surjective homomorphisms of local rings. Then for $1 < q < \pi(R)$ there is an exact sequence of vector spaces over $R/\underline{m}$

$$V_{q+1}(\bar{R}, \bar{\bar{R}}) \to V_q(R, \bar{R}) \to V_q(R, \bar{\bar{R}}) \to V_q(\bar{R}, \bar{\bar{R}}) \to$$

$$\to V_{q-1}(R, \bar{R}) \to V_{q-1}(R, \bar{\bar{R}}) \to V_{q-1}(\bar{R}, \bar{\bar{R}}) \to \dots$$

$$\dots \to V_1(\bar{R}, \bar{\bar{R}}) \to 0 .$$

<u>Proof.</u> Let X^* be an acyclic closure of $R \to \bar{R}$ and let ϵ be the augmentation of X^*. Let X' be the (R)-algebra X^* equipped with the obvious augmentation $\epsilon' : X' \to \bar{\bar{R}}$. Let A be the augmented (R)-algebra $\bar{R} \to \bar{\bar{R}}$. ϵ may be regarded as a homomorphism of augmented (R)-algebras

$$\begin{tikzcd} X' \arrow[r, "\epsilon"] \arrow[d, "\epsilon'"'] & \bar{R} \arrow[d] \\ \bar{\bar{R}} \arrow[r, equal] & \bar{\bar{R}} \end{tikzcd}$$

inducing an isomorphism $\tilde{\epsilon}_* : \tilde{H}(X') \to \tilde{H}(A)$. By (1.9.7) there exist acyclic closures

$$Y = X' < \dots S_i \dots ; dS_i = s_i > \text{ and}$$

$$A^* = A < \dots T_i \dots ; dT_i = t_i > \text{ of } X' \text{ and } A$$

respectively and an extension of ϵ to a homomorphism of (R)-algebras

$$\epsilon^* : Y \to A^*$$

satisfying $\epsilon^*(S_i) = T_i$. Y is a free (R)-algebra ,

and A^* is a free $(\bar{R})$ -algebra. We observe that for $q \geq 1$:

$$\# N_q(Y|R) = \# N_q(Y|X') + \# N_q(X'|R)$$

$$= \# N_q(A^*|A) + \# N_q(X^*|R)$$

We note that if F is a free (R) -algebra with basis $\{U_j\}_{j \in J}$ then the set of residue classes of U_j in $F/D(F)$ is a basis for $F/D(F)$ as an $R/\underline{m}$ -module. Using this, it is easily seen that the (R) -algebra homomorphisms

$$X^* \subset Y \xrightarrow{\epsilon^*} A^*$$

induce an exact sequence of complexes

(5) $0 \to X^*/D(X^*) \to Y/D(Y) \to A^*/D(A^*) \to 0$

We have isomorphisms of graded vector spaces

$$H(X^*/D(X^*)) = X^*/D(X^*) \approx V(R\ ,\ \bar{R})$$

$$H(A^*/D(A^*)) = A^*/D(A^*) \approx V(\bar{R}\ ,\ \bar{\bar{R}})$$

Since Y is a free acyclic extension of $R \to \bar{\bar{R}}$ we have $H_q(Y/D(Y)) \approx V_q(R, \bar{\bar{R}})$ for $q < \pi$ by (3.2.3). Now the exact sequence in the theorem follows from the exact homology sequence of (5).

<u>Remark.</u> We do not know if the restriction on q in the exact sequence of (3.2.4) is necessary. A result much similar to (3.2.4) can be found in [An, Proposition 18.2].

§3. <u>Application to the deviations</u> $\varepsilon_q(R)$. <u>Behaviour under flat extensions of the ring.</u>

We give first two corollaries of the theorems in the previous section.

<u>Corollary 3.3.1</u>

(i) Let $R \to \bar{R} \to \bar{\bar{R}}$ be surjective homomorphisms of local rings. Then

$$\nu_q(R, \bar{\bar{R}}) \leq \nu_q(R, \bar{R}) + \nu_q(\bar{R}, \bar{\bar{R}}) \quad \text{for all } q > 0$$

(ii) Let $f: R' \to R$ be a homomorphism of local rings $(R', \underline{m}')$ and $(R, \underline{m})$. Suppose that $f(\underline{m}') \subset \underline{m}$ and suppose that R is R'-flat. Then

$$\varepsilon_q(R) \leq \varepsilon_q(R') + \varepsilon_q(R/\underline{m}'R) \quad \text{for all} \quad q \geq 0$$

<u>Proof.</u> Concerning (i), let Y be as in the proof of (3.2.4). Let $q > 0$. By (3.2.3) we have an injection of vector spaces

$$V_q(R, \bar{\bar{R}}) \to H_q(Y/D(Y))$$

From the exact homology sequence of (5) in the proof of (3.2.4) we obtain an exact sequence

$$V_q(R, \bar{R}) \to H_q(Y/D(Y)) \to V_q(\bar{R}, \bar{\bar{R}})$$

The inequality follows, considering the dimensions of the above vector spaces.

Concerning (ii), we put $k = R/\underline{m}$ and $k' = R'/\underline{m}'$, substitute $\bar{R} = R/m'R$ and $\bar{\bar{R}} = k$ in (i) and obtain

$$\varepsilon_q(R) = \nu_{q+1}(R, k) \leq \nu_{q+1}(R, R/\underline{m}'R) + \nu_{q+1}(R/\underline{m}'R, k)$$

Moreover $\nu_{q+1}(R\ ,\ R/\underline{m}'R) = \nu_{q+1}(R'\ ,\ k')$ for all q . Indeed, if X is an acyclic closure of $R' \to k'$ then by (1.9.8) $X \otimes_{R'} R$ is an acyclic closure of $R \to k' \otimes_{R'} R$ which can be identified with the augmented (R) -algebra $R \to R/\underline{m}'R$. The inequality now follows.

<u>Corollary 3.3.2</u> If R is a factor ring of a regular local ring $\tilde{R}$, then

(1) $\varepsilon_q(R) = \nu_q(\tilde{R}\ ,\ R)$ for $1 < q < \pi-1$

(ii) $\varepsilon_{\pi-1}(R) \geq \nu_{\pi-1}(\tilde{R}\ ,\ R)$ for $\pi < \infty$.

(iii) $\varepsilon_1(R) - \varepsilon_o(R) = \nu_1(\tilde{R}\ ,\ R) - \varepsilon_o(\tilde{R})$.

(iv) $\varepsilon_1(R) \leq \nu_1(\tilde{R}\ ,\ R)$.

<u>Proof.</u> We apply (3.2.4) to the surjective ring homomorphisms $\tilde{R} \to R \to k$ and consider the vectorspace dimension over k of the terms in the exact sequence in (3.2.4). Since $\tilde{R}$ is regular we have $V_q(\tilde{R}\ ,\ k) = 0$ for $q > 1$. Indeed, the Koszul complex generated

over $\tilde{R}$ by a minimal set of generators for $\tilde{\underline{m}}$ is an acyclic closure of $\tilde{R} \to k$. The relations (i) - (iii) follows immediately. Since $\tilde{R} \to R$ is surjective we have $\epsilon_o(R) \leq \epsilon_o(\tilde{R})$. Hence (iv) follows from (iii) .

<u>Theorem 3.3.5</u> (Gulliksen) Let R be a factor ring of a regular local ring and let $R_{\underline{p}}$ be the localization of R with respect to a prime ideal $\underline{p}$ in R . Then (i) $\epsilon_q(R_{\underline{p}}) \leq \epsilon_q(R)$ for $q < \pi(R) - 1$.

<u>Proof.</u> Let $f : \tilde{R} \to R$ be a surjective homomorphism, where $\tilde{R}$ is a regular local ring. We may assume without loss of generality that $\ker f \subset \tilde{\underline{m}}^2$, i.e. $\epsilon_o(\tilde{R}) = \epsilon_o(R)$. Let $\tilde{\underline{p}}$ be the inverse image of $\underline{p}$ in $\tilde{R}$. f induces a surjective homomorphism $\tilde{R}_{\tilde{\underline{p}}} \to R_{\underline{p}}$. Since $\tilde{R}$ is regular so is the localization $\tilde{R}_{\tilde{\underline{p}}}$, see [No' Chapter 9, Theorem 25]. Hence

$$\epsilon_o(R_{\underline{p}}) \leq \epsilon_o(\tilde{R}_{\tilde{\underline{p}}} = \operatorname{Dim} \tilde{R}_{\tilde{\underline{p}}} \leq \operatorname{Dim} \tilde{R} = \epsilon_o(\tilde{R}) = \epsilon_o(R) \quad .$$

Thus (i) has been established for $q = 0$.

In the rest of the proof let $1 \leq q < \pi(R) - 1$. Let $X = \tilde{R} < \dots , S_i \dots >$ be an acyclic closure of $\tilde{R} \to R$. Since $\tilde{R}_{\tilde{\underline{p}}}$ is $\tilde{R}$ -flat and we have a canonical isomorphism of rings $R \otimes_{\tilde{R}} \tilde{R}_{\tilde{\underline{p}}} \approx R_{\underline{p}}$,

$$\bar{X} = X \otimes_{\tilde{R}} \tilde{R}_{\tilde{\underline{p}}} \approx \tilde{R}_{\tilde{\underline{p}}} < \dots S_i' , \dots >$$ is a free

acyclic extension of the canonical augmented $(\tilde{R}_{\tilde{\underline{p}}})$ -algebra $\tilde{R}_{\tilde{\underline{p}}} \to R_{\underline{p}}$. Since $\tilde{R}_{\tilde{\underline{p}}}$ is regular, and $\pi(R_{\underline{p}}) \geq \pi(R)$, (3.3.2 (i) and (iv)) give

$$\epsilon_q(R_{\underline{p}}) \leq \nu_q(\tilde{R}_{\tilde{\underline{p}}} , R_{\underline{p}})$$

Letting $K = \tilde{R}_{\tilde{\underline{p}}}/\tilde{p}\tilde{R}_{\underline{p}}$ and $k = R/\underline{m}$ (3.2.3) gives

$$\nu_q(\tilde{R}_{\tilde{\underline{p}}} , R_{\underline{p}}) \leq \dim_k H_q(\bar{X}/D(\bar{X}))$$

$$\leq \dim_K(\bar{X}/D(\bar{X}))_q = \dim_k(X/D(X))_q$$

$$= \nu_q(\tilde{R} , R) .$$

Since $\epsilon_o(\tilde{R}) = \epsilon_o(R)$, it follows from (3.3.2(i) and (iii)) that

$$\nu_q(\tilde{R}, R) = \epsilon_q(R)$$

Hence $$\epsilon_q(R_{\underline{p}}) \leq \epsilon_q(R) .$$

<u>Remark.</u> In view of the characterization in (1.4.13) of regular local rings in terms of $\epsilon_1(R)$, the above theorem may be considered as a generalization of the theorem mentioned in the introduction stating that the class of regular local rings is closed under localizations with respect to prime ideals. For $q = 0$ the above formula (i) has been established for arbitrary rings. See Theorem 2.2 in [Va]. It seems likely that the inequality is true in general with no restrictions on q . See also [An, Proposition 28.6]

We close this section by relating some of the deviations to the homology algebra of the Koszul complex.

<u>Proposition 3.3.4</u> Let E be the Koszul complex generated over R by a minimal set of generators for $\underline{m}$. Then

(i) $\varepsilon_1(R) = \dim_k H_1(E)$

(ii) $\varepsilon_2(R) = \dim_k H_2(E)/H_1(E)^2$

(iii) (Levin, Sakuma, Okuyama)

$$\varepsilon_3(R) = \dim_k H_3(E)/H_1(E)H_2(E) + \binom{\varepsilon_1}{2} - \dim_k H_1(E)^2$$

<u>Proof</u> By (3.1.2) the deviations are invariant under $\underline{m}$-adic completion, and by (1.9.8) so is the homology algebra $H(E)$. We may therefore assume that R is complete, hence by (1.4.14) we may assume that $R = \tilde{R}/\underline{a}$ where $\tilde{R}$ is a regular ring and $\underline{a}$ is contained in the square of the maximal ideal $\tilde{\underline{m}}$ in $\tilde{R}$. Letting $\tilde{E}$ be the Koszul complex generated over $\tilde{R}$ by a minimal set of generators for $\tilde{\underline{m}}$ we have an isomorphism of $\tilde{R}$-algebras

$$E \approx \tilde{E} \otimes_{\tilde{R}} R$$

Since $\tilde{R}$ is regular, $\tilde{E}$ is an $\tilde{R}$-free resolution of the $\tilde{R}$-module k see (1.4.13). Let P be an acyclic closure of the augmented $\tilde{R}$-algebra $\tilde{R} \to R$. The $\tilde{R}$-algebra homomorphisms

$$\tilde{E} \underset{\tilde{R}}{\otimes} R \xleftarrow{1_{\tilde{E}} \otimes \epsilon_P} \tilde{E} \underset{\tilde{R}}{\otimes} P \xrightarrow{\epsilon_{\tilde{E}} \otimes 1_P} k \underset{\tilde{R}}{\otimes} P$$

induce k-algebra isomorphisms

$$H(\tilde{E} \otimes R) \xleftarrow{\approx} H(\tilde{E} \otimes P) \xrightarrow{\approx} H(k \underset{\tilde{R}}{\otimes} P)$$

This follows from a well known spectral sequence argument since $\tilde{E}$ and P are $\tilde{R}$-free resolutions. We have therefore k-algebra isomorphisms

$$H(E) \approx H(k \underset{\tilde{R}}{\otimes} P) \approx H(P/\underline{\tilde{m}}P) \quad .$$

Choose a basis for the extension $\tilde{R} \to P$ and Let $T_1, \ldots, T_t$, $S_1, \ldots, S_s$ and $U_1, \ldots, U_u$ denote the adjoined variables of degree 1, 2 and 3 respectively.

Since $\underline{a} \subset \tilde{\underline{m}}^2$ we have $\epsilon_o(R) = \epsilon_o(\tilde{R})$ hence it follows from (3.3.2 (iii)) that $t = \epsilon_1(R)$. From (3.3.2 (i) and (iii)) we obtain $s = \epsilon_2(R)$. Let $\bar{T}_i$, $\bar{S}_j$, $\bar{U}_\ell$ denote the residue classes of T_i , S_j and U_ℓ in the k -algebra $\bar{P} = P/\underline{m}P$. Note that $B_q(P) \subset \underline{m}P_q$ for $q = 0 , 1$. Hence

$$B_q(\bar{P}) = 0 \quad \text{for} \quad q = 0 , 1$$

Hence $\bar{T}_1 , \cdots , \bar{T}_t$ is a basis for $H_1(\bar{P}) = \bar{P}_1$ and (i) follows. We have $\bar{P}_2 = \bar{P}_1{}^2 \oplus \coprod_{j=1}^{s} kS_j$, moreover $Z_2(\bar{P}) = \bar{P}_2$. Since P is an acyclic closure of $\tilde{R} \to R$ it follows, from the definition, that $B_2(\bar{P}) \subset \bar{P}_1{}^2$. Hence

$$H_2(\bar{P})/H_1(\bar{P})^2 = \bar{P}_2/\bar{P}_1{}^2 \approx \coprod_{j=1}^{s} kS_j$$

Now (ii) follows.

A general proof of (iii) is given in Chapter four, where it follows from (4.4.3). However, if k has characteristic $\neq 2$, we can give the following simple proof as an application of (3.3.2) :

With the same notations as above, put $Z_q = Z_q(\bar{P})$ and let L be the submodule of $\bar{P}_3$ generated by $\bar{U}_1, \ldots, \bar{U}_u$. We have $\bar{P}_3 = \bar{P}_1\bar{P}_2 \oplus L$. Since $\dot{Z}_q = \bar{P}_q$ for $q = 1, 2$, we have $dL = B_2(\bar{P})$. Moreover $B_3(\bar{P}) \subset \bar{P}_1\bar{P}_2$ (by property (ii) of acyclic closure), hence

$$L \cap Z_3 \approx Z_3/Z_1Z_2 \approx H_3(\bar{P})/H_1(\bar{P})H_2(\bar{P})$$

Therefore

(1) $L \approx H_3(\bar{P})/\ H_1(\bar{P})H_2(\bar{P}) \oplus B_2(\bar{P})$

Since $B_2(\bar{P}) \subset \bar{P}_1^2 = Z_1^2$ we have

$$H_1(\bar{P})^2 = Z_1{}^2/B_2(\bar{P})$$

Since $\dim_k Z_1{}^2 = \binom{\varepsilon_1}{2}$ it follows that

(2) $\dim_k B_2(\bar{P}) = \binom{\varepsilon_1}{2} - \dim_k H_1(\bar{P})^2$

By definition we have $\dim_k L = \nu_3(\tilde{R}, R)$. If k has characteristic $\neq 2$ it follows from (3.3.2(i)) that $\varepsilon_3 = \nu_3(\tilde{R}, R) = \dim_k L$. In view of (1) and (2) the proof is now complete.

§4. Behaviour of the Poincaré series upon reduction of the ring modulo a principal ideal.

Proposition 3.4.1 Let x be a non-zerodivisor in R. Put $\bar{R} = R/xR$. Let $R \to R'$ be a surjective ringhomomorphism, suppose that $xR' = 0$ and consider the induced homomorphism $\bar{R} \to R'$. Then

$$\nu_q(R\ ,\ R') = \nu_q(\bar{R}\ ,\ R') \quad \text{for} \quad q \geq 3$$

$$\nu_2(R\ ,\ R') - \nu_1(R\ ,\ R') = \nu_2(\bar{R}\ ,\ R') - \nu_1(\bar{R}\ ,\ R') - 1$$

Proof Apply (3.2.4) to the canonical ring-homomorphisms $R \to \bar{R} \to R'$ and let X^* and Y be as in the proof of (3.2.4). Since x is a non-zero-divisor, we have (1) $X^* = R < U;\ dU = x >$

where U is a variable of degree 1. Hence $X^*_q = 0$ for $q > 1$. By definition of acyclic closure we have $B(Y) \subset C(Y) + \underline{m}Y + X^*$. In particular $B_{2r}(Y) \subset C(Y)_{2r} + \underline{m}Y_{2r}$ for all r. It follows from

(3.2.3) that $V_q(R\ ,\ R') \approx H_q(Y/D(Y)$ for all q . Hence it follows from (5) in the proof of (3.2.4) that the long exact sequence in (3.2.4) exists without restrictions on q . The relations in the proposition now follows from that sequence using the fact that

$$V_1(R\ ,\ \bar{R}) \approx k$$

$$V_q(R\ ,\ \bar{R}) = 0 \quad \text{for} \quad q \geq 2$$

The last two lines follows from (1) .

<u>Corollary 3.4.2</u> Let x be a non-zerodivisor in R . Put $\bar{R} = R/xR$.

(i) (Tate) If $x \in \underline{m}^2$ then

$$P(\bar{R}) = \frac{P(R)}{1-z^2}$$

(ii) If $x \in \underline{m} \setminus \underline{m}^2$ then

$$P(R) = (1+z)P(\bar{R})$$

Proof Putting $R' = R/\underline{m}$ we obtain from the previous proposition $\epsilon_q(R) = \epsilon_q(\bar{R})$ for $q \geq 2$, and $\epsilon_1(R) - \epsilon_o(R) = \epsilon_1(\bar{R}) - \epsilon_o(\bar{R}) - 1$. If $x \in \underline{m}^2$ then $\epsilon_o(R) = \epsilon_o(\bar{R})$ hence $\epsilon_1(\bar{R}) = \epsilon_1(R) + 1$ If $x \in \underline{m} \setminus \underline{m}^2$ then $\epsilon_o(\bar{R}) = \epsilon_o(R) - 1$ hence $\epsilon_1(R) = \epsilon_1(\bar{R})$.

Corollary 3.4.3 (Tate, Zariski) Let R be a local complete intersection. Then the Poincaré series has the form

$$P(R) = \frac{(1+z)^n}{(1-z^2)^m}$$

where $n = \epsilon_o(R)$ and $m = \epsilon_o(R) - \operatorname{Dim} R$.

Proof The deviations and consequently the Poincaré series are invariant under $\underline{m}$-adic completion of R. Moreover if R is a local complete intersection, then so is the completion of R. We may therefore assume that R is complete, and by (1.4.14) $R = \tilde{R}/\underline{a}$ where $\tilde{R}$ is a regular local ring and $\underline{a} \subset \tilde{\underline{m}}^2$.

Put $n = \varepsilon_o(R)$ and $m = \varepsilon_o(R) - \mathrm{Dim}\, R$. $\underline{a}$ is by (1.5.3) generated by an $\tilde{R}$ -sequence with m elements. m times application of (3.4.2 (i)) gives

$$P(R) = \frac{P(\tilde{R})}{(1-z^2)^m}$$

The corollary now follows since $\tilde{R}$ is regular with embedding dimension $\varepsilon_o(\tilde{R}) = \varepsilon_o(R) = n$ so $P(\tilde{R}) = (1+z)^n$. The last formula follows from (3.4.2 (ii)) . Indeed, $\tilde{m}$ is generated by an $\tilde{R}$ -sequence of length n . n times application of (3.4.2 (ii)) gives $P(\tilde{R}) = (1+z)^n P(\tilde{R}/\tilde{\underline{m}})$ where $P(\tilde{R}/\tilde{m}) = 1$.

<u>Remark</u> Given a local ring R , it is possible to find a finite set of surjective ring homomorphisms $R = R_o \to R_1 \to \dots \to R_s = R/\underline{m}$ such that for $0 \leq i < s$ $\ker (R_i \to R_{i+1})$ is generated by an element x_i which is either a non-zerodivisor or anihilated by the maximal ideal $\underline{m}_i$ in R_i . Indeed, if codim $R_i > 0$ then choose a non-zerodivisor x_i in $\underline{m}_i$ and put $R_{i+1} = R_i / x_i R_i$. If codim $R_i = 0$ and $\underline{m}_i \neq 0$

then $0 : \underline{m}_i \neq 0$. This time choose $0 \neq x_i \in 0 : \underline{m}_i$ and put $R_{i+1} = R_i / x_i R_i$ maximal ideal. In view of corollary (3.4.2) the problem of computing the Poincaré series of a given ring R would be solved if we knew how it behaves upon reduction modulo an element anihilated by the maximal ideal. However, this problem is far from being solved. The following proposition is a partial result.

<u>Proposition 3.4.4</u> Let $u \in \underline{m} \setminus \underline{m}^2$ and suppose that $\underline{m}u = 0$. Put $\bar{R} = R/uR$. We have

$$P(\bar{R}) = \frac{P(R)}{1+zP(R)}$$

<u>Proof</u>. Let $\bar{X}$, $\bar{d}$ be a minimal resolution of the $\bar{R}$ -module $k = \bar{R}/\bar{\underline{m}}$. In each homogeneous component $\bar{X}_q$ choose an $\bar{R}$ -free basis $\{g_{q,1}, \dots, g_{q,b_q}\} =$

$= G_q$. Let X be the graded R-module where X_q is the free R-module generated by G_q . Let $u, t_1, \dots, t_{n-1}$ be a minimal set of generators

for $\underline{m}$. Put $\underline{a} = (t_1 , \cdots , t_{n-1})$. We have $\underline{m} = (u) \in \underline{a}$. The canonical map $R \to \bar{R}$ induces a bijection $h : \underline{a} \to \bar{\underline{m}}$. We now define a differential d on X as follows: For each generator $g_{q+1,\, i}$ define

$$dg_{q+1,\, i} = \sum_{j=1}^{b_q} h^{-1}(\bar{r}_j)\, g_{q\, ,\, j} \quad \text{where}$$

$$\bar{d}g_{q+1,\, i} = \sum_{j=1}^{b_q} \bar{r}_j\, g_{q\, ,\, j} \quad , \quad \bar{r}_j \in \bar{R} \ .$$

All this means is that the obvious map $X \to \bar{X}$ commutes with the differentials.

We shall now define a minimal resolution Y of the R-module k . Put $Y_o = X_o$ and define inductively

$$Y_{i+1} = X_{i+1} \oplus \coprod_{p+q=i} X_p \otimes Y_q \quad \text{for} \quad i > 0 \ .$$

We have an obvious embedding $X \to Y$. We will now extend the differential on X to a differential on Y . Let $x \otimes y \in X_p \otimes Y_q$, $p + q = i$. If $q < i$ define $d(x \otimes y) = (dx) \otimes y$. If $q = i$ we define $d\,(x \otimes y) = uxy$ having identified X_o with

the ring R . One easily shows that Y now is a minimal resolution of $k = R/\underline{m}$. It follows that

$$P(R) = P(\bar{R}) + z\, P(\bar{R})\, P(R) \quad .$$

<u>Remark</u> It has been conjectured by Kaplansky and Serre independently that $P(R)$ always represents a rational function. This is still an open question and is one of central problems in the theory of homology of local rings. We return to this question in the next Chapter.

§5. Characterizations of local complete intersections.

In this section R , $\underline{m}$ denotes a local noetherian ring with residue class field $k = R/\underline{m}$. We refer the reader to the definition of local complete intersections given in (1.5.1)

<u>Theorem 3.5.1</u> The following statements are equivalent:

(i) R is a local complete intersection.

(ii) $\varepsilon_2(R) = 0$ (Assmus, Scheja)

(iii) $\varepsilon_3(R) = 0$ (Gulliksen)

(iv) There exist non-negative integers m, n such that

$$P(R) = \frac{(1+z)^n}{(1-z^2)^m}$$

<u>Proof</u> Since the deviations and consequently the Poincaré series are invariant under $\underline{m}$-adic completion of R, and since R is a local complete intersection if and only if the completion is, we may assume that R is complete. By (1.4.14) we have $R = \tilde{R}/\underline{a}$ where $\tilde{R}$ is a regular local ring and $\underline{a}$ is an ideal in $\tilde{R}$ contained in $\tilde{\underline{m}}^2$.

The implication (i) $\Rightarrow$ (iv) was proved in (3.4.3). The implications (iv) $\Rightarrow$ (ii) and (iv) $\Rightarrow$ (iii) are trivial since the product representation of the Poincaré series is unique. See remark after (3.1.3).

Concerning the implications (ii) $\Rightarrow$ (i) and

(iii) $\Rightarrow$ (i) we let P be an acyclic closure of the augmented $(\tilde{R})$ -algebra $\tilde{R} \to R$. We consider the usual filtration $\{F_q P\}_{q \geq 0}$ on P. Put $\tilde{E} = F_1 P$ and $\tilde{F} = F_2 P$. $\tilde{E}$ is the Koszulcomplex generated over $\tilde{R}$ by a minimal set of generators for $\underline{a}$.

If $\varepsilon_2(R) = 0$ then by (3.3.2) we have $\nu_2(\tilde{R}, R) = 0$. It follows that $H_1(\tilde{E}) = 0$. By (1.4.8) $\underline{a}$ is generated by an $\tilde{R}$ -sequence. Hence R is a local complete intersection. See (1.5.3) .

Let us now suppose that $\varepsilon_3(R) = 0$. By (3.3.2) we have $\nu_3(\tilde{R}, R) = 0$, hence $H_2(\tilde{F}) = 0$. We will show that also in this case $\underline{a}$ is generated by an $\tilde{R}$ -sequence. Since $\tilde{R}$ is regular $\underline{a}$ has finite homological dimension over $\tilde{R}$, (cf. (1.4.12)), hence by (1.4.9) it suffices to show that $H_1(\tilde{E})$ is $\tilde{R}/\underline{a}$ -free. If $H_1(\tilde{E}) = 0$ there is nothing more to show. We may therefore assume that $H_1(\tilde{E}) \neq 0$. We have $\tilde{F} = \tilde{E} < S_1, \cdots, S_n ; dS_i = s_i >$ where $s_1, \cdots, s_n$ are cycles in $\tilde{E}_1$ representing a minimal

set of generators for $H_1(\tilde{E})$.

Suppose that we are given a relation

$$\Sigma r_i s_i = de \quad \text{where} \quad r_i \in \tilde{R} \quad \text{and} \quad e \in \tilde{E}_2$$

Then

$$\Sigma r_i S_i - e \in Z_2(\tilde{F}) = B_2(\tilde{F}) \subset \tilde{E}_2 \oplus \coprod_{i=1}^{n} \underline{a}\, S_i$$

Because of the direct sum it follows that $r_i \in \underline{a}$ for $i = 1, \ldots, n$. We conclude that $H_1(\tilde{E})$ is $R/\underline{a}$ -free.

<u>Theorem 3.5.2</u> (Assmus) Let X^* be an acyclic closure of the augmented (R) -algebra $R \to R/\underline{m}$. Put $E = F_1X^*$ and $F = F_2X^*$. Then the following statements are equivalent:

(i) R is a local complete intersection

(ii) $H(E)$ is the exterior algebra over $H_1(E)$

(iii) $H_2(E) = H_1(E)^2$

(iv) $H_2(F) = 0$

(v) $H_q(F) = 0$ for $q > 0$.

Proof Since the k-algebras $H(E)$ and $H(F)$ are invariant under $\underline{m}$-adic completion, see (1.9.8), we may assume that R is complete. Let $\tilde{R}$ and $\underline{a}$ be as in the previous proof.

(i) implies by (1.5.3) that $\underline{a}$ is generated by an $\tilde{R}$-sequence. Let $a_1, \dots, a_\varepsilon$ be a minimal set of generators for $\underline{a}$ and let $P = \tilde{R} < T_1, \dots, T_\varepsilon ; dT_i = a_i >$. By (1.4.8) P is acyclic, hence P is an acyclic closure of the augmented $(\tilde{R})$-algebra $\tilde{R} \to R$. By the proof of (3.3.4) we have an isomorphism of k-algebras

$$H(E) \approx H(P/\underline{m}P) = P/\underline{m}P = k < \bar{T}_1, \dots, \bar{T}_n >$$

where $\bar{T}_i$ is the residue class of T_i. Hence $H(E)$ is the exterior algebra over $H_1(E)$.

(ii) $\Rightarrow$ (iii) is trivial. By (3.3.4) we have $\dim H_2(F) = \varepsilon_2(R) = \dim H_2(E)/H_1(E)^2$, hence the implication (iii) $\Rightarrow$ (iv) is valid.

If $H_2(F) = 0$ then $\varepsilon_2(R) = 0$ hence by the previous theorem there exist $n, m \geq 0$ such that

$$P(R) = \frac{(1+z)^n}{(1-z^2)^m}$$

We conclude that $H_q(F) = 0$ for $q > 0$. Indeed, if there exists a q_o such that $H_{q_o}(F) \neq 0$ then $F \neq X^*$. Thus the extension $R \subset X^*$ has a basis element of degree ≥ 3. Hence there would exist a $q_1 \geq 2$ such that $\varepsilon_{q_1} \neq 0$. This contradicts the above formula for $P(R)$.

Finally from (v) it follows that $\varepsilon_2(R) = 0$ hence R is a complete intersection by the previous theorem.

<u>Remark</u> We do not know if there exist rings other than local complete intersections for which some of the higher deviations $\varepsilon_q (q \geq 4)$ vanish.

<u>Conjecture</u> (Gulliksen) If there exists an integer q_o such that $\varepsilon_q(R) = 0$ for all $q > q_o$, then R is a local complete intersection.

<u>Theorem 3.5.3 (Wiebe)</u> Let E be the Koszul complex generated over R by a minimal set of generators for $\underline{m}$. Let $n = \varepsilon_o(R)$. Then R is a local complete intersection if and only if $H_1(E)^n \neq 0$.

<u>Proof</u> Let $E = R < T_1, \ldots, T_n ; dT_i = t_i >$ where $t_1, \ldots, t_n$ is a minimal set of generators for $\underline{m}$. We have $H_1(E)^n = Z_1(E)^n = \delta_o(\underline{m})T_1 \ldots T_n$ where $\delta_o(\underline{m})$ is the Fitting ideal of $\underline{m}$. By [Wi, Satz 3] R is a local complete intersection if and only if $\delta_o(\underline{m}) \neq 0$.

CHAPTER FOUR

ON THE RATIONALITY OF THE POINCARÉ SERIES

In this chapter, we discuss the various cases in which $P(R)$ is known to be a rational function.

§1. Eagon's resolution.

The minimal algebra resolution of k requires in general the adjunction of an infinite number of variables and thus yields little information about the rationality of the Poincaré series. In the following resolution, discovered by J.A. Eagon and unpublished, the adjunctions are well under control, but in general the resolution is not minimal.

Let K be any complex of free R-modules of finite type such that $H_i(K)$ is a k-space for each $i > 0$ and such that $H_o(K) = k$. For example, we could take $K = E$,

the Koszul complex of R or $K = E\langle S_1,\ldots,S_m; dS_i = z_i\rangle$ where the z_i are cycles in E.

Now, take free modules X_i, $i = 1,2,\ldots$ such that $X_i \otimes k \approx H_i(K)$. We define a sequence of complexes $Y^o, Y^1, Y^2, \ldots$ inductively. $Y^o = K$ and, assuming that Y^n is defined,

$$Y_i^{n+1} = Y_{i+1}^n \oplus Y_o^h \otimes X_i \quad , \quad i > 0$$

$$Y_o^{n+1} = Y_1^n$$

We have a differential $d: K_i \to K_{i-1}$. To define d on $Y_i^1 = K_{i+1} \oplus K_o \otimes X_i$; $K_o \otimes X_i$ is free so there exists $\alpha: K_o \otimes X_i \to Z_i(K)$ making the diagram

$$\begin{tikzcd} & K_o \otimes X_i \arrow[dl, "\alpha"'] \arrow[d] \\ Z_i(K) \arrow[r] & H_i(K) \approx k \otimes X_i \end{tikzcd}$$

commute. Then for $x \in K_o$, $y \in X_i$, define

$$d(x \otimes y) = \alpha(x \otimes y) \in Z_i(K)$$

$d^2 = 0$ and $d(Y_i^1) = Z_i(K)$. To continue this process we need

<u>Lemma 4.1.1</u>. Assume that d has been defined on Y^n such that $H_i(Y^n) \approx H_o(Y^n) \otimes X_i$. Then d can be extended to Y^{n+1} such that

$$H_i(Y^{n+1}) \approx H_o(Y^{n+1}) \otimes X_i .$$

<u>Proof</u>. As above, we can lift the map

$$Y_o^n \otimes X_i \to H_o(Y^n) \otimes X_i \approx H_i(Y^n)$$

to $\alpha: Y_o^n \otimes X_i \to Z_i(Y^n)$. We then get the following commutative diagram

$$\begin{tikzcd}
 & 0 & 0 & 0 & \\
0 \arrow[r] & B_o(Y^n) \otimes X_i \arrow[r] \arrow[u] & Y_o^n \otimes X_i \arrow[r] \arrow[u] \arrow[dr, "\alpha"] & H_i(Y^n) \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & Z_i(Y^{n+1}) \arrow[r] \arrow[u] & Y_i^{n+1} \arrow[r] \arrow[u] & Z_i(Y^n) \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & Z_{i+1}(Y^n) \arrow[r] \arrow[u] & Y_{i+1}^n \arrow[r] \arrow[u] & B_i(Y^n) \arrow[r] \arrow[u] & 0 \\
 & 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

The rows are exact and the last two columns are exact, so the 9-lemma assures us that the first column is exact.

By construction, $Z_{i+1}(Y^n) = B_i(Y^{n+1})$ and since

$0 \to Z_i(Y^n) \to Y_1^n \to B_o(Y^n) \to 0$ is exact, $B_o(Y^n) \approx H_o(Y^{n+1})$.

Hence the exactness of the first column tells us that

$$H_i(Y^{n+1}) \approx H_o(Y^{n+1}) \otimes X_i$$

Note that $d(Y_i^{n+1}) \subset Y_i^n$. So now we can conclude that

$$\cdots \to Y_o^{n+1} \to Y_o^n \to \cdots \to Y_o^1 \to Y_o^o \to k$$

is a free resolution of k . We even get as a bonus that

$$\cdots \to Y_i^{n+1} \to Y_i^n \to \cdots \to Y_i^1 \to Y_i^o \to B_{i-1}(K) \to 0$$

is a free resolution of $B_{i-1}(K)$.

The diagram contains one additional piece of information. For $y \in Y_o^n$, $x \in X_i$,

$$\alpha(y \otimes x) = dy \otimes x + (-1)^n \beta(y,x)$$

where $\beta(y,x) \in Y_{i+1}^{n-1}$. This follows easily by chasing the diagram. The reason for the $(-1)^n$ will become evident later. Note that $\beta(y,x)$ need not be a cycle.

Now, of course, the question is when is $\cdots \to Y_o^{n+1} \to Y_o^n \to \cdots \to Y_o^o \to k$ a minimal resolution

of k ? First we produce a necessary condition.

<u>Theorem 4.1.2</u>. If $\cdots \to Y_o^{n+1} \to Y_o^n \to \cdots \to Y_o^o \xrightarrow{\varepsilon} k \to 0$ is a minimal resolution of k, then one can choose β such that $\beta(y,x) \in K$.

<u>Proof</u>. Since the resolution is minimal, $Z_i(Y^n) \subset \underline{m}Y^n$. We proceed by induction on n. For $n = 1$, the assertion is obvious. For $n = 2$, $Y_i^2 = Y_{i+1}^1 + Y_o^1 \otimes X_i$ and if $y \in Y_o^1$, $x \in X_i$,

$$d(y \otimes x) = (dy) \otimes x - \beta(y,x) .$$

So $d\beta(y,x) = \beta(dy,x) \in \underline{m}\, Z_1(Y^o) \subset B_1(Y^o)$ and, consequently, $\beta(y,x)$ may be chosen in Y^o. Here we needed that $H_1(Y^o)$ is a k-space.

Now for the induction step, assume that $\beta(Y_o^{n-1} \otimes X_i) \subset Y^o$. We must show that for $y \in Y_o^n$, $x \in X_i$, $d\beta(y,x) \in B(Y^o)$. But $d\beta(y,x) = \beta(dy,x)$ and by the induction hypothesis, $\beta(dy,x)$ is a cycle in Y^o. Hence there is some $y' \in Y'$ with $dy' = (-1)^n\beta(dy,x)$.

Then $d(dy \otimes x + y') = 0$ so $dy \otimes x + y' \in Z(Y^n)$.
But then $y' \in \underline{m}\, Y^1$ so $d\beta(y,x) = (-1)^n\, dy' \in \underline{m}\, Z(Y^o) \subset$
$\subset B(Y^o)$.

To reverse the theorem, conditions on K are required. In particular, these conditions are met if K is the Koszul complex.

<u>Theorem 4.1.3</u>. Suppose that

$$\underline{m}^2\, K \cap B(K) \subset \underline{m}\, B(K) .$$

Then if one can choose $\beta(y,x) \in K$, the resolution
$\ldots \to Y_o^{n+1} \to Y_o^n \to \ldots \to Y_o^1 \to Y_o^o \to k \to 0$ is minimal.

<u>Proof</u>. The problem reduces to showing inductively that $\beta(y,x) \in \underline{m}\, K$ for $y \in Y_o^n$, $x \in X_i$. In fact, it suffices to show that $\beta(dy,x) \in \underline{m}\, B(K)$. Assume that for $y \in Y_o^{n-1}$, $x \in X_i$, $\beta(y,x) \in \underline{m}\, K$. Then for $y \in Y_o^n$, $dy \in \underline{m}\, Y_o^{n-1}$ by induction, so $\beta(dy,x) \in \underline{m}^2 K \cap B(K) \subset \underline{m}\, B(K)$.

Theorems (4.1.2) and (4.1.3) are based on ideas of Golod [Go].

§2. Massey operations and a theorem of Golod.

In this section K will denote the Koszul complex of R. We will exhibit conditions on K which make it possible to choose $\beta(y,x) \in K$.

First, Y_o can be given a graded K-module structure in the following way. Each Y_i^n is a direct summand of Y_{i-1}^{n+1} which
is in turn a direct summand of Y_{i-2}^{n+2}, etc., and finally a direct summand of Y_o^{n+i}. K is, of course, a graded K-module. This provides a multiplication

$$K_i \otimes Y_j^o \longrightarrow Y_{i+j}^o \subset Y_j^i \subset Y_o^{i+j}$$

Let $z = w + y \otimes x \in Y_j^1$, $\lambda \in K_i$. Then define
$\lambda z = \lambda w + (\lambda y) \otimes x \in Y_{j+1}^i \oplus Y_o^i \otimes X_j = Y_j^{i+1} \subset Y_o^{i+j+1}$.
In general,

$$K_i \otimes Y_j^n \longrightarrow Y_j^{i+n} \subset Y_o^{i+n+j}$$

<u>Lemma 4.2.1</u>. It is possible to choose $\beta(y,x)$ such that for $\lambda \in K$, $\beta(\lambda y,x) = \lambda\beta(y,x)$. Also if d is the differential on Y_o determined by this choice of β, then

$$d(\lambda w) = (d\lambda)w + (-1)^i \lambda dw$$

for $\lambda \in K_i$, $w \in Y_j^n$.

<u>Proof</u>. Assume that for $i + n < m$,

$d(\lambda w) = (d\lambda)w + (-1)^i \lambda\, dw$, $\lambda \in K_i$, $w \in Y_j^n$.

Since $Y_j^{n+1} = Y_{j+1}^n + Y_o^n \otimes X_j$, we need only show that for $y \in Y_o^n$, $x \in X_j$, $\lambda \in K_i$,

$$d(\lambda(y \otimes x)) = (d\lambda)(y \otimes x) + (-1)^i \lambda d(y \otimes x) .$$

But $d(\lambda(y \otimes x)) = d(\lambda y \otimes x) = d(\lambda y) \otimes x + (-1)^{i+n}\beta(\lambda y,x)$

= (by induction) $(d\lambda)(y \otimes x) + (-1)^i(\lambda\, dy) \otimes x$

$+ (-1)^{i+n} \beta(\lambda y,x)$. Thus what we must show is that β can be chosen so that $\beta(\lambda y,x) = \lambda\beta(y,x)$. This in turn will be true if

$$d\beta(\lambda y,x) = d(\lambda\beta(y,x)) .$$

By induction

$$d(\lambda\beta(y,x)) = (d\lambda)\beta(y,x) + (-1)^i \lambda\beta(dy,x) .$$

On the other hand,

$$d\beta(\lambda y,x) = \beta(d(\lambda y),x) = \beta((d\lambda)y + (-1)^i \lambda\, dy,\ x)$$
$$= \beta((d\lambda)y,x) + (-1)^i \beta(\lambda\, dy,x)$$

which, by induction,

$$= (d\lambda)\beta(y,x) + (-1)^i \lambda\beta(dy,x) .$$

<u>Definition 4.2.2</u>. Suppose z_1,z_2 are cycles in K. Then their product z_1z_2 is again a cycle. The class of z_1z_2 in $H(K)$ depends only on the homology classes of z_1 and z_2. This product makes $H(K)$ into a ring. We will now define a sequence of operations of $H(K)$ called the <u>Massey operations</u>.

Suppose $z_1,z_2,z_3 \in Z(K)$ such that z_1z_2 and z_2z_3 are both in $B(K)$. Take $\lambda_1,\lambda_2 \in K$ with $d\lambda_1 = z_1z_2$ and $d\lambda_2 = z_2z_3$. It is then easily checked that $z_1\lambda_2 + \lambda_1z_3$ is a cycle in K, and if $H(K)$ has trivial multiplication, then the homology class of $\gamma(z_1,z_2,z_3) = z_1\lambda_2 + \lambda_1z_3$ is a well defined function of the homology classes of z_1,z_2,z_3. $\gamma(z_1,z_2,z_3)$ is called the ternary Massey operation.

Similarly if all ternary Massey operations on K

vanish, we can define the fourth order Massey operation as follows. Let $z_1, z_2, z_3, z_4 \in Z(K)$, and choose $y(z_1,z_2)$, $y(z_3,z_4)$, $y(z_1,z_2,z_3)$, and $y(z_2,z_3,z_4)$ in K with $dy(z_1,z_2) = z_1 z_2$, $dy(z_3,z_4) = z_3 z_4$, $dy(z_1,z_2,z_3) = \gamma(z_1,z_2,z_3)$ and $dy(z_2,z_3,z_4) = \gamma(z_2,z_3,z_4)$. Then define

$$\gamma(z_1,z_2,z_3,z_4) = z_1\, y(z_2,z_3,z_4) + (-1)^{\deg z_1 + 1}\, y(z_1,z_2)y(z_3,z_4) + (-1)^{\deg z_1 + \deg z_2}\, y(z_1,z_2,z_3)z_4 .$$

Assume now that all $(n-1)$-st order Massey operations vanish and for $2 \le i \le n-1$, let $y(z_1,\ldots,z_i) \in K$ with

$$dy(z_1,\ldots,z_i) = \gamma(z_1,\ldots,z_i) .$$

Then define the n-th order Massey operation to be

$$\gamma(z_1,\ldots,z_n) = z_1 y(z_2,\ldots,z_n) + \sum_{i=2}^{n-2} (-1)^{\sum_{j=1}^{i-1}(\deg z_j + 1)}\, y(z_1,\ldots,z_i)y(z_{i+1},\ldots,z_n) + (-1)^{\sum_{j=1}^{n-2}(\deg z_j + 1)}\, y(z_1,\ldots,z_{n-1})z_n$$

Theorem 4.2.2 (Golod [Go]) . The Eagon resolution is minimal if and only if all the Massey operations on $H(K)$ vanish.

Proof. We know from 4.1.2 and 4.1.3 that the Eagon resolution is minimal if and only if $\beta(dy,x) \in B(K)$ for $y \in Y_o^n$, $x \in X_i$.

To prove the theorem, then, we need only identify the $\beta(dy,x)$ with the Massey operations.

For example, let $z,z' \in Z_i(K)$, $Z_j(K)$ respectively. Then by definition of β , there exist $x \in X_j$, $\lambda \in K_{j+1}$ such that

$$d(\lambda + 1 \otimes x) = z'$$

So $zz' = z(d\lambda + \beta(1,x)) = zd\lambda + \beta(z,x)$ by lemma 4.2.2. Hence $\beta(z,x) \in B(K)$ if and only if $zz' \in B(K)$.

Since $B(K)$ is an ideal in $Z(K)$ and β is K-linear in the first variable, we will have $d\beta(y,x) \in B(K)$ for all y,x if and only if $d\beta(1 \otimes x_1 \otimes \dots \otimes x_n, x_{n+1}) \in B(K)$ for all $x_i \in X$ with $d(1 \otimes x_i) = z_i \in Z(K)$.

It is easily checked by induction that

$$d(1\otimes x_1\otimes\ldots\otimes x_n) = z_1\otimes x_2\otimes\ldots\otimes x_n$$

$$+ \sum_{i=2}^{n-1} (-1)^{\sum_{j=1}^{i-1}(\deg z_j+1)} \beta(1\otimes x_1\otimes\ldots\otimes x_{i-1},x_i)\otimes x_{i+1}\otimes\ldots\otimes x_n$$

$$+ (-1)^{\sum_{j=1}^{n-1}(\deg z_j+1)} \beta(1\otimes x_1\otimes\ldots\otimes x_{n-1},x_n)$$

Then assuming that $\beta(1\otimes x_1\otimes\ldots\otimes x_{i-1},x_i) \in K$ for $2 \leq i \leq n$,

$$d\beta(1\otimes x_1\otimes\ldots\otimes x_n,x_{n+1}) = \beta(d(1\otimes x_1\otimes\ldots\otimes x_n),x_{n+1})$$

$$= z_1\beta(1\otimes x_2\otimes\ldots\otimes x_n,x_{n+1})$$

$$\sum_{i=2}^{n-1} (-1)^{\sum_{j=1}^{i-1}(\deg z_j+1)} \beta(1\otimes x_1\otimes\ldots\otimes x_{i-1},x_i)\beta(1\otimes x_{i+1}\otimes\ldots\otimes x_n,x_{n+1})$$

$$+ (-1)^{\sum_{j=1}^{n-1}(\deg z_j+1)} \beta(1\otimes x_1\otimes\ldots\otimes x_{n-1},x_n)z_{n+1}$$

But this is precisely the formula for the $(n+1)$-st order Massey operation $\gamma(z_1,\ldots,z_{n+1})$.

<u>Corollary 4.2.4</u> (Golod). Let $c_i = \dim_k H_i(K)$, $n = \varepsilon_o(R)$.

Then $P(R) \leq \dfrac{(1+z)^n}{1 - \sum_{i=1}^{n} c_i z^{i+1}}$

and equality holds if and only if all the Massey operations vanish on K.

<u>Proof</u>. Equality holds if and only if

$$b_m = c_1 b_{m-2} + c_2 b_{m-3} + \ldots + c_n b_{m-n-1} + \binom{n}{m}$$

i.e. if and only if Eagon's resolution is minimal.

The inequality was first proved by Serre using the spectral sequence

$$\mathrm{Tor}^R(k, \mathrm{Tor}^{\tilde{R}}(k,R)) \Longrightarrow \mathrm{Tor}^{\tilde{R}}(k,k)$$

associated with the ring homomorphism $\tilde{R} \to R$ where $\tilde{R}$ is a regular local ring of dimension n mapping into R.

<u>Definition 4.2.5</u>. We call a local ring R a <u>Golod ring</u> if all the Massey operations on the Koszul complex of R vanish.

<u>Remark</u>. One way to show that a ring R is a Golod ring is to show that for any two cycles, $z_1, z_2 \in Z(K)$, there

exist homologous cycles $z_1', z_2' \in Z(K)$ with $z_1' z_2' = 0$. To see this, first observe that if S is a complete set of representatives for $H(K)$ (i.e. one cycle for each homology class), then R will be a Golod ring if and only if $d\beta(1 \otimes x_1 \otimes x_1 \otimes \ldots \otimes x_{n-1}, x_n) \in B(K)$ where $d(1 \otimes x_i) \in S$. Now suppose that for any $z_1, z_2 \in S$, $z_1 z_2 = 0$. Then we can assume that $\beta(1 \otimes x_1, x_2) = 0$ for $d(1 \otimes x_i) \in S$ and hence that $\gamma(z_1, z_2, z_3) = 0$ for $z_i \in S$. This, in turn, means we can assume that $\beta(1 \otimes x_1 \otimes x_2, x_3) = 0$, etc.

§3. Applications of Golod's theorem

Two Relations

Lemma 4.3.1. (Shamash). Let R be a regular local ring, K its Koszul complex, and $x \in \underline{m}$. Suppose $z_1, z_2 \in K$ such that $dz_i \in (x)K$, $i = 1,2$. Then there exist $y_1, y_2 \in K$ with $z_i - y_i \in B(K)$, and $y_1 y_2 = 0$.

Proof. Let $dz_i = xw_i$. Then $xdw_i = 0$ so since x is a nonzero-divisor, $dw_i = 0$. Let $T \in K_1$ with $dT = x$. Then $d(Tw_i) = xw_i$, so $z_i - Tw_i \in Z(K) = B(K)$ since K is acyclic. Clearly $Tw_1 Tw_2 = 0$.

Theorem 4.3.2. (Shamash [Sh]). Let $\tilde{R}$ be a regular local ring of dimension n, and suppose that $R = \tilde{R}/x\underline{b}$ where $x \in \tilde{\underline{m}}$ and $\underline{b}$ is an ideal in $\tilde{R}$. Then R is a Golod ring.

Proof. Let $\tilde{K}$ be the Koszul complex of $\tilde{R}$. Then $K = \tilde{K} \otimes_{\tilde{R}} R$ is the Koszul complex of R. Cycles in K are represented by elements of $\tilde{K}$ which bound in xK. Now apply lemma 4.3.1.

Theorem 4.3.3. (Shamash [Sh]). Suppose $\underline{a}$ is an ideal in $\tilde{R}$ generated by two elements $x,y \in \tilde{\underline{m}}^2$. Then $R = \tilde{R}/\underline{a}$ is either a local complete intersection or a Golod ring.

Proof. Either x,y is an $\tilde{R}$-sequence or since $\tilde{R}$ is a unique factorization domain, x and y have a common non-unit factor.

In both cases, the Poincaré series is a rational function. If R is a complete intersection, then $P(R) = \dfrac{(1+z)^n}{(1-z^2)^2}$. Otherwise

Theorem 4.3.4. (Gulliksen). Under the same hypotheses as in Theorem 4.3.3., if R is not a complete intersection, then

$$P(R) = \frac{(1+z)^{n-1}}{1-z-z^2}$$

Proof. Since in this case, R is a Golod ring,

$$P(R) = \frac{(1+z)^n}{1-c_1z^2-c_2z^3-\dots-c_nz^{n+1}}$$

where $c_i = \dim_k H_i(K)$. But $H_i(K) \approx \mathrm{Tor}_i^{\tilde{R}}(k,R)$ so $H_i(K)$ can be computed by resolving R over $\tilde{R}$. Let z be a greatest common divisor of x and y . Then $(x,y) = z(x',y')$, where x',y' is an $\tilde{R}$-sequence. Since z is a nonzero-divisor in $\tilde{R}$, $(x,y) \approx (x',y')$. Hence $H(K) \approx E \otimes k$ where $E = \tilde{R}\langle T_1,T_2; dT_1 = x', dT_2 = y'\rangle$. It follows that $c_1 = 2$, $c_2 = 1$ and $c_i = 0$ for $i > 2$. Then

$$P(R) = \frac{(1+z)^n}{1-2z^2-z^3} = \frac{(1+z)^{n-1}}{1-z-z^2}$$

Two Generators

Theorem 4.3.5. (Scheja [Sc]). Let $\tilde{R}$ be a two dimensional regular local ring and $\underline{a}$ an ideal in $\tilde{\underline{m}}^2$. Then $R/\underline{a}$ is either a local complete intersection or a Golod ring. (Proof from Shamash [Sh]).

Proof. Let K be the Koszul complex of R . We can assume that $\underline{a}$ has grade 2 for otherwise $0: \underline{m} = 0$ and $(Z_1(K))^2 \subset (0:\underline{m})K_2 = 0$ which implies that R is a Golod ring.

Let x_1, x_2 be a maximal $\tilde{R}$-sequence in a. Then it is known [Z.S, II, p.395] that the ideal $\underline{b} = (x_1, x_2)$ is irreducible and $\dim(\underline{b}:\tilde{\underline{m}})/\underline{b} = 1$. So in $R' = \tilde{R}/\underline{b}$, $(0:\underline{m}')$ is a minimal ideal and is therefore contained in any non-zero ideal of R'. If $\underline{b} = \underline{a}$, R is a local complete intersection. Otherwise, $\underline{a}/\underline{b} \neq 0$ in R' and $\underline{b}: \tilde{\underline{m}} \subset \underline{a}$.

Now assume that R is not a local complete intersection, and let $x_1, x_2 \in \underline{a}$. We must show that if $\tilde{z}_1, \tilde{z}_2 \in \tilde{K}_1$ with $d\tilde{z}_i = x_i$, $i = 1,2$, then $\tilde{z}_1\tilde{z}_2 \in \underline{a}\,\tilde{K}_2$ so that $z_1 z_2 = 0$ in K.

<u>Case 1</u>. x_1 and x_2 are relatively prime.

Then x_1, x_2 are a maximal $\tilde{R}$-sequence in $\underline{a}$ so $(x_1, x_2): \tilde{\underline{m}} \subset \underline{a}$ and

$$\tilde{z}_1\tilde{z}_2 \in ((x_1, x_2): \tilde{\underline{m}})\,\tilde{K}_2 \subset \underline{a}\,\tilde{K}_2 .$$

<u>Case 2</u>. x_1 and x_2 have a common non-unit divisor. Then by Lemma 4.3.1, there exist cycles $z_1', z_2' \in K_1$ with $z_1' \sim z_1$, $z_2' \sim z_2$ and $z_1' z_2' = 0$. Then $z_1 z_2 \sim z_1' z_2' = 0$, and since $B_2(K) = 0$, $z_1 z_2 = 0$.

For the sake of completeness, we include the cases of one generator and one relation. If $\underline{m}$ is generated by a single element, then every ideal is principal so this case is included in the case of one relation. If $\underline{a}$ has one generator, R is a local complete intersection with $P(R) = \dfrac{(1+z)^n}{1-z^2}$.

<u>Theorem 4.3.6</u>. (Golod). $k[x_1,\ldots,x_n]/(x_1,\ldots,x_n)^d$ is a Golod ring.

<u>Proof</u>. It is easily seen that

$$Z_i(K) = (x_1,\ldots,x_n)^{d-1} K_i + B_i(K) .$$

The theorem follows.

To summarize, the Poincaré series is known to be a rational function if R is a local complete intersection or a Golod ring. These two cases include between them the cases where $\underline{m}$ has two generators or R is defined by two relations. In addition, a new instance has been discovered by H. Wiebe.

<u>Theorem</u> (Wiebe [Wi]). Let R be a local Gorenstein ring

with $\varepsilon_o(R) = n$ and homological codimension c. If R is not a local complete intersection and $c \geq n - 3$, then

$$P(R) = \frac{(1+z)^n}{1-rz^2-rz^3+z^5}, \quad r = \varepsilon_1(R).$$

§4. <u>An Exact Couple</u>.

Again K is the Koszul complex of a local ring R. The short exact sequences

$$0 \longrightarrow B_p(K) \longrightarrow Z_p(K) \longrightarrow H_p(K) \longrightarrow 0$$

induce exact homology triangles

$$\begin{tikzcd}
\mathrm{Tor}^R(k, B(K)) \arrow[rr, "i"] & & \mathrm{Tor}^R(k, Z(K)) \arrow[dl, "j"] \\
& \mathrm{Tor}^R(k, H(K)) \arrow[ul, "h"] &
\end{tikzcd}$$

Then from the sequences

$$0 \longrightarrow Z_{p+1}(K) \longrightarrow K_{p+1} \longrightarrow B_p(K) \longrightarrow 0$$

we get isomorphisms

$$\mathrm{Tor}_p(k, B_q(K)) \xrightarrow{\partial} \mathrm{Tor}_{p-1}(k, Z_{q+1}(K))$$

for $p \geq 1$, $q \geq 0$.

Now for $p,q \geq 0$, define

$$D_{p,q} = \mathrm{Tor}_p(k, B_q(K))$$

$$E_{p,q} = \mathrm{Tor}_p(k, H_q(K)) \qquad \text{and}$$

$$f: D_{p,q} \longrightarrow D_{p+1,q-1} \quad \text{by} \quad f = \partial^{-1} i, \ q \geq 1$$

$$g: D_{p,q} \longrightarrow E_{p-1,q+1} \quad \text{by} \quad g = j\partial \ , \quad p \geq 1$$

If p or $q < 0$, define $E_{p,q} = 0$ and

$$D_{p,q} = \begin{cases} 0 & \text{if } q < 0 \\ E_{o,o} & \text{if } p < 0 \text{ and } p+q = -1 \\ D_{o,p+q} & \text{if } p < 0 \text{ and } p+q \neq -1 . \end{cases}$$

Then if we define

$$E_{o,o} \xrightarrow{1} D_{-1,o}$$

and $D_{p,q} \xrightarrow{1} D_{p+1,q-1}$ if $p < 0$

and $E_{o,q} \xrightarrow{0} D_{-1,q}$

we get an exact couple $\mathcal{C}$

$$\begin{tikzcd} D \arrow[rr, "f"] & & D \arrow[dl, "g"] \\ & E \arrow[ul, "h"] & \end{tikzcd}$$

where f has bidegree $(1,-1)$, g has bidegree $(-1,1)$ and h $(-1,0)$.

We can then [Ma,p.336] form the derived couple

$$\mathcal{C}^2 : \quad \begin{tikzcd} D^2 \arrow[rr, "f_2"] & & D^2 \arrow[dl, "g_2"] \\ & E^2 \arrow[ul, "h_2"] & \end{tikzcd}$$

where $D^2_{p,q} = f\, D_{p-1,q+1}$ and $E^2_{p,q}$ is defined as the homology of the complex

$$E_{p+2,q-1} \xrightarrow{gh} E_{p,q} \xrightarrow{gh} E_{p-2,q+1}$$

f_2 is the map induced by f, $g_2(fd) = gd + ghE$ for $d \in D$, and $h_2(e + ghE) = he$, $e \in E$. The derived couple is again exact. Iteration produces a sequence of derived couples

$$\mathcal{C} = \mathcal{C}^1, \mathcal{C}^2, \mathcal{C}^3, \ldots$$

<u>Theorem 4.4.1.</u> Let $n = \varepsilon_o(R)$. Then

$$E^{n+1}_{p,q} = 0 \quad \text{if} \quad q \neq 0$$

and

$$E^{n+1}_{p,o} \approx K_p \otimes_R k$$

<u>Proof</u>. We use formula 5.3 from [Ma, p.336].

$$E^r = h^{-1}(f^{r-1}D)/g(\operatorname{Ker} f^{r-1}) .$$

Since $f^n: D_{p,q} \longrightarrow D_{p+n,q-n}$, $f^n = 0$ if $q < n$.

Hence for $0 < q \leq n$

$$E^{n+1}_{p,q} = h^{-1}(0)/g(D_{p+1,q-1}) = 0$$

by exactness.

To compute $E^{n+1}_{p,o}$

$$0 = D_{p+1,-1} \xrightarrow{g} E_{p,o} \xrightarrow{h} D_{p-1,o} \xleftarrow{f^n} D_{p-1-n,n}$$

So $E^{n+1}_{p,o} = h^{-1}(f^n D_{p-1-n,n})$.

Recall now the definition of the conjugate of an element in $\mathrm{Tor}^R(k,k)$. Let X be a minimal algebra resolution of k . K is then identified with the sub-algebra of X generated by the elements of degree one. If

$$x \otimes 1 + \ldots + 1 \otimes y$$

is a cycle in $X \underset{R}{\otimes} X$, of degree m , then the class of y in $\mathrm{Tor}^R_m(k,k)$ is the <u>conjugate</u> of the class of x . $(\bar{y} = C(\bar{x}))$.

In particular, if

$$x \otimes 1 + \ldots + 1 \otimes y$$

is a cycle of degree m in $X \otimes K$, then $C(\bar{x}) \in K \underset{R}{\otimes} k = \Lambda$, the subalgebra of $\mathrm{Tor}^R(k,k)$ generated by the elements of degree 1. Since $C^2 = 1$ and $C(\Lambda) = \Lambda$, this shows that $\bar{x} \in \Lambda$.

We return to the computation of $E^{n+1}_{p,o}$. If $p \geq n+1$,

$$D_{p-1-n,n} = \mathrm{Tor}_{p-1-n}(k, B_n(K)) = 0$$

so $E^{n+1}_{p,o} = 0$.

If $p < n+1$,

$$D_{p-1-n,n} = D_{o,p-1} = \mathrm{Tor}_o(k, B_{p-1}(K)) = B_{p-1}(K) \otimes k .$$

Then

$$E_{p,o} \xrightarrow[\approx]{h} D_{p-1,o} \xleftarrow{f^{p-1}} D_{o,p-1} \xleftarrow{1} D_{p-1-n,n}$$

Now, by the definition of f, the map

$$\Lambda^p \approx D_{o,p-1} \xrightarrow{f^{p-1}} D_{p-1,o} \xrightarrow{h^{-1}} E_{p,o} = \mathrm{Tor}^R_p(k,k)$$

is up to a sign difference, the conjugation.

So $$K_p \underset{R}{\otimes} k \approx \Lambda^p \xrightarrow[\approx]{C} E^{n+1}_{p,o}$$

We wish to take advantage of the $H(K)$-module structure of $E_{p,q}$. In an obvious way

$$E_{p,q} \otimes H_r(K) \longrightarrow E_{p,q+r}$$

<u>Theorem 4.4.2.</u> The map $gh: E_{p,q} \longrightarrow E_{p-2,q+1}$ is $H(K)$-linear.

<u>Proof</u>. gh breaks down as follows

$$\mathrm{Tor}_p(k,H_q(K)) \xrightarrow{h} \mathrm{Tor}_{p-1}(k,B_q(K))$$

$$\xrightarrow[\approx]{\partial} \mathrm{Tor}_{p-2}(k,Z_{q+1}(K)) \xrightarrow{j} \mathrm{Tor}_{p-2}(k,H_q(K)) .$$

We will show that for $\bar{u} \in E_{p,o}$, $\bar{z} \in H_q(K)$ $gh(\bar{u} \otimes \bar{z}) = (gh(\bar{u}))\bar{z}$.

Let X be a minimal algebra resolution of k over R, and let $u \in X_p$, $z \in Z_q(K)$ with classes $\bar{u} \in X_p \otimes k = \mathrm{Tor}_p^R(k,k)$ and $\bar{z} \in H_q(K)$. Since h is a connecting homomorphism, we compute $h(\bar{u} \otimes \bar{z})$ by expressing $du \otimes z$ as an element of $X_{p-1} \otimes B_q(K)$.

Suppose $du = \Sigma\, c_i v_i$ with $c_i \in \underline{m}$, $v_i \in X_{p-1}$.

Then $du \otimes z = \Sigma\, v_i \otimes c_i z \in X_{p-1} \otimes B_q(K)$. ∂ is now computed by "integrating" the second factor and "differentiating" the first.

$du \otimes z = \Sigma\, v_i \otimes c_i z = \Sigma\, v_i \otimes d(C_i z)$ where $C_i \in X_1$ with $dC_i = c_i$. So the image of $\Sigma\, dv_i \otimes C_i z$ in $\mathrm{Tor}_{p-2}(k, Z_{q+1}(K))$ is $\partial h(\bar{u} \otimes \bar{z})$ and its image in $\mathrm{Tor}_{p-2}(k, H_{q+1}(K))$ is $gh(\bar{u} \otimes \bar{z})$. However, the same argument shows that $gh(\bar{u})$ = the image in $\mathrm{Tor}_{p-2}(k, H_{q+1}(K))$ of $\Sigma\, dv_i \otimes C_i$. This establishes the linearity.

We can use the exact couple to obtain explicit expressions for the Betti numbers b_i , $0 \leq i \leq 4$ in terms of the deviations $\epsilon_o = n$, $\epsilon_1 = \dim H_1(K)$, $\epsilon_2 = \dim H_2(K)/(H_1(K))^2$, and the homology algebra $H(K)$.

<u>Theorem 4.4.3.</u> (Levin)

$$b_o = 1 \ , \ b_1 = \epsilon_o \ , \ b_2 = \binom{n}{2} + \epsilon_1, \ b_3 = \binom{n}{3} + \epsilon_o\epsilon_1 + \epsilon_2$$

and

$$b_4 = \binom{n}{4} + \epsilon_o\epsilon_2 + \binom{n}{2}\epsilon_1 + \dim H_3(K)/H_1(K)H_2(K)$$
$$+ \epsilon_1^2 - \dim(H_1(K))^2 \ .$$

Proof. The expressions for b_o, b_1, b_2, b_3 are well known. To compute $b_4 = \dim_k E^1_{4,o}$, note that $E^r_{p,q} \longrightarrow E^r_{p-r-1,q+r}$ and by Theorem 4.4.1, $E^5_{p,q} = 0$ for $q \neq 0$, and $E^5_{p,o} = \Lambda^p$. Since E^5 is the homology of E^4, $E^4_{p,q} = E^5_{p,q}$ for $0 \leq p \leq 4$. This, in turn, yields information about E^3, namely, $E^3_{2,1} = E^3_{1,2} = E^3_{1,1} = 0$ and

$$0 \longrightarrow E^4_{4,o} \longrightarrow E^3_{4,o} \longrightarrow E^3_{o,3} \longrightarrow 0$$

is exact.

Then $E^2_{2,1} = E^2_{1,1} = 0$ and

$$0 \longrightarrow E^3_{4,o} \longrightarrow E^2_{4,o} \longrightarrow E^2_{1,2} \longrightarrow 0$$

is exact. Finally,

$$0 \longrightarrow E^2_{4,o} \longrightarrow E^1_{4,o} \longrightarrow E^1_{2,1} \longrightarrow E^1_{o,2}$$

is exact. All we have to do now is compute $E^3_{o,3}$, $E^2_{1,2}$ and $\mathrm{Ker}(E^1_{2,1} \longrightarrow E^1_{o,2})$. $E^2_{1,2}$ is the homology of the complex

$$E^1_{3,1} \longrightarrow E^1_{1,2} \longrightarrow 0$$

Since $E^1_{3,o} \longrightarrow E^1_{1,1}$ is an epimorphism $(E^2_{1,1} = 0)$. Theorem 4.4.2 gives that the image of $E^1_{3,1}$ in $E^1_{1,2}$ is $\approx \mathrm{Tor}_1(k,k) \otimes (H_1(K))^2$. So

$$E^2_{1,2} \approx \mathrm{Tor}_1(k,k) \otimes (H_2(K)/(H_1(K))^2) .$$

Similarly, since $E^1_{2,o} \longrightarrow E^1_{o,1}$ is an epimorphism, the image of $E^1_{2,1}$ in $E^1_{o,2}$ is $(H_1(K))^2$. So

$$\dim \mathrm{Ker}(E^1_{2,1} \longrightarrow E^1_{o,2}) = b_2\varepsilon_1 - \dim(H_1(K))^2 .$$

We compute $E^3_{o,3}$ from the exact sequences

$$E^1_{2,2} \longrightarrow E^1_{o,3} \longrightarrow E^2_{o,3} \longrightarrow 0$$

and $E^2_{3,1} \longrightarrow E^2_{o,3} \longrightarrow E^3_{o,3} \longrightarrow 0$.

The same sort of analysis gives

$$E^2_{o,3} = H_3(K)/H_1(K)H_2(K)$$

We claim that $E^2_{3,1} \longrightarrow E^2_{o,3}$ is the zero map. This will prove that $E^3_{o,3} = E^2_{o,3}$.

Let $u = \sum_i e_i \otimes \bar{z}_i \in E^1_{3,1}$ represent an element of $E^2_{3,1}$. To compute $g_2h_2(u)$, we use the diagram

$$\begin{tikzcd}
E^1_{3,1} \arrow[r, "h"] & D^1_{2,1} \arrow[r, "g"] & E^1_{1,2} \\
 & & \mathrm{Tor}_1(k,Z_2(K)) \arrow[ul, "\partial"'] \\
 & D^1_{1,2} \arrow[uu, "f"] \arrow[ur, "i"] \arrow[r, "g"] & E^1_{o,3}
\end{tikzcd}$$

Since u represents an element of $E^2_{3,1}$, $gh(u) = 0$ in $E^1_{1,2}$ so $h(u) = f(v)$, $v \in D^1_{1,2}$ then $g_2h_2(u) =$ the class of $g(v)$ in $E^1_{o,3}$.

Let X be a minimal algebra resolution of k. Then $\partial^{-1}h(e_i)$ in $\mathrm{Tor}_1(k,Z_1(K))$ is represented by a cycle of the form $\sum_j T_j \otimes y_{ij} \in X_1 \underset{R}{\otimes} Z_1(K)$ where the T_j are a basis for $X_1 = K_1$ and $dT_j = t_j \in \underline{m}$. This means that $\sum_j t_j y_{ij} = 0$ so that $\sum_j T_j y_{ij} \in Z_2(K)$.

Then, by linearity, $\partial^{-1}h(u)$ is represented by the cycle $\sum_{i,j} T_j \otimes y_{ij}z_i \in X_1 \otimes Z_2(K)$. Since $\partial^{-1}h(u)$ is in the image of i, each $\sum_i y_{ij}z_j = d\lambda_j$ for some $\lambda_j \in K_3$. Then $\sum_j t_j\lambda_j$ is a cycle in $Z_3(K)$ whose image in $E^2_{o,3}$ is $g_2h_2(u)$.

But $\sum_j t_j\lambda_j = \sum_j (d(T_j\lambda_j) + T_j d\lambda_j)$

$= d(\sum_j T_j\lambda_j) + \sum_{j,i} T_j y_{ij} z_j \in B_3(K) + Z_2(K)Z_1(K)$,

and thus $\sum_j t_j\lambda_j$ represents the zero class in $E^2_{o,3}$.

Putting all the information together, one gets the desired expression for b_4 .

From the formula for b_4 , it is easy to compute the third deviation

$$\epsilon_3(R) = \dim H_3(K)/H_1(K)H_2(K) + \binom{\epsilon_1}{2} + \dim H_1(K)^2$$

This is the expression promised in (3.3.4 (iii)). The same result has also been recently obtained by Sakuma and Okuyama.

We conclude this section with a theorem about the dual algebra $\mathrm{Ext}_R(k,k)$.

<u>Theorem 4.4.4</u>. (Levin) Let R be a Golod ring with $\epsilon_o(R) = n$. Then the algebra $\mathrm{Ext}_R(k,k)$ with the Yoneda product is generated by elements of degree $\leq n + 1$. In particular, it is finitely generated.

<u>Proof</u>. We show that the non-zero primitive elements (see Ch.Two) of $\mathrm{Tor}^R(k,k)$ all have degree $\leq n + 1$. Referring to the notation in §2, it is sufficient to show

that the class of $1 \otimes x_1 \otimes x_2 \otimes \ldots \otimes x_m$ in $\mathrm{Tor}^R(k,k)$ is not primitive if $m > 1$. An element $\lambda \otimes x_1 \otimes \ldots \otimes x_m$ in Y_o $(\lambda \in K)$ is said to have <u>weight</u> m. Filter Y_o by letting $F_m(Y^o)$ be all R-linear combinations of elements of weight $\leq m$. Then it is easily verified that (by abuse of notation) the component of $\Delta(1 \otimes x_1 \otimes x_2 \otimes \ldots \otimes x_m)$ in $\mathrm{Tor}_2(k,k) \otimes \mathrm{Tor}(k,k)$ is congruent modulo $\mathrm{Tor}_2(k,k) \otimes (F_{m-2}(Y^o) \otimes k)$ to

$$\pm (C(1 \otimes x_1)) \otimes ((1 \otimes x_2 \otimes \ldots \otimes x_m)$$

and is therefore different from zero.

On the basis of this evidence, it might be conjectured that the primitive elements always have degree $\leq n + 1$. However in the following counterexample $\epsilon_o(R) = 3$ and $\mathrm{Tor}_5^R(k,k)$ contains a non-zero primitive element.

Let $R = k[[t_1,t_2,t_3]]/(t_1^3,t_2^3,t_3^3,t_1^2t_2t_3,t_1t_2^2t_3)$.

A minimal algebra resolution of k over R must contain the (R)-algebra

$$R\langle T_1,T_2,T_3,S_1,S_2,S_3,S_4,S_5;\ dT_i = t_i\ ;\ dS_i = t_i^2T_i\ ,$$
$$1 \leq i \leq 3\ ;\ dS_4 = t_1t_2t_3T_1,\ dS_5 = t_1t_2t_3T_2\rangle\ .$$

It is easily checked that the products $(dS_i)(dS_j)$, $i \leq i,j \leq 3$ are non-trivial (not 2-boundaries), and $(dS_1)(dS_2)\, S_3$ is a 4-cycle. Suppose $dU = (dS_1)(dS_2)S_3$. We claim that U has non-zero class in $\mathrm{Tor}_5(k,k)$ and is primitive.

We can use the exact couple to show that U has non-zero class. Since $\epsilon_o(R) = 3$, $E^3_{2,2} = 0$. This shows that $E^2_{5,o} \longrightarrow E^2_{2,2}$ is an epimorphism. One checks that the class of U belongs to $E^2_{5,o}$ and its image in $E^2_{2,2}$ is $\neq 0$ since all the $(dS_i)(dS_j)$ $1 \leq i, j \leq 3$ are non-trivial.

To show that the class of U is primitive, suppose $dV = (dS_1)(dS_3)\, S_2$. Then

$$U \otimes 1 - t_1^2 t_2 T_2 \otimes T_1 T_2 S_3 + t_1^2\, S_2 \otimes T_1 S_3$$

$$+ t_1^2 t_3 (T_3 S_2) \otimes T_1 T_3 - t_1 t_3^2\, T_1 T_3 S_2 \otimes T_1$$

$$- 1 \otimes V$$

is a cycle defining $\Delta(U)$.

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(*) Lemma 1 is false. A correction is given in a later volume.