HIGHER HOMOTOPIES AND GOLOD RINGS

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ABSTRACT. We study the homological algebra of an $R = Q/I$-module M using A_∞-structures on Q-projective resolutions of R and M. We use these higher homotopies to construct an R-projective bar resolution of M, Q-projective resolutions for all R-syzygies of M, and describe the differentials in the Avramov spectral sequence for M. These techniques apply particularly well to Golod modules over local rings. We characterize R-modules that are Golod over Q as those with minimal A_∞-structures. This gives a construction of the minimal resolution of every module over a Golod ring, and it also follows that if the inequality traditionally used to define Golod modules is an equality in the first $\dim Q+1$ degrees, then the module is Golod, where no bound was previously known. We also relate A_∞-structures on resolutions to Avramov's obstructions to the existence of a dg-module structure. Along the way we give new, shorter, proofs of several classical results about Golod modules.

1. Introduction

Let R be an algebra over a commutative ring Q, and M an R-module. If R and M are projective as Q-modules, the bar resolution gives a canonical R-projective resolution of M. To generalize to the case that R or M is not projective over Q, we have to replace R, respectively M, by a Q-projective resolution with an algebra, respectively module, structure. Indeed, let us assume that $R = Q/I$ is a cyclic Q-algebra, a case of interest in commutative algebra. In this case, Iyengar showed in [22] that if $A \xrightarrow{\simeq} R$ is a Q-projective resolution with a differential graded (dg) algebra structure, and $G \xrightarrow{\simeq} M$ is Q-projective resolution with a dg A-module structure, then the bar resolution on A and G gives an R-projective resolution of M. Recall that a differential graded algebra/module, is an algebra/module-over-that-algebra object in the category of Q-chain complexes.

In this paper we generalize Iyengar's result by constructing an R-projective resolution of M using A_∞, or "dg up-to-coherent-homotopy", structures on Q-projective resolutions. The distinction of A_∞-machinery is that every Q-projective resolution of R has an A_∞-algebra structure and every Q-projective resolution of M has an A_∞ A-module structure, by [9, §2.4][1]. This is particularly useful in the study of local rings, where we are interested in minimal resolutions. Avramov showed that the minimal Q-free resolution of R need not have a dg-algebra structure, or if it does, the minimal Q-free resolution of M need not have a dg-module structure over it [3]. While Q-free resolutions with dg structures always exist, the dg bar resolution on a non-minimal Q-resolution is non-minimal over R. Thus using A_∞-structures removes a non-vanishing obstruction to a dg bar resolution being minimal.

[1]This follows the classical spirit of A_∞-machinery, but because A is not augmented and the ground ring is a not a field, it does not follow from arguments in the literature.

When Q is local Noetherian and M is finitely generated, there is a well-known upper bound on the size of a minimal R-free resolution of M, called the Golod bound. The A_∞-bar resolution on minimal Q-free resolutions realizes this bound. When the bound is achieved M is by definition a Golod module over Q. Summarizing the previous two paragraphs, we have the following.

Theorem A. *Let Q be a commutative ring, R a cyclic Q-algebra, and M an R-module. Let A and G be Q-projective resolutions of R and M, respectively. There is an R-projective bar resolution of M built from A, G and A_∞-structures on these resolutions. If Q is local Notherian and M is finitely generated and A, G are minimal, the bar resolution is minimal if and only if M is a Golod module over Q.*

In particular the minimality of the bar resolution does not depend on the A_∞-structures chosen.

The Golod bound first appeared in [15], and has been well-studied since. In particular it is known that Golod modules occur with some frequency. To make this precise, we recall some terminology. The map $Q \to R$ is a Golod morphism when the residue field of R is a Golod module over Q. The ring R is a Golod ring if there is a Golod morphism $Q \to R$ with Q regular local. Levin showed that for a local ring $(Q, \mathfrak{n})$, the map $Q \to Q/\mathfrak{n}^k$ is Golod for all $k \gg 0$ [27, 3.15]. This was expanded upon by Herzog, Welker and Yassemi, who showed that if Q is regular and I is any ideal, then Q/I^k is Golod for all $k \gg 0$ [20], and more recently by Herzog and Huneke [19, Theorem 2.3] who showed that if Q is a standard graded polynomial algebra over a field of characteristic zero and I is any graded ideal, then $R = Q/I^k$ is a Golod ring for all $k \geq 2$ (there is an obvious analogue of Golodness for standard graded rings and graded modules over them). Finally, Lescot showed that the $\dim Q$-th syzygy of every module over a Golod ring is a Golod module [25].

The interpretation of Golodness in terms of A_∞-structures in Theorem A is quite useful. The first new result is the following.

Corollary 1. *If Q is regular and the Golod bound is an equality in the first* $\dim Q +$ 1-*degrees, then M is a Golod module.*

There was no previous bound of this type known and the proof follows surprisingly easily from the structure of the bar resolution. Indeed, an A_∞-structure on A, respectively A_∞ A-module structure on G, is given by a sequence of maps

$$\widetilde{m}_n : A_{\geq 1}^{\otimes n} \to A_{\geq 1} \quad \text{respectively} \quad \widetilde{m}_n^G : A_{\geq 1}^{\otimes n-1} \otimes G \to G$$

for all $n \geq 1$, where $\widetilde{m}_n$ has degree $2n - 1$ and $\widetilde{m}_n^G$ has degree $2n - 2$. Since we assume Q is regular, A and G have length at most $\dim Q$. By degree considerations there are only finitely many non-zero maps in any A_∞-structure on A or G, and by construction these all appear as direct summands in the first $\dim Q + 1$-degrees of the bar resolution of M. Thus if the bar resolution is minimal in these degrees, it must be minimal in all degrees. Corollary 1 follows almost immediately.

Using the other direction of Theorem A, if M is a Golod module, the bar resolution is a minimal R-free resolution constructed from the finite data of A, G and their higher homotopy maps. Coupled with Lescot's result mentioned above that every module over a Golod ring has a syzygy that is a Golod module, we have the following.

Corollary 2. *If R is a Golod ring, there is a finite construction of the minimal free resolution of every finitely generated R-module.*

This adds a large new class of rings for which an explicit construction of minimal resolutions exist. And let us emphasize this is a construction. Finding the non-zero maps $\widetilde{m}_n$ and $\widetilde{m}_n^G$ in A_∞-structures on A and G can be implemented on a computer, using e.g. [16] (although it is currently only feasible for small examples). We also give a new proof of Lescot's result in 6.14, using techniques developed earlier in the paper, so Corollary 2 is self-contained.

In addition to the the bar resolution, we introduce two other constructions that use A_∞-structures to study R-modules. To describe the first, we let $\Omega_R^1(M)$ be the first syzygy of M over R.

Proposition B. *Let Q be a commutative ring, R a cyclic Q-algebra, and M an R-module. Let A and G be Q-projective resolutions of R and M with A_∞ algebra and A_∞ A-module structures, respectively. There is a complex $\mathrm{syz}_R^1(G)$, built from A, G and A_∞-structures on these complexes, with a canonical A_∞ A-module structure and a quasi-isomorphism*

$$\mathrm{syz}_R^1(G) \xrightarrow{\simeq} \Omega_R^1(M).$$

Thus $\mathrm{syz}_R^1(G)$ is a Q-projective resolution of the first R-syzygy of M.

This is a special case of a well-known procedure for calculating a Q-projective resolution of an R-syzygy of M using a mapping cone. The novelty here is that we do not have to choose a lift of the multiplication map $R \otimes M \to M$ to Q-free resolutions: it is contained in the A_∞-structure. And since the new Q-resolution has an A_∞ A-module structure, one can iterate and construct canonical Q-projective resolutions of all R-syzygies of M. If Q is local and A, G are minimal, these new resolutions are not necessarily minimal, but are for Golod modules. Thus we have a closed description of the minimal Q-free resolution of every R-syzygy of a Golod module, in terms of A_∞-structures on A and G.

The second construction uses A_∞-structures to describe the differentials of the Avramov spectral sequence. This spectral sequence, introduced in [3] using an Eilenberg-Moore type construction, transfers information from Q to R, the opposite direction of the standard change of rings spectral sequence. Specifically, we have the following.

Theorem C. *Let Q be local with residue field k and A and G minimal Q-free resolutions of R and M, respectively. A_∞-structures on A and G describe the differentials in the Avramov spectral sequence for M:*

$$E^2_{p,q} = \left(\mathrm{Tor}_p^{\mathrm{Tor}_*^Q(R,k)}(\mathrm{Tor}_*^Q(M,k), k)\right)_q \Rightarrow \mathrm{Tor}_{p+q}^R(M,k).$$

While the precise description of the differentials is technical, as a corollary we can describe the relation of A_∞-structures to the obstructions constructed in [3]. To state this, assume that A is a dg-algebra. Avramov constructed a group which vanishes if G has a dg A-module structure. This is equivalent to the existence of an A_∞ A-module structure $(\widetilde{m}_n^G)$ with $\widetilde{m}_n^G = 0$ for all $n \geq 3$. Thus if the obstruction is nonzero, then $\widetilde{m}_n^G$ is nonzero for some $n \geq 3$ in any A_∞ A-module structure on G. But in fact, using Theorem C we show more is true.

Corollary 3. *If Avramov's obstruction is non-zero, then in any A_∞ A-module structure $(\widetilde{m}_n^G)$ on G, $\widetilde{m}_n^G$ contains a unit for some $n \geq 3$.*

Thus these obstructions only detect the existence of a dg-structure modulo the maximal ideal of R. This is motivated by a strategy sketched in [3] to prove the Buchsbaum-Eisenbud rank conjecture [7], where the key step is finding an obstruction theory that detects precisely when G has a dg-module structure. It would be enough to do this in the case A is a Koszul complex.

The three tools described above (the lettered results) work well in conjunction. We use them here to give new, short, proofs of the following four classical results on Golod modules.

Corollary 4. *Let $(Q, \mathfrak{n}, k)$ be a regular local ring, $R = Q/I$ with $I \subseteq \mathfrak{n}^2$, and M a finitely generated R-module.*

(1) *If M is a Golod module, then its first syzygy $\Omega^1_R(M)$ is also Golod.*
(2) *If M is a Golod module, then R is a Golod ring.*
(3) *If R is Golod, then $\Omega^{d+1}_R(M)$ is a Golod module, where $d = \dim Q$.*
(4) *M is Golod if and only if the change of rings maps*
$$\mathrm{Tor}^Q_*(M, k) \to \mathrm{Tor}^R_*(M, k) \quad \mu : \mathrm{Tor}^Q_*(\Omega^1_R(M), k) \to \mathrm{Tor}^R_*(\Omega^1_R(M), k)$$
are injective.

The first is due to Levin [28], the second, independently, to Lescot [25] and Levin [28], the third Lescot [25], and the fourth Levin [27]. Let us quickly sketch our proofs. It follows from Theorem A that if M is Golod, then any A_∞-structures $\widetilde{m}, \widetilde{m}^G$ are minimal. The A_∞-structure maps on the Q-free resolution $\mathrm{syz}^1_R(G)$ of $\Omega^1_R(M)$ given in Proposition B are built from $\widetilde{m}$ and $\widetilde{m}^G$, thus are also minimal, and so $\Omega^1_R(M)$ is Golod, again by Theorem A. Using the computations of Theorem C, we show in Theorem 6.13 that R is Golod if and only if $\widetilde{m}$ is minimal (i.e. this implies that the A_∞ A-module structure on the resolution of k is minimal). Coupled with Theorem A, this makes it clear that if M is a Golod module, then R is a Golod ring. To prove the third result, we show that for $n > \dim Q$, the nth iterate of the syzygy resolution of Proposition B has an A_∞ A-module structure with higher maps constructed entirely from $\widetilde{m}$ (i.e. we have iterated the maps of $\widetilde{m}^G$ into the differential). So if R is Golod, then $\widetilde{m}$ is minimal, and this implies the higher homotopies for a resolution of $\Omega^n_R(M)$ are also minimal when $n > \dim Q$, so the syzygy module is Golod. The proof of the fourth result uses the fact that the change of rings maps are edge maps in the Avramov spectral sequence.

To end the introduction, let us discuss the context and future directions of this work. Dg-techniques have been used been widely in homological commutative algebra, see [2] and its references, but A_∞-machinery has been used sparingly, if at all. One probable reason is that unless $Q \to R$ is the identity map, the resolution A is not augmented over Q. And until recently, most parts of the A_∞-machinery required an augmentation. Positselski showed in [31] how to compensate for a lack of augmentation by including a curvature term on the bar construction. It is not made explicit below, but this is the key to using A_∞-structures as we do. There are further details on this setup in [9, §2].

The results of the paper generalize in a straightforward way to the case $Q \to R$ is module finite[2] and the basic setup of the bar resolution generalizes to the case R is an arbitrary A_∞-algebra over Q. We assume that R is cyclic to simplify the definition of A_∞-algebra on a Q-projective resolution of R; in general the curvature term is non-zero in higher degrees, and this complicates notation, especially in

[2]This assumption is necessary to discuss the difference in the size of minimal Q and R free resolutions of an R-module M in the way we do here, in particular for a Golod bound to exist.

definitions. There is a very interesting generalization in a different direction. Here we are implicitly using the twisting cochain $\operatorname{Bar} A \to A$ (see e.g. [30] and [21] for classical references on dg-bar constructions and twisting cochains, and [9] for generalizations of the definitions and basic results to A_∞-objects and curvature on the coalgebra side) called the universal twisting cochain. When R is Golod, our results show that the universal twisting cochain is the smallest *acyclic* twisting cochain, i.e. one that preserves all the homological data of R. The main goal of [9] is to give tools to find acyclic twisting cochains $C \to A$, with C smaller than $\operatorname{Bar} A$. For instance, when R is a complete intersection and A is the Koszul complex on defining equations of R, there is a twisting cochain $C \to A$ with C the divided powers on $A_1[1]$, and this explains the polynomial behavior of infinite resolutions over complete intersections. We view this paper as the Golod chapter in a series, with further details on complete intersections to follow soon. The next natural class of rings to consider are codimension 3 Gorenstein rings (every codimension 2 ring is complete intersection or Golod). We give a running example of a specific codimension 3 Gorenstein ring that we hope may hint at the possibilities of the machinery.

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2. Conventions

By Q we denote a commutative ring and we always work over this ring. In particular, all unmarked tensor and hom functors are defined over Q. For graded modules M, N, let $\operatorname{Hom}(M,N)$ be the graded module with degree i component $\prod_{j\in\mathbb{Z}} \operatorname{Hom}(M_j, N_{j+i})$. We use homological indexing for complexes, so differentials lower degree. If M and N are complexes, $\operatorname{Hom}(M,N)$ is a complex with differential $d^{\operatorname{Hom}}(f) = fd_M - (-1)^{|f|}d_N f$. A map of chain complexes is a cycle in $\operatorname{Hom}(M,N)$ and a morphism of complexes is a degree zero map of chain complexes. If M is a complex, $M[1]$ is the complex with $M[1]_n = M_{n-1}$ and $d_{M[1]} = -d_M$. We set

$$s : M \to M[1]$$

to be the degree 1 map with $s(m) = m$. When evaluating tensor products of homogeneous maps we use the sign convention $(f \otimes g)(x \otimes y) = (-1)^{|g||x|} f(x) \otimes g(y)$. If M, N are complexes, $M \otimes N$ is a complex with differential $d_M \otimes 1 + 1 \otimes d_N$. All elements of graded objects are assumed to be homogeneous.

If $f : M \to N$ is a morphism of complexes, $\operatorname{cone}(f)$ is the complex with underlying module $M[1] \oplus N$ and differential

$$d^{\operatorname{cone}(f)} = \begin{pmatrix} d^{M[1]} & 0 \\ fs^{-1} & d^N \end{pmatrix} : M[1] \oplus N \to M[1] \oplus N.$$

If M is a complex and x a cycle in M, $\operatorname{cls}(x)$ is the class of x in $H_*(M)$.

Throughout R is a cyclic Q-algebra, M an R-module, and

$$A \xrightarrow{\simeq} R \qquad G \xrightarrow{\simeq} M$$

Q-projective resolutions. We assume that $A_0 = Q$ and set $A_+ = A_{\geq 1}$. We do not assume that R or M has finite projective dimension over Q.

3. A-infinity structures on resolutions

In this section we recall the definition of A_∞-algebra and module, give a short proof that such structures exist as claimed in the introduction, and use them to construct an R-projective bar resolution of M. For an introduction to A_∞-algebras, see [23]. We use some definitions and very basic results from [9], but emphasize that none of the heavy machinery of that paper is used here.

We denote the differential of the complex A as d^A. In the following definition we view d_1^A as a degree -2 map of chain complexes $A_+[1] \to A_0$ and make the identifications $A_0 \otimes A_+[1] = A_+[1] = A_+[1] \otimes A_0$.

Definition 3.1. An *A_∞-algebra structure on A* is a sequence of degree -1 maps $m = (m_n) = (m_n)_{n\geq 1}$, with $m_n : A_+[1]^{\otimes n} \to A_+[1]$, satisfying:

$$m_1 = d^{A[1]}|_{A_+[1]},$$

$$m_1 m_2 + m_2(m_1 \otimes 1 + 1 \otimes m_1) = d_1^A \otimes 1 - 1 \otimes d_1^A,$$

$$\text{and} \qquad \sum_{i=1}^{n} \sum_{j=0}^{n-i} m_{n-i+1}(1^{\otimes j} \otimes m_i \otimes 1^{\otimes n-i-j}) = 0 \text{ for all } n \geq 3.$$

If Q is graded, I is homogeneous and A is a graded projective resolution, m is a *graded A_∞-algebra structure* if each m_n preserves internal degrees.

Remarks. We note the following:

(1) Using $A_+[1]$ instead of A_+ removes signs from the definition.
(2) Using A_+ instead of A ensures that our A_∞-algebras are *strictly unital*. Indeed, let 1_A be an element of A_0 that maps to $1 \in R$. If (m_n) is an A_∞-structure on A, then one can uniquely extend each m_n to a map $m'_n : A[1]^{\otimes n} \to A[1]$ such that $m'_1 = d^{A[1]}$, $m'_2(a \otimes 1_A) = (-1)^{|a|+1} a$ and $m'_2(1_A \otimes a) = -a$ for all $a \in A$, and m'_n vanishes on any term containing 1_A, for $n \geq 3$.
(3) The map $d_1^A \otimes 1 - 1 \otimes d_1^A$ is a *curvature term*, introduced by Positselski, accounting for the fact that A is not augmented. See [31, §7] and [9, 2.2.8] for further details.
(4) To match up with notation from the introduction, set $\widetilde{m}_n = s m_n (s^{-1})^{\otimes n}$.

Example 3.2. Let $i : A_+ \hookrightarrow A$ and $p : A \to A_+$ be the canonical maps. If A has a dg-algebra structure with multiplication μ, then

$$m_2 = sp\mu(is^{-1} \otimes is^{-1}) : A_+[1] \otimes A_+[1] \to A_+[1]$$

and $m_n = 0$ for $n \geq 3$, is an A_∞-algebra structure on A.

Definition 3.3. Let (m_n) be an A_∞-algebra structure on A. An *A_∞ A-module structure on G* is a sequence of degree -1 maps $m^G = (m_n^G) = (m_n^G)_{n\geq 1}$, with $m_n^G : A_+[1]^{\otimes n-1} \otimes G \to G$, satisfying:

$$m_1^G = d^G,$$

$$m_1^G m_2^G + m_2^G(m_1 \otimes 1 + 1 \otimes m_1^G) = d_1^A \otimes 1,$$

$$\sum_{i=1}^{n} \sum_{j=0}^{n-i} m^G_{n-i+1}(1^{\otimes j} \otimes m_i \otimes 1^{\otimes n-i-j}) = 0 \text{ for all } n \geq 3,$$

where the m_i in the last line is m_i^G when $j = n - i$. If Q, R, M are graded and A, G are graded resolutions, m^G is a *graded* A_∞ *A-module structure* if each m_n^G preserves internal degrees.

Example 3.4. If A is a dg-algebra and G is a dg A-module with structure map $\mu^G : A \otimes G \to G$, then

$$m_2^G = \mu^G(is^{-1} \otimes 1) : A_+[1] \otimes G \to G$$

and $m_n^G = 0$ for $n \geq 3$ is an A_∞ A-module structure on G.

Example 3.5. Let $R = Q/(f)$ with f a non-zero divisor and set $A = 0 \to Q \xrightarrow{f} Q \to 0$. Then A is a Q-projective resolution of R and $m_n = 0$ for all n is an A_∞-algebra structure on A. If m^G is an A_∞ A-module structure on G, identify m_n^G with a degree -1 map $G[2n-2] \to G$, e.g. a degree $2n-3$ map $G \to G$. Then m_2^G is a homotopy for multiplication by f and for $n \geq 3$,

$$\sum_{i=1}^{n} m_{n-i+1}^G m_i^G = 0.$$

Thus an A_∞ A-module structure on G is the same as a system of higher homotopies in the sense of [32] and [11], where $\sigma_i = m_{i+1}^G$.

It follows from [9, Theorem 2.4.5] that there always exists an A_∞-algebra structure on A and an A_∞ A-module structure on G. We give a shorter proof below for the special case we are working with in this paper. This proof also gives an algorithm for constructing A_∞-structures by computer, and shows that if R, M, A and G are graded, then graded A_∞-structures on A and G exist.

Proposition 3.6. *, R a cyclic Q-algebra, M an R-module, and*

$$A \xrightarrow{\simeq} R \quad G \xrightarrow{\simeq} M$$

Q-projective resolutions. There exists an A_∞-algebra structure on A and an A_∞ A-module structure on G.

Proof. We construct the sequence (m_n) inductively. Set $m_1 = d^{A[1]}|_{A_+[1]}$. To construct m_2, consider the degree –2 map

$$\alpha = d_1^A \otimes 1 - 1 \otimes d_1^A : A_+[1] \otimes A_+[1] \to A_+[1],$$

where d_1^A is viewed as a degree –2 map $A_+[1] \to A_0$ and we have identified $A_0 \otimes A_+[1] \cong A_+[1] \cong A_+[1] \otimes A_0$. Since d_1^A is a map of chain complexes, $d_1^A \otimes 1 - 1 \otimes d_1^A$ is also. The map α has the form:

$$\begin{tikzcd}
0 & A_1 \otimes A_1 \arrow[l] \arrow[d, "d_1^A \otimes 1 - 1 \otimes d_1^A"'] & \begin{smallmatrix} A_1 \otimes A_2 \\ A_2 \otimes A_1 \end{smallmatrix} \arrow[l] \arrow[d, "\begin{pmatrix} d_1^A \otimes 1 \\ -1 \otimes d_1^A \end{pmatrix}"'] & \begin{smallmatrix} A_1^{\otimes 3} \\ A_1 \otimes A_3 \\ A_2 \otimes A_2 \\ A_3 \otimes A_1 \end{smallmatrix} \arrow[l] \arrow[d, "\begin{pmatrix} 0 \\ d_1^A \otimes 1 \\ 0 \\ -1 \otimes d_1^A \end{pmatrix}"'] & \dots \arrow[l] \\
0 & A_1 \arrow[l] & A_2 \arrow[l] & A_3 \arrow[l] & \dots \arrow[l]
\end{tikzcd}$$

The first non-zero component induces the zero map in homology and thus by the classical lifting result [10, V.1.1] is nullhomotopic. Take any homotopy as m_2.

Given $m_1, \dots, m_{n-1}$, for $n \geq 3$, we use the obstruction lemma of Lefévre-Hasegawa [24, Appendix B], modified for strictly unital algebras in [9, §2.3], to construct m_n. By [9, 2.3.3],

$$r(m|_{n-1}) := \sum_{i=2}^{n-1} \sum_{j=0}^{n-i} m_{n-i+1}(1^{\otimes j} \otimes m_i \otimes 1^{\otimes n-i-j})$$

is a degree -1 cycle in the complex $\mathrm{Hom}(A_+[1]^{\otimes n}, A_+[1])$, where $A_+[1]^{\otimes n}$ has the usual tensor differential, and a map m_n satisfies the A_∞-relations if and only if

$$d^{\mathrm{Hom}}(m_n) + r(m|_{n-1}) = 0.$$

But for $n \geq 3$,

$$H_{-1} \mathrm{Hom}(A_+[1]^{\otimes n}, A_+[1]) \cong H_{n-2} \mathrm{Hom}(A_+^{\otimes n}, I) = 0$$

since $\mathrm{Hom}(A_+^{\otimes n}, I)$ is concentrated in non-positive degrees. Thus $r(m|_{n-1})$ is a boundary. Pick any map in the preimage for m_n.

To put an A_∞ A-module structures on G, set $m_1^G = d^G$ and let m_2^G be a homotopy for the null-homotopic map $d_1^A \otimes 1 : A_+[1] \otimes G \to G$. Then one proceeds analogously as for A, using the analogue of the obstruction theory for modules given in [9, §3.3]. □

By [9, Theorem 2.4.5] the A_∞-structures of Proposition 3.6 are unique up to homotopy. We will need the following weaker results, whose proofs are clear.

Corollary 3.7. *Using the notation of 3.6, the following are equivalent.*

(1) *there exists an A_∞-algebra structure (m_n) on A with $m_3 = 0$;*
(2) *there exists an A_∞-algebra structure (m_n) on A with $m_n = 0$ for all $n \geq 3$;*
(3) *A has a dg-algebra structure.*

Corollary 3.8. *Assume that A has a dg-algebra structure. The following are equivalent.*

(1) *there exists an A_∞ A-module structure (m_n^G) on G with $m_3^G = 0$;*
(2) *there exists an A_∞ A-module structure (m_n^G) on G with $m_n^G = 0$ for all $n \geq 3$;*
(3) *G has a dg A-module structure.*

Question. It would be interesting to compute A_∞-structures in special cases, e.g. in Example 3.11 below, or for the resolutions Srinivasan shows in [33] have no dg-algebra structure. More ambitiously, if A or G has a combinatorial structure, e.g. if A is a cellular resolution [5], can one give a corresponding combinatorial description of the A_∞-structures?

In another direction, we ask what restrictions do A_∞-structures place on the Boij-Soederberg decompositions [13] of A and G, and vice versa?

Example 3.9. If A and G have length 2, then by degree considerations, $m_n = 0 = m_n^G$ for all $n \geq 2$, i.e A is a dg-algebra and G is a dg A-module. If A has length 3, then by [7, 1.3], A is a dg-algebra.

This next example shows that G can have non-zero m_3^G when A, G both have length 3. I do not know if it is possible to put a dg A-module structure on this complex.

Example 3.10. Let k be a field and $Q = k[[x,y,z]]$. Let $I = (x^2, -yz, xy + z^2, -xz, y^2)$ be the ideal generated by the submaximal Pfaffians of the alternating matrix

$$\phi = \begin{pmatrix} 0 & y & 0 & 0 & z \\ -y & 0 & x & z & 0 \\ 0 & -x & 0 & y & 0 \\ 0 & -z & -y & 0 & x \\ -z & 0 & 0 & -x & 0 \end{pmatrix},$$

and set $R = Q/I$. By [7], R is a codimension 3 Gorenstein ring and the minimal Q-free resolution of R is

$$A: \quad 0 \leftarrow Q \xleftarrow{d_1} Q^5 \xleftarrow{\phi} Q^5 \xleftarrow{d_1^{\mathrm{Tr}}} Q \leftarrow 0$$

with $d_1 = \begin{pmatrix} x^2 & -yz & xy+z^2 & -xz & y^2 \end{pmatrix}$. Let $(a_1, \ldots, a_5)$, $(b_1, \ldots, b_5)$ and (c_1) be bases for A_1, A_2 and A_3, respectively. By [7, 4.1], A is a graded-commutative dg-algebra with multiplication table

$$a_1a_2 = xb_3 + yb_5 \quad a_1a_3 = -xb_2 - zb_5 \quad a_1a_4 = xb_5 \quad a_1a_5 = -yb_2 + zb_3 - xb_4$$
$$a_2a_3 = xb_1 - zb_4 \quad a_2a_4 = zb_3 \quad a_2a_5 = yb_1 \quad a_3a_4 = -zb_2 + yb_5$$
$$a_3a_5 = -zb_1 - yb_4 \quad a_4a_5 = xb_1 + yb_3 \quad a_ib_j = \delta_{ij}c_1 \text{ for } 1 \le i,j \le 5.$$

(See also [2, 2.1.3] for details on this construction.) We consider the dg-algebra A as an A_∞-algebra, as in Example 3.2.

Let $K \xrightarrow{\simeq} k$ be the Koszul complex over Q. Since K is the Q-free resolution of the R-module k, it has an A_∞ A-module structure. To construct one explicitly, we consider the following map of complexes.

$$\begin{tikzcd} A: & 0 & Q \arrow[l] \arrow[d, "="'] & Q^5 \arrow[l] \arrow[d, "\alpha_1"'] & Q^5 \arrow[l] \arrow[d, "\alpha_2"'] & Q \arrow[l] \arrow[d, "\alpha_3"'] & 0 \arrow[l] \\ K: & 0 & Q \arrow[l] & Q^3 \arrow[l, "d_1^K"] & Q^3 \arrow[l, "d_2^K"] & Q \arrow[l, "d_3^K"] & 0 \arrow[l] \end{tikzcd}$$

with

$$\alpha_1 = \begin{pmatrix} x & 0 & 0 & 0 & 0 \\ 0 & 0 & x & 0 & y \\ 0 & -y & z & -x & 0 \end{pmatrix} \quad \alpha_2 = \begin{pmatrix} 0 & -x & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -x \\ y & 0 & 0 & 0 & 0 \end{pmatrix} \quad \alpha_3 = \begin{pmatrix} xy \end{pmatrix},$$

where

$$d_1^K = \begin{pmatrix} x & y & z \end{pmatrix} \quad d_2^K = \begin{pmatrix} -y & -z & 0 \\ x & 0 & -z \\ 0 & x & y \end{pmatrix} \quad d_3^K = \begin{pmatrix} z \\ -y \\ x \end{pmatrix}.$$

Set $m_1^K = d^K$ and

$$m_2^K = \mu^K(-\alpha i s^{-1} \otimes 1) : A_+[1] \otimes K \to K,$$

where $\mu^K : K \otimes K \to K$ is multiplication. Using that α is a map of complexes and that $m_1 = -(d^A_{\ge 2} - d^A_1)$, one checks that $m_1^K m_2^K + m_2^K(m_1 \otimes 1 + 1 \otimes m_1^K) = d_1^A \otimes 1$.

We now calculate $m_2^K(m_2 \otimes 1 + 1 \otimes m_2^K)$. For degree reasons, the only possible non-zero component is $A_1 \otimes A_1 \otimes K_0 \to K_2$. On this component, we have

$$m_2^K(m_2 \otimes 1 + 1 \otimes m_2^K) = -\alpha_2 \mu^A + \mu^K(\alpha_1 \otimes \alpha_1).$$

The matrix of this map with respect to the basis $a_i \otimes a_j$ of $A_1 \otimes A_1$, ordered so that the first two elements are $a_3 \otimes a_4$ and $a_4 \otimes a_3$, is

$$\begin{pmatrix} -xy & xy & 0 & 0 \dots \\ xy & -xy & 0 & 0 \dots \\ -x^2 & x^2 & 0 & 0 \dots \end{pmatrix}.$$

Thus, defining $m_3^K : A_+[1]^{\otimes 2} \otimes K \to K$ on the component $A_1 \otimes A_1 \otimes K_0 \to K_3$ by

$$m_3^K = \begin{pmatrix} x & -x & 0 & 0 \dots \end{pmatrix},$$

and zero elsewhere, gives

$$m_3^K(m_1 \otimes 1^{\otimes 2} + 1 \otimes m_1 \otimes 1 + 1^{\otimes 2} \otimes m_1^K) + m_1^K m_3^K = m_1^K m_3^K = -m_2^K(m_2 \otimes 1 + 1 \otimes m_2^K),$$

and so m^K is an A_∞ A-module structure on K.

Example 3.11. Let $Q = k[[x, y, z, w]]$, $R = Q/(x^2, xy, yz, zw, w^2)$ and A be the minimal Q-free resolution of R. Then $\operatorname{pd}_Q R = 4$, and by [2, 2.3.1] there is no dg-algebra structure on A. Thus every A_∞-algebra structure on A has nonzero m_3.

The proof of the above uses the obstuctions of [3]; we relate these to A_∞-structures in Section 5.

Definition 3.12. Let $\operatorname{Bar} A$ be the module $\bigoplus_{n \geq 0} A_+[1]^{\otimes n}$ and write $[x_1|\dots|x_n]$ for the element $s(x_1) \otimes \dots \otimes s(x_n) \in A_+[1]^{\otimes n} \subseteq \operatorname{Bar} A$. The *bar resolution* on A and G has underlying module

$$R \otimes \operatorname{Bar} A \otimes G = \bigoplus_{n \geq 0} R \otimes A_+[1]^{\otimes n} \otimes G$$

and differential

$$d([x_1|\dots|x_n] \otimes g) = R \otimes \left(\sum_{i=1}^{n} \sum_{j=0}^{i-1} [\underline{x}_1|\dots|\underline{x}_j|m_i([x_{j+1}|\dots|x_{j+i}])|\dots|x_n] \otimes g \right.$$
$$\left. + \sum_{i=0}^{n} [\underline{x}_1|\dots|\underline{x}_{n-i+1}] \otimes m_i^G([x_{n-i+2}|\dots|x_n] \otimes g) \right)$$

where $\underline{x} = (-1)^{|x|+1} x$. Let $\epsilon_G : G \to M$ be the augmentation of the resolution. Define a map

$$\epsilon_{\mathrm{Bar}} : R \otimes \operatorname{Bar} A \otimes G \to M$$

by $\epsilon_{\mathrm{Bar}}([\,] \otimes g) = \epsilon_G(g)$ and $\epsilon_{\mathrm{Bar}}([x_1|\dots|x_n] \otimes g) = 0$ for $n \geq 1$.

Remark. The module $\operatorname{Bar} A = \bigoplus_{n \geq 0} A_+[1]^{\otimes n}$ has a graded coalgebra structure (the tensor coalgebra) and a coderivation induced by the A_∞-algebra structure on A. Together with the Q-linear map $\operatorname{Bar} A \twoheadrightarrow A_{1+} \xrightarrow{d_1} Q = A_0$, they form a curved dg-coalgebra called the *bar construction* or bar coalgebra of A.

Theorem 3.13. *Let Q be a commutative ring, R a cyclic Q-algebra, M an R-module, and*

$$A \xrightarrow{\simeq} R \quad G \xrightarrow{\simeq} M$$

Q-projective resolutions with A_∞-algebra and A_∞ A-module structures, respectively. The bar resolution is a complex of projective R-modules and the map

$$\epsilon_{\mathrm{Bar}} : R \otimes \operatorname{Bar} A \otimes G \to M$$

makes it an R-projective resolution of M.

When A is a dg-algebra and G is a dg A-module, this is [22, 1.4].

Proof. The results of [9, §7] show that $R \otimes \operatorname{Bar} A \otimes G$ is a complex with homology M. One can also check directly, using the definition of A_∞algebra and module, that it is a complex. And to show it is exact, one can use an analogous proof as in [22, 1.4], by filtering the complex by the number of bars and considering the resulting spectral sequence. $\square$

Example 3.14. Continuing Example 3.5, $R \otimes \operatorname{Bar} A \otimes G$ is $R \otimes -$ applied to:

$$0 \leftarrow G_0 \xleftarrow{m_1^G} G_1 \xleftarrow{\begin{pmatrix} m_2^G & m_1^G \end{pmatrix}} \begin{matrix} G_0 \\ G_2 \end{matrix} \xleftarrow{\begin{pmatrix} m_2^G & m_1^G \\ m_1^G & 0 \end{pmatrix}} \begin{matrix} G_1 \\ G_3 \end{matrix} \xleftarrow{\begin{pmatrix} m_3^G & m_2^G & m_1^G \\ m_2^G & m_1^G & 0 \end{pmatrix}} \begin{matrix} G_0 \\ G_2 \\ G_4 \end{matrix} \leftarrow \dots$$

This is the resolution constructed from a system of higher homotopies in [32, 3.1] and [11, §7], for hypersurfaces.

Example 3.15. Continuing Example 3.10, $R \otimes \operatorname{Bar} A \otimes K$ is $R \otimes -$ applied to:

$$0 \leftarrow K_0 \xleftarrow{m_1^K} K_1 \xleftarrow{\begin{pmatrix} m_2^K & m_1^K \end{pmatrix}} \begin{matrix} A_1 \otimes K_0 \\ K_2 \end{matrix} \xleftarrow{\begin{pmatrix} m_1 \otimes 1 & 1 \otimes m_1^K & 0 \\ m_2^K & m_2^K & m_1^K \end{pmatrix}} \begin{matrix} A_2 \otimes K_0 \\ A_1 \otimes K_1 \\ K_3 \end{matrix}$$

$$\xleftarrow{\begin{pmatrix} m_2 \otimes 1 & m_1 \otimes 1 & 1 \otimes m_1^K & 0 \\ 1 \otimes m_2^K & 0 & m_1 \otimes 1 & 1 \otimes m_1^K \\ m_3^K & m_2^K & m_2^K & m_2^K \end{pmatrix}} \begin{matrix} A_1^{\otimes 2} \otimes K_0 \\ A_3 \otimes K_0 \\ A_2 \otimes K_1 \\ A_1 \otimes K_2 \end{matrix} \xleftarrow{d_5} \begin{matrix} A_2 \otimes A_1 \otimes K_0 \\ A_1 \otimes A_2 \otimes K_0 \\ A_1^{\otimes 2} \otimes K_1 \\ A_3 \otimes K_1 \\ A_2 \otimes K_2 \\ A_1 \otimes K_3 \end{matrix} \leftarrow \dots$$

with

$$d_5 = \begin{pmatrix} m_1 \otimes 1^{\otimes 2} & 1 \otimes m_1 \otimes 1 & 1^{\otimes 2} \otimes m_1^K & 0 & 0 & 0 \\ m_2 \otimes 1 & m_2 \otimes 1 & 0 & 1 \otimes m_1^K & 0 & 0 \\ 1 \otimes m_2^K & 0 & m_2 \otimes 1 & m_1 \otimes 1 & 1 \otimes m_1^K & 0 \\ 0 & 1 \otimes m_2^K & 1 \otimes m_2^K & 0 & m_1 \otimes 1 & 1 \otimes m_1^K \end{pmatrix}.$$

For later use, we note the maps $d_1, \dots, d_4$ above are minimal, and d_5 has rank 1 after tensoring with k. The non-minimality is due to the surjective multiplication maps $A_1 \otimes A_2 \to A_3 = Q$ and $A_2 \otimes A_1 \to A_3 = Q$ ($a_i b_i = c_i$, in the notation of 3.10).

Remark. There is a general construction of resolutions given in [9, §7] that takes as further input an acyclic twisting cochain $\tau : C \to A$. Theorem 3.13 is implicitly using the universal twisting cochain $\tau_A : \operatorname{Bar} A \to A$. If R is a codimension c complete intersection, there is an acyclic twisting cochain $\tau : C \to A$ with C the divided powers coalgebra on a rank c free Q-module (the dual of C is the symmetric algebra) and A the Koszul complex resolving R over Q. The resolution of [9, §7] applied to this acyclic twisting cochain recovers the standard resolution for complete intersections constructed in [11, §7].

4. Q-projective resolutions of R-syzygies

In this section we fix an A_∞-algebra structure $m = (m_n)$ on A and an A_∞ A-module structure $m^G = (m_n^G)$ on G. Using these, we construct Q-projective resolutions of all R-syzygies of M. This construction requires no further choices once the initial A_∞-structures on A and G are fixed. If Q is local (or graded), these resolutions will not in general be minimal. This construction was inspired by the "box complex" defined in [12, §8].

Definition 4.1. The *first syzygy* of M over R is

$$\Omega_R^1(M) := \ker(R \otimes G_0 \to M).$$

This is usually defined as the kernel of any surjection from a projective R-module to M, but since we have fixed a surjection from a projective, we use it.

Lemma 4.2. *Consider the complex* $(A_+[1] \otimes G_0, m_1 \otimes 1)$. *The degree zero map*

$$\phi = sm_2^G|_{A_+[1]\otimes G_0} : A_+[1] \otimes G_0 \to G_+[1],$$

is a morphism of complexes and there is a quasi-isomorphism

$$\mathrm{cone}(\phi)[-2] \xrightarrow{\simeq} \Omega_R^1(M),$$

thus $\mathrm{cone}(\phi)[-2]$ *is a* Q*-projective resolution of* $\Omega_R^1(M)$.

Proof. It follows from the definition of A_∞ A-module that ϕ is a morphism of complexes. Since $A_+[1] \otimes G_0$ is concentrated in non-negative homological degrees, to finish the proof it is enough to show that $H_i(\mathrm{cone}(\phi)[-2]) = 0$ for $i \neq 0$ and $H_0(\mathrm{cone}(\phi)[-2]) \cong \Omega_R^1(M)$. To calculate the homology we define a map

$$\phi' : A[1] \otimes G_0 \to G[1]$$

whose restriction to $A_+[1] \otimes G_0$ is ϕ by setting $\phi'(1_A \otimes g) = g$, where 1_A is a basis element of $A_0 = Q$. Note that $\mathrm{cone}(\phi)$ is homotopic to $\mathrm{cone}(\phi')$. The complex $A[1] \otimes G_0$ has homology $R[1] \otimes G_0$ in degree 1, is zero elsewhere, and the homology map $H(\phi')$ is a surjection $R[1] \otimes G_0 \to M[1]$. It follows from the homology long exact sequence for the triangle

$$A[1] \otimes G_0 \xrightarrow{\phi'} G[1] \to \mathrm{cone}(\phi') \to$$

that $H_2(\mathrm{cone}(\phi')) \cong \Omega_R^1(M)$ and $H_i(\mathrm{cone}(\phi')) = 0$ for $i \neq 2$. $\square$

Definition 4.3. The *first syzygy complex of* G, denoted $\mathrm{syz}_R^1(G)$, is $\mathrm{cone}(\phi)[-2]$.

Remark. The proof shows that for any map of complexes $A_+ \otimes G_0 \to G_+$ lifting the surjection $R \otimes G_0 \to M$, the cone is a Q-projective resolution of $\Omega_R^1(M)$. But the A_∞-structure gives a canonical lift. Moreover, 4.6 will show that the A_∞-structures on A and G determine a canonical A_∞structure on $\mathrm{syz}_R^1(G)$, and thus we can iterate taking syzygies of G without making further choices.

Example 4.4. Let Q, R, A and G be as in Example 3.5. Then $\mathrm{syz}_R^1(G)$ is

$$0 \leftarrow G_1 \xleftarrow{\begin{pmatrix} m_2^G & -m_1^G \end{pmatrix}} \begin{matrix} G_0 \\ G_2 \end{matrix} \xleftarrow{\begin{pmatrix} 0 \\ -m_1^G \end{pmatrix}} G_3 \xleftarrow{-m_1^G} G_4 \leftarrow \ldots$$

Example 4.5. Let Q, R, A, and K be as in Example 3.10. Then $\operatorname{syz}^1_R(G)$ is

$$0 \leftarrow K_1 \xleftarrow{\begin{pmatrix} m_2^K & -m_1^K \end{pmatrix}} \begin{matrix} A_1 \otimes K_0 \\ K_2 \end{matrix} \xleftarrow{\begin{pmatrix} -m_1 \otimes 1 & 0 \\ m_2^K & -1 \otimes m_1^K \end{pmatrix}} \begin{matrix} A_2 \otimes K_0 \\ K_3 \end{matrix} \xleftarrow{\begin{pmatrix} -m_1 \otimes 1 \\ m_2^K \end{pmatrix}} A_3 \otimes K_0 \leftarrow 0.$$

Proposition 4.6. *The maps*

$$m_n^{\mathrm{syz}} : A_+[1]^{\otimes n-1} \otimes \operatorname{syz}^1_R(G) \to \operatorname{syz}^1_R(G)$$

$$m_n^{\mathrm{syz}} = \begin{pmatrix} -s^{-1}(m_n \otimes 1)(1 \otimes s) & 0 \\ s^{-1} m_{n+1}^G (1 \otimes s) & -s^{-1} m_n^G (1 \otimes s) \end{pmatrix}$$

are an A_∞ A-module structure on $\operatorname{syz}^1_R(G)$.

Proof. We first construct an A_∞-structure on $\operatorname{cone}(\phi)$. Write

$$\operatorname{cone}(\phi) = (A_+[1] \otimes G_0)[1] \oplus G_+[1].$$

The module $H = A_+[1] \otimes G_0$ is an A_∞ A-module via the maps $m_{n-1}^H = m_n \otimes 1$. The module $G_+[1]$ is an A_∞ A-module via the maps $-s m_n^G|_{A_+[1]^{\otimes n-1} \otimes G_+}(1^{\otimes n-1} \otimes s^{-1})$ since $m^G(G_+) \subseteq G_+$. Moreover, the maps

$$\varphi_n = s(m_{n+1}^G|_{A_+[1]^{\otimes n-1} \otimes G_0}) : A_+[1]^{\otimes n-1} \otimes (A_+[1] \otimes G_0) \to G_+[1]$$

form a morphism of A_∞ A-modules $\varphi : H \to G_+[1]$, see [9, §3] for the definition, and thus the maps

$$m'_n : A_+[1]^{\otimes n-1} \otimes \operatorname{cone}(\phi) \cong (A_+[1]^{\otimes n} \otimes G_0[1]) \oplus (A_+[1]^{\otimes n-1} \otimes G_+[1])$$

$$\to (A_+[1] \otimes G_0)[1] \oplus G_+[1]$$

defined as

$$m'_n = \begin{pmatrix} -s(m_n \otimes 1)(1 \otimes s^{-1}) & 0 \\ s m_{n+1}^G (1 \otimes s^{-1}) & -s m_n^G (1 \otimes s^{-1}) \end{pmatrix}, \tag{4.7}$$

are an A_∞ A-module structure on $\operatorname{cone}(\varphi_1) = \operatorname{cone}(\phi)$ by [9, 5.4.4]. Shifting this A_∞ A-module twice, see [9, 4.3.12], the maps

$$m_n^{\mathrm{syz}} = s^{-2} m'_n (1 \otimes s^2) : A_+[1]^{\otimes n-1} \otimes \operatorname{cone}(\phi)[-2] \to \operatorname{cone}(\phi)[-2]$$

are an A_∞ A-module structure on $\operatorname{cone}(\varphi)[-2] = \operatorname{cone}(\phi)[-2]$. Note that the shifts s^{-2} and $1 \otimes s^2$ do not affect signs as they have degee -2. $\square$

Definition 4.8. Define $\operatorname{syz}^n_R(G)$ inductively as $\operatorname{syz}^1_R(\operatorname{syz}^{n-1}_R(G))$, where $\operatorname{syz}^{n-1}_R(G)$ has the A_∞ A-module structure given by 4.6.

Example 4.9. Continuing Example 4.4, write $\operatorname{syz}^1_R(G) = G_0[1] \oplus G_+[-1]$. Then

$$m_n^{\mathrm{syz}} = \begin{pmatrix} 0 & 0 \\ m_{n+1}^G & -m_n^G \end{pmatrix}.$$

Using this, one checks that $\operatorname{syz}^2_R(G)$ is the box complex of [12, §7].

Example 4.10. Continuing Example 4.5, set $H = \operatorname{syz}^1_R(K)$. The map

$$m_2^H : A_1[1] \otimes H \to H$$

is given by

$$\begin{tikzcd}
0 & A_1 \otimes K_1 \arrow[l] \arrow[d, "\left[\begin{smallmatrix} 0 \\ -m_2^K \end{smallmatrix}\right]"] & \begin{smallmatrix} A_1 \otimes A_1 \otimes K_0 \\ A_1 \otimes K_2 \end{smallmatrix} \arrow[l] \arrow[d, "\left[\begin{smallmatrix} -m_2 \otimes 1 & 0 \\ m_3^K & -m_2^K \end{smallmatrix}\right]"] & \begin{smallmatrix} A_1 \otimes A_2 \otimes K_0 \\ A_1 \otimes K_3 \end{smallmatrix} \arrow[l] \arrow[d, "\left[\begin{smallmatrix} m_2 \otimes 1 & 0 \end{smallmatrix}\right]"] & A_1 \otimes A_3 \otimes K_0 \arrow[l] \\
K_1 & \begin{smallmatrix} A_1 \otimes K_0 \\ K_2 \end{smallmatrix} \arrow[l] & \begin{smallmatrix} A_2 \otimes K_0 \\ K_3 \end{smallmatrix} \arrow[l] & A_3 \otimes K_0 \arrow[l] & 0 \arrow[l]
\end{tikzcd}$$

and $m_2^H : A_2[1] \otimes H \to H$ is

$$\begin{tikzcd}
 & 0 & A_2 \otimes K_1 \arrow[l] \arrow[d, "\left[\begin{smallmatrix} 0 \\ -m_2^K \end{smallmatrix}\right]"] & \begin{smallmatrix} A_2 \otimes A_1 \otimes K_0 \\ A_2 \otimes K_2 \end{smallmatrix} \arrow[l] \arrow[d, "\left[\begin{smallmatrix} -m_2 \otimes 1 & 0 \end{smallmatrix}\right]"] & \dots \arrow[l] \arrow[d, "0"] \\
K_1 & \begin{smallmatrix} A_1 \otimes K_0 \\ K_2 \end{smallmatrix} \arrow[l] & \begin{smallmatrix} A_2 \otimes K_0 \\ K_3 \end{smallmatrix} \arrow[l] & A_3 \otimes K_0 \arrow[l] & 0 \arrow[l]
\end{tikzcd}$$

Both $m_2 : A_3[1] \otimes H \to H$ and m_3^H are zero.

The map m_n^G occurs as a component of $m_{n-1}^{\mathrm{syz}_R^1(G)}$. The result below generalizes this. It implies almost immediately the classical result of Lescot that all modules over a Golod ring have a Golod syzygy; see 6.14.

Corollary 4.11. *Let G be a complex with finite length c, and set $H := \mathrm{syz}_R^{c+1}(G)$ with the A_∞ A-module structure m^H defined above. For $n \geq 2$, the structure maps m_n^H are defined entirely in terms of the structure maps m_n of A, e.g. the maps m_n^G only factor into m_1^H.*

Proof. Let H_j be the A_∞ A-module $\mathrm{syz}_R^j(G)$, defined in 4.8. The idea of the proof is that if k is the unique integer such that m_n^G occurs as a component of $m_k^{H_i}$, then as j increases, k decreases.

For $H_1 = \mathrm{syz}_R^1(G) = (A_+[1] \otimes G_0) \oplus G_+[-1]$, it follows from the formula

$$m_k^{H_1} = \pm \begin{bmatrix} -m_k \otimes 1 & 0 \\ m_{k+1}^G & -m_k^G \end{bmatrix}$$

that only terms G_i, and not $A_i \otimes G_0$, receive a component m_k^G. We show by induction this holds for all H_j. Assume that it holds for some H_j. We have $H_{j+1} = (A_+[1] \otimes (H_j)_0) \oplus (H_j)_+$ and

$$m_k^{H_{j+1}} = \pm \begin{bmatrix} -m_k \otimes 1 & 0 \\ m_{k+1}^{H_j} & -m_k^{H_j} \end{bmatrix}.$$

By induction, the only summands in $(H_j)_+$ receiving m_n^G for some n, are G_i for some i, and so we see this holds for $(H_{j+1})_+$ as well.

To finish the proof, note that if G_i occurs as a summand of the complex H_j, then $j \geq i$. Thus if $G_j = 0$ for $j > c$, then by the above, we see that m_n^G is not a component of $m_k^{H_{c+1}}$, for $k \geq 2$. $\square$

5. Avramov spectral sequence and obstructions

The standard change of rings spectral sequence for the ring map $Q \to R$ transfers homological information from R to Q, see e.g. [10, XVI, §5]. Avramov constructed a spectral sequence in [3] that transfers information from Q to R. Iyengar gave a second construction in [22] using the dg-bar resolution. We adapt Iyengar's arguments to construct the spectral sequence using the A_∞-bar resolution, and then show how the higher homotopies on A and G describe the differentials. The fact that A_∞-structures can be used to describe differentials in Eilenberg-Moore type sequences first appears in [34].

The spectral sequence depends heavily on the algebra/module structures on Tor groups given by Cartan-Eilenberg's $\pitchfork$-product. Before we construct the spectral sequence, we recall the definition of the product and relate it to A_∞-structures.

For the next definition, we suspend the assumptions and notation used previously.

Definition 5.1. ([10, §XI]; see also [4, Theorem 1.1].) Let Q be a commutative ring and let R, S be Q-algebras. For M a left R-module and N a left S-module, there is a homogeneous map

$$\pitchfork\colon \mathrm{Tor}_*^Q(R,S) \otimes \mathrm{Tor}_*^Q(M,N) \to \mathrm{Tor}_*^Q(M,N)$$

that makes $T := \mathrm{Tor}_*^Q(R,S)$ a graded Q-algebra and $\mathrm{Tor}_*^Q(M,N)$ a graded T-module. If $A \xrightarrow{\simeq} R$ and $G \xrightarrow{\simeq} M$ are Q-projective resolutions, let $\mu : A \otimes G \to G$ be a morphism of complexes lifting the multiplication map $R \otimes M \to M$. Then $\pitchfork$ is defined as

$$H_*(A \otimes S) \otimes H_*(G \otimes N) \xrightarrow{\kappa} H_*(A \otimes S \otimes G \otimes N) \cong$$

$$H_*(A \otimes G \otimes S \otimes N) \xrightarrow{H_*(m \otimes \mu_N)} H_*(G \otimes N),$$

where $\kappa(\mathrm{cls}(x) \otimes \mathrm{cls}(y)) = \mathrm{cls}(x \otimes y)$ is the Kunneth map and $\mu_N : S \otimes N \to N$ is multiplication.

We now return to previous assumptions. There is an obvious extension and shift of $m_2^G : A_+[1] \otimes G \to G$ to a morphism

$$\mu : A \otimes G \to G$$

that lifts the multiplication $R \otimes M \to M$. Using the definition of $\pitchfork$ above, we have:

Lemma 5.2. *Let Q be a commutative ring, R a cyclic Q-algebra, S a Q-algebra, M an R-module and N a left S-module. Let A and G be Q-projective resolutions of R and M, with A_∞-algebra and module structures, respectively. The map*

$$\pitchfork\colon \mathrm{Tor}_*^Q(R,S) \otimes \mathrm{Tor}_*^Q(M,N) \to \mathrm{Tor}_*^Q(M,N),$$

is equal to the map

$$H_*(A \otimes S) \otimes H_*(G \otimes N) \xrightarrow{\kappa} H_*(A \otimes S \otimes G \otimes N) \cong$$

$$H_*(A \otimes G \otimes S \otimes N) \xrightarrow{H_*(\mu \otimes \mu_P)} H_*(G \otimes N),$$

where μ is constructed from m_2^G as above.

We now construct the spectral sequence. Let S be an R-algebra and N a left S-module. Consider the R-projective resolution $R \otimes \operatorname{Bar} A \otimes G \xrightarrow{\simeq} M$ constructed in 3.13. Set

$$X = (R \otimes \operatorname{Bar} A \otimes G) \otimes_R N \cong (\operatorname{Bar} A \otimes S) \otimes_S (G \otimes N).$$

We filter X by the number of bars, so

$$F_p X = \bigoplus_{m \leq p} (A_+[1] \otimes S)^{\otimes m} \otimes_S (G \otimes N).$$

This gives a first quadrant spectral sequence converging to $H_*(X) \cong \operatorname{Tor}_*^R(M, N)$. Setting $\overline{A} = A \otimes S$, we have

$$E^0_{p,q} = \left(\overline{A}_+[1]^{\otimes p} \otimes_S (G \otimes N)\right)_{p+q} \cong \left(\overline{A}_+^{\otimes p} \otimes_S (G \otimes N)\right)_q.$$

If either $H_*(\overline{A}) \cong \operatorname{Tor}_*^Q(R, S)$ or $H_*(G \otimes N) \cong \operatorname{Tor}_*^Q(M, N)$ is flat over S, e.g. if S is a field, then by the Kunneth formula

$$E^1_{p,q} \cong \left(\operatorname{Tor}_+^Q(R, S)^{\otimes p} \otimes_S \operatorname{Tor}_*^Q(M, N)\right)_q.$$

The complex

$$\ldots E^1_{p,*} \to E^1_{p-1,*} \to \ldots \to E^1_{1,*} \to E^1_{0,*} \cong \operatorname{Tor}_*^Q(M, N) \to 0,$$

using Lemma 5.2, is isomorphic to

$$S \otimes_{\operatorname{Tor}_*^Q(R,S)} \left(\operatorname{Tor}_*^Q(R, S) \otimes_S \operatorname{Bar}\left(\operatorname{Tor}_*^Q(R, S)\right) \otimes_S \operatorname{Tor}_*^Q(M, N)\right)$$

where $\operatorname{Tor}_*^Q(R, S) \otimes_S \operatorname{Bar}\left(\operatorname{Tor}_*^Q(R, S)\right) \otimes_S \operatorname{Tor}_*^Q(M, N)$ is the classical bar resolution of $\operatorname{Tor}_*^Q(M, N)$ as a graded module over $\operatorname{Tor}_*^Q(R, S)$. If $\operatorname{Tor}_*^Q(M, N)$ is projective over S, the bar resolution is a projective resolution, and we can make the further identification

$$E^2_{p,q} \cong \left(\operatorname{Tor}_p^{\operatorname{Tor}_*^Q(R,S)}(\operatorname{Tor}_*^Q(M, N), S)\right)_q.$$

We have proved the following. The arguments are adapted from [22].

Theorem 5.3. *Let Q be a commutative ring, R a cyclic Q-algebra, S an R-algebra, M an R-module and N an S-module. There is a spectral sequence*

$$E^0_{p,q} \cong \left(\overline{A}_+[1]^{\otimes p} \otimes_S (G \otimes N)\right)_{p+q} \cong \left(\overline{A}_+^{\otimes p} \otimes_S (G \otimes N)\right)_q$$

$$\Rightarrow \operatorname{Tor}^R_{p+q}(M, N)$$

with $E^0_{p,q} = 0$ for $p > q$ or $q \leq 0$. If $\operatorname{Tor}_*^Q(M, N)$ is projective over S, then

$$E^2_{p,q} \cong \left(\operatorname{Tor}_p^{\operatorname{Tor}_*^Q(R,S)}(\operatorname{Tor}_*^Q(M, N), S)\right)_q,$$

where the algebra and module structure on $\operatorname{Tor}_*^Q(R, S)$, respectively $\operatorname{Tor}_*^Q(M, N)$, is the one given in Definition 5.1.

Now we describe the differentials using the A_∞-structures on A and G. The general idea is that the maps m_i, m_i^G, for $1 \leq i \leq r + 1$, combine to form the differential on the rth page.

We keep the above notation. Write the differential of X restricted to F_pX as $d = t_1 + t_2 + \ldots + t_{p+1}$ with

$$t_i : E_p^0 \to E_{p-i+1}^0 = \sum_{j=0}^{p-i+1} 1^{\otimes j} \otimes \overline{m}_i \otimes 1^{\otimes p-i-j+1},$$

where $\overline{m}_i = m_i \otimes R$ for $j < p-i+1$ and $\overline{m}_i = m_i^G \otimes N$ for $j = p-i+1$ (the indexing is chosen so that t_i involves the maps $\overline{m}_i$).

Following the construction of a spectral sequence from a filtration, after e.g. [29], set $Z_p^r = F_pX \cap d^{-1}F_{p-r}X, B_p^r = d(Z_{p+r}^r)$, and $E_p^r = Z_p^r/(Z_{p-1}^{r-1} + B_p^{r-1})$. An element

$$x = \sum_{k=0}^{p} x_k \in \bigoplus_{k=0}^{p} E_k^0 = F_pX$$

is in Z_p^r if and only if

$$\sum_{i=1}^{n} t_i(x_{p-n+i}) = \sum_{i=1}^{n} \sum_{j=0}^{p-i+1} 1^{\otimes j} \otimes \overline{m}_i \otimes 1^{\otimes p-i-j+1}(x_{p-n+1}) = 0 \tag{5.4}$$

for $n = 1, \ldots, r$. The differential d^r on $\mathrm{cls}(y) \in E_p^r$ with $y \in E_p^0$ is given by $d^r(\mathrm{cls}(y)) = \mathrm{cls}(d(x))$ for any $x = \sum_{i=0}^{p} x_i \in Z_p^r$ with $x_p = y$. But also note that $d(x_{p-r-1} + \ldots + x_0) \in B_{p-r}^{r-1}$ and so $\mathrm{cls}(d(x)) = \mathrm{cls}(d(\sum_{i=p-r}^{p} x_i))$. Finally, note that x is in Z_p^r if and only if $\sum_{i=p-r}^{p} x_i$ is in Z_p^r.

We have proved the following. The argument is adapted from [17, 5.6].

Theorem 5.5. *Let Q be a commutative ring, $R = Q/I$, S an R-algebra, M an R-module and N an S-module. Let A and G be Q-projective resolutions of R and M, with A_∞ algebra and module structures m and m^G, respectively. Set $\overline{A} = A \otimes S$, $\overline{m}_i = m_i \otimes S$ and $\overline{m}_i^G = m_i^G \otimes S$.*

In the Avramov spectral sequence

$$E_{p,q}^0 \cong \left(\overline{A}_+^{\otimes p} \otimes_S (G \otimes N)\right)_q \Rightarrow \mathrm{Tor}_{p+q}^R(M, N),$$

an element $x \in E_p^0$ survives to E_p^r if and only if there exist $x_i \in E_i^0$ for $i = p-r, \ldots, p-1$ such that (5.4) holds with $x_p = x$. In this case $d^r(x)$ is the image of

$$\sum_{i=1}^{r+1} \sum_{j=0}^{p-r} 1^{\otimes j} \otimes \overline{m}_i \otimes 1^{\otimes p-r-j-1}(x_{p-r-1+i})$$

in E_{p-r}^r.

Remark. The theorem shows that the differential on the rth page can be written in terms of $m_i \otimes S$ for $i \le r+1$. In particular, if $m_i \otimes S = 0$ for $i \le r+1$, then $E^0 = E^1 = \ldots = E^r$ in the spectral sequence.

Of particular importance, the theorem gives information on edge maps.

Corollary 5.6. *The edge maps*

$$E_{0,q}^1 = \mathrm{Tor}_q^Q(M, N) \to \mathrm{Tor}_q^R(M, N)$$

are the change of rings map for Tor. This map takes the image of the multiplication

$$\pitchfork\colon \left(\mathrm{Tor}_+^Q(R, S) \otimes \mathrm{Tor}_*^Q(M, N)\right)_q \to \mathrm{Tor}_q^Q(M, N)$$

to zero, and the induced map

$$\frac{\operatorname{Tor}_q^Q(M,N)}{(\operatorname{Tor}_+^Q(R,S)\cdot \operatorname{Tor}^Q(M,N))_q} \to \operatorname{Tor}_q^R(M,N)$$

factors through the edge map

$$E^2_{0,q} \to \operatorname{Tor}_q^R(M,N),$$

giving a commutative diagram

$$\begin{tikzcd}
E^1_{0,q} \arrow[d] \arrow[dr] & \\
\frac{\operatorname{Tor}_q^Q(M,N)}{(\operatorname{Tor}_+^Q(R,S)\cdot \operatorname{Tor}^Q(M,N))_q} \arrow[r] \arrow[d] & \operatorname{Tor}_q^R(M,N). \\
E^2_{0,q} \arrow[ur] &
\end{tikzcd}$$

Proof. We have $E^1_{0,q} = H_q(G\otimes N) \cong \operatorname{Tor}_q^Q(M,N)$. The edge maps on E^1 are induced by $G \hookrightarrow R \otimes \operatorname{Bar} A \otimes G$. This is a morphism of complexes from a Q-projective to R-projective resolution of M, lifting the identity on M. The change of rings map on Tor is induced by any such map.

Lemma 5.2 and the description of differentials show that the multiplication $\pitchfork$: $\operatorname{Tor}_+^Q(R,S) \otimes \operatorname{Tor}_*^Q(M,N) \to \operatorname{Tor}_*^Q(M,N)$ factors through $E^1_{1,*} = H_*(\overline{A} \otimes_S (G \otimes N)) \xrightarrow{d^1} E^1_{0,*} = \operatorname{Tor}_*^Q(M,N)$, and the result follows. $\square$

We now specialize to the case $(Q,\mathfrak{n},k)$ is local and Noetherian, M is finitely generated, and $S=k$ is the residue field. Recall that

Definition 5.7. A Q-linear map $\phi : E \to F$ is *minimal* if $\phi(E) \subseteq \mathfrak{n}F$. A Q-free resolution is minimal if each differential is minimal.

Every finitely generated module has a minimal free resolution, and it is unique up to isomorphism. We will abuse language and speak of the minimal free resolution.

Let $A \xrightarrow{\simeq} R$ and $G \xrightarrow{\simeq} M$ be minimal Q-free resolutions. Set

$$\overline{A} = A \otimes k \cong \operatorname{Tor}^Q(R,k) \quad \text{and} \quad \overline{G} = G\otimes k \cong \operatorname{Tor}^Q(M,k).$$

In the spectral sequence of Theorem 5.3, with $S = N = k$, we have

$$E^2_{p,q} = \left(\operatorname{Tor}_p^{\operatorname{Tor}_*^Q(R,k)}(\operatorname{Tor}_*^Q(M,k),k)\right)_q \Rightarrow \operatorname{Tor}^R_{p+q}(M,k). \tag{5.8}$$

Let $\nu_q : \mathrm{Tor}_q^Q(M,k) \to \mathrm{Tor}_q^R(M,k)$ be the change of rings maps. By 5.6, we have the following diagram

$$\begin{tikzcd} E^1_{0,q} \arrow[r, "\nu_q"] \arrow[d, two heads] & \mathrm{Tor}_q^R(M,k) \\ E^2_{0,q} \arrow[d, two heads] & \\ \vdots \arrow[d, two heads] & \\ E^r_{0,q} \arrow[uuur, "\alpha"'] & \end{tikzcd}$$

where α is injective and $r = \lfloor (q-1)/2 \rfloor$. (The formula for r comes from the fact that $E^0_{p,q} = 0$ for $q < p$.) Thus ν_q is an injection if and only if each vertical map is an isomorphism if and only if d^i is zero on $E^i_{i,q-i+1}$ for all $2 \le i \le r$. From the description of the differentials d^i, and the fact that $\overline{m}_1 = 0 = \overline{m}_1^G$, we have the following.

Corollary 5.9. *Let A be a minimal Q-free resolution of R with an A_∞ algebra structure and G a minimal Q-free resolution of M with an A_∞ A-module structure m^G. Let*

$$\nu_q : \mathrm{Tor}_q^Q(M,k) \to \mathrm{Tor}_q^R(M,k)$$

be the change of rings map.

(1) *If the maps*

$$(\overline{m}_n^G)_{q+1} = (m_n^G \otimes k)_{q+1} : \left(A_+^{\otimes n-1}[1] \otimes G\right)_{q+1} \otimes k \to G_q \otimes k$$

are zero for all n, then ν_q is injective.

(2) *If ν_q is injective, then $(\overline{m}_2^G)_{q+1} = 0$.*

Corollary 5.10. *The change of rings maps*

$$\nu : \mathrm{Tor}_*^Q(M,k) \to \mathrm{Tor}_*^R(M,k) \quad \nu' : \mathrm{Tor}_*^Q(\Omega_R^1(M),k) \to \mathrm{Tor}_*^R(\Omega_R^1(M),k)$$

are injective if and only if $\overline{m}_n = 0 = \overline{m}_n^G$ for all $n \ge 1$.

The proof is a pleasant combination of Section 4 and Corollary 5.9.

Proof. If $\overline{m}_n$ and $\overline{m}_n^G$ are zero, the differentials of the spectral sequence are zero, and so the edge morphism $\mathrm{Tor}_*^Q(M,k) \to \mathrm{Tor}_*^R(M,k)$ is injective. Since m_2^G is minimal, the syzygy complex $\mathrm{syz}_R^1(G)$ defined in 4.3 is minimal, and $\overline{m_n^{\mathrm{syz}}}$ is zero for all n, by definition. Thus there are no nonzero differentials in the spectral sequence for $\Omega_R^1(M)$, and so $\mathrm{Tor}_*^Q(\Omega_R^1(M),k) \to \mathrm{Tor}_*^R(\Omega_R^1(M),k)$ is also injective.

If ν is injective, then $\overline{m}_2^G = 0$ by 5.9. Thus $\mathrm{syz}_R^1(G)$ is a minimal resolution of $\Omega_R^1(M)$ and we can apply 5.9. This shows that if ν' is injective, then $\overline{m_2^{\mathrm{syz}}} = 0$. It follows from Definition 4.7 that $\overline{m}_2 = 0$. Thus $E^1 = E^2$, and we can repeat the argument to show that $\overline{m}_3, \overline{m}_3^G = 0$, in which case $E^2 = E^3$, etc. □

Avramov defined

$$o_q(M) = \ker\left(\frac{\mathrm{Tor}_q^Q(M,k)}{(\mathrm{Tor}_+^Q(R,k) \cdot \mathrm{Tor}^Q(M,k))_q} \to \mathrm{Tor}_q^R(M,k) \right)$$

and showed that if A is a dg-algebra, i.e. $m_n = 0$ for all $n \geq 3$, and if

$$o(M) := \bigoplus_{q \geq 1} o_q(M) \neq 0,$$

then the minimal free Q-free resolution G of M has no structure of a dg A-module [3, 1.2]. We generalize this, slightly, below.

Corollary 5.11. *Let $(Q, \mathfrak{n}, k)$ be a local ring, R a cyclic Q-algebra, and M a finitely generated R-module. Let A and G be minimal Q-free resolutions of R and M, with A_∞ algebra and module structures m and m^G, respectively. Set $\overline{m}_n = m_n \otimes k$ and $\overline{m}^G_n = m^G_n \otimes k$, and assume that $\overline{m}_n = 0$ for $n \geq 3$, e.g. A is a dg-algebra.*

The obstruction $o_q(M)$ vanishes if and only if the differential $d^r : E^r_{r,q-r+1} \to E^r_{0,q}$, which is induced by $(\overline{m}^G_{r+1})_{q+1}$, is zero for all $r \geq 2$. In particular, if $\overline{m}^G_r = 0$ for all $r \geq 3$, then $o(M) = 0$.

The proof follows from considering the edge map

$$\beta_q : E^2_{0,q} = \frac{\mathrm{Tor}^Q_q(M,k)}{(\mathrm{Tor}^Q_+(R,k) \cdot \mathrm{Tor}^Q(M,k))_q} \to \mathrm{Tor}^R_q(M,k),$$

the diagram

$$\begin{tikzcd}
E^2_{0,q} \arrow[r, "\beta_q"] \arrow[d, two heads] & \mathrm{Tor}^R_q(M,k) \\
E^3_{0,q} \arrow[d, two heads] & \\
\vdots \arrow[d, two heads] & \\
E^r_{0,q} \arrow[uuur, "\alpha"'] &
\end{tikzcd}$$

and the description of the differentials given in 5.5.

This recovers Avramov's result, since if $o(M) \neq 0$, then for any A_∞-structure m^G, there exists $n \geq 3$ with $\overline{m}^G_n \neq 0$. In particular $m^G_n \neq 0$, so by Corollary 3.8, G does not have a dg A-module structure.

6. Golod maps

In this section $(Q, \mathfrak{n}, k)$ is a local Noetherian ring, R is a cyclic Q-algebra and $\varphi : Q \to R$ is the projection map.

and M is a finitely generated R-module. We let $\varphi : Q \to R$ be projection.

Definition 6.1. An A_∞-algebra structure m, or A_∞ A-module structure m^G, is minimal if m_n, or m^G_n, is minimal for all $n \geq 2$.

We now further assume that $A \xrightarrow{\simeq} R$ and $G \xrightarrow{\simeq} M$ are minimal Q-free resolutions. In this section we determine when there exist minimal A_∞-structures A and G, or equivalently, when the bar resolution is minimal. We need the following notation on minimal free resolutions.

Definition 6.2. The *Poincare series of M* is the generating function

$$\mathrm{P}^R_M(t) := \sum_{n \geq 0} \dim_k \mathrm{Tor}^R_n(M,k) t^n.$$

This applies to any local ring, so e.g. $\mathrm{P}^Q_M(t)$ also makes sense.

Lemma 6.3. *There is a degree-wise inequality of power series*

$$\mathrm{P}^R_M(t) \preccurlyeq \frac{\mathrm{P}^Q_M(t)}{1 - t\left(\mathrm{P}^Q_R(t) - 1\right)}. \tag{6.4}$$

This is an equality if, respectively only if, the resolution 3.13 is minimal for some, respectively every, choice of A_∞-structures on A, G.

We refer to the above as the Golod bound for M.

Proof. The free resolution of 3.13 has Poincare series the rational function on the right. Since the minimal R-free resolution of M is a subcomplex of this resolution, the inequality follows. If we can choose minimal A_∞-structures, then the resolution 3.13 is clearly minimal, and so the inequality must be an equality. If the inequality is an equality, then for any choice of A_∞-structures the corresponding bar resolution has the same Poincare series as the minimal resolution and so must be minimal. □

Remark. The inequality (6.4) classical, first appearing in print in [15] in case $M = k$ (and is credited to Serre there). The Eagon resolution of the residue field, see [18, 4.1], realizes the bound for this module, while Gokhale [14] shows the existence of a resolution realizing the bound for any module, but he does not give an explicit description of the differentials. To our knowledge, the resolution of Theorem 3.13 is the first explicit realization of the Golod bound for every finitely generated module.

Definition 6.5. A finitely generated R-module M is *φ-Golod* if (6.4) is an equality. If k is a φ-Golod module, then φ is a *Golod homomorphism*.

Remark. Levin first defined Golod modules [26] and Golod homomorphisms [27]. The definitions were inspired by Golod [15]. For more information, and history on Golod homomorphisms see [1] and [2, §5].

Example 6.6. Let $R = Q/(f)$ with $f \in \mathfrak{n}^2$ a non-zero divisor. Let K be the Koszul complex on a minimal set of generators of $\mathfrak{n}$ over Q. The construction of [8, 2.2] gives a homotopy $s : K \to K$ for multiplication by f on K such that s is minimal and $(s)^2 = 0$. Following 3.5, $m^K_2 = s$ and $m^K_n = 0$ for $n \geq 3$ is an A_∞ A-module structure on K, and it is clear the corresponding bar resolution of k is minimal. Thus $Q \to R$ is Golod.

Example 6.7. If is often the case that quotients of powers of ideals are Golod. Levin shows that for a local ring $(Q, \mathfrak{n})$, the map $Q \to Q/\mathfrak{n}^r$ is Golod for all $r \gg 0$, [27, 3.15], while Herzog, Welker and Yassemi show that if $(Q, \mathfrak{n})$ is regular, or a standard graded polynomial ring, then for any ideal I, graded if Q is the polynomial ring, the map $Q \to Q/I^r$ is Golod for all $r \gg 0$ [20, 4.1]. When Q is a standard graded polynomial ring over a field of characteristic zero, Herzog and Huneke show that $Q \to Q/I^r$ is Golod for all graded ideals I and $r \geq 2$ [19, Theorem 2.3].

Theorem 6.8. *Let $(Q, \mathfrak{n}, k)$ be a local Noetherian ring, $\varphi : Q \to R$ a surjective ring map, and M a finitely generated R-module. Let $A \xrightarrow{\simeq} R$ and $G \xrightarrow{\simeq} M$ be minimal Q-free resolutions. The following are equivalent:*

1. *M is φ-Golod;*
2. *the bar resolution 3.13 is minimal for some (respectively, every) A_∞-structure on A and G;*

(3) *there exist minimal A_∞-structures on A and G (respectively, every such structure is minimal);*
(4) *there exist Q-free resolutions A' and G' of R and Q, respectively, that are not necessarily minimal, but have minimal A_∞-structures;*
(5) *the Avramov spectral sequence 5.3, with $S = N = k$, collapses on the first page;*
(6) *the change of rings maps*

$$\mathrm{Tor}^Q(M,k) \to \mathrm{Tor}^R(M,k) \quad \mathrm{Tor}^Q(\Omega^1_R(M),k) \to \mathrm{Tor}^R(\Omega^1_R(M),k)$$

are injective.

Proof. First note both conditions in 2 are equivalent by Lemma 6.3. This also shows 1 and 2 are equivalent. Let m, m^G be A_∞-structures and assume 2 is true, so the bar resolution tensored with k is zero. Using the fact that a morphism $\begin{pmatrix} f_1 & f_2 \\ f_3 & f_4 \end{pmatrix}$ is zero if and only if f_i is zero for all i, we see that $m_n \otimes k = 0$ and $m_n^G \otimes k = 0$ for all $n \geq 1$. Thus the A_∞-structures are minimal, and so 2 implies the stronger assertion of 3. The weaker assertion of 3 tautologically implies 4, and 4 implies 5 by the description of the differentials of the spectral sequence in Theorem 5.5. The ranks on the first page of the Avramov spectral sequence are the same as the bound in 6.3, thus the sequence collapses on the first page if and only if the bound is an equality, i.e. 5 and 1 are equivalent. Finally, 3 and 6 are equivalent by 5.10. $\square$

The equivalence of 1 and 6 was first proved by Levin in [28]. The next result is also proved there. We give a short proof using the syzygy of a resolution defined in 4.3.

Corollary 6.9. *If M is φ-Golod, then $\Omega^1_R(M)$ is also φ-Golod.*

Proof. Let m, m^G be A_∞-structures. These are minimal by Theorem 6.8. By 4.6, $\Omega^1_R(M)$ has a Q-free resolution with minimal A_∞-structures, thus is φ-Golod. $\square$

The following answers a question I learned from Eisenbud. There was no previously bound of this type known, even in case Q is regular.

Corollary 6.10. *Assume that $c = \max\{\mathrm{pd}_Q R, \mathrm{pd}_Q M - 1\}$ is finite. If (6.4) is an equality up to the coefficient of t^{c+1}, then M is φ-Golod.*

Proof. If equality holds in degrees $\leq c+1$, then for any choice of A_∞-structures the differentials d_n of the bar resolution 3.13 must be minimal for $n \leq c+2$, since the minimal free resoution is a subcomplex of the bar resolution. As in the proof of 6.8, this means that every map occurring as a summand in degrees $\leq c+2$ is minimal. But since $A_c \otimes G_0$ is a summand in degree $c+1$, all maps $m_n : A_+[1]^{\otimes n} \to A_+[1]$ must have occured in degrees $\leq c+2$ as terms $m_n \otimes 1_{G_0}$. Also, since G_{c+1} occurs in degree $c+1$, all maps m_n^G must have occurred in degrees $\leq c+2$. Thus m_n and m_n^G are minimal for all n and so the entire bar resolution is minimal, and M is φ-Golod. $\square$

Golodness cannot be verified in fewer degrees, by the following:

Example 6.11. Let Q, R be as in 3.10. By 3.15, the Poincare series of k starts

$$\mathrm{P}^R_k(t) = 1 + 3t + 8t^2 + 21t^3 + 55t^4 + \dots,$$

however

$$\frac{\mathrm{P}_k^Q(t)}{1-t\left(\mathrm{P}_R^Q(t)-1\right)} = \frac{(1+t)^3}{1-t(5t+5t^2+t^3)} = 1+3t+8t^2+21t^3+56t^4+\dots$$

and so k is not φ-Golod. Since R is a codimension 3 Gorenstein ring, one could also use a result of Wiebe, [35, Satz 9], that shows

$$\mathrm{P}_k^R(t) = \frac{(1+t)^3}{1-5t^2-5t^3+t^5}.$$

For the rest of the paper, we assume that $(Q,\mathfrak{n},k)$ is a regular local ring, $R \cong Q/I$ with $I \subseteq \mathfrak{n}^2$, and $\varphi : Q \to R$ is projection. Any local ring that is the quotient of a regular local ring can be written in this form; see [2, §4].

Definition 6.12. A finitely generated R-module is Golod if it is φ-Golod. The local ring R is a Golod ring if the residue field k is a Golod module.

Remark. Whether M is Golod does not depend on the presentation $R = Q/I$. Indeed, if $R \cong Q'/I'$ with $I' \subseteq (\mathfrak{n}')^2$, then we have $\mathrm{P}_Q^M(t) = \mathrm{P}_{Q'}^M(t)$ for any finitely generated R-module M, and so the bound 6.3 is an equality for Q if and only if it is an equality for Q'. To see the equality of Poincare series, if K is the Koszul complex on a minimal generating set of $\mathfrak{n}$ and K' for $\mathfrak{n}'$, then $K \otimes R \cong K' \otimes_{Q'} R$, since both are Koszul complexes on a minimal generating set of the maximal ideal of R. So we have $\mathrm{Tor}_*^Q(M,k) \cong H_*(K \otimes M) \cong H_*(K' \otimes_{Q'} M) \cong \mathrm{Tor}_*^{Q'}(M,k)$.

Theorem 6.13. *Let $(Q,\mathfrak{n},k)$ be a regular local ring, $I \subseteq \mathfrak{n}^2$ an ideal, and $R = Q/I$. Let $A \xrightarrow{\simeq} R$ be the minimal free Q-resolution. The following are equivalent:*

1. *R is Golod;*
2. *there exists a non-zero Golod R-module;*
3. *there exists a minimal A_∞-structure on A (respectively, every A_∞-structure on A is minimal);*
4. *the change of rings map for the maximal ideal $\mathfrak{m}$ of R,*
$$\mathrm{Tor}^Q(\mathfrak{m},k) \to \mathrm{Tor}^R(\mathfrak{m},k),$$
is injective;
5. *the inequality (6.4), for the module $M = k$, is an equality up to the coefficient of t^{e+1}, where $e = \dim Q$.*

Proof. Set K to be the Koszul complex on $\mathfrak{n}$ with an A_∞ A-module structure m^K. Since $I \subseteq \mathfrak{n}^2$, the change of rings map $\nu : \mathrm{Tor}^Q(k,k) \to \mathrm{Tor}^R(k,k)$ is injective. This is well known. To see it, one can minimally resolve k over R by adjoining higher degree dg-variables to the dg R-algebra $K \otimes R$, as in [2, §6].

For the implications, we have $1 \Rightarrow 2$ by definition, and $2 \Rightarrow 3$ by 6.8. If 3 holds, then $m \otimes k = 0$, and arguing as in the proof of 5.10, the injectivity of ν implies that $m^K \otimes k = 0$. Thus by 5.9 applied to $M = k$, 4 holds. Since ν is always injective, we have $4 \Rightarrow 1$ by 5.10 and 6.8. Finally, $1 \iff 5$ holds by 6.10. □

The equivalence of 1 and 4 was first proved in [27], and that of 1 and 2 in [28] and [25], independently.

Remark. Golod rings are also characterized as having a homotopy Lie algebra that is free in degrees ≥ 2, or as having trivial Massey products on the homology of the

Koszul complex, or those for which Eagon's resolution of the residue field is minimal; see [1, 18]. There seems to be interesting, but delicate, connections between these three characterizations and A_∞-structures. We plan to return to this in future work.

The following was first proved by Lescot in [25]. We give a short proof using the syzygy construction of a free resolution.

Corollary 6.14. *Let R be a Golod ring, M a finitely generated R-module, and let $c = \operatorname{pd}_Q M$. Then $\Omega_R^{c+1}(M)$ is a Golod module.*

Proof. By the above theorem, A has a minimal A_∞-structure. Thus by 4.11 there is a (possibly non-minimal) Q-free resolution of $\Omega_R^{c+1}(M)$ with a minimal A_∞ A-module structure, and so by 6.8.(4), it is a Golod module. □

By Theorem 6.13, to show R is Golod it is enough to show that $\overline{m}_n = 0$ for all n. This is connected to a folklore question. There is an isomorphism of graded k-algebras $H_*(K \otimes R) \cong \overline{A}$, where K is the Koszul complex on $\mathfrak{n}$, the maximal ideal of Q. There is no known example of a non-Golod ring R with trivial multiplication on $H_*(K \otimes R)$. The question is whether such non-Golod rings exist. In terms of A_∞-algebras, this is the following.

Question. Let $R = Q/I$ be a local ring, with $(Q, \mathfrak{n})$ a regular local ring and $I \subseteq \mathfrak{n}^2$. Let A be the minimal Q-free resolution of R with A_∞-algebra structure (m_n). If $\overline{m}_2 = 0$, is $\overline{m}_n = 0$ for all $n \geq 2$?

By [6], this is true when Q is a graded polynomial ring and I a monomial ideal.

Example 6.15. Theorem 6.13 gives an easy proof of a result of Shamash [32]. Let $(Q, \mathfrak{n})$ be a regular local ring, J be an ideal in Q, and $f \in \mathfrak{n}$ a nonzero element. Then $R = Q/(f \cdot J)$ is Golod. Indeed, let $B \xrightarrow{\simeq} Q/J$ be the minimal Q-free resolution. Since f is a non-zero divisor, multiplication by f gives an isomorphism of Q-modules

$$J \xrightarrow{\cong} f \cdot J.$$

Thus we can construct a minimal Q-free resolution of R by setting $A_+ = B_+$, $A_0 = Q$ and $d_1^A = f \cdot d_1^B$. If $m_n^B : B_+[1]^{\otimes n} \to B_+[1]$ is an A_∞-algebra structure on B, then

$$m_n^A = f^{n-1} m_n^B : A_+[1]^{\otimes n} \to A_+[1]$$

is an A_∞-algebra structure on A. In particular, m_n^A is minimal for all n, so R is Golod.

Corollary 6.16. *If R is a Golod local ring, then one can construct the minimal R-free resolution of every finitely generated module in finitely many steps.*

Proof. Let M be a finitely generated R-module. By 6.14, a syzygy N of M is a Golod module. We can then construct the finite minimal Q-free resolutions of R and N, and the finitely many maps in A_∞-structures on these. By 6.8, the resolution 3.13 using these A_∞-structures is the minimal resolution of N. □

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