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The Poincaré Series of a Local Ring II*

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1. (Introduction) The purpose of this paper is to prove that the Poincaré Series of a Local Ring is rational provided that certain conditions are satisfied on the Koszul complex. These conditions are satisfied for example if the ring is a complete intersection or if the Koszul complex has trivial Massey operations. The methods I have used are new and leave considerable room for improvement. It is likely that a sharpening of these methods will eliminate some, if not all, of the conditions that I require in this paper.

Section (2) is introductory in character and indicates the outline of the proof. In Sections (3) and (4) I state the conditions that I need on the Koszul complex. Section (5) contains the proof of the main theorem of the paper. In Section (6) I give some examples and work out the Poincaré Series of some rings whose rationality had not been previously known. The notation is as in Shamash [3].

2. *Notation.* Let E be a graded free R-module such that $E^i = 0$ for $i \leqslant 1$ and $i > n+1$, and $\dim(E^i) = \epsilon_{i-1}$ for $1 < i \leqslant n+1$. Let $\underline{T}(E)$ denote the tensor algebra of E. Let $u_1{}^i,..., u^i_{\epsilon_{i-1}}$ be a base for E^i and let $T_1^{i-1},..., T^{i-1}_{\epsilon_{i-1}}$, be elements of K^{i-1} whose images in the homology complex form a base for $H^{i-1}(\mathscr{K})$. An element of $\mathscr{K} \otimes \underline{T}(E)$ of the form $x = x_1 \otimes \cdots \otimes x_n$ where for all i, x_i is some $u_k{}^j$ will be called a U-basic element, and n will be called its weight. An element of $\mathscr{K} \otimes \underline{T}(E)$ of the form $x_1 \otimes \cdots \otimes x_n$, where x_1 is a homogeneous element of $\mathscr{K}$ and for $2 \leqslant i \leqslant n$, x_i is some $u_k{}^j$, will be called a K-basic element and its weight is defined to be n. Every homogeneous element z of $\mathscr{K} \otimes \underline{T}(E)$ of degree j can clearly be written as a linear sum of basic elements of degree j. However, because of the definition of K-basic elements z can be written in many ways as a linear sum of basic elements with non-zero coefficients. But among all such ways of writing z choose the one with the fewest number of terms and

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with the property that the coefficients of the K-basic elements are 1. There is clearly one and only one such. It will be called the canonical factorization of z into basic elements. Suppose then that $z = \sum_{k=1}^{s} r_k z_k$ is its canonical factorization. We define the weight of z to be $\max\{\text{weight}(z_k)\}_{k=1,s}$. In the set $\{z_k\}_{k=1,s}$ let $\{\bar{z}_\ell\}_{\ell=1,t}$ be the largest subset each of whose elements has weight equal to the weight of z. Let $\{\bar{r}_\ell\}_{\ell=1,t}$ be the corresponding coefficients in the canonical factorization of z. Then $\bar{z} = \sum_{\ell=1}^{t} \bar{r}_\ell \bar{z}_\ell$ will be called the heavy part of z. Finally, for $i \geqslant 0$ let R^i denote the free R-module of $\mathscr{K} \otimes \underset{\sim}{T}(E)$ of degree i.

THEOREM 1. *We can define a differentiation* $d^i : R^i \to R^{i-1}$ *for all* $i > 0$ *such that*

$$\cdots \longrightarrow R^i \xrightarrow{d^i} R^{i-1} \longrightarrow \cdots \longrightarrow R^1 \xrightarrow{d^1} R^0 \xrightarrow{\epsilon} \underset{\sim}{k}$$

is a resolution for $\underset{\sim}{k}$.

Proof. We assume by induction that for all $j \leqslant i$, d has been defined on R^j such that the following properties are satisfied:

1) If x is an element of K^j then $d^j(x)$ is just the differentiation on the Koszul complex.

2) If x is some $u_k{}^j$, $d(x) = T_k^{j-1}$.

3) If $x = x_1 \otimes \cdots \otimes x_n$ is a U-basic element then

$$d(x) = d(x_1 \otimes \cdots \otimes x_{n-1}) \otimes x_n + \gamma(x_1 \otimes \cdots \otimes x_n)$$

with the waight of $\gamma(x_1 \otimes \cdots \otimes x_n) < n$.

4) If $x = x_1 \otimes \cdots \otimes x_n$ is a K-basic element, then

$$d(x) = d(x_1) \otimes x_2 \otimes \cdots \otimes x_n + (-1)^{\deg x_1} x_1 \otimes d(x_2 \otimes \cdots \otimes x_n).$$

5) $d^j d^{j-1} = 0$.

Remark 1. It is an immediate consequence of (3) that if x is U-basic, $d(x) = d(x_1) \otimes x_2 \otimes \cdots \otimes x_n +$ some element of weight $< n$.

Remark 2. Suppose x is K-basic. Define

$$\gamma(x) = (-1)^{\deg x_1} x_1 \otimes \gamma(x_2 \otimes \cdots \otimes x_n).$$

Then $d(x)$ clearly equals $d(x_1 \otimes \cdots \otimes x_{n-1}) \otimes x_n + \gamma(x)$. Suppose now that z is any element of R^j and suppose that $\sum_{k=1}^{s} r_k z_k$ is its canonical factorization. We then define $\gamma(z) = \sum_{k=1}^{s} r_k \gamma(z_k)$.

Before we define d^{i+1} we observe the following lemmas which are consequences of the inductive hypotheses:

LEMMA 1. *If z is a homogeneous element of $\mathscr{K} \otimes T(E)$ and x is a homogeneous element of the Koszul complex such that* $\deg(x \otimes z) \leqslant i$, *then* $d(x \otimes z) = d(x) \otimes z + (-1)^{\deg x}\, x \otimes dz$.

Proof. By linearity we can assume that $z = x_1 \otimes \cdots \otimes x_n$ is a basic element. If x_1 is not an element of the Koszul complex, then letting

$$\begin{aligned} z' &= x_2 \otimes \cdots \otimes x_n\,, \\ d(x \otimes z) &= d(x \otimes x_1) \otimes z' + (-1)^{\deg(x \otimes x_1)}\,(x \otimes x_1) \otimes dz' \\ &= dx \otimes x_1 \otimes z' + (-1)^{\deg x}\, x \otimes dx_1 \otimes z' \\ &\quad + (-1)^{\deg x + \deg x_1}\,(x \otimes x_1) \otimes dz' \\ &= dx \otimes z + (-1)^{\deg x}\, x \otimes dz. \end{aligned}$$

LEMMA 2. *If $x = x_1 \otimes \cdots \otimes x_n$ is a basic element then $d(x) = d(x_1) \otimes x_2 \otimes \cdots \otimes x_n$ plus some element of weight $< n$.*

Proof. If x is U-basic then lemma (2) is just Remark (1). If x is K-basic then the lemma follows from hyp. (4) and the case that x is U-basic.

LEMMA 3. *Let z be a cycle in R^j of weight n, and let $z = \sum_{\ell=1}^{t} \bar{r}_\ell \bar{z}_\ell$ be its heavy part. Assume that $\bar{r}_\ell$ is a unit for all ℓ. Then for all ℓ, if $\bar{z}_\ell = x_1^\ell \otimes \cdots \otimes x_n^\ell$, x_1^ℓ is a cycle in $\mathscr{K}$.*

Proof. By lemma (2), $d(z) = \sum_{\ell=1}^{t} d(x_1^\ell) \otimes x_2^\ell \otimes \cdots \otimes x_n^\ell$ plus some term of smaller weight. Thus if z is a cycle, $\sum_{\ell=1}^{t} d(x_1^\ell) \otimes x_2^\ell \otimes \cdots \otimes x_n^\ell = 0$ which clearly implies that for all $1 \leqslant \ell \leqslant t$, x_1^ℓ is in $\mathscr{K}$ and is a cycle.

LEMMA 4. *For $j < i$, suppose that z in R^j is of weight n. Then we can find a $w \in R^{j+1}$ of weight $n + 1$ such that $z - dw$ has weight $\leqslant n$ and the heavy part of $z - dw$ has units for coefficients.*

Proof. Suppose $\bar{z} = \sum_{\ell=1}^{t} \bar{r}_\ell \bar{z}_\ell$ and suppose that $\bar{r}_1, \ldots, \bar{r}_u$ are non-units but that $\bar{r}_{u+1}, \ldots, \bar{r}_t$ are units. Now since $\bar{r}_k \in \underline{m}$ for all $1 \leqslant k \leqslant u$, we can find $\tilde{r}_k \in K^1$ such that $d^1(\tilde{r}_k) = \bar{r}_k$. Define $w = \sum_{k=1}^{u} \tilde{r}_k \otimes \bar{z}_k$. It is clear then that either $z - dw$ has weight $< n$, or else the coefficients of its heavy part are non-units. In the first case we continue the process until the desired result is obtained.

LEMMA 5. *For $j < i$, suppose that z is a cycle in R^j of weight n. Then if the coefficients of $\bar{z}$ are all units we can find an element w in R^{j+1} of weight n such that $z - dw$ has weight $< n$. If the coefficients of $\bar{z}$ contain non-units, we can find an element w in R^{j+1} of weight $n + 1$ such that $z - dw$ has weight $< n$.*

Proof. By Lemma 4 we can assume that the coefficients of $\bar{z}$ are all units. Then Lemma 5 is an immediate consequence of Lemmas 2 and 3.

Remark. Lemma 5 clearly implies that d is exact at R^j for $j < i$.

We can now return to the definition of d^{i+1} and it clearly suffices to define it on basic elements so that hypotheses (3-5) are satisfied. If x is a K-basic element we define $d(x)$ to satisfy hypothesis (4) and then by inductive hypothesis (5) and Lemma 1 it clearly follows that $d^i d^{i+1} = 0$. Suppose then that $x = x_1 \otimes \cdots \otimes x_n$ is U-basic and we can clearly assume that $n > 2$. Then from inductive hypotheses (3) and (5) and Remarks 2 and 1 preceding Lemma 1 we have that

$$\begin{aligned} d(d(x_1 \otimes \cdots \otimes x_{n-1}) \otimes x_n) &= \gamma(d(x_1 \otimes \cdots \otimes x_{n-1}) \otimes x_n) \\ &= \gamma(T_1 \otimes x_2 \otimes \cdots \otimes x_n) + \gamma(z) \end{aligned}$$

where z is an element of weight $\leqslant n - 1$. Thus by inductive hypothesis (3) and Remark 2, $d(d(x_1 \otimes \cdots \otimes x_{n-1}) \otimes x_n)$ either has weight $\leqslant n - 2$ or its heavy part is $T_1 \otimes \gamma(x_2 \otimes \cdots \otimes x_n)$. Now if $d(d(x_1 \otimes \cdots \otimes x_{n-1}) \otimes x_n)$ has weight $\leqslant n - 2$, we can by Lemma (5) find a $\gamma(x_1 \otimes \cdots \otimes x_n)$ in R^i of weight $\leqslant n - 1$ such that

$$d(\gamma(x_1 \otimes \cdots \otimes x_n)) = -d(d(x_1 \otimes \cdots \otimes x_{n-1}) \otimes x_n).$$

Defining then $d(x_1 \otimes \cdots \otimes x_n) = d(x_1 \otimes \cdots \otimes x_{n-1}) \otimes x_n + \gamma(x_1 ,\ldots, x_n)$ inductive hypotheses (3) and (5) will clearly be satisfied. Otherwise $d(d(x_1 \otimes \cdots \otimes x_{n-1}) \otimes x_n) - d(x_1 \otimes \gamma(x_2 \otimes \cdots \otimes x_n))$ has weight $\leqslant n - 2$. Then since $x_1 \otimes \gamma(x_2 \otimes \cdots \otimes x_n)$ has weight $\leqslant n - 1$ we can satisfy hypothesis (3) and (5) as above by applying Lemma 5. This completes the proof of Theorem 1.

Given any resolution of a module M, I wish now to describe a method for obtaining a minimal resolution. Let R be a local ring and M an R-module. Let $\cdots R^i \xrightarrow{d^i} R^{i-1} \to \cdots \to R^0 \xrightarrow{\epsilon} M$ be any resolution of M. For any $i > 0$, consider the map $\bar{d}^{i+1} : (R^{i+1}/\underline{m}R^{i+1}) \to R^i/\underline{m}R^i$. Let $f_1^{i+1},\ldots, f_{\alpha(i)}^{i+1}$ be elements of $R^{i+1}/\underline{m}R^{i+1}$ which form a base for a supplement to the kernel of $\bar{d}^{i+1}$. Let F^{i+1} denote the free direct summand of R^{i+1} spanned by $f_1^{i+1},\ldots, f_{\alpha(i)}^{i+1}$. Then $d(F^{i+1})$ is a free direct summand of R^i. We denote this module by H^i. For $i \geqslant 1$, it is clear that $F^i \cap H^i$ is trivial and that $F^i \oplus H^i$ is a free direct summand of R^i and we define $\tilde{R}^i = R^i/(F^i \oplus H^i)$. We define $\tilde{R}^0 = R^0/H^0$. d^i induces a map $\tilde{d}^i$ of $\tilde{R}^i \to \tilde{R}^{i-1}$ and ϵ induces a map $\tilde{\epsilon}$ of $\tilde{R}^0$ to M.

PROPOSITION 1. $\cdots \tilde{R}^{i+1} \xrightarrow{d^{i+1}} \tilde{R}^i \to \cdots \to \tilde{R}^0 \xrightarrow{\epsilon} M$ *is a minimal resolution for* M.

Proof. It is clear that $\tilde{d}^i\tilde{d}^{i+1} = 0$ and $\epsilon\tilde{d}^0 = 0$. For $i \geqslant 1$ suppose that $\tilde{x} \in \tilde{R}^i$ is such that $\tilde{d}^i(x) = 0$. Let x be an element of R^i whose image in $\tilde{R}^i$ is $\tilde{x}$. Then $d^i(x) = y + z$ with $y \in H^{i-1}$ and $z \in F^{i-1}$. Since $d^{i-1}(H^{i-1}) = d^{i-1}d^i(F^i) = 0$, and since d^{i-1} is isomorphic on F^{i-1} we have that $z = 0$. Let $y' \in F^i$ be such that $d^i(y') = y$. Then $d^i(x - y') = 0$ so there exists $z \in R^{i+1}$ such that $d(z) = x - y$. Then if $\tilde{z}$ denotes the image of z in $\tilde{R}^{i+1}$, $\tilde{d}(\tilde{z}) = \tilde{x}$ which completes the proof of exactness. The minimality of the resolution is clear.

COROLLARY 1. $\mathrm{Tor}_R{}^i(M, k) \leqslant \dim(R^i)$.

COROLLARY 2. (*Serre*) $P_R(\underline{k}) \leqslant [(1 + z)^n/1 - \epsilon_1 z^2 - \cdots - \epsilon_n z^{n+1}]$. *The inequality is on each coefficient.*

Proof. Corollary 2 follows immediately from Corollary 1 and Theorem 1.

Remark 1. The above result of Serre is unpublished. Golod [1] asserts that it was proved using spectral sequences.

Remark 2. I conjectured in Shamash [3] that a minimal resolution for $\underline{k}$ can be given a structure of an associative algebra isomorphic to $\mathscr{K} \otimes \underline{T}(E)/\underline{a}$ where $\underline{a}$ is a finitely generated ideal. By using Theorem 1 and the above proposition the conjecture reduces to choosing the F^i so that the module $\sum_{i\geqslant 0} F^i \oplus H^i$ is a finitely generated ideal of $\mathscr{K} \otimes T(E)$. Let $\bar{F}^i$ and $\bar{H}^i$ denote the images of F^i and H^i in $\mathscr{K} \otimes \underline{T}(E)/\underline{m}(\mathscr{K} \otimes \underline{T}(E)) = \overline{\mathscr{K} \otimes T(E)}$. A somewhat simpler problem is to choose the F^i so that $\sum_{i\geqslant 0} \bar{F}^i \oplus \bar{H}^i$ is a finitely generated ideal $\underline{a}$ of $\overline{\mathscr{K} \otimes \underline{T}(E)}$. It is clear that $P(\overline{\mathscr{K} \otimes \underline{T}(E)}/\underline{a}) = P_R(k)$. Thus finding the ideal $\underline{a}$ would prove the rationality of $\underline{k}$. This is what I propose to do in this paper given certain conditions on the Koszul complex. These conditions will be stated in the next two sections.

3. *Notation.* Let $\mathfrak{A}$ be an alternating graded algebra over the field $\underline{k}$. We assume that $\mathfrak{A}^i$ is of finite dimension α_i for all i and that $\alpha_i = 0$ for $i \leqslant 0$ and $i > n$. Let B be any base of $\mathfrak{A}$ consisting of homogeneous elements. Then B is a finite set consisting of say r elements. Let σ be an isomorphism from the set of integers $\{1, \ldots, r\}$ to B subject to the property that for $i \leqslant j$ $\deg(\sigma_i) \leqslant \deg(\sigma_j)$. Then σ defines an ordering on B. We denote $\sigma(i)$ by b_i. Let B^* denote the set of all sequences $(b_{s_1}, \ldots, b_{s_a})$ with $b_{s_i} \in B$ and with the properties that $a \leqslant n$ and for all i, $1 \leqslant i \leqslant a - 1$, $s_i \leqslant s_{i+1}$. B^* is thus a finite set. If $x = (b_{s_1}, \ldots, b_{s_a})$ and $y = (b_{t_1}, \ldots, b_{t_c})$ are two elements of B^*, we will set $x < y$ provided either (1) there exists an integer i such that $b_{s_i} < b_{t_i}$ and for all $j < i$ $b_{s_j} = b_{t_j}$, or, (2) $a < c$ and $b_{s_i} = b_{t_i}$ for all i,

$1 \leqslant i \leqslant a$. This defines a total linear ordering on B^*. If $x = (b_{s_1}, ..., b_{s_a}) \in B^*$, we define $\sigma^*(x) = b_{s_1}' \otimes \cdots \otimes b_{s_n} \in \mathcal{O}\!\ell$.

PROPOSITION 2. *There exists a subset $\bar{B} = \{\bar{b}_1, ..., \bar{b}_r\}$ of B^* such that*

(1) *$\{\sigma^*(\bar{b}_1), ..., \sigma^*(\bar{b}_r)\}$ forms a base of $\mathcal{O}\!\ell$.*

(2) *$\bar{b}_i < \bar{b}_{i+1}$ for all i, $\quad 1 \leqslant i \leqslant r$*

(3) *If $B' = \{b_1', ..., b_r'\}$ is another subset of B^* having properties* (1) *and* (2), *then $b_i' \geqslant \bar{b}_i$, for all i, $\quad 1 \leqslant i \leqslant r$.*

Proof. We define $\bar{B}$ as follows: $\bar{b}_1 = b_1$ and by induction supposing that $\bar{b}_1, ..., \bar{b}_j$ have been defined we define $\bar{b}_{j+1}$ to be the smallest element of B^* such that $\sigma^*(\bar{b}_{j+1})$ is not contained in the subspace of $\mathcal{O}\!\ell$ generated by $\sigma^*(\bar{b}_1), ..., \sigma^*(\bar{b}_j) = \langle \sigma^*(\bar{b}_1), ..., \sigma^*(\bar{b}_j) \rangle$. Suppose now that B' is another subset of B^* having properties (1) and (2). It is clear that $\bar{b}_1 \leqslant b_1'$. Suppose by induction that we have shown that $\bar{b}_j \leqslant b_j'$ for all j, $1 \leqslant j < i$. Then if $\langle \sigma^*(\bar{b}_1), ..., \sigma^*(\bar{b}_{i-1}) \rangle = \langle \sigma^*(b_1'), ..., \sigma^*(b_{i-1}') \rangle$ we have by the definition of $\bar{B}$ that $\bar{b}_i \leqslant b_i'$. If on the other hand $\langle \sigma^*(\bar{b}_1), ..., \sigma^*(\bar{b}_{i-1}) \rangle \neq \langle \sigma^*(b_1'), ..., \sigma^*(b_{i-1}') \rangle$ then for some j, $1 \leqslant j \leqslant i - 1$, $b_j' \in \langle \bar{b}_1, ..., \bar{b}_{i-1} \rangle$ so that $\bar{b}_i \leqslant b_j' < b_i'$.

Remark. It is easily seen that among all subsets of B^* whose image under σ^* form a base of $\mathcal{O}\!\ell$, $\bar{B}$ is characterized by the property that for every $x \in \bar{B}$, if $\sigma^*(x)$ is contained in the subspace $\langle \sigma^*(y_1), ..., \sigma^*(y_m) \rangle$ where $y_j \in B^*$ for all j, $1 \leqslant j \leqslant m$, then for some i, $1 \leqslant i \leqslant m$, $x \leqslant y_i$.

Notation. Let $b_\alpha \in B$ and $x = (b_{s_1}, ..., b_{s_a}) \in B^*$. We will write $b_\alpha * x$ to be $(b_{s_1}, ..., b_{s_i}, b_\alpha, b_{s_{i+1}}, ..., b_{s_a})$ where i is the largest integer such that $\alpha \geqslant s_i$. Thus $\sigma(b_\alpha * x) = \pm b_\alpha \otimes \sigma(x)$. We set $\deg(x) = \deg(b_{s_1}) + \cdots + \deg(b_{s_a})$.

PROPOSITION 3. *Suppose that $(b_{s_1}, ..., b_{s_a})$ is an element of $\bar{B}$, then for all j, $1 \leqslant j \leqslant a$, $(b_{s_1}, ..., b_{s_{j-1}}, b_{s_{j+1}}, ..., b_{s_a})$, denoted by $x^{\hat{j}}$, is also an element of $\bar{B}$.*

Proof. We observe first the following lemma:

LEMMA 6. *Let $b \in B$ and let x and y be elements of B^* such that* $\deg(x) = \deg(y)$. *Then if $x < y$, $b*x < b*y$.*

Proof. Suppose that $x = (b_{i_1}, ..., b_{i_m})$ and $y = (b_{j_1}, ..., b_{j_n})$. Let α be the largest integer such that $b \geqslant b_{i_\alpha}$. Suppose first we have for all $k \leqslant \alpha$ $b_{i_k} = b_{j_k}$. Then since $x < y$ and $\deg(x) = \deg(y)$, $\alpha \neq m$. Furthermore $b < b_{i_{\alpha+1}} \leqslant b_{j_{\alpha+1}}$. Thus $b * x < b * y$ follows from $x < y$. Suppose then that for some $k \leqslant \alpha$ we have that $b_{j_k} > b_{i_k}$. Then if $b \geqslant b_{j_k}$. $b * x < b * y$ easily follows from $x < y$. Whereas if $b < b_{j_k}$, then $b * x < b * y$ follows from $b > b_{i_\alpha} \geqslant b_{i_k}$.

This proves the lemma. To now prove the proposition, suppose that $x^{\hat{j}}$ is not an element of $\bar{B}$. Then there exist elements $y_1, \ldots, y_n \in \bar{B}^*$ such that $\sigma^*(x^{\hat{j}}) \in \langle \sigma^*(y_1), \ldots, \sigma^*(y_n) \rangle$ and $x^{\hat{j}} > y_i$ for all i, $1 \leqslant i \leqslant n$. Now $\sigma^*(b_{i_j} * x^{\hat{j}}) = \sigma^*(x)$ is clearly contained in $\langle \sigma^*(b_{i_j} * y_1), \ldots, \sigma^*(b_{i_j} * y_n) \rangle$. Furthermore by the above lemma we have for all k, $1 \leqslant k \leqslant m$, that $x = b_{i_j} * x^{\hat{j}} > b_{i_j} * y_k$ and this contradicts the fact that $\sigma^*(x)$ was assumed to be in $\bar{B}$. This completes the proof of Proposition 3.

Definition. $\mathfrak{A}$ will be called excellent if there exists a base B of $\mathfrak{A}$ such that for any $x = (b_{i_1}, \ldots, b_{i_n}) \in B^*$ we have that $x \in \bar{B}$ provided that $x^{\hat{j}} \in \bar{B}$ for all j, $1 \leqslant j \leqslant i_n$. B is then called an excellent base for $\mathfrak{A}$. $\mathfrak{A}$ will be called B-excellent if there exists an excellent base B for $\mathfrak{A}$ such that whenever b_1, b_2, and b_3 are elements of B with (b_1, b_2) and $(b_2, b_3) \in \bar{B}$ then $(b_1, b_3) \in \bar{B}$.

Remark 1. If $\mathfrak{A}$ is B-excellent then it follows from the assumption that B is an excellent base that if $(b_{i_1}, \ldots, b_{i_n})$ is an element of $\bar{B}$ and (b_{i_n}, b_j) is an element of $\bar{B}$, so that in particular $i_1 < i_2 < \cdots < i_n < j$, then $(b_{i_1}, \ldots, b_{i_n}, b_j)$ is an element of $\bar{B}$.

Remark 2. Suppose B is an excellent base for $\mathfrak{A}$. Then often a simple renumbering of B will yield a base B' such that $\mathfrak{A}$ is B'-excellent.

Remark 3. If R is a complete intersection or if the Koszul complex for R has trivial products, then it is clear that the Koszul complex for R is B-excellent for every choice of base B. In Section (6) I will give an example of a Koszul complex wich is not excellent.

4. Some Massey Type Operations. Let $Z(\mathscr{K})$ be the submodule of cycles in $\mathscr{K}$. We have the usual map $\bar{\eta} : Z(\mathscr{K}) \to H(\mathscr{K})$. Now let T be any set of homogeneous cycles in $\mathscr{K}$ such that $\bar{\eta}(T)$ which we will denote by B, forms a base in $H(\mathscr{K})$. For every $k \in \underline{k}$ let r_k be an element of R whose image in $R/\underline{m}$ is k and we assume that $r_0 = 0$ and $r_1 = 1$. This defines in an obvious way a map $\eta : H(\mathscr{K}) \to Z(\mathscr{K})$ such that $\eta\bar{\eta}$ is the identity on T. If z is any cycle we define $\rho_T(z) = \eta\bar{\eta}(z)$ and $\gamma_T(z) = z$ so that $\gamma_T(z)$ is homologous to $\rho_T(z)$. If z_1 and z_2 are any two homogeneous cycles we define

$$y_T(z_1, z_2) = (-1)^{\deg(z_1)+1} \gamma_T(z_1) \otimes \gamma_T(z_2) + (-1)^{\deg(z_1)+2} \rho_T(z_1 \otimes z_2)$$

$y_T(z_1, z_2)$ is clearly a boundary and we define $\gamma_T(z_1, z_2)$ to be some homogeneous element such that $d\gamma_T(z_1, z_2) = y_T(z_1, z_2)$. Now suppose that $z_1, z_2, \ldots, z_n$ are any n homogeneous cycles such that y_T and γ_T have been defined on any subset of $n-1$ cycles with $d\gamma_T = \gamma_T$. Then we define inductively,

$$y_T(z_1, \dots, z_n) = \sum_{i=1}^{n-1} (-1)^{\sum_{j=1}^{i} \deg(z_j)+i} \gamma_T(z_1, \dots, z_i) \otimes \gamma_T(z_{i+1}, \dots, z_n)$$

$$+ \sum_{i=1}^{n-1} (-1)^{\sum_{j=1}^{i} \deg(z_j)+(i+1)}$$

$$\times \gamma_T(z_1, \dots, z_{i-1}, \rho_T(z_i \otimes z_{i+1}), z_{i+2}, \dots, z_n)$$

It is easily checked by induction that $y_T(z_1, \dots, z_n)$ is a cycle, and if it is a boundary we define $\gamma_T(z_1, \dots, z_n)$ to be some homogeneous element such that $d\gamma_T(z_1, \dots, z_n) = y_T(z_1, \dots, z_n)$.

DEFINITION. If for all integers n, γ_T is defined on all n-tuples of homogeneous cycles then we say that $B = \bar{\eta}(T)$ is nice.

EXAMPLES. Examples of Koszul complexes with nice bases occur when (1) R is a complete intersection, (2) $\mathscr{K}$ has trivial Massey operations, (3) $\dim(\underline{m}/\underline{m}^2) = 3$, or more generally $H^i(\mathscr{K})$ is trivial for $i > 3$.

The conditions that we shall hypothesize on the Koszul Complex may now be stated as follows: there exists a base B for which $H(\mathscr{K})$ is B-excellent and such that $\bar{B}$ is nice.

Let $B = \bar{\eta}(T)$ be a nice base for $H(\mathscr{K})$. We will first give a new proof of Theorem 1 which will display the γ's that occur more explicitly. Let $T^i = \{T_1{}^i, \dots, T_{\epsilon_i}^i\}$ be the subset of T of elements of degree i. As in Section 2, we suppose that $u_1^{i+1}, \dots, u_{\epsilon_i}^{i+1}$ is a base for E^i. We define a map $\eta^* : H^i(\mathscr{K}) \to E^{i+1}$ as follows:

$$\eta^*\left(\sum_{j=1}^{\epsilon_i} k_j \bar{\eta}(T_j{}^i)\right) = \sum_{j=1}^{\epsilon_i} r_{k_j} u_j^{i+1}.$$

For all i and j, we define $d^i(u_j{}^i) = T_j^{i-1}$. For any i, j, k, and ℓ we define $\gamma(u_j{}^i, u_\ell{}^k) = \eta^*(\bar{\eta}(du_j{}^i) \otimes \bar{\eta}d(u_\ell{}^k))$. Now suppose that $x = x_1 \otimes \cdots \otimes x_n$ is a U-Basic element of weight $\geqslant 2$. We define, inductively,

$$d(x) = d(x_1 \otimes \cdots \otimes x_{n-1}) \otimes x_n$$

$$+ (-1)^{\deg(x_1)+\cdots+\deg(x_{n-1})} x_1 \otimes \cdots \otimes x_{n-2} \otimes \gamma(x_{n-1}, x_n)$$

$$+ \gamma_T(dx_1, \dots, dx_n). \tag{1}$$

Whereas if x is K-basic we define

$$d(x) = d(x_1) \otimes x_2 \otimes \cdots \otimes x_n + (-1)^{\deg x_1} x_1 \otimes d(x_2 \otimes \cdots \otimes x_n). \tag{2}$$

It is immediately seen that Eq. (1) is equivalent to

$$\begin{aligned} d(x) = \sum_{i=1}^{n} \gamma_T(dx_1, \ldots, dx_i) \otimes x_{i+1} \otimes \cdots \otimes x_n \\ + \sum_{i=1}^{n-1} (-1)^{\sum_{j=1}^{i} \deg(x_j)} x_1 \otimes \cdots \otimes x_{i-1} \otimes \gamma(x_i, x_{i+1}) \otimes x_{i+2} \otimes \cdots \otimes x_n \end{aligned} \tag{3}$$

It is easily checked by induction that $d^2 = 0$. Moreover, since d then clearly satisfies properties (1-5) in the proof of Theorem 1 we obtain from the remark following Lemma 5 that d is exact.

Now if $B = \bar{\eta}(T)$, then we have that $\bar{B} = \bar{\eta}(\bar{T})$ where $\bar{T}$ corresponds, in the obvious fashion, to T as $\bar{B}$ corresponds to B. We assume that $\bar{B} = \bar{\eta}(\bar{T})$ is nice.

LEMMA 7[1]. *Let $x = x_1 \otimes \cdots \otimes x_n$ be a U-basic element of weight $n \geqslant 2$. Then we can find elements $\gamma_{\bar{T}}(x)$ such that the following properties are satisfied:*

(1) *$\gamma_{\bar{T}}(x)$ is a suitable choice for $\gamma_{\bar{T}}(dx_1, \ldots, dx_n)$,* i.e. *$d\gamma_{\bar{T}}(x) = y_{\bar{T}}(dx_1, \ldots, dx_n)$*

(2) *$\gamma_{\bar{T}}(x) \in \underline{m}\mathscr{K}$*

(3) *Suppose $\{x^i\}_{i=1,\ell}$ are distinct U-basic elements of the same weight and the same degree. Then for any set $\{r_i \in R\}_{i=1,\ell}$, if there exists a cylce z such that $\sum_{i=1}^{\ell} r_i \gamma_{\bar{T}}(x_i) + z \in \underline{m}^2$, we have that $\sum_{i=1}^{\ell} r_i \gamma_{\bar{T}}(x_i) \in \underline{m}^2$, or equivalently, $z \in \underline{m}^2$.*

Proof. We note first that if $n = 2$, then since $\rho_{\bar{T}}(dx_1 \otimes dx_2)$ is contained in $\underline{m}^2\mathscr{K}$ by the definition of $\bar{B}$, we have that $y_{\bar{T}}(dx_1, dx_2) \in m^2\mathscr{K}$ so that by the injectivity of $d : \mathscr{K}/\underline{m}\mathscr{K} \to \underline{m}\mathscr{K}/\underline{m}^2\mathscr{K}$ (see Serre [2], p. IV-50), $\gamma_{\bar{T}}(dx_1, dx_2) \in \underline{m}\mathscr{K}$. Thus for $n = 2$ properties (1) and (2) are satisfied. We proceed by induction on n assuming that

A. $\gamma_{\bar{T}}(x)$ has been defined for all x of weight $\geqslant 2$ and $< n$ such that properties (1-3) are satisfied. (For $n = 2$, this condition is of course vacuous.)

B. We can find $\gamma_{\bar{T}}(x)$ for all x of weight n such that properties (1) and (2) are satisfied.

[1] In my original manuscript lemma (7) merely asserted that $\gamma_{\bar{T}} \in m\mathscr{K}$. The proof was short and wrong. The referee pointed out the error and I supplied the new proof.

We wish to find first, for all x of weight n, $\gamma_T(x)$ such that properties (1), (2) and (3) are satisfied.

Let P denote the set of all U-basic elements of weight n and of the same (fixed) degree. P being a finite set there certainly exists some total linear ordering on it which we choose in an arbitrary manner. Thus we can write $P = \{x^1,..., x^t\}$ where for all i, $x^i = x_1{}^i \otimes \cdots \otimes x_n{}^i$ with $x_j{}^i$ some U-basic element of weight 1. Let s be any integer $< t$ and let us assume that we have defined $\gamma_T(x_i)$ for $1 \leqslant i \leqslant s$ such that the following properties are satisfied.

(1′) $d\gamma_T(x^i) = y_T(x^i)$

(2′) $\gamma_T(x^i) \in \underline{m}\mathscr{K}$

(3′) If $\sum_{i=1}^{s} r_i\gamma_T(x^i) + z \in m^2$ for $r_i \in R$ and z a cycle, then $z \in \underline{m}^2$.

We wish to find $\gamma_T(x^{s+1})$ such that the above three properties are satisfied with s replaced by $s + 1$. We observe first of all that, since by our inductive hypothesis (B) there exists a $\gamma_T(x^{s+1}) \in \underline{m}\mathscr{K}$ such that properties (1′) and (2′) are satisfied, then from the injectivity of $d : \mathscr{K}/\underline{m}\mathscr{K} \to \underline{m}\mathscr{K}/\underline{m}^2\mathscr{K}$ we have that for any choice of $\gamma_T(x^{s+1})$ for which property (1′) is satisfied, property (2′) is automatically satisfied. We thus first choose $\gamma_T(x^{s+1})$ in an arbitrary way so that (1′) is satisfied and we assume that we have an equation of the form

$$\sum_{i=1}^{s+1} r_i\gamma_T(x^i) + z \in \underline{m}^2 \quad \text{for} \quad r_i \in R \text{ and } z \text{ a cycle.}$$

If $r_{s+1} \in \underline{m}$, then since $\gamma_T(x^{s+1}) \in \underline{m}$ we obtain that $\sum_{i=1}^{s} r_i\gamma_T(x^i) + z \in \underline{m}^2$ and so by our inductive hypothesis (3′), we obtain $z \in \underline{m}^2$. Suppose then that $r_{s+1} \notin \underline{m}$ so that we can assume thar $r_{s+1} = 1$. Since the choice of $\gamma_T(x^{s+1})$ is arbitrary up to a cycle, we can with a change of notation assume that we have the equation

$$\sum_{i=1}^{s} r_i\gamma_T(x^i) + \gamma_T(x^{s+1}) \in m^2. \tag{*1}$$

$\gamma_T(x^{s+1})$ has now been firmly chosen and will remain fixed and we wish to show that (3′) is now satisfied with s replaced by $s + 1$. Suppose then that we have

$$\sum_{i=1}^{s+1} r_i'\gamma_T(x^i) + z' \in \underline{m}^2 \quad \text{with} \quad r_i' \in R \text{ and } z' \text{ a cycle.} \tag{*2}$$

Arguing as above we can assume that $r'_{s+1} = 1$. Subtracting equation (*2) from equation (*1) we obtain that

$$\sum_{i=1}^{s} (r_i - r_i')\,\gamma_T(x^i) - z' \in m^2.$$

By our inductive hypothesis (3′) we obtain that $z' \in \underline{m}^2$. This completes the inductive proof that $\gamma_{\bar{T}}(x)$ can be chosen for all x of weight n such that properties (1-3) are satisfied, i.e. inductive hypothesis (A) is satisfied with n replaced by $n + 1$.

To now complete the proof of the lemma we need to show that if x is U-Basic of weight $n + 1$ we can find $\gamma_{\bar{T}}(x)$ so that properties (1) and (2) are satisfied. Having defined $\gamma_{\bar{T}}(x)$ for all U-basic elements of weight n, we have in effect defined $\gamma_{\bar{T}}(\bar{z}_1, \dots, \bar{z}_n)$ for all n-tuples $(\bar{z}_1, \dots, \bar{z}_n)$ of elements of $\bar{T}$. If $z_1, \dots, z_n$ are any n-cycles we can define $\gamma_{\bar{T}}(\rho_{\bar{T}}(z_1), \dots, \rho_{\bar{T}}(z_n))$ by multilinearity and we immediately check that

$$d\gamma_{\bar{T}}(\rho_{\bar{T}}(z_1), \dots, \rho_{\bar{T}}(z_n)) = y_{\bar{T}}(\rho_{\bar{T}}(z_1), \dots, \rho_{\bar{T}}(z_n)).$$

Now let $x = x_1 \otimes \cdots \otimes x_{n+1}$ be a U-basic element of weight $n + 1$. Then

$$\begin{aligned} y_{\bar{T}}(dx_1, \dots, dx_{n+1}) &= \sum_{i=1}^{n} (-1)^{\sum_{j=1}^{i} \deg(z_j) + i} \gamma_{\bar{T}}(dx_1, \dots, dx_i) \otimes \gamma_{\bar{T}}(dx_{i+1}, \dots, dx_{n+1}) \\ &\quad + \sum_{i=1}^{n} (-1)^{\sum_{j=1}^{i} \deg(z_j) + (i+1)} \\ &\quad \times \gamma_{\bar{T}}(dx_1, \dots, dx_{i-1}, \rho_{\bar{T}}(dx_i \otimes dx_{i+1}), dx_{i+2}, \dots, dx_n). \end{aligned}$$

Now

$$\sum_{i=1}^{n} (-1)^{\sum_{j=1}^{i} \deg(z_j) + i} \gamma_{\bar{T}}(dx_1, \dots, dx_i) \otimes \gamma_{\bar{T}}(dx_{i+1}, \dots, dx_{n+1})$$

is clearly in $\underline{m}^2$, whereas by the multilinearity of $\gamma_{\bar{T}}$ just described we have that

$$\begin{aligned} &\sum_{i=1}^{n} (-1)^{\sum_{j=1}^{i} \deg(z_j) + i + 1} \gamma_{\bar{T}}(dx_1, \dots, dx_{i-1}, \rho_{\bar{T}}(dx_i \otimes dx_{i+1}), \dots, dx_{m+1}) \\ &\qquad = \sum_{i=1}^{s} r_i \gamma_{\bar{T}}(x^i); \end{aligned}$$

where for $1 \leqslant i \leqslant s$, $r_i \in R$ and x^i is some U-basic element of weight n. Thus we have that $\sum_{i=1}^{s} r_i \gamma_{\bar{T}}(x^i) - y_{\bar{T}}(x) \in \underline{m}^2$. But $y_{\bar{T}}(x)$ is a cycle, so by our inductive hypothesis (A), we obtain $y_{\bar{T}}(x) \in \underline{m}^2$. Thus by the injectivity of $d : \mathscr{K}/mK \to \underline{m}\mathscr{K}/m^2\mathscr{K}$, $\gamma_{\bar{T}}(x) \in \underline{m}$. This proves that inductive hypothesis (B) is satisfied with n replaced by $n + 1$ and thus completes the proof of the lemma.

Conjecture. Let $z_1, \ldots, z_n$ be elements of $\bar{T}$ such that for some i, $1 \leqslant i \leqslant n-1$, $z_i \otimes z_{i+1}$ is not a boundary. Then $\gamma_{\bar{T}}(z_1, \ldots, z_n)$ can be chosen to be 0.

THEOREM 2. *Assume that $\bar{B}$ is nice. Let d be the differentiation on $\mathscr{K} \otimes T(E)$ given by equations* (1) *and* (2) *with respect to the nice base $\bar{B}$. Let $\bar{d}$ and $\bar{\gamma}$ be the maps induced on $\mathscr{K} \otimes \underline{T}(E)/\underline{m}\mathscr{K} \otimes \underline{T}(E)$. Then if $x = x_1 \otimes \cdots \otimes x_n$ is K-basic,*

$$\bar{d}(x) = (-1)^{\deg x_1} x_1 \otimes d(x_2 \otimes \cdots \otimes x_n) \tag{4}$$

whereas if $x = x_1 \otimes \cdots \otimes x_n$ is U-basic,

$$\bar{d}(x) = \sum_{i=1}^{n-1} (-1)^{\sum_{j=1}^{i-1} \deg(x_j)} x_1 \otimes \cdots \otimes x_{i-1} \otimes \bar{\gamma}(x_i, x_{i+1}) \otimes \cdots \otimes x_n \tag{5}$$

Proof. This is an immediate consequence of Eq. (2) and (3) and the previous lemma.

Remark. We will now drop the assumption that $\bar{B}$ is nice and assume only that Eq. (4) and (5) are satisfied. We will assume furthermore that $\mathscr{K}$ is B-excellent.

Notation. The elements of $\bar{T} \cap T$ will be called the primary elements of $\bar{T}$ Then every element of $\bar{T}$ is the tensor product of primary elements. Now the ordering of $\bar{T}$ induces an ordering of $\bar{T} \cap T$ which with this induced ordering we will write as $T_1, \ldots, T_n$. Every element of $\bar{T} = T_{i_1} \otimes \cdots \otimes T_{i_r}$ with $i_j < i_k$ for $j < k$. We will denote such an element by $T_{(i_1, \ldots, i_r)}$. Then if $T_{(j_1, \ldots, j_s)}$ is another element we have that $T_{(i_1, \ldots, i_r)} < T_{(j_1, \ldots, j_s)} \Leftrightarrow (i_1, \ldots, i_r) < (j_1, \ldots, j_s)$ with the lexicographic ordering. For every element $T_{(i_1, \ldots, i_r)}$ in $\bar{T}$ we define $u_{(i_1, \ldots, i_r)}$ to be the basis element of E such that $d(u_{(i_1, \ldots, i_r)}) = T_{(i_1, \ldots, i_r)}$. This induces an ordering on a base of E which we shall call the distinguished base. $u_1, \ldots, u_n$ will be called the primary elements of this base. We wish to define the $\bar{F}^i$ and $\bar{H}^i$ as in Section (2) so that $\sum_{i=0}^{\infty} \bar{F}^i \oplus \bar{H}^i$ is a finitely generated ideal of $\mathscr{K} \otimes \underline{T}(E)/\underline{m} \ \mathscr{K} \otimes \underline{T}(E)$.

LEMMA 8. *It suffices to find a homogeneous subspace F of $\underline{T}(E)/\underline{m}\underline{T}(E)$ such that* (1) *$\bar{d}$ is injective on F,* (2) *$\bar{d}(\underline{T}(E)/\underline{m}T(E)) \subseteq \bar{d}(F)$, and* (3) *$F \oplus \bar{d}(F)$ is a finitely generated ideal of $\underline{T}(E)/\underline{m}\underline{T}(E)$.*

Proof. This is an immediate consequence of Eq. (4).

Thus it remains to define F. Changing notation we let E denote $E/\underline{m}E$, $\underline{T}(E)$ denote $\underline{T}(E)/\underline{m}\underline{T}(E)$, and we let $u_{(i_1, \ldots, i_r)}$ denote the image of $u_{(i_1, \ldots, i_r)}$ in $E/\underline{m}E$. Suppose now that $x = x_1 \otimes x_2$ is a basic element of $T(E)$ of weight 2.

x will be called a bad element if $x_1 = u_j$ is a primary element and x_2 is some $u_{(i_1,\ldots,i_r)}$ such that $u_{(i_1,\ldots,i_r,j)}$ is also a distinguished base element of E. In particular j must be $>i_r$. If $x = x_1 \otimes \cdots \otimes x_n$ is a basic element of weight n (possibly equal to 1), then x will be called good provided (1) each x_i is primary, and (2) there does not exist a j, $1 \leqslant j \leqslant n-1$, such that $x_j \otimes x_{j+1}$ is bad. x will be called bad, if $x_1 \otimes \cdots \otimes x_{n-1}$ is good and $x_{n-1} \otimes x_n$ is a bad element (of weight 2). x will be called semi-good if (1) $x_1 \otimes \cdots \otimes x_{n-1}$ is good (2) x_n is not a primary element, and (3) x is not bad, i.e. $x_{n-1} \otimes x_n$ is not bad; or if $n = 1$ and x_n is not a primary element. We now define inductively the following subspace of $T(E)$:

$$E_i = E \otimes E \otimes \cdots \otimes E \qquad i\text{-times}$$

F_i = submodule of E_i generated by the bad elements

G_i = submodule of E_i generated by the good elements

H_i = submodule of E_i generated by the semi-good elements:

$$F(i) = (F(i-1) \otimes E) \oplus (F_{i-1} \otimes E), \qquad \text{where} \quad F(1) = 0,$$

$$H(i) = (H(i-1 \otimes E) \oplus (H_{i-1} \otimes E), \qquad \text{where} \quad H(1) = 0.$$

LEMMA 9. (a) $G_i \otimes E = F_{i+1} \oplus G_{i+1} \oplus H_{i+1}$,

(b) $E_i = F_i \oplus G_i \oplus H_i \oplus F(i) \oplus H(i)$.

Proof. Obvious.

LEMMA 10. *For all* $i > 2$, $\dim(F(i)) = \dim(H(i-1))$.

Proof. d is clearly an isomorphism from F_2 onto H_1. Then since $H(\mathscr{K})$ is $\bar{B}$-excellent we obtain from Remark 1 following the definition of this property that this $1-1$ correspondence extends to a $1-1$ correspondence between F_i and H_{i-1} which clearly extends to a $1-1$ correspondence between $F(i)$ and $H(i-1)$.

LEMMA 11. $\bar{d}(E_i) \subseteq F_{i-1} \oplus F(i-1) \oplus H_{i-1} \oplus H(i-1)$.

Proof. This is an immediate consequence of Lemma (9) and Eq. (5).

LEMMA 12. $\bar{d}(F_i \oplus F(i)) + (F_{i-1} \oplus F(i-1)) \supseteq H_{i-1} \oplus H(i-1)$.

Proof. Below.

COROLLARY 1. *$\bar{d}$ is injective on $F_i \oplus F(i)$ and*

$$\bar{d}(F_i \oplus F(i)) \cap F_{i-1} \oplus F(i-1) = 0.$$

Proof. This is an immediate consequence of Lemma 10 and the fact that $(F_{i-1} \oplus F(i-1)) \cap (H_{i-1} \oplus H(i-1)) = 0$.

COROLLARY 2. $\bar{d}(E(i)) = \bar{d}(F_i \oplus F(i))$.

Proof. This follows from Lemma 11, Corollary 1 and the fact that $\bar{d}^2 = 0$.

Proof of Lemma 12.

A. It suffices by induction to show that $L.H.S.$ contains H_{i-1}. For suppose x is a basic element of $F(i)$, say $x = x_1 \otimes \cdots \otimes x_i$ with each x_j a distinguished base element of E. $x_1 \otimes \cdots \otimes x_{i-1}$ is then either an element of $F(i-1)$ or of F_{i-1}. By Eq. (5) $\bar{d}(x) = \bar{d}(x_1 \otimes \cdots \otimes x_{i-1}) \otimes x_i + x_1 \otimes \cdots \otimes x_{i-2} \otimes \bar{\gamma}(x_{i-1}, x_i)$. Now if $x_1 \otimes \cdots \otimes x_{i-1} \in F(i-1)$, then $x_1 \otimes \cdots \otimes x_{i-2} \otimes \bar{\gamma}(x_{i-1}, x_i)$ is clearly contained in $F(i-1)$. If on the other hand $x_1 \otimes \cdots \otimes x_{i-1}$ is an element of F_{i-1}, then since $H(\mathscr{K})$ is B-excellent $x_1 \otimes \cdots \otimes x_{i-2} \otimes \bar{\gamma}(x_{i-1}, x_i)$ is contained either in F_{i-1} or H_{i-1}. Thus in all cases we obtain that $\bar{d}(x_1 \otimes \cdots \otimes x_{i-1}) \otimes x_i$ is contained in $L.H.S.$, which proves by induction that $H(i-1)$ is contained in $L.H.S.$

B. In order to prove that the hypothesis of (A) is satisfied we need to modify the ordering on the distinguished base of E. Suppose $u_{(i_1,\ldots,i_r)}$ and $u_{(i_1,\ldots,i_r,j_1,\ldots,j_s)}$ are two distinguished elements then we define $u_{(i_1,\ldots,i_r)}$ to be $> u_{(i_1,\ldots,i_r,j_1,\ldots,j_s)}$ which is the reverse of what we had. Otherwise, the ordering stays the same. This modified ordering is clearly still linear. The modification is introduced so that the following lemma holds.

LEMMA 13. *Suppose $x_1 \otimes x_2$ is a basic element of weight 2, which is not bad. Then $\bar{\gamma}(x_1 \otimes x_2)$ is a linear sum of elements of the form $\bar{\gamma}(y \otimes z)$ where $y \otimes z$ is a basic bad element of weight 2 and either $z < x_2$ or $z = x_2$ and $y < x_1$.*

Proof. This is an immediate consequence of the definitions.

With respect to this new ordering on the distinguished elements of E, we order the basic elements of $F_i \oplus F(i)$ with the induced lexicographic ordering of E_i starting from the right. i.e., if $x = x_1 \otimes \cdots \otimes x_i$ and $y = y_1 \otimes \cdots \otimes y_i$ are two basic elements then $x < y \Leftrightarrow$ there exists a j such that $x_j < y_j$ and for $k > j$, $x_k = y_k$. We then prove the following:

LEMMA 14. *Suppose $x = x_1 \otimes \cdots \otimes x_i$ is a basic element in F_i or $F(i)$. Then for all j, $1 \leqslant j \leqslant i$, $x_1 \otimes \cdots \otimes x_{j-1} \otimes \bar{\gamma}(x_j, x_{j+1}) \otimes \cdots \otimes x_i$ is in $F_{i-1} \oplus F(i-1) + \bar{d}(W)$ where W consists of the subspace spanned by all basic elements of $F_i \oplus F(i)$ which are $\leqslant x$.*

Proof of Lemma 14. We proceed by induction on the order of x. Suppose first that $x \in F_i$. We will use the inductive hypothesis to show that for $j < i - 1$, the conclusion of the lemma holds. Then since $x \in W$ the result will hold for $j = i - 1$ by the definition of $\bar{d}(x)$. So suppose $j < i - 1$. Then $x_j \otimes x_{j+1}$ is not bad. Thus by Lemma 13, $\bar{\gamma}(x_j, x_{j+1})$ is a linear sum of certain $\bar{\gamma}(y_k, z_k)$ where $y_k \otimes z_k$ is a bad element of weight 2 and $y_k \otimes z_k < x_j \otimes x_{j+1}$. Thus applying the inductive hypothesis to every $x_1 \otimes x_2 \otimes \cdots \otimes x_{j-1} \otimes y_k \otimes z_k \otimes x_{j+1} \otimes \cdots \otimes x_i$, we have that $x_1 \otimes \cdots \otimes x_{j-1} \otimes \bar{\gamma}(y_k, z_k) \otimes x_{j+2} \otimes \cdots \otimes x_i$ is contained in *L.H.S.* and and so $x_1 \otimes \cdots \otimes x_{j-1} \otimes \bar{\gamma}(x_j, x_{j+1}) \otimes \cdots \otimes x_i$ is contained in *L.H.S.* This completes the proof for $x \in F_i$. Suppose now for some $k < i$, $x_1 \otimes \cdots \otimes x_k \in F_k$ so that $x_{k-1} \otimes x_k$ is a bad element of weight 2. Then for $j > k$, we clearly have that

$$x_1 \otimes \cdots \otimes x_{j-1} \otimes \bar{\gamma}(x_j, x_{j+1}) \otimes \cdots \otimes x_i \in F(i-1).$$

If $j < k - 1$ the result follows as in the case $x \in F_i$. Consider now the case $j = k$. There are two possibilities.

(1) $x_k \otimes x_{k+1}$ is not bad. Then we obtain the results as above by using Lemma 13.

(2) $x_k \otimes x_{k+1}$ is bad. In that case since $H(K)$ is $\bar{B}$-excellent, we have that $x_{k-1} \otimes \bar{\gamma}(x_k, x_{k+1})$ is bad so that $x_1 \otimes \cdots \otimes x_{k-1} \otimes \bar{\gamma}(x_k, x_{k+1}) \otimes \cdots \otimes x_i \in F_{i-1} \oplus F(i-1) + \bar{d}(W)$, which completes the proof of Lemma 14. To now show that the hypotheses of (A) are satisfied so as to complete the proof of Lemma 12, simply apply Lemma 14 to the case $x \in F_i$ and $j = i - 1$.

THEOREM 3. *Assuming that equations* (4) *and* (5) *hold and that* $H(\mathscr{K})$ *is* $\bar{B}$*-excellent, then* $P_R(\underline{k})$ *is a rational function.*

Proof. We apply Lemma 8 to $F = \bigcup_{i=1}^{\infty} (F(i) \oplus F_i)$. Then by Corollary 1 to Lemma 12, $\bar{d}$ is injective on F, and by Corollary 2 to Lemma 12, $\bar{d}(\underline{T}(E)) \subseteq \bar{d}(F)$. Furthermore, if H denotes $\bigcup_{i=1}^{\infty} (H_i \oplus H(i))$ we obtain from Lemma 12 that $F \oplus \bar{d}(F) = F \oplus H$. Now $F \oplus H$ is immediately seen to be the (two-sided) ideal generated by the bad elements of weight 2 and the distinguished base elements of E which are not primary.

6. (Some Examples).

EXAMPLE 1. If $R = S/\underline{b}$ is a complete intersection with $\underline{b}$ minimally generated by d elements then $R/\underline{m} \otimes_R \mathscr{K} \otimes_R \underline{T}(E)/\underline{a}$, with $\underline{a}$ as in Theorem 3, has a structure of a "polynomial ring" in d-variables $\{x_1, \ldots, x_d\}$ with coefficients in $\mathscr{K}/\underline{m}\mathscr{K}$ and such that multiplication of the variables is given by the relations $x_i x_j = 0$ for $i > j$. Thus a typical monomial in the ring is of the form $c \prod_{i=1}^{d} x_i^{\alpha(i)}$ where $c \in \mathscr{K}/\underline{m}\mathscr{K}$ and $\alpha(i)$ is some integer ≥ 0 for all i.

EXAMPLE 2. Let $R = \underline{k}[[x_1, \ldots, x_n]]$ for $n \geqslant 3$. Let $\underline{b} = (x_1^2, \ldots, x_{n-1}^2)$ and let $\underline{a} = (x_1^2, x_2^2, \ldots, x_{n-1}^2, x_1x_2 \cdots x_n)$. Let $S = R/\underline{a}$. We will show that S satisfies the hypotheses of Theorem 3. Let $\mathscr{K}$ denote the Koszul complex for R and $\bar{\mathscr{K}} = R/\underline{a} \otimes_R \mathscr{K}$ that for S. Let $e_1, \ldots, e_n \in K^1$ be such that $d^1(e_i) = x_i$. Then a base for $H^1(\bar{\mathscr{K}})$ is given by the images in $H^1(\bar{\mathscr{K}})$ of $A_1^1 = x_1e_1$, $A_2^1 = x_2e_2, \ldots, A_{n-1}^1 = x_{n-1}e_{n-1}$ and $B_1^1 = x_1 \cdots x_{n-1}e_n$. Now for $1 \leqslant i \leqslant n$ define the set

$$A^i = \left\{A_1^i, \ldots, A^i_{\binom{n-1}{i}}\right\}$$

to be the set of all elements of the form $x_{j_1} \cdots x_{j_i}e_{j_1} \otimes \cdots \otimes e_{j_i}$ with $j_1 < j_2 < \cdots < j_i$ and let

$$B^i = \left\{B_1^i, \ldots, B^i_{\binom{n-1}{i-1}}\right\}$$

be the set of all elements of the form $x_1 \cdots x_{n-1}e_{j_1} \otimes \cdots \otimes e_{j_{i-1}} \otimes e_n$ with $j_1 < j_2 < \cdots < j_{i-1}$. It is clear that such elements are cycles and we will show that the image of $A^i \cup B^i$ in $H^i(\bar{\mathscr{K}})$ forms a base for $H^i(\bar{\mathscr{K}})$. Notice that the image of A^i in $H^i(R/\underline{b} \otimes_R \mathscr{K})$ is a base. We show first that the image of $A^i \cup B^i$ in $H^i(\bar{\mathscr{K}})$ is a linearly independent set. So suppose we have an equation of the form

$$\sum_j \alpha_j A_j^i + \sum_j \beta_j B_j^i + dv + w = 0 \tag{1}$$

with $\alpha_j, \beta_j \in \underline{k}$, $v \in K^{i+1}$ and $x \in \underline{a}K^i$. We have to show that $\alpha_j \in \underline{m}$ and $\beta_j \in \underline{m}$ for all j. Applying d^i to the above equation we obtain

$$\sum_j \alpha_j\, d(A_j^i) + \sum_j \beta_j\, d(B_j^i) + dw = 0.$$

Now $dw \in \underline{m}\underline{a}K^{i-1}$ and it is clear that $d(A_j^i) \in \underline{m}\underline{a}K^{i-1}$. Furthermore it is easily seen that the images of $d(B_j^i) \in \underline{a}K^{i-1}/\underline{m}\underline{a}K^{i-1}$ is a linearly independent set. Thus reading the equation $\sum_j \alpha_j d(A_j^i) + \sum_j \beta_j d(B_j^i) + dw = 0$ in $\underline{a}K^{i-1}/\underline{m}\underline{a}K^{i-1}$ we obtain $\beta_j \in \underline{m}$ for all j. Thus upon changing v if necessary Eq. (1) becomes

$$\sum_j \alpha_j A_j^i + d(v) + w = 0. \tag{2}$$

If not all the α_j were in $\underline{m}$ then Eq. (2) states that there exists some $A \in K^i$ whose image in $H^i(R/\underline{b} \otimes_R \mathscr{K})$ is not equal to 0 but whose image in $H^i(R/\underline{a} \otimes_R \mathscr{K})$ is equal to 0, and we will show that this is impossible. Suppose then that A is such an element. Then we have that $A = d(z) +$

$(x_1 \cdots x_{n-1}x_n)\, w + v$ for some $z \in K^{i+1}$, $w \in K^i$ and $v \in \underline{b}K^i$. Now $d(A)$ is by assumption in $\underline{b}K^{i-1}$ so since $dv \in \underline{b}K^{i-1}$ we obtain $x_n(x_1 \cdots x_{n-1})\, dw \in \underline{b}K^{i-1}$. Now x_n is $R/\underline{b}$-regular so that $x_1 \cdots x_{n-1}\, dw \in \underline{b}K^{i-1}$. Letting now $z' = z - x_1 \cdots x_{n-1}e_n \otimes w$ we obtain that $A = d(z') + v'$ with $v' \in \underline{b}K^{i-1}$ contradicting the assumption that A is not a boundary in $R/\underline{b} \otimes_R \mathscr{K}$. This completes the proof that the images of A^i and B^i in $H^i(\bar{\mathscr{K}})$ form a linearly independent set and it remains to show that they generate $H^i(\bar{\mathscr{K}})$. So suppose $T \in K^i$ is such that $dT \in \underline{a}K^{i-1}$. Then $d(T) = x_1 \cdots x_n w + v$ with $v \in \underline{b}K^{i-1}$. Thus $x_n(x_1 \cdots x_{n-1}\, dw) + dv = 0$. Since x_n is $R/\underline{b}$-regular we have that $x_1 \cdots x_{n-1}\, dw \in \underline{b}K^{i-1}$. Thus $d(T - x_1 \cdots x_{n-1}e_n \otimes w) \in \underline{b}K^{i-1}$ and so $T - x_1 \cdots x_{n-1}e_n \otimes w$ is homologous in $R/\underline{b} \otimes_R \mathscr{K}$ and so certainly in $\bar{\mathscr{K}}$ to some linear combination of elements in B^i. On the other hand $x_1 \cdots x_{n-1}e_n \otimes w$ is clearly equal in $R/\underline{a} \otimes_R \mathscr{K}$ to some linear combination of elements in B^i. This gives that T is homologous in $\bar{\mathscr{K}}$ to some linear combination of elements in $A^i \cup B^i$. Now it is clear that the image in $H(\bar{\mathscr{K}})$ of $\bigcup_{i=1}^{n} (A^i \cup B^i)$ is a base B whose derived base $\bar{B}$ is just B. $H(\bar{\mathscr{K}})$ is then clearly B-excellent. Furthermore B is clearly a nice base and so we we can apply theorem (3) to obtain after an easy calculation that,

$$P_R(k) = \frac{(1+z)^n}{1 - \epsilon_1 z^2 - \epsilon_2 z^3 - \cdots - \epsilon_n z^{n+1} + \sum_{j=2}^{n-1} (-1)^j \binom{n-1}{j} z^{2j}}$$

where ϵ_1 denotes the dimension of $H^1(\bar{K})$ and for $n \geqslant 2$, ϵ_i denotes $\dim(H^i(\bar{K})) - \dim(H^i(R/b \otimes_R K))$. In particular if $n = 3$ we obtain

$$P_R(k) = \frac{(1+z)^3}{1 - 3z^2 - 3z^3 - z^4 + z^4} = \frac{(1+z)^3}{1 - 3z^2 - 3z^3}$$

EXAMPLE 3. I want to finally give an example of a Koszul complex which is not excellent. Let $R = k[[x, y, z]]/\underline{a}$ where $\underline{a}$ is the ideal generated by x^2, y^2, z^2, and xyz. Then if e_1, e_2, and e_3 are such that $d(e_1) = x$, $d(e_2) = y$ and $d(e_3) = z$, the cycles $T_1 = xe_1$, $T_2 = ye_2$ and $T_3 = ze_3$ are such that $T_1 \otimes T_2$, $T_1 \otimes T_3$, and $T_2 \otimes T_3$ are linearly independent in the homology complex but $T_1 \otimes T_2 \otimes T_3 = 0$.

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References

1. E. S. Golod, Homologies of some local rings, *Dokl. Akad. Nauk. SSSR* **144** (1962), 479–482.
2. J.-P. Serre, "Algèbre Locale Multiplicités," (rédigé par Pierre Gabriel), Springer-Verlag.
3. J. Shamash, The Poincaré Series of a local ring, *J. Algebra* **12** (1969), 453–470.