Ann Univ Ferrara (2008) 54:37–59
DOI 10.1007/s11565-008-0037-1

Weierstrass points and their impact in the study of algebraic curves: a historical account from the "Lückensatz" to the 1970s

Andrea Del Centina

Received: 26 October 2006 / Accepted: 4 April 2007 / Published online: 11 March 2008

Abstract In this note we give a historical account of the origin and the development of the concept of Weierstrass point. We also explain how Weierstrass points have contributed to the study of compact Riemann surfaces and algebraic curves in the century from Weierstrass' statement of the gap theorem to the 1970s. In particular, we focus on the seminal work of Hürwitz that raised questions which are at the center of contemporary research on this topic.

Keywords Riemann surfaces · Algebraic curves · Weierstrass point · Origin and development of the concept · Impact in the study of algebraic curves

Mathematics Subject Classification (2000) 01A · 14H

1 Introduction

The study of Weierstrass points on a compact Riemann surface X, i.e. of those points $P \in X$ such that $i(gP) > 0$ (g genus of X), is a fascinating topic in the theory of compact Riemann surfaces and algebraic projective curves, rich in geometrical applications and still a flourishing area of research.

The aim of this paper is to give a historical account of the origins and of the development of the concept of Weierstrass point, as well as an explanation of how the study of Weierstrass points contributed to the understanding of some aspects of the geometry of Riemann surfaces and algebraic curves during the century from K. Weierstrass' statement of the *Lückensatz* (or Gap Theorem) to the 1970s. We will also sketch proofs of some of the main results presented.

A. Del Centina (✉)
Dipartimento di Matematica, Università di Ferrara, via Machiavelli 45, 44100 Ferrara, Italy
e-mail: cen@dns.unife.it

The history of Weierstrass points during this century may be divided into three main periods: the first from the early 1860s to the publication in 1902 of the treatise of Hensel and Landsberg [32]; the second from this event to the 1950s, and the third from the publication in 1959 of Rauch's paper [65] onward.

The first period is that of the development of the theory, of Noether's generalization of the gap theorem, of the first applications to the study of algebraic curves, and, in particular, of the fundamental results by A. Hürwitz, such as the wronskian method and the bound on the order of the automorphisms group of an algebraic curve (see Sects. 2–4).

In the middle period it seems that geometers did not pay much attention to Weirstrass points and related questions and, in fact, during this time only a few results or papers can be listed; we recall, e.g., the functional equation that Severi gave in [76], and the extension of the concept of Weierstrass points in positive characteristic due to Schmidt [71] (see Sect. 5).

The third period is that of revival. After the second world war the lectures on Riemann surfaces delivered by L. Ahlfors, at Harvard, and L. Bers, in New York, strongly renewed interest in this subject and then on Weierstrass points: old questions were revisited and new problems came onto the scene (see Sects. 6, 7).

This work does not pretend to be exhaustive. For instance we have only mentioned, if not left completely out, many questions and results concerning positive characteristic, the modular group and higher dimension. We apologize in advance to anyone whose favorite topic has been skipped.

In the following, unless otherwise stated, X will denote a compact Riemann surface of genus g, or, equivalently, a smooth algebraic projective curve over the complex field $\mathbb{C}$. For all others symbols not defined in the following, we refer to [30].

2 The origins of the theory of Weierstrass points

In this section we will explain how Weierstrass points came into being and their first application, due to M. Noether, in the study of algebraic curves.

The history of Weierstrass points is not marked by a precise starting date because it is not clear when Weierstrass stated and proved his *Lückensatz* (or "gap" theorem) but one can argue that probably it was in the early 1860s. This theorem can be stated as follows: *For each $P \in X$, there are exactly g integers $\alpha_i(P)$ with*

$$1 = \alpha_1(P) < \cdots < \alpha_g(P) \leq 2g - 1$$

such that there exist no meromorphic function on X having a pole at P, of multiplicity $\alpha_i(P)$, as its only singularity (see [81, 3, pp. 432–433]).

The numbers $\alpha_1(P), \ldots, \alpha_g(P)$ are called *gaps* of X at P; the number g, which is independent of the choice of P, was taken by Weierstrass as the definition of the genus of X.

The gap theorem appeared for the first time in the dissertation of Weierstrass' student Schottky (Berlin, 1875, and after published as [72]). This theorem was generalized by

Noether in 1882 to the case of a sequence of points $P_1, P_2, \ldots$ on X, instead of the constant sequence $P, P, \ldots$ [57] (see [25, p. 79] for a modern version of this theorem).

Let $G(P)$ denote the set of gaps at P, the set $N(P) := \mathbb{N} \backslash G(P)$ is called the set of *non-gaps*: for each $n \in N(P)$ there exists on X a meromorphic function having a pole of order n at P as its only singularity. Weierstrass studied the case when X is hyperelliptic, i.e. when there exists a (unique) degree two morphism f of X onto $\mathbb{P}^1$. In this case if $P \in X$ is one of the $2g+2$ ramification points of f (see formula 2.2), then either P is a double pole of f, or a double pole for $(f - f(p))^{-1}$(by considering f as a meromorphic function on X). Hence for each such point P the first non-gap is 2, so all even numbers greater than 2 are also non-gaps and we have $(\alpha_1(P), \ldots, \alpha_g(P)) = (1, 3, \ldots, 2g-1)$. In particular each of the $2g+2$ ramification points of f is a Weierstrass point.

Weierstrass [81, 3, pp 297–307] stated another important result. Let m be the first (i.e. the smallest) non-gap at P, n the next relatively prime non-gap, and denote by x and y two meromorphic functions on X with a pole at P of order m and n, respectively, as their only singularity. Then one has: *x and y satisfy the irreducible equation*

$$y^m + A_1(x)y^{m-1} + \cdots + A_{m-1}(x)y + A_m(x) = 0,$$

where the $A_i(x)$'s are polynomials in x with $\deg A_i(x) \leq n_i/m$ *for $i < m$ and* $\deg A_m(x) = n$, *and X is defined by this equation.*

Weierstrass also mentioned this equation in the letter he wrote to H.A. Schwarz on the 3rd of October 1875 (see [81, 2, p. 239]). This equation soon became known as the "Weierstrass Normalformen" [Weierstrass normal form] of X (for a proof see also [6,32] and the recent [38]).

By using the Riemann–Roch theorem it is easy to see that

$$(\alpha_1(P), \ldots, \alpha_g(P)) \neq (1, \ldots, g)$$

if and only if

$$i(gP) > 0, \tag{2.1}$$

where $i()$, as is customary, denotes the index of speciality of the divisor in the brackets, i.e. P is a Weierstrass point if and only if $i(gP) > 0$.

Denote by W the set of Weierstrass points of X. Again the Riemann–Roch theorem implies that if $g \leq 1$, then $W = \emptyset$. From now on we will suppose $g \geq 2$.

Let C be a plane curve of degree d (whose singularities are at most nodes) of genus g given by $f = 0$. Brill and Noether on page 302 of their joint paper [9] wrote: "wir schreiben der adjungirten Curve $(d-3)$ Ordnung, die gegebene curve f in eine Punkte g-punktig zu treffen" [we impose to the adjoint curve of degree $d-3$ that it intersect the plane curve f in a g-fold point] and in a footnote they added: "Dieser Weg ist vor längerer Zeit schon von Herrn Weierstrass eingeschlagen worden" [this way has been already traced since a long time ago by M. Weierstrass]. This footnote, and what they also wrote in the introduction of the afore mentioned paper, suggests that they were aware of what Weierstrass was developing in his lectures in Berlin. It is well known that Weierstrass was quite popular, and many people—not only from Germany—attended his lectures. By applying the (very general) de Jonquières formula concerning multiple contacts of plane curves [16] (see below), Brill and Noether computed the number of adjoint curves as above, finding $g^3 - g$. Although they do not say it explicitly, this fact implies that W is nonempty, finite and has cardinality $\leq g^3 - g$. They also remarked that equality holds when, for all points P in W, the gap sequence $(\alpha_1(P), \ldots, \alpha_g(P))$ is $(1, \ldots, g-1, g+1)$.

The set W is then a set of distinguished points on the curve and in fact Noether called it "Werthsysteme" [distinguished system (of points)]. Only many years later the individual points of W were named "points de Weierstrass" [Weierstrass points] by Haure in [31].

The formula of de Jonquière gives the total number N of divisors D of a linear series g_n^r on the curve C of genus g, consisting of one point of multiplicity m_1, one of multiplicity $m_2, \ldots,$ one of multiplicity m_t, where $\sum_{i=1}^t m_i = n$ and $\sum_{i=1}^t (m_i - 1) = r$, and where α_1 points have one multiplicity, α_2 another, etc. This formula can be put in the form:

$$N = \frac{\prod_i (g-i+1)m_i}{\alpha_1!\alpha_2!\cdots} \times \left[t! + (t-1)!g\sum_i (k_i - 1) \right.$$
$$\left. + (t-2)!g(g-1)\sum_{i,j}(k_i-1)(k_j-1) + \cdots + \prod_i (g-i+1)(k_i-1) \right]$$

(see [15, pp 284–288]). Several proofs of this formula are known, but the first rigorous proof seems to be due to MacDonald [47] (see also [78]). For the divisors D dominating $(r+1)P$ (i.e. for D's containing P at least $r+1$ times) we have $t = n - r$ and

$$m_1 = r+1, \quad m_2 = \cdots = m_{n-r-1} = 1, \quad \alpha_1 = 1, \quad \alpha_2 = n - r - 1,$$

so the formula of de Jonquières gives [74, p. 87]:

$$N = (r+1)(n + rg - r), \tag{2.2}$$

and for the canonical series g_{g-2}^{g-1}, we get $N = g^3 - g$, i.e. the maximal number of distinct Weierstrass points on a curve.

We note that Roch already in 1866 [69], by using the formula for the numbers of zeros of a suitable theta-function, came to the same result of Brill and Noether. Roch's approach, which we sketch now, was inspired by Sect. 12 of Riemann's paper on Abelian functions [67]. Suppose that the curve C is defined by the equation $f(x, y) = 0$ (f of degree n with respect to y) and consider g independent Abelian integrals of the first kind

$$u_i = \int_{P_0}^{P} \frac{\varphi_{n-3}^{(i)}}{\partial f/\partial y} dx, \quad i = 1, \dots, g,$$

(here φ_{n-3} denotes an adjoint polynomial of f). Let (I, Ω) be the associated period matrix with respect to a symplectic basis $a_1, \dots, a_g, b_1, \dots, b_g$ of the first homology group of the desingularization X of C, and construct a Riemann theta-function of g arguments $\theta(u_1, \dots, u_g)$. Let $P_1, \dots, P_m$ be variable points and Q a fixed point on C; denote by $u_i^{(k)}$ and c_i, respectively, the values of the above integrals u_i for $P = P_k$, $k = 1, \dots, m$, and $P = Q$. The theta-function of g arguments

$$\theta\left(\sum_{k=1}^{m} u_1^{(k)} - c_1, \dots, \sum_{k=1}^{m} u_g^{(k)} - c_g \right)$$

vanishes only if either the points $P_1, \dots, P_m$ belong to the same adjoint curve (which implies $i(P_1 + \cdots + P_m) > 0$), or one of them coincides with Q. Then Roch defined the theta-function $\vartheta := \theta(mu_1 - c_1, \dots, mu_g - c_g)$, which results from the previous one when $P_1 = \cdots = P_m = P$, and he computed (following Riemann) the number of zeros of ϑ via the integral

$$\int_{\delta X} d \log \vartheta,$$

where $\delta X = \prod a_i b_i a_i^{-1} b_i^{-1}$, getting $m^2 g$ zeros [69, p. 179]. Since ϑ has a zero of multiplicity m in Q [69, p. 184], it follows that ϑ has other $m^2 g - m$ zeros: in particular for $m = g$ this means that there are at most $g^3 - g$ Weierstrass points on C, since if P is such a zero of ϑ we have $i(gP) > 0$. For the sake of exactness we have to say that Roch did not mention "Weierstrass points" explicitly (see also Haure's paper at p.126).

The formula (2.2) can also be deduced from one that Veronese gave in [80, p. 201] for the number of $(n+1)$-osculating hyperplanes to a curve C of genus p and degree m in $\mathbb{P}^n$:

$$\begin{aligned} w^{(n-2)} = {} & (n+1)(m-n) - nR \\ & - [(n-1)w_1 + \cdots + w_1^{(n-2)}] + n(n+1)p, \end{aligned}$$

where $w_1, \dots, w_1^{(n-2)}$ and R are numbers associated, in a suitable manner, to the singularities of C. In the case that C is a canonical curve, this formula immediately gives (2.2) (after a change of notation), since $w_1, \dots, w_1^{(n-2)}$ and R are all zero, inasmuch as C is smooth. In particular Weierstrass points are hyperosculation points of the canonical curve (non-hyperelliptic case). We notice that Veronese's idea of studying $(r+1)$-osculating hyperplanes to a non-degenerate curve C of degree n in $\mathbb{P}^r$ will lead to the modern concept of "generalized Weierstrass points" (see Sect. 7).

Noether used the existence of Weierstrass points in the study of the geometry of a curve. In [55] he gave an algebraic proof (based on elimination theory) of the theorem of Schwarz [73], according to which the group of automorphisms $\mathrm{Aut}(X)$ of a curve X of genus $g > 1$ is discrete. One year later in the addendum [56], Noether showed that $\mathrm{Aut}(X)$ is actually finite. This theorem is commonly awarded to Schwarz but actually he only proved that $\mathrm{Aut}(X)$ cannot be continuous. It seems that the first proof of the finiteness of $\mathrm{Aut}(X)$ is due to F. Klein (see [62, p. 16]).

The key point in Noether's proof is that the "Werthsysteme" W on a plane curve must be transformed, under a birational map of the plane into itself, into an analogous set W' of the transformed curve. This idea was subsequently used also by Hürwitz to get a very simple proof of Schwarz's theorem (see the next section).

3 Hürwitz's seminal paper

The paper *Über algebraische Gebilde mit eindeutigen Trasformationen in sich* by Hürwitz [34] is a milestone in the theory of Weierstrass points and of their applications into the study of algebraic curves. In this paper, in fact, Hürwitz introduced the Wronskian method (which, in modern language, allows us to look at W as the zero-locus of a section of a certain bundle, see Sect. 7); gave a precise definition of "multiplicity" or, as we say today, of "weight" of a Weierstrass point; defined the concept of "higher-order" Weierstrass points. Moreover many questions on Weierstrass points were raised which were rediscovered many years later, and became of great interest (see [66] and the recent [21]). This section is mainly devoted to describing Hürwitz's paper following, as much as possible, his arguments and notation.

Let X be of genus $g > 1$, and $u_1(u), u_2(u), \ldots, u_g(u)$ be linearly independent integrals of the first kind on X, written in local coordinate u at P. The determinant

$$\Delta_u = \begin{vmatrix} \frac{du_1}{du} & \cdots & \frac{du_g}{du} \\ \vdots & & \vdots \\ \frac{d^g u_1}{du^g} & \cdots & \frac{d^g u_g}{du^g} \end{vmatrix},$$

is not identically zero. By changing the basis, the determinant is multiplied by a non zero constant, and by changing the local coordinate at P one gets

$$\Delta_u = \Delta_t \left(\frac{dt}{du}\right)^{g(g+1)/2}.$$

Since $\frac{dt}{du}$ does not vanish, the vanishing order of Δ_u at P does not depend on the choice of the local coordinate. At this point Hürwitz wrote: "Wir wollen nun für u insbesondere ein überall endliches Integral wälen. Dann stellt die Determinante Δ_u eine eindeutige algebraische Funktion der Fläche X vor" [Now we choose for u an integral which is everywhere finite. Then the determinant Δ_u represents a well defined algebraic function of X] and added "Die Gleichung (2) zeigt nun, dass Δ_u nur an den $2g-2$ Nullstellen von du unendlich wird und zwar an jeder von der Ordnung

$g(g+1)/2$. Die Gesamtordnung des Unendlichewerdens beträgt also $(g-1)g(g+1)$, und ebenso gross ist also auch die Gesamtordnung des Verschwindens. Nun wird aber $\frac{dt}{du}$ an keiner Stelle Null; also folgt: Es gibt stets eine endliche Zahl von Stellen P an welchen die Determinante Δ_t verschwindet." [The equation (2) now shows that Δ_u becomes infinite, and precisely of order $g(g+1)/2$, only at the $2g-2$ zeros of du. The total order of infinities then amounts to $(g-1)g(g+1)$, so the same is true for the total order of vanishing. Since $\frac{dt}{du}$ never vanishes, it follows that there always exist a finite number of places P where the determinant Δ_t vanishes] [34, p. 396]. Let $P_1, P_2, \ldots, P_r$ be these points, and $m_1, m_2, \ldots, m_r$ be the order of vanishing of Δ_t at $P_1, P_2, \ldots, P_r$ respectively. Then we have:

$$m_1 + m_2 + \cdots + m_r = (g-1)g(g+1).$$

At this point Hürwitz wrote: "Es handelt sich jetzt um eine nähere Untersuchung der Zahlen $m_1, m_2, \ldots, m_r$" [Now we can study in depth the numbers $m_1, m_2, \ldots, m_r$]. Let P be a fixed point of X. In a neighborhood of it we can write

$$\begin{gathered} u_1 = t^{\rho_1} + \cdots \\ u_2 = t^{\rho_2} + \cdots \\ \vdots \\ u_g = t^{\rho_g} + \cdots , \end{gathered}$$

with $0 < \rho_1 < \rho_2 < \cdots < \rho_g$. By substituting into Δ_t, we get

$$ct^{\rho_1+\rho_2+\cdots+\rho_g-g(g+1)/2} = ct^m.$$

Hence

$$m = \rho_1 + \rho_2 + \cdots + \rho_g - g(g+1)/2, \tag{3.1}$$

this is the "multiplicity" of P, or the *weight* of P.

Hürwitz observed that $\rho_1, \rho_2, \ldots, \rho_g$ are precisely the gaps at P, so that if $P \notin \{P_1, P_2, \ldots, P_r\}$, then $m = 0$ and $\rho_1 = 1, \rho_2 = 2, \ldots, \rho_g = g$; moreover, if $P \in \{P_1, P_2, \ldots, P_r\}$, then

$$1 = \rho_1 < \rho_2 < \cdots < \rho_g \leq 2g - 1.$$

At page 397 of his paper, Hürwitz remarked that if $\alpha, \beta \in N(P) = \mathbb{N}\backslash G(P)$ and f_α, f_β are two functions having, respectively, a pole of order α and β at P as their only singularities, then the function $f_\alpha f_\beta$ has a pole of order $\alpha + \beta$ at P as its only singularity, and in the next two pages he proved that X is hyperelliptic if and only if the smallest non-gap is 2 (as we have already seen in the previous section). Thus in this case and only in this case, each Weierstrass point has weight $g(g-1)/2$. These points are precisely the $2g+2$ fixed points of the *hyperelliptic involution* f, which (as can be proved easily) is the only automorphism of X that leaves all the Weierstrass points fixed.

As a consequence of these facts one has that the number r of distinct Weierstrass points on a nonhyperelliptic X is strictly greater than $2g + 2$. At the end of Sect. 2, Hürwitz also remarked: "Eine genaue Untersuchung würde übrigens nicht nur festzustellen haben, welche Werte die Zahl r annehmen kann, sondern auch welche Zahlesysteme $\rho_1, \rho_2, \ldots, \rho_g$ für die einzeln Stellen möglich sind" [A more in-depth analysis should extablish not only which values the number r can assume, but also which systems of numbers $\rho_1, \rho_2, \ldots, \rho_g$ are actually possible for a given point], and observed that the minimum number of Weierstrass points on "Eine ebene Kurve 4 Ordnung" [a plane (non singular) quartic] is 12 and gave the example of the curve $x_1^4 + x_2^4 + x_3^4 = 0$.

In Sect. 3, Hürwitz introduced the space of k-differentials as the space of forms of order k on the space of holomorphic differentials on X , and, chosen a basis for it, say $\psi_1, \ldots, \psi_\varkappa$, he defined the determinant $\Delta_\varkappa$ similarly as he did for Δ_u. He was led to the concept of what we now call *Weierstrass points of order* k, or k-*Weierstrass points*. More precisely, a point P of X is said to be a k-Weierstrass point if there exist a k -differential vanishing at P of order at least $\varkappa$. Then he proved that the set W_k of k-Weierstrass points is finite of cardinality $(2k-1)^2 g(g-1)^2$. Higher-order Weierstrass points will become of great interest in the 1970s and we shall return to this argument later on.

In the second part of his paper (from Sect. 5 onward) Hürwitz gave a proof of his famous formula (today known as Riemann–Hürwitz's formula):

$$2\widetilde{p} - 2 = R + n(2p - 2), \tag{3.2}$$

for the genus $\widetilde{p}$ of a Riemann surface that is a covering, of degree n and total ramification R, of a Riemann surface of genus p. Then (see Sect. 7) he proved the bound $|\mathrm{Aut}(X)| \leq 84(g-1)$. Before we reproduce (with slight modifications) his proof, let us recall Hürwitz's argument (which he gave in Sect. 1) in order to show the finiteness of $\mathrm{Aut}(X)$. If $\alpha \in \mathrm{Aut}\,(X)$, then for every point $p \in X$, p and $\alpha(p)$ have the same gap sequence; in particular if $p \in W$ we have also $\alpha(p) \in W$. So it is clear that to every α corresponds a permutation on W and we have a group homomorphism $\lambda : \mathrm{Aut}(X) \to \mathrm{Perm}(W)$. Suppose that $\alpha \neq \mathrm{id}_X$ has s fixed points, and let $P_1, \ldots, P_{g+1}$ be distinct points, none of which is fixed for α. By the Riemann–Roch theorem there exists a meromorphic function f on X having precisely $P_1, \ldots, P_g$ as poles. Then the meromorphic function $f - f \circ \alpha$ has $2g + 2$ poles, and therefore $2g + 2 \geq s$. Suppose X is not hyperelliptic; in this case the set W has cardinality $r > 2g + 2$, then λ is injective and $|\mathrm{Aut}(X)| \leq r!$. If X is hyperelliptic, $\ker \lambda = \langle f \rangle$ and then $\mathrm{Aut}(X)$ is isomorphic to a subgroup of $\mathrm{Perm}(W)$ and hence is finite. Now, for simplicity, we put $G = \mathrm{Aut}(X)$, we denote Y the compact Riemann surface which is the quotient of X by the finite group G, and $\pi : X \to Y$ the quotient map. Suppose that there are N branch points $y_1, \ldots, y_N$ in Y, with π having multiplicity k_n at the $|G|/k_n$ points above y_n, $n = 1, \ldots, N$. Then by (3.2) we have

$$
\begin{aligned}
2g-2 &= |G|\left(2g(Y)-2+\sum_{n=1}^{N}\frac{|G|}{k_n}(k_n-1)\right)\\
&= |G|\left(2g(Y)-2+\sum_{n=1}^{N}\left(1-\frac{1}{k_n}\right)\right).
\end{aligned}
$$

From this formula we get: if $g(Y)\geq 2$ then $2g-2\geq 2\,|G|$ and so $|G|\leq g-1$; if $g(Y)=1$ then $2g-2=|G|\sum_{n=1}^{N}(1-\frac{1}{k_n})\geq |G|/2$ and so $|G|\leq 4(g-1)$; finally if $g(X/G)=0$, then

$$2g-2=|G|\left(N-2-\sum_{n=1}^{N}\frac{1}{k_n}\right).$$

Since $g\geq 2$, it follows that $N\geq 3$. Now if $N\geq 5$, we have $2g-2\geq |G|\,(3-5/2)$ and then $|G|\leq 4(g-1)$. If $N=4$, we have $2g-2=|G|\,(2-\sum_{n=1}^{N}\frac{1}{k_n})$ and $|G|$ is maximum for $k_1=2, k_2=2, k_3=2, k_4=3$, hence in this case $|G|=12(g-1)$, so in general we have $|G|\leq 12(g-1)$. If $N=3$, we get $2g-2=|G|\,(1-\sum_{n=1}^{N}\frac{1}{k_n})$ and $|G|$ is maximum for $k_1=2, k_2=2, k_3=7$, hence in this case $|G|=84(g-1)$, so in general we have $|G|\leq 84(g-1)$.

Let us remark that the maximum is reached by the famous *Klein quartic* , which is the genus three curve defined in $\mathbb{P}^2$ by

$$x_0^3x_1+x_1^3x_2+x_2^3x_0=0.$$

4 Other contributions in the first period

One year after Hürwitz's seminal paper appeared, Segre published his long memoir [74]. That memoir can be seen as the first treatise on curves from the geometric point of view. Here an entire section is devoted to an exposition of the known results on multiple points of a linear series, mainly based on the De Jonquière formula. In particular on p. 89 of his paper, Segre considered a point P that is at least of multiplicity m_1 for the ∞^{r-1} divisors in the g_n^r containing P, at least of multiplicity m_2 for ∞^{r-2} divisors,..., of multiplicity m_{r-1} for the ∞^1 divisors, and of multiplicity m_r for the only one containing rP; and proved (by induction on r) that the presence of such a point decreases the number N of $(r+1)$-ple points of the g_n^r by

$$m_1+\cdots+m_r-\frac{r(r+1)}{2}\cdot \tag{4.1}$$

Here Segre was near to introducing the concept of "weight" of those points that in the future were to be called "generalized Weierstrass points" (see Sect. 7.1). Hürwitz's seminal paper is cited for the first time by Segre in a footnote in article 43. In this article Segre rediscovered the formula of Veronese we have given in Sect. 2, and in the footnote he remarked that the same computation appeared, for the case of the canonical series, in the paper of Hürwitz. In the subsequent article Segre discussed the case of *sextactic points*, i.e. those points P on a plane curve C of genus g, for which there exists a conic γ having a contact of order 6 with C at P. We notice that this notion was introduced long before Segre by Cayley in [11]. For smooth plane quartics these points correspond to 2-Weierstrass points (see also the end of this section). Segre cited Hürwitz's paper a second time when he presented Weierstrass' *Lückensatz* as an application of the Riemann–Roch theorem; in a footnote he stressed the fact that in that paper Hürwitz came to the remarkable result that, aside from the hyperelliptic case, a curve has strictly more than $2g+2$ Weierstrass points. A result, in Segre's words, that allows one to deduce a very nice and simple proof of the finiteness of $\mathrm{Aut}(X)$ (see [75, footnote 7]).

The question of how to find a lower bound for the number of Weierstrass points on a non-hyperelliptic curve (that Hürwitz posed in his paper), may have attracted Segre. In fact a few years later he published a short note on this subject [75]. Here, following Haure's suggestion, Segre talked of "punti di Weierstrass" (Weierstrass points) and proved, by means of a projective argument involving the canonical curve, that the maximal weight of a Weierstrass point on a nonhyperelliptic curve C of genus g is $(g-1)(g-2)/2+1$, and so that C must have at least

$$2g+6+\frac{8(g-3)}{g(g-3)+4}$$

distinct Weierstrass points. Segre remarked that a careful analysis of his method should have lead to a better bound. This was actually performed by Cipolla in [14], where it is shown that if $g>7$, then

$$w(P)\leq\frac{p^2-5p+12}{2}.$$

The same problem was studied more than 70 years later by T. Kato and the result of Cipolla was improved (see Sect. 7.3).

The work of Hürwitz strongly influenced Haure, in fact in his already cited paper [31] he wrote: "je développe... conformément à des indications données par M. Hürwitz, une méthode pour former des tableaux d'ordres manquants [I develop... according to the indications given by M. Hürwitz, a method for making a table of the gaps]". The first two of the five chapters of Haure's long paper are devoted to the exposition of the following arguments: Weierstrass gap theorem and Noether's generalization; the definition of Weierstrass points (we recall that it was Haure who gave them this name); the introduction of the Wronskian. In chapter three, Haure developed a method for determining the gaps at a given point when the smallest gap is fixed. In chapter four he gave a formula for the number of parameters on which the Weierstrass normal form of a curve of genus g having a Weierstrass point with a fixed gap sequence depends, and provided a list of the possible gap sequences for $g=3,\ldots,7$. In the last chapter he applied his method to space curves.

The list of gap sequences given by Haure was not completely correct. For instance, for $g=3$ the sequence $(1,2,5)$ corresponding to a point of weight 2 is missing, for

$g = 6$ the sequence $(1, 2, 3, 5, 7, 10)$ should contain 11 instead of 10 (this sequence corresponds to a ramification point of a bielliptic cover, see Sect. 7, in particular [36, 37, 44]). Nevertheless we like to stress the fact that in his paper Haure raised questions concerning Weierstrass points were to become of great interest many years later (see Sect. 7): "la définition d'une classe de courbes possédant un point de Weierstrass A d'espéce déterminée" [the definition (by means of the Weierstrass normal form) of a family of curves having a Weierstrass point A of a fixed kind (i.e. with a fixed gap sequence)], and the problem of finding the number of parameters on which such a family depends (that he computed for $g \le 7$).

The question of determining the number of parameters on which the class of Riemann surfaces with assigned Weierstrass normal form depends, were also considered by Hensel and Landsberg in their treatise [32]. Here they considered the problem of "moduli" taking (incorrectly) into account only the first non-gap n, and next relatively prime non-gap $n + r$, so they found that the number of parameters is given by $2g - 3 + n - \rho$, where ρ denotes the index of speciality of $(n + r)P$ (see Sects. 6, 7). We remark that on page 493 of their book, Hensel and Landsberg make explicit, by using the new algebraic terminology, the fact that: "Das System $\mathfrak{H}$ [$= \mathbb{N} \setminus G(P)$] der vorhandenen Ordnungzahlen bildet einen additive Modul" [The system $\mathfrak{H}$ of existing orders form an additive module].

Concluding this section we like to recall two others papers. Weierstrass points of weight two on a non-singular plane quartic C, are points P for which the tangent t_P to C at P is such that $t_P \cdot C = 4P$. In the old literature, these points were called *undulation points*.

The first attempt to classify smooth plane quartic, according to the number of undulation points they contain, is due to Masoni in [49], where he studied the problem using the methods of plane projective geometry. In particular Masoni (incorrectly) claimed that a curve with 12 undulation points must be isomorphic to the quartic of Fermat: $x_1^4 + x_2^4 + x_3^4 = 0$. This paper was rediscovered, and amended in some part, by Edge in [20] almost 70 years later (see also [79]), but the problem of finding, up to isomorphisms, all plane quartics with 12 undulations points remained open. This question was solved only in 1977 by Kuribayashi and Komiya; in fact they proved in [40], that (up to isomorphisms) there are essentially two distinct quartics with this property: they are the Fermat quartic and the one defined by the equation $x^4 + y^4 + z^4 + 3(x^2y^2 + y^2z^2 + x^2z^2) = 0$.

The second paper we like to mention is one by Wiman. In [83] Wiman showed, among others things, that the fixed points of an involution acting on a smooth plane quartic are either 2-Weierstrass points (sextactic points) or Weierstrass points of weight two (undulation points). Wiman's paper anticipates future researches by Lewittes and other authors of the early 1960s (see the end of the next section and Sect. 7.3).

5 The "middle period"

As we already said in the introduction, in the first half of the XXth century, geometers seemed to pay not (too much) attention to questions concerning Weierstrass points. So very few results on this subject were published in this period.

However, in 1924 two papers appeared, one by Rosati [70] and one by Chisini [13], both concerning the proof of Noether's generalization of the gap theorem we have mentioned in Sect. 2. Rosati amended the proof given by Noether, while Chisini provided a new simple proof of the same result.

Wierstrass points were briefly treated by Severi in his treatise of 1926 [76], but he gave the following functional interpretation of the formula (2, 2):

$$H \equiv (r+1)D + \binom{r+1}{2} K, \tag{5.1}$$

where H denotes the divisor of $(r+1)$-tuple points in a g_n^r, taken with their right multiplicity, D a divisor of the g_n^r and K a canonical divisor (see also Sect. 7).

Also J. L. Coolidge devoted (only) a few pages of his book on plane curves [15] to the gap theorem and Weierstrass points (developed from the algebraic point of view), but he gave a proof of Segre's result that we mentioned in the previous section. In this latter context looking for the 'sexstactic' points, as an example illustrating the theory, he began with the funny phrase: "it is hard to keep away from sex these days".

In 1939 Schmidt [71] extended, using the Wronskian, the concept of Weierstrass point for curves defined over a field of positive characteristic.

After the second world war the lectures on Riemann surfaces delivered by L. Ahlfors, at Harvard [4], and L. Bers, in New York [7], strongly renewed interest in Riemann surfaces and Weierstrass points. In 1955, Weyl published a new edition (in English) [82] of his fundamental book *Die Idee der riemanschen Flächen* that had appeared in 1913. Here Weyl took into account the changes that had occurred in Topology between the two world wars, and, as he remarked in the introduction, his book was inspired by Chevalley's *Introduction to the theory of algebraic functions of one variable*, which had appeared 4 years earlier [12]. Anyway, in Weyl's book Weierstrass points are not treated and the gap theorem is referred as "an amusing application of the Riemann–Roch theorem" [82, p. 139]. In 1957, Springer published his *Introduction to Riemann surfaces* [77], which soon became a classic, but here Weierstrass points are still introduced in the same way as Hürwitz did in his seminal paper.

It was known since Riemann's paper [67], that a point in the set $\mathfrak{M}_g$ of isomorphisms classes of Riemmann surfaces of genus $g > 1$ depends on $3g - 3$ parameters or "moduli". An important step toward a new approach to some of the old problems on Weierstrass points, especially those regarding moduli, was the introduction of a complex structure on $\mathfrak{M}_g$, and on the set $\mathfrak{T}_g$ of equivalence classes of Teichmüller surfaces of genus g, i.e. Riemann surfaces of genus g equipped with a topological isomorphism onto a fixed Riemann surface. This was done mainly by the work of Rauch in 1955 (see [63,64]), and Ahlfors (see [3] and also [8]). This fact allowed one to put questions about "parameters" into the right setting. But the crucial turning point was a paper by Rauch that appeared in 1959.

6 The revival of Weierstrass points

In his paper [65], Rauch studied Riemann surfaces possessing a Weierstrass point P with first non-gap n. This study, as we have seen, formerly considered by

Hürwitz (Haure and Hensel–Landsberg). In fact in the introduction of his paper Rauch wrote: "The inspiration for the present research came from a statement on the number of constants in the normal form of the defining equation of an algebraic curve due to Weierstrass, Schwarz, Landsberg" [65, p. 546]. More precisely, Rauch proved that the Riemann surfaces (Teichmüller surfaces) of a fixed genus g, having a Weierstrass point whose first non-gap is n, form a complex-analytic (possibly disconnected) subvariety of $\mathfrak{M}_g$ ($\mathfrak{T}_g$). This variety has dimension $n + 2g - 3$ if $n + 1$ is a gap (*general case*), and has dimension $n + 2g - 4$ otherwise. The main tools used in the proof (based on viewing X as an n-sheeted covering of the Riemann sphere) are variational formulae for the periods and the "principle of non-degeneracy", according to which: *A sufficiently small shift of branch points does not lower the smallest member of a Weierstrass sequence* [65, p. 559].

Along the same line are the two papers by Farkas [22,23]. In particular in the latter it is shown that if $g \geq 4$, then a Weierstrass point $P \in X$ whose first non-gap is 3, has 4 as a gap. This result was first generalized by Horiuchi in [33], where, under the assumption $g > r(r-1)/2$, $1 < r < g$, he proved that if r is the first non-gap, then $r + 1$ is a gap. Then Jenkins in [35], showed that if h is the first non-gap at P, and q is relatively prime to h, then q is a gap if $g > (h-1)(q-1)/2$.

We observe that both Rauch and Farkas left open the question as to whether the subvarieties they were considering were empty or not. This question was given a partial answer by the paper [48], where Maclachlan applied results of [45] (which we will discuss below), in order to show that if $\theta = \{\gamma_1, \dots, \gamma_g\}$ is a set of integer such that $0 < \gamma_1 < \cdots < \gamma_g < 2g$, $\mathbb{Z}^+ \backslash \theta$ is closed under addition and starts with 3, then there exist X of genus g with a Weierstrass point whose gap sequence is θ.

The study of Riemann surfaces with Weierstrass points whose first non-gap is n were reconsidered, from another point of view, some years later by E. Arbarello (see Sect. 7.2).

Again Hürwitz's work inspired many of the questions studied by Lewittes in his paper [45], based on his Ph.D. thesis (advisor Rauch). After a concise review of known theorems and concepts on Weirstrass points and having observed that the maximal number of fixed points of an automorphism on a non-hyperelliptic curve is $2g$ (which improves the bound given by Hürwitz); he discussed the representation of $h \in \mathrm{Aut}(X)$ as a linear transformation of the space of holomorphic differentials and computed the dimension of the invariant subspace under the action of a subgroup $H \subseteq \mathrm{Aut}(X)$. Then Lewittes examined the diagonal form of the matrix of the representation and proved the following very remarkable result: *If an automorphism φ of a compact Riemann surface X has a fixed point that is not a Weierstrass point, then φ admits at least two and at most four fixed points* [45, Theorem 6]. Let $\nu(\varphi)$ denote the number of fixed points of φ ; then Lewittes theorem implies that if $\nu(\varphi) \geq 5$, then all fixed points of φ are Weierstrass points. To get this result, Lewittes made use of representation theory but, in a note at page 746 of his paper, he remarked that R.D.M. Accola pointed out to him that it was possible to derive it directly from the Riemann–Hürwitz relation. Following Accola's idea, the proof goes as follows (for instance [2, p. 52]). Suppose that an automorphism φ of order n has 5 or more fixed points, then the quotient map $\pi : X \to X' := X/\langle\varphi\rangle$ has degree n, and, if P_0 is a fixed point, $\mathrm{mult}_{P_0}(\pi) = n$. So for the total ramification R (see Sect. 3) we have

$$R = \sum_{P \in X} (\mathrm{mult}_P(\pi) - 1)$$
$$\geq \sum_{P \in X, \varphi(P)=P} (\mathrm{mult}_P(\pi) - 1) \geq 5(n-1) > 4(n-1).$$

Let g' be the genus of $X/\langle\varphi\rangle$ and $P_0' = \pi(P_0)$, then by the Riemann–Roch theorem we have $l((g'+1)P_0) > 1$, so there exists a meromorphic function f on X' having P_0 as its only pole, with $\mathrm{ord}_{P_0}(f) \geq -(g'+1)$, whence $\mathrm{ord}_P(f \circ \pi) \geq -n(g'+1)$. From

$$\sum_{P \in X} (\mathrm{mult}_P(\pi) - 1) > 4(n-1),$$

by the Riemann–Hürwitz formula we get

$$2g - 2 = n(2g' - 2) + \sum_{P \in X} (\mathrm{mult}_P(\pi) - 1) > 2(ng' + n - 2)$$

and so $g+1 > n(g'+1)$. Finally $\mathrm{ord}_P(f \circ \pi) > -(g+1)$. In other words $l(gP_0) > 1$, and P_0 is a Weierstrass point.

One of the first results in this direction is by Maclachlam [48]. Here, by using results of [45], the author proved the existence of compact Riemann surfaces having a Weierstrass point with any assigned Weierstrass sequence beginning with 3.

We end this section by recalling the innovative lectures R.C. Gunning gave in Princeton during the academic year 1965–1966, in which the technique of line bundles played an important role and the Wronskian is obtained as a section of the $g(g-1)/2$th tensor power of the canonical line bundle $\mathscr{K}$ (see [29]). We think that this work has strongly influenced the way to look at Weierstrass points since then.

7 The 1970s, new problems come into the scene

After 1970, the discoveries and the new tools introduced in the field of Analytical and Algebraic varieties during the previous 50 years, allowed some of the problems left behind for many years in the study of Riemann surfaces and Algebraic curves do be revisited. Among these were those left by Hürwitz, Haure, Hensel and Landsberg and others, namely: study "the existence of Weierstrass points with prescribed semigroup of non-gaps" (or more simply with a given first non-gap); determine "the dimensions of the irreducible components" of the loci in the moduli space of smooth curves they define; find "the relations between Wierstrass points and automorphisms", along the line traced by Hürwitz (this problem had already reemerged in the early 1960's in the work of Lewittes); and finally, to study " the inflectional points" in relation with Segre's formula and Severi's functional equation. These problems were largely studied in the

1970s, but also new problems came into the scene and the literature on Weierstrass points increased very quickly.

7.1 Generalized Weierstrass points

A new interest in higher order Weierstrass points arise after Lipman Bers conjectured that the set $W(\mathscr{K}) := \bigcup_k W_k$ is dense in X. This conjecture was proved in 1971 by B.A. Olsen, who actually showed a more general result concerning those points that are now commonly known as "generalized Weierstrass points" (for the origin of this name see the end of this subsection). In order to state Olsen's theorem, we require some preparation.

Suppose X is of genus $g \geq 2$, and let $\mathscr{L}$ be a holomorphic line bundle on X of positive degree (i.e. $c(\mathscr{L}) > 0$) that admits holomorphic sections which have no common zeros. Let $\gamma(\mathscr{L})$ denote the complex dimension of the space of holomorphic sections of $\mathscr{L}$ (we observe that to give such an $\mathscr{L}$ on X is equivalent to giving a complete linear series g^r_n without fixed points, with $n = c(\mathscr{L})$ and $r = \gamma(\mathscr{L}) - 1$). For all k's, the kth tensor power $\mathscr{L}^k$ of $\mathscr{L}$ admits holomorphic sections which have no common zeros. Olsen called a point $P \in X$ an $\mathscr{L}^k$-*special point*, if there exists a nonzero holomorphic section of $\mathscr{L}^k$ having at P a zero of order $\geq \gamma(\mathscr{L}^k)$ (in case $\mathscr{L}$ is the canonical bundle $\mathscr{K}$ this notion coincides with that of k-Weierstrass point). Then, denoting by $W(\mathscr{L}^k)$ the set of all $\mathscr{L}^k$-special points on X, Olsen put

$$W(\mathscr{L}) = \bigcup_k W(\mathscr{L}^k),$$

and proved that: *$W(\mathscr{L})$ is dense in X for any $\mathscr{L}$ as above* [58, Theorem 2].

Let us sketch the proof of Olsen's striking result. Denote by J the Jacobian of X, equipped with a choice of an embedding $F : X \to J$, and let ϑ be a Riemann's theta-function with respect to the period matrix (I, Z) for J, namely $\vartheta(w) = \sum \exp 2\pi i({}^t aZa + 2{}^t aw)$. The map F can be extended by linearity to a map from the group of divisors on X into J (since J is an Abelian group). Such a map, with a slight abuse of notation, is still denoted by F. Let W^{g-1} be the image under F of all positive divisors of degree $g - 1$ on X. Then, by the theorem of Riemann for the divisor θ of ϑ in J, we have $\theta = W^{g-1} + \kappa(F)$, where $\kappa(F)$ is the Riemann's constant depending only on F. Now the theorems of Riemann–Roch and Abel imply that $P \in X$ is $\mathscr{L}^k$-special if and only if:

$$\gamma(\mathscr{L}^k)F(P) \in [\theta + F(\mathscr{L}^k) + \kappa(F)],$$

where $F(\mathscr{L}^k)$ denotes the image (under F) of the divisor of any (nonzero) holomorphic section of $\mathscr{L}^k$. Then one has only to show that for any $P \in X$ and any neighborhood U_P of P, there exists a sufficiently large integer k such that:

$$\gamma(\mathscr{L}^k)F(U_P) \cap [\theta + F(\mathscr{L}^k) + \kappa(F)] \neq \emptyset.$$

Fix a covering map $\mathbb{C}^g \to J$, and assume that the embedding $F : X \to J$ is made with P as base point. Then F induces a map from the unit disk U_P into $\mathbb{C}^g$ such that $P \mapsto 0$. Let H be the complex line which is the image of $\psi : \mathbb{C} \to \mathbb{C}^g$ given by $t \mapsto F'(P)t$. Denote by B a limit point in J for the sequence $F(\mathscr{L}^k) + \kappa(F)$. Then $\theta + B$ is the divisor of the (non-degenerate) theta-function $\vartheta_B(t) = \vartheta(t - B)$. Now a fundamental step toward the proof is that a (non-degenerate) theta-function restricted to H has infinitely many zeros on H [53, Theorem 1], so $\theta + B$ intersects H at some point. If k_i is a sequence of integer such that $F(\mathscr{L}^{k_i}) + \kappa(F)$ converges to B, il follows that

$$\gamma(\mathscr{L}^{k_i})F(U_P) \cap [\theta + F(\mathscr{L}^{k_i}) + \kappa(F)] \neq \emptyset,$$

and the theorem is proved.

Mumford independently showed that the set $W(\mathscr{K})$ is dense in X , and also that its points are "evenly distributed" (with respect to the Bergman metric) on X [52, p. 11], but his proof was never published (see also [54]).

The classical theme of the $(r + 1)$-tuple points of a linear series g_n^r (and then the one of the inflectional points of a projective curve in the sense of De Jonquières and Veronese) was reconsidered by G. Galbura in the light of the theory of line bundles, 2 years after the publication of Olsen's paper.

Let $G = g_n^r$ be a linear series without fixed points on X, and $\mathscr{L}$ the associated holomorphic line bundle: the g_n^r corresponds to a $(r + 1)$-dimensional subspace A of the space $H^0(X, \mathscr{L})$ of holomorphic sections of $\mathscr{L}$. Suppose $\{\varphi_{ij}\}$ are the transition functions of $\mathscr{L}$ with respect to the complex atlas $\{U_i\}$. Let $\{s_0, \ldots, s_r\}$ be a basis for A, then the Wronskian $W_i(s_0, \ldots, s_r)$ is holomorphic on U_i and the order of vanishing of W_i at $P \in U_i$ does not depend on the basis $\{s_0, \ldots, s_r\}$, nor on the local chart U_i. Now if z_i is a local coordinate in U_i and s_l is represented by the collection $\{f_{li}\}$ of holomorphic functions, then we have

$$W_i = \left| \frac{d^k f_{li}}{dz_i^k} \right| = \varphi_{ij}^{r+1} \left(\frac{dz_j}{dz_i} \right)^{r(r+1)/2} W_j$$

[27, Theorem 1]. Hence (W_i, U_i) represents a section W of a holomorphic line bundle $\mathscr{M}$ isomorphic to $\mathscr{L}^{\otimes r+1} \otimes \mathscr{K}$, and both $\mathscr{M}$ and the divisor of W only depend on A and not on the basis $\{s_0, \ldots, s_r\}$. It is clear that

$$\deg W = n(r + 1) + \frac{r(r + 1)}{2}(2g - 2) = (r + 1)(n + rg - r))$$

(which coincide with (2.2)). This formula was translated by Galbura into Severi's functional relation (2.4), by considering the divisors associated to the respective line bundles [27, Theorem 1]. He also gave a proof of Segre's result (4.1): chose a basis $\{s_0, \ldots, s_r\}$, such that $s_0, \ldots, s_r$ have zeros at P of orders $0 = m_0 < m_1 < \cdots < m_r$, respectively, then to show that W has a zero at P of order

$$v_P = m_1 + \cdots + m_r - \frac{r(r+1)}{2} = \sum_{i=0}^{r}(m_i - i)$$

is just a matter of computation of determinants.

Accola in his paper [1], studied the points P for which $v_P > 0$, and gave them the name of *generalized Weierstrass points* for $G = g^r_n$ (a name which is commonly used today for these points, see [2,51]); he also introduced for v_P the symbol $w_G(P)$, which he called the G-*weight* of P. Moreover Accola proved the bound

$$w_G(P) \leq \frac{(g - i(G))(g - i(G) + 1)}{2},$$

and that equality holds only if the curve is hyperelliptic.

With a note at the end of his paper, Galbura remarked on the possibility of an extension of his theorem (following the line traced in [71]) to curves defined over a field of positive characteristic. Galbura's observation became later Laksov's point of departure toward a new definition of "generalized Weiertrass points", and the extension of Galbura's results to complete linear series on curves defined over a field of any characteristic [41] (see also [42]).

We end this part recalling a beautiful idea due to D. Mumford. By using k-differentials one can define, similarly to what happens for the canonical map, the kth-canonical map $\mathscr{C}_k : X \to \mathbb{P}^N$, so that the k-Weierstrass points of X are the hyperosculation points of the embedded kth-canonical curve $\mathscr{C}_k(X)$. There is a bijective correspondence between the isomorphism classes of Riemann surfaces and the projective isomorphisms classes of kth-canonical curves. According to this, Mumford suggested a characterization of kth-canonical curves by means of the k-Weierstrass points, in order to define global parameters for the moduli space $\mathfrak{M}_g$ [52, p. 30] (see also [60]).

7.2 Existence of Weierstrass points of given type

As we know, Hürwitz's original question in his paper of 1893, was about the existence of Riemann surfaces having a Weiertstrass point whenever

$$\mathbb{H} := \mathbb{N} \setminus \{\alpha_1, \ldots, \alpha_g\},$$

where $0 < \alpha_1 < \cdots < \alpha_g < 2g$, satisfies the semigroup conditions. Three years later, Haure incorrectly gave some restrictions, and in 1902 Hensel and Landsberg incorrectly proved there were none. The questions of determining non-emptiness and dimensions for of the irreducible components of the loci that particular Weierstrass points define in the "moduli space of smooth curves", were difficult and probably impossible to answer at that time. Things changed after more then half a century. In 1960, Ahlfors gave to the set of isomorphism classes $\mathfrak{M}_g$ of Riemann surfaces a structure of $(3g-3)$-dimensional analytical space [3]; in 1963, Mayer and Mumford made an important step toward a new compactification $\overline{\mathfrak{M}}_g$ of $\mathfrak{M}_g$—which followed that Satake constructed in 1956—by introducing the fundamental notion of "stable curves" (see [53, p. 228]). After the works of Mayer [50] and Deligne–Mumford [17], the questions above became approachable.

The first problem to be studied was (the particular) case of Weierstrass points whose first non-gap is a fixed number $n \geq 2$. A Riemann surface with such a Weierstarss point can be identified with an n-sheeted cover of $\mathbb{P}^1$ having a point of total ramification. Inspired by Fulton's construction of the Hürwitz space $H^{n,w}$, whose points parameterized the set of n-sheeted coverings of $\mathbb{P}^1$ with w simple ramification points [26], Arbarello introduced in [5] a Weiertrass space $WH^{n,w}$, whose points parameterized the set of *simple Weierstrass coverings of type* (n, w), i.e. n-sheeted coverings X of $\mathbb{P}^1$ with w branch points such that there exist $x \in X$ whose ramification index is n (while any other ramification point $x' \in X \setminus \{x\}$ is simple). The space $WH^{n,w}$ then defines an analytic subvariety of $\overline{\mathfrak{M}}_g$ whose closure is denoted $\overline{W}_{n,g}$. Notice that $\overline{W}_{n,g}$ contains the space of moduli of genus g curves having a Weierstrass point whose first non-gap is n. Arbarello first proved that $\overline{W}_{n,g}$ is irreducible and of dimension $2g+n-3$. It was the point of view of the moduli space $\overline{\mathfrak{M}}_g$, instead of the Teichmüller space $\mathfrak{T}_g$ that allowed Arbarello to prove the irreducibility; in fact, as he remarked in the paper, is not clear at all that the preimages of the $\overline{W}_{n,g}$'s in $\mathfrak{T}_g$ are connected. He also noticed that the dimensionality statement implicitly gives that the first case of Rauch's theorem is actually the "generic case". Moreover, by using a degeneration argument that show that any simple Weierstrass covering of type $(n-1, w-1)$ can be thought of as a limit of simple Weierstrass coverings of type (n, w), Arbarello proved the existence of the following filtration

$$\overline{W}_{2,g} \subset \overline{W}_{3,g} \subset \cdots \subset \overline{W}_{g-1,g} \subset \overline{W}_{g,g} = \overline{\mathfrak{M}}_g.$$

We notice that the study of subvarieties of $\overline{\mathfrak{M}}_g$ defined by the existence of certain type of exceptional Weierstrass points was continued some years later by S. Diaz with a series of papers starting with his 1982 Ph.D. thesis at Brown University. He applied Kodaira–Spencer theory of first order deformations of the curve together with a line bundle on it, in order to get information on the dimensions of these subvarieties and on how many of such Weierstrass points are present on certain types of curves (see for instance [18, 19]).

Hürwitz's original question was first studied for particular values of the genus. If $g = 4$ there are six possible gap sequences, and H.C. Pinkham in 1974 proved that for each of them there exists a Riemann surface with a Weierstrass point having the appropriate gap sequences [61]. So he gave a positive answer to the question when $g = 4$. There were others positive answers to this question; for instance that given by Rim and Vitulli in [68] in the case of a "negatively graded" semigroup $\mathbb{H}$. Finally Buchweitz in 1980 in [10] showed that *not every semigroup* $\mathbb{H}$ *occurs*, by means of the following example: $g = 16$ and gap sequence $\{1, 2, \ldots, 12, 19, 21, 24, 25\}$. His surprising simple argument is as follows: if such a point P exists on X, then X would have holomorphic differentials vanishing to orders $0, \ldots, 11, 18, 20, 23, 24$ at P, and thus would have quadratic differentials vanishing to every order between 0 and 48 except possibly 37, 39 and 45; since $49 - 3 = 46 > H^0(X, \omega_X^2) = 3g - 3 = 45$, this is impossible (see also [21, p. 499]).

Kato in [36] discussed the existence of Weiertstrass points whose first non-gap is three. Here, by using also the Weierstrass normal form (see his [38]), he proved that there are no Riemann surfaces of genus 4 with 5, respectively 11, Weierstrass points whose gap sequence is $(1, 2, 4, 7)$, respectively $(1, 2, 4, 5)$. For $g \geq 5$ he proved that there are at most $g+2$ Weierstrass points whose first non-gap is 3. To achieve this latter result, he makes use of the fact that there are only three permissible gap sequences with first non gap 3 (from his paper [37]). He also shows that at most two, among such gap sequences, can coexist on the same Riemann surface. See [39] for the case of first non-gap 4.

We end this subsection by remarking that, for $g = 4$, the question of describing the moduli of Riemann surfaces which have a Weierstrass point with a specified non-gap sequence was studied in 1980 by Lax in [44]. In that paper, by using some results published in his [43], he showed in particular that the Riemann surfaces of genus 4 having a Weierstrass point with gap sequence $(1, 2, 3, 6)$ (resp. $(1, 2, 3, 7)$) depend on 8 (respectively 7) moduli. This result pointed out the deficiency in Hensel and Landsberg argument for computing moduli for a given gap sequence: in fact (as we said before) they only considered the first non-gap and the next relatively prime non-gap, thus obtaining in both cases 8 moduli. We observe, on the contrary, that the computation on the number of parameters made by Haure was correct, at least for $g = 4$.

7.3 Weierstrass points and automorphisms

The interest in relations between the automorphisms of a Riemann surfaces and its Weierstrass points, reemerged after the publication of Lewittes' paper in 1963, and this aspect of the theory of Riemann surfaces was very much investigated in the 1970s. A first paper to be mentioned is the one by Farkas [24], even if it is not directly connected with Weierstrass points. In this paper he proved that the maximal number of fixed points of an automorphism on a non-hyperelliptic Riemann surface of genus g is $2g - 1$ (so improving Lewittes' bound), and that this result is the best possible as the Riemann surface (of genus three) of equation $w^3 = (z - z_1^2)(z - z_2) \cdots (z - z_5)$ shows.

In 1978, Accola proved that if T is an automorphism of order n of a compact Riemann surface X with $\nu(T) \geq 3$ (i.e. T has at least three fixed points), then each fixed point of T is a Weierstrass point of order n. This result, which was published in a more general form several years later in [1] (see also [28]), generalizes the theorem of Lewittes. Probably in the same year Farkas and Kra showed that under the same hypothesis each fixed point of T is a q-Weierstrass point for each $q \equiv 1 (\text{mod } n)$, $q > 1$ (this result was later strengthened and included in their book [25]).

Still in 1978, I. Guerrero took into account the case in which $\nu(T) = 1, 2$ [28]. He proved the following: *if* $\nu(T) = 1$, *then the fixed point is an ordinary Weierstrass point* *except for a very special case (that he describes) in which the point is a* 2*-Weierstrass point, while if* $\nu(T) = 2$ *it is possible that the two points miss the dense set* $\cup_q W_q$. The latter fact is proved by a counterexample, suggested by Accola [28, p. 216], which is as follows. Suppose that T has prime order, that $E = X/T$ has genus one and that P_1, P_2 are the two fixed point of T. Denote a_1, a_2 the images of P_1, P_2 in $E \simeq J(E)$, then P_1, P_2 are q-Weierstrass points for some $q \geq 1$ if and only if $a_1 - a_2$ is a rational point on E.

Kato, in his paper [37], studied Weierstrass points of maximal weight on non-hyperelliptic Riemann surfaces and proved that if $g \geq 3$, then the weight $w(P)$ of each point $P \in X$ satisfies the conditions

$$0 \leq w(P) \leq \frac{g(g-1)}{2}$$

if $g = 3, 4, 6, 7, 9, 10$, or

$$0 \leq w(P) \leq \frac{g^2 - 5g + 10}{2}$$

in the other cases. He found examples of Riemann surfaces having Weierstrass points of maximal weight, and also proved the following: *let* $g = 8$ *or* $g \geq 11$*, and suppose that there exists* $P \in X$ *such that* $w(P)$ *is maximal, then* X *is elliptic–hyperelliptic* (i.e. there exists a holomorphic degree two map of X onto a torus; these curves are more often called *bielliptic*). We notice that, apart from some technical lemmas, the key points in the proof are the use of the Weierstrass normal form and the following result that he published in [38]: *for* $\widetilde{g} \geq 0$*, let* $k'_1 < k'_2 < \cdots < k'_i < \cdots$ *be a sequence of integers closed under addition and such that* $k'_{\widetilde{g}+1} = 2\widetilde{g} + i$ $(i = 0, 1, 2, \ldots)$*, then if for a point* $P \in X$ *there exists an integer* $\alpha > 1$ *such that* $N(P) = \{k_i\}$ *satisfies the condition* $k_i = \alpha k'_i$ $(i = 1, \ldots, l; \max\{3\widetilde{g} + 2, 5\widetilde{g} - 4\})$*, then* X *is an* α*-sheeted covering of a Riemann surface of genus* $\widetilde{g}$.

From the 1980s onward, the literature on Weierstrass points and related questions started to grow very fast, and in the last 20 years the number of papers which have appeared on this subject are already more than three times what has been published in the previous thirty.

Acknowledgments We would like to express our warmest thanks to Ryutaro Horiuchi, who provided us with an English translation of Kato's paper [38], to Steven Kleiman, who read a first draft of the paper, for his helpful comments, and to Anthony Geramita who helped us to improve the use of English in the final version.

References

1. Accola, R.D.M.: On generalized Weierstrass points on Riemann surfaces. In: Modular functions in Analysis and Number theory. Lectures Notes in Mathematics and Statistics, pp. 1–19. Pittsburg (1983)
2. Accola, R.D.M.: Topics in the theory of Riemann surfaces. In: Lectures Notes in Mathematics, vol. 1595. Springer, Berlin (1955)
3. Ahlfors, L.: The complex analytic structure of the space of closed Riemann surfaces. In: Analytic functions, pp. 45–66. Princeton, NJ (1960)

4. Ahlfors, L., Sario, L.: Riemann Surfaces. Princeton University Press, Princeton (1960)
5. Arbarello, E.: Weierstrass points and moduli of curves. Compos. Math. **29**, 325–342 (1974)
6. Baker, H.F.: Abel's theorem and the allied theory of theta functions. Cambridge (1897)
7. Bers, L.: Riemann surfaces. Courant Institut of Math. Sciences, NYU (1958)
8. Bers, L.: The space of Riemann surfaces. In: Proceedings of the International Congress of Mathematicians, pp. 349–361. Edinburg (1958)
9. Brill, A., Noether, M.: Über die algebraischen Functionen und ihre Anwendung in der Geometrie. Math. Ann. VII (1873)
10. Buchweitz, R.O.: On Zariski's criterion for equisingularity and non smoothable monomial curves (1980, preprint)
11. Cayley, A.: Sextactic points of a plane curve. Lond. Philos. Trans. **155** (1865)
12. Chevalley, C.: Introduction to the theory of algebraic functions of one variable. In: Mathematical Surveys, vol. 6. American Mathematical Society, NY (1961)
13. Chisini, O.: Intorno alla dimostrazione di un teorema di Noether. Boll. Un. Mat. It. **3**(197), 197–200 (1924)
14. Cipolla, I.: Sul numero dei punti di Weierstrass fra loro distinti di una curva algebrica. Rend. R. Ac. Lincei, 5a, Serie **XIV**, 210–214 (1905)
15. Coolidge, J.L.: A treatise on algebraic plane curve. Clarendon Press, Oxford (1931)
16. De Jonquières, J.: Mémoire sur les contacts multiples d'ordre quelconque des courbes... Jour. für Reine Und Angew. Math. **66**, 289–321 (1866)
17. Deligne, P., Mumford, D.: The irreducibility of the space of stable curves of given genus. Publ. Math. I.H.E.S. **36**, 75–110 (1969)
18. Diaz, S.: Tangent spaces in moduli via deformations with applications to Weierstrass points. Duke Math. J. **51**, 905–922 (1984)
19. Diaz, S.: Moduli of curves with two exceptional Weierstrass points. J. Differ. Geom. **20**, 471–478 (1984)
20. Edge, W.L.: A plane quartic with eight undulation points. Edinb. Math. Proc. **8**(2), 147–162 (1950)
21. Eisenbud, D., Harris, J.: Existence, decomposition and limits of certain Weierstrass points. Invent. Math. **87**, 495–515 (1987)
22. Farkas, H.M.: Special divisors and analytic subvarieties of Teichmüller space. Am. J. Math. **88**, 881–901 (1966)
23. Farkas, H.M.: Weierstrass points and analytic submanifolds of Teichmüller space. Proc. Am. Math. Soc. **20**, 35–38 (1969)
24. Farkas, H.M.: Remarks on automorphisms of compact Riemann surfaces. In: Discontinous groups and Riemann surfaces. Proc. Conf. Univ. Maryland, College Park 1973, pp. 121–144. AMS, vol. 79, Princeton University Press, Princeton (1974)
25. Farkas, H.M., Kra, I.: Riemann surfaces, GTM 71. Springer, Berlin (1980)
26. Fulton, W.: Hürwitz scheme and irreducibility of moduli of algebraic curves. Ann. Math. **90**, 542–575 (1969)
27. Galbura, G.: Il wronskiano di un sistema di sezioni di un fibrato vettoriale di rango 1 sopra una curva algebrica ed il relativo divisore di Brill-Severi. Ann. di Mat. Pura E Appl. **98**, 349–355 (1974)
28. Guerrero, I.: Automorphisms of compact Riemann surfaces and Weierstrass points. In: Proceedings of "Riemann surfaces and related topics". Stony Brook 1978, pp. 215–224. Princeton University Press, Princeton (1980)
29. Gunning, R.C.: Lectures on Riemann surfaces. Princeton Ac. Press, Princeton (1966)
30. Hartshorne, R.: Algebraic geometry, GTM 52. Spinger, Berlin (1977)
31. Haure M.M.: Recherches sur les points de Weierstrass d'une curbe plane algébriques. Ann. École Nor. Sup. **13**, 115–196 (1896)
32. Hensel, K., Landsberg, G.: Theorie der algebraischen Funktionen einer variabeln und ihre anwendung auf algebraische Kurven und abelsche Integrale. Leipzig (1902)
33. Horiuchi, R.: A note on a paper of Farkas. Proc. Jpn Acad. **45**, 859–860 (1969)
34. Hürwitz, A.: Über algebraische Gebilde mit eindeutigen Trasformationen in sich. Math. Ann. **41**, 391–430 (1893)
35. Jenkins, J.A.: Some remars on Weierstrass points. Proc. Am. Math. Soc. **44**, 121–122 (1974)
36. Kato, T.: On Weierstrass points whose first non-gaps are three. Jour. für Reine Und Angew. Math. **316**, 99–109 (1979)
37. Kato, T.: Non-hyperelliptic Weierstrass points of maximal weight. Math. Ann. **239**, 141–147 (1979)
38. Kato, T.: Weierstrass normal form of a Riemann surface and its applications (in Japanese). Sûgaku **32**, 73–75 (1980)

39. Komeda, J.: On Weierstrass points whose first non-gaps are four. J. Reine Angew. Math. **341**, 68–86 (1983)
40. Kuribayshi, I., Komya, K.: On Weierstrass points of non-hyperelliptic compact Riemann surfaces of genus three. Hiroshima Math. J. **7**, 743–768 (1977)
41. Laksov, D.: Weierstrass points on curves. In: Young tableaux and Schur functions in algebra and geometry, Torun 1980. Astérisque 87–88, pp. 221–242 (1981)
42. Laufer, H.: On generalized Weierstrass points and rings with no prime elements. In: Riemann surfaces and related problems, 1978. Ann. of Math. Studies. Princeton University Press, Princeton (1981)
43. Lax, R.F.: Weierstrass points of the universal curve. Math. Ann. **216**, 34–42 (1975)
44. Lax, R.F.: Gap sequences and moduli in genus 4. Math. Zeitschrift **175**, 67–75 (1980)
45. Lewittes, J.: Automorphisms of compact Riemann surfaces. Am. J. Math. **85**, 734–752 (1963)
46. Macbeath, A.M.: On a theorem of Hürwitz. Proc. Glasgow Math. Soc. **5**, 90–96 (1961)
47. MacDonald, I.: Symmetric products of an algebraic curve. Topology **1**, 319–343 (1962)
48. Maclaclhan, C.: A bound for the number of automorphisms of a compact Riemann surface. J. Lond. Math. Soc. **44**, 265–272 (1969)
49. Masoni, U.: Sopra alcune curve del quarto ordine dotate di punti di ondulazione. Rend. R. Acc. Napoli **21**, 45–69 (1882)
50. Mayer, A.: Compactification of the variety of moduli of curves. Notes of the Inst. for Adv. Study, pp. 6–15 (1969)
51. Miranda, R.: Algebraic curves and Riemann surfaces. In: Graduate Studies in Mathematics, vol. 5. AMS, Providence (1995)
52. Mumford, D.: Curves and their Jacobians. University of Michigan Press, Ann Arbor (1975)
53. Mumford, D., Fogarty, J., Kirwan, F.: Geometric invariant theory. In: Erg. der Math. und ihrer Grenzgebiete, vol. 34. Springer, Berlin (1994)
54. Neeman, A.: The distribution of Weierstrass points on a compact Riemann surface. Ann. Math. **120**, 317–328 (1984)
55. Noether, M.: Note über die algebraischen Curven, welche eine Schaar eindeutiger Transformationen in sich zulassen. Math. Ann. XX:59–62 (1882)
56. Noether, M.: Nachtrag zur "Note über die algebraischen Curven, welche eine Schaar eindeutiger Transformationen in sich zulassen. Math. Ann. XXI:138–140 (1883)
57. Noether, M.: Beweis und Erweiterung eines algebraisch-functionen-theoretischen Satzes des Herrn Weierstrass. Jour. für Reine Und Angew. Math. **97**, 224–229 (1884)
58. Olsen, B.A.: On higher order Weierstrass points. Ann. Math. **95**, 357–364 (1972)
59. Petersson, H.: Über Weirstrasspunkte und die expliziten Darstellungen der automorphen Formen von reeller Dimension. Math. Z. **52**, 32–59 (1950)
60. Pflaum, U.: The canonical constellation of k–Weierstrass points. Manusc. Math. **59**, 21–34 (1987)
61. Pinkham, H.C.: Deformation of algebraic varieties with G_m action. Astérisque 20, Paris Soc. Mathématique de France (1974)
62. Poincaré, H.: Sur un théorème de M. Fuchs. C. R. Ac. Sc Paris **99**, 75–77 (1884)
63. Rauch, H.E.: On the transcendental moduli of algebraic Riemann surfaces. Proc. Natl. Acad. Sci. USA **41**, 42–48 (1955)
64. Rauch, H.E.: On moduli in conformal mapping. Proc. Natl. Acad. Sci. USA **41**, 176–180 (1955)
65. Rauch, H.E.: Weierstrass points, branch points and moduli of Riemann surfaces. Comm. Pure Appl. Math. **12**, 543–560 (1959)
66. Rauch, H.E.: Variational methods in the problem of the moduli of Riemann surfaces. In: Contributions to Functions Theory, pp. 17–40. Tata Inst. Bombay (1960)
67. Riemann, B.: Theorie der Abelschen Functionen. Jour. für Reine Und Angew. Math. **54**, 115–155 (1857)
68. Rim, D.S., Vitulli, D.S.: Weierstrass points and monomial curves. J. Algebra **48**, 454–476 (1977)
69. Roch, G.: Über Theta-Functionen vielfacher Argumente. Jour. für Reine Und Angew. Math. **66**, 177–184 (1866)
70. Rosati, C.: Sopra un teorema di Noether. Boll. Un. Mat. It. 3 **197**, 162–167 (1924)
71. Schmidt, F.K.: Zur arithmetischen Theorie der algebraischen Functionen II. Math. Zeitschrift **45**, 75–96 (1939)
72. Schottky, F.H.: Über die conforme Abbildung mehrfach zusammenhängender ebene Flächen. Jour. für Reine Und Angew. Math. **83**, 300–351 (1877)
73. Schwarz, A.: Über diejenigen algebraischen Gleichungen zwischen zwei veränderlichen... Jour. für Reine Und Angew. Math. **87**, 139–145 (1879)

74. Segre, C.: Introduzione alla geometria sopra un ente algebrico semplicemente infinito. In: Annali Mat. Pura Appl. serie II, tomo, vol. XXII, pp. 42–142 (1894)
75. Segre, C.: Intorno ai punti di Weierstass di una curva algebrica. Atti Reale Acc. Lincei, Rendiconti Serie V **VIII**, 89–91 (1899)
76. Severi, F.: Trattato di Geometria algebrica, I. Bologna (1926)
77. Springer, G.: Introduction to Riemann surfaces. Addison-Wesley, Reading (1957)
78. Vainsencher, I.: Counting divisors with prescribed singularities. Trans. Am. Math. Soc. **267**, 399–422 (1981)
79. Vermeulen, A.M.: Weierstrass points of weight two on curves of genus three. Thesis Universitait van Amsterdam (1983)
80. Veronese, G.: Behandlugung der projectivischen Verhältnisse der Räume von verschiedenen Dimensionen... Math. Ann. **XIX**, 161–234 (1881)
81. Weierstrass, K.: Mathematische Werke, 7 vols (1894–1927) Berlin, reprint by G. Olms, Hildesheim (1967)
82. Weyl, H.: The concept of Riemann surface. Addison-Wesley, Reading (1955)
83. Wieman, A.: Zur Theorie der endlichen Gruppen von birationalen Trasformationen in der Ebene. Math. Ann. **98**, 195–210 (1897)