Miller Professorship application.

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**Introduction**

I am applying for a Miller professorship because, after a period of administrative work as Director of the Mathematical Sciences Research Institute (MSRI), a position I occupied for most of the years 1997-2022 (in which I felt that I accomplished my major goals, extending and greatly improving the physical facility and raising a very significant endowment) I have discovered a novel realm of phenomena in an area of mathematics that has been studied from a different point of view for about 70 years. I can guess at far more of the picture than I can establish mathematically, and I would like to have a period in which I can pursue this new line of thought with the rigor and intensity that I think it deserves.

In addition to the freedom to devote time to pure research, I would look forward to the connections with other faculty and postdocs provided by the Miller Foundation. I have had 36 successful PhD students so far, and my connections with them—in many cases still ongoing—are among the most satisfying aspects of my career. I also organized structured and much appreciated mentoring arrangements for the approximately 35 postdoctoral fel- lows who spend time at MSRI each year. I would view the opportunities to take part in the Miller arrangements for mentoring postdocs as a welcome extension of that activity.

Here is some background on my project. My fundamental area of study is algebraic geometry: the qualitative study of geometric forms defined by polynomial equations. Ex- amples of such forms that are familiar to every high school student are the curves in the plane that we call circle, ellipse, parabola, and hyperbola – they are called the “conic sec- tions” because they can all be constructed by slicing a circular cone. These are all part of one family algebraically too, defined by equations involving only linear and quadratic terms. They were intensely studied by mathematicians from the time of Euclid until the first third of the 19th century. Nowadays algebraic geometry is concerned with a far wider class of objects, often embedded in high-dimensional spaces, and this has led to wide field of applications from differential geometry and mathematical physics to artificial vision to robotic motion planning.

Underpinning much work of this kind is the need to understand solutions of systems of linear equations of a certain kind. Again high-school students typically learn to solve two

linear equations in two unknowns, say ax+by = c and dx+ey = f, and what they learn is actually more sophisticated than what was known to mathematicians before about 1700. But the modern theory that can be used in algebraic geometry requires the solution of such equations when the coefficients a,b,c,d,e,f are themselves varying (perhaps as polynomials in further variables) and the solutions to be found are also systems of polynomials. Such equations were first seriously studied by David Hilbert, the greatest mathematician of his day, around 1890, and became the center of a great deal of research starting in the 1950s. From the 1960s on, people began to understand the need to study situations where the coefficients and the solutions come from even more exotic domains, rings of functions on other algebraic varieties. In this context the complete resolution of one system of equations leads inevitably to an infinite sequence of further equations and solutions called in total an infinite free resolution.

It is in the context of these infinite free resolutions that my new observations, conjecture and theorems fall. Over the last 70 years, most of the work done in this area (and there has been a lot) has focused only on the numbers of independent solutions of each successive system of equations. Modern computational algebra and fast computers allow one to look much more deeply at the successive systems. Using that power my collaborators and I have seen many unexpected phenomena. In some cases, we have already been able to prove that what we are observing are general phenomena; but our observations greatly outstrip our current proofs. This is the area on which I would like to focus during a year as Miller professor.

**Technical descriptions**

The coefficients and solutions to the linear equations of which I spoke in the previous para- graphs are taken from a commutative Noetherian ring *R* (although the noncommutative case, particularly that of group algebras in finite characteristic, is also very interesting and open). The equations are usually presented as a homomorphism between finitely generated free *R*-modules *ϕ*1 : *F*1 *→ F*0 (in applications generally a presentation matrix for a module or ideal of interest). A *free resolution* is then a sequence of maps of free modules

*· · ·*  ... *Fi ϕi*... *Fi−*1 ... *· · ·*  ... *F*2 *ϕ*2... *F*1 *ϕ*1... *F*0

such that *ϕi*+1 has image exactly the kernel of *ϕi* for each *i*.

If we write each *ϕi* as a matrix, representing a set of homogeneous linear equations, the columns of *ϕi*+1 generate the solutions of the equations *ϕi*. In the circumstances of most interest (local or positively graded *R*) there is a good notion of a minimal set of generators, and if the columns of each *ϕi*+1 are minimal generators of the solutions to the equations *ϕi*, then the whole resolution is uniquely determined by *ϕ*1, and represents the complete solution of these equations. We will assume from now on that we have this minimality and uniqueness. It is a theorem of David Hilbert that the modules *Fi* are then all finitely generated.

If for *i* sufficiently large the *Fi* vanish, the result is called a *finite free resolution*, and in this case, on which I have published many papers, much of the work done concerns the nature of the maps *ϕi* and various kinds of equations that connect the entries of the matrices that appear there. For example, if *R* = *k*[*x*1*, . . . , xn*], where *k* is a field such as the real or complex numbers, the *Fi* must vanish for *i > n*, independent of *ϕ*1 – also a result of David Hilbert. In one of my earliest papers on the subject (joint with David Buchsbaum), I gave a new explanation for this, showing that the ideals in *R* generated by the entries of *ϕi* must grow, by a certain measure, with *i*, and for *i > n* there are no ideals as large as would be required.

In the infinite case the measure of the size of ideals (the *depth*, or *grade*) which is fundamental in the finite case, is unavailable in typical situations, and no such result is known. Instead, most of the work done concerns the rate of growth of the number of generators of *Fi* as *i* increases. **Modern computational methods** (based on the technique of Gr¨obner bases, and coded in the computer algebra system *Macaulay2*, of which I am a developer) have played a major role in this work. They allow one to look in detail at many examples of (finite parts) of infinite resolutions, and it seems that there are rich new phenomena waiting to be established.

As a Miller professor I would intensely investigate the following three closely related phenomena that I have observed in extensive computations and proven abstractly in cer- tain ranges. Each of them points toward the presence of undiscovered structures in the resolutions:

# Conjecture 1: Let *F* be a free resolution as above, and let *Ji* be the ideal generated by entries of a matrix representing *ϕi*. When *i* is sufficiently large, *Ji*+2 = *Ji*.

Partly through work of mine with Dao, this conjecture has been verified in two broad classes of examples, but our observations suggest sitll wider applicability.

# Conjecture 2: Let *R* be an Artinian local ring of embedding dimension *d* with residue field *k* and let *F* be the minimal free resolution of *k*. Let *S* be the set of positive integers such that *k* appears as a direct summand of the image of *ϕi*. Then *S* consists of all integers *≥* some integer *S*0 *possibly excepting d* + 1.

We have many examples showing that if true, this conjecture is sharp; all these patterns appear and we have proven the conjecture for two important classes of rings *R*.

**Problem 3: Let** *R* **be a Noetherian local ring with residue field** *k***. It was shown by John Tate and Tor Gulliksen that the minimal free resolution of** *k* **has the structure of a free commutative differential graded algebra. The construction is much more general, but the case of the residue field is the *only* one where it is known to be minimal. I recently found an unexpected family of examples. How far can this be extended?**