BEILINSON MONAD

1. Setup

We index homologically throughout. Fix a toric variety X with Cox ring $S = k[x_0, \dots, x_n]$ graded by $\mathrm{Pic}(X) = \mathbb{Z}^{\oplus r}$. Let $\mathcal{F}$ be a coherent sheaf on X. Write E for the Koszul dual of S. We equip E with a $\mathrm{Pic}(X) \times \mathbb{Z}$-grading such that $|e_i| = (-|x_i|, 1)$. Write $\omega := E^\vee$.

Remark 1.1. Before getting started, we record the following elementary observations. Of course, ω is an E-module with k-basis given by exterior polynomials in the e_i^*. Note that $|e_i^*| = (|x_i|, -1)$. The action of E on ω is by contraction. The $x_i \in S$ are also duals of the e_i, but we use different notation for the basis of ω to prevent confusion.

Another technical note: all E-modules are right modules. In particular, entries of matrices over E act on the right. This is also Macaulay2's convention. Note that this is the only way to make sense of the definition of the $\mathbf{R}$-functor in [EFS03]; if we apply the definition to a left E-module M, the maps in the complex $\mathbf{R}(M)$ are not E-linear. Nevertheless, sometimes we will multiply elements of E-modules on the left by elements of e (for instance, in the definition of the $\mathbf{L}$-functor below). When we do this, here is what we mean. When we write em for $e \in E$ and $m \in M$, where M is a right E-module, we mean $(-1)^{|e||m|}me$, where $|-|$ denotes the degree with respect to the second (standard) grading.

2. The Tate resolution

Define $\mathcal{O}_{X \times E}$ to be the sheaf of algebras on X given by

$$U \mapsto \Gamma(U, \mathcal{O}_X) \otimes_k E.$$

Let $\mathrm{Com}(X \times E)$ denote the category of complexes of $\mathcal{O}_{X \times E}$-modules.

Example 2.1. Define an object $\kappa^*(\mathcal{F}) \in \mathrm{Com}(X \times E)$ given by

$$\kappa^*(\mathcal{F})_i = \bigoplus_{l \in \mathrm{Pic}(X)} \mathcal{F}(-\ell) \otimes \omega(\ell, -i)$$

with differential given by $\sum_i x_i \otimes e_i$.

Fix once and for all an affine open cover $\{V_0, \dots, V_t\}$ of X. Given a sheaf $\mathcal{M}$ on X, denote by $\mathcal{C}^{\mathcal{M}}$ the Čech complex of $\mathcal{M}$ corresponding to this open cover. Recall that we're indexing homologically, so the Čech complex is concentrated in nonpositive degrees.

Given $\mathcal{G} \in \mathrm{Com}(X \times E)$, define a bicomplex as follows. The p^{th} column is given by $\mathcal{C}^{\mathcal{G}_p}$, but with the vertical differential multiplied by $(-1)^p$. The horizontal differential is induced by the differential on $\mathcal{G}$. Denote by $\tau_*(\mathcal{G}) \in \mathrm{Com}(E)$ the direct sum totalization of this bicomplex. In other words, $\tau_*(\mathcal{G})$ is given as follows: take an explicit Čech model of the derived global sections of the complex $\mathcal{G}$, considered as a complex of $\mathcal{O}_X$-modules, and then remember the E-module structure on the terms of the resulting complex.

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Let's look more closely at $(\tau_* \circ \kappa^*)(\mathcal{F})$. The bicomplex $\mathcal{B}$ such that

$$\mathrm{Tot}^{\oplus}(\mathcal{B}) = (\tau_* \circ \kappa^*)(\mathcal{F})$$

looks like

$$\begin{tikzcd}
 & {} \arrow[d] & & {} \arrow[d] & \\
\cdots & 0 \arrow[l] \arrow[d] & & 0 \arrow[ll] \arrow[d] & \cdots \arrow[l] \\
\cdots & \bigoplus_{\ell \in \mathrm{Pic}(X)} \mathcal{C}_0^{\mathcal{F}(-\ell)} \otimes \omega(\ell, -p) \arrow[l] \arrow[d] & & \bigoplus_{\ell \in \mathrm{Pic}(X)} \mathcal{C}_0^{\mathcal{F}(-\ell)} \otimes \omega(\ell, -(p-1)) \arrow[ll] \arrow[d] & \cdots \arrow[l] \\
 & \vdots \arrow[d] & & \vdots \arrow[d] & \\
\cdots & \bigoplus_{\ell \in \mathrm{Pic}(X)} \mathcal{C}_{-t}^{\mathcal{F}(-\ell)} \otimes \omega(\ell, -p) \arrow[l] \arrow[d] & & \bigoplus_{\ell \in \mathrm{Pic}(X)} \mathcal{C}_{-t}^{\mathcal{F}(-\ell)} \otimes \omega(\ell, -(p-1)) \arrow[ll] \arrow[d] & \cdots \arrow[l] \\
\cdots & 0 \arrow[l] \arrow[d] & & 0 \arrow[ll] \arrow[d] & \cdots, \arrow[l] \\
 & {} & & {} &
\end{tikzcd}$$

where the differential in the p^{th} column is the Čech differential, multiplied by $(-1)^p$, and the horiztonal differential is induced by the differential on $\kappa^*(\mathcal{F})$. We note that $\mathcal{B}$ is a bicomplex in the category of E-modules.

Choose k-vector spaces $G^{\ell}_{(p,q)}$ for all $-t \leq q \leq 0$, $p \in \mathbb{Z}$, and $\ell \in \mathrm{Pic}(X)$ such that

$$\mathcal{C}_q^{\mathcal{F}(-\ell)} = G^{\ell}_{(p,q)} \oplus d_{\mathrm{vert}} G^{\ell}_{(p,q+1)} \oplus H_q^{-\ell},$$

where $H_q^{\ell} \cong H^{-q}(X, \mathcal{F}(-\ell))$ (in this formula, we take $G^{\ell}_{(p,1)} := 0$ for all p and ℓ). We therefore have an E-linear splitting

$$\mathcal{B}_{p,q} = \bigoplus_{\ell \in \mathrm{Pic}(X)} (G^{\ell}_{(p,q)} \oplus d_{\mathrm{vert}} G^{\ell}_{(p,q+1)} \oplus H^{\ell}_q) \otimes \omega(\ell, -p)$$

of each column of $\mathcal{B}$. Applying [EFS03, Lemma 3.5], we conclude that $(\tau_*\kappa^*)(\mathcal{F})$ is homotopy equivalent to a complex of the form
(1)

$$\cdots \to \bigoplus_{q=0}^{t} \bigoplus_{\ell \in \mathrm{Pic}(X)} H^q(X, \mathcal{F}(-\ell)) \otimes \omega(\ell, -n-q) \to \bigoplus_{q=0}^{t} \bigoplus_{\ell \in \mathrm{Pic}(X)} H^{-q}(X, \mathcal{F}(-\ell)) \otimes \omega(\ell, -(n-1)-q) \to \cdots;$$

see the statement of [EFS03, Lemma 3.5] for an explicit formula for the differential in (1) in terms of the choices of splittings above.

Definition 2.2. The *Tate resolution of* $\mathcal{F}$, denoted $\mathrm{Tate}(\mathcal{F})$, is the complex (1).

Proposition 2.3. $\mathrm{Tate}(\mathcal{F})$ *is exact.*

Proof. The bicomplex $\mathcal{B}$ above is bounded and has exact rows, so this follows from an easy spectral sequence argument. $\square$

We highlight the following observation:

$$\dim_k H^q(X, \mathcal{F}(-\ell)) = \# \text{ of copies of } \omega(\ell, -n-q) \text{ appearing in } \mathrm{Tate}(\mathcal{F})_n.$$

3. The BGG functors L and R

We start with

$$\mathbf{L} : \mathrm{Com}(E) \to \mathrm{Com}(S).$$

Note that this is the same definition as in [EFS03]. For an E-module M concentrated in degree 0, $\mathbf{L}(M)$ is the complex with

$$\mathbf{L}(M)_q = \bigoplus_{d \in \mathrm{Pic}(X)} M_{(-d,-q)} \otimes_k S(d)$$

and differential

$$m \otimes s \mapsto \sum_{i=0}^{n} e_i m \otimes x_i s. \tag{2}$$

For a general complex $(C, \partial) \in \mathrm{Com}(E)$, we form the bicomplex

(3)

$$\begin{tikzcd}
 & \arrow[d] & \arrow[d] & \\
\cdots & \mathbf{L}(C_p)_q \arrow[l, "\partial"'] \arrow[d] & \mathbf{L}(C_{p+1})_q \arrow[l, "\partial"'] \arrow[d] & \cdots \arrow[l, "\partial"'] \\
\cdots & \mathbf{L}(C_p)_{q-1} \arrow[l, "\partial"'] \arrow[d] & \mathbf{L}(C_{p+1})_{q-1} \arrow[l, "\partial"'] \arrow[d] & \cdots \arrow[l, "\partial"'] \\
 & {} & {} &
\end{tikzcd}$$

and apply $\mathrm{Tot}^{\oplus}(-)$, where the vertical differential $\mathbf{L}(C_p)_q \to \mathbf{L}(C_p)_{q-1}$ is the dual Koszul map (2) multiplied by $(-1)^p$.

The functor

$$\mathbf{R} : \mathrm{Com}(S) \to \mathrm{Com}(E)$$

is defined as follows. Given $M \in \mathrm{Com}(S)$, define a bicomplex $\mathcal{B}$ with

$$\mathcal{B}_{p,q} = \bigoplus_{d \in \mathrm{Pic}(X)} (M_q)_d \otimes \omega(-d, -p)$$

here, $(M_q)_d$ denotes the internal degree d part of the q^{th} term of M. The horizontal differential is given by

$$m \otimes e \mapsto \sum_{i=0}^{n} x_i m \otimes e e_i,$$

and the p^{th} vertical differential is the map induced by the differential on M multiplied by $(-1)^p$. Notice that the rows are 1-periodic, up to twisting by $E(0,1)$. Define $\mathbf{R}(M) = \mathrm{Tot}^{\oplus}(\mathcal{B})$.

4. The U-functor

Let $\mathcal{L}(C)$ denote the bicomplex of $\mathcal{O}_X$-modules given by applying the associated sheaf functor to the bicomplex (3). Let $\mathcal{L}'(C)$ be the sub-bicomplex of $\mathcal{L}(C)$ given by taking summands of the form $C_{p,(-d,-q)} \otimes_k \mathcal{O}(d)$ with d effective. From now on, we'll write "$d \geq 0$" for "d effective". Here, p denotes homological degree, and $(-d,-q)$ denotes internal degree. We define a functor

$$\mathbf{U} : \mathrm{Com}(E) \to \mathrm{Com}(\mathbb{P})$$

to be given by $C \mapsto \mathrm{Tot}^{\oplus}(\mathcal{L}'(C))$.

♣♣♣ Michael: [Why $\mathrm{Tot}^{\oplus}$ and not Tot^{Π}? Why $d \geq 0$ and not $d \leq 0$? Give clean conceptual explanation for definition of $\mathbf{U}$-functor.]

Proposition 4.1. *The above definition of the* $\mathbf{U}$*-functor agrees with Daniel's.*

Proof. Daniel's definition is given by

$$\omega(i,j) \mapsto \mathcal{L}(\omega_{\leq i})(i)[-j].$$

(We're abusing notation here by identifying the 1-column bicomplex $\mathcal{L}(\omega_{\leq i})(i)$ with its totalization.) We have

$$\begin{aligned}
(\mathcal{L}(\omega_{\leq i})(i)[-j])_q &= \mathcal{L}(\omega_{\leq i})_{-j+q}(i) \\
&= \bigoplus_d (\omega_{\leq i})_{(-d,j-q)} \otimes \mathcal{O}(d+i) \\
&= \bigoplus_d (\omega_{\leq i})_{(i-d,j-q)} \otimes \mathcal{O}(d) \\
&= \bigoplus_{d\geq 0} (\omega_{\leq i})_{(i-d,j-q)} \otimes \mathcal{O}(d) \\
&= \bigoplus_{d\geq 0} \omega_{(i-d,j-q)} \otimes \mathcal{O}(d) \\
&= \mathbf{U}(\omega(i,j))_q.
\end{aligned}$$

And of course the maps in both complexes are identical as well. □

Question 4.2. *Does the* $\mathbf{U}$*-functor preserve homotopy? Check this.*

5. Computing sheaf cohomology over toric varieties

We start by describing a way of computing the linear part of the original [EFS03] Tate resolution over $\mathbb{P}^n$.

5.1. Classical setting.

In this subsection, when I write $\mathbf{L}$ and $\mathbf{R}$, I mean the versions of these maps from [EFS03]. Take a finitely generated S-module M. Choose $e \gg 0$ so that $\mathbf{R}(M_{\geq e})$ is exact. Let K be the kernel of the 0^{th} differential of $\mathbf{R}(M_{\geq e})$. Set

$$K^* = \mathrm{Hom}_E(K, E).$$

It follows from the Reciprocity Theorem [EFS03, Theorem 3.7(b)] that the natural map

$$K^* \to (\mathbf{R} \circ H_*(-) \circ \mathbf{L})(K^*)$$

is the linear part of a minimal injective resolution. Dualizing, we get the linear part of a minimal free resolution

$$(\mathbf{R} \circ H_*(-) \circ \mathbf{L})(K^*)^* \to K$$

Note that $K \cong K^{**}$ because every finitely generated E-module is MCM and hence reflexive. It follows that there is a canonical map

$$(\mathbf{R} \circ H_*(-) \circ \mathbf{L})(K^*)^* \to \mathbf{R}(M_{\geq e})$$

whose mapping cone is the linear part of the Tate resolution of $\widetilde{M}$. One could therefore compute the sheaf cohomology of $\widetilde{M}$ by reading off the Betti numbers from the above complex.

Example 5.1. Take $X = \mathbb{P}^1$ and $M = S/(x_1)$. We can take $e = 0$ in this case, and $\mathbf{R}(M)$ is the complex

$$\omega \xrightarrow{e_0} \omega(-1) \xrightarrow{e_0} \cdots$$

concentrated in nonpositive degrees. We have

$$K = k \cdot 1 \oplus k \cdot e_1^* \subseteq \omega. \tag{4}$$

An easy computation shows that K^* is a cyclic module concentrated in degrees $-2, -3$ whose annihilator is the ideal generated by e_0 in E.

Applying $\mathbf{L}$ to K^*, we get the complex

$$0 \to S(2) \xrightarrow{x_1} S(3) \to 0$$

concentrated in degrees $-2, -3$. Taking homology, we get the object $S(3)/x_1$ concentrated in degree -3. Applying $\mathbf{R}$ to this, we get

$$\omega(3) \xrightarrow{e_0} \omega(2) \xrightarrow{e_0} \cdots$$

concentrated in nonpositive degrees. Dualizing, we get

$$\cdots \xrightarrow{e_0} \omega^*(-2) \xrightarrow{e_0} \omega^*(-3) \tag{5}$$

concentrated in nonnegative degrees. It's easy to check that there is an isomorphism

$$\omega \cong \omega^*(-4)$$

(the left side is a rank 1 free E-module with generator in degree 2, and the right side is a rank 1 free E-module with generator in degree -2). Rewriting (5), we get

$$\cdots \xrightarrow{e_0} \omega(2) \xrightarrow{e_0} \omega(1) \tag{6}$$

There is a morphism of complexes from (4) to (6)

$$\begin{tikzcd}
\cdots \arrow[r, "e_0"] & \omega(2) \arrow[r, "e_0"] & \omega(1) \arrow[d, "e_0"] & & \\
& & \omega \arrow[r, "e_0"] & \omega(-1) \arrow[r, "e_0"] & \cdots
\end{tikzcd}$$

and the mapping cone of this morphism gives the Tate resolution:

$$\cdots \xrightarrow{e_0} \omega(1) \xrightarrow{e_0} \omega \xrightarrow{e_0} \omega(-1) \xrightarrow{e_0} \cdots$$

of $\widetilde{M}$ (I'm missing some signs). In fact, since the Tate resolution happens to be linear in this case, we get the entire Tate resolution.

5.2. **Toric setting.** We now mimic the above construction in the toric setting. Let X be a toric variety as above, and let M be an S-module. Choose $e \gg 0$ so that $\mathbf{R}(M_{\geq e})$ is exact. Recall that the complex $\mathbf{R}(M_{\geq e})$ is the complex given as follows:

$$\mathbf{R}(M_{\geq e})_p = \bigoplus_{d \geq e} (M_{\geq e})_d \otimes \omega(-d, -p),$$

with differential $m \otimes e \mapsto \sum_{i=0}^{n} x_i m \otimes ee_i$. Take K to be the kernel of the restriction of the differential to $M_e \otimes \omega(-e, 0)$.

Claim 5.2. *There is a natural map*

$$f : ((\mathbf{R} \circ H_*(-) \circ \mathbf{L})(K^*))^* \to \mathbf{R}(M_{\geq e})$$

such that $\operatorname{cone}(f)$ *is the linear part of the Tate resolution of* $\widetilde{M}$.

Example 5.3. Take $X = \mathbb{P}(1, m)$ and $M = S/(x_0)$. We can take $e = 0$ in this case, and $\mathbf{R}(M)$ is the complex

$$\cdots \to \bigoplus_{d \geq 0} \omega(-dm, -1) \to \bigoplus_{d \geq 0} \omega(-dm, 0) \to \bigoplus_{d \geq 0} \omega(-dm, 1) \to \cdots, \tag{7}$$

where the differential is given by multiplication by e_1. We have

$$K = k \cdot 1 \oplus k \cdot e_0^* \subseteq \omega(0, 0).$$

An easy computation shows K^* is cyclic and concentrated in degrees $-m-1$, $-m-2$. The complex $\mathbf{L}(K^*)$ looks like

$$0 \to S(m+1) \xrightarrow{x_0} S(m+2) \to 0,$$

and its homology is $S(m+2)/x_0$ concentrated in degree -3. Applying $\mathbf{R}$ to this, and reindexing, we get the complex

$$\cdots \to \bigoplus_{d \leq -1} \omega((d+2)m+2, -4) \to \bigoplus_{d \leq -1} \omega((d+2)m+2, -3) \to \bigoplus_{d \leq -1} \omega((d+2)m+2, -2) \to \cdots,$$

with $\bigoplus_{d \leq -1} \omega((d+2)m+2, -3)$ in homological degree 0, and with differential again given by multiplication by e_1. Dualizing, we get the complex

$$\cdots \to \bigoplus_{d \leq -1} \omega^*(-(d+2)m-2, 2) \to \bigoplus_{d \leq -1} \omega^*(-(d+2)m-2, 3) \to \bigoplus_{d \leq -1} \omega^*(-(d+2)m-2, 4) \to \cdots$$

Now note that $\omega(2m+2, -4) \cong \omega^*$, so we may rewrite this as

$$\cdots \to \bigoplus_{d \leq -1} \omega(-dm, -2) \to \bigoplus_{d \leq -1} \omega(-dm, -1) \to \bigoplus_{d \leq -1} \omega(-dm, 0) \to \cdots. \tag{8}$$

We have a morphism of complexes from (8) to (7)

$$\begin{tikzcd}
\cdots \arrow[r] & \bigoplus_{d \leq -1} \omega(-dm, -2) \arrow[r] \arrow[d] & \bigoplus_{d \leq -1} \omega(-dm, -1) \arrow[r] \arrow[d] & \bigoplus_{d \leq -1} \omega(-dm, 0) \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & \bigoplus_{d \geq 0} \omega(-dm, -1) \arrow[r] & \bigoplus_{d \geq 0} \omega(-dm, 0) \arrow[r] & \bigoplus_{d \geq 0} \omega(-dm, 1) \arrow[r] & \cdots
\end{tikzcd}$$

given as follows: each $\omega(m, i)$ maps to $\omega(0, i+1)$ via multiplication by e_1, and the rest of the summands are sent to 0. The mapping cone of this morphism is the complex

$$\cdots \to \bigoplus_{d \in \mathbb{Z}} \omega(-dm, -1) \to \bigoplus_{d \in \mathbb{Z}} \omega(-dm, 0) \to \bigoplus_{d \in \mathbb{Z}} \omega(-dm, 1) \to \cdots .$$

with differential given by multiplication by $\pm e_1$ (I do not vouch for all the signs in the above calculation, by the way). A quick look at our definition of the Tate resolution confirms that this complex coincides with the Tate resolution of $\widetilde{M}$. Since f is a quasi-isomorphism, the cone is exact, so we're getting the entire Tate resolution, not just the minimal part, as expected.

6. Beilinson monad

Recall that $\mathrm{Tate}(\mathcal{F})$ is homotopy equivalent to the complex $(\tau_* \circ \kappa^*(\mathcal{F}))$, which has the form

$$\cdots \to T(0, -1) \to T \to T(0, 1) \to \cdots ,$$

where $T = \bigoplus_{i=0}^{t} \bigoplus_{\ell \in \mathrm{Pic}(X)} \mathcal{C}_{-i}^{\mathcal{F}(-\ell)} \otimes \omega(\ell, -i)$. The (p, q) term of $\mathcal{L}'((\tau_* \circ \kappa^*(\mathcal{F})))$ is

$$\bigoplus_{d \geq 0} T(0, -p)_{(-d, -q)} \otimes \mathcal{O}(d) = \bigoplus_{d \geq 0} T_{(-d, -p-q)} \otimes \mathcal{O}(d),$$

and so

$$(\mathbf{U} \circ \tau_* \circ \kappa^*)(\mathcal{F})_m = \bigoplus_{p \in \mathbb{Z}} \bigoplus_{d \geq 0} T_{(-d, -m)} \otimes \mathcal{O}(d).$$

Equip each $(\mathbf{U} \circ \tau_* \circ \kappa^*)(\mathcal{F}))_m$ with a $k[u]$-module structure determined by the following "shift" operation: if $t = (\ldots, t_{-1}, t_0, t_1, \ldots) \in \bigoplus_{p \in \mathbb{Z}} \bigoplus_{d \geq 0} T_{(-d, -m)} \otimes \mathcal{O}(d)$,

$$u(t)_p = (-1)^{m-p} t_{p-1}.$$

Proposition 6.1. *The differential on $(\mathbf{U} \circ \tau_* \circ \kappa^*)(\mathcal{F}))$ is $k[u]$-linear.*

Proof. I'm writing down the proof to make sure I got the sign right in the definition of the u-action. We prove that the action of u commutes with both horizontal and vertical differentials. Write d_T for the differential on $(\mathbf{U} \circ \tau_* \circ \kappa^*)(\mathcal{F}))$ and d_K for the dual Koszul differential. We have

$$\begin{aligned} d_{\mathrm{hor}}(u \cdot (\ldots, t_{-1}, t_0, t_1, \ldots)) &= d_{\mathrm{hor}}(\ldots, (-1)^{m-1} t_{-2}, (-1)^m t_{-1}, (-1)^{m-1} t_0, \ldots) \\ &= (\ldots, (-1)^m d_T(t_{-3}), (-1)^{m-1} d_T(t_{-2}), (-1)^m d_T(t_{-1}), \ldots) \\ &= u \cdot (\ldots, d_T(t_{-2}), d_T(t_{-1}), d_T(t_0), \ldots) \\ &= u \cdot d_{\mathrm{hor}}(\ldots, t_{-1}, t_0, t_1, \ldots), \end{aligned}$$

and

$$\begin{aligned} d_{\mathrm{ver}}(u \cdot (\ldots, t_{-1}, t_0, t_1, \ldots)) &= d_{\mathrm{ver}}(\ldots, (-1)^{m-1} t_{-2}, (-1)^m t_{-1}, (-1)^{m-1} t_0, \ldots) \\ &= (\ldots, (-1)^m d_K(t_{-2}), (-1)^m d_K(t_{-1}), (-1)^m d_K(t_0), \ldots) \\ &= u \cdot (\ldots, -d_K(t_{-1}), d_K(t_0), -d_K(t_1), \ldots) \\ &= u \cdot d_{\mathrm{ver}}(\ldots, t_{-1}, t_0, t_1, \ldots). \end{aligned}$$

$\square$

So, $(\mathbf{U} \circ \tau_* \circ \kappa^*)(\mathcal{F}))$ is a complex of $X \times \mathbb{A}^1$-modules. In fact, since the action of u is invertible, $(\mathbf{U} \circ \tau_* \circ \kappa^*)(\mathcal{F}))$ is a complex of $X \times \mathbb{G}_m$-modules. Define

$$\mathbf{BM}(\mathcal{F}) := (\mathbf{U} \circ \tau_* \circ \kappa^*)(\mathcal{F}))/(u-1).$$

We have an isomorphism

$$\mathbf{BM}(\mathcal{F})_m \cong \bigoplus_{d \geq 0} T_{(-d,-m)} \otimes \mathcal{O}(d) \tag{9}$$

given by representing each class in $\mathbf{BM}(\mathcal{F})_m$ by an element concentrated in the $p = 0$ summand of $\bigoplus_{p \in \mathbb{Z}} \bigoplus_{d \geq 0} T_{(-d,-m)} \otimes \mathcal{O}(d)$. Via this isomorphism, the differential on $\mathbf{BM}(\mathcal{F})$ is given by $ud_T + d_K$, where d_T is the Tate differential and d_K is the dual Koszul differential.

Theorem 6.2. *The complex* $\mathbf{BM}(\mathcal{F})$ *is a monad with homology* $\mathcal{F}$.

Example 6.3. Take $X = \mathbb{P}(w)$, with w some positive integer, and take $\mathcal{F} = \mathcal{O}$. So X is a stacky point. If we set $T = \bigoplus_{i \in \mathbb{Z}} \omega(i, 0)$, then

$$(\tau_* \circ \kappa^*)(\mathcal{O})) = \cdots \xrightarrow{e} T(0,-1) \xrightarrow{e} T \xrightarrow{e} T(0,1) \xrightarrow{e} \cdots$$

We therefore have

$$(\mathbf{U} \circ \tau_* \circ \kappa^*)(\mathcal{O}))_m = \bigoplus_{p \in \mathbb{Z}} \bigoplus_{d \geq 0} \bigoplus_{i \in \mathbb{Z}} \omega_{(i-d,-m)} \otimes \mathcal{O}(d) = \begin{cases} \bigoplus_{p \in \mathbb{Z}} \bigoplus_{d \geq 0} \omega_{(0,0)} \otimes \mathcal{O}(d), & m = 0; \\ \bigoplus_{p \in \mathbb{Z}} \bigoplus_{d \geq 0} \omega_{(w,-1)} \otimes \mathcal{O}(d), & m = 1; \\ 0, & \text{else.} \end{cases}$$

Taking coinvariants of the $\mathbb{G}_m$-action and applying the isomorphism (9) gives the complex

$$0 \to \bigoplus_{d \geq 0} \omega_{(w,-1)} \otimes \mathcal{O}(d) \xrightarrow{\begin{pmatrix} -1 & 0 & 0 & 0 & \dots \\ x & -1 & 0 & 0 & \dots \\ 0 & x & -1 & 0 & \dots \\ 0 & 0 & x & -1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}} \bigoplus_{d \geq 0} \omega_{(0,0)} \otimes \mathcal{O}(d) \to 0$$

whose homology is $\mathcal{O}$ in degree 0 and 0 elsewhere, as expected.

7. Proof of Theorem 6.2

Let $\mathcal{R} \in \mathrm{Com}(X \times X)$ be Daniel's resolution of the diagonal. To prove Theorem 6.2, it suffices to give a homotopy equivalence

$$\mathbf{BM}(\mathcal{F}) \simeq \mathbf{R}\pi_{2*}(\pi_1^* \mathcal{F} \otimes \mathcal{R}).$$

The notation $\mathbf{R}\pi_{2*}$ is misleading: we are not working at the level of derived categories. Rather, we use the notation $\mathbf{R}\pi_{2*}$ to denote the Čech model for $\mathbf{R}\pi_{2*}$ induced by the affine open cover $\{V_0, \dots, V_t\}$ of X chosen above.

The rough idea is to define a map

$$\mathrm{id} \times \overline{\mathbf{U}} : \mathrm{Com}(X \times E) \dashrightarrow \mathrm{Com}(X \times X)$$

such that the diagram

$$\begin{tikzcd}
 & \mathrm{Com}(X \times E) \arrow[r, "\tau_*"] \arrow[d, dashed, "\mathrm{id} \times \overline{\mathbf{U}}"] & \mathrm{Com}(E) \arrow[d, "\overline{\mathbf{U}}"] \\
\mathrm{coh}(X) \arrow[ur, "\kappa^*"] \arrow[dr, "\pi_1^*(-)\otimes \mathcal{R}"'] & & \\
 & \mathrm{Com}(X \times X) \arrow[r, "\mathbf{R}\pi_{2*}"] & \mathrm{Com}(X)
\end{tikzcd}$$

commutes up to homotopy, where $\overline{\mathbf{U}}$ denotes the functor given by applying the $\mathbf{U}$-functor and modding out by the relation $u-1$, as discussed above. Here is how to define $\mathrm{id} \times \overline{\mathbf{U}}$ on the image of κ^* (which is all we need). Recall that

$$\kappa^*(\mathcal{F})_i = \bigoplus_{l \in \mathrm{Pic}(X)} \mathcal{F}(-\ell) \otimes \omega(\ell, -i)$$

and the differential is the dual Koszul map. We apply "$\mathrm{id} \times \mathbf{U}$" to $\kappa^*(\mathcal{F})$ to get the complex whose m^{th} term is

$$\bigoplus_{p \in \mathbb{Z}} \bigoplus_{d \geq 0} \bigoplus_{l \in \mathrm{Pic}(X)} \omega_{(\ell-d,-m)} \otimes \mathcal{F}(-\ell) \boxtimes \mathcal{O}(d)$$

with differential $\sum_{i=0}^n e_i \otimes x_i + (-1)^p e_i \otimes y_i$. This complex has $\mathbb{G}_m$-action just as $(\mathbf{U} \circ \tau_* \circ \kappa^*)(\mathcal{O}))$ does. Taking coinvariants and applying an isomorphism similar to (9), we arrive at the complex with m^{th} term

$$\bigoplus_{d \geq 0} \bigoplus_{l \in \mathrm{Pic}(X)} \omega_{(\ell-d,-m)} \otimes \mathcal{F}(-\ell) \boxtimes \mathcal{O}(d)$$

and m^{th} differential $\sum_{i=0}^n (-1)^m e_i \otimes x_i + e_i \otimes y_i$.

Proposition 7.1. $(\mathrm{id} \times \overline{\mathbf{U}})(\kappa^*(\mathcal{O}))$ *coincides with Daniel's resolution of the diagonal.*

Proof. Fill in. Should be easy. Also, I think the argument Daniel uses to show his complex is a resolution of the diagonal works to prove $(\mathrm{id} \times \overline{\mathbf{U}})(\kappa^*(\mathcal{O}))$ is as well. □

Remark 7.2. Suppose $X = \mathbb{P}(w_0, \ldots, w_n)$. Set $w = \sum w_i$. Suppose we change the definition of $(\mathrm{id} \times \overline{\mathbf{U}})(\kappa^*(\mathcal{O}))$ slightly so that we get a complex that looks like this:

(10)
$$0 \to \bigoplus_{0 \leq d \leq w-1} \bigoplus_{0 \leq l \leq d} \omega_{(d-\ell,-(n-1))} \otimes \mathcal{O}(-d,\ell) \to \cdots \to \bigoplus_{0 \leq d \leq w-1} \bigoplus_{0 \leq l \leq d} \omega_{(d-\ell,0)} \otimes \mathcal{O}(-d,\ell) \to 0.$$

Here are the changes:

- restricted the ranges on d and ℓ
- changed $\omega_{(l-d,-m)}$ to $\omega_{(d-l,-m)}$, and
- changed $\mathcal{O}(-\ell, d)$ to $\mathcal{O}(-d,\ell)$.

I claim that (10) is Canonaco-Karp's resolution of the diagonal. I checked this for $\mathbb{P}(1,2)$ and it's correct on the nose.

The first change in the list above amounts to changing the definitions of κ and $\mathbf{U}$ so that the "irrelevant" Koszul complexes are removed. The second change arises from our choice to take $d \geq 0$ summands rather than $d \leq 0$ in the definition of the $\mathbf{U}$-functor; this choice seems to create a conflict with Canonaco-Karp. Not sure if this was just an arbitrary choice or if one is better than the other. The third change just switches the order of the tensor factors, keeping the second change in mind.

Example 7.3. Let's check this for $X = \mathbb{P}(w)$. Reading off the formula, we get

$$0 \to \bigoplus_{d\geq 0} \omega_{(w,-1)} \otimes \mathcal{O}(d, -(d+w)) \xrightarrow{\begin{pmatrix} y & 0 & 0 & 0 & \dots \\ -x & y & 0 & 0 & \dots \\ 0 & -x & y & 0 & \dots \\ 0 & 0 & -x & y & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}} \bigoplus_{d\geq 0} \omega_{(0,0)} \otimes \mathcal{O}(d, -d) \to 0.$$

The homology is $\mathcal{O} \oplus \cdots \oplus \mathcal{O}(w-1, -(w-1))$, and this is indeed the diagonal in this case (this is not entirely trivial to check). Notice this precisely recovers Daniel's "1-variable" example when $w = 1$.

Proposition 7.4. $(\mathbf{R}\pi_{2*} \circ (\mathrm{id} \times \overline{\mathbf{U}}) \circ \kappa^*)(\mathcal{F}) = (\overline{\mathbf{U}} \circ \tau_* \circ \kappa^*)(\mathcal{F})$.

Proof. Let's start by computing the right hand side. As above, set

$$T = \bigoplus_{i=0}^{t} \bigoplus_{\ell \in \mathrm{Pic}(X)} \mathcal{C}_{-i}^{\mathcal{F}(-\ell)} \otimes \omega(\ell, -i),$$

so that

$$(\overline{\mathbf{U}} \circ \tau_* \circ \kappa^*)(\mathcal{F}) = \overline{\mathbf{U}}(\cdots \to T(0,-1) \to T \to T(0,1) \to \cdots).$$

We have

$$\begin{aligned}(\overline{\mathbf{U}} \circ \tau_* \circ \kappa^*)(\mathcal{F})_m &= \bigoplus_{d\geq 0} T_{(-d,-m)} \otimes \mathcal{O}(d) \\ &= \bigoplus_{d\geq 0} \bigoplus_{i=0}^{t} \bigoplus_{\ell \in \mathrm{Pic}(X)} \omega_{(l-d,-i-m)} \otimes \mathcal{C}_{-i}^{\mathcal{F}(-\ell)} \otimes \mathcal{O}(d),\end{aligned}$$

and the differential is $ud_T + d_K$. On the other hand $(\mathbf{R}\pi_{2*} \circ (\mathrm{id} \times \overline{\mathbf{U}}) \circ \kappa^*)(\mathcal{F})$ is the sheaf given as follows. Let W be an open set in X. We abuse notation slightly and write $\mathcal{C}^{\mathcal{O}(d)|_W}$ for the Čech complex on $\mathcal{O}(d)|_W$ corresponding to the open cover

$$\{V_0 \cap W, \dots, V_t \cap W\}.$$

We recall that the natural map

$$\mathcal{C}^{\mathcal{F}(-\ell)} \otimes \mathcal{C}^{\mathcal{O}(d)|_W} \to \mathcal{C}^{\mathcal{F}(-\ell) \boxtimes \mathcal{O}(d)|_W}$$

is a homotopy equivalence, where the target is the Čech complex associated to the open cover $\{V_i \times (V_j \cap W)\}_{0\leq i,j\leq t}$. Form a bicomplex with p^{th} column given by

$$\bigoplus_{d\geq 0} \bigoplus_{\ell \in \mathrm{Pic}(X)} \omega_{(l-d,-p)} \otimes \mathcal{C}^{\mathcal{F}(-\ell)} \otimes \mathcal{C}^{\mathcal{O}(d)|_W},$$

with vertical differential multiplied by $(-1)^p$, and horizontal differential induced by the differential on $((\mathrm{id} \times \overline{\mathbf{U}}) \circ \kappa^*)(\mathcal{F})$. Applying $\mathrm{Tot}^{\oplus}$ to this complex gives the value of the

$(\mathbf{R}\pi_{2*} \circ (\mathrm{id} \times \overline{\mathbf{U}}) \circ \kappa^*)(\mathcal{F})$ at W, up to homotopy equivalence. Explicitly, the value at W is the complex whose m^{th} term is

$$\bigoplus_{i=0}^{t^2} \bigoplus_{d \geq 0} \bigoplus_{\ell \in \mathrm{Pic}(X)} \omega_{(l-d,-i-m)} \otimes (\mathcal{C}^{\mathcal{F}(-\ell)} \otimes \mathcal{C}^{\mathcal{O}(d)|_W})_{-i},$$

with the induced differential. It suffices to check that if we sheafify the presheaf

$$W \mapsto \bigoplus_{i=0}^{t^2} \omega_{(l-d,i-n)} \otimes (\mathcal{C}^{\mathcal{F}(-\ell)} \otimes \mathcal{C}^{\mathcal{O}(d)|_W})_{-i},$$

we get

$$\bigoplus_{i=0}^{t} \omega_{(l-d,i-n)} \otimes \mathcal{C}^{\mathcal{F}(-\ell)}_{-i} \otimes \mathcal{O}(d).$$

And of course we need to check the differentials coincide as well. Need to fill in the rest of the details, but I think this is clear. □

Proof of Theorem 6.2. Combine Propositions 7.1 and 7.4. □

References

[EFS03] D. Eisenbud, G. Floystad, and F.-O. Schreyer, *Sheaf cohomology and free resolutions over exterior algebras*, Transactions of the American Mathematical Society **355** (2003), no. 11, 4397–4426.