DIFFERENTIAL MODULES

The main goals this note are to develop the algebraic foundations we will need for studying differential modules. In particular, we have two keys goals. The first is to develop a theory of minimal free resolutions for differential modules, which is the focus of Section 2). The second goal is to use differential modules to extend the BGG functors to the toric setting, which is the focus of Section 3). Section 3 does not depend on Section 2.

1. Notation/Terminology

In this section, we lay out key definitions regarding differential modules over a ring. Let A be an abelian group, and let R be an A-graded ring.

Definition 1.1. A *differential R-module* is an A-graded R-module M equipped with a homogeneous endomorphism ∂ such that $\partial^2 = 0$. We write $|M|$ for the underlying graded R-module of D. A *morphism* $M \to M'$ of differential R-modules is a degree 0 morphism $|M| \to |M'|$ that commutes with the differentials.

We let $\mathrm{Mod}(R)$ denote the category of A-graded R-modules and $\mathrm{DM}(R)$ the category of differential R-modules. We denote by $H(M)$ the homology of a differential module M. A morphism of differential modules is a *quasi-isomorphism* if it induces an isomorphism on homology.

There are natural notions of homotopy and mapping in the category of DM-modules. In particular: a *homotopy* of morphisms $f, f' : M \to M'$ of differential R-modules is a morphism $h : |M| \to |M'|$ such that $f - f' = h\partial + \partial' h$. The *mapping cone* of a morphism $f : M \to M'$ of differential modules is the differential module $(M \oplus M', \begin{pmatrix} -\partial & 0 \\ f & \partial' \end{pmatrix})$.

We are interested in differential modules equipped with a filtration, in the following sense (see also [ABI07]).

Definition 1.2. A *flag* is a differential R-module (M, ∂) equipped with a filtration $\mathcal{F}^\bullet M$ such that

- $\mathcal{F}^i M \subseteq \mathcal{F}^{i-1} M$
- $\partial(\mathcal{F}^i M) \subseteq \mathcal{F}^{i+1} M$,
- $\bigcup_i \mathcal{F}^i M = M$, and
- $\mathcal{F}^{>0} M = 0$.

We say a flag is *locally finitely generated* if each piece of the associated graded is finitely generated. A *split flag* is a differential module (M, ∂) equipped with a decomposition $M = \bigoplus_{i \in \mathbb{Z}} M^i$ such that the filtration $\mathcal{F}^i M = \bigoplus_{j > i} M^j$ makes (M, ∂) a flag.

Remark 1.3. A split flag (M, ∂) such that $\partial(M^i) \subseteq M^{i+1}$ is the same thing as a complex of R-modules that is 0 in positive degrees.

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2. Free covers

Definition 2.1. Let $M \in \mathrm{DM}(R)$, and let $F \in \mathrm{DM}(R)$ be a split flag such that $|F|$ is free. We say a morphism $\epsilon : F \to M$ of differential modules is a *free cover* if

(1) ϵ is a quasi-isomorphism (of differential modules), and
(2) $F^0 \hookrightarrow F \xrightarrow{\epsilon} M$ factors through $\ker(\partial_M) \hookrightarrow M$.

A free cover is something like a free resolution of a differential module.[1] The main idea of this section is something like "minimal free covers exist and are unique", and we aim to prove this in as general of a context as possible. This is captured in Theorem 2.5.

But first, just as in classical homological algebra, it is essential to observe that morphisms of differential modules may be lifted to free covers in a unique way, up to homotopy. More generally, we have the following, which is a DM analogue of lifting morphisms from a projective complex to an exact complex. ♣♣♣ Daniel: [Note: as in the classical case, since the proof builds the lift iteratively, it really seems to require a flag structure on F.]

Proposition 2.2. *Let $M, N \in \mathrm{DM}(R)$, and suppose we have morphisms $\epsilon : F \to M$, $\eta : G \to N$ of differential modules, where F is a split flag such that $|F|$ is projective, and η is a quasi-isomorphism. Assume the composition*

$$F^0 \hookrightarrow F \xrightarrow{\epsilon} M$$

factors through $\ker(\partial_M) \hookrightarrow M$. *Given a morphism $f : M \to N$ of differential modules, there exists a morphism*

$$\widetilde{f} : \mathrm{cone}(\epsilon) \to \mathrm{cone}(\eta)$$

of differential modules of the form $\begin{pmatrix} \alpha & 0 \\ \gamma & f \end{pmatrix}$. *Moreover, $\widetilde{f}$ is unique up to homotopy.*

♣♣♣ Michael: [This generalizes to differential modules in any abelian category. In particular, we can prove it simultaneously in the graded and local cases, if we want.] ♣♣♣ Daniel: [We struggled to get this proof right. And something about it still feels unsatisfying, at least to me, as it is much more complicated than the corresponding proof for chain complexes.]

Proof. Set $F' := \mathrm{cone}(\epsilon)$ and $G' := \mathrm{cone}(\eta)$. We begin by defining $g_0 : F^0 \to G'$ such that the map

$$\widetilde{f_0} : F^0 \oplus M \to G'$$

given by $(x, m) \mapsto g_0(x) + (0, f(m))$ is a morphism of differential modules, where $F^0 \oplus M$ is equipped with the differential $\begin{pmatrix} 0 & 0 \\ \epsilon & \partial_M \end{pmatrix}$, i.e. the restriction of $\partial_{F'}$ to $F^0 \oplus M$. We have a diagram

$$\begin{tikzcd} & G' \arrow[d, "\partial_{G'}"] \\ F^0 \arrow[r, "\beta"] & \mathrm{im}(\partial_{G'}) = \ker(\partial_{G'}), \end{tikzcd}$$

[1] We have chosen the phrase "free cover" instead of "free resolution" because we will occasionally be transforming complexes into DM-modules, and we felt the phrase "resolution" could cause confusion. But we don't love the name.

where $\beta(x) = (0, (f\epsilon)(x))$. Note that β does indeed land in $\ker(\partial_{G'})$: we have

$$(\partial_{G'}\beta)(x) = (0, (\partial_N f\epsilon)(x)) = (0, (f\partial_M\epsilon)(x)) = 0;$$

the last equality holds since the map $F^0 \hookrightarrow F \xrightarrow{\epsilon} M$ factors through $\ker(\partial_M) \hookrightarrow M$. Since F^0 is projective, we get an induced map

$$g_0 : F^0 \to G'$$

making the diagram commute. One easily checks that g_0 has the desired property: if $(x, m) \in F^0 \oplus M$,

$$\begin{aligned}(\widetilde{f}_0\partial_{F'})(x, m) &= (0, (f\epsilon)(x)) + (0, (f\partial_M)(m)) \\ &= \beta(x) + (0, (\partial_N f)(m)) \\ &= (\partial_{G'}g_0)(x) + \partial_{G'}(0, f(m)) \\ &= (\partial_{G'}\widetilde{f}_0)(x, m).\end{aligned}$$

Now, suppose we have

$$g_i : F^{\geq i} \to G'$$

for all $i > -n$ (where $n > 0$) such that

- the map $\widetilde{f}_i : F^{\geq i} \oplus M \to G'$ given by $(x, m) \mapsto g_i(x) + (0, f(m))$ is a morphism of differential modules (where $F^{\geq i} \oplus M$ is equipped with the differential given by the restriction of $\partial_{F'}$), and
- $g_i|_{F^{\geq j}} = g_j$ for all $j > i$ (set $g_{>0} := 0$).

We have a diagram

$$\begin{tikzcd} & G' \arrow[d, "\partial_{G'}"] \\ F^{-n} \arrow[r, "\gamma"] & \mathrm{im}(\partial_{G'}) = \ker(\partial_{G'}), \end{tikzcd}$$

where $\gamma(x) = (\widetilde{f}_{-(n-1)}\partial_{F'})(x, 0)$; the map γ lands in $\ker(\partial_{G'})$, since

$$(\partial_{G'}\widetilde{f}_{-(n-1)}\partial_{F'})(x, 0) = (\widetilde{f}_{-(n-1)}\partial_{F'}\partial_{F'})(x, 0) = 0.$$

Since F^{-n} is projective, we obtain a map $\widetilde{\gamma} : F^{-n} \to G'$ making the diagram commute. We define $g_{-n} : F^{\geq -n} \to G'$ to be the map

$$\begin{pmatrix} g_{-(n-1)} & \widetilde{\gamma} \end{pmatrix} : F^{\geq -(n-1)} \oplus F^{-n} \to G'.$$

We now verify that the map

$$\widetilde{f}_{-n} : F^{\geq -n} \oplus M \to G',$$

given by $(x,m) \mapsto g_{-n}(x) + (0, f(m))$, is a morphism of differential modules. Let $(x,m) \in F^{\geq -n} \oplus M$. We have:

$$\begin{aligned}
(\widetilde{f}_{-n}\partial_{F'})(x,m) &= g_{-n}(-\partial_F(x)) + (0, (f\epsilon)(x) + (f\partial_M)(m)) \\
&= \widetilde{f}_{-n}(-\partial_F(x), \epsilon(x)) + (0, (f\partial_M)(m)) \\
&= (\widetilde{f}_{-n}\partial_{F'})(x,0) + (0, (f\partial_M)(m)) \\
&= (\partial_{G'}\widetilde{f}_{-n})(x,0) + \widetilde{f}_{-n}(0, \partial_M(m)) \\
&= (\partial_{G'}\widetilde{f}_{-n})(x,0) + (\widetilde{f}_{-n}\partial_{F'})(0,m) \\
&= (\partial_{G'}\widetilde{f}_{-n})(x,m).
\end{aligned}$$

To construct $\widetilde{f}$, we let g be the colimit of the g_i and take $\widetilde{f} : F' \to G'$ to be given by $(x,m) \mapsto g(x) + (0, f(m))$.

We now show $\widetilde{f}$ is unique up to homotopy. Without loss, assume $f = 0$; we will show $\widetilde{f}$ is null homotopic. We again proceed by induction. We have a diagram

$$\begin{tikzcd}
 & G' \arrow[d, "\partial_{G'}"] \\
F^0 \arrow[r, "g_0"] & \ker(\partial_{G'}),
\end{tikzcd}$$

since $(\partial_{G'}g_0)(x) = \beta(x) = 0$ for all $x \in F^0$. Since F^0 is projective, we obtain a map $s_0 : F^0 \to G'$ making the diagram commute. Suppose we have maps $s_i : F^{\geq i} \to G'$ for $i > -n$ (where $n > 0$) such that

- $g_i = \partial_{G'}s_i - s_{i+1}\partial_F$ (set $s_{>0} := 0$), and
- $s_i|_{F^{\geq j}} = s_j$ for all $j > i$.

In particular, let's record the relation

$$g_{-(n-1)} = \partial_{G'}s_{-(n-1)} - s_{-(n-2)}\partial_F. \tag{1}$$

We have a diagram

$$\begin{tikzcd}
 & G' \arrow[d, "\partial_{G'}"] \\
F^{\geq -n} \arrow[r, "g_{-n}+s_{-(n-1)}\partial_F"] & \ker(\partial_{G'}),
\end{tikzcd}$$

since, by (1), we have

$$\begin{aligned}
\partial_{G'}(g_{-n} + s_{-(n-1)}\partial_F) &= \partial_{G'}g_{-n} + (g_{-(n-1)} + s_{-n-2}\partial_F)\partial_F \\
&= \partial_{G'}g_{-n} + g_{-(n-1)}\partial_F,
\end{aligned}$$

and

$$\begin{aligned}
(\partial_{G'}g_{-n})(x) &= (\partial_{G'}\widetilde{f}_{-n})(x,0) \\
&= (\widetilde{f}_{-n}\partial_{F'})(x,0) \\
&= \widetilde{f}_{-n}(-\partial_F(x), \epsilon(x)) \\
&= -(g_{-(n-1)}\partial_F)(x).
\end{aligned}$$

Define $s_{-n} : F^{\geq -n} \to G'$ making the diagram commute. Let s denote the colimit of the s_i. We have

$$g = \partial_{G'} s - s\partial_F.$$

Now take $h : F' \to G'$ to be the map given by $(x, m) \mapsto s(x)$, and observe that

$$\begin{aligned} \widetilde{f}(x, m) &= g(x) \\ &= (\partial_{G'} s)(x) - (s\partial_F)(x) \\ &= (\partial_{G'} h)(x, m) + (h\partial_{F'})(x, m). \end{aligned}$$

□

2.1. **Minimal free covers.** We now aim to prove the existence and uniqueness of minimal free covers. To get the definition right, we will assume $A = \mathbb{Z}^{\oplus r}$ for some $r \geq 1$. Further, we require that

- R_0 is a field, and
- The set $\{a \in A : R_a \neq 0\}$ forms a pointed cone in A.

Set $R_+ := \bigoplus_{a \in A \setminus \{0\}} R_a$. We will call a morphism $f : M \to N$ of A-graded R-modules *minimal* if $f(M) \subseteq R_+ N$. ♣♣♣ Michael: [At the moment, we still don't know how to prove existence and uniqueness in the local case, as opposed to the graded case. But many of the pieces go through immediately.]

Definition 2.3. Let $M \in \mathrm{DM}(R)$. We say a free cover $F \to M$ is *minimal* if ∂_F is minimal. We say the free cover is *locally finitely generated* if the flag F is.

Our next goal is to prove that minimal free covers of finitely generated differential R-modules exist and are unique up to isomorphism. Here, we have to be careful what we mean: given $M \in \mathrm{DM}(R)$, an *isomorphism of free covers* $F \xrightarrow{\epsilon} M$, $G \xrightarrow{\eta} M$ is an isomorphism

$$\mathrm{cone}(\epsilon) \xrightarrow{\cong} \mathrm{cone}(\eta)$$

of differential modules. This is a more general notion than a commutative triangle

$$\begin{tikzcd} F \arrow[r, "\epsilon"] \arrow[d, "\cong"'] & M \\ G \arrow[ur, "\eta"'] & \end{tikzcd}$$

The reader might find it surprising that our notion of an isomorphism of free covers does not involve the flag structures on F and F'. This is necessary for uniqueness of minimal free covers, as the following example illustrates.

Example 2.4. Let M be a finitely generated A-graded R-module. Let

$$\cdots \to F^{-2} \to F^{-1} \to F^0 \to D \to 0$$

be a minimal A-graded free resolution of $|M|$. Then we can build a minimal free cover $G \to D$ with $G^i = F^i$ for $i \geq 0$. But we could also choose $H \to D$ where $H^i = F^{i+5}$ or F^{2i}.

Here is a more exotic example. Let $A = \mathbb{Z}$ and $R = k[x, y]$, where k is a field and $|x| = 1 = |y|$. Take $M = R/(x, y)$ with trivial differential. Define a flag F as follows:

- $F^0 = R$,

- $F^{-1} = R(-1)$,
- $F^{-2} = R(-1)$,
- $F^{-3} = R(-2)$, and
- the differential is given by

$$F^{-1} \oplus F^{-2} \xrightarrow{\begin{pmatrix} x & y \end{pmatrix}} F^0$$

and

$$F^{-3} \xrightarrow{\begin{pmatrix} -y \\ x \end{pmatrix}} F^{-1} \oplus F^{-2}.$$

Of course, F is just the Koszul complex, considered as a split flag with a strange flag structure. The natural map $F \to M$ is a minimal free cover of M.

Theorem 2.5. *Let $M \in \mathrm{DM}(R)$, and assume $|M|$ is finitely generated. M admits a locally finitely generated minimal free cover that is unique up to isomorphism.*

Proof. Choose representatives in M of a minimal generating set for $H(M)$, say of cardinality r_0 (if $H(M) = 0$, $r_0 = 0$). Use them to give a map of differential modules

$$d_0 : F^0 := R^{\oplus r_0} \to M$$

inducing a surjection on homology, where F^0 is equipped with the trivial differential. Next, choose representatives in $F^0 \oplus M$ of a minimal generating set for $H(\mathrm{cone}(d_0))$, say of cardinality r_1, and use them to give a map

$$d_1 : F^{-1} := R^{\oplus r_1} \to \mathrm{cone}(d_0)$$

inducing a surjection on homology, where F^{-1} is equipped with the trivial differential. We will write d_1 as the matrix $\begin{pmatrix} d_1^{F^0} \\ d_1^M \end{pmatrix}$. Notice that $d_1^{F^0}$ is minimal, i.e. the lifts of the minimal generating set for $H(\mathrm{cone}(d_0))$ lie in $R_+F^0 \oplus M$. Continue in this manner to obtain a split flag

$$F = \cdots \oplus F^{-2} \oplus F^{-1} \oplus F^0.$$

The differential ∂_F is given by the matrix

$$\begin{pmatrix}
\vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\
\cdots & 0 & 0 & \cdots & 0 & 0 & 0 \\
\cdots & d_n^{F^{-(n-1)}} & 0 & \cdots & 0 & 0 & 0 \\
\cdots & d_n^{F^{-(n-2)}} & d_{n-1}^{F^{-(n-2)}} & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\
\cdots & d_n^{F^{-1}} & d_{n-1}^{F^{-1}} & \cdots & d_2^{F^{-1}} & 0 & 0 \\
\cdots & d_n^{F^0} & d_{n-1}^{F^0} & \cdots & d_2^{F^0} & d_1^{F^0} & 0
\end{pmatrix},$$

where the $d_j^{F^{-i}}$, for $0 \le i < j$, are defined in the same way as $d_1^{F^0}$ was defined above. Since the $d_i^{F^j}$ are minimal, so is ∂_F. There is a quasi-isomorphism $\epsilon : F \to M$ given by the infinite row matrix

$$\epsilon : F \xrightarrow{\begin{pmatrix} \cdots & d_n^M & d_{n-1}^M & \cdots & d_1^M & d_0 \end{pmatrix}} M.$$

Let $\eta : G \to M$ be another minimal free cover of M. Set $F' := \operatorname{cone}(\epsilon)$ and $G' := \operatorname{cone}(\eta)$. We now show $F' \cong G'$. Applying Proposition 2.2 to the identity map on M, we get maps $f : F' \to G'$ and $g : G' \to F'$ such that fg is homotopic to $\mathrm{id}_{G'}$ and gf is homotopic to $\mathrm{id}_{F'}$. Choose an endomorphism h of $|F'|$ such that

$$\mathrm{id}_{F'} - gf = h\partial_{F'} + \partial_{F'} h. \tag{2}$$

Write

$$F = \bigoplus_{a \in A} R(a)^{\oplus i_a},$$

Equip the grading group A with the increasing filtration

$$A_n = \{a \in A : R_a \not\subseteq (R_+)^n\},$$

and set

$$F_{\geq n} := \bigoplus_{a \in A_n} R(a)^{\oplus i_a} \subseteq F.$$

Notice that any homogeneous endomorphism of F' maps $F_{\geq n} \oplus M$ into itself.

It follows from (2) that $\det(gf) = 1$ modulo the maximal ideal R_+, since $\partial_{F'}$ is minimal. The same is true of the restriction $(gf)_{\geq n}$ of gf to $F_{\geq n} \oplus M$. It follows immediately from the graded version of Nakayama's Lemma that $(gf)_{\geq n}$ is surjective, and hence it must be injective as well (we're using here that $F_{\geq n} \oplus M$ is finitely generated). Taking a colimit as $n \to \infty$, we conclude gf is an isomorphism. Similarly, fg is an isomorphism. ☐

♣♣♣ Daniel: [Our current thought is that examples like those above regarding the non-uniqueness flag structures suggest that minimal free covers can only be unique if we ignore the flag structure. But we might also try to go in the other direction: perhaps by only considering flag structures with some additional structure, we could also get uniqueness that way.]

3. BGG for differential modules

Tjhe goal of this section is to define variants of $\mathbf{R}$ and $\mathbf{L}$ functors from [EFS03], but in the context of differential modules. The main result is a similar equivalence of categories. Fix $n \geq 1$, and let k be a field. Let W be an $(n+1)$-dimensional k-vector space with basis $\{e_0, \dots, e_n\}$, and denote the corresponding basis of $V = W^*$ by $\{x_0, \dots, x_n\}$. Set $S = \operatorname{Sym}(V)$ and $E = \bigwedge W$. Let $d_0, \dots, d_n \in A = \mathbb{Z}^{\oplus r}$, and equip S with an A-grading by setting $|x_i| = d_i$. Equip E with a grading by setting $|e_i| = -|x_i|$. Denote by $\mathfrak{m}_S$ (resp. $\mathfrak{m}_E$) the homogeneous maximal ideal of S (resp E).

Let M be an A-graded S-module. If $a \in A$ and $m \in M_a$ is nonzero, we define the *total degree* of m to be

$$||m|| := \max\{i : m \in \mathfrak{m}_S^{i+1} M\}.$$

Define the total degree similarly for A-graded E-modules.

We have a functor

$$\mathbf{R} : \mathrm{DM}(S) \to \mathrm{DM}(E)$$

given as follows. Let $E^\vee$ denote the k-linear dual of E. If $(M, d_M) \in \mathrm{DM}(S)$ and $a \in A$,

$$\mathbf{R}(M, d_M)_a = E^\vee \otimes_k M_a,$$

and the differential ∂ on $\mathbf{R}(M, d_M)$ is given by

$$\partial(f \otimes m) = \sum_{i=0}^{n} f(- \wedge e_i) \otimes x_i m + (-1)^{||m||} f \otimes d_M(m).$$

Going the other direction, we have a functor

$$\mathbf{L} : \mathrm{DM}(E) \to \mathrm{DM}(S)$$

given by

$$\mathbf{L}(N, d_N)_a = S \otimes_k N_{-a},$$

with differential ∂ on $\mathbf{L}(N, d_N)$ given by

$$\partial(s \otimes n) = \sum_{i=0}^{n} s x_i \otimes n e_i + (-1)^{||n||} s \otimes d_N(n).$$

Proposition 3.1. $\mathbf{L}$ *is left adjoint to* $\mathbf{R}$.

Proof. We have

$$\begin{aligned}
\mathrm{Hom}_S(\mathbf{L}(N), M) &= \mathrm{Hom}_S(S \otimes_K N, M) \\
&\cong \mathrm{Hom}_K(N, M) \\
&\cong \mathrm{Hom}_E(N, \mathrm{Hom}_K(E, M)) \\
&= \mathrm{Hom}_E(N, \mathbf{R}(M)).
\end{aligned}$$

We need only check that this isomorphism determines a bijection between maps that respect the A-grading and preserve the differentials. This follows exactly as in [EFS03, Proposition 2.1]. $\square$

Given an A-graded ring R, let $\mathrm{D}_{\mathrm{diff}}(R)$ denote the derived category of differential R-modules, as defined in [Rou06, Section 4.1.2]. Let $\mathrm{D}^b_{\mathrm{diff}}(R)$ denote the bounded derived category of differential R-modules, i.e. the subcategory of $\mathrm{D}_{\mathrm{diff}}(R)$ given by differential modules whose homology is finitely generated.

Proposition 3.2. $\mathbf{R}$ *induces an equivalence*

$$\mathrm{D}_{\mathrm{diff}}(S) \xrightarrow{\simeq} \mathrm{D}_{\mathrm{diff}}(E).$$

We also have an induced equivalence

$$\mathrm{D}^b_{\mathrm{diff}}(S) \xrightarrow{\simeq} \mathrm{D}^b_{\mathrm{diff}}(E).$$

Before we start the proof, we recall the following definition:

Definition 3.3 ([ABI07] Section 1.9)**.** Let R be a commutative ring, let (X, d_X) be a complex of R-modules, and let (M, d_M) be a differential R-module. We define $X \boxtimes_R M$ to be the differential module $\bigoplus_{n \in \mathbb{Z}} X_n \otimes_R M$ with differential

$$x \otimes m \mapsto d_X(x) \otimes m + (-1)^{|x|} x \otimes d_M(m).$$

Proof of Proposition 3.2. Let K denote the Koszul complex

$$\cdots \to S \otimes_k \bigwedge^2 W \to S \otimes_k W \to S \to 0.$$

It suffices to show there are natural isomorphisms

$$K \boxtimes_k - \cong \mathbf{LR}(-) \text{ and } K \boxtimes_k - \cong \mathbf{RL}(-),$$

and this is straightforward. $\square$

References

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