EXAMPLES OF RESOLUTIONS OF THE DIAGONAL IN WEIGHTED PROJECTIVE SPACE

Below are computations of resolutions of the diagonal over $\mathbb{P}(1,2)$ and $\mathbb{P}(1,1,2)$. I highlighted the answers in blue if you want to just skip to them.

1. $\mathbb{P}(1,2)$

Let x and y be the coordinates on $\mathbb{P} = \mathbb{P}(1,2)$, with $|x| = 1$ and $|y| = 2$. Let $\mathcal{K}$ denote the Koszul complex

$$0 \to \mathcal{O}(-3) \to \mathcal{O}(-1) \oplus \mathcal{O}(-2) \to \mathcal{O} \to 0$$

on $\mathbb{P}$, indexed cohomologically, where $\mathcal{O}$ is in degree 0. Following the notation in [CK08, Section 3], we define complexes $\mathcal{M}_i \subseteq \mathcal{K}(-i)$ for $i = 0, -1, -2$ in the following way:

- $\mathcal{M}_0 = \mathcal{O}$,
- $\mathcal{M}_{-1} = \mathcal{O} \xrightarrow{x} \mathcal{O}(1)$,
- $\mathcal{M}_{-2} = \mathcal{O}(1) \oplus \mathcal{O} \xrightarrow{\begin{pmatrix} x & y \end{pmatrix}} \mathcal{O}(2)$.

In Canonaco-Kemp's resolution of the diagonal, the $\mathcal{M}_i$ play the role that the powers of the cotangent bundle play in Beilinson's resolution of the diagonal on projective space.

Let x, y, x', y' be the coordinates on $\mathbb{P} \times \mathbb{P}$, with $|x| = 1 = |x'|$ and $|y| = 2 = |y'|$. We start by setting

$$\mathcal{R}_0 := \mathcal{O} \boxtimes \mathcal{M}_0 = \mathcal{O} \in \mathrm{D}^b(\mathbb{P} \times \mathbb{P}).$$

Then we will build complexes $\mathcal{R}_{-1}$ and $\mathcal{R}_{-2}$ using an iterated mapping cone construction. The complex $\mathcal{R}_{-2}$ will be our resolution of the diagonal.

We define a map

$$\alpha_{-1} : \mathcal{O}(-1) \boxtimes \mathcal{M}_{-1}[-1] \to \mathcal{R}_0$$

of complexes in the following way:

$$
\begin{array}{ccccc}
0 \longrightarrow & \mathcal{O}(-1,0) & \xrightarrow{-x'} & \mathcal{O}(-1,1) & \longrightarrow 0 \\
& \downarrow{-x} & & \downarrow & \\
0 \longrightarrow & \mathcal{O} & \longrightarrow 0 & \longrightarrow 0 &
\end{array}
$$

(the sign in the top differential is coming from the shift by -1). Set $\mathcal{R}_{-1} := \mathrm{cone}(\alpha_{-1})$. That is, $\mathcal{R}_{-1}$ is the complex

$$0 \to \mathcal{O}(-1,0) \xrightarrow{\begin{pmatrix} x' \\ -x \end{pmatrix}} \mathcal{O}(-1,1) \oplus \mathcal{O} \to 0,$$

concentrated in degrees -1 and 0.

Next, we define a map

$$\alpha_{-2} : \mathcal{O}(-2) \boxtimes \mathcal{M}_{-2}[-1] \to \mathcal{R}_{-1}$$

to be given by

$$
\begin{array}{ccccccc}
0 & \longrightarrow & \mathcal{O}(-2,1) \oplus \mathcal{O}(-2,0) & \xrightarrow{\begin{pmatrix} -x' & -y' \end{pmatrix}} & \mathcal{O}(-2,2) & \longrightarrow & 0 \\
& & \downarrow{\scriptsize\begin{pmatrix} -x & 0 \\ 0 & -y \end{pmatrix}} & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O}(-1,0) & \xrightarrow{\begin{pmatrix} x' \\ -x \end{pmatrix}} & \mathcal{O}(-1,1) \oplus \mathcal{O} & \longrightarrow & 0.
\end{array}
$$

Our resolution of the diagonal is $\mathrm{cone}(\alpha_{-2})$. Explicitly, it's the complex

$$
0 \to \mathcal{O}(-2,1) \oplus \mathcal{O}(-2,0) \oplus \mathcal{O}(-1,0) \xrightarrow{\begin{pmatrix} x' & y' & 0 \\ -x & 0 & x' \\ 0 & -y & -x \end{pmatrix}} \mathcal{O}(-2,2) \oplus \mathcal{O}(-1,1) \oplus \mathcal{O} \to 0
$$

concentrated in degrees -1 and 0.

2. $\mathbb{P}(1,1,2)$

Let x, y, z be coordinates on $\mathbb{P} = \mathbb{P}(1,1,2)$, with $|x| = 1 = |y|$ and $|z| = 2$. Let $\mathcal{K}$ denote the Koszul complex

$$
0 \to \mathcal{O}(-4) \to \mathcal{O}(-3) \oplus \mathcal{O}(-3) \oplus \mathcal{O}(-2) \to \mathcal{O}(-1) \oplus \mathcal{O}(-1) \oplus \mathcal{O}(-2) \to \mathcal{O} \to 0,
$$

as above. This time, we have complexes $\mathcal{M}_i \subseteq \mathcal{K}(-i)$ for $i = 0, -1, -2, -3$:

- $\mathcal{M}_0 = \mathcal{O}$,

- $\mathcal{M}_{-1} = \mathcal{O} \oplus \mathcal{O} \xrightarrow{\begin{pmatrix} x & y \end{pmatrix}} \mathcal{O}(1)$,

- $\mathcal{M}_{-2} = \mathcal{O} \xrightarrow{\begin{pmatrix} -y \\ x \\ 0 \end{pmatrix}} \mathcal{O}(1) \oplus \mathcal{O}(1) \oplus \mathcal{O} \xrightarrow{\begin{pmatrix} x & y & z \end{pmatrix}} \mathcal{O}(2)$,

- $\mathcal{M}_{-3} = \mathcal{O}(1) \oplus \mathcal{O} \oplus \mathcal{O} \xrightarrow{\begin{pmatrix} -y & -z & 0 \\ x & 0 & -z \\ 0 & x & y \end{pmatrix}} \mathcal{O}(2) \oplus \mathcal{O}(2) \oplus \mathcal{O}(1) \xrightarrow{\begin{pmatrix} x & y & z \end{pmatrix}} \mathcal{O}(3)$.

Let x, y, z, x', y', z' be the coordinates on $\mathbb{P} \times \mathbb{P}$, with the obvious degrees. As above, we set

$$
\mathcal{R}_0 = \mathcal{O} \boxtimes \mathcal{M}_0 = \mathcal{O} \in \mathrm{D}^b(\mathbb{P} \times \mathbb{P}).
$$

We have a map

$$
\alpha_{-1} : \mathcal{O}(-1) \boxtimes \mathcal{M}_{-1}[-1] \to \mathcal{R}_0
$$

of complexes given by

$$
\begin{array}{ccccccc}
0 & \longrightarrow & \mathcal{O}(-1,0) \oplus \mathcal{O}(-1,0) & \xrightarrow{\begin{pmatrix} -x' & -y' \end{pmatrix}} & \mathcal{O}(-1,1) & \longrightarrow & 0 \\
& & \downarrow{\scriptsize\begin{pmatrix} -x & -y \end{pmatrix}} & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O} & \longrightarrow & 0 & \longrightarrow & 0.
\end{array}
$$

We set $\mathcal{R}_{-1}$ to be $\mathrm{cone}(\alpha_{-1})$, i.e. the complex

$$0 \to \mathcal{O}(-1,0) \oplus \mathcal{O}(-1,0) \xrightarrow{\begin{pmatrix} x' & y' \\ -x & -y \end{pmatrix}} \mathcal{O}(-1,1) \oplus \mathcal{O} \to 0,$$

concentrated in degrees -1 and 0.

Next, we have a map

$$\alpha_{-2} : \mathcal{O}(-2) \boxtimes \mathcal{M}_{-2}[-1] \to \mathcal{R}_{-1}$$

given by

$$
\begin{array}{ccccccc}
0 & \longrightarrow & \mathcal{O}(-2,0) & \xrightarrow{\begin{pmatrix} y' \\ -x' \\ 0 \end{pmatrix}} & \mathcal{O}(-2,1) \oplus \mathcal{O}(-2,1) \oplus \mathcal{O}(-2,0) & \xrightarrow{\begin{pmatrix} -x' & -y' & -z' \end{pmatrix}} & \mathcal{O}(-2,2) & \longrightarrow 0 \\
& & \Big\downarrow{\begin{pmatrix} y \\ -x \end{pmatrix}} & & \Big\downarrow{\begin{pmatrix} -x & -y & 0 \\ 0 & 0 & -z \end{pmatrix}} & & \Big\downarrow & \\
0 & \longrightarrow & \mathcal{O}(-1,0) \oplus \mathcal{O}(-1,0) & \xrightarrow[\begin{pmatrix} x' & y' \\ -x & -y \end{pmatrix}]{} & \mathcal{O}(-1,1) \oplus \mathcal{O} & \longrightarrow & 0 & \longrightarrow 0.
\end{array}
$$

We take $\mathcal{R}_{-2} = \mathrm{cone}(\alpha_{-2})$, which is the complex

$$0 \to \mathcal{O}(-2,0) \xrightarrow{\begin{pmatrix} -y' \\ x' \\ 0 \\ y \\ -x \end{pmatrix}} \mathcal{O}(-2,1) \oplus \mathcal{O}(-2,1) \oplus \mathcal{O}(-2,0) \oplus \mathcal{O}(-1,0) \oplus \mathcal{O}(-1,0)$$

$$\xrightarrow{\begin{pmatrix} x' & y' & z' & 0 & 0 \\ -x & -y & 0 & x' & y' \\ 0 & 0 & -z & -x & -y \end{pmatrix}} \mathcal{O}(-2,2) \oplus \mathcal{O}(-1,1) \oplus \mathcal{O}$$

$$\to 0,$$

concentrated in degrees $-2, -1, 0$.

Finally, we compute $\mathcal{R}_{-3}$, which is our resolution of the diagonal. We need to construct a map

$$\alpha_{-3} : \mathcal{O}(-3) \boxtimes \mathcal{M}_{-3}[-1] \to \mathcal{R}_{-2}.$$

To shorten notation, let $C_{-2}, C_{-1},$ and C_0 denote the terms of $\mathcal{R}_{-2}$. Our map α_{-3} is given by

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & 0 & \longrightarrow & \mathcal{O}(-3) \boxtimes \mathcal{M}_{-3}^{-2} & \longrightarrow & \mathcal{O}(-3) \boxtimes \mathcal{M}_{-3}^{-1} & \longrightarrow & \mathcal{O}(-3) \boxtimes \mathcal{M}_{-3}^{0} & \longrightarrow 0 \\
& & \Big\downarrow & & \Big\downarrow{\alpha_{-3}^{-1}} & & \Big\downarrow{\alpha_{-3}^{0}} & & \Big\downarrow & \\
0 & \longrightarrow & \mathcal{R}_{-2}^{-2} & \longrightarrow & \mathcal{R}_{-2}^{-1} & \longrightarrow & \mathcal{R}_{-2}^{0} & \longrightarrow & 0 & \longrightarrow 0,
\end{array}
$$

where

$$\alpha_{-3}^{-1} = \begin{pmatrix} y & 0 & 0 \\ -x & 0 & 0 \\ 0 & -x & -y \\ 0 & z & 0 \\ 0 & 0 & z \end{pmatrix},$$

and

$$\alpha_{-3}^{0} = \begin{pmatrix} -x & -y & 0 \\ 0 & 0 & -z \\ 0 & 0 & 0 \end{pmatrix}.$$

Our resolution of the diagonal is $\mathrm{cone}(\alpha_{-3})$. Explicitly, it's the complex

$$0 \to (\mathcal{O}(-3) \boxtimes \mathcal{M}_{-3}^{-2}) \oplus \mathcal{R}_{-2}^{-2} \xrightarrow{\begin{pmatrix} -y' & -z' & 0 & 0 \\ x' & 0 & -z' & 0 \\ 0 & x' & y' & 0 \\ y & 0 & 0 & -y' \\ -x & 0 & 0 & x' \\ 0 & -x & -y & 0 \\ 0 & z & 0 & y \\ 0 & 0 & z & -x \end{pmatrix}} (\mathcal{O}(-3) \boxtimes \mathcal{M}_{-3}^{-1}) \oplus \mathcal{R}_{-2}^{-1}$$

$$\xrightarrow{\begin{pmatrix} x' & y' & z' & 0 & 0 & 0 & 0 & 0 \\ -x & -y & 0 & x' & y' & z' & 0 & 0 \\ 0 & 0 & -z & -x & -y & 0 & x' & y' \\ 0 & 0 & 0 & 0 & 0 & -z & -x & -y \end{pmatrix}} (\mathcal{O}(-3) \boxtimes \mathcal{M}_{-3}^{0}) \oplus \mathcal{R}_{-2}^{0} \to 0.$$

References

[CK08] A. Canonaco and R. L. Karp, *Derived autoequivalences and a weighted Beilinson resolution*, Journal of Geometry and Physics **58** (2008), no. 6, 743–760.