

BEILINSON MONAD

1. THE \mathbf{L} -FUNCTOR

We work over weighted projective space for now. So $S = k[x_0, \dots, x_n]$ with $|x_i| = w_i$. Denote by \mathbb{P} the associated weighted projective space. We equip E with the $\mathbb{Z} \times \mathbb{Z}$ -grading such that $|e_i| = (-w_i, 1)$. We recall the definition of the functor

$$\mathbf{L} : \text{Com}(E) \rightarrow \text{Com}(S)$$

from Daniel's notes. Here, $\text{Com}(E)$ denotes the category of complexes of $\mathbb{Z} \times \mathbb{Z}$ -graded E -modules, and $\text{Com}(S)$ is the category of complexes of \mathbb{Z} -graded S -modules. For M an E -module concentrated in degree 0, $\mathbf{L}(M)$ is the complex with

$$\mathbf{L}(M)_q = \bigoplus_d M_{(-d, -q)} \otimes_k S(d)$$

and differential

$$(1) \quad m \otimes s \mapsto \sum_{i=0}^n e_i m \otimes x_i s.$$

♣♣♣ **Michael:** [I'm using homological indexing so that comparing to M2 will be easier.] For a general complex $(C, \partial) \in \text{Com}(E)$, we form the bicomplex

$$(2) \quad \begin{array}{ccccccc} & & \downarrow & & \downarrow & & \\ \cdots & \xleftarrow{\partial} & \mathbf{L}(C_p)_q & \xleftarrow{\partial} & \mathbf{L}(C_{p+1})_q & \xleftarrow{\partial} & \cdots \\ & & \downarrow & & \downarrow & & \\ \cdots & \xleftarrow{\partial} & \mathbf{L}(C_p)_{q-1} & \xleftarrow{\partial} & \mathbf{L}(C_{p+1})_{q-1} & \xleftarrow{\partial} & \cdots \\ & & \downarrow & & \downarrow & & \end{array}$$

and apply $\text{Tot}^\oplus(-)$. Note that the vertical differential $\mathbf{L}(C_p)_q \rightarrow \mathbf{L}(C_p)_{q-1}$ is the dual Koszul map (1) multiplied by $(-1)^p$.

Let $\mathcal{L}(C)$ denote the bicomplex of $\mathcal{O}_{\mathbb{P}}$ -modules given by applying the associated sheaf functor to the bicomplex (2). Let $\mathcal{L}'(C)$ be the sub-bicomplex of $\mathcal{L}(C)$ given by taking summands of the form $C_{p,(-d,-q)} \otimes_k \mathcal{O}(d)$ with $d \geq 0$; here, p denotes homological degree, and $(-d, -q)$ denotes internal degree.

Proposition 1.1. *Let $\mathbf{U} : \text{Com}(E) \rightarrow \text{Com}(\mathbb{P})$ be the \mathbf{U} -functor from Daniel's notes. The bicomplex whose totalization is $\mathbf{U}(C)$ coincides with $\mathcal{L}'(C)$.*

Proof. Clear from definitions. □

Date: May 14, 2020.

Let \mathcal{M} be a coherent $\mathcal{O}_{\mathbb{P}}$ -module, and let $(\mathcal{T}, \partial) \in \text{Com}(E)$ be its Tate complex. The goal is to prove

Theorem 1.2. $H_n(\mathbf{U}(\mathcal{T})) \cong \mathcal{M}$ for all n .

Remark 1.3. Before getting started, we record the following elementary observations. Set $\omega_E := E^\vee$. ω_E is an E -module with k -basis given by exterior polynomials in the e_i^* . Note that $|e_i^*| = (w_i, -1)$, while $|e_i| = (-w_i, 1)$. The action of E on ω_E is by contraction. Note that the $x_i \in S$ are also duals of the e_i , but we use different notation for the basis of ω_E to prevent confusion.

Proof when $w_i = 1$ for all i . The Tate module T is a direct sum of copies of $\omega_E(i, -i)$ for $i \in \mathbb{Z}$. We have

$$\mathbf{L}(\omega_E(i, -i))_q = \bigoplus_d \omega_{(i-d, -i-q)} \otimes \mathcal{O}(d).$$

A nonzero summand must satisfy $-i - q = -a$ and $i - d = a$ for some $a \in \{0, \dots, 2^{n+1}\}$, i.e. $-q = d$. So, forming the bicomplex $\mathcal{L}'(\mathcal{T})$ amounts to applying the associated sheaf functor to the bicomplex (2) and chopping off the rows with $q > 0$. By page 142 of Weibel, we have a spectral sequence

$$E_{pq}^2 = H_p^h H_q^v(\mathcal{L}'(\mathcal{T})) \Rightarrow H_{p+q}(\mathbf{U}(\mathcal{T}))$$

which collapses on page 2 to row $q = 0$, since the columns are exact elsewhere. So, it suffices to show

$$H_p^h H_0^v(\mathcal{L}'(\mathcal{T})) \cong \mathcal{M}$$

for all p . But this is clear, since each $H_0^v(\mathcal{L}'(\mathcal{T}))_p$ is just the result of applying the Ω -functor to T , and we know the homology of this complex is \mathcal{M} in each degree, from Eisenbud-Floystad-Schreyer. \square

Remark 1.4. In the above argument, notice that we also have a spectral sequence

$$H_p^v H_q^h(\mathcal{L}'(\mathcal{T})) \Rightarrow H_*(\text{Tot}^\Pi(\mathcal{L}'(\mathcal{T})))$$

(Weibel page 143). Since the rows in $\mathcal{L}'(\mathcal{T})$ are exact, this shows $H_*(\text{Tot}^\Pi(\mathcal{L}'(\mathcal{T}))) = 0$. So we certainly want direct sum totalization in the definition of \mathbf{U} !

Some weighted projective examples:

Example 1.5. Take $S = k[x_0]$ with $|x_0| = m$ and $\mathcal{M} = \mathcal{O}$. So $\mathbb{P} = [\text{Spec}(k)/(\mathbb{Z}/m)]$. The Tate complex is

$$\dots \xrightarrow{\partial} T \xrightarrow{\partial} T \xrightarrow{\partial} \dots,$$

where $T = \bigoplus_{i \in \mathbb{Z}} \omega_E(mi, -i)$, and ∂ is a matrix with e_0 on the subdiagonal and 0 elsewhere. Let's compute $\mathbf{L}(T)_q$.

$$\begin{aligned} \mathbf{L}(T(mj, -j))_q &= \mathbf{L}\left(\bigoplus_{i \in \mathbb{Z}} \omega_E(mi, -i)\right)_q \\ &= \bigoplus_{d \in \mathbb{Z}} \bigoplus_{i \in \mathbb{Z}} \omega_E(mi, -i)_{(-d, -q)} \otimes S(d) \\ &= \bigoplus_{d \in \mathbb{Z}} \bigoplus_{i \in \mathbb{Z}} \omega_{(mi-d, -i-q)} \otimes S(d). \end{aligned}$$

There are two nonzero summands:

- $i = -q$ and $d = -mq$
- $i = -q + 1$ and $d = -mq$.

We conclude:

$$\mathbf{L}(T)_q = (\omega_{(0,0)} \otimes S(-mq)) \oplus (\omega_{(-m,1)} \otimes S(-mq)).$$

The bicomplex (2) therefore looks like

$$\begin{array}{ccccccc}
& & \downarrow & & \downarrow & & \\
\cdots & \xleftarrow{\partial} & (\omega_{(0,0)} \otimes S(-m)) \oplus (\omega_{(m,-1)} \otimes S(-m)) & \xleftarrow{\partial} & (\omega_{(0,0)} \otimes S(-m)) \oplus (\omega_{(m,-1)} \otimes S(-m)) & \xleftarrow{\partial} & \cdots \\
& & \downarrow & & \downarrow & & \\
\cdots & \xleftarrow{\partial} & (\omega_{(0,0)} \otimes S) \oplus (\omega_{(m,-1)} \otimes S) & \xleftarrow{\partial} & (\omega_{(0,0)} \otimes S) \oplus (\omega_{(m,-1)} \otimes S) & \xleftarrow{\partial} & \cdots \\
& & \downarrow & & \downarrow & & \\
\cdots & \xleftarrow{\partial} & (\omega_{(0,0)} \otimes S(m)) \oplus (\omega_{(m,-1)} \otimes S(m)) & \xleftarrow{\partial} & (\omega_{(0,0)} \otimes S(m)) \oplus (\omega_{(m,-1)} \otimes S(m)) & \xleftarrow{\partial} & \cdots, \\
& & \downarrow & & \downarrow & &
\end{array}$$

where the vertical maps are all given by $\begin{pmatrix} 0 & \pm x_0 \\ 0 & 0 \end{pmatrix}$, and the horizontal maps are all $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.

Thus, $\mathcal{L}'(\mathcal{T})$ looks like

$$\begin{array}{ccccccc}
& & \downarrow & & \downarrow & & \\
\cdots & \xleftarrow{\quad} & 0 & \xleftarrow{\quad} & 0 & \xleftarrow{\quad} & \cdots \\
& & \downarrow & & \downarrow & & \\
\cdots & \xleftarrow{\partial} & (\omega_{(0,0)} \otimes \mathcal{O}) \oplus (\omega_{(m,-1)} \otimes \mathcal{O}) & \xleftarrow{\partial} & (\omega_{(0,0)} \otimes \mathcal{O}) \oplus (\omega_{(m,-1)} \otimes \mathcal{O}) & \xleftarrow{\partial} & \cdots \\
& & \downarrow & & \downarrow & & \\
\cdots & \xleftarrow{\partial} & (\omega_{(0,0)} \otimes \mathcal{O}(m)) \oplus (\omega_{(m,-1)} \otimes \mathcal{O}(m)) & \xleftarrow{\partial} & (\omega_{(0,0)} \otimes \mathcal{O}(m)) \oplus (\omega_{(m,-1)} \otimes \mathcal{O}(m)) & \xleftarrow{\partial} & \cdots. \\
& & \downarrow & & \downarrow & &
\end{array}$$

By page 142 of Weibel, we have a spectral sequence

$$E_{pq}^2 = H_p^h H_q^v(\mathcal{L}'(\mathcal{T})) \Rightarrow H_{p+q}(\mathbf{U}(\mathcal{T})).$$

This spectral sequence collapses to row $q = 0$ at page 2, and we easily conclude

$$H_n(\mathbf{U}(\mathcal{T})) \cong \mathcal{O}$$

for all n , as expected.

Example 1.6. Take $S = k[x_0, x_1]$ with $|x_0| = 1$ and $|x_1| = m$. Let $\mathcal{M} = \mathcal{O}/(x_0)$. The Tate complex looks like this:

$$\cdots \xrightarrow{\partial} T \xrightarrow{\partial} T \xrightarrow{\partial} \cdots,$$

where $T = \bigoplus_{i \in \mathbb{Z}} \omega_E(mi, -i)$, and ∂ is a matrix with e_1 on the subdiagonal. Let's compute $\mathbf{L}(T)_q$.

$$\begin{aligned} \mathbf{L}(T)_q &= \mathbf{L}\left(\bigoplus_{i \in \mathbb{Z}} \omega_E(mi, -i)\right)_q \\ &= \bigoplus_{d \in \mathbb{Z}} \bigoplus_{i \in \mathbb{Z}} \omega_E(mi, -i)_{(-d, -q)} \otimes S(d) \\ &= \bigoplus_{d \in \mathbb{Z}} \bigoplus_{i \in \mathbb{Z}} \omega_{(mi-d, -i-q)} \otimes S(d). \end{aligned}$$

This time, there are 4 nonzero summands:

- $i = -q$ and $d = -mq$
- $i = -q + 1$ and $d = m(1 - q) - 1$
- $i = -q + 1$ and $d = -mq$
- $i = -q + 2$ and $d = m(1 - q) - 1$

We conclude:

$$\mathbf{L}(T)_q = ((\omega_{(0,0)} \oplus \omega_{(m,-1)}) \otimes S(-mq)) \oplus ((\omega_{(1,-1)} \oplus \omega_{(m+1,-2)}) \otimes S(-mq + m - 1)).$$

To ease notation, set $V = (\omega_{(0,0)} \oplus \omega_{(m,-1)})$ and $W = (\omega_{(1,-1)} \oplus \omega_{(m+1,-2)})$. The bicomplex (2) looks like

$$\begin{array}{ccccccc} & & \downarrow & & \downarrow & & \\ \cdots & \xleftarrow{\partial} & (V \otimes S(-m)) \oplus (W \otimes S(-1)) & \xleftarrow{\partial} & (V \otimes S(-m)) \oplus (W \otimes S(-1)) & \xleftarrow{\partial} & \cdots \\ & & \downarrow & & \downarrow & & \\ \cdots & \xleftarrow{\partial} & (V \otimes S) \oplus (W \otimes S(m-1)) & \xleftarrow{\partial} & (V \otimes S) \oplus (W \otimes S(m-1)) & \xleftarrow{\partial} & \cdots \\ & & \downarrow & & \downarrow & & \\ \cdots & \xleftarrow{\partial} & (V \otimes S(m)) \oplus (W \otimes S(2m-1)) & \xleftarrow{\partial} & (V \otimes S(m)) \oplus (W \otimes S(2m-1)) & \xleftarrow{\partial} & \cdots, \\ & & \downarrow & & \downarrow & & \end{array}$$

where the vertical maps are all given by

$$\begin{pmatrix} 0 & \pm x_1 & \pm x_0 & 0 \\ 0 & 0 & 0 & \mp x_1 \\ 0 & 0 & 0 & \pm x_0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

and the horizontal maps are

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Thus, $\mathcal{L}'(\mathcal{T})$ looks like

$$\begin{array}{ccccccc}
& & \downarrow & & \downarrow & & \\
\cdots & \longleftarrow & 0 & \longleftarrow & 0 & \longleftarrow & \cdots \\
& & \downarrow & & \downarrow & & \\
\cdots & \xleftarrow{\partial} & (V \otimes \mathcal{O}) \oplus (W \otimes \mathcal{O}(m-1)) & \xleftarrow{\partial} & (V \otimes \mathcal{O}) \oplus (W \otimes \mathcal{O}(m-1)) & \xleftarrow{\partial} & \cdots \\
& & \downarrow & & \downarrow & & \\
\cdots & \xleftarrow{\partial} & (V \otimes \mathcal{O}(m)) \oplus (W \otimes \mathcal{O}(2m-1)) & \xleftarrow{\partial} & (V \otimes \mathcal{O}(m)) \oplus (W \otimes \mathcal{O}(2m-1)) & \xleftarrow{\partial} & \cdots \\
& & \downarrow & & \downarrow & &
\end{array}$$

As in the previous example, we have a spectral sequence

$$E_{pq}^2 = H_p^h H_q^v(\mathcal{L}'(\mathcal{T})) \Rightarrow H_{p+q}(\mathbf{U}(\mathcal{T})),$$

and it collapses to row $q = 0$ at page 2. One easily checks that $H_0^v(\mathcal{L}'(\mathcal{T}))_p$ is free of rank 2 for all p , with basis

$$b_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, b_2 = \begin{pmatrix} 0 \\ x_0 \\ -x_1 \\ 0 \end{pmatrix}.$$

The horizontal map ∂ kills b_1 and sends b_2 to $x_0 b_1$. We therefore have

$$H_n(\mathbf{U}(\mathcal{T})) \cong \mathcal{O}/(x_0)$$

for all n , as expected.