BEILINSON MONAD

1. The U-functor

We work over weighted projective space for now. So $S = k[x_0, ..., x_n]$ with $|x_i| = w_i$. Denote by \mathbb{P} the associated weighted projective space. We equip E with the $\mathbb{Z} \times \mathbb{Z}$ -grading such that $|e_i| = (-w_i, 1)$. We recall the definition of the functor

$$L: Com(E) \to Com(S)$$

from Daniel's notes. Here, $\operatorname{Com}(E)$ denotes the category of complexes of $\mathbb{Z} \times \mathbb{Z}$ -graded E-modules, and $\operatorname{Com}(S)$ is the category of complexes of \mathbb{Z} -graded S-modules.

Remark 1.1. All E-modules are right modules. In particular, entries of matrices over E act on the right. This is also Macaulay2's convention. Note that this is the only way to make sense of the definition of the **R**-functor in the EFS paper; if we apply the definition to a left E-module M, the maps in the complex $\mathbf{R}(M)$ are not E-linear. Nevertheless, sometimes we will multiply elements of E-modules on the left by elements of e (for instance, in the definition of the **L**-functor). When we do this, here is what we mean. When we write em for $e \in E$ and $m \in M$, where M is a right E-module, we mean $(-1)^{|e||m|}me$, where |-| denotes the degree with respect to the second (standard) grading.

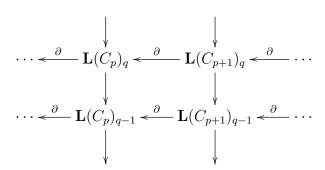
For M an E-module concentrated in degree 0, L(M) is the complex with

$$\mathbf{L}(M)_q = \bigoplus_d M_{(-d,-q)} \otimes_k S(d)$$

and differential

$$(1) m \otimes s \mapsto \sum_{i=0}^{n} e_i m \otimes x_i s.$$

******* Michael: [I'm using homological indexing so that comparing to M2 will be easier.] For a general complex $(C, \partial) \in \text{Com}(E)$, we form the bicomplex



and apply $\operatorname{Tot}^{\oplus}(-)$. Note that the vertical differential $\mathbf{L}(C_p)_q \to \mathbf{L}(C_p)_{q-1}$ is the dual Koszul map (1) multiplied by $(-1)^p$.

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Let $\mathcal{L}(C)$ denote the bicomplex of $\mathcal{O}_{\mathbb{P}}$ -modules given by applying the associated sheaf functor to the bicomplex (2). Let $\mathcal{L}'(C)$ be the sub-bicomplex of $\mathcal{L}(C)$ given by taking summands of the form $C_{p,(-d,-q)} \otimes_k \mathcal{O}(d)$ with $d \geq 0$ here, p denotes homological degree, and (-d,-q) denotes internal degree. We define a functor

$$U : Com(E) \to Com(\mathbb{P})$$

to be given by $C \mapsto \operatorname{Tot}^{\oplus}(\mathcal{L}'(C))$. See Remark 1.5 for why we use direct sum totalization rather than direct product, and see Remark 2.3 for why we truncate by taking summands with $d \geq 0$ rather than $d \leq 0$.

Proposition 1.2. The above definition of the U-functor agrees with Daniel's.

Proof. Daniel's definition is given by

$$\omega(i,j) \mapsto \mathcal{L}(\omega_{\leq i})(i)[-j].$$

(We're abusing notation here by identifying the 1-column bicomplex $\mathcal{L}(\omega_{\leq i})(i)$ with its totalization.) We have

$$(\mathcal{L}(\omega_{\leq i})(i)[-j])_{q} = \mathcal{L}(\omega_{\leq i})_{-j+q}(i)$$

$$= \bigoplus_{d} (\omega_{\leq i})_{(-d,j-q)} \otimes \mathcal{O}(d+i)$$

$$= \bigoplus_{d} (\omega_{\leq i})_{(i-d,j-q)} \otimes \mathcal{O}(d)$$

$$= \bigoplus_{d\geq 0} (\omega_{\leq i})_{(i-d,j-q)} \otimes \mathcal{O}(d)$$

$$= \bigoplus_{d\geq 0} \omega_{(i-d,j-q)} \otimes \mathcal{O}(d)$$

$$= \mathbf{U}(\omega(i,j))_{q}.$$

And of course the maps in both complexes are identical as well.

Let M be a finitely generated S-module, and let $(\mathcal{T}, \partial) \in \text{Com}(E)$ be its Tate complex. We recall that \mathcal{T} is a complex of the form

$$\cdots \xrightarrow{\partial} T \xrightarrow{\partial} T \xrightarrow{\partial} \cdots$$

where T is a direct sum of twists of $\omega := E^{\vee}$. The goal is to prove

Theorem 1.3. $H_n(\mathbf{U}(\mathcal{T})) \cong \widetilde{M}$ for all n.

Remark 1.4. Before getting started, we record the following elementary observations. Of course, ω is an E-module with k-basis given by exterior polynomials in the e_i^* . Note that $|e_i^*| = (w_i, -1)$, while $|e_i| = (-w_i, 1)$. The action of E on ω is by contraction. The $x_i \in S$ are also duals of the e_i , but we use different notation for the basis of ω to prevent confusion.

Proof when $w_i = 1$ for all i. The Tate module T is a direct sum of copies of $\omega(-i, i)$ for $i \in \mathbb{Z}$. We have

$$\mathbf{L}(\omega(-i,i))_q = \bigoplus_{\substack{d \\ 2}} \omega_{(-i-d,i-q)} \otimes \mathcal{O}(d).$$

A nonzero summand must satisfy i-q=-a and -i-d=a for some $a\in\{0,\ldots,n+1\}$, i.e. -q=d. So, forming the bicomplex $\mathcal{L}'(\mathcal{T})$ amounts to applying the associated sheaf functor to the bicomplex (2) and chopping off the rows with q > 0. By page 142 of Weibel, we have a spectral sequence

$$E_{pq}^2 = H_p^h H_q^v(\mathcal{L}'(\mathcal{T})) \Rightarrow H_{p+q}(\mathbf{U}(\mathcal{T}))$$

that collapses on page 2 to row q=0, since the columns are exact elsewhere. So, it suffices to show

$$H_p^h H_0^v(\mathcal{L}'(\mathcal{T})) \cong \mathcal{M}$$

for all p. But this is clear, since each $H_0^v(\mathcal{L}'(\mathcal{T}))_p$ is just the result of applying the Ω -functor to T, and we know the homology of this complex is \mathcal{M} in each degree, from Eisenbud-Floystad-Schreyer.

Remark 1.5. Let \mathcal{T} be as in the above proof. The rows of $\mathcal{L}'(\mathcal{T})$ are exact as well. Since the rows in $\mathcal{L}'(\mathcal{T})$ are 0 for q>0, we have a spectral sequence

$$H_p^v H_q^h(\mathcal{L}'(\mathcal{T})) \Rightarrow H_{p+q}(\mathrm{Tot}^{\Pi}(\mathcal{L}'(\mathcal{T})))$$

(Weibel page 143). It follows that $H_*(\operatorname{Tot}^{\Pi}(\mathcal{L}'(\mathcal{T}))) = 0$. This is why we take the direct sum totalization in the definition of U(C); otherwise, U applied to the Tate complex would give 0.

2. Examples in weighted projective space

Example 2.1. Take $S = k[x_0]$ with $|x_0| = m$ and $\mathcal{M} = \mathcal{O}$. So $\mathbb{P} = [\operatorname{Spec}(k)/(\mathbb{Z}/m)]$. The Tate complex is

$$\cdots \xrightarrow{\partial} T \xrightarrow{\partial} T \xrightarrow{\partial} \cdots,$$

where $T = \bigoplus_{i \in \mathbb{Z}} \omega_E(-mi, i)$, and ∂ is given by multiplication by e_0 . We have:

$$\mathbf{L}(T)_{q} = \mathbf{L}(\bigoplus_{i \in \mathbb{Z}} \omega(-mi, i))_{q}$$

$$= \bigoplus_{d \in \mathbb{Z}} \bigoplus_{i \in \mathbb{Z}} \omega(-mi, i)_{(-d, -q)} \otimes S(d)$$

$$= \bigoplus_{d \in \mathbb{Z}} \bigoplus_{i \in \mathbb{Z}} \omega_{(-mi-d, i-q)} \otimes S(d).$$

There are two nonzero summands:

- i = q and d = -mq
- i = 1 q and d = -mq.

We conclude:

$$\mathbf{L}(T)_q = (\omega_{(0,0)} \otimes S(-mq)) \oplus (\omega_{(-m,1)} \otimes S(-mq)).$$

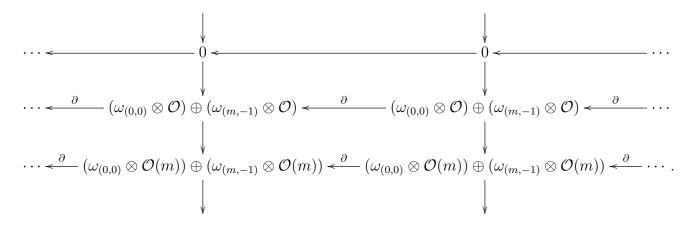
The bicomplex (2) therefore looks like

$$\cdots \stackrel{\partial}{\longleftarrow} (\omega_{(0,0)} \otimes S(-m)) \oplus (\omega_{(m,-1)} \otimes S(-m)) \stackrel{\partial}{\longleftarrow} (\omega_{(0,0)} \otimes S(-m)) \oplus (\omega_{(m,-1)} \otimes S(-m)) \stackrel{\partial}{\longleftarrow} \cdots$$

$$\cdots \stackrel{\partial}{\longleftarrow} (\omega_{(0,0)} \otimes S) \oplus (\omega_{(m,-1)} \otimes S) \stackrel{\partial}{\longleftarrow} (\omega_{(0,0)} \otimes S) \oplus (\omega_{(m,-1)} \otimes S) \stackrel{\partial}{\longleftarrow} \cdots$$

$$\cdots \stackrel{\partial}{\longleftarrow} (\omega_{(0,0)} \otimes S(m)) \oplus (\omega_{(m,-1)} \otimes S(m)) \stackrel{\partial}{\longleftarrow} (\omega_{(0,0)} \otimes S(m)) \oplus (\omega_{(m,-1)} \otimes S(m)) \stackrel{\partial}{\longleftarrow} \cdots$$

where the vertical maps are all given by $\begin{pmatrix} 0 & \pm x_0 \\ 0 & 0 \end{pmatrix}$, and the horizontal maps are all $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Thus, $\mathcal{L}'(\mathcal{T})$ looks like



By page 142 of Weibel, we have a spectral sequence

$$E_{pq}^2 = H_p^h H_q^v(\mathcal{L}'(\mathcal{T})) \Rightarrow H_{p+q}(\mathbf{U}(\mathcal{T})).$$

This spectral sequence collapses to row q=0 at page 2, and we easily conclude

$$H_n(\mathbf{U}(\mathcal{T})) \cong \mathcal{O}$$

for all n, as expected.

Example 2.2. Take $S = k[x_0, x_1]$ with $|x_0| = 1$ and $|x_1| = m$. Let $\mathcal{M} = \mathcal{O}/(x_0)$. The Tate complex looks like this:

$$\cdots \xrightarrow{\partial} T \xrightarrow{\partial} T \xrightarrow{\partial} \cdots,$$

where $T = \bigoplus_{i \in \mathbb{Z}} \omega(-mi, i)$, and ∂ given by multiplication by e_1 . We have:

$$\mathbf{L}(T)_{q} = \mathbf{L}(\bigoplus_{i \in \mathbb{Z}} \omega(-mi, i))_{q}$$

$$= \bigoplus_{d \in \mathbb{Z}} \bigoplus_{i \in \mathbb{Z}} \omega(-mi, i)_{(-d, -q)} \otimes S(d)$$

$$= \bigoplus_{d \in \mathbb{Z}} \bigoplus_{i \in \mathbb{Z}} \omega_{(-mi - d, i - q)} \otimes S(d).$$

This time, there are 4 nonzero summands:

- i = q and d = -mq
- i = q 1 and d = m(1 q) 1
- i = q 1 and d = -mq
- i = q 2 and d = m(1 q) 1

We conclude:

$$\mathbf{L}(T)_q = ((\omega_{(0,0)} \oplus \omega_{(m,-1)}) \otimes S(-mq)) \oplus ((\omega_{(1,-1)} \oplus \omega_{(m+1,-2)}) \otimes S(m(1-q)-1)).$$

To ease notation, set $V = (\omega_{(0,0)} \oplus \omega_{(m,-1)})$ and $W = (\omega_{(1,-1)} \oplus \omega_{(m+1,-2)})$. The bicomplex (2) looks like

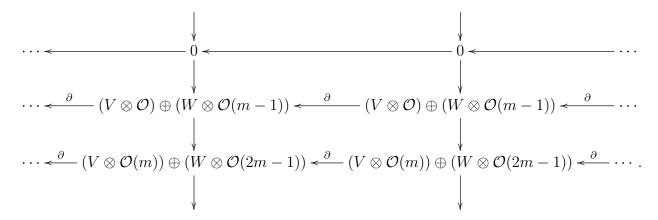
where the vertical maps are all given by

$$\begin{pmatrix} 0 & \pm x_1 & \pm x_0 & 0 \\ 0 & 0 & 0 & \pm x_0 \\ 0 & 0 & 0 & \mp x_1 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

and the horizontal maps are

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Thus, $\mathcal{L}'(\mathcal{T})$ looks like



As in the previous example, we have a spectral sequence

$$E_{pq}^2 = H_p^h H_q^v(\mathcal{L}'(\mathcal{T})) \Rightarrow H_{p+q}(\mathbf{U}(\mathcal{T})).$$

The spectral sequence collapses to row q = 0 at page 2. One easily checks that $H_0^v(\mathcal{L}'(\mathcal{T}))_p$ is free of rank 2 for all p, with basis

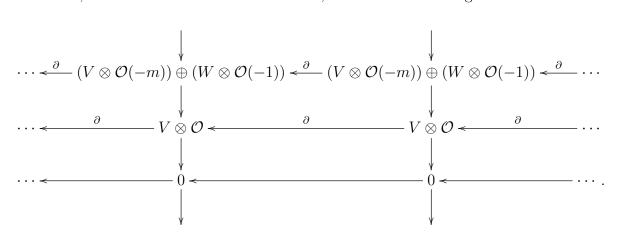
$$b_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, b_2 = \begin{pmatrix} 0 \\ x_0 \\ -x_1 \\ 0 \end{pmatrix}.$$

The horizontal map ∂ kills b_1 and sends b_2 to x_0b_1 . We therefore have

$$H_n(\mathbf{U}(\mathcal{T})) \cong \mathcal{O}/(x_0)$$

for all n, as expected.

Remark 2.3. In the last example, let's suppose that we defined $\mathcal{L}'(-)$ by chopping off the d > 0 terms, rather than the d < 0 terms. Also, assume m > 1. We get:



In this setting, it only makes sense to define the U-functor using a direct product totalization; the direct sum totalization will give the wrong answer, by the reasoning in Remark 1.5. As

above, we have a spectral sequence

$$E_{pq}^2 = H_p^h H_q^v(\mathcal{L}'(\mathcal{T})) \Rightarrow H_{p+q}(\mathbf{U}(\mathcal{T})).$$

We have

$$H_0^v(\mathcal{L}'(\mathcal{T}))_p = \mathcal{O}/(x_1) \oplus \mathcal{O}/(x_0)$$

and

$$H_1^v(\mathcal{L}'(\mathcal{T}))_p = \mathcal{O}(-1)$$

for all p. The rest of the vertical homology is obviously 0, since the columns in $\mathcal{L}(\mathcal{T})$ are exact. We therefore have

$$H_p^h H_q^v(\mathcal{L}'(\mathcal{T})) = \begin{cases} \mathcal{O}/(x_1), & q = 0\\ \mathcal{O}(-1), & q = 1\\ 0 & \text{else} \end{cases}$$

We get a long exact sequence

$$\cdots \to H_{p+1}(\mathbf{U}(\mathcal{T})) \to \mathcal{O}/(x_1) \to \mathcal{O}(-1) \to H_p(\mathbf{U}(\mathcal{T})) \to \cdots$$

The map $\mathcal{O}/(x_1) \to \mathcal{O}(-1)$ is obviously 0, which means each $H_p(\mathbf{U}(\mathcal{T}))$ contains a copy of $\mathcal{O}(-1)$. In particular, $H_p(\mathbf{U}(\mathcal{T})) \neq \mathcal{O}/(x_0)$, so our output is wrong!