TORIC TATE RESOLUTIONS

Contents

To do:

- Show definitions of Tate resolution coincide and recover the classical definition for $\mathbb{P}^n$ (up to folding).
- Flesh out alternative Tate resolution section. In particular, give the definition in the case of a general toric variety and check it in examples.
- Flesh out properties of **L** and **R**
- Finite degree window for **U**-functor, **R**-functor, κ^*.

1. Setup

We index homologically throughout. Fix a toric variety X with Cox ring $S = k[x_0, \dots, x_n]$ graded by $\mathrm{Pic}(X) = \mathbb{Z}^{\oplus r}$. Let $\mathcal{F}$ be a coherent sheaf on X. Write E for the Koszul dual of S. We equip E with a $\mathrm{Pic}(X) \times \mathbb{Z}$-grading such that $|e_i| = (-|x_i|, 1)$. Write $\omega := E^\vee$.

Remark 1.1. Before getting started, we record the following elementary observations. Of course, ω is an E-module with k-basis given by exterior polynomials in the e_i^*. Note that $|e_i^*| = (|x_i|, -1)$. The action of E on ω is by contraction. The $x_i \in S$ are also duals of the e_i, but we use different notation for the basis of ω to prevent confusion.

Another technical note: all E-modules are right modules. In particular, entries of matrices over E act on the right. This is also Macaulay2's convention. Note that this is the only way

Date: August 4, 2020.

to make sense of the definition of the **R**-functor in [EFS03]; if we apply the definition to a left E-module M, the maps in the complex $\mathbf{R}(M)$ are not E-linear. Nevertheless, sometimes we will multiply elements of E-modules on the left by elements of e (for instance, in the definition of the **L**-functor below). When we do this, here is what we mean. When we write em for $e \in E$ and $m \in M$, where M is a right E-module, we mean $(-1)^{|e||m|}me$, where $|-|$ denotes the degree with respect to the second (standard) grading.

2. The Tate resolution

Define $\mathcal{O}_{X\times E}$ to be the sheaf of algebras on X given by

$$U \mapsto \Gamma(U, \mathcal{O}_X) \otimes_k E.$$

Let $\mathrm{Com}(X \times E)$ denote the category of complexes of $\mathcal{O}_{X\times E}$-modules.

Example 2.1. Define an object $\kappa^*(\mathcal{F}) \in \mathrm{Com}(X \times E)$ given by

$$\kappa^*(\mathcal{F})_i = \bigoplus_{l \in \mathrm{Pic}(X)} \mathcal{F}(-\ell) \otimes \omega(\ell, -i)$$

with differential given by $\sum_i x_i \otimes e_i$.

Fix once and for all an affine open cover $\{V_0, \ldots, V_t\}$ of X. Given a sheaf $\mathcal{M}$ on X, denote by $\mathcal{C}^{\mathcal{M}}$ the Čech complex of $\mathcal{M}$ corresponding to this open cover. Recall that we're indexing homologically, so the Čech complex is concentrated in nonpositive degrees.

Given $\mathcal{G} \in \mathrm{Com}(X \times E)$, define a bicomplex as follows. The p^{th} column is given by $\mathcal{C}^{\mathcal{G}_p}$, but with the vertical differential multiplied by $(-1)^p$. The horizontal differential is induced by the differential on $\mathcal{G}$. Denote by $\tau_*(\mathcal{G}) \in \mathrm{Com}(E)$ the direct sum totalization of this bicomplex. In other words, $\tau_*(\mathcal{G})$ is given as follows: take an explicit Čech model of the derived global sections of the complex $\mathcal{G}$, considered as a complex of $\mathcal{O}_X$-modules, and then remember the E-module structure on the terms of the resulting complex.

Let's look more closely at $(\tau_* \circ \kappa^*)(\mathcal{F})$. The bicomplex $\mathcal{B}$ such that

$$\mathrm{Tot}^{\oplus}(\mathcal{B}) = (\tau_* \circ \kappa^*)(\mathcal{F})$$

looks like

$$
\begin{tikzcd}
 & {} \arrow[d] & {} \arrow[d] & \\
\cdots & 0 \arrow[l] \arrow[d] & 0 \arrow[l] \arrow[d] & \cdots \arrow[l] \\
\cdots & \bigoplus_{\ell \in \mathrm{Pic}(X)} \mathcal{C}_0^{\mathcal{F}(-\ell)} \otimes \omega(\ell, -p) \arrow[l] \arrow[d] & \bigoplus_{\ell \in \mathrm{Pic}(X)} \mathcal{C}_0^{\mathcal{F}(-\ell)} \otimes \omega(\ell, -(p-1)) \arrow[l] \arrow[d] & \cdots \arrow[l] \\
 & \vdots \arrow[d] & \vdots \arrow[d] & \\
\cdots & \bigoplus_{\ell \in \mathrm{Pic}(X)} \mathcal{C}_{-t}^{\mathcal{F}(-\ell)} \otimes \omega(\ell, -p) \arrow[l] \arrow[d] & \bigoplus_{\ell \in \mathrm{Pic}(X)} \mathcal{C}_{-t}^{\mathcal{F}(-\ell)} \otimes \omega(\ell, -(p-1)) \arrow[l] \arrow[d] & \cdots \arrow[l] \\
\cdots & 0 \arrow[l] \arrow[d] & 0 \arrow[l] \arrow[d] & \cdots, \arrow[l] \\
 & {} & {} &
\end{tikzcd}
$$

where the differential in the p^{th} column is the Čech differential, multiplied by $(-1)^p$, and the horiztonal differential is induced by the differential on $\kappa^*(\mathcal{F})$. We note that $\mathcal{B}$ is a bicomplex in the category of E-modules.

Choose k-vector spaces $G^{\ell}_{(p,q)}$ for all $-t \leq q \leq 0$, $p \in \mathbb{Z}$, and $\ell \in \mathrm{Pic}(X)$ such that

$$\mathcal{C}_q^{\mathcal{F}(-\ell)} = G^{\ell}_{(p,q)} \oplus d_{\mathrm{vert}} G^{\ell}_{(p,q+1)} \oplus H_q^{-\ell},$$

where $H_q^{\ell} \cong H^{-q}(X, \mathcal{F}(-\ell))$ (in this formula, we take $G^{\ell}_{(p,1)} := 0$ for all p and ℓ). We therefore have an E-linear splitting

$$\mathcal{B}_{p,q} = \bigoplus_{\ell \in \mathrm{Pic}(X)} (G^{\ell}_{(p,q)} \oplus d_{\mathrm{vert}} G^{\ell}_{(p,q+1)} \oplus H_q^{\ell}) \otimes \omega(\ell, -p)$$

of each column of $\mathcal{B}$. Applying [EFS03, Lemma 3.5], we conclude that $(\tau_* \kappa^*)(\mathcal{F})$ is homotopy equivalent to a complex of the form

(1)

$$\cdots \to \bigoplus_{q=0}^{t} \bigoplus_{\ell \in \mathrm{Pic}(X)} H^q(X, \mathcal{F}(-\ell)) \otimes \omega(\ell, -n-q) \to \bigoplus_{q=0}^{t} \bigoplus_{\ell \in \mathrm{Pic}(X)} H^{-q}(X, \mathcal{F}(-\ell)) \otimes \omega(\ell, -(n-1)-q) \to \cdots;$$

see the statement of [EFS03, Lemma 3.5] for an explicit formula for the differential in (1) in terms of the choices of splittings above.

Definition 2.2. The *Tate resolution of* $\mathcal{F}$, denoted $\mathrm{Tate}(\mathcal{F})$, is the complex (1).

Proposition 2.3. $\mathrm{Tate}(\mathcal{F})$ *is exact.*

Proof. The bicomplex $\mathcal{B}$ above is bounded and has exact rows, so this follows from an easy spectral sequence argument. $\square$

We highlight the following observation:

$$\dim_k H^q(X, \mathcal{F}(-\ell)) = \#\text{ of copies of } \omega(\ell, -n-q) \text{ appearing in } \mathrm{Tate}(\mathcal{F})_n.$$

3. The BGG functors $\mathbf{L}$ and $\mathbf{R}$

3.1. **Definitions.** We start with

$$\mathbf{L} : \mathrm{Com}(E) \to \mathrm{Com}(S).$$

This is the same definition as in [EFS03]. For an E-module M concentrated in degree 0, $\mathbf{L}(M)$ is the complex with

$$\mathbf{L}(M)_q = \bigoplus_{d \in \mathrm{Pic}(X)} M_{(-d,-q)} \otimes_k S(d)$$

and differential

$$m \otimes s \mapsto \sum_{i=0}^{n} e_i m \otimes x_i s. \tag{2}$$

For a general complex $(C, \partial) \in \mathrm{Com}(E)$, we form the bicomplex

(3)

$$\begin{tikzcd}
 & {} \arrow[d] & {} \arrow[d] & \\
\cdots & \mathbf{L}(C_p)_q \arrow[l, "\partial"'] \arrow[d] & \mathbf{L}(C_{p+1})_q \arrow[l, "\partial"'] \arrow[d] & \cdots \arrow[l, "\partial"'] \\
\cdots & \mathbf{L}(C_p)_{q-1} \arrow[l, "\partial"'] \arrow[d] & \mathbf{L}(C_{p+1})_{q-1} \arrow[l, "\partial"'] \arrow[d] & \cdots \arrow[l, "\partial"'] \\
 & {} & {} &
\end{tikzcd}$$

and apply $\mathrm{Tot}^{\oplus}(-)$, where the vertical differential $\mathbf{L}(C_p)_q \to \mathbf{L}(C_p)_{q-1}$ is the dual Koszul map (2) multiplied by $(-1)^p$.

The functor

$$\mathbf{R} : \mathrm{Com}(S) \to \mathrm{Com}(E)$$

is not the same as in [EFS03]. It's defined as follows. Given $M \in \mathrm{Com}(S)$, define a bicomplex $\mathcal{B}$ with

$$\mathcal{B}_{p,q} = \bigoplus_{d \in \mathrm{Pic}(X)} (M_q)_d \otimes \omega(-d, -p)$$

here, $(M_q)_d$ denotes the internal degree d part of the q^{th} term of M. The horizontal differential is given by

$$m \otimes e \mapsto \sum_{i=0}^{n} x_i m \otimes e e_i,$$

and the p^{th} vertical differential is the map induced by the differential on M multiplied by $(-1)^p$. Notice that the rows are 1-periodic, up to twisting by $E(0,1)$. Define $\mathbf{R}(M) = \mathrm{Tot}^{\oplus}(\mathcal{B})$.

Remark 3.1. Let $\mathbf{R}_{\mathrm{EFS}}$ denote the original $\mathbf{R}$-functor from [EFS03]. When $X = \mathbb{P}^n$, our $\mathbf{R}$-functor factors as

$$\mathrm{Mod}(S) \xrightarrow{\mathbf{R}_{\mathrm{EFS}}} \mathrm{Lin}_{\mathrm{Free}}(E) \xrightarrow{\mathrm{Fold}} \mathrm{Lin}_{\mathrm{Free}}^{\mathrm{per}}(E) \tag{4}$$

where

- $\mathrm{Lin}_{\mathrm{Free}}(E)$ is the subcategory of $\mathrm{Com}(E)$ given by linear complexes of free modules.
- $\mathrm{Lin}_{\mathrm{Free}}^{\mathrm{per}}(E)$ is the subcategory of $\mathrm{Lin}_{\mathrm{Free}}(E)$ given by complexes that satisfy

$$C_i = C_{i-1} \otimes E(0,1)$$

  and such that all the differentials are identical,
- given a complex C in $\mathrm{Lin}_{\mathrm{Free}}(E)$, $\mathrm{Fold}(C)$ is the complex with

$$(\mathrm{Fold}(C))_i = \bigoplus_{j \in \mathbb{Z}} C_j \otimes E(j, -i)$$

  and the obvious differential.

3.2. **The $\overline{\mathbf{L}}$-functor.** We will be particularly interested in applying $\mathbf{L}$ to objects in $C \in \mathrm{Com}(E)$ of the form

$$\cdots \to N(0,-1) \to N \to N(0,1) \to \dots,$$

e.g. complexes of the form $\mathbf{R}(M)$ with M an S-module. The (p,q) term of the bicomplex whose totalization is $\mathbf{L}(C)$ is

$$\bigoplus_{d \geq 0} N(0,-p)_{(-d,-q)} \otimes S(d) = \bigoplus_{d \in \mathrm{Pic}(X)} N_{(-d,-p-q)} \otimes S(d),$$

and so

$$\mathbf{L}(C)_m = \bigoplus_{p \in \mathbb{Z}} \bigoplus_{d \in \mathrm{Pic}(X)} N_{(-d,-m)} \otimes S(d).$$

Equip each $\mathbf{L}(C)_m$ with a $k[u]$-module structure determined by the following "shift" operation: if $t = (\dots, t_{-1}, t_0, t_1, \dots) \in \bigoplus_{p \in \mathbb{Z}} \bigoplus_{d \geq 0} N_{(-d,-m)} \otimes S(d)$,

$$u(t)_p = (-1)^{m-p} t_{p-1}.$$

Proposition 3.2. *The differential on $\mathbf{L}(C)$ is $k[u]$-linear.*

Proof. I'm writing down the proof to make sure I got the sign right in the definition of the u-action. We prove that the action of u commutes with both horizontal and vertical differentials. Write d_C for the differential on C and d_K for the dual Koszul differential. We have

$$\begin{aligned} d_{\mathrm{hor}}(u \cdot (\dots, t_{-1}, t_0, t_1, \dots)) &= d_{\mathrm{hor}}(\dots, (-1)^{m-1} t_{-2}, (-1)^m t_{-1}, (-1)^{m-1} t_0, \dots) \\ &= (\dots, (-1)^m d_T(t_{-3}), (-1)^{m-1} d_T(t_{-2}), (-1)^m d_T(t_{-1}), \dots) \\ &= u \cdot (\dots, d_T(t_{-2}), d_T(t_{-1}), d_T(t_0), \dots) \\ &= u \cdot d_{\mathrm{hor}}(\dots, t_{-1}, t_0, t_1, \dots), \end{aligned}$$

and

$$\begin{aligned} d_{\mathrm{ver}}(u\cdot(\dots,t_{-1},t_0,t_1,\dots)) &= d_{\mathrm{ver}}(\dots,(-1)^{m-1}t_{-2},(-1)^m t_{-1},(-1)^{m-1}t_0,\dots) \\ &= (\dots,(-1)^m d_K(t_{-2}),(-1)^m d_K(t_{-1}),(-1)^m d_K(t_0),\dots) \\ &= u\cdot(\dots,-d_K(t_{-1}),d_K(t_0),-d_K(t_1),\dots) \\ &= u\cdot d_{\mathrm{ver}}(\dots,t_{-1},t_0,t_1,\dots). \end{aligned}$$

□

So, $\mathbf{L}(C)$ is a complex of $\mathcal{O}_{X\times\mathbb{A}^1}$-modules. In fact, since the action of u is invertible, $\mathbf{L}(C)$ is a complex of $\mathcal{O}_{X\times\mathbb{G}_m}$-modules. Define

$$\overline{\mathbf{L}}(C) := \mathbf{L}(C)/(u-1).$$

We have an isomorphism

$$\overline{\mathbf{L}}(C)_m \cong \bigoplus_{d\in\mathrm{Pic}(X)} N_{(-d,-m)}\otimes S(d) \tag{5}$$

given by representing each class in $\overline{\mathbf{L}}(C)_m$ by an element concentrated in the $p=0$ summand of $\bigoplus_{p\in\mathbb{Z}}\bigoplus_{d\in\mathrm{Pic}(X)} N_{(-d,-m)}\otimes S(d)$. Via this isomorphism, the m^{th} differential on $\overline{\mathbf{L}}(C)$ is given by $(-1)^m d_C+\sum_{i=0}^n e_i\otimes x_i$. ♣♣♣ Michael: [I would not bet my life on this sign being right. But I think it is.]

4. Alternative definition of the Tate resolution

We start by describing a way of computing the linear part of the original [EFS03] Tate resolution over $\mathbb{P}^n$.

4.1. **Classical setting.** In this subsection, when I write $\mathbf{L}$ and $\mathbf{R}$, I mean the versions of these maps from [EFS03]. Take a finitely generated S-module M. Choose $e\gg 0$ so that $\mathbf{R}(M_{\geq e})$ is exact. Let K be the kernel of the 0^{th} differential of $\mathbf{R}(M_{\geq e})$. Set

$$K^* = \mathrm{Hom}_E(K,E).$$

It follows from the Reciprocity Theorem [EFS03, Theorem 3.7(b)] that the natural map

$$K^*\to(\mathbf{R}\circ H_*(-)\circ\mathbf{L})(K^*)$$

is the linear part of a minimal injective resolution. Dualizing, we get the linear part of a minimal free resolution

$$(\mathbf{R}\circ H_*(-)\circ\mathbf{L})(K^*)^*\to K^{**}$$

Note that $K\cong K^{**}$ because every finitely generated E-module is MCM and hence reflexive. It follows that there is a canonical map

$$(\mathbf{R}\circ H_*(-)\circ\mathbf{L})(K^*)^*\to\mathbf{R}(M_{\geq e})$$

whose mapping cone is the linear part of the Tate resolution of $\widetilde{M}$. One could therefore compute the sheaf cohomology of $\widetilde{M}$ by reading off the Betti numbers from the above complex.

Example 4.1. Take $X = \mathbb{P}^1$ and $M = S/(x_1)$. We can take $e = 0$ in this case, and $\mathbf{R}(M)$ is the complex

$$\omega \xrightarrow{e_0} \omega(-1) \xrightarrow{e_0} \cdots$$

concentrated in nonpositive degrees. We have

$$K = k \cdot 1 \oplus k \cdot e_1^* \subseteq \omega. \tag{6}$$

An easy computation shows that K^* is a cyclic module concentrated in degrees $-2, -3$ whose annihilator is the ideal generated by e_0 in E.

Applying $\mathbf{L}$ to K^*, we get the complex

$$0 \to S(2) \xrightarrow{x_1} S(3) \to 0$$

concentrated in degrees $-2, -3$. Taking homology, we get the object $S(3)/x_1$ concentrated in degree -3. Applying $\mathbf{R}$ to this, we get

$$\omega(3) \xrightarrow{e_0} \omega(2) \xrightarrow{e_0} \cdots$$

concentrated in nonpositive degrees. Dualizing, we get

$$\cdots \xrightarrow{e_0} \omega^*(-2) \xrightarrow{e_0} \omega^*(-3) \tag{7}$$

concentrated in nonnegative degrees. It's easy to check that there is an isomorphism

$$\omega \cong \omega^*(-4)$$

(ω is a rank 1 free E-module with generator in degree 2, and ω^* is a rank 1 free E-module with generator in degree -2). Rewriting (7), we get

$$\cdots \xrightarrow{e_0} \omega(2) \xrightarrow{e_0} \omega(1) \tag{8}$$

There is a morphism of complexes from (6) to (8)

$$\begin{tikzcd} \cdots \arrow[r, "e_0"] & \omega(2) \arrow[r, "e_0"] & \omega(1) \arrow[d, "e_0"] & & \\ & & \omega \arrow[r, "e_0"] & \omega(-1) \arrow[r, "e_0"] & \cdots \end{tikzcd}$$

and the mapping cone of this morphism gives the linear part of the Tate resolution:

$$\cdots \xrightarrow{e_0} \omega(1) \xrightarrow{e_0} \omega \xrightarrow{e_0} \omega(-1) \xrightarrow{e_0} \cdots$$

of $\widetilde{M}$ (I'm missing some signs). In fact, since the Tate resolution happens to be linear in this case, we get the entire Tate resolution.

4.2. **Toric setting.** We now wish to mimic the above construction in the toric setting. We start with weighted projective space, where everything is simpler. Let $X = \mathbb{P}(w_0, \ldots, w_n)$, and let M be an S-module.

4.3. **Key difficulty.** Let $r = \operatorname{reg}(M)$. Over projective space, there is a quasi-isomorphism

$$H_*(\mathbf{R}(M_{\geq r})) \hookrightarrow \mathbf{R}(M_{\geq r});$$

this is true whether we consider the [EFS03] version of the $\mathbf{R}$-functor or our 1-periodic version. This is a crucial aspect of the construction of the Tate resolution described above; it allows us to build the Tate resolution by resolving the homology of $\mathbf{R}(M_{\geq r})$, rather than the entire complex.

Unfortunately, this is not true over weighted projective space. Before looking at a counterexample, we record the following useful observation (cf. [EFS03, Proposition 2.3(b)]):

Proposition 4.2. *For any S-module M, we have*

$$H_0(\mathbf{R}(M))_{(i,j)} = \operatorname{Tor}^S_{-j}(k, M)_i.$$

for all $i, j \in \mathbb{Z}$.

Proof. Fill in. $\square$

Example 4.3. Take $X = \mathbb{P}(1, 2)$ and $M = S/(x_0^2 - x_1)$. The regularity of M is 1. $M_{\geq 1}$ has a free resolution given by

$$0 \leftarrow S(-1) \xleftarrow{x_0^2 - x_1} S(-3) \leftarrow 0.$$

It follows from Proposition 4.2 that $H_0(\mathbf{R}(M_{\geq 1}))$ is 2-dimensional, with basis in degrees $(1, 0)$ and $(3, -1)$. Note that M_d is 1-dimensional for all $d \geq 0$, with basis given by $x_1^{d/2}$ when d is even and $x_0 x_1^{(d-1)/2}$ when d is odd. So, we have

$$\mathbf{R}(M_{\geq 1})_i = \bigoplus_{d \in \mathbb{Z}} M_d \otimes \omega(-d, -i) = \bigoplus_{d \geq 1} \omega(-d, -i).$$

It's easy to check that

$$1^* x_0 \in M_1 \otimes \omega(-1, 0)$$

and

$$e_1^* x_0 - e_0^* x_1 \in M_1 \otimes \omega(-1, 0) \oplus M_2 \otimes \omega(-2, 0)$$

give a basis for $H_0(\mathbf{R}(M_{\geq 1}))$; the point is that no cycle involving x_0 can be a boundary. But notice that $e_1^* x_0 - e_0^* x_1$ and $1^* x_0$ do not form a submodule ($1^* x_1$ is missing, because it's a boundary). Therefore, there is no quasi-isomorphism

$$H_*(\mathbf{R}(M_{\geq 1})) \hookrightarrow \mathbf{R}(M_{\geq 1}).$$

So, to adapt the [EFS03] construction of the Tate resolution to the toric setting, we can't resolve the (dual of the) homology of $\mathbf{R}(M_{\geq r})$. Fortunately, there is a way around this.

4.4. **Alternative definition of the Tate resolution.** The first step is an analogue of (part of) the Reciprocity Theorem from [EFS03]:

Claim 4.4. *For any differential E-module N, i.e. any object in $\operatorname{Com}(E)$ of the form*

$$\cdots \xrightarrow{\partial} N(0, -1) \xrightarrow{\partial} N \xrightarrow{\partial} N(0, 1) \xrightarrow{\partial} \cdots,$$

there is a canonical quasi-isomorphism

$$N \xrightarrow{\simeq} (\mathbf{R} \circ \overline{\mathbf{L}})(N).$$

Proof. Fill in. $\square$

Claim 4.5. *For any differential E-module N such that $\overline{\mathbf{L}}(N)_i = 0$ for $i \gg 0$, the complex $(\mathbf{R} \circ \overline{\mathbf{L}})(N)$ is homotopy equivalent to a* **minimal** *complex of the form*

$$\cdots \to H_*(0,-1) \to H_* \to H_*(0,1) \to \cdots ,$$

where H_q is the q^{th} vertical homology of the bicomplex whose totalization is $(\mathbf{R} \circ \overline{\mathbf{L}})(N)$. Moreover, the linear part of this complex is induced by the horizontal differential in the bicomplex (i.e. the $\mathbf{R}$-functor differential).

Proof. I think this should follow immediately from [EFS03, Lemma 3.5]. □

Now, let M be a finitely generated S-module with regularity r, and take $N = \mathbf{R}(M_{\geq r})^\vee$, where

$$(-)^\vee = \operatorname{Hom}_E(-, E).$$

Proposition 4.6. *$\overline{\mathbf{L}}(N)$ is concentrated in homological degrees $-(2n+2), \ldots, -(n+1)$.*

Proof. First, notice that $\omega = E(-w, n+1)$, and so $\omega(-d, 0)^\vee = E(w+d, -n-1)$. In particular, each $\omega(-d,0)^\vee$ is 0 in degrees (i,j) such that $j < n+1$ and $j > 2n+2$. Therefore, the same is true of

$$(\mathbf{R}(M)_0)^\vee = \bigoplus_{d \in \mathbb{Z}} M_d \otimes \omega(-d, 0)^\vee.$$

Now just apply the definition of the $\overline{\mathbf{L}}$-functor. □

Remark 4.7. The bad news is that the terms of $\overline{\mathbf{L}}(N)$ will almost always be infinitely generated.

In particular, $\overline{\mathbf{L}}(N)$ is bounded, and so Claim (4.5) applies to N. Let $\operatorname{Tail}(N)$ denote the minimal complex obtained in Claim 4.5. Combining Claims 4.4 and 4.5, we get a composition

$$\mathbf{R}(M_{\geq r})^\vee = N \xrightarrow{\simeq} (\mathbf{R} \circ \overline{\mathbf{L}})(N) \xrightarrow{\simeq} \operatorname{Tail}(N).$$

of quasi-isomorphisms. Dualizing, and noting that E is self-injective, we arrive at a quasi-isomorphism

$$\operatorname{Tail}(N)^\vee \xrightarrow{\simeq} \mathbf{R}(M_{\geq r}). \tag{9}$$

Claim 4.8. *The mapping cone of (9) is the Tate resolution of M.*

Proof. Might be able to use uniqueness of minimal free covers. For this, we would have to show $\operatorname{Tail}(N)^\vee$ is a flag, which I think is true. But I'm worried that our uniqueness result won't apply, because the differential module we're resolving isn't finitely generated. Instead, it might be better to just directly show that the ranks of the modules in (9) compute sheaf cohomology, as in [EFS03]. □

Corollary 4.9. *There is a canonical map*

$$\mathbf{R}(H_*(\overline{\mathbf{L}}(N))^\vee \to \mathbf{R}(M_{\geq r})$$

whose mapping cone is the linear part of the Tate resolution of M.

Proof. As modules, $\mathbf{R}(H_*(\overline{\mathbf{L}}(N))) = \operatorname{Tail}(N)$. The canonical map is just the map (9). □

4.5. **Computing sheaf cohomology.** Assume Claim 4.8 is true. We recall that we can read the sheaf cohomology of $\mathcal{F} := \widetilde{M}$ off of its Tate resolution in the following way:

$$\dim_k H^q(X, \mathcal{F}(\ell)) = \#\text{ of copies of } \omega(-\ell, -n-q) \text{ appearing in } \mathrm{Tate}(\mathcal{F})_n.$$

Notice that n doesn't appear on the left hand side. So in practice, we can just take $n = 0$.

Proposition 4.10. *As in the discussion in Subsection 4.4, let $N = \mathbf{R}(M_{\geq r})^\vee$. Recall that w denotes the sum of the weights of the variables in S, and $n+1$ is the number of variables in S. We have:*

$$\dim_k H^q(X, \mathcal{F}(\ell)) = \begin{cases} \dim_k H_{q-2(n+1)+1}(\overline{\mathbf{L}}(N))_{-\ell-2w}, & q > 0, \text{ or } q = 0 \text{ and } \ell < r; \\ \dim_k M_\ell, & q = 0 \text{ and } \ell \geq r \end{cases}$$

Proof. By Corollary 4.9, to compute the sheaf cohomology of $\mathcal{F}$, we just need to understand $\mathbf{R}(M)_0$ and $\mathbf{R}(H_*(\overline{\mathbf{L}}(N)))^\vee_{-1}$. This just amounts to unravelling the definitions of the $\overline{\mathbf{L}}$ and $\mathbf{R}$ functors. □

Remark 4.11. It wasn't obviously necessary in the construction of the Tate resolution to take the truncation of M at $\mathrm{reg}(M)$. But it must be necessary either to prove that our two definitions of the Tate resolution coincide or that this alternative definition of the Tate resolution computes sheaf cohomology.

For instance, take the example $X = \mathbb{P}^0$ and $M = S$. We have

$$\mathbf{R}(M) = \cdots \xrightarrow{0} E \xrightarrow{0} E(0,1) \xrightarrow{0} E(0,2) \xrightarrow{0} \cdots,$$

where $E(0,1)$ is in homological degree 0. Dualizing, we get

$$N = \mathbf{R}(M)^\vee = \cdots \xrightarrow{0} E(0,-2) \xrightarrow{0} E(0,-1) \xrightarrow{0} E \xrightarrow{0} \cdots.$$

Therefore, $\overline{\mathbf{L}}(N)$ is the complex

$$0 \to S \xrightarrow{x} S(1) \to 0$$

concentrated in homological degrees $-1, -2$. So, we have $H_{-2}(\overline{\mathbf{L}}(N)) = S(1)/x$, implying that

$$\dim_k H^1(X, \mathcal{O}(-1)) = 1,$$

which is absurd.

4.6. **Examples.**

Example 4.12. Let's go back to the setting of Example 4.3, so $X = \mathbb{P}(1,2)$ and $M = S/(x_0^2 - x_1)$. Let's compute the cohomology of $\mathcal{F}(\ell) = \widetilde{M(\ell)}$ for all ℓ using Proposition 4.10. Note: the answer we're after is

$$\dim_k H^0(X, \mathcal{F}(\ell)) = 1$$

for all $\ell \in \mathbb{Z}$.

The regularity of M is 1. In this case, we have

$$\omega = E(-3, 2).$$

Thus, $\mathbf{R}(M_{\geq 1})$ is the complex

$$\cdots \to \bigoplus_{d \geq 1} E(-3-d, 1) \to \bigoplus_{d \geq 1} E(-3-d, 2) \to \bigoplus_{d \geq 1} E(-3-d, 3) \to \cdots$$

with $\bigoplus_{d\geq 1} E(-3-d,2)$ in homological degree 0, with differential

$$\begin{pmatrix} 0 & 0 & 0 & 0 & \cdots \\ e_1 & 0 & 0 & 0 & \cdots \\ e_2 & e_1 & 0 & 0 & \cdots \\ 0 & e_2 & e_1 & 0 & \cdots \\ 0 & 0 & e_2 & e_1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}.$$

Dualizing, we get

$$\cdots \to \bigoplus_{d\geq 1} E(3+d,-3) \to \bigoplus_{d\geq 1} E(3+d,-2) \to \bigoplus_{d\geq 1} E(3+d,-1) \to \cdots$$

with differential

$$\begin{pmatrix} 0 & e_1 & e_2 & 0 & 0 & \cdots \\ 0 & 0 & e_1 & e_2 & 0 & \cdots \\ 0 & 0 & 0 & e_1 & e_2 & \cdots \\ 0 & 0 & 0 & 0 & e_1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}.$$

♣♣♣ Michael: [The signs are off in everything that follows, because there should be signs in the transpose of a matrix over a skew-symmetric ring. Figure this out later.] We have

$$\dim_k E(3+d,-2)_{(i,j)} = \begin{cases} 1, & (i,j) = (-d-3,2), (-d-4,3), (-d-5,3), (-d-6,4) \\ 0 & \text{else.} \end{cases}$$

So, $\overline{\mathbf{L}}(N,\partial)$ is a complex of the form

$$0 \to \bigoplus_{d\geq 1} S(d+3) \to \bigoplus_{d\geq 1} S(d+4) \oplus S(d+5) \to \bigoplus_{d\geq 1} S(d+6) \to 0,$$

concentrated in homological degrees $-2,-3,-4$ (notice that this squares with Proposition 4.6). We need to compute the Hilbert function of the homology of this complex.

Before we do this, let's note that, by Proposition 4.10, we should get

- $H_i\mathbf{L}(N,\partial) = 0$ for $i \neq -3$,
- $\dim_k H_{-3}\mathbf{L}(N,\partial)_{-\ell-6} = 0$ for $\ell \geq 1$ and 1 for $\ell < 1$. ♣♣♣ Michael: [So the homology probably should be $S(6)/x_1$ in degree -3.]

The $(-2)^{\text{th}}$ differential is the matrix

$$\begin{pmatrix} \begin{pmatrix} x_0 \\ x_1 \end{pmatrix} & \begin{pmatrix} 1 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ -1 \end{pmatrix} & 0 & \cdots \\ 0 & \begin{pmatrix} x_0 \\ x_1 \end{pmatrix} & \begin{pmatrix} 1 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ -1 \end{pmatrix} & \cdots \\ 0 & 0 & \begin{pmatrix} x_0 \\ x_1 \end{pmatrix} & \begin{pmatrix} 1 \\ 0 \end{pmatrix} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{pmatrix}.$$

The $(-3)^{\text{th}}$ differential is the matrix

$$\begin{pmatrix} \begin{pmatrix} x_1 & -x_0 \end{pmatrix} & \begin{pmatrix} 0 & -1 \end{pmatrix} & \begin{pmatrix} 1 & 0 \end{pmatrix} & 0 & \cdots \\ 0 & \begin{pmatrix} x_1 & -x_0 \end{pmatrix} & \begin{pmatrix} 0 & -1 \end{pmatrix} & \begin{pmatrix} 1 & 0 \end{pmatrix} & \cdots \\ 0 & 0 & \begin{pmatrix} x_1 & -x_0 \end{pmatrix} & \begin{pmatrix} 0 & -1 \end{pmatrix} & \cdots \\ \vdots & \vdots & \vdots & \vdots, & \end{pmatrix}$$

♣♣♣ Michael: [The signs are obviously wrong. The matrices don't compose to 0.] Clearly

$$H_{-2}\mathbf{L}(N, \partial) = 0,$$

and it follows from the graded Nakayama's Lemma that

$$H_{-4}\mathbf{L}(N, \partial) = 0.$$

5. Beilinson monad

5.1. **Definition of the Beilinson monad.** Let $\mathcal{L}(C)$ denote the bicomplex of $\mathcal{O}_X$-modules given by applying the associated sheaf functor to the bicomplex (3). Let $\mathcal{L}'(C)$ be the sub-bicomplex of $\mathcal{L}(C)$ given by taking summands of the form $C_{p,(-d,-q)} \otimes_k \mathcal{O}(d)$ with d effective. From now on, we'll write "$d \geq 0$" for "d effective". Here, p denotes homological degree, and $(-d, -q)$ denotes internal degree. We define a functor

$$\mathbf{U} : \mathrm{Com}(E) \to \mathrm{Com}(\mathbb{P})$$

to be given by $C \mapsto \mathrm{Tot}^{\oplus}(\mathcal{L}'(C))$.

♣♣♣ Michael: [Why $\mathrm{Tot}^{\oplus}$ and not Tot^{Π}? Why $d \geq 0$ and not $d \leq 0$? Give clean conceptual explanation for definition of $\mathbf{U}$-functor.]

Proposition 5.1. *The above definition of the* $\mathbf{U}$*-functor agrees with Daniel's.*

Proof. Daniel's definition is given by

$$\omega(i, j) \mapsto \mathcal{L}(\omega_{\leq i})(i)[-j].$$

(We're abusing notation here by identifying the 1-column bicomplex $\mathcal{L}(\omega_{\leq i})(i)$ with its totalization.) We have

$$\begin{aligned} (\mathcal{L}(\omega_{\leq i})(i)[-j])_q &= \mathcal{L}(\omega_{\leq i})_{-j+q}(i) \\ &= \bigoplus_d (\omega_{\leq i})_{(-d, j-q)} \otimes \mathcal{O}(d+i) \\ &= \bigoplus_d (\omega_{\leq i})_{(i-d, j-q)} \otimes \mathcal{O}(d) \\ &= \bigoplus_{d \geq 0} (\omega_{\leq i})_{(i-d, j-q)} \otimes \mathcal{O}(d) \\ &= \bigoplus_{d \geq 0} \omega_{(i-d, j-q)} \otimes \mathcal{O}(d) \\ &= \mathbf{U}(\omega(i, j))_q. \end{aligned}$$

And of course the maps in both complexes are identical as well. $\square$

Question 5.2. *Does the* $\mathbf{U}$*-functor preserve homotopy? Check this.*

Just as with the $\mathbf{L}$-functor, applying the $\mathbf{U}$-functor to a 1-periodic complex (e.g. the Tate resolution of a sheaf) yields a complex with some extra structure. In more detail: given an object C in $\mathrm{Com}(E)$ of the form

$$\cdots \to T(0,-1) \to T \to T(0,1) \to \cdots,$$

the (p,q) term of $\mathcal{L}'(C)$ is

$$\bigoplus_{d\geq 0} T(0,-p)_{(-d,-q)} \otimes \mathcal{O}(d) = \bigoplus_{d\geq 0} T_{(-d,-p-q)} \otimes \mathcal{O}(d),$$

and so

$$\mathbf{U}(C)_m = \bigoplus_{p\in\mathbb{Z}} \bigoplus_{d\geq 0} T_{(-d,-m)} \otimes \mathcal{O}(d).$$

Just as in the definition of the $\overline{\mathbf{L}}$-functor above, equip each $\mathbf{U}(C)_m$ with a $k[u]$-module structure determined by the following "shift" operation: if $t = (\dots, t_{-1}, t_0, t_1, \dots) \in \bigoplus_{p\in\mathbb{Z}} \bigoplus_{d\geq 0} T_{(-d,-m)} \otimes \mathcal{O}(d)$,

$$u(t)_p = (-1)^{m-p} t_{p-1}.$$

An argument identical to the one in Proposition 3.2 shows that the differential on $\mathbf{U}(C)$ is $k[u]$-linear.

So, $\mathbf{U}(C)$ is a complex of $\mathcal{O}_{X\times\mathbb{G}_m}$-modules. Define

$$\overline{\mathbf{U}}(C) := \mathbf{U}(C)/(u-1).$$

We have an isomorphism

$$\overline{\mathbf{U}}(C)_m \cong \bigoplus_{d\geq 0} T_{(-d,-m)} \otimes \mathcal{O}(d) \tag{10}$$

given by representing each class in $\mathbf{BM}(\mathcal{F})_m$ by an element concentrated in the $p=0$ summand of $\bigoplus_{p\in\mathbb{Z}} \bigoplus_{d\geq 0} T_{(-d,-m)} \otimes \mathcal{O}(d)$. Via this isomorphism, the differential on $\mathbf{BM}(\mathcal{F})$ is given by $(-1)^m d_C + d_K$, where d_T is the Tate differential and d_K is the dual Koszul differential.

Definition 5.3. Let $\mathcal{F} \in \mathrm{coh}(X)$. The *Beilinson monad* of $\mathcal{F}$ is the complex

$$\mathbf{BM}(\mathcal{F}) := (\overline{\mathbf{U}} \circ \tau_* \circ \kappa^*)(\mathcal{F}).$$

Theorem 5.4. *The complex* $\mathbf{BM}(\mathcal{F})$ *is a monad with homology* $\mathcal{F}$.

Example 5.5. Take $X = \mathbb{P}(w)$, with w some positive integer, and take $\mathcal{F} = \mathcal{O}$. So X is a stacky point. If we set $T = \bigoplus_{i\in\mathbb{Z}} \omega(i,0)$, then

$$(\tau_* \circ \kappa^*)(\mathcal{O})) = \cdots \xrightarrow{e} T(0,-1) \xrightarrow{e} T \xrightarrow{e} T(0,1) \xrightarrow{e} \cdots$$

We therefore have

$$(\mathbf{U} \circ \tau_* \circ \kappa^*)(\mathcal{O}))_m = \bigoplus_{p\in\mathbb{Z}} \bigoplus_{d\geq 0} \bigoplus_{i\in\mathbb{Z}} \omega_{(i-d,-m)} \otimes \mathcal{O}(d) = \begin{cases} \bigoplus_{p\in\mathbb{Z}} \bigoplus_{d\geq 0} \omega_{(0,0)} \otimes \mathcal{O}(d), & m=0; \\ \bigoplus_{p\in\mathbb{Z}} \bigoplus_{d\geq 0} \omega_{(w,-1)} \otimes \mathcal{O}(d), & m=1; \\ 0, & \text{else.} \end{cases}$$

Taking coinvariants of the $\mathbb{G}_m$-action and applying the isomorphism (10) gives the complex

$$0 \to \bigoplus_{d\geq 0} \omega_{(w,-1)} \otimes \mathcal{O}(d) \xrightarrow{\begin{pmatrix} -1 & 0 & 0 & 0 & \dots \\ x & -1 & 0 & 0 & \dots \\ 0 & x & -1 & 0 & \dots \\ 0 & 0 & x & -1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}} \bigoplus_{d\geq 0} \omega_{(0,0)} \otimes \mathcal{O}(d) \to 0$$

whose homology is $\mathcal{O}$ in degree 0 and 0 elsewhere, as expected.

5.2. **Proof of Theorem 5.4.** Let $\mathcal{R} \in \mathrm{Com}(X \times X)$ be Daniel's resolution of the diagonal. To prove Theorem 5.4, it suffices to give a homotopy equivalence

$$\mathbf{BM}(\mathcal{F}) \simeq \mathbf{R}\pi_{2*}(\pi_1^* \mathcal{F} \otimes \mathcal{R}).$$

The notation $\mathbf{R}\pi_{2*}$ is misleading: we are not working at the level of derived categories. Rather, we use the notation $\mathbf{R}\pi_{2*}$ to denote the Čech model for $\mathbf{R}\pi_{2*}$ induced by the affine open cover $\{V_0, \dots, V_t\}$ of X chosen above.

The rough idea is to define a map

$$\mathrm{id} \times \overline{\mathbf{U}} : \mathrm{Com}(X \times E) \dashrightarrow \mathrm{Com}(X \times X)$$

such that the diagram

$$\begin{tikzcd} & \mathrm{Com}(X \times E) \arrow[r, "\tau_*"] \arrow[d, dashed, "\mathrm{id} \times \overline{\mathbf{U}}"] & \mathrm{Com}(E) \arrow[d, "\overline{\mathbf{U}}"] \\ \mathrm{coh}(X) \arrow[ur, "\kappa^*"] \arrow[r, "\pi_1^*(-)\otimes\mathcal{R}"'] & \mathrm{Com}(X \times X) \arrow[r, "\mathbf{R}\pi_{2*}"] & \mathrm{Com}(X) \end{tikzcd}$$

commutes up to homotopy. Here is how to define $\mathrm{id} \times \overline{\mathbf{U}}$ on the image of κ^* (which is all we need). Recall that

$$\kappa^*(\mathcal{F})_i = \bigoplus_{l \in \mathrm{Pic}(X)} \mathcal{F}(-\ell) \otimes \omega(\ell, -i)$$

and the differential is the dual Koszul map. We apply "$\mathrm{id} \times \mathbf{U}$" to $\kappa^*(\mathcal{F})$ to get the complex whose m^{th} term is

$$\bigoplus_{p\in\mathbb{Z}} \bigoplus_{d \geq 0} \bigoplus_{l \in \mathrm{Pic}(X)} \omega_{(\ell-d,-m)} \otimes \mathcal{F}(-\ell) \boxtimes \mathcal{O}(d)$$

with differential $\sum_{i=0}^n e_i \otimes x_i + (-1)^p e_i \otimes y_i$. This complex has $\mathbb{G}_m$-action just as $(\mathbf{U} \circ \tau_* \circ \kappa^*)(\mathcal{O})$) does. Taking coinvariants and applying an isomorphism similar to (10), we arrive at the complex with m^{th} term

$$\bigoplus_{d \geq 0} \bigoplus_{l \in \mathrm{Pic}(X)} \omega_{(\ell-d,-m)} \otimes \mathcal{F}(-\ell) \boxtimes \mathcal{O}(d)$$

and m^{th} differential $\sum_{i=0}^n (-1)^m e_i \otimes x_i + e_i \otimes y_i$.

Proposition 5.6. $(\mathrm{id}\times\overline{\mathbf{U}})(\kappa^*(\mathcal{O}))$ *coincides with Daniel's resolution of the diagonal.*

Proof. Fill in. Should be easy. Also, I think the argument Daniel uses to show his complex is a resolution of the diagonal works to prove $(\mathrm{id}\times\overline{\mathbf{U}})(\kappa^*(\mathcal{O}))$ is as well. $\square$

Remark 5.7. Suppose $X = \mathbb{P}(w_0, \dots, w_n)$. Set $w = \sum w_i$. Suppose we change the definition of $(\mathrm{id}\times\overline{\mathbf{U}})(\kappa^*(\mathcal{O}))$ slightly so that we get a complex that looks like this:

(11)

$$0 \to \bigoplus_{0\le d\le w-1}\bigoplus_{0\le l\le d} \omega_{(d-\ell,-(n-1))}\otimes\mathcal{O}(-d,\ell) \to \cdots \to \bigoplus_{0\le d\le w-1}\bigoplus_{0\le l\le d} \omega_{(d-\ell,0)}\otimes\mathcal{O}(-d,\ell) \to 0.$$

Here are the changes:

- restricted the ranges on d and ℓ
- changed $\omega_{(l-d,-m)}$ to $\omega_{(d-l,-m)}$, and
- changed $\mathcal{O}(-\ell,d)$ to $\mathcal{O}(-d,\ell)$.

I claim that (11) is Canonaco-Karp's resolution of the diagonal. I checked this for $\mathbb{P}(1,2)$ and it's correct on the nose.

The first change in the list above amounts to changing the definitions of κ and $\mathbf{U}$ so that the "irrelevant" Koszul complexes are removed. The second change arises from our choice to take $d \ge 0$ summands rather than $d \le 0$ in the definition of the $\mathbf{U}$-functor; this choice seems to create a conflict with Canonaco-Karp. Not sure if this was just an arbitrary choice or if one is better than the other. The third change just switches the order of the tensor factors, keeping the second change in mind.

Example 5.8. Let's check this for $X = \mathbb{P}(w)$. Reading off the formula, we get

$$0 \to \bigoplus_{d\ge 0}\omega_{(w,-1)}\otimes\mathcal{O}(d,-(d+w)) \xrightarrow{\begin{pmatrix} y & 0 & 0 & 0 & \dots \\ -x & y & 0 & 0 & \dots \\ 0 & -x & y & 0 & \dots \\ 0 & 0 & -x & y & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}} \bigoplus_{d\ge 0}\omega_{(0,0)}\otimes\mathcal{O}(d,-d) \to 0.$$

The homology is $\mathcal{O}\oplus\cdots\oplus\mathcal{O}(w-1,-(w-1))$, and this is indeed the diagonal in this case (this is not entirely trivial to check). Notice this precisely recovers Daniel's "1-variable" example when $w = 1$.

Proposition 5.9. $(\mathbf{R}\pi_{2*}\circ(\mathrm{id}\times\overline{\mathbf{U}})\circ\kappa^*)(\mathcal{F}) = (\overline{\mathbf{U}}\circ\tau_*\circ\kappa^*)(\mathcal{F})$.

Proof. Let's start by computing the right hand side. As above, set

$$T = \bigoplus_{i=0}^{t}\bigoplus_{\ell\in\mathrm{Pic}(X)} \mathcal{C}_{-i}^{\mathcal{F}(-\ell)}\otimes\omega(\ell,-i),$$

so that

$$(\overline{\mathbf{U}}\circ\tau_*\circ\kappa^*)(\mathcal{F}) = \overline{\mathbf{U}}(\cdots\to T(0,-1)\to T\to T(0,1)\to\cdots).$$

We have

$$\begin{aligned}(\overline{\mathbf{U}} \circ \tau_* \circ \kappa^*)(\mathcal{F})_m &= \bigoplus_{d \geq 0} T_{(-d,-m)} \otimes \mathcal{O}(d) \\ &= \bigoplus_{d\geq 0} \bigoplus_{i=0}^{t} \bigoplus_{\ell \in \operatorname{Pic}(X)} \omega_{(l-d,-i-m)} \otimes \mathcal{C}^{\mathcal{F}(-\ell)}_{-i} \otimes \mathcal{O}(d),\end{aligned}$$

and the differential is $ud_T + d_K$. On the other hand $(\mathbf{R}\pi_{2*} \circ (\operatorname{id} \times \overline{\mathbf{U}}) \circ \kappa^*)(\mathcal{F})$ is the sheaf given as follows. Let W be an open set in X. We abuse notation slightly and write $\mathcal{C}^{\mathcal{O}(d)|_W}$ for the Čech complex on $\mathcal{O}(d)|_W$ corresponding to the open cover

$$\{V_0 \cap W, \dots, V_t \cap W\}.$$

We recall that the natural map

$$\mathcal{C}^{\mathcal{F}(-\ell)} \otimes \mathcal{C}^{\mathcal{O}(d)|_W} \to \mathcal{C}^{\mathcal{F}(-\ell) \boxtimes \mathcal{O}(d)|_W}$$

is a homotopy equivalence, where the target is the Čech complex associated to the open cover $\{V_i \times (V_j \cap W)\}_{0 \leq i,j \leq t}$. Form a bicomplex with p^{th} column given by

$$\bigoplus_{d \geq 0} \bigoplus_{\ell \in \operatorname{Pic}(X)} \omega_{(l-d,-p)} \otimes \mathcal{C}^{\mathcal{F}(-\ell)} \otimes \mathcal{C}^{\mathcal{O}(d)|_W},$$

with vertical differential multiplied by $(-1)^p$, and horizontal differential induced by the differential on $((\operatorname{id} \times \overline{\mathbf{U}}) \circ \kappa^*)(\mathcal{F})$. Applying $\operatorname{Tot}^{\oplus}$ to this complex gives the value of the $(\mathbf{R}\pi_{2*} \circ (\operatorname{id} \times \overline{\mathbf{U}}) \circ \kappa^*)(\mathcal{F})$ at W, up to homotopy equivalence. Explicitly, the value at W is the complex whose m^{th} term is

$$\bigoplus_{i=0}^{t^2} \bigoplus_{d \geq 0} \bigoplus_{\ell \in \operatorname{Pic}(X)} \omega_{(l-d,-i-m)} \otimes (\mathcal{C}^{\mathcal{F}(-\ell)} \otimes \mathcal{C}^{\mathcal{O}(d)|_W})_{-i},$$

with the induced differential. It suffices to check that if we sheafify the presheaf

$$W \mapsto \bigoplus_{i=0}^{t^2} \omega_{(l-d,i-n)} \otimes (\mathcal{C}^{\mathcal{F}(-\ell)} \otimes \mathcal{C}^{\mathcal{O}(d)|_W})_{-i},$$

we get

$$\bigoplus_{i=0}^{t} \omega_{(l-d,i-n)} \otimes \mathcal{C}^{\mathcal{F}(-\ell)}_{-i} \otimes \mathcal{O}(d).$$

And of course we need to check the differentials coincide as well. Need to fill in the rest of the details, but I think this is clear. □

Proof of Theorem 5.4. Combine Propositions 5.6 and 5.9. □

References

[EFS03] D. Eisenbud, G. Floystad, and F.-O. Schreyer, *Sheaf cohomology and free resolutions over exterior algebras*, Transactions of the American Mathematical Society **355** (2003), no. 11, 4397–4426.