EXAMPLES OF RESOLUTIONS OF THE DIAGONAL IN WEIGHTED PROJECTIVE SPACE

Below are computations of resolutions of the diagonal over $\mathbb{P}(1,2)$ and $\mathbb{P}(1,1,2)$. I highlighted the answers in blue if you want to just skip to them.

1. $\mathbb{P}(1,2)$

Let x and y be the coordinates on $\mathbb{P} = \mathbb{P}(1,2)$, with $|x| = 1$ and $|y| = 2$. Let $\mathcal{K}$ denote the Koszul complex

$$0 \to \mathcal{O}(-3) \to \mathcal{O}(-1) \oplus \mathcal{O}(-2) \to \mathcal{O} \to 0$$

on $\mathbb{P}$, indexed cohomologically, where $\mathcal{O}$ is in degree 0. Following the notation in [CK08, Section 3], we define complexes $\mathcal{M}_i \subseteq \mathcal{K}(-i)$ for $i = 0, -1, -2$ in the following way:

- $\mathcal{M}_0 = \mathcal{O}$,
- $\mathcal{M}_{-1} = \mathcal{O} \xrightarrow{x} \mathcal{O}(1)$,
- $\mathcal{M}_{-2} = \mathcal{O}(1) \oplus \mathcal{O} \xrightarrow{\begin{pmatrix} x & y \end{pmatrix}} \mathcal{O}(2)$.

In Canonaco-Kemp's resolution of the diagonal, the $\mathcal{M}_i$ play the role that the powers of the cotangent bundle play in Beilinson's resolution of the diagonal on projective space.

Let x, y, x', y' be the coordinates on $\mathbb{P} \times \mathbb{P}$, with $|x| = 1 = |x'|$ and $|y| = 2 = |y'|$. We start by setting

$$\mathcal{R}_0 := \mathcal{O} \boxtimes \mathcal{M}_0 = \mathcal{O} \in \mathrm{D}^b(\mathbb{P} \times \mathbb{P}).$$

Then we will build complexes $\mathcal{R}_{-1}$ and $\mathcal{R}_{-2}$ using an iterated mapping cone construction. The complex $\mathcal{R}_{-2}$ will be our resolution of the diagonal.

We define a map

$$\alpha_{-1} : \mathcal{O}(-1) \boxtimes \mathcal{M}_{-1}[-1] \to \mathcal{R}_0$$

of complexes in the following way:

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{O}(-1,0) \arrow[r, "-x'"] \arrow[d, "-x"] & \mathcal{O}(-1,1) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{O} \arrow[r] & 0 \arrow[r] & 0
\end{tikzcd}$$

(the sign in the top differential is coming from the shift by -1). Set $\mathcal{R}_{-1} := \mathrm{cone}(\alpha_{-1})$. That is, $\mathcal{R}_{-1}$ is the complex

$$0 \to \mathcal{O}(-1,0) \xrightarrow{\begin{pmatrix} x' \\ -x \end{pmatrix}} \mathcal{O}(-1,1) \oplus \mathcal{O} \to 0,$$

concentrated in degrees -1 and 0.

Next, we define a map

$$\alpha_{-2} : \mathcal{O}(-2) \boxtimes \mathcal{M}_{-2}[-1] \to \mathcal{R}_{-1}$$

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to be given by

$$\begin{tikzcd}
 & 0 \arrow[r] \arrow[d] & \mathcal{O}(-2,1)\oplus\mathcal{O}(-2,0) \arrow[r, "\begin{pmatrix}-x' & -y'\end{pmatrix}"] \arrow[d, "\begin{pmatrix}-x & 0\\ 0 & -y\end{pmatrix}"] & \mathcal{O}(-2,2) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{O}(-1,0) \arrow[r, "\begin{pmatrix}x'\\ -x\end{pmatrix}"] & \mathcal{O}(-1,1)\oplus\mathcal{O} \arrow[r] & 0. &
\end{tikzcd}$$

Our resolution of the diagonal is $\mathrm{cone}(\alpha_{-2})$. Explicitly, it's the complex

$$0 \to \mathcal{O}(-2,1)\oplus\mathcal{O}(-2,0)\oplus\mathcal{O}(-1,0) \xrightarrow{\begin{pmatrix}x' & y' & 0\\ -x & 0 & x'\\ 0 & -y & -x\end{pmatrix}} \mathcal{O}(-2,2)\oplus\mathcal{O}(-1,1)\oplus\mathcal{O} \to 0$$

concentrated in degrees -1 and 0.

2. $\mathbb{P}(1,1,2)$

Let x, y, z be coordinates on $\mathbb{P} = \mathbb{P}(1,1,2)$, with $|x| = 1 = |y|$ and $|z| = 2$. Let $\mathcal{K}$ denote the Koszul complex

$$0 \to \mathcal{O}(-4) \to \mathcal{O}(-3)\oplus\mathcal{O}(-3)\oplus\mathcal{O}(-2) \to \mathcal{O}(-1)\oplus\mathcal{O}(-1)\oplus\mathcal{O}(-2) \to \mathcal{O} \to 0,$$

as above. This time, we have complexes $\mathcal{M}_i \subseteq \mathcal{K}(-i)$ for $i = 0, -1, -2, -3$:

- $\mathcal{M}_0 = \mathcal{O}$,
- $\mathcal{M}_{-1} = \mathcal{O}\oplus\mathcal{O} \xrightarrow{\begin{pmatrix}x & y\end{pmatrix}} \mathcal{O}(1)$,
- $\mathcal{M}_{-2} = \mathcal{O} \xrightarrow{\begin{pmatrix}y\\ -x\\ 0\end{pmatrix}} \mathcal{O}(1)\oplus\mathcal{O}(1)\oplus\mathcal{O} \xrightarrow{\begin{pmatrix}x & y & z\end{pmatrix}} \mathcal{O}(2)$,
- $\mathcal{M}_{-3} = \mathcal{O}(1)\oplus\mathcal{O}\oplus\mathcal{O} \xrightarrow{\begin{pmatrix}y & z & 0\\ -x & 0 & z\\ 0 & -x & -y\end{pmatrix}} \mathcal{O}(2)\oplus\mathcal{O}(2)\oplus\mathcal{O}(1) \xrightarrow{\begin{pmatrix}x & y & z\end{pmatrix}} \mathcal{O}(3)$.

Let x, y, z, x', y', z' be the coordinates on $\mathbb{P}\times\mathbb{P}$, with the obvious degrees. As above, we set

$$\mathcal{R}_0 = \mathcal{O}\boxtimes\mathcal{M}_0 = \mathcal{O} \in \mathrm{D}^b(\mathbb{P}\times\mathbb{P}).$$

We have a map

$$\alpha_{-1} : \mathcal{O}(-1)\boxtimes\mathcal{M}_{-1}[-1] \to \mathcal{R}_0$$

of complexes given by

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{O}(-1,0)\oplus\mathcal{O}(-1,0) \arrow[r, "\begin{pmatrix}-x' & -y'\end{pmatrix}"] \arrow[d, "\begin{pmatrix}-x & -y\end{pmatrix}"] & \mathcal{O}(-1,1) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{O} \arrow[r] & 0 \arrow[r] & 0.
\end{tikzcd}$$

We set $\mathcal{R}_{-1}$ to be $\operatorname{cone}(\alpha_{-1})$, i.e. the complex

$$0 \to \mathcal{O}(-1,0) \oplus \mathcal{O}(-1,0) \xrightarrow{\begin{pmatrix} x' & y' \\ -x & -y \end{pmatrix}} \mathcal{O}(-1,1) \oplus \mathcal{O} \to 0,$$

concentrated in degrees -1 and 0.

Next, we have a map

$$\alpha_{-2} : \mathcal{O}(-2) \boxtimes \mathcal{M}_{-2}[-1] \to \mathcal{R}_{-1}$$

given by

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{O}(-2,0) \arrow[r, "\begin{pmatrix} -y' \\ x' \\ 0 \end{pmatrix}"] \arrow[d, "\begin{pmatrix} y \\ -x \end{pmatrix}"] & \mathcal{O}(-2,1) \oplus \mathcal{O}(-2,1) \oplus \mathcal{O}(-2,0) \arrow[r, "\begin{pmatrix} -x' & -y' & -z' \end{pmatrix}"] \arrow[d, "\begin{pmatrix} -x & -y & 0 \\ 0 & 0 & -z \end{pmatrix}"] & \mathcal{O}(-2,2) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{O}(-1,0) \oplus \mathcal{O}(-1,0) \arrow[r, "\begin{pmatrix} x' & y' \\ -x & -y \end{pmatrix}"] & \mathcal{O}(-1,1) \oplus \mathcal{O} \arrow[r] & 0 \arrow[r] & 0.
\end{tikzcd}$$

We take $\mathcal{R}_{-2} = \operatorname{cone}(\alpha_{-2})$, which is the complex

$$\begin{aligned} 0 \to \mathcal{O}(-2,0) &\xrightarrow{\begin{pmatrix} y' \\ -x' \\ 0 \\ y \\ -x \end{pmatrix}} \mathcal{O}(-2,1) \oplus \mathcal{O}(-2,1) \oplus \mathcal{O}(-2,0) \oplus \mathcal{O}(-1,0) \oplus \mathcal{O}(-1,0) \\ &\xrightarrow{\begin{pmatrix} x' & y' & z' & 0 & 0 \\ -x & -y & 0 & x' & y' \\ 0 & 0 & -z & -x & -y \end{pmatrix}} \mathcal{O}(-2,2) \oplus \mathcal{O}(-1,1) \oplus \mathcal{O} \\ &\to 0, \end{aligned}$$

concentrated in degrees $-2, -1, 0$.

Finally, we compute $\mathcal{R}_{-3}$, which is our resolution of the diagonal. We need to construct a map

$$\alpha_{-3} : \mathcal{O}(-3) \boxtimes \mathcal{M}_{-3}[-1] \to \mathcal{R}_{-2}.$$

To shorten notation, let C_{-2}, C_{-1}, and C_0 denote the terms of $\mathcal{R}_{-2}$. Our map α_{-3} is given by

$$\begin{tikzcd}
0 \arrow[r] & 0 \arrow[r] \arrow[d] & \mathcal{O}(-3) \boxtimes \mathcal{M}_{-3}^{-2} \arrow[r] \arrow[d, "\alpha_{-3}^{-1}"] & \mathcal{O}(-3) \boxtimes \mathcal{M}_{-3}^{-1} \arrow[r] \arrow[d, "\alpha_{-3}^{0}"] & \mathcal{O}(-3) \boxtimes \mathcal{M}_{-3}^{0} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{R}_{-2}^{-2} \arrow[r] & \mathcal{R}_{-2}^{-1} \arrow[r] & \mathcal{R}_{-2}^{0} \arrow[r] & 0 \arrow[r] & 0,
\end{tikzcd}$$

where

$$\alpha_{-3}^{-1} = \begin{pmatrix} y & 0 & 0 \\ -x & 0 & 0 \\ 0 & -x & -y \\ 0 & z & 0 \\ 0 & 0 & z \end{pmatrix},$$

and

$$\alpha_{-3}^{0} = \begin{pmatrix} -x & -y & 0 \\ 0 & 0 & -z \\ 0 & 0 & 0 \end{pmatrix}.$$

[The diagram doesn't quite commute, because some signs are off in the entries of these matrices. In particular, the matrices in the resolution of the diagonal are off by a few signs. I'll fix this.] Our resolution of the diagonal is cone(α_{-3}). Explicitly, it's the complex

$$0 \to (\mathcal{O}(-3) \boxtimes \mathcal{M}_{-3}^{-2}) \oplus \mathcal{R}_{-2}^{-2} \xrightarrow{\begin{pmatrix} y' & z' & 0 & 0 \\ -x' & 0 & z' & 0 \\ 0 & -x' & -y' & 0 \\ y & 0 & 0 & -y' \\ -x & 0 & 0 & x' \\ 0 & -x & -y & 0 \\ 0 & z & 0 & y \\ 0 & 0 & z & -x \end{pmatrix}} (\mathcal{O}(-3) \boxtimes \mathcal{M}_{-3}^{-1}) \oplus \mathcal{R}_{-2}^{-1}$$

$$\xrightarrow{\begin{pmatrix} x' & y' & z' & 0 & 0 & 0 & 0 & 0 \\ -x & -y & 0 & -x' & -y' & -z' & 0 & 0 \\ 0 & 0 & -z & -x & -y & 0 & x' & y' \\ 0 & 0 & 0 & 0 & 0 & -z & -x & -y \end{pmatrix}} (\mathcal{O}(-3) \boxtimes \mathcal{M}_{-3}^{0}) \oplus \mathcal{R}_{-2}^{0} \to 0.$$

References

[CK08] A. Canonaco and R. L. Karp, *Derived autoequivalences and a weighted Beilinson resolution*, Journal of Geometry and Physics **58** (2008), no. 6, 743–760.