CLASS AND RANK OF DIFFERENTIAL MODULES

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Abstract. A differential module is a module equipped with a square-zero endomorphism. This structure underpins complexes of modules over rings, as well as differential graded modules over graded rings. We establish lower bounds on the class—a substitute for the length of a free complex—and on the rank of a differential module in terms of invariants of its homology. These results specialize to basic theorems in commutative algebra and algebraic topology. One instance is a common generalization of the equicharacteristic case of the New Intersection Theorem of Hochster, Peskine, P. Roberts, and Szpiro, concerning complexes over commutative noetherian rings, and of a theorem of G. Carlsson on differential graded modules over graded polynomial rings.

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Introduction

This paper has its roots in a confluence of ideas from commutative algebra and algebraic topology. Similarities between two series of results and conjectures in these fields were discovered and efficiently exploited by Gunnar Carlsson more than twenty years ago. On the topological side they dealt with finite CW complexes admitting free torus actions; on the algebraic one, with finite free complexes with homology of finite length. However, no single statement—let alone common proof—covers even the basic case of modules over polynomial rings.

In this paper we explore the commonality of the earlier results and prove that broad generalizations hold for all commutative algebras over fields. They include

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both Carlsson's theorems on differential graded modules over graded polynomial rings and the New Intersection Theorem for local algebras, due to Hochster, Peskine, P. Roberts, and Szpiro. They also suggest precise statements about matrices over commutative rings, that imply conjectures on free resolutions, due to Buchsbaum, Eisenbud, and Horrocks, and conjectures on the structure of complexes with almost free torus actions, due to Carlsson and Halperin. These conjectures are among the fundamental open questions on both sides of this narrative.

The focus here is on a simple construct: a module over an associative ring R, equipped with an R-linear endomorphism of square zero. We call these data a *differential R-module*. They are part of the structure underlying the familiar and ubiquitous notions of complex or differential graded module. Differential modules as such appeared already five decades ago in Cartan and Eilenberg's treatise [10], where they are assigned mostly didactic functions. Our goal is to establish that these basic objects are of considerable interest in their own right.

To illustrate the direction and scope of the generality so gained, take a complex

$$P = \quad 0 \longrightarrow P_l \xrightarrow{\partial_l} P_{l-1} \longrightarrow \cdots \longrightarrow P_1 \xrightarrow{\partial_1} P_0 \longrightarrow 0$$

of finite free modules over a ring R. The module $P = \bigoplus_n P_n$ with endomorphism $\delta = \bigoplus_n \partial_n$ is a differential R-module $P_{\mathbf{\Delta}}$. With respect to an obvious choice of basis for the underlying free module, δ is represented by a block triangular matrix

$$A = \begin{bmatrix} 0 & A_{01} & 0 & \dots & 0 & 0 \\ 0 & 0 & A_{12} & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & A_{l-1,l} \\ 0 & 0 & 0 & \dots & 0 & 0 \end{bmatrix} \quad \text{with} \quad A^2 = 0\,,$$

Results on finite free complexes are equivalent to statements about such matrices.

The key contention, supported by our results, is that such statements should extend in suitable form to *any* strictly upper triangular matrix

$$A = \begin{bmatrix} 0 & A_{01} & A_{02} & \dots & A_{0\,l-1} & A_{0\,l} \\ 0 & 0 & A_{12} & \dots & A_{1\,l-1} & A_{1\,l} \\ 0 & 0 & 0 & \dots & A_{2\,l-1} & A_{2\,l} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & A_{l-1\,l} \\ 0 & 0 & 0 & \dots & 0 & 0 \end{bmatrix} \quad \text{with} \quad A^2 = 0\,,$$

Matrices of this type arise from sequences of submodules

$$\{F^n\} = \quad 0 \subseteq F^0 \subseteq F^1 \subseteq \cdots \subseteq F^l = D$$

in a differential R-module D with endomorphism δ, satisfying for every n the conditions: F^n/F^{n-1} is free of finite rank and $\delta(F^n) \subseteq F^{n-1}$. We say that $\{F^n\}$ is a *free differential flag* with $(l+1)$ folds in D. When D admits such a flag we say that its *free class* is at most l, and write $\operatorname{free\,class}_R D \le l$; else, we set $\operatorname{free\,class}_R D = \infty$. The *projective class* of D is defined analogously, and is denoted $\operatorname{proj\,class}_R D$. Note that if D has finite free (respectively, projective) class, then it is necessarily finitely generated and free (respectively, projective).

The *homology* of D is the R-module $\mathrm{H}(D) = \mathrm{Ker}(\delta)/\mathrm{Im}(\delta)$. A central result of this paper links the size of its annihilator, $\mathrm{Ann}_R \mathrm{H}(D)$, to the class of D by a

Class Inequality. *Let R be a noetherian commutative algebra over a field and D a finitely generated differential R-module. One then has*

$$\operatorname{proj\,class}_R D \geq \operatorname{height} I \quad \textit{where} \quad I = \mathrm{Ann}_R \mathrm{H}(D)\,.$$

The example $D = K_{\Delta}$, where K is the Koszul complex on d elements generating an ideal of height d, shows that the inequality cannot be strengthened in general.

For the differential module P_{Δ} defined above one has

$$l \geq \operatorname{proj\,class}_R P_{\Delta} \quad \text{and} \quad \mathrm{H}(P_{\Delta}) = \bigoplus_n \mathrm{H}_n(P)\,,$$

so the New Intersection Theorem follows from the Class Inequality. The hypothesis that R contain a field is due to the use in our proof of Hochster's big Cohen-Macaulay modules [18]. The conclusion holds whenever such modules exist, in particular, when $\dim R \leq 3$, see [19], or when R is Cohen-Macaulay. P. Roberts proved that the Intersection Theorem holds for all noetherian commutative rings R, see [21], and we conjecture that so does the Class Inequality.

The Class Inequality, a result about commutative rings in general, has its origin in the study of free actions of the group $(\mathbb{Z}/2\mathbb{Z})^d$ on a CW complex X. Carlsson [7] proved that over a polynomial ring $S = \mathbb{F}_2[x_1, \dots, x_d]$ a *differential graded* module C with $\operatorname{rank}_{\mathbb{F}_2} \mathrm{H}(C)$ finite and non-zero has $\operatorname{free\,class}_S C \geq d$ and used this result to produce obstructions for such actions. In [8] he conjectured that $\operatorname{rank}_S C \geq 2^d$ always holds, and showed that a positive answer implies $\sum_n \operatorname{rank}_{\mathbb{F}_2} \mathrm{H}_n(X, \mathbb{F}_2) \geq 2^d$. This is a counterpart to Halperin's question as to whether an almost free action of a d-dimensional real torus forces $\sum_n \operatorname{rank}_{\mathbb{Q}} \mathrm{H}_n(X, \mathbb{Q}) \geq 2^d$, see [15].

Carlsson [9] verified his conjecture for $d \leq 3$, and Allday and Puppe [1] proved that $\operatorname{rank}_S C \geq 2d$ always holds. We subsume these results into the following

Rank Inequalities. *Let R be a commutative noetherian ring, D a differential R-module of finite free class, and set $d = \operatorname{height} \mathrm{Ann}_R \mathrm{H}(D)$. One then has*

$$\operatorname{rank}_R D \geq \begin{cases} 2d & \textit{when } d \leq 3 \textit{ or } R \textit{ is an algebra over a field;} \\ 8 & \textit{when } d \geq 3 \textit{ and } R \textit{ is a unique factorization domain}\,. \end{cases}$$

We conjecture that an inequality $\operatorname{rank}_R D \geq 2^d$ always holds. If it does, it will settle the conjectures of Carlsson and Halperin, and will go a long way towards confirming a classical conjecture of Buchsbaum, Eisenbud, and Horrocks: Over a local ring R, a *free resolution* P of a non-zero R-module of finite length and finite projective dimension should satisfy $\operatorname{rank}_R P_n \geq \binom{\dim R}{n}$ for each n; see [5], [16].

The Rank Inequality is proved in Section 5. The Class Inequality is established in Section 4, via a version for local rings treated in Section 3. The environment for these arguments is *homological algebra of differential modules*. References for the basic formal properties needed to lay out our arguments and a language fit to express our results are lacking. In Sections 1 and 2 we close this gap, guided by the well understood models of complexes and of differential graded modules.

Such a transfer of technology encounters subtle obstacles. Complexes and differential graded modules are endowed with gradings for which the differential is homogeneous. The use of these gradings in homological arguments is so instinctive and pervasive that intuition may falter when they are not available.

A filtration with adequate properties can sometimes compensate for the absence of a grading. This observation led us to the concept of differential flags. Their study earned unexpected dividends. One is an elementary description of matrices that admit a certain standard form over a local ring. To express it succinctly, let 0_r and 1_r denote the $r \times r$ zero and identity matrices, respectively.

Standard Forms. *Let $A = (a_{ij})$ be a $2r \times 2r$ strictly upper triangular matrix with entries in a commutative local ring R. There is a matrix $U \in \mathrm{GL}(2r, R)$ such that*

$$UAU^{-1} = \begin{bmatrix} 0_r & 1_r \\ 0_r & 0_r \end{bmatrix}$$

if, and only if, the solutions in R of the linear system of equations

$$\sum_{j=1}^{2r} a_{ij} x_j = 0 \quad \textit{for} \quad i = 1, \dots, 2r$$

are precisely the R-linear combinations of the columns of A.

This result appears in Section 6. It is a consequence of a theorem proved in Section 2 for arbitrary associative rings: If D is a differential module that admits a projective flag, then $\mathrm{H}(D) = 0$ implies that D is *contractible*; that is, $D \cong C \oplus C$ with differentiation the map $(c', c'') \mapsto (c'', 0)$. Such a statement is needed to get homological algebra started. The corresponding result for bounded complexes of projective modules holds for trivial reasons. On the other hand, not every differential module D with finitely generated projective underlying module and $\mathrm{H}(D) = 0$ is contractible. Simple examples are given in Section 1.

Our migration to the category of differential modules from the more familiar environment of complexes was motivated in part by the investigation in [4] of "levels" in derived categories. The treatment of differential modules presented here is intentionally lean, concentrating just on what is actually needed to present and prove the results. A detailed analysis of the homological or homotopical machinery that differential modules are susceptible of will follow in [3].

1. Differential modules

In this paper R is an associative ring, and rings act on their modules from the left. Right R-modules are identified with modules over R°, the opposite ring of R.

In this section we provide background on differential modules. Under the name 'modules with differentiation' the concept goes back to the monograph of Cartan and Eilenberg [10], where it appears twice: in Ch. IV, §§1,2 preceding the introduction of complexes, and in Ch. XV, §§1–3 in the construction of spectral sequences.

1.1. *Differential modules.* A *differential R-module* is an R-module D equipped with an R-linear map $\delta^D \colon D \to D$, called the *differentiation* of D, satisfying $(\delta^D)^2 = 0$. Sometimes we say a pair (D, δ) is a differential module, implying $\delta = \delta^D$.

A *morphism* of differential R-modules is a homomorphism $\phi\colon D \to E$ of R-modules that commutes with the differentiations: $\delta^E \circ \phi = \phi \circ \delta^D$. If D is a direct summand, as a differential module, of a differential R-module E, we say D is a *retract* of E; this distinguishes it from direct summands of the R-module E.

Note that the category of differential R-modules can be identified with the category of modules over $R[\varepsilon]$, the ring of dual numbers over R, see [10, Ch. IV]. In particular, the following assertions hold: $\mathrm{Ker}(\phi)$ and $\mathrm{Coker}(\phi)$ are differential

R-modules; differential R-modules and their morphisms form an abelian category, denoted $\Delta(R)$; this category has arbitrary limits and colimits; the formation of products, coproducts, and filtered colimits are exact operations.

1.2. *Homology.* For every differential R-module D set

$$\mathrm{B}(D) = \mathrm{Im}(\delta^D) \qquad \text{and} \qquad \mathrm{Z}(D) = \mathrm{Ker}(\delta^D)\,.$$

These submodules of D satisfy $\mathrm{B}(D) \subseteq \mathrm{Z}(D)$ because $\delta^2 = 0$. The quotient module

$$\mathrm{H}(D) = \mathrm{Z}(D)/\,\mathrm{B}(D)$$

is the *homology* of D. We say D is *acyclic* when $\mathrm{H}(D) = 0$.

Homology is a functor from $\Delta(R)$ to the category of R-modules. It commutes with products, coproducts, and filtered colimits. A *quasi-isomorphism* is a morphism of differential modules that induces an isomorphism in homology; the symbol $\simeq$ indicates quasi-isomorphisms, while $\cong$ is reserved for isomorphisms.

Every exact sequence of morphisms of differential R-modules

$$0 \to D \to D' \to D'' \to 0$$

induces a *homology exact triangle* of homomorphisms of R-modules

(1.2.1)
$$\begin{tikzcd} \mathrm{H}(D) \arrow[rr] & & \mathrm{H}(D') \arrow[dl] \\ & \mathrm{H}(D'') \arrow[ul] & \end{tikzcd}$$

For a proof, see [10, Ch. VI, (1.1)] and the remark following it.

The *suspension* of a differential R-module (D, δ) is the differential R-module

(1.2.2)
$$\Sigma D = (D, -\delta)\,.$$

Suspension is an automorphism of $\Delta(R)$ of order two; one has

$$\mathrm{H}(\Sigma D) \cong \mathrm{H}(D)\,.$$

Let $\phi\colon D \to E$ be a morphism of differential R-modules. The pair

(1.2.3)
$$\mathsf{cone}(\phi) = \big(D \oplus E\,,\ (d,e) \mapsto (-\delta^D(d), \delta^E(e) + \phi(d))\big)$$

is a differential module, called the *cone* of ϕ.

Obvious morphisms define an exact sequence of differential R-modules

(1.2.4)
$$0 \to E \to \mathsf{cone}(\phi) \to \Sigma D \to 0\,.$$

Given the isomorphism $\mathrm{H}(D) \cong \mathrm{H}(\Sigma D)$, it is readily verified that the homology exact triangle associated with this exact sequence yields an exact triangle:

(1.2.5)
$$\begin{tikzcd} \mathrm{H}(D) \arrow[rr, "\mathrm{H}(\phi)"] & & \mathrm{H}(E) \arrow[dl] \\ & \mathrm{H}(\mathsf{cone}(\phi)) \arrow[ul] & \end{tikzcd}$$

Thus, ϕ is a quasi-isomorphism if and only if $\mathsf{cone}(\phi)$ is acyclic.

1.3. *Compression.* Let $\mathsf{C}(R)$ denote the category of complexes over R with chain maps of degree 0 as morphisms. We display complexes in the form

$$X = \qquad \cdots \longrightarrow X_n \xrightarrow{\partial^X_n} X_{n-1} \longrightarrow \cdots$$

The *compression* of a complex X is the differential R-module

$$X_{\Delta} = \left(\bigoplus_{n\in\mathbb{Z}} X_n, \bigoplus_{n\in\mathbb{Z}} \partial_n^X \right).$$

Compression defines a functor $\mathsf{C}(R) \to \Delta(R)$. It preserves exact sequences and quasi-isomorphisms, commutes with colimits, suspensions, and cones, and satisfies

$$\mathrm{H}(X_{\Delta}) = \bigoplus_{n\in\mathbb{Z}} \mathrm{H}_n(X)\,.$$

Compression identifies complexes as *graded differential modules*, but offers no help in studying differential modules. A functor in the opposite direction does:

1.4. *Expansion.* The *expansion* of a differential R-module D is the complex

$$D_{\bullet} = \quad \cdots \longrightarrow D \xrightarrow{\delta^D} D \xrightarrow{\delta^D} D \longrightarrow \cdots$$

Expansion is a functor $\Delta(R) \to \mathsf{C}(R)$ that commutes with limits, colimits, suspensions, and cones, and preserves exact sequences. Since one has

$$\mathrm{H}_n(D_{\bullet}) = \mathrm{H}(D) \quad \text{for every} \quad n \in \mathbb{Z}\,,$$

expansion preserves quasi-isomorphisms as well.

1.5. *Contractibility.* A differential module is *contractible* if it is isomorphic to

$$(C \oplus C, \delta) \quad \text{with} \quad \delta(c', c'') = (c'', 0)\,.$$

It is evident that every contractible differential R-module is acyclic.

In certain cases acyclicity implies contractibility.

Remark 1.6. Assume that R is *regular*, in the sense that every R-module has finite projective dimension. If D is an acyclic differential R-module, such that the underlying R-module D is projective, then D is contractible.

Indeed, the hypothesis $\mathrm{H}(D_{\bullet}) = 0$ yields an exact sequence of R-modules

$$0 \to \mathrm{Im}(\delta^D) \longrightarrow D \xrightarrow{\delta^D} D \longrightarrow \cdots \longrightarrow D \xrightarrow{\delta^D} D \longrightarrow \mathrm{Im}(\delta^D) \to 0$$

containing $\operatorname{proj\,dim}_R \mathrm{Im}(\delta^D)$ copies of D. It follows that $\mathrm{Im}(\delta^D)$ is projective.

Choosing a splitting σ of the surjection $D \to \mathrm{Im}(\delta^D)$, set

$$D' = \mathrm{Ker}(\delta^D) \quad \text{and} \quad D'' = \mathrm{Im}(\sigma)\,.$$

Thus, $D = D' \oplus D''$ and $\delta^D|_{D''}$ defines an isomorphism $D'' \cong D'$.

Examples of non-contractible acyclic differential modules exist, even with finite free underlying R-module, over rings that are close to being regular.

Example 1.7. The ring $R = k[x, y, z]/(x^2 + yz)$, where k is a field and $k[x, y, z]$ a polynomial ring, is a hypersurface and a normal domain with isolated singularity. Let D be the differential module with underlying module R^2, defined by the matrix

$$A = \begin{bmatrix} x & y \\ z & -x \end{bmatrix}$$

with $A^2 = 0$, see Section 6. Either by direct computation or by using Eisenbud's [13] technique of matrix factorizations, it is easy to check that D is acyclic. However, D is not contractible: $\mathrm{Im}(\delta^D) \subseteq (x, y, z)D$ implies $\mathrm{Im}(\delta^D)$ is not a direct summand.

The next result assesses the gap between acyclicity and contractibility.

Recall that when X and Y are complexes of R-modules $\operatorname{Hom}_R(X,Y)$ denotes the complex of $\mathbb{Z}$-modules with

$$\operatorname{Hom}_R(X,Y)_n = \prod_{i\in\mathbb{Z}} \operatorname{Hom}_R(X_i, Y_{n+i})$$
$$\partial(\vartheta) = \delta^Y\vartheta - (-1)^{|\vartheta|}\vartheta\delta^X$$

In particular, in $\operatorname{Hom}_R(X,Y)$ the cycles of degree 0 are the morphisms of complexes $X \to Y$, and two cycles are in the same homology class if and only if they are homotopic chain maps.

Proposition 1.8. *For a differential R-module D the following are equivalent.*

(i) *D is contractible.*

(ii) $\operatorname{Im}(\delta) = \operatorname{Ker}(\delta)$ *and the following exact sequence of R-modules is split:*

$$0 \longrightarrow \operatorname{Ker}(\delta) \longrightarrow D \xrightarrow{\ \pi\ } \operatorname{Im}(\delta) \longrightarrow 0\,.$$

(iii) $\operatorname{H}(\operatorname{Hom}_R(D_\bullet, D_\bullet)) = 0$.

Proof. (i) $\implies$ (iii). We may assume D has the form (1.5). The maps $\chi_n\colon D \to D$ given by $\chi_n(c', c'') = (0, c')$ for each $n \in \mathbb{Z}$ then satisfy $\mathrm{id}^D = \delta\chi_n + \chi_{n-1}\delta$. If $\alpha\colon D_\bullet \to D_\bullet$ is a cycle of degree i, then $\delta\alpha_n = (-1)^i\alpha_{n-1}\delta$ holds for all n. Thus, $\chi'_n = \chi_{n+i}\alpha_n\colon D \to D$ define a homomorphism $\chi'\colon D_\bullet \to D_\bullet$ such that $\alpha = \partial(\chi')$.

(iii) $\implies$ (ii). As $\operatorname{H}_0(\operatorname{Hom}_R(D_\bullet, D_\bullet))$ vanishes, the identity map $\mathrm{id}^{D_\bullet}$ is homotopic to 0, so there are homomorphisms $\chi_n\colon D \to D$ of R-modules, satisfying $\mathrm{id}^D = \delta\chi_n + \chi_{n-1}\delta$ for each $n \in \mathbb{Z}$. Fix some n and set $\chi = \chi_n$. One has

$$\operatorname{Ker}(\delta) = \delta\chi(\operatorname{Ker}(\delta)) + \chi_{n-1}\delta(\operatorname{Ker}(\delta)) = \delta\chi(\operatorname{Ker}(\delta)) \subseteq \operatorname{Im}(\delta)\,.$$

This implies $\operatorname{Im}(\delta) = \operatorname{Ker}(\delta)$. The map $\varepsilon = \delta\chi$ satisfies $\delta = \varepsilon\delta$. Thus, for $E = \operatorname{Im}(\varepsilon)$ one gets $E \subseteq \operatorname{Im}(\delta) \subseteq E$, hence $E = \operatorname{Im}(\delta)$. One also has $\varepsilon^2 = \varepsilon$, so for every $e \in E$ the map $\sigma = \chi|_E\colon E \to D$ satisfies $\delta\sigma(e) = e$, hence the sequence in (ii) splits.

(ii) $\implies$ (i). The argument at the end of Remark 1.6 applies. $\square$

No natural differentiation on tensor products of differential modules commutes with expansion, defined in 1.4. The absence of tensor products on the category of differential modules seriously limits the applicability of standard technology. A more frugal structure, defined below, provides a partial remedy to that situation.

1.9. *Tensor products.* Let R' be an associative ring and X a complex of R'-R° bimodules. The *tensor product* of X and $D \in \Delta(R)$ is the differential R'-module

$$X \boxtimes_R D = \left(\bigoplus_{n\in\mathbb{Z}} (X_n \otimes_R D)\,,\ x\otimes d \mapsto \partial^X(x)\otimes d + (-1)^{|x|} x \otimes \delta^D(d)\right)$$

where $|x|$ denotes the degree of x. Tensor product defines a functor

$$-\boxtimes_R -\colon \mathsf{C}(R' \otimes_{\mathbb{Z}} R^\circ) \times \Delta(R) \longrightarrow \Delta(R')\,.$$

Whenever needed, modules or bimodules are considered to be complexes concentrated in degree 0. Thus, $M \boxtimes_R D$ is defined for every R'-R°-bimodule M; as it is equal to $(M \otimes_R D, M \otimes_R \delta)$, we sometimes write $M \otimes_R D$ in place of $M \boxtimes_R D$.

Tensor products commute with colimits on both sides. We collect some further properties, using equalities to denote canonical isomorphisms. For every complex X of R'-R°-bimodules and every differential R-module D, one has

$$(X \boxtimes_R D)_{\bullet} = X \otimes_R D_{\bullet} \qquad \text{in} \quad \mathsf{C}(R') \tag{1.9.1}$$

$$\mathrm{H}(X \boxtimes_R D) = \mathrm{H}_n(X \otimes_R D_{\bullet}) \qquad \text{for each} \quad n \in \mathbb{Z} \tag{1.9.2}$$

For every complex W of R''-R'° bimodules one has

$$(W \otimes_{R'} X) \boxtimes_R D = W \boxtimes_{R'} (X \boxtimes_R D) \qquad \text{in} \quad \Delta(R'') \tag{1.9.3}$$

$$W \boxtimes_{R'} X_{\Delta} = (W \otimes_{R'} X)_{\Delta} \qquad \text{in} \quad \Delta(R'') \tag{1.9.4}$$

For the differential R-module $R = (R, 0)$ one has

$$X \boxtimes_R R = X_{\Delta} \qquad \text{in} \quad \Delta(R') \tag{1.9.5}$$

$$R \boxtimes_R D = D \qquad \text{in} \quad \Delta(R) \tag{1.9.6}$$

For all morphisms $\vartheta \in \mathsf{C}(R' \otimes_{\mathbb{Z}} R^{\circ})$ and $\phi \in \Delta(R)$ one has

$$\mathsf{cone}(\vartheta \boxtimes_R D) = \mathsf{cone}(\vartheta) \boxtimes_R D \qquad \text{in} \quad \Delta(R') \tag{1.9.7}$$

$$\mathsf{cone}(X \boxtimes_R \phi) \cong X \boxtimes_R \mathsf{cone}(\phi) \qquad \text{in} \quad \Delta(R') \tag{1.9.8}$$

We need to track exactness properties of tensor products. Evidently, if D is contractible, for each complex X of R'-R°-bimodules the differential R'-module $X \boxtimes_R D$ also is contractible, and hence acyclic. The next result shows, in particular, that $(X \boxtimes_R -)$ preserves acyclicity under the expected hypotheses on X.

Proposition 1.10. *Let X and Y be bounded below complexes of R'-R°-bimodules, such that the R°-modules X_i and Y_i are flat for all $i \in \mathbb{Z}$.*

(1) *The following functor preserves exact sequences and quasi-isomorphisms*

$$(X \boxtimes_R -) \colon \Delta(R) \longrightarrow \Delta(R') .$$

(2) *A quasi-isomorphism $\vartheta \colon X \to Y$ in $\mathsf{C}(R' \otimes_{\mathbb{Z}} R^{\circ})$ induces for each differential R-module D a quasi-isomorphism of differential R'-modules*

$$\vartheta \boxtimes_R D \colon X \boxtimes_R D \longrightarrow Y \boxtimes_R D .$$

Proof. (1) The functor $(X \boxtimes_R -)$ preserves exact sequences because the complex X consists of flat R°-modules. On the other hand, a morphism ϕ of differential modules is a quasi-isomorphism if and only if its cone is acyclic, see (1.2.5). In view of the isomorphism (1.9.8), to finish the proof it suffices to show that if a differential module D is acyclic, then so is $X \boxtimes_R D$.

For each integer n define a subcomplex of X as follows:

$$X_{\leqslant n} = \quad \cdots \longrightarrow 0 \longrightarrow 0 \longrightarrow X_n \longrightarrow X_{n-1} \longrightarrow \cdots$$

It fits into an exact sequence of complexes of right R-modules

$$0 \longrightarrow X_{\leqslant n-1} \longrightarrow X_{\leqslant n} \longrightarrow \Sigma^n X_n \longrightarrow 0 .$$

This sequence induces an exact sequence of differential modules

$$0 \longrightarrow X_{\leqslant n-1} \boxtimes_R D \longrightarrow X_{\leqslant n} \boxtimes_R D \longrightarrow (\Sigma^n X_n) \boxtimes_R D \longrightarrow 0 .$$

As the R°-module X_n is flat, one has

$$\mathrm{H}((\Sigma^n X_n) \boxtimes_R D) \cong X_n \otimes_R \mathrm{H}(D) = 0 .$$

Since $X_{\leqslant n} = 0$ holds for $n \ll 0$, by induction we may assume $\mathrm{H}(X_{\leqslant n-1} \boxtimes_R D) = 0$ holds as well. The exact sequence above yields $\mathrm{H}(X_{\leqslant n} \boxtimes_R D) = 0$, so using the equality $X = \bigcup_n X_{\leqslant n}$ and the exactness of colimits one obtains

$$\mathrm{H}(X \boxtimes_R D) = \mathrm{H}(\mathrm{colim}_n(X_{\leqslant n} \boxtimes_R D)) = \mathrm{colim}_n \mathrm{H}(X_{\leqslant n} \boxtimes_R D) = 0\,.$$

(2) Note that $W = \mathsf{cone}(\vartheta)$ is a bounded below complex of flat R°-modules with $\mathrm{H}(W) = 0$. Arguing as in (1), one sees it suffices to prove $W \boxtimes_R D$ is acyclic. By (1.9.2) this is equivalent to proving that the complex $W \otimes_R D_{\bullet}$ is acyclic.

Set $(-)^{\vee} = \mathrm{Hom}_{\mathbb{Z}}(-, \mathbb{Q}/\mathbb{Z})$. As the $\mathbb{Z}$-module $\mathbb{Q}/\mathbb{Z}$ is faithfully injective, it suffices to prove $\mathrm{H}(W \otimes_R D_{\bullet})^{\vee} = 0$. The exactness of $(-)^{\vee}$ yields $\mathrm{H}(W^{\vee}) \cong \mathrm{H}(W)^{\vee} = 0$. It also implies that each $(W_i)^{\vee}$ is an injective R-module, because W_i is a flat R°-module. Thus, $W^{\vee}$ is acyclic, bounded above complex of injective R-modules, and so it is contractible. This explains the equality in the sequence

$$\mathrm{H}(W \otimes_R D_{\bullet})^{\vee} \cong \mathrm{H}((W \otimes_R D_{\bullet})^{\vee}) \cong \mathrm{H}(\mathrm{Hom}_R(D_{\bullet}, W^{\vee})) = 0\,.$$

Exactness of $(-)^{\vee}$ yields the first isomorphism and adjointness the second. □

Finding properties of differential modules that guarantee exactness of tensor products is a more delicate matter. It is discussed in the next section.

2. Differential flags

Throughout this section R denotes an associative ring. We introduce and study classes of differential R-modules that conform to classical homological intuition.

2.1. *Flags.* Let F be a differential R-module. A *differential flag* in F is a family $\{F^n\}$ of R-submodules satisfying the conditions

$$F^n \subseteq F^{n+1}\,, \quad F^{-1} = 0\,, \quad \bigcup_{i \in \mathbb{Z}} F^i = F\,, \quad \text{and} \tag{2.1.1}$$

$$\delta(F^n) \subseteq F^{n-1} \tag{2.1.2}$$

for each $n \in \mathbb{Z}$. Condition (2.1.2) implies F^n is a differential submodule of F and

$$F_n = F^n / F^{n-1} \tag{2.1.3}$$

is a differential module with trivial differentiation; we call it the nth *fold* of $\{F^n\}$. The flag defines for each $n \in \mathbb{Z}$ an exact sequence of differential R-modules

$$0 \longrightarrow F^{n-1} \xrightarrow{\iota^{n-1}} F^n \longrightarrow F_n \longrightarrow 0\,. \tag{2.1.4}$$

Properties of the folds of a flag affect the character of a differential module to a stronger degree than do properties of the underlying module.

2.2. *Types of flags.* A flag $\{F^n\}$ in F is *free* (respectively, *projective*, *flat*) if every fold F_n has the corresponding property; these conditions are progressively weaker.

When $\{F^n\}$ is a projective flag one has $F \cong \bigoplus_{n=0}^{\infty} F_n$ as R-modules.

When $\{F^n\}$ is a flat flag (2.1.4) induces an exact sequence of differential modules

$$0 \longrightarrow X \boxtimes_R F^{n-1} \longrightarrow X \boxtimes_R F^n \longrightarrow X \boxtimes_R F_n \longrightarrow 0 \tag{2.2.1}$$

for every $X \in \mathsf{C}(R^{\circ})$. Its homology exact triangle (1.2.1) has the form

$$
\begin{tikzcd}
\mathrm{H}(X \boxtimes_R F^{n-1}) \arrow[rr, "\mathrm{H}(X\boxtimes_R \iota^{n-1})"] & & \mathrm{H}(X \boxtimes_R F^{n}) \arrow[dl] \\
& \mathrm{H}(X) \boxtimes_R F_n \arrow[ul] &
\end{tikzcd}
\tag{2.2.2}
$$

because the differential module F_n is flat and has zero differentiation.

Theorem 2.3. *Let D be a retract of a differential R-module F that admits a projective flag $\{F^n\}$.*

If D is acyclic, then D is contractible.

Proof. By Proposition 1.8, it suffices to show that $\operatorname{Hom}_R(D_\bullet, D_\bullet)$ is acyclic. It is a retract of $\operatorname{Hom}_R(F_\bullet, D_\bullet)$, so we prove that $\mathrm{H}(X) = 0$ implies $\mathrm{H}(\operatorname{Hom}_R(F_\bullet, X)) = 0$.

First we show that $\operatorname{Hom}_R((F^n)_\bullet, X)$ is acyclic by induction on n. Each sequence (2.1.4) yields an exact sequence of complexes of R-modules

$$0 \longrightarrow (F^{n-1})_\bullet \xrightarrow{(\iota^{n-1})_\bullet} (F^n)_\bullet \longrightarrow (F_n)_\bullet \longrightarrow 0\,.$$

Since $F^n = 0$ for $n < 0$, we may assume $\mathrm{H}(\operatorname{Hom}_R((F^{n-1})_\bullet, X)) = 0$ for some $n \geq 0$. Each ι_i^{n-1} is split, so $\pi^n = \operatorname{Hom}_R((\iota^{n-1})_\bullet, X)$ is surjective, hence the sequence

$$0 \to \operatorname{Hom}_R((F_n)_\bullet, X) \to \operatorname{Hom}_R((F^n)_\bullet, X) \xrightarrow{\pi^n} \operatorname{Hom}_R((F^{n-1})_\bullet, X) \to 0$$

is exact. The complex $(F_n)_\bullet$ has zero differential and F_n is projective, so one obtains

$$
\begin{aligned}
\mathrm{H}(\operatorname{Hom}_R((F_n)_\bullet, X)) &= \mathrm{H}\Big(\prod_{i\in\mathbb{Z}} \Sigma^i \operatorname{Hom}_R(F_n, X)\Big) \\
&= \prod_{i\in\mathbb{Z}} \Sigma^i \mathrm{H}(\operatorname{Hom}_R(F_n, X)) \\
&= 0\,.
\end{aligned}
$$

The exact sequence and the induction hypothesis yield

$$\mathrm{H}(\operatorname{Hom}_R((F^n)_\bullet, X)) = 0\,.$$

The first isomorphism below comes from (2.1.1), the second is standard:

$$
\begin{aligned}
\mathrm{H}(\operatorname{Hom}_R(F_\bullet, X)) &= \mathrm{H}(\operatorname{Hom}_R(\operatorname{colim}_n (F^n)_\bullet, X)) \\
&= \mathrm{H}(\lim_n \operatorname{Hom}_R((F^n)_\bullet, X))\,.
\end{aligned}
$$

Since the limit is taken over the surjective morphisms π^n and each complex in the inverse system is acyclic, the limit complex is acyclic. $\square$

The following result complements Proposition 1.10.

Recall that $\mathsf{C}(R' \otimes_{\mathbb{Z}} R^{\circ})$ is the category of complexes of R'-R°-bimodules; let $\mathsf{C}_+(R' \otimes_{\mathbb{Z}} R^{\circ})$ be its full subcategory consisting of bounded below complexes.

Proposition 2.4. *Let D and E be retracts of differential R-modules with flat flags.*

(1) *The following functor preserves exact sequences and quasi-isomorphisms:*

$$(- \boxtimes_R D)\colon \mathsf{C}(R') \otimes_{\mathbb{Z}} R^{\circ}) \longrightarrow \Delta(R')\,.$$

(2) *A quasi-isomorphism $\phi\colon D \to E$ of differential R-modules induces for each $X \in \mathsf{C}_+(R' \otimes_{\mathbb{Z}} R^\circ)$ a quasi-isomorphism of differential R'-modules*

$$X \boxtimes_R \phi\colon X \boxtimes_R D \longrightarrow X \boxtimes_R E\,.$$

Proof. (1) The functor $-\boxtimes_R D$ preserves exact sequences because the R-module D is flat. It remains to verify that $-\boxtimes_R D$ preserves quasi-isomorphisms. As D is a retract of a differential module F with a flat flag, it suffices to prove that if ϑ is a quasi-isomorphism of complexes, then $\vartheta \boxtimes_R F$ is one of differential modules.

Recall that a morphism of complexes or of differential modules is a quasi-isomorphism if and only if its cone is acyclic. Therefore, in view of the isomorphism $\mathsf{cone}(\vartheta \boxtimes_R F) = \mathsf{cone}(\vartheta) \boxtimes_R F$, see (1.9.7), it suffices to prove that for each acyclic complex X, the differential module $X \boxtimes_R F$ is acyclic.

Let $\{F^n\}$ be a flat flag in F. We prove $\mathrm{H}(X \boxtimes_R F^n) = 0$ by induction on n. This is obvious for $n < 0$, as one then has $F^n = 0$. If $X \boxtimes_R F^{n-1}$ is acyclic for some $n \geq 0$, then the exact triangle (2.2.2) yields $X \boxtimes_R F^n$ is acyclic, as desired. The flag $\{F^n\}$ in F induces a flag $\{X \boxtimes_R F^n\}$ in $X \boxtimes_R F$, so one gets

$$\begin{aligned}
\mathrm{H}(X \boxtimes_R F) &= \mathrm{H}(\operatorname{colim}_n (X \boxtimes_R F^n)) \\
&= \operatorname{colim}_n \mathrm{H}(X \boxtimes_R F^n) \\
&= \operatorname{colim}_n 0 \\
&= 0\,.
\end{aligned}$$

(2) Choose a quasi-isomorphism $\rho\colon W \to X$, where W is a bounded below complex of flat R°-modules. In the commutative diagram

$$\begin{tikzcd}
W \boxtimes_R F \arrow[r, "W\boxtimes_R \phi"] \arrow[d, "\rho\boxtimes_R D"'] & W \boxtimes_R E \arrow[d, "\rho\boxtimes_R E"] \\
X \boxtimes_R D \arrow[r, "X\boxtimes_R \phi"] & X \boxtimes_R E
\end{tikzcd}$$

both vertical maps are quasi-isomorphisms by (1), and $W \boxtimes_R \phi$ is a quasi-isomorphism by Proposition 1.10(1). Thus, $X \boxtimes_R \phi$ is a quasi-isomorphism, as desired. $\square$

We show by example that the the flag structure of F is essential for the validity of the preceding theorem. In fact, its conclusion may fail even if the differential module involved is free as an R-module.

Example 2.5. Set $R = \mathbb{Z}/(4)$. A projective resolution of $k = R/(2)$ is given by

$$X = \quad \cdots \longrightarrow R \xrightarrow{2} R \xrightarrow{2} R \longrightarrow 0 \longrightarrow 0 \longrightarrow \cdots$$

The differential R-module $D = (R, 2\cdot\mathrm{id}^R)$ has $\mathrm{H}(D) = 0$, therefore $\mathrm{H}(X \boxtimes_R D) = 0$, see Proposition 1.10.(1). However, $\mathrm{H}(k \boxtimes_R D) = k$, so the map $\vartheta \boxtimes_R D$ induced by the augmentation $\vartheta\colon X \to k$ is not a quasi-isomorphism.

Each flag in a differential module naturally gives rise to a spectral sequence, see [10, Ch. XV, §§1–3]. It is used to prove Theorem 5.2.

2.6. *Spectral sequences.* Let $\{F^n\}$ be a flag in a differential R-module F. For each $r \geq 1$ the rth page of the spectral sequence is a family ${}^r\mathrm{E}\{F^n\}$ of r complexes

$${}^r\mathrm{E}^p = \quad \cdots \longrightarrow {}^r\mathrm{E}_{i+r} \xrightarrow{\partial_{i,i+r}} {}^r\mathrm{E}_i \xrightarrow{\partial_{i-r,i}} {}^r\mathrm{E}_{i-r} \longrightarrow \cdots$$

of R-modules, where $p = 0, 1, \dots, r-1$ and $i \equiv p \mod r$. The first page is

$$ {}^1\mathrm{E}_i^0 = F_i \quad \text{and} \quad \partial_{i-1,i}(x + F^{i-1}) = \delta(x) + F^{i-2}\,. $$

Successive pages of the spectral sequence are linked by equalities

$$ {}^{r+1}\mathrm{E}_i = \mathrm{Ker}(\partial_{i-r,i})/\,\mathrm{Im}(\partial_{i,i+r}) \quad \text{for each pair} \quad (r,i) \in \mathbb{N} \times \mathbb{N}\,. $$

Evidently when $r \geq i+1$ one has $\partial_{i-r,i} = 0$ and there is a surjective system

$$ {}^{r}\mathrm{E}_i \longrightarrow {}^{r+1}\mathrm{E}_i \longrightarrow {}^{r+2}\mathrm{E}_i \longrightarrow \cdots $$

One sets ${}^{\infty}\mathrm{E}_i = \mathrm{colim}_r\, {}^{r}\mathrm{E}_i$. For each integer i, let

$$ \mathrm{H}(F)^i = \mathrm{Im}(\mathrm{H}(F^i) \to \mathrm{H}(F))\,, $$

where the arrow is induced by the inclusion $F^i \subseteq F$. The spectral sequence strongly converges to $\mathrm{H}(F)$, in the sense that there are isomorphisms of R-modules

$$ \begin{aligned} &\mathrm{H}(F)^i/\mathrm{H}(F)^{i-1} \cong {}^{\infty}\mathrm{E}_i \quad \text{for each } i \geq 0 \\ &\mathrm{H}(F)^{-1} = 0 \quad \text{and} \quad \bigcup_{i\in\mathbb{Z}} \mathrm{H}(F)^i = \mathrm{H}(F)\,. \end{aligned} \tag{2.6.1} $$

If $F^l = F$ for some $l \geq 0$, then one has ${}^{\infty}\mathrm{E}_i = 0$ for $i \notin [0,l]$, hence

$$ {}^{\infty}\mathrm{E}_i = {}^{r}\mathrm{E}_i \quad \text{for} \quad r \geq \max_{0\leqslant i\leqslant l}\{i, l-i\} + 1\,. \tag{2.6.2} $$

Convergence of the spectral sequence above transfers information from $\mathrm{E}^r\{F^n\}$ to $\mathrm{H}(F)$. For instance, if the R-module $\bigoplus_i \mathrm{H}_i(\mathrm{E}^r\{F^n\})$ has finite length for some r, then so does the R-module $\mathrm{H}(F)$. However, this fact does not reduce the study of differential modules to that of complexes. The reason is that properties of $\mathrm{H}(F)$, the primary invariant of F, rarely translate into usable information about the pages of the spectral sequence. An explicit example is given next.

Example 2.7. Set $R = k[x,y]/(x^2, xy)$. By Section 6 the matrix

$$ A = \begin{bmatrix} 0 & x & y & 0 \\ 0 & 0 & x & y \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} $$

defines a differential module F. The complex ${}^1\mathrm{E}\{F^n\}$ has the form

$$ \cdots \longrightarrow 0 \longrightarrow R^2 \xrightarrow{[x\ y]} R \xrightarrow{x} R \longrightarrow 0 \longrightarrow \cdots $$

The length of $\mathrm{H}(F)$ is finite, because $\mathfrak{p} = (x)$ is the only non-maximal ideal of R, the local ring $R_{\mathfrak{p}}$ is equal to the field $k(y)$ and $\mathrm{rank}_{k(y)}(A_{\mathfrak{p}}) = 2$. On the other hand, one has ${}^2\mathrm{E}_0^0\{F^n\} \cong k[y]$, and this module has infinite length.

Next we define invariants that are central to this paper. The terminology is modelled on the usage of 'class' in group theory to measure the shortest length of a filtration with subquotients of a certain type, such as in a 'nilpotent group of class l'. The length of a 'solvable' free differential graded modules over graded polynomial ring, introduced in [7, Def. 9], is related to its free class, defined below.

2.8. *Class.* We define the *flat class* of a differential R-module F to be the number

$$\operatorname{flat\,class}_R F = \inf\left\{ l \in \mathbb{N} \;\middle|\; \begin{matrix} F \text{ admits a flat flag} \\ \{F^n\} \text{ with } F^l = F \end{matrix} \right\}$$

The *projective class* of F over R, denoted $\operatorname{proj\,class}_R M$, and its *free class*, denoted $\operatorname{free\,class}_R F$, are defined similarly. We list some simple properties of these invariants. In statements valid for any flavor of the definition, we let $\mathcal{P}$- stand for either 'flat', 'projective', or 'free', and let $\mathcal{P}\text{-class}_R M$ denote the corresponding number.

(1) $\mathcal{P}\text{-class}_R F = \infty$ if and only if F admits no finite $\mathcal{P}$-flag.
(2) $\mathcal{P}\text{-class}_R F = 0$ if and only if F is a $\mathcal{P}$-module and $\delta^F = 0$.
(3) If F is a contractible, non-zero, projective (respectively, flat) module over R, then $\operatorname{proj\,class}_R F = 1$ (respectively, $\operatorname{flat\,class}_R F = 1$).
(4) If F' and F'' are differential R-modules, then flat and projective class satisfy

$$\mathcal{P}\text{-class}_R(F' \oplus F'') = \max\{\mathcal{P}\text{-class}_R F' \,,\, \mathcal{P}\text{-class}_R F''\}\,.$$

(5) If $0 \to F \to F' \to F'' \to 0$ is an exact sequence of differential R-modules, then

$$\mathcal{P}\text{-class}_R F' \leq \mathcal{P}\text{-class}_R F + \mathcal{P}\text{-class}_R F'' + 1\,.$$

(6) If $F = P_{\Delta}$, where P is a non-zero, bounded below complex of $\mathcal{P}$-modules, then

$$\mathcal{P}\text{-class}_R F \leq \operatorname{card}\{i \in \mathbb{Z} \mid P_i \neq 0\} - 1\,.$$

(7) For every F the following inequalities hold:

$$\operatorname{flat\,class}_R F \leq \operatorname{proj\,class}_R F \leq \operatorname{free\,class}_R F\,.$$

(8) When F is finitely generated and R is noetherian, one has

$$\operatorname{flat\,class}_R F = \operatorname{proj\,class}_R F\,.$$

(9) If R is an IBN ring, see Section 6, and F has a free flag, then

$$\operatorname{free\,class}_R F \leq \operatorname{rank}_R F\,.$$

(10) If $R \to S$ is a homomorphism of rings, then

$$\mathcal{P}\text{-class}_S(S \otimes_R F) \leq \mathcal{P}\text{-class}_R F\,.$$

Indeed, (6) follows from (5) and (2). For (8), note that if $\{F^n\}$ is a flat flag in F, then each fold F_n is finitely presented, hence it is projective. Ranks of free modules, when defined, are additive in exact sequences: this gives (9). The other assertions follow directly from the definition of class.

A moment of reflection shows why non-trivial lower bounds on $\mathcal{P}\text{-class}_R F$ may be hard to obtain. One method for obtaining such bounds is given by the following technical result, distilled from the proof of [7, Thm. 16].

Proposition 2.9. *Consider a sequence of complexes*

$$X^s \xrightarrow{\vartheta^s} X^{s-1} \longrightarrow \cdots \longrightarrow X^0 \xrightarrow{\vartheta^0} X$$

of R°-modules with the following property:

(a) $$\mathrm{H}(\vartheta^n) = 0 \quad \textit{for} \quad n = 0, 1, \dots, s\,.$$

If $\pi\colon F \to D$ is a morphism of differential R-modules satisfying the condition

(b) $$\mathrm{H}(\vartheta \boxtimes_R \pi) \neq 0 \quad \textit{for} \quad \vartheta = \vartheta^0 \circ \cdots \circ \vartheta^s\,,$$

then the inequality below holds:

$$\operatorname{flat\,class}_R F \geq s+1\,.$$

Proof. Let $\{F^n\}$ be a flat flag with $F^s = F$, and let $\iota^n\colon F^n \to F^{n+1}$ denote the inclusions of differential submodules. Form the morphisms of complexes

$$\begin{aligned} \theta^n &= \vartheta^0 \circ \cdots \circ \vartheta^n \colon X^n \longrightarrow X \quad \text{for } n = 0, \ldots, s\,; \\ \theta^{-1} &= \mathrm{id}^X \colon X \longrightarrow X\,, \end{aligned}$$

and the morphisms of differential modules

$$\phi^n = \pi \circ \iota^s \circ \cdots \circ \iota^n \colon F^n \longrightarrow D \quad \text{for } n = -1, \ldots, s\,.$$

By descending induction on n we will show that the map

$$\mathrm{H}(\theta^n \boxtimes_R \phi^n)\colon \mathrm{H}(X^n \boxtimes_R F^n) \to \mathrm{H}(X \boxtimes_R D)$$

is non-zero for each integer $n \in [-1, s]$. This contradicts $F^{-1} = 0$.

One has $F^s = F$, so (b) is the desired assertion for $n = s$. Assume that it holds for some $n \in [0, s]$. The exact triangle (2.2.2) yields a commutative ladder

$$\begin{tikzcd}
\mathrm{H}(X^n \boxtimes_R F^{n-1}) \arrow[r, "\mathrm{H}(\vartheta^n \boxtimes_R F^{n-1})"'] \arrow[d, "\mathrm{H}(X^n \boxtimes_R \iota^{n-1})"'] & \mathrm{H}(X^{n-1} \boxtimes_R F^{n-1}) \arrow[d, "\mathrm{H}(X^{n-1} \boxtimes_R \iota^{n-1})"] \\
\mathrm{H}(X^n \boxtimes_R F^n) \arrow[r, "\mathrm{H}(\vartheta^n \boxtimes F^n)"'] \arrow[d] & \mathrm{H}(X^{n-1} \boxtimes_R F^n) \arrow[d] \\
\mathrm{H}(X^n) \boxtimes_R F_n \arrow[r, "\mathrm{H}(\vartheta^n) \boxtimes_R F_n"'] & \mathrm{H}(X^{n-1}) \boxtimes_R F_n
\end{tikzcd}$$

with exact rows. As $\mathrm{H}(\vartheta^n) \boxtimes_R F_n = 0$ holds by condition (a), one has an inclusion

$$\operatorname{Im} \mathrm{H}(X^{n-1} \boxtimes_R \iota^{n-1}) \supseteq \operatorname{Im} \mathrm{H}(\vartheta^n \boxtimes_R F^n)\,.$$

In view of the definitions of θ^n and ϕ^n and of the induction hypothesis, it yields

$$\begin{aligned} \operatorname{Im} \mathrm{H}(\theta^{n-1} \boxtimes_R \phi^{n-1}) &= \mathrm{H}(\theta^{n-1} \boxtimes_R \phi^n)(\operatorname{Im} \mathrm{H}(X^{n-1} \boxtimes_R \iota^{n-1})) \\ &\supseteq \mathrm{H}(\theta^{n-1} \boxtimes_R \phi^n)(\operatorname{Im} \mathrm{H}(\vartheta^n \boxtimes_R F^n)) \\ &= \operatorname{Im} \mathrm{H}(\theta^n \boxtimes_R \phi^n) \\ &\neq 0\,. \end{aligned}$$

The induction step is now complete, and so the proposition is proved. $\square$

3. Class inequality. I

For most of this section $(R, \mathfrak{m}, k)$ is a *local ring*, meaning that R is commutative and noetherian, $\mathfrak{m}$ is its unique maximal ideal, and $k = R/\mathfrak{m}$ its residue field.

The next theorem is the main step towards establishing the Class Inequality announced in the introduction. It is in some respects sharper than the global version, see Theorem 4.1. The proof is given at the end of the section.

Theorem 3.1. *Let $(R, \mathfrak{m}, k)$ be a local ring, F a differential R-module, and D a retract of F such that the R-module* $\mathrm{H}(D)$ *has non-zero finite length.*

When R has a big Cohen-Macaulay module one has:

$$\operatorname{flat\,class}_R F \geq \dim R\,.$$

We pause to recall existence results for big Cohen-Macaulay modules, and to discuss antecedents of the theorem.

3.2. *Big Cohen–Macaulay modules.* Recall that an R-module M is *big Cohen-Macaulay* if for some system of parameters $\boldsymbol{x} = x_1, \dots, x_d$ of R the element x_i is not a zero divisor on $M/(x_1, \dots, x_{i-1})M$ for $i = 1, \dots, d$ and $M \neq (\boldsymbol{x})M$. It is not known whether every R has such a module, but many important cases are covered:

Big Cohen-Macaulay modules exist when R contains a field as a subring, due to a celebrated construction of Hochster [18]. They exist also over all local rings of dimension at most 3: the difficult case of dimension 3 is settled by Hochster [19] using Heitmann's proof of the direct summand conjecture in dimension 3. Any Cohen-Macaulay ring R is a big Cohen-Macaulay module over itself.

We recall a fundamental result in commutative algebra:

Remark 3.3. The New Intersection Theorem reads: Let R be a local ring and let

$$P = \quad \cdots \longrightarrow 0 \longrightarrow P_l \longrightarrow \cdots \longrightarrow P_0 \longrightarrow 0 \longrightarrow \cdots$$

be a complex of finite free modules with $P_0 \neq 0 \neq P_l$; if P is not exact, and $\operatorname{length}(\mathrm{H}_n(P))$ is finite for each n, then $l \geq \dim R$.

Hochster, Peskine and Szpiro, and P. Roberts established the theorem when R has a big Cohen-Macaulay module. Čech complexes play a role in all these proofs, cf. the discussion in [20, pp. 82–86]. In mixed characteristic the theorem was proved by Roberts, using local Chern classes. His monograph [21] contains detailed arguments and develops the necessary intersection theory. The technology powering this portion of the proof has no analog for differential modules at present.

Theorem 3.1 contains the New Intersection Theorem for rings with big Cohen-Macaulay modules, see Remark 2.8(6), and vastly generalizes the next result.

Remark 3.4. Carlsson's theorem, [7, Thm. 16], may be stated as follows: Let R be a polynomial ring in d variables of positive degree over a field k and F a differential module with finitely generated graded free underlying module and δ^F homogeneous of degree -1; every homogeneous free flag $\{F^n\}$ in F then has $F^d \neq F$.

To prove Theorem 3.1 we transplant an idea from [7], see Proposition 2.9, utilize Čech complexes, see 3.7, and introduce two novel ingredients.

One is the determination of a framework for stating and proving a common generalization of the two theorems above; it is given by differential modules with flat flags. Their properties are put to full use: almost every result established in Sections 1 and 2 participates in the proofs of Theorems 3.1 and 3.5.

A second new ingredient is Theorem 3.5 below, a homological version of Nakayama's Lemma. A similar statement for bounded below complexes follows easily by inspecting the augmentation map to the non-vanishing homology module of lowest degree. In the ungraded world of differential modules such a map simply does not exist, so a completely new approach is needed. To this end we adapt a *dévissage* procedure introduced by Dwyer, Greenlees, and Iyengar [12, §5].

Theorem 3.5. *Let $(R,\mathfrak{m},k)$ be a local ring and M an R-module with $\mathfrak{m}M \neq M$. Let D be a retract of a differential R-module F that admits a flat flag.*

If $\mathrm{H}(D)$ is finitely generated and $M \otimes_R D$ is acyclic, then D is acyclic.

If, in addition, F admits a projective flag, then D is contractible.

Proof. The second assertion follows from the first one and Theorem 2.3.

We prove the first assertion in four steps.

Step 1. $k \boxtimes_R D$ is acyclic.

Let $Y \to M$ be a flat resolution. For $V = \mathrm{H}(k \otimes_R Y)$ one has $V_0 = M/\mathfrak{m}M \neq 0$, so k is a direct summand of V, hence it suffices to prove $V \boxtimes_R D$ is acyclic.

Let $X \to k$ be a flat resolution. As $k \otimes_R Y$ is a complex of k-vector spaces, one may choose the first one of the quasi-isomorphisms below:

$$V \xrightarrow{\ \simeq\ } k \otimes_R Y \xleftarrow{\ \simeq\ } X \otimes_R Y\,.$$

The second one is standard. Proposition 2.4(1) and formula (1.9.3) now yield

$$\begin{aligned}\mathrm{H}(V \boxtimes_R D) &\cong \mathrm{H}((k \otimes_R Y) \boxtimes_R D)\\ &\cong \mathrm{H}((X \otimes_R Y) \boxtimes_R D)\\ &= \mathrm{H}(X \boxtimes_R (Y \boxtimes_R D))\,.\end{aligned}$$

Proposition 2.4(1) also gives the first quasi-isomorphism below:

$$Y \boxtimes_R D \xrightarrow{\ \simeq\ } M \boxtimes_R D \xrightarrow{\ \simeq\ } 0\,.$$

The second one is our hypothesis. From Proposition 1.10(1) we get isomorphisms

$$\mathrm{H}(X \boxtimes_R (Y \boxtimes_R D)) \cong \mathrm{H}(X \boxtimes_R (M \boxtimes_R D)) \cong \mathrm{H}(X \boxtimes_R 0) = 0\,.$$

The two chains of isomorphisms yield $\mathrm{H}(V \boxtimes_R D) = 0$, as desired.

Step 2. $L \boxtimes_R D$ is acyclic for each R-module L of finite length.

We induce on $\mathrm{length}_R L$. When it is 1 one has $L \cong k$, so the desired result was established in Step 1. For $\mathrm{length}_R L \geq 2$ there is an exact sequence of R-modules $0 \to k \to L \to L' \to 0$. It induces an exact sequence of differential R-modules

$$0 \longrightarrow k \boxtimes_R D \longrightarrow L \boxtimes_R D \longrightarrow L' \boxtimes_R D \longrightarrow 0\,.$$

As $\mathrm{length}_R L' = \mathrm{length}_R L - 1$, the induction hypothesis and the exact triangle (1.2.5) yield $\mathrm{H}(L \boxtimes_R D) = 0$. This completes the proof of step 2.

Step 3. $W \boxtimes_R D$ is acyclic for each bounded complex W of R-modules, such that the R-module $\mathrm{H}_h(W)$ has finite length for each h.

Set $i = \inf\{h \,|\, \mathrm{H}_h(W) \neq 0\}$. The inclusion into W of the subcomplex

$$\cdots \longrightarrow W_{i+2} \longrightarrow W_{i+1} \longrightarrow \mathrm{Ker}(\partial_i) \longrightarrow 0 \longrightarrow 0 \longrightarrow \cdots$$

is a quasi-isomorphism, so by Proposition 2.4(1) we may assume $W_h = 0$ for $h < i$.

Set $H = \Sigma^i \mathrm{H}_i(W)$, let $\pi \colon W \to H$ be the augmentation, and let j be the number of non-zero homology modules of W. The exact sequence of complexes

$$0 \longrightarrow \mathrm{Ker}(\pi) \longrightarrow W \longrightarrow H \longrightarrow 0$$

shows $\mathrm{Ker}(\pi)$ has $j-1$ non-vanishing homology modules of finite length. Since one has $\mathrm{H}(H \boxtimes_R D) = 0$ by Step 2, the exact sequence of differential modules

$$0 \longrightarrow \mathrm{Ker}(\pi) \boxtimes_R D \longrightarrow W \boxtimes_R D \longrightarrow H \boxtimes_R D \longrightarrow 0$$

yields a homology triangle (1.2.1), from which $\mathrm{H}(W \boxtimes_R D) = 0$ follows by induction.

Step 4. D is acyclic.

Let K be the Koszul complex on a finite generating set for the maximal ideal of R. Step 3 shows that $K \boxtimes_R D$ is acyclic, hence so is D, by Lemma 3.6 below. □

Lemma 3.6. *Let $(R, \mathfrak{m}, k)$ be a local ring and K the Koszul complex on elements $x_1, \ldots, x_e$ in $\mathfrak{m}$. Let D be a differential R-module with $\mathrm{H}(D)$ finitely generated.*

The module $\mathrm{H}(K \boxtimes_R D)$ is then finitely generated.

Furthermore, one has $\mathrm{H}(K \boxtimes_R D) = 0$ if and only if $\mathrm{H}(D) = 0$.

Proof. Recall that K is a tensor product $K' \otimes_R K''$, where K' and K'' are Koszul complexes on the sequences x_1 and $x_2, \ldots, x_e$, respectively. Thus, one has a canonical isomorphism $K \boxtimes_R D = K' \boxtimes_R (K'' \boxtimes_R D)$ of differential modules, see (1.9.3). By induction, it suffices to prove the lemma for $e = 1$, so we set $x = x_1$.

The Koszul complex on x is the cone of $x\,\mathrm{id}^R$. Using (1.9.7) and (1.9.6) one gets

$$\mathsf{cone}(x\,\mathrm{id}^R) \boxtimes_R D = \mathsf{cone}((x\,\mathrm{id}^R) \boxtimes_R D) = \mathsf{cone}(x\,\mathrm{id}^D)\,.$$

Thus, the exact triangle (1.2.5) gives an exact sequence of R-modules

$$0 \longrightarrow \mathrm{H}(D)/x\mathrm{H}(D) \longrightarrow \mathrm{H}(K \boxtimes_R D) \longrightarrow \mathrm{H}(D)$$

The assertions of the lemma follow, with a nod from Nakayama for the last one. □

We describe a final tool needed for the proof of Theorem 3.1.

3.7. *Čech complexes.* For each element $x \in R$, the localization map $R \to R_x$ defines a complex of R-modules with R in degree 0, as follows:

$$C(x) = \quad \cdots \longrightarrow 0 \longrightarrow R \longrightarrow R_x \longrightarrow 0 \longrightarrow \cdots$$

Let $\boldsymbol{x} = x_1, \ldots, x_{s+1}$ be a sequence of elements in R. Set

$$C^n = \begin{cases} R & \text{if } n = 0 \\ C(x_1) \otimes_R \cdots \otimes_R C(x_n) & \text{if } n \geq 1. \end{cases} \tag{3.7.1}$$

The complex C^n is concentrated between degrees $-n$ and 0. It is the modified Čech complex on $x_1, \ldots, x_n$; see [6, §3.5].

Since $C^{n+1} = C^n \otimes_R C(x_{n+1})$, the exact sequence of complexes

$$0 \longrightarrow \Sigma^{-1} R_{x_{n+1}} \longrightarrow C(x_{n+1}) \longrightarrow R \longrightarrow 0$$

induces an exact sequence of bounded complexes of flat R-modules:

$$0 \longrightarrow \Sigma^{-1} C^n \otimes_R R_x \longrightarrow C^{n+1} \xrightarrow{\varepsilon^{n+1}} C^n \longrightarrow 0\,. \tag{3.7.2}$$

Proof of Theorem 3.1. Set $s = \dim R - 1$. Let M be a big Cohen-Macaulay R-module and let $\boldsymbol{x} = x_1, \ldots, x_{s+1}$ be a system of parameters that forms an M-regular sequence, see (3.2). Replacing the x_i with their powers, we assume $\boldsymbol{x}\mathrm{H}(D) = 0$.

For the entire proof we fix the complexes $X^n = M \otimes_R C^{n+1}$, obtained for $n = -1, \ldots, s$ from (3.7.1), and use (3.7.2) to define morphisms

$$\vartheta^n \colon X^n \xrightarrow{M \otimes_R \varepsilon^n} X^{n-1} \quad \text{for} \quad n = 0, \ldots, s\,.$$

The M-regularity of $\boldsymbol{x}$ implies $\mathrm{H}_i(X^n) = 0$ for $i \neq n$, see [6, (3.5.6) and (1.6.16)]. Therefore, one has inclusions

$$\mathrm{Im}(\mathrm{H}_i(\vartheta^n)) \subseteq \mathrm{H}_i(X^{n-1}) = 0\,.$$

They yield equalities

$$\mathrm{H}(\vartheta^n) = 0 \quad \text{for} \quad n = 0, \dots, s\,. \tag{a}$$

By hypothesis, there is a split epimorphism $\pi\colon F \to D$ of differential R-modules, where $\mathrm{H}(D)$ has finite non-zero length. Proposition 2.9 shows that the condition

$$\mathrm{H}(\vartheta \boxtimes_R \pi) \neq 0 \quad \text{for} \quad \vartheta^0 \circ \cdots \circ \vartheta^s\,. \tag{b}$$

gives the desired conclusion: $\operatorname{flat\,class}_R F \geq s+1$. We verify (b) in three steps.

Step 1. $(X^n \otimes_R R_{x_n}) \boxtimes_R D$ is acyclic for $n = -1, \dots, s$.

Fix n, set $x = x_n$, and choose a flat resolution $Y \to M$. The arrow below

$$Y \otimes_R C^{n+1} \otimes_R R_x \longrightarrow M \otimes_R C^{n+1} \otimes_R R_x = X^n \otimes_R R_x$$

is a quasi-isomorphism because $C^{n+1} \otimes_R R_x$ is a bounded complex of flat modules. By Proposition 2.4(1), it induces the first quasi-isomorphism of differential modules

$$\begin{aligned}(X^n \otimes_R R_x) \boxtimes_R D &\simeq (Y \otimes_R C^{n+1} \otimes_R R_x) \boxtimes_R D \\ &\cong (Y \otimes_R C^{n+1}) \boxtimes_R (R_x \otimes_R D)\,.\end{aligned}$$

The isomorphism is due to associativity. As R_x is R-flat and $x \cdot \mathrm{H}(D) = 0$, we get $\mathrm{H}(R_x \otimes_R D) \cong R_x \otimes_R \mathrm{H}(D) = 0$. That is, the map $R_x \otimes_R D \to 0$ is a quasi-isomorphism. As $Y \otimes_R C^{n+1}$ is a bounded below complex of flat modules, it induces

$$(Y \otimes_R C^{n+1}) \boxtimes_R (R_x \otimes_R D) \simeq (Y \otimes_R C^{n+1}) \boxtimes_R 0 = 0\,,$$

see Proposition 1.10. The proof of Step 1 is now complete.

Step 2. $\mathrm{H}(\vartheta^n \boxtimes_R D)\colon \mathrm{H}(X^n \boxtimes_R D) \to \mathrm{H}(X^{n-1} \boxtimes_R D)$ is bijective for $n = 0, \dots, s$.

The complexes in the exact sequence (3.7.2) consist of flat R-modules, so

$$0 \longrightarrow \Sigma^{-1}(X^{n-1} \otimes_R R_x) \longrightarrow X^n \xrightarrow{\vartheta^n} X^{n-1} \longrightarrow 0$$

is an exact sequence of complexes of R-modules. The induced sequence

$$0 \to \Sigma^{-1}(X^{n-1} \otimes_R R_x) \boxtimes_R D \to X^n \boxtimes_R D \xrightarrow{\vartheta^n \boxtimes_R D} X^{n-1} \boxtimes_R D \to 0$$

of differential R-modules is exact. Its homology exact triangle, (1.2.1) and the result of Step 1 imply that $\mathrm{H}(\vartheta^n \boxtimes_R D)$ is bijective. Step 2 is complete.

Step 3. Condition (b) holds.

Indeed, the map $\mathrm{H}(\vartheta \boxtimes_R \pi)$ factors as a composition

$$\mathrm{H}(X^s \boxtimes_R F) \xrightarrow{\mathrm{H}(X^s \boxtimes_R \pi)} \mathrm{H}(X^s \boxtimes_R D) \xrightarrow{\mathrm{H}(\vartheta \boxtimes_R D)} \mathrm{H}(X^{-1} \boxtimes_R D)\ .$$

The first map above is surjective because π is a split epimorphism of differential modules. The second map is a composition of isomorphisms, due to the result of Step 2. Finally, the module $\mathrm{H}(X^{-1} \boxtimes_R D)$ is not zero, by Theorem 3.5. □

4. Class inequality. II

In this section we prove global, relative versions of the results in Section 3. The theorem below can be compared with Theorem 3.1, through Remark 2.8(8).

Theorem 4.1. *Let $R \to S$ be a homomorphism of commutative noetherian rings. Let F be a finitely generated differential R-module and D a retract of F. Let $\mathfrak{q}$ be a prime ideal in S minimal over IS, where $I = \operatorname{Ann}_R \mathrm{H}(D)$.*

When $S_\mathfrak{q}$ has a big Cohen-Macaulay module one has

$$\operatorname{proj\,class}_R F \geq \dim S_\mathfrak{q} \,.$$

An inequality $\operatorname{proj\,class}_R F \geq \dim S_\mathfrak{q} - 1$ *holds in general.*

Recall a notion introduced by Hochster [17]: for an ideal I in R, set

$$\operatorname{super\,height} I = \sup \left\{ \operatorname{height}(IS) \,\middle|\, \begin{matrix} R \to S \text{ is a homomorphism} \\ \text{of rings and } S \text{ is noetherian} \end{matrix} \right\} \,.$$

Evidently, one has $\operatorname{super\,height} I \geq \operatorname{height} I$, whence the notation.

Every local ring containing a field has a big Cohen-Macaulay module, cf. (3.2), so the next result is essentially a reformulation of part of the theorem. The second inequality in it implies the Class Inequality, stated in the introduction.

Corollary 4.2. *If R is a commutative noetherian ring, F a finitely generated differential R-module, D a retract of F, and $I = \operatorname{Ann}_R \mathrm{H}(D)$, then one has*

$$\operatorname{proj\,class}_R F \geq \operatorname{super\,height} I - 1 \,.$$

When R is an algebra over a field a stronger inequality holds:

$$\operatorname{proj\,class}_R F \geq \operatorname{super\,height} I \,.$$

If $\dim R \leq 3$ *holds, or if R is Cohen-Macaulay, then one has*

$$\operatorname{proj\,class}_R F \geq \operatorname{height} I \,. \qquad \square$$

Hochster's motivation for introducing super heights was to prove the following homological generalization of Krull's Principal Ideal Theorem: If R is a noetherian ring containing a field, then every finitely generated R-module M satisfies

$$\operatorname{super\,height}(\operatorname{Ann}_R M) \leq \operatorname{proj\,dim}_R M \,;$$

see [17], also [6, (9.4.4)]. In view of Remark 2.8(6), this result may be recovered by applying the corollary to $F = P_\Delta$, where P is a projective resolution of M, of length equal to $\operatorname{proj\,dim}_R M$.

Examples show that the last two inequalities in Corollary 4.2 are sharp:

Example 4.3. Let R be a commutative noetherian algebra over a field. For each integer d with $0 \leq d \leq \dim R$, there exists a differential R-module F for which

$$\operatorname{proj\,class}_R F = d = \operatorname{super\,height} I \quad \text{where } I = \operatorname{Ann}_R \mathrm{H}(F).$$

Indeed, fix such a d. One can then find an ideal I of height d generated by a set $\boldsymbol{x}$ with d elements. Let K be the Koszul complex on $\boldsymbol{x}$, and set $F = K_\Delta$. Since $I\mathrm{H}_i(K) = 0$ for each i and $\mathrm{H}_0(K) = R/I$, one has $\operatorname{Ann}_R \mathrm{H}(F) = I$. Remark 2.8(6) implies the first inequality below:

$$d \geq \operatorname{proj\,class}_R F \geq \operatorname{super\,height} I \geq \operatorname{height} I = d \,.$$

The second one is given by Corollary 4.2; the last one holds always.

To prove Theorem 4.1 we need information on how the support of the homology of differential modules is affected by base change. The next result provides a complete and completely satisfying answer in the presence of projective flags.

Theorem 4.4. *Let $R \to S$ be a homomorphism of noetherian commutative rings and D a differential R-module with $\mathrm{H}(D)$ finitely generated.*

If D is a retract of some differential module admitting a projective flag, then

$$\operatorname{Supp}_S \mathrm{H}(S \otimes_R D) = \operatorname{Supp}_S(S/IS) \quad \textit{where} \quad I = \operatorname{Ann}_R \mathrm{H}(D)\,.$$

When, in addition, the S-module $S \otimes_R D$ is finitely generated, the homology module $\mathrm{H}(S \otimes_R D)$ has finite length over S if and only if the ring S/IS is artinian.

Proof. Let $\mathfrak{q}$ be a prime ideal of S, set $\mathfrak{p} = R \cap \mathfrak{q}$, and let $R_\mathfrak{p} \to S_\mathfrak{q}$ be the induced local homomorphism. One then has isomorphisms

$$\mathrm{H}(S \otimes_R D)_\mathfrak{q} \cong \mathrm{H}((S \otimes_R D)_\mathfrak{q}) \cong \mathrm{H}(S_\mathfrak{q} \otimes_{R_\mathfrak{p}} D_\mathfrak{q})$$

of $S_\mathfrak{q}$-modules. Since the differential $R_\mathfrak{p}$-module $D_\mathfrak{p}$ and the $R_\mathfrak{p}$-module $S_\mathfrak{q}$ satisfy the hypotheses of Theorem 3.5; it shows that $\mathrm{H}(S_\mathfrak{q} \otimes_{R_\mathfrak{p}} D_\mathfrak{q}) = 0$ holds if and only if $\mathrm{H}(D_\mathfrak{p}) = 0$ does, that is to say, if and only if $\mathrm{H}(D)_\mathfrak{p} = 0$. As $\mathrm{H}(D)$ is finite over R, the last condition is equivalent to $\mathfrak{p} \not\supseteq I$, which is tantamount to $\mathfrak{q} \not\supseteq IS$. We have now established that the S-modules $\mathrm{H}(S \otimes_R D)$ and S/IS, have the same support.

If $S \otimes_R D$ is finitely generated over S, then so is $\mathrm{H}(S \otimes_R D)$, hence its length is finite if and only if its support consists of maximal ideals. This support being equal to that of S/IS, the last condition is equivalent to the ring S/IS being artinian. □

Proof of Theorem 4.1. When $S_\mathfrak{q}$ has a big Cohen-Macaulay module, one gets

$$\operatorname{proj\,class}_R F \geq \operatorname{proj\,class}_{S_\mathfrak{q}}(S_\mathfrak{q} \otimes_R F)$$

from Remark 2.8(10). The differential $S_\mathfrak{q}$-module $S_\mathfrak{q} \otimes_R D$ is a retract of $S_\mathfrak{q} \otimes_R F$. Since the length of $S_\mathfrak{q}/IS_\mathfrak{q}$ is non-zero and finite, Theorem 4.4 implies that the same holds for the length of $\mathrm{H}(S_\mathfrak{q} \otimes_R D)$. Therefore, Theorem 3.1 yields

$$\operatorname{proj\,class}_{S_\mathfrak{q}}(S_\mathfrak{q} \otimes_R F) \geq \dim S_\mathfrak{q}\,.$$

The preceding inequalities yield $\operatorname{proj\,class}_R F \geq \dim S_\mathfrak{q}$, as needed.

Next we drop the assumption on $\mathfrak{q}$. Let p denote the characteristic of the residue field of $S_\mathfrak{q}$. The ring $S' = S/pS$ is an algebra over the prime field of characteristic p, so it has a big Cohen-Macaulay module, see Remark 3.2. The already established part of the theorem yields the first inequality:

$$\operatorname{proj\,class}_R F \geq \dim S' \geq \dim S_\mathfrak{q} - 1\,.$$

The second one holds always. This completes the proof. □

5. Rank inequalities

The results of Sections 3 and 4 provide lower bounds on the projective class of a differential module in terms of invariants of its homology. Here we provide similar bounds for the rank of a differential module admitting a finite free flag.

Theorem 5.1. *Let R be a commutative noetherian algebra over a field.*

If D is a retract of a differential R-module F that admits a finite free flag, then

$$\operatorname{rank}_R F \geq 2(\text{super height}\, I) \quad \textit{where} \quad I = \operatorname{Ann}_R \mathrm{H}(D)\,.$$

Remark. Not surprisingly, the proof shows that over every noetherian ring one has an inequality $\operatorname{rank}_R F \geq 2(\text{super height}\, I) - 2$.

When the height of $\operatorname{Ann}_R \mathrm{H}(F)$ is at least 5, this remark implies $\operatorname{rank}_R F \geq 8$. For smaller heights we have the following result.

Theorem 5.2. *Let R be a commutative noetherian ring, F a differential R-module, and set $I = \operatorname{Ann}_R \mathrm{H}(F)$ and $d = \operatorname{height} I$.*

If F admits a finite free flag, then one has inequalities:

(5.2.1) $\qquad \operatorname{rank}_R F \geq 2d \quad$ *when* $\quad d \leq 3\,;$

(5.2.2) $\qquad \operatorname{rank}_R F \geq 8 \quad$ *when* $\quad d \geq 3$ *and R is a UFD.*

For differential graded modules with finite length homology over graded polynomial rings, see Remark 3.4 for details, our theorem specializes to [9, Thm. 2]. Theorems 5.1 and 5.2 are proved at the end of the section. Together, they contain the Rank Inequality, stated in the introduction, and suggest the following:

Conjecture 5.3. Let R be a local ring and F a differential R-module with a finite free flag. If $\mathrm{H}(F)$ has non-zero finite length, then

$$\operatorname{rank}_R F \geq 2^d \quad \text{for} \quad d = \dim R\,.$$

This is in line with several results and open problems in algebra and topology:

Remark 5.4. Buchsbaum and Eisenbud [5, (1.4)], and Horrocks [16, Pbl. 24] conjectured that if P is a finite free resolution of a module of finite length over a local ring R of dimension d, then $\operatorname{rank}_R P_n \geq \binom{d}{n}$. These inequalities predict

$$\operatorname{rank}_R(P_\Delta) = \sum_n \operatorname{rank}_R P_n \geq 2^d\,,$$

as does Conjecture 5.3. For $d \leq 4$ the Buchsbaum-Eisenbud-Horrocks conjecture follows from the Generalized Principal Ideal Theorem. For equicharacteristic rings Avramov and Buchweitz [2, (1)] use Evans and Griffith's Syzygy Theorem [14] to prove $\operatorname{rank}_R(P_\Delta) \geq \frac{3}{2}(d-1)^2+8$ for $d \geq 5$; thus, for $d = 5$ one has $\operatorname{rank}_R(P_\Delta) \geq 2^5$.

Remark 5.5. Let X be a finite CW complex. Halperin [15, (1.4)] asked: If the torus $(\mathbb{R}/\mathbb{Z})^d$ acts with finite isotropy groups on X, does then one have

$$\sum_n \operatorname{rank}_{\mathbb{Q}} \mathrm{H}_n(X;\mathbb{Q}) \geq 2^d\,?$$

In a similar vein, Carlsson [8, Conj. I.3] conjectured that if the elementary abelian group $(\mathbb{Z}/2\mathbb{Z})^d$ acts freely on X, then one has

$$\sum_n \operatorname{rank}_{\mathbb{F}_2} \mathrm{H}_n(X;\mathbb{F}_2) \geq 2^d\,.$$

It is known that if Conjecture 5.3 holds for differential graded modules over a graded polynomial ring in d variables over a field, then so do the inequalities above; see the remarks after [8, Conj. II.2]. In particular, Theorem 5.2 implies certain results of Allday and Puppe; see [1, (1.4.21) and (4.4.3)(1)].

We note that the conclusion of Theorems 5.2 and 5.1 may fail if the hypothesis on F is weakened from admitting a free flag to just being free as an R-module:

Example 5.6. Let $R = k[x,y]$ be a polynomial ring over a field k and D the differential R module given by the square-zero matrix

$$A = \begin{bmatrix} xy & -x^2 \\ y^2 & -xy \end{bmatrix}$$

A straightforward calculation yields

$$\operatorname{Ker}(\delta) = R\begin{bmatrix} x \\ y \end{bmatrix} \quad \text{and} \quad \operatorname{Im}(\delta) = (x,y)R\begin{bmatrix} x \\ y \end{bmatrix}$$

Therefore, one has $I = (x,y)$, as well as $\operatorname{rank}_R F = 2 < 4 = 2 \operatorname{height} I$.

The proofs of the preceding theorems require preparation.

When R is a domain, R_0 its field of fractions, and M an R-module one sets $\operatorname{rank}_R M = \operatorname{rank}_{R_0}(R_0 \otimes_R M)$. For a matrix A with entries in R let $\operatorname{rank}_R(A)$ denote its rank over R_0. Rank is used to form Euler-Poincaré characteristics of complexes. Only vestigial versions of Euler's formula hold in the absence of gradings.

Remark 5.7. If R is an integral domain and D is a finitely generated differential R-module, then the following hold:

$$\operatorname{rank}_R D = 2\operatorname{rank}_R(\delta^D) \iff \operatorname{rank}_R \mathrm{H}(D) = 0 \tag{5.7.1}$$

$$\operatorname{rank}_R D \equiv \operatorname{rank}_R \mathrm{H}(D) \pmod 2\,. \tag{5.7.2}$$

Indeed, both formulas result from the equality

$$\operatorname{rank}_R D = \operatorname{rank}_R \mathrm{H}(D) + 2\operatorname{rank}_R \operatorname{Im}(\delta^D)$$

obtained by additivity of rank from the exact sequences of R-modules

$$0 \longrightarrow \operatorname{Ker}(\delta^D) \longrightarrow D \longrightarrow \operatorname{Im}(\delta^D) \longrightarrow 0$$

$$0 \longrightarrow \operatorname{Im}(\delta^D) \longrightarrow \operatorname{Ker}(\delta^D) \longrightarrow \mathrm{H}(D) \longrightarrow 0\,.$$

Lemma 5.8. *If R is a domain and F is a finitely generated differential R-module with* $\operatorname{free\,class}_R F = l < \infty$*, then one has*

$$\operatorname{rank}_R F \geq 2l\,. \tag{5.8.1}$$

Moreover, if $\{F^n\}$ is a free flag with $F^l = F$, then the following hold:

$$\operatorname{rank}_R(F_n) \geq \begin{cases} 1 & \text{for } n = 0 \text{ or } n = l \\ 2 & \text{for } n = 1, \dots, l-1 \end{cases} \tag{5.8.2}$$

$$\delta(F^n) \not\subseteq F^{n-2} \quad \text{for} \quad n = 1, \dots, l\,. \tag{5.8.3}$$

Proof. Assuming (5.8.3) fails for some n, one finds in F a new flag $\{G^i\}$ by setting

$$G^i = \begin{cases} F^i & \text{for } i \leq n-2 \\ F^{i+1} & \text{for } i \geq n-1 \end{cases}$$

It implies $\operatorname{free\,class}_R F < l$, contradicting our hypothesis. Thus, (5.8.3) holds.

Assume next that (5.8.2) fails for some n. We show that (5.8.3) fails for some j, and thus draw a contradiction. For $n = 0$, $n = l$, or $F_n = 0$, one may take $j = n+1$. For $n \in [1, l-1]$ the complex (2.6) has the form

$$\cdots \longrightarrow F_{n+1} \xrightarrow{\partial_{n,n+1}} R \xrightarrow{\partial_{n-1,n}} F_{n-1} \longrightarrow \cdots$$

Since R is a domain, either $\partial_{n+1} = 0$ or $\partial_n = 0$; set $j = n+1$ or $j = n$, respectively.

Finally, formula (5.8.1) follows from the computation

$$\begin{aligned}\operatorname{rank}_R F &= \operatorname{rank}_R(F_0) + \sum_{n=1}^{l-1} \operatorname{rank}_R(F_n) + \operatorname{rank}_R(F_l)\\ &\geq 1 + (l-1)2 + 1\\ &= 2l\,,\end{aligned}$$

where the inequality comes from the already established formula (5.8.2). □

Proof of Theorem 5.1. Let $R \to S$ be a homomorphism to a noetherian ring S, such that $\operatorname{super\,height} I = \operatorname{height}(IS)$. Pick a prime ideal $\mathfrak{q}$ in S, minimal over IS, then a prime ideal $\mathfrak{p} \subseteq \mathfrak{q}$, such that $\operatorname{height}(IS) = \dim(S_\mathfrak{q}) = \dim(S_\mathfrak{q}/\mathfrak{p}S_\mathfrak{q})$.

Let R' denote the local domain $S_\mathfrak{q}/\mathfrak{p}S_\mathfrak{q}$. The differential graded R'-module $D' = R' \otimes_R D$ is a retract of $F' = R' \otimes_R F$, so the following inequalities

$$\operatorname{rank}_{R'} F' \geq 2 \operatorname{free\,class}_{R'} F' \geq 2 \operatorname{proj\,class}_{R'} F' \geq 2 \dim R'$$

are given by formula (5.8.1), Remark 2.8(10), and Theorem 3.1. The desired result follows, as one has $\operatorname{rank}_R F = \operatorname{rank}_{R'} F'$ and $\dim R' = \operatorname{super\,height} I$. □

Proof of Theorem 5.2. Recall that F is a differential R-module admitting a finite free flag, $\operatorname{Ann}_R \mathrm{H}(F) = I \neq R$, and $d = \operatorname{height} I$.

When $d = 1$ the desired inequality $\operatorname{rank}_R F \geq 2d$ clearly holds.

When $d = 2$ or $d = 3$ pick prime ideals $\mathfrak{q} \supseteq I$ and $\mathfrak{p} \subseteq \mathfrak{q}$ such that $\operatorname{height}(\mathfrak{q}/\mathfrak{p}) = 2$, and set $S = R_\mathfrak{q}/\mathfrak{p}R_\mathfrak{q}$. Note that $\operatorname{length}_S \mathrm{H}(S \otimes_R F)$ is finite and non-zero, by Theorem 4.4, and that S has a big Cohen-Macaulay module, since $\dim S \leq 3$. Applying successively Remark 2.8(9), formula (5.8.1), and Theorem 4.1 we obtain

$$\operatorname{rank}_R F \geq \operatorname{rank}_S(S \otimes_R F) \geq 2 \operatorname{proj\,class}_S(S \otimes_R F) \geq 2 \cdot d\,.$$

This completes the proof of (5.2.1).

Assume $d \geq 3$ and R is a UFD. Let $\mathfrak{q}$ be a prime ideal containing I such that $\dim R_\mathfrak{q} = d$. Since R is a domain, (5.8.1) yields the first inequality below

$$\operatorname{rank}_R F \geq 2 \operatorname{proj\,class}_R F \geq 2 \cdot \min\{3, d-1\} = 6\,.$$

The second is due to Theorem 4.1, because when $d = 3$ the ring $R_\mathfrak{q}$ has a big Cohen-Macaulay module, see 3.2. Formula (5.7.2) rules out $\operatorname{rank}_R F = 7$, so to finish the proof of the inequality $\operatorname{rank}_R F \geq 8$ it remains to show $\operatorname{rank}_R F \neq 6$.

Replacing R with $R_\mathfrak{q}$, for the rest of the proof we assume that R is a local UFD, $\dim R \geq 3$, and $\mathrm{H}(F)$ has finite non-zero length. We assume $\operatorname{rank}_R F = 6$ and draw a contradiction. Set $l = \operatorname{free\,class}_R F$ and let $\{F^n\}$ be a free flag with $F^l = F$. Lemma 5.8 implies that the first page of the spectral sequence in Remark 2.6 is

$${}^1\mathrm{E}^0\{F^n\} = \cdots \longrightarrow 0 \longrightarrow R \xrightarrow{\partial_{23}} R^2 \xrightarrow{\partial_{12}} R^2 \xrightarrow{\partial_{01}} R \longrightarrow 0 \longrightarrow \cdots$$

with $\partial_{n,n+1} \neq 0$ for $n = 0, 1, 2$. Let A_n denote the matrix of $\partial_{n,n+1}$ in some bases; $\partial_{23} \neq 0$ implies $\operatorname{rank}_R A_1 = 1$. Let J be the ideal generated by the entries of A_1.

If $J = R$, then (possibly after changing bases) one can find $x, z \neq 0$ such that

$$A_2 = \begin{bmatrix} 0 \\ z \end{bmatrix}, \quad A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad A_0 = \begin{bmatrix} 0 & x \end{bmatrix}$$

It follows that on the second page the only non-trivial complex is

$${}^2\mathrm{E}^0\{F^n\} = \quad \cdots \longrightarrow 0 \longrightarrow R/(z) \xrightarrow{\partial_{02}} R/(x) \longrightarrow 0 \longrightarrow \cdots$$

where ∂_{02} is induced by multiplication with some $y \in (x : z)$. The condition $l = 3$ implies ${}^3\mathrm{E}\{F^n\} = \mathrm{E}^\infty\{F^n\}$, see (2.6.2), so (2.6.1) yields an exact sequence

$$0 \longrightarrow R/(x,y) \longrightarrow \mathrm{H}(F) \longrightarrow (x : y)/(z) \longrightarrow 0$$

of R-modules. Since R is a UFD, the ideal $(x : y)$ is generated by $x' = x/v$ where $v = \gcd(x,y)$. Thus, the module $(x : y)/(z)$ is isomorphic to $R/(w)$, where $w = z/x'$. The hypothesis $\mathrm{H}(F) \neq 0$ implies $R/(w) \neq 0$ or $R/(x,y) \neq 0$. In the first case we get $\dim_R R/(w) = \dim R - 1 \geq 2$, in the second $\dim_R R/(x,y) \geq \dim R - 2 \geq 1$. Either inequality contradicts the hypothesis that $\mathrm{H}(F)$ has finite length.

If $J \neq R$, choose a prime ideal $\mathfrak{p}$ minimal over J, and let S denote the field $R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}$. Corollary A.3 implies $\operatorname{height}\mathfrak{p} \leq 2$, so one has $\mathfrak{p} \not\supseteq I$, and thus $\mathrm{H}(F_\mathfrak{p}) = 0$. Theorem 3.5 now yields $\mathrm{H}(S \otimes_R F) = 0$. The first page of the spectral sequence ${}^r\mathrm{E}^p = {}^r\mathrm{E}^p\{S \otimes_R F^n\}$ associated to the flag $\{S \otimes_R F^n\}$ is the complex

$${}^1\mathrm{E}^0 = \qquad \cdots \longrightarrow 0 \longrightarrow S \xrightarrow{\partial_{23}} S^2 \xrightarrow{0} S^2 \xrightarrow{\partial_{01}} S \longrightarrow 0 \longrightarrow \cdots$$

of vector spaces over S. The two complexes on the second page ${}^2\mathrm{E}$ are

$$\begin{aligned} {}^2\mathrm{E}^0 &= \qquad \cdots \longrightarrow 0 \longrightarrow \operatorname{Coker}(\partial_{23}) \xrightarrow{\partial_{02}} \operatorname{Coker}(\partial_{01}) \longrightarrow 0 \longrightarrow \cdots \\ {}^2\mathrm{E}^1 &= \qquad \cdots \longrightarrow 0 \longrightarrow \operatorname{Ker}(\partial_{23}) \xrightarrow{\partial_{13}} \operatorname{Ker}(\partial_{01}) \longrightarrow 0 \longrightarrow \cdots \end{aligned}$$

Counting ranks over S, one verifies the following assertions: ${}^3\mathrm{E}^2_2 \neq 0$ when $\partial_{23} = 0$, or when $\partial_{23} \neq 0$ and $\partial_{01} \neq 0$ hold simultaneously; ${}^3\mathrm{E}^1_1 \neq 0$ when $\partial_{23} \neq 0$, but $\partial_{01} = 0$. Now (2.6.2) yields ${}^\infty\mathrm{E}^2_2 = {}^3\mathrm{E}^2_2$ and ${}^\infty\mathrm{E}^1_1 = {}^3\mathrm{E}^1_1$, so one has $\mathrm{H}(S \otimes_R F) \neq 0$ by (2.6.1). This new contradiction completes the proof of (5.2.2). □

6. Square-zero matrices

Our primary purpose in this short section is to record matrix versions of theorems proved earlier in the text. As a side benefit we get a framework for describing examples. We assume that R is an IBN (= invariant basis number) associative ring: in every finitely generated free R-module any two bases have the same number of elements, see [11]. Commutative rings and left noetherian rings have this property.

Let $\mathrm{M}_s(R^\circ)$ be the ring of $s \times s$ matrices over R°. Let E be the R-module of $s \times 1$ matrices with entries in R and $e_i \in E$ the matrix with 1 in the ith row and 0 elsewhere. For each $A = (a_{ij}) \in \mathrm{M}_s(R^\circ)$, define $\varepsilon_A \in \operatorname{End}_R(E)$ by the condition

$$\varepsilon_A(e_j) = \sum_{i=1}^{s} a_{ij} e_i \quad \text{for} \quad j = 1, \dots, s$$

The map $A \mapsto \varepsilon_A$ is an isomorphism of rings $\mathrm{M}_s(R^\circ) \cong \operatorname{End}_R(E)$. The standard operations of $\mathrm{M}_s(R^\circ)$ turns E into a bimodule, so $\operatorname{Ker}(A) = \{e \in E \,|\, Ae = 0\}$ is an R-submodule. Let $\operatorname{Im}(A)$ be the R-submodule of E spanned by the columns of A.

The map $A \mapsto (E, \varepsilon_A)$ induces a bijection between conjugacy classes of square-zero matrices in $\mathrm{M}_s(R^\circ)$ and isomorphism classes of free differential R-modules of rank s; one has $\mathrm{H}(E, \varepsilon_A) \cong \operatorname{Ker}(A)/\operatorname{Im}(A)$. A matrix A with $A^2 = 0$ is conjugated to a strictly upper triangular matrix if and only if (E, ε_A) is a free flag.

A ring is *projective-free* if it has the IBN property and its finitely generated projectives are free, see [11]. The remarks above translate Theorem 2.3 into the next statement, where 0_r and 1_r denote the $r \times r$ zero and identity matrices, respectively.

Theorem 6.1. *Let R be a projective-free ring and let $A = (a_{ij})$ be an $s \times s$ matrix with entries in R. The following conditions are equivalent.*

(i) *$s = 2r$ and A is conjugated to the matrix* $\begin{bmatrix} 0_r & 1_r \\ 0_r & 0_r \end{bmatrix}$.

(ii) *A is conjugated to some strictly upper triangular matrix, and $(x_1, \dots, x_s) \in R^s$ is a solution of the system of equations*

$$\sum_{j=1}^{s} a_{ij} x_j = 0 \quad \textit{for} \quad i = 1, \dots, s$$

if and only if $x_j = \sum_{i=1}^{s} a_{ji} c_i$ for fixed $c_1, \dots, c_s \in R$ and $j = 1, \dots, s$. □

Let B be an $s \times s$ strictly upper triangular matrix with entries in R. A *block partition* of B in *l steps* is a sequence of integers $1 \leq s_0 < \cdots < s_l = s$, such that

$$(b_{uv})_{\substack{s_i \leq u < s_{i+1} \\ s_j \leq v < s_{j+1}}} = 0 \quad \text{for all} \quad i \geq j\,.$$

When such a partition exists one has

$$l \geq \operatorname{free\,class}_R(E, \varepsilon_B) \geq \operatorname{proj\,class}_R(E, \varepsilon_B)\,.$$

Thus, Theorems 4.1, 5.1, and 5.2 translate into:

Theorem 6.2. *Let R be a commutative noetherian algebra containing a field and let A be an $s \times s$ matrix with entries in R, such that $A^2 = 0$. Let I denote the ideal* $\operatorname{Ann}_R \operatorname{H}(A)$*, assume $I \neq R$, and set $d =$* $\operatorname{height} I$.

When A is conjugated to a strictly upper triangular matrix B, every block partition of B has at least d steps, and the inequality $s \geq 2d$ holds.

When, in addition, R is a UFD and $d \geq 3$ holds, one has $s \geq 8$. □

Appendix A. Ranks of matrices

In this appendix R is a commutative ring, A is an $m \times n$ matrix with entries in R, and $I_r(A)$ denotes the ideal in R generated by all $r \times r$ minors of A. As usual, we set $I_0(A) = R$ and $I_r(A) = 0$ for $r > \min\{m, n\}$, so that $I_r(A) \supseteq I_{r+1}(A)$ holds for all $r \geq 0$. We compare two notions of rank for A.

Remark A.1. The *determinantal rank* of A, denoted $\operatorname{det\,rank}_R A$, is the largest integer $r \geq 0$ such that $I_r(A) \neq 0$. The *inner rank* of A, denoted $\operatorname{inn\,rank}_R A$, is the least integer $s \geq 0$ such that A can be written as a product $A = A'A''$ with an $n \times s$ matrix A' and an $s \times m$ matrix A'', see [11]. Standard linear algebra yields

$$\operatorname{det\,rank}_R A \leq \operatorname{inn\,rank}_R A$$

and shows that equality holds when R is a field. It follows that when R is an integral domain the ranks coincide if and only if $\operatorname{inn\,rank}_R A = \operatorname{inn\,rank}_K A$, where K is the field of fractions of R. Domains over which this holds for all A have been described by multiple conditions; in particular, by the property that the kernel of every homomorphisms $R^n \to R^m$ is a union of free submodules, see [11, (5.5.9)].

The curious result below complements the criterion for agreement of ranks.

Proposition A.2. *An integral domain R is factorial if and only if every non-empty set of principal ideals of R has a maximal element, and each matrix A over R with* $\operatorname{det\,rank}_R A \leq 1$ *satisfies* $\operatorname{det\,rank}_R A = \operatorname{inn\,rank}_R A$.

From Krull's Principal Ideal Theorem one obtains an immediate consequence:

Corollary A.3. *If R is a noetherian UFD and* $\det\operatorname{rank}_R A = 1$*, then either*

$$I_1(A) = R \quad \textit{or} \quad \operatorname{height} I_1(A) \leq \max\{m, n\}\,. \qquad \square$$

Proof of Proposition A.2. Let first R be a factorial domain. The maximality condition on principal ideals holds by [11, (0.9.4)], so we let A be an $m \times n$ matrix over R with $\det\operatorname{rank}_R A \leq 1$ and show that its determinantal and inner ranks coincide.

There is nothing to prove if $\det\operatorname{rank}_R A = 0$ or if ether m or n is equal to 1, so we assume $A \neq 0$ and $\min\{m, n\} \geq 2$. Let B denote the $(m-1) \times n$ matrix consisting of the first $m-1$ rows in A. If $B = 0$, then $A = A'A''$ holds for the $m \times 1$ matrix A', transpose to $[0 \ \cdots \ 0 \ 1]$, and the $1 \times n$ matrix A'' with $a_j = a_{mj}$. We further assume $B \neq 0$ and induce on m. If $m = 2$, then the matrix has the form

$$A = \begin{bmatrix} x_1 & \cdots & x_n \\ y_1 & \cdots & y_n \end{bmatrix}$$

with some $x_h \neq 0$. As $\det\operatorname{rank}_R(A) \leq 1$, the rows of A are proportional over the field of fractions R_0; in other words, $y_j = (p/q)x_j$ for some $p/q \in R_0$ and $1 \leq j \leq n$. As R is a UFD, one may assume p, q are relatively prime. One can then find $y'_1, \dots, y'_n \in R$ satisfying $y_j = py'_j$ for each j. This implies $x_j = qy'_j$, hence

$$A = \begin{bmatrix} q \\ p \end{bmatrix} \cdot \begin{bmatrix} y'_1 & \cdots & y'_n \end{bmatrix}$$

With the base case settled, assume $m \geq 3$ and the result holds for matrices with $m-1$ rows. As one has $\det\operatorname{rank}_R(B) \leq 1$, the induction hypothesis yields matrices B' and B'', of size $(m-1) \times 1$ and $1 \times n$ respectively, such that $B = B'B''$. Set

$$B''' = \begin{bmatrix} b'_1 & 0 \\ \vdots & \vdots \\ b'_{m-1} & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} b''_1 & \cdots & b''_n \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}$$

Evidently, $A = B'''C$. Now $B \neq 0$ implies $B' \neq 0$, so one gets $\det\operatorname{rank}_R(C') \leq 1$. The basis of the induction shows $C = C'C''$ for a 2×1 matrix C' and a $1 \times n$ matrix C''. The matrices $A' = B'''C'$ and $A'' = C''$ provide the desired decomposition.

Let now R be a domain whose principal ideals satisfy the maximality condition and over which every 2×2 matrix with zero determinant has inner rank 1. To prove that R is factorial it suffices to show that for all $a, b \in R$ the ideal $(a) \cap (b)$ is principal, see [11, (0.9.4)]. By hypothesis, there is an $u \in R$ such that (ua) is maximal among principal ideals in $(a) \cap (b)$. We are going to prove $(a) \cap (b) = (ua)$.

Indeed, for each element $xa \in (a) \cap (b)$ there is a unique $x' \in R$ satisfying $xa = x'b$. In view of the equalities $x/x' = b/a = u/u'$, the first matrix below has determinantal rank 1, so the hypothesis on the ring yields a factorization

$$\begin{bmatrix} u & u' \\ x & x' \end{bmatrix} = \begin{bmatrix} c \\ d \end{bmatrix} \begin{bmatrix} y & z \end{bmatrix} = \begin{bmatrix} cy & cz \\ dy & dz \end{bmatrix}$$

Using the first rows of the matrices one gets $cya = ua = u'b = czb$, hence $ya = zb$. Thus, there are inclusions of ideals $(ua) \subseteq (ya) = (zb) \subseteq (a) \cap (b)$; the maximality of (ua) implies $(ua) = (ya)$. Using this equality and the second rows of the matrices, one now obtains $xa = dya \in (dua) \subseteq (ua)$, as desired. $\square$

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