BEILINSON MONAD

1. Setup

We index homologically throughout. Fix a toric variety X with Cox ring $S = k[x_0, \ldots, x_n]$ graded by $\mathrm{Pic}(X) = \mathbb{Z}^{\oplus r}$. Let $\mathcal{F}$ be a coherent sheaf on X. Write E for the Koszul dual of S. We equip E with a $\mathrm{Pic}(X) \times \mathbb{Z}$-grading such that $|e_i| = (-|x_i|, 1)$. Write $\omega := E^\vee$.

Remark 1.1. Before getting started, we record the following elementary observations. Of course, ω is an E-module with k-basis given by exterior polynomials in the e_i^*. Note that $|e_i^*| = (|x_i|, -1)$. The action of E on ω is by contraction. The $x_i \in S$ are also duals of the e_i, but we use different notation for the basis of ω to prevent confusion.

Another technical note: all E-modules are right modules. In particular, entries of matrices over E act on the right. This is also Macaulay2's convention. Note that this is the only way to make sense of the definition of the $\mathbf{R}$-functor in the EFS paper; if we apply the definition to a left E-module M, the maps in the complex $\mathbf{R}(M)$ are not E-linear. Nevertheless, sometimes we will multiply elements of E-modules on the left by elements of e (for instance, in the definition of the $\mathbf{L}$-functor below). When we do this, here is what we mean. When we write em for $e \in E$ and $m \in M$, where M is a right E-module, we mean $(-1)^{|e||m|}me$, where $|-|$ denotes the degree with respect to the second (standard) grading.

2. The Tate resolution

Define $\mathcal{O}_{X \times E}$ to be the sheaf of algebras on X given by

$$U \mapsto \Gamma(U, \mathcal{O}_X) \otimes_k E.$$

Let $\mathrm{Com}(X \times E)$ denote the category of complexes of $\mathcal{O}_{X \times E}$-modules.

Example 2.1. Define an object $\kappa^*(\mathcal{F}) \in \mathrm{Com}(X \times E)$ given by

$$\kappa^*(\mathcal{F})_i = \bigoplus_{\ell \in \mathrm{Pic}(X)} \mathcal{F}(-\ell) \otimes \omega(\ell, -i)$$

with differential given by $\sum_i x_i \otimes e_i$.

Fix once and for all an affine open cover $\{V_0, \ldots, V_t\}$ of X. Given a sheaf $\mathcal{M}$ on X, denote by $\mathcal{C}^{\mathcal{M}}$ the Čech complex of $\mathcal{M}$ corresponding to this open cover. Recall that we're indexing homologically, so the Čech complex is concentrated in nonpositive degrees.

Given $\mathcal{G} \in \mathrm{Com}(X \times E)$, define a bicomplex as follows. The p^{th} column is given by $\mathcal{C}^{\mathcal{G}_p}$, but with the vertical differential multiplied by $(-1)^p$. The horizontal differential is induced by the differential on $\mathcal{G}$. Denote by $\tau_*(\mathcal{G}) \in \mathrm{Com}(E)$ the direct sum totalization of this bicomplex. In other words, $\tau_*(\mathcal{G})$ is given as follows: take an explicit Čech model of the derived global sections of the complex $\mathcal{G}$, considered as a complex of $\mathcal{O}_X$-modules, and then remember the E-module structure on the terms of the resulting complex.

Date: July 6, 2020.

Definition 2.2. The *Tate resolution of* $\mathcal{F}$, denoted $\mathrm{Tate}(\mathcal{F})$, is given by $(\tau_* \circ \kappa^*)(\mathcal{F})$.

Remark 2.3. Probably this isn't the right definition; instead, we want to replace this with a homotopy equivalent complex using Lemma 3.5 of EFS. But for the purpose of the Beilinson monad theorem, I think this distinction is irrelevant (assuming the **U**-functor preserves homotopy).

Remark 2.4. Set

$$T(\mathcal{F}) := \bigoplus_{i=0}^{t} \bigoplus_{l \in \mathrm{Pic}(X)} \mathcal{C}_{-i}^{\mathcal{F}(-l)} \otimes \omega(l, -i).$$

Unravelling the definitions, one sees that $\mathrm{Tate}(\mathcal{F})$ has the form

$$\cdots \xrightarrow{\partial} T(0,-1) \xrightarrow{\partial} T \xrightarrow{\partial} T(0,1) \xrightarrow{\partial} \cdots .$$

3. The U-functor

We recall the definition of the functor

$$\mathbf{L} : \mathrm{Com}(E) \to \mathrm{Com}(S)$$

from Daniel's notes. For an E-module M concentrated in degree 0, $\mathbf{L}(M)$ is the complex with

$$\mathbf{L}(M)_q = \bigoplus_{d \in \mathrm{Pic}(X)} M_{(-d,-q)} \otimes_k S(d)$$

and differential

$$m \otimes s \mapsto \sum_{i=0}^{n} e_i m \otimes x_i s. \tag{1}$$

For a general complex $(C, \partial) \in \mathrm{Com}(E)$, we form the bicomplex

$$\begin{tikzcd}
 & \arrow[d] & \arrow[d] & \\
\cdots & \mathbf{L}(C_p)_q \arrow[l, "\partial"'] \arrow[d] & \mathbf{L}(C_{p+1})_q \arrow[l, "\partial"'] \arrow[d] & \cdots \arrow[l, "\partial"'] \\
\cdots & \mathbf{L}(C_p)_{q-1} \arrow[l, "\partial"'] \arrow[d] & \mathbf{L}(C_{p+1})_{q-1} \arrow[l, "\partial"'] \arrow[d] & \cdots \arrow[l, "\partial"'] \\
 & {} & {} &
\end{tikzcd} \tag{2}$$

and apply $\mathrm{Tot}^{\oplus}(-)$, where the vertical differential $\mathbf{L}(C_p)_q \to \mathbf{L}(C_p)_{q-1}$ is the dual Koszul map (1) multiplied by $(-1)^p$.

Let $\mathcal{L}(C)$ denote the bicomplex of $\mathcal{O}_X$-modules given by applying the associated sheaf functor to the bicomplex (2). Let $\mathcal{L}'(C)$ be the sub-bicomplex of $\mathcal{L}(C)$ given by taking summands of the form $C_{p,(-d,-q)} \otimes_k \mathcal{O}(d)$ with d effective. From now on, we'll write "$d \geq 0$" for "d effective". Here, p denotes homological degree, and $(-d,-q)$ denotes internal degree. We define a functor

$$\mathbf{U} : \mathrm{Com}(E) \to \mathrm{Com}(\mathbb{P})$$

to be given by $C \mapsto \mathrm{Tot}^{\oplus}(\mathcal{L}'(C))$.

♣♣♣ Michael: [Why $\mathrm{Tot}^{\oplus}$ and not Tot^{Π}? Why $d \geq 0$ and not $d \leq 0$? Give clean conceptual explanation for definition of $\mathbf{U}$-functor.]

Proposition 3.1. *The above definition of the* $\mathbf{U}$*-functor agrees with Daniel's.*

Proof. Daniel's definition is given by

$$\omega(i,j) \mapsto \mathcal{L}(\omega_{\leq i})(i)[-j].$$

(We're abusing notation here by identifying the 1-column bicomplex $\mathcal{L}(\omega_{\leq i})(i)$ with its totalization.) We have

$$\begin{aligned}
(\mathcal{L}(\omega_{\leq i})(i)[-j])_q &= \mathcal{L}(\omega_{\leq i})_{-j+q}(i) \\
&= \bigoplus_d (\omega_{\leq i})_{(-d,j-q)} \otimes \mathcal{O}(d+i) \\
&= \bigoplus_d (\omega_{\leq i})_{(i-d,j-q)} \otimes \mathcal{O}(d) \\
&= \bigoplus_{d\geq 0} (\omega_{\leq i})_{(i-d,j-q)} \otimes \mathcal{O}(d) \\
&= \bigoplus_{d\geq 0} \omega_{(i-d,j-q)} \otimes \mathcal{O}(d) \\
&= \mathbf{U}(\omega(i,j))_q.
\end{aligned}$$

And of course the maps in both complexes are identical as well. □

Question 3.2. *Does the* $\mathbf{U}$*-functor preserve homotopy? Check this.*

4. Beilinson monad

Recall that $\mathrm{Tate}(\mathcal{F})$ is of the form

$$\cdots \to T(0,-1) \to T \to T(0,1) \to \cdots,$$

where T is as described in Remark 2.4. The (p,q) term of $\mathcal{L}'(\mathrm{Tate}(\mathcal{F}))$ is

$$\bigoplus_{d\geq 0} T(0,-p)_{(-d,-q)} \otimes \mathcal{O}(d) = \bigoplus_{d\geq 0} T_{(-d,-p-q)} \otimes \mathcal{O}(d),$$

and so

$$\mathbf{U}(\mathrm{Tate}(\mathcal{F}))_m = \bigoplus_{p\in\mathbb{Z}} \bigoplus_{d\geq 0} T_{(-d,-m)} \otimes \mathcal{O}(d).$$

Equip each $\mathbf{U}(\mathrm{Tate}(\mathcal{F}))_m$ with a $k[u]$-module structure determined by the following "shift" operation: if $t = (\ldots, t_{-1}, t_0, t_1, \ldots) \in \bigoplus_{p\in\mathbb{Z}} \bigoplus_{d\geq 0} T_{(-d,-m)} \otimes \mathcal{O}(d)$,

$$u(t)_p = (-1)^{m-p} t_{p-1}.$$

Proposition 4.1. *The differential on* $\mathbf{U}(\mathrm{Tate}(\mathcal{F}))$ *is* $k[u]$*-linear.*

Proof. I'm writing down the proof to make sure I got the sign right in the definition of the u-action. We prove that the action of u commutes with both horizontal and vertical differentials. Write d_T for the differential on $\mathrm{Tate}(\mathcal{F})$ and d_K for the dual Koszul differential. We have

$$\begin{aligned} d_{\mathrm{hor}}(u\cdot(\dots,t_{-1},t_0,t_1,\dots)) &= d_{\mathrm{hor}}(\dots,(-1)^{m-1}t_{-2},(-1)^m t_{-1},(-1)^{m-1}t_0,\dots) \\ &= (\dots,(-1)^m d_T(t_{-3}),(-1)^{m-1}d_T(t_{-2}),(-1)^m d_T(t_{-1}),\dots) \\ &= u\cdot(\dots,d_T(t_{-2}),d_T(t_{-1}),d_T(t_0),\dots) \\ &= u\cdot d_{\mathrm{hor}}(\dots,t_{-1},t_0,t_1,\dots), \end{aligned}$$

and

$$\begin{aligned} d_{\mathrm{ver}}(u\cdot(\dots,t_{-1},t_0,t_1,\dots)) &= d_{\mathrm{ver}}(\dots,(-1)^{m-1}t_{-2},(-1)^m t_{-1},(-1)^{m-1}t_0,\dots) \\ &= (\dots,(-1)^m d_K(t_{-2}),(-1)^m d_K(t_{-1}),(-1)^m d_K(t_0),\dots) \\ &= u\cdot(\dots,-d_K(t_{-1}),d_K(t_0),-d_K(t_1),\dots) \\ &= u\cdot d_{\mathrm{ver}}(\dots,t_{-1},t_0,t_1,\dots). \end{aligned}$$

□

So, $\mathbf{U}(\mathrm{Tate}(\mathcal{F}))$ is a complex of $X\times\mathbb{A}^1$-modules. In fact, since the action of u is invertible, $\mathbf{U}(\mathrm{Tate}(\mathcal{F}))$ is a complex of $X\times\mathbb{G}_m$-modules. Define

$$\mathbf{BM}(\mathcal{F}) := \mathbf{U}(\mathrm{Tate}(\mathcal{F}))/(u-1).$$

We have an isomorphism

$$\mathbf{BM}(\mathcal{F})_m \cong \bigoplus_{d\geq 0} T_{(-d,-m)}\otimes\mathcal{O}(d) \tag{3}$$

given by representing each class in $\mathbf{BM}(\mathcal{F})_m$ by an element concentrated in the $p=0$ summand of $\bigoplus_{p\in\mathbb{Z}}\bigoplus_{d\geq 0} T_{(-d,-m)}\otimes\mathcal{O}(d)$. Via this isomorphism, the differential on $\mathbf{BM}(\mathcal{F})$ is given by ud_T+d_K, where d_T is the Tate differential and d_K is the dual Koszul differential.

Theorem 4.2. *The complex $\mathbf{BM}(\mathcal{F})$ is a monad with homology $\mathcal{F}$.*

Example 4.3. Take $X=\mathbb{P}(w)$, with w some positive integer, and take $\mathcal{F}=\mathcal{O}$. So X is a stacky point. If we set $T=\bigoplus_{i\in\mathbb{Z}}\omega(i,0)$, then

$$\cdots \xrightarrow{e} T(0,-1) \xrightarrow{e} T \xrightarrow{e} T(0,1) \xrightarrow{e} \cdots$$

is the Tate resolution of $\mathcal{O}$. We therefore have

$$\mathbf{U}(\mathrm{Tate}(\mathcal{O}))_m = \bigoplus_{p\in\mathbb{Z}}\bigoplus_{d\geq 0}\bigoplus_{i\in\mathbb{Z}}\omega_{(i-d,-m)}\otimes\mathcal{O}(d) = \begin{cases} \bigoplus_{p\in\mathbb{Z}}\bigoplus_{d\geq 0}\omega_{(0,0)}\otimes\mathcal{O}(d), & m=0; \\ \bigoplus_{p\in\mathbb{Z}}\bigoplus_{d\geq 0}\omega_{(w,-1)}\otimes\mathcal{O}(d), & m=1; \\ 0, & \text{else.}\end{cases}$$

Taking coinvariants of the $\mathbb{G}_m$-action and applying the isomorphism (3) gives the complex

$$0\to\bigoplus_{d\geq 0}\omega_{(w,-1)}\otimes\mathcal{O}(d) \xrightarrow{\begin{pmatrix} -1 & 0 & 0 & 0 & \dots \\ x & -1 & 0 & 0 & \dots \\ 0 & x & -1 & 0 & \dots \\ 0 & 0 & x & -1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}} \bigoplus_{d\geq 0}\omega_{(0,0)}\otimes\mathcal{O}(d)\to 0$$

whose homology is $\mathcal{O}$ in degree 0 and 0 elsewhere, as expected.

5. Proof of Theorem 4.2

Let $\mathcal{R} \in \mathrm{Com}(X \times X)$ be Daniel's resolution of the diagonal. To prove Theorem 4.2, it suffices to give a homotopy equivalence

$$\mathbf{BM}(\mathcal{F}) \simeq \mathbf{R}\pi_{2*}(\pi_1^*\mathcal{F} \otimes \mathcal{R}).$$

The notation $\mathbf{R}\pi_{2*}$ is misleading: we are not working at the level of derived categories. Rather, we use the notation $\mathbf{R}\pi_{2*}$ to denote the Čech model for $\mathbf{R}\pi_{2*}$ induced by the affine open cover $\{V_0, \dots, V_t\}$ of X chosen above.

The rough idea is to define a map

$$\mathrm{id} \times \overline{\mathbf{U}} : \mathrm{Com}(X \times E) \dashrightarrow \mathrm{Com}(X \times X)$$

such that the diagram

$$\begin{tikzcd} & \mathrm{Com}(X \times E) \arrow[r, "\tau_*"] \arrow[d, dashed, "\mathrm{id} \times \overline{\mathbf{U}}"] & \mathrm{Com}(E) \arrow[d, "\overline{\mathbf{U}}"] \\ \mathrm{coh}(X) \arrow[ur, "\kappa^*"] \arrow[r, "\pi_1^*(-)\otimes\mathcal{R}"'] & \mathrm{Com}(X \times X) \arrow[r, "\mathbf{R}\pi_{2*}"] & \mathrm{Com}(X) \end{tikzcd}$$

commutes up to homotopy, where $\overline{\mathbf{U}}$ denotes the functor given by applying the $\mathbf{U}$-functor and modding out by the relation $u - 1$, as discussed above. Here is how to define $\mathrm{id} \times \overline{\mathbf{U}}$ on the image of κ^* (which is all we need). Recall that

$$\kappa^*(\mathcal{F})_i = \bigoplus_{\ell \in \mathrm{Pic}(X)} \mathcal{F}(-\ell) \otimes \omega(\ell, -i)$$

and the differential is the dual Koszul map. We apply "$\mathrm{id} \times \mathbf{U}$" to $\kappa^*(\mathcal{F})$ to get the complex whose m^{th} term is

$$\bigoplus_{p \in \mathbb{Z}} \bigoplus_{d \geq 0} \bigoplus_{\ell \in \mathrm{Pic}(X)} \omega_{(\ell - d, -m)} \otimes \mathcal{F}(-\ell) \boxtimes \mathcal{O}(d)$$

with differential $\sum_{i=0}^n e_i \otimes x_i + (-1)^p e_i \otimes y_i$. This complex has $\mathbb{G}_m$-action just as the Tate resolution does. Taking coinvariants and applying an isomorphism similar to (3), we arrive at the complex with m^{th} term

$$\bigoplus_{d \geq 0} \bigoplus_{\ell \in \mathrm{Pic}(X)} \omega_{(\ell - d, -m)} \otimes \mathcal{F}(-\ell) \boxtimes \mathcal{O}(d)$$

and m^{th} differential $\sum_{i=0}^n (-1)^m e_i \otimes x_i + e_i \otimes y_i$.

Proposition 5.1. $(\mathrm{id} \times \overline{\mathbf{U}})(\kappa^*(\mathcal{O}))$ *coincides with Daniel's resolution of the diagonal.*

Proof. Fill in. Should be easy. $\square$

Remark 5.2. Suppose $X = \mathbb{P}(w_0, \dots, w_n)$. Set $w = \sum w_i$. Suppose we change the definition of $(\mathrm{id} \times \overline{\mathbf{U}})(\kappa^*(\mathcal{O}))$ slightly so that we get a complex that looks like this:

$$(4)\quad 0 \to \bigoplus_{0\le d\le w-1} \bigoplus_{0\le l\le d} \omega_{(d-\ell,-(n-1))} \otimes \mathcal{O}(-d,\ell) \to \cdots \to \bigoplus_{0\le d\le w-1} \bigoplus_{0\le l\le d} \omega_{(d-\ell,0)} \otimes \mathcal{O}(-d,\ell) \to 0.$$

Here are the changes:

- restricted the ranges on d and ℓ
- changed $\omega_{(l-d,-m)}$ to $\omega_{(d-l,-m)}$, and
- changed $\mathcal{O}(-\ell, d)$ to $\mathcal{O}(-d,\ell)$.

I claim that (4) is Canonaco-Karp's resolution of the diagonal. I checked this for $\mathbb{P}(1,2)$ and it's correct on the nose.

The first change in the list above amounts to changing the definitions of κ and $\mathbf{U}$ so that the "irrelevant" Koszul complexes are removed. The last two changes arise from our choice to take $d \ge 0$ summands rather than $d \le 0$ in the definition of the $\mathbf{U}$-functor; this seems to create a conflict with Canonaco-Karp. Not sure if this was just an arbitrary choice or if one is better than the other.

Example 5.3. Let's check this for $X = \mathbb{P}(w)$. Reading off the formula, we get

$$0 \to \bigoplus_{d\ge 0} \omega_{(w,-1)} \otimes \mathcal{O}(d, -(d+w)) \xrightarrow{\begin{pmatrix} y & 0 & 0 & 0 & \dots \\ -x & y & 0 & 0 & \dots \\ 0 & -x & y & 0 & \dots \\ 0 & 0 & -x & y & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}} \bigoplus_{d\ge 0} \omega_{(0,0)} \otimes \mathcal{O}(d,-d) \to 0.$$

The homology is $\mathcal{O} \oplus \cdots \oplus \mathcal{O}(w-1, -(w-1))$, and this is indeed the diagonal in this case (this is not entirely trivial to check). Notice this precisely recovers Daniel's "1-variable" example when $w = 1$.

Proposition 5.4. $(\mathbf{R}\pi_{2*} \circ (\mathrm{id} \times \overline{\mathbf{U}}) \circ \kappa^*)(\mathcal{F}) = (\overline{\mathbf{U}} \circ \mathrm{Tate})(\mathcal{F})$.

Proof. Let's start by computing the right hand side. As in Remark 2.4, let

$$T = \bigoplus_{i=0}^{t} \bigoplus_{l \in \mathrm{Pic}(X)} \mathcal{C}_{-i}^{\mathcal{F}(-l)} \otimes \omega(l, -i),$$

so that

$$(\overline{\mathbf{U}} \circ \mathrm{Tate})(\mathcal{F}) = \overline{\mathbf{U}}(\cdots \to T(0,-1) \to T \to T(0,1) \to \cdots).$$

We have

$$\begin{aligned}(\overline{\mathbf{U}} \circ \mathrm{Tate})(\mathcal{F})_m &= \bigoplus_{d\ge 0} T_{(-d,-m)} \otimes \mathcal{O}(d) \\ &= \bigoplus_{d\ge 0} \bigoplus_{i=0}^{t} \bigoplus_{l\in \mathrm{Pic}(X)} \omega_{(l-d,-i-m)} \otimes \mathcal{C}_{-i}^{\mathcal{F}(-l)} \otimes \mathcal{O}(d),\end{aligned}$$

and the differential is $ud_T + d_K$. On the other hand $(\mathbf{R}\pi_{2*} \circ (\mathrm{id} \times \overline{\mathbf{U}}) \circ \kappa^*)(\mathcal{F})$ is the sheaf given as follows. Let W be an open set in X. We abuse notation slightly and write $\mathcal{C}^{\mathcal{O}(d)|_W}$ for the Čech complex on $\mathcal{O}(d)|_W$ corresponding to the open cover

$$\{V_0 \cap W, \ldots, V_t \cap W\}.$$

We recall that the natural map

$$\mathcal{C}^{\mathcal{F}(-l)} \otimes \mathcal{C}^{\mathcal{O}(d)|_W} \to \mathcal{C}^{\mathcal{F}(-l) \boxtimes \mathcal{O}(d)|_W}$$

is a homotopy equivalence, where the target is the Čech complex associated to the open cover $\{V_i \times (V_j \cap W)\}_{0 \le i,j \le t}$. Form a bicomplex with p^{th} column given by

$$\bigoplus_{d \ge 0} \bigoplus_{l \in \mathrm{Pic}(X)} \omega_{(l-d,-p)} \otimes \mathcal{C}^{\mathcal{F}(-l)} \otimes \mathcal{C}^{\mathcal{O}(d)|_W},$$

with vertical differential multiplied by $(-1)^p$, and horizontal differential induced by the differential on $((\mathrm{id} \times \overline{\mathbf{U}}) \circ \kappa^*)(\mathcal{F})$. Applying $\mathrm{Tot}^{\oplus}$ to this complex gives the value of the $(\mathbf{R}\pi_{2*} \circ (\mathrm{id} \times \overline{\mathbf{U}}) \circ \kappa^*)(\mathcal{F})$ at W, up to homotopy equivalence. Explicitly, the value at W is the complex whose m^{th} term is

$$\bigoplus_{i=0}^{t^2} \bigoplus_{d \ge 0} \bigoplus_{l \in \mathrm{Pic}(X)} \omega_{(l-d,-i-m)} \otimes (\mathcal{C}^{\mathcal{F}(-l)} \otimes \mathcal{C}^{\mathcal{O}(d)|_W})_{-i},$$

with the induced differential. It suffices to check that if we sheafify the presheaf

$$W \mapsto \bigoplus_{i=0}^{t^2} \omega_{(l-d,i-n)} \otimes (\mathcal{C}^{\mathcal{F}(-l)} \otimes \mathcal{C}^{\mathcal{O}(d)|_W})_{-i},$$

we get

$$\bigoplus_{i=0}^{t} \omega_{(l-d,i-n)} \otimes \mathcal{C}^{\mathcal{F}(-l)}_{-i} \otimes \mathcal{O}(d).$$

And of course we need to check the differentials coincide as well. Need to fill in the rest of the details, but I think this is clear. $\square$

Proof of Theorem 4.2. Combine Propositions 5.1 and 5.4. $\square$