THE TATE RESOLUTION AND A_∞-OPERATIONS

Contents

Here are two ideas for how to recover the higher degree parts of the Tate differential from an A_∞-algebra. The first one is simpler and has a good chance of working. The second is the one involving the tensor coalgebra; I'm now pretty sure this one doesn't work. But, I included the second approach in case aspects of it are useful.

1. First approach

I'm afraid I'm going need a bit of abstract nonsense. I promise this is really necessary to understand the idea!

1.1. Dg-categories.

Say we're working over a field k. A *differential graded (dg) category $\mathcal{C}$ over k* is a category with some extra structure: the morphism sets are not just sets, but complexes of k-vector spaces. The composition rule

$$\mathrm{Hom}_{\mathcal{C}}(Y, Z) \otimes_k \mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{C}}(X, Z)$$

is also required to be a morphism of complexes.

Example 1.1. Say $\mathcal{C}$ has a single object X. The complex $\mathrm{End}_{\mathcal{C}}(X)$ has an algebra structure given by composition. The fact that composition gives a morphism of complexes amounts, in this case, to the Leibniz rule for $\mathrm{End}_{\mathcal{C}}(X)$; in other words, a dg-category with a single object is exactly the data of a dg k-algebra.

Given any dg-category $\mathcal{C}$, we obtain an ordinary category $H^0(\mathcal{C})$ with the same objects as $\mathcal{C}$ and such that

$$\mathrm{Hom}_{H^0(\mathcal{C})}(X, Y) = H^0 \mathrm{Hom}_{\mathcal{C}}(X, Y).$$

In many cases, $H^0 \mathrm{Hom}_{\mathcal{C}}(X, Y)$ is triangulated, and in fact "most" familiar triangulated categories arise in this way. $H^0(\mathcal{C})$ is called the *homotopy category* of $\mathcal{C}$.

Date: January 10, 2021.

Example 1.2. Our key example is this one. Let X be a scheme, and define a dg-category $\mathrm{D}^{\mathrm{b}}_{\mathrm{dg}}(X)$ with objects given by bounded complexes of coherent sheaves on X and morphism complexes between two objects $\mathcal{F}$ and $\mathcal{G}$ given by $\mathbf{R}\Gamma\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F},\mathcal{G})$. The homotopy category of $\mathrm{D}^{\mathrm{b}}_{\mathrm{dg}}(X)$ is the usual (triangulated) bounded derived category $\mathrm{D}^{\mathrm{b}}(X)$.

One can also form a dg derived category of any differential graded algebra A, which we denote by $\mathrm{D}_{\mathrm{dg}}(A)$. This is just the category of dg-A-modules localized along quasi-isomorphisms (let's gloss over how one localizes a dg-category...). It is also possible to make sense of a *bounded* derived category $\mathrm{D}^{\mathrm{b}}_{\mathrm{dg}}(A)$ of a dga, but I will gloss over this as well. The homotopy category of $\mathrm{D}^{\mathrm{b}}_{\mathrm{dg}}(A)$ is a triangulated category which we denote by $\mathrm{D}^{\mathrm{b}}(A)$.

A *dg-functor* is just an ordinary functor such that the maps on morphisms are chain maps. Such a functor is called an *equivalence* if

- it is essentially surjective after passing to homotopy categories, and
- it induces quasi-isomorphisms on morphism complexes.

This generalizes the notion of a quasi-isomorphism of dg-algebras. An equivalence of dg-categories induces an equivalence on homotopy categories, by definition.

1.2. **Relating the Tate resolution to an A_∞-algebra.** It follows from Beilinson's resolution of the diagonal that the object $\mathcal{O}(-n) \oplus \cdots \oplus \mathcal{O}$ in the triangulated category $\mathrm{D}^{\mathrm{b}}(\mathbb{P}^n)$ is a *generator*. That is, the smallest triangulated subcategory of $\mathrm{D}^{\mathrm{b}}(\mathbb{P}^n)$ that contains the object $\mathcal{O}(-n) \oplus \cdots \oplus \mathcal{O}$ and is closed under direct summands is $\mathrm{D}^{\mathrm{b}}(\mathbb{P}^n)$ itself.

Denote the object $\mathcal{O}(-n) \oplus \cdots \oplus \mathcal{O}$ by $\mathcal{G}$, and let A denote the dg-algebra $\mathrm{End}_{\mathrm{D}^{\mathrm{b}}_{\mathrm{dg}}(\mathbb{P}^n)}(\mathcal{G})$. In more prosaic terms, A is just the dg-algebra $\mathbf{R}\Gamma\mathcal{E}nd_{\mathcal{O}_{\mathbb{P}^n}}(\mathcal{G})$. Notice that

$$H^i\mathbf{R}\Gamma\mathcal{E}nd_{\mathcal{O}_{\mathbb{P}^n}}(\mathcal{G}) = \bigoplus_{j=0}^{n} H^i(\mathbb{P}^n, \mathcal{G}(j)).$$

By general abstract nonsense, the fact that $\mathcal{O}(-n) \oplus \cdots \oplus \mathcal{O}$ is a generator implies that there is an equivalence of dg-categories

$$\mathrm{D}^{\mathrm{b}}_{\mathrm{dg}}(\mathbb{P}^n) \xrightarrow{\simeq} \mathrm{D}^{\mathrm{b}}_{\mathrm{dg}}(A)$$

that sends an object $\mathcal{F}$ to the dg-A-module $\mathrm{Hom}_{\mathrm{D}^{\mathrm{b}}_{\mathrm{dg}}(\mathbb{P}^n)}(\mathcal{O}(-n) \oplus \cdots \oplus \mathcal{O}, \mathcal{F})$. For a reference, see, for instance, [Dyc11, Theorem 5.1].

Now, let E denote the exterior algebra that is Koszul dual to the coordinate ring of $\mathbb{P}^n$, and let $K^{\mathrm{ex}}(E)$ denote the homotopy category of exact complexes of finitely generated free E-modules. We recall that there is an equivalence of categories

$$\Omega : K^{\mathrm{ex}}(E) \xrightarrow{\simeq} \mathrm{D}^{\mathrm{b}}(\mathbb{P}^n),$$

defined in [EFS03]; this equivalence sends the Tate resolution of a sheaf $\mathcal{F}$ to a Beilinson monad of $\mathcal{F}$. Putting everything together, we conclude:

Theorem 1.3. *There is an equivalence*

$$K^{\mathrm{ex}}(E) \xrightarrow{\simeq} \mathrm{D}^{\mathrm{b}}(A)$$

that sends the Tate resolution $T(\mathcal{F})$ of a sheaf $\mathcal{F}$ to the dg-A-module

$$\mathrm{Hom}_{\mathrm{D}^{\mathrm{b}}_{\mathrm{dg}}(\mathbb{P}^n)}(\mathcal{G}, (\Omega \circ T)(\mathcal{F})) = \mathbf{R}\Gamma\mathcal{H}om_{\mathcal{O}_{\mathbb{P}^n}}(\mathcal{G}, (\Omega \circ T)(\mathcal{F})).$$

Finally, we recall that the usual theory of A_∞-algebras tells us that there is an A_∞-structure on

$$H^*(A) = \bigoplus_{j=0}^{n} H^*(\mathbb{P}^n, \mathcal{G}(j))$$

such that there is a quasi-isomorphism

$$f : H^*(A) \xrightarrow{\simeq} A$$

of A_∞-algebras. Now, *I think* that there is a similar statement for any dg-A-module M. That is, I hope the following claim is true:

Claim 1.4. *We can equip $H^*(M)$ with an A_∞-module structure over $H^*(A)$ such that there is a quasi-isomorphism*

$$H^*(M) \xrightarrow{\simeq} M$$

of A_∞ $H^*(A)$-modules (here, M is considered as an A_∞ $H^*(A)$-module via restriction of scalars).

I haven't found a reference for this claim yet, which worries me a bit. But, if the claim is true, Theorem 1.3 tells us that the Tate resolution of a sheaf $\mathcal{F}$ can be completely recovered from an A_∞-module structure on

$$H^*\mathbf{R}\Gamma\mathcal{H}om_{\mathcal{O}_{\mathbb{P}^n}}(\mathcal{G}, (\Omega \circ T)(\mathcal{F})) = \bigoplus_{j=0}^{n} H^*(\mathbb{P}^n, \mathcal{F}(j)),$$

which is roughly what we've expect all along.

2. Second approach

2.1. A_∞ **operations and the tensor coalgebra.** Let R be a ring and M an R-module. The tensor coalgebra $T^c(M)$ is the sum of all the tensor powers of M, equipped with the comultiplication

$$x_1 \otimes \cdots \otimes x_r \mapsto \sum_{i=1}^{r} (x_1 \otimes \cdots \otimes x_i) \otimes (x_{i+1} \otimes \cdots \otimes x_r) \in T^c(M) \otimes T^c(M).$$

We recall that a *coderivation* on an R-coalgebra (C, δ) is an R-linear map $d : C \to C$ satisfying the co-Leibniz rule $\Delta \circ d = (d \otimes 1 + 1 \otimes d) \circ \Delta$. A *dg-coalgebra over R* is a $\mathbb{Z}$-graded R-coalgebra equipped with a square zero coderivation of degree -1.

Suppose S is a $\mathbb{Z}$-graded R-module. Then $T^c(S)$ inherits a $\mathbb{Z}$-grading, and the datum of an A_∞-algebra structure on S is exactly the datum of a square 0 coderivation on $T^c(S)$. In more detail: any coderivation d on $T^c(S)$ is determined by its projection $T^c(S) \xrightarrow{d} T^c(S) \twoheadrightarrow S$ onto the first tensor power. So, a coderivation is the same thing as a degree -1 map $d : T^c(S) \to S$. The components of this map are the higher multiplications on S, and the requirement that $d^2 = 0$ encodes the A_∞ relations.

2.2. **Perturbations and A_∞-structures.** Let R be a ring, and suppose A is an augmented, connected, unital differential graded R-algebra. Moreover, suppose A contracts onto its homology, i.e. there are maps

$$\pi : A \leftrightarrows H_*(A) : \iota$$

of graded R-modules such that $\pi\iota = \mathrm{id}_{H_*(A)}$ and $\iota\pi$ is homotopic to id_A. We get induced maps of tensor coalgebras

$$T^c(\pi) : T^c(A) \leftrightarrows T^c(H_*(A)) : T^c(\iota). \tag{1}$$

We observe that $T^c(A)$ is again a dg-coalgebra with differential induced by the one on A:

$$x_1 \otimes \cdots \otimes x_r \mapsto \sum_{i=1}^{r} \pm x_1 \otimes \cdots \otimes d(x_i) \otimes \cdots \otimes x_r$$

(note: this is the coderivation determined by the degree -1 map $T^c(A) \to A$ given by $x_1 \otimes \cdots \otimes x_r \mapsto \sum_{i=1}^{r} \pm d(x_i)$). We will abuse notation and call this induced differential d. The homotopy between $\iota\pi$ and id_A induces a homotopy between $T^c(\iota)T^c(\pi)$ and $\mathrm{id}_{T^c(A)}$. That is, the maps in (1) give a contraction of $T^c(A)$ onto $T^c(H_*(A))$ (where the latter has no differential).

Now, suppose we perturb the differential on $T^c(A)$ via some second differential ∂ so that $d+\partial$ remains a coderivation. For instance, we could do this by perturbing the differential on A so that A remains a dga. The so-called Coalgebra Perturbation Lemma ([HK91] (2.1_*)) says that the differential on $T^c(H_*(A))$ obtained from the usual perturbation lemma is a coderivation. We therefore obtain an induced A_∞ structure on $H_*(A)$.

Example 2.1. Suppose R is a field. Then one of course one can always give an R-linear contraction of A onto $H_*(A)$. We can perturb the differential on $T^c(A)$ by adding in the *bar differential*

$$a_1 \otimes \cdots \otimes a_r \mapsto (\sum_{i=1}^{r-1} \pm a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_r) \pm (a_r a_1 \otimes \cdots \otimes a_{r-1}).$$

We obtain from this an A_∞-structure on $H_*(A)$. Moreover, $H_*(A)$ is quasi-isomorphic to A as an A_∞-algebra (the point is that the A_∞-structure on A induced by the perturbed coderivation on $T^c(A)$ yields the original dg-algebra structure on A). That is, we recover the usual statement that a dga over a field can be recovered from its homology equipped with certain A_∞-operations.

In summary: a perturbation ∂ of the coderivation d on $T^c(A)$ *such that $d+\partial$ is also a coderivation* determines an A_∞-algebra structure on $H_*(A)$. In particular, a perturbation ∂ of the differential on A *such that $d+\partial$ is also a derivation* determines such a structure.

2.3. **The Tate differential and A_∞ operations.** We wish to apply the above formalism to the Tate resolution. Let X be a toric variety, and let $\mathcal{K}$ denote the exact differential module $\bigoplus_{\ell \in \mathrm{Pic}(X)} \mathcal{O}(-\ell) \otimes_k \omega(\ell, 0)$ with differential $\sum_{i=0}^{n} x_i \otimes e_i$.

For the purpose of making it more transparent that the above dg-methods extend to our setting, expand $\mathcal{K}$ into a 1-periodic complex:

$$\cdots \to \mathcal{K}(0,1) \to \mathcal{K} \to \mathcal{K}(0,-1) \to \cdots . \tag{2}$$

Form a bicomplex $\mathcal{B}$ whose columns are the Čech complexes of the terms of (2), with trivial horizontal differentials. Notice that the columns of $\mathcal{B}$ split E-linearly. The complex $A = \mathrm{Tot}^{\oplus}(\mathcal{B})$ therefore contracts onto its homology. Note also that the Čech cup product makes A a dga over the ring E. ♣♣♣ Michael: [Or maybe ω and not E...of course the difference just amounts to a grading twist. A possibly more significant issue is that A is not a connected dga, because it's 1-periodic.]

We can perturb the differential on A by adding in the horizontal differentials. The result remains a dga. To see this, recall the general fact that, given two complexes $\mathcal{F}$ and $\mathcal{G}$ of sheaves, there is a map

$$\mathcal{C}(\mathcal{F}) \otimes \mathcal{C}(\mathcal{G}) \to \mathcal{C}(\mathcal{F} \otimes \mathcal{G})$$

of Čech complexes ♣♣♣ Michael: [tensoring differential modules is subtle, so this may require some more care]. Now note that we have a natural map $\mathcal{C}(\mathcal{K} \otimes \mathcal{K}) \to \mathcal{C}(\mathcal{K})$, since $\mathcal{K}$ is a dga.

Applying the formalism in the previous section, we obtain an induced A_∞-structure on the homology of A. Note that, of course, $H_*(A)$ is just the sheaf cohomology of the twists of $\mathcal{O}$ in each degree.

Finally, recall that the perturbed differential on $H_*(A)$ is the Tate differential. On the other hand, the A_∞-operations we constructed are built out of the corresponding perturbation of the differential on $T^c(H_*(A))$. So, these A_∞-operations should be encoding the Tate differential. There is just a little fuzziness here concerning the relationship between the perturbed differential on $H_*(A)$ and the perturbed differential on $T^c(H_*(A))$. Modulo this, it's clear how the above A_∞-operations are induced by the Tate differential.

To perform this construction with a sheaf other than $\mathcal{O}$, we need a version of the results in the previous section for dg- and A_∞-*modules*, rather than algebras. I suspect this is possible, but I can't seem to find it written down (perhaps because it's too obvious to write down...).

♣♣♣ Michael: [This approach doesn't work. I worked out an example, and the A_∞-algebra one gets has m_1 given by the entire Tate differential, not just the linear part. So this clearly isn't helpful.]

References

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