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1. The U-functor

We work over weighted projective space for now. So $S = k[x_0, \dots, x_n]$ with $|x_i| = w_i$. Denote by $\mathbb{P}$ the associated weighted projective space. We equip E with the $\mathbb{Z} \times \mathbb{Z}$-grading such that $|e_i| = (-w_i, 1)$. We recall the definition of the functor

$$\mathbf{L} : \mathrm{Com}(E) \to \mathrm{Com}(S)$$

from Daniel's notes. Here, $\mathrm{Com}(E)$ denotes the category of complexes of $\mathbb{Z} \times \mathbb{Z}$-graded E-modules, and $\mathrm{Com}(S)$ is the category of complexes of $\mathbb{Z}$-graded S-modules.

Remark 1.1. All E-modules are right modules. In particular, entries of matrices over E act on the right. This is also Macaulay2's convention. Note that this is the only way to make sense of the definition of the $\mathbf{R}$-functor in the EFS paper; if we apply the definition to a left E-module M, the maps in the complex $\mathbf{R}(M)$ are not E-linear. Nevertheless, sometimes we will multiply elements of E-modules on the left by elements of e (for instance, in the definition of the $\mathbf{L}$-functor). When we do this, here is what we mean. When we write em for $e \in E$ and $m \in M$, where M is a right E-module, we mean $(-1)^{|e||m|}me$, where $|-|$ denotes the degree with respect to the second (standard) grading.

For M an E-module concentrated in degree 0, $\mathbf{L}(M)$ is the complex with

$$\mathbf{L}(M)_q = \bigoplus_d M_{(-d,-q)} \otimes_k S(d)$$

and differential

$$m \otimes s \mapsto \sum_{i=0}^{n} e_i m \otimes x_i s. \tag{1}$$

♣♣♣ Michael: [I'm using homological indexing so that comparing to M2 will be easier.] For a general complex $(C, \partial) \in \mathrm{Com}(E)$, we form the bicomplex

(2)
$$\begin{tikzcd}
 & {} \arrow[d] & {} \arrow[d] & \\
\cdots & \mathbf{L}(C_p)_q \arrow[l, "\partial"'] \arrow[d] & \mathbf{L}(C_{p+1})_q \arrow[l, "\partial"'] \arrow[d] & \cdots \arrow[l, "\partial"'] \\
\cdots & \mathbf{L}(C_p)_{q-1} \arrow[l, "\partial"'] \arrow[d] & \mathbf{L}(C_{p+1})_{q-1} \arrow[l, "\partial"'] \arrow[d] & \cdots \arrow[l, "\partial"'] \\
 & {} & {} &
\end{tikzcd}$$

and apply $\mathrm{Tot}^{\oplus}(-)$. Note that the vertical differential $\mathbf{L}(C_p)_q \to \mathbf{L}(C_p)_{q-1}$ is the dual Koszul map (1) multiplied by $(-1)^p$.

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Let $\mathcal{L}(C)$ denote the bicomplex of $\mathcal{O}_{\mathbb{P}}$-modules given by applying the associated sheaf functor to the bicomplex (2). Let $\mathcal{L}'(C)$ be the sub-bicomplex of $\mathcal{L}(C)$ given by taking summands of the form $C_{p,(-d,-q)} \otimes_k \mathcal{O}(d)$ with $d \geq 0$ here, p denotes homological degree, and $(-d, -q)$ denotes internal degree. We define a functor

$$\mathbf{U} : \mathrm{Com}(E) \to \mathrm{Com}(\mathbb{P})$$

to be given by $C \mapsto \mathrm{Tot}^{\oplus}(\mathcal{L}'(C))$. See Remark 1.5 for why we use direct sum totalization rather than direct product, and see Remark 2.3 for why we truncate by taking summands with $d \geq 0$ rather than $d \leq 0$.

Proposition 1.2. *The above definition of the* $\mathbf{U}$*-functor agrees with Daniel's.*

Proof. Daniel's definition is given by

$$\omega(i,j) \mapsto \mathcal{L}(\omega_{\leq i})(i)[-j].$$

(We're abusing notation here by identifying the 1-column bicomplex $\mathcal{L}(\omega_{\leq i})(i)$ with its totalization.) We have

$$\begin{aligned}
(\mathcal{L}(\omega_{\leq i})(i)[-j])_q &= \mathcal{L}(\omega_{\leq i})_{-j+q}(i) \\
&= \bigoplus_d (\omega_{\leq i})_{(-d,j-q)} \otimes \mathcal{O}(d+i) \\
&= \bigoplus_d (\omega_{\leq i})_{(i-d,j-q)} \otimes \mathcal{O}(d) \\
&= \bigoplus_{d\geq 0} (\omega_{\leq i})_{(i-d,j-q)} \otimes \mathcal{O}(d) \\
&= \bigoplus_{d\geq 0} \omega_{(i-d,j-q)} \otimes \mathcal{O}(d) \\
&= \mathbf{U}(\omega(i,j))_q.
\end{aligned}$$

And of course the maps in both complexes are identical as well. □

Let M be a finitely generated S-module, and let $(\mathcal{T}, \partial) \in \mathrm{Com}(E)$ be its Tate complex. We recall that $\mathcal{T}$ is a complex of the form

$$\cdots \xrightarrow{\partial} T \xrightarrow{\partial} T \xrightarrow{\partial} \cdots,$$

where T is a direct sum of twists of $\omega := E^{\vee}$. The goal is to prove

Theorem 1.3. $H_n(\mathbf{U}(\mathcal{T})) \cong \widetilde{M}$ *for all* n.

Remark 1.4. Before getting started, we record the following elementary observations. Of course, ω is an E-module with k-basis given by exterior polynomials in the e_i^*. Note that $|e_i^*| = (w_i, -1)$, while $|e_i| = (-w_i, 1)$. The action of E on ω is by contraction. The $x_i \in S$ are also duals of the e_i, but we use different notation for the basis of ω to prevent confusion.

Proof when $w_i = 1$ *for all* i. The Tate module T is a direct sum of copies of $\omega(-i, i)$ for $i \in \mathbb{Z}$. We have

$$\mathbf{L}(\omega(-i,i))_q = \bigoplus_d \omega_{(-i-d,i-q)} \otimes \mathcal{O}(d).$$

A nonzero summand must satisfy $i-q=-a$ and $-i-d=a$ for some $a\in\{0,\dots,n+1\}$, i.e. $-q=d$. So, forming the bicomplex $\mathcal{L}'(\mathcal{T})$ amounts to applying the associated sheaf functor to the bicomplex (2) and chopping off the rows with $q>0$. By page 142 of Weibel, we have a spectral sequence

$$E^2_{pq}=H^h_pH^v_q(\mathcal{L}'(\mathcal{T}))\Rightarrow H_{p+q}(\mathbf{U}(\mathcal{T}))$$

that collapses on page 2 to row $q=0$, since the columns are exact elsewhere. So, it suffices to show

$$H^h_pH^v_0(\mathcal{L}'(\mathcal{T}))\cong\mathcal{M}$$

for all p. But this is clear, since each $H^v_0(\mathcal{L}'(\mathcal{T}))_p$ is just the result of applying the Ω-functor to T, and we know the homology of this complex is $\mathcal{M}$ in each degree, from Eisenbud-Floystad-Schreyer. □

Remark 1.5. Let $\mathcal{T}$ be as in the above proof. The rows of $\mathcal{L}'(\mathcal{T})$ are exact as well. Since the rows in $\mathcal{L}'(\mathcal{T})$ are 0 for $q>0$, we have a spectral sequence

$$H^v_pH^h_q(\mathcal{L}'(\mathcal{T}))\Rightarrow H_{p+q}(\mathrm{Tot}^{\Pi}(\mathcal{L}'(\mathcal{T})))$$

(Weibel page 143). It follows that $H_*(\mathrm{Tot}^{\Pi}(\mathcal{L}'(\mathcal{T})))=0$. This is why we take the direct sum totalization in the definition of $\mathbf{U}(C)$; otherwise, $\mathbf{U}$ applied to the Tate complex would give 0.

2. Examples in weighted projective space

Example 2.1. Take $S=k[x_0]$ with $|x_0|=m$ and $\mathcal{M}=\mathcal{O}$. So $\mathbb{P}=[\mathrm{Spec}(k)/(\mathbb{Z}/m)]$. The Tate complex is

$$\cdots\xrightarrow{\partial}T\xrightarrow{\partial}T\xrightarrow{\partial}\cdots,$$

where $T=\bigoplus_{i\in\mathbb{Z}}\omega_E(-mi,i)$, and ∂ is given by multiplication by e_0. We have:

$$\begin{aligned}\mathbf{L}(T)_q&=\mathbf{L}(\bigoplus_{i\in\mathbb{Z}}\omega(-mi,i))_q\\&=\bigoplus_{d\in\mathbb{Z}}\bigoplus_{i\in\mathbb{Z}}\omega(-mi,i)_{(-d,-q)}\otimes S(d)\\&=\bigoplus_{d\in\mathbb{Z}}\bigoplus_{i\in\mathbb{Z}}\omega_{(-mi-d,i-q)}\otimes S(d).\end{aligned}$$

There are two nonzero summands:

- $i=q$ and $d=-mq$
- $i=1-q$ and $d=-mq$.

We conclude:

$$\mathbf{L}(T)_q=(\omega_{(0,0)}\otimes S(-mq))\oplus(\omega_{(-m,1)}\otimes S(-mq)).$$

The bicomplex (2) therefore looks like

$$
\begin{tikzcd}
 & {} \arrow[d] & & {} \arrow[d] & \\
\cdots & (\omega_{(0,0)} \otimes S(-m)) \oplus (\omega_{(m,-1)} \otimes S(-m)) \arrow[l, "\partial"'] \arrow[d] & & (\omega_{(0,0)} \otimes S(-m)) \oplus (\omega_{(m,-1)} \otimes S(-m)) \arrow[ll, "\partial"'] \arrow[d] & \cdots \arrow[l, "\partial"'] \\
\cdots & (\omega_{(0,0)} \otimes S) \oplus (\omega_{(m,-1)} \otimes S) \arrow[l, "\partial"'] \arrow[d] & & (\omega_{(0,0)} \otimes S) \oplus (\omega_{(m,-1)} \otimes S) \arrow[ll, "\partial"'] \arrow[d] & \cdots \arrow[l, "\partial"'] \\
\cdots & (\omega_{(0,0)} \otimes S(m)) \oplus (\omega_{(m,-1)} \otimes S(m)) \arrow[l, "\partial"'] \arrow[d] & & (\omega_{(0,0)} \otimes S(m)) \oplus (\omega_{(m,-1)} \otimes S(m)) \arrow[ll, "\partial"'] \arrow[d] & \cdots , \arrow[l, "\partial"'] \\
 & {} & & {} &
\end{tikzcd}
$$

where the vertical maps are all given by $\begin{pmatrix} 0 & \pm x_0 \\ 0 & 0 \end{pmatrix}$, and the horizontal maps are all $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Thus, $\mathcal{L}'(\mathcal{T})$ looks like

$$
\begin{tikzcd}
 & {} \arrow[d] & & {} \arrow[d] & \\
\cdots & 0 \arrow[l] \arrow[d] & & 0 \arrow[ll] \arrow[d] & \cdots \arrow[l] \\
\cdots & (\omega_{(0,0)} \otimes \mathcal{O}) \oplus (\omega_{(m,-1)} \otimes \mathcal{O}) \arrow[l, "\partial"'] \arrow[d] & & (\omega_{(0,0)} \otimes \mathcal{O}) \oplus (\omega_{(m,-1)} \otimes \mathcal{O}) \arrow[ll, "\partial"'] \arrow[d] & \cdots \arrow[l, "\partial"'] \\
\cdots & (\omega_{(0,0)} \otimes \mathcal{O}(m)) \oplus (\omega_{(m,-1)} \otimes \mathcal{O}(m)) \arrow[l, "\partial"'] \arrow[d] & & (\omega_{(0,0)} \otimes \mathcal{O}(m)) \oplus (\omega_{(m,-1)} \otimes \mathcal{O}(m)) \arrow[ll, "\partial"'] \arrow[d] & \cdots . \arrow[l, "\partial"'] \\
 & {} & & {} &
\end{tikzcd}
$$

By page 142 of Weibel, we have a spectral sequence

$$E^2_{pq} = H^h_p H^v_q(\mathcal{L}'(\mathcal{T})) \Rightarrow H_{p+q}(\mathbf{U}(\mathcal{T})).$$

This spectral sequence collapses to row $q = 0$ at page 2, and we easily conclude

$$H_n(\mathbf{U}(\mathcal{T})) \cong \mathcal{O}$$

for all n, as expected.

Example 2.2. Take $S = k[x_0, x_1]$ with $|x_0| = 1$ and $|x_1| = m$. Let $\mathcal{M} = \mathcal{O}/(x_0)$. The Tate complex looks like this:

$$\cdots \xrightarrow{\partial} T \xrightarrow{\partial} T \xrightarrow{\partial} \cdots ,$$

where $T = \bigoplus_{i\in\mathbb{Z}} \omega(-mi, i)$, and ∂ given by multiplication by e_1. We have:

$$\begin{aligned}\mathbf{L}(T)_q &= \mathbf{L}(\bigoplus_{i\in\mathbb{Z}} \omega(-mi,i))_q \\ &= \bigoplus_{d\in\mathbb{Z}}\bigoplus_{i\in\mathbb{Z}} \omega(-mi,i)_{(-d,-q)} \otimes S(d) \\ &= \bigoplus_{d\in\mathbb{Z}}\bigoplus_{i\in\mathbb{Z}} \omega_{(-mi-d,i-q)} \otimes S(d).\end{aligned}$$

This time, there are 4 nonzero summands:

- $i = q$ and $d = -mq$
- $i = q-1$ and $d = m(1-q)-1$
- $i = q-1$ and $d = -mq$
- $i = q-2$ and $d = m(1-q)-1$

We conclude:

$$\mathbf{L}(T)_q = ((\omega_{(0,0)} \oplus \omega_{(m,-1)}) \otimes S(-mq)) \oplus ((\omega_{(1,-1)} \oplus \omega_{(m+1,-2)}) \otimes S(m(1-q)-1)).$$

To ease notation, set $V = (\omega_{(0,0)} \oplus \omega_{(m,-1)})$ and $W = (\omega_{(1,-1)} \oplus \omega_{(m+1,-2)})$. The bicomplex (2) looks like

$$\begin{tikzcd}
 & {} \arrow[d] & & {} \arrow[d] & \\
\cdots & (V\otimes S(-m)) \oplus (W\otimes S(-1)) \arrow[l, "\partial"'] \arrow[d] & & (V\otimes S(-m)) \oplus (W\otimes S(-1)) \arrow[ll, "\partial"'] \arrow[d] & \cdots \arrow[l, "\partial"'] \\
\cdots & (V\otimes S) \oplus (W\otimes S(m-1)) \arrow[l, "\partial"'] \arrow[d] & & (V\otimes S) \oplus (W\otimes S(m-1)) \arrow[ll, "\partial"'] \arrow[d] & \cdots \arrow[l, "\partial"'] \\
\cdots & (V\otimes S(m)) \oplus (W\otimes S(2m-1)) \arrow[l, "\partial"'] \arrow[d] & & (V\otimes S(m)) \oplus (W\otimes S(2m-1)) \arrow[ll, "\partial"'] \arrow[d] & \cdots, \arrow[l, "\partial"'] \\
 & {} & & {} &
\end{tikzcd}$$

where the vertical maps are all given by

$$\begin{pmatrix} 0 & \pm x_1 & \pm x_0 & 0 \\ 0 & 0 & 0 & \pm x_0 \\ 0 & 0 & 0 & \mp x_1 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

and the horizontal maps are

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Thus, $\mathcal{L}'(\mathcal{T})$ looks like

$$\begin{tikzcd}
 & {} \arrow[d] & {} \arrow[d] & \\
\cdots & 0 \arrow[l] & 0 \arrow[l] & \cdots \arrow[l] \\
\cdots & (V \otimes \mathcal{O}) \oplus (W \otimes \mathcal{O}(m-1)) \arrow[l, "\partial"'] \arrow[from=u] & (V \otimes \mathcal{O}) \oplus (W \otimes \mathcal{O}(m-1)) \arrow[l, "\partial"'] \arrow[from=u] & \cdots \arrow[l, "\partial"'] \\
\cdots & (V \otimes \mathcal{O}(m)) \oplus (W \otimes \mathcal{O}(2m-1)) \arrow[l, "\partial"'] \arrow[from=u] \arrow[d] & (V \otimes \mathcal{O}(m)) \oplus (W \otimes \mathcal{O}(2m-1)) \arrow[l, "\partial"'] \arrow[from=u] \arrow[d] & \cdots . \arrow[l, "\partial"'] \\
 & {} & {} &
\end{tikzcd}$$

As in the previous example, we have a spectral sequence

$$E^2_{pq} = H^h_p H^v_q(\mathcal{L}'(\mathcal{T})) \Rightarrow H_{p+q}(\mathbf{U}(\mathcal{T})).$$

The spectral sequence collapses to row $q = 0$ at page 2. One easily checks that $H^v_0(\mathcal{L}'(\mathcal{T}))_p$ is free of rank 2 for all p, with basis

$$b_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, b_2 = \begin{pmatrix} 0 \\ x_0 \\ -x_1 \\ 0 \end{pmatrix}.$$

The horizontal map ∂ kills b_1 and sends b_2 to $x_0 b_1$. We therefore have

$$H_n(\mathbf{U}(\mathcal{T})) \cong \mathcal{O}/(x_0)$$

for all n, as expected.

Remark 2.3. In the last example, let's suppose that we defined $\mathcal{L}'(-)$ by chopping off the $d > 0$ terms, rather than the $d < 0$ terms. Also, assume $m > 1$. We get:

$$\begin{tikzcd}
 & {} \arrow[d] & {} \arrow[d] & \\
\cdots & (V \otimes \mathcal{O}(-m)) \oplus (W \otimes \mathcal{O}(-1)) \arrow[l, "\partial"'] \arrow[d] & (V \otimes \mathcal{O}(-m)) \oplus (W \otimes \mathcal{O}(-1)) \arrow[l, "\partial"'] \arrow[d] & \cdots \arrow[l, "\partial"'] \\
\cdots & V \otimes \mathcal{O} \arrow[l, "\partial"'] \arrow[d] & V \otimes \mathcal{O} \arrow[l, "\partial"'] \arrow[d] & \cdots \arrow[l, "\partial"'] \\
\cdots & 0 \arrow[l] \arrow[d] & 0 \arrow[l] \arrow[d] & \cdots . \arrow[l] \\
 & {} & {} &
\end{tikzcd}$$

In this setting, it only makes sense to define the $\mathbf{U}$-functor using a direct product totalization; the direct sum totalization will give the wrong answer, by the reasoning in Remark 1.5. As above, we have a spectral sequence

$$E^2_{pq} = H^h_p H^v_q(\mathcal{L}'(\mathcal{T})) \Rightarrow H_{p+q}(\mathbf{U}(\mathcal{T})).$$

We have

$$H^v_0(\mathcal{L}'(\mathcal{T}))_p = \mathcal{O}/(x_1) \oplus \mathcal{O}/(x_0)$$

and

$$H^v_1(\mathcal{L}'(\mathcal{T}))_p = \mathcal{O}(-1)$$

for all p. The rest of the vertical homology is obviously 0, since the columns in $\mathcal{L}(\mathcal{T})$ are exact. We therefore have

$$H^h_p H^v_q(\mathcal{L}'(\mathcal{T})) = \begin{cases} \mathcal{O}/(x_1), & q = 0 \\ \mathcal{O}(-1), & q = 1 \\ 0 & \text{else} \end{cases}$$

We get a long exact sequence

$$\cdots \to H_{p+1}(\mathbf{U}(\mathcal{T})) \to \mathcal{O}/(x_1) \to \mathcal{O}(-1) \to H_p(\mathbf{U}(\mathcal{T})) \to \cdots .$$

The map $\mathcal{O}/(x_1) \to \mathcal{O}(-1)$ is obviously 0, which means each $H_p(\mathbf{U}(\mathcal{T}))$ contains a copy of $\mathcal{O}(-1)$. In particular, $H_p(\mathbf{U}(\mathcal{T})) \neq \mathcal{O}/(x_0)$, so our output is wrong!