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1. The U-functor

We work over weighted projective space for now. So $S = k[x_0, \dots, x_n]$ with $|x_i| = w_i$. Denote by $\mathbb{P}$ the associated weighted projective space. We equip E with the $\mathbb{Z} \times \mathbb{Z}$-grading such that $|e_i| = (-w_i, 1)$. We recall the definition of the functor

$$\mathbf{L} : \mathrm{Com}(E) \to \mathrm{Com}(S)$$

from Daniel's notes. Here, $\mathrm{Com}(E)$ denotes the category of complexes of $\mathbb{Z} \times \mathbb{Z}$-graded E-modules, and $\mathrm{Com}(S)$ is the category of complexes of $\mathbb{Z}$-graded S-modules. For M an E-module concentrated in degree 0, $\mathbf{L}(M)$ is the complex with

$$\mathbf{L}(M)_q = \bigoplus_d M_{(-d,-q)} \otimes_k S(d)$$

and differential

$$m \otimes s \mapsto \sum_{i=0}^{n} e_i m \otimes x_i s. \tag{1}$$

♣♣♣ Michael: [I'm using homological indexing so that comparing to M2 will be easier.] For a general complex $(C, \partial) \in \mathrm{Com}(E)$, we form the bicomplex

$$\begin{tikzcd}
 & {} \arrow[d] & {} \arrow[d] & \\
\cdots & \mathbf{L}(C_p)_q \arrow[l, "\partial"'] \arrow[d] & \mathbf{L}(C_{p+1})_q \arrow[l, "\partial"'] \arrow[d] & \cdots \arrow[l, "\partial"'] \\
\cdots & \mathbf{L}(C_p)_{q-1} \arrow[l, "\partial"'] \arrow[d] & \mathbf{L}(C_{p+1})_{q-1} \arrow[l, "\partial"'] \arrow[d] & \cdots \arrow[l, "\partial"'] \\
 & {} & {} &
\end{tikzcd} \tag{2}$$

and apply $\mathrm{Tot}^{\oplus}(-)$. Note that the vertical differential $\mathbf{L}(C_p)_q \to \mathbf{L}(C_p)_{q-1}$ is the dual Koszul map (1) multiplied by $(-1)^p$.

Let $\mathcal{L}(C)$ denote the bicomplex of $\mathcal{O}_{\mathbb{P}}$-modules given by applying the associated sheaf functor to the bicomplex (2). Let $\mathcal{L}'(C)$ be the sub-bicomplex of $\mathcal{L}(C)$ given by taking summands of the form $C_{p,(-d,-q)} \otimes_k \mathcal{O}(d)$ with $d \geq 0$ here, p denotes homological degree, and $(-d, -q)$ denotes internal degree. We define a functor

$$\mathbf{U} : \mathrm{Com}(E) \to \mathrm{Com}(\mathbb{P})$$

to be given by $C \mapsto \mathrm{Tot}^{\oplus}(\mathcal{L}'(C))$. See Remark 1.4 for why we use direct sum totalization rather than direct product, and see Remark 1.7 for why we truncate by taking summands with $d \geq 0$ rather than $d \leq 0$.

Date: May 20, 2020.

Proposition 1.1. *The above definition of the* $\mathbf{U}$*-functor agrees with Daniel's.*

Proof. Daniel's definition is given by

$$\omega_E(i,j) \mapsto \mathcal{L}(\omega_{\leq i})(i)[-j].$$

We have

$$\begin{aligned}
(\mathcal{L}(\omega_{\leq i})(i)[-j])_q &= \mathcal{L}(\omega_{\leq i})_{-j+q}(i) \\
&= \bigoplus_d (\omega_{\leq i})_{(-d,j-q)} \otimes \mathcal{O}(d+i) \\
&= \bigoplus_d (\omega_{\leq i})_{(i-d,j-q)} \otimes \mathcal{O}(d) \\
&= \bigoplus_{d\geq 0} (\omega_{\leq i})_{(i-d,j-q)} \otimes \mathcal{O}(d) \\
&= \bigoplus_{d\geq 0} \omega_{(i-d,j-q)} \otimes \mathcal{O}(d) \\
&= \mathbf{U}(\omega(i,j))_q.
\end{aligned}$$

And of course the maps in both complexes are identical as well. $\square$

Let M be a finitely generated S-module, and let $(\mathcal{T}, \partial) \in \mathrm{Com}(E)$ be its Tate complex. We recall that $\mathcal{T}$ is a complex of the form

$$\cdots \xrightarrow{\partial} T \xrightarrow{\partial} T \xrightarrow{\partial} \cdots,$$

where T is a direct sum of twists of $\omega_E := E^\vee$. The goal is to prove

Theorem 1.2. $H_n(\mathbf{U}(\mathcal{T})) \cong \widetilde{M}$ *for all* n*.*

Remark 1.3. Before getting started, we record the following elementary observations. Of course, ω_E is an E-module with k-basis given by exterior polynomials in the e_i^*. Note that $|e_i^*| = (w_i, -1)$, while $|e_i| = (-w_i, 1)$. The action of E on ω_E is by contraction. The $x_i \in S$ are also duals of the e_i, but we use different notation for the basis of ω_E to prevent confusion.

Proof when $w_i = 1$ *for all* i. The Tate module T is a direct sum of copies of $\omega_E(i, -i)$ for $i \in \mathbb{Z}$. We have

$$\mathbf{L}(\omega_E(i,-i))_q = \bigoplus_d \omega_{(i-d,-i-q)} \otimes \mathcal{O}(d).$$

A nonzero summand must satisfy $-i-q = -a$ and $i-d = a$ for some $a \in \{0, \ldots, n+1\}$, i.e. $-q = d$. So, forming the bicomplex $\mathcal{L}'(\mathcal{T})$ amounts to applying the associated sheaf functor to the bicomplex (2) and chopping off the rows with $q > 0$. By page 142 of Weibel, we have a spectral sequence

$$E^2_{pq} = H^h_p H^v_q(\mathcal{L}'(\mathcal{T})) \Rightarrow H_{p+q}(\mathbf{U}(\mathcal{T}))$$

that collapses on page 2 to row $q = 0$, since the columns are exact elsewhere. So, it suffices to show

$$H^h_p H^v_0(\mathcal{L}'(\mathcal{T})) \cong \mathcal{M}$$

for all p. But this is clear, since each $H^v_0(\mathcal{L}'(\mathcal{T}))_p$ is just the result of applying the Ω-functor to T, and we know the homology of this complex is $\mathcal{M}$ in each degree, from Eisenbud-Floystad-Schreyer. $\square$

Remark 1.4. Let $\mathcal{T}$ be as in the above proof. The rows of $\mathcal{L}'(\mathcal{T})$ are exact as well. Since the rows in $\mathcal{L}'(C)$ are 0 for $q < 0$, we have a spectral sequence

$$H^v_p H^h_q(\mathcal{L}'(\mathcal{T})) \Rightarrow H_{p+q}(\mathrm{Tot}^{\Pi}(\mathcal{L}'(\mathcal{T})))$$

(Weibel page 143). It follows that $H_*(\mathrm{Tot}^{\Pi}(\mathcal{L}'(\mathcal{T}))) = 0$. This is why we take the direct sum totalization in the definition of $\mathbf{U}(C)$; otherwise, $\mathbf{U}$ applied to the Tate complex would give 0.

Some weighted projective examples:

Example 1.5. Take $S = k[x_0]$ with $|x_0| = m$ and $\mathcal{M} = \mathcal{O}$. So $\mathbb{P} = [\mathrm{Spec}(k)/(\mathbb{Z}/m)]$. The Tate complex is

$$\cdots \xrightarrow{\partial} T \xrightarrow{\partial} T \xrightarrow{\partial} \cdots,$$

where $T = \bigoplus_{i \in \mathbb{Z}} \omega_E(mi, -i)$, and ∂ is a matrix with e_0 on the subdiagonal and 0 elsewhere. Let's compute $\mathbf{L}(T)_q$.

$$\begin{aligned}
\mathbf{L}(T(mj,-j))_q &= \mathbf{L}(\bigoplus_{i\in\mathbb{Z}} \omega_E(mi,-i))_q \\
&= \bigoplus_{d\in\mathbb{Z}} \bigoplus_{i\in\mathbb{Z}} \omega_E(mi,-i)_{(-d,-q)} \otimes S(d) \\
&= \bigoplus_{d\in\mathbb{Z}} \bigoplus_{i\in\mathbb{Z}} \omega_{(mi-d,-i-q)} \otimes S(d).
\end{aligned}$$

There are two nonzero summands:

- $i = -q$ and $d = -mq$
- $i = -q + 1$ and $d = -mq$.

We conclude:

$$\mathbf{L}(T)_q = (\omega_{(0,0)} \otimes S(-mq)) \oplus (\omega_{(-m,1)} \otimes S(-mq)).$$

The bicomplex (2) therefore looks like

$$\begin{tikzcd}
 & {} \arrow[d] & & {} \arrow[d] & \\
\cdots & (\omega_{(0,0)} \otimes S(-m)) \oplus (\omega_{(m,-1)} \otimes S(-m)) \arrow[l, "\partial"'] \arrow[d] & & (\omega_{(0,0)} \otimes S(-m)) \oplus (\omega_{(m,-1)} \otimes S(-m)) \arrow[ll, "\partial"'] \arrow[d] & \cdots \arrow[l, "\partial"'] \\
\cdots & (\omega_{(0,0)} \otimes S) \oplus (\omega_{(m,-1)} \otimes S) \arrow[l, "\partial"'] \arrow[d] & & (\omega_{(0,0)} \otimes S) \oplus (\omega_{(m,-1)} \otimes S) \arrow[ll, "\partial"'] \arrow[d] & \cdots \arrow[l, "\partial"'] \\
\cdots & (\omega_{(0,0)} \otimes S(m)) \oplus (\omega_{(m,-1)} \otimes S(m)) \arrow[l, "\partial"'] \arrow[d] & & (\omega_{(0,0)} \otimes S(m)) \oplus (\omega_{(m,-1)} \otimes S(m)) \arrow[ll, "\partial"'] \arrow[d] & \cdots, \arrow[l, "\partial"'] \\
 & {} & & {} &
\end{tikzcd}$$

where the vertical maps are all given by $\begin{pmatrix} 0 & \pm x_0 \\ 0 & 0 \end{pmatrix}$, and the horizontal maps are all $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Thus, $\mathcal{L}'(\mathcal{T})$ looks like

$$\begin{tikzcd}
 & {} \arrow[d] & & {} \arrow[d] & \\
\cdots & 0 \arrow[l] \arrow[d] & & 0 \arrow[ll] \arrow[d] & \cdots \arrow[l] \\
\cdots & (\omega_{(0,0)} \otimes \mathcal{O}) \oplus (\omega_{(m,-1)} \otimes \mathcal{O}) \arrow[l, "\partial"'] \arrow[d] & & (\omega_{(0,0)} \otimes \mathcal{O}) \oplus (\omega_{(m,-1)} \otimes \mathcal{O}) \arrow[ll, "\partial"'] \arrow[d] & \cdots \arrow[l, "\partial"'] \\
\cdots & (\omega_{(0,0)} \otimes \mathcal{O}(m)) \oplus (\omega_{(m,-1)} \otimes \mathcal{O}(m)) \arrow[l, "\partial"'] \arrow[d] & & (\omega_{(0,0)} \otimes \mathcal{O}(m)) \oplus (\omega_{(m,-1)} \otimes \mathcal{O}(m)) \arrow[ll, "\partial"'] \arrow[d] & \cdots . \arrow[l, "\partial"'] \\
 & {} & & {} &
\end{tikzcd}$$

By page 142 of Weibel, we have a spectral sequence

$$E^2_{pq} = H^h_p H^v_q(\mathcal{L}'(\mathcal{T})) \Rightarrow H_{p+q}(\mathbf{U}(\mathcal{T})).$$

This spectral sequence collapses to row $q = 0$ at page 2, and we easily conclude

$$H_n(\mathbf{U}(\mathcal{T})) \cong \mathcal{O}$$

for all n, as expected.

Example 1.6. Take $S = k[x_0, x_1]$ with $|x_0| = 1$ and $|x_1| = m$. Let $\mathcal{M} = \mathcal{O}/(x_0)$. The Tate complex looks like this:

$$\cdots \xrightarrow{\partial} T \xrightarrow{\partial} T \xrightarrow{\partial} \cdots ,$$

where $T = \bigoplus_{i \in \mathbb{Z}} \omega_E(mi, -i)$, and ∂ is a matrix with e_1 on the subdiagonal. Let's compute $\mathbf{L}(T)_q$.

$$\begin{aligned}
\mathbf{L}(T)_q &= \mathbf{L}(\bigoplus_{i\in\mathbb{Z}} \omega_E(mi, -i))_q \\
&= \bigoplus_{d\in\mathbb{Z}} \bigoplus_{i\in\mathbb{Z}} \omega_E(mi, -i)_{(-d,-q)} \otimes S(d) \\
&= \bigoplus_{d\in\mathbb{Z}} \bigoplus_{i\in\mathbb{Z}} \omega_{(mi-d,-i-q)} \otimes S(d).
\end{aligned}$$

This time, there are 4 nonzero summands:

- $i = -q$ and $d = -mq$
- $i = -q + 1$ and $d = m(1 - q) - 1$
- $i = -q + 1$ and $d = -mq$
- $i = -q + 2$ and $d = m(1 - q) - 1$

We conclude:

$$\mathbf{L}(T)_q = ((\omega_{(0,0)} \oplus \omega_{(m,-1)}) \otimes S(-mq)) \oplus ((\omega_{(1,-1)} \oplus \omega_{(m+1,-2)}) \otimes S(-mq + m - 1)).$$

To ease notation, set $V = (\omega_{(0,0)} \oplus \omega_{(m,-1)})$ and $W = (\omega_{(1,-1)} \oplus \omega_{(m+1,-2)})$. The bicomplex (2) looks like

$$\begin{tikzcd}
 & {} \arrow[d] & & {} \arrow[d] & \\
\cdots & (V \otimes S(-m)) \oplus (W \otimes S(-1)) \arrow[l, "\partial"'] \arrow[d] & & (V \otimes S(-m)) \oplus (W \otimes S(-1)) \arrow[ll, "\partial"'] \arrow[d] & \cdots \arrow[l, "\partial"'] \\
\cdots & (V \otimes S) \oplus (W \otimes S(m-1)) \arrow[l, "\partial"'] \arrow[d] & & (V \otimes S) \oplus (W \otimes S(m-1)) \arrow[ll, "\partial"'] \arrow[d] & \cdots \arrow[l, "\partial"'] \\
\cdots & (V \otimes S(m)) \oplus (W \otimes S(2m-1)) \arrow[l, "\partial"'] \arrow[d] & & (V \otimes S(m)) \oplus (W \otimes S(2m-1)) \arrow[ll, "\partial"'] \arrow[d] & \cdots, \arrow[l, "\partial"'] \\
 & {} & & {} &
\end{tikzcd}$$

where the vertical maps are all given by

$$\begin{pmatrix} 0 & \pm x_1 & \pm x_0 & 0 \\ 0 & 0 & 0 & \mp x_1 \\ 0 & 0 & 0 & \pm x_0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

and the horizontal maps are

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Thus, $\mathcal{L}'(\mathcal{T})$ looks like

$$\begin{tikzcd}
 & {} \arrow[d] & & {} \arrow[d] & \\
\cdots & 0 \arrow[l] \arrow[d] & & 0 \arrow[ll] \arrow[d] & \cdots \arrow[l] \\
\cdots & (V \otimes \mathcal{O}) \oplus (W \otimes \mathcal{O}(m-1)) \arrow[l, "\partial"'] \arrow[d] & & (V \otimes \mathcal{O}) \oplus (W \otimes \mathcal{O}(m-1)) \arrow[ll, "\partial"'] \arrow[d] & \cdots \arrow[l, "\partial"'] \\
\cdots & (V \otimes \mathcal{O}(m)) \oplus (W \otimes \mathcal{O}(2m-1)) \arrow[l, "\partial"'] \arrow[d] & & (V \otimes \mathcal{O}(m)) \oplus (W \otimes \mathcal{O}(2m-1)) \arrow[ll, "\partial"'] \arrow[d] & \cdots. \arrow[l, "\partial"'] \\
 & {} & & {} &
\end{tikzcd}$$

As in the previous example, we have a spectral sequence

$$E^2_{pq} = H^h_p H^v_q(\mathcal{L}'(\mathcal{T})) \Rightarrow H_{p+q}(\mathbf{U}(\mathcal{T})).$$

The spectral sequence collapses to row $q = 0$ at page 2. One easily checks that $H_0^v(\mathcal{L}'(\mathcal{T}))_p$ is free of rank 2 for all p, with basis

$$b_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, b_2 = \begin{pmatrix} 0 \\ x_0 \\ -x_1 \\ 0 \end{pmatrix}.$$

The horizontal map ∂ kills b_1 and sends b_2 to $x_0 b_1$. We therefore have

$$H_n(\mathbf{U}(\mathcal{T})) \cong \mathcal{O}/(x_0)$$

for all n, as expected.

Remark 1.7. In the last example, let's suppose that we defined $\mathcal{L}'(-)$ by chopping off the $d \geq 0$ terms, rather than the $d \leq 0$ terms. Also, assume $m > 1$. We'd get:

$$\begin{tikzcd}
 & {} \arrow[d] & & {} \arrow[d] & \\
\cdots & (V \otimes \mathcal{O}(-m)) \oplus (W \otimes \mathcal{O}(-1)) \arrow[l, "\partial"'] \arrow[d] & & (V \otimes \mathcal{O}(-m)) \oplus (W \otimes \mathcal{O}(-1)) \arrow[ll, "\partial"'] \arrow[d] & \cdots \arrow[l, "\partial"'] \\
\cdots & V \otimes \mathcal{O} \arrow[l, "\partial"'] \arrow[d] & & V \otimes \mathcal{O} \arrow[ll, "\partial"'] \arrow[d] & \cdots \arrow[l, "\partial"'] \\
\cdots & 0 \arrow[l] \arrow[d] & & 0 \arrow[ll] \arrow[d] & \cdots \arrow[l] \\
 & {} & & {} &
\end{tikzcd}$$

In this setting, the $\mathbf{U}$-functor is defined using a direct product totalization, for the reason described in Remark 1.4. As above, we have a spectral sequence

$$E^2_{pq} = H^h_p H^v_q(\mathcal{L}'(\mathcal{T})) \Rightarrow H_{p+q}(\mathbf{U}(\mathcal{T})).$$

We have

$$H^v_0(\mathcal{L}'(\mathcal{T}))_p = \mathcal{O}/(x_1)$$

and

$$H^v_1(\mathcal{L}'(\mathcal{T}))_p = 0$$

for all p. The rest of the vertical homology is obviously 0, since the columns in $\mathcal{L}(\mathcal{T})$ are exact. The induced horizontal maps on the vertical homology are all 0, so we have

$$H_n(\mathbf{U}(\mathcal{T})) = \mathcal{O}/(x_1)$$

for all n. This is wrong!