U-FUNCTOR

1. The $\mathbf{L}$-functor(s)

Let $S = k[x_0, x_1, x_2]$ be the Cox ring of $\mathbb{P}(1,1,2)$ and E be the dual exterior algebra with variables e_0, e_1, e_2 with degrees $(-1;1), (-1;1)$, and $(-2;1)$.

Given an E-module M, we define a free complex of S-modules $\mathbf{L}(M)$

$$\cdots \to \mathbf{L}(M)_{j+1} \to \mathbf{L}(M)_j \to \mathbf{L}(M)_{j-1} \to \cdots$$

where in homological degree j we have $\mathbf{L}(M)_j = \bigoplus_d S(d) \otimes_k M_{-d;-j}$ (so the "extra grading" reproduces the homological degree). The differential is $\sum_i x_i \otimes e_i$. This yields a functor:

$$\mathbf{L}\colon \operatorname{Mod}(E) \to \operatorname{Cpx}(S)$$

Remark 1.1. Note what happens under twist: $M(a;b)_{-d;-j} = M_{-d+a;-j+b}$ and thus $\mathbf{L}(M(a;b))_j = \bigoplus_d S(d) \otimes M_{-d+a;-j-b}$ and thus $\mathbf{L}(M(a;b)) = \mathbf{L}(M)(a)[-b]$.

This is the effect of $\mathbf{L}$ on a module. But now imagine that M has a nontrivial differential $\partial\colon M \to M(0;1)$. Throughout, we will try to represent differential modules as unfolded complexes:

$$\cdots \xrightarrow{\partial} M(0;-1) \xrightarrow{\partial} M \xrightarrow{\partial} M(0;1) \xrightarrow{\partial} \cdots$$

so as to account for the appropriate twists in grading. We claim that the differential ∂ induces a map of complexes $\mathbf{L}(\partial)\colon \mathbf{L}(M)[1] \to \mathbf{L}(M)$ which squares to zero. To check this, we choose an element $m \in M_{-d;-j}$ which yields a generator $1 \otimes m \in \mathbf{L}(M)_j$. By the degree of ∂, we have $\partial(m) \in M_{-d;-j+1}$. We can thus define a map $\mathbf{L}(M)[1] \to \mathbf{L}(M)$ by $1 \otimes m \mapsto 1 \otimes \partial(m) \in \mathbf{L}(M)_{j-1}$. Checking that this is a map of complexes amount to checking that

$$\sum x_i \otimes e_i \partial(m) = \sum x_i \otimes \partial(e_i m)$$

which is simply the fact that ∂ was an E-module map. The fact that $\mathbf{L}(\partial)$ squares to zero is also immediate.

In summary:

Proposition 1.2. *There is a functor which $\mathbf{L}_{\mathrm{DM}}$ which sends a differential E-module*

$$\cdots \xrightarrow{\partial} M(0;-1) \xrightarrow{\partial} M \xrightarrow{\partial} M(0;1) \xrightarrow{\partial} \cdots$$

to a "differential complex":

$$\cdots \xrightarrow{\mathbf{L}\partial} \mathbf{L}(M)[1] \xrightarrow{\mathbf{L}\partial} \mathbf{L}(M) \xrightarrow{\mathbf{L}\partial} \mathbf{L}(M)[-1] \xrightarrow{\mathbf{L}\partial} \cdots$$

At the functorial level: we write $\mathrm{DM}_{(0;1)}(E)$ for the category of differential E-modules where the differential shifts the degree by $(0;1)$. And write $\mathrm{DC}_{[1]}(S)$ for the category of differential complexes, where the differential shifts the homological *degree by 1. Then $\mathbf{L}_{\mathrm{DM}}$ is a functor:*

$$\mathbf{L}_{\mathrm{DM}}\colon \mathrm{DM}_{(0;1)}(E) \to \mathrm{DC}_{[1]}(S).$$

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Taking the homology of an element in $\mathrm{DC}_{[1]}(S)$ will yield a complex over S, and thus an element of the derived category $D(S)$; in particular, if M is a differential module, then the homology of $\mathbf{L}_{\mathrm{DM}}(M)$ has a homological grading.

In a previous file, we constructed the functor $\mathbf{R}$ which (using this new notation) went from the derived category $D(S) \to \mathrm{DM}_{(0;1)}(E)$. The key claim is something like:

Proposition 1.3. *The (zeroth) homology of $(\mathbf{L}_{\mathrm{DM}} \circ \mathbf{R})(M)$–which is itself a complex of S-modules–is quasi-isomorphic to M.*

Sketch of proof. To simplify notation, we write $M\underline{\otimes}_k\omega_E$ for the graded tensor product. Then we separately track the "extra" grading. Thus $\mathbf{R}(M)$ is the differential module:

$$\cdots \to M\underline{\otimes}_k\omega_E(0;-1) \to M\underline{\otimes}_k\omega_E \to M\underline{\otimes}_k\omega_E(0;1) \to \cdots$$

Each term is simply a free E-module. For an E-module N, we write $N_{*;i}$ for the degree $(*;i)$ piece, where $*$ can be anything. And we write $N_{*,i}\underline{\otimes}_k S$ for the graded tensor products $\oplus_{a\in\mathbb{Z}} N_{-a,i} \otimes_k S(a)$. Applying the $\mathbf{L}$ functor to $M\underline{\otimes}_k\omega_E$ yields a complex

$$\mathbf{L}(M\underline{\otimes}_k\omega_E) = [(M\underline{\otimes}_k\omega_E)_{*;0}\underline{\otimes}S \to (M\underline{\otimes}_k\omega_E)_{*;1}\underline{\otimes}S \to \cdots \to (M\underline{\otimes}_k\omega_E)_{*;w}\underline{\otimes}S]$$

Thus, applying $\mathbf{L}_{\mathrm{DM}}$ to the differential above yields a double complex

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & (M\underline{\otimes}_k\omega_E)_{*;0}\underline{\otimes}S \arrow[r] \arrow[d] & (M\underline{\otimes}_k\omega_E)_{*;1}\underline{\otimes}S \arrow[r] \arrow[d] & \cdots \\
0 \arrow[r] \arrow[d] & (M\underline{\otimes}_k\omega_E)_{*;0}\underline{\otimes}S \arrow[r] \arrow[d] & (M\underline{\otimes}_k\omega_E)_{*;1}\underline{\otimes}S \arrow[r] \arrow[d] & {} \arrow[d] & \cdots \\
(M\underline{\otimes}_k\omega_E)_{*;0}\underline{\otimes}S \arrow[r] & (M\underline{\otimes}_k\omega_E)_{*;1}\underline{\otimes}S \arrow[r] & (M\underline{\otimes}_k\omega_E)_{*;2}\underline{\otimes}S & {} & \cdots
\end{tikzcd}$$

By associativity of tensor products, each column becomes

$$M\underline{\otimes}_k(\omega_{*;0}\underline{\otimes}S) \to M\underline{\otimes}_k(\omega_{*;1}\underline{\otimes}S) \to \cdots = M\underline{\otimes}_k\Big((\mathbf{L}_{\mathrm{DM}} \circ \mathbf{R})(k)\Big)$$

So it suffices to check $(\mathbf{L}_{\mathrm{DM}} \circ \mathbf{R})(k)$ is quasi-isomorphic to k. But $\mathbf{R}(k) = \omega_E$ and

$$\mathbf{L}(\omega_E) = \Big[(\omega_E)_{4,0} \otimes S(-4) \to (\omega_E)_{3,1} \otimes S(-3) \oplus (\omega_E)_{2,1} \otimes S(-2) \to \cdots\Big]$$

is the Koszul complex, which is quasi-isomorphic to k as desired. ♣♣♣ Daniel: [Need to check now that we've changed indexing...] □

2. Truncated Koszul complexes

Let K be the Koszul complex on x_0, x_1, x_2

$$0 \to \mathcal{O}(-4) \to \mathcal{O}(-2) \oplus \mathcal{O}^2(-3) \to \mathcal{O}(-1)^2 \oplus \mathcal{O}(-2) \to \underline{\mathcal{O}} \to 0.$$

Throughout, we will indicate homological degree zero by underlining it (as above). We parametrize bases from right to left as: $\{x_0x_1x_2\}, \{x_1x_2, x_0x_2, x_0x_1\}, \{x_2, x_1, x_0\}, \{1\}$. In this way, the differential on K is $\sum_i x_i \otimes e_i$ acting via contraction. So our basis of the Koszul complex is in natural bijection with the basis of ω_E.

For any degree $d \in \mathbb{Z}$ we define K_d as the subcomplex consisting of line bundles of degrees between 0 and $-d$. We do not yet impose any homological shifts. Thus

$$K_0 = [\underline{\mathcal{O}}] \qquad K_1 = [\mathcal{O}(-1)^2 \to \underline{\mathcal{O}}] \qquad K_2 = [\mathcal{O}(-2) \to \mathcal{O}(-1)^2 \oplus \mathcal{O}(-2) \to \underline{\mathcal{O}}]$$

and

$$K_3 = [\mathcal{O}(-2) \oplus \mathcal{O}(-3)^2 \to \mathcal{O}(-1)^2 \oplus \mathcal{O}(-2) \to \underline{\mathcal{O}}]$$

Of course $K_d = 0$ if $d < 0$ and K_d is quasi-isomorphic to zero if $d \geq 4$.

We want to realize each K_d as $\mathbf{L}(-)$ for some module. We write $\omega_{\leq d}$ for the submodule of ω_E consisting of degrees $\leq d$. For an E-module M, define $\mathbf{L}(M)$ as before (a complex of S-moduules), and we write $\widetilde{\mathbf{L}}(M)$ for the corresponding complex of sheaves on $\mathbb{P}(1,1,2)$ Thus for instance $\omega_{\leq 0} = k$ (the socle) while $\omega_{\leq 4}$ is all of ω_E. We note that

$$K_d = \widetilde{\mathbf{L}}(\omega_{\leq d})$$

Lemma 2.1. *If $f \in E_{-a;j}$ and d is any integer (though we are most interested in $d = 0, 1, \dots, w-1$) then multiplication by f induces each of the following maps:*

(1) $\omega_E(a; -j) \to \omega_E$
(2) $\omega_{\leq d}(a; -j) \to \omega_{\leq d-a}$
(3) $\mathbf{L}(\omega_{\leq -d}(a; -j)) = \mathbf{L}(\omega_{\leq -d})(a)[j] \to \mathbf{L}(\omega_{\geq -d-a})$
(4) $K_d(a)[j] \to K_{d+a}$

Proof. Mostly obvious, though the twist/shift stuff follows from Remark 1.1. $\square$

Example 2.2. Let's consider the element e_0 which has degree $(-1; 1)$. This induces a map $\omega_E(1; -1) \to \omega_E$. And thus a map of complexes:

(1)
$$\begin{tikzcd} \widetilde{\mathbf{L}}(\omega_{\leq 1})(1)[1] : \arrow[d] & \underline{\mathcal{O}^2} \arrow[r] \arrow[d, "{[1,0]}"] & \mathcal{O}(1) \\ \widetilde{\mathbf{L}}(E_{\leq 0}) : & \underline{\mathcal{O}} & \end{tikzcd}$$

If we had chosen the element e_1, we would have obtained a similar map, except the vertical arrow would be $[0, -1]$.

Each e_0 and e_1 also induce maps on other complexes. For instance, $e_1 : \omega_{\leq 2}(1; -1) \to \omega_{\leq 1}$ and thus induces a map

(2)
$$\begin{tikzcd} K_2(2)[1] = \widetilde{\mathbf{L}}(\omega_{\leq 3})(1)[1] \arrow[d] & \mathcal{O} \arrow[r] \arrow[d, "{[1,0]}"] & \underline{\mathcal{O}(1)^2 \oplus \mathcal{O}} \arrow[r] \arrow[d, "{[0,-1,0]}"] & \mathcal{O}(2) \\ K_1(1) = \widetilde{\mathbf{L}}(\omega_{\leq 2}) : & \mathcal{O}^2 \arrow[r] & \underline{\mathcal{O}(1)} & \end{tikzcd}$$

3. The U-functor(s)

As with the $\mathbf{L}$-functor, we will define the $\mathbf{U}$ functor on (free) modules first, and then on modules with a differential. For modules, we claim:

Corollary 3.1. *There is a functor*

$$\mathbf{U}\colon \operatorname{Mod}_{free}(E) \to \operatorname{Cpx}(\mathbb{P}^n)$$

determined by $\omega_E(d;-i) \mapsto K_d(d)[i] = \widetilde{\mathbf{L}}(\omega_{\leq d})(d)[i]$. *Note in particular that for any free module* F, $\mathbf{U}(F(0;-1)) = \mathbf{U}(F)[1]$.

Proof. The key point is the above lemma… ☐

Example 3.2. To streamline notation, let's write $\mathbf{U}(d;i) := \mathbf{U}(\omega_E(d;i))$. Thus we have four basic complexes (up to shifts):

$$\mathbf{U}(0;0) = \left(\underline{\mathcal{O}}\right) \qquad \mathbf{U}(1;0) = \left(\mathcal{O}^2 \to \underline{\mathcal{O}(1)}\right) \qquad \mathbf{U}(2;0) = \left(\mathcal{O} \to \mathcal{O}(1)^2 \oplus \mathcal{O} \to \underline{\mathcal{O}(2)}\right)$$

and

$$\mathbf{U}(3;0) = \left(\mathcal{O}(1) \oplus \mathcal{O}^2 \to \mathcal{O}(2)^2 \oplus \mathcal{O}(1) \to \underline{\mathcal{O}(3)}\right)$$

The element e_0 induces maps $\mathbf{U}(d;i) \to \mathbf{U}(d-1;i+1)$ for all d. So there are three interesting induced maps:

$$\begin{tikzcd}
\mathbf{U}(1;-1) & = & \underline{\mathcal{O}^2} \arrow[r] \arrow[d] & \mathcal{O}(1) \\
\mathbf{U}(0;0) & = & \underline{\mathcal{O}} &
\end{tikzcd}$$

and

$$\begin{tikzcd}
\mathbf{U}(2;-1) & = & \mathcal{O} \arrow[r] \arrow[d] & \underline{\mathcal{O}(1)^2 \oplus \mathcal{O}} \arrow[r] \arrow[d] & \mathcal{O}(2) \arrow[d] \\
\mathbf{U}(1;0) & = & \mathcal{O}^2 \arrow[r] & \underline{\mathcal{O}(1)} \arrow[r] & 0
\end{tikzcd}$$

and

$$\begin{tikzcd}
\mathbf{U}(3;-1) & = & 0 \arrow[r] \arrow[d] & \mathcal{O}(1) \oplus \mathcal{O}^2 \arrow[r] \arrow[d] & \underline{\mathcal{O}(2)^2 \oplus \mathcal{O}(1)} \arrow[r] \arrow[d] & \mathcal{O}(3) \arrow[d] \\
\mathbf{U}(2;0) & = & \mathcal{O} \arrow[r] & \mathcal{O}(1)^2 \oplus \mathcal{O} \arrow[r] & \underline{\mathcal{O}(2)} \arrow[r] & 0
\end{tikzcd}$$

So imagine that our Tate window is $\omega_E(2;0) \oplus \omega_E(1;0) \oplus \omega_E$ where the differential is

$$\begin{pmatrix} 0 & e_0 & 0 \\ 0 & 0 & e_0 \\ 0 & 0 & 0 \end{pmatrix}$$

Then the induced complex should be $\mathbf{U}(2;-1) \oplus \mathbf{U}(1;-1) \oplus \mathbf{U}(0;-1) \to \mathbf{U}(2;0) \oplus \mathbf{U}(1;0) \oplus \mathbf{U}(0;0)$, and thus be a map of complexes of the form:

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & \mathcal{O} \arrow[r] \arrow[d] & \underline{\mathcal{O}^3 \oplus \mathcal{O}(1)^2} \arrow[r] \arrow[d] & \mathcal{O} \oplus \mathcal{O}(1) \oplus \mathcal{O}(2) \arrow[d] \\
\mathcal{O} \arrow[r] & \mathcal{O}^3 \oplus \mathcal{O}(1)^2 \arrow[r] & \underline{\mathcal{O} \oplus \mathcal{O}(1) \oplus \mathcal{O}(2)} \arrow[r] & 0
\end{tikzcd}$$

Example 3.3. Consider the (homogeneous) map of free E-modules $e_0 : \omega_E(1;-1) \to \omega_E$ as in (1). Applying $\mathbf{U}$ yields a map of complexes, as in (1) but with a global twist by $-w = -4$:

(3)
$$\begin{tikzcd} \widetilde{\mathbf{L}}(E_{\leq -3})(-3)[1] : \arrow[d] & \underline{\mathcal{O}^2} \arrow[r] \arrow[d, "{[1,0]}"] & \mathcal{O} \\ \widetilde{\mathbf{L}}(E_{\leq -4})(-4) : & \underline{\mathcal{O}} & \end{tikzcd}$$

As with the $\mathbf{L}$-functor, we can apply this $\mathbf{U}$-functor to a module which also has a differential. Take a free E-module F (it is helpful to imagine that F is a Tate resolution, or the Beilinson window within a Tate resolution) together with a differential of degree $(0;1)$. We view this as a periodic complex:

(4)
$$\cdots \xrightarrow{\partial} F(0;-1) \xrightarrow{\partial} F \xrightarrow{\partial} F(0;1) \xrightarrow{\partial} \cdots$$

From functoriality, applying $\mathbf{U}$ yields a periodic complex of complexes

(5)
$$\cdots \xrightarrow{\mathbf{U}\partial} \mathbf{U}(F)[1] \xrightarrow{\mathbf{U}\partial} \mathbf{U}(F) \xrightarrow{\mathbf{U}\partial} \mathbf{U}(F)[-1] \xrightarrow{\partial} \cdots$$

If we take the zeroth homology, we thus get a complex, or an element of $\mathrm{D}(\mathbb{P}^n)$. This is the functor we want:

Proposition 3.4. *There exists a functor*

$$\mathbf{U}_{\mathrm{DM}} \colon \mathrm{DM}_{(0;1)}(E) \to \mathrm{D}(\mathbb{P}^n)$$

which takes as input a periodic complex of the form (4) *and as output, the zeroth homology of the complex* (5)*. Moreover, we claim* $\mathbf{U}_{\mathrm{DM}}(\mathbf{T}\mathcal{E})$ *is quasi-isomorphic to* $\mathcal{E}$*.*

From the other file "TateDM.pdf", we can define the Tate module $\mathbf{T}(\mathcal{E})$, which is an exact differential module where the underlying module is $\oplus_{i,d} H^i(\mathcal{E}(-d)) \otimes_k \omega_E(d;-i)$.

Example 3.5. For $\mathcal{O}$ on $\mathbb{P}(1,1,2)$, the Tate module has underlying module:

$$H^2_* \oplus H^0_* = \bigoplus_{d\leq -4} H^2(\mathcal{O}(d)) \otimes_k \omega_E(-d;2) \oplus \bigoplus_{d\geq 0} H^0(\mathcal{O}(d)) \otimes_k \omega_E(-d;0)$$

Concretely this is:

$$\cdots \omega_E(6;-2)^4 \oplus \omega_E(5;-2)^2 \oplus \omega_E(4;-2) \oplus \boxed{\omega_E(0;0)} \oplus \omega_E(-1;0)^2 \oplus \omega_E(-2;0) \cdots$$

We put a box around the terms in the Beilinson window. Since the restriction of the differential of $\mathbf{T}\mathcal{E}$ to this Beilinson is just zero, we will move onto the next example.

The Tate module for $\mathcal{O}(1)$ is the above twisted by $E(1;0)$, namely it is:

$$\cdots \omega_E(7;-2)^4 \oplus \omega_E(6;-2)^2 \oplus \omega_E(5;-2) \oplus \boxed{\omega_E(1;0) \oplus \omega_E(0;0)^2} \oplus \omega_E(-1;0) \cdots$$

Set $F = \omega_E(1;0) \oplus \omega_E(0;0)^2$. We can apply the functor $\mathbf{U}$ and we get the complex

$$\mathbf{U}(\omega_E(1;0) \oplus \omega_E(0;0)^2) = \quad \mathcal{O}^2 \to \underline{\mathcal{O}(1) \oplus \mathcal{O}^2}.$$

But F inherits a differential $F(0;-1) \to F$ from $\mathbf{T}(\mathcal{O}(1))$. We get:

(6)
$$\cdots \to \omega_E(1;-1) \oplus \omega_E(0;-1)^2 \xrightarrow{\partial} \omega_E(1;0) \oplus \omega_E(0;0)^2 \xrightarrow{\partial} \omega_E(1;1) \oplus \omega_E(0;1)^2 \xrightarrow{\partial} \cdots$$

where each differential looks like:

$$\begin{array}{c} \\ \omega_E(1;0) \\ \omega_E(0;0) \end{array} \begin{array}{c} \begin{array}{cc} \omega_E(1;-1) & \omega_E(0;-1)^2 \end{array} \\ \begin{pmatrix} 0 & 0 \\ [e_0, e_1]^t & 0 \end{pmatrix} \end{array}$$

Applying the $\mathbf{U}$-functor yields a double complex, whose rows correspond to the terms of (6):

$$\begin{tikzcd}
\mathbf{U}(\omega_E(1;-1) \oplus \omega_E(0;-1)^2) \arrow[d] & = & & & \underline{\mathcal{O}^2} \arrow[r] \arrow[d] & \mathcal{O}(1) \oplus \mathcal{O}^2 \arrow[d] \\
\mathbf{U}(\omega_E(1;0) \oplus \omega_E(0;0)^2) \arrow[d] & = & & \mathcal{O}^2 \arrow[r] \arrow[d] & \underline{\mathcal{O}(1) \oplus \mathcal{O}^2} \arrow[r] \arrow[d] & 0 \\
\mathbf{U}(\omega_E(1;1) \oplus \omega_E(0;1)^2) & = & \mathcal{O}^2 \arrow[r] & \mathcal{O}(1) \oplus \mathcal{O}^2 \arrow[r] & \underline{0} &
\end{tikzcd}$$

Taking homology of the vertical arrows will yield an element of $\mathrm{D}(\mathbb{P}^n)$ which is what we defined as $\mathbf{U}_{\mathrm{DM}}(\mathbf{T}\mathcal{O}(1))$. Since the vertical arrows identify the copies of $\mathcal{O}^2$, the homology of the vertical arrows (in the middle row) is the complex with a copy of $\mathcal{O}(1)$ in homological degree 0.

4. Proofs?

♣♣♣ Daniel: [Everything below here is a mess. Just working notes so I don't have to recompute some of this stuff.]. The connection of K_d with $\widetilde{\mathbf{L}}(E_{\leq -d})$ should let us prove that $\mathbf{U}$ is a well-defined functor. To check that $\mathbf{U}_{\mathrm{DM}}(\mathbf{T}\mathcal{E})$ is quasi-isomorphic to $\mathcal{E}$, it probably suffices to check this on generators for the derived category, such as these complexes $\widetilde{\mathbf{L}}(E_{\leq -d})(d-w)$. Would it be enough to check that the Tate module of $\widetilde{\mathbf{L}}(E_{\leq -d})(d-w)$ is $E_{\leq -d}$, maybe up to an appropriate twist?

Imagine that M is a module such that $\widetilde{M}(-i)$ has no higher cohomology for $i = 0, 1, 2, \dots, w-1$ (the Beilinson Window). Then within the Beilinson window, $\mathbf{T}(\widetilde{M})$ agrees with $\mathbf{R}(M)$ and so we can compute $\mathbf{UT}(\widetilde{M})$ entirely in terms of $\mathbf{R}(M)$. What we end up getting is that the total complex of the following

$$\begin{tikzcd}
M_{-3} \otimes K_3(3) \arrow[d] & = & M_{-3} \otimes (\mathcal{O}(3) \leftarrow \mathcal{O}(2)^2 \oplus \mathcal{O}(1) \leftarrow \mathcal{O}(1) \oplus \mathcal{O}^2) \arrow[d] \\
M_{-2} \otimes K_2(2) \arrow[d] & = & M_{-2} \otimes (\mathcal{O}(2) \leftarrow \mathcal{O}(1)^2 \oplus \mathcal{O} \leftarrow \mathcal{O}) \arrow[d] \\
M_{-1} \otimes K_1(1) \arrow[d] & = & M_{-1} \otimes (\mathcal{O}(1) \leftarrow \mathcal{O}^2) \arrow[d] \\
M_0 \otimes K_0 & = & M_0 \otimes \mathcal{O}
\end{tikzcd}$$

Of course there is also a differential of (shifted) complexes on this, but we will ignore that and focus on just the object itself, because that seems to be sufficient in the case where M

was a free module. Rewriting according to line bundles we get something like...

$$\begin{array}{c}
\mathcal{O}(3) \otimes (M_{-3}) \\
\big\uparrow \\
\mathcal{O}(2) \otimes \big(M_{-2} \leftarrow M_{-3}^2\big) \\
\big\uparrow \\
\mathcal{O}(1) \otimes \big(M_{-1} \leftarrow M_{-2}^2 \oplus M_{-3} \leftarrow M_{-3}\big) \\
\big\uparrow \\
\mathcal{O} \otimes \big(M_0 \leftarrow M_{-1}^2 \oplus M_{-2} \leftarrow M_{-2} \oplus M_{-3}^2\big)
\end{array}$$

If $M = S(j)$ then these complexes in parentheses are like the strands of the Koszul complex. In particular, if $M = S(j)$ with $0 \le j < 4$ then we get

$$\begin{array}{c}
\mathcal{O}(3) \otimes (S_{j-3}) = \mathcal{O}(3) \otimes (S/\mathfrak{m})_{j-3} \\
\big\uparrow \\
\mathcal{O}(2) \otimes \big(S_{j-2} \leftarrow S_{j-3}^2\big) = \mathcal{O}(2) \otimes (S/\mathfrak{m})_{j-2} \\
\big\uparrow \\
\mathcal{O}(1) \otimes \big(S_{j-1} \leftarrow S_{j-2}^2 \oplus S_{j-3} \leftarrow S_{j-3}\big) = \mathcal{O}(1) \otimes (S/\mathfrak{m})_{j-1} \\
\big\uparrow \\
\mathcal{O} \otimes \big(S_j \leftarrow S_{j-1}^2 \oplus S_{j-2} \leftarrow S_{j-2} \oplus S_{j-3}^2\big) = \mathcal{O} \otimes (S/\mathfrak{m})_j
\end{array}$$

But since $(S/\mathfrak{m})_0 = k$ and $(S/\mathfrak{m})_j = 0$ for $j \neq 0$, we see that $\mathcal{O}(j) \mapsto \mathcal{O}(j)$ under this map.

For an arbitrary M, you could imagine that $M_{-4} \neq 0$ and so on. We could also choose to include $M_{-4} \otimes K_4(4)$ and so in the above; since $K_4(4)$ is irrelevant, this wouldn't change anything in the derived category. But it would provide a more complete picture, filling in the "missing terms" in the above. We would get a complex whose strands are

$$\oplus_{i \ge 0}\Big(\mathcal{O}(i) \otimes (M_{-i} \leftarrow M_{-i-1}^2 \oplus M_{-i-2} \leftarrow M_{-i-2} \oplus M_{-i-3}^2 \leftarrow M_{-i-4})\Big)$$

So if $M = S(j)$ for $j \ge 0(?)$ we would get $\mathbf{UT}\mathcal{O}(j) = \oplus_{i \ge 0}\mathcal{O}(i) \otimes (S/\mathfrak{m})_{j-i} = \mathcal{O}(j)$. This would seem to have potential to work in the more general setting.

References

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