TITLE

Contents

1. Introduction

Notational conventions:

- We index homologically throughout.

2. Differential modules

Let A be an abelian group, and let R be an A-graded ring (in particular, A could be 0). All modules over R are right modules. ♣♣♣ Michael: [We work with right modules because our main example will be $R = E$, and in Macaulay2, entries of matrices over E act on the right. This is the same reason I'm working with homological indexing as opposed to cohomological: I'm trying to match M2.]

Definition 2.1. Let $a \in A$. A *degree a differential R-module* is a pair (D, ∂_D), where D is an A-graded module, and

$$\partial : D \to D(a)$$

is an R-linear map such that $\partial^2 = 0$. When the fixed element a of A is clear, we will just call (D, ∂_D) a *differential module*. A morphism $(D, \partial) \to (D', \partial')$ of degree a differential modules is a map $f : D \to D'$ satisfying $f \circ \partial = \partial' \circ f$.

For the rest of section, fix an element $a \in A$. Let $\mathrm{DM}(R, a)$ denote the category of degree a differential R-modules. The *homology* of an object $(D, \partial) \in \mathrm{DM}(R, a)$ is the subquotient

$$\ker(\partial : D \to D(a) / \operatorname{im}(\partial : D(-a) \to D),$$

denoted $H(D, \partial)$. A morphism in $\mathrm{DM}(R, a)$ is a *quasi-isomorphism* if it induces an isomorphism on homology. A *homotopy* of morphisms $f, f' : (D, \partial) \to (D', \partial')$ in $\mathrm{DM}(R, a)$ is a

Date: November 9, 2020.

morphism $h : D \to D'(-a)$ of A-graded R-modules such that $f - f' = h\partial + \partial' h$. The *mapping cone* of a morphism $f : (D, \partial') \to (D', \partial')$ in $\mathrm{DM}(R, a)$ is the object $(D \oplus D'(-a), \begin{pmatrix} -\partial & 0 \\ f & \partial' \end{pmatrix})$.

The main goal of this section is to develop a theory of minimal free resolutions for differential modules. The following dictionary provides an easy shorthand for understanding our main results. We note that part of the contribution of this note is providing definitions for many of the objects in the righthand column.

Complexes	**Differential modules**
Bounded above complex	Flag differential module
Perfect complex	Finite flag differential module
Projective resolution	Projective flag resolution
Minimal free resolution	Minimal free resolution
	(i.e. free, minimal and flag retract)

We will prove, for instance: a map of differential modules modules induces a map on projective flag resolutions, which is unique up to homotopy; over a local or graded ring, every finitely generated differential module admits a minimal free resolution, which is unique up to isomorphism; and so on.

2.1. **Expansion.** Let $\mathrm{Com}_{\mathrm{per}}(R, a)$ denote the category of complexes of A-graded R-modules satisfying

$$D[j] = D(-a)$$

for all $j \in \mathbb{Z}$, with morphisms given by maps of complexes that are identical in each homological degree. There is an equivalence of categories

$$\mathrm{Ex} : \mathrm{DM}(R, a) \xrightarrow{\simeq} \mathrm{Com}_{\mathrm{per}}(R, a)$$

given by sending the differential module (D, ∂) to the "expanded" complex

$$\cdots \xrightarrow{\partial} D(-a) \xrightarrow{-\partial} D \xrightarrow{\partial} D(a) \xrightarrow{-\partial} \cdots .$$

The above notions of homology, quasi-isomorphism, homotopy, and mapping cone for differential modules all correspond to the usual notions via the equivalence Ex.

Remark 2.2. Our notion of expansion of a differential module is slightly different from Avramov-Buchweitz-Iyengar's in [ABI07, Section 1.4]: for them, the differentials in the expanded complex are all identical, while ours differ by a sign.

2.2. **Projective flag resolutions.** We are interested in differential modules equipped with a filtration, in the following sense (cf. [ABI07, 2.1]).

Definition 2.3 (cf. [ABI07] Section 2.1). A *flag* is an object $(D, \partial) \in \mathrm{DM}(R, a)$ equipped with a filtration $\mathcal{F}_\bullet D$ such that

- $\mathcal{F}_i D \subseteq \mathcal{F}_{i+1} D$
- $\partial(\mathcal{F}_i D) \subseteq \mathcal{F}_{i-1} D$,
- $\bigcup_i \mathcal{F}_i D = D$, and
- $\mathcal{F}_{<0} D = 0$.

We say a flag is *locally finitely generated* if each component of the associated graded module is finitely generated. A *split flag* is a differential module (D, ∂) equipped with a decomposition

$D = \bigoplus_{j \in \mathbb{Z}} D_j$ such that the filtration $\mathcal{F}_i D = \bigoplus_{j<i} D_j$ makes (D, ∂) a flag. A *projective (resp. free) split flag* is a split flag such that each D_j is projective (resp. free).

Remark 2.4. A split flag (D, ∂) such that $\partial(D_i) \subseteq D_{i+1}$ is the same thing as a chain complex of R-modules that is concentrated in nonnegative degrees.

Definition 2.5. Let $(D, \partial_D) \in \mathrm{DM}(R, a)$, and let $(P, \partial_P) \in \mathrm{DM}(R, a)$ be a projective (resp. free) split flag. A quasi-isomorphism $\epsilon : (P, \partial_P) \to (D, \partial_D)$ is called a *projective flag resolution* (resp. *free flag resolution*). A projective (resp. free) flag resolution is called *locally finitely generated* if the flag P is such.

Proposition 2.6. *Every $(D, \partial) \in \mathrm{DM}(R, a)$ admits a free flag resolution.*

Proof. Clear using "killing cycles" or Cartan-Eilenberg. ♣♣♣ Michael: [Fill in.] ☐

As in classical homological algebra, morphisms of differential modules may be lifted to projective flag resolutions in a unique way, up to homotopy. More generally, we have the following

Proposition 2.7. *Let $(D, \partial_D), (D', \partial_{D'}) \in \mathrm{DM}(R, a)$, and suppose we have morphisms $\epsilon : (P, \partial_P) \to (D, \partial_D)$, $\epsilon' : (P', \partial_{P'}) \to (D', \partial_{D'})$, where (P, ∂_P) is a projective split flag, and ϵ' is a quasi-isomorphism. Given a morphism $f : (D, \partial_D) \to (D', \partial_{D'})$ of differential modules, there exists a morphism*

$$\widetilde{f} : \mathrm{cone}(\epsilon) \to \mathrm{cone}(\epsilon')$$

of differential modules of the form

$$\begin{pmatrix} \alpha & 0 \\ \rho & f \end{pmatrix}. \tag{1}$$

In particular, the entry $\alpha : P \to P'$ of (1) is a morphism of differential modules. Moreover, given two such lifts

$$\widetilde{f}_1 = \begin{pmatrix} \alpha_1 & 0 \\ \rho_1 & f \end{pmatrix}, \widetilde{f}_2 = \begin{pmatrix} \alpha_2 & 0 \\ \rho_2 & f \end{pmatrix} : \mathrm{cone}(\epsilon) \to \mathrm{cone}(\epsilon'),$$

there is a homotopy

$$h = \begin{pmatrix} h_1 & 0 \\ h_2 & 0 \end{pmatrix} : P \oplus D \to P'(-a) \oplus D'(-a).$$

In particular, h_1 is a homotopy between α_2 and α_1.

Remark 2.8. It need not be the case that $\epsilon' \alpha = f \epsilon$. For instance, ♣♣♣ Michael: [Fill in.]

Proof. Set $\widetilde{P} := \mathrm{cone}(\epsilon)$ and $\widetilde{P}' := \mathrm{cone}(\epsilon')$. We begin by defining $g_0 : P_0 \to \widetilde{P}'$ such that the map

$$\widetilde{f}_0 : P_0 \oplus D \to \widetilde{P}'$$

given by $(p, d) \mapsto g_0(p) + (0, f(d))$ is a morphism of differential modules, where $P_0 \oplus D$ is equipped with the differential $\begin{pmatrix} 0 & 0 \\ \epsilon & \partial_D \end{pmatrix}$, i.e. the restriction of $\partial_{\widetilde{P}}$ to $P_0 \oplus D$. We have a diagram

$$\begin{tikzcd} & \widetilde{P}' \arrow[d, "\partial_{\widetilde{P}'}"] \\ P_0 \arrow[r, "\beta"] & \operatorname{im}(\partial_{\widetilde{P}'}) = \ker(\partial_{\widetilde{P}'}), \end{tikzcd}$$

where $\beta(p) = (0, (f\epsilon)(p))$. Note that β does indeed land in $\ker(\partial_{\widetilde{P}'})$: we have

$$(\partial_{\widetilde{P}'}\beta)(p) = (0, (\partial_{D'} f\epsilon)(p)) = (0, (f\partial_D\epsilon)(p)) = 0;$$

the last equality holds since $\partial_P|_{P_0} = 0$, and $\epsilon\partial_P = \partial_D\epsilon$. Since P_0 is projective, we get an induced map

$$g_0 : P_0 \to \widetilde{P}'$$

making the diagram commute. One easily checks that g_0 has the desired property: if $(p, d) \in P_0 \oplus D$,

$$\begin{aligned}(\widetilde{f}_0\partial_{\widetilde{P}})(p, d) &= (0, (f\epsilon)(p)) + (0, (f\partial_D)(d)) \\ &= \beta(p) + (0, (\partial_{D'} f)(d)) \\ &= (\partial_{\widetilde{P}'} g_0)(p) + \partial_{\widetilde{P}'}(0, f(d)) \\ &= (\partial_{\widetilde{P}'}\widetilde{f}_0)(p, d).\end{aligned}$$

Now, suppose $n > 0$, and assume we have

$$g_i : P_{\leq i} \to \widetilde{P}'$$

for all $i < n$, such that

- the map $\widetilde{f}_i : P_{\leq i} \oplus D \to \widetilde{P}'$ given by $(p, d) \mapsto g_i(p) + (0, f(d))$ is a morphism of differential modules (where $P_{\leq i} \oplus D$ is equipped with the differential given by the restriction of $\partial_{\widetilde{P}}$), and
- $g_i|_{P_{\leq j}} = g_j$ for all $j < i$.

We have a diagram

$$\begin{tikzcd} & \widetilde{P}' \arrow[d, "\partial_{\widetilde{P}'}"] \\ P_n \arrow[r, "\gamma"] & \operatorname{im}(\partial_{\widetilde{P}'}) = \ker(\partial_{\widetilde{P}'}), \end{tikzcd}$$

where $\gamma(p) = (\widetilde{f}_{n-1}\partial_{\widetilde{P}})(p, 0)$; the map γ lands in $\ker(\partial_{\widetilde{P}'})$, since

$$(\partial_{\widetilde{P}'}\widetilde{f}_{n-1}\partial_{\widetilde{P}})(p, 0) = (\widetilde{f}_{n-1}\partial_{\widetilde{P}}\partial_{\widetilde{P}})(p, 0) = 0.$$

Since P_n is projective, we obtain a map $\widetilde{\gamma} : P_n \to \widetilde{P}'$ making the diagram commute. We define $g_n : P_{\leq n} \to \widetilde{P}'$ to be the map

$$\begin{pmatrix} g_{n-1} & \widetilde{\gamma} \end{pmatrix} : P_{\leq n-1} \oplus P_n \to \widetilde{P}'.$$

We now verify that the map

$$\widetilde{f}_n : P_{\leq n} \oplus D \to \widetilde{P}',$$

given by $(p,d) \mapsto g_n(p)+(0,f(d))$, is a morphism of differential modules. Let $(p,d) \in P_{\le n} \oplus D$. We have:

$$\begin{aligned}
(\widetilde{f}_n \partial_{\widetilde{P}})(p,d) &= g_n(-\partial_P(p)) + (0, (f\epsilon)(p) + (f\partial_D)(d)) \\
&= \widetilde{f}_n(-\partial_P(p), \epsilon(p)) + (0, (\partial_{D'} f)(d)) \\
&= (\widetilde{f}_n \partial_{\widetilde{P}})(p,0) + (\partial_{\widetilde{P}'} \widetilde{f}_n)(0,d),
\end{aligned}$$

so it suffices to show

$$(\widetilde{f}_n \partial_{\widetilde{P}})(p,0) = (\partial_{\widetilde{P}'} \widetilde{f}_n)(p,0).$$

To see this, write $p = p' + p''$, where $p' \in P_{\le n-1}$ and $p'' \in P_n$, and notice that

$$\begin{aligned}
(\widetilde{f}_n \partial_{\widetilde{P}})(p,0) &= (\widetilde{f}_{n-1} \partial_{\widetilde{P}})(p,0) \\
&= (\widetilde{f}_{n-1} \partial_{\widetilde{P}})(p',0) + (\widetilde{f}_{n-1} \partial_{\widetilde{P}})(p'',0) \\
&= (\partial_{\widetilde{P}'} \widetilde{f}_{n-1})(p',0) + \gamma(p'') \\
&= (\partial_{\widetilde{P}'} \widetilde{f}_n)(p',0) + (\partial_{\widetilde{P}'} g_n)(p'') \\
&= (\partial_{\widetilde{P}'} \widetilde{f}_n)(p',0) + (\partial_{\widetilde{P}'} \widetilde{f}_n)(p'',0) \\
&= (\partial_{\widetilde{P}'} \widetilde{f}_n)(p,0).
\end{aligned}$$

Let g be the colimit of the g_i, and take $\widetilde{f} : \widetilde{P} \to \widetilde{P}'$ to be given by $(p,d) \mapsto g(p) + (0, f(d))$. We now show our lift $\widetilde{f}$ is unique up to homotopy. Without loss, assume $f = 0$; we will show $\widetilde{f}$ is null homotopic. We again proceed by induction. We have a diagram

$$\begin{tikzcd}
 & \widetilde{P}' \arrow[d, "\partial_{\widetilde{P}'}"] \\
P_0 \arrow[r, "g_0"] & \ker(\partial_{\widetilde{P}'}),
\end{tikzcd}$$

since $(\partial_{\widetilde{P}'} g_0)(p) = \beta(p) = 0$ for all $p \in P_0$. Since P_0 is projective, we obtain a map $s_0 : P_0 \to \widetilde{P}'$ making the diagram commute. Let $n > 0$, and suppose we have maps $s_i : P_{\le i} \to \widetilde{P}'$ for $i < n$ such that

- $g_i = \partial_{\widetilde{P}'} s_i - s_{i-1} \partial_P$ (set $s_{<0} := 0$), and
- $s_i|_{P_{\le j}} = s_j$ for all $j < i$.

In particular, let's record the relation

(2) $$g_{n-1} = \partial_{\widetilde{P}'} s_{n-1} - s_{n-2} \partial_P.$$

We have a diagram

$$\begin{tikzcd}
 & \widetilde{P}' \arrow[d, "\partial_{\widetilde{P}'}"] \\
P_{\le n} \arrow[r, "g_n + s_{n-1}\partial_P"] & \ker(\partial_{\widetilde{P}'}),
\end{tikzcd}$$

since, by (2), we have

$$\begin{aligned}\partial_{\widetilde{P}'}(g_n + s_{n-1}\partial_P) &= \partial_{\widetilde{P}'}g_n + (g_{n-1} + s_{n-2}\partial_P)\partial_P \\ &= \partial_{\widetilde{P}'}g_n + g_{n-1}\partial_P,\end{aligned}$$

and

$$\begin{aligned}(\partial_{\widetilde{P}'}g_n)(p) &= (\partial_{\widetilde{P}'}\widetilde{f}_n)(p, 0) \\ &= (\widetilde{f}_n\partial_{\widetilde{P}})(p, 0) \\ &= \widetilde{f}_n(-\partial_P(p), \epsilon(p)) \\ &= -(g_{n-1}\partial_P)(p).\end{aligned}$$

Define $s_n : P_{\leq n} \to \widetilde{P}'$ making the diagram commute. Let s denote the colimit of the s_i. We have

$$g = \partial_{\widetilde{P}'}s - s\partial_P.$$

Now take $h : \widetilde{P} \to \widetilde{P}'$ to be the map given by $(p, d) \mapsto s(p)$, and observe that

$$\begin{aligned}\widetilde{f}(p, d) &= g(p) \\ &= (\partial_{\widetilde{P}'}s)(p) - (s\partial_P)(p) \\ &= (\partial_{\widetilde{P}'}h)(p, d) + (h\partial_{\widetilde{P}})(p, d).\end{aligned}$$

$\square$

2.3. Ext **and** Tor**.** Given $(D, \partial_D) \in \mathrm{DM}(R, a)$ and an R-module N, we define $D \otimes_R N$ to be the differential module $(D \otimes_R N, \partial_D \otimes 1)$. Define $\mathrm{Hom}_R(D, N)$ and $\mathrm{Hom}_R(N, D)$ similarly.

Let M and N be R-modules, and let F be a free flag resolution of M. We define

$$\mathrm{Tor}^R_{\mathrm{DM}}(M, N) = H(F \otimes_R N)$$

and

$$\mathrm{Ext}_R^{\mathrm{DM}}(M, N) = H(\mathrm{Hom}_R(F, N)).$$

It follows easily from Proposition 2.7 that these definitions do not depend on the choice of free flag resolution.

Remark 2.9. We can extend these definitions to allow one of the modules to have a nontrivial differential, but not both. The problem is there is no way to take a tensor product or internal Hom of differential modules. ♣♣♣ Michael: [Rewrite this more cleanly.]

2.4. **Minimal free resolutions.** From now on, assume either that

(1) the grading group A is trivial and R is local, or
(2) the set $\bigoplus_{a \neq 0} R_a$ is a maximal ideal of R.

♣♣♣ Michael: [Are these the right assumptions? Double check. We just need graded Nakayama.]
Denote the unique (homogeneous, in the second case) maximal ideal by $\mathfrak{m}$. We say a morphism $f : M \to N$ of A-graded R-modules is *minimal* if $f(M) \subseteq \mathfrak{m}N$.

We now wish to define a notion of minimal free resolutions for differential R-modules. It is tempting to define such a resolution to be a minimal free flag resolution. But, we will see in Example 2.16 that such resolutions do not exist in general. Instead, we proceed as follows.

Definition 2.10. A *trivial* differential R-module is a direct sum of objects in $\mathrm{DM}(R, a)$ of the form

$$R(b) \oplus R(b-a) \xrightarrow{\begin{pmatrix} 0 & 0 \\ u & 0 \end{pmatrix}} R(b+a) \oplus R(b)$$

for some $b \in A$, where u is a unit.

Proposition 2.11. *Let (F, ∂_F) be either a finitely generated free differential module or a locally finitely generated free split flag. There is an automorphism A of F such that*

$$(F, A\partial_F A^{-1}) = (T, \partial_T) \oplus (M, \partial_M),$$

where (T, ∂_T) is trivial and (M, ∂_M) is minimal.

Proof. Suppose first that F is finitely generated. Choose a basis of F, and view ∂_F as a matrix with respect to this basis. If ∂_F has no unit entries, then it is minimal and we are done. Otherwise, the condition $\partial_F^2 = 0$ forces ∂_F to have a unit entry u that does not lie on the diagonal. Without loss of generality, we can assume that this entry is in the first column and second row. Let B_1 be the matrix corresponding to the row operations that zero out all other entries in the first column of ∂_F. This is an identity matrix, except in the second column. It follows that $B_1 \partial_F B_1^{-1}$ has the form:

$$B_1 \partial_F B_1^{-1} = \begin{pmatrix} 0 & a'_{1,2} & a'_{1,3} & \cdots \\ u & a_{2,2} & a_{2,3} & \cdots \\ 0 & a'_{3,2} & a'_{3,3} & \cdots \\ 0 & a'_{4,2} & a'_{4,3} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Let B_2 the matrix corresponding the column operations which zero out all the other entries in the second row of ∂_F. This is an identity matrix, except in the top row. It follows that $B_2^{-1} B_1 \partial_F B_1^{-1} B_2$ has the form

$$B_2^{-1} B_1 \partial_F B_1^{-1} B_2 = \begin{pmatrix} 0 & a''_{1,2} & a''_{1,3} & \cdots \\ u & 0 & 0 & \cdots \\ 0 & a''_{3,2} & a''_{3,3} & \cdots \\ 0 & a''_{4,2} & a''_{4,3} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

The first column of ∂_F^2 equals the second column of ∂_F multiplied by u. Since $\partial_F^2 = 0$, this means that the entire second column is zero. Similarly, the second row of ∂_F^2 is the first row of ∂_F multiplied by u, and thus the entire first row of ∂_F must be zero. We conclude that

$$(F, B_2^{-1} B_1 \partial_F B_1^{-1} B_2) = (T, \partial_T) \oplus (D, \partial_D),$$

where T is a rank 2 free R-module, and $\partial_T = \begin{pmatrix} 0 & 0 \\ u & 0 \end{pmatrix}$. Now apply the same argument to (D, ∂_D). Since F is finitely generated, this process eventually terminates.

Suppose now that F is a locally finitely generated free split flag. We can apply the above argument to each summand F_i, yielding automorphisms A_i such that $(F_i, A_i \partial_F A_i^{-1}) = (M_i, \partial_{M_i}) \oplus (T_i, \partial_{T_i})$ for all i. Moreover, the above argument shows that we can choose the trivial summands to be compatible for increasing i: that is, we may assume that there are inclusions $T_i \to T_{i+1}$ for all i, such that

$$\begin{tikzcd} T_i \arrow[r] \arrow[d] & F_i \arrow[d] \\ T_{i+1} \arrow[r] & F_{i+1} \end{tikzcd}$$

commutes for all i. These diagrams induce maps $M_i \to M_{i+1}$, which may not be inclusions. ♣♣♣ Daniel: [This will happen if, say, the "killing cycles" algorithm produces a non-minimal resolution. We should have an example of that to refer to here. In this case, there will be a unit from a summand of F_{i+1} to a summand of M_i.] We let (M, ∂_M) be the colimit of the (M_i, ∂_{M_i}), and similarly for (T, ∂_T). Since colimits commute with coproducts, $F = T \oplus M$. It is clear from the construction that T is trivial. Since each ∂_{M_i} factors through $\mathfrak{m}M_i$, $\operatorname{colim} \partial_{M_i}$ factors through $\mathfrak{m} \operatorname{colim} M_i = \mathfrak{m}M$. □

Definition 2.12. Let $(D, \partial_D) \in \mathrm{DM}(R, a)$. A *stably free flag resolution* of (D, ∂_D) is a free differential module (G, ∂_G) such that there is a free flag resolution

$$(F, \partial_F) \xrightarrow{\epsilon} (D, \partial_D)$$

and a trivial differential module (T, ∂_T) satisfying $(F, \partial_F) = (G, \partial_G) \oplus (T, \partial_T)$. We say (G, ∂_F) is a *minimal stably free flag resolution* if ∂_G is minimal. We will shorten "minimal stably free flag resolution" to "minimal free resolution" from now on. ♣♣♣ Daniel: [Should we call this a "flag retract" or something?]♣♣♣ Michael: [I think not, because retract just means summand of a flag, but we want a summand of a flag *whose complement is trivial*. So I think "stable" really is the right term here.]

Proposition 2.13. *Every $(D, \partial_D) \in \mathrm{DM}(R, a)$ has a minimal free resolution.*

Proof. Combine Propositions 2.6 and 2.11. □

To prove uniqueness of minimal free resolutions, we will need the following

Lemma 2.14. *Let $(M, \partial_M), (M', \partial_{M'})$ be minimal differential modules. If a morphism*

$$f : (M, \partial_M) \to (M', \partial_{M'})$$

factors through a trivial differential module, f is minimal.

Proof. Suppose we have a factorization

$$(M, \partial_M) \xrightarrow{g} (T, \partial_T) \xrightarrow{h} (M', \partial_{M'})$$

of f, where (T, ∂_T) is trivial. Let $m \in M$, and choose a basis $\{e_i\}_{i \in I}$ of T. We can write $g(m)$ as

$$r_1 e_{i_1} + \cdots + r_n e_{i_n}.$$

Suppose $r_j \notin \mathfrak{m}$. Since $r_1 \partial_T(e_{i_1}) + \cdots + r_n \partial_T(e_{i_n}) = \partial_T(g(m)) = g(\partial_M(m)) \in \mathfrak{m}T$, and ∂_T is a matrix with at most a single 1 in each row and 0's elsewhere, we have $\partial_T(e_{i_j}) = 0$. Using that T is exact, choose an element $t \in T$ such that $e_{i_j} = \partial_T(t)$. Then

$$h(e_{i_j}) = h(\partial_T(t)) = \partial_{M'}(h(t)) \in \mathfrak{m}M'.$$

We conclude that $f(m) \in \mathfrak{m}M'$. □

Theorem 2.15. *Let $(D, \partial_D) \in \mathrm{DM}(R, a)$, and let*

$$\begin{tikzcd}
(M, \partial_M) \oplus (T, \partial_T) \arrow[dr, "\epsilon"] & & (M', \partial_{M'}) \oplus (T', \partial_{T'}) \arrow[dl, "\epsilon'"'] \\
& (D, \partial_D) &
\end{tikzcd}$$

be locally finite free flag resolutions, where (M, ∂_M), $(M', \partial_{M'})$ are minimal and (T, ∂_T), $(T', \partial_{T'})$ are trivial. There is an isomorphism $(M, \partial_M) \cong (M', \partial_{M'})$. In particular, minimal free resolutions of differential modules are unique up to isomorphism.

Proof. Applying Proposition 2.7 to the identity map on D, we may choose morphisms

$$\alpha = \begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{pmatrix} : (M, \partial_M) \oplus (T, \partial_T) \to (M', \partial_{M'}) \oplus (T', \partial_{T'})$$

$$\alpha' = \begin{pmatrix} \alpha_1' & \alpha_2' \\ \alpha_3' & \alpha_4' \end{pmatrix} : (M', \partial_{M'}) \oplus (T', \partial_{T'}) \to (M, \partial_M) \oplus (T, \partial_T)$$

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$$h = \begin{pmatrix} s_1 & s_2 \\ s_3 & s_4 \end{pmatrix} : M \oplus T \to M(0, -1) \oplus T(0, -1)$$

$$h' = \begin{pmatrix} s_1' & s_2' \\ s_3' & s_4' \end{pmatrix} : M' \oplus T' \to M'(0, -1) \oplus T'(0, -1)$$

such that

$$\alpha'\alpha - \mathrm{id}_F = h\partial_{M \oplus T} + \partial_{M \oplus T} h$$

$$\alpha\alpha' - \mathrm{id}_{F'} = h'\partial_{M' \oplus T'} + \partial_{M' \oplus T'} h'$$

Reading off the top-left entry of the matrices on each side of these equations, we get

$$\alpha_1'\alpha_1 + \alpha_2'\alpha_3 - \mathrm{id}_F = h_1\partial_M + \partial_M h_1$$

$$\alpha_1\alpha_1' + \alpha_2\alpha_3' - \mathrm{id}_{F'} = h_1'\partial_{M'} + \partial_{M'} h_1'.$$

By Lemma 2.14, $\alpha_2'\alpha_3$ and $\alpha_2\alpha_3'$ are minimal. We conclude that $\alpha_1'\alpha_1 = \mathrm{id}_M$ and $\alpha_1\alpha_1' = \mathrm{id}_{M'}$ modulo $\mathfrak{m}$.

Now, assume R is nontrivially graded. By the graded version of Nakayama's Lemma and the local finiteness of M and M', it follows that $\alpha_1'\alpha_1$ and $\alpha_1\alpha_1'$ are automorphisms. In particular, α_1 is injective and surjective. In the case where R is trivially graded... ♣♣♣ Michael: [Not sure how this is going to work in non-graded case. May just need to assume M and M' are finitely generated in this case.] □

Example 2.16. We now give an example of a differential module with no minimal free flag resolution. Take $A = \mathbb{Z}$, $a = 2$, and $R = k[x, y]$, where $|x| = 1 = |y|$. Let $D = R^{\oplus 2}$, and take

$$\partial_D : R^{\oplus 2} \to R(2)^{\oplus 2}$$

to be

$$\begin{pmatrix} xy & -x^2 \\ y^2 & -xy \end{pmatrix}.$$

Since (D, ∂_D) does not admit a flag structure ([ABI07]), it suffices, by Theorem 2.15, to show that (D, ∂_D) is the minimal free resolution of itself.

We use the Cartan-Eilenberg construction to produce a free flag resolution of (D, ∂_D). The cycles are the rank 1 free submodule of R^2 generated by $\begin{pmatrix} x \\ y \end{pmatrix}$, so Z is resolved by $G := [R(-1)]$. The boundaries B are the image of the above matrix, and so $B(-2)$ is resolved by $H := [R(-2)^2 \leftarrow R(-3)]$. Using this, we can produce a Cartan-Eilenberg resolution of (D, ∂_D) given by

$$F = G_0 \oplus H_0 \oplus H_1(2) = R(-1) \oplus R(-2) \oplus R(-2) \oplus R(-1),$$

$$\partial_F = \begin{pmatrix} 0 & -y & -x & 1 \\ 0 & 0 & 0 & -x \\ 0 & 0 & 0 & y \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

and

$$\epsilon = \begin{pmatrix} x & -1 & 0 & 0 \\ y & 0 & 1 & 0 \end{pmatrix} : F \to D.$$

Now, take $A = \begin{pmatrix} -1 & 0 & 0 & 0 \\ x & -1 & 0 & 0 \\ y & 0 & 1 & 0 \\ 0 & -y & -x & -1 \end{pmatrix}$, so that

$$A\partial_F A^{-1} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & xy & -x^2 & 0 \\ 0 & y^2 & -xy & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Observe that $(F, A\partial_F A^{-1}) \cong (F, \partial_F)$ is a direct sum of (D, ∂_D) and a trivial object.

Example 2.17. In the category of complexes over a graded or local ring, any complex with bounded homology will admit a minimal free resolution. If you start with a complex which is free, minimal, and bounded, then it will equal its own minimal free resolution. However, if the original complex was free and minimal, but not bounded above, then it will generally not equal its minimal free resolution. For instance, if $R = k[x]/(x^2)$ and one considers the complex F:

$$0 \to R \xrightarrow{x} R(1) \xrightarrow{x} R(2) \xrightarrow{x} \cdots,$$

then the minimal free resolution will be the complex F':

$$\cdots \xrightarrow{x} R(-3) \xrightarrow{x} R(-2) \xrightarrow{x} R(-1) \to 0.$$

And of course F' is not isomoprhic to F.

A similar phenomenon occurs for differential modules: namely one can find a free, minimal differential module which is not isomorphic to its minimal free resolution. Essentially the same example works. Consider the differential module (D, ∂_D) obtained from the complex F by forgetting the homological grading. Let (D', ∂'_D) be the differential module obtained from F' in a similar way. The homological grading on F' can realize F' as a free, flag and the quasi-isomorphism of complexes $F' \xrightarrow{x} F$ shows that (D', ∂'_D) is a flag, free resolution of D. Of course, D' is not isomorphic to D because the underlying graded modules are distinct. By the uniqueness of minimal free resolutions, it follows that D is not a minimal free resolution of itself.

Remark 2.18. Results similar to these are proven in Avramov-Foxby-Halperin's unpublished notes, but with boundedness assumptions that are not satisfied in our setting. ♣♣♣ Michael: [Fill this in.]

3. The toric BGG correspondence

From now on:

- k is a field.
- A is an abelian group, and $w_0, \dots, w_n \in A$.
- V is a vector space of dimension $n+1$ equipped with a basis $\{e_0, \dots, e_n\}$.
- W denotes the dual of V, equipped with the dual basis $\{x_0, \dots, x_n\}$.
- $S = \mathrm{Sym}_k(W)$, equipped with the A-grading given by $|x_i| = w_i$
- $E = \Lambda(V)$, equipped with the $A \times \mathbb{Z}$-grading given by $|e_i| = (-|x_i|, 1)$.
- Unless otherwise specified, all E-modules are right modules. In particular, entries of matrices over E act on the right. This is also Macaulay2's convention. There is an equivalence of categories

$$\{\text{Right } E - \text{mod}\} \to \{\text{Left } E - \text{mod}\}$$

  given by sending a right E-module M to the left E-module M with $em = (-1)^{|e||m|}me$, where $| - |$ denotes the degree with respect to the $\mathbb{Z}$-grading (i.e. the standard grading). We will sometimes use this equivalence to translate between left and right E-modules.

3.1. **The BGG adjunction.** Define a functor

$$\widetilde{\mathbf{L}} : \mathrm{Com}(E) \to \mathrm{Com}(S)$$

as follows. For an E-module N concentrated in degree 0, $\widetilde{\mathbf{L}}(N)$ is the complex with

$$\widetilde{\mathbf{L}}(N)_q = \bigoplus_{d \in A} N_{(-d,-q)} \otimes_k S(d)$$

and differential

$$m \otimes s \mapsto \sum_{i=0}^{n} e_i m \otimes x_i s. \tag{3}$$

For a general complex $(C, \partial) \in \mathrm{Com}(E)$, we form the bicomplex

$$\begin{tikzcd}
 & {} \arrow[d] & {} \arrow[d] & \\
\cdots & \widetilde{\mathbf{L}}(C_p)_q \arrow[l, "\partial"'] \arrow[d] & \widetilde{\mathbf{L}}(C_{p+1})_q \arrow[l, "\partial"'] \arrow[d] & \cdots \arrow[l, "\partial"'] \\
\cdots & \widetilde{\mathbf{L}}(C_p)_{q-1} \arrow[l, "\partial"'] \arrow[d] & \widetilde{\mathbf{L}}(C_{p+1})_{q-1} \arrow[l, "\partial"'] \arrow[d] & \cdots \arrow[l, "\partial"'] \\
 & {} & {} &
\end{tikzcd} \tag{4}$$

and apply $\mathrm{Tot}^{\oplus}(-)$, where the vertical differential $\widetilde{\mathbf{L}}(C_p)_q \to \widetilde{\mathbf{L}}(C_p)_{q-1}$ is the dual Koszul map (3) multiplied by $(-1)^p$.

The functor

$$\mathbf{R} : \mathrm{Com}(S) \to \mathrm{Com}(E)$$

is defined as follows. For an S-module M concentrated in degree 0, $\mathbf{R}(M)$ is the complex with

$$\mathbf{R}(M)_p = \bigoplus_{d \in A} M_d \otimes \omega(-d, -p),$$

where ω denotes the E-module $\mathrm{Hom}_k(E, k)$. Note that $\omega \cong E(-w, n+1)$, where w is the sum of the degrees of the variables of $S = k[x_0, \dots, x_n]$. The p^{th} differential is given by

$$m \otimes e \mapsto (-1)^p \sum_{i=0}^{n} x_i m \otimes e e_i,$$

♣♣♣ Michael: [Notice the sign.] Given $(C, \partial) \in \mathrm{Com}(S)$, we form a bicomplex

(5)

$$\begin{tikzcd}
 & {} \arrow[d, "\partial"] & {} \arrow[d, "\partial"] & \\
\cdots & \mathbf{R}(C_q)_p \arrow[l] \arrow[d, "\partial"] & \mathbf{R}(C_q)_{p+1} \arrow[l] \arrow[d, "\partial"] & \cdots \arrow[l] \\
\cdots & \mathbf{R}(C_{q-1})_p \arrow[l] \arrow[d, "\partial"] & \mathbf{R}(C_{q-1})_{p+1} \arrow[l] \arrow[d, "\partial"] & \cdots \arrow[l] \\
 & {} & {} &
\end{tikzcd}$$

and apply $\mathrm{Tot}^{\oplus}(-)$.

We observe that the functor $\mathbf{R} : \mathrm{Com}(S) \to \mathrm{Com}(E)$ factors through the natural map $\mathrm{Com}_{\mathrm{per}}(E, (0,1)) \to \mathrm{Com}(E)$ (see Section 2.1 for the definition of $\mathrm{Com}_{\mathrm{per}}(E, (0,1))$). We may therefore consider $\mathbf{R}$ as taking values in $\mathrm{DM}(E, (0,1))$. From now on, we will denote the category $\mathrm{DM}(E, (0,1))$ by just $\mathrm{DM}(E)$.

♣♣♣ Michael: [Need to double check all the indices in this remark, but I think it's okay.]

Remark 3.1. Suppose $A = \mathbb{Z}$, and $w_i = 1$ for all i, so that S is equipped with the standard grading. Then any object in $\mathrm{DM}(E)$ whose underlying module is free can be identified with a complex of $\mathbb{Z}$-graded E-modules. In more detail: let $\mathrm{Com}^{\mathrm{Free}}_{\mathbb{Z}}(E)$ denote the category of complexes of $\mathbb{Z}$-graded E-modules, where E is equipped with the grading given by $|e_i| = -1$. Let $\mathrm{DM}^{\mathrm{Free}}(E)$ denote the full subcategory of $\mathrm{DM}(E)$ given by objects whose underlying module is free. We now describe an equivalence of categories

$$\mathrm{Com}^{\mathrm{Free}}_{\mathbb{Z}}(E) \xrightarrow{\simeq} \mathrm{DM}^{\mathrm{Free}}(E).$$

Given a complex $C \in \mathrm{Com}^{\mathrm{Free}}_{\mathbb{Z}}(E)$, write $C_n = \bigoplus_{a \in \mathbb{Z}} E(a)^{\oplus r_{n,a}}$. We send C to the object (D, ∂_D), where $D = \bigoplus_{a, n \in \mathbb{Z}} E(a, n-a)^{\oplus r_{n,a}}$, and ∂_D is induced by the differential on C. The inverse is given as follows. Let $(D, \partial_D) \in \mathrm{DM}^{\mathrm{Free}}(E)$, and write $D = \bigoplus E(a,i)^{\oplus r_{a,i}}$. The associated complex C is given by $C_n = \bigoplus_{i-a=n} E(a,i)^{\oplus r_{a,i}}$ with differential induced by ∂_D. Notice that, since every element of E has degree $(-d, d)$ for some $0 \le d \le n$, and ∂_D is a map from D to $D(0,1)$, each summand $E(a,i)$ can only map to summands of the form $E(a-d, i+d-1)$ for some $0 \le d \le n$. It follows that the differential on our complex C maps each C_n to C_{n-1}.

Now, if $D \in \mathrm{DM}(E)$, then it makes sense to apply the functor $\widetilde{\mathbf{L}}$ to $\mathrm{Ex}(D) \in \mathrm{Com}(E)$. The object $\widetilde{\mathbf{L}}(\mathrm{Ex}(D))$ has some extra structure arising from the periodicity of $\mathrm{Ex}(D)$ which we now wish to exploit. The (p, q) term of the bicomplex whose totalization is $\widetilde{\mathbf{L}}(\mathrm{Ex}(D))$ is

$$\bigoplus_{d \geq 0}(D(0, -p))_{(-d,-q)} \otimes S(d) = \bigoplus_{d \in A}(D)_{(-d,-p-q)} \otimes S(d),$$

and so

$$\widetilde{\mathbf{L}}(\mathrm{Ex}(D))_m = \bigoplus_{p \in \mathbb{Z}} \bigoplus_{d \in A}(D)_{(-d,-m)} \otimes S(d).$$

Notice that the summands do not depend on p. Equip each $\widetilde{\mathbf{L}}(D)_m$ with a $k[u]$-module structure determined by the following "shift" operation: if $t = (\dots, t_{-1}, t_0, t_1, \dots) \in \bigoplus_{p\in\mathbb{Z}} \bigoplus_{d\in A}(D_0)_{(-d,-m)} \otimes S(d)$,

$$u(t)_p = (-1)^{m-p} t_{p-1}.$$

Proposition 3.2. *The differential on $\widetilde{\mathbf{L}}(D)$ is $k[u]$-linear.*

♣♣♣ Michael: [I only wrote this proof out to make sure it's correct. Probably this should be omitted in the paper.]

Proof. I'm writing down the proof to make sure I got the sign right in the definition of the u-action. We prove that the action of u commutes with both horizontal and vertical differentials. Write d_D for the differential on D and d_K for the dual Koszul differential. We have

$$\begin{aligned} d_{\mathrm{hor}}(u \cdot (\dots, t_{-1}, t_0, t_1, \dots)) &= d_{\mathrm{hor}}(\dots, (-1)^{m-1}t_{-2}, (-1)^m t_{-1}, (-1)^{m-1}t_0, \dots) \\ &= (\dots, (-1)^m d_T(t_{-3}), (-1)^{m-1} d_T(t_{-2}), (-1)^m d_T(t_{-1}), \dots) \\ &= u \cdot (\dots, d_T(t_{-2}), d_T(t_{-1}), d_T(t_0), \dots) \\ &= u \cdot d_{\mathrm{hor}}(\dots, t_{-1}, t_0, t_1, \dots), \end{aligned}$$

and

$$\begin{aligned} d_{\mathrm{ver}}(u \cdot (\dots, t_{-1}, t_0, t_1, \dots)) &= d_{\mathrm{ver}}(\dots, (-1)^{m-1}t_{-2}, (-1)^m t_{-1}, (-1)^{m-1}t_0, \dots) \\ &= (\dots, (-1)^m d_K(t_{-2}), (-1)^m d_K(t_{-1}), (-1)^m d_K(t_0), \dots) \\ &= u \cdot (\dots, -d_K(t_{-1}), d_K(t_0), -d_K(t_1), \dots) \\ &= u \cdot d_{\mathrm{ver}}(\dots, t_{-1}, t_0, t_1, \dots). \end{aligned}$$

□

Define

$$\mathbf{L}(D) := \widetilde{\mathbf{L}}(D)/(u - 1).$$

We have an isomorphism

$$\mathbf{L}(D)_m \cong \bigoplus_{d \in A}(D_0)_{(-d,-m)} \otimes S(d) \tag{6}$$

given by representing each class in $\mathbf{L}(D)_m$ by an element concentrated in the $p = 0$ summand of $\bigoplus_{p\in\mathbb{Z}} \bigoplus_{d\in A}(D_0)_{(-d,-m)} \otimes S(d)$. Via this isomorphism, the m^{th} differential on $\mathbf{L}(D)$ is given by $(-1)^m d_D + \sum_{i=0}^n e_i \otimes x_i$.

Proposition 3.3. *There is an adjunction*

$$\mathbf{L} : \mathrm{DM}(E) \leftrightarrows \mathrm{Com}(S) : \mathbf{R}.$$

Proof. To start, let M be an S-module, and let D be a differential E-module. Let N be the homological degree 0 term of D. We have

$$\begin{aligned}\mathrm{Hom}_S(\mathbf{L}(D)_q, M) &= \mathrm{Hom}_S(\bigoplus_{d\in A}(D_0)_{(-d,-q)} \otimes_k S(d), M) \\ &= \prod_{d\in A} \mathrm{Hom}_k((D_0)_{(-d,-q)}, M_{-d})\end{aligned}$$

Now, let M be a complex of S-modules. To avoid confusion with subscripts, I'm going to denote the homological degree q term of M by M^{-q}. By the above reasoning, $\mathrm{Hom}_{\mathrm{Com}(S)}(\mathbf{L}(D), M)$ is the submodule of

$$\prod_{d\in A, q\in\mathbb{Z}} \mathrm{Hom}_k((D_0)_{(-d,-q)}, M^{-q}_{-d})$$

given by morphisms that commute with the differentials. On the other hand, $\mathrm{Hom}_{\mathrm{Com}(E)}(D, \mathbf{R}(M))$ is the submodule of

$$\begin{aligned}\mathrm{Hom}_E(D_0, \mathbf{R}(M)_0) &= \mathrm{Hom}_E(D_0, \bigoplus_{d\in A, q\in\mathbb{Z}} M_d^{-q} \otimes \omega(-d, q)) \\ &= \prod_{d\in A, q\in\mathbb{Z}} \mathrm{Hom}_E(D_0, \mathrm{Hom}_k(E(d,-q), M_d^{-q})) \\ &= \prod_{d\in A, q\in\mathbb{Z}} \mathrm{Hom}_k(D_0(d,-q), M_d^{-q}) \\ &= \prod_{d\in A, q\in\mathbb{Z}} \mathrm{Hom}_k((D_0)_{(d,-q)}, M_d^{-q})\end{aligned}$$

given by morphisms that commute with the differentials. Note that the second equality holds because each $M_d^{-q} \otimes \omega(-d, q)$ is 0 in all but finitely many degrees. Reindexing by replacing d with $-d$, we get

$$\prod_{d\in A, q\in\mathbb{Z}} \mathrm{Hom}_k((D_0)_{(-d,-q)}, M^{-q}_{-d}),$$

as desired. We just need to check that the requirements imposed by compatibility with the differentials coincide ♣♣♣ Michael: [I haven't checked this yet.] □

3.2. **Computing the homology of L and R.** We record the following useful observation (cf. [EFS03, Proposition 2.3(b)]).

Proposition 3.4. *Let $P \in \mathrm{Com}(S)$ and $D \in \mathrm{DM}(E)$, and let K denote the Koszul complex on the variables of S.*

(a) *There is an equality*

$$\mathbf{R}(P)_0 = P \otimes_S K$$

of k-vector spaces. Moreover,

$$H_0(\mathbf{R}(P))_{(a,j)} = H_{-j}(P \otimes^{\mathbb{L}}_S k)_a.$$

In particular, if P is concentrated in degree 0, $H_0(\mathbf{R}(P))_{(a,j)} = \mathrm{Tor}^S_{-j}(P,k)_a$

(b) *Let $w = \sum_{i=0}^n |x_i| \in A$, and recall that $n+1$ is the number of variables in S. There is a canonical isomorphism*

$$(\mathbf{L}(D)_i)_a \cong (\underline{\mathrm{Hom}}_{\mathrm{DM}}(\mathbf{R}(S)^\vee, D)_0)_{(w+a,-i-n-1)}$$

for all $a \in A$ and $i \in \mathbb{Z}$. Moreover,

$$H_i(\mathbf{L}(D))_a \cong H_0(\underline{\mathrm{Hom}}_{\mathrm{DM}}(\mathbf{R}(S)^\vee, D))_{(w+a,-i-n-1)}.$$

♣♣♣ Michael: [We probably want to call the right side $\mathrm{Ext}^{\mathrm{DM}}_E(k,D)_{(w+a,-i-n-1)}$.]

Proof. We can view K as the complex of k-vector spaces with homological degree j component $\bigoplus_{a\in A} S \otimes_k \omega_{(a,-j)}$ and differential given by multiplication by $\sum_{i=0}^n x_i \otimes e_i$. We have:

$$\begin{aligned}
(\mathbf{R}(P)_0)_{(a,j)} &= (\bigoplus_{i\in\mathbb{Z}} \mathbf{R}(P_i)_{-i})_{(a,j)} \\
&= (\bigoplus_{i\in\mathbb{Z}} \bigoplus_{d\in A} (P_i)_d \otimes_k \omega(-d,i))_{(a,j)} \\
&= \bigoplus_{i\in\mathbb{Z}} \bigoplus_{d\in A} (P_i)_d \otimes_k \omega_{(a-d,i+j)} \\
&= (\bigoplus_{i\in\mathbb{Z}} \bigoplus_{d\in A} P_i \otimes_k \omega_{(d,i+j)})_a \\
&= (\bigoplus_{i\in\mathbb{Z}} P_i \otimes_S K_{-i-j})_a \\
&= ((P \otimes_S K)_{-j})_a
\end{aligned}$$

This equality identifies 0-cyles in $\mathbf{R}(P)_{(a,j)}$ with $-j$-cycles in $(P \otimes_S K)_a$, and similarly for boundaries. This proves (a). As for (b), we have:

$$\begin{aligned}
(\mathbf{L}(D)_i)_a &= (\bigoplus_{d\in A} (D_0)_{(-d,-i)} \otimes_k S(d))_a \\
&= \bigoplus_{d\in A} (D_0)_{(-d,-i)} \otimes_k S_{d+a} \\
&= \bigoplus_{d\in A} S_d \otimes_k (D_0)_{(-d+a,-i)} \\
&= \bigoplus_{d\in A} S_d \otimes_k D_0(-w-d, n+1)_{(w+a,-i-n-1)} \\
&= (\bigoplus_{d\in A} S_d \otimes_k \omega(-d,0) \otimes_E D_0)_{(w+a,-i-n-1)} \\
&= (\mathbf{R}(S)_0 \otimes D_0)_{(w+a,-i-n-1)} \\
&\cong (\underline{\mathrm{Hom}}_{\mathrm{DM}}(\mathbf{R}(S)^\vee, D)_0)_{(w+a,-i-n-1)}.
\end{aligned}$$

The i^{th} differential on $\mathbf{L}(D)$ is $(-1)^i d_D + \sum_{i=0}^n e_i \otimes x_i$, while the 0^{th} differential on $\mathbf{R}(S) \otimes D$ is $d_D - \sum_{i=0}^n e_i \otimes x_i$. The signs aren't the same, but we can tweak the vertical differentials in the bicomplex defining $\mathbf{L}$ so these the two differentials agree up to a sign, in which case the i-cycles in the source will coincide with 0-cycles in the target, and similarly for boundaries. ♣♣♣ Michael: [Concerning the signs: we're using two facts here. First, a morphism of bicomplexes that induces a quasi-isomorphism on vertical homology induces a quasi-isomorphism on totalizations. Second, given a chain complex (C, d), we can define a chain complex with same underlying module C and i^{th} differential $(-1)^i d_i$, and it is quasi-isomorphic to (C, d) in an obvious way. The isomorphism is 4-periodic cycle given by $\mathrm{id}, \mathrm{id}, -\mathrm{id}, -\mathrm{id}$]. ☐

Remark 3.5. $\mathbf{R}(M)$ does *not* seem to be the 1-periodic folding of $M \otimes_S K$, as complexes of $A \times \mathbb{Z}$-graded k-vector spaces. The degrees are off. Note that $M \mapsto M \otimes_S K$ is Baranovsky's BGG functor; he thinks of $M \otimes_S K$ as a dg-module over the dg-algebra K. So it's not clear how exactly to relate Baranovsky's BGG functor to ours.

3.3. An equivalence on derived categories.

Definition 3.6. We will say a complex of free modules over S is *linear* if each of its differentials can be expressed as a matrix whose entries are elements of W. Let $\mathrm{Lin}(S)$ denote the category of linear free complexes of S-modules.

The following is an analogue of [EFS03, Proposition 2.1], and it follows from essentially the same proof:

Proposition 3.7. *The induced functor* $\mathbf{L} : \mathrm{Mod}(E) \to \mathrm{Lin}(S)$ *is an equivalence.*

♣♣♣ Daniel: [I am tempted to define "linear complexes on S" as anything in the image of $\mathbf{L} : \mathrm{Mod}(E) \to D(S)$. Is this too restrictive?] ♣♣♣ Michael: [This proposition seems to imply that your suggestion is reasonable. The situation for linear DM's over E isn't as clean though. The problem is that the image of $\mathbf{R}$ only includes differential modules whose underlying modules are of the form $\bigoplus_{d \in A} \omega(d, 0)^{\oplus r_d}$. That is, the twist in the auxiliary degree is always 0. It seems sort of unnatural to define a "linear differential module" over E to have this form.

I think this is okay; we sort of rigged this BGG correspondence so that things work out cleanly for the symmetric side, but things are kind of wonky on the exterior side (for instance, we only consider E-modules with this extra auxiliary grading).]

Proof. The inverse is given as follows. We first note that, for any $e \in V = \mathrm{Hom}_k(W, k)$ and k-vector space U, there is an induced map

$$e : W \otimes U \to U$$

given by $w \otimes u \mapsto e(w)u$. Now, let

$$\cdots \xrightarrow{d} \bigoplus_{d \in A} S(d) \otimes_k N_{d,i} \xrightarrow{d} \bigoplus_{d \in A} S(d) \otimes_k N_{d,i-1} \xrightarrow{d} \cdots$$

be an object in $\mathrm{Lin}(S)$. Let $N = \bigoplus_{i \in \mathbb{Z}, d \in A} N_{d,i}$. Define an E-module structure on N as follows. If $n \in N_{i,d}$ and $e \in V$, $e \cdot n = e(d(n)) \in \bigoplus_{d \in A} N_{d,i-1}$; this makes sense since $d(N_{d,i}) \subseteq \bigoplus_{d \in A} W(d) \otimes_k N_{d,i-1}$. We consider N as an $A \times \mathbb{Z}$-graded E-module by defining $N_{(d,i)} = N_{-d,-i}$. The relation $d^2 = 0$ implies the relations N must satisfy to be an E-module. ♣♣♣ Michael: [double check this to make sure.] ☐

Proposition 3.8. *For any* $C \in \mathrm{Com}(S)$*, the counit of adjunction*

$$(\mathbf{L} \circ \mathbf{R})(C) \to C$$

is a surjective quasi-isomorphism. For any $D \in \mathrm{DM}(E)$, the unit of adjunction

$$D \to (\mathbf{R} \circ \mathbf{L})(D)$$

is an injective quasi-isomorphism.

♣♣♣ Michael: [There might be a simpler proof of this.]

Proof. I'll sketch a proof of the second statement. The first should be similar. Injectivity should follow from the same argument as in [EFS03, Corollary 2.7]. Next, note that every object in $\mathrm{DM}(E)$ is a filtered colimit of objects whose terms are finitely generated. To see this, choose a family $\{N_i\}_{i \in I}$ of finitely generated submodules of D_0 such that $D_0 = \operatorname{colim} N_i$. For each i, define a differential submodule D_i of D whose terms are $N_i + \partial(N_i)$ and with the induced differential. Then $D = \operatorname{colim} D_i$.

The functor $\mathbf{L}$ obviously commutes with colimits, since it is a left adjoint. I think $\mathbf{R}$ commutes with colimits as well, but this requires proof (this holds in the [EFS03] setting); it should boil down to E being finite dimensional over k. Since filtered colimits are exact, we can assume the terms of D are finitely generated. Thus, $\mathbf{L}(D)$ is bounded, and so the bicomplex whose totalization is $(\mathbf{R} \circ \mathbf{L})(D)$ has only finitely many nonzero rows. The horizontal homology will be easy to understand, since the rows are just $\mathbf{R}$ applied to free modules. I think for degree reasons (the auxiliary degree that is), the associated spectral sequence degenerates at page 2, and the result should immediately follow. □

Let $\mathrm{D^b}(S)$ denote the bounded derived category of S, and let $\mathrm{D}^b_{\mathrm{DM}}(E)$ denote the derived category of the subcategory of $\mathrm{DM}(E)$ whose objects have finitely-generated homology in each degree.

Corollary 3.9. *The induced maps*

$$\mathbf{L} : \mathrm{D}^b_{\mathrm{DM}}(E) \leftrightarrows \mathrm{D^b}(S) : \mathbf{R}$$

are inverse equivalences.

Given any $A \times \mathbb{Z}$-graded ring R, define a functor

$$\mathrm{Fold} : \mathrm{Com}(R) \to \mathrm{DM}(R, a)$$

given by

$$(\mathrm{Fold}(C))_i = \bigoplus_{j \in \mathbb{Z}} C_j(0, j - i)$$

and with i^{th} differential given by $(-1)^i d^C$.

Now, suppose $X = \mathbb{P}^n$. We end this section by explaining the relationship between the equivalence in the above Corollary and the classical BGG equivalence. One easily checks that

$$\mathrm{Fold} \circ \mathbf{R}_{\mathrm{EFS}} = \mathbf{R}.$$

It follows that Fold determines an equivalence

$$\mathrm{D^b}(E) \xrightarrow{\simeq} \mathrm{D}^b_{\mathrm{DM}}(E).$$

So one way to phrase our approach to toric BGG is as follows: first reinterpret the classical BGG correspondence as involving differential E-modules, and then observe that this interpretation has an obvious generalization to the toric setting.

We note that functor Fold : $\mathrm{Com}(E) \to \mathrm{DM}(E)$ is not itself an equivalence. Here is how to see this. The category $\mathrm{Com}(E)$ can be interpreted as the category of dg-modules over the dg-algebra E with trivial differential. Consider the dg-algebra $E[t, t^{-1}]$, where $|t| = (0, 1)$. The category $\mathrm{DM}(E)$ is just the category of dg-modules over $E[t, t^{-1}]$. There is a canonical morphism

$$E \to E[t, t^{-1}]$$

of dg-algebras, and the functor Fold is just extension of scalars along this morphism. This functor has a right adjoint given by restriction of scalars, which amounts to the forgetful functor $\mathrm{DM}(E) \to \mathrm{Com}(E)$. This right adjoint is obviously not an equivalence; it's far from essentially surjective. It follows that Fold is not an equivalence either.

♣♣♣ Daniel: [We need to state and prove reciprocity theroem. For that, we'll need to define an injective resolution of differential modules. That's not difficult, but it does need to be added.]

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