TATE DIFFERENTIAL MODULES

1. Overview

Our overarching idea is to realize the Tate resolution as a sort of pushforward in a category of differential modules. This perspective provides additional flexibility, allowing us to easily construct "Tate modules" in contexts–like with toric varieties–where we know that the categories of complexes/resolutions is insufficient. On $\mathbb{P}^n$, the key diagram is:

$$\begin{tikzcd}
\mathrm{Coh}(\mathbb{P}^n_k) \arrow[r, "\kappa^*"] \arrow[dr, "\mathbf{T}"'] & \mathrm{DM}_{\mathrm{free}}(\mathbb{P}^n_E) \arrow[d, "\tau_*"] \\
& \mathrm{DM}^\circ(E)
\end{tikzcd}$$

The arrow κ^* sends $\mathcal{E}$ to the differential $\mathcal{O}_{\mathbb{P}^n_E}$-module which is essentially the Koszul complex with respect to $x_0, x_1, \ldots, x_n$; see Section 3. The vertical arrow τ_* is a variant of the pushforward functor $R\pi_*$ where $\pi : \mathbb{P}^n \to \mathrm{Spec}(k)$, but with a few key subtleties:

(1) We are pushing forward onto a non-commutative ring E;
(2) We want a differential module up to homotopy, not up to quasi-isomoprhism;
(3) There are different possible definitions for the "pushforward" of a DM-module.

These will be discussed in detail in Section 4, but in short: (1) poses no significant obstacle; (2) relies on the fact that the underlying module of $\kappa^*\mathcal{E}$ is free as a E-module; (3) is where the novelty lies. The $\mathbf{T}$ arrow is the [EFS03] construction which sends $\mathcal{E}$ to its Tate resolution $\mathbf{T}\mathcal{E}$, but then considering that complex as a differential module. The alternate construction is the claim that the above diagram commutes:

Theorem 1.1. *We have $\mathbf{T}\mathcal{E} = \tau_*\kappa^*\mathcal{E}$; that is, $\tau_*\kappa^*\mathcal{E}$ is the Tate resolution of $\mathcal{E}$.*

The key is **not** that $\tau_*\kappa^*\mathcal{E}$ is a simpler way to construct the Tate resolution. In fact, on $\mathbb{P}^n$ the opposite will be true, as $\mathbf{T}\mathcal{E}$ is simply obtained by taking a minimal free resolution, whereas constructing $\tau_*\kappa^*\mathcal{E}$ requires the development of some additional algebraic tools. The value is that the formal properties of $\tau_*\kappa^*\mathcal{E}$ are easier to understand from this perspective, and this will allow us to vary the construction in a number of different ways. Our main goal is to obtain variants for other toric varieties, but even on $\mathbb{P}^n$ we get new constructions.

2. Notation

2.1. **Differential modules.** Throughout $\mathrm{DM}(R)$ will denote the category of differential modules over R; so objects $D \in \mathrm{DM}(R)$ are differential R-modules, that is an R-module D together with a differential $\partial : D \to D$; and morphisms are morphisms of the underlying module, which commute with the differentials. We write $|D|$ for the underlying module, when we wish to emphasize its distinction from the differential module. We generally write ∂ for all differentials, or ∂_D when needed for clarity. If R is graded, then we will assume that

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so is $|D|$ and so is ∂. $\mathrm{H}(D)$ will denote the homology of the differential module and is thus a functor $\mathrm{H} : \mathrm{DM}(R) \to \mathrm{Mod}(R)$. We will use $\mathrm{DM}^{\circ}(R)$ to denote the category of differential modules up to homotopy.

Remark 2.1. Let $\mathrm{D}(R)$ denoted the derived category of R-modules and $\mathrm{D}(R)^*$ denote the subcategory that is invaraint under homological shifts; that is, it is the subcategory generated by complexes where all objects and all morphisms are the same. The equivalence sends the element $M \in \mathrm{DM}(R)$ to the unfolded complex

$$\cdots \xrightarrow{\partial} M \xrightarrow{\partial} M \xrightarrow{\partial} M \xrightarrow{\partial} \cdots$$

The category $\mathrm{DM}(R)$ is naturally isomorphic to $\mathrm{D}(R)^*$ and it will be helpful to occasionally move view $\mathrm{DM}(R)$ as this subcategory of $\mathrm{D}(R)$.[1]

2.2. **Polynomial rings and exterior algebras.** S will denote the coordinate ring of projective space or some other simplicial, projective toric variety X; as such S will be a polynomial ring with variables $x_0, \dots, x_n$ and graded by $\mathrm{Pic}(X)$. E will denote an exterior algebra, generally on $n+1$ variables $e_0, \dots, e_n$, dual to the x_i. We will give E a $\mathrm{Pic}(X) \oplus \mathbb{Z}$ grading, denoted by $E(\ell; j)$ with $\ell \in \mathrm{Pic}(X)$ and $j \in \mathbb{Z}$. The extra $\mathbb{Z}$-grading will be the "standard" grading on the exterior algebra, where $\bigwedge^j$ has degree j. This "extra" will only be relevant when we move beyond the standard $\mathbb{P}^n$ example.

2.3. **Module categories.** $\mathrm{Mod}(\mathbb{P}^n_E)$ will stand for the category of $\mathcal{O}_{\mathbb{P}^n} \otimes_k E$-modules, that is $\mathcal{O}_{\mathbb{P}^n}$-modules which admit an E-module structure as well. For instance $\mathcal{O}_{\mathbb{P}^n} \otimes_k E \in \mathrm{Mod}(\mathbb{P}^n_E)$. $\mathrm{Mod}_{\mathrm{free}}(\mathbb{P}^n_E)$ will denote the subcategory of such modules which are free as E-modules; every such object is a direct sum of objects of the form $\mathcal{F} \otimes_k E(\ell; i)$ for some $\ell \in \mathbb{Z}$ and some $i \in \mathbb{Z}$. The categories $\mathrm{D}(\mathbb{P}^n_E), \mathrm{DM}(\mathbb{P}^n_E)$ and so on will be defined in the obvious ways.

For a simplicial toric variety X, we use the analogous definitions for the categories $\mathrm{Mod}(X)$, $\mathrm{DM}(X), \mathrm{Mod}(X_E), \mathrm{DM}(X_E)$ and so on.

3. Construction of κ^*

Given a (coherent) sheaf $\mathcal{E}$ on $\mathbb{P}^n$ we want $\kappa^*\mathcal{E}$ to be the differential $\mathcal{O}_{\mathbb{P}^n_E}$-module given by the Koszul complex of $\mathcal{E}$ summed over all twists $\ell \in \mathbb{Z}$. Precisely, we will define $\kappa^*\mathcal{E}$ as the differential module whose underlying sheaf is $\oplus_{\ell \in \mathbb{Z}} \mathcal{E}(-\ell) \otimes_k E(\ell; 0)$ and where the differential is given by $\sum x_i \otimes e_i$. Note that, by forgetting the E-module structure and just considering this as a differential module on $\mathbb{P}^n$, this is just a direct sum of the differential modules obtain from Koszul complexes. Note that the differential mixes the copies of the direct summand. For instance it sends $\mathcal{E} \otimes E$ to $\mathcal{E}(1) \otimes E(-1)$ and so on.

If we write $\mathcal{F} = \oplus_{\ell \in \mathbb{Z}} \mathcal{E}(\ell) \otimes_k E(\ell; 0)$ for the underlying module, then the differential goes: $\partial : \mathcal{F} \to \mathcal{F} \otimes_k E(0; 1)$. The unfolded complex for $\kappa^*\mathcal{E}$ thus looks like:

$$\cdots \xrightarrow{\partial} \mathcal{F} \otimes_k E(0; -1) \xrightarrow{\partial} F \xrightarrow{\partial} \mathcal{F} \otimes_k E(0; 1) \xrightarrow{\partial} \cdots$$

[1] Grading imposes lots of delicate issues which we have not fully pinned down. For instance, if R is graded, then we need to be more careful with this defintion, as we we could want to send M to $\cdots \xrightarrow{\partial} M \xrightarrow{\partial} M(1) \xrightarrow{\partial} M(2) \xrightarrow{\partial} \cdots$ which is only invariant under shift up to an appropriate degree twist.

4. Construction of τ_*

The main result of this section is the following. The proof is identical in the case of $\mathbb{P}^n$ or the case of a simplicial toric variety. So we prove it in general, but it may be useful to work the case of $\mathbb{P}^n$ to understand it.

Lemma 4.1. *Let X be a toric variety over k with Cox ring S, graded by $\mathrm{Pic}(X)$. Let E be the exterior algebra on the dual variables, with the $\mathrm{Pic}(X) \oplus \mathbb{Z}$-grading described above.*

(1) There is an additive functor: $\pi_* \colon \mathrm{DM}(X) \to \mathrm{DM}^\circ(k)$ which preserves exactness and where $|\pi_*\mathcal{D}| = \oplus_{i=0}^{\dim X} H^i(X, |\mathcal{D}|)$, namely: the underlying module of $\pi_*\mathcal{D}$ is the total sheaf cohomology of the underlying module of $\mathcal{D}$.

(2) There is an additive functor: $\tau_* \colon \mathrm{DM}_{\mathrm{free}}(\mathbb{P}^n_E) \to \mathrm{DM}^\circ(E)$ which preserves exactness and where the underlying module of $\tau_*\mathcal{D}$ satisfies: If $|\mathcal{D}| = \oplus_{\ell \in \mathrm{Pic}(X)} \mathcal{E}_\ell \otimes_k E(\ell; 0)$ then

$$|\tau_*\mathcal{D}| = \bigoplus_{i=0}^{\dim X} \bigoplus_{\ell \in \mathrm{Pic}(X)} H^i(\mathcal{E}_\ell) \otimes_k E(\ell; -i). \tag{1}$$

Proof. We first prove (1). Start with a differential X-module whose underlying sheaf is $\mathcal{F}$. Following Remark 2.1, we can realize this in $\mathrm{D}(\mathbb{P}^n_E)$ as

$$\cdots \xrightarrow{\partial} \mathcal{F} \xrightarrow{\partial} \mathcal{F} \xrightarrow{\partial} \mathcal{F} \xrightarrow{\partial} \cdots$$

Let $C^0 \to C^1 \to \cdots \to C^n$ be the Cech resolution for $\mathcal{F}$, with C^i an S-module. We build a double complex F^i_j where $F^i_j = C^i_j$ (in particular F^i_j only depends on j); the columns are Cech resolutions and the horizontal maps are localizations of the differential ∂.

Each column of the double complex is the same, and taking vertical homology in the i'th spot gives $H^i(X, \mathcal{F})$. By choosing a splitting of $H^i(\mathcal{E}_\ell)$ into the Cech complex for $\mathcal{F}$, we get a splitting for the vertical homology in this double complex. We can then apply [EFS03, Lemma 3.5], to get a complex of E-modules:

$$\cdots \xrightarrow{\partial} G \xrightarrow{\partial} G \xrightarrow{\partial} G \xrightarrow{\partial} \cdots$$

where G in column 0 is

$$G = \oplus_{i=0}^{\dim X} \mathrm{H}^i(F^*_i) = \oplus_{i=0}^{\dim X} H^i(\mathcal{F}).$$

A different splitting will induce the same complex, up to homotopy (since both would be homotopic to the original total complex, again by [EFS03, Lemma 3.5]). Applying Remark 2.1 in reverse, G induces a well-defined element of $\mathrm{DM}^\circ(k)$ as desired.

Exactness is nearly immediate from the construction. By construction, the ith homology of the complex

$$\cdots \xrightarrow{\partial} \pi_*\mathcal{D} \xrightarrow{\partial} \pi_*\mathcal{D} \xrightarrow{\partial} \pi_*\mathcal{D} \xrightarrow{\partial} \cdots$$

in $\mathrm{D}(k)$ is the hypercohomology of the complex

$$\cdots \xrightarrow{\partial} \mathcal{D} \xrightarrow{\partial} \mathcal{D} \xrightarrow{\partial} \mathcal{D} \xrightarrow{\partial} \cdots$$

But by the standard hypercohomology spectral sequence $E_2^{i,j} = H^i(\mathrm{H}^j -) \Rightarrow H^{i+j}(-)$, if (the unfolded complex for) $\mathcal{D}$ is exact then so is (the unfolded complex for) $\pi_*\mathcal{D}$.

The proof of (2) is nearly identical as the proof of (1), though we need to track the gradings in more detail. Starting with an object $\mathcal{F} := \oplus_\ell \mathcal{E}_\ell \otimes_k E(\ell;0)$ from $\mathrm{DM}_{\mathrm{free}}(\mathbb{P}^n_E)$, we get the unfolded complex:

$$\cdots \xrightarrow{\partial} \oplus_\ell \mathcal{E}_\ell \otimes_k E(\ell;-1) \xrightarrow{\partial} \oplus_\ell \mathcal{E}_\ell \otimes_k E(\ell;0) \xrightarrow{\partial} \oplus_\ell \mathcal{E}_\ell \otimes_k E(\ell;1) \xrightarrow{\partial} \cdots$$

For each ℓ, we take a separate Cech resolution of $\mathcal{E}_\ell$, tensor with $-\otimes_k E(\ell;0)$, and then take the direct sum of these to get one of our vertical columns, C_0^*. We then let $C_i^* = C_0^* \otimes_E E(0;i)$. We again build a double complex from these, where the columns are twists of Cech resolutions, and the horizontal maps are localizations of the differential ∂.

Each column is the same (up to a twist of the E-part), and taking vertical homology in the i'th spot gives $\mathrm{H}(C_j^*) = \oplus_\ell H^i(\mathcal{E}_\ell) \otimes_k E(\ell;-j)$. Since the vertical differentials are defined entirely over k, we can still choose a splitting of the homology into the Cech complex, and thus obtain a splitting for the vertical homology in this double complex. We apply [EFS03, Lemma 3.5], to get a complex of E-modules:

$$\cdots \xrightarrow{\partial} G(0;-1) \xrightarrow{\partial} G \xrightarrow{\partial} G(0;1) \xrightarrow{\partial} \cdots$$

where G in homological degree 0 is

$$G = \oplus_{i=0}^{\dim X} \mathrm{H}^i(F_i^*) = \oplus_{i=0}^{\dim X} \oplus_{\ell \in \mathrm{Pic}(X)} H^i(\mathcal{E}_\ell) \otimes_k E(\ell;-i).$$

We thus obtain a well-defined element of $\mathrm{DM}^\circ(E)$ as desired. To check exactness, we can consider the commutative square

$$\begin{tikzcd}
\mathrm{DM}_{\mathrm{free}}(\mathbb{P}^n_E) \arrow[r] \arrow[d, "\tau_*"'] & \mathrm{DM}(\mathbb{P}^n_k) \arrow[d, "\pi_*"'] \\
\mathrm{DM}^\circ(E) \arrow[r] & \mathrm{DM}^\circ(k)
\end{tikzcd}$$

where the horizontal arrows simply forget the E-module structure. Since forgetful maps respect exactness, exactness for part (2) follows from exactness for part (1). ☐

Example 4.2. Let's work on $\mathbb{P}^1$. Let $\mathcal{D} := \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}^2_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-2)$ with the Koszul differential on $\mathcal{D}$, so that $\mathcal{D}$ is exact. Let $C^0 \to C^1$ be the Cech resolution of $\mathcal{D}$. Under the construction above, we end up with a total complex $\cdots \xrightarrow{\partial} G \xrightarrow{\partial} G \xrightarrow{\partial} G \xrightarrow{\partial} \cdots$ where

$$G \cong H^0(\mathbb{P}^1, \mathcal{O}) \oplus H^1(\mathbb{P}^1, \mathcal{O}(-2)).$$

While G is isomorphic to the total cohomology of $C^\bullet$, note that the H^0 and H^1 come from different copies of $C^\bullet$ in the total complex. Namely, if we are looking at the copy of G in position 0, then the H^0 is the 0'th homology of the $C^\bullet$ in column 0 whereas the H^1 part is the 1'st homology of $C^\bullet$ in column 1. That is to say, if we write $C_i^\bullet$ for the Cech complex in column i, and we write G_i for the copy of G in position i, then:

$$G_i = \mathrm{H}^0(C_i^\bullet) \oplus \mathrm{H}^1(C_{i+1}^\bullet).$$

So to get the map $H^1 \to H^0$ we would require at least 3 copies of the Cech complex. This explains why we can't just define the differential as "the pushforward of the map ∂".

Remark 4.3. There are other ways to pushforward a DM-module which would give fundamentally different answers. Given a DM-module $\mathcal{D}$ on $\mathbb{P}^n_k$, the k-module $H^0(\mathcal{D})$ together with the map $H^0(\partial)$ forms a DM-module over k. In fact, for any i, we get a DM-module on

$H^i(\mathcal{D})$ with differential $H^i(\partial)$. Of course, since it is well known that the functor $H^0(-)$ does not necessarily preserve exactness properties, one might want to define the "right" derived functor version of this for DM-modules.

Another possibility is to work with the functor $R\pi_*$. We could then obtain an DM-module structure on $\oplus_{i=0}^n H^i(\mathcal{D})$ via the map $R\pi_*(\partial)$ which is defined a priori in the dervied category, but which lifts to the homotopy category. However, in this case $R\pi_*(\partial)$ would simply be the direct sum of the maps $H^i(\partial)$; in particular, it would not induce any morphisms between say $H^1(\mathcal{D})$ and $H^0(\mathcal{D})$, which are crucial to the exactness of the Tate resolution. And so this would not send an exact DM-module to an exact DM-module. To see those higher maps, it seems to be essential to push forward the "unfolded" DM module as above.

Sketch of proof of Theorem 1.1. Let $\mathcal{E}$ be a coherent sheaf on $\mathbb{P}^n$ and let M be a graded S-module such that $\widetilde{M} = \mathcal{E}$. Both $\tau_*\kappa^*\mathcal{E}$ $\mathbf{T}\mathcal{E}$ agree with $\mathbf{R}M$ in sufficiently high degrees. Since both $\tau_*\kappa^*\mathcal{E}$ and $\mathbf{T}\mathcal{E}$ are exact, the uniqueness of minimal free covers (as in DM.pdf) implies that they are isomorphic. ☐

5. Tate Modules on Toric Varieties

The proof of Theorem 1.1 will go through essentailly verbatim in the toric case. Let X be a simplicial toric variety with Cox ring $S = k[x_0, \dots, x_n]$ and Picard group $\mathbb{Z}^r$. Let E be the dual exterior algebra with variables $e_0, \dots, e_n$ graded by $\mathbb{Z}^r \oplus \mathbb{Z}$ where the "extra" copy of $\mathbb{Z}$ is the standard grading on an exterior algebra. For any subset $I \subseteq \{0, \dots, n\}$ let $\mathbf{x}_I = \{x_i | i \in I\}$. Consider:

$$\begin{tikzcd} \mathrm{Coh}(X) \arrow[r, "\kappa_I^*"] \arrow[dr, "\mathbf{T}_I"] & \mathrm{DM}_{\mathrm{free}}(X_E) \arrow[d, "\tau_*"] \\ & \mathrm{DM}(E) \end{tikzcd}$$

When I is the entire set $I = \{0, \dots, n\}$ then we omit the I and write simply κ^* and $\mathbf{T}$. The arrow τ_* was defined in Lemma 4.1. The arrow κ_I^* is defined entirely analogously to the $\mathbb{P}^n$-case.

Definition 5.1. For $\mathcal{E} \in \mathrm{Mod}(X)$ and $I \subseteq \{0, \dots, n\}$ we define $\kappa_I^*\mathcal{E} \in \mathrm{DM}(X_E)$ as the differential module where the underlying module is $\oplus_{\ell \in \mathrm{Pic}(X)} \mathcal{E}(-\ell) \otimes_k E(\ell; 0)$ and where the differential is $\sum_{i \in I} x_i \otimes e_i$.

And the arrow $\mathbf{T}_I$ is simply defined as the composition $\tau_*\kappa_I^*$.

NOTE: even if E only involves some of the variables, it is convenient define $\mathbf{T}_I\mathcal{E}$ over the exterior algebra E on all of the dual variables, so that we can compare $\mathbf{T}_I$ for various I without a need to change rings.

Example 5.2. Let $X = \mathbb{P}^1 \times \mathbb{P}^1$ with Cox ring $S = k[x_0, x_1, x_2, x_3]$ and irrelevant ideal $(x_0, x_2) \cap (x_1, x_3)$. Let $I = \{x_0, x_2\}$. Then $\tau_*\kappa_I^*(\mathcal{E})$ is essentially the Tate resolution of $\mathcal{E}$ but only considering the horizontal differentials. This was shown to be exact in [EES15].

Theorem 5.3. *If $V(\mathbf{x}_I) = \emptyset$ then $\mathbf{T}_I(\mathcal{E})$ is exact.*

In particular, choosing $I = \{0, \dots, n\}$, we get that $\mathbf{T}(\mathcal{E})$ is an exact differential module whose underlying module is

$$|\mathbf{T}(\mathcal{E})| = \bigoplus_{i=0}^{\dim X} \bigoplus_{\ell \in \mathrm{Pic}(X)} H^i(X, \mathcal{E}(-\ell)) \otimes_k E(\ell; -i).$$

Proof. If $V(\mathbf{x}_I) = \emptyset$ then the Koszul of $\mathcal{E}$ with respect to $\mathbf{x}_I$ is exact, and thus $\kappa_I^* \mathcal{E}$ is also exact; exactness of $\tau_* \kappa_I^* \mathcal{E}$ then follows from Lemma 4.1. The statement on underlying modules also follows from Lemma 4.1. ☐

Remark 5.4. Our need to work with differential modules was first observed by considering what the $\mathbf{R}$-functor from [EFS03] should be in the toric case. However, with this new construction, we can now more easily understand properties about $\mathbf{R}$ in that case.

Definition 5.5. Define a functor $\mathbf{R}_I$ from graded S-modules to $\mathrm{DM}(E)$ via:

$$\mathbf{R}_I M := \oplus_{\ell \in \mathbb{Z}^r} M_\ell \otimes_k E(\ell; 0)$$

with differential equal to multipilcation by $\sum_{i \in I} x_i \otimes e_i$. If M is the graded S-module corresponding to some sheaf $\mathcal{E}$, then this is essentially just $H^0 \kappa_I^* \mathcal{E}$. If I is the entire set $\{0, 1, \dots, n\}$ then we omit the I and write it as simply $\mathbf{R}M$.

For $\ell, m \in \mathrm{Pic}(X)$, we say that $\ell \geq m$ is $\ell - m$ is effective.

Corollary 5.6. *Let M be a multigraded S-module and choose I so that $V(\mathbf{x}_I) = \emptyset$. For any $m \gg 0$, $\mathbf{R}_I M$ is exact in degrees $\ell \geq m$.*

Proof. $\mathbf{T}_I(\mathcal{E})$ and $\mathbf{R}_I M$ agree in high degrees, since higher cohomology vanishes in high degrees, and the statement is then immediate. ☐

Note the reversal: in [EFS03, EES15], it is first shown that $\mathbf{R}M$ is exact in high degrees, and then this is used to construct the Tate resolution; by contrast, we first construct an exact Tate DM-module, and then using this to conclude that $\mathbf{R}M$ is exact in high degrees.

Corollary 5.7. *Let M be a multigraded S-module whose corresponding sheaf is $\mathcal{E}$. If $V(\mathbf{x}_I) = \emptyset$ then $\mathbf{T}_I(\mathcal{E})$ is a minimal free cover $\mathbf{R}_I M$.*

Proof. $\mathbf{T}_I(\mathcal{E})$ and $\mathbf{R}_I M$ agree in high degrees, and since $\mathbf{T}_I(\mathcal{E})$ and minimal free covers are unique, the statement follows. (A bit more to line up here like we may need some sort of boundedness of the homology of $\mathbf{R}_I M$. We will double back when the algebraic stuff is complete.) ☐

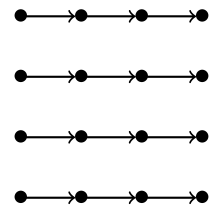
6. Algorithm for Tate Modules on Hirzebruch Surfaces (Sketch)

♣♣♣ Daniel: [Starting in this section, things get sketchier.]

We have an algorithm for computing Tate DM-modules on a Hirzebruch surface and it appears to give the correct answer. This was the code we sent over the summer. Below we give the sketch of this construction. However, we emphasize that we have not worked out a general algorithm for constructing Tate DM-modules on a toric variety.

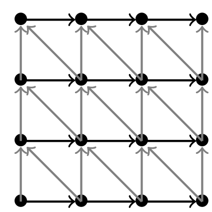
Let X be a Hirzebruch surface (the degree does not really matter to the overview of the construction). Let S be the Cox ring with irrelevant ideal $(x_0, x_2) \cap (x_1, x_3)$. Let M be any finitely generated $\mathbb{Z}^2$-graded S-module and $\mathcal{E} = \widetilde{M}$ the corresponding coherent sheaf.

(1) Since $V(x_0, x_2) = \emptyset$, we can apply Theorem **??** to see that $\tau_* \kappa_{02}^* \mathcal{E}$ is exact. Since $\tau_* \kappa_{02}^* \mathcal{E}$ and $\mathbf{R}_{02} M$ agree in high degrees, it follows[2] that $\mathbf{R}_{02} M$ is exact in degree α for any $\alpha = (\alpha_1, \alpha_2) \gg 0$. Here is a sketchy graphic of $\mathbf{R}_{02}$ in high degrees:



The maps are horizontal because the entries all have degree $(-1, 0)$. Of $\mathbf{R}_{02} M$ should continue indefinitely up and to the right.

(2) The differential ∂ on $\mathbf{R} M$ can be written as a sum of two maps of E-modules: $\partial = \partial_{02} + \partial_{13}$ where ∂_{02} is the differential on $\mathbf{R}_{02} M$ and ∂_{13} is the differential on $\mathbf{R}_{13} M$. Since $\partial^2 = 0$, it follows from an elementary computation that ∂_{13} induces a morphism of DM-modules $\partial_{13} : \mathbf{R}_{02} M \to \mathbf{R}_{02} M$. The morphism ∂_{13} has entries of degree $(0, 1)$ and $(-1, 1)$ so it looks like the grey arrows here:



(3) Since the elements x_0, x_2 have degree $(1, 0)$, $\mathbf{R}_{02} M$ splits as a direct sum of DM-modules, with one strand in degrees $(*, j)$ for each $j \geq \alpha_2$. Let F be a minimal free cover $\mathbf{R}_{02} M$. F also splits as a direct sum, with one strand in degrees $(*, j)$ for each $j \geq \alpha_2$. In other words, F is a direct sum of minimal free extensions, one for each row $(*, j)$, which extends indefinitely to the left and right. By uniqueness of minimal free covers, these agree with the strands of $\tau_* \kappa_{02}^* \widetilde{M}$ for all $j \geq \alpha_2$.

(4) We lift ∂_{13} to a morphism $\widetilde{\partial}_{13} : F \to F$. We claim that we can do this in a way where $(\widetilde{\partial}_{13})^2 = 0$; certainly this is homotopic to zero by uniqueness of lifts, and (DETAILS MISSING). We have now filled in the upper half-plane, in essence, with entires extending arbitrarily to the left and right and up, all above row α_2.

(5) Now we switch perspectives consider F with the new differential $\widetilde{\partial}_{13}$. Call this DM-module F'. We observe that F' equals $\tau_* \kappa_{13}^* \widetilde{M}$ in degrees $(*, j)$ with $j \geq \alpha_2$. (We also observe that ∂^F is now a map of DM-modules $F' \to F'$.) We note that F' is exact in degrees $(*, j)$ where $j \geq \alpha_2$.

(6) Let G be a minimal free cover of F'. This extends us to a minimal free cover supported in all degrees of $\mathbb{Z}^2$. We also lift the morphism ∂^F. We claim that G now equals $\tau_* \kappa_{13}^* \mathcal{E}$ in all degrees. For computing sheaf cohomology, the module G is sufficient; if we want the full Tate module, then we replace the differential of G by something like the sum of the differential of G and the lift of ∂^F.

[2]Note the reverse implication from the standard construction!

7. Other Stuff to do

Beyond the algorithm for a general, we also don't have any idea of how to relate this to:

- Resolutions of the diagonal of X in $X \times X$;
- "Beilinson monads" on X;
- Free monads and resolutions of truncations (e.g [EES15, Proposition 1.7(1)], which could yield some interesting corollaries for virtual resolutions.)

Even on weighted projective space, where there is a resolution of the diagonal, much of this seems rather subtle.

8. Generalized Tate Complexes on $\mathbb{P}^n$ (Sketchy)

Let $\mathbf{f} = (f_0, \ldots, f_r)$ be any set of homogeneous polynomials. Let E be the exterior algebra on $e_0, \ldots, e_r$ where $\deg(e_i) = -f_i$. Consider:

$$\begin{tikzcd}
\mathrm{Coh}(\mathbb{P}^n_k) \arrow[r, "\kappa_{\mathbf{f}}^*"] \arrow[dr, "\mathbf{T_f}"] & \mathrm{DM}_{\mathrm{free}}(\mathbb{P}^n_E) \arrow[d, "\tau_*"] \\
& \mathrm{DM}(E)
\end{tikzcd}$$

The arrow $\kappa_{\mathbf{f}}^*$ sends $\mathcal{E}$ maps to the differential $\mathcal{O}_{\mathbb{P}^n_E}$-module which is the Koszul complex on $\mathcal{E}$ with respect to $f_0, f_1, \ldots, f_r$, summed over all twists. Note that this may or may not be an exact differential module. This construction allows us to define a "Tate module" $\mathbf{T_f}\mathcal{E}$ for any $\mathbf{f}$. The differential module $\mathbf{T_f}\mathcal{E}$ will be exact if $V(f_0, \ldots, f_r) = \emptyset$; in general its homology will be related to the Koszul homology of $f_0, \ldots, f_r$ on $\mathcal{E}$.

Definition 8.1. We can define a functor $\mathbf{R_f}$ from graded S-modules to $\mathrm{DM}(E)$ which is entirely parallel to the standard defintion of $\mathbf{R}$. Namely, $\mathbf{R_f}M := \oplus_\ell M_\ell \otimes_k E(\ell; 0)$ with differential equal to multipilcation by $\sum f_i \otimes e_i$. If $M = H^0_*\mathcal{E}$ for some sheaf $\mathcal{E}$, then this is just $H^0\kappa_{\mathbf{f}}^*\mathcal{E}$.

We first consider cases where $V(\mathbf{f}) = \emptyset$.

Proposition 8.2. *If $V(f_0, \ldots, f_r) = \emptyset$ then $\tau_*\kappa_{\mathbf{f}}^*\mathcal{E}$ is exact.*

Proof. Follows from Lemma 4.1. ☐

Example 8.3. Let $\mathbf{f} = (x_0, x_1^2)$ and let E be the exterior algebra on two variables e_0, e_1 of degrees $-1, -2$. We get that $\tau_*\mathcal{O}_{\mathbb{P}^1}$ is:

$$\begin{tikzcd}
\cdots \arrow[r] \arrow[rr, bend left] & E(-3)^2 \arrow[r] \arrow[drr] & E(-2) \arrow[drr] & & & & & \\
& & & E \arrow[r] \arrow[rr, bend right] & E(1)^2 \arrow[r] \arrow[rr, bend left] & E(2)^3 \arrow[r] \arrow[rr, bend right] & E(3)^4 \arrow[r] & \cdots
\end{tikzcd}$$

where the arrows indicate entries in the differential. If you label the bases with the natural monomial basis, then the simple horizontal arrows correspond to multiplication by x_0, the curvy ones correspond to multiplication by x_1^2 and the diagonal ones correspond to multiplication by $x_0x_1^2$. So for instance, if the basis of $E(-3)^2$ is $\frac{1}{x_0^2x_1}$ and $\frac{1}{x_0x_1^2}$ then the differential sends $1 \otimes \frac{1}{x_0^2x_1} \mapsto e_0 \otimes \frac{1}{x_0x_1} \in E(-2)$ plus $1 \otimes \frac{1}{x_0x_1^2} \mapsto e_0e_1 \otimes 1 \in E$. If $f : \mathbb{P}^1 \to \mathbb{P}(1,2)$ is the map given by $(x_0, x_1) \mapsto (x_0, x_1^2)$ then this is the Tate resolution for $f_*\mathcal{O}_{\mathbb{P}^1}$ on $\mathbb{P}(1,2)$.

We are not sure what the applications are when $V(f_0, \ldots, f_r) \neq \emptyset$, but it seems interesting that you can construct a sort-of Tate resolution from any $\mathbf{f}$ whatsoever; they will not always be exact, but the homology will encode something about the homology of the sheaf mod $\mathbf{f}$.

Proposition 8.4. *For any* $\mathbf{f}$ *(i.e. omitting the assumption that* $V(\mathbf{f}) = \emptyset$*), we have*

$$\mathrm{H}(\tau_* \kappa_{\mathbf{f}}^* \mathcal{E}) = H^*(\mathrm{H}\kappa_{\mathbf{f}}^* \mathcal{E}).$$

Proof. ♣♣♣ Daniel: [This proof was written before we started worrying about the appropriate twists on E. So it'll need to be fixed. But the rought idea seems to be correct.] Let $C^\bullet$ be the Cech complex for $\mathcal{E}$. Consider the double complex $C^\bullet \otimes_k E$ with horizontal differential given by the Koszul differential $\sum_i f_i \otimes e_i$ and the vertical differential given by the Cech differential. Up to quasisomorphism, the unfolded complex for $\tau_* \kappa_{\mathbf{f}}^* \mathcal{E}$ is the total complex of $C^\bullet \otimes_k E$. Since the homology of this total complex is cyclic, it suffices to the compute the homology in degree 0.

We compute this by running the spectral on the double complex, starting with the horizontal arrows. After taking homology with respect to the horizontal arrows, we get a double complex with a vertical differential where the rows correspond to Cech resolution of the unfolded complex of sheaves

$$\cdots \to \kappa_{\mathbf{f}}^* \mathcal{E} \to \kappa_{\mathbf{f}}^* \mathcal{E} \to \kappa_{\mathbf{f}}^* \mathcal{E} \to \cdots .$$

Taking homology of these Cech resolutions, we get to an E^2-page where $E_2^{i,j} = H^i \mathrm{H} \kappa_{\mathbf{f}}^* \mathcal{E}$. But the differentials must be zero now: if V is the vector space $e_0, \ldots, e_n$, then on page E^k, the differentials would lie in $\wedge^{-k+1} V$. So the spectral sequence collapses at the E^2-page and we get the desired result. □

Example 8.5. Let $\mathbf{f} = (x_0)$ on $\mathbb{P}^1$. The differential will not be exact. Here, E is just the exterior algebra on one variables e_0. We get that $|\tau_* \mathcal{O}_{\mathbb{P}^1}|$ is the direct sum of the modules:

$$\begin{tikzcd}
\cdots \arrow[r] & E(-3)^2 \arrow[r] & E(-2) \arrow[dr] & & & & \\
& & & E \arrow[r] & E(1)^2 \arrow[r] & E(2)^3 \arrow[r] & E(3)^4 \arrow[r] & \cdots
\end{tikzcd}$$

The differential is the direct sum of arrows, which are given by the following matrices (from left to right)

$$\begin{pmatrix} e_0 & 0 \end{pmatrix}, \begin{pmatrix} 0 \end{pmatrix}, \begin{pmatrix} e_0 \\ 0 \end{pmatrix}, \begin{pmatrix} e_0 & 0 \\ 0 & e_0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} e_0 & 0 & 0 \\ 0 & e_0 & 0 \\ 0 & 0 & e_0 \\ 0 & 0 & 0 \end{pmatrix},$$

These compose to zero. Taking homology we get seem to get

$$\begin{tikzcd}
\cdots \arrow[r] & k(-3) \arrow[r] & k(-2) \arrow[dr] & & & & \\
& & & k(-1) \arrow[r] & k \arrow[r] & k(1) \arrow[r] & k(2) \arrow[r] & \cdots
\end{tikzcd}$$

(the arrows don't really mean anything here). Note that these precisely correspond to the cohomology groups of $\mathcal{O}_{\mathbb{P}^1}/(x_0))$, which is a single point.

References

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