Contents

Chapter 6

Hyperelliptic curves and curves of genus 2 and 3

6A. Hyperelliptic curves

Recall that a hyperelliptic curve C is a curve of genus ≥ 2 admitting a map $\pi : C \to \mathbb{P}^1$ of degree 2. We met hyperelliptic curves in Chapter 2 and proved that the canonical map from C is the composition of π with the embedding of $\mathbb{P}^1$ in $\mathbb{P}^{g-1}$ as a rational normal curve, showing in particular that π is unique up to automorphisms of $\mathbb{P}^1$.

We used this to show that every special linear series on a hyperelliptic curve is a sum of a multiple of the unique g^1_2 plus basepoints. We will begin this chapter with an explicit construction of hyperelliptic curves and use it to give a concrete computation of the canonical series, reproving what we did in Chapter 2. Then we will consider the projective embeddings of curves of genus 2 (which are all hyperelliptic) and genus 3.

There will be a further discussion of hyperelliptic curves in Chapter 17.

The equation of a hyperelliptic curve. Because the degree of the canonical map is 2, each point in $\mathbb{P}^1$ has either two distinct preimages, or only one; in the latter case, this point is a ramification point with ramification index 1; that is, the map is given in terms of local analytic coordinates on C and $\mathbb{P}^1$ by $z \mapsto z^2$. In particular, both the ramification divisor and the branch divisor (as defined in Chapter 2) are reduced. By Hurwitz's formula there are exactly $2g+2$ branch points in $\mathbb{P}^1$. These points determine the curve:

Theorem 6.1. *There is a unique smooth projective hyperelliptic curve C expressible as a 2-sheeted cover of $\mathbb{P}^1$ branched over any given set of $2g+2$ distinct points $\{q_1, \dots, q_{2g+2}\}$.*

Proof. We will exhibit such a curve, leaving the proof of uniqueness to Section 6B. If the coordinate of the point $q_i \in \mathbb{P}^1$ is λ_i, we take for C the smooth projective model of the affine curve

$$C^\circ = \left\{(x, y) \in \mathbb{A}^2 \;\middle|\; y^2 = \prod_{i=1}^{2g+2}(x - \lambda_i)\right\}.$$

Note that we're choosing a coordinate x on $\mathbb{P}^1$ with the point $x = \infty$ at infinity not among the q_i, so that the preimage of $\infty \in \mathbb{P}^1$ is two points $r, s \in C$. Concretely, we see that as $x \to \infty$, the ratio y^2/x^{2g+2} approaches 1, so that

$$\lim_{x\to\infty} \frac{y}{x^{g+1}} = \pm 1.$$

The two possible values of this limit correspond to the two points $r, s \in C$. $\square$

The curve C thus constructed is *not* simply the closure of the affine curve $C^\circ \subset \mathbb{A}^2$ in either $\mathbb{P}^2$ or $\mathbb{P}^1 \times \mathbb{P}^1$: as you can see from a direct examination of the equation, each of these closures will be singular at the (unique) point at infinity.

To give a smooth projective model of a hyperelliptic curve C with given branch divisor, we divide the $2g + 2$ branch points into two sets of the same size, $\{q_1, \dots, q_{g+1}\}$ and $\{q_{g+2}, \dots, q_{2g+2}\}$. We can then take C to be the closure in $\mathbb{P}^1 \times \mathbb{P}^1$ of the locus

$$\left\{(x, y) \in \mathbb{A}^2 \;\middle|\; y^2 \prod_{i=1}^{g+1}(x - \lambda_i) = \prod_{i=g+2}^{2g+2}(x - \lambda_i)\right\};$$

in projective coordinates, this is

$$C = \left\{((X_0, X_1), (Y_0, Y_1)) \in \mathbb{P}^1 \times \mathbb{P}^1 \;\middle|\; Y_1^2 \prod_{i=1}^{g+1}(X_1 - \lambda_i X_0) = Y_0^2 \prod_{i=g+2}^{2g+2}(X_1 - \lambda_i X_0)\right\}.$$

To see that $C \subset \mathbb{P}^1 \times \mathbb{P}^1$ is smooth we note that it is a curve of bidegree $(2, g+1)$ in $\mathbb{P}^1 \times \mathbb{P}^1$, and the formula for the genus of a curve in $\mathbb{P}^1 \times \mathbb{P}^1$ derived in Example 2G tells us that such a curve has arithmetic genus g, and thus no singular points.

From this model, we deduce:

Corollary 6.2. *If C is a hyperelliptic curve and $p_1, \dots, p_{2g+2} \in C$ are the ramification points of the unique degree 2 map $C \to \mathbb{P}^1$, then for any division of $\{1, \dots, 2g + 2\}$ into two sets A, B of cardinality $g + 1$,*

$$\sum_{i\in A} p_i \sim \sum_{i\in B} p_i.$$

Proof. The abstract curve $C \subset \mathbb{P}^1 \times \mathbb{P}^1$ above is independent of the choice of A and B, since in any case the projection to the first factor is ramified at the same set $p_1, \ldots, p_{2g+2}$. Given the representation above, the sets $\{p_i \mid i \in A\}$ and $\{p_i \mid i \in B\}$ are preimages of $(0, 1)$ and $(1, 0)$ in the second factor. $\square$

The map $\iota : C \to C$ that exchanges the two points in each reduced fiber of the map $C \to \mathbb{P}^1$ and fixes the ramification points is algebraic: in terms of the last representation of C, it is given by $((X_0, X_1), (Y_0, Y_1)) \mapsto ((X_0, X_1), (Y_0, -Y_1))$. The map ι is called the *hyperelliptic involution* on C.

Differentials on a hyperelliptic curve. We can give a pleasantly concrete description of the differentials, and thus the canonical linear system, on a hyperelliptic curve C by working with the affine model $C^\circ = V(f) \subset \mathbb{A}^2$, where

$$f(x, y) = y^2 - \prod_{i=1}^{2g+2} (x - \lambda_i).$$

We will again denote the two points at infinity (that is, the two points of $C \setminus C^\circ$) by r and s; for convenience, we'll denote the divisor $r + s$ by D. We write $\pi : C \to \mathbb{P}^1$ for the morphism that, on C°, sends $(x, y) \in C$ to x.

We can construct a differential form on C by following the proof of Hurwitz's theorem in Chapter 2. Let dx denote the usual differential on $\mathbb{P}^1$ having a double pole at infinity, and consider $\pi^* dx$ on C. The function x is regular on C°, and is a local parameter over points other than the λ_i; from the local description of the map π, we see that $\pi^* dx$ is regular on C° with simple zeros at the ramification points $q_i = (\lambda_i, 0)$. Since dx has a double pole at the point at $\infty \in \mathbb{P}^1$ and π is a local isomorphism near r and s, the differential $\pi^* dx$ has double poles at the points r and s. Thus the canonical divisor of C is

$$K_C \sim (dx) \sim R - 2D,$$

where R denotes the ramification divisor, in this case the sum of the ramification points.

How can we find differentials that are regular everywhere on C? If we divide dx by x^2 (or any quadratic polynomial in x) to kill the poles we introduce new poles in the finite part C° of C.

Instead, we want to multiply dx by a rational function with zeros at r and s, but whose poles occur only at the points where dx has zeroes — that is, the points λ_i. A natural choice is the reciprocal of the partial derivative $f_y := \partial f / \partial y = 2y$, which vanishes at the points q_i, and has a pole of order $g + 1$ at each of the points r and s (reason: y/x^{g+1} approaches ± 1 as x goes to infinity, and x has a pole of order 1 at $\infty \in \mathbb{P}^1$ and thus also at each of r, s). In other words, as long as $g \geq 1$, the differential

$$\omega = \pi^* \left(\frac{dx}{f_y} \right)$$

is regular, with divisor

$$(\omega) = (g-1)r + (g-1)s = (g-1)D.$$

The remaining regular differentials on C are now easy to find: Since x has only a simple pole at the two points at infinity we can multiply ω by any x^k with $k = 0, 1, \ldots, g-1$. This gives us g differentials

$$\omega, x\omega, \ldots, x^{g-1}\omega$$

that are independent, and so form a basis for $H^0(K_C)$.

With this description of the differentials, we can see clearly why the canonical map of a hyperelliptic curves is degree 2 onto a rational normal curve, as proved in Chapter 2: the relations on $\omega, x\omega, \ldots, x^{g-1}\omega$ are the relations on x^i, and we see that the canonical image is the rational normal curve of degree $g-1$.

6B. Branched covers with specified branching

Given a curve B and points $p_1, \ldots, p_b$ in B, what are the branched covers $\pi : C \to B$ of degree d with specified branching over each of the points p_i, up to isomorphism over B? We will reduce this question to the classification of topological covering spaces of the complement $U = B \setminus \Delta$; we will then use properties of the fundamental group of U to enumerate such covering spaces. We will prove the uniqueness of hyperelliptic curves with specified branch points at the end of this section as a special case of a general analysis of branched covers.

Theorem 6.3. *Let B be a smooth curve, let $\Delta \subset B$ be a finite set of points, and let $U := B \setminus \Delta$. If $\pi^\circ : V \to U$ is a topological covering space then V may be given the structure of a Riemann surface in a unique way so that the map π° is holomorphic; and V may be compactified to a compact Riemann surface C in a unique way such that the map π° extends to a holomorphic map $\pi : C \to B$.*

Proof. The space V inherits the structure of a complex manifold from U because if $D \subset U$ is any simply connected coordinate chart, then the preimage $(\pi^\circ)^{-1}(D)$ is a disjoint union of d copies of D, and we may use them as coordinate charts on V.

To compactify V we observe that if $D^* = \{z \in \mathbb{C} \mid 0 < |z| < 1\}$ is a punctured disc, then the map $z \mapsto z^n$ on the unit disk restricts to a connected n-fold covering space $D^* \to D^*$. Since $\pi_1(D^*) = \mathbb{Z}$, any connected covering space E of degree n is homeomorphic to this one by a homeomorphism inducing the identity on the target of π. If we define a holomorphic structure on E by pulling back the one on D, then this homeomorphism is biholomorphic.

Thus if D_i is a small neighborhood of the point $p_i \in B$ biholomorphic to a disc, then the preimage in V of the punctured disc $D_i^* := D_i \cap U$ is a disjoint union of punctured discs $E_{i,j}^*$. The maps $E_{i,j}^* \to D_i^*$ are homeomorphic to the maps $z \mapsto z^{n_{i,j}}$ of the punctured unit disc for some $n_{i,j}$. Because of the way the holomorphic structure of V is defined, the maps $E_{i,j}^* \to D_i^*$ are actually holomorphic. Thus they extend holomorphically to maps of the full disks $E_{i,j} \to D_i$ and $V \cup \bigcup E_{i,j}$ is a compact Riemann surface in a unique way. □

The problem of classifying smooth curves C that have a map $\pi : C \to B$ of degree d thus becomes one of classifying covering spaces of U.

Branched covers of $\mathbb{P}^1$. We continue with the notation $U = B \setminus \Delta$, now supposing that $B = \mathbb{P}^1_{\mathbb{C}}$, the Riemann sphere. Again, let $\pi : V \to U$ be a covering space.

Choose a basepoint $p_0 \in U$, and draw simple, nonintersecting arcs γ_i joining p_0 to p_i in U. If Σ is the complement of the union of these arcs in the sphere, then the preimage of Σ in V will be the disjoint union of d copies of Σ, called the *sheets* of the cover; label these $\Sigma_1, \dots, \Sigma_d$.

Given U, elementary homotopy theory asserts the existence of a bijection between coverings $V \to U$ of U of degree d (up to homeomorphisms of V fixing U) and group homomorphisms

$$M : \pi_1(U, p_0) \to S_d,$$

to the symmetric group on d letters, up to inner automorphisms of S_d. The map M is called the *monodromy* of the covering: given $V \to U$ and a labeling of the d sheets of V over the point p_0, the value of M at a loop β in U based at p_0 is the permutation of the points of $\pi^{-1}(p_0)$ given by sending a point $q \in \pi^{-1}(p_0)$ to the endpoint of the unique lift of β starting at q. A permutation σ of the labels of the sheets leads to a map M' equal to the composition of M with conjugation by σ.

A convenient set of generators of $\pi_1(U, p_0)$ is the set of paths β_i indicated in Figure 6.1: starting at p_0, going out along the arc γ_i until just short of p_i, going once around p_i and then going back to p_0 along the same path γ_i. The fundamental group of U is the free group generated by the paths $\beta_1, \dots \beta_b$ modulo the relation $\prod_{i=1}^{b} \beta_i = 1$ which comes from the fact that the sphere minus the part enclosed by the paths β_i is contractible.

Given a degree d covering space V and a labeling of the d sheets over the point p_0, let τ_i be the permutation of $\{1, 2, \dots, d\}$ corresponding to the path β_i.

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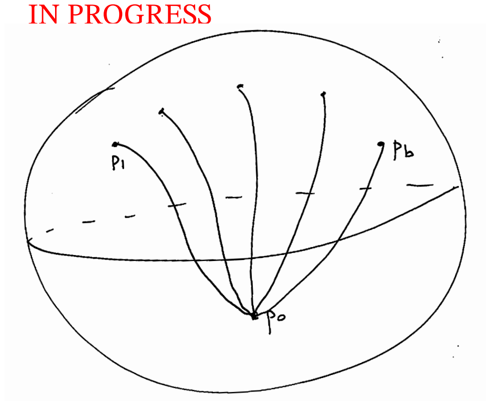

Figure 6.1. Generators for the fundamental group of a multiply punctured sphere

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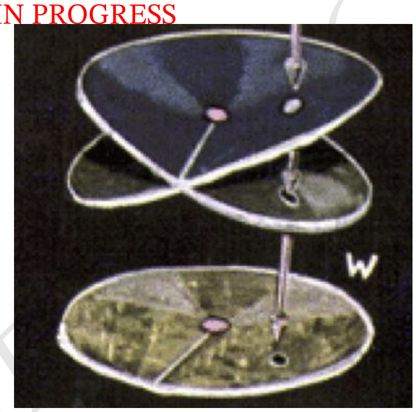

Figure 6.2. Local picture of a simple branch point $z \mapsto z^2$. [Drawing by George Francis to be replaced after discussion]

The space V is connected if and only if the τ_i generate a transitive subgroup of S_d. The case $d = 2$ is illustrated in Figure 6.2.

If we start from a map of Riemann surfaces $C \to \mathbb{P}^1$ that is simply branched, and take V to be the complement of the set of branch points, then each τ_i is a transposition.

Summarizing we have proven:

Lemma 6.4. *Let $p_1, \dots, p_b \in \mathbb{P}^1$ be any b distinct points. There is a natural bijection between*

(1) *the set of simply branched covers $\pi : C \to \mathbb{P}^1$ of degree d, branched over the points p_i, up to isomorphism over $\mathbb{P}^1$; and*
(2) *the set of b-tuples of transpositions $\tau_1, \dots, \tau_b \in S_d$ satisfying the conditions:*

(a) $\prod \tau_i$ *is the identity; and*
(b) $\tau_1, \ldots, \tau_b$ *generate a transitive subgroup of* S_d,

modulo simultaneous conjugation by S_d. □

Proof of the uniqueness statement in Theorem 6.1. In the case of double covers of $\mathbb{P}^1$ that is relevant to hyperelliptic curves, we note that there is only one transposition in S_2. Thus there is a unique double cover of $\mathbb{P}^1$ with given branch points $p_1, \ldots, p_b$. (The product condition shows again that the number of branch points must be even.) This completes the proof of Theorem 6.1. □

Example 6.5. In contrast to the situation of double covers of $\mathbb{P}^1$, there are generally many branched covers of specified degree greater than 2 or with given branch points and given conjugacy classes of the local monodromy. The number of these is called the *Hurwitz number* of the configuration, and its computation in general is the subject of a large and active literature; see for example [?].

To illustrate this, we can use Lemma 6.4 to count the number of degree 3 branched covers $C \to \mathbb{P}^1$ with given simple branch points, using that fact that every odd permutation $\tau \in S_3$ is a transposition. Thus if b is even and $\tau_1, \ldots, \tau_{b-1} \in S_3$ are arbitrary transpositions, then the product $\tau_1 \cdots \cdots \tau_{b-1}$ is also a transposition. It follows that the number of ordered b-tuples of transpositions $\tau_1, \ldots, \tau_b \in S_3$ with $\prod \tau_i$ equal to the identity is 3^{b-1}. The requirement that the group generated by the τ_i is transitive eliminates just the three cases where all the τ_i are equal. The group S_3 acts on the set of b-tuples of permutations without stabilizing any b-tuple, so every cover corresponds to exactly 6 sequences $\tau_1, \ldots, \tau_b$. In sum, the number of simply branched 3-sheeted covers of $\mathbb{P}^1$ with specified branch points $q_1, \ldots, q_b \in \mathbb{P}^1$ is

$$\frac{3^{b-1} - 3}{6} = \frac{3^{b-2} - 1}{2}.$$

One can use a similar strategy to count covers in other cases, when the target has higher genus and/or the degree of the covering is larger, but the combinatorics becomes more complicated.

6C. Curves of genus 2

Since curves of genus 2 are hyperelliptic, everything we said above applies to them; in particular, the canonical map $\phi_K : C \to \mathbb{P}^1$ on a curve of genus 2 is the expression of C as a double cover of $\mathbb{P}^1$, simply branched over 6 points in $\mathbb{P}^1$, which are unique up to automorphisms of $\mathbb{P}^1$.

In this section, we'll consider other maps from hyperelliptic curves C to projective space, starting with maps $C \to \mathbb{P}^1$. See for example [?] for a treatment of certain embeddings of hyperelliptic curves of all genera.

Maps of C to $\mathbb{P}^1$. The curve C has a unique degree 2 morphism to $\mathbb{P}^1$ associated to the canonical system $|K_C|$. But there are many other morphisms to $\mathbb{P}^1$. For example, there is a 2-parameter family of maps of degree 3:

Let $\mathcal{L}$ be an invertible sheaf of degree 3 on C. Since $3 > 2g-2$, the Riemann–Roch theorem tells us immediately that $h^0(\mathcal{L}) = 2$, and there are two possibilities:

(1) If the linear system $|\mathcal{L}|$ has a basepoint $p \in C$, then $h^0(\mathcal{L}(-p)) = 2$, and hence $\mathcal{L}$ must be of the form $\mathcal{L} = K_C(p)$. Conversely, if $\mathcal{L} = K_C(p)$, then $h^0(\mathcal{L}(-p)) = h^0(\mathcal{L})$, which is to say p is a basepoint of $|\mathcal{L}|$. There is a 1-parameter family of such $\mathcal{L}$.

(2) If $\mathcal{L}$ is not of the form $\mathcal{L} = K_C(p)$, then $|\mathcal{L}|$ does not have a basepoint, and so defines a degree 3 map $\phi_{\mathcal{L}} : C \to \mathbb{P}^1$.

Since the variety $\mathrm{Pic}_3(C)$ has dimension $g = 2$ the general invertible sheaf of degree 3 is of the second kind, and this gives a 2-parameter family of such maps.

There are plenty of higher-degree maps as well: an invertible sheaf of degree $d \geq 4 = 2g$ is basepoint free, and gives a map to $\mathbb{P}^{d-2}$, from which we can project in many ways to $\mathbb{P}^1$.

Maps of C to $\mathbb{P}^2$. Next consider maps of a curve C of genus 2 to the plane. By the Riemann–Roch theorem, an invertible sheaf $\mathcal{L}$ of degree 4 on C has $h^0(\mathcal{L}) = 3$ and is basepoint free by Corollary 2.19, so the linear system $|\mathcal{L}|$ gives a morphism $\phi_{\mathcal{L}} : C \to \mathbb{P}^2$. The invertible sheaf $\mathcal{L} \otimes \omega_C^{-1}$ is either ω_C or nonspecial; in either case, by the Riemann–Roch theorem, it has at least one section, so we may write $\mathcal{L} \otimes \omega_C^{-1} = \mathcal{O}_C(p+q)$ for some points p, q. There are two possibilities:

(1) If $p + q = K_C$, then $\mathcal{L} = \omega_C^2$. Since the elements of $H^0(\omega_C)$ may be written as $\omega, x\omega$, the map

$$\mathrm{Sym}^2 H^0(\omega_C) \to H^0(\mathcal{L})$$

is injective, and since both sides are 3-dimensional vector spaces, they are equal. In other words, every divisor $D \sim 2K_C$ is the sum of two divisors $D_1, D_2 \in |K_C|$. We conclude that the map $\phi_{\mathcal{L}}$ is the composition of the canonical map $\phi_K : C \to \mathbb{P}^1$ with the Veronese embedding $\nu_2 : \mathbb{P}^1 \to \mathbb{P}^2$ of $\mathbb{P}^1$ as a conic in the plane and the map $\phi_{\mathcal{L}}$ is generically 2-to-1 onto the conic.

(2) If $p + q \neq K_C$ then $h^0(p+q) = 1$, so the pair p, q is unique. Furthermore, $h^0(\mathcal{L} - p) = 2 = h^0(\mathcal{L}(-p-q))$ so $H^0(\mathcal{L}(-p)) = H^0(\mathcal{L}(-q))$ and $\phi_{\mathcal{L}}(p) = \phi_{\mathcal{L}}(q)$. By the genus formula, the δ invariant of this point must be 1. By Exercise 2-11 this is a node (if $p \neq q$) or cusp (if $p = q$).

Thus for $\mathcal{L}$ in an open subset of $\mathrm{Pic}_4(C)$ the image is a quartic with a node; for a one-dimensional locus in $\mathrm{Pic}_4(C)$, the image is a quartic with a cusp; and for one point in $\mathrm{Pic}_4(C)$ the image is a conic.

Embeddings in $\mathbb{P}^3$. By Corollary 2.19 any invertible sheaf $\mathcal{L}$ of degree 5 is very ample. Write $\phi_{\mathcal{L}} : C \to \mathbb{P}^3$ for the map given by the complete linear system $|L|$. Since $\phi_{\mathcal{L}}$ is an embedding, we'll also denote the image $\phi_{\mathcal{L}}(C) \subset \mathbb{P}^3$ by C and write $\mathcal{O}_C(1)$ for $\mathcal{L}$.

What degree surfaces in $\mathbb{P}^3$ contain the curve C? We start with degree 2, and consider the restriction map

$$H^0(\mathcal{O}_{\mathbb{P}^3}(2)) \to H^0(\mathcal{O}_C(2)) = H^0(\mathcal{L}^2).$$

The space on the left has dimension 10; by the Riemann–Roch theorem we have $h^0(\mathcal{L}^2) = 2 \cdot 5 - 2 + 1 = 9$. It follows that C lies on a quadric surface Q. Since C is not contained in a plane or a union of planes, any quadric containing C is irreducible; if there were more than one such, Bézout's theorem would imply that $\deg C \le 4$. Thus Q is unique.

We might ask at this point: is Q smooth or a quadric cone? The answer depends on the choice of invertible sheaf $\mathcal{L}$.

Proposition 6.6. *Let $C \subset \mathbb{P}^3$ be a smooth curve of degree 5 and genus 2, and set $\mathcal{L} = \mathcal{O}_C(1)$. The unique quadric Q containing C is singular if and only if*

$$\mathcal{L} \cong K^2(p)$$

for some point $p \in C$; in this case, the point p maps to the vertex of Q.

Proof. Suppose first that $\mathcal{L} \cong K^2(p)$ for some $p \in C$. Then $\mathcal{L}(-p) \cong K^2$, so that the map $\pi : C \to \mathbb{P}^2$ given by projection from p is the map $\phi_{K^2} : C \to \mathbb{P}^2$ given by the square of the canonical sheaf. As we have seen, the map ϕ_{K^2} is two-to-one onto a conic $E \subset \mathbb{P}^2$, so that the curve C lies on the cone Q over E with vertex p, and this is the unique quadric surface containing C.

On the other hand, if $\mathcal{L}$ is not of the form $K^2(p)$, then we can write

$$\mathcal{L} = K \otimes \mathcal{M},$$

where by hypothesis $\mathcal{M}$ is not of the form $K(p)$. We are in case (2) on page 118; that is, the pencil $|\mathcal{M}|$ gives a degree 3 map $C \to \mathbb{P}^1$.

This gives us a way of factoring the map $\phi_{\mathcal{L}} : C \to \mathbb{P}^3$: we have maps $\phi_K : C \to \mathbb{P}^1$ of degree 2 and $\phi_{\mathcal{M}} : C \to \mathbb{P}^1$ of degree 3, and we can compose their product with the Segre embedding $\sigma : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^3$:

$$C \xrightarrow{\phi_K \times \phi_{\mathcal{M}}} \mathbb{P}^1 \times \mathbb{P}^1 \xrightarrow{\sigma} \mathbb{P}^3.$$

This description of the map $\phi_{\mathcal{L}}$ shows that C *is a curve of type* $(2,3)$ *on a smooth quadric* $Q \subset \mathbb{P}^3$, completing the proof of Proposition 6.6. $\square$

The variety $\mathrm{Pic}_5(C)$ has dimension 2, while the sheaves $K^2(p)$ form a one-dimensional subfamily. Thus for a general invertible sheaf $\mathcal{L} \in \mathrm{Pic}_5(C)$ the unique quadric Q containing $\phi_{\mathcal{L}}(C)$ is smooth.

The ideal of a quintic space curve of genus 2. Continuing the discussion above, let $C \subset \mathbb{P}^3$ be a smooth quintic curve of genus 2. To describe a minimal set of generators of the homogeneous ideal $I(C) \subset \mathbb{C}[x_0, x_1, x_2, x_3]$ we look at the restriction map

$$H^0(\mathcal{O}_{\mathbb{P}^3}(3)) \to H^0(\mathcal{O}_C(3)).$$

Since the dimensions of these spaces are 20 and $15 - 2 + 1 = 14$ respectively, we see that the vector space of cubics vanishing on C has dimension at least 6. The subspace of cubics divisible by Q has dimension 4. It follows that there are at least two cubics vanishing on C that are linearly independent modulo those vanishing on Q.

We can identify these cubics geometrically. Suppose first that Q is smooth, so that C is a curve of type $(2, 3)$ on Q. In that case, if $L \subset Q$ is any line of the first ruling, the sum $C + L$ is the complete intersection of Q with a cubic S_L, unique modulo the ideal of Q; conversely, if S is any cubic containing C but not containing Q, the intersection $S \cap Q$ will be the union of C and a line L of the first ruling; thus $S = S_L$ modulo $I(Q)$. A similar argument applies in case Q is a cone, and L is any line of the (unique) ruling of Q. In Exercise 6-6 you may show that there are no more cubics containing C.

The dimension of the family of genus 2 curves. Each of the types of maps that we described from a curve C of genus 2 to projective space suggests a way to compute the dimension of the family of genus 2 curves, and indeed, as we will explain in Chapter 8, there is a moduli space of this dimension.

First, every curve of genus 2 is uniquely expressible as a double cover of $\mathbb{P}^1$ branched at six points, modulo the group PGL_2 of automorphisms of $\mathbb{P}^1$. The space of such double covers has dimension 6, and $\dim \mathrm{PGL}_2 = 3$, and since the group acts with finite stabilizers this gives a family of dimension $6 - 3 = 3$.

Also, each curve of genus 2 is expressible as a 3-sheeted cover of $\mathbb{P}^1$ (with eight branch points) in a 2-dimensional family of ways. As we saw in Section 6B, such a triple cover is determined up to a finite number of choices by its branch divisor, so the space of such triple covers has dimension 8; modulo PGL_2 it has dimension 5, and since every curve is expressible as a triple cover in a two-dimensional family of ways, we arrive again at a family of dimension $5 - 2 = 3$.

We've also seen that each curve of genus 2 can be realized as (the normalization of) a plane quartic with a node in a 2-dimensional family of ways. The space of plane quartics has dimension 14; the family of those with a node has codimension one (Proposition 8.14) and hence dimension 13. Since the automorphism group PGL_3 of $\mathbb{P}^2$ has dimension 8, this suggests that the family of nodal plane quartics modulo PGL_3 has dimension 5. Finally, since every curve of genus 2 corresponds to a 2-parameter family of such curves, this again suggests a family of dimension $5 - 2 = 3$.

Finally, a curve of genus 2 may be realized as a quintic curve in $\mathbb{P}^3$ in a two-parameter family of ways. To count the dimension of the family of such curves, note that each one lies on a unique quadric Q. We can assume for this purpose that Q is smooth, since the singular quadrics and curves on them occur in codimension 1. The curve C is of type $(2, 3)$ on Q. Thus to specify such a curve we have to specify Q (9 parameters) and then a bihomogeneous polynomial of bidegree $(2, 3)$ on $Q \cong \mathbb{P}^1 \times \mathbb{P}^1$ up to scalars; these have $3 \cdot 4 - 1 = 11$ parameters. Thus there is a 20-dimensional family of such divisors; modulo the automorphism group PGL_4 of $\mathbb{P}^3$, this is a 5-dimensional family. Again, every abstract curve C of genus 2 corresponds to a 2-parameter family of these curves modulo PGL_4, so once more this suggests a family of dimension $5 - 2 = 3$.

6D. Curves of genus 3

Let C be a smooth projective curve of genus 3. Since we have already discussed hyperelliptic curves, we will assume that C is not hyperelliptic. By Theorem 2.27, the canonical map $\phi_K : C \to \mathbb{P}^2$ embeds C as a smooth plane quartic curve. Conversely, by Proposition 2.8 any smooth plane curve of degree 4 has genus 3 and is embedded by the complete canonical series.

Since the space of plane quartic curves is 14-dimensional and $\mathrm{PGL}(3)$ has dimension 8, this suggests that there is a 6-dimensional family of curves of genus 3, and in Chapter 8 we will see that this is indeed the case.

Other representations of a curve of genus 3. Since we have assumed that C is not hyperelliptic there is no degree 2 cover of $\mathbb{P}^1$. On the other hand, there are degree 3 covers: if $\mathcal{L} \in \mathrm{Pic}_3(C)$ is an invertible sheaf of degree 3 then, by the Riemann–Roch theorem, we have

$$h^0(\mathcal{L}) = \begin{cases} 2 & \text{if } \mathcal{L} \cong K - p \text{ for some point } p \in C, \\ 1 & \text{otherwise.} \end{cases}$$

There is thus a 1-dimensional family of representations of C as a 3-sheeted cover of $\mathbb{P}^1$. These are visible directly from the canonical model: a degree 3 map $\phi_{K-p} : C \to \mathbb{P}^1$ is the composition of the canonical embedding $\phi_K : C \to \mathbb{P}^2$ with a projection from p, as illustrated in Figure 6.3.

There are other representations of C as the normalization of a plane curve. By Clifford's theorem C has no g^2_3, and the canonical system is the only g^2_4, but there are plenty of models as plane quintic curves: by Proposition 1.11, if $\mathcal{L}$ is any invertible sheaf of degree 5, the linear system $|\mathcal{L}|$ will be a basepoint free g^2_5 as long as L is not of the form $K + p$, so that $\phi_{\mathcal{L}}$ maps C birationally onto a plane

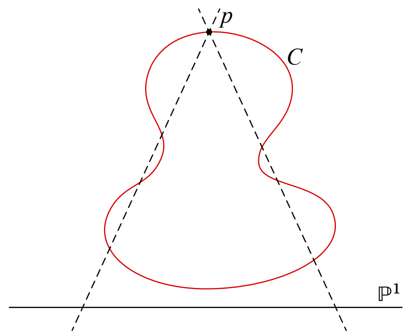

Figure 6.3. Expression of a plane quartic C of genus 3 as a 3-sheeted cover of $\mathbb{P}^1$ by projecting the canonical model from a point on it.

quintic curve $C_0 \subset \mathbb{P}^2$. These can also be described geometrically in terms of the canonical model: any such invertible sheaf $\mathcal{L}$ is of the form $2K - p - q - r$ for some trio of points $p, q, r \in C$ that are not collinear in the canonical model, and we see that C_0 is obtained from the canonical model of C by applying a Cremona transform with respect to the points p, q and r, that is, by applying the birational transformation of the plane defined by the linear series of conics through p, q, r.

Proposition 1.11 implies that a divisor D of degree 6 is very ample if and only if it is not of the form $K + p + q$ for any $p, q \in C$ and since the family of invertible sheaves on C has dimension 3, we see that a general invertible sheaf of degree 6 is very ample (indeed, this is a simple case of Theorem 5.13).

If $C \subset \mathbb{P}^3$ is a curve of genus 3 embedded as a curve of degree 6, then C cannot lie on a singular quadric since by Example 2G it would have to be a complete intersection of the quadric with a cubic, and then such a curve has genus 4. If C lies on a smooth quadric in class (a, b) then a or b would be 2, so C would be hyperelliptic, and conversely any curve in class $(2, 4)$ is a hyperelliptic curve of genus 3, degree 6.

Thus if C is not hyperelliptic, then C does not lie on a quadric surface. We have $h^0(\mathcal{O}_{\mathbb{P}^3}(3)) = 20$ while, by the Riemann–Roch formula, $h^0(\mathcal{O}_C(3)) = 18 - 3 + 1 = 16$, so C lies on (at least) 4 independent cubics. Each of these cubics must be irreducible, so any two of them intersect in a curve of degree 9 containing C and another component or components D of degree totaling 3. By Bertini's theorem if we choose two *general* cubics containing C, then each of the components of D will be smooth. We shall see in Theorem 16.1 that the arithmetic genus of D must be 0; thus D must be a twisted cubic curve. The ideal of the twisted cubic is generated by the 2×2 minors of a matrix of the form

$$\begin{pmatrix} \ell_0 & \ell_1 & \ell_2 \\ \ell_1 & \ell_2 & \ell_3 \end{pmatrix}$$

where the ℓ_i are linear forms, and it follows that the two cubics can be written as the two 3×3 minors involving the first two rows of a matrix of the form

$$\begin{pmatrix} \ell_0 & \ell_1 & \ell_2 \\ \ell_1 & \ell_2 & \ell_3 \\ \ell_4 & \ell_5 & \ell_6 \\ \ell_7 & \ell_8 & \ell_9 \end{pmatrix}$$

where $\ell_4, \ldots, \ell_9$ are linear forms as well. From the Hilbert–Burch theorem (Corollary 18.12) one can show that the ideal of C is generated by the four 3×3 minors of this matrix, whose columns generate the syzygies of the ideal of the curve.

6E. Theta characteristics

In this section we sketch the algebraic theory of theta characteristics, starting with the case of curves of genus 3.

Suppose that $C \subset \mathbb{P}^2$ is a smooth plane curve. A *bitangent* to C is a line $L \subset \mathbb{P}^2$ that is either tangent to C at two distinct points, or has contact of order ≥ 4 with C at a point. Alternatively, we can say that a bitangent corresponds to an effective divisor of degree 2 on C such that $2D$ is contained in the intersection of C with a line $L \subset \mathbb{P}^2$.

A naive dimension count suggests that a smooth plane curve should have a finite number of bitangents (it's one condition on a line $L \in (\mathbb{P}^2)^*$ to be tangent to C, so it should be two conditions for it to be bitangent). Indeed, this is the case; by Bézout's theorem a conic or cubic curve cannot have any bitangents, but as we will show in Section 13A3 every smooth curve of degree $d \geq 4$ has

$$12\binom{d+1}{4} - 4d(d-2),$$

counted with appropriate multiplicities — for example, a line simply tangent to C at 3 points counts as three bitangents. Accordingly, a smooth plane quartic has 28 bitangents (see the drawing by Plücker in Figure 20.5, which shows a special case in which the 28 bitangents are all realized over $\mathbb{R}$).

The bitangents to a smooth plane quartic C (a canonical curve of genus 3) have a special significance: since $4 = 2 \times 2$, if $D = p + q$ is a bitangent, then the divisor $2D$ comprises the complete intersection of C with a line; in other words, we have a linear equivalence

$$2D \sim K_C$$

or equivalently the invertible sheaf $\mathcal{O}_C(D)$ is a square root of the canonical sheaf of C. Because of their appearance in the theory of theta functions, Riemann named the square roots of the canonical sheaf *theta characteristics*.

How many such square roots are there? If $\mathcal{L}$ and $\mathcal{M}$ are invertible sheaves with $\mathcal{L}^2 = \mathcal{M}^2 = K$, then $\mathcal{L}$ and $\mathcal{M}$ differ by an invertible sheaf of order 2; that is,

$$\mathcal{M} = \mathcal{L} \otimes \mathcal{F}, \quad \text{where} \quad \mathcal{F} \otimes \mathcal{F} \sim \mathcal{O}_C.$$

In other words, $\mathcal{F}$ is an invertible sheaf of degree 0 and, having fixed $\mathcal{L}$, the other sheaf, $\mathcal{M}$, corresponds to a point of order 2 in the Picard group $\mathrm{Pic}_0(C)$. Since we've seen that $\mathrm{Pic}_0(C) = \mathrm{Jac}(C)$ is a complex torus of dimension $g = 3$ — the quotient of $\mathbb{C}^3$ by a lattice $\Lambda \cong \mathbb{Z}^6$ — we see that there are $2^6 = 64$ such invertible sheaves, and thus, given that there is some invertible sheaf $\mathcal{L}$ satisfying $\mathcal{L}^2 \cong K_C$, there are exactly $64 = 2^{2g}$ of them.

The reader will have noticed that the number 64 of theta characteristics does not agree with the number 28 of bitangents. The reason is that bitangents correspond to *effective* divisors D with $2D \sim K$, while a theta characteristic $\mathcal{L}$ may have $h^0(\mathcal{L}) = 0$, that is, may not correspond to an effective divisor. This situation also occurs in other genera. What can we say about the dimensions $h^0(\mathcal{L})$ of the space of sections of the theta characteristics on C?

There is a beautiful partial answer to this question, which can be deduced from a remarkable fact: the dimension $h^0(\mathcal{L})$ of the space of sections of a theta characteristic mod 2 is invariant in families. We will now sketch the necessary results; see [?] and [?] for a full treatment.

Theorem 6.7. *Let $\mathcal{C} \to B$ be a family of smooth curves, and $\mathcal{L}_b$ a family of theta characteristics on the curves in this family — in other words, an invertible sheaf $\mathcal{L}$ on $\mathcal{C}$ such that $(\mathcal{L}|_{C_b})^2 \cong K_{C_b}$ for each $b \in B$. If $f : B \to \mathbb{Z}/2$ is defined by*

$$f(b) = h^0(\mathcal{L}|_{C_b}) \pmod 2,$$

then f is locally constant.

We say that a theta characteristic $\mathcal{L}$ is *even* or *odd* according to the parity of $h^0(\mathcal{L})$. Given the irreducibility of the space of smooth irreducible curves of genus g (which we'll discuss in Chapter 8), Theorem 6.7 suggests that all curves of genus g have the same number of even (equivalently, of odd) theta characteristics, and this is in fact the case.

Theorem 6.8. *If C is a curve of genus g, then of the 2^{2g} theta characteristics on C there are $2^{g-1}(2^g + 1)$ even theta characteristics and $2^{g-1}(2^g - 1)$ odd theta characteristics.*

Using Theorem 6.7 and the connectedness of the moduli space of curves, Theorem 6.8 is reduced to the case when C is hyperelliptic. We will compute the number of theta characteristics in the hyperelliptic case later in this section (page 126).

For a nonhyperelliptic curve C of genus 3, the dimension $h^0(\mathcal{L})$ of a theta

replacing "Note that in the case of"

characteristic $\mathcal{L}$ cannot be ≥ 2, so the odd theta characteristics are exactly the effective theta characteristics, and this says that there are $2^{g-1}(2^g - 1) = 28$ effective theta characteristics corresponding to the 28 bitangents.

We will present a proof of Theorem 6.7 using an ingenious construction of Mumford's, after explaining the necessary facts about quadratic forms in an even number of variables.

Cheerful Fact 6.9. Suppose that V is a $2n$-dimensional complex vector space with a nondegenerate bilinear form Q. An *isotropic subspace* for Q is a subspace $\Lambda \subset V$ such that $Q(\Lambda, \Lambda) = 0$.

(1) The maximal isotropic subspaces for Q have dimension n.
(2) The set of maximal isotropic subspaces for Q is a subvariety of the Grassmannian $G(n, V)$, of dimension $\binom{n}{2}$, that has exactly two connected components.
(3) If $\Lambda, \Lambda' \subset V$ are any two maximal isotropic subspaces, then

$$\dim(\Lambda \cap \Lambda') \equiv n \pmod 2 \iff \Lambda, \Lambda' \text{ belong to the same ruling.}$$

A proof is given in [?, pp. 735–740].

Remark 6.10. The first assertion in Cheerful Fact 6.9 is elementary: since the map $\widetilde{Q} : V \xrightarrow{\cong} V^*$ associated to the form Q carries an isotropic subspace to its annihilator, there can't be an isotropic subspace of dimension $> n$; and similarly if $\Lambda \subset V$ is any isotropic subspace of dimension $< n$ we can include Λ in a larger isotropic subspace by adding any vector v with $\overline{Q}(v, v) = 0$ for the induced bilinear form $\overline{Q}$ on $\operatorname{ann}(\Lambda)/\Lambda$.

The second and third assertions are less elementary, but the reader may already have seen the first two nontrivial cases of each:

Example 6.11. When $n = 2$ the form Q corresponds to a smooth quadric surface in $\mathbb{P}^3$, and the lines on this surface correspond to the isotropic 2-planes in $\mathbb{C}^4$. There are two rulings by lines, and lines of opposite rulings meet in a point, while lines of the same ruling are either disjoint or equal.

Example 6.12. When $n = 3$, the Grassmannian $\mathbb{G}(1, 3)$, in its Plücker embedding, is a smooth quadric in $\mathbb{P}^5$. The isotropic subspaces in the two distinct components are easy to describe: in one component they are the projective 2-plane of lines containing a given point $p \in \mathbb{P}^3$. In the other component they are the planes corresponding to the lines contained in a given plane $H \subset \mathbb{P}^3$. These families visibly satisfy property (3) above. See Exercise 6-9.

Proof of Theorem 6.7. Suppose that C is a smooth curve of genus g, and let $\mathcal{L}$ be an invertible sheaf on C with $\mathcal{L}^2 \cong K_C$ — that is, a theta characteristic. Choose a divisor $D = p_1 + \cdots + p_n$ of degree $n > g - 1$ consisting of distinct points, and let V be the $2n$-dimensional vector space

$$V := H^0(\mathcal{L}(D)/\mathcal{L}(-D)).$$

From the exact sequence

$$0 \to \mathcal{L}(-D) \to \mathcal{L}(D) \to \mathcal{L}(D)/\mathcal{L}(-D) \to 0$$

we see that the sheaf $\mathcal{L}(D)/\mathcal{L}(-D)$ is supported on D, with stalk isomorphic to $\mathcal{O}_p/\mathfrak{m}_{C,p}^2$ of dimension 2 at each $p \in D$. We can define a bilinear form on V by setting

$$Q(\sigma, \tau) := \sum_i \operatorname{Res}_{p_i}(\sigma\tau)$$

where we use the isomorphism $\mathcal{L}^2 \cong K_C$ to identify the product $\sigma\tau$ with a rational differential.

We now introduce two isotropic subspaces for Q. The first is

$$\Lambda := H^0(\mathcal{L}/\mathcal{L}(-D)),$$

which is isotropic because the product of two of its elements corresponds to a regular differential, and so has no residues. Second, we set

$$\Lambda' := \operatorname{im}\big(H^0(\mathcal{L}(D)) \to H^0(\mathcal{L}(D)/\mathcal{L}(-D))\big).$$

Since $H^0(\mathcal{L}(-D)) = 0$, the map is injective and according to the Riemann–Roch theorem we have $h^0(\mathcal{L}(D)) = n$, so this is again an n-dimensional subspace of V; it's isotropic because the sum of the residues of a global rational differential on C is 0. Finally,

$$H^0(\mathcal{L}) \cong \Lambda \cap \Lambda',$$

and Theorem 6.7 follows. □

Counting theta characteristics (proof of Theorem 6.8). One way to count the number of odd and even theta characteristics on a curve of genus g is to describe them explicitly in the case of a hyperelliptic curve and use Theorem 6.7 to deduce the corresponding statements for any smooth curve of genus g. The reader may wish to try a relatively simple case in Exercise 6-7 before looking at the general case below. We start with some preliminary calculations:

Lemma 6.13. *For any positive integer n, we have*

(1) $$\sum_{k=0}^{n}\binom{2n}{2k} = \sum_{k=0}^{n-1}\binom{2n}{2k+1} = 2^{2n-1},$$

(2) $$\sum_{k=0}^{n}\binom{4n}{4k} = 2^{4n-2} + (-1)^n 2^{2n-1}, \quad \sum_{k=0}^{n-1}\binom{4n}{4k+2} = 2^{4n-2} - (-1)^n 2^{2n-1},$$

(3) $$\sum_{k=0}^{n}\binom{4n+2}{4k+1} = 2^{4n} + (-1)^n 2^{2n}, \quad \sum_{k=0}^{n-1}\binom{4n}{4k+3} = 2^{4n} - (-1)^n 2^{2n}.$$

Proof. Equality (1) is elementary; by the binomial theorem, we have

$$2^{2n} = (1+1)^{2n} = \sum_{l=0}^{2n}\binom{2n}{l} \quad \text{and} \quad 0 = (1-1)^{2n} = \sum_{l=0}^{2n}(-1)^l\binom{2n}{l},$$

and taking the sum and the difference of these two equations yields theresult.

The equalities in (2) follow similarly by applying the binomial theorem to the expression $(1+i)^{4n} = (-1)^n 2^{2n}$. Equating the real parts, we have

$$\sum_{k=0}^{n}\binom{4n}{4k} - \sum_{k=0}^{n-1}\binom{4n}{4k+2} = (-1)^n 2^{2n},$$

while by (1) we have

$$\sum_{k=0}^{n}\binom{4n}{4k} + \sum_{k=0}^{n-1}\binom{4n}{4k+2} = 2^{4n-1}.$$

Taking the sum and difference of these equations yields the desired formulas.

For (3) we apply the binomial theorem to the expression $(1+i)^{4n+2} = (-1)^n 2^{2n+1} i$. Equating the imaginary parts, this gives

$$\sum_{k=0}^{n}\binom{4n+2}{4k+1} - \sum_{k=0}^{n-1}\binom{4n+2}{4k+3} = (-1)^n 2^{2n+1},$$

whereas by (1),

$$\sum_{k=0}^{n}\binom{4n+2}{4k+1} + \sum_{k=0}^{n-1}\binom{4n+2}{4k+3} = (-1)^n 2^{4n+1},$$

and as before taking the sum and difference yields the result. □

We will count the number of theta characteristics on a hyperelliptic curve in terms of sums of subsets of the ramification points, so we need to know what linear equivalences exist among sums of these subsets:

Lemma 6.14. *Let C be the hyperelliptic curve of genus g expressed as a 2-sheeted cover of $\mathbb{P}^1$ with ramification points $p_1, \dots, p_{2g+2}$. The divisor class of any half of the ramification points is equal to the divisor class of the other half, but there are no smaller relations. More precisely, let I_1, I_2 be subsets of $\{1, \dots 2g+2\}$ and set*

$$D_i = \sum_{j \in I_i} p_j.$$

The divisors D_1, D_2 are linearly equivalent if and only if $I_1 = I_2$ or they have the same cardinality $g+1$ and $I_1 \cup I_2 = \{1, \dots, 2g+2\}$.

Proof. The "if" part is simply Corollary 6.2 above.

For the "only if" part, subtracting whatever points D_1 and D_2 have in common we may suppose that $I_1 \cap I_2 = \emptyset$. If $D_1 \sim D_2$, it follows at once that they have the same degree, $d \leq g+1$, and we must show that either $d = 0$ or $d = g+1$.

We have $D_1 \sim D_2$ if and only if $D_1 + D_2 \equiv 2D_1$. If $d \leq g$ we have $r(2D_1) = d$: for $d < g$ this is the extremal case of Clifford's theorem, while for $d = g$ this follows simply from the Riemann–Roch formula. Thus in case $d \leq g$ every divisor in $|2D_1|$ is a sum of d fibers of the 2 to 1 map of C to $\mathbb{P}^1$, and for such a divisor to be a sum of distinct points p_i the degree d must be 0, concluding the argument. ☐

Returning now to the counting, let C be the hyperelliptic curve of genus g, expressed as a 2-sheeted cover of $\mathbb{P}^1$, with ramification points $p_1, \dots, p_{2g+2}$.

First of all, if we denote the class of the unique g^1_2 on C by E, and D is any theta characteristic, then $D + E$ will be effective, and so we can write

$$D \sim mE + F$$

with $-1 \leq m \leq (g-1)/2$ and F the sum of $g - 1 - 2m$ distinct points p_i. This representation is unique unless $m = -1$; in that case, we note that the sum of $g+1$ of the branch points of C is linearly equivalent to the sum of the other $g+1$ by Corollary 6.2. Thus the total number of theta characteristics is a sum of binomial coefficients; if g is odd, it is

$$\binom{2g+2}{0} + \binom{2g+2}{2} + \binom{2g+2}{4} + \cdots + \binom{2g+2}{g-1} + \frac{1}{2}\binom{2g+2}{g+1}$$

and similarly if g is even it is

$$\binom{2g+2}{1} + \binom{2g+2}{3} + \binom{2g+2}{5} + \cdots + \binom{2g+2}{g-1} + \frac{1}{2}\binom{2g+2}{g+1}.$$

In either case, we are adding up every other entry in the $(2g+2)$-nd row of Pascal's triangle, starting from the left and ending up with one half of the middle term. This sum is exactly one half of the sum of every other entry in the whole row; by the first part of Lemma 6.13 this equals $\frac{1}{4} \cdot 2^{2g+2} = 2^{2g}$.

Finally, we can add up the number of even and odd theta characteristics separately simply by taking every other term in the sums above; using equalities (2) and (3) in Lemma 6.13 (in case g is odd and even, respectively) we can conclude that C has $2^{g-1}(2^g - 1)$ odd theta characteristics and $2^{g-1}(2^g + 1)$ even theta characteristics. By Theorem 6.7 and the connectedness of the space of smooth irreducible curves of genus g, this count then holds for all curves of genus g, establishing Theorem 6.8. □

It is also possible to describe the configurations of odd and even theta characteristics as subsets of the set S of all theta characteristics, which as we've seen is a principal homogeneous space for the group $\mathrm{Jac}(C)_2 \cong (\mathbb{Z}/2\mathbb{Z})^{2g}$ of points of order 2 on the Jacobian. This leads to an alternative proof of Theorem 6.8 as in [?].

Cheerful Fact 6.15. There is more to say about the configuration of theta characteristics. For example: As noted, if we choose any theta characteristic on a curve C, we may identify the set S^- of odd theta characteristics with a subset of the group $\mathrm{Jac}(C)_2$ of points of order 2 on the Jacobian of C. We might expect that some 4-tuples of these points will add up to 0 in $\mathrm{Jac}(C)$; in other words, there should exist some 4-tuples $\mathcal{L}_1, \dots, \mathcal{L}_4 \in S^-$ such that

$$\mathcal{L}_1 \otimes \cdots \otimes \mathcal{L}_4 = 2K_C.$$

What this means in the case of genus $g = 3$ is that among the 28 bitangents to a smooth plane quartic curve C, there are some subsets of 4 whose eight points of tangency form the intersection of C with a plane conic. From the more detailed knowledge of the configuration S^- we can say how many. Indeed, the number was first found by Salmon [?]; it is 315.

6F. Exercises

Exercise 6-1. We have seen that a curve C of genus $g = 1$ is expressible as a 2-sheeted cover of $\mathbb{P}^1$ branched over four points; that is, as the smooth projective curve associated to the affine curve $C^\circ \subset \mathbb{A}^2$ given by $y^2 - \prod_{i=1}^4 (x - \lambda_i)$. Show that the closure $\overline{C^\circ}$ of $C^\circ \subset \mathbb{A}^2$ in either $\mathbb{P}^2$ or $\mathbb{P}^1 \times \mathbb{P}^1$ consists of the union of C° with one additional point, with that point a tacnode of $\overline{C^\circ}$ in either case.

Hint: In either case the complement $\overline{C^\circ} \setminus C^\circ$ consists of a single point, with two points of C mapping to it; now use the genus formula in either $\mathbb{P}^2$ or $\mathbb{P}^1 \times \mathbb{P}^1$.

Exercise 6-2. Find the number of 3-sheeted covers $C \to \mathbb{P}^1$ of genus g with simple branching except for one point of total ramification (that is, one point with just a single preimage point.)

Hint: Such a cover is specified by giving $2g + 2$ transpositions, not all equal, whose product is a nontrivial 3-cycle, modulo simultaneous conjugation.

We have already worked out the number of such tuples whose product is the identity; just subtract.

Exercise 6-3. Let C be a curve of genus g. How many unramified double covers of C are there?

Hint: Topologically, such covers are in 1-1 correspondence with subgroups of index 2 in $\pi_1(C)$; and such a subgroup is necessarily the preimage of a subgroup of index 2 in the abelianization $H_1(C, \mathbb{Z}) \cong \mathbb{Z}^{2g}$.

Exercise 6-4. Show that unramified double covers of a smooth curve C are in one-to-one correspondence with invertible sheaves $\mathcal{L}$ on C such that $\mathcal{L}^2 \cong \mathcal{O}_C$, that is with the 2-torsion points of $\mathrm{Jac}(C)$.

Hint: If $f : X \to C$ is an unramified double cover, consider the direct image $f_*(\mathcal{O}_X)$. This is a locally free sheaf of rank 2 on C, on which the group $\mathbb{Z}/2$ acts; the $+1$-eigenspace is the structure sheaf $\mathcal{O}_C$, and the -1-eigenspace is an invertible sheaf $\mathcal{L}$ on C such that $\mathcal{L}^2 \cong \mathcal{O}_C$.

Exercise 6-5. Let E be a curve of genus 1, and $q_1, \dots, q_b \in E$. How many double covers $C \to E$ are there branched over the q_i?

Hint: By our analysis, to specify such a cover, we have to specify the monodromy around representative loops generating $H_1(E, \mathbb{Z}) \cong \mathbb{Z}^2$; thus there are four possibilities.

Exercise 6-6. In this exercise, we ask you to complete the earlier description of the ideal of a quintic space curve of genus 2, keeping the notation of page 120.

Show that for any pair of lines L, L' of the appropriate ruling of Q, the three polynomials Q, S_L and $S_{L'}$ generate the homogeneous ideal $I(C)$. Find relations among them. Write out the minimal resolution of $I(C)$.

Hint: Choose any line $M \subset Q$ of the opposite ruling, and look at the linear forms H, H' on $\mathbb{P}^3$ vanishing on $L \cup M$ and $L' \cup M$.

Exercise 6-7. Let C be a curve of genus 2, expressed as a 2-sheeted cover of $\mathbb{P}^1$ with ramification points $p_1, \dots, p_6$. In this exercise we will count the number of even and odd theta characteristics. The text contains the count for a hyperelliptic curve of any genus; we offer the case of genus 2 as a warmup.

(1) Show that the theta characteristics on C are either of the form $\mathcal{L} = \mathcal{O}_C(p_i)$ or of the form $\mathcal{L} = \mathcal{O}_C(p_i + p_j - p_k)$ with i, j, k distinct.

(2) Show that in the first case we have $h^0(\mathcal{L}) = 1$, and in the second case we have $h^0(\mathcal{L}) = 0$.

(3) Finally, show that there are six of the former kind, and 10 of the latter, making $2^4 = 16$ in all.

Hint: If $h^0(\mathcal{L}) = 0$, we have $h^0(\mathcal{L}(p_k)) = 1$ for any ramification point p_k; show that the unique effective divisor in $|\mathcal{L}(p_k)|$ must be the sum of two ramification points.

Exercise 6-8. Let C be a curve of genus 2 and let $\mathcal{L} \in \mathrm{Pic}_4(C)$ be an invertible sheaf of the form $\mathcal{L} = K_C(p+q)$ with $p \neq q$ and $p + q \nsim K_C$ as in 2. Show that

(1) $h^0(\mathcal{L}(-2r)) = 1$ for any point $r \in C$, and

(2) $h^0(\mathcal{L}(-2p-2q)) = 0$.

Deduce from this that the map $\phi_{\mathcal{L}}$ is an immersion, and that the tangent lines to the two branches of $\phi_{\mathcal{L}}(C)$ at the point $\phi_{\mathcal{L}}(p) = \phi_{\mathcal{L}}(q)$ are distinct, meaning the point $\phi_{\mathcal{L}}(p) = \phi_{\mathcal{L}}(q)$ is a node of $\phi_{\mathcal{L}}(C)$.

Hint: For the first part (which implies that the map $\phi_{\mathcal{L}}$ is an immersion), observe that $h^0(\mathcal{L} \otimes K_C^{-1}) = 1$, meaning p and q are unique. The second part says that the images of the differential $d\phi_{\mathcal{L}}$ at p and q are distinct.

Exercise 6-9. We can represent any line in $\mathbb{P}^3$ as the row space of a 2×4 matrix by choosing 2 points on the line and using their coordinates as the rows. The *Plücker coordinates* of the line are the six 2×2 minors

$$\{p_{i,j}\}_{0\le i<j\le 3}$$

of this matrix. They are independent, up to a common scalar multiple, of the two points chosen, and define the *Plücker embedding* of the Grassmannian $\mathbb{G}(1,3)$ in $\mathbb{P}^5$.

The minors $p_{i,j}$ satisfy a nonsingular quadratic equation: if we stack two copies of the 2×2 matrix to produce a 4×4 matrix, its determinant is zero, and the Laplace expansion of this determinant is the *Plücker equation*

$$p_{0,1}p_{2,3} - p_{0,2}p_{1,3} + p_{0,3}p_{1,2} = 0.$$

(1) Show that the quadratic form $Q = p_{0,1}p_{2,3} - p_{0,2}p_{1,3} + p_{0,3}p_{1,2}$ is nonsingular, and deduce that it generates the ideal of $\mathbb{G}(1,3)$ in $\mathbb{P}^5$.

(2) Write the bilinear form corresponding to Q as the determinant of a matrix, and deduce that two points in $\mathbb{G}(1,3)$ correspond to vectors that pair to 0 if and only if they correspond to lines that intersect.

(3) Deduce that a maximal isotropic subspace for Q corresponds either to the set of lines containing a given point or the set of lines contained in a given plane; and that two such sets of lines of the same typemeet in a single point or coincide.