Personalities of Curves

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An essential construction in studying a curve C is the association to a given divisor of degree d of a invertible sheaf of that degree—in other words, the map

$$\mu : \{\text{effective divisors of degree } d\} \longrightarrow \{\text{invertible sheaves of degree } d\}.$$

A priori, this is a map of sets. But it is a fundamental fact that both sets may be given the structures of algebraic varieties in a natural way, so that the map between them is regular. The geometry of this map governs the geometry of the curve in many ways.

One part is relatively easy: the divisors on a smooth curve are parametrized by the *symmetric powers* $C^{(d)}$ of the curve C, described in Section symmetric section 1.1. By contrast, the parametrization of the set of invertible sheaves on C of a given degree by the points of an algebraic variety $\mathrm{Pic}^d(C)$ is a major undertaking, one that historically brought complex analysis and algebraic geometry together. We'll describe the original construction of the varieties $\mathrm{Pic}^d(C)$ by complex analysis in Section picard section 1.3 below, and touch briefly on the algebraic constructions.

The fact that invertible sheaves of a given degree of a curve C are parametrized by the points of a variety $\mathrm{Pic}^d(C)$ has many consequences. For example, applying dimension theory to $\mathrm{Pic}^d(C)$, we will show in Theorem g+3 theorem 1.2.3 that every curve can be embedded in projective space as a curve of degree $g + 3$.

1.1 Symmetric products

ymmetric section

If G is a finite group acting by automorphisms on an affine scheme $X := \operatorname{Spec} A$ then X/G is by definition $\operatorname{Spec}(A^G)$, the spectrum of the ring A^G of invariant elements of A. It is a basic theorem of commutative algebra that the map $X \to X/G$ induced by the inclusion of rings is finite, and the fibers of the map $X \to X/G$ are actually the orbits of G (see for example [**?**, Theorem ***]), something that often fails when G is infinite. Since the map $X \to X/G$ is finite, $\dim X/G = \dim X$. The construction commutes with the passage to G-invariant open affine sets, and thus passes to more general schemes—and in particular to projective schemes (see exercise **??**)—as well. quotient of projective

Exercise 1.1.1. Let G be a finite group acting on a quasi-projective scheme X. Show that there is a finite covering of X by invariant open affine sets. (Hint: consider the sum of the G-translates of a very ample divisor.)

For any variety X
((I guess we need to say in the intro that varieties are quasi-projective...))
we define the d-th symmetric power of X to be the quotient of the Cartesian product X^d of d copies of X by the action of the group of all permutations of the factors. The resulting variety X^d/S_d is called the *d-th symmetric power*, or *d-th symmetric product*, of X, denoted $X^{(d)}$.

For example, if $X = \mathbb{A}^1$ then $X^d = \mathbb{A}^d$, and the ring of invariants of the symmetric group acting on $\mathcal{O}_{\mathbb{A}^d} = k[x_1, \ldots, x_d]$ by permuting the variables is generated by the d elementary symmetric functions, which generate a polynomial subring. Since the symmetric functions of the roots of a polynomial are the coefficients of the polynomial, we may identify the scheme X^d with $\mathbb{A}^d$. ([**?**, Exercises 1.6, 13.2-13.4])

If $X = \mathbb{P}^1$ we can observe that on the product $(\mathbb{P}^1)^d$, taking the homogenesous coordinates of the i-th copy of $\mathbb{P}^1$ to be (s_i, t_i), the multilinear symmetric functions of degree d,

$$s_0 t_1 t_2 \cdots t_d, \ldots, s_0 s_1 \cdots s_d$$

localize on each of the standard affine open sets $(\mathbb{A}^1)^d = \mathbb{A}^d$ to the usual ordinary symmetric functions, and define an isomorphism $\operatorname{Sym}^d(\mathbb{P}^1) \to \mathbb{P}^d$. Again, we may think of this map as taking a d-tuple of points to the homogeneous form of degree d vanishing on it, which is unique up to scalars.

Note that this argument does not say anything about the symmetric products of $\mathbb{A}^2$, which are in fact singular—see Exercise sym2A2 1.1.2.

Since an effective divisor of degree d on a curve C is an unordered d-tuple of points on C, with repetitions allowed, it corresponds to a point in the *dth symmetric power* $C^{(d)}$.

There is one aspect of the symmetric powers that is special to the case of curves:

sym2A2 **Proposition 1.1.2.** *If X is a smooth curve then each symmetric power $X^{(d)}$ is smooth.*

Proof. The general case follows from the case of $\mathbb{A}^1$ because locally analytically the action of the symmetric group on C^d is the same as for $\mathbb{A}^1$: If $\overline{p} \in X^{(d)}$, then it suffices to show that the quotient of an invariant formal neighborhood of the preimage $p_1, \ldots, p_s$ of overline p is smooth. After completing the local rings, we get an action of the symmetric group G on the product of the completions of X at the p_i, and this depends only on the orbit structure of G acting on $\{p_1, \ldots, p_s\}$. Thus it would be the same for some orbit of points on $\mathbb{A}^1$. □

By contrast, if $\dim X \geq 2$ then the symmetric powers $X^{(d)}$ are singular for all $d \geq 2$.

Exercise 1.1.3. 1. We say that a group G acts freely on X if $gx = gy$ only when $g = 1$ or $x = y$. Show that if G is a finite group acting freely on a smooth affine variety X then the quotient X/G is smooth.

2. Let $X = (\mathbb{A}^2)^2$ and let $G = \mathbb{Z}/2$ act on X by permuting the two copies of $\mathbb{A}^2$; algebraically, $(\mathbb{A}^2)^2 = \operatorname{Spec} S$, with $S = k[x_1, x_2, y_1, y_2]$ and the nontrivial element $\sigma \in G$ acts by $\sigma(x_i) = y_i$.

3. . Show that G acts freely on the complement of the diagonal, but fixes the diagonal pointwise.

4. Show that the algebra S^G has dimension 4 and is generated by the 5 elements

$$f_1 = x_1 + y_1, f_2 = x_2 + y_2, g_1 = x_1y_1, g_2 = x_2y_2, h = x_1y_2 + x_2y_1,$$

perhaps by appropriately modifying the steps given in [**?**, Exercise 1.6].

5. Show that h^2 lies in the subring generated by $f_1, \dots, f_4$, and thus $S^{(2)}$ is a hypersurface, singular along the codimension 2 subset $f_1 = f_2 = 0$, which is the image of the diagonal subset of the cartesian product $(\mathbb{A}^2)^2$.

niversal divisor

Exercise 1.1.4 (The universal divisor of degree d)**.** Let C be a smooth projective curve, and $C^{(d)}$ its dth symmetric power. Show that the locus

$$\mathcal{D} := \{(D,p) \in C^{(d)} \times C \mid p \in D\}$$

is a closed subvariety of the product $C^{(d)} \times C$, whose fiber over any point $D \in C^{(d)}$ is the divisor $D \subset C$.
((How do our readers do this? We need to have proven the universal property of the symmetric product – the fine moduli space for invariant divisors of degree d.))

The variety $\mathcal{D}$ is called the universal divisor on C by virtue of the fact that for any family of divisors of degree d on C—that is, a scheme B and a subscheme $\mathcal{E} \subset B \times C$ flat of degree d over B, there is a unique morphism $\phi : B \to C^{(d)}$ such that $\mathcal{E}$ is the pullback via ϕ of $\mathcal{D} \subset C^{(d)} \times C$. Indeed, this amounts to saying that $C^{(d)}$ is the *Hilbert scheme* parametrizing subschemes of C of degree d. These statements are not generally true for higher-dimensional varieties; see Chapter ?? and especially Exercise ??

hilbert scheme chapter symmetric power vs Hilbert sche

1.2 Jacobians

To construct $\mathrm{Pic}^d(C)$ we start with $d = 0$, and identify $\mathrm{Pic}^0(C)$ with the *Jacobian* $J(C)$ of C using *abelian integrals* and the classical topology. This produces a complex manifold rather than an algebraic variety, but has the virtue of being relatively concrete.
((I think we should make the following into a formal theorem– in the characterization section — which maybe doesn't exist yet?))
The Jacobian $J(C)$ is in fact a projective variety, and may be constructed purely algebraically—so that, for example, if the curve C is defined over a given field K then $J(C)$ will be defined over K as well. The search for such a construction was one of the driving forces of algebraic geometry in the first half of the 20th century, giving rise to the notion of abstract algebraic varieties. See for example [?] [Kleiman must have something for this].

The goal of the 19th century mathematicians who first described abelian integrals was to make sense of integrals of algebraic functions. In the early development of calculus, mathematicians figured out how to evaluate explicitly integrals such as

$$\int_{t_0}^{t} \frac{dx}{\sqrt{x^2+1}}.$$

Such integrals can be thought of as path integrals of meromorphic differentials on the Riemann surface associated to the equation $y^2 = x^2 + 1$. This surface is isomorphic to $\mathbb{P}^1$, meaning that x and y can be expressed as rational functions of a single variable z; making the corresponding change of variables transformed the integral into one of the form

$$\int_{s_0}^{s} R(z)dz,$$

with R a rational function, and such integrals are readily evaluated by the technique of partial fractions.

When they tried to extend this to similar-looking integrals like

$$\int_{t_0}^{t} \frac{dx}{\sqrt{x^3+1}},$$

which arises when one studies the length of an arc of an elipse and was thus called an elliptic integral, they were stymied. The reason gradually emerged: the problem is that the Riemann surface associated to the equation $y^2 = x^3+1$ is not $\mathbb{P}^1$, but rather a curve of genus 1, and so has nontrivial homology group $H_1(C, \mathbb{Z}) \cong \mathbb{Z}^2$. In particular, if one expresses this "function" of t as a path integral, then the value depends on a choice of path; it is defined only modulo a lattice $\mathbb{Z}^2 \subset \mathbb{C}$. This implies that the inverse function is a doubly periodic meromorphic function on $\mathbb{C}$, and not an elementary function. Many new special functions, such as the Weierstrass $\mathcal{P}$-function were studied as a result. The name "elliptic curve" arose from these considerations too.

Once this case was understood, the next step was to extend the theory to path integrals of holomorphic differentials on curves of arbitrary genus. One problem is that the dependence of the integral on the choice of path is much worse; the set of homology classes of paths between two points $p_0, p \in C$ is identified with $H_1(C, \mathbb{Z}) \cong \mathbb{Z}^{2g}$ rather than $\mathbb{Z}^2$. The Jacobian arises when one considers the integrals of *all* holomorphic differentials on C simultaneously.

To express the resulting construction in relatively modern terms, let C be a smooth projective curve of genus g over $\mathbb{C}$, and let ω_C be the sheaf of differential forms on C. We will consider C as a complex manifold. Every meromorphic differential form is in fact algebraic [**?**], and we consider ω_C as a sheaf in the analytic topology.

We consider the space $V = H^0(\omega_C)^*$ of linear functions on the space of differentials $H^0(\omega_C)$. Integration over a closed loop in C defines a linear function on 1-forms, so that we have a map

$$\iota : \mathbb{Z}^{2g} = H_1(C, \mathbb{Z}) \ \to\ H^0(\omega_C)^* \cong H^1(\mathcal{O}_C) = \mathbb{C}^g.$$

Using Hodge theory[1] one can show that ι induces an injective map of vector spaces

$$\mathbb{R} \otimes H_1(C, \mathbb{Z}) = H_1(C, \mathbb{R}) \to H^0(\omega_C)^*$$

The complex structure on $H^0(\omega_C)^*$ yields a complex analytic structure on the quotient $\mathbb{C}^g/(\iota(\mathbb{Z}^{2g})$, which is thus a torus of real dimension $2g$. We call this quotient, with its structure as a g-dimensional complex manifold, the Jacobian of C, denoted

$$J(C) \ = \ V/\Lambda.$$

The point of this construction is that for any pair of points $p, q \in C$, the expression $\int_q^p$ describes a linear functional on $H^0(\omega_C)$, defined up to functionals obtained by integration over closed loops, and thus a point of $J(C)$. Thus, for example, if we choose a "base point" $q \in C$, we get a holomorphic map

$$\mu \ :\ C \ \to\ J(C); \quad p \mapsto \int_q^p$$

Having chosen a base point $q \in C$ as above, we get for each $d \geq 0$ the *Abel-Jacobi* map

$$\mu_d \ :\ C^{(d)} \ \to\ J(C),$$

[1] By Hodge theory

$$H^1(C, \mathbb{C}) \cong H^1(C, \mathcal{O}_C) \oplus \overline{H^1(C, \mathcal{O}_C)}$$

where the bar denotes complex conjugation $H^1(C, \mathbb{C})$, and the map ι is the composition of the natural inclusion with the projection to the first summand. Now $H_1(C, \mathbb{C}) = \mathbb{C} \otimes_{\mathbb{Z}} H_1(C, \mathbb{Z})$, so any basis of $H_1(C, \mathbb{Z})$ maps to a basis of $H^1(C, \mathbb{C})$ invariant under conjugation in $H^1(C, \mathbb{C})$—See Voisin [] or Griffiths-Harris []. If there were a real dependence relation among elements of the image of this basis under ι, then it the same relation would hold after complex conjugation and thus hold on the image of the basis in $H_1(C, \mathbb{C})$, a contradiction.

defined by

$$\mu_d(p_1 + \cdots + p_d) \;=\; \sum \int_q^{p_i} .$$

When there is no ambiguity about d, we will denote them by μ. and we we define $\mu(-D)$ to be $-\mu(D)$. The map μ is a group homomorphism in the sense that if D, E are divisors, then $\mu(D+E) = \mu(D) + \mu(E)$; this is immediate when the divisors are effective, and follows in general because the group of divisors is a free group. The connection between the discussion above and the geometry of linear series is made by *Abel's theorem*:

Theorem 1.2.1. *Two divisors $D, D' \in C^{(d)}$ on C are linearly equivalent if and only if $\mu(D) = \mu(D')$; in other words, the fibers of μ_d are the complete linear systems of degree d on C.*

See [**?**, Section 2.2] for a complete proof; we will just prove the "only if" part. This was in fact the only part proved by Abel; the converse, which is substantially more subtle, was proved by Clebsch.

Proof of "only if". Suppose that D and D' are linearly equivalent; that is, $\mathcal{O}_C(D) \cong \mathcal{O}_C(D')$. Call this invertible sheaf $\mathcal{L}$, and suppose that D and D' are the zero divisors of sections $\sigma, \sigma' \in H^0(\mathcal{L})$. Taking linear combinations of σ and σ', we get a pencil $\{D_\lambda\}_{\lambda \in \mathbb{P}^1}$ of divisors on C, with

$$D_\lambda \;=\; V(\lambda_0 \sigma + \lambda_1 \sigma'),$$

universal divisor

and by Exercise 1.1.4 this corresponds to a regular map $\alpha : \mathbb{P}^1 \to C^{(d)}$.

Consider now the composition

$$\phi = \mu \circ \alpha \;:\; \mathbb{P}^1 \;\to\; J(C).$$

Now, $J(C)$ is the quotient of the complex vector space $V = H^0(\omega_C)^*$ by a discrete lattice. If z is any linear functional on V, then, the differential dz on V descends to a global holomorphic 1-form on the quotient $J(C)$, so that the regular one-forms on $J(C)$ generate the cotangent space to $J(C)$ at every point. But for any 1-form ω on $J(C)$, the pullback $\phi^*\omega$ is a global holomorphic 1-form on $\mathbb{P}^1$, and hence identically zero. It follows that the differential $d\phi$ vanishes identically, and hence (since we are in characteristic 0) that ϕ is constant; thus $\mu(D) = \mu(D')$. □

Abel's Theorem goes surprisingly far to describe the Jacobian. The first statement of the following Corollary suggests how to describe the structure of the Jacobian algebraically, and was used by Andre Weil in the first such construction.

Corollary 1.2.2. *If C is a smooth curve of genus g then the Abel-Jacobi map $\mu_g : C^{(g)} \to J(C)$ is a surjective birational map. More generally, μ_d is generically injective for $d \le g$ and surjective for $d \ge g$.*

Proof. For $d \le g = \dim H^0(\omega_C)$, a divisor D that is the sum of d general points $p_1, \ldots, p_d \in C$ will impose independent vanishing conditions on the sections of ω_C, and thus

$$h^1\mathcal{O}_C(D) = h^0(\omega_C(-D)) = g - d,$$

by Serre duality. Using this, the Riemann-Roch formula gives $h^0\mathcal{O}_C(D) = 1$, so the fiber of μ_d consists of a single point, proving generic injectivity. In particular when $d = g$, the image of μ_d has dimension g, and since $C^{(g)}$ is compact, the image is closed, so it must be equal to $J(C)$.

Similarly, if $d \ge g$, we will have $h^0(\omega_C(-D)) = 0$ and hence $h^0(\mathcal{O}_C(D)) = d-g+1$. Since this is the affine dimension, the linear series $|D|$ has dimension $d - g = \dim C^{(d)} - \dim J(C)$, and again it follows that μ_d is surjective. □

1.2.1 Applications to linear series

To illustrate some of the power of Abel's theorem, we will use it to prove a basic result:

g+3 theorem **Theorem 1.2.3.** *Let C be a smooth projective curve of genus g. If $D \in C_{g+3}$ is a general divisor of degree $g+3$ on C, then D is very ample. In particular, every curve of genus g may be embedded in $\mathbb{P}^3$ as a curve of degree $g+3$.*

We proved in Theorem very ample ?? that *every divisor* of degree $\ge 2g+1$ is very ample; the difference here is that we are taking a *general* divisor. This result is sharp in the sense that hyperelliptic curves, for example, cannot be embedded in projective space as curves of any degree less than $g+3$, as we'll see in Chapter ScrollsChapter ??. However, if we consider only general divisors on general curves, we can do still better: "most" curves of genus g can in fact be embedded in $\mathbb{P}^3$ as curves of degree $d = \lceil 3g/4 \rceil + 3$ [].

Proof. If D is general of degree $g+3$ we have $h^0(\mathcal{O}_C(D)) = 4$. To show that it is very ample, we have to show that

1. for any point $p \in C$, we have $h^0(\mathcal{O}_C(D-p)) = 3$ (that is, $|D|$ has no base points, and so defines a regular map $\phi_D : C \to \mathbb{P}^3$); and

2. for any pair of points $p, q \in C$, we have $h^0(\mathcal{O}_C(D-p-q)) = 2$.

The second of these assertions immediately implies the first, and this is what we will prove.

Now let D be an arbitrary divisor of degree $g+3$. To say that $h^0(\mathcal{O}_C(D-p-q)) \geq 3$ is equivalent, by the Riemann-Roch theorem, to the condition $h^0(\omega_C(-D+p+q)) \geq 1$; fixing a divisor $K_C \in |\omega_C|$, this is the condition that there exists an effective divisor E of degree $g-3$ linearly equivalent to a divisor in $K_C - D + p + q|$.

Now consider the map

$$\nu : C^{(g-3)} \times C^{(2)} \to J(C)$$

given by

$$\nu : (E, F) \mapsto \mu_{2g-2}(K_C) - \mu_{g-3}(E) + \mu_2(p+q),$$

where the $+$ and $-$ on the right refer to the group law on $J(C)$.

By what we have just said, and Abel's theorem, the divisor D fails to be very ample only if $\mu(D) \in \mathrm{Im}(\nu)$. But the source $C^{(g-3)} \times C^{(2)}$ of ν has dimension $g-3+2 = g-1$, and so its image in $J(C)$ must be a proper subvariety; since μ_{g+3} is dominant, the image of a general divisor $D \in C^{(g-3)}$ is a general point of $J(C)$ and thus will not lie in $\mathrm{Im}(\nu)$. $\square$

Thus Abel's theorem, which was born out of an effort to evaluate calculus integrals, winds up proving a basic fact in the theory of algebraic curves!

((we said early on that we don't need to know that $J(C)$ is algebraic; for the present purposes, it's enough to know that $J(C)$ is a complex torus of dimension g. But in that case we do need to know that if $f : X \to Y$ is a holomorphic map of compact complex manifolds with $\dim X < \dim Y$, then $f(X)$ is a proper analytic subvariety of Y. we also need to know that the group law is algebraic. We

need to have a formal statement of the existence as an algebraic group. Will be taken care of by the Characterization section, yet to be written.))

1.3 Picard varieties

picard section

The modern treatment of the Picard variety, due to Grothendieck and his school, defines the Picard variety as the solution to a universal problem. If B is any scheme, then a *family of invertible sheaves on* X *over* B is an invertible sheaf on $X \times B$, flat over B. We often minimize the impact of B on this definition by saying that two invertible sheaves on X over B are equivalent if they differ by an invertible sheaf pulled back from B, and we write $Pic(X/B)$ for the quotient abelian group:

$$Pic(X/B) := \frac{\{\text{Invertible sheaves on } X \times B, \text{ flat over } B\}}{\{\text{Invertible sheaves pulled back from } B\}}.$$

Note that $Pic(X/B)$ is a contravariant functor of B: if $B' \to B$ is a morphism, then we can pull invertible sheaves on B back to B', and also pull invertible sheaves on $X \times B$ back to $X \times B'$. These two pullback maps induce a homomorphism of abelian groups $Pic(X/B) \to Pic(X/B')$. The Picard scheme of X, if it exists, is the scheme that represents this functor:

Definition 1.3.1. If X is a projective scheme over $\mathbb{C}$ then a *Picard scheme of* X, denoted $\mathrm{Pic}_{X/\mathbb{C}}$ is a scheme with a natural isomorphism of functors $\mathrm{Mor}(B, \mathrm{Pic}_{X/\mathbb{C}}) \cong Pic(X/B)$.

Theorem 1.3.2. *If* X *is a projective variety, then* $\mathrm{Pic}_{X/\mathbb{C}}$ *exists.*

We will sketch the proof of this result in the case when X is a smooth curve below; for detailed references and variations, see [**?**].

From the definition we see that the identity morphism of $\mathrm{Pic}_{X/\mathbb{C}}$ corresponds to a family $\mathcal{P}$ of invertible sheaves over $\mathrm{Pic}_{X/\mathbb{C}}$, that is, to an invertible sheaf $\mathcal{P}$ on $X \times \mathrm{Pic}_{X/\mathbb{C}}$, flat over $\mathrm{Pic}_{X/\mathbb{C}}$, and well-defined up to tensoring with an invertible sheaf pulled back from $\mathrm{Pic}_{X/\mathbb{C}}$. By the Yoneda lemma, the naturality of the isomorphism $\mathrm{Mor}(B, \mathrm{Pic}_{X/\mathbb{C}}) \cong Pic(X/B)$ means that the family of invertible sheaves on $X \times B$ correponding to a given morphism

$\phi : B \to \mathrm{Pic}_{X/\mathbb{C}}$, is, up to the pullback of an invertible sheaf on B, the pullback of $\mathcal{P}$ along the map $X \times \phi : X \times B \to X \times \mathrm{Pic}_{X/\mathbb{C}}$; this is the *universal property* of the Poincaré sheaf.

For example, if $\mathcal{L}$ is any invertible sheaf on X, then $\mathcal{L}$ may be regarded as a family of sheaves over a closed point p, so there is a unique morphism $p \to \mathrm{Pic}_{X/\mathbb{C}}$ such that $\mathcal{L}$ is the pullback of $\mathcal{P}$ under the induced map $X = X \times p \to X \times \mathrm{Pic}_{X/\mathbb{C}}$; more colloquially, the closed points of $\mathrm{Pic}_{X/\mathbb{C}}$ correspond to the invertible sheaves on X.

If X is a smooth curve, then each invertible sheaf has a degree; and since the degree is constant in any family of invertible sheaves over a connected curve, $\mathrm{Pic}_{X/\mathbb{C}}$ is a disjoint union of spaces $\mathrm{Pic}_{d,X/\mathbb{C}}$, and the restriction of $\mathcal{P}$ to $X \times \mathrm{Pic}_{d,X/\mathbb{C}}$ is a family of invertible sheaves of degree d in the sense that for every point $p \in \mathrm{Pic}_{d,X/\mathbb{C}}$ the restriction of $\mathcal{P}$ to $X = X \times p$ is a sheaf of degree d.

The functorial description of $\mathrm{Pic}_{X/\mathbb{C}}$ makes it easy to prove a number of properties:

Theorem 1.3.3. *If X is a smooth curve, then* $\mathrm{Pic}_{X/\mathbb{C}}$ *is smooth and the tangent space to* $\mathrm{Pic}_{X/\mathbb{C}}$ *at any closed point is isomorphic to $H^1(\mathcal{O}_X)$.*

Proof. We first prove the smoothness. Since we are working over an algebraically closed field, it is enough to check the criterion of formal smoothness [**?**]: given an affine scheme B and a subscheme B' such that $\mathcal{I}^2_{B'/B} = 0$, we must show that any map $B' \to X$ extends to a map $B \to X$. But a map $B' \to X$ corresponds to an invertible sheaf on $X \times B'$, and similarly for B, so we must show that every invertible sheaf on $X \times B'$ extends to an invertible sheaf on $X \times B$. We will use the identification of the group of invertible sheaves on a space with the first cohomology of the multiplicative group of invertible functions on the space.

Note that $\mathcal{I}_{B'/B}$ is supported on B', and Let N be the pullback of $\mathcal{I}_{B'/B}$ to $X \times B'$. From the exact sequence

$$0 \to N \to \mathcal{O}_{X\times B} \to \mathcal{O}_{X\times B'} \to 0$$

we deduce an exact sequence of multiplicative groups

$$1 \to (1+N) \to \mathcal{O}^*_{X\times B} \to \mathcal{O}^*_{X\times B'} \to 1.$$

Since $\mathcal{I}_(B'/B)^2 = 0$, the sheaf of multiplicative groups, $(1+N)$ is isomorphic to the sheaf of additive groups N, and we get a long exact sequence in cohomology

$$H^0\mathcal{O}^*_{X\times B} \to H^0\mathcal{O}^*_{X\times B'} \to H^1\mathcal{O}_X \to H^1(\mathcal{O}^*_{X\times T}) \to H^1(\mathcal{O}^*_{X^*\times B'}) \to H^2(N)$$

Since B is affine and X is 1-dimensional, $H^2(N) = 0$, proving that we can extend invertible sheaves.

Rcall that the tangent space to a scheme Y at a closed point $p : \operatorname{Spec} k \to Y$ is the set of extensions of p to a morphism $\operatorname{Spec} k[\epsilon]/\epsilon^2 \to Y$. From the exact sequence above we see that the set of extensions is $H^1(\mathcal{O}_X)$. □

We can also use the universal property to prove properness:

Corollary 1.3.4. *X is a nonsingular projective curve, then $\operatorname{Pic}_{X/\mathbb{C}}$ is proper over $\mathbb{C}$.*

Proof. We use the valuative criterion of properness. Thus we consider $D := \operatorname{Spec} R$, where R is a discrete valuation ring, and an invertible sheaf $\mathcal{L}$ on $X \times U$, where U is the generic point of D, and we must show that $\mathcal{L}$ extends to $X \times D$.

Choose a rational section of $\mathcal{L}$, and let ℓ be the associated Cartier divisor of zeros and poles. Because X is smooth, the closure $\overline{\ell}$ of ℓ in $X \times D$ is again a Cartier divisor, and the invertible sheaf associated to $\overline{\ell}$ is an extension of $\mathcal{L}$. □

Since the points of $\operatorname{Pic}_{X/\mathbb{C}}$ correspond to invertible sheaves, it is not surprising that $\operatorname{Pic}_{X/\mathbb{C}}$ is an abelian algebraic group in a natural way:

Exercise 1.3.5. Write $\pi_{1,2}$ and $\pi_{1,3}$ for the projections

$$X \times \operatorname{Pic}_{X/\mathbb{C}} \times \operatorname{Pic}_{X/\mathbb{C}} \to X \times \operatorname{Pic}_{X/\mathbb{C}}$$

onto the $(1,2)$ and $(1,3)$ factors, respectively. The map $\operatorname{Pic}_{X/\mathbb{C}} \times \operatorname{Pic}_{X/\mathbb{C}} \to \operatorname{Pic}_{X/\mathbb{C}}$ corresponding to the family $\pi_{1,2}^*\mathcal{P} \otimes \pi_{1,3}^*\mathcal{P}$ makes $\operatorname{Pic}_{X/\mathbb{C}}$ into an abelian algebraic group, with inverse operation $\operatorname{Pic}_{X/\mathbb{C}} \to \operatorname{Pic}_{X/\mathbb{C}}$ corresponding to the sheaf $\mathcal{P}^{-1}$.

Exercise 1.3.6. Show that if X is an irreducible curve with a smooth closed point p over a field k, then tensoring with $cO_X(p)$ induces isomorphisms $\mathrm{Pic}_{d,X/k} \to \mathrm{Pic}_{d+1,X/k}$ for all d. Without assuming the existence of a smooth closed point, show that $\mathrm{Pic}_{d,X/k} \to \mathrm{Pic}_{d+2g-2,X/k}$, where g is the genus of X.

Cheerful Fact 1.3.1. There are smooth curves over certain fields such that $\mathrm{Pic}_{d,X/k}$ is not isomorphic to $\mathrm{Pic}_{e,X/k}$, unless $d \equiv \pm e \pmod{2g-2}$ [**?**].

In parallel with the functor $B \mapsto Pic(X/B)$ we define the functor of relative divisors:

$$Div(X/B) := \text{Divisors in } X \times B, \text{ flat over } B\}$$

((DE rev to here 3/13/22))

1.4 Differential of the Abel-Jacobi map

In this section we will describe the differential $d\mu$ of the Abel-Jacobi map $\mu : C_d \to J(C)$; this yield a sharper form of Abel's theorem.

((clarify the structure of the next few pages: what is *the* theorem, what are special cases to get an intuitive feel.))
To start, suppose $D = p_1 + \cdots + p_d$ is a divisor consisting of d distt oints on our curve C. Since the quotient map $C^d \to C_d$ is unramified at D, the tangent space to C_d at the point D is naturally identified with the tangent space to C^d at $(p_1, \ldots, p_d)$; that is, the direct sum of the tangent spaces to C at the points p_i:

$$T_D(C_d) = \bigoplus T_{p_i}(C).$$

On the other hand, the tangent space to $J(C)$ at the image point $\mu(D)$ is the vector space $H^0(\omega_C)^*$ of which $J(C)$ is a quotient (as it is at every point!). The differential $d\mu_D$ is thus a linear map

$$\bigoplus T_{p_i}(C) \longrightarrow H^0(\omega_C)^*,$$

and the transpose of this a linear map

$$H^0(\omega_C) \longrightarrow \bigoplus T^*_{p_i}(C).$$

This last map is easy to describe: since the map μ is given by

$$\mu_d(p_1 + \cdots + p_d) \;=\; \sum \int_q^{p_i},$$

we can differentiate under the integral sign to conclude that *the codifferential d^*_μ is the map*

$$\begin{aligned} H^0(\omega_C) &\to \bigoplus T^*_{p_i}(C) \\ \omega &\mapsto (\omega(p_1), \ldots, \omega(p_d)). \end{aligned}$$

There is a natural extension of this to the case of non-reduced divisors D, that is, divisors with repeated points. We first need a description of the tangent space to C_d at the point D:

ct tangent space

Proposition 1.4.1. *The tangent and cotangent spaces to C_d at the point corresponding to an arbitrary divisor $D = \sum a_i p_i$ are naturally identified with $H^0(\mathcal{O}_C(D)/\mathcal{O}_C)$ and $H^0(\omega_C/\omega_C(-D))$ respectively.*

This is not a proof. ideas: first, C_d is a Hilbert scheme, and the tangent space at a divisor D is thus $H^0 N_{D/C}$. Now the normal bundle is the dual of the conormal bundle $O(-D)/O(-2D)$; and the pairing $O(-D) \times O(D) \to O$ induces a perfect pairing $H^0(\mathcal{O}_C(D)/\mathcal{O}_C)$ with $H^0(O(-D)/O(-2D))$, so the former is the tangent space.

Note that we have a natural pairing between the spaces $H^0(\mathcal{O}_C(D)/\mathcal{O}_C)$ and $H^0(\omega_C/\omega_C(-D))$, given by sending (f, ω) to $\sum_i Res_{p_i}(f\omega)$. Note also that the term "natural" has a precise meaning here: if we let

$$\mathcal{D} = \{(D, p) \in C_d \times C \;\mid\; p \in D\}$$

be the universal effective divisor of degree d on C, the proposition says that the cotangent sheaf $T^*_{C_d}$ is the direct image $\alpha_*(\beta^*\omega_C/\beta^*\omega_C(-\mathcal{D}))$, where α and β are the projections of $C_d \times C$ onto the two factors.

□

Given Proposition 1.4.1, we can extend our earlier statement to the

Proposition 1.4.2. *The codifferential $d\mu^*$ of the Abel-Jacobi map is the natural restriction map*

$$H^0(\omega_C) \longrightarrow H^0(\omega_C/\omega_C(-D)).$$

Now, note that the codimension of the image of $d\mu^*$—equivalently, the dimension of the kernel of the differential $d\mu$—is by the geometric Riemann-Roch theorem exactly the dimension of the fiber of C_d over the point $\mu(D) \in J(C)$. In other words, the fibers of μ are smooth, and in particular reduced. Thus we can think of Proposition 1.4.2 as a strengthening of the Abel-Clebsch theorem: while Abel and Clebsch show that the fibers of μ are complete linear series set-theoretically, we see from the above that it is in fact true scheme-theoretically.

1.5 Further consequences

One consequence of the description of the Jacobian and the Abel-Jacobi map of a curve C is that *the set of linear series on C of given degree d and dimension r can be given the structure of a scheme in its own right.*

((all this needs a base point, and some care to state the universal property precisely. OK to state it intuitively, then translate))
To start with, we can define $W^r_d(C) \subset \mathrm{Pic}^d(C)$ to be the set of invertible sheaves $L \in \mathrm{Pic}^d(C)$ such that $h^0(L) \geq r+1$. We can see readily that this is a Zariski closed subset of $\mathrm{Pic}^d(C)$, for example by pointing out that it is exactly the locus where the fiber dimension of the Abel-Jacobi map $\mu : C_d \to \mathrm{Pic}^d(C)$ is at least r; this is closed by upper-semicontinuity of fiber dimension.

Note that among the subvarieties W^r_d are the images W^0_d of the Abel-Jacobi maps; in other words, the locus of *effective* divisor classes of degree d. The superscript is often omitted, meaning W^0_d is usually written W_d.

Moreover, the subsets $W^r_d \subset \mathrm{Pic}^d(C)$ can be given the structure of a scheme, in a natural way. One way to characterize this scheme structure is to say that the scheme W^r_d represents the functor of families of invertible sheaves $L \in \mathrm{Pic}^d(C)$ on C with $h^0(L) \geq r+1$.

In the other construction, we can actually parametrize the set of linear series g^r_d on C: that is, there is a scheme $G^r_d(C)$ parametrizing pairs (L, V) with $L \in \mathrm{Pic}^d(C)$ and $V \subset H^0(L)$ a subspace of dimension $r+1$. Again, the scheme structure may be characterized by saying that $G^r_d(C)$ represents the functor of families of linear series on C. Note that the natural map $G^r_d \to W^r_d$ is an isomorphism over the dense open subset $W^r_d \setminus W^{r+1}_d$, and more generally its fiber over a point of $W^s_d \setminus W^{s+1}_d$ is a copy of the Grassmannian $\mathbb{G}(r, s)$.

1.5.1 Examples in low genus

Genus 2

There is not a lot going on here, but there are a couple observations to make. First of all, the map $\mu_1 : C \to J(C)$ embeds the curve C in $J(C)$. Secondly, the map $\mu_2 : C_2 \to J(C)$ is an isomorphism except along the locus $\Gamma \subset C_2$ of divisors of the unique g^1_2 on C; in other words, *the symmetric square C_2 of C is the blow-up of $J(C)$ at a point.*
((true that the fiber is $\mathbb{P}^1$, but is that enough? Maybe so for a birational map of smooth surfaces, but does the reader know this?))

Exercise 1.5.1. Let $C \subset J(C)$ be the image of the Abel-Jacobi map μ_1. Show that the self-intersection of the curve C is 2,

1. by applying the adjunction formula to $C \subset J(C)$; and

2. by calculating the self-intersection of its preimage $C + p \subset C_2$ and using the geometry of the map μ_2.

1.5.2 Genus 3

1.5.3 Genus 4

In genus 4 we encounter for the first time a scheme $W^r_d(C)$ that is neither of the form W_d or $K - W_e$. This is the subscheme $W^1_3(C)$ parametrizing g^1_3s on C.

1.5.4 Genus 5

Want: for general curve C of genus 5, the scheme $W^1_4(C)$ is smooth & irreducible; but when C becomes trigonal, $W^1_4(C)$ becomes reducible, with one component of the form $W^1_3 + C$ and the other $K - W^1_3 - C$.

1.6 Martens' theorem and variants

The general theorems we have described so far dealing with linear series on a curve C, like the Riemann-Roch and Clifford theorems, have to do with the existence or non-existence of linear series on C. Now that we've seen how to parametrize the set of linear series on C by the varieties $W^r_d(C)$, we can ask more quantitative questions: for example, what can the dimension of $W^r_d(C)$ be? One basic result, for example, is the following.

Theorem 1.6.1 (Martens' theorem)**.** *If C is any smooth projective curve of genus g, then for any d and g we have*

$$\dim(W^r_d(C)) \leq d - 2r;$$

moreover, if we have equality for any $r > 0$ and $d < 2g-2$ the curve C must be hyperelliptic.

Note that if C is hyperelliptic with $g^1_2 = |D|$, we have

$$W^r_d(C) \supset W_{d-2r}(C) + \mu(rD).$$

(In fact, as we'll see in the following chapter, this is an equality.) Since this has dimension $d-2r$, we see that Martens' theorem is sharp. Note also that Clifford's theorem is a special case of Martens' theorem!

Proof. □

There are extensions of Martens' theorem to the case $\dim(W^r_d(C)) = d - 2r - 1$ (Mumford) and $d - 2r - 2$ (Keem).

1.7 The full Brill-Noether theorem

The existence of the varieties $W_d^r(C)$ parametrizing linear series on an arbitrary curve C allows us to strengthen Clifford's theorem to Martens' theorem. If we ask about linear series on a *general* curve, similarly, it allows us to give a much more detailed version of the Brill-Noether theorem: instead of saying when there exists a g_d^r on a general curve C, we can give the dimension of the variety $W_d^r(C)$ parametrizing such linear series, and we can also talk about the geometry of a general linear series on C. We collect the basic facts into the

Theorem 1.7.1 (Brill-Noether theorem, omnibus version)**.** *Let C be a general curve of genus g. If we set $\rho = g - (r+1)(g-d+r)$, then*

1. $\dim(W_d^r(C)) = \rho$;
2. *the singular locus of $W_d^r(C)$ is exactly $W_d^{r+1}(C)$;*
3. *if $\rho > 0$ then $W_d^r(C)$ is irreducible;*
4. *if $\rho = 0$ then the variety $\mathcal{W}_d^r$ is irreducible;*
5. *if L is any invertible sheaf on C, the map*
$$H^0(L) \otimes H^0(\omega_C L^{-1}) \longrightarrow H^0(\omega_C)$$
is injective;
6. *if $|D|$ is a general g_d^r on C, then*
   (a) *if $r \geq 3$ then D is very ample; that is, the map $\phi_D : C \to \mathbb{P}^2$ embeds C in $\mathbb{P}^r$;*
   (b) *if $r = 2$ the map $\phi_D : C \to \mathbb{P}^2$ gives a birational embedding of C as a nodal plane curve; and*
   (c) *if $r = 1$, the map $\phi_D : C \to \mathbb{P}^2$ expresses C as a simply branched cover of $\mathbb{P}^1$.*

1.8 The Torelli theorem

((consider making this a cheerful fact. or exercise?))
In the examples above, we see that a lot of information about a curve C is encoded in the geometry of its Jacobian. In fact, we can make this official: we have the celebrated

Theorem 1.8.1 (Torelli)**.** *A curve C is determined by the pair $(J(C), \Theta)$.*

Proof. In fact, there are many ways of reconstructing a curve from its Jacobian; this one is due to Andreotti, and makes essential use of our description of the differential of the Abel-Jacobi map.

A key fact is that the Jacobian $J(C) = H^0(\omega_C)^*/H_1(C, \mathbb{Z})$ is a torus, and so has trivial tangent bundle, with fiber $H^0(\omega_C)^*$ at every point. What this means is that if $X \subset J(C)$ is a smooth, k-dimensional subvariety, we have a *Gauss map*

$$\mathcal{G} : X \to G(k, g) = G(k, H^0(\omega_C)^*),$$

sending a point $x \in X$ to its tangent plane $T_x X \subset T_x J(C) = H^0(\omega_C)^*$; more generally, if X is singular then $\mathcal{G}$ will be a rational map. In particular, if $X = \Theta = W_{g-1}$, we get a rational map

$$W_{g-1} \longrightarrow \mathbb{P}^{g-1} = \mathbb{P}(H^0(\omega_C))$$

between two $g-1$-dimensional varieties, and it is the geometry of this map from which we can recover the curve C.

To start with, let's identify an open subset of W_{g-1} where the Gauss map is defined. This is not hard: a point $L \in W_{g-1} \setminus W^1_{g-1}$ is the image of a unique point $D \in C_{g-1}$ under the map μ, and moreover we've seen that the differential $d\mu$ is injective at D; it follows that L is a smooth point of W_{g-1}.

Moreover, we've identified the tangent space to W_{g-1} at $L = \mu(D)$: as we saw, the differential $d\mu : T_D(C_{g-1}) \to T_L(J) = H^0(\omega_C)^*$ is just the transpose of the evaluation map $H^0(\omega_C) \to H^0(\omega_C(-D))$, and it follows that the tangent space to W_{g-1} at the point L is the hyperplane in $H^0(\omega_C)^*$ dual to the unique differential vanishing on D. To put it another way: if we think of C as canonically embedded in $\mathbb{P}(H^0(\omega_C)^*)$, then by geometric Riemann-Roch the divisor D will span a hyperplane in $\mathbb{P}(H^0(\omega_C)^*)$, and the

Gauss map $\mathcal{G}$ sends L to the point in the dual projective space $\mathbb{P}(H^0(\omega_C))$ corresponding to that hyperplane.

Cheerful Fact 1.8.1. We have shown that the open subset $W_{g-1} \setminus W^1_{g-1}$ is contained in the smooth locus of W_{g-1}. In fact, they are equal; that is, W^1_{g-1} is exactly the singular locus of W_{g-1}. This is a special case of the beautiful *Riemann singularity theorem*, which says that for any point $L \in W_{g-1}$, the multiplicity $\mathrm{mult}_L(W_{g-1}) = h^0(L)$. For a proof of the Riemann singularity theorem, see for example [GH].

((David – can we find another reference for the RST? The proof in [GH] is clear but somewhat sketchy; I don't have a copy handy, but as I recall it implicitly assumes that the tangent cone is generically reduced.))

We are now in a position to describe the Gauss map

$$\mathcal{G} : W_{g-1} \longrightarrow \mathbb{P}(H^0(\omega_C))$$

explicitly in terms of the geometry of the canonical curve $C \subset \mathbb{P}(H^0(\omega_C)^*)$. To start, let $p \in \mathbb{P}(H^0(\omega_C))$ be a general point, dual to a general hyperplane $H \subset \mathbb{P}(H^0(\omega_C)^*)$. The hyperplane H will intersect the canonical curve C transversely in $2g-2$ points $p_1, \dots, p_{2g-2}$; these points will be in linear general position (in particular, any $g-1$ of them will be linearly independent and so span H). It follows that the fiber of $\mathcal{G}$ over the point H will consist of the invertible sheaves $L = \mathcal{O}_C(p_{\alpha_1} + \cdots + p_{\alpha_{g-1}})$, where $p_{\alpha_1}, \dots, p_{\alpha_{g-1}}$ is any subset of $g-1$ of the points p_i; in particular, we see that the degree of the map $\mathcal{G}$ is

$$\deg(\mathcal{G}) = \binom{2g-2}{g-1}.$$

The next question is, where does this analysis fail—in other words, for which hyperplanes $H \subset \mathbb{P}H^0(\omega_C)^*$ does the fiber of $\mathcal{G}$ not consist of $\binom{2g-2}{g-1}$ points, or equivalently, what is the branch divisor of the map $\mathcal{G}$? The answer is, the analysis above fails in two cases: when the points $p_1, \dots, p_{2g-2}$ are not in linear general position—specifically, when some $g-1$ of the points p_i fail to be linearly independent; and when the hyperplane H is not transverse to

C, so that the hyperplane section $H \cap C$ consists of fewer than $2g-2$ distinct points.

The first of these occurs in codimension 2 in $\mathbb{P}H^0(\omega_C)$, and so does not contribute any components to the branch divisor of $\mathcal{G}$. It follows that the branch divisor of the map $\mathcal{G}$ is exactly the locus of hyperplanes $H \subset H^0(\omega_C)^*$ tangent to the canonical curve C; in other words, *the branch divisor of $\mathcal{G}$ is the hypersurface in $\mathbb{P}H^0(\omega_C)$ dual to the canonical curve $C \subset \mathbb{P}H^0(\omega_C)^*$.*

Now we can invoke the fact that the dual of the dual of a variety $X \subset \mathbb{P}^n$ is X itself (see for example [3264] or something by Kleiman). We thus have a way of recovering the curve C from the data of the pair (J, W_{g-1}): simply put, *the curve C is the dual of the branch divisor of the Gauss map on W_{g-1}*, and the Torelli theorem is proved.

$\square$

The Torelli theorem for curves was the first instance of a class of theorems, called *Torelli theorems*, to the effect that certain classes of varieties are determined to some degree by their Hodge structure; there are, for example, Torelli theorems of varying strength for K3 surfaces, cubic threefolds and fourfolds and hypersurfaces in $\mathbb{P}^n$.

1.9 Additional topics

A couple of topics that would naturally go here, if we have the inclination and space.

1.9.1 Theta characteristics

Basically: introduce the notion of theta-characteristic (= square root of the canonical bundle), and prove the invariance of $h^0(\mathcal{L})$ mod 2. Describe the configuration of theta-characteristics on a given curve C as a principal homogeneous space for the group $J(C)[2] \cong (\mathbb{Z}/2)^{2g}$ of torsion of order 2 in the Jacobian.

Example: bitangents to a plane quartic; distinguished triples of bitangents

1.9.2 Intermediate Jacobians and the irrationality of cubic threefolds

First, describe the intermediate Jacobians $J(X)$ of higher-dimensional varieties X by analogy with the case of curves; introduce the Abel-Jacobi maps from parameter spaces of cycles on X to $J(X)$.

Application: show that the intermediate Jacobian of a cubic threefold is not the Jacobian of a curve by calculating the degree of the Gauss map on the theta-divisor and showing it's not 70 (which by the calculation above it would be if $J(X)$ were a Jacobian). Deduce irrationality of X.

I know this is a bit of a stretch for the current volume, but I'd really like to include it if at all possible: the proof in Clemens-Griffiths is a mess, and this is much simpler