With DE corrections

ch 4 (curves of $g = 2,3$)

Contents

Chapter 1

Hyperelliptic curves and curves of genus 2 and 3

and 3 chapter

1.1 Hyperelliptic Curves

In the world of curves, hyperelliptic curves are outliers: they behave differently from other curves, and the techniques used to analyze them are different from the techniques used for more general curves. Many theorems about curves contain the hypothesis "non-hyperelliptic," with the corresponding result for hyperelliptic curves arrived at directly by ad hoc methods. Because the methods of this section will not be used in other cases, it could be skipped in first reading

There will be a further discussion of hyperelliptic curves in Chapter ??, focussing on the algebra and geometry of their projective embeddings; the analysis here will cover most of the questions we'll be asking about curves in general in the next four chapters.

ScrollsChapter

Is there going to be a scrolls chapter?

1.1.1 The equation of a hyperelliptic curve

By definition, a hyperelliptic curve C is one admitting a degree two map $\pi : C \to \mathbb{P}^1$. Because the degree is only 2, each point in $\mathbb{P}^1$ has either two distinct preimages, or one point of simple ramification. There can be no higher ramification, so at all but finitely many points $p \in C$ the map π is a local isomorphism ("local" here in the complex analytic/classical or étale topology, not the Zariski topology!); at any other point $p \in C$, the map is given in terms of local analytic coordinates on C and $\mathbb{P}^1$ simply by $z \mapsto z^2$. In particular, both the ramification divisor and the branch divisor (as defined in Chapter linear systems chapter ??) are reduced. Thus by the Riemann-Hurwitz formula there are exactly $2g+2$ branch points $q_1, \ldots, q_{2g+2} \in \mathbb{P}^1$. These points determine the curve:

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ptic existence

Theorem 1.1.1. *There is a unique smooth projective hyperelliptic curve C expressible as a 2-sheeted cover of $\mathbb{P}^1$ branched over any set of $2g+2$ distinct points.*

Proof

We can easily construct such a curve, postponing for a moment the uniqueness: If the coordinate of the point $p_i \in \mathbb{P}^1$ is λ_i, it is the smooth projective model of the affine curve

$$C^\circ = \{(x,y) \in \mathbb{A}^2 \mid y^2 = \prod_{i=1}^{2g+2}(x - \lambda_i)\}.$$

Note that we're choosing a coordinate x on $\mathbb{P}^1$ with the point $x = \infty$ at infinity not among the q_i, so that the pre-image of $\infty \in \mathbb{P}^1$ is two points $r, s \in C$. Concretely, we see that as $x \to \infty$, the ratio $y^2/x^{2g+2} \to 1$, so that

$$\lim_{x\to\infty} \frac{y}{x^{g+1}} = \pm 1;$$

the two possible values of this limit correspond to the two points $r, s \in C$.

It's worth pointing out that C is *not* simply the closure of the affine curve $C^\circ \subset \mathbb{A}^2$ in either $\mathbb{P}^2$ or $\mathbb{P}^1 \times \mathbb{P}^1$: as you can see from a direct examination of the equation, each of these closures will be singular at the (unique) point at infinity.

Completion of the proof of Theorem hyperelliptic existence *1.1.1.* The proof (in characteristic 0) of uniqueness follows from elementary algebraic topology:

First, a punctured 2-disk has fundamental group $\mathbb{Z}$ and the unique n-sheeted covering is again a punctured disk; regarding these disks as neighborhoods of the origin in $\mathbb{C}^2$, the covering map can be taken to be $x \mapsto x^n$. This map can of course be extended (by the same formula) to a map analytic also at the origin, with ramification index (by definition) $n - 1$.

Now suppose that $\Gamma = \{p_1, \ldots, p_d\}$ is the desired branch divisor. Globally, if γ_i is a small loop around p_i then the abelianization of the fundamental group π of the d-times punctured sphere

$$S' := \mathbb{P}^1 \setminus \Gamma$$

is its first homology group,

$$H := H_1(S', \mathbb{Z}) = \frac{\oplus \mathbb{Z} \cdot \gamma_i}{\mathbb{Z} \cdot \sum_i [\gamma_i]}$$

((Insert "lollipop picture".))
Since $\mathbb{Z}/2$ is abelian, a degree 2 unramified covering of S' corresponds to a map $H \to \mathbb{Z}/2$, and this map must send $2\gamma_i$ to 0 for $i = 1 \ldots d$. There is such a map if and only if d is even, and in this case the map is unique.

Summarizing: there is, a unique degree 2 topological covering $C' \to \mathbb{P}^1 \setminus \Gamma$ by a surface C' that extends to a ramified covering of $\rho : C \to \mathbb{P}^1$, simply ramified over the points of Γ, as long as the number of ramification points is even.

A triangulation of $\mathbb{P}^1$ with V vertices including the points of Γ, E edges, and F triangles must have

$$V - E + F = \chi_{\text{top}}(S^2) = 2.$$

It lifts to a triangulation of C with $2V - d$ vertices, $2E$ edges, and $2F$ faces, so

$$\chi_{\text{top}}(C) = 2V - d - 2E + 2F = 4 - d,$$

so if $d = 2g + 2$ then $\chi_{\text{top}}(C) = 2 - 2g$, so C is a surface of genus g.

Though given as a topological surface, the map ρ is a local homeomorphism at every point not in the preimage of Γ, so C inherits a unique complex structure from the requirement that ρ be holomorphic; thus C is actually a smooth algebraic curve of genus g.
((we had better say topological and algebraic genus are the same in the intro.))

Need to assert GAGA somewhere

Say here that the deck trans is thus a holomorphic involution — and a alg. invol.

□

Exercise 1.1.2. In the case $g = 1$, show that the closure $\overline{C^\circ}$ of $C^\circ \subset \mathbb{A}^2$ in either $\mathbb{P}^2$ or $\mathbb{P}^1 \times \mathbb{P}^1$ consists of the union of C° with one additional point, with that point a tacnode of $\overline{C^\circ}$ in either case.

It is also possible to give a projective model of the hyperelliptic curve C with given branch divisor: if we divide the points $q_1, \dots, q_{2g+2} \in \mathbb{P}^1$ into two sets of $g+1$—say, for example, $q_1, \dots, q_{g+1}$ and $q_{g+2}, \dots, q_{2g+2}$—then C is the closure in $\mathbb{P}^1 \times \mathbb{P}^1$ of the locus

$$\{(x,y) \in \mathbb{A}^2 \mid y^2 \prod_{i=1}^{g+1}(x-\lambda_i) = \prod_{i=g+2}^{2g+2}(x-\lambda_i)\};$$

in projective coordinates, this is

$$C = \{(X,Y) \in \mathbb{P}^1 \times \mathbb{P}^1 \mid Y_1^2 \prod_{i=1}^{g+1}(X_1-\lambda_i X_0) = Y_0^2 \prod_{i=g+2}^{2g+2}(X_1-\lambda_i X_0)\}.$$

(No local analysis is needed to see that $C \subset \mathbb{P}^1 \times \mathbb{P}^1$ is smooth: it is a curve of bidegree $(2, g+1)$ in $\mathbb{P}^1 \times \mathbb{P}^1$, and the formula for the genus of a curve in $\mathbb{P}^1 \times \mathbb{P}^1$ derived in Section rational quartics section ?? tells us that such a curve has arithmetic genus g.)

was this done? =>

1.1.2 Differentials on a hyperelliptic curve

We can give a very concrete description of the differentials, and thus the canonical linear series, on a hyperelliptic curve C by working with the affine model $C^\circ = V(f) \subset \mathbb{A}^2$, where

$$f(x,y) = y^2 - \prod_{i=1}^{2g-2}(x-\lambda_i).$$

2g+2 ?

We will again denote the two points at infinity—that is, the two points of $C \setminus C^\circ$ by r and s; for convenience, we'll denote the divisor $r+s$ by D.

To start, consider the simple differential dx on C. (Technically, we should write this as π^*dx, since we mean the pullback to C of the differential dx on $\mathbb{P}^1$, but for simplicity of notation we'll suppress the π^*.) The function x is regular on C°, and is a local parameter over points other than the λ_i; from the local description of the map π, we see that dx is regular on C° with simple zeros at the ramification points $q_i = (\lambda_i, 0)$. But it does not extend to a regular differential on all of C: it will have double poles at r and s. This can be seen directly: the differential dx extends to a rational differential on

$\mathbb{P}^1$, and in terms of the local coordinate $w = 1/x$ around the point $x = \infty$ on $\mathbb{P}^1$, we have

$$dx = d\left(\frac{1}{w}\right) = \frac{-dw}{w^2}$$

so dx has a double pole at the point at ∞; since the map π is a local isomorphism near r and s the pullback of dx to C likewise has double poles at the points r and s.

We could also see that dx must have poles by degree considerations: as we said, dx has $2g + 2$ zeros and no poles in C°, while the degree of K_C is $2g - 2$, meaning that there must be a total of four poles at the points r and s. In any event, we have an expression for the canonical divisor class on C: denoting by $R = q_1 + \cdots + q_{2g+2}$ the sum of the ramifications points of π, we have

$$K_C \sim (dx) \sim R - 2D;$$

this is a case of the Riemann-Hurwitz of Chapter ??. linear systems chapter

So, given that dx has poles at r and s, how do we find regular differentials on C? One thing to do would be simply to divide by x^2 (or any quadratic polynomial in x) to kill the poles. But that just introduces new poles in the finite part C° of C. Instead, we want to multiply dx by a rational function with zeros at p and q, but *whose poles occur only at the points where dx has zeroes*—that is, the points q_i. A natural choice is simply the reciprocal of the partial derivative $f_y = \partial f/\partial y = 2y$, which vanishes exactly at the points q_i, and has correspondingly a pole of order $g + 1$ at each of the points r and s (reason: the involution $y \to -y$ fixes C° and x, and exchanges the points r and s). In other words, as long as $g \geq 1$, the differential

$$\omega = \frac{dx}{f_y}$$

is regular, with divisor

$$(\omega) = (g-1)r + (g-1)s = (g-1)D.$$

The remaining regular differentials on C are now easy to find: Since x has only a simple pole at the two points at infinity we can multiply ω by any x^k with $k = 0, 1, \ldots, g - 1$. Since this gives us g independent differentials, we see that the differentials

$$\omega, x\omega, \ldots, x^{g-1}\omega$$

form a basis for $H^0(K_C)$.

liptic special

1.1.3 The canonical map of a hyperelliptic curve

Given that a basis for $H^0(K_C)$ is given by

$$H^0(K_C) = \langle \omega, x\omega, \ldots, x^{g-1}\omega \rangle,$$

we see that the canonical map $\phi : C \to \mathbb{P}^{g-1}$ is given by $[1, x, \ldots, x^{g-1}]$. In other words, the canonical map ϕ is simply the composition of the map $\pi : C \to \mathbb{P}^1$ with the Veronese embedding $\mathbb{P}^1 \hookrightarrow \mathbb{P}^{g-1}$ of $\mathbb{P}^1$ into $\mathbb{P}^{g-1}$ as a rational normal curve of degree $g-1$.

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Note that as a consequence of this fact, we see that *a hyperelliptic curve C has a unique linear series g^1_2 of degree 2 and dimension 1*, that is, a unique map of degree 2 to $\mathbb{P}^1$. Finally, we can give an explicit description of special linear series on a hyperelliptic curve: if $D = \sum p_i$ is any effective divisor on C, we can pair up points p_i that are conjugate under the involution ι exchanging sheets of the degree 2 map $C \to \mathbb{P}^1$; each conjugate pair is a divisor of the unique g^1_2 on C, and so we can write

$$D \sim r \cdot g^1_2 + q_1 + \cdots + q_{d-2r},$$

where no two of the points q_i are conjugate under ι. Now the geometric form of the Riemann-Roch formula tells us that the dimension $r(D)$ of the complete linear series $|D|$ is exactly r, so that in fact

$$|D| = |r \cdot g^1_2| + + q_1 + \cdots + q_{d-2r};$$

K

that is, the points q_i are base points of the linear series D.

One key observation is that, according to this analysis, *no special linear series on a hyperelliptic curve can be very ample*; the map associated to any special series factors through the degree 2 map $C \to \mathbb{P}^1$. This is in marked contrast to the case of non-hyperelliptic curves, for which the embeddings of minimal degree in projective space are given by special linear series.

of genus ≥ 2

1.2 Curves of genus 2

Since curves of genus 2 are hyperelliptic, everything we said above applies to them; in particular, the canonical map $\phi_K : C \to \mathbb{P}^1$ on a curve of genus 2 is simply the expression of C as a double cover of $\mathbb{P}^1$. In this section, we'll consider other maps from C to projective space, starting with the simplest possible projective embedding of C.

1.2.1 Maps of C to $\mathbb{P}^1$

enus 2 pencil

Of course, C may be expressed as a degree 2 cover of $\mathbb{P}^1$ by taking the map associated to the canonical series $|K_C|$. But what about other degrees? For example, can we express C as a three-sheeted cover of $\mathbb{P}^1$?

The answer is "yes," and in fact we can do so in many ways. First start with a line bundle L of degree 3 on C. Riemann-Roch tells us immediately that $h^0(L) = 2$, and we see that there are two possibilities:

1. First, if the linear series $|L|$ has a base point $p \in C$, then $h^0(L(-p)) = 2$, and hence L must be of the form $L = K_C(p)$. Conversely, if $L = K_C(p)$, then $h^0(L(-p)) = h^0(L)$, which is to say p is a base point of $|L|$.
2. On the other hand, if L is not of the form $L = K_C(p)$, then $|L|$ does not have a base point, and so defines a degree 3 map $\phi_L : C \to \mathbb{P}^1$.

Do both possibilities occur? Certainly the first does; there's a one-parameter family of line bundles of the form $K_C(p)$. But we know that the variety $\operatorname{Pic}^3(C)$ is 2-dimensional, so we see that the general line bundle of degree 3 does give an expression of C as a 3-sheeted cover of $\mathbb{P}^1$; in fact there exists a 2-parameter family of such maps.

1.2.2 Maps of C to $\mathbb{P}^2$

Let's move on to consider maps of our curve C of genus 2 to the plane. By Riemann-Roch, a line bundle L of degree 4 on C will have $h^0(L) = 3$; and since $h^0(L(-p)) = 2$ for any point $p \in C$ (again by Riemann-Roch), we see that the linear series $|L|$ will give a regular map $\phi_L : C \to \mathbb{P}^2$. We ask now about the geometry of this map.

This again depends on the choice of L. This time there are three possibilities:

1. First, suppose $L = K_C^2$ is simply the square of the canonical line bundle on C. We have then a map

$$\operatorname{Sym}^2 H^0(K_C) \to H^0(L);$$

since both sides are 3-dimensional vector spaces and the map is injective, we have equality here; in other words, every divisor $D \sim K_C^2$ is the sum of two divisors $D_1, D_2 \in |K_C|$ in the canonical series. To express this in terms of the map ϕ_L, it says simply that that map ϕ_L is the composition of the canonical map $\phi_K : C \to \mathbb{P}^1$ with the Veronese embedding $\nu_2 : \mathbb{P}^1 \to \mathbb{P}^2$ of $\mathbb{P}^1$ as a conic curve in the plane. In other words, the map ϕ_L is generically 2-to-1 onto a conic in the plane.

2. Suppose now that L is not equal to K_C^2; equivalently, $M = L \otimes K_C^{-1}$ is a line bundle of degree 2 other than the canonical bundle. We have then $h^0(M) = 1$, so that M is the line bundle $\mathcal{O}_C(p+q)$ associated to a pair of points $p, q \in C$; in other words, $L = K_C(p+q)$ for a unique pair of points $p, q \in C$.

   Let's consider next the general case $p \neq q$. In this case, every section of L vanishing at p vanishes at q and vice versa, so that $\phi_L(p) = \phi_L(q)$. At the same time, for any effective divisor $D = r + s$ of degree 2 on C other than $p + q$, we have $h^0(L(-D)) = 1$, so apart from the fact that $\phi_L(p) = \phi_L(q)$, the map ϕ_L is an embedding. We'll see in Exercise ?? below that in fact the point $\phi_L(p) = \phi_L(q)$ is a node of the image curve $\phi_L(C)$; so to summarize: in this case ($L = K_C(p+q)$, with $p \neq q$ and $p+q \neq K_C$), the map $\phi_L : C \to \mathbb{P}^2$ is a birational embedding of C as a quartic plane curve with one node, the node being the common image of p and q.

3. Finally, the remaining case is where $L = K_C(2p)$, where $p \in C$ is any point such that $2p \not\sim K_C$. This behaves much like the preceding case, but here the map ϕ_L is one-to-one with vanishing differential at p, and the image curve $\phi_L(C)$ has correspondingly a cusp at the point $\phi_L(p)$.

To summarize: the map $\phi_L : C \to \mathbb{P}^2$ associated to a line bundle L of degree 4 on C is either

1. Two-to-one onto a plane conic curve, if $L = K_C^2$;

2. Birational onto a plane quartic curve with a cusp, if $L = K_C(2p)$ with $2p \not\sim K_C$; and

3. Birational onto a plane quartic curve with a node, if $L = K_C(p+q)$ with $p \neq q$ and $p + q \not\sim K_C$.

Note that the last case is the "general" one, meaning it holds for L in an open subset of $\mathrm{Pic}^4(C)$; the second case holds for a one-dimensional locus in $\mathrm{Pic}^4(C)$, and the first case holds for just one point in $\mathrm{Pic}^4(C)$.

Exercise 1.2.1. Let $L \in \mathrm{Pic}^4(C)$ be a line bundle of the form $L = K_C(p+q)$ with $p \neq q$ and $p+q \not\sim K_C$. Show that

1. $h^0(L(-2p)) = h^0(L(-2q)) = 1$, and
2. $h^0(L(-2p-2q)) = 0$.

Deduce from this that the map ϕ_L is an immersion, and that the tangent lines to the two branches of $\phi_L(C)$ at the point $\phi_L(p) = \phi_L(q)$ are distinct, meaning the point $\phi_L(p) = \phi_L(q)$ is a node of $\phi_L(C)$.

Embeddings in $\mathbb{P}^3$ make this a subsection

So far we have not found any embeddings of C in projective space; but that's about to change: if $L \in \mathrm{Pic}^5(C)$ is any line bundle of degree 5, by Corollary degree 2g+1 embedding ??, it is very ample and gives an embedding of C in $\mathbb{P}^3$. ~~Let's consider now what we can say about the geometry of the image curve.~~

Write ~~So: for the following, let L be any line bundle of degree 5 on our curve C, and~~ $\phi_L : C \to \mathbb{P}^3$ for the embedding given by the complete linear system $|L|$. ~~By a mild abuse of language,~~ We'll also denote the image $\phi_L(C) \subset \mathbb{P}^3$ by C.

~~The first question to ask is once more,~~ What degree surfaces in $\mathbb{P}^3$ contain the curve C? We start with degree 2, where we consider the restriction map

$$H^0(\mathcal{O}_{\mathbb{P}^3}(2)) \to H^0(\mathcal{O}_C(2)) = H^0(L^2).$$

The space on the left has dimension 10 ~~as always~~; ~~on the right~~, By the Riemann-Roch ~~tells us that~~ the $h^0(L^2) = 2 \cdot 5 - 2 + 1 = 9$. It follows that C must lie on a quadric surface Q; and by Bezout's the that Q is unique (since C can't lie on a union of planes, any quadric containing C must be irreducible; if there were more than one such, Bezout would imply that $\deg(C) \leq 4$).

We might ask at this point: is Q smooth or a quadric cone? The answer depends on the choice of line bundle L.

is 2 embedding

Proposition 1.2.2. *Let $C \subset \mathbb{P}^3$ be a smooth curve of degree 5 and genus 2 and $Q \subset \mathbb{P}^3$ the unique quadric containing C. If ~~$L = \mathcal{O}_C(1) \in \mathrm{Pic}^5(C)$, then~~ Q is singular if and only if we have*

$$\mathcal{O}_C(1)\, \cancel{L} \cong K^2(p)$$

for some point $p \in C$; in this case, the point p is the vertex of Q.

The fam $\mathrm{Pic}^5(C)$ has dim 2, and the subfamily of the $K^2(p)$ forms a

~~Note that there is a 2-parameter family of line bundles of degree 5 on C, of which~~ a one-dimensional subfamily ~~are of the form $K^2(p)$, conforming~~ ~~So that~~ Thus ~~to our naive expectation that~~ "in general" Q should be smooth, and that it should become singular in codimension 1.

Proof. First, suppose that the line bundle $L \cong K^2(p)$ for some $p \in C$. Then $L(-p) \cong K^2$, ~~meaning~~ so that the map $\pi : C \to \mathbb{P}^2$ given by projection from p is the map $\phi_{K^2} : C \to \mathbb{P}^2$ given by the square of the canonical bundle. As we've seen, the map ϕ_{K^2} is two-to-one onto a conic curve $E \subset \mathbb{P}^2$, ~~and~~ so ~~we see that~~ the curve C lies on the cone Q over E with vertex p, and this is the unique quadric surface containing C.

Next, let's consider the case where L is not of the form $K^2(p)$. Set $M = LK^{-1}$, so that we can write

$$L = K \otimes M,$$

where by hypothesis M is not of the form $K(p)$. As we saw in Section 1.2.1, genus 2 pencil this means that the pencil $|M|$ gives a degree 3 map $C \to \mathbb{P}^1$.

This gives us a way of factoring the map $\phi_L : C \to \mathbb{P}^3$: we have maps $\phi_K : C \to \mathbb{P}^1$ of degree 2 and $\phi_M : C \to \mathbb{P}^1$ of degree 3, and we can compose their product with the Segre embedding $\sigma : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^3$:

$$\begin{tikzcd}
 & \mathbb{P}^1 & & \\
C \arrow[ur, "\phi_K"] \arrow[rr, "\phi_K \times \phi_M"] \arrow[dr, "\phi_M"'] & & \mathbb{P}^1 \times \mathbb{P}^1 \arrow[ul] \arrow[dl] \arrow[r, "\sigma"] & \mathbb{P}^3 \\
 & \mathbb{P}^1 & &
\end{tikzcd}$$

This gives us a description of the map ϕ_L that shows us immediately that *C is a curve of type $(2, 3)$ on a smooth quadric $Q \subset \mathbb{P}^3$*, completing the proof of Proposition genus 2 embedding 1.2.2.

$\square$

Whether the quadric Q is smooth or not, we can describe a minimal set of generators of the homogeneous ideal $I(C) \subset \mathbb{C}[x_0, x_1, x_2, x_3]$ similarly. First, we look at the restriction map

$$H^0(\mathcal{O}_{\mathbb{P}^3}(3)) \to H^0(\mathcal{O}_C(3));$$

since the dimensions of these spaces are 20 and $15 - 2 + 1 = 14$ respectively, we see that vector space of cubics vanishing on C has dimension at least 6. Four of these are already accounted for: we can take the defining equation of Q and multiply it by any of the linear forms on $\mathbb{P}^3$; we conclude, accordingly, that *there are at least two cubics vanishing on C linearly independent modulo those vanishing on Q.*

In fact, we can prove the existence of these cubics geometrically, and show that there are no more than 2 linearly independent modulo the ideal of Q. Suppose first that Q is smooth, so that C is a curve of type $(2, 3)$ on Q. In that case, if $L \subset Q$ is any line of the first ruling, the sum $C + L$ is the complete intersection of Q with a cubic S_L, unique modulo the ideal of Q; conversely, if S is any cubic containing C but not containing S, the intersection $S \cap Q$ will be the union of C and a line L of the first ruling; thus, mod $I(Q)$, $S = S_L$. A similar argument applies in case Q is a cone, and L is any line of the (unique) ruling of Q.

Exercise 1.2.3. Show that for any pair of lines L, L' of the appropriate ruling of Q, the three polynomials Q, S_L and $S_{L'}$ generate the homogeneous ideal $I(C)$. Find relations among them. Write out the minimal resolution of $I(C)$. How?

The dimension of M_2 via maps to projective space

We remark here that each of the maps we've described from a curve C of genus 2 to projective space gives us a way of finding the dimension of the moduli space M_2 of curves of genus 2.

To start, we know that every curve C of genus 2 is uniquely expressible as a double cover of $\mathbb{P}^1$ branched at six points, modulo the group PGL_2 of automorphisms of $\mathbb{P}^1$. The space of such double covers has dimension 6, and $\dim(PGL_2) = 3$, so we may conclude that $\dim(M_2) = 6 - 3 = 3$.

Similarly, we've seen that a curve C of genus 2 is expressible as a 3-sheeted cover of $\mathbb{P}^1$ (with eight branch points) in a 2-dimensional family of ways. Such a triple cover is determined up to a finite number of choices by its branch divisor, so the space of such triple covers has dimension 8; modulo PGL_2 it has dimension 5, and since every curve is expressible as a triple cover in a two-dimensional family of ways, we arrive again at $\dim M_2 = 5 - 2 = 3$.

true but obvious?

We've also seen that C can be realized as (the normalization of) a plane quartic curve with a node in a 2-dimensional family of ways. The space of plane quartics has dimension 14; the family of those with a node has codimension one and hence dimension 13. Since the automorphism group PGL_3 of $\mathbb{P}^2$ has dimension 8, we see that the family of nodal plane quartics modulo PGL_3 has dimension 5, and since every curve of genus 2 corresponds to a 2-parameter family of such curves, we have $\dim M_2 = 5 - 2 = 3$.

Finally, a curve of genus 2 may be realized as a quintic curve in $\mathbb{P}^3$ in a two-parameter family of ways. To count the dimension of the family of such curves, note that each one lies on a unique quadric Q, and is of type $(2, 3)$ on Q. Thus to specify such a curve we have to specify Q (9 parameters) and then a bihomogeneous polynomial of bidgree $(2, 3)$ on $Q \cong \mathbb{P}^1 \times \mathbb{P}^1$ up to scalars; these have $3 \cdot 4 - 1 = 11$ parameters. Altogether, then, there is a 20-dimensional family of such curves; modulo the automorphism group PGL_4 of $\mathbb{P}^3$, this is a 5-dimensional family. Again, every abstract curve C of genus 2 corresponds to a 2-parameter family of these curves modulo PGL_4, so once more we have $\dim M_2 = 5 - 2 = 3$.

1.3 Curves of genus 3

If C be a smooth projective curve of genus 3. The is an immediate bifurcation into two cases, hyperelliptic and non-hyperelliptic curves; we will discuss hyperelliptic curves of any genus in Section ??, and so for the following we'll assume C is nonhyperellitic. By our general theorem ??, this means that the canonical map $\phi_K : C \to \mathbb{P}^2$ embeds C as a smooth plane quartic curve; and should be in Ch 1

curve

conversely, by adjunction any smooth plane of degree 4 has genus 3 and is canonical (that is, $\mathcal{O}_C(1) \cong K_C$).
((maybe a reference to the plane curve chapter for differentials etc?))

Note that this gives us a way to determine the dimension of the moduli space M_3 of smooth curves of genus 3: if $\mathbb{P}^{14}$ is the space of all plane quartic curves, and $U \subset \mathbb{P}^{14}$ the open subset corresponding to smooth curves, we have a dominant map $U \to M_3$ whose fibers are isomorphic to the 8-dimensional affine group PGL_3. (Actually, the fiber over a point $[C] \in M_3$ is isomorphic to the quotient of PGL_3 by the automorphism group of C; but since $Aut(C)$ is finite this is still 8-dimensional.) We conclude, therefore, that

$$\dim M_3 = 14 - 8 = 6.$$

What about other linear series on C, and the corresponding models of C? To start with, by hypothesis C has no g^1_2s; that is, it is not expressible as a 2-sheeted cover of $\mathbb{P}^1$. On the other hand, it is expressible as a 3-sheeted cover: if $L \in \mathrm{Pic}^3(C)$ is a line bundle of degree 3, by Riemann-Roch we have

$$h^0(L) = \begin{cases} 2, & \text{if } L \cong K - p \text{ for some point } p \in C;\text{ and} \\ 1 & \text{otherwise.} \end{cases}$$

There is thus a 1-dimensional family of representations of C as a 3-sheeted cover of $\mathbb{P}^1$. In fact, these are plainly visible from the canonical model: the degree 3 map $\phi_{K-p} : C \to \mathbb{P}^1$ is just the composition of the canonical embedding $\phi_K : C \to \mathbb{P}^2$ with the projection from the point p.

There are of course other representations of C as the normalization of a plane curve. By Riemann-Roch, C will have no g^2_3s and the canonical series is the only g^2_4, but there are plenty of models as plane quintic curves: by Proposition ??, if L is any line bundle of degree 5, the linear series $|L|$ will be a base-point-free g^2_5 as long as L is not of the form $K + p$, so that ϕ_L maps C birationally onto a plane quintic curve $C_0 \subset \mathbb{P}^2$. But these can also be described geometrically in terms of the canonical model: any such line bundle L is of the form $2K - p - q - r$ for some trio of points $p, q, r \in C$ that are not colinear in the canonical model, and we see correspondingly that C_0 is obtained from the canonical model of C by applying a Cremona transform with respect to the points p, q and r.

This is nice! but refers things we have not explained

We can also embed C in $\mathbb{P}^3$ as a smooth sextic curve by Proposition ** ??; in fact, a line bundle $L \in \mathrm{Pic}^6(C)$ of degree 6 will be very ample if and only if it is not of the form $K + p + q$ for any $p, q \in C$. One cheerful fact in this connection is that these curves are determinantal:

Exercise 1.~~3.1.~~ ~~Let C~~ $\subset \mathbb{P}^3$ be a smooth non-hyperelliptic curve of degree 3 and genus 6. Show that there exists a 3×4 matrix M of linear forms on $\mathbb{P}^3$ such that

$$C = \{p \in \mathbb{P}^3 \mid \mathrm{rank}(M(p)) \leq 2\}.$$

needs a hint.