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Plane algebraic curves with many cusps, with an appendix by Eugenii Shustin

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Abstract The maximum number $k(d)$ of cusps on a plane algebraic curve of degree d is an open classical problem that dates back to the nineteenth century. A related open problem is the asymptotic value ($a.v.$) of the number of cusps on plane curves, that is $a.v. = \limsup_{d\to\infty} k(d)/d^2$. In this paper, we improve the best known lower bound for the asymptotic value by constructing curves with the largest known number of cusps for infinitely many degrees. Some particular curves of relatively low degree with many cusps are constructed too. The "Appendix" to this paper is devoted to the case of degree 11 and it is due to E. Shustin.

Keywords Algebraic curves · Ordinary cusps · Asymptotic value

Mathematics Subject Classification (2010) 14H50

1 Introduction

In this paper, a *cusp* of an irreducible plane algebraic curve is a singularity defined locally by $x^2 + y^3 = 0$. Usually, it is called a double point of type A_2, or an ordinary cusp, but we omit the adjective "ordinary" for short. We are interested in irreducible plane algebraic curves with only cusps as singularities.

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Let $k(d)$ be the maximum number of cusps on a plane curve of degree d with only cusps as singularities. Even if $k(d)$ has been studied for over than one and a half century, the exact value of $k(d)$ is only known for $d \leq 8$ and $d = 10$, see some references in Sect. 2.

A related open problem is to find

$$a.v. = \limsup_{d\to\infty} \frac{k(d)}{d^2},$$

which we call the *asymptotic value* of the number of cusps on plane curves.

According to our knowledge of the literature, the best known upper bound of $a.v.$ is

$$a.v. \leq \frac{125+\sqrt{73}}{432} = 0,30912963\ldots$$

and it is due to Langer [18]. See Sect. 2 for other previous upper bounds.

On the other hand, up to now, the best known lower bound of $a.v.$ is

$$a.v. \geq \frac{283}{960} = 0.294791667\ldots,$$

it is due to Kulikov [16], and it is based on the ideas contained in the thesis [21] of the second author. It has been obtained by constructing plane curves with many cusps for infinitely many degrees. See Sect. 2 for other previous lower bounds of $a.v.$ and Sect. 3 for the ideas used for these constructions.

In this paper, we construct plane curves of degree $d = 4\cdot 3^m$, $m \geq 0$, which for $m \geq 2$ improve the largest known number of cusps on curves of degree d, and we get the following lower bound:

$$a.v. \geq \frac{2567}{8640} = 0.297106481\ldots$$

See Remarks 8 and 9 in Sect. 4 for other better lower bounds under some mild assumptions which we believe to be true but we are not able to prove.

Our constructions start from a curve of degree 4 with three cusps. We then run the same process by starting from curves of other degrees, namely 3, 5 and 6, with the maximum number of cusps, and we improve the largest known number of cusps for other infinitely many degrees. However, these other constructions do not imply a better value of $a.v.$

In the “Appendix”, E. Shustin shows the existence of a plane curve of degree 11 with 30 cusps, which proves that $k(11) \geq 30$, while $k(11) \leq 31$ follows by Varchenko’s results in [30].

In this paper, we do not deal with other interesting problems concerning plane curves with cusps, which are still active research subjects nowadays, like for example, the properties of the variety parametrizing plane curves with (nodes and) cusps, bounds for the number of cusps in terms of the genus, rational curves with unibranched singularities (non-ordinary cusps), branch curves of generic projections of surfaces, cf. [3,4,7,14,23,28,29] for very recent papers.

Let us resume very briefly the content of this paper.

In Sect. 2, we recall classical and more recent results about the maximum number $k(d)$ of cusps on plane curves of degree d and about the asymptotic value.

In Sect. 3, we describe the main ideas used for constructing plane curves with many cusps for infinitely many degrees, and finally in Sect. 4, we explain our new constructions.

The “Appendix” by E. Shustin is devoted to the case of plane curves of degree 11.

We work over the field $\mathbb{C}$ of complex numbers, even if our constructions can be done over any algebraically closed field of characteristic zero.

2 Some known results

A classical upper bound for $k(d),\ d \geq 7$, coming from Plücker formulas is

$$k(d) \leq d(d-2)/3,$$

and it is due to Lefschetz [19]. Recall that Plücker formulas assume that a curve and its dual have only nodes and cusps as singularities, but we will construct curves whose dual has higher order singularities. For generalized Plücker formulas, see [9, 17].

We collect known results and references about $k(d)$ for small values of d in Table 1, where bold numbers mean that the upper bound of $k(d)$ equals the lower bound.

2.1 Historical note

The dual of a quartic curve with 1 cusp and 1 node is a septic curve with 10 cusps and 4 nodes. Zariski [32] proved that there is no octic curve with 16 cusps and that there is no septic curve with 11 cusps. He remarks that the dual of a sextic curve with 7 cusps and 1 node is a septic curve with 10 cusps and 3 nodes. In later papers, Zariski proved that the nodes on the septic curve can be smoothed independently, which implies the existence of a septic curve with 10 cusps and no node.

Recall that in 1882, at p. 89 of [2], Crone claimed (in a context which is not clear to all the authors) that there is no sextic curve with 7 cusps and 1 node. Crone is cited also in the Additions of Enriques–Chisini's book [6, vol. 1, p. 709], concerning the problem of the existence of plane curves with given Plückerian characters. We notice that an easy equation of a sextic curve with 7 cusps and 1 node can be given in the form $F_3^2 + F_2^3 = 0$ (cf. [26]) where the cubic curve $C_3 : F_3 = 0$ and the conic $C_2 : F_2 = 0$ are given by

$$F_3(x, y) = 1 + xy + xy^2 - 3/2x^2y + x^3 - 61/24y^2 + y^3, \quad F_2(x, y) = -1 + xy - x^2 + y^2.$$

The sextic $F_3^2 + F_2^3 = 0$ has 6 cusps at $C_3 \cap C_2$, one cusp at the origin (0, 0) and one node at the point at infinity of the x-axis.

Therefore, a definitive negative answer to the question of the existence of curves with given Plückerian characters was given by the above mentioned paper [32] by Zariski.

Concerning curves of degree 8, Manara [20] proved the existence of an octic curve with 14 cusps and 2 nodes. Since the number of cusps is smaller than three times the degree, Zariski's

Table 1 Known upper and lower bounds to $k(d)$ for small d

d	$k(d) \leq$		Realized cusps	
3	**1**		**1**	Diocles
4	**3**		**3**	Euler, Steiner
5	**5**	Plücker	**5**	Del Pezzo [5]
6	**9**	Plücker	**9**	Dual of a smooth cubic
7	**10**	Zariski [32]	**10**	Cf. historical note 2.1
8	**15**	Zariski [32]	**15**	Shustin [27]
9	21	Lefschetz [19], Varchenko [30]	20	Shustin [27]
10	**26**	Lefschetz [19], Varchenko [30]	**26**	Koelman [15]
11	31	Varchenko [30]	30	Shustin ["Appendix" to this paper]
12	40	Lefschetz [19], Ivinskis [13]	39	Hirano [11]

results imply that the 2 nodes can be smoothed. A direct construction of an octic curve with 14 cusps and no node is due to Hirano [11]. Finally, Shustin [27] proved the existence of an octic curve with 15 cusps.

In the 1980s, Varchenko [30] gave the upper bound $k(d) \leq B(d)/2$, where $B(d)$ is the number of pairs (k_1, k_2) such that k_1, k_2 are integers with $0 < k_i < d$, $i = 1, 2$, and $[d/6] + 1 < k_1 + k_2 \leq 7d/6$, which implies that $a.v. \leq 23/72 = 0.3194\ldots$ for the asymptotic value. In the same years, Ivinskis [13] and Hirzebruch [12] gave the following upper bound:

$$k(d) \leq d(5d - 6)/16,$$

for even $d \geq 6$ (cf. also Sakai [24]), which implies that $a.v. \leq 5/16 = 0.3125$ for the asymptotic value. Up to now, the best upper bound for $a.v.$ has been obtained by Langer [18] and it is mentioned in the Sect. 1.

On the other hand, lower bounds for $k(d)$ can be obtained by constructing plane curves with many cusps. Recent constructions for infinitely many degrees are due to Ivinskis [13], Hirano [11] and few years ago by Paccagnan [21] and Kulikov [16].

Ivinskis constructed curves of degree $d = 6r$ with $9r^2 = d^2/4$ cusps by taking (r^2)-fold coverings of a sextic curve with 9 cusps, which implies $a.v. \geq 1/4 = 0.25$. Hirano then constructed curves of degree $d = 2 \cdot 3^m$ with $9(9^m - 1)/8 = 9d^2/32$ cusps, which implies $a.v. \geq 9/32 = 0.28125$. By improving their ideas in his thesis [21], the second author of this paper claimed in [22] that $a.v. \geq 1019/3456 = 0,294849537\ldots$ under some mild assumptions, described in Remark 9 in Sect. 4, which have not been proven. Kulikov [16] instead proved the lower bound $a.v. \geq 283/960 = 0.294791667\ldots$ In Sect. 3, we describe the ideas used in these constructions.

We collect the known upper and lower bounds for $a.v.$ in Table 2.

Concerning some relatively small values of d, we compare the results about $k(d)$ by Ivinskis, Hirano, Kulikov and this paper in Table 3, where the numbers marked with an

Table 2 Known upper and lower bounds for $a.v$

$a.v. \geq$		$a.v. \leq$	
Ivinskis	$1/4 = 0.25$	Plücker–Lefschetz	$1/3 = 0.3333\ldots$
Hirano	$9/32 = 0.28125$	Varchenko	$23/72 = 0.3194\ldots$
Kulikov	$283/960 = 0.29479\ldots$	Hirzebruch–Ivinskis	$5/16 = 0.3125$
This paper	$2567/8640 = 0.29716\ldots$	Langer	$(125 + \sqrt{73})/432 = 0.3091\ldots$

Table 3 Comparison of results by Ivinskis, Hirano, Kulikov and this paper

d	$k(d) \leq$		Realized cusps			
	Lefschetz	Ivinskis	Ivinskis	Hirano	Kulikov	This paper
12	40	40	36	39	39	39
18	96	94	81	90		90
36	408	391	324	360*	372	375
54	936	891	729	819		837
108	3816	3604	2916	3249*	3411	3438

asterisk (*) are not explicitly written by Hirano [11], but they can be obtained by the same arguments therein.

3 Main ideas

When $X_d : F_d(x_0, x_1, \ldots, x_n) = 0$ is a hypersurface in $\mathbb{P}^n$, the equation $F_{rd}(x_0, \ldots, x_n) = F_d(x_0^r, x_1^r, \ldots, x_n^r) = 0$ defines a (r^n)-fold covering X_{rd} of X_d which is ramified over the coordinate hyperplanes and totally ramified over the coordinate points. If X_d has a singularity of a given type at a point P off the coordinate hyperplanes, then X_{rd} has r^n singularities of the same type at the r^n points over P.

This fact has been classically used to produce hypersurfaces with many singular points of the same type, see for example, the papers by Segre [25] and Gallarati [8] regarding surfaces in $\mathbb{P}^3$ with isolated singular points.

From now on, we deal only with plane curves having cusps.

Ivinskis [13] uses the above r^2-fold coverings of a sextic curve with 9 cusps.

Let x_0, x_1, x_2 be the homogeneous coordinates of the projective plane $\mathbb{P}^2$. Let A_1, A_2, A_3 be the coordinate points $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$, respectively. Set $T : x_0x_1x_2 = 0$, which we call the *fundamental triangle*. Let $V = \{A_1, A_2, A_3\}$ be the set of vertices of T.

Let $C_d : F_d = 0$ be an irreducible algebraic curve of degree d in $\mathbb{P}^2$, where $F_d \in \mathbb{C}[x_0, x_1, x_2]$ is a homogeneous polynomial of degree d. If $C_d \cap V = \emptyset$, we say that $C_{3d} : F_{3d}(x_0, x_1, x_2) = F_d(x_0^3, x_1^3, x_2^3) = 0$ is the *standard 9-fold covering* of C_d ramified over T.

We say that a line ℓ is a *proper m-tangent* to C_d if ℓ is tangent to C_d at m distinct smooth points of C_d and ℓ meets C_d in other $d - 2m$ distinct points. If $m = 1$ [resp., $m = 2$], we say that ℓ is a *proper tangent* [resp., *bitangent*] to C_d.

Hirano [11] remarks that, if one side ℓ of T is a proper m-tangent line to C_d at points off V, the standard 9-fold covering C_{3d} of C_d then has $3m$ cusps over the m points of tangency of ℓ to C_d. Note that the cusps of C_{3d}, coming from either cusps of C_d or points of tangency of T to C_d, are distinct.

In his thesis [21], the second author remarks that, if t is a proper m-tangent line to C_d at a point $P \notin T$ and t passes through a vertex of T, the standard ninefold covering C_{3d} of C_d has three proper $3m$-tangent lines over t, each one passing through the same vertex of T.

The proofs of the above statements are straightforward. For the reader's convenience, we sketch the proofs here, considering the affine coordinates $x = x_1/x_0,\ y = x_2/x_0$.

Following Hirano, if the point P of tangency of Y to C_d is $(x, y) = (0, b)$, then

$$F_d(x, y, 1) = x + (y - b)^2 + \cdots, \qquad F_d(x^3, y^3, 1) = x^3 + (y^3 - b)^2 + \cdots$$

Concerning the second author's remark, if $P = (r, s)$ is the point of tangency, then

$$F_d(x, y, 1) = sx - ry + (x - r)^2 + \cdots, \quad F_d(x^3, y^3, 1) = sx^3 - ry^3 + (x^3 - r)^2 + \cdots$$

4 Constructions

Start from a smooth conic $C_2 : F_2(x_0, x_1, x_2) = 0$ which is tangent to $T : x_0x_1x_2 = 0$ at three distinct points (different from vertices). By applying the standard Cremona quadratic transformation $(x_0, x_1, x_2) \mapsto (x_1x_2, x_0x_2, x_0x_1)$, one gets a quartic curve $C_4 : F_2(x_1x_2, x_0x_2, x_0x_1) = 0$ with three cusps. Change then coordinates in $\mathbb{P}^2$ in such a way that (cf. Fig. 1).

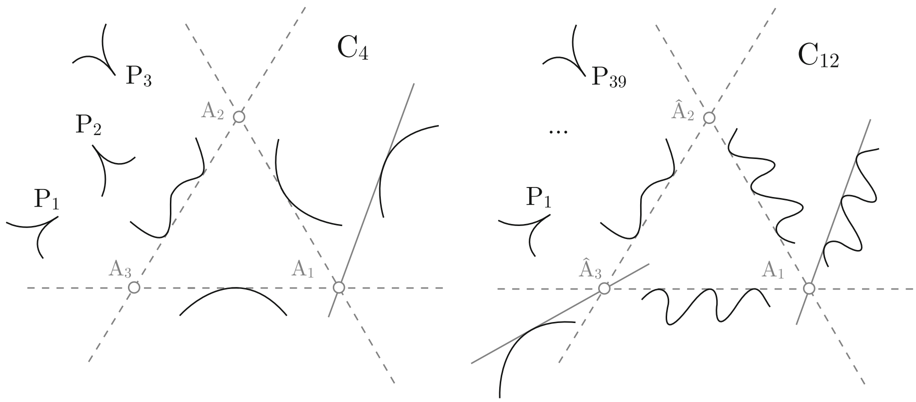

Fig. 1 The curves C_4 and C_{12}

- the sides of T are the bitangent line to C_4 and two proper tangent lines to C_4,
- the curve C_4 does not pass through the vertices of T.

By suitably choosing the conic and the tangent lines, we get that C_4 has the cusps at the points $(1, 8, 27)$, $(1, 1, -64)$ and $(1, -27, 1)$ and its equation is

$$\begin{aligned} F_4 =& -252420168x_0x_1x_2^2 - 480335856x_0x_1^2x_2 + 977700152x_0x_1^3 + 45532564x_0x_2^3 \\ & -12794804022x_0^2x_1x_2 + 21001409286x_0^2x_1^2 + 1677705414x_0^2x_2^2 + 120649804704x_0^3x_1 \\ & + 4578709212x_0^3x_2 + 187278x_1x_2^3 + 4546269x_1^2x_2^2 + 1199562x_1^3x_2 - 492013864969x_0^4 \\ & + 14480427x_1^4 + 352947x_2^4 = 0. \end{aligned}$$

The standard ninefold covering C_{12} of C_4 ramified over T then has 39 cusps and no other singularity: 27 cusps come from the 3 cusps of C_4, and the other 12 cusps come from the 4 points of tangency to C_4 lying on T. This construction is due to Hirano. We recall that Plücker formulas imply $k(12) \leq 40$ but no example of curve with degree 12 and 40 cusps is known.

Following the second author, we remark that there exists a proper tangent line ℓ to C_4 at a point off T and passing through A_1, which produces three proper 3-tangent lines ℓ_1, ℓ_2, ℓ_3 to C_{12} passing through A_1. The equation of ℓ is $403x_1 - 119x_2 = 0$, thus $\ell_1\ell_2\ell_3\colon 403x_1^3 - 119x_2^3 = 0$.

By using computer algebra software, one can compute the dual curve $\tilde{C}_{15}$ of degree 15 to C_{12} and one can check that C_{12} has 18 proper bitangent lines. We choose one of them and we call it r. Setting ρ a root of the polynomial $33605209 + 3240523t^3 + 199927t^6$, the equation of r is $179707\rho x_0 - 179707x_1 + (-121737 + 1183\rho^3)x_2 = 0$.

Remark 1 A classical method to compute the dual of a plane curve is as follows. Let $F(x, y) = 0$ be the affine equation of degree d of a plane curve C_d. Replacing y with $\alpha x + \beta$ in F and computing the discriminant of $F(x, \alpha x + \beta)$ with respect to x, one finds a polynomial of degree $d(d-1)$ in the unknowns α, β which is a multiple of the affine equation of the dual curve to C_d. See [1] for a recent paper dealing with the problem of computing the dual of a plane curve.

The equations of C_{12} and of its dual, and the coordinates of the cusps of C_{12}, are relatively simple because we conveniently chose the coordinates of the cusps of C_4.

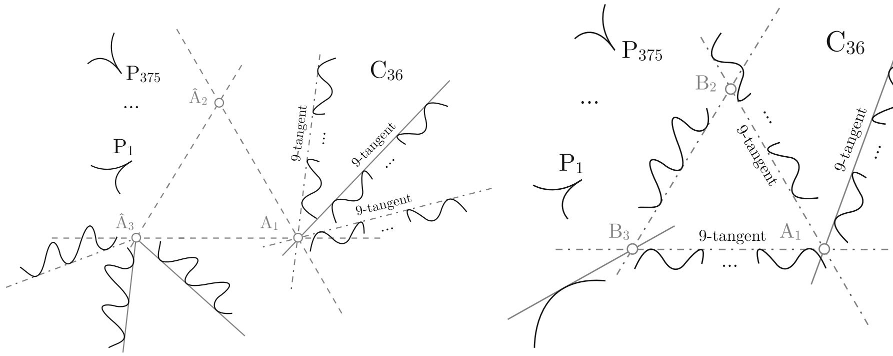

Fig. 2 The curve C_{36} with respect to the triangles T' and T''

Setting T' the triangle whose sides are ℓ_1, ℓ_2 (which are proper 3-tangent lines to C_{12}) and r (a proper bitangent line to C_{12}), one checks that C_{12} does not pass through the vertices of T'. Note that $\ell_1 \cap \ell_2 = A_1$. Furthermore, there exists a proper tangent line t to C_{12} at a point off T' and passing through the vertex $\hat{A}_3 = r \cap \ell_1 \notin C_{12}$, cf. Fig. 1. The equation of t is too complicated to be written here.

Recall that, according to the definition, proper m-tangent lines to a curve C_d do not pass through any singularity of C_d, in particular through the cusps of C_d.

The standard ninefold covering C_{36} of C_{12} ramified over T' has degree 36 with 375 cusps and no other singularity: 351 cusps come from the 39 cusps of C_{12}, and 24 cusps come from the 8 points of tangency to C_{12} lying on T'. This example shows the following:

Lemma 2 *When $d = 36$, one has $k(36) \geq 375$.*

Hirano's [11] arguments give $k(36) \geq 360$, and Kulikov [16] showed that $k(36) \geq 372$.

Note that the proper 3-tangent line ℓ_3 to C_{12}, which passes through $A_1 \notin C_{12}$, produces three proper 9-tangent lines $\ell'_1, \ell'_2, \ell'_3$ to C_{36}, each one passing through A_1, and the proper tangent line t to C_{12}, passing through $\hat{A}_3$, produces three proper 3-tangent lines to C_{36} passing through $\hat{A}_3$; call r' one of them.

Now, set T'' the triangle whose sides are ℓ'_1, ℓ'_2, r', cf. Fig. 2. Note that $A_1 = \ell'_1 \cap \ell'_2 \in T''$. The standard ninefold covering C_{108} of C_{36} ramified over T'' has degree 108 with 3438 cusps and no other singularity: 3375 cusps come from the 375 cusps of C_{36}, and 63 cusps come from the 21 points of tangency to C_{36} on T''. Therefore, the following lemma holds.

Lemma 3 *When $d = 108$, one has $k(108) \geq 3438$.*

The previous known lower bound was $k(108) \geq 3411$ by Kulikov [16].

The proper 9-tangent line ℓ'_3 to C_{36} produces three proper 27-tangent lines $\ell''_1, \ell''_2, \ell''_3$ to C_{108}, and we will choose ℓ''_1, ℓ''_2 as sides of the new fundamental triangle T'''.

Remark 4 Choose a general point $P \in \ell''_1$. Then,

- there are lines r''_1, r''_2 passing through P such that r''_i, $i = 1, 2$, is a proper tangent line to C_{108} at a point distinct from P;
- C_{108} does not pass through the vertices of the quadrangle with sides $\ell''_1, \ell''_2, r''_1, r''_2$.

Indeed, the point P is the dual of a general line ℓ_P in the dual plane passing through the point P''_1, dual of the line ℓ''_1. Setting $\tilde{C}$ the dual curve of C_{108}, the line ℓ_P passes through smooth points of $\tilde{C}$ off P''_1, which are the dual of proper tangent lines to C_{108}.

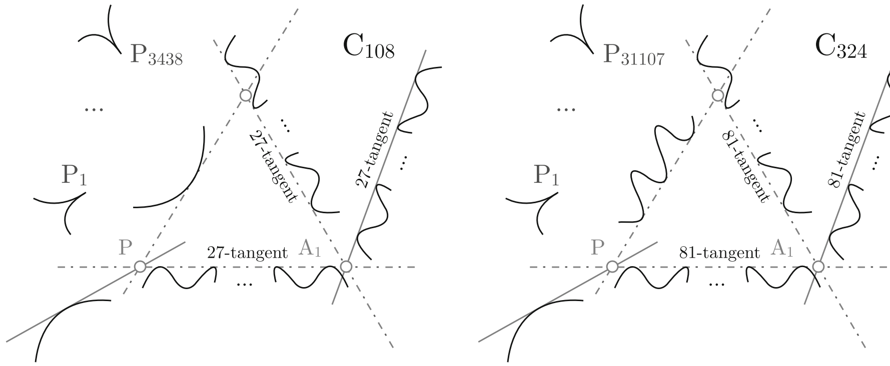

Fig. 3 The curves C_{108} with 3438 cusps and C_{324} with 31107 cusps

Let T''' be the triangle with sides $\ell_1'', \ell_2'', r_1''$, cf. Fig. 3. The standard ninefold covering C_{324} of C_{108} ramified over T''' then has

$$3438 \cdot 9 + (27 + 27 + 1)3 = 31107 \text{ cusps}$$

and no other singularity. Note that the proper tangent line r_2'' to C_{108} at a point off T''' and passing through $P = \ell_1'' \cap r_1''$ produces three proper 3-tangent lines to C_{324}, at points off T''', passing through P.

Remark 5 Our computers are not able to show that there exists a proper tangent line to C_{36} at a point off T'' and passing through a vertex of T'' different from A_1. This would imply that there exists a proper 3-tangent line to C_{108} which we could use as a side of the triangle T''' instead of the proper tangent line r_1'' and that would produce more cusps of C_{324}.

For the same reason, we are not able to check if we could replace r_1'' in the triangle T''' with a proper bitangent line to C_{108} that also would produce more cusps of C_{324}.

At the next step, choose the triangle whose sides are two proper 81-tangent lines to C_{324} and a proper 3-tangent line to C_{324}. We then get a curve C_{972} of degree 972 with

$$31107 \cdot 9 + (81 + 81 + 3)3 = 280458 \text{ cusps}$$

and no other singularity, cf. Fig. 4.

Go on similarly: for each odd positive integer n, choose a triangle T_n whose sides are two proper $(27 \cdot 3^n)$-tangent lines and a proper 3-tangent line to $C_{108\cdot 3^n}$. If k_n denotes the number of cusps of $C_{108\cdot 3^n}$, the standard ninefold covering $C_{108\cdot 3^{n+1}}$ of $C_{108\cdot 3^n}$ ramified over T_n has $9k_n + (2 \cdot 27 \cdot 3^n + 3)3$ cusps and no other singularity.

For each even positive integer n, we can choose two proper $(27 \cdot 3^n)$-tangent lines $\bar{\ell}_1, \bar{\ell}_2$ to $C_{108\cdot 3^n}$ as sides of the new fundamental triangle T_n. A general point $P \in \bar{\ell}_1$ is such that there exist two proper tangent lines $\bar{r}_1, \bar{r}_2$ to $C_{108\cdot 3^n}$ at points off P and passing through P with the property that $C_{108\cdot 3^n}$ does not pass through the vertices of the triangle T_n with sides $\bar{\ell}_1, \bar{\ell}_2, \bar{r}_1$, cf. Remark 4. If k_n denotes the number of cusps of $C_{108\cdot 3^n}$, the standard ninefold covering $C_{108\cdot 3^{n+1}}$ of $C_{108\cdot 3^n}$ ramified over T_n has $9k_n + (2 \cdot 27 \cdot 3^n + 1)3$ cusps and no other singularity.

The proper tangent line $\bar{r}_2$ to $C_{108\cdot 3^n}$ produces proper 3-tangent lines to $C_{108\cdot 3^{n+1}}$ which will be used as one side of the fundamental triangle for the subsequent step.

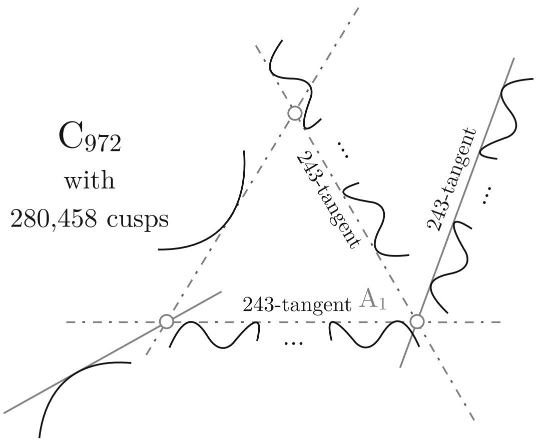

Fig. 4 The curve C_{972} with 280458 cusps

This process gives a curve of degree $108 \cdot 3^{2n}$ for each n with

$$9^{2n} \cdot 3438 + 2 \cdot 27 \cdot 3 \sum_{i=0}^{2n-1} \left(9^i \cdot 3^{2n-1-i}\right) + 3 \cdot 3 \sum_{i=0}^{n-1} 9^{2i} + 3 \cdot 9 \sum_{i=0}^{n-1} 9^{2i}$$
$$= 9^{2n} \cdot 3438 + 2 \cdot 27 \cdot 9^n \frac{9^n - 1}{3 - 1} + 36 \frac{9^{2n} - 1}{9^2 - 1} = \left(3465 + \frac{36}{80}\right) 9^{2n} - 27 \cdot 9^n - \frac{36}{80}$$

cusps and no other singularity. Therefore, the following theorem holds.

Theorem 6 *For each positive integer n, one has*

$$k(108 \cdot 9^n) \geq \frac{69309}{20} 9^{2n} - 27 \cdot 9^n - \frac{9}{20}.$$

□

The above process gives the largest known number of cusps on curves of degree $108 \cdot 3^m$ for odd m too. We leave those computations to the reader.

Corollary 7 *The asymptotic value $a.v.$ of the number of cusps on plane curves is*

$$a.v. \geq \frac{69309}{20 \cdot 108^2} = \frac{2567}{8640} = 0.297106481\ldots$$

□

The above lower bound of $a.v.$ is better that Kulikov's lower bound 283/960 and that of the second author in [22], which was 1019/3456.

Remark 8 If, at each step, one could check that there exists a proper bitangent line that can be chosen as one side of the fundamental triangle, together with the two proper $(27 \cdot 3^n)$-tangent lines, one would then get the lower bound

$$a.v. \geq \frac{13863}{4 \cdot 108^2} = \frac{4621}{15552} = 0.297132202\ldots$$

Remark 9 If one could choose as one side of the fundamental triangle a proper 3-tangent line also at each step with even n, one would then get the improved lower bound

$$a.v. \geq \frac{27729}{8 \cdot 108^2} = \frac{1027}{3456} = 0.297164352\dots$$

Further constructions Let $C_3 : F_3(x_0, x_1, x_2) = 0$ be a cubic curve with a cusp. We choose the fundamental triangle $T : x_0x_1x_2 = 0$ in such a way that the sides of T are proper tangent lines to C_3 at points different from the vertices of T and moreover such that, for each vertex P of T, there is a further tangent line to C_3 at a point $\notin T$ and passing through P. The corresponding C_9 has 18 cusps (cf. Hirano [11]) and nine proper 3-tangent lines. Assuming three of them as new fundamental triangle, the curve C_{27} has $18 \cdot 9 + 27 = 189$ cusps.

Next, we consider Del Pezzo quintic C_5 with 5 cusps, cf. [5]. We choose the vertices and the sides of the fundamental triangle T with the same properties as before for the cubic C_3. The corresponding C_{15} has 54 cusps, plus at least nine proper 3-tangent lines. The corresponding C_{45} has $54 \cdot 9 + 27 = 513$ cusps.

Similarly, starting from a C_6 with 9 cusps, we obtain a C_{54} with $90 \cdot 9 + 27 = 837$ cusps, and so on.

By running the same process of the previous construction, we get the following

Theorem 10 *For each positive integer n, one has*

$$k(45 \cdot 9^n) \geq \frac{10449}{20} 9^{2n} - 9 \cdot 9^n - \frac{9}{20}, \qquad k(54 \cdot 9^n) \geq \frac{16929}{20} 9^{2n} - 9 \cdot 9^n - \frac{9}{20},$$
$$k(81 \cdot 9^n) \geq \frac{35829}{20} 9^{2n} - 27 \cdot 9^n - \frac{9}{20}.$$

More precisely, we improve the largest known number of cusps on plane curves of degree d with $d = 3 \cdot 3^n, 5 \cdot 3^n, 6 \cdot 3^n$, for each $n \geq 2$.

Remark 11 The irreducibility of all the constructed curves can be proved simply noting that they have only cusps as singularities.

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Appendix: Plane curve of degree 11 with 30 cusps

This appendix is devoted to the proof of the following:

Proposition 12 *There exists a complex plane curve of degree* 11 *with* 30 *cusps as its only singularities.*

Proof We present an explicit construction based on the Viro's patchworking method [31] adapted for singular algebraic varieties in [27]. For the reader's convenience, we shortly explain here the main idea of the method, referring to [27] for all details. We want to find a plane algebraic curve given by a polynomial with Newton polygon $\Delta \subset \mathbb{R}^2$ (a non-degenerate convex lattice polygon), which has a prescribed collection of isolated singularities in the torus $(\mathbb{C}^*)^2$. Our construction starts with the following combinatorial and algebraic data:

- a convex, piecewise linear, integral-valued function $\nu : \Delta \to \mathbb{R}$, whose linearity domains $\Delta_1, \dots, \Delta_N \subset \Delta$ are non-degenerate convex lattice polygons;

- polynomials $F_k(x, y) = \sum_{(i,j)\in\Delta_k} a_{ij}x^i y^j$ with Newton polygons Δ_k, $k = 1, \dots, N$, respectively, which define curves $C_1, \dots, C_N$ with certain isolated singularities in $(\mathbb{C}^*)^2$.

Then, we consider a family of curves $C_{(t)}$, $t \in (\mathbb{C}, 0)$, $t \neq 0$, with Newton polygon Δ given by

$$F_{(t)}(x, y) = \sum_{(i,j)\in\Delta} (a_{ij} + O(t))t^{\nu(i,j)}x^i y^j,$$

and we claim that, for t close to zero, the singular points of $C_{(t)}$ in $(\mathbb{C}^*)^2$ are in one-to-one correspondence with the disjoint union of singular points of $C_1, \dots, C_N$ in $(\mathbb{C}^*)^2$, and the corresponding points are of the same type. The task is to define suitable coefficients $a_{ij} + O(t)$, $(i, j) \in \Delta$. Given $k = 1, \dots, N$, the restriction of $\nu(i, j)$ to Δ_k is a linear function $\alpha i+\beta j+\gamma$. The singular points in $(\mathbb{C}^*)^2$ of the curve $C_{(t),k}$ defined by the polynomial $F_{(t),k}(x, y) := t^{-\gamma} F_{(t)}(xt^{-\alpha}, yt^{-\beta})$ are those of $C_{(t)}$ moved in $(\mathbb{C}^*)^2$ by the transformation $(x, y) \mapsto (xt^{-\alpha}, yt^{-\beta})$. On the other hand,

$$F_{(t),k} = \sum_{(i,j)\in\Delta} (a_{ij} + O(t))t^{\nu(i,j)-\alpha i-\beta j-\gamma}x^i y^j = \sum_{(i,j)\in\Delta_k} a_{ij}x^i y^j + O(t) = F_k(x, y)+O(t),$$

that is, for small $|t|$, the curve $C_{(t),k}$ is a small deformation of C_k in a compact neighborhood of the singular points of C_k in $(\mathbb{C}^*)^2$. So, we require that the curves $C_{(t),k}$, $t \in (\mathbb{C}, 0)$ lie inside the germ M_k of the family of curves with Newton polygon Δ, which are close to C_k and have singularities of the same type in a neighborhood of the singular points of C_k. These requirements for all $k = 1, \dots, N$ can be reduced to a soluble implicit function problem with respect to the unknown coefficients $a_{ij} + O(t)$, $(i, j) \in \Delta$, provided that, first, for each $k = 1, \dots, N$, the germ M_k is smooth and intersects transversally with an affine space spanned by some coefficients $(i, j) \in \delta_k \subset \Delta_k \cap \mathbb{Z}^2$, and, second, the sets $\delta_1, \dots, \delta_N$ are disjoint and their collection is linearly ordered (cf. [27, Theorem 3.1 and Theorem 4.1(1)]). The order is provided by an orientation of the adjacency graph Γ of the polygons $\Delta_1, \dots, \Delta_N$ such that no oriented cycle occurs. In turn, the deformation theory of singularities provides a sufficient condition for the above smoothness and transversality which for curves with ordinary cusps reads (cf. [27, Theorem 4.1, first inequality in (1)]):

$$\begin{pmatrix}\text{number of cusps}\\ \text{of } C_k \text{ in } (\mathbb{C}^*)^2\end{pmatrix} \le \begin{pmatrix}\text{number of integral points in } \partial\Delta_k \text{ that}\\ \text{do not lie on those edges of } \Delta_k \text{ which}\\ \text{correspond to the arcs of } \Gamma \text{ oriented inward } \Delta_k\end{pmatrix} \qquad (1)$$

We are ready to construct a curve of degree 11 having 30 cusps. Consider the subdivision of the Newton triangle $\Delta = \mathrm{Conv}\{(0, 0), (11, 0), (0, 11)\}$ of a general bivariate polynomial of degree 11 into convex lattice polygons shown in the left hand side of Fig. 5. Clearly, this subdivision lifts to a graph of a convex piecewise linear function $\nu : \Delta \to \mathbb{R}$ (we use only its existence, not any explicit formula).

We claim that there exist polynomials $F_1, F_2, F_3, G \in \mathbb{C}[x, y]$ with Newton pentagons P_1, P_2, P_3 and Newton triangle T, respectively, such that each of F_1, F_2, F_3 defines a plane curve with 11 cusps in $(\mathbb{C}^*)^2$, and G defines a plane curve with 3 cusps in $(\mathbb{C}^*)^2$, and furthermore, for any $i = 1, 2, 3$, the truncations of F_i and G on the edge $P_i \cap T$ coincide.

Having the above polynomials F_1, F_2, F_3, G and orienting the adjacency graph as shown in the right-hand side of Fig. 5, we derive the existence of the required curve from [27, Theorem 3.1 and Theorem 4.1(1)], since the condition (1) reads as $12 > 9$ for each of F_1, F_2, F_3 and as $11 > 3$ for G.

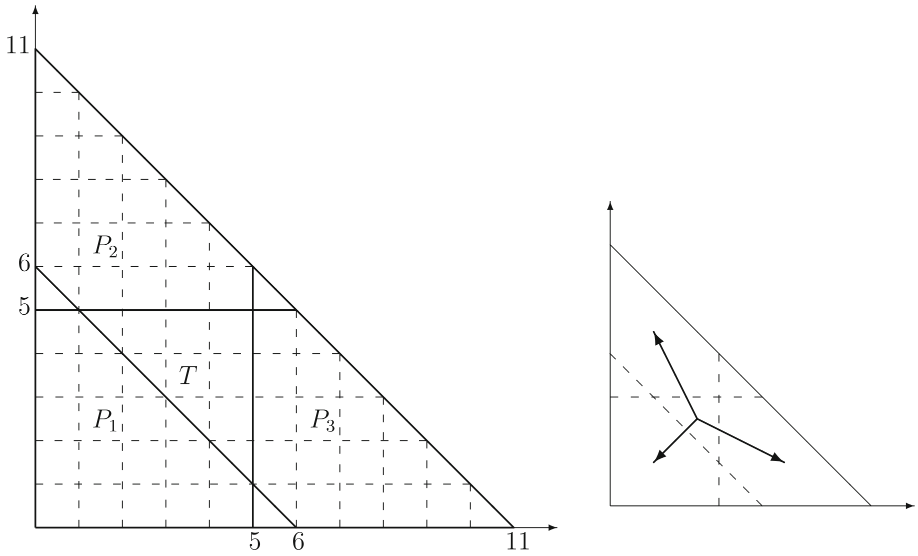

Fig. 5 Subdivision and adjacency graph

Now, we take F_1, F_2, F_3 to be defining plane sextic curves with 9 cusps (dual to plane smooth cubics), and G to be defining a plane quartic curve with 3 cusps (dual to a plane nodal cubic curve). The only remaining step is to find polynomials as above, whose truncations on common edges of their Newton polygons respectively coincide. This can be done, using the following fact: Let $\mathcal{P}_{6,9}$, $\mathcal{P}_{4,3}$ be the spaces of plane sextic curves with 9 cusps, respectively, plane quartic curves with 3 cusps, crossing a given fixed line L transversally, then the maps

$$\pi_6 : \mathcal{P}_{6,9} \to \mathrm{Sym}^6(L), \quad C \mapsto C \cap L, \qquad \pi_4 : \mathcal{P}_{4,3} \to \mathrm{Sym}^4(L), \quad C \mapsto C \cap L,$$

are dominant. Indeed,

$$\mathrm{codim}_{|\mathcal{O}_{\mathbb{P}^2}(6)|}\mathcal{P}_{6,9} = 2 \cdot 9 = 18, \quad \mathrm{codim}_{|\mathcal{O}_{\mathbb{P}^2}(4)|}\mathcal{P}_{4,3} = 2 \cdot 3 = 6, \tag{2}$$

whereas

$$\mathrm{codim}_{|\mathcal{O}_{\mathbb{P}^2}(6)|}\pi_6^{-1}(\omega_6) = 2 \cdot 9 + 6 = 24, \quad \omega_6 \in \mathrm{Sym}^6(L), \tag{3}$$

$$\mathrm{codim}_{|\mathcal{O}_{\mathbb{P}^2}(6)|}\pi_4^{-1}(\omega_4) = 2 \cdot 3 + 4 = 10, \quad \omega_4 \in \mathrm{Sym}^4(L). \tag{4}$$

Relations (2)–(4) can immediately be extracted from [10]: (2) from Corollary 6.3 with the inequalities $17 > 9$ and $11 > 3$ as sufficient conditions, (3) and (4) from Theorem 6.1 applied to the plane blown up at ω_6, resp. ω_4, with the inequalities $17 - 6 = 11 > 9$ and $11 - 4 = 7 > 3$ as sufficient conditions. □

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