Personalities of Curves

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Chapter 1

Curves on Scrolls

ScrollsChapter

1.1 Hyperelliptic curves

((maybe start a new Chapter?))

In our early encounters with curves, we frequently assumed that the curve we were considering was non-hyperelliptic, since the behavior of hyperelliptic curves is so atypical. In this section, we'll describe the geometry of hyperelliptic curves.

1.1.1 Basic models of hyperelliptic curves

((move this section to ch 2; add discussion of adjoints— perhaps as exercises?))

We start by establishing some basic facts about hyperelliptic curves. Many of these follow from general theorems like Riemann-Roch; but since they can be established by direct examination we will carry that out here.

Suppose C is a smooth, projective hyperelliptic curve of genus $g \geq 2$. By definition, C admits a degree 2 map $\pi : C \to \mathbb{P}^1$; and as we've observed (**??**)

this map is unique.

By Riemann-Hurwitz,
((attibution?))
the map $\pi : C \to \mathbb{P}^1$ will have $2g+2$ distinct simple branch points, say $\lambda_1, \dots, \lambda_{2g-2} \in \mathbb{P}^1$. An open subsect C° of C can then be realized as the smooth projective completion of the affine curve given as

$$C^\circ = \left\{(x,y) \in \mathbb{A}^2 \;\mid\; y^2 = \prod_{i=1}^{2g+2}(x-\lambda_i)\right\}.$$

((if two of the λ_i coinncide, then the curve develops a singular point. Much of what we will do carries over to the singular case.))

((say the smooth model has 2 points at ∞.))
Note that if we simply take the closure of this locus in $\mathbb{P}^2$, the resulting curve will be highly singular at the point $[1,0,0]$, as can be seen either directly by making an appropriate change of variables, or by invoking the genus formula for plane curves: if the closure were smooth, it would have genus $\binom{2g+1}{2}$. We can, however, complete the curve simply in $\mathbb{P}^1 \times \mathbb{P}^1$, for example by setting
((this is a rabbit from a hat. Consider either saying that by the previous section, if there's an emb in P3 then its on P1 x P1 as a divisor of type 2,g+1; and then "finding" this embedding as below; or moving this page to the early place where hyperelliptic curves are first mentioned.))

$$y' = \frac{y}{\prod_{i=1}^{g+1}(x-\lambda_i)};$$

we can then write the equation of a still smaller open subset of C as

$$y'^2 \cdot \prod_{i=1}^{g+1}(x-\lambda_i) \;=\; \prod_{i=g+2}^{2g+2}(x-\lambda_i).$$

If we now take the closure of this locus in $\mathbb{P}^1 \times \mathbb{P}^1$, we get a curve of type $(2, g+1)$ on $\mathbb{P}^1 \times \mathbb{P}^1$; this curve is smooth, as can be seen again either directly in coordinates or by invoking the genus formula for curves on $\mathbb{P}^1 \times \mathbb{P}^1$. In other words,

$$C = V\Big(Y_0^2 \cdot \prod_{i=1}^{g+1}(X_1 - \lambda_i X_0) - Y_1^2 \cdot \prod_{i=g+2}^{2g+2}(X_1 - \lambda_i X_0)\Big)$$

Next, let's describe the space of regular differentials on C. For this, it's convenient to work with the affine model $C^\circ = V(f) \subset \mathbb{A}^2$, where

$$f(x, y) = y^2 - \prod_{i=1}^{2g-2}(x - \lambda_i).$$

We'll denote the two points at infinity—that is, the two points of $C \setminus C^\circ$—as p and q.

To start, consider the simple differential $dx \in \Omega_{C^\circ/k}$. This is clearly regular on C°, with zeros at the ramification points $r_i = (\lambda_i, 0)$. But it does not extend to a regular differential on all of C: it will have double poles at p and q, as can be seen either directly or by degree considerations: as we said, dx has $2g + 2$ zeros, while the degree of K_C is $2g - 2$, meaning that there must be poles at the points p and q.

To kill these poles, we can of course divide by x^2 (or any quadratic polynomial in x). But that just introduces new poles in the finite part C° of C. Instead, we want to multiply dx by a rational function with zeros at p and q, but *whose poles occur only at the points where dx has zeroes*—that is, the points r_i. A natural choice is simply the reciprocal of the partial derivative $f_y = \partial f/\partial y = 2y$, which vanishes exactly at the points r_i, and has correspondingly a pole of order $g + 1$ at each of the points p and q (reason: the involution $y \to -y$ fixes C° and x), and exchanges the points p, q. In other words, the differential

$$\omega = \frac{dx}{f_y}$$

is regular, with divisor

$$(\omega) = (g - 1)p + (g - 1)q.$$

The remaining regular differentials on C are now easy to find: Since x has only a simple pole at the two points at infinity
((say why.))

we can multiply ω by any x^k with $k = 0, 1, \ldots, g-1$. Since this gives us g independent differentials, these form a basis for $H^0(K_C)$.

1) special linear series are mult g^1_2+basepoints. 2) Given an embedding, there's a union of lines. If the embedding is complete, we get a matrix...that defines the union of lines. Scrolls in all dimensions as unions of spans of divisors.

1.1.2 General embeddings of degree genus+3

It's a divisor on a quadric in $\mathbb{P}^3$ of type $(2, g+1)$

1.2 Trigonal curves

1.2.1 Special linear series on trigonal curves

In analyzing special linear series on a hyperelliptic curve, we made crucial use of the facts that the canonical image of a hyperelliptic curve is a rational normal curve, and that any collection of points on a rational normal curve $C \subset \mathbb{P}^n$ either are linearly independent or span $\mathbb{P}^n$. In a similar (though necessarily less complete) way, we can use the fact that the canonical image of a trigonal curve lies on a rational normal surface scroll to describe special linear series on it.

Lemma 1.2.1. *Let $S = S_{a,b} \subset \mathbb{P}^n$ be a rational normal surface scroll. Any hyperplane section $H \cap S$ consists of the union of a rational normal curve E, which is a section of the scroll, and a union of lines of the ruling of the scroll.*

Note that the curve E must be a reduced component of $S \cap H$, but the lines L_i may coincide, i.e., may be non-reduced components of the intersection. In the following proof, we'll assume for clarity that the lines L_i are distinct (that is, $S \cap H$ is reduced); we leave it as an exercise to rewrite the proof to accommodate the remaining cases.

Proof. Let $F \in \mathrm{Pic}(S)$ be the class of a line of the ruling. Since $F^2 = 0$ and $H \cdot F = 1$, exactly one of the components of $S \cap H$ must have intersection number 1 with F; all other components must have intersection number 0 with F and so must be lines of the ruling.

It remains to show that the unique component E of $H \cap S$ having intersection number 1 with F is a rational normal curve. This can be seen directly, but there's a shortcut. Suppose that we have

$$S \cap H = E \cup L_1 + \cdots + L_k,$$

so that in particular $\deg(E) = n - 1 - k$. Since each of the lines L_i of the ruling must meet C, we have that

$$\begin{aligned} n - 1 &= \dim(\overline{S \cap H}) \\ &\leq \dim(\overline{E}) + k \\ &\leq (n - 1 - k) + k \\ &= n - 1. \end{aligned}$$

We conclude that $\dim(\overline{E}) = n - k - 1$, and hence that E is a rational normal curve. □

Note that if $S = S_{a,b}$ with $a \leq b$, we must have either $0 \leq k \leq a$ or $k = b$: as soon as $k > a$, the span of the lines L_i will contain the directrix of the scroll, and so must consist of the union of the directrix with $n - 1 - a = b$ lines.

Now let C be a trigonal curve of genus $g \geq 5$, embedded in
P^{g-1} as a canonical curve, and let S be the scroll containing C. We want to describe special linear series $\mathcal{D} = |D|$. If our linear series has base points, we can delete them; so we'll assume that $|D|$ and $|K - D|$ are base point free. Note that this implies that both $r(D) \geq 1$ and $r(K - D) \geq 1$. In addition, it follows by Bertini that a general divisor $D \in \mathcal{D}$ is reduced, that is, consists of distinct points $p_1, \ldots, p_d$.

Now, the first hypothesis, that $r(D) \geq 1$, says that the points $p_1, \ldots, p_d$ are linearly dependent. The second hypothesis, that $r(K - D) \geq 1$, says that the points p_i span a subspace of codimension at least 2 in $\mathbb{P}^{g-1}$ They therefore lie on at least a pencil of hyperplanes; let H be a general hyperplane containing D.

. is effective, says that the divisor D lies in a hyperplane section $C \cap H$; let H be a general such hyperplane. At the same time canonical image lies on a 2-dim scroll (non -subcanonical embedding only on 3-dim scrolls). embedding of a trigonal curve lies on the same scroll.Stratification of trigonal curves by Maroni invariants. Dimensions via automorphism groups of scrolls.

1.3 Castelnuovo's Theorem

(Statement only)
((we'll need the existence of smooth curves in given classes — base point freeness of certain divisor classes on the scroll. Theorem: bpf iff they meet both a,b rational normal curves positively. reference to Montreal? better to make a tex file of the essential bit and put it in, as appendix. or ACGH?))