Personalities of Curves

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Contents

Chapter 1

Linear Systems

Morphisms of a smooth curve C (or indeed of any scheme) to a projective space are conveniently studied using the closely related notions of Divisors, linear systems and invertible sheaves.

1.1 Morphisms to projective space, and families of Cartier divisors

Let $\phi : C \to \mathbb{P}^r$ be a morphism from a smooth curve C. If $H \subset \mathbb{P}^r$ is a hyperplane that does not contain $\phi(C)$, then the preimage of $\phi(C) \cap H$ is a finite sets of points on C, with multiplicities when H is tangent to $\phi(C)$ or passes through a singular point of $\phi(C)$. Such a set of points with non-negative integer multiplicities is called an *effective divisor* on C; more generally, a *divisor* (sometimes called a *Weil divisor*)on a scheme X is an integral linear combination of codimension 1 subvarieties, and it is called *effective* if the coefficients are all non-negative. The divisors that arise as the pullbacks of general hyperplanes are special: since a hyperplane is defined by just one equation, which is locally given by the vanishing of a function, the pullback of a hyperplane will be locally defined by the vanishing of a single function that is a nonzerodivisor; that is, it is an *effective Cartier divisor*. See [**?**, pp. 140-146] for more information; on a smooth curve every divisor is Cartier, so the difference between Weil and Cartier divisors will not be an issue for us.)

The word "local" scattered through the previous paragraph is needed because, if X is a projective variety, then the only algebraic functions $X \to \mathbb{C}$ are constant functions. (Proof: the image of a projective variety is again projective, and the only projective subvarieties of an affine variety are points.)

If we are given the family of divisors on C that are the preimages of the intersections of hyperplanes with $\phi(C)$, we can recover the morphism ϕ set-theoretically: it takes a point $p \in C$ to the point of projective space that is the intersection of those hyperplanes whose preimages contain p.

The relationship of two divisors on C that are preimages of intersections of $\phi(C)$ with hyperplanes is simple to describe: If hyperplanes $H, H' \subset \mathbb{P}^r$ are defined by the linear form h, h' then $1/h$ has a simple pole along E—we may say that it "vanishes along E" to degree -1. In this sense the divisor $H - H'$ on $\mathbb{P}^n$ is defined by the rational function $\lambda = h'/h$. If neither H nor H' contain C then the pullback of λ is a well-defined, nonzero rational function on C, and the divisor $\phi^{-1}(\phi(C) \cap H') - \phi^{-1}(\phi(C) \cap H)$ is defined by the pullback $\phi^*(\lambda) := \lambda \circ f$. Thus the divisors arising from a given morphism to $\mathbb{P}^r$ differ by the divisors of zeros minus poles of rational functions on C.

If C is a smooth curve then the local ring $\mathcal{O}_{C,p}$ of C at a point p is a discrete valuation ring, and if π is a generator of the maximal ideal of $\mathcal{O}_{C,p}$, then any rational function λ on C can be expressed uniquely as $u\pi^k$ where $u \in \mathcal{O}_{C,p}$ is a unit and $k \in \mathbb{Z}$. We say that the *order* of λ at p, and write $k = \operatorname{ord}_p \lambda$. We associate λ to the divisor

$$(\lambda) := \sum_{p \in C} (\operatorname{ord}_p \lambda) p.$$

The *class group* of C is defined to be the the group of divisors on C modulo the divisors of rational functions. Thus the divisors on C that are preimages of intersections of $\phi(C)$ with different hyperplanes all belong to the same *divisor class*, and form a linear system in the sense of the following section.

1.2 Morphisms and linear systems

We want to understand morphisms to $\mathbb{P}^r$ more than set-theoretically, and we want to be able to produce them from data on C. For this we use the notion of linear system (sometimes called linear series).

Definition 1.2.1. A *linear system* on a scheme X is a pair $\mathcal{V} = (\mathcal{L}, V)$ where $\mathcal{L}$ is an invertible sheaf on X and V is a vector space of global sections of $\mathcal{L}$.

We will spend the next pages unpacking this notion. Our goal is to explain and prove:

Theorem 1.2.2. *There is a natural bijection between the set of nondegenerate morphisms $\phi : C \to \mathbb{P}^r$ modulo PGL_{r+1}, and basepoint-free linear systems of dimension r on C.*

Here "nondegenerate" means the image of the morphism ϕ is not contained in any hyperplane.

1.2.1 Invertible sheaves

Recall first that a *coherent sheaf* $\mathcal{L}$ on a scheme X may be defined by giving

- An open affine cover $\{U_i\}$ of X;
- For each i, a finitely generated $\mathcal{O}_X(U_i)$-module L_i;
- For each i, j, an isomorphism $\sigma_{i,j} : L_i \mid_{U_i \cap U_j} \to L_j \mid_{U_i \cap U_j}$ satisfying the compatibility conditions $\sigma_{j,k}\sigma_{i,j} = \sigma_{i,k}$.

A *global section* of $\mathcal{L}$ is a family of elements $t_i \in F_i$ such that $\sigma_{i,j} t_i = t_j$. Such a section may be realized as the image of the constant function 1 under a homomorphism of sheaves $\mathcal{O}_X \to \mathcal{L}$. By Theorem [**?**, Thm III.5.2] the space $H^0(\mathcal{L})$ of global sections is a finite-dimensional vector space. For example, $H^0(\mathcal{O}_X) = \mathbb{C}$ because the only globally defined functions on X are the constant functions.

The coherent sheaf $\mathcal{L}$ is said to be an *invertible sheaf* on X if there is an open cover as above with the additional property that $F_i \cong \mathcal{O}_X(U_i)$, the free module on one generator.

If $\sigma \in H^0\mathcal{L}$ is a global section of an invertible sheaf on X, and $p \in X$ is a point, then $\sigma(p)$ is in the stalk of $\mathcal{L}$ at p, a module isomorphic to $\mathcal{O}_{X,p}$. Since the isomorphism is not canonical, σ does not define a function on X at p; but since any two isomorphisms differ by a unit in $\mathcal{O}_{X,p}$, the vanishing locus, denoted $(\sigma)_0$ of σ *is* a well-defined subscheme of X. Moreover, if X is integral, then the ratio of two global sections is a well-defined rational function, so the divisor class of $(\sigma)_0$ is independent of the choice of σ.

Proposition 1.2.3. *The invertible sheaves on X form a group under $\otimes_X$, called the* Picard group of X*, denoted* $\mathrm{Pic}(X)$.

Proof. If $\mathcal{F}, \mathcal{G}$ are invertible sheaves then so are $\mathcal{F} \otimes_X \mathcal{G}$ and $\mathrm{Hom}_X(\mathcal{F}, \mathcal{G})$, as one sees immediately by restricting to the open sets where $\mathcal{F}$ and $\mathcal{G}$ are isomorphic to $\mathcal{O}_X$. Moreover the natural isomorphisms

$$\mathcal{F}(U) \otimes_X \mathrm{Hom}(\mathcal{F}(U), \mathcal{O}_X(U)) \to \mathcal{O}_X(U) \quad s \otimes f \mapsto f(s)$$

patch together to define a global isomorphism

$$\mathcal{F} \otimes_X \mathrm{Hom}(\mathcal{F}, \mathcal{O}_X) \to \mathcal{O}_X$$

justifying the definition $\mathcal{F}^{-1} := \mathrm{Hom}(\mathcal{F}, \mathcal{O}_X)$ and thus the name "invertible sheaf". □

If $D \subset X$ is an effective divisor, then we define $\mathcal{O}_X(-D)$ to be the ideal sheaf of D. If D is locally defined by the vanishing of a (locally defined) nonzerodivisor in $\mathcal{O}_X$, (that is, D is a Cartier divisor), then $\mathcal{O}_X(-D)$ is an invertible sheaf. We write $\mathcal{O}_X(D)$ for the inverse, $\mathcal{O}_X(-D)^{-1}$. The dual of the inclusion $\mathcal{O}_X(-D) \subset \mathcal{O}_X$ is a map $\mathcal{O}_X \to \mathcal{O}_X(D)$ sending the global section $1 \in \mathcal{O}_X$ to a section $\sigma \in \mathcal{O}_X(D)$ that vanishes precisely on D.

Example 1.2.4 (Invertible sheaves on $\mathbb{P}^r$)**.** If $H \subset \mathbb{P}^r$ is a hyperplane defined by the vanishing of a linear form $\ell = \ell(x_0, \dots x_r)$ then the ideal sheaf $\mathcal{O}_{\mathbb{P}^r}(-1) := \mathcal{I}_{H/\mathbb{P}^r} \subset \mathcal{O}_{\mathbb{P}^r}$ is generated on the open affine set $U_i := \{x_i \neq 0\} \cong \mathbb{A}^r$ by ℓ/x_i, and is thus an invertible sheaf. Moreover, if H' is the hyperplane defined by another linear form ℓ', then

$$\frac{\ell'}{\ell} \cdot \mathcal{I}_{H/\mathbb{P}^r} = \mathcal{I}_{H'/\mathbb{P}^r}$$

((check that this is out notation for ideal sheaf))
so the sheaves $\mathcal{I}_{H/\mathbb{P}^r}$ and $\mathcal{I}_{H'/\mathbb{P}^r}$ are isomorphic, justifying the name $\mathcal{O}_{\mathbb{P}^r}(-1)$.

The p-th tensor power of $\mathcal{O}_{\mathbb{P}^r}(-1)$ is called $\mathcal{O}_{\mathbb{P}^r}(-d)$; it is isomorphic to the ideal sheaf of any hypersurface of degree d. Because polynomials satisfy the unique factorization property, every effective divisor $D \subset \mathbb{P}^r$ is a hypersurface of some degree d, so $\mathcal{O}_{\mathbb{P}^r}(-D) \cong \mathcal{O}_{\mathbb{P}^r}(-d)$. Note that if $d > 0$ then $H^0(\mathcal{O}_{\mathbb{P}^r}(-D)) = 0$, since it may be realized as the sheaf of locally defined functions vanishing on D, and there are no such globally defined functions except 0.

We take $\mathcal{O}_{\mathbb{P}^r}(d)$ to be the inverse of $\mathcal{O}_{\mathbb{P}^r}(-d)$. If D is the hypersurface defined by a form F of degree d, then $\mathcal{O}_{\mathbb{P}^r}(-D)$ is generated on U_i by $F/(x_i^d)$, so $\mathcal{O}_{\mathbb{P}^r}(D)$ is generated on U_i by x_i^d/F. Starting from the inclusion $\mathcal{O}_{\mathbb{P}^r}(-D) \subset \mathcal{O}_{\mathbb{P}^r}$ and taking inverses, we see that $\mathcal{O}_{\mathbb{P}^r} \subset \mathcal{O}_{\mathbb{P}^r}(D)$ and the global section $1 \in H^0(\mathcal{O}_{\mathbb{P}^r}) \subset H^0(\mathcal{O}_{\mathbb{P}^r}(D))$, restricted to U_i, is $F/(x_0^d)$ times the local generator of $\mathcal{O}_{\mathbb{P}^r}(D)$ and thus vanishes on D. Because every rational function on $\mathbb{P}^r$ has degree 0, and any two global sections differ by a rational function, it follows that every global section of $\mathcal{O}_{\mathbb{P}^r}(d)$ vanishes on a divisor of degree d. Thus we may identify $H^0(\mathcal{O}_{\mathbb{P}^r}(d))$ with the $\binom{n+d}{n}$-dimensional vector space of forms of degree d on $\mathbb{P}^r$.

The proof of Theorem 1.2.2 is contained in the material of the next two subsections:

1.2.2 The morphism to projective space coming from a linear system

For any $\mathbb{C}$-vector space V of dimension $r+1$ with basis $x_0, \dots, x_r$, we write $\mathrm{Sym}(V) \cong \mathbb{C}[x_0, \dots, x_r]$ for the symmetric algebra on V, and $\mathbb{P}(V) \cong \mathbb{P}^r_{\mathbb{C}}$ to be the projective space $\mathrm{Proj}(\mathrm{Sym}(V))$, which is naturally isomorphic to the space of lines in V^*. Note that the isomorphism $\mathbb{P}(V) \cong \mathbb{P}^r_{\mathbb{C}}$ is well-defined up to the action of $\mathrm{Aut}(\mathbb{P}^r) = PGL(r+1)$.

Given a linear system $\mathcal{V} := (\mathcal{L}, V)$ of dimension r on a scheme X, where $\mathcal{L}$ is an invertible sheaf on X and $V = \langle \sigma_0, \dots \sigma_r \rangle$ is a vector space of global sections, we define the *base locus* of $\mathcal{V}$ to be the closed subscheme

$$B_{\mathcal{V}} := \bigcap_{i=0}^{r} \{\sigma_i = 0\}.$$

Let $W := X \setminus B_{\mathcal{V}}$ be the open subscheme where not all sections σ_i vanish.

For any point $q \in W$ we may choose an open neighborhood $W' \subset W$ of q, and an identification

$$t : \mathcal{L}\mid_{W'} \xrightarrow{\cong} \mathcal{O}_{W'}$$

and define $\phi_{\mathcal{V}} : W' \to \mathbb{P}(V)$ by

$$W' \ni p \mapsto \big(t(\sigma_0(p)), \dots, t(\sigma_r(p))\big) \in \mathbb{P}(V).$$

This is a morphism on W'. A change of neighborhoods W' or of identifications t would multiply each value $t(\sigma_i(p))$ by a unit, the same one for each i, and thus the construction would define the same morphism. It follows that the morphisms defined on different W' agree on overlaps, and thus define a morphism $W \to \mathbb{P}(V) \cong \mathbb{P}^r$. This is the reason that the dimension of $\mathcal{V}$ is defined to be $r = \dim V - 1$ instead of $\dim V$.

The most useful linear series are those that define morphisms defined on all of X. This happens when $B_{\mathcal{V}} = \emptyset$, that is, for every point $q \in X$, there is a section $\sigma \in V$ such that σ does not vanish at x. In this case we say that $(\mathcal{L}, \mathcal{V})$ is *basepoint free.*

Example 1.2.5. The morphism from $\mathbb{P}^r$ defined by the complete linear system $|\mathcal{O}_{\mathbb{P}^r}(d)|$ has target $\mathbb{P}^{\binom{r+d}{r}}$, and takes a point $x_0, \dots x_r$ to the point whose coordinates are all the monomials of degree d in $x_0, \dots x_r$. It is called the *d-th Veronese morphism* of $\mathbb{P}^r$. For example on $\mathbb{P}^1$, this has the form

$$(x_0, x_1) \mapsto (x_0^d,\ x_0^{d-1}x_1,\ \dots, x_1^d).$$

The image of $\mathbb{P}^1$ under this morphism is called the *rational normal curve* of degree d; in the case $d = 2$ is the *plane conic*, and if $d = 3$ it is called the *twisted cubic*. Veronese himself studied the image of $\mathbb{P}^2$ by the Veronese morphism of degree 2 now simply called *the Veronese surface.*

Exercise 1.2.6. Show that there is no non-constant morphism $\mathbb{P}^r \to \mathbb{P}^s$ when $s < r$ by showing that any nontrivial linear system of dimension $< r$ has a non-empty base locus.

1.2.3 The linear system coming from a morphism to projective space

Conversely, suppose that we are given a morphism $\phi : X \to \mathbb{P}^r$. With notation as in Example 1.2.4 we may choose an open affine cover $W_{i,j}$ of X such that

$\phi(W_{i,j}) \subset U_j$. Composing the regular functions $x_0/x_j, \dots, x_r/x_j$ with ϕ we get functions $\sigma_0, \dots, \sigma_r$ on $W_{i,j}$. The function σ_j, is the image under $\phi^* : \mathcal{O}_{U_j} \to \mathcal{O}_{W_{i,j}}$ of the function $x_j/x_j = 1$ on U_j, so it $\sigma_j = 1 \in \mathcal{O}_{W_{i,j}}$. In particular, the module $\mathcal{L}_{\phi^{-1}(U_j)}$ generated by the rational functions

$$\{(\sigma_i)_{\phi^{-1}(U_j)} = \phi^*(x_i/x_j)\}_{0 \le i \le n}$$

is a free $\mathcal{O}_{W_{i,j}}$-module on 1 generator. On the preimage of $U_j \cap U_k$ these sections differ by the common unit $\phi^*(x_k/x_j)$, and thus the collection of these modules defines an invertible sheaf $\mathcal{L}$ on X together with an $r+1$-dimensional space of global sections $\mathcal{V} := \langle \sigma_0, \dots \sigma_r \rangle$ that forms a basepoint free linear system. Note that the subscheme $\{\sigma_k = 0\} \subset W_{i,j}$ is the scheme-theoretic preimage of the the hyperplane $\{x_k = 0\} \subset \mathbb{P}^r$. This completes the explanation and proof of Theorem 1.2.2

1.2.4 More about linear systems

Let $\mathcal{V} = (\mathcal{L}, V)$ be a linear sysytem on X. The linear system is said to be *complete* if $V = H^0(\mathcal{L})$; in this case it is sometimes denoted $|\mathcal{L}|$. If $\mathcal{L} \cong \mathcal{O}_C(D)$), we also write it as $|D|$. The *dimension of* $\mathcal{V}$ is $\dim V - 1$. If D is any divisor on C we write $r(D)$ for the dimension of the complete linear series $|D|$; that is, $r(D) = h^0(\mathcal{O}_C(D)) - 1$. Finally, a linear system of dimension 1 is called a *pencil*, a linear system of dimension 2 is called a *net* and, less commonly, a three-dimensional linear system is called a *web*.

A linear system $\mathcal{V} = (\mathcal{L}, V)$ is called *basepoint free* if it defines a morphism to $\mathbb{P}(V)$, or equivalently if the the sections in V generate $\mathcal{L}$ locally at each point of X. It is called *very ample* if it is basepoint-free and defines an embedding. If D is a Cartier divisor on X, then we say that D is *very ample* if the complete linear system $|D|$ is versy ample, and we say that D is *ample* if mD is very ample for some integer $m > 0$.

Given a linear system $\mathcal{V} = (\mathcal{L}, V)$ and an effective divisor D on C, we'll set

$$V(-D) := \{\sigma \in V \mid \sigma(D) = 0\}.$$

The difference $\dim V - \dim V(-D)$ is called the *number of conditions imposed by D on the linear system $\mathcal{V}$*; we say that D *imposes independent conditions* on $\mathcal{V}$ if $\dim V - \dim V(-D) = \deg(D)$.

Via the correspondence of Theorem 1.2.2, the statements about the geometry of a morphism $\phi : C \to \mathbb{P}^r$ can be formulated as statements about the relevant linear systems. We will see this in many instances throughout this book. It will be most convenient to formulate this in terms of the vector space $H^0(\mathcal{L})$ of global sections of $\mathcal{L}$, and we write $h^0(\mathcal{L})$ for the dimension of this vector space. Here is a first example:

Proposition 1.2.7. *[?, Thm. IV.3.1] Let $\mathcal{L}$ be an invertible sheaf on a smooth curve C. The complete linear system $|\mathcal{L}|$ is base-point-free iff*

$$h^0(\mathcal{L}(-p)) = h^0(\mathcal{L}) - 1 \quad \forall p \in C;$$

and in this case the associated morphism $\phi_{\mathcal{L}}$ is an embedding, so $|\mathcal{L}|$ is very ample, iff

$$h^0(\mathcal{L}(-p-q)) = h^0(\mathcal{L}) - 2 \quad \forall p, q \in C.$$

Proof. The statement $h^0(\mathcal{L}(-p-q)) = h^0(\mathcal{L}) - 2$ for $p \neq q$ implies that $\phi_{\mathcal{L}}(p) \neq \phi_{\mathcal{L}}(q)$. The tangent space of C at p is $(\mathcal{I}_C(p)/\mathcal{I}_C(p)^2)^*$, so the condition that there is a section of $\mathcal{L}$ that vanishes at p, but does not vanish to order 2, implies that the differential $d\phi_{\mathcal{L}}$ is injective at p. $\square$

((this uses a lot: even given the identification of the tangent space with m/m^2 we really only get an analytic isomorphism. To deduced the algebraic one we'd need a finiteness principal: projective maps with finite fibers are finite. Should we say some of this??))

We can also relate the geometry of the morphism associated to a incomplete linear system $V \subset H^0(\mathcal{L})$ to the geometry of the morphism associated to the complete linear system $|\mathcal{L}|$. In general, if $V \subset W \subset H^0(\mathcal{L})$ are a pair of nested linear systems, we have a linear map $W^* \to V^*$ dual to the inclusion $V \hookrightarrow W$, and a corresponding linear projection $\pi : \mathbb{P}W^* \dashrightarrow \mathbb{P}V^*$, with indeterminacy locus the subspace $\mathbb{P}(Ann(V)) \subset \mathbb{P}W^*$. In this case, we have

$$\phi_V = \pi \circ \phi_W;$$

that is, we have the diagram

$$\begin{tikzcd} & \mathbb{P}W^* \arrow[d, dashed, "\pi"] \\ C \arrow[ur, "\phi_W"] \arrow[r, "\phi_V"] & \mathbb{P}V^*. \end{tikzcd}$$

Note that in this case, given that W is base-point-free, the condition that V be base-point-free is equivalent to saying that the center $\mathbb{P}(Ann(V))$ of the projection π is disjoint from $\phi_W(C)$.

By way of language, we will say that a curve $C \subset \mathbb{P}^r$ embedded by a complete linear series is *linearly normal*; this is equivalent to saying that the restriction map

$$H^0(\mathcal{O}_{\mathbb{P}^r}(1)) \to H^0(\mathcal{O}_C(1))$$

is surjective, which is in turn equivalent to saying that C is not the regular projection of a nondegenerate curve $\tilde{C} \subset \mathbb{P}^{r+1}$.

Exercise 1.2.8. Extend the statement of Proposition 1.2.7 to incomplete linear systems; that is, prove that the morphism associated to a linear system $(\mathcal{L}, V)$ is an embedding iff

$$\dim\big(V \cap H^0(\mathcal{L}(-p-q))\big) = \dim V - 2 \quad \forall p, q \in C.$$

Exercise 1.2.9. An automorphism of $\mathbb{P}^r$ takes hyperplanes to hyperplanes. Deduce that it is given by the linear system $\mathcal{V} = \mathcal{O}_{\mathbb{P}^r}(1), H^0(\mathcal{O}_{\mathbb{P}^r}(1))$, and use this to show that $\operatorname{Aut}\mathbb{P}^r = PGL(r+1)$.

Exercise 1.2.10. Show that, if $s < r$, then the image of any morphism $\mathbb{P}^r \to \mathbb{P}^s$ is a single point.

For another example of the relationship between linear series on curves and morphisms of curves to projective space, consider a smooth curve $C \subset \mathbb{P}^r$ embedded in projective space, and assume that C is linearly normal. If $\phi : C \to C$ is any automorphism, we can ask whether ϕ is induced by an automorphism of $\mathbb{P}^r$; in other words, does there exist an automorphism $\Phi : \mathbb{P}^r \to \mathbb{P}^r$ such that $\Phi(C) = C$ and $\Phi|_C = \phi$? The answer is expressed in the following exercise.

Exercise 1.2.11. In the circumstances above, the automorphism ϕ is induced by an automorphism of $\mathbb{P}^r$ if and only if ϕ carries the invertible sheaf $\mathcal{O}_C(1)$ to itself; that is, $\phi^*(\mathcal{O}_C(1)) = \mathcal{O}_C(1)$.

Example 1.2.12. Consider the morphism of $\mathbb{P}^1 to \mathbb{P}^d$ given by the complete linear system $|\mathcal{O}_{\mathbb{P}^1}(d)|$; this is called the *rational normal curve*. Since there is a unique invertible sheaf of each degree n on C, and the curve is linearly normal, we see that *every automorphism of a rational normal curve $C \subset \mathbb{P}^d$ is projective*, so "the" rational normal curve of degree d is well-defined up to an automorphism of $\mathbb{P}^d$. A similar statement holds for the image of any Veronese morphism.

If $\mathcal{L}, \mathcal{L}'$ are linear systems on a smooth curve C and $D = (\sigma)_0, D' = (\sigma')_0$ are the divisors of zeros of sections of $\mathcal{L}$ and $\mathcal{L}'$ respectively, then $D + D'$ is the divisor of zeros of the section $\sigma \otimes \sigma'$ of $\mathcal{L} \otimes \mathcal{L}'$.

We often want to consider sections of a given invertible sheaf $\mathcal{L}$ with bounded singularities: if $D = \sum m_i p_i$ is a divisor, we define the invertible sheaf $\mathcal{L}(D)$ to be the sheaf of rational sections σ of $\mathcal{L}$ satisfying $\operatorname{ord}_{p_i}(\sigma) \geq -m_i$ for all i; as a line bundle, this is the same as $\mathcal{L} \otimes \mathcal{O}_C(D)$.

If $\phi : X \to \mathbb{P}^r$ is a generically finite morphism, then the *degree of ϕ* is the number of points in the preimage of a general point of $\phi(X)$. Thus, for example, if $D := \sum_{p \in C} n_p p$ is a divisor on a smooth curve, and the linear system $|D|$ is basepoint free, then the degree of the morphism associated to $|D|$ is $\deg D := \sum_{p \in C} n_p$.

1.2.5 The most interesting linear system

The most important invertible sheaf on a smooth variety X is the sheaf of global sections of the top exterior power of the the cotangent bundle of X, called the canonical sheaf ω_X of X (for canonical sheaves more generally, see Chapter **??**). A section of ω_X is thus a differential form of degree equal to the dimension of X, and the divisor class of such a form is usually denoted K_C.

Theorem 1.2.13. *The canonical sheaf of $\mathbb{P}^r$ is $\mathcal{O}_{\mathbb{P}^n}(-r-1)$.*

Proof. Let $x_0, \ldots, x_r$ be the projective coordinates on $\mathbb{P}^r$ and let $U = \mathbb{P}^r \setminus H$ be the affine open set where $x_0 \neq 0$. Thus $U \cong \mathbb{A}^r$ with coordinates $z_{1:=}x_1/x_0, \ldots, z_r := x_r/x_0$. The space of r-dimensional differential forms on U is spanned by $d(x_1/x_0) \wedge \cdots \wedge d(x_r/x_0)$, which is regular everywhere in U. In view of the formula

$$d\frac{x_i}{x_0} = \frac{x_0 dx_i - x_i dx_0}{x_0^2}$$

we get

$$d(x_1/x_0) \wedge \cdots \wedge d(x_r/x_0) = \frac{dx_1 \wedge \cdots \wedge dx_r}{x_0^r} - \sum_{i=1}^{r} x_i \frac{dx_1 \wedge \cdots \wedge \widehat{d_{x_i}} \wedge \cdots \wedge dx_r}{x_0^{r+1}}$$

which has a pole of order $r+1$ along the locus H defined by x_0. Thus the divisor of this differential form is $-(r+1)H$, and this is the canonical class. $\square$

Cheerful Fact 1.2.1. A different derivation: there is a short exact sequence of sheaves of differentials, called the Euler sequence:

$$0 \to \Omega_{\mathbb{P}^r} \to \mathcal{O}_{\mathbb{P}^r}^{r+1}(-1) \to \mathcal{O}_{\mathbb{P}^r} \to 0.$$

. Taking exterior powers, we see that

$$\bigwedge^{r} \Omega_{\mathbb{P}^r} \otimes \bigwedge^{1} \mathcal{O}_{\mathbb{P}^r} = \bigwedge^{r+1} (\mathcal{O}_{\mathbb{P}^r}^{r+1}(-1)) = \mathcal{O}_{\mathbb{P}^r}(-r-1).$$

Computations of the canonical sheaf on a variety usually involve comparing the variety to another variety, such as projective space, where the canonical sheaf is already known. The most useful results of this type are the *adjunction formula* and the *Hurwitz' Theorem.*

Proposition 1.2.14. *(Adjunction Formula) Let X be a variety that is a Cartier divisor on a variety Y. If the canonical divisor of S is K_Y, then K_X is the restriction to X of the divisor $K_Y + X$.*

This is a special case of [**?**, ****].

Proof. There is an exact sequence of sheaves

$$0 \to \mathcal{I}_{X/Y}\mid_X \to \Omega_Y\mid_X \to \Omega_X \to 0$$

where Ω_X is the sheaf of differential forms on X (see [**?**, Theorem ***]), and $\mathcal{I}_{X/Y}\mid_X = \mathcal{O}_Y(-X)\mid_X = \mathcal{O}_X(-X)$. The proposition follows by taking top exterior powers. □

Corollary 1.2.15. *If $C \subset \mathbb{P}^2$ is a smooth plane curve of degree d, then $\omega_C = \mathcal{O}_C(d-3)$; more generally, if $X \subset \mathbb{P}^r$ is a complete intersection of hypersurfaces of degrees $d_1, \ldots, d_c$ then $\omega_X = \mathcal{O}_X(\sum_i d_i - r - 1)$.*

Given a (nonconstant) morphism $f : C \to X$ of smooth projective curves, the Riemann-Hurwitz formula computes the canonical sheaf C in terms of that of X and the local geometry f. To do this we define the *ramification index* of f at p, denoted $\mathrm{ram}(f, p)$, by the formula of divisors

$$f^{-1}(f(p)) = \sum_{p \in C | f(p) = q} (\mathrm{ram}(f, p) + 1) \cdot p$$

In terms of a suitable choice of local coordinates z on C around p and w on X around $f(p)$, we can write the morphism as $z \mapsto w = z^m$ for some integer $m > 0$, and $\mathrm{ram}(f, p) = m - 1$.

It follows from complex analysis (or the separability of field extensions in characteristic 0) that there are only finitely many points on C where $\mathrm{ram}(f, p) \neq 0$ (this would be false in characteristic > 0 in the case where the induced extension of fraction fields was inseparable.) Thus we may define the *ramification divisor* of f to be the divisor

$$R = \sum_{p \in C} \mathrm{ram}(f, p) \cdot p \ \in \ \mathrm{Div}(C).$$

and the *branch divisor* to be

$$B = \sum_{q \in X} \Big(\sum_{p \in f^{-1}(q)} \mathrm{ram}(f, p) \Big) \cdot q \ \in \ \mathrm{Div}(X).$$

Note that R and B have the same degree $\sum_{p \in C} \mathrm{ram}(f, p)$.

Theorem 1.2.16. *(Hurwitz' Theorem) [**?**, ****] If $f : C \to X$ is a non-constant morphism of smooth curves, with ramification divisor R, then*

$$\omega_C = f^* \omega_X(-R).$$

Proof. Choose a rational 1-form ω on X, and $\eta = f^*(\omega)$ be its pullback to C. For simplicity, we will assume that the zeroes and poles of ω lie outside the branch divisor B, so that ω will be regular and nonzero at each branch point. (Since we have the freedom to multiply by any rational function on X we can certainly find such a form, and in any event the calculation goes through without this assumption, albeit with more complicated notation.)

Since the zeroes of ω lie outside the branch divisor B, for every zero of ω of multiplicity m we have exactly d zeroes of η, each with multiplicity m; and likewise for the poles of ω. Meanwhile, at every point of B, the form ω is regular and nonzero. At a point p where (locally) f has the form $z \mapsto w = z^e$ and $\omega = dw$, ηdz we have $\eta = z^{e-1}dz$; that is η has a zero of multiplicity $\mathrm{ram}(f,p)$ at p. Thus the divisor K_C of η is $K_C = df^*(K_X) + R$. $\square$

Example 1.2.17. Let V be the vector space of homogeneous polynomials of degree d in two variables; that is, $V = H^0(\mathcal{O}_{\mathbb{P}^1}(d))$. In the projectivization $\mathbb{P}(V^*) \cong \mathbb{P}^d$, let Δ be the locus of polynomials with a repeated factor. Since Δ is defined by the vanishing of the discriminant, it is a hypersurface. What is its degree?

To answer this, let $W^* \subset V^*$ be a general 2-dimensional linear subspace—that is, a general pencil of forms of degree d on $\mathbb{P}^1$. The linear system $\mathcal{W} = (\mathcal{O}_{\mathbb{P}^d}, W^*)$ defines a morphism $\phi_{\mathcal{W}} : \mathbb{P}^1 \to \mathbb{P}(W) \cong \mathbb{P}^1$ and the fiber over the point of $\mathbb{P}(W)$ corresponding to a form f of degree d is the divisor $f = 0 \subset \mathbb{P}^1$. Thus the locus of polynomials in W with a multiple root is the branch locus of $\phi_{\mathcal{W}}$, where we count an m-fold root $m-1$ times. By Hurwitz' formula, the degree of the branch locus B of a degree d morphism from $\mathbb{P}^1$ to $\mathbb{P}^1$ is

$$\deg B = \deg \omega_{\mathbb{P}^1} - d \deg \omega_{\mathbb{P}^1} = 2d - 2.$$

Cheerful Fact 1.2.2. A famous result asserted by Franchetta and proved by **** is that the canonical sheaf (and its powers) are the *only* sheaves that can be chosen uniformly among all, or even almost all, smooth curves. For a more precise statement, see ****.

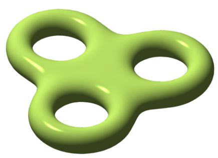
1.3 Genus, Riemann-Roch and Serre Duality

We will henceforward assume that the reader is acquainted with sheaf cohomology, at least sufficiently to write $H^i(X;(F)$ or $H^i(\mathcal{F})$ (our preferred form) without blushing. If D is a divisor on a scheme X we will often abbreviate $H^i(\mathcal{O}_X(D))$ to $H^i(D)$, and we write $h^i(\mathcal{F})$ or $h^i(D)$ for $\dim_{\mathbb{C}} H^i(\mathcal{F})$ or $\dim_{\mathbb{C}} H^i(D)$. Because $h^i(\mathcal{F})$, for $i > 0$, often appears as a kind of "error term" in formulas when one would like to compute $H^0(\mathcal{F})$, vanishing theorems have an important place in all of algebraic and analytic geometry. We will use the simplest of these often:

Theorem 1.3.1. *(Serre Vanishing Theorem) If $\mathcal{F}$ is a coherent sheaf on $\mathbb{P}^n$ then $H^i(\mathcal{F}(d)) = 0$ for all $i > 0$ and $d \gg 0$.*

1.3.1 The genus of a curve

The sole topological invariant of a smooth projective curve C is its genus. We can think of C as a submanifold of the complex projective space $\mathbb{P}^r(\mathbb{C})$ with the classical topology; as such, it is a compact, oriented surface, and its genus is the rank of its first integral homology, $H^1(C;\mathbb{Z})$—informally, the "number of holes":



**** Riemann Surface of genus 3, from Wikimedia ****

Of course this definition does not apply to curves over fields other than $\mathbb{C}$, and doesn't relate the genus to the algebra of the curve. However, we can relate the topological genus of a curve directly to its topological Euler characteristic $\chi_{top}(C) = 2 - 2g$. By the Hopf index theorem, the topological Euler characteristic is the degree of the tangent sheaf, or equivalently, minus the degree of the cotangent sheaf ω_C; that is, $\deg K_C = 2g - 2$, and thus

$$g(C) = \frac{\deg(K_C)}{2} + 1.$$

(This formula serves to define the genus of a smooth projective curve over any field).

Other characterizations of the genus require more machinery to establish. We will give some here, and use tools from the following section to prove equivalence.

1. $g(C)$ is the dimension of the vector space of regular 1-forms (that is, global sections of the cotangent sheaf) on C.

2. The (Zariski) Euler characteristic of the structure sheaf of C is $\chi(\mathcal{O}_C) = h^0(\mathcal{O}_C) - h^1(\mathcal{O}_C)$. Since $h^0(\mathcal{O}_C) = 1$,
$$g(C) = 1 - \chi(\mathcal{O}_C).$$
Recall that if $X \subset \mathbb{P}^r = \mathbb{P}(V)$ is any projective scheme, the *homogeneous coordinate ring* of X is the ring $S/I(X)$ where $S = \operatorname{Sym} V \cong \mathbb{C}[x_0, \dots, x_r]$ and $I(V) \subset S$ is the ideal of homogeneous forms that vanish on X.

3. Suppose that $C \subset \mathbb{P}^r = \mathbb{P}(V)$ is a smooth curve of degree d with homogeneous coordinate ring S_C, then the function $d \mapsto \dim_{\mathbb{C}}(S_C)_d$ is equal to a polynomial function $p_C(m)$ for large d. We have:
$$p_C(m) = dm - g + 1,$$
so $g(C) = 1 - p_C(0)$.

1.3.2 The Riemann-Roch Theorem

To prove that these formulas for the genus are correct, we use the Riemann-Roch Theorem and Serre duality (sometimes called Kodaira-Serre duality, since Kodaira was responsible for the analytic version.)

Theorem 1.3.2 (Riemann-Roch Theorem). *If C is a smooth, connected projective curve of genus g, and D a divisor of degree d on C then*

$$h^0(D) = d - g + 1 + h^0(K_C - D).$$

For example, if we take $D = 0$, this tells us that $h^0(K) = g$, proving the characterization (1) above. Also, since $h^0(D) = 0$ for any divisor D of negative degree, the formula gives the dimension of $h^0(D)$ when $\deg D$ is large:

Corollary 1.3.3. *For any divisor of degree $d \geq 2g - 1$, we have*

$$h^0(D) = d - g + 1.$$

Using this, we can apply Proposition 1.2.7 to show that all high degree divisors come from embeddings:

Corollary 1.3.4. *Let D be a divisor of degree d on a smooth, connected projective curve of genus g. If $d \geq 2g$, the complete linear series $|D|$ is base point free; and if $d \geq 2g + 1$ the associated morphism $\phi_D : C \to \mathbb{P}^{d-g}$ is an embedding, so that D is the preimage of the intersection of C with a hyperplane in $\mathbb{P}^{d-g}$.*

Since the complement of a hyperplane in projective space is an affine space, we get an affine embedding result too:

Corollary 1.3.5. *If C is any smooth, connected projective curve and $\emptyset \neq \Gamma \subset C$ a finite subset then $C \setminus \Gamma$ is affine.*

Proof. Let D be the divisor defined by Γ By Corollary 1.3.4 a high multiple of D is very ample, and gives an embedding $\phi : C \to \mathbb{P}^n$ such that the preimage of the intersection of C with some hyperplane H is a multiple of D. It follows that $C \setminus \Gamma$ is embedded in $\mathbb{P}^n \setminus H$. □

We can use Theorem 1.3.2in the simple case of Corollary 1.3.3 to determine the Hilbert polynomial of a projective curve. To do this, let $C \subset \mathbb{P}^r$ be a smooth curve of degree d and genus g, and consider the exact sequence of sheaves

$$0 \longrightarrow \mathcal{I}_{C/\mathbb{P}^r}(m) \longrightarrow \mathcal{O}_{\mathbb{P}^r}(m) \longrightarrow \mathcal{O}_C(m) \longrightarrow 0$$

and the corresponding exact sequence

$$H^0(\mathcal{O}_{\mathbb{P}^r}(m)) \xrightarrow{\rho_m} H^0(\mathcal{O}_C(m)) \longrightarrow H^1(\mathcal{I}_{C/\mathbb{P}^r}(m)) \longrightarrow 0.$$

The *Hilbert function* h_C of C is defined by

$$h_C(m) = \dim_{\mathbb{C}}(S_C)_m = \operatorname{rank}(\rho_m).$$

By Theorem 1.3.1 we have $H^1(\mathcal{I}_{C/\mathbb{P}^r}(m)) = 0$ for large m, so $h_C(m) = h^0(\mathcal{O}_C(m))$, for large m, which, by the Riemann-Roch Theorem , equals $md - g + 1$, again for large m. Thus, the Hilbert polynomial of $C \subset \mathbb{P}^r$ is $p_C(m) = dm - g + 1$, establishing the characterization (3).

The Riemann-Roch formula does *not* give us a formula for the dimension $h^0(D)$ when $h^0(K_C - D) > 0$; such divisors D are called *special divisors*, or *special divisor classes*. The existence or non-existence of divisors D with given $h^0(D)$ and $h^1(D)$ often serves to distinguish one curve from another, and will be an important part of our study.

Cheerful Fact 1.3.1. Classically, the dimension $h^0(K_C - D) = h^1(D)$ was called the *superabundance* of D: the idea was that a divisor of degree d had, at a minimum, $d - g + 1$ sections and $h^1(D)$ represented the number of "extra" sections. Even though the introduction of cohomology was still almost a century away, the ranks of cohomology groups h^1 had classical names, often involving the term superabundance—a premonition of the Riemann-Roch theorem in general.

Cheerful Fact 1.3.2. If k is a field that is not algebraically closed there may be genus 0 curves that are not isomorphic to $\mathbb{P}^1$. However, they must be"forms" of $\mathbb{P}^1$ in the sense that they become isomorphic to $\mathbb{P}^1$ after extension of scalars to the algebraic closure $\overline{k}$ of k. The unique example with $k = \mathbb{R}$ is the conic $x^2 + y^2 + z^2 = 0$. Indeed, any form of $\mathbb{P}^1$ over any field k can all be embedded in $\mathbb{P}^2_k$ (by using the anti-canonical linear system.

The curve $\mathbb{P}^1_k$ itself may be described as the scheme of left ideals of k-vector-space dimension 1 in the ring of 2×2 matrices over k (such an ideal can be embedded in the matrix ring as a linear combination of the 2 columns in an appropriate sense). More generally, any scheme that is a form of $\mathbb{P}^1$ over k may be described as the scheme of 1-dimensional left ideals in a central simple (= Azumaya) algebra over k—though as a set this scheme has no k-rational points unless the algebra is the algebra of 2×2 matrices!

1.3.3 Serre duality

In general, if $\mathcal{F}$ and $\mathcal{G}$ are coherent sheaves on a scheme X, we have for every i and j a cup product map

$$H^i(\mathcal{F}) \otimes H^j(\mathcal{G}) \to H^{i+j}(\mathcal{F} \otimes \mathcal{G}).$$

Theorem 1.3.6 (Serre Duality)**.** *Let C be a smooth connected projective curves with canonical divisor K. We have*

$$h^1(K) = 1$$

and the cup product map

$$H^1(D) \otimes H^0(K-D) \to H^1(K)$$

is a perfect pairing; that is, it induces a natural isomorphism

$$H^1(D) = H^0(K-D)^*.$$

1.3.4 A partial proof

Combining Theorem 1.3.2and Serre Duality we get:

Corollary 1.3.7. *If C is a smooth, connected projective curve and D is a divisor on C then*

$$\chi(\mathcal{O}_C(C)) := h^0(D) - h^1(D) = d - g + 1$$

or in other words, for any invertible sheaf $\mathcal{L}$ of degree d on C,

$$\chi(\mathcal{L}) = d - g + 1$$

which is pretty easy to prove. To see this, observe that for any invertible sheaf $\mathcal{L}$ on C and any point $p \in C$ we have an exact sequence of sheaves

$$0 \to \mathcal{L}(-p) \to \mathcal{L} \to \mathcal{L}_p \to 0.$$

It follows that $\chi(\mathcal{L}(-p)) = \chi(\mathcal{L}) - 1$, so that Riemann-Roch for $\mathcal{L}$ is equivalent to Riemann-Roch for $\mathcal{L}(-p)$. Since any divisor can be obtained from 0 by adding and subtracting points, the Riemann-Roch formula for an arbitrary $\mathcal{L}$ follows from the special case $\mathcal{L} = \mathcal{O}_C$.

1.3.5 Clifford's theorem

Theorem 1.3.8. *Let C be a curve of genus g and $\mathcal{L}$ a line bundle of degree $d \leq 2g-2$. Then*

$$r(\mathcal{L}) \leq \frac{d}{2}.$$

Moreover, if equality holds then we must have either

1. *$d = 0$ and $\mathcal{L} = \mathcal{O}_C$;*

2. *$d = 2g-2$ and $\mathcal{L} = K_C$; or*

3. *C is hyperelliptic, and $|\mathcal{L}|$ is a multiple of the g^1_2 on C.*

Proof. The proof of Clifford rests on a very basic construction and observation.

To start, let $\mathcal{D} = (\mathcal{L}, V)$ and $\mathcal{E} = (\mathcal{M}, W)$ be two linear series on a curve C. By the *sum* $\mathcal{D} + \mathcal{E}$ of $\mathcal{D}$ and $\mathcal{E}$, we will mean the pair

$$\mathcal{D} + \mathcal{E} = (\mathcal{L} \otimes \mathcal{M}, U)$$

where $U \subset H^0(\mathcal{L} \otimes \mathcal{M})$ is the subspace generated by the image of $V \otimes W$, under the multiplication/cup product map $H^0(\mathcal{L}) \otimes H^0(\mathcal{M}) \to H^0(\mathcal{L} \otimes \mathcal{M})$—in other words, it's the subspace of the complete linear series $|\mathcal{L} \otimes \mathcal{M}|$ spanned by divisors of the form $D + E$, with $D \in \mathcal{D}$ and $E \in \mathcal{E}$.

The observation is a simple one:

Lemma 1.3.9. *If $\mathcal{D}$ and $\mathcal{E}$ are two nonempty linear series on a curve C, then*

$$\dim(\mathcal{D} + \mathcal{E}) \geq \dim \mathcal{D} + \dim \mathcal{E}.$$

(To see this, we observe that to say $\dim \mathcal{D} \geq m$ means exactly that we can find a divisor $D \in \mathcal{D}$ containing any given m points of C; since $\mathcal{D} + \mathcal{E}$ contains all pairwise sums $D + E$ with $D \in \mathcal{D}$ and $E \in \mathcal{E}$, we can certainly find a divisor $F \in cD + \mathcal{E}$ containing any given $\dim \mathcal{D} + \dim \mathcal{E}$ points of C.)

Given this lemma, the proof of Clifford follows simply by applying it to the pair $|\mathcal{L}|$ and $|K_C \otimes \mathcal{L}^{-1}|$: by Riemann-Roch, we have

$$r(K_C \otimes \mathcal{L}^{-1}) = r(\mathcal{L}) + g - d - 1$$

and so we deduce that

$$g = r(K_C) + 1 \geq r(\mathcal{L}) + r(K_C \otimes \mathcal{L}^{-1}) + 1 \geq 2r(\mathcal{L}) + g - d;$$

hence $r(\mathcal{L}) \leq d/2$.

Our proof of the second half of Clifford rests on a basic fact about the geometry of hyperplane sections of a curve in projective space (Proposition **??**); we'll defer it until we've established that fact. ∎

1.4 The canonical morphism

Given the central role played by the canonical divisor class, it is natural to look at the geometry of the morphism $\phi_K : C \to \mathbb{P}^{g-1}$ associated to the complete canonical series $|K|$. By the Riemann-Roch theorem, $h^0(K) = g(C)$, so $|K|$ cannot define a non-constant morphism unless $g(C) \geq 2$, and cannot define an embedding unless $g(C) \geq 3$.

Definition 1.4.1. A curve C of genus $g \geq 2$ is said to be *hyperelliptic* if there exists a morphism $f : C \to \mathbb{P}^1$ of degree 2.

Proposition 1.4.2. *The canonical morphism $\phi_K : C \to \mathbb{P}^{g-1}$ is an embedding if and only if C is not hyperelliptic.*

Proof. By Corollary 1.3.4 we have to show that for any pair of points $p, q \in C$ we have

$$h^0(K_C(-p-q)) = h^0(K_C) - 2 = g - 2.$$

Applying the Riemann-Roch Theorem we see that this would fail if and only if $h^0(\mathcal{O}_C(p+q)) \geq 2$ for some $p, q \in C$, and by Lemma 1.4.3 $|p+q|$ would define a degree 2 morphism to $\mathbb{P}^1$. □

Lemma 1.4.3. *Let C be a smooth, projective curve of genus $g \geq 2$. Any invertible sheaf of degree 2 on C defines a morphism to $\mathbb{P}^1$. In particular, if $g(C) = 2$ then the canonical series $|K_C|$ defines a 2 to 1 morphism to $\mathbb{P}^1$.*

Proof. If this happens, we claim that $\mathcal{O}_C(p+q)$ is basepoint free, so that C is hyperelliptic. To finish the proof, by Corollary 1.3.4 it suffices to show that an invertible sheaf $\mathcal{L}$ of degree 1 on C must have $h^0(\mathcal{L}) \leq 1$.

Suppose that σ_0, σ_1 were two linearly independent sections of $\mathcal{L}$. Each σ_i vanishes at a unique point p_i. If $p_0 = p_1$ then a linear combination of σ_0, σ_1 would be a section vanishing to order ≥ 2, which is impossible, so $\mathcal{L}$ is basepoint free, and defines a degree 1 morphism $C \to \mathbb{P}^1$. Such a morphism must be an isomorphism (because $\mathbb{P}^1$ is normal), contradicting $g(C) \geq 2$. □

((the following argument is only set-theoretic. Admit this or make it precise))
Note that if C is hyperelliptic, the morphism ϕ_K factors through the degree 2 morphism $\pi : C \to \mathbb{P}^1$: if $\{p, q\} \subset C$ is a fiber of this morphism, we have $h^0(\mathcal{O}_C(p+q)) = 2$ and hence $\phi_K(p) = \phi_K(q)$. The image of the morphism ϕ_K is a nondegenerate curve of degree $g-1$ in $\mathbb{P}^{g-1}$, which we will see is a *rational normal curve*. This observation implies in particular that if C is hyperelliptic of genus $g \geq 2$, then the invertible sheaf $\mathcal{L}$ of degree 2 with $h^0(\mathcal{L}) = 2$ is in fact unique.

Among curves with $g \geq 3$ the hyperelliptic curves are very special: in the family of all curves, as we'll see, they comprise a closed subvariety. Also, the behavior of linear series and morphisms on a hyperelliptic curve is very different from that of series on a general curve; when we discuss the geometry of curves of low genus in the Chapter **??**, we will exclude the hyperelliptic case, and deal with this case in a separate chapter.

For non-hyperelliptic curves, however, the geometry of the canonical morphism, and its image, the canonical curve, are the keys to understanding the curve. We'll see this in detail in many cases in the following chapter; for now, we mention one highly useful result along these lines.

((add here: canonical series on plane curves cut by $|\mathcal{O}_{\mathbb{P}^2}(d-3)|$; consequence that no smooth plane curve can be hyperelliptic))

((maybe move initial discussion of hyperelliptic curves from Ch. 6 to a section here))

((maybe add to this chapter: differentials on plane curves C, possibly with nodes or more general singularities; adjoint conditions; algorithm for determining the complete linear system associated to a divisor D on C))

1.4.1 The geometric Riemann-Roch theorem

Let's state this first in a relatively simple case: let C be a nonhyperelliptic curve, embedded in $\mathbb{P}^{g-1}$ by its canonical series and let $D = p_1 + \cdots + p_d$ be a divisor consisting of d distinct points; let $\overline{D}$ be the span of the points $p_i \in C \subset \mathbb{P}^{g-1}$. Since the hyperplanes in $\mathbb{P}^{g-1}$ containing $\{p_1, \ldots, p_d\}$ correspond (up to scalars) to sections of K_C vanishing at all the points p_i, we see that

$$h^0(K_C - D) = g - 1 - \dim \overline{D}.$$

Plugging this into the Riemann-Roch formula, we arrive at the statement

$$r(D) = d - 1 - \dim \overline{D};$$

or in other words, *the dimension of the linear series $|D|$ in which the divisor D moves is equal to the number of linear relations on the points p_i on the canonical curve.* Thus, for example, if $D = p_1 + p_2 + p_3$, we see that D moves in a pencil if and only if the points p_i are collinear.

We can extend this statement to the case of arbitrary effective divisors D (and even hyperelliptic curves) if we define our terms correctly. To do this, suppose $f : C \to \mathbb{P}^d$ is any morphism, and $D \subset C$ any divisor. We define the *span* of $f(D)$ to be the intersection

$$\overline{f(D)} = \bigcap_{H \mid f^{-1}(H) \supset D} H$$

of all hyperplanes in $\mathbb{P}^d$ whose preimage in C contains D.

Theorem 1.4.4 (Geometric Riemann-Roch Theorem)**.** *If C is any curve of genus $g \geq 2$, $\phi : C \to \mathbb{P}^{g-1}$ its canonical morphism and $D \subset C$ any effective divisor of degree d, then*

$$r(D) = d - 1 - \dim \overline{\phi(D)}.$$

Chapter 2

Curves of genus 0 and 1

In this chapter, we'll begin our project of describing curves in projective space with the simplest cases, that of curves of genus 0 and 1. Despite the relative simplicity of these curves, there are many interesting statements to make about the geometry of their embeddings in $\mathbb{P}^r$, as well as many conjectures and open problems.

One reason for restricting our attention (for now!) to the cases $g = 0$ and 1 is that the divisor class theory is particularly simple in these cases. Specifically, on a curve of genus 0, there is a unique invertible sheaf of given degree d; and on a curve of genus 1 all invertible sheaves of given degree d are congruent modulo the automorphism group of the curve. Thus, in regard to the geometry of the associated maps to projective space, all invertible sheaves of given degree d behave in the same way. By contrast, on a curve C of higher genus there are many different divisor classes of given degree, and to describe their various geometries we need to introduce and describe the space $\mathrm{Pic}^d(C)$ parametrizing these invertible sheaves. We will do that in the following chapter, and then return in Chapter 4 to the geometry of curves of genera 2 and 3 in projective space.

Our knowledge of the geometry of curves becomes increasingly less complete as the genus increases, and 6, as we shall see, is a natural turning point; we will consider the case of curves of genus , 5 and 6 in Chapter 5

2.1 Curves of genus 0

As we saw in more generality in Example 1.2.4, there is for each $d \in \mathbb{Z}$ a unique invertible sheaf $\mathcal{O}_{\mathbb{P}^1}(d)$ of degree d on $\mathbb{P}^1$. To compute $H^0(\mathcal{O}_{\mathbb{P}^1}(d))$ directly, let $D = z_1 + z_2 + \cdots + z_d$ be a divisor of degree d and suppose that the coordinates are chosen so that none of the z_i are at infinity. The sections of $\mathcal{O}_{\mathbb{P}^1}(D)$ are the rational functions with poles only at the z_i. In affine coordinates, identifying the z_i with complex numbers, these can each be written

$$\frac{g(z)}{(z-z_1)(z-z_2)\cdots(z-z_d)}$$

with $\deg(g) \leq d$, the condition that infinity is not a pole. We see that these form a vector space of dimension $d+1$.

((the following result is such a good exercise in the correspondence of linear systems and maps and divisors, maybe move it to Ch 2?))

By the Riemann-Roch Theorem , any invertible sheaf of degree d on a curve of genus 0, like $\mathbb{P}^1$, has at least $d+1$ sections. In fact (over $\mathbb{C}$, or any algebraically closed field), this characterizes $\mathbb{P}^1$:

Theorem 2.1.1. *Let C be a reduced, irreducible projective curve and let $\mathcal{L}$ be an invertible sheaf of degree d on C. If $\mathrm{h}^0(\mathcal{L}) \geq d+1$ then $C \cong \mathbb{P}^1$, so $\mathcal{L} \cong \mathcal{O}_{\mathbb{P}^1}(d)$, and $\mathrm{h}^0(\mathcal{L}) = d+1$.*

Proof. Let $p_1, \dots p_{d-1}$ be general points of C, and set $\mathcal{L}' := \mathcal{L}(-p_1 - \cdots - p_{d-1})$. From the correspondence between divisors and invertible sheaves, we see that the degree of $\mathcal{L}'$ is 1. Since $\mathcal{L}$ is locally isomorphic to the sheaf of functions on C, the condition of vanishing at a point imposes at most 1 linear condition on the global sections of $\mathcal{L}$, and thus $H^0(\mathcal{L}') \geq 2$, so we may assume from the outset that $d = 1$.

The linear system $(\mathcal{L}, H^0(\mathcal{L}))$ cannot have any base points, since otherwise after subtracting one, we would get an invertible sheaf of degree ≤ 0 with two independent global sections. Again by the correspondence with divisors, neither of these sections could vanish at any point of C, so their ratio would be a non-constant function defined everywhere on C, a contradiction.

Thus we see that the linear system $(\mathcal{L}, H^0(\mathcal{L}))$ defines a morphism $\phi : C \to \mathbb{P}^1$ of degree 1 whose fibers—the divisors defined by sections of $\mathcal{L}$ are of degree 1. Thus if $p \in C$ is the preimage of $q \in \mathbb{P}^1$, the induced map of local rings $\phi^* : \mathcal{O}_{\mathbb{P}^1,q} \to \mathcal{O}_{C,p}$ is a finite, birational map. Since $\mathcal{O}_{\mathbb{P}^1,q}$ is integrally closed, this is an isomorphism. Thus ϕ is an isomorphism, as required. $\square$

Note that we used the algebraic closure of the ground field in choosing points on C.

Corollary 2.1.2. *Every smooth curve C of genus 0 over an algebraically closed field is isomorphic to $\mathbb{P}^1$.*

Proof. By the Riemann-Roch Theorem , any linear system $\mathcal{L}$ of degree d on C has $h^0\mathcal{L} \geq d+1$. $\square$

Note that all the above depends fundamentally on the algebraic closure of the ground field: over a non-algebraically closed field, a curve C of genus 0 need not have any points, or any line bundles of odd degree (since the canonical bundle K_C has degree -2, there do necessarily exist line bundles of every even degree; thus an arbitrary curve of genus 0 is isomorphic to a conic plane curve). The classification of curves of genus 0 over non-algebraically closed fields is a subject that goes back to Gauss.

2.2 Rational Normal Curves

Recall from Example 1.2.5 that the image of the d-th *Veronese map* $\phi_d : \mathbb{P}^1 \to \mathbb{P}(H^0((\mathcal{O}_{\mathbb{P}^1}(d)) \cong \mathbb{P}^d$ is called the *rational normal curve* of degree d. Rational normal curves are probably the most ubiquitous curves in projective space; they have many unique properties, and are extremal in many respects. We will accordingly take a few pages and list some of the special properties of rational normal curves.

Rational normal curves have minimal degree

The first is a characterization of rational normal curves as having smallest possible degree among irreducible, nondegenerate curves:

Proposition 2.2.1. *If C is a nondegenerate curve in $\mathbb{P}^d$ then $\deg C \geq d$, with equality if and only if C is a rational normal curve.*

Proof. By the correspondence between morphisms and linear systems, the invertible sheaf $\mathcal{L}$ corresponding to the morphism $C \hookrightarrow \mathbb{P}^d$ has degree d and $h^0(\mathcal{L}) \geq d+1$. The conclusion follows from Theorem 2.1.1. □

We will see more generally that, if X is a non-degenerate variety in $\mathbb{P}^d$ of dimension k, then $\deg(X) \geq d-k+1$; and we will describe the varieties that achieve the minimum in Section **??**.

Independence of points on a rational normal curve

The points on a rational normal curve are "as independent as possible:"

Proposition 2.2.2. *If $C \subset \mathbb{P}^d$ is a rational normal curve of degree d and $\Gamma \subset C$ is a subscheme of length $\ell \leq d+1$, then Γ lies on no plane of dimension $< \ell$. In particular, any $m \leq d+1$ distinct points on a rational normal curve $C \subset \mathbb{P}^d$ are linearly independent.*

The rational normal curve is the unique curve with this property, as we shall see in Chapter 6.

Proof. We can reduce to the case $\ell = d+1$ by adding points to Γ, so it suffices to do that case, which follows at once from Bezout's Theorem. $\square$

In the case of distinct points it is easy to make a direct argument: In affine coordinates chosen so that none of the points are at infinity we can identify the points $\lambda_1, \ldots, \lambda_{d+1} \in C \cong \mathbb{P}^1$ with complex numbers, and the statement (for $\ell = d+1$) is tantamount to the nonvanishing of the Vandermonde determinant

$$\begin{vmatrix} 1 & \lambda_1 & \lambda_1^2 & \ldots & \lambda_1^d \\ 1 & \lambda_2 & \lambda_2^2 & \ldots & \lambda_2^d \\ \vdots & & & & \vdots \\ 1 & \lambda_{d+1} & \lambda_{d+1}^2 & \ldots & \lambda_{d+1}^d \end{vmatrix}.$$

Rational normal curves are projectively normal

We say that a smooth curve $C \subset \mathbb{P}^d$ is *projectively normal* if the restriction map

$$H^0(\mathcal{O}_{\mathbb{P}^d}(m)) \ \to \ H^0(\mathcal{O}_C(m))$$

is surjective for every m. We'll this property it in many settings, in particular the discussion of *liaison* in Chapter **??**. Since every monomial of degree md on $\mathbb{P}^1$ is a product of m monomials of degree d, we see that the rational normal curve is projectively normal.

The equations defining a rational normal curve

It is easy to write down equations that define a rational normal curve. Choosing a basis s, t for the linear forms on $\mathbb{P}^1$, we can write

$$\phi_d : (s,t) \mapsto (s^d, s^{d-1}t, \ldots t^d)$$

from which we see that C lies in the zero locus of the homogeneous quadratic polynomial $z_i z_j - z_k z_l$ for every $i+j = k+l$. As a convenient way to package these, we can realize these forms the 2×2 minors of the matrix

$$M \ = \begin{pmatrix} z_0 & z_1 & \ldots & z_{d-1} \\ z_1 & z_2 & \ldots & z_d \end{pmatrix}.$$

Note that if we substitute $s^i t^{(d-i)}$ for z_i and identify $H^0(\mathcal{O}_{\mathbb{P}^1}(i)$ with $\mathbb{C}[s,t]_i$, this becomes the multiplication table

$$H^0(\mathcal{O}_{\mathbb{P}^1}(i)) \times H^0(\mathcal{O}_{\mathbb{P}^1}(d-i-1)) \to H^0(\mathcal{O}_{\mathbb{P}^1}(d));$$

we shall see a general version of this in Chapter **??**.

In fact, the minors of this matrix generate the ideal of forms on $\mathbb{P}^d$ vanishing on C. For this result see for example [**?**, ****]. We can immediately prove two slightly weaker results:

First, C is set-theoretically defined by the 2×2 minors of M. Explicitly, suppose that $p = (z_0, \ldots, z_d) \in \mathbb{P}^d$ is any point, and all the polynomials Q_{ijkl} above vanish at p. If $z_0 = 0$, then from the vanishing of $\det \begin{pmatrix} z_0 & z_1 \\ z_1 & z_2 \end{pmatrix}$ we see that $z_1 = 0$, and similarly we have $z_2 = \cdots = z_{d-1} = 0$; this the point $p = (0, \ldots, 0, 1)$, which is a point on the rational normal curve. On the other hand, if $z_0 \neq 0$, set $\lambda = z_1/z_0$; we see in turn that $z_2/z_1 = \cdots = z_d/z_{d-1} = \lambda$; thus $p = (1, \lambda, \ldots, \lambda^d)$, again a point of the rational normal curve.

Second, the $\binom{d}{2}$ distinct 2×2 minors of M are linearly independent, as one can see by first factoring out x_0 and x_d and noting that the resulting minors generate the square of the maximal ideal in $\mathbb{C}[x_1, \ldots, x_{d-1}]$. Note that the restriction map

$$H^0(\mathcal{O}_{\mathbb{P}^d}(2)) \;\to\; H^0(\mathcal{O}_C(2)) = H^0(\mathcal{O}_{\mathbb{P}^1}(2d))$$

is surjective because every monomial of degree $2d$ on $\mathbb{P}^1$ is a product of two monomials of degree d. Comparing dimensions, we see that the dimension of the kernel—that is, the space of quadratic polynomials on $\mathbb{P}^d$ vanishing on C—has dimension

$$\binom{d+2}{2} - (2d+1) \;=\; \binom{d}{2}.$$

In fact, this gives us another characterization of rational normal curves as extremal: rational normal curves lie on more quadrichypersurfaces than any other irreducible, nondegenerate curve in $\mathbb{P}^d$.

Proposition 2.2.3. *If $C \subset \mathbb{P}^d$ is any irreducible, nondegenerate curve, then*

$$h^0(\mathcal{I}_{C/\mathbb{P}^d}(2)) \leq \binom{d}{2};$$

and if equality holds then C is a rational normal curve

Proof. Consider the restriction of the quadrics containing C to a general hyperplane $H \cong \mathbb{P}^{d-1} \subset \mathbb{P}^d$, and let $\Gamma = H \cap C$. We have exact sequence:

$$0 \to \mathcal{I}_{C/\mathbb{P}^d}(1) \to \mathcal{I}_{C/\mathbb{P}^d}(2) \to \mathcal{I}_{\Gamma/\mathbb{P}^{d-1}}(2) \to 0.$$

Since C is nondegenerate, $h^0(\mathcal{I}_{C/\mathbb{P}^d}(1)) = 0$, and since $\deg C \geq d$, the hyperplane section Γ of C must contain at least d linearly independent points. Since linearly independent points impose independent conditions on quadrics, we have

$$h^0(\mathcal{I}_{\Gamma/\mathbb{P}^{d-1}}(2)) \leq h^0(\mathcal{O}_{\mathbb{P}^{d-1}}(2)) - d,$$

establishing the desired inequality. □

Exercise 2.2.4. Establish the analogous statement for hypersurfaces of any degree d; that is, no irreducible, nondegenerate curve in $\mathbb{P}^r$ lies on more hypersurfaces of degree d than the rational normal curve.

Exercise 2.2.5. Prove directly the special case $r = 3$: that the twisted cubic is the unique irreducible, nondegenerate space curve lying on three quadrics. (Hint: if $C \subset \mathbb{P}^3$ is such a curve lying on three quadrics, what must be the intersection of two of the quadrics containing C?)

Rational normal curves are projectively homogeneous

Another important property of rational normal curves $C \subset \mathbb{P}^d$ is that they are *projectively homogeneous*: the subgroup G of the automorphism group PGL_{d+1} of automorphisms of $\mathbb{P}^d$ that carries C to itself acts transitively on C. More generally, every $\mathbb{P}^r$ is a homogeneous variety in the sense that $\operatorname{Aut}\mathbb{P}^r$ acts transitively. If σ is an automorphism then, because $\mathcal{O}_{\mathbb{P}^r}(d)$ is the unique invertible sheaf of degree d on $\mathbb{P}^r$, we have $\sigma^*\mathcal{O}_{\mathbb{P}^r}(d) = \mathcal{O}_{\mathbb{P}^r}(d)$ so σ induces an automorphism ϕ on $H^0(\mathcal{O}_{\mathbb{P}^r}(d))$, and an automorphism $\overline{\phi}$ on the ambient space $\mathbb{P}H^0(\mathcal{O}_{\mathbb{P}^r}(d))$ of the target of the d-th Veronese map. If α is a rational function with divisor D, then $\phi(\alpha) = \alpha \circ \sigma$ has divisor $\sigma^{-1}(D)$, so $\overline{\phi}^{-1}$ induces σ on $\mathbb{P}^r$.

The rational normal curve $C \subset \mathbb{P}^r$ can also be characterized among irreducible, nondegenerate curves as the unique projectively homogeneous curve in $\mathbb{P}^r$, as we shall see in Chapter 6.

Exercise 2.2.6. Let $\mathbb{P}^1 \hookrightarrow C \subset \mathbb{P}^3$ be a twisted cubic. Show that the normal bundle $\mathcal{N}_{C/\mathbb{P}^3}$ (defined to be the quotient of the restriction $T_{\mathbb{P}^3}|_C$ to C of the tangent bundle of $\mathbb{P}^3$ by the tangent bundle T_C) is

$$\mathcal{N}_{C/\mathbb{P}^3} \cong \mathcal{O}_{\mathbb{P}^1}(5) \oplus \mathcal{O}_{\mathbb{P}^1}(5)$$

Exercise 2.2.7. Let $\mathbb{P}^1 \hookrightarrow C \subset \mathbb{P}^d$ be a rational normal curve. Show that the normal bundle $\mathcal{N}_{C/\mathbb{P}^d}$ is

$$\mathcal{N}_{C/\mathbb{P}^d} \cong \bigoplus_{i=1}^{d-1} \mathcal{O}_{\mathbb{P}^1}(d+2).$$

Exercise 2.2.8. In the situation of the preceding problem, the set of direct summands of $\mathcal{N}_{C/\mathbb{P}^d}$ is a projective space $\mathbb{P}^{d-2}$. How does the group of automorphisms of $\mathbb{P}^d$ carrying C to itself act on this $\mathbb{P}^{d-2}$?

2.2.1 Other rational curves

What about other rational curves in projective space? There are many other embeddings of $\mathbb{P}^1$ in $\mathbb{P}^r$ other than the rational normal curve, and we'll talk now about some of these.

The first thing to say is that, since any linear series $\mathcal{D}$ of degree d on $\mathbb{P}^1$ is a subseries of the complete series $|\mathcal{O}_{\mathbb{P}^1}(d)|$, we see that *any rational curve*

$C \subset \mathbb{P}^r$ of degree d is a projection of a rational normal curve in $\mathbb{P}^d$. Slightly more generally, any map $\phi : \mathbb{P}^1 \to \mathbb{P}^r$ of degree d is given as

$$z \mapsto (f_0(z), \dots, f_r(z))$$

for some $(r+1)$-tuple of polynomials f_α of degree d on $\mathbb{P}^1$, which is to say it is the composition of the embedding $\phi_d : \mathbb{P}^1 \to \mathbb{P}^d$ of $\mathbb{P}^1$ as a rational normal curve with a linear projection $\pi : \mathbb{P}^d \to \mathbb{P}^r$.

Given how easy it is to describe rational curves in projective space in this way, it is in some ways surprising how many open questions there are about such curves. We'll talk more about some of these questions in the following section; for now, we will try to give a sense of what we can say about such curves by considering one of the first and simplest cases: smooth rational curves of degree 4 in $\mathbb{P}^3$.

So: let $C \subset \mathbb{P}^3$ be a smooth, nondegenerate curve of degree 4 and genus 0 in $\mathbb{P}^3$. To describe the geometry of C, the first thing to determine is what surfaces it lies on—that is, what degree polynomials on $\mathbb{P}^3$ vanish on C.
((what's proven is that C lies on a smooth quadric and the ideal needs at least 3 additional 3-ic generators. State in advance, and maybe do better.))
To start with, we can ask: does C lie on a quadric surface? To answer this, we consider again the restriction map

$$H^0(\mathcal{O}_{\mathbb{P}^3}(2)) \to H^0(\mathcal{O}_C(2)) = H^0(\mathcal{O}_{\mathbb{P}^1}(8)).$$

Here the vector space on the left—homogeneous quadratic polynomials on $\mathbb{P}^3$—has dimension 10, while the one on the right, either by Riemann-Roch or by direct examination, has dimension 9. We conclude that *the curve C must lie on at least one quadric surface $Q \subset \mathbb{P}^3$.*

Since C is irreducible and nondegenerate, it can't lie on a union of planes, so the quadric Q must either be smooth or a cone over a conic curve. We'll see in a moment that the latter case can't occur, so let's assume for now that Q is smooth.

The natural follow-up question is, what is the class of C in the Picard group of Q? We know that $Q \cong \mathbb{P}^1 \times \mathbb{P}^1$, with the fibers of the two projections appearing as lines of the two rulings of Q. Lines L and M of the two rulings generate the Picard group [**?**, ***], so that we must have $C \sim aL + bM$ for some a, b (in other words, in terms of the isomorphism $Q \cong \mathbb{P}^1 \times \mathbb{P}^1$, C is the zero locus of a bihomogeneous polynomial of bidegree (a, b)), and we ask what a and b are. The choices are limited: since C is a quartic curve, we must have $a+b = 4$. Adjunction [**?**, ***] tells us which must be the case: the genus formula for curves on Q tells us that the genus of a smooth curve of class (a, b) on Q has genus $(a-1)(b-1)$, whence the class of our curve C must be $(1, 3)$ (for a suitable ordering of the two rulings).

It follows in particular that *Q is the unique quadric containing C.* One way to see this is that since C has class $(1, 3)$ it meets the lines of the first ruling three times; if Q' is any quadric containing C, then, it must contain all these lines and hence must equal Q. Equivalently, we may consider the exact sequence

$$0 \to \mathcal{I}_{C/Q}(2) \to \mathcal{O}_Q(2) \to \mathcal{O}_C(2) \to 0.$$

If C has class $L + 3M$, we have $\mathcal{I}_{C/Q}(2) = \mathcal{O}_Q(L - M)$. Since this bundle has negative degree on every line of the first ruling, it has no sections; hence the restriction map $H^0(\mathcal{O}_Q(2)) \to H^0(\mathcal{O}_C(2))$ is injective and so there are no quadrics in $\mathbb{P}^3$ containing C other than Q.

We can also describe the rest of the ideal of C similarly. For example, to find the cubic polynomials vanishing on C we consider the restriction map

$$H^0(\mathcal{O}_{\mathbb{P}^3}(3)) \ \to \ H^0(\mathcal{O}_C(3)) = H^0(\mathcal{O}_{\mathbb{P}^1}(12)).$$

The dimensions of these two vector spaces being 20 and 13 respectively, we see that C must lie on at least 7 cubics; four of these are simply products of Q with linear forms, and so we see that C must lie on at least three cubics modulo those containing Q. Indeed, these are easy to spot: if L and L' are any two lines of the first ruling, the divisor $C + L + L'$ has class $(3,3)$ on Q and hence is the intersection of Q with a cubic surface. As $L + L'$ varies in a two-dimensional linear series, we get three cubics containing C modulo those containing Q. Conversely, any cubic containing C (but not containing Q) will intersect Q in the union of C with a curve of type $(2,0)$ on Q, which is to say the sum of two lines of the first ruling, so these are all the cubics containing C.

((at least state that these are the generators. And state the set-theoretic intersection problem.))

Finally, we have to show that the quadric containing the curve C cannot be a cone over a conic plane curve. The key question here is whether or not C contains the vertex p of the cone: if not, the same adjunction-based calculation shows that C must have genus 1; while a parity argument (how many times does C meet a line of the ruling of Q?) shows that if a curve $C \subset Q$ of even degree contains p it must be singular there.
((this is pretty fast, compared to the level in the rest of the Ch. let's fill it in.))

Exercise 2.2.9. Find all possible Hilbert functions of smooth rational quintic curves $C \subset \mathbb{P}^3$. (There are only two, depending on whether or not C lies on a quadric, so this isn't so bad.)

Exercise 2.2.10. Every g^3_4 on $\mathbb{P}^1$ is uniquely expressible as a sum of the g^1_1 and a g^1_3

Exercise 2.2.11. There is a 1-parameter family of rational quartic curves in $\mathbb{P}^3$ up to projective equivalence. (Finding the invariants is a nice problem, which we should talk about. This is the cross-ratio of the roots of the quartic in 2 variables corresponding to the projection center.)

2.2.2 Further problems (open and otherwise) concerning rational curves in projective space

To begin with, we should remark that this one example of a non-linearly normal rational curve in projective space is misleading in that we can give such a complete description. For general d and r, we have no idea what may be the Hilbert function of a rational curve of degree d in $\mathbb{P}^r$. Indeed, even in the limited case of $r = 3$, our knowledge gives out around $d = 9$.

We can, however, say some things about a *general* rational curve $C \subset \mathbb{P}^r$ of given degree d. To make sense of this, let $C_0 \subset \mathbb{P}^d$ be a rational normal curve of degree d. As we've said, any rational curve of degree d in $\mathbb{P}^r$ is the projection $\pi_\Lambda(C_0)$ of C_0 from a $(d-r-1)$-plane $\Lambda \subset \mathbb{P}^d$. If we let $\mathbb{G} = \mathbb{G}(d-r-1, d)$ be the Grassmannian of $(d-r-1)$-planes in $\mathbb{P}^d$, and we let $U \subset \mathbb{G}$ be the open subset of planes disjoint from the secant variety of C_0, we have a family of rational curves in $\mathbb{P}^r$ parametrized by U and including every smooth rational curve $C \subset \mathbb{P}^r$ of degree d. Thus in particular we can talk about a *general rational curve* of degree d and genus g in $\mathbb{P}^r$, and ask about its geometry.

This is, in fact, still largely uncharted waters. Consider, for example, one of the most basic questions we might ask: what is the Hilbert function of a general rational curve $C \subset \mathbb{P}^r$ of degree d? As in the example, this is tantamount to looking at the restriction map

$$\rho_m : H^0(\mathcal{O}_{\mathbb{P}^r}(m) \to H^0(\mathcal{O}_C(m)) = H^0(\mathcal{O}_{\mathbb{P}^1}(md)).$$

Equivalently, we're asking: if V is a general $(r+1)$-dimensional vector space of homogeneous polynomials of degree d, what is the dimension of the space of polynomials spanned by m-fold products of polynomials in V? We might naively guess that the answer is, "as large as possible," meaning that the rank of ρ_m is $\binom{m+r}{r}$ when that number is less than $md + 1$, and equal to $md + 1$ when it is greater—in other words, the map ρ_m is either injective or surjective for each m.

This, it turns out, is true, but it is only relatively recently known: the case $g = 0$, as here, was done by Ballico in **** (??), and the analogous statement for curves of arbitrary genus, which we will describe in Chapter **??**, was proved in 2019 by Eric Larson.

The secant plane conjecture

Another question we may ask about a curve in projective space is what secant planes it has. To frame the question, let's start with some language: given a smooth curve $C \subset \mathbb{P}^r$, we say that an e-secant s-plane to C is an s-plane $\Lambda \cong \mathbb{P}^s \subset \mathbb{P}^r$ such that the intersection $\Lambda \cap C$ has degree $\geq e$; if we exclude degenerate cases (for example, where $\Lambda \cap C$ fails to span Λ), this is the same as saying we have a divisor $D \subset C$ of degree e whose span is contained in an s-plane.

Do we expect a curve $C \subset \mathbb{P}^r$ to have any e-secant s-planes? The set of s-planes in $\mathbb{P}^r$ is parametrized by the Grassmannian $\mathbb{G} = \mathbb{G}(s, r)$, which had dimension $(s+1)(r-s)$. Inside $\mathbb{G}$, the locus of planes that meet C has codimension $r-s-1$ (the locus of planes containing a given point of C has codimension $r-s$); so our naive expectation might be that the locus of e-secant s-planes would have codimension $e(r-s-1)$ in $\mathbb{G}$. Thus we would expect a curve $C \subset \mathbb{P}^r$ to have e-secant s-planes when

$$e \;\leq\; (s+1)\frac{r-s}{r-s-1}.$$

Is this true of a general rational curve? For most e, r and s, we don't know!

2.3 Curves of genus 1

We cannot begin to describe everything that has been said or done with curves of genus 1, or *elliptic curves*[1]. They appeared, in the second half of the 19th century, as key objects in the developing subjects of geometry, number theory and complex analysis, and the literature is correspondingly rich. Though all curves of genus 0 are isomorphic to $\mathbb{P}^1$ and on a given curve of genus 0 all divisors of a given degree are linearly equivalent, neither of the analogous statements holds true for curves of genus 1. The ways in which 19th century geometers dealt with this fact has shaped much of algebraic geometry.

Specifically, classical geometers observed that there was a one-parameter family of curves of genus 1 up to isomorphism, and that on a given curve of genus 1 there was a one-dimensional family of divisors up to linear equivalence. These were perhaps the earliest examples of *moduli spaces*, and they were ultimately generalized to the moduli space M_g of curves of genus g, and the Picard variety $\mathrm{Pic}^d(C)$ parametrizing divisors of degree d on a given curve C up to linear equivalence.

Here we will focus on the geometric side, and try to describe maps of genus 1 curves to projective space. As a sort of through-line for our discussion, we will try to indicate in each case how the given projective model of a curve E of genus 1 gives rise to the expectation that there is a one-parameter family of curves of genus 1 up to isomorphism. For any d, the automorphism group of E acts transitively on the invertible sheaves of degree d on E. In other words, if $\phi, \phi' : E \to \mathbb{P}^r$ are two maps given by complete linear series $|L|$ and $|L'|$ of degree d on E, then there exists automorphisms $\alpha : \mathbb{P}^r \to \mathbb{P}^r$ and $\beta : E \to E$ such that $\phi' \circ \beta = \alpha \circ \phi$. In particular, if ϕ and ϕ' are embeddings—as will be the case when $d \geq 3$—then their images are projectively equivalent. As for the business of parametrizing invertible sheaves on a given curve C, we will take that up in the next chapter, and see it applied in the case of curves of genus $g \geq 2$ in Chapter 4.

[1]Technically, an elliptic curve is a smooth curve of genus 1 with a distinguished point, called the *origin*.

2.3.1 Double covers of $\mathbb{P}^1$

Let E be a smooth projective curve of genus 1. If L is any invertible sheaf of degree 1 on E, the Riemann-Roch Theorem says that $h^0(L) = 1$, so if we're looking for nonconstant maps to projective space we have to go to degree 2 and higher.

To start with, suppose L is an invertible sheaf of degree 2 on E. By the Riemann-Roch Theorem , $h^0(L) = 2$ and the linear series $|L|$ is base point free, so we get a map $\phi : E \to \mathbb{P}^1$ of degree 2. By the Riemann-Hurwitz Theorem , the map ϕ will have 4 branch points; by the remark above, these four points are determined, up to automorphisms of $\mathbb{P}^1$ by the curve E, and are independent of the choice of L. After composing with an automorphism of $\mathbb{P}^1$ we can take these four points to be $0, 1, \infty$ and λ for some $\lambda \neq 0, 1 \in \mathbb{C}$. Since there is a unique double cover of $\mathbb{P}^1$ with given branch divisor (see **??**) it follows that $E \cong E_\lambda$, where E_λ is the curve given by the affine equation

$$y^2 = x(x-1)(x-\lambda).$$

When are two curves E_λ and $E_{\lambda'}$ isomorphic? By what we've said, this will be the case if and only if there is an automorphism of $\mathbb{P}^1$ carrying the points $\{0, 1, \infty, \lambda\}$ to $\{0, 1, \infty, \lambda'\}$, in any order. This will be the case if and only if λ and λ' belong to the same orbit under the action of the group $G \cong S_3 \subset PGL(3)$ of automorphisms of $\mathbb{P}^1$ permuting the three points $0, 1$ and ∞. Direct computation shows that the orbit of λ is

$$\lambda' \in \{\lambda,\ 1-\lambda,\ \frac{1}{\lambda},\ \frac{1}{1-\lambda},\ \frac{\lambda-1}{\lambda},\ \frac{\lambda}{\lambda-1}\}.$$

Now, the quotient of $\mathbb{P}^1$ by the action of G is again isomorphic to $\mathbb{P}^1$ by Luroth's theorem, which means that the field of rational functions on $\mathbb{P}^1$ invariant under G is again a purely transcendental extension $K(j)$; explicitly, we can take

$$j \ = \ 256 \cdot \frac{\lambda^2 - \lambda + 1}{\lambda^2(\lambda-1)^2}.$$

(the factor of 256 is there for arithmetic reasons). In any case, we see explicitly that there is a unique smooth projective curve of genus 1 for each value of j; in particular, the family of all such curves is parametrized by a curve.

2.3.2 Plane cubics

Moving from degree 2 to degree 3, let L be an invertible sheaf of degree 3 on E. We see from Corollary 1.3.4 that the sections of L give an embedding of E as a smooth plane cubic curve; conversely, the genus formula tells us that a smooth plane cubic curve indeed has genus 1.

We won't delve into the geometry of plane cubics, except to point out that once more we can use this representation to argue that the isomorphism classes of elliptic curves form a 1-dimensional family. To see this, observe that the space of homogeneous polynomials of degree 3 in three variables is 10-dimensional, and the space of plane cubic curves is correspondingly parametrized by $\mathbb{P}^9$; the locus of smooth curves is a Zariski open subset of this $\mathbb{P}^9$. On the other hand, by what we've said, two plane cubics are isomorphic iff they are congruent under the group PGL_3 of automorphisms of $\mathbb{P}^2$. Since the group PGL_3 has dimension 8, we would expect that the family of such curves up to isomorphism has dimension 1.

2.3.3 Quartics in $\mathbb{P}^3$

Again, let E be a smooth projective curve of genus 1, and consider now the embedding of E into $\mathbb{P}^3$ given by the sections of an invertible sheaf L of degree 4. The first question we might ask is what polynomial equations in $\mathbb{P}^3$ cut out the image, and as before we will do this by looking at the restriction map

$$\rho_2 \ : \ H^0(\mathcal{O}_{\mathbb{P}^3}(2)) \ \rightarrow \ H^0(\mathcal{O}_E(2)) = H^0(L^2).$$

The space on the right—the space of homogeneous polynomials of degree 2 in four variables—has dimension 10, while by Riemann-Roch the space $H^0(L^2)$ has dimension 8. It follows that E lies on at least two linearly independent quadrics Q and Q'. Since E does not lie in any plane, neither Q nor Q' can be reducible; thus by Bezout's Theoremwe see that

$$E = Q \cap Q'$$

is the complete intersection of two quadrics in $\mathbb{P}^3$. Moreover, we also see from the Lasker-Noether "AF+BG" theorem that the kernel of ρ_2 is exactly the span of Q and Q'. Thus E determines a point in the Grassmannian $G(2, H^0(\mathcal{O}_{\mathbb{P}^3}(2))) = G(2,10)$ of pencils of quadrics; and by Bertini's Theorem, a Zariski open subset of that Grassmannian correspond to smooth quartic curves of genus 1. We can use this to once more calculate the dimension of the family of curves of genus 1: the Grassmannian $G(2,10)$ has dimension 16, while the group PGL_4 of automorphisms of $\mathbb{P}^3$ has dimension 15, so we may conclude that the family of curves of genus 1 up to isomorphism has dimension 1.

Projective normality II

((maybe this should be part of the homological algebra development much later))
Observe that last two cases (cubic and quartic genus 1 curves) are projectively normal; extend this to arbitrary smooth complete intersections.

Exercise: $C \subset Q \subset \mathbb{P}^3$ of class (a,b) is projectively normal iff $|a-b| \leq 1$.

Chapter 3

Jacobians

An essential construction in studying a curve C is the association to a given divisor of degree d of a invertible sheaf of that degree—in other words, the map

$$\mu : \{\text{effective divisors of degree } d\} \longrightarrow \{\text{invertible sheaves of degree } d\}.$$

A priori, this is a map of sets. But it is a fundamental fact that both sets may be given the structures of algebraic varieties in a natural way, so that the map between them is regular. The geometry of this map governs the geometry of the curve in many ways.

One part is relatively easy: the divisors on a smooth curve are parametrized by the *symmetric powers* $C^{(d)}$ of the curve C, described in Section 3.1. By contrast, the parametrization of the set of invertible sheaves on C of a given degree by the points of an algebraic variety $\mathrm{Pic}^d(C)$ is a major undertaking, one that historically brought complex analysis and algebraic geometry together. We'll describe the original construction of the varieties $\mathrm{Pic}^d(C)$ by complex analysis in Section 3.3 below, and touch briefly on the algebraic constructions.

The fact that invertible sheaves of a given degree of a curve C are parametrized by the points of a variety $\mathrm{Pic}^d(C)$ has many consequences. For example, applying dimension theory to $\mathrm{Pic}^d(C)$, we will show in Theorem 3.2.3 that every curve can be embedded in projective space as a curve of degree $g + 3$.

3.1 Symmetric products

If G is a finite group acting by automorphisms on an affine scheme $X := \mathrm{Spec}\, A$ then X/G is by definition $\mathrm{Spec}(A^G)$, the spectrum of the ring A^G of invariant elements of A. It is a basic theorem of commutative algebra that the map $X \to X/G$ induced by the inclusion of rings is finite, and the fibers of the map $X \to X/G$ are actually the orbits of G (see for example [**?**, Theorem ***]), something that often fails when G is infinite. Since the map $X \to X/G$ is finite, $\dim X/G = \dim X$. The construction commutes with the passage to G-invariant open affine sets, and thus passes to more general schemes—and in particular to projective schemes (see exercise **??**)—as well.

Exercise 3.1.1. Let G be a finite group acting on a quasi-projective scheme X. Show that there is a finite covering of X by invariant open affine sets. (Hint: consider the sum of the G-translates of a very ample divisor.)

For any variety X
((**I guess we need to say in the intro that varieties are quasi-projective...**
))
we define the d-th symmetric power of X to be the quotient of the Cartesian product X^d of d copies of X by the action of the group of all permutations of the factors. The resulting variety X^d/S_d is called the *d-th symmetric power*, or *d-th symmetric product*, of X, denoted $X^{(d)}$.

For example, if $X = \mathbb{A}^1$ then $X^d = \mathbb{A}^d$, and the ring of invariants of the symmetric group acting on $\mathcal{O}_{\mathbb{A}^d} = k[x_1, \dots, x_d]$ by permuting the variables is generated by the d elementary symmetric functions, which generate a polynomial subring. Since the symmetric functions of the roots of a polynomial are the coefficients of the polynomial, we may identify the scheme X^d with $\mathbb{A}^d$. ([**?**, Exercises 1.6, 13.2-13.4])

If $X = \mathbb{P}^1$ we can observe that on the product $(\mathbb{P}^1)^d$, taking the homogenesous coordinates of the i-th copy of $\mathbb{P}^1$ to be (s_i, t_i), the multilinear symmetric functions of degree d,

$$s_0 t_1 t_2 \cdots t_d, \dots, s_0 s_1 \cdots s_d$$

localize on each of the standard affine open sets $(\mathbb{A}^1)^d = \mathbb{A}^d$ to the usual ordinary symmetric functions, and define an isomorphism $\mathrm{Sym}^d(\mathbb{P}^1) \to \mathbb{P}^d$. Again, we may think of this map as taking a d-tuple of points to the homogeneous form of degree d vanishing on it, which is unique up to scalars.

Note that this argument does not say anything about the symmetric products of $\mathbb{A}^2$, which are in fact singular—see Exercise 3.1.2.

Since an effective divisor of degree d on a curve C is an unordered d-tuple of points on C, with repetitions allowed, it corresponds to a point in the *dth symmetric power* $C^{(d)}$.

There is one aspect of the symmetric powers that is special to the case of curves:

Proposition 3.1.2. *If X is a smooth curve then each symmetric power $X^{(d)}$ is smooth.*

Proof. The general case follows from the case of $\mathbb{A}^1$ because locally analytically the action of the symmetric group on C^d is the same as for $\mathbb{A}^1$: If $\overline{p} \in X^{(d)}$, then it suffices to show that the quotient of an invariant formal neighborhood of the preimage $p_1, \dots, p_s$ of overline p is smooth. After completing the local rings, we get an action of the symmetric group G on the product of the completions of X at the p_i, and this depends only on the orbit structure of G acting on $\{p_1, \dots, p_s\}$. Thus it would be the same for some orbit of points on $\mathbb{A}^1$. $\square$

By contrast, if $\dim X \geq 2$ then the symmetric powers $X^{(d)}$ are singular for all $d \geq 2$.

Exercise 3.1.3. 1. We say that a group G acts freely on X if $gx = gy$ only when $g = 1$ or $x = y$. Show that if G is a finite group acting freely on a smooth affine variety X then the quotient X/G is smooth.

2. Let $X = (\mathbb{A}^2)^2$ and let $G = \mathbb{Z}/2$ act on X by permuting the two copies of $\mathbb{A}^2$; algebraically, $(\mathbb{A}^2)^2 = \operatorname{Spec} S$, with $S = k[x_1, x_2, y_1, y_2]$ and the nontrivial element $\sigma \in G$ acts by $\sigma(x_i) = y_i$.

3. . Show that G acts freely on the complement of the diagonal, but fixes the diagonal pointwise.

4. Show that the algebra S^G has dimension 4 and is generated by the 5 elements

$$f_1 = x_1 + y_1, f_2 = x_2 + y_2, g_1 = x_1 y_1, g_2 = x_2 y_2, h = x_1 y_2 + x_2 y_1,$$

perhaps by appropriately modifying the steps given in [**?**, Exercise 1.6].

5. Show that h^2 lies in the subring generated by $f_1, \dots, f_4$, and thus $S^{(2)}$ is a hypersurface, singular along the codimension 2 subset $f_1 = f_2 = 0$, which is the image of the diagonal subset of the cartesian product $(\mathbb{A}^2)^2$.

Exercise 3.1.4 (The universal divisor of degree d)**.** Let C be a smooth projective curve, and $C^{(d)}$ its dth symmetric power. Show that the locus

$$\mathcal{D} := \{(D, p) \in C^{(d)} \times C \mid p \in D\}$$

is a closed subvariety of the product $C^{(d)} \times C$, whose fiber over any point $D \in C^{(d)}$ is the divisor $D \subset C$.
((How do our readers do this? We need to have proven the universal property of the symmetric product – the fine moduli space for invariant divisors of degree d.))

The variety $\mathcal{D}$ is called the universal divisor on C by virtue of the fact that for any family of divisors of degree d on C—that is, a scheme B and a subscheme $\mathcal{E} \subset B \times C$ flat of degree d over B, there is a unique morphism $\phi : B \to C^{(d)}$ such that $\mathcal{E}$ is the pullback via ϕ of $\mathcal{D} \subset C^{(d)} \times C$. Indeed, this amounts to saying that $C^{(d)}$ is the *Hilbert scheme* parametrizing subschemes of C of degree d. These statements are not generally true for higher-dimensional varieties; see Chapter **??** and especially Exercise **??**

3.2 Jacobians

To construct $\mathrm{Pic}^d(C)$ we start with $d = 0$, and identify $\mathrm{Pic}^0(C)$ with the *Jacobian* $J(C)$ of C using *abelian integrals* and the classical topology. This produces a complex manifold rather than an algebraic variety, but has the virtue of being relatively concrete.

((I think we should make the following into a formal theorem– in the characterization section — which maybe doesn't exist yet?))

The Jacobian $J(C)$ is in fact a projective variety, and may be constructed purely algebraically—so that, for example, if the curve C is defined over a given field K then $J(C)$ will be defined over K as well. The search for such a construction was one of the driving forces of algebraic geometry in the first half of the 20th century, giving rise to the notion of abstract algebraic varieties. See for example [**?**] [Kleiman must have something for this].

The goal of the 19th century mathematicians who first described abelian integrals was to make sense of integrals of algebraic functions. In the early development of calculus, mathematicians figured out how to evaluate explicitly integrals such as

$$\int_{t_0}^{t} \frac{dx}{\sqrt{x^2+1}}.$$

Such integrals can be thought of as path integrals of meromorphic differentials on the Riemann surface associated to the equation $y^2 = x^2 + 1$. This surface is isomorphic to $\mathbb{P}^1$, meaning that x and y can be expressed as rational functions of a single variable z; making the corresponding change of variables transformed the integral into one of the form

$$\int_{s_0}^{s} R(z)dz,$$

with R a rational function, and such integrals are readily evaluated by the technique of partial fractions.

When they tried to extend this to similar-looking integrals like

$$\int_{t_0}^{t} \frac{dx}{\sqrt{x^3+1}},$$

which arises when one studies the length of an arc of an elipse and was thus called an elliptic integral, they were stymied. The reason gradually emerged: the problem is that the Riemann surface associated to the equation $y^2 = x^3 + 1$ is not $\mathbb{P}^1$, but rather a curve of genus 1, and so has nontrivial homology group $H_1(C, \mathbb{Z}) \cong \mathbb{Z}^2$. In particular, if one expresses this "function" of t as a path integral, then the value depends on a choice of path; it is defined only modulo a lattice $\mathbb{Z}^2 \subset \mathbb{C}$. This implies that the inverse function is a doubly periodic meromorphic function on $\mathbb{C}$, and not an elementary function. Many new special functions, such as the Weierstrass $\mathcal{P}$-function were studied as a result. The name "elliptic curve" arose from these considerations too.

Once this case was understood, the next step was to extend the theory to path integrals of holomorphic differentials on curves of arbitrary genus. One problem is that the dependence of the integral on the choice of path is much worse; the set of homology classes of paths between two points $p_0, p \in C$ is identified with $H_1(C, \mathbb{Z}) \cong \mathbb{Z}^{2g}$ rather than $\mathbb{Z}^2$. The Jacobian arises when one considers the integrals of *all* holomorphic differentials on C simultaneously.

To express the resulting construction in relatively modern terms, let C be a smooth projective curve of genus g over $\mathbb{C}$, and let ω_C be the sheaf of differential forms on C. We will consider C as a complex manifold. Every meromorphic differential form is in fact algebraic [**?**], and we consider ω_C as a sheaf in the analytic topology.

We consider the space $V = H^0(\omega_C)^*$ of linear functions on the space of differentials $H^0(\omega_C)$. Integration over a closed loop in C defines a linear function on 1-forms, so that we have a map

$$\iota : \mathbb{Z}^{2g} = H_1(C, \mathbb{Z}) \ \to \ H^0(\omega_C)^* \cong H^1(\mathcal{O}_C) = \mathbb{C}^g.$$

Using Hodge theory[1] one can show that ι induces an injective map of vector spaces

$$\mathbb{R} \otimes H_1(C, \mathbb{Z}) = H_1(C, \mathbb{R}) \to H^0(\omega_C)^*$$

The complex structure on $H^0(\omega_C)^*$ yields a complex analytic structure on the quotient $\mathbb{C}^g/(\iota(\mathbb{Z}^{2g}))$, which is thus a torus of real dimension $2g$. We call this quotient, with its structure as a g-dimensional complex manifold, the Jacobian of C, denoted

$$J(C) \ = \ V/\Lambda.$$

The point of this construction is that for any pair of points $p, q \in C$, the expression $\int_q^p$ describes a linear functional on $H^0(\omega_C)$, defined up to functionals obtained by integration over closed loops, and thus a point of $J(C)$. Thus, for example, if we choose a "base point" $q \in C$, we get a holomorphic map

$$\mu \ : \ C \ \to \ J(C); \quad p \mapsto \int_q^p$$

Having chosen a base point $q \in C$ as above, we get for each $d \geq 0$ the *Abel-Jacobi* map

$$\mu_d \ : \ C^{(d)} \ \to \ J(C),$$

[1] By Hodge theory

$$H^1(C, \mathbb{C}) \cong H^1(C, \mathcal{O}_C) \oplus \overline{H^1(C, \mathcal{O}_C)}$$

where the bar denotes complex conjugation $H^1(C, \mathbb{C})$, and the map ι is the composition of the natural inclusion with the projection to the first summand. Now $H_1(C, \mathbb{C}) = \mathbb{C} \otimes_{\mathbb{Z}} H_1(C, \mathbb{Z})$, so any basis of $H_1(C, \mathbb{Z})$ maps to a basis of $H^1(C, \mathbb{C})$ invariant under conjugation in $H^1(C, \mathbb{C})$— See Voisin [] or Griffiths-Harris []. If there were a real dependence relation among elements of the image of this basis under ι, then it the same relation would hold after complex conjugation and thus hold on the image of the basis in $H_1(C, \mathbb{C})$, a contradiction.

defined by

$$\mu_d(p_1 + \cdots + p_d) \;=\; \sum \int_q^{p_i} .$$

When there is no ambiguity about d, we will denote them by μ. and we we define $\mu(-D)$ to be $-\mu(D)$. The map μ is a group homomorphism in the sense that if D, E are divisors, then $\mu(D+E) = \mu(D) + \mu(E)$; this is immediate when the divisors are effective, and follows in general because the group of divisors is a free group. The connection between the discussion above and the geometry of linear series is made by *Abel's theorem*:

Theorem 3.2.1. *Two divisors $D, D' \in C^{(d)}$ on C are linearly equivalent if and only if $\mu(D) = \mu(D')$; in other words, the fibers of μ_d are the complete linear systems of degree d on C.*

See [**?**, Section 2.2] for a complete proof; we will just prove the "only if" part. This was in fact the only part proved by Abel; the converse, which is substantially more subtle, was proved by Clebsch.

Proof of "only if". Suppose that D and D' are linearly equivalent; that is, $\mathcal{O}_C(D) \cong \mathcal{O}_C(D')$. Call this invertible sheaf $\mathcal{L}$, and suppose that D and D' are the zero divisors of sections $\sigma, \sigma' \in H^0(\mathcal{L})$. Taking linear combinations of σ and σ', we get a pencil $\{D_\lambda\}_{\lambda \in \mathbb{P}^1}$ of divisors on C, with

$$D_\lambda \;=\; V(\lambda_0 \sigma + \lambda_1 \sigma'),$$

and by Exercise 3.1.4 this corresponds to a regular map $\alpha : \mathbb{P}^1 \to C^{(d)}$.

Consider now the composition

$$\phi = \mu \circ \alpha \;:\; \mathbb{P}^1 \;\to\; J(C).$$

Now, $J(C)$ is the quotient of the complex vector space $V = H^0(\omega_C)^*$ by a discrete lattice. If z is any linear functional on V, then, the differential dz on V descends to a global holomorphic 1-form on the quotient $J(C)$, so that the regular one-forms on $J(C)$ generate the cotangent space to $J(C)$ at every point. But for any 1-form ω on $J(C)$, the pullback $\phi^*\omega$ is a global holomorphic 1-form on $\mathbb{P}^1$, and hence identically zero. It follows that the differential $d\phi$ vanishes identically, and hence (since we are in characteristic 0) that ϕ is constant; thus $\mu(D) = \mu(D')$. □

Abel's Theorem goes surprisingly far to describe the Jacobian. The first statement of the following Corollary suggests how to describe the structure of the Jacobian algebraically, and was used by Andre Weil in the first such construction.

Corollary 3.2.2. *If C is a smooth curve of genus g then the Abel-Jacobi map $\mu_g : C^{(g)} \to J(C)$ is a surjective birational map. More generally, μ_d is generically injective for $d \leq g$ and surjective for $d \geq g$.*

Proof. For $d \le g = \dim H^0(\omega_C)$, a divisor D that is the sum of d general points $p_1, \dots, p_d \in C$ will impose independent vanishing conditions on the sections of ω_C, and thus

$$h^1\mathcal{O}_C(D) = h^0(\omega_C(-D)) = g - d,$$

by Serre duality. Using this, the Riemann-Roch formula gives $h^0\mathcal{O}_C(D) = 1$, so the fiber of μ_d consists of a single point, proving generic injectivity. In particular when $d = g$, the image of μ_d has dimension g, and since $C^{(g)}$ is compact, the image is closed, so it must be equal to $J(C)$.

Similarly, if $d \ge g$, we will have $h^0(\omega_C(-D)) = 0$ and hence $h^0(\mathcal{O}_C(D)) = d - g + 1$. Since this is the affine dimension, the linear series $|D|$ has dimension $d - g = \dim C^{(d)} - \dim J(C)$, and again it follows that μ_d is surjective. □

3.2.1 Applications to linear series

To illustrate some of the power of Abel's theorem, we will use it to prove a basic result:

Theorem 3.2.3. *Let C be a smooth projective curve of genus g. If $D \in C_{g+3}$ is a general divisor of degree $g+3$ on C, then D is very ample. In particular, every curve of genus g may be embedded in $\mathbb{P}^3$ as a curve of degree $g+3$.*

We proved in Theorem 1.2.7 that *every divisor* of degree $\ge 2g+1$ is very ample; the difference here is that we are taking a *general* divisor. This result is sharp in the sense that hyperelliptic curves, for example, cannot be embedded in projective space as curves of any degree less than $g+3$, as we'll see in Chapter **??**. However, if we consider only general divisors on general curves, we can do still better: "most" curves of genus g can in fact be embedded in $\mathbb{P}^3$ as curves of degree $d = \lceil 3g/4 \rceil + 3$ [].

Proof. If D is general of degree $g+3$ we have $h^0(\mathcal{O}_C(D)) = 4$. To show that it is very ample, we have to show that

1. for any point $p \in C$, we have $h^0(\mathcal{O}_C(D-p)) = 3$ (that is, $|D|$ has no base points, and so defines a regular map $\phi_D : C \to \mathbb{P}^3$); and
2. for any pair of points $p, q \in C$, we have $h^0(\mathcal{O}_C(D-p-q)) = 2$.

The second of these assertions immediately implies the first, and this is what we will prove.

Now let D be an arbitrary divisor of degree $g+3$. To say that $h^0(\mathcal{O}_C(D-p-q)) \ge 3$ is equivalent, by the Riemann-Roch theorem, to the condition $h^0(\omega_C(-D+p+q)) \ge 1$; fixing a divisor $K_C \in |\omega_C|$, this is the condition that there exists an effective divisor E of degree $g-3$ linearly equivalent to a divisor in $K_C - D + p + q|$.

Now consider the map

$$\nu : C^{(g-3)} \times C^{(2)} \ \to\ J(C)$$

given by

$$\nu : (E, F) \ \mapsto\ \mu_{2g-2}(K_C) - \mu_{g-3}(E) + \mu_2(p+q),$$

where the $+$ and $-$ on the right refer to the group law on $J(C)$.

By what we have just said, and Abel's theorem, the divisor D fails to be very ample only if $\mu(D) \in \mathrm{Im}(\nu)$. But the source $C^{(g-3)} \times C^{(2)}$ of ν has dimension $g-3+2 = g-1$, and so its image in $J(C)$ must be a proper subvariety; since μ_{g+3} is dominant, the image of a general divisor $D \in C^{(g-3)}$ is a general point of $J(C)$ and thus will not lie in $\mathrm{Im}(\nu)$. □

Thus Abel's theorem, which was born out of an effort to evaluate calculus integrals, winds up proving a basic fact in the theory of algebraic curves!

((we said early on that we don't need to know that $J(C)$ is algebraic; for the present purposes, it's enough to know that $J(C)$ is a complex torus of dimension g. But in that case we do need to know that if $f : X \to Y$ is a holomorphic map of compact complex manifolds with $\dim X < \dim Y$, then $f(X)$ is a proper analytic subvariety of Y. we also need to know that the group law is algebraic. We need to have a formal statement of the existence as an algebraic group. Will be taken care of by the Characterization section, yet to be written.))

3.3 Picard varieties

The modern treatment of the Picard variety, due to Grothendieck and his school, defines the Picard variety as the solution to a universal problem. If B is any scheme, then a *family of invertible sheaves on X over B* is an invertible sheaf on $X \times B$, flat over B. We often minimize the impact of B on this definition by saying that two invertible sheaves on X over B are equivalent if they differ by an invertible sheaf pulled back from B, and we write $Pic(X/B)$ for the quotient abelian group:

$$Pic(X/B) := \frac{\{\text{Invertible sheaves on } X \times B, \text{ flat over } B\}}{\{\text{Invertible sheaves pulled back from } B\}}.$$

Note that $Pic(X/B)$ is a contravariant functor of B: if $B' \to B$ is a morphism, then we can pull invertible sheaves on B back to B', and also pull invertible sheaves on $X \times B$ back to $X \times B'$. These two pullback maps induce a homomorphism of abelian groups $Pic(X/B) \to Pic(X/B')$. The Picard scheme of X, if it exists, is the scheme that represents this functor:

Definition 3.3.1. If X is a projective scheme over $\mathbb{C}$ then a *Picard scheme of X*, denoted $\mathrm{Pic}_{X/\mathbb{C}}$ is a scheme with a natural isomorphism of functors $\mathrm{Mor}(B, \mathrm{Pic}_{X/\mathbb{C}}) \cong Pic(X/B)$.

Theorem 3.3.2. *If X is a projective variety, then $\mathrm{Pic}_{X/\mathbb{C}}$ exists.*

We will sketch the proof of this result in the case when X is a smooth curve below; for detailed references and variations, see [**?**].

From the definition we see that the identity morphism of $\mathrm{Pic}_{X/\mathbb{C}}$ corresponds to a family $\mathcal{P}$ of invertible sheaves over $\mathrm{Pic}_{X/\mathbb{C}}$, that is, to an invertible sheaf $\mathcal{P}$ on $X \times \mathrm{Pic}_{X/\mathbb{C}}$, flat over $\mathrm{Pic}_{X/\mathbb{C}}$, and well-defined up to tensoring with an invertible sheaf pulled back from $\mathrm{Pic}_{X/\mathbb{C}}$. By the Yoneda lemma, the naturality of the isomorphism $\mathrm{Mor}(B, \mathrm{Pic}_{X/\mathbb{C}}) \cong Pic(X/B)$ means that the family of invertible sheaves on $X \times B$ correponding to a given morphism $\phi : B \to \mathrm{Pic}_{X/\mathbb{C}}$, is, up to the pullback of an invertible sheaf on B, the pullback of $\mathcal{P}$ along the map $X \times \phi : X \times B \to X \times \mathrm{Pic}_{X/\mathbb{C}}$; this is the *universal property* of the Poincaré sheaf.

For example, if $\mathcal{L}$ is any invertible sheaf on X, then $\mathcal{L}$ may be regarded as a family of sheaves over a closed point p, so there is a unique morphism $p \to \mathrm{Pic}_{X/\mathbb{C}}$ such that $\mathcal{L}$ is the pullback of $\mathcal{P}$ under the induced map $X = X \times p \to X \times \mathrm{Pic}_{X/\mathbb{C}}$; more colloquially, the closed points of $\mathrm{Pic}_{X/\mathbb{C}}$ correspond to the invertible sheaves on X.

If X is a smooth curve, then each invertible sheaf has a degree; and since the degree is constant in any family of invertible sheaves over a connected curve, $\mathrm{Pic}_{X/\mathbb{C}}$ is a disjoint union of spaces $\mathrm{Pic}_{d,X/\mathbb{C}}$, and the restriction of $\mathcal{P}$ to $X \times \mathrm{Pic}_{d,X/\mathbb{C}}$ is a family of invertible sheaves of degree d in the sense that for every point $p \in \mathrm{Pic}_{d,X/\mathbb{C}}$ the restriction of $\mathcal{P}$ to $X = X \times p$ is a sheaf of degree d.

The functorial description of $\mathrm{Pic}_{X/\mathbb{C}}$ makes it easy to prove a number of properties:

Theorem 3.3.3. *If X is a smooth curve, then $\mathrm{Pic}_{X/\mathbb{C}}$ is smooth and the tangent space to $\mathrm{Pic}_{X/\mathbb{C}}$ at any closed point is isomorphic to $H^1(\mathcal{O}_X)$.*

Proof. We first prove the smoothness. Since we are working over an algebraically closed field, it is enough to check the criterion of formal smoothness [**?**]: given an affine scheme B and a subscheme B' such that $\mathcal{I}^2_{B'/B} = 0$, we must show that any map $B' \to X$ extends to a map $B \to X$. But a map $B' \to X$ corresponds to an invertible sheaf on $X \times B'$, and similarly for B, so we must show that every invertible sheaf on $X \times B'$ extends to an invertible sheaf on $X \times B$. We will use the identification of the group of invertible sheaves on a space with the first cohomology of the multiplicative group of invertible functions on the space.

Note that $\mathcal{I}_{B'/B}$ is supported on B', and Let N be the pullback of $\mathcal{I}_{B'/B}$ to $X \times B'$. From the exact sequence

$$0 \to N \to \mathcal{O}_{X \times B} \to \mathcal{O}_{X \times B'} \to 0$$

we deduce an exact sequence of multiplicative groups

$$1 \to (1+N) \to \mathcal{O}^*_{X\times B} \to \mathcal{O}^*_{X\times B'} \to 1.$$

Since $\mathcal{I}_(B'/B)^2 = 0$, the sheaf of multiplicative groups, $(1+N)$ is isomorphic to the sheaf of additive groups N, and we get a long exact sequence in cohomology

$$H^0\mathcal{O}^*_{X\times B} \to H^0\mathcal{O}^*_{X\times B'} \to H^1\mathcal{O}_X \to H^1(\mathcal{O}^*_{X\times T}) \to H^1(\mathcal{O}^*_{X^*\times B'}) \to H^2(N)$$

Since B is affine and X is 1-dimensional, $H^2(N) = 0$, proving that we can extend invertible sheaves.

Rcall that the tangent space to a scheme Y at a closed point $p : \operatorname{Spec} k \to Y$ is the set of extensions of p to a morphism $\operatorname{Spec} k[\epsilon]/\epsilon^2 \to Y$. From the exact sequence above we see that the set of extensions is $H^1(\mathcal{O}_X)$. $\square$

We can also use the universal property to prove properness:

Corollary 3.3.4. *X is a nonsingular projective curve, then $\operatorname{Pic}_{X/\mathbb{C}}$ is proper over $\mathbb{C}$.*

Proof. We use the valuative criterion of properness. Thus we consider $D := \operatorname{Spec} R$, where R is a discrete valuation ring, and an invertible sheaf $\mathcal{L}$ on $X \times U$, where U is the generic point of D, and we must show that $\mathcal{L}$ extends to $X \times D$.

Choose a rational section of $\mathcal{L}$, and let ℓ be the associated Cartier divisor of zeros and poles. Because X is smooth, the closure $\overline{\ell}$ of ℓ in $X \times D$ is again a Cartier divisor, and the invertible sheaf associated to $\overline{\ell}$ is an extension of $\mathcal{L}$. $\square$

Since the points of $\operatorname{Pic}_{X/\mathbb{C}}$ correspond to invertible sheaves, it is not surprising that $\operatorname{Pic}_{X/\mathbb{C}}$ is an abelian algebraic group in a natural way:

Exercise 3.3.5. Write $\pi_{1,2}$ and $\pi_{1,3}$ for the projections

$$X \times \operatorname{Pic}_{X/\mathbb{C}} \times \operatorname{Pic}_{X/\mathbb{C}} \to X \times \operatorname{Pic}_{X/\mathbb{C}}$$

onto the $(1,2)$ and $(1,3)$ factors, respectively. The map $\operatorname{Pic}_{X/\mathbb{C}} \times \operatorname{Pic}_{X/\mathbb{C}} \to \operatorname{Pic}_{X/\mathbb{C}}$ corresponding to the family $\pi^*_{1,2}\mathcal{P} \otimes \pi^*_{1,3}\mathcal{P}$ makes $\operatorname{Pic}_{X/\mathbb{C}}$ into an abelian algebraic group, with inverse operation $\operatorname{Pic}_{X/\mathbb{C}} \to \operatorname{Pic}_{X/\mathbb{C}}$ corresponding to the sheaf $\mathcal{P}^{-1}$.

Exercise 3.3.6. Show that if X is an irreducible curve with a smooth closed point p over a field k, then tensoring with $cO_X(p)$ induces isomorphisms $\operatorname{Pic}_{d,X/k} \to \operatorname{Pic}_{d+1,X/k}$ for all d. Without assuming the existence of a smooth closed point, show that $\operatorname{Pic}_{d,X/k} \to \operatorname{Pic}_{d+2g-2,X/k}$, where g is the genus of X.

Cheerful Fact 3.3.1. There are smooth curves over certain fields such that $\operatorname{Pic}_{d,X/k}$ is not isomorphic to $\operatorname{Pic}_{e,X/k}$, unless $d \equiv \pm e \pmod{2g-2}$ [?].

In parallel with the functor $B \mapsto Pic(X/B)$ we define the functor of relative divisors:

$$Div(X/B) := \text{Divisors in } X \times B, \text{ flat over } B\}$$

((DE rev to here 3/13/22))

3.4 Differential of the Abel-Jacobi map

In this section we will describe the differential $d\mu$ of the Abel-Jacobi map $\mu : C_d \to J(C)$; this yield a sharper form of Abel's theorem.

((clarify the structure of the next few pages: what is *the* theorem, what are special cases to get an intuitive feel.))
To start, suppose $D = p_1 + \cdots + p_d$ is a divisor consisting of d distt oints on our curve C. Since the quotient map $C^d \to C_d$ is unramified at D, the tangent space to C_d at the point D is naturally identified with the tangent space to C^d at $(p_1, \ldots, p_d)$; that is, the direct sum of the tangent spaces to C at the points p_i:

$$T_D(C_d) = \bigoplus T_{p_i}(C).$$

On the other hand, the tangent space to $J(C)$ at the image point $\mu(D)$ is the vector space $H^0(\omega_C)^*$ of which $J(C)$ is a quotient (as it is at every point!). The differential $d\mu_D$ is thus a linear map

$$\bigoplus T_{p_i}(C) \longrightarrow H^0(\omega_C)^*,$$

and the transpose of this a linear map

$$H^0(\omega_C) \longrightarrow \bigoplus T^*_{p_i}(C).$$

This last map is easy to describe: since the map μ is given by

$$\mu_d(p_1 + \cdots + p_d) \; = \; \sum \int_q^{p_i},$$

we can differentiate under the integral sign to conclude that *the codifferential d^*_μ is the map*

$$\begin{aligned} H^0(\omega_C) &\to \bigoplus T^*_{p_i}(C) \\ \omega &\mapsto (\omega(p_1), \ldots, \omega(p_d)). \end{aligned}$$

There is a natural extension of this to the case of non-reduced divisors D, that is, divisors with repeated points. We first need a description of the tangent space to C_d at the point D:

Proposition 3.4.1. *The tangent and cotangent spaces to C_d at the point corresponding to an arbitrary divisor $D = \sum a_i p_i$ are naturally identified with $H^0(\mathcal{O}_C(D)/\mathcal{O}_C)$ and $H^0(\omega_C/\omega_C(-D))$ respectively.*

This is not a proof. ideas: first, C_d is a Hilbert scheme, and the tangent space at a divisor D is thus $H^0 N_{D/C}$. Now the normal bundle is the dual of the conormal bundle $O(-D)/O(-2D)$; and the pairing $O(-D)\times O(D) \to O$ induces a perfect pairing $H^0(\mathcal{O}_C(D)/\mathcal{O}_C)$ with $H^0(O(-D)/O(-2D))$, so the former is the tangent space.

Note that we have a natural pairing between the spaces $H^0(\mathcal{O}_C(D)/\mathcal{O}_C)$ and $H^0(\omega_C/\omega_C(-D))$, given by sending (f, ω) to $\sum_i Res_{p_i}(f\omega)$. Note also that the term "natural" has a precise meaning here: if we let

$$\mathcal{D} = \{(D, p) \in C_d \times C \mid p \in D\}$$

be the universal effective divisor of degree d on C, the proposition says that the cotangent sheaf $T^*_{C_d}$ is the direct image $\alpha_*(\beta^*\omega_C/\beta^*\omega_C(-\mathcal{D}))$, where α and β are the projections of $C_d \times C$ onto the two factors.

□

Given Proposition 3.4.1, we can extend our earlier statement to the

Proposition 3.4.2. *The codifferential $d\mu^*$ of the Abel-Jacobi map is the natural restriction map*

$$H^0(\omega_C) \longrightarrow H^0(\omega_C/\omega_C(-D)).$$

Now, note that the codimension of the image of $d\mu^*$—equivalently, the dimension of the kernel of the differential $d\mu$—is by the geometric Riemann-Roch theorem exactly the dimension of the fiber of C_d over the point $\mu(D) \in J(C)$. In other words, the fibers of μ are smooth, and in particular reduced. Thus we can think of Proposition 3.4.2 as a strengthening of the Abel-Clebsch theorem: while Abel and Clebsch show that the fibers of μ are complete linear series set-theoretically, we see from the above that it is in fact true scheme-theoretically.

3.5 Further consequences

One consequence of the description of the Jacobian and the Abel-Jacobi map of a curve C is that *the set of linear series on C of given degree d and dimension r can be given the structure of a scheme in its own right.*

((all this needs a base point, and some care to state the universal property precisely. OK to state it intuitively, then translate))
To start with, we can define $W^r_d(C) \subset \mathrm{Pic}^d(C)$ to be the set of invertible sheaves $L \in \mathrm{Pic}^d(C)$ such that $h^0(L) \geq r + 1$. We can see readily that this is a Zariski

closed subset of $\mathrm{Pic}^d(C)$, for example by pointing out that it is exactly the locus where the fiber dimension of the Abel-Jacobi map $\mu : C_d \to \mathrm{Pic}^d(C)$ is at least r; this is closed by upper-semicontinuity of fiber dimension.

Note that among the subvarieties W^r_d are the images W^0_d of the Abel-Jacobi maps; in other words, the locus of *effective* divisor classes of degree d. The superscript is often omitted, meaning W^0_d is usually written W_d.

Moreover, the subsets $W^r_d \subset \mathrm{Pic}^d(C)$ can be given the structure of a scheme, in a natural way. One way to characterize this scheme structure is to say that the scheme W^r_d represents the functor of families of invertible sheaves $L \in \mathrm{Pic}^d(C)$ on C with $h^0(L) \geq r+1$.

In the other construction, we can actually parametrize the set of linear series g^r_d on C: that is, there is a scheme $G^r_d(C)$ parametrizing pairs (L, V) with $L \in \mathrm{Pic}^d(C)$ and $V \subset H^0(L)$ a subspace of dimension $r+1$. Again, the scheme structure may be characterized by saying that $G^r_d(C)$ represents the functor of families of linear series on C. Note that the natural map $G^r_d \to W^r_d$ is an isomorphism over the dense open subset $W^r_d \setminus W^{r+1}_d$, and more generally its fiber over a point of $W^s_d \setminus W^{s+1}_d$ is a copy of the Grassmannian $\mathbb{G}(r, s)$.

3.5.1 Examples in low genus

Genus 2

There is not a lot going on here, but there are a couple observations to make. First of all, the map $\mu_1 : C \to J(C)$ embeds the curve C in $J(C)$. Secondly, the map $\mu_2 : C_2 \to J(C)$ is an isomorphism except along the locus $\Gamma \subset C_2$ of divisors of the unique g^1_2 on C; in other words, *the symmetric square C_2 of C is the blow-up of $J(C)$ at a point.*
((true that the fiber is $\mathbb{P}^1$, but is that enough? Maybe so for a birational map of smooth surfaces, but does the reader know this?))

Exercise 3.5.1. Let $C \subset J(C)$ be the image of the Abel-Jacobi map μ_1. Show that the self-intersection of the curve C is 2,

1. by applying the adjunction formula to $C \subset J(C)$; and

2. by calculating the self-intersection of its preimage $C + p \subset C_2$ and using the geometry of the map μ_2.

3.5.2 Genus 3

3.5.3 Genus 4

In genus 4 we encounter for the first time a scheme $W^r_d(C)$ that is neither of the form W_d or $K - W_e$. This is the subscheme $W^1_3(C)$ parametrizing g^1_3s on C.

3.5.4 Genus 5

Want: for general curve C of genus 5, the scheme $W^1_4(C)$ is smooth & irreducible; but when C becomes trigonal, $W^1_4(C)$ becomes reducible, with one component of the form $W^1_3 + C$ and the other $K - W^1_3 - C$.

3.6 Martens' theorem and variants

The general theorems we have described so far dealing with linear series on a curve C, like the Riemann-Roch and Clifford theorems, have to do with the existence or non-existence of linear series on C. Now that we've seen how to parametrize the set of linear series on C by the varieties $W^r_d(C)$, we can ask more quantitative questions: for example, what can the dimension of $W^r_d(C)$ be? One basic result, for example, is the following.

Theorem 3.6.1 (Martens' theorem)**.** *If C is any smooth projective curve of genus g, then for any d and g we have*

$$\dim(W^r_d(C)) \leq d - 2r;$$

moreover, if we have equality for any $r > 0$ and $d < 2g - 2$ the curve C must be hyperelliptic.

Note that if C is hyperelliptic with $g^1_2 = |D|$, we have

$$W^r_d(C) \supset W_{d-2r}(C) + \mu(rD).$$

(In fact, as we'll see in the following chapter, this is an equality.) Since this has dimension $d - 2r$, we see that Martens' theorem is sharp. Note also that Clifford's theorem is a special case of Martens' theorem!

Proof. □

There are extensions of Martens' theorem to the case $\dim(W^r_d(C)) = d - 2r - 1$ (Mumford) and $d - 2r - 2$ (Keem).

3.7 The Torelli theorem

((consider making this a cheerful fact. or exercise?))
In the examples above, we see that a lot of information about a curve C is encoded in the geometry of its Jacobian. In fact, we can make this official: we have the celebrated

Theorem 3.7.1 (Torelli)**.** *A curve C is determined by the pair $(J(C), \Theta)$.*

Proof. In fact, there are many ways of reconstructing a curve from its Jacobian; this one is due to Andreotti, and makes essential use of our description of the differential of the Abel-Jacobi map.

A key fact is that the Jacobian $J(C) = H^0(\omega_C)^*/H_1(C, \mathbb{Z})$ is a torus, and so has trivial tangent bundle, with fiber $H^0(\omega_C)^*$ at every point. What this means is that if $X \subset J(C)$ is a smooth, k-dimensional subvariety, we have a *Gauss map*

$$\mathcal{G} : X \to G(k, g) = G(k, H^0(\omega_C)^*),$$

sending a point $x \in X$ to its tangent plane $T_x X \subset T_x J(C) = H^0(\omega_C)^*$; more generally, if X is singular then $\mathcal{G}$ will be a rational map. In particular, if $X = \Theta = W_{g-1}$, we get a rational map

$$W_{g-1} \longrightarrow \mathbb{P}^{g-1} = \mathbb{P}(H^0(\omega_C))$$

between two $g-1$-dimensional varieties, and it is the geometry of this map from which we can recover the curve C.

To start with, let's identify an open subset of W_{g-1} where the Gauss map is defined. This is not hard: a point $L \in W_{g-1} \setminus W^1_{g-1}$ is the image of a unique point $D \in C_{g-1}$ under the map μ, and moreover we've seen that the differential $d\mu$ is injective at D; it follows that L is a smooth point of W_{g-1}.

Moreover, we've identified the tangent space to W_{g-1} at $L = \mu(D)$: as we saw, the differential $d\mu : T_D(C_{g-1}) \to T_L(J) = H^0(\omega_C)^*$ is just the transpose of the evaluation map $H^0(\omega_C) \to H^0(\omega_C(-D))$, and it follows that the tangent space to W_{g-1} at the point L is the hyperplane in $H^0(\omega_C)^*$ dual to the unique differential vanishing on D. To put it another way: if we think of C as canonically embedded in $\mathbb{P}(H^0(\omega_C)^*)$, then by geometric Riemann-Roch the divisor D will span a hyperplane in $\mathbb{P}(H^0(\omega_C)^*)$, and the Gauss map $\mathcal{G}$ sends L to the point in the dual projective space $\mathbb{P}(H^0(\omega_C))$ corresponding to that hyperplane.

Cheerful Fact 3.7.1. We have shown that the open subset $W_{g-1} \setminus W^1_{g-1}$ is contained in the smooth locus of W_{g-1}. In fact, they are equal; that is, W^1_{g-1} is exactly the singular locus of W_{g-1}. This is a special case of the beautiful *Riemann singularity theorem*, which says that for any point $L \in W_{g-1}$, the multiplicity $\mathrm{mult}_L(W_{g-1}) = h^0(L)$. For a proof of the Riemann singularity theorem, see for example [GH].

((David – can we find another reference for the RST? The proof in [GH] is clear but somewhat sketchy; I don't have a copy handy, but as I recall it implicitly assumes that the tangent cone is generically reduced.))

We are now in a position to describe the Gauss map

$$\mathcal{G} : W_{g-1} \longrightarrow \mathbb{P}(H^0(\omega_C))$$

explicitly in terms of the geometry of the canonical curve $C \subset \mathbb{P}(H^0(\omega_C)^*)$. To start, let $p \in \mathbb{P}(H^0(\omega_C))$ be a general point, dual to a general hyperplane $H \subset \mathbb{P}(H^0(\omega_C)^*)$. The hyperplane H will intersect the canonical curve C transversely in $2g-2$ points $p_1, \ldots, p_{2g-2}$; these points will be in linear general position (in particular, any $g-1$ of them will be linearly independent and so span H). It follows that the fiber of $\mathcal{G}$ over the point H will consist of the invertible sheaves $L = \mathcal{O}_C(p_{\alpha_1} + \cdots + p_{\alpha_{g-1}})$, where $p_{\alpha_1}, \ldots, p_{\alpha_{g-1}}$ is any subset of $g-1$ of the points p_i; in particular, we see that the degree of the map $\mathcal{G}$ is

$$\deg(\mathcal{G}) = \binom{2g-2}{g-1}.$$

The next question is, where does this analysis fail—in other words, for which hyperplanes $H \subset \mathbb{P}H^0(\omega_C)^*$ does the fiber of $\mathcal{G}$ not consist of $\binom{2g-2}{g-1}$ points, or equivalently, what is the branch divisor of the map $\mathcal{G}$? The answer is, the analysis above fails in two cases: when the points $p_1, \ldots, p_{2g-2}$ are not in linear general position—specifically, when some $g-1$ of the points p_i fail to be linearly independent; and when the hyperplane H is not transverse to C, so that the hyperplane section $H \cap C$ consists of fewer than $2g-2$ distinct points.

The first of these occurs in codimension 2 in $\mathbb{P}H^0(\omega_C)$, and so does not contribute any components to the branch divisor of $\mathcal{G}$. It follows that the branch divisor of the map $\mathcal{G}$ is exactly the locus of hyperplanes $H \subset H^0(\omega_C)^*$ tangent to the canonical curve C; in other words, *the branch divisor of $\mathcal{G}$ is the hypersurface in $\mathbb{P}H^0(\omega_C)$ dual to the canonical curve $C \subset \mathbb{P}H^0(\omega_C)^*$.*

Now we can invoke the fact that the dual of the dual of a variety $X \subset \mathbb{P}^n$ is X itself (see for example [3264] or something by Kleiman). We thus have a way of recovering the curve C from the data of the pair (J, W_{g-1}): simply put, *the curve C is the dual of the branch divisor of the Gauss map on W_{g-1}*, and the Torelli theorem is proved.

□

The Torelli theorem for curves was the first instance of a class of theorems, called *Torelli theorems*, to the effect that certain classes of varieties are determined to some degree by their Hodge structure; there are, for example, Torelli theorems of varying strength for K3 surfaces, cubic threefolds and fourfolds and hypersurfaces in $\mathbb{P}^n$.

3.8 Additional topics

A couple of topics that would naturally go here, if we have the inclination and space.

3.8.1 Theta characteristics

Basically: introduce the notion of theta-characteristic (= square root of the canonical bundle), and prove the invariance of $h^0(\mathcal{L})$ mod 2. Describe the configuration of theta-characteristics on a given curve C as a principal homogeneous space for the group $J(C)[2] \cong (\mathbb{Z}/2)^{2g}$ of torsion of order 2 in the Jacobian.

Example: bitangents to a plane quartic; distinguished triples of bitangents

3.8.2 Intermediate Jacobians and the irrationality of cubic threefolds

First, describe the intermediate Jacobians $J(X)$ of higher-dimensional varieties X by analogy with the case of curves; introduce the Abel-Jacobi maps from parameter spaces of cycles on X to $J(X)$.

Application: show that the intermediate Jacobian of a cubic threefold is not the Jacobian of a curve by calculating the degree of the Gauss map on the theta-divisor and showing it's not 70 (which by the calculation above it would be if $J(X)$ were a Jacobian). Deduce irrationality of X.

I know this is a bit of a stretch for the current volume, but I'd really like to include it if at all possible: the proof in Clemens-Griffiths is a mess, and this is much simpler

Chapter 4

Hyperelliptic curves and curves of genus 2 and 3

4.1 Hyperelliptic Curves

In the world of curves, hyperelliptic curves are outliers: they behave differently from other curves, and the techniques used to analyze them are different from the techniques used for more general curves. Many theorems about curves contain the hypothesis "non-hyperelliptic," with the corresponding result for hyperelliptic curves arrived at directly by ad hoc methods. Because the methods of this section will not be used in other cases, it could be skipped in first reading

There will be a further discussion of hyperelliptic curves in Chapter **??**, focussing on the algebra and geometry of their projective embeddings; the analysis here will cover most of the questions we'll be asking about curves in general in the next four chapters.

4.1.1 The equation of a hyperelliptic curve

By definition, a hyperelliptic curve C is one admitting a degree two map $\pi : C \to \mathbb{P}^1$. Because the degree is only 2, each point in $\mathbb{P}^1$ has either two distinct preimages, or one point of simple ramification. There can be no higher ramification, so at all but finitely many points $p \in C$ the map π is a local isomorphism ("local" here in the complex analytic/classical or étale topology, not the Zariski topology!); at any other point $p \in C$, the map is given in terms of local analytic coordinates on C and $\mathbb{P}^1$ simply by $z \mapsto z^2$. In particular, both the ramification divisor and the branch divisor (as defined in Chapter **??**) are reduced. Thus by the Riemann-Hurwitz formula there are exactly $2g+2$ branch points $q_1, \dots, q_{2g+2} \in \mathbb{P}^1$. These points determine the curve:

Theorem 4.1.1. *There is a unique smooth projective hyperelliptic curve C expressible as a 2-sheeted cover of $\mathbb{P}^1$ branched over any set of $2g+2$ distinct points.*

We can easily construct such a curve, postponing for a moment the uniqueness: If the coordinate of the point $p_i \in \mathbb{P}^1$ is λ_i, it is the smooth projective model of the affine curve

$$C^\circ = \big\{(x,y) \in \mathbb{A}^2 \ \mid \ y^2 = \prod_{i=1}^{2g+2} (x - \lambda_i)\big\}.$$

Note that we're choosing a coordinate x on $\mathbb{P}^1$ with the point $x = \infty$ at infinity not among the q_i, so that the pre-image of $\infty \in \mathbb{P}^1$ is two points $r, s \in C$. Concretely, we see that as $x \to \infty$, the ratio $y^2/x^{2g+2} \to 1$, so that

$$\lim_{x \to \infty} \frac{y}{x^{g+1}} \ = \ \pm 1;$$

the two possible values of this limit correspond to the two points $r, s \in C$.

It's worth pointing out that C is *not* simply the closure of the affine curve $C^\circ \subset \mathbb{A}^2$ in either $\mathbb{P}^2$ or $\mathbb{P}^1 \times \mathbb{P}^1$: as you can see from a direct examination of the equation, each of these closures will be singular at the (unique) point at infinity.

Completion of the proof of Theorem 4.1.1. The proof (in characteristic 0) of uniqueness follows from elementary algebraic topology:

First, a punctured 2-disk has fundamental group $\mathbb{Z}$ and the unique n-sheeted covering is again a punctured disk; regarding these disks as neighborhoods of the origin in $\mathbb{C}^2$, the covering map can be taken to be $x \mapsto x^n$. This map can of course be extended (by the same formula) to a map analytic also at the origin, with ramification index (by definition) $n - 1$.

Now suppose that $\Gamma = \{p_1, \ldots, p_d\}$ is the desired branch divisor. Globally, if γ_i is a small loop around p_i then the abelianization of the fundamental group π of the d-times punctured sphere

$$S' := \mathbb{P}^1 \setminus \Gamma$$

is its first homology group,

$$H := H_1(S', \mathbb{Z}) = \frac{\oplus \mathbb{Z} \cdot \gamma_i}{\mathbb{Z} \cdot \sum_i [\gamma_i]}$$

((Insert "lollipop picture".))
Since $\mathbb{Z}/2$ is abelian, a degree 2 unramified covering of S' corresponds to a map $H \to \mathbb{Z}/2$, and this map must send $2\gamma_i$ to 0 for $i = 1 \ldots d$. There is such a map if and only if d is even, and in this case the map is unique.

Summarizing: there is, a unique degree 2 topological covering $C' \to \mathbb{P}^1 \setminus \Gamma$ by a surface C' that extends to a ramified covering of $\rho : C \to \mathbb{P}^1$, simply ramified over the points of Γ, as long as the number of ramification points is even.

A triangulation of $\mathbb{P}^1$ with V vertices including the points of Γ, E edges, and F triangles must have

$$V - E + F = \chi_{\text{top}}(S^2) = 2.$$

It lifts to a triangulation of C with $2V - d$ vertices, $2E$ edges, and $2F$ faces, so

$$\chi_{\text{top}}(C) = 2V - d - 2E + 2F = 4 - d,$$

so if $d = 2g + 2$ then $\chi_{\text{top}}(C) = 2 - 2g$, so C is a surface of genus g.

Though given as a topological surface, the map ρ is a local homeomorphism at every point not in the preimage of Γ, so C inherits a unique complex structure from the requirement that ρ be holomorphic; thus C is actually a smooth algebraic curve of genus g.
((we had better say topological and algebraic genus are the same in the intro.))

□

Exercise 4.1.2. In the case $g = 1$, show that the closure $\overline{C^\circ}$ of $C^\circ \subset \mathbb{A}^2$ in either $\mathbb{P}^2$ or $\mathbb{P}^1 \times \mathbb{P}^1$ consists of the union of C° with one additional point, with that point a tacnode of $\overline{C^\circ}$ in either case.

It is also possible to give a projective model of the hyperelliptic curve C with given branch divisor: if we divide the points $q_1, \dots, q_{2g+2} \in \mathbb{P}^1$ into two sets of $g+1$—say, for example, $q_1, \dots, q_{g+1}$ and $q_{g+2}, \dots, q_{2g+2}$—then C is the closure in $\mathbb{P}^1 \times \mathbb{P}^1$ of the locus

$$\left\{(x, y) \in \mathbb{A}^2 \;\mid\; y^2 \prod_{i=1}^{g+1}(x - \lambda_i) = \prod_{i=g+2}^{2g+2}(x - \lambda_i)\right\};$$

in projective coordinates, this is

$$C \;=\; \left\{(X, Y) \in \mathbb{P}^1 \times \mathbb{P}^1 \;\mid\; Y_1^2 \prod_{i=1}^{g+1}(X_1 - \lambda_i X_0) = Y_0^2 \prod_{i=g+2}^{2g+2}(X_1 - \lambda_i X_0)\right\}.$$

(No local analysis is needed to see that $C \subset \mathbb{P}^1 \times \mathbb{P}^1$ is smooth: it is a curve of bidegree $(2, g+1)$ in $\mathbb{P}^1 \times \mathbb{P}^1$, and the formula for the genus of a curve in $\mathbb{P}^1 \times \mathbb{P}^1$ derived in Section **??** tells us that such a curve has arithmetic genus g.)

4.1.2 Differentials on a hyperelliptic curve

We can give a very concrete description of the differentials, and thus the canonical linear series, on a hyperelliptic curve C by working with the affine model

$C^\circ = V(f) \subset \mathbb{A}^2$, where

$$f(x,y) = y^2 - \prod_{i=1}^{2g-2}(x - \lambda_i).$$

We will again denote the two points at infinity—that is, the two points of $C \setminus C^\circ$ by r and s; for convenience, we'll denote the divisor $r + s$ by D.

To start, consider the simple differential dx on C. (Technically, we should write this as $\pi^* dx$, since we mean the pullback to C of the differential dx on $\mathbb{P}^1$, but for simplicity of notation we'll suppress the π^*.) The function x is regular on C°, and is a local parameter over points other than the λ_i; from the local description of the map π, we see that dx is regular on C° with simple zeros at the ramification points $q_i = (\lambda_i, 0)$. But it does not extend to a regular differential on all of C: it will have double poles at r and s. This can be seen directly: the differential dx extends to a rational differential on $\mathbb{P}^1$, and in terms of the local coordinate $w = 1/x$ around the point $x = \infty$ on $\mathbb{P}^1$, we have

$$dx = d\left(\frac{1}{w}\right) = \frac{-dw}{w^2}$$

so dx has a double pole at the point at ∞; since the map π is a local isomorphism near r and s the pullback of dx to C likewise has double poles at the points r and s.

We could also see that dx must have poles by degree considerations: as we said, dx has $2g + 2$ zeros and no poles in C°, while the degree of K_C is $2g - 2$, meaning that there must be a total of four poles at the points r and s. In any event, we have an expression for the canonical divisor class on C: denoting by $R = q_1 + \cdots + q_{2g+2}$ the sum of the ramifications points of π, we have

$$K_C \sim (dx) \sim R - 2D;$$

this is a case of the Riemann-Hurwitz of Chapter **??**.

So, given that dx has poles at r and s, how do we find regular differentials on C? One thing to do would be simply to divide by x^2 (or any quadratic polynomial in x) to kill the poles. But that just introduces new poles in the finite part C° of C. Instead, we want to multiply dx by a rational function with zeros at p and q, but *whose poles occur only at the points where dx has zeroes*—that is, the points q_i. A natural choice is simply the reciprocal of the partial derivative $f_y = \partial f/\partial y = 2y$, which vanishes exactly at the points q_i, and has correspondingly a pole of order $g + 1$ at each of the points r and s (reason: the involution $y \to -y$ fixes C° and x, and exchanges the points r and s). In other words, as long as $g \geq 1$, the differential

$$\omega = \frac{dx}{f_y}$$

is regular, with divisor

$$(\omega) = (g-1)r + (g-1)s = (g-1)D.$$

The remaining regular differentials on C are now easy to find: Since x has only a simple pole at the two points at infinity we can multiply ω by any x^k with $k = 0, 1, \ldots, g-1$. Since this gives us g independent differentials, we see that the differentials

$$\omega, x\omega, \ldots, x^{g-1}\omega$$

form a basis for $H^0(K_C)$.

4.1.3 The canonical map of a hyperelliptic curve

Given that a basis for $H^0(K_C)$ is given by

$$H^0(K_C) = \langle \omega, x\omega, \ldots, x^{g-1}\omega \rangle,$$

we see that the canonical map $\phi : C \to \mathbb{P}^{g-1}$ is given by $[1, x, \ldots, x^{g-1}]$. In other words, the canonical map ϕ is simply the composition of the map $\pi : C \to \mathbb{P}^1$ with the Veronese embedding $\mathbb{P}^1 \hookrightarrow \mathbb{P}^{g-1}$ of $\mathbb{P}^1$ into $\mathbb{P}^{g-1}$ as a rational normal curve of degree $g-1$.

Note that as a consequence of this fact, we see that *a hyperelliptic curve C has a* unique *linear series g^1_2 of degree 2 and dimension 1*, that is, a unique map of degree 2 to $\mathbb{P}^1$. Finally, we can give an explicit description of special linear series on a hyperelliptic curve: if $D = \sum p_i$ is any effective divisor on C, we can pair up points p_i that are conjugate under the involution ι exchanging sheets of the degree 2 map $C \to \mathbb{P}^1$; each conjugate pair is a divisor of the unique g^1_2 on C, and so we can write

$$D \sim r \cdot g^1_2 + q_1 + \cdots + q_{d-2r},$$

where no two of the points q_i are conjugate under ι. Now the geometric form of the Riemann-Roch formula tells us that the dimension $r(D)$ of the complete linear series $|D|$ is exactly r, so that in fact

$$|D| = |r \cdot g^1_2| + +q_1 + \cdots + q_{d-2r};$$

that is, the points q_i are base points of the linear series D.

One key observation is that, according to this analysis, *no special linear series on a hyperelliptic curve can be very ample*; the map associated to any special series factors through the degree 2 map $C \to \mathbb{P}^1$. This is in marked contrast to the case of non-hyperelliptic curves, for which the embeddings of minimal degree in projective space are given by special linear series.

4.2 Curves of genus 2

Since curves of genus 2 are hyperelliptic, everything we said above applies to them; in particular, the canonical map $\phi_K : C \to \mathbb{P}^1$ on a curve of genus 2 is simply the expression of C as a double cover of $\mathbb{P}^1$. In this section, we'll consider other maps from C to projective space, starting with the simplest possible projective embedding of C.

4.2.1 Maps of C to $\mathbb{P}^1$

Of course, C may be expressed as a degree 2 cover of $\mathbb{P}^1$ by taking the map associated to the canonical series $|K_C|$. But what about other degrees? For example, can we express C as a three-sheeted cover of $\mathbb{P}^1$?

The answer is "yes," and in fact we can do so in many ways. First start with a line bundle L of degree 3 on C. Riemann-Roch tells us immediately that $h^0(L) = 2$, and we see that there are two possibilities:

1. First, if the linear series $|L|$ has a base point $p \in C$, then $h^0(L(-p)) = 2$, and hence L must be of the form $L = K_C(p)$. Conversely, if $L = K_C(p)$, then $h^0(L(-p)) = h^0(L)$, which is to say p is a base point of $|L|$.
2. On the other hand, if L is not of the form $L = K_C(p)$, then $|L|$ does not have a base point, and so defines a degree 3 map $\phi_L : C \to \mathbb{P}^1$.

Do both possibilities occur? Certainly the first does; there's a one-parameter family of line bundles of the form $K_C(p)$. But we know that the variety $\mathrm{Pic}^3(C)$ is 2-dimensional, so we see that the general line bundle of degree 3 does give an expression of C as a 3-sheeted cover of $\mathbb{P}^1$; in fact there exists a 2-parameter family of such maps.

4.2.2 Maps of C to $\mathbb{P}^2$

Let's move on to consider maps of our curve C of genus 2 to the plane. By Riemann-Roch, a line bundle L of degree 4 on C will have $h^0(L) = 3$; and since $h^0(L(-p)) = 2$ for any point $p \in C$ (again by Riemann-Roch), we see that the linear series $|L|$ will give a regular map $\phi_L : C \to \mathbb{P}^2$. We ask now about the geometry of this map.

This again depends on the choice of L. This time there are three possibilities:

1. First, suppose $L = K_C^2$ is simply the square of the canonical line bundle on C. We have then a map

$$\mathrm{Sym}^2 H^0(K_C) \to H^0(L);$$

since both sides are 3-dimensional vector spaces and the map is injective, we have equality here; in other words, every divisor $D \sim K_C^2$ is the sum of two divisors $D_1, D_2 \in |K_C|$ in the canonical series. To express this in terms of the map ϕ_L, it says simply that that map ϕ_L is the composition of the canonical map $\phi_K : C \to \mathbb{P}^1$ with the Veronese embedding $\nu_2 : \mathbb{P}^1 \to \mathbb{P}^2$ of $\mathbb{P}^1$ as a conic curve in the plane. In other words, the map ϕ_L is generically 2-to-1 onto a conic in the plane.

2. Suppose now that L is not equal to K_C^2; equivalently, $M = L \otimes K_C^{-1}$ is a line bundle of degree 2 other than the canonical bundle. We have then $h^0(M) = 1$, so that M is the line bundle $\mathcal{O}_C(p+q)$ associated to a pair of points $p, q \in C$; in other words, $L = K_C(p+q)$ for a unique pair of points $p, q \in C$.

   Let's consider next the general case $p \neq q$. In this case, every section of L vanishing at p vanishes at q and vice versa, so that $\phi_L(p) = \phi_L(q)$. At the same time, for any effective divisor $D = r + s$ of degree 2 on C other than $p + q$, we have $h^0(L(-D)) = 1$, so apart from the fact that $\phi_L(p) = \phi_L(q)$, the map ϕ_L is an embedding. We'll see in Exercise **??** below that in fact the point $\phi_L(p) = \phi_L(q)$ is a node of the image curve $\phi_L(C)$; so to summarize: in this case ($L = K_C(p+q)$, with $p \neq q$ and $p + q \neq K_C$), the map $\phi_L : C \to \mathbb{P}^2$ is a birational embedding of C as a quartic plane curve with one node, the node being the common image of p and q.

3. Finally, the remaining case is where $L = K_C(2p)$, where $p \in C$ is any point such that $2p \not\sim K_C$. This behaves much like the preceding case, but here the map ϕ_L is one-to-one with vanishing differential at p, and the image curve $\phi_L(C)$ has correspondingly a cusp at the point $\phi_L(p)$.

To summarize: the map $\phi_L : C \to \mathbb{P}^2$ associated to a line bundle L of degree 4 on C is either

1. Two-to-one onto a plane conic curve, if $L = K_C^2$;

2. Birational onto a plane quartic curve with a cusp, if $L = K_C(2p)$ with $2p \not\sim K_C$; and

3. Birational onto a plane quartic curve with a node, if $L = K_C(p+q)$ with $p \neq q$ and $p + q \not\sim K_C$.

Note that the last case is the "general" one, meaning it holds for L in an open subset of $\mathrm{Pic}^4(C)$; the second case holds for a one-dimensional locus in $\mathrm{Pic}^4(C)$, and the first case holds for just one point in $\mathrm{Pic}^4(C)$.

Exercise 4.2.1. Let $L \in \mathrm{Pic}^4(C)$ be a line bundle of the form $L = K_C(p+q)$ with $p \neq q$ and $p + q \not\sim K_C$. Show that

1. $h^0(L(-2p)) = h^0(L(-2q)) = 1$, and
2. $h^0(L(-2p-2q)) = 0$.

Deduce from this that the map ϕ_L is an immersion, and that the tangent lines to the two branches of $\phi_L(C)$ at the point $\phi_L(p) = \phi_L(q)$ are distinct, meaning the point $\phi_L(p) = \phi_L(q)$ is a node of $\phi_L(C)$.

Embeddings in $\mathbb{P}^3$

So far we have not found any embeddings of C in projective space; but that's about to change: if $L \in \mathrm{Pic}^5(C)$ is any line bundle of degree 5, by Corollary 1.3.4, it is very ample and gives an embedding of C in $\mathbb{P}^3$. Let's consider now what we can say about the geometry of the image curve.

So: for the following, let L be any line bundle of degree 5 on our curve C, and $\phi_L : C \to \mathbb{P}^3$ the embedding given by the complete linear system $|L|$. By a mild abuse of language, we'll also denote the image $\phi_L(C) \subset \mathbb{P}^3$ by C.

The first question to ask is once more, what degree surfaces in $\mathbb{P}^3$ contain the curve C? We start with degree 2, where we consider the restriction map

$$H^0(\mathcal{O}_{\mathbb{P}^3}(2)) \to H^0(\mathcal{O}_C(2)) = H^0(L^2).$$

The space on the left has dimension 10 as always; on the right, Riemann-Roch tells us that $h^0(L^2) = 2 \cdot 5 - 2 + 1 = 9$. It follows that C must lie on a quadric surface Q; and by Bezout that Q is unique (since C can't lie on a union of planes, any quadric containing C must be irreducible; if there were more than one such, Bezout would imply that $\deg(C) \leq 4$).

We might ask at this point: is Q smooth or a quadric cone? The answer depends on the choice of line bundle L.

Proposition 4.2.2. *Let $C \subset \mathbb{P}^3$ be a smooth curve of degree 5 and genus 2 and $Q \subset \mathbb{P}^3$ the unique quadric containing C. If $L = \mathcal{O}_C(1) \in \mathrm{Pic}^5(C)$, then Q is singular if and only if we have*

$$L \cong K^2(p)$$

for some point $p \in C$; in this case, the point p is the vertex of Q.

Note that there is a 2-parameter family of line bundles of degree 5 on C, of which a one-dimensional subfamily are of the form $K^2(p)$, conforming to our naive expectation that "in general" Q should be smooth, and that it should become singular in codimension 1.

Proof. First, suppose that the line bundle $L \cong K^2(p)$ for some $p \in C$. Then $L(-p) \cong K^2$, meaning that the map $\pi : C \to \mathbb{P}^2$ given by projection from p is the map $\phi_{K^2} : C \to \mathbb{P}^2$ given by the square of the canonical bundle. As we've seen, the map ϕ_{K^2} is two-to-one onto a conic curve $E \subset \mathbb{P}^2$, and so we see that the curve C lies on the cone Q over E with vertex p, and this is the unique quadric surface containing C.

Next, let's consider the case where L is not of the form $K^2(p)$. Set $M = LK^{-1}$, so that we can write

$$L = K \otimes M,$$

where by hypothesis M is not of the form $K(p)$. As we saw in Section 4.2.1, this means that the pencil $|M|$ gives a degree 3 map $C \to \mathbb{P}^1$.

This gives us a way of factoring the map $\phi_L : C \to \mathbb{P}^3$: we have maps $\phi_K : C \to \mathbb{P}^1$ of degree 2 and $\phi_M : C \to \mathbb{P}^1$ of degree 3, and we can compose their product with the Segre embedding $\sigma : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^3$:

$$\begin{tikzcd}
 & \mathbb{P}^1 & & \\
C \arrow[ur, "\phi_K"] \arrow[rr, "\phi_K \times \phi_M"] \arrow[dr, "\phi_M"'] & & \mathbb{P}^1 \times \mathbb{P}^1 \arrow[ul] \arrow[dl] \arrow[r, "\sigma"] & \mathbb{P}^3 \\
 & \mathbb{P}^1 & &
\end{tikzcd}$$

This gives us a description of the map ϕ_L that shows us immediately that *C is a curve of type $(2,3)$ on a smooth quadric $Q \subset \mathbb{P}^3$*, completing the proof of Proposition 4.2.2.

□

Whether the quadric Q is smooth or not, we can describe a minimal set of generators of the homogeneous ideal $I(C) \subset \mathbb{C}[x_0, x_1, x_2, x_3]$ similarly. First, we look at the restriction map

$$H^0(\mathcal{O}_{\mathbb{P}^3}(3)) \to H^0(\mathcal{O}_C(3));$$

since the dimensions of these spaces are 20 and $15 - 2 + 1 = 14$ respectively, we see that vector space of cubics vanishing on C has dimension at least 6. Four of these are already accounted for: we can take the defining equation of Q and multiply it by any of the linear forms on $\mathbb{P}^3$; we conclude, accordingly, that *there are at least two cubics vanishing on C linearly independent modulo those vanishing on Q.*

In fact, we can prove the existence of these cubics geometrically, and show that there are no more than 2 linearly independent modulo the ideal of Q.

Suppose first that Q is smooth, so that C is a curve of type $(2,3)$ on Q. In that case, if $L \subset Q$ is any line of the first ruling, the sum $C+L$ is the complete intersection of Q with a cubic S_L, unique modulo the ideal of Q; conversely, if S is any cubic containing C but not containing S, the intersection $S \cap Q$ will be the union of C and a line L of the first ruling; thus, mod $I(Q)$, $S = S_L$. A similar argument applies in case Q is a cone, and L is any line of the (unique) ruling of Q.

Exercise 4.2.3. Show that for any pair of lines L, L' of the appropriate ruling of Q, the three polynomials Q, S_L and $S_{L'}$ generate the homogeneous ideal $I(C)$. Find relations among them. Write out the minimal resolution of $I(C)$.

The dimension of M_2 via maps to projective space

We remark here that each of the maps we've described from a curve C of genus 2 to projective space gives us a way of finding the dimension of the moduli space M_2 of curves of genus 2.

To start, we know that every curve C of genus 2 is uniquely expressible as a double cover of $\mathbb{P}^1$ branched at six points, modulo the group PGL_2 of automorphisms of $\mathbb{P}^1$. The space of such double covers has dimension 6, and $\dim(PGL_2) = 3$, so we may conclude that $\dim(M_2) = 6 - 3 = 3$.

Similarly, we've seen that a curve C of genus 2 is expressible as a 3-sheeted cover of $\mathbb{P}^1$ (with eight branch points) in a 2-dimensional family of ways. Such a triple cover is determined up to a finite number of choices by its branch divisor, so the space of such triple covers has dimension 8; modulo PGL_2 it has dimension 5, and since every curve is expressible as a triple cover in a two-dimensional family of ways, we arrive again at $\dim M_2 = 5 - 2 = 3$.

We've also seen that C can be realized as (the normalization of) a plane quartic curve with a node in a 2-dimensional family of ways. The space of plane quartics has dimension 14; the family of those with a node has codimension one and hence dimension 13. Since the automorphism group PGL_3 of $\mathbb{P}^2$ has dimension 8, we see that the family of nodal plane quartics modulo PGL_3 has dimension 5, and since every curve of genus 2 corresponds to a 2-parameter family of such curves, we have $\dim M_2 = 5 - 2 = 3$.

Finally, a curve of genus 2 may be realized as a quintic curve in $\mathbb{P}^3$ in a two-parameter family of ways. To count the dimension of the family of such curves, note that each one lies on a unique quadric Q, and is of type $(2,3)$ on Q. Thus to specify such a curve we have to specify Q (9 parameters) and then a bihomogeneous polynomial of bidgree $(2,3)$ on $Q \cong \mathbb{P}^1 \times \mathbb{P}^1$ up to scalars; these have $3 \cdot 4 - 1 = 11$ parameters. Altogether, then, there is a 20-dimensional family of such curves; modulo the automorphism group PGL_4 of $\mathbb{P}^3$, this is a 5-dimensional family. Again, every abstract curve C of genus 2 corresponds to a 2-parameter family of these curves modulo PGL_4, so once more we have $\dim M_2 = 5 - 2 = 3$.

4.3 Curves of genus 3

If C be a smooth projective curve of genus 3. The is an immediate bifurcation into two cases, hyperelliptic and non-hyperelliptic curves; we will discuss hyperelliptic curves of any genus in Section **??**, and so for the following we'll assume C is nonhyperellitic. By our general theorem **??**, this means that the canonical map $\phi_K : C \to \mathbb{P}^2$ embeds C as a smooth plane quartic curve; and conversely, by adjunction any smooth plane of degree 4 has genus 3 and is canonical (that is, $\mathcal{O}_C(1) \cong K_C$).
((maybe a reference to the plane curve chapter for differentials etc?))

Note that this gives us a way to determine the dimension of the moduli space M_3 of smooth curves of genus 3: if $\mathbb{P}^{14}$ is the space of all plane quartic curves, and $U \subset \mathbb{P}^{14}$ the open subset corresponding to smooth curves, we have a dominant map $U \to M_3$ whose fibers are isomorphic to the 8-dimensional affine group PGL_3. (Actually, the fiber over a point $[C] \in M_3$ is isomorphic to the quotient of PGL_3 by the automorphism group of C; but since $Aut(C)$ is finite this is still 8-dimensional.) We conclude, therefore, that

$$\dim M_3 = 14 - 8 = 6.$$

What about other linear series on C, and the corresponding models of C? To start with, by hypothesis C has no g^1_2s; that is, it is not expressible as a 2-sheeted cover of $\mathbb{P}^1$. On the other hand, it is expressible as a 3-sheeted cover: if $L \in \mathrm{Pic}^3(C)$ is a line bundle of degree 3, by Riemann-Roch we have

$$h^0(L) = \begin{cases} 2, & \text{if } L \cong K - p \text{ for some point } p \in C\text{; and} \\ 1 & \text{otherwise.} \end{cases}$$

There is thus a 1-dimensional family of representations of C as a 3-sheeted cover of $\mathbb{P}^1$. In fact, these are plainly visible from the canonical model: the degree 3 map $\phi_{K-p} : C \to \mathbb{P}^1$ is just the composition of the canonical embedding $\phi_K : C \to \mathbb{P}^2$ with the projection from the point p.

There are of course other representations of C as the normalization of a plane curve. By Riemann-Roch, C will have no g^2_3s and the canonical series is the only g^2_4, but there are plenty of models as plane quintic curves: by Proposition **??**, if L is any line bundle of degree 5, the linear series $|L|$ will be a base-point-free g^2_5 as long as L is not of the form $K + p$, so that ϕ_L maps C birationally onto a plane quintic curve $C_0 \subset \mathbb{P}^2$. But these can also be described geometrically in terms of the canonical model: any such line bundle L is of the form $2K-p-q-r$ for some trio of points $p, q, r \in C$ that are not colinear in the canonical model, and we see correspondingly that C_0 is obtained from the canonical model of C by applying a Cremona transform with respect to the points p, q and r.

We can also embed C in $\mathbb{P}^3$ as a smooth sextic curve by Proposition **??**; in fact, a line bundle $L \in \mathrm{Pic}^6(C)$ of degree 6 will be very ample if and only if it is not of the form $K + p + q$ for any $p, q \in C$. One cheerful fact in this connection is that these curves are determinantal:

Exercise 4.3.1. Let $C \subset \mathbb{P}^3$ be a smooth non-hyperelliptic curve of degree 3 and genus 6. Show that there exists a 3×4 matrix M of linear forms on $\mathbb{P}^3$ such that

$$C = \{p \in \mathbb{P}^3 \mid \mathrm{rank}(M(p)) \leq 2\}.$$

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Chapter 5

Curves of genus 4, 5 and 6

5.1 Curves of genus 4

As in the case of curves of genus 3, the study of curves of genus 4 bifurcates immediately into two cases: hyperelliptic and non-hyperelliptic; again, we will study the geometry of hyperelliptic curves in Chapter **??** and focus here on the nonhyperelliptic case.

In genus 4 we have a question that the elementary theory based on the Riemann-Roch formula cannot answer: are nonhyperelliptic curves of genus 4 expressible as three-sheeted covers of $\mathbb{P}^1$? The answer will emerge from our analysis in Proposition 5.1.2 below.

Let C be a non-hyperelliptic curve of genus 4. We start by considering the canonical map $\phi_K : C \hookrightarrow \mathbb{P}^3$, which embeds C as a curve of degree 6 in $\mathbb{P}^3$. We identify C with its image, and investigate the homogeneous ideal $I = I_C$ of equations it satisfies. As in previous cases we may try to answer this by considering the restriction maps
((replaced K_C^m with mK_C.))

$$r_m : \mathrm{H}^0(\mathcal{O}_{\mathbb{P}^3}(m)) \;\to\; \mathrm{H}^0(\mathcal{O}_C(m)) = \mathrm{H}^0(mK_C).$$

For $m = 1$, this is by construction an isomorphism; that is, the image of C is non-degenerate (not contained in any plane).

For $m = 2$ we know that $\mathrm{h}^0(\mathcal{O}_{\mathbb{P}^3}(2)) = \binom{5}{3} = 10$, while by the Riemann-Roch Theorem we have

$$\mathrm{h}^0(\mathcal{O}_C(2)) = 12 - 4 + 1 = 9.$$

This shows that the curve $C \subset \mathbb{P}^3$ must lie on at least one quadric surface Q. The quadric Q must be irreducible, since any any reducible and/or non-reduced quadric must be a union of planes, and thus cannot contain an irreducible non-degenerate curve. If $Q' \neq Q$ is any other quadric then, by Bézout's Theorem,

$Q \cap Q'$ is a curve of degree 4 and thus could not contain C. From this we see that Q is unique, and it follows that r_2 is surjective.

What about cubics? Again we consider the restriction map

$$r_3 : \mathrm{H}^0(\mathcal{O}_{\mathbb{P}^3}(3)) \ \to\ \mathrm{H}^0(\mathcal{O}_C(3)) = \mathrm{H}^0(3K_C).$$

The space $\mathrm{H}^0(\mathcal{O}_{\mathbb{P}^3}(3))$ has dimension $\binom{6}{3} = 20$, while the Riemann-Roch Theorem shows that

$$\mathrm{h}^0(\mathcal{O}_C(3)) = 18 - 4 + 1 = 15.$$

It follows that the ideal of C contains at least a 5-dimensional vector space of cubic polynomials. We can get a 4-dimensional subspace as products of the unique quadratic polynomial F vanishing on C with linear forms—these define the cubic surfaces containing Q. Since $5 > 4$ we conclude that the curve C lies on at least one cubic surface S not containing Q. Bézout's Theorem shows that the curve $Q \cap S$ has degree 6; thus it must be equal to C.

Let $G = 0$ be the cubic form defining the surface S. By Lasker's Theorem the ideal (F, G) is unmixed, and thus is equal to the homogeneous ideal of C. Putting this together, we have proven the first statement of the following result:

Theorem 5.1.1. *The canonical model of any nonhyperelliptic curve of genus 4 is a complete intersection of a quadric $Q = V(F)$ and a cubic surface $S = V(G)$ meeting along nonsingular points of each. Conversely, any smooth curve that is the intersection of a quadric and a cubic surface in $\mathbb{P}^3$ is the canonical model of a nonhyperelliptic curve of genus 4.*

Proof. Let $C = Q \cap S$ with Q a quadric and S a cubic. Because C is nonsingular and a complete intersection, both S and Q must be nonsingular at every point of their intersection Applying the Adjunction Formula to $Q \subset \mathbb{P}^3$ we get

$$\omega_Q = (\omega_{\mathbb{P}^3} \otimes \mathcal{O}_{\mathbb{P}^3}(2))|_Q = \mathcal{O}_Q(-4+2) = \mathcal{O}_Q(-2).$$

Applying it again to C on Q, and noting that $\emptyset_Q(C) = \emptyset_Q(3)$, we get

$$\omega_C = ((\omega_Q \otimes \mathcal{O}_3(3))|_C = \mathcal{O}_C(-2+3) = \mathcal{O}_C(1)$$

as required. $\square$

We can now answer the question we asked at the outset, whether a nonhyperelliptic curve of genus 4 can be expressed as a three-sheeted cover of $\mathbb{P}^1$. This amounts to asking if there are any divisors D on C of degree 3 with $r(D) \geq 1$; since we can take D to be a general fiber of a map $\pi : C \to \mathbb{P}^1$, we can for simplicity assume $D = p + q + r$ is the sum of three distinct points.

By the geometric Riemann-Roch theorem, a divisor $D = p + q + r$ on a canonical curve $C \subset \mathbb{P}^{g-1}$ has $r(D) \geq 1$ if and only if the three points $p, q, r \in C$ are colinear. If three points $p, q, r \in C$ lie on a line $L \subset \mathbb{P}^3$ then the quadric Q

would meet L in at least three points, and hence would contain L. Conversely, if L is a line contained in Q, then the divisor $D = C \cap L = S \cap L$ on C has degree 3. Thus we can answer our question in terms of the family of lines contained in Q.

Any smooth quadric is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$, and contains two families of lines, or *rulings*. On the other hand, any singular quadric is a cone over a plane conic, and thus has just one ruling. By the argument above, the pencils of divisors on C cut out by the lines of these rulings are the g^1_3s on C. This proves:

Proposition 5.1.2. *A nonhyperelliptic curve of genus 4 may be expressed as a 3-sheeted cover of $\mathbb{P}^1$ in either one or two ways, depending on whether the unique quadric containing the canonical model of the curve is singular or smooth.*

((include this?))
(One might ask why the non-singularity of the cubic surface S plays no role. However, G is determined only up to a multiple of F, and it follows that the linear series of cubics in the ideal I_C has only base points along C. Bertini's Theorem says that a general element of this series will be nonsingular away from C; and since any every irreducible cubic in the family must be nonsingular along C, it follows that the general such cubic is nonsingular.)

A curve expressible as a 3-sheeted cover of $\mathbb{P}^1$ is called *trigonal*; by the analyses of the preceding sections, we have shown that *every curve of genus $g \leq 4$ is either hyperelliptic or trigonal.*

We can also describe the lowest degree plane models of nonhyperelliptic curves C of genus 4. We can always get a plane model of degree 5 by projecting C from a point p of the canonical model of C. Moreover, the Riemann-Roch Theorem shows that if D is a divisor of degree 5 with $r(D) = 2$ then, $\mathrm{h}^0(K - D) = 1$. Thus D is of the form $K - p$ for some point $p \in C$, and the map to $\mathbb{P}^2$ corresponding to D is π_p. These maps $\pi_p : C \to \mathbb{P}^2$ have the lowest possible degree (except for those whose image is contained in a line) because, by Clifford's Theorem a nonhyperelliptic curve of genus 4 cannot have a g^2_4.

We now consider the singularities of the plane quintic $\pi_p(C)$. Suppose as above that $C = Q \cap S$, with Q a quadric. If a line L through p meets C in p plus a divisor of degree ≥ 2 then, as we have seen, L must lie in Q. All other lines through p meet C in at most a single points, so π_p whose images are thus nonsingular points of $\pi(C)$, and π_C is one-to-one there. Moreover, a line that met C in > 3 points would have to lie in both the quadric and the cubic containing C, and therefore would be contained in C. Since C is irreducible there can be no such line.

We distinguish two cases:

1. Q is nonsingular: In this case there are two lines L_1, L_2 on Q that pass through p; they meet C in p plus divisors E_1 and E_2 of degree 2. If E_i

consists of distinct points, then, since the tangent planes to the quadric along L_i are all distinct $\pi(C)$ will have a node at their common image.

((do we expect the reader to know this about quadrics, or should we prove it? or should we argue that since there are two distinct branches and a plane quintic of genus 2 can have only the equivalent of two double points, these must be simple?? The first option is probably better.))

On the other hand, if E_i consists of a double point $2q$ (that is, L_i is tangent to C at $q \neq p$, or meets C 3 times at $q = p$), then $\pi(C)$ will have a cusp at the corresponding image point. In either case, $\pi(C)$ has two distinct singular points, each either a node or a cusp. The two g^1_3s on C correspond to the projections from these singular points.

2. Q is a cone: In this case, since the curve cannot pass through the singular point of Q there is a unique line $L \subset Q$ that passes through p. Let $p + E$ be the divisor on C in which this line meets C. The tangent planes to Q along L are all the same. Thus if $E = q_1 + q_2$ consists of two distinct points, the image $\pi_p(C)$ will have two smooth branches sharing a common tangent line at $\pi_p(q_1) = \pi_p(q_2)$. Such a point is called a *tacnode* of $\pi_p(C)$. On the other hand, if $E = 2q$, that is, if L meets C tangentially at one point $q \neq p$ (or meets C 3 times at p) then the image curve will have a higher order cusp, called a *ramphoid cusp*. In either case, the one g^1_3 on C is the projection from the unique singular point of $\pi(C)$.

((add pictures illustrating some of the possibilities above.))

5.2 Curves of genus 5

We consider now nonhyperelliptic curves of genus 5. There are now two questions that cannot be answered by simple application of the Riemann-Roch Theorem:

1. Is C expressible as a 3-sheeted cover of $\mathbb{P}^1$? In other words, does C have a g^1_3?
2. Is C expressible as a 4-sheeted cover of $\mathbb{P}^1$? In other words, does C have a g^1_4?

As we'll see, all other questions about the existence or nonexistence of linear series on C can be answered by the Riemann-Roch Theorem.

As in the preceding case, the answers can be found through an investigation of the geometry of the canonical model $C \subset \mathbb{P}^4$ of C. This is an octic curve in $\mathbb{P}^4$, and as before the first question to ask is what sort of polynomial equations define C. We start with quadrics, by considering the restriction map

$$r_2 : \mathrm{H}^0(\mathcal{O}_{\mathbb{P}^4}(2)) \ \to \ \mathrm{H}^0(\mathcal{O}_C(2)).$$

On the left, we have the space of homogeneous quadratic polynomials on $\mathbb{P}^4$, which has dimension $\binom{6}{4} = 15$, while by the Riemann-Roch Theorem the target is a vector space of dimension

$$2 \cdot 8 - 5 + 1 = 12.$$

We deduce that C lies on at least 3 independent quadrics. We will see in the course of the following analysis that it is exactly 3; that is, r_2 is surjective.) Since C is irreducible and, by construction, does not lie on a hyperplane, each of the quadrics containing C is irreducible, and thus the intersection of any two is a surface of degree 4. There are now two possibilities: The intersection of (some) three quadrics $Q_1 \cap Q_2 \cap Q_3$ containing the curve is 1-dimensional; or every such intersection is two dimensional.

We first consider the case where $Q_1 \cap Q_2 \cap Q_3$ is 1-dimensional. By the principal ideal theorem the intersection has no 0-dimensional components. By Bézout's Theorem the intersection is a curve of degree 8, and since C also has degree 8 we must have $C = Q_1 \cap Q_2 \cap Q_3$. Lasker's Theorem then shows that the three quadrics Q_i generate the whole homogeneous ideal of C.

We can now answer the first of our two questions for curves of this type. As in the genus 4 case the geometric Riemann-Roch Theorem implies that C has a g^1_3 if and only if the canonical model of C contains 3 colinear points or, more generally, meets a line L in a divisor of 3 points. When C is the intersection of quadrics, this cannot happen, since the line L would have to be contained in all the quadrics that contain C and $L \subset C$, which is absurd. Thus, in this case, C has no g^1_3.

What about g^1_4s? Again invoking the geometric Riemann-Roch Theorem, a divisor of degree 4 moving in a pencil lies in a 2-plane; so the question is, does $C \subset \mathbb{P}^4$ contain a divisor of degree 4, say $D = p_1 + \cdots + p_4 \subset C$, that lies in a plane Λ? Supposing this is so, we consider the restriction map

$$\mathrm{H}^0(\mathcal{I}_{C/\mathbb{P}^4}(2)) \ \to \ \mathrm{H}^0(\mathcal{I}_{D/\Lambda}(2)).$$

By hypothesis, the left hand space is 3-dimensional; but any four noncolinear points in the plane impose independent conditions on quadrics,
((this is a scheme of length 4; how is the reader supposed to cope with this if we don't assume the notion of a scheme, at least a finite one? And does the reader really know this fact about schemes of length 4 in the plane?))
so that the right hand space is 2-dimensional. It follows that Λ *must be contained in one of the quadrics Q containing C.*

The quadrics in ¶^{4} that contain 2-planes are exactly the singular quadrics: such a quadric is a cone over a quadric in ¶^{3}, and it is ruled by the (one or two) families of 2-planes it contains, which are the cones over the (one or two) rulings of the quadric in ¶^{3}. The argument above shows that the existence of a g^1_4s on C in this case implies the existence of a singular quadric containing C.

Conversely, suppose that $Q \subset \mathbb{P}^4$ is a singular quadric containing $C = Q_1 \cap Q_2 \cap Q_3$. Now say $\Lambda \subset Q$ is a 2-plane. If Q' and Q'' are "the other two quadrics" containing C, we can write

$$\Lambda \cap C = \Lambda \cap Q' \cap Q'',$$

from which we see that $D = \Lambda \cap C$ is a divisor of degree 4 on C, and so has $r(D) = 1$ by the geometric Riemann-Roch Theorem. Thus, the rulings of singular quadrics containing C cut out on C pencils of degree 4; and every pencil of degree 4 on C arises in this way.

Does C lie on singular quadrics? There is a $\mathbb{P}^2$ of quadrics containing C—a 2-plane in the space $\mathbb{P}^{14}$ of quadrics in $\mathbb{P}^4$—and the family of singular quadrics consists of a hypersurface of degree 5 in $\mathbb{P}^{14}$–called the *discriminant* hypersurface. By Bertini's Theorem, not every quadric containing C is singular. Thus the set of singular quadrics containing C is a plane curve B cut out by a quintic equation. So C does indeed have a g^1_4, and is expressible as a 4-sheeted cover of $\mathbb{P}^1$. In sum, we have proven:

Proposition 5.2.1. *Let $C \subset \mathbb{P}^4$ be a canonical curve, and assume C is the complete intersection of three quadrics in $\mathbb{P}^4$. Then C may be expressed as a 4-sheeted cover of $\mathbb{P}^1$ in a one-dimensional family of ways, and there is a map from the set of g^1_4s on C to a plane quintic curve B, whose fibers have cardinality 1 or 2.*

((could the "quintic curve" be reducible/multiple? Just a line?))
Of course, we can go further and ask about the geometry of the plane curve B and how it relates to the geometry of C; a fairly exhaustive list of possibilities is given in [**?**] [ACGH]. But that's enough for now.

In the second possibility above, that the canonical curve $C \subset \mathbb{P}^4$ is not a complete intersection; we will see in *** that the the intersection of the quadrics containing C is two-dimensional: a rational normalscroll; and C is trigonal, that is, a 3-sheeted cover of $\mathbb{P}^1$.

5.3 Curves of genus 6

Canonical model lies on at least 6 quadrics.

To prove projective quadratic normality, use general position: the general hyperplane section is 10 points in $\mathbb{P}^4$ 8 of them lie on the union of two hyperplanes – which won't contain the rest – so they impose exactly 9 conditions.

Prove monodromy of hyperplane sections is the symmetric group. Do this carefully. Explain the correspondence between monodromy and Galois theory.

Deduce projective normality from quadratic normality.

At this point, we're stuck: we still don't know what linear series exist on our curve, or much about the geometry of the canonical model. But if we invoke Brill-Noether, we have both: the curve has a g^2_6, which gives us a plane model as a sextic (with only double points, since no g^1_3s); the canonical series on the curve is cut out by cubics passing through the double points, which embeds the (blow-up of the) plane as a del Pezzo surface in ¶^{5}, of which the canonical curve is a quadric section. Also, use the count of g^2_6s on C to deduce the uniqueness of the del Pezzo.

Chapter 6

Inflections and Brill Noether

In this concluding chapter, we want to introduce one more aspect of the geometry of linear series on curves, the *inflectionary points* of a linear system, and use it to give a proof of at least half of the classical Brill-Noether theorem.

Inflectionary points in general are a direct generalization of the notion of flex point of a smooth plane curve to curves in higher-dimensional space. Just as a point $p \in C$ on a smooth plane curve $C \subset \mathbb{P}^2$ is called a *flex point* if there is a line $L \subset \mathbb{P}^2$ having contact of order 3 or more with C at p, a point on a smooth, nondegenerate curve $C \subset \mathbb{P}^r$ will be called an inflectionary point if there is a hyperplane $H \subset \mathbb{P}^r$ having contact of order $r+1$ or more with C at p. This notion can be extended to arbitrary linear series on smooth curves (as opposed to very ample ones); we'll see below that every linear series has finitely many inflectionary points, and how to count them.

6.1 Inflection points, Plücker formulas and Weierstrass points

6.1.1 Definitions

To start with the definition: let C be a smooth projective curve of genus g, and $\mathcal{D} = (\mathcal{L}, V)$ a g^r_d on C; that is, let $\mathcal{L} \in \mathrm{Pic}^d(C)$ be a line bundle of degree d on C and $V \subset H^0(\mathcal{L})$ an $(r+1)$-dimensional vector space of sections.

For any point $p \in C$, we can find a basis $\sigma_0, \ldots, \sigma_r$ of V consisting of sections vanishing to different orders at p (just start with any basis and if two elements vanish to the same order, replace one with a linear combination of the two vanishing to strictly higher order; since the order $\mathrm{ord}_p(\sigma)$ of any section at p is bounded above by d, this process must terminate). The set

$$\{\mathrm{ord}_p(\sigma) \mid \sigma \neq 0 \in V\}$$

thus has cardinality $r+1$, and we can write it as

$$\{\mathrm{ord}_p(\sigma) \mid \sigma \neq 0 \in V\} = \{a_0, \dots, a_r\} \ \text{ with } \ a_0 < a_1 < \cdots < a_r;$$

the sequence $a_i = a_i(\mathcal{D}, p)$ is called the *vanishing sequence* of $\mathcal{D}$ at p. Finally, since $a_i \geq i$, we can set $\alpha_i = \alpha_i(\mathcal{D}, p) = a_i - i$; the sequence $0 \leq \alpha_0 \leq \alpha_1 \leq \cdots \leq \alpha_r$ is called the *ramification sequence* of $\mathcal{D}$ at p, and we say that p is an *inflectionary point* of the linear series $\mathcal{D}$ if $(\alpha_0, \dots, \alpha_r) \neq (0, \dots, 0)$. (Note that if $\mathcal{D}$ is very ample, so that it may be viewed at the linear series cut on C by hyperplanes for some embedding $C \subset \mathbb{P}^r$, then this coincides with the notion above: p is an inflectionary point if there is a hyperplane $H \subset \mathbb{P}^r$ having contact of order $r+1$ or more with C at p.)

Finally, we define the *weight* of an inflectionary point to be the sum

$$w(\mathcal{D}, p) = \sum_{i=0}^{r} \alpha_i(\mathcal{D}, p).$$

6.1.2 The Plücker formula

There are two essential facts about the inflectionary points of a linear series $\mathcal{D} = (\mathcal{L}, V)$ on a smooth curve C.

The first is simply that *not every point of C is an inflectionary point.* This may seem obvious—try to imagine a plane curve in which every point is a flex!—but in fact it's false in characteristic p: there exist what are called "strange curves," smooth curves in $\mathbb{P}^r$ such that every point is inflectionary. Luckily, we are in characteristic 0 here, so we don't have to worry about this phenomenon.

The second is that the total number of inflectionary points, properly counted, is determined by d, g and r. Here "properly counted" means that we count an inflectionary point $p \in C$ w times, where $w = w(\mathcal{D}, p)$ is the weight of p; the actual formula, called the *Plücker formula*, is

$$\sum_{p \in C} w(\mathcal{D}, p) \ = \ (r+1)d + r(r+1)(g-1). \tag{6.1}$$

Proofs of these two statements can be found in a variety of sources; see for example [] (3264?).

It should be emphasized that the Plücker formula, while extremely useful (as we'll see in the remainder of this chapter), leaves many questions unanswered: we don't know, for example, what combinations of inflectionary points are possible, or what the behavior of the inflectionary points on a suitably general curve may be.

6.1.3 Weierstrass points

As with any extrinsic invariant of a curve in projective space, we can derive an intrinsic invariant of an abstract curve by applying the notion of inflectionary point to the canonical linear series.

We define a *Weierstrass point* of a curve C to be an inflectionary point of the canonical linear series $|K_C|$. This amounts to saying a point p is a Weierstrass point if there exists a canonical differential on C vanishing to order g or more at p; by Riemann-Roch, this is tantamount to the condition that $h^0(\mathcal{O}_C(gp)) \geq 2$, or in other words to saying that there exists a rational function on C, regular away from p and having a pole of order g or less at p.

We can similarly characterize all the inflectionary indices of the canonical series at a point. We see from Riemann-Roch that for any $k \geq 0$, there exists a rational function on C, regular on $C \setminus \{p\}$ and having a pole of order exactly k at p—that is,

$$h^0(\mathcal{O}_C(kp)) > h^0(\mathcal{O}_C((k-1)p))—$$

if and only if

$$h^0(K_C(-kp)) = h^0(K_C((-k+1)p));$$

that is, if and only if there does *not* exist a regular differential on C with a zero of order exactly $k-1$ at p. To give the classical terminology, we see from the above that there will exist exactly g values of k such that there does *not* exist a rational function on C with a pole of order exactly k at p; these are called the *gap values* of the point $p \in C$, and by the above they comprise exactly the vanishing sequence of the canonical series $|K_C|$ at p, shifted by 1. Moreover, it is clear that the complement in $\mathbb{N}$ of the gap values—that is, the set of k such that there *does* exist a rational function on C with a pole of order exactly k at p—forms a semigroup, called the *Weierstrass semigroup* of $p \in C$. Finally, the *weight* w_p of a Weierstrass point $p \in C$ is defined to be the weight $w(|K_C|, p)$ of p as an inflectionary point of the canonical series.

From the general theory of ramification above, we see that a general point p on any curve C has gap sequence $(1, 2, \dots, g)$, and correspondingly the semigroup $W_p = (0, g+1, g+2, \dots)$. A Weierstrass point is called *normal* if it has weight 1; this is tantamount to saying that the gap sequence is $(1, 2, \dots, g-1, g+1)$, or that the semigroup is $(0, g, g+2, g+3, \dots)$. (The full Brill-Noether theorem tells us that a general curve C has only normal Weierstrass points; this will be a consequence of Theorem 6.4.5 below.) Finally, the Plücker formula tells us the total weight of the Weierstrass points on a given curve C: plugging in, we have

$$\sum_{p \in C} w(|K_C|, p) \;=\; g(2g-2) + (g-1)g(g-1) \;=\; g^3 - g.$$

and hence

Theorem 6.1.1. *The sum of the weights of the Weierstrass points on a curve C of genus g is*

$$\sum_{p \in C} w_p = g^3 - g.$$

There is still much we don't know about Weierstrass points in general. Most notably, we don't know what semigroups of finite index in $\mathbb{N}$ occur as Weierstrass semigroups; an example of Buchweitz shows that not all semigroups occur, but there are also positive results, such as the statement ([EH]) that every semigroup of weight $w \leq g/2$ occurs, and its refinement and strengthening by Pflueger ([**?**]).

6.2 Finiteness of the automorphism group

As an application of just a rudimentary knowledge of Weierstrass points, we will deduce a fundamental fact: that the automorphism group of a curve of genus $g \geq 2$ is finite. The idea behind the argument is simple: because the Weierstrass points of a curve C are intrinsically defined, *any automorphism of C must carry Weierstrass points to Weierstrass points.* Since there are only finitely many Weierstrass points, then, it will suffice to show that the subgroup of $Aut(C)$ of automorphisms of C that fix all the Weierstrass points individually is finite. In fact, the following two lemmas establish a strong version of this:

Lemma 6.2.1. *Let C be a smooth projective curve of genus $g \geq 2$, and $f : C \to C$ an automorphism of C.*

1. *If f has $2g + 3$ distinct fixed points, then f is the identity; and*
2. *If f has $2g + 2$ distinct fixed points, then either f is the identity or C is hyperelliptic and f is the hyperelliptic involution.*

Proof. There are two possible arguments here, one invoking the classical topology and applying the Lefschetz fixed point formula and the other more algebrao-geometric.

For the first, we recall the definition of the *Lefschetz number* of a map $f : M \to M$ of a compact oriented real n-manifold M. This is the alternating sum of the traces of the action of f on $H^i(X, \mathbb{C})$:

$$L(f) := \sum_{i=0}^{n} \operatorname{Trace}\left(f^* : H^i(X, \mathbb{C}) \to H^i(X, \mathbb{C})\right).$$

The Lefschetz fixed point formula then says that if f has isolated fixed points, the number of those points, properly counted, is equal to $L(f)$.

In the situation of a smooth projective curve over $\mathbb{C}$, any automorphism other than the identity has isolated fixed points, and since the map is orientation-preserving each fixed point contributes positively to the total; thus the number of distinct fixed points is at most $L(f)$.

Now suppose $f : C \to C$ is any automorphism. Of necessity, f acts as the identity on $H^0(C, \mathbb{C})$ and $H^2(C, \mathbb{C})$, so if we want to bound $L(f)$ we just have to say something about the action of f on $H^1(C, \mathbb{C})$. To do this, note that the action of f on $H^1(C, \mathbb{C})$ respects the *Hodge decomposition*

$$H^1(C, \mathbb{C}) = H^0(K_C) \oplus H^1(\mathcal{O}_C).$$

Moreover, the action of f on $H^0(K_C)$ preserves the definite Hermitian inner product

$$H(\eta, \phi) = \int_C \eta \wedge \overline{\phi},$$

and it follows that *the eigenvalues of the action of f on $H^0(K_C)$ are all complex numbers of absolute value 1*, and likewise for the action on $H^1(\mathcal{O}_C)$. The absolute value of the trace of $f^* : H^1(C, \mathbb{C}) \to H^1(C, \mathbb{C})$ is thus at most 2g, and hence

$$L(f) \leq 2 + 2g,$$

proving the stated inequality in general.

Finally, if we have equality then f must act as -1 on $H^1(C, \mathbb{C})$, and it follows (again from Lefschetz) that f^2 is the identity; applying Riemann-Hurwitz to the map from C to the quotient $B = C/\langle f \rangle$ we may deduce that $B = \mathbb{P}^1$, so C is hyperelliptic and f the hyperelliptic involution. □

An alternative, more algebraic argument for the lemma may be given using the intersection pairing on the surface $S = C \times C$ and applying the index theorem for surfaces. To carry this out, let Δ and $\Gamma \subset S$ be the diagonal and the graph of f respectively, and let Φ_1 and $\Phi_2 \subset S$ be fibers of the two projection maps; let $\delta, \gamma, \varphi_1$ and $\varphi_2 \in N(S)$ be the classes of these curves in the Neron-Severi group of S. We are trying to estimate the intersection number $b = \delta \cdot \gamma$.

We know all the other pairwise intersection number of these classes: the ones involving φ_1 or φ_2 are obvious; we have

$$\delta^2 = 2 - 2g$$

and since the automorphism $id_C \times f : C \times C \to C \times C$ carries Δ to Γ, we see that $\gamma^2 = 2 - 2g$ as well.

We can now apply the index theorem for surfaces to deduce our inequality. To keep things relatively simple, let's introduce two new classes: set

$$\delta' = \delta - \varphi_1 - \varphi_2 \quad \text{and} \quad \gamma' = \gamma - \varphi_1 - \varphi_2,$$

so that δ' and γ' are orthogonal to the class $\varphi_1 + \varphi_2$. Since $\varphi_1 + \varphi_2$ has positive self-intersection, the index theorem tells us that the intersection pairing must be negative definite on the span $\langle \delta', \gamma' \rangle \subset N(S)$. In particular, the determinant of the intersection matrix

	δ'	γ'
δ'	$-2g$	$b-2$
γ'	$b-2$	$-2g$

(where again $b = \gamma \cdot \delta$) must be nonnegative, from which our inequality follows.

Having established an upper bound on the number of fixed points an automorphism f of C (other than the identity) may have, it remains to find a lower bound on the number of distinct Weierstrass points; this is the content of the next lemma.

Lemma 6.2.2. *If C is a smooth projective curve of genus $g \geq 2$, then C has at least $2g+2$ distinct Weierstrass points; and if it has exactly $2g+2$ Weierstrass points it is hyperelliptic.*

Proof. Let $p \in C$ be any point, and $w_1 = w_1(p), \ldots, w_g = w_g(p)$ the ramification sequence of the canonical series $|K_C|$ at p. By definition,

$$h^0(K_C(-(w_i+i)p)) = g - i.$$

Applying Clifford's theorem we have

$$g - i \leq \frac{2g-2-w_i-i}{2} + 1;$$

solving, we see that

$$w_i \leq i$$

and hence

$$w_p \leq \binom{g}{2}$$

where w_p is the total weight of p as a Weierstrass point. Since the total weight of the Weierstrass points on C is $g^3 - g$ by Plücker, we see that the number of distinct Weierstrass points must be at least

$$\frac{g^3-g}{\binom{g}{2}} = 2g+2.$$

Finally, by the strong form of Clifford, equality here implies that the curve is hyperelliptic. □

6.3 Proof of (half of) the Brill-Noether theorem

In its most basic form, the Brill-Noether theorem asserts for any d, g and r that

1. if $\rho(g,r,d) := g - (r+1)(g-d+r) \geq 0$, then every curve C of genus g possesses as g^r_d; and

2. if $\rho < 0$ then a general curve of genus g does not possess a g^r_d.

The first part, often called the "existence half" of Brill-Noether, was originally proved by Kempf ([]) and Kleiman-Laksov ([]); both proofs relied on an application of the Thom-Porteous formula to a particular map of vector bundles on the Jacobian of C. An account of this argument may also be found in Appendix A of [].

We will now apply the notion of inflectionary points, and the Plücker formula in particular, to deduce the second half of the statement above—the "nonexistence half." We will then go back and deduce some of the other parts of the full Brill-Noether statement from the same set-up.

The basic approach here is to consider a family of curves $\pi : \mathcal{C} \to B$, where

1. B is a smooth curve, with distinguished point $0 \in B$;
2. for all $b \neq 0 \in B$, the fiber $C_b = \pi^{-1}(b)$ is a smooth, projective curve of genus g; and
3. the fiber C_0 over 0 is a rational curve with g ordinary cusps.

We will establish the

Lemma 6.3.1. *If $\mathcal{C} \to B$ is a family of curves as above, then for general $b \in B$ the fiber C_b does not possess a g^r_d with $\rho < 0$.*

Once we establish the existence of such a family, we deduce the basic

Theorem 6.3.2. *A general curve C of genus g does not possess a g^r_d with $\rho(g, r, d) < 0$.*

The basic outline of the argument is by contradiction, but straightforward: we assume that the general curve C_b in the family does have a g^r_d, consider what the limit of those g^r_ds might look like and, using our knowledge of the relatively simple curve C_0, arrive at a contradiction. By way of notation, let $B^\circ = B \setminus \{0\}$ and let $\mathcal{C}^\circ = \pi^{-1}(B^\circ)$ be the complement in $\mathcal{C}$ of the special fiber. The proof proceeds essentially in four/five steps.

Step 0: Existence of such a family

Lemma 6.3.3. *For each g, there exists a family $\mathcal{C} \to B$ of curves with B smooth and one-dimensional; C_b a smooth curve of genus g for $b \neq 0 \in B$ and C_0 a rational curve with g cusps.*

Step 1: Finding a family of g^r_ds over B°

Suppose now that the general curve C_b in the family does have a line bundle of degree d with $r+1$ sections. The first thing to observe is that, possibly after a base change, we can pick out one such line bundle $\mathcal{L}_b$ for each $b \neq 0$, varying regularly with b; or, in other words *there exists a line bundle $\mathcal{L}^\circ$ on the complement $\mathcal{C}^\circ$ of the special fiber such that*

$$\deg(\mathcal{L}^\circ|_{C_b}) = d \quad \text{and} \quad h^0(\mathcal{L}^\circ|_{C_b}) \geq r+1$$

for all $b \neq 0 \in B$.
((need to give argument for this assertion))

Step 2: Extending the line bundle $\mathcal{L}^\circ$ to a sheaf on all of $\mathcal{C}$

Next, we want to extend $\mathcal{L}^\circ$ to a sheaf on all of $\mathcal{C}$. We claim that *there exists a torsion-free sheaf $\mathcal{L}$ on all of $\mathcal{C}$ such that $\mathcal{L}|_{\mathcal{C}^\circ} \cong \mathcal{L}^\circ$.*

To see this, we choose an auxiliary line bundle $\mathcal{M}$ on $\mathcal{C}$ with relative degree $e > d+2g$ (for example, embed $\mathcal{C}$ is projective space and take $\mathcal{M} = \mathcal{O}_{\mathcal{C}}(m)$ for large m); in keeping with our notational conventions, let $\mathcal{M}^\circ$ be the restriction of $\mathcal{M}$ to $\mathcal{L}^\circ$. Consider the line bundle

$$\mathcal{N}^\circ = (\mathcal{L}^\circ)^* \otimes \mathcal{M}^\circ.$$

The bundle $\mathcal{N}^\circ$ has lots of sections: the direct image is locally free of rank $e-g+1 > 0$, and after restricting to an open neighborhood of $0 \in B$ we can assume it's generated by them. Choose a section σ of $\mathcal{N}^\circ$; let $D^\circ \subset \mathcal{C}^\circ$ be its zero divisor, and let $D \subset \mathcal{C}$ be the closure of D° in $\mathcal{C}$. Now, away from C_0 we can write

$$\mathcal{L}^\circ = (\mathcal{N}^\circ)^* \otimes \mathcal{M}^\circ = \mathcal{I}_{D^\circ/\mathcal{C}^\circ} \otimes \mathcal{M}^\circ$$

and accordingly the sheaf

$$\mathcal{L} := \mathcal{I}_{D/\mathcal{C}} \otimes \mathcal{M}$$

is the desired sheaf. Note that this need not be locally free: the total space $\mathcal{C}$ of our family may not be smooth at the cusps of the special fiber C_0 (even if the family we originally started with had smooth total space, the base change called for in Step 1 would yield a family with total space singular at the cusps of C_0), and if D passes through any of these points it need not be Cartier.

In sum, if the general fiber C_b of our family has a g^r_d, we can conclude that the special fiber C_0 has a torsion-free sheaf $\mathcal{L}_0$ with

$$c_1(\mathcal{L}_0) = d;$$

and, by upper-semicontinuity of cohomolgy,

$$h^0(\mathcal{L}_0) \geq r+1.$$

Step 3: Local description of the sheaf $\mathcal{L}_0$

The next question is, what does $\mathcal{L}_0$ look like if it's not locally free? Here we have a basic lemma:

Lemma 6.3.4. *Let p be a cusp of a curve C. If $\mathcal{F}$ is a torsion-free sheaf on C, then in a neighborhood of p in C the sheaf $\mathcal{F}$ is either locally free or isomorphic to the ideal sheaf $\mathcal{I}_{p/C}$ of p in C.*

Proof. Consider the endomorphism ring of $\mathcal{F}$, and note that it is commutative and integral over $\mathcal{O}_{p/C}$; thus it is either $\mathcal{O}_{p/C}$ or it's integral closure $\mathcal{O}_{p/\tilde{C}}$. In the latter case it is free over $\mathcal{O}_{p/\tilde{C}} \cong \mathcal{I}_{p/C}$. In the former case....
((complete the argument))

□

Exercise 6.3.5. 1. Show that the conclusion of Lemma 6.3.4 holds in case p is a node of C

2. Show by example that the conclusion of Lemma 6.3.4 is false in case p is either a tacnode or a triple point of C.

Step 4: Applying the Plücker formula

Now, back to our family $\pi : \mathcal{C} \to B$ of curves. We have assumed that for some d and r with $\rho(g, r, d) < 0$ the general curve C_b has a g^r_d, and deduced that the special fiber C_0 has a rank 1 torsion-free sheaf $\mathcal{L}_0$ of degree d with at least $r + 1$ sections; we now have to derive from this a contradiction.

To see most clearly where this contradiction comes from, let's start with the simplest case: where $\mathcal{L}_0$ is indeed locally free. In this case, let $\nu : C^\nu \cong \mathbb{P}^1 \to C$ be the normalization of C and let $q_1, \ldots, q_g \in \mathbb{P}^1$ be the points lying over the cusps of C_0. We have

$$\nu^*(\mathcal{L}) \cong \mathcal{O}_{\mathbb{P}^1}(d)$$

and

$$V = \nu^*(H^0(\mathcal{L}_0)) \subset H^0(\mathcal{O}_{\mathbb{P}^1}(d))$$

is an $(r+1)$-dimensional space of sections. (If $H^0(\mathcal{L}_0) > r + 1$, just choose any $(r+1)$-dimensional subspace.)

Now, given that any section $\sigma \in V \subset H^0(\mathcal{O}_{\mathbb{P}^1}(d))$ is pulled back from the cuspidal curve C, we see that σ *cannot vanish to order exactly 1 at the point $q_i \in \mathbb{P}^1$ lying over any of the cusps of C_0*. It follows that for each i the ramification index

$$\alpha_1(V, q_i) \geq 1$$

and hence in general $\alpha_1(q_i, V) \geq 1$ for all $i \geq 1$. In particular, the weight of the inflectionary point q_i for the linear series V satisfies

$$w(V, q_i) \geq r$$

and correspondingly

$$\sum_{i=1}^{g} w(V, q_i) \geq rg$$

But the Plücker formula **??** tells us that the total weight of all inflectionary points for the series V is

$$\sum_{p \in \mathbb{P}^1} w(V, p) = (r+1)(d-r)$$

and there's our contradiction: by the hypothesis that

$$\rho(g, r, d) := g - (r+1)(g - d + r) < 0$$

we have $rg > (r+1)(d-r)$.

Finally, the case where $\mathcal{L}_0$ is not locally free is if anything even easier. Suppose now that the sheaf $\mathcal{L}$ fails to be locally free at l of the cusps of C_0, say $\nu(p_1), \dots, \nu(p_l)$. Again, we can pull $\mathcal{L}$ back to $\mathbb{P}^1$; again we have

$$\nu^*(\mathcal{L}_0) \cong \mathcal{O}_{\mathbb{P}^1}(d);$$

and again we pull back section of $\mathcal{L}$ to arrive at a linear system

$$V = \nu^*(H^0(\mathcal{L}_0)) \subset H^0(\mathcal{O}_{\mathbb{P}^1}(d))$$

of degree d and genus g on $\mathbb{P}^1$. The only difference here is that sections of V all vanish at $p_1, \dots, p_l$, so that we have

$$w(V, p_k) \geq \begin{cases} r+1 & \text{if } k \leq l; \text{ and} \\ r & \text{if } k > l. \end{cases}$$

so that

$$\sum_{k=1}^{g} w(V, p_k) \geq rg + l$$

and our contradiction is even more of a contradiction!

6.4 Corollaries and extensions of our proof

We have proved a bare-bones version of the nonexistence half of Brill-Noether; in particular, the argument does not a priori tell us anything about the geometry of $W^r_d(C)$ for a general curve C, or the geometry of C as embedded by a general linear system. In the final section of this chapter, we'll see how we can deduce stronger forms of Brill-Noether from the basic set-up above.

6.4.1 Brill-Noether with inflection

Let's start with the low-hanging fruit. Since the basic ingredient of the proof of Theorem 6.3.2 is the Plücker formula, the argument tells us something about the inflectionary behavior of linear series on a general curve C as well.

We start with a definition.

Definition 6.4.1. Let C be a smooth curve of genus g and $p_1, \dots, p_n \in C$ distinct points of C. If $\mathcal{D} = (L, V)$ is a linear system on C of degree d and dimension r, we define the *adjusted Brill-Noether number* of $\mathcal{D}$ relative to the points p_k to be

$$\rho(\mathcal{D}; p_1, \dots, p_k) := g - (r+1)(g-d+r) - \sum_{k=1}^{n} w(\mathcal{D}, p_k).$$

In these terms, we have the basic (but powerful) extension of the Brill-Noether theorem proved above:

Theorem 6.4.2. *Let $(C; p_1, \dots, p_n)$ be a general n-pointed curve of genus g (that is, let C be a general curve and $p_1, \dots, p_n \in C$ general points; equivalently, let $(C; p_1, \dots, p_n)$ correspond to a general point of $M_{g,n}$). If $\mathcal{D}$ is any linear system on C, then*

$$\rho(\mathcal{D}; p_1, \dots, p_k) \geq 0.$$

Proof. The proof is just an extension of the argument for Lemma 6.3.1. To start, let $\mathcal{C} \to B$ be a family of curves as in the proof of Lemma 6.3.3. Let $\sigma_1, \dots, \sigma_n : B \to \mathcal{C}$ be sections of $\mathcal{C} \to B$ with $\sigma_k(0)$ a smooth point of C_0 for all k (such sections can always be found after passing to an étale open neighborhood of $0 \in B$). Exactly as in the proof of Lemma 6.3.1, if the general curve C_b in our family admits a g^r_d $\mathcal{D}$ with

$$\rho(\mathcal{D}; \sigma_1(b), \dots, \sigma_n(b)) < 0$$

we can choose a family $\{\mathcal{D}_b\}$ of such linear series on the fibers C_b for $b \neq 0$ and, taking limits, we arrive at a g^r_d $\mathcal{D}_0$ on $\mathbb{P}^1$ with

$$w(\mathcal{D}_0, q_i) \geq r$$

for each of the g points $q_i \in \mathbb{P}^1$ lying over the cusps of C_0, and in addition

$$w(\mathcal{D}_0, r_k) \geq w(\mathcal{D}_b, \sigma_k(b))$$

where $r_k \in \mathbb{P}^1$ is the point in $\mathbb{P}^1$ lying over $\sigma_k(0) \in C_0$. Adding up, we have

$$\begin{aligned}\sum_{i=1}^{g} w(\mathcal{D}_0, q_i) + \sum_{k=1}^{n} w(\mathcal{D}_0, r_i) &\geq rg + \sum_{k=1}^{n} w(\mathcal{D}_b, \sigma_k(b)) \\ &> rg + g - (r+1)(g-d+r) = (r+1)(d-r)\end{aligned}$$

since we assumed that

$$\rho(\mathcal{D}_b; \sigma_1(b), \ldots, \sigma_n(b)) = g - (r+1)(g-d+r) - \sum_{k=1}^{n} w(\mathcal{D}_b, \sigma_k(b)) < 0.$$

But as before the Plücker formula for $\mathbb{P}^1$ tells us that

$$\sum_{p \in \mathbb{P}^1} w(\mathcal{D}_0, p) = (r+1)(d-r),$$

a contradiction. □

6.4.2 Brill-Noether with dimension

Theorem 6.4.2 might at first glance seem relevant only to problems involving inflection, but in fact in can be used to prove results that have nothing to do with inflection points. For example, one consequence is the stronger form of Brill-Noether:

Theorem 6.4.3. *If C is a general curve of genus g, then for any d and r with $\rho(g, r, d) \geq 0$,*

$$\dim W^r_d(C) = \rho(g, r, d).$$

Proof. The basic idea of the proof is simple: basically, we argue that if we had a $(\rho+1)$-dimensional family of $g^r_d s$ on C, then we could find one with nonzero ramification at $\rho+1$ general points of C, violating Theorem 6.4.2.

This idea is easier to implement after specializing, so once more we go back to our family $\mathcal{C} \to B$ of smooth curves specializing to a g-cuspidal curve C_0, with normalization $\mathbb{P}^1$. The basic lemma is:

Lemma 6.4.4. *Let Σ be a complete curve and let $\{\mathcal{D}_\lambda\}_{\lambda \in \Sigma}$ be a (nonconstant) family of $g^r_d s$ on $\mathbb{P}^1$ parametrized by Σ. If $p \in \mathbb{P}^1$ is any fixed point, then for at least one $\lambda \in \Sigma$ we have $w(\mathcal{D}_\lambda, p) > 0$.*

Proof. Embed $\mathbb{P}^1$ in $\mathbb{P}^d$ as a rational normal curve of degree d. Given a $(d-r-1)$-plane $\Lambda \subset \mathbb{P}^d$, the hyperplanes in $\mathbb{P}^d$ containing Λ cut out a g^r_d $\mathcal{D}_\Lambda$ on $\mathbb{P}^1$, and indeed every g^r_d on $\mathbb{P}^1$ can be described in this way for a unique Λ. The g^r_ds on $\mathbb{P}^1$ are thus parametrized by the Grassmannian $\mathbb{G}(d-r-1, d)$, and we can think of Σ as a complete curve in $\mathbb{G}(d-r-1, d)$.

Consider now the hyperplanes $H \subset \mathbb{P}^d$ such that the divisor $H \cap \mathbb{P}^1$ has multiplicity $\geq r+1$ at p. These correspond to points in a linear space of codimension $r+1$ in $(\mathbb{P}^d)^*$; in particular, their intersection is an r-plane $\Omega \subset \mathbb{P}^d$, called the *osculating plane* to the rational normal curve at p. The condition that a g^r_d $\mathcal{D}_\Lambda$ have non-zero ramification at p—in other words, that $\mathcal{D}_\Lambda$ contains a divisor with multiplicity $\geq r+1$ at p—is simply that $\Lambda \cap \Omega \neq \emptyset$. But the set of such Λ is a hyperplane section of $\mathbb{G}(d-r-1, d)$ under trhe Plücker embedding; in particular, any complete curve $\Sigma \subset \mathbb{G}(d-r-1, d)$ must intersect it. □

Given this lemma, the proof of Theorem 6.4.3 proceeds as follows. We know from the basic dimension estimates of Chapter **??** that $\dim W^r_d(C) \geq \rho(g,r,d)$ for any C; we have to show that we cannot have $\dim W^r_d(C) > \rho(g,r,d)$ for a general curve C. We argue as follows:

First: if it were the case that $\dim W^r_d(C) > \rho(g,r,d)$ for a general curve C, we would have, after specializing and pulling back to $\mathbb{P}^1$, at least a $(\rho+1)$-dimensional family of g^r_ds on $\mathbb{P}^1$, all of which had ramification weight at least r at the points q_i of $\mathbb{P}^1$ lying over the cusps of C_0.

Secondly, we pick any $\rho+1$ points $r_k \in \mathbb{P}^1$ other than the q_i. Applying Lemma 6.4.4 repeatedly, we find that there is at least a ρ-dimensional subfamily of g^r_ds having nonzero ramification at p_1, a $(\rho-1)$-dimensional subfamily of g^r_ds having nonzero ramification at p_1 and p_2, and so on; ultimately, we conclude that there is a g^r_d $\mathcal{D}$ on $\mathbb{P}^1$ with ramification index at least r at each q_i and nonzero ramification index at each r_k.

Finally, we observe that the linear series $\mathcal{D}$ has total ramification at least

$$rg + \rho + 1 = (r+1)(d-r) + 1$$

at the points q_i and r_k, once more violating the Plücker formula. □

We can combine Theorem 6.4.3 and Theorem 6.4.2 into one theorem, more complicated but more inclusive:

Theorem 6.4.5. *Let C be a smooth curve of genus g and $p_1, \dots, p_n \in C$ distinct points; for $k = 1, \dots, n$ let $\alpha^k = (\alpha^k_0, \dots \alpha^k_r)$ be a nondecreasing sequence of nonnegative integers, and let*

$$G^r_d(p_1, \dots, p_n; \alpha^1, \dots, \alpha^n) = \{\mathcal{D} \in G^r_d(D) \mid \alpha_i(\mathcal{D}, p_k) \geq \alpha^k_i\}.$$

If $(C, p_1, \dots, p_n)$ is a general n-pointed curve, then either $G^r_d(p_1, \dots, p_n; \alpha^1, \dots, \alpha^n)$ is empty or

$$\dim G^r_d(p_1, \dots, p_n; \alpha^1, \dots, \alpha^n) = \rho(g,r,d) - \sum_{k+1}^{n} \sum_{i=0}^{r} \alpha^k_i.$$

Finally, we can combine this last theorem with a little dimension-counting to deduce a simple fact:

Theorem 6.4.6. *If $\mathcal{D}$ is a general g^r_d on a general curve, then $\mathcal{D}$ has only simple ramification; that is,*

$$w(\mathcal{D}, p) \leq 1 \quad \textit{for all } p \in C.$$

Note that applying this in case $d = 2g-2$ and $r = g-1$, we arrive at the statement made earlier: that a general curve C of genus g has only normal Weierstrass points!

Chapter 7

Hilbert Schemes I: Examples

In Chapter **??**, we looked at curves of low genus and described the linear systems on them; that is, their maps to (and in particular their embeddings in) projective space. In this chapter we'll ask a more refined question: can we describe the family of all such curves in projective space?

((Add a section on basics of the Hilbert scheme explaining why Hilbert schemes; the universal property; and the tangent space **I think this should go in Chapter 6—we should have a "cast of characters" section there, where we introduce all the moduli spaces we'll be dealing with))**

Denote by $\mathcal{H} = \mathcal{H}_{g,r,d}$ the Hilbert scheme parametrizing subschemes of $\mathbb{P}^r$ with Hilbert polynomial $p(m) = dm - g + 1$ (which includes smooth curves of degree d and genus g in $\mathbb{P}^r$), and by $\mathcal{H}^\circ \subset \mathcal{H}$ the open subset parametrizing smooth, irreducible, nondegenerate curves $C \subset \mathbb{P}^r$ (called the *restricted Hilbert scheme*).

Three basic questions about the schemes $\mathcal{H}^\circ$ are:

- Is $\mathcal{H}^\circ$ irreducible? and
- What is its dimension or dimensions?
- Where is it smooth, and where is it singular?

Of course, there are many more questions about the geometry of $\mathcal{H}^\circ$: for example, what is the closure $\overline{\mathcal{H}^\circ} \subset \mathcal{H}$ in the whole Hilbert scheme? (In other words, when is a subscheme $X \subset \mathbb{P}^r$ with Hilbert polynomial $dm - g + 1$ *smoothable*, in the sense that it is the flat limit of a family of smooth curves?) What is the Picard group of $\mathcal{H}^\circ$ or of its closure? We will for the most part not address these, though we will indicate the answers in special cases.

We'll limit ourselves in this chapter to looking at curves in $\mathbb{P}^3$. Most of the questions we raise in what follows could be asked, and many of them answered, in $\mathbb{P}^r$ for any $r \geq 3$, but for the most part the $r = 3$ case is enough to give us the flavor. We will start with curves of the lowest possible degree:

7.1 Degree 3

The smallest possible degree of an irreducible, nondegenerate curve $C \subset \mathbb{P}^3$ is 3. Any irreducible, nondegenerate curve $C \subset \mathbb{P}^3$ of degree 3 is a twisted cubic, so that in this case $\mathcal{H}^\circ$ is the parameter space for twisted cubics.

Proposition 7.1.1. *The open subset $\mathcal{H}^\circ$ of the Hilbert scheme $\mathcal{H}_{0,3,3}$ parametrizing twisted cubics is irreducible of dimension 12.*

Proof. There are in fact several ways of establishing this statement. To start with the simplest, let $C_0 \subset \mathbb{P}^3$ be any given twisted cubic, and consider the family of translates of C_0 by automorphisms $A \in \mathrm{PGL}_4$ of $\mathbb{P}^3$: that is, the family

$$\mathcal{C} = \{(A, p) \in \mathrm{PGL}_4 \times \mathbb{P}^3 \mid p \in A(C_0)\}.$$

Via the projection $\pi : \mathcal{C} \to \mathrm{PGL}_4$, this is a family of twisted cubics, and so it induces a map

$$\phi : \mathrm{PGL}_4 \to \mathcal{H}^\circ.$$

Since every twisted cubic is a translate of C_0, this is surjective, with fibers isomorphic to the stabilizer of C_0, that is, the subgroup of PGL_4 of automorphisms of $\mathbb{P}^3$ carrying C_0 to itself. By the discussion in Section **??**, every automorphism of C_0 is induced by an automorphism of $\mathbb{P}^3$, so the stabilizer is isomorphic to PGL_2 and thus has dimension 3. Since PGL_4 is irreducible of dimension 15, we conclude that $\mathcal{H}^\circ$ *is irreducible of dimension 12.* □

Exercise 7.1.2. Use an analogous argument to show that the restricted Hilbert scheme $\mathcal{H}^\circ \subset \mathcal{H}_{0,r,r}$ of rational normal curves $C \subset \mathbb{P}^r$ is irreducible of dimension $r^2 + 2r - 3$.

Second proof of Proposition 7.1.1

The argument above for Proposition 7.1.1 is based on a rather special fact, that all irreducible nondegenerate cubic curves $C \subset \mathbb{P}^3$ are translates of one another. There is another, less ad-hoc way of arriving at the conclusion above, called the method of *liaison*, or *linkage*, which we'll now describe. While it is more involved, it is more broadly applicable, at least in $\mathbb{P}^3$.

The idea behind this approach is the fact the intersection of any two distinct quadrics $Q, Q' \supset C$ containing a twisted cubic curve C has degree 4 and is unmixed; therefore it is the union of C and a line $L \subset \mathbb{P}^3$.

Conversely, suppose that $L \subset \mathbb{P}^3$ is any line and Q, Q' two general quadrics containing L; write the intersection $Q \cap Q'$ as a union $L \cup C$. Since smooth quadrics contain lines a general quadric containing L is smooth. The quadric Q' will intersect it in a curve of type $(2,2)$, so the curve C will have class $(2,1)$ or $(1,2)$. The quadrics Q' containing L cut out on Q the complete linear system of curves of type $(2,1)$, which has no base locus, so Bertini's theorem tells us that C will be smooth, so that the intersection $Q \cap Q' = L \cup C$ will be the union of L and a twisted cubic. This suggests that we set up an incidence correspondence: let $\mathbb{P}^9$ denote the projective space of quadrics in $\mathbb{P}^3$, and consider

$$\Phi = \{(C, L, Q, Q') \in \mathcal{H}^\circ \times \mathbb{G}(1,3) \times \mathbb{P}^9 \times \mathbb{P}^9 \mid Q \cap Q' = C \cup L\}.$$

We'll analyze Φ by considering the projection maps to $\mathcal{H}^\circ$ and $\mathbb{G}(1,3)$; that is, by looking at the diagram

$$\begin{tikzcd} & \Phi \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ \mathcal{H}^\circ & & \mathbb{G}(1,3) \end{tikzcd}$$

Consider first the projection map $\pi_2 : \Phi \to \mathbb{G}(1,3)$ on the second factor. By what we just said, the fiber over any point $L \in \mathbb{G}(1,3)$ is an open subset of $\mathbb{P}^6 \times \mathbb{P}^6$, where $\mathbb{P}^6$ is the space of quadrics containing L; it follows that Φ is irreducible of dimension $4 + 2 \times 6 = 16$. Going down the other side, we see that the map $\pi_1 : \Phi \to \mathcal{H}^\circ$ is surjective, with fiber over every curve C an open subsets of $\mathbb{P}^2 \times \mathbb{P}^2$, where $\mathbb{P}^2$ is the projective space of quadrics containing C; we conclude again that $\mathcal{H}^\circ$ *is irreducible of dimension 12.*

We'll see below several more instances of the application of liaison to the study of curves in $\mathbb{P}^3$. It should be said, though, that the method is largely limited to curves in $\mathbb{P}^3$ (and subvarieties $X \subset \mathbb{P}^r$ of codimension 2 in general); for example, you can't use it to do Exercise 7.1.2 for $r \geq 4$.

Third proof of Proposition 7.1.1

Yet another proof of Proposition 7.1.1 is based on a remarkable fact about twisted cubics, described in the next proposition; the application to $\mathcal{H}^\circ$ is carried out in the following exercise. In fact, the proposition here applies more generally to *rational normal curves*, and we'll state it in that generality.

Proposition 7.1.3. *If $p_1, \dots, p_{n+3} \in \mathbb{P}^n$ are any $n+3$ points in $\mathbb{P}^n$ in* linear general position, *that is, with no $n+1$ lying in a hyperplane, then there exists a unique rational normal curve $C \subset \mathbb{P}^n$ containing them.*

Proof. To start, we observe that there is an automorphism $\Phi : \mathbb{P}^n \to \mathbb{P}^n$ carrying the points $p_1, \dots, p_{n+1}$ to the coordinate points $[0, \dots, 0, 1, 0, \dots, 0] \in \mathbb{P}^n$; denote the images of the remaining two points p_{n+2} and p_{n+3} by $[\alpha_0, \dots, \alpha_n]$ and $[\beta_0, \dots, \beta_n]$. We consider maps $\mathbb{P}^1 \to \mathbb{P}^n$ given in terms of an inhomogeneous coordinate z on $\mathbb{P}^1$ by

$$z \mapsto \left[\frac{\alpha_0}{z-\nu_0}, \frac{\alpha_1}{z-\nu_1}, \dots, \frac{\alpha_n}{z-\nu_n}\right]$$

with $\nu_0, \dots, \nu_n$ any distinct scalars, and $\alpha_0, \dots, \alpha_n$ any nonzero scalars. Clearing denominators, we see that the image of such a map is a rational normal curve, and it passes through the $n+1$ coordinate points of $\mathbb{P}^n$, which are the images of the points $z = \nu_0, \dots, \nu_n \in \mathbb{P}^1$. Moreover, the image of the point $z = \infty$ at infinity is the point $[\alpha_0, \dots, \alpha_n]$; and we can adjust the values of $\nu_0, \dots, \nu_n$ so that the image of the point $z = 0$ is $[\beta_0, \dots, \beta_n]$. This proves existence; we'll leave uniqueness as the following exercise. $\square$

Exercise 7.1.4. Show that if $C, C' \subset \mathbb{P}^n$ are two rational normal curves and $\#(C \cap C') \geq n+3$, then $C = C'$. (Hint: use induction on n.)

There is another way to prove Proposition 7.1.3 that may provide more insight (it actually produces the equations defining the rational normal curve through the points $p_1, \dots, p_{n+3}$); this is described in [**?**].

There are also a number of further statements and open problems involving generalizations of this construction. For example, in the statement of Proposition 7.1.3, we can generalize the points $p_1, \dots, p_{n+3} \in \mathbb{P}^n$ to an arbitrary *curvilinear scheme* $\Gamma \subset \mathbb{P}^n$, where by curvilinear scheme we mean a 0-dimensional scheme with Zariski tangent space of dimension at most 1 at every point (equivalently, such that every irreducible component of Γ is isomorphic to $\operatorname{Spec} K[\epsilon]/(\epsilon^k)$ for some k). In this setting the condition of "linear general position" is generalized to the condition that for any hyperplane $H \subset \mathbb{P}^n$ we have $\deg(\Gamma \cap H) \leq n+1$; and it's shown in [**?**] that the statement of Proposition 7.1.3 holds in this greater generality.

For an open problem related to Proposition 7.1.3, let's return to $\mathbb{P}^3$ and suppose $\mathcal{H}^\circ$ is any component of the restricted Hilbert scheme parametrizing curves of degree d and genus g in $\mathbb{P}^3$; say the dimension $\dim \mathcal{H}^\circ = 2m$. A straightforward dimension count then shows that if $p_1, \dots, p_m \in \mathbb{P}^3$ are general points, then there will be a finite number of curves in this component containing the points p_i; Proposition 7.1.3 asserts that in case $\mathcal{H}^\circ$ parametrizes twisted cubics, that number is 1. The question is, are there any other components of the restricted Hilbert scheme for which the number is similarly 1, other than components parametrizing complete intersections of two surfaces of the same degree?

In any case, returning to the case $n = 3$, we see that if $p_1, \dots, p_6 \in \mathbb{P}^3$ are any six points, with no four lying in a plane, then there is a unique twisted cubic containing all six; as promised, we can use this somewhat esoteric fact to deduce the dimension of the Hilbert scheme parametrizing twisted cubics.

Exercise 7.1.5. Consider the incidence correspondence

$$\Phi = \{(p_1, \dots, p_6, C) \in (\mathbb{P}^3)^6 \times \mathcal{H}^\circ \mid p_1, \dots, p_6 \in C\}.$$

Use the result above to show that $\mathcal{H}^\circ$ is irreducible of dimension 12. More generally, use Proposition 7.1.3 to give a second proof of Exercise 7.1.2.

7.1.1 Tangent spaces to Hilbert schemes

As we've said, our descriptions of Hilbert schemes of curves is primarily concerned with issues like the irreducibility and dimension of the restricted Hilbert scheme $\mathcal{H}^\circ$. Nonetheless, it is worth pointing out that we have at least one useful tool for answering questions about the smoothness or singularity of the restricted Hilbert scheme. In practice, it's very often the case that we can describe the Zariski tangent space $T_{[C]}\mathcal{H}^\circ$ to the Hilbert scheme at a point $[C] \in \mathcal{H}^\circ$, via the identification of $T_{[C]}\mathcal{H}^\circ$ with the space $H^0(\mathcal{N}_{C/\mathbb{P}^3})$ of global sections of the normal sheaf of C in $\mathbb{P}^3$, described in Section **??**. In particular, we'll see in Section 8.4 below how to exhibit an everywhere nonreduced component of the restricted Hilbert scheme.

To illustrate how this may go, the following exercise gives a very simple and basic example.

Exercise 7.1.6. Let $C \cong \mathbb{P}^1 \subset \mathbb{P}^3$. Show that the normal bundle $\mathcal{N}_{C/\mathbb{P}^3} \cong \mathcal{O}_{\mathbb{P}^1}(5)^{\oplus 2}$; that is, the normal bundle of a twisted cubic is the direct sum of two line bundles of degree 5. Use this to prove that the restricted Hilbert scheme $\mathcal{H}^\circ$ of twisted cubics is everywhere smooth.

7.1.2 Extraneous components

Although $\mathcal{H}^\circ$ is open in the Hilbert scheme $\mathcal{H} = \mathcal{H}_{3m+1}(\mathbb{P}^3)$, its closure is not all of $\mathcal{H}$! There is a second irreducible component of $\mathcal{H}$, of dimension 15. This is an example of what is called an *extraneous component* of the Hilbert scheme; they are components of the Hilbert scheme whose general point does *not* correspond to a smooth, irreducible nondegenerate curve $C \subset \mathbb{P}^n$. They are the bane of anyone who works with Hilbert schemes; and while choosing to work just with the locus $\mathcal{H}^\circ \subset \mathcal{H}$ means that we won't be dealing with them directly, it's worth describing their behavior in at least the case of twisted cubics.

To start, observe that any plane cubic $C \subset \mathbb{P}^2 \subset \mathbb{P}^3$ has Hilbert polynomial $p(m) = 3m$. If $p \in \mathbb{P}^3 \setminus C$ is any point not on C, then, the union $C' = C \cup \{p\}$ is a subscheme of $\mathbb{P}^3$ with Hilbert polynomial $3m + 1$, and so corresponds to a point of $\mathcal{H}$.

Now, let $\mathcal{H}' \subset \mathcal{H}$ be the open subset corresponding to unions $C' = C \cup \{p\}$ of a plane cubic and a point. By an argument analogous to the one given in [**?**] for plane conics, the Hilbert scheme $\mathcal{H}_{3m}$ is a $\mathbb{P}^9$-bundle over the dual projective space $(\mathbb{P}^3)^*$, and so in particular is irreducible of dimension 12; the locus $\mathcal{H}'$ is then an open subset of the product $\mathcal{H}_{3m} \times \mathbb{P}^3$, and so is irreducible of dimension 15.

Exercise 7.1.7. Show that the Hilbert scheme $\mathcal{H}_{3m+1}$ is indeed the union of the closures of the loci $\mathcal{H}^\circ$ and $\mathcal{H}'$ above (in other words, any subscheme of $\mathbb{P}^3$ with Hilbert polynomial $3m + 1$ is either a flat limit of twisted cubics, or a flat limit of subschemes of the form $C \cup \{p\}$ with C a plane cubic).

Given this, we conclude that the Hilbert scheme $\mathcal{H}_{3m+1}$ consists of two irreducible components: one, the closure of the locus $\mathcal{H}^\circ$ of twisted cubics, which has dimension 12; and a second, the closure of $\mathcal{H}'$, of dimension 15.

One further question: given that the Hilbert scheme $\mathcal{H}_{3m+1}$ consists of two irreducible components, it's natural to ask what their intersection is. The answer is suggested by an example in [**?**, II.3.4], where we take a general twisted cubic $C \subset \mathbb{P}^3$ and apply the family of linear maps $A_t : \mathbb{P}^3 \to \mathbb{P}^3$ given by

$$A_t = \begin{pmatrix} t & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix};$$

we see there that the flat limit $\lim_{t \to 0} A_t(C)$ is a nodal plane cubic, with a spatial embedded point of multiplicity 1 at the node. In fact, the intersection of the two components is exactly the closure of this locus, as the following exercise asks you to show.

Exercise 7.1.8. Show that the locus Σ of schemes X consisting of a nodal plane cubic curve C with a spatial embedded point of multiplicity 1 at the node is dense in the intersection $\overline{\mathcal{H}^\circ} \cap \overline{\mathcal{H}'}$.

Extraneous components in general

While we'll largely ignore the extraneous components of the Hilbert schemes that we'll be dealing with here, it's worth taking a moment out and seeing how they arise, and how numerous they are.

It starts already in dimension 0, actually. Let $\mathcal{H} = \mathcal{H}_d(\mathbb{P}^n)$ be the Hilbert scheme of subschemes of $\mathbb{P}^n$ with Hilbert polynomial the constant d. We have an open subset $\mathcal{H}^\circ \subset \mathcal{H}$ whose points correspond to reduced d-tuples of points in $\mathbb{P}^n$, and this open subset is easy to describe: it's just the complement of the diagonal in the dth symmetric power of $\mathbb{P}^n$. The closure of this open set will be called the *principal component* of $\mathcal{H}$.

You might think this would be all of the Hilbert scheme $\mathcal{H}$, but as the name suggests, it's not in general. Iarrobino in [**?**] first proved for any $n \geq 3$ and any sufficiently large d the existence of components of $\mathcal{H}_d(\mathbb{P}^n)$ having dimension strictly larger than dn—in particular, whose general point corresponded to a nonreduced subscheme of $\mathbb{P}^n$. Other such examples have been found (ref?); in general, no one knows how many irreducible components the Hilbert scheme $\mathcal{H} = \mathcal{H}_d(\mathbb{P}^n)$ has, or what their dimensions might be.

And that in turn infects the Hilbert schemes of curves. For example, if we're looking at the Hilbert scheme $\mathcal{H}_{dm-g+1}$ parametrizing curves of degree d and genus g in $\mathbb{P}^3$, we'll have a component whose general point corresponds to a union of a plane curve of degree d and $\binom{d-1}{2} - g$ points; moreover, if Γ is any irreducible component of the Hilbert scheme of zero-dimensional subschemes of degree $\binom{d-1}{2} - g$ in $\mathbb{P}^3$, there'll be a component of $\mathcal{H}_d(\mathbb{P}^n)$ whose general point corresponds to a union of a plane curve of degree d and the subscheme corresponding to a general point of Γ. And of course we can replace the plane curves in this construction with any component of the Hilbert scheme of curves of degree d and genus $g' > g$; in addition, there may also be components of $\mathcal{H}_{dm-g+1}$ whose general point corresponds to a subscheme of $\mathbb{P}^3$ with an embedded point—we don't know (see the paper by Dawei Chen and Scott Nollet, at https://arxiv.org/abs/0911.2221).

Bottom line, it's a mess. For many g, d the Hilbert scheme $\mathcal{H}_{dm-g+1}(\mathbb{P}^3)$ has many components. In most cases no one knows how many, or what their dimensions are. For that reason, we'll henceforth focus exclusively on the restricted Hilbert scheme, and ignore the extraneous components as much as possible.

7.2 Linkage

As the second proof of Proposition 7.1.1 suggests, when the union of two curves C and D forms a complete intersection we can use this fact to relate the geometry of their respective Hilbert schemes. This is a technique we'll use repeatedly. One thing we need in order to apply it is a formula relating the genera of the curves C and D. This is one aspect of the general theory of *liaison*, or *linkage*, of curves in $\mathbb{P}^3$.

Theorem 7.2.1. *Let $C \subset \mathbb{P}^3$ be a purely 1-dimensionsional subscheme of degree c, and let $S = V(F)$ and $T = V(G)$ be surfaces of degrees s and t containing C and having no common component. If $D \subset \mathbb{P}^3$ is the subscheme defined by $\mathcal{I}_D = (F, G) : \mathcal{I}_C$ then D is purely one-dimensional and $\mathcal{I}_C = (F, G) : \mathcal{I}_D$.*

Furthermore, if we denote by d the degree of D, then we have $c + d = st$ and

$$p_a(C) - p_a(D) = \frac{s+t-4}{2}(c-d); \tag{7.1}$$

In words, the difference between the genera of C and D is proportional to the difference in their degrees, with constant of proportionality $(s + t - 4)/2$.

We will prove Theorem **??** in its full generality in Chapter **??**, using a homological algebra argument. For now, we'll give a simple proof by intersection theory in a case sufficient for our needs in this chapter, and postpone the general proof to Chapter **??**. For this, assume that C and $D \subset \mathbb{P}^3$ are smooth curves of degrees c and d with no common components. Let $S = V(F)$ and $T = V(G)$ be surfaces of degrees s and t respectively, such that that $C \cup D = S \cap T$ is a complete intersection, and assume in addition that S smooth. In this situation, Bézout's Theorem tells us that $c+d = st$; we want a formula relating the genera $g = p_a(C)$ and $h = p_a(D)$ of C and D.

To do this, we work in the Chow ring of S. By adjunction, the canonical divisor class of S is $K_S = (s - 4)H$, where H denotes the hyperplane class on S, so that by adjunction

$$2g - 2 = (C \cdot C) + (K_S \cdot C) = C \cdot C + (s - 4)c,$$

or in other words,

$$(C \cdot C) = 2g - 2 - (s - 4)c.$$

Next, since $C \cup D$ is a complete intersection of S with a surface of degree t, we have $C + D \sim tH$. Thus we have

$$(C \cdot D) = (C \cdot (tH - C)) = tc - (C \cdot C) = tc - 2g + 2 + (s - 4)c$$

and similarly

$$(D \cdot D) = (D \cdot (tH - C)) = td - tc + 2g - 2 - (s - 4)c.$$

Finally, we can apply the adjunction formula to D to arrive at

$$2h - 2 = (D \cdot D) + (K_S \cdot D) = (s - 4)d + td - tc + 2g - 2 - (s - 4)c.$$

Collecting terms, we can write this in the convenient form

$$h - g = \frac{s+t-4}{2}(d-c); \tag{7.2}$$

We will see this formula used repeatedly in this chapter, and as we indicated it will be discussed as part of the larger theory of liaison for space curves in Chapter **??**. For now, you should just take a moment and reassure yourself that the right hand side of (7.2) is indeed an integer!

7.3 Degree 4

By Clifford's Theorem an irreducible nondegenerate curve of degree 4 in $\mathbb{P}^3$ must have genus 0 or 1; we consider these cases in turn.

7.3.1 Genus 0

We can deal with rational quartics by a slight variant of the first method we used to deal with twisted cubics. A rational curve of degree 4 is the image of a map $\phi_F : \mathbb{P}^1 \to \mathbb{P}^3$ given by a four-tuple $F = (F_0, F_1, F_2, F_3)$ with $F_i \in H^0(\mathcal{O}_{\mathbb{P}^1}(4))$. The space of all such four-tuples up to scalars is a projective space of dimension $4 \times 5 - 1 = 19$; let $U \subset \mathbb{P}^{19}$ be the open subset of four-tuples such that the map ϕ is a nondegenerate embedding. We then have a surjective map $\pi : U \to \mathcal{H}^\circ$, whose fiber over a point C is the space of maps with image C. Since any two such maps differ by an automorphism of $\mathbb{P}^1$—that is, an element of PGL_2—the fibers of π are three-dimensional; we conclude that $\mathcal{H}^\circ_{0,3,4}$ *is irreducible of dimension 16.*

The same analysis can be used on rational curves of any degree d: the space U of nondegenerate embeddings $\mathbb{P}^1 \to \mathbb{P}^3$ of degree d is an open subset of the projective space $\mathbb{P}^{4(d+1)-1}$ of four-tuples of homogeneous polynomials of degree d on $\mathbb{P}^1$ modulo scalars; and the fibers of the corresponding map $U \to \mathcal{H}^\circ_{dm+1}$ are copies of PGL_2. This yields the

Proposition 7.3.1. *The open set $\mathcal{H}^\circ \subset \mathcal{H}_{0,3,d}$ parametrizing smooth, irreducible nondegenerate rational curves $C \subset \mathbb{P}^3$ is irreducible of dimension $4d$.*

Exercise 7.3.2. Give an argument for Proposition 7.3.1 in case $d = 4$ using linkage.

One further remark. Following our discussion of twisted cubics, we were able to see in Exercise 7.1.6 that the restricted Hilbert scheme of twisted cubics is smooth by identifying the normal bundle of a twisted cubic and determining the dimension of its space of global sections. In fact, the same is true for rational curves of any degree, as the following exercise shows.

Exercise 7.3.3. Let $C \cong \mathbb{P}^1 \subset \mathbb{P}^3$ be a smooth rational curve of any degree d.

1. Show that $h^1(\mathcal{N}_{C/\mathbb{P}^3}) = 0$; that is, the normal bundle of C is nonspecial.
2. Using this, the Riemann-Roch formula for vector bundles on a curve and Proposition 7.3.1, show that the Hilbert scheme $\mathcal{H}$ is smooth at the point $[C]$.

We should point out that, in contrast to the case of twisted cubics, smooth rational curves in $\mathbb{P}^r$ of the same degree may have different normal bundles. This gives an interesting stratification of the restricted Hilbert scheme of rational curves; see [**?**] for a discussion.

7.3.2 Genus 1

As we saw in Section **??**, a quartic curve $C \subset \mathbb{P}^3$ of genus 1 is the intersection of two quadric surfaces, and by Lasker's theorem, every quadric containing C is a linear combination of those two. Conversely, the intersection of two general quadrics in $\mathbb{P}^3$ is a quartic curve of genus 1. We can thus construct a family of quartics of genus 1: let $V = H^0(\mathcal{O}_{\mathbb{P}^3}(2))$ be the 10-dimensional vector space of homogeneous quadric polynomials in $\mathbb{P}^3$ and $G(2, V)$ the Grassmannian of 2-planes in V, and consider the incidence correspondence

$$\Gamma = \{(\Lambda, p) \in G(2, V) \times \mathbb{P}^3 \mid F(p) = 0 \; \forall \; F \in \Lambda\}.$$

The fiber of Γ over a point $\Lambda \in G(2, V)$ is thus the base locus of the pencil of quadrics represented by Λ; let $B \subset G(2, V)$ be the Zariski open subset over which the fiber is smooth, irreducible and nondegenerate of dimension 1. By the universal property of Hilbert schemes, the family $\pi_1 : \Gamma_B \to U$ induces a map $\phi : B \to \mathcal{H}^\circ$ that is one-to-one on points; it follows that the reduced subscheme of $\mathcal{H}^\circ$ is birational to an open subset of the Grassmannian $G(2, 10)$, and we conclude that $\mathcal{H}^\circ_{1,3,4}$ *is irreducible of dimension 16.* Exercise 7.3.4 shows that this map is actually an isomorphism.

Exercise 7.3.4. Let $C = Q \cap Q' \subset \mathbb{P}^3$ be a smooth curve of degree 4 and genus 1. Identify the normal bundle $\mathcal{N}_{C/\mathbb{P}^3}$ of C, and use this to conclude that $\mathcal{H}^\circ_{1,3,4}$ is itself reduced, and even smooth, and thus isomorphic to an open subset of the Grassmannian $G(2, 10)$.

The argument here—where we constructed a family $\mathcal{C} \to B$ of curves of given type, and then invoked the universal property of the Hilbert scheme to get a map $B \to \mathcal{H}$ is typical in analyses of Hilbert schemes. Here here are two slightly more general cases:

Exercise 7.3.5. Let $m \geq n > 0$ be two positive integers. Show that the locus $U_{n,m} \subset \mathcal{H}^\circ$ of curves $C \subset \mathbb{P}^3$ that are smooth complete intersections of surfaces of degrees n and m is an open subset of the Hilbert scheme.

Exercise 7.3.6. Consider the locus $U_{n,n} \subset \mathcal{H}^\circ$ of curves $C \subset \mathbb{P}^3$ that are smooth complete intersections of two surfaces of degrees n. Show that $U_{n,n}$ is isomorphic to an open subset of the Grassmannian $G(2, H^0(\mathcal{O}_{\mathbb{P}^3}(n)))$.

7.4 Degree 5

Let $C \subset \mathbb{P}^3$ be a smooth, irreducible, nondegenerate quintic curve of genus g. By Clifford's theorem the bundle $\mathcal{O}_C(1)$ must be nonspecial, so by the Riemann-Roch theorem we must have $0 \leq g \leq 2$. We have already seen that the space $\mathcal{H}^\circ_{5m+1}$ of rational quintic curves is irreducible of dimension 20. We will treat the case $g = 2$ in detail, and leave the case $g = 1$ as an exercise. This case will be covered in a different way in Section 7.6.

7.4.1 Genus 2

We have considered curves of genus 2 in Section **??**. To recap the analysis, let $C \subset \mathbb{P}^3$ be a smooth, irreducible, nondegenerate curve of degree 5 and genus 2. By the Riemann-Roch theorem, $h^0(\mathcal{O}_C(2)) = 10 - 2 + 1 = 9 < h^0(\mathcal{O}_{\mathbb{P}^3}(2)) = 10$ so the restriction map

$$H^0(\mathcal{O}_{\mathbb{P}^3}(2)) \to H^0(\mathcal{O}_C(2))$$

has a kernel. Since $\deg C = 5 > 2 \times 2$, the curve C cannot lie on two independent quadrics; thus C lies on a unique quadric surface Q. Similarly, the restriction map

$$H^0(\mathcal{O}_{\mathbb{P}^3}(3)) \to H^0(\mathcal{O}_C(3))$$

has at least a 6-dimensional kernel; since cubics of the form LQ span only a 4-dimensional space, we see that C lies on a cubic surface S not containing Q. The intersection $Q \cap S$ has degree 6, and is thus the union of C and a line. If Q is smooth then, in terms of the isomorphism $Q \cong \mathbb{P}^1 \times \mathbb{P}^1$, we can say C is a curve of type $(2,3)$ on the quadric Q. Note that conversely if $L \subset \mathbb{P}^3$ is a line and Q and $S \subset \mathbb{P}^3$ are general quadric and cubic surfaces containing L, and if we write

$$Q \cap S = L \cup C$$

then the curve C is a curve of type $(2,3)$ on the quadric Q and hence, by the adjunction formula, a quintic of genus 2.

This suggests two ways of describing the family $\mathcal{H}^\circ \subset \mathcal{H}_{5m-1}$ of such curves. First, we can use the fact that C is linked to a line to make an incidence correspondence

$$\Psi = \{(C, L, Q, S) \in \mathcal{H}^\circ \times \mathbb{G}(1,3) \times \mathbb{P}^9 \times \mathbb{P}^{19} \mid Q \cap S = C \cup L\},$$

where the $\mathbb{P}^9$ (respectively, $\mathbb{P}^{19}$) is the space of quadric (respectively, cubic) surfaces in $\mathbb{P}^3$. Given a line $L \in \mathbb{G}(1,3)$, the space of quadrics containing L is a $\mathbb{P}^6$, and the space of cubics containing L is a $\mathbb{P}^{15}$; thus the fiber of the projection $\pi_2 : \Psi \to \mathbb{G}(1,3)$ over L is an open subset of $\mathbb{P}^6 \times \mathbb{P}^{15}$, and we see that Ψ *is irreducible of dimension* $4 + 6 + 15 = 25$.

On the other hand, the fiber of Ψ over a point $C \in \mathcal{H}^\circ$ is an open subset of the $\mathbb{P}^5$ of cubics containing C; and we conclude that $\mathcal{H}^\circ$ *is irreducible of dimension* 20.

Exercise 7.4.1. Let $C \subset \mathbb{P}^3$ be a smooth curve of degree 5 and genus 2, and assume that the quadric surface Q containing C is smooth. From the exact sequence

$$0 \to \mathcal{N}_{C/Q} \to \mathcal{N}_{C/\mathbb{P}^3} \to \mathcal{N}_{Q/\mathbb{P}^3}|_C \to 0,$$

calculate $h^0()$ and deduce that $\mathcal{H}^\circ_{2,3,5}$ *is smooth at the point* $[C]$. Does this conclusion still hold if Q is singular?

Another, in some ways more direct, approach to describing the restricted Hilbert scheme $\mathcal{H}^\circ_{2,3,5}$ would be to use the fact that the quadric surface Q containing a quintic curve $C \subset \mathbb{P}^3$ of genus 2 is unique. We thus have a map

$$\mathcal{H}^\circ \to \mathbb{P}^9,$$

whose fiber over a point $Q \in \mathbb{P}^9$ is the space of quintic curves of genus 2 on Q.

The problem is, the space of quintic curves of genus 2 on a given quadric Q is not in general irreducible: for a general, and thus smooth quadric Q it consists of the disjoint union of the open subsets of smooth elements in the two linear series of curves of type $(2,3)$ and $(3,2)$ on Q, each of which is a $\mathbb{P}^{11}$. We can conclude immediately that $\mathcal{H}^\circ$ is of pure dimension 20; but to conclude that it is irreducible we need to verify that, in the family of all smooth quadric surfaces, the monodromy exchanges the two rulings.
((refer to the place—earlier—where monodromy is discussed, and say this follows from the irreducibility of an appropriately modified incidence correspondence. Do this example where the monodromy if first discussed, too.))
This is not hard: it amounts to the assertion that the family

$$\Gamma = \{(Q, L) \in \mathbb{P}^9 \times \mathbb{G}(1,3) \mid L \subset Q\}$$

is irreducible, which can be seen via projection on the second factor.

Exercise 7.4.2. Show that a smooth, irreducible, nondegenerate curve $C \subset \mathbb{P}^3$ of degree 5 and genus 1 is residual to a rational quartic in the complete intersection of two cubics, and use the result of subsection 7.3.1 to deduce that the space of genus 1 quintics is irreducible of dimension 20.

7.5 Degree 6

Again the Clifford and Riemann-Roch theorems suffice to compute the possible genera of a curve of degree 6. To start with, if the line bundle $\mathcal{O}_C(1)$ is nonspecial, then by the Riemann-Roch theorem we have $g \le 3$. Suppose on the other hand that $\mathcal{O}_C(1)$ is special. Since $h^0(\mathcal{O}_C(1)) \ge 4$, we have equality in Clifford's theorem, and either C is hyperelliptic and $\mathcal{O}_C(1)$ is a multiple of the g^1_2 or C is a canonically embedded curve of genus 4. The first case cannot occur, since no special multiple of the hyperelliptic series of degree $\le 2g-2$ can be very ample; thus C must be a canonical curve of genus 4. In sum, by applying Clifford's Theorem and the Riemann-Roch Theorem, we see that a smooth irreducible, nondegenerate curve of degree 6 in $\mathbb{P}^3$ has genus at most 4.

Exercise 7.5.1.
1. Show that all genera $g \le 4$ do occur; that is, there exists a smooth irreducible, nondegenerate curve of degree 6 and genus g in $\mathbb{P}^3$ for all $g \le 4$.

2. What is the largest possible genus of a smooth irreducible, nondegenerate curve $C \subset \mathbb{P}^3$ of degree $d = 7$? Can you do this with Clifford and Riemann-Roch, or do you need to invoke Castelnuovo?

The cases of genera 0, 1 and 2 are covered under Proposition 7.6.1, leaving us the cases $g = 3$ and 4. Both are well-handled by the Cartesian approach of describing their ideals.

7.5.1 Genus 4

As we've seen in Section?? a canonical curve of genus 4 is the complete intersection of a (unique) quadric Q and a cubic surface S. We thus have a map

$$\alpha : \mathcal{H}^\circ \longrightarrow \mathbb{P}^9$$

sending a curve C to the quadric Q containing it. Moreover, the fibers of this map are open subsets of the projective space $\mathbb{P}V$, where V is the quotient

$$V = \frac{H^0(\mathcal{O}_{\mathbb{P}^3}(3))}{H^0(\mathcal{I}_{Q/\mathbb{P}^3}(3))}$$

of the space of all cubic polynomials modulo cubics containing Q. Since this vector space has dimension 16, the fibers of α are irreducible of dimension 15, and we deduce that *the space* $\mathcal{H}^\circ_{6m-3}$ *is irreducible of dimension 24.*

In fact, Exercise 7.3.6 can be generalized in this way to smooth complete intersections of surfaces of any degree:

Exercise 7.5.2. As before, let $U_{n,m} \subset \mathcal{H}^\circ$ be the locus of curves $C \subset \mathbb{P}^3$ that are smooth complete intersections of surfaces of degrees n and m. In case $m > n$, show that $U_{m,n}$ is isomorphic to an open subset of a projective bundle over the projective space $\mathbb{P}(H^0(\mathcal{O}_{\mathbb{P}^3}(n))) \cong \mathbb{P}^{\binom{n+3}{3}-1}$ of surfaces of degree n, with fiber over the point $[S] \in \mathbb{P}(H^0(\mathcal{O}_{\mathbb{P}^3}(n)))$ the projective space $\mathbb{P}(H^0(\mathcal{O}_{\mathbb{P}^3}(m))/H^0(\mathcal{I}_{S/\mathbb{P}^3}(m)) \cong \mathbb{P}^{\binom{m+3}{3}-\binom{m-n+3}{3}-1}$.

7.5.2 Genus 3

We leave this to the reader to complete as follows:

Exercise 7.5.3. Let C be a curve of degree 6 and genus 3, and assume that C does not lie on any quadric surface. Show that C is residual to a twisted cubic in the complete intersection of two cubic surfaces, and use this to deduce that the space of such curves is irreducible of dimension 24.

Exercise 7.5.4. Now let C again be a curve of degree 6 and genus 3, but now assume that C *does* lie on a quadric surface Q. Show that such a curve is a flat limit of curves of the type described in the last exercise, and conclude that $\mathcal{H}^\circ_{3,3,6}$ is irreducible of dimension 24. (Hint: Let L, Q and F denote a general linear form, a general quadratic form and a general cubic form, and consider the pencil of surfaces $S_t = V(tF + LQ) \subset \mathbb{P}^3$ specializing from the cubic surface $V(F)$ the to reducible cubic $V(LQ)$.)

7.6 Why $4d$?

The sharp-eyed reader will have noticed that, in every case analyzed so far, the Hilbert scheme parametrizing smooth curves of degree d and genus g in $\mathbb{P}^3$ has dimension $4d$. While this is not the case in general (we will see shortly an example where it fails), $4d$ is indeed the "expected dimension" from certain points of view. In the following subsections we'll describe two such computations. For the remainder of this section, we will step outside $\mathbb{P}^3$ and consider, more generally, the restricted Hilbert scheme $\mathcal{H}^\circ$ of smooth, irreducible, nondegenerate curves in $\mathbb{P}^r$.

7.6.1 Estimating $\dim \mathcal{H}^\circ$ by Brill-Noether

One method of estimating the dimension of $\mathcal{H}^\circ$ is a generalization of the proof of Proposition 7.3.1, with two additional wrinkles: First, since not all line bundles of degree d on a curve C of genus $g > 0$ are linearly equivalent, we must invoke the Picard variety $\mathrm{Pic}_d(C)$ parametrizing line bundles of degree d on a given curve C, discussed in Chapter 3. Second, since not all curves of genus $g > 0$ are isomorphic, we must involve the moduli space M_g parametrizing abstract curves of genus g, discussed in Chapter **??**.

To begin with a simple example, let $\mathcal{H}^\circ$ again be the space of smooth, irreducible, nondegenerate curves $C \subset \mathbb{P}^3$ of degree 5 and genus 2. By the property of M_2 as a coarse moduli space, we get a map

$$\mu : \mathcal{H}^\circ \longrightarrow M_2.$$

To analyze the fiber $\Sigma_C = \mu^{-1}(C)$ of the map μ over a point $C \in M_2$ we first use the map

$$\nu : \Sigma_C \longrightarrow \mathrm{Pic}_5(C),$$

obtained by sending a point in Σ_C to the line bundle $\mathcal{O}_C(1)$. Proposition **??**, implies that any line bundle of degree 5 on a curve of genus 2 is very ample, so this map is surjective. Note that $h^0(\mathcal{L}) = 4$, so the linear series giving the embedding is complete. Thus, once we have specified the abstract curve C, and the line bundle $\mathcal{L} \in \mathrm{Pic}_5(C)$ the embedding is determined by giving a basis for $H^0(\mathcal{L})$, up to scalars. In other words, each fiber of ν is isomorphic to PGL_4. We can now work our way up from M_2:

- We know that M_2 is irreducible of dimension 3.

- It follows that the space of pairs $(C, \mathcal{L})$ with $C \in M_2$ a smooth curve of genus 2 and $\mathcal{L} \in \mathrm{Pic}_5(C)$ is irreducible of dimension $3 + 2 = 5$; and finally
- It follows that $\mathcal{H}^\circ$ is irreducible of dimension $5 + 15 = 20$.

In fact, this approach applies to a much wider range of examples: whenever $d \geq 2g + 1$ and $r \leq d - g$, we can look at the tower of spaces

$$\begin{tikzcd} \mathcal{H}^\circ = \mathcal{H}^\circ_{dm-g+1}(\mathbb{P}^r) \arrow[d] \\ \mathcal{P}_{d,g} = \{(C, \mathcal{L}) \mid \mathcal{L} \in \mathrm{Pic}_d(C)\} \arrow[d] \\ M_g. \end{tikzcd}$$

Exactly as in the special case $(d, g, r) = (5, 2, 3)$ above, we can work our way up the tower:

- M_g is irreducible of dimension $3g - 3$;
- it follows from the fact that the Picard variety is irreducible of dimension g that $\mathcal{P}_{d,g}$ is irreducible of dimension $3g - 3 + g = 4g - 3$; and finally
- since the fibers of $\mathcal{H}^\circ \to \mathcal{P}_{d,g}$ consist of $(r+1)$-tuples of linearly independent sections of $\mathcal{L}$ (mod scalars), and $h^0(\mathcal{L}) = d - g + 1$, it follows that $\mathcal{H}^\circ$ is irreducible of dimension $\dim(\mathcal{P}_{d,g}) + (r+1)(d-g+1) = 4g - 3 + (r+1)(d-g+1) - 1$.

In sum, we have the

Proposition 7.6.1. *Whenever $d \geq 2g+1$, the space $\mathcal{H}^\circ$ of smooth, irreducible, nondegenerate curves $C \subset \mathbb{P}^r$ is either empty (if $d - g < r$) or irreducible of dimension $4g - 3 + (r+1)(d-g+1) - 1$; in particular, if $r = 3$, the dimension of $\mathcal{H}^\circ$ is $4d$.*

Exercise 7.6.2. By analyzing the geometry of linear series of degrees $2g - 1$ and $2g$ on a curve of genus g, extend Proposition 7.6.1 to the cases $d = 2g - 1$ and $2g$. What goes wrong if $d \leq 2g - 2$?

Proposition 7.6.1 gives a simple and clean answer to our basic questions about the dimension and irreducibility of the restricted Hilbert scheme $\mathcal{H}^\circ$ in case $d \geq 2g - 1$. But what happens outside of this range? In fact, we can use Brill-Noether theory to modify this analysis to extend this beyond the range $d \geq 2g + 1$.

Basically, what's different in general is that the map $\mathcal{H}^\circ \to \mathcal{P}_{d,g}$ is no longer dominant; rather, over a point $[C] \in M_g$, its image is open in the subvariety $W^r_d(C) \subset \mathrm{Pic}_d(C)$ parametrizing line bundles $\mathcal{L}$ on C of degree d with at least $r+1$ sections. Now, as long as the Brill-Noether number $\rho(d, g, r)$ is nonnegative, the Brill-Noether theorem tells us that for a general curve C, the variety $W^r_d(C)$ has dimension ρ, and (assuming $r \geq 3$) the general point of $W^r_d(C)$ corresponds to a very ample line bundle with exactly $r+1$ sections. In this situation, there is a unique component of $\mathcal{H}_0 \subset \mathcal{H}^\circ$ dominating M_g, and the map $\mathcal{H}^\circ \to \mathcal{P}_{d,g}$ carries this component to a subvariety $\mathcal{W}^r_d \subset \mathcal{P}_{d,g}$ of dimension $3g - 3 + \rho$. In sum, then, we have the basic theorem

Theorem 7.6.3. *Let g, d and r be any nonnegative integers, with Brill-Noether number $\rho(g, r, d) = g - (r+1)(g - d + r) \geq 0$. There is then a unique component $\mathcal{H}_0$ of the restricted Hilbert scheme $\mathcal{H}^\circ_{g,r,d}$ dominating the moduli space M_g; and this component has dimension*

$$\dim \mathcal{H}_0 = 3g - 3 + \rho + (r+1)^2 - 1 = 4g - 3 + (r+1)(d - g + 1) - 1.$$

The component $\mathcal{H}_0$ identified in Theorem 7.6.3 is called the *principal component* of the Hilbert scheme; there may be others as well, of possibly different dimension, and we do not know precisely for which d, g and r these occur. Finally, in case $\rho < 0$, the Brill-Noether theorem tells us only that there is no component of $\mathcal{H}^\circ_{g,r,d}$ dominating M_g; we'll discuss some of the outstanding questions in this range in Section 8.5 below.

7.6.2 Estimating $\dim \mathcal{H}^\circ$ by the Euler characteristic of the normal bundle

It is interesting to compare the estimate of $\dim \mathcal{H}^\circ$ above with what we get from deformation theory. Let $\mathcal{H}$ be a component of the scheme $\mathcal{H}^\circ$, with $C \subset \mathbb{P}^r$ a curve corresponding to a general point $[C]$ of $\mathcal{H}$.

We start with the idea that the dimension of the scheme $\mathcal{H}$ is approximated by the dimension of its Zariski tangent space $T_{[C]}\mathcal{H}$ at a general point $[C]$. In Section **??** we saw that the tangent space to $\mathcal{H}$ at $[C]$ is the space $H^0(\mathcal{N}_{C/\mathbb{P}^r})$ of global sections of the normal bundle $\mathcal{N} = \mathcal{N}_{C/\mathbb{P}^r}$. We can think of the dimension $h^0(\mathcal{N})$ as approximated by the Euler characteristic $\chi(\mathcal{N})$, with "error term" $h^1(\mathcal{N})$ coming from its first cohomology group.

Given these two approximations, we arrive at a number we can compute. From the exact sequence

$$0 \to T_C \to T_{\mathbb{P}^r}|_C \to \mathcal{N} \to 0$$

we deduce that

$$\begin{aligned} c_1(\mathcal{N}) &= c_1(T_{\mathbb{P}^r}|_C) - c_1(T_C) \\ &= (r+1)d - (2-2g). \end{aligned}$$

Now we can apply the Riemann-Roch Theorem for vector bundles on curves ([**?**, Theorem ???]) to conclude that

$$\begin{aligned} \chi(\mathcal{N}) &= c_1(\mathcal{N}) - \operatorname{rank}(\mathcal{N})(g-1) \\ &= (r+1)d - (r-3)(g-1). \end{aligned}$$

Note that our two "estimates" are actually inequalities. But, unfortunately, they go in opposite directions: we have

$$\dim \mathcal{H} \leq \dim T_{[C]}\mathcal{H},$$

but

$$\dim T_{[C]}\mathcal{H} \geq \chi(\mathcal{N}).$$

Nonetheless, one can show that if $C \subset \mathbb{P}^r$ is a smooth curve then the versal deformation space of $C \subset \mathbb{P}^r$ has dimension at least $\chi(\mathcal{N})$. If we consider the family of Picard varieties over the family of smooth curves in a neighborhood of C and we can deduce that for any component of $\mathcal{H}^\circ$ containing C we have

$$\dim \mathcal{H}^\circ \ \geq \ (r+1)d - (r-3)(g-1)$$

7.6.3 They're the same!

Proposition 7.6.1 suggests that the "expected dimension" of the restricted Hilbert scheme $\mathcal{H}^\circ$ of curves of degree d and genus g in $\mathbb{P}^r$ should be

$$h(g,r,d) := 4g - 3 + (r+1)(d-g+1) - 1.$$

But the calculation immediately above suggests it should be $(r+1)d - (r-3)(g-1)$. Which is it? The answer is both: they're the same number!

Chapter 8

Hilbert Schemes II: Counterexamples

In the preceding chapter, we described a number of examples of Hilbert schemes, and observed some patterns in their behavior: in each case the restricted Hilbert scheme $\mathcal{H}^\circ$ parametrizing smooth, irreducible and nondegenerate curves was irreducible of the "expected dimension" $h(g,r,d) := 4g-3+(r+1)(d-g+1)-1$. In fact, Theorem 7.6.3 tells us that these patterns persist, for those components of $\mathcal{H}^\circ$ dominating the moduli space M_g.

But what about other components of the Hilbert scheme—components with $\rho(g,r,d) < 0$, or for that matter components with $\rho(g,r,d) \geq 0$ that simply don't dominate M_g? In fact, none of the patterns we've observed so far hold in general, and the first thing we'll do in this chapter is to give some examples, culminating with Mumford's celebrated example of a component of the restricted Hilbert scheme that is everywhere non-reduced.

We will close the chapter by discussing some intriguing conjectures suggested by Brill-Noether theory and by observed behavior in small cases.

8.1 Degree 8

We start with an example of a component of the restricted Hilbert scheme $\mathcal{H}^\circ$ whose dimension is strictly greater than $h(g,r,d)$, the space $\mathcal{H}^\circ = \mathcal{H}^\circ_{9,3,8}$ of smooth, irreducible, nondegenerate curves of degree 8 and genus 9. Let C be such a curve, and consider the restriction map

$$\rho_2 : H^0(\mathcal{O}_{\mathbb{P}^3}(2)) \longrightarrow H^0(\mathcal{O}_C(2)).$$

The source of ρ_2 has dimension 10, but the Riemann-Roch Theorem

$$h^0(\mathcal{O}_C(2)) = \begin{cases} 9, & \text{if } \mathcal{O}_C(2) \cong K_C; \\ 8, & \text{if } \mathcal{O}_C(2) \not\cong K_C \end{cases}$$

admits two possibilities for the dimension of target of ρ_2. However, if $h^0(\mathcal{O}_C(2))$ were 8 then C would lie on two distinct quadrics Q and Q'. Since C is non-degenerate, it cannot lie on any irreducible quadrics; thus Q and Q' would have to be irreducible, which would violate Bézout's Theorem. We deduce that $\mathcal{O}_C(2) \cong K_C$, and thus that C lies on a unique quadric surface Q (which must be irreducible since C is irreducible and doesn't lie on a plane).

Similarly, C cannot lie on any cubic not containing Q. Moving on to quartics, we look again at the restriction map

$$\rho_4 : H^0(\mathcal{O}_{\mathbb{P}^3}(4)) \longrightarrow H^0(\mathcal{O}_C(4)).$$

The dimensions here are, respectively, 35 and $4 \cdot 8 - 9 + 1 = 24$; and we deduce that C lies on at least an 11-dimensional vector space of quartic surfaces. On the other hand, only a 10-dimensional vector subspace of these vanish on Q; and so we conclude that C *lies on a quartic surface not containing* Q. It follows from Bézout's Theorem that $C = Q \cap S$. By Lasker's Theorem, the ideal (Q, S) is saturated, so it is equal to the homogeneous ideal of C. Thus $\ker(\rho_4)$ has dimension exactly 11, and S is unique modulo quartics vanishing on Q.

From these facts it is easy to compute the dimension of $\mathcal{H}^\circ$. This is a special case of Exercise 7.5.2, but just to say it: associating to C the unique quadric on which it lies gives a map $\mathcal{H}^\circ \to \mathbb{P}^9$ with dense image, and each fiber is an open subset of the projective space $\mathbb{P}V$, where V is the 25-dimensional vector space

$$V = \frac{H^0(\mathcal{O}_{\mathbb{P}^3}(4))}{H^0(\mathcal{I}_{Q/\mathbb{P}^3}(4))}.$$

It follows that *the space* $\mathcal{H}^\circ_{8m-8}(\mathbb{P}^3)$ *is irreducible of dimension 33*—one larger than the "expected" $4d$.

8.2 Degree 9

For the next example, consider the space $\mathcal{H}^\circ = \mathcal{H}^\circ_{9m-9}(\mathbb{P}^3)$ of curves of degree 9 and genus 10. Once more, to describe such a curve C, we look to the restriction maps $\rho_m : H^0(\mathcal{O}_{\mathbb{P}^3}(m)) \longrightarrow H^0(\mathcal{O}_C(m))$. The Riemann-Roch Theorem tells us that

$$h^0(\mathcal{O}_C(2)) = \begin{cases} 10, & \text{if } \mathcal{O}_C(2) \cong K_C \text{ ("the first case,") and} \\ 9, & \text{if } \mathcal{O}_C(2) \not\cong K_C \text{ ("the second case.")} \end{cases}$$

Unlike the the situation in degree 8, both are possible; we'll analyze each.

1. Suppose first that C does not lie on any quadric surface (so that we are necessarily in the first case above), and consider the map $\rho_3 : H^0(\mathcal{O}_{\mathbb{P}^3}(3)) \to H^0(\mathcal{O}_C(3))$. By the Riemann-Roch Theorem, the dimension of the target is $3 \cdot 9 - 10 + 1 = 18$, from which we conclude that C lies on at least a pencil of cubic surfaces. Since C lies on no quadrics, all of these cubic surfaces must be irreducible, and it follows by Bézout's Theorem that the intersection of two such surfaces is exactly C. At this point, Lasker's Theorem assures us that C lies on exactly two cubics.

By Exercise 7.3.6, then, the space $\mathcal{H}_1^\circ$ of curves of this type is thus an open subset of the Grassmannian $G(2, 20)$ of pencils of cubic surfaces, which is irreducible of dimension 36.

2. Next, suppose that C does lie on a quadric surface $Q \subset \mathbb{P}^3$; let $\mathcal{H}_2^\circ \subset \mathcal{H}^\circ$ be the locus of such curves. In this case, we claim two things:

a. Q must be smooth; and

b. C must be a curve of type $(3, 6)$ on Q

For part (a), we claim that in fact *a smooth, irreducible nondegenerate curve C of degree 9 lying on a singular quadric must have genus 12.* We can see this by observing that Q must be a cone over a smooth conic curve, and so its blow-up at the vertex is the Hirzebruch surface $\mathbb{F}_2$, with the directrix $E \subset \mathbb{F}_2$ the exceptional divisor of the blowup, and a line L of the ruling of $\mathbb{F}_2$ the proper transform of a line lying on Q. The pullback to $\mathbb{F}_2$ of the hyperplane class has intersection number 1 with L and 0 with E, from which it follows that its class must be $H = 2L + E$

Now, the proper transform $\tilde{C}$ of C in $\mathbb{F}_2$ has intersection number 1 with E, since C passes through the vertex of Q and is smooth there; given this, and the fact that it has intersection number 9 with $H = 2L + E$, we can deduce that the class of $\tilde{C}$ is $9L + 4E$. Now, we know that $K_{\mathbb{F}_2} = -2E - 4L$; by adjunction we deduce that the genus of C is 12.

For the second part, once we know that Q is smooth, the genus formula on Q tells us immediately that C must be of type $(3, 6)$ or $(6, 3)$. Now, since the quadric Q containing C is unique, by Bézout, we have a map $\mathcal{H}_2^\circ \to \mathbb{P}^9$ associating to each curve C of this type the unique quadric containing it. The fiber of this map over a given quadric Q is the disjoint union of open subsets of the projective spaces $\mathbb{P}^{27}$ parametrizing curves of type $(3, 6)$ and $(6, 3)$ on Q, and we see that the locus $\mathcal{H}_2^\circ$ again has dimension 36.

Exercise 8.2.1. While the above argument does not prove that the locus $\mathcal{H}_2^\circ$ is irreducible (in the absence of a monodromy argument), we can see that it's irreducible via a liaison argument: we're saying that a curve C of the second type is residual to a union of three skew lines in the intersection of a quadric and a sextic curve. Carry out this argument to establish that $\mathcal{H}_2^\circ$ is indeed irreducible.

In sum, there are two types of smooth, irreducible, nondegenerate curves $C \subset \mathbb{P}^3$ of degree 9 and genus 10: type 1, which are complete intersections of two cubics; and type 2, which are curves of type $(3,6)$ on a quadric surface. Moreover, the family of curves of each type is irreducible of dimension 36; and we conclude that *the space $\mathcal{H}^\circ_{9m-9}(\mathbb{P}^3)$ is reducible, with two components of dimension 36.*

Exercise 8.2.2. In the preceding argument, we used a dimension count to conclude that a general curve of type 1 could not be a specialization of a curve of type 2, and vice versa. Prove these assertions directly: specifically, argue that

1. by upper-semicontinuity of $h^0(\mathcal{I}_{C/\mathbb{P}^3}(2))$, argue that a curve C not lying on a quadric cannot be the specialization of curves C_t lying on quadrics; and

2. show that for a general curve of type $(3,6)$ on a quadric, $K_C \not\cong \mathcal{O}_C(2)$, and deduce that a general curve of type 2 is not a specialization of curves of type 1.

Exercise 8.2.3. Let Σ_1 and $\Sigma_2 \subset \mathcal{H}^\circ_{9m-9}(\mathbb{P}^3)$ be the loci of curves of types 1 and 2 respectively.

1. What is the intersection of the closures of Σ_1 and Σ_2 in $\mathcal{H}^\circ_{9m-9}(\mathbb{P}^3)$?

2. What is the intersection of the closures of Σ_1 and Σ_2 in the whole Hilbert scheme $\mathcal{H}_{9m-9}(\mathbb{P}^3)$?

8.3 Special components in the nonspecial range

If we ignore the finer points of the Brill-Noether theorem and focus just on the statement about the dimension and irreducibility of the variety of linear series on a curve, we can express it in a simple form: according to Theorem 7.6.3 *Any component of the restricted Hilbert scheme $\mathcal{H}^\circ$ of curves of degree d and genus g that dominates the moduli space M_g has the expected dimension*

$$h(g,r,d) = 4g-3+(r+1)(d-g+1)-1 = (r+1)d-(r-3)(g-1)$$

as calculated in Section 7.6 above; and, in the Brill-Noether range (that is, when the Brill-Noether number $\rho(g,r,d) \geq 0$ is nonnegative), there exists a unique such component.

If we restrict further to the nonspecial range $d \geq g+r$, we don't need the ghosts of Brill or Noether to tell us this: if $\mathcal{L}$ is a general line bundle of degree d on a general curve C of genus g, and $V \subset H^0(\mathcal{L})$ a general $(r+1)$-dimensional subspace, the linear system V will embed the curve C as a nondegenerate curve of degree d in $\mathbb{P}^r$, and the curve obtained in this way will comprise an irreducible component of the restricted Hilbert scheme.

But that doesn't mean that there aren't other components of the restricted Hilbert scheme, even in the nonspecial range! In this section, we'll construct an example of this: a component of the restricted Hilbert scheme $\mathcal{H}^\circ_{g,r,d}$, with $d \geq g + r$, that does not dominate M_g and indeed has the wrong dimension.

For our example, we'll take $d = 28$, $g = 21$ and $r = 7$. Again, a general line bundle $\mathcal{L}$ of degree 28 on a general curve C of genus 21 will be very ample (we could invoke the Brill-Noether theorem for this, but it follows from the more elementary argument for Theorem **??**). Curves of genus 21 embedded in $\mathbb{P}^7$ in this way comprise a component $\mathcal{H}_0$ of the Hilbert scheme $\mathcal{H}^\circ_{21,7,28}$ having the expected dimension

$$h(21, 7, 28) = 4g - 3 + (r + 1)(d - g + 1) - 1 = 144.$$

But here's another way to construct a curve of degree 28 and genus 21 in $\mathbb{P}^7$, that will produce a larger family of such curves! To start with, let's restrict to the trigonal locus in M_{21}; that is, we'll assume the curve C is trigonal. (This immediately cuts down on our degrees of freedom, but we'll make up for it in the choice of linear system.)

We now want to look at the line bundle residual to 4 times the g^1_3 on C; that is, if $\mathcal{M}$ is the line bundle of degree 3 on C having two sections, we take $\mathcal{L} = K_C \otimes \mathcal{M}^{-4}$. We first need to calculate the dimension of the space of sections of $\mathcal{L}$, and to show that this bundle is in fact very ample; these will be special cases of the following lemma.

Lemma 8.3.1. *Let C be a general trigonal curve of genus g, $\mathcal{M}$ the line bundle of degree 3 on C having two sections, and $\mathcal{L} = K_C \otimes \mathcal{M}^{-l}$.*

1. *If $l \leq g/2$, then $h^0(\mathcal{L}) = g - 2l$; and*
2. *If $l \leq (g - 4)/2$, then $\mathcal{L}$ is very ample.*

Proof. Both statements follow from our description of the geometry of canonical models of trigonal curves, carried out in **??**. We observed there that a trigonal canonical curve lies on a rational normal scroll S, and that *if C is general, then the scroll S is balanced.* The linear system $|\mathcal{L}| = |K_C \otimes \mathcal{M}^{-l}|$ is then cut out by hyperplanes in $\mathbb{P}^{g-1}$ containing any l chosen lines from the ruling of S; and the first part follows from the fact that *on a balanced scroll $S \subset \mathbb{P}^r$, any $(r + 1)/2$ lines of the ruling are linearly independent.*

The second part follows similarly, when we observe that if $r \geq 5$, $S \subset \mathbb{P}^r$ is any balanced rational normal scroll, and $L \subset S$ any line of the ruling, then the projection $\pi_L : S \to \mathbb{P}^{r-2}$, while a priori only rational, in fact extends to a regular map on all of S, embedding S as a balanced scroll in $\mathbb{P}^{r-2}$. Restricting to any curve $C \subset S$, it follows that π_L gives an embedding of C in $\mathbb{P}^{r-2}$ as well. □

Getting back to our present example, what we see is that if C is a general trigonal curve of genus 21 with $g^1_3 = |\mathcal{M}|$, and $\mathcal{L} = K_C \otimes \mathcal{M}^{-4}$, then the line bundle $\mathcal{L}$ embeds C as a curve of degree $2g-2-12 = 28$ in $\mathbb{P}^{12}$. Now we consider the projection of the image curve in $\mathbb{P}^{12}$ to $\mathbb{P}^7$. The family of such projections is parametrized by an open subset of the Grassmannian $\mathbb{G}(4, 12)$, which has dimension 40. We thus have $2g + 1 = 43$ degrees of freedom in choosing the general trigonal curve C, and another 40 degrees of freedom in choosing the projection (that is, the subseries $g^7_{28} \subset |\mathcal{L}|$); together these determine the image curve $C \subset \mathbb{P}^7$ up to automorphisms of $\mathbb{P}^7$. In sum, we see that the family $\mathcal{H}_1$ of curves $C \subset \mathbb{P}^7$ described in this way has dimension

$$\dim \mathcal{H}_1 = 43 + 40 + 63 = 146.$$

In particular, $\mathcal{H}_1$ cannot be in the closure of $\mathcal{H}_0$. Thus, even though we are in the nonspecial range $d \geq g+r$, there is at least one other irreducible component of the restricted Hilbert scheme, which maps to a proper subvariety of M_g and has dimension strictly greater than the expected.

8.4 Degree 14: Mumford's example

In many of the analyses above, we've been able to use the identification of the tangent space to the Hilbert scheme $\mathcal{H}$ at a point $[C]$ with the space $H^0(\mathcal{N}_{C/\mathbb{P}^3})$ of global sections of the normal bundle of C to tell whether the Hilbert scheme was smooth or singular at the point $[C]$. What's more, in every case where we carried this out, the conclusion was that the restricted Hilbert scheme $\mathcal{H}^\circ$ at least was smooth.

Does this pattern persist? The answer is a resounding "no:" in this section, we'll analyze an example, first discovered by Mumford, of an entire irreducible component of $\mathcal{H}^\circ$ that is everywhere singular, that is, everywhere nonreduced.

The example is the Hilbert scheme $\mathcal{H}^\circ = \mathcal{H}^\circ_{24,3,14}$ parametrizing smooth, irreducible curves C of degree 14 and genus 24 in $\mathbb{P}^3$. We shall analyze this example in out usual way, and examine three irreducible components of $\mathcal{H}^\circ$, one of which will be the celebrated Mumford component.

We will begin as always by analyzing the possible degrees of generators of the ideal of C, for $C \subset \mathbb{P}^3$ a smooth, irreducible curve of degree 14 and genus 24. By applying the genus formula for plane curves and curves on quadrics we see that C cannot lie in a plane or on a quadric. By Bézout's Theorem, C cannot lie on both a cubic and a quartic hypersurface, though we shall see that both possibilities are realized.

For $m \geq 3$ let $\rho_m : H^0(\mathcal{O}_{\mathbb{P}^3}(m)) \longrightarrow H^0(\mathcal{O}_C(m))$ be the natural maps. We will proceed by computing the size of the kernel of ρ_m for $m \geq 3$.

For $m \geq 4$, the line bundle $\mathcal{O}_C(m)$ has degree $> 2g - 2 = 46$, so the Riemann-Roch Theorem gives an exact value of $h^0(\mathcal{O}_C(m))$. However, when

m	$h^0(\mathcal{O}_C(m))$	$h^0(\mathcal{O}_{\mathbb{P}^3}(m))$
3	19, 20 or 21	20
4	33	35
5	47	56
6	61	84

Table 8.1: Postulation table

$m = 3$ we have

$$h^0(\mathcal{O}_C(3)) = 42 - 24 + 1 + h^0(K_C(-3)).$$

Since $d - g + 1 = 14 - 24 + 1$ is negative, C is embedded in $\mathbb{P}^3$ by a special linear series, and it follows from Section 4.1.3 that C is not hyperelliptic. The special line bundle $K_C(-3)$ has degree $46 - 42 = 4$ so, by Clifford's Theorem in the non-hyperelliptic case, $h^0(K_C(-3)) \leq 2$. Thus $h^0(\mathcal{O}_C(3)) = 19, 20$ or 21.

The "postulation table" (8.1) collects the dimensions of the source and target of ρ_m for $m = 3, \ldots, 6$.

8.4.1 Case 1: C does not lie on a cubic surface

Proposition 8.4.1. *The locus $\mathcal{H}_1 \subset \mathcal{H}^\circ$ parameterizing curves not lying on a cubic surface is dense in an irreducible component of $\mathcal{H}^\circ$. It has dimension 56, and is generically smooth.*

The proof of this proposition will occupy us for several pages. Let C be curve in $\mathcal{H}_1$. Table 8.1 shows that C lies on at least two linearly independent quartic surfaces S and S'; and since C does not lie on any surface of smaller degree, neither can be reducible. It follows that the intersection $S \cap S'$ must consist of the union of the curve C and a curve D of degree 2. The linkage formula (7.2) says that

$$p_a(C) - p_a(D) = (14 - 2)\frac{4 + 4 - 4}{2} = 24,$$

so D has arithmetic genus 0. Note that the proof above of formula (7.2) requires that at least one of the quartic surfaces containing C is smooth, which we don't a priori know in this setting; to apply it we need to invoke the more general Theorem **??** from Chapter **??**.

We can now invoke the following lemma:

Lemma 8.4.2. *A subscheme $D \subset \mathbb{P}^3$ of dimension 1, degree 2 and arithmetic genus 0 (that is, $\chi(\mathcal{O}_D) = 1$) is necessarily a plane conic; that is, the complete intersection of a plane and a quadric.*

We remark that the need to prove a lemma like this is one of the drawbacks of the method of liaison: even if we are a priori interested just in smooth, irreducible and nondegenerate curves in $\mathbb{P}^3$, applying liaison can lead to singular and/or nonreduced curves. There are some restrictions—by Theorem **??**, for example, says that a curve residual to a pure-dimensional scheme in a complete intersection is pure dimensional. For the present case, knowing even this is unnecessary because a general curve in $\mathcal{H}_1$ lies on a smooth quartic surface.

Proof of Lemma 8.4.2. Let $H \subset \mathbb{P}^3$ be a general plane, and set $\Gamma = C \cap H$. This is a scheme of dimension 0 and degree 2 in $H \cong \mathbb{P}^2$, which is then either the union of two reduced points, or a single nonreduced point isomorphic to $\operatorname{Spec} k[\epsilon]/(\epsilon^2)$. Either way, we observe that the restriction map $H^0(\mathcal{O}_{\mathbb{P}^3}(m)) \to H^0(\mathcal{O}_\Gamma(m))$ is surjective for all $m \geq 1$, and hence the map $H^0(\mathcal{O}_C(m)) \to H^0(\mathcal{O}_\Gamma(m))$ is as well. It follows that

$$h^0(\mathcal{O}_C(m)) \geq h^0(\mathcal{O}_C(m-1)) + 2$$

for all $m \geq 1$; since we know by hypothesis that $h^0(\mathcal{O}_C(m)) = 2m + 1$ for m large, we may conclude that $h^0(\mathcal{O}_C(1)) \leq 3 < h^0(\mathcal{O}_{\mathbb{P}^3}(1))$—in other words, the scheme C must be contained in a plane. It is thus a plane conic, without embedded points since any embedded points would mean $p_a(C) < 0$. □

Conversely, if C is any curve residual to a conic D in the complete intersection of two quartics, it must have degree 14 and genus 24, and by Bézout's Theorem it cannot lie on a cubic surface. We can thus compute the dimension of the family $\mathcal{H}_1$ of smooth curves of degree 14 and genus 24 not lying on a cubic surface via the incidence correspondence

$$\Phi = \{(C, D, S, S') \in \mathcal{H}^\circ \times \mathcal{H}_D \times \mathbb{P}^{34} \times \mathbb{P}^{34} \mid S \cap S' = C \cup D\}.$$

where $\mathcal{H}_D$ denotes the Hilbert scheme of plane conics. The Hilbert scheme $\mathcal{H}_D$ is irreducible of dimension 8 (this is a special case $m = 1$, $n = 2$ of Exercise 7.5.2); and for any conic $D = V(L, Q)$ given as the complete intersection of the plane $V(L)$ and the quadric $V(Q)$, Lasker's Theorem says that the homogeneous ideal of $D \subset \mathbb{P}^3$ is generated by L and Q; this allows us to see that the space of quartic surfaces containing D is a linear subspace of $\mathbb{P}^{34}$ of dimension 26. The fibers of Φ over $\mathcal{H}_D$ are thus open subsets of $\mathbb{P}^{25} \times \mathbb{P}^{25}$, and we deduce that Φ is irreducible of dimension 58.

Exercise 8.4.3. The general members of the family of quartic surfaces containing a smooth conic are themselves smooth.
((give a hint))

The general members of the family of quartic surfaces containing a smooth conic are themselves smooth, so we see from considering C, D as divisors on a smooth quartic, as in the derivation of the linkage formula, that $(C \cdot D) = 10$.

It follows that any quartic surface containing C must contain D as well and so, by Lasker's Theorem, must be a linear combination of S and S'. The fibers of Φ over its image in $\mathcal{H}_C$ are thus open subsets of $\mathbb{P}^1 \times \mathbb{P}^1$. The condition of not lying on a cubic surface is open, so $\mathcal{H}_1$ is dense in an irreducible component of $\mathcal{H}^\circ$ of dimension 56.

8.4.2 Tangent space calculations

It remains to show that $\mathcal{H}_1$ is generically smooth. To do this, we have to show that, at a general point $[C] \in \mathcal{H}_1$, the dimension of the Zariski tangent space $H^0(\mathcal{N}_{C/\mathbb{P}^3})$ is 56. Let S be a smooth quartic surface containing C, and consider the exact sequence

$$0 \to \mathcal{N}_{C/S} \to \mathcal{N}_{C/\mathbb{P}^3} \to \mathcal{N}_{S/\mathbb{P}^3}|_C \to 0. \tag{8.1}$$

The bundle $\mathcal{N}_{S/\mathbb{P}^3}|_C \cong \mathcal{O}_C(4)$, which is nonspecial; we have $h^0(\mathcal{O}_C(4)) = 33$ and $h^1(\mathcal{O}_C(4)) = 0$. By the adjunction formula applied to S we see that $K_S = \mathcal{O}_S$, and applying the formula again on S we see that $\mathcal{N}_{C/S} \cong K_C$. Thus $h^0(\mathcal{N}_{C/S}) = 24$ and $h^1(\mathcal{N}_{C/S}) = 1$.

From the long exact sequence in cohomology associated to the sequence (*) we see that there are two possibilities for the dimension of $H^0(\mathcal{N}_{C/\mathbb{P}^3})$: 56 and 57, depending on whether the map $H^0(\mathcal{N}_{C/\mathbb{P}^3}) \to H^0(\mathcal{N}_{S/\mathbb{P}^3}|_C)$ is surjective or of corank 1.

To settle this question, we need to invoke a basic fact about deformations of subschemes of a given scheme. For this discussion, let Z be an arbitrary fixed scheme, and $X \subset Y \subset Z$ a nested pair of subschemes. We can ask two questions:

1. Given a first-order deformation $\tilde{Y} \subset \operatorname{Spec} k[\epsilon]/(\epsilon^2) \times Z$ of Y in Z, does there exist a first-order deformation $\tilde{X} \subset \operatorname{Spec} k[\epsilon]/(\epsilon^2) \times Z$ of X contained in it? and
2. Given a first-order deformation $\tilde{X} \subset \operatorname{Spec} k[\epsilon]/(\epsilon^2) \times Z$ of X in Z, does there exist a first-order deformation $\tilde{Y} \subset \operatorname{Spec} k[\epsilon]/(\epsilon^2) \times Z$ of Y containing it?

The answer is a basic fact from deformation theory. Let α, β be the natural maps in the following diagram:

$$\begin{tikzcd} H^0(\mathcal{N}_{X/Z}) \arrow[r, "\alpha"] & H^0(\mathcal{N}_{Y/Z}|_X) \\ & H^0(\mathcal{N}_{Y/Z}). \arrow[u, "\beta"'] \end{tikzcd}$$

Lemma 8.4.4. *The first-order deformation of X corresponding to the global section $\sigma \in H^0(\mathcal{N}_{X/Z})$ is contained in the first-order deformation of Y corresponding to the global section $\tau \in H^0(\mathcal{N}_{Y/Z})$ if and only if $\alpha(\sigma) = \beta(\tau)$. In particular, every first-order deformation of Y contains a first-order deformation of X if and only if $\operatorname{im}(\beta) \subset \operatorname{im}(\alpha)$.*

For a proof of this lemma, see Chapter 6 of [**?**].

We apply this construction to $Z = \mathbb{P}^3$, $Y = S \subset \mathbb{P}^3$ a smooth quartic surface, and $X = D \subset S$ a smooth plane conic curve. We start with the sequence

$$0 \to \mathcal{N}_{D/S} \to \mathcal{N}_{D/\mathbb{P}^3} \to \mathcal{N}_{S/\mathbb{P}^3}|_D \to 0.$$

Identifying D with $\mathbb{P}^1$, we have by adjunction that $\mathcal{N}_{D/S} \cong \mathcal{O}_{\mathbb{P}^1}(-2)$, and S being a quartic, we have $\mathcal{N}_{S/\mathbb{P}^3}|_D \cong \mathcal{O}_{\mathbb{P}^1}(8)$. Moreover, since D is the complete intersection of a quadric and a plane, we have $\mathcal{N}_{D/\mathbb{P}^3} \cong \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(4)$, so that the sequence above looks like

$$0 \to \mathcal{O}_{\mathbb{P}^1}(-2) \to \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(4) \to \mathcal{O}_{\mathbb{P}^1}(8) \to 0$$

Now, we know that $H^1(\mathcal{O}_{\mathbb{P}^1}(-2)) \neq 0$, while $H^1(\mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(4)) = 0$, so we conclude by Lemma 8.4.4 that the map $H^0(\mathcal{N}_{D/\mathbb{P}^3}) \to H^0(\mathcal{N}_{S/\mathbb{P}^3}|_D)$ cannot be surjective; in other words, there exist first-order deformations of S that contain no first-order deformation of D.

The same argument works if D is the union of two lines meeting at a point.

We need to introduce one more element into the argument, which is expressed in the following proposition.

Proposition 8.4.5. *Let S be a smooth quartic surface, and C and $D \subset S$ a pair of curves forming the complete intersection of S with another quartic surface S', with D a plane conic curve. A first-order deformation $\tilde{S}$ of S contains a first-order deformation of C if and only if it contains a first-order deformation of D.*

Proof. The key ingredient is the observation that $H^1(\mathcal{O}_S(D)) = H^1(\mathcal{O}_S(C)) = 0$. What this says is that a first-order deformation $\tilde{S}$ of S contains a first-order deformation of D if and only if it contains a first-order deformation of the line bundle $\mathcal{L} = \mathcal{O}_S(D)$; that is, if and only if there exists a line bundle $\tilde{\mathcal{L}}$ on $\tilde{S}$ such that $\mathcal{L}|_S \cong \mathcal{O}_S(D)$, and likewise for C. But the existence of a line bundle $\tilde{\mathcal{L}}$ on $\tilde{S}$ extending $\mathcal{O}_S(D)$ is equivalent to the existence of a line bundle $\tilde{\mathcal{M}}$ on $\tilde{S}$ extending $\mathcal{O}_S(C)$, since they're related by $\tilde{\mathcal{M}} = \mathcal{O}_{\tilde{S}}(4) \otimes \tilde{\mathcal{L}}$. $\square$

Now, going back to the exact sequence (8.1), we have shown that there exist first-order deformations of S that contain no first-order deformations of C; thus the sequence (8.1) is not exact on global sections, and hence the dimension $h^0(\mathcal{N}_{C/\mathbb{P}^3}) = 56$, showing that $\mathcal{H}_1$ is generically smooth.

8.4.3 What's going on here?

We should take a moment to give some background for the argument above. The basic idea is built on a striking fact about curves on surfaces in $\mathbb{P}^3$, called the *Noether-Lefschetz theorem.*

Theorem 8.4.6 (Noether-Lefschetz). *If $S \subset \mathbb{P}^3$ is a very general surface of degree $d \geq 4$ in $\mathbb{P}^3$, and $C \subset S$ is any curve, then C is a complete intersection $S \cap T$ with S.*

Thus, for example, a very general quartic surface contains no lines, conics or twisted cubics—facts you can readily establish for yourself via a standard dimension count, as the following exercises suggest.

Exercise 8.4.7. Let $\mathbb{G}(1,3)$ be the Grassmannian of lines in $\mathbb{P}^3$, let $\mathbb{P}^{19}$ denote the space of quartic surfaces $S \subset \mathbb{P}^3$, and consider the incidence correspondence

$$\Gamma = \{(S, L) \in \mathbb{P}^{19} \times \mathbb{G}(1,3) \mid L \subset S\}$$

Calculate the dimension of Γ, and deduce in particular that the projection map $\Gamma \to \mathbb{P}^{19}$ cannot be dominant.

Exercise 8.4.8. In the preceding exercise, replace the Grassmannian $\mathbb{G}(1,3)$ with the restricted Hilbert schemes $\mathcal{H}^\circ$ parametrizing conics and twisted cubics, and carry out the analogous calculation to deduce that a general quartic surface $S \subset \mathbb{P}^3$ contains no conics or twisted cubics. What goes wrong when we replace $\mathcal{H}^\circ$ with the restricted Hilbert scheme of curves of higher degree?

In fact, calculations like the one suggested in these exercises were how Noether first came to propose Theorem 8.4.6; it was not until Lefschetz that a complete proof was given.

In these terms, we can identify the crucial ingredient in the proof of the generic smoothness part of Proposition 8.4.1 as a strengthened form of the Noether-Lefschetz Theorem:

Theorem 8.4.9 (Deformation Noether-Lefschetz). *If $S \subset \mathbb{P}^3$ is a smooth surface of degree $d \geq 4$, and $C \subset S$ is any curve that is not a complete intersection with S, then there exists a first-order deformation $\tilde{S}$ of S that does not contain a first-order deformation of C.*

We will prove this by ad-hoc methods in the case of interest to us here; the proof of the general case, given in ****, uses Hodge theory.

8.4.4 Case 2: C lies on a cubic surface S

Now suppose that C is a smooth irreducible curve of degree 14 and genus 24 that *does* lie on a cubic surface S. Bézout's Theorem tells us that S is unique, and we will restrict ourselves to the open subset $\mathcal{H}_2 \subset \mathcal{H}^\circ \setminus \mathcal{H}_1$ where the surface S is smooth, which in fact is dense—see [?].

Bézout's Theorem tells us that C cannot lie on a quartic surface not containing S. If C lay on a quintic surface not containing S then C would be residual to a line in the complete intersection of S and the quintic, and the liaison formula **??** would tell us that

$$g(C) = (14-1)\frac{3+5-4}{2} = 26,$$

a contradiction, so C lies on no quintic surface.

On the other hand, Table 8.1 tells us that there is at least a $84 - 61 =$ 23-dimensional vector space of sextic polynomials vanishing on C, only a 20-dimensional subspace of which can vanish on S. Thus there is a $\mathbb{P}^2$ of sextic surfaces containing C but not containing S, and, choosing one of them we can write

$$S \cap T = C \cup D$$

with T a sextic surface and D a curve of degree 4. The liaison formula tells us that

$$g(C) - g(D) = (14-4)\frac{3+6-4}{2} = 25,$$

so the arithmetic genus of D is -1. We will henceforth take T to be general among sextics containing C, so that D will be a general member of the (at least) 2-dimensional linear system cut on S by sextics containing C.

Proposition 8.4.10. *D must either be (a) the disjoint union of a line and a twisted cubic on S; or (b) a union of two disjoint conics on S.*

Exercise 8.4.11. (Guided exercise to prove this proposition: first, D cannot have multiple components; then, must be disconnected.)

Since neither of the cases described in Proposition 8.4.10 is a specialization of the other, we conclude that the locus $\mathcal{H}_2$ is the union of two disjoint loci $\mathcal{H}_2'$ and $\mathcal{H}_2''$ corresponding to these two cases. We consider these in turn.

Exercise 8.4.12. (Guided exercise to prove this AND deduce that $\mathcal{H}_2'$ and $\mathcal{H}_2''$ are irreducible, either by the incidence correspondences or by monodromy.)

Case $2'$: D is the disjoint union of a twisted cubic and a line

Proposition 8.4.13. *The locus $\mathcal{H}_2' \subset \mathcal{H}^\circ$ parameterizing curves C residual to the disjoint union of a line and a twisted cubic in the complete intersection of a sextic and a smooth cubic surface is an irreducible component of $\mathcal{H}^\circ$. It has dimension 56, and is generically smooth.*

Proof. Let $\mathcal{H}$ be the locus in the Hilbert scheme $\mathcal{H}_{-1,3,4}$ corresponding to disjoint unions of twisted cubics and lines, and consider the correspondence

$$\Phi = \{(C, D, S, T) \in \mathcal{H}_2' \times \mathcal{H} \times \mathbb{P}^{19} \times \mathbb{P}^{83} \mid S \cap T = C \cup D\}.$$

We have $\dim \mathcal{H} = 16$, and by Proposition 8.4.15 the fiber of Φ over a point $[D] \in \mathcal{H}$ is an open subset of the product $\mathbb{P}^5 \times \mathbb{P}^{37}$; so we see that Φ is irreducible of dimension 58. The fibers of Φ over $\mathcal{H}_2'$ are 2-dimensional, and we conclude that $\mathcal{H}_2'$ is irreducible of dimension 56.

Finally, we calculate the dimension of the Zariski tangent space $H^0(\mathcal{N}_{C/\mathbb{P}^3})$ to $\mathcal{H}_2'$ at a general point $[C]$. We do this, as before, by considering the exact sequence associated to the inclusion of C in S:

$$0 \to \mathcal{N}_{C/S} \to \mathcal{N}_{C/\mathbb{P}^3} \to \mathcal{N}_{S/\mathbb{P}^3}|_C \to 0$$

Here there is no ambiguity about the first term: by adjunction, the degree of the normal bundle of C in S is 60, which is greater than $2g(C) - 2 = 46$; so $h^1(\mathcal{N}_{C/S}) = 0$ and $h^0(\mathcal{N}_{C/S}) = 37$.

On the other hand, $\mathcal{N}_{S/\mathbb{P}^3}|_C \cong \mathcal{O}_C(3)$, and from Table 8.1, we see that $h^0(\mathcal{O}_C(3))$ can a priori be 19, 20 or 21. We will use the explicit description of C to show that, in this case, $h^0(\mathcal{O}_C(3)) = 19$.

For this purpose, let L and T denote the line component and the twisted cubic component of D respectively; and let H denote the hyperplane class on S. From the adjunction formula we can compute the self-intersection numbers of these curves on S as $(L \cdot L) = -1$ and $(T \cdot T) = 1$. Since $C \sim 6H - D$ on S, we have

$$(C \cdot L) = ((6H - L - T) \cdot L) = 7; \quad \text{and} \quad (C \cdot T) = ((6H - L - T) \cdot L) = 17$$

In other words, the curves L and T intersect C in divisors E_L and E_T of degrees 7 and 17 respectively. By Serre duality,

$$h^1(\mathcal{O}_C(3)) = h^0(K_C(-3))$$

and by adjunction,

$$K_C(-3) = K_S(C)(-3)|_C = \mathcal{O}_S(-H + 6H - D - 3H)|_C = \mathcal{O}_C(2)(-E_L - E_T).$$

Now, the quadrics in $\mathbb{P}^3$ cut out on C the complete linear series $|\mathcal{O}_C(2)|$,
((Could be proven by using the representation of a cubic surface as a blowup of the plane.))
so $h^1(\mathcal{O}_C(3))$ is the dimension of the space of quadratic polynomials vanishing on E_L and E_T. But E_L consists of seven points on the line L, so any quadric containing E_L contains L; and likewise since E_T has degree $17 > 2 \cdot 3$, any quadric containing E_T contains T. Since no quadric contains the disjoint union of a line and a twisted cubic, we conclude that $h^1(\mathcal{O}_C(3)) = 0$ and $h^0(\mathcal{O}_C(3)) = 19$.

Putting this all together, we conclude that $h^0(\mathcal{N}_{C/\mathbb{P}^3}) = 56$; so the component $\mathcal{H}_2'$ of the Hilbert scheme $\mathcal{H}^\circ$ is generically smooth of dimension 56. $\square$

Case $2''$: D is the disjoint union of two conics

Proposition 8.4.14. *The locus $\mathcal{H}_2'' \subset \mathcal{H}^\circ$ parameterizing curves C residual to the disjoint union of two conics in the complete intersection of a sextic and a smooth cubic surface is an irreducible component of $\mathcal{H}^\circ$. It has dimension 56, but is non-reduced: its tangent space at a generic point has dimension 57.*

Proof. The analysis this case follows the same path as the preceding until the very last step, where the residual curve D is the disjoint union of two conic curves rather than the disjoint union of a line and a twisted cubic. What difference does this make? Both the disjoint union of two conic curves and the disjoint union of a line and a twisted cubic are curves of degree 4 and arithmetic genus -1, so they both have Hilbert polynomial $p(m) = 4m + 2$. The difference is that they do not have the same Hilbert function, according to the following proposition:

Proposition 8.4.15. *Let E be the disjoint union of two conic curves in $\mathbb{P}^3$ and E' the disjoint union of a line and a twisted cubic. Let $h(m)$ and $h'(m)$ be their respective Hilbert functions, and $p(m) = 4m + 2$ their common Hilbert polynomial.*

1. *For all $m \neq 3$, we have $h(m) = h'(m)$; and both are equal to $p(m) = 4m+2$ for $m \geq 3$; but*
2. *$h(2) = 9$, while $h'(2) = 10$ (in other words, E lies on a unique quadric surface, while E' is not contained in any quadric surface).*

Proof. Let S be the homogeneous coordinate ring of $\mathbb{P}^3$, and let $I_E = I_{Q_1} \cap I_{Q_2}$ be the homogeneous ideal of E, where the I_{Q_i} are the homogeneous ideals of the two disjoint conics. Similarly, let $I_{E'} = I_L \cap I_T$ be the homogeneous ideal of E', where I_L is the homogeneous ideal of a line and I_T is the homogeneous ideal of a disjoint twisted cubic. We have exact sequences

$$
\begin{aligned}
0 \to & S/I_E \to S/I_{Q_1} \oplus S/I_{Q_2} \to S/(I_{Q_1} + I_{Q_2}) \to 0 \\
0 \to & S/I_{E'} \to S/I_L \oplus S/I_T \to S/(I_L + I_T) \to 0.
\end{aligned}
$$

Writing h_Q, h_L, h_T for the Hilbert functions of Q, L and T respectively, we have

$$
\begin{aligned}
h_Q(m) &= 2m + 1 \\
h_L(m) &= m + 1 \\
h_T(m) &= 3m + 1
\end{aligned}
$$

for all $m \geq 0$.

Because each of E, E' is a disjoint union, the rings $U := S/(I_{Q_1} + I_{Q_2})$ and $V := S/(I_L + I_T)$ have finite length. We claim that $U \cong k[x, y]/(q_1, q_2)$ is a complete intersection of 2 quadrics while $V \cong k[x, y]/(x^2, xy, y^2)$. It follows that the dimensions of the homogeneous components of U in degrees $0, 1, 2, 3\ldots$ are $1, 2, 1, 0\ldots$ while those of V are $1, 2, 0, 0\ldots$. Together with the computation above, this will prove the Proposition.

To analyze U, let write $I_{Q_i} = (\ell_i, q_i)$ where the ℓ_i are linear forms and the q_i are quadratic forms. Since $I_{Q_1} + I_{Q_2}$ has finite length, the four forms ℓ_1, ℓ_2, q_1, q_2 must be a regular sequence. Working modulo (ℓ_1, ℓ_2) we see that U is isomorphic to a complete intersection of 2 quadrics in 2 variables, as claimed.

To prove that V has the given Hilbert function, it suffices to show that the degree 2 part of V is 0. Since the Hilbert function of $S/I_L \oplus S/I_T$ is $4m + 2$, this is equivalent to showing that the degree 2 part of $S/I_{L\cup T}$ is 10-dimensional; that is, that no quadric vanishes on both L and T. Since T spans $\mathbb{P}^3$ and is irreducible, the quadric must be irreducible. By **** the residual $L\cup T$ to C is unmixed, and it follows that T is unmixed and spans $\mathbb{P}^3$.

We claim that if a line and a curve of degree 3 and genus 0 lie on any quadric, then they meet: If the quadric is smooth then T would have class $(1, 2)$ and the line would have to have class $(1, 0)$ or $(0, 1)$ both of which meet T. If the quadric is an irreducible cone, then we note that every curve meets every line on the cone. If T lies on the union of two planes then T has components in both planes and thus meets any line in one of them; and finally if T lies on a double plane, then the line would meet T_{red}. Thus $T \cup L$ cannot lie on a quadric, and we are done. □

To return to the proof of Proposition 8.4.14, let $\mathcal{H}$ now be the locus in the Hilbert scheme $\mathcal{H}_{-1,3,4}$ corresponding to disjoint unions of two conics, and consider the correspondence

$$\Phi = \{(C, D, S, T) \in \mathcal{H}_2'' \times \mathcal{H} \times \mathbb{P}^{19} \times \mathbb{P}^{83} \mid S \cap T = C \cup D\}.$$

Once more we have $\dim \mathcal{H} = 16$, and the fiber of Φ over a point $[D] \in \mathcal{H}$ is again an open subset of the product $\mathbb{P}^5 \times \mathbb{P}^{37}$ (unions of two disjoint conics imposes the same number of conditions on cubics and sextics as the disjoint union of a line and a twisted cubic); so we see that Φ is irreducible of dimension 58. The fibers of Φ over $\mathcal{H}''$ are 2-dimensional, and we conclude that $\mathcal{H}''$ is irreducible of dimension 56.

The calculation of the dimension of the Zariski tangent space $H^0(\mathcal{N}_{C/\mathbb{P}^3})$ to $\mathcal{H}''$ at a general point $[C]$ also proceeds as in the last case: we start with the exact sequence

$$0 \to \mathcal{N}_{C/S} \to \mathcal{N}_{C/\mathbb{P}^3} \to \mathcal{N}_{S/\mathbb{P}^3}|_C \to 0.$$

Again, the line bundle $\mathcal{N}_{C/S}$ has degree 60 and so is nonspecial with $h^1(\mathcal{N}_{C/S}) = 0$ and $h^0(\mathcal{N}_{C/S}) = 37$.

However, the determination of the cohomology of the third term, $\mathcal{N}_{S/\mathbb{P}^3}|_C \cong \mathcal{O}_C(3)$ is different. Let Q and Q' be the two conics comprising the residual curve D; and let H denote the hyperplane class on S. The planes P, P' spanned by Q and Q' respectively meet in a line L. Since L contains the scheme of length 4 of intersection with $Q \cup Q'$, it is contained in S. Thus the curves Q and Q' are linearly equivalent on S, so we can write the class of C on S as $6H - 2Q \sim 4H + 2L$.

Since $Q \cap Q' = \emptyset$ we have $Q \cdot Q = 0$; and since $C \sim 6H - 2Q$ on S, we have

$$(C \cdot Q) = ((6H - 2Q) \cdot Q) = 12.$$

In other words, the curves Q and Q' intersect C in divisors E_Q and $E_{Q'}$ of degree 12. As before, we can write

$$h^1(\mathcal{O}_C(3)) = h^0(K_C(-3)) = h^0(\mathcal{O}_C(2)(-E_Q - E_{Q'}))$$

and using again the completeness of the linear series cut out on C by quadrics, we see that $h^1(\mathcal{O}_C(3))$ *is the dimension of the space of quadratic polynomials vanishing on* E_Q *and* $E_{Q'}$; again, since $12 > 2 \cdot 2$, this is the same as the space of quadrics containing the two curves Q and Q'.

Here is where the stories diverge: we saw in Proposition 8.4.15 that whereas there is no quadric containing the disjoint union of a line and a twisted cubic, there is indeed a unique quadric containing the union of two given disjoint conics, namely, the union of the planes of the conics. Thus $h^1(\mathcal{O}_C(3)) = 1$ so $h^0(\mathcal{O}_C(3)) = 20$ and correspondingly $h^0(\mathcal{N}_{C/\mathbb{P}^3}) = 57$. □

What's going on here?

What accounts for the different behaviors of curves in cases $2'$ and $2''$? Here is one explanation:

To start, let C be a curve corresponding to a general point of $\mathcal{H}'_2$. As we've seen, we have

$$h^1(\mathcal{O}_C(3)) = 0 \quad \text{and} \quad h^0(\mathcal{O}_C(3)) = 19,$$

so we see already from Table 8.1 that C must lie on a cubic surface. Moreover, by upper-semicontinuity, the same is true of any deformation of C, and so in an étale neighborhood of $[C]$ the Hilbert scheme looks like a projective bundle over the space of cubic surfaces.

By contrast, if C is the curve corresponding to a general point of $\mathcal{H}''_2$, we have

$$h^1(\mathcal{O}_C(3)) = 1 \quad \text{and} \quad h^0(\mathcal{O}_C(3)) = 20.$$

In other words, C is not forced to lie on a cubic surface, it just chooses to do so! The "extra" section of the normal bundle corresponds to a first-order deformation of C that is not contained in any deformation of S. If we could extend these deformations to arbitrary order, we would arrive at a family of curves whose general member lay in the first component $\mathcal{H}_1$; but we know that a general point of $\mathcal{H}''$ is not in the closure of $\mathcal{H}_1$, and so *these deformations of* C *must be obstructed.*

One note: it may seem that the phenomenon described in this last example—a component of the Hilbert scheme that is everywhere nonreduced, even though the objects parametrized are perfectly nice smooth, irreducible curves in $\mathbb{P}^3$—represents a pathology, and indeed, it was first described by David Mumford, in a paper entitled "Pathologies"! But, as Ravi Vakil has shown, it is to be expected: Vakil shows that *every* complete local ring over an algebraically closed field, up to adding power series variables, occurs as the completion of the local ring of a Hilbert scheme of smooth curves—that is, in effect, every singularity is possible. (reference to Vakil's paper, and more precise statement of Ravi's theorem).

8.5 Open problems

8.5.1 Brill-Noether in low codimension

If we ignore the finer points of the Brill-Noether theorem and focus just on the statement about the dimension and irreducibility of the variety of linear series on a curve, we can express it in a simple form: according to Theorem 7.6.3 *Any component of the restricted Hilbert scheme $\mathcal{H}^\circ$ of curves of degree d and genus g that dominates the moduli space M_g has the expected dimension*

$$h(g,r,d) = 4g-3+(r+1)(d-g+1)-1 = (r+1)d-(r-3)(g-1)$$

as calculated in Section 7.6 above.

Now, we saw in Section 8.1 an example of a component of the Hilbert scheme violating this dimension estimate, and it's not hard to produce lots of similar examples: components of the Hilbert scheme that parametrize complete intersections, or more generally determinantal curves, have in general dimension larger than the Hilbert number $h(g,r,d)$, and the following exercise gives a way of generating many more.

Exercise 8.5.1. Let $\mathcal{H}^\circ$ be a component of the Hilbert scheme parametrizing curves of degree d and genus g in $\mathbb{P}^3$ that dominates the moduli space M_g. For $s,t \gg d$, let $\mathcal{K}^\circ$ be the family of smooth curves residual to a curve $C \in \mathcal{H}^\circ$ in a complete intersection of surfaces of degrees s and t.

1. Show that $\mathcal{K}^\circ$ is open and dense in a component of the Hilbert scheme of curves of degree $st-d$ and the appropriate genus.

2. Calculate the dimension of $\mathcal{K}^\circ$, and in particular show that it is strictly greater than $h(g,r,d)$.

So it may seem that the issue is settled: components of the Hilbert scheme dominating M_g have the expected dimension; others don't in general. But there is an observed phenomenon that suggests more may be true: that components of

$\mathcal{H}^\circ$ whose image in M_g have low codimension still have the expected dimension $h(g, r, d)$.

The cases with codimension ≤ 2 are already known: In [**?**], it is shown that if $\Sigma \subset M_g$ is any subvariety of codimension 1, then the curve C corresponding to a general point of Σ has no linear series with Brill-Noether number $\rho < -1$; and Edidin in [**?**] proves the analogous (and much harder) result for subvarieties of codimension 2. Indeed, looking over the examples we know of components of the Hilbert scheme whose dimension is strictly greater than the expected $h(g, r, d)$, there are none whose image in M_g has codimension less than $g - 4$. We could therefore make the conjecture:

Conjecture 8.5.2. *If $\mathcal{K} \subset \mathcal{H}^\circ_{d,g,r}$ is any component of a restricted Hilbert scheme, and the image of $\mathcal{K}$ in M_g has codimension $\leq g - 4$, then $\dim \mathcal{K} = h(g, r, d)$.*

8.5.2 Maximally special curves

Most of Brill-Noether theory, and the theory of linear systems on curves in general, centers on the behavior of linear series on a general curve. The opposite end of the spectrum is also interesting, and we may ask: How special a linear series on a special curve can be?

To make such a question precise, let $\tilde{M}^r_{g,d} \subset M_g$ be the closure of the image of the map $\phi : \mathcal{H}^\circ_{d,g,r} \to M_g$ sending a curve to its isomorphism class.

1. What is the smallest possible dimension of $\mathcal{H}^\circ_{d,g,r}$?

2. What is the smallest possible dimension of $\tilde{M}^r_{g,d}$?

3. Modifying the last question slightly, let $M^r_{g,d} \subset M_g$ be the closure of the locus of curves C that possess a g^r_d (in other words, we are dropping the condition that the g^r_d be very ample). We can ask what is the smallest possible dimension of $M^r_{g,d}$?

One might suppose that the most special curves, from the point of view of questions 2 and 3, are hyperelliptic curves but the locus in M_g of hyperelliptic curves has dimension $2g - 1$. What about smooth plane curves? That's better – in the sense that the locus in M_g of smooth plane curves has dimension asymptotic to g, as the following exercise will show – but there are still a lot of them.

Exercise 8.5.3.

1. Let $C \subset \mathbb{P}^2$ be a smooth plane curve of degree d. Show that the g^2_d cut by lines on C is unique; that is, $W^2_d(C)$ consists of one point.

2. Using this, find the dimension of the locus of smooth plane curves in M_g.

Can we do better? Well, in $\mathbb{P}^3$ we can consider the locus of smooth complete intersections of two surfaces of degree m. As we saw in Exercise 7.3.5, these comprise an open subset $\mathcal{H}^\circ_{ci}$ of the Hilbert scheme of curves of degree $d = m^2$, and genus g given by the relation

$$2g - 2 = \deg K_C = m^2(2m - 4),$$

or, asymptotically,

$$g \sim m^3.$$

Moreover, the dimension of this component of the Hilbert scheme is easy to compute, since as we saw in Exercise 7.3.6 that it is isomorphic to an open subset of the Grassmannian $G(2, \binom{m+3}{3})$, and so has dimension

$$2(\binom{m+3}{3} - 2) \ \sim \ \frac{m^3}{3}$$

Finally, we observe that if $C \subset \mathbb{P}^r$ is a complete intersection curve of genus $g > 1$, the canonical bundle K_C is a positive power of $\mathcal{O}_C(1)$, and by Lasker's Theorem C is linearly normal. In particular, for a given abstract curve C there are only finitely many embeddings of C in projective space $\mathbb{P}^r$ as a complete intersection, up to PGL_{r+1}; in other words, the fibers of $\mathcal{H}^\circ_{ci}$ over M_g have dimension $\dim(PGL_{r+1}) = r^2 + 2r$.

Thus, we have a sequence of components of the restricted Hilbert scheme $\mathcal{H}^\circ$ whose images in M_g have dimension tending asymptotically to $g/3$.

The following exercise suggests why we chose complete intersections of surfaces of the same degree.

Exercise 8.5.4. Consider the locus of curves $C \subset \mathbb{P}^3$ that are complete intersections of a quadric surface and a surface of degree m. Show that these comprise components of the restricted Hilbert scheme, and that their images in moduli have dimension asymptotically approaching g as $m \to \infty$.

More generally, we can consider complete intersections of $r-1$ hypersurfaces of degree m in $\mathbb{P}^r$; in a similar fashion we can calculate that their images in M_g have dimension asympotically approaching $2g/r!$ as $m \to \infty$, as we ask you to verify in the following exercise.

Exercise 8.5.5. Consider the locus $\mathcal{H}^\circ_{ci}$, in the Hilbert scheme $\mathcal{H}^\circ$, of smooth, irreducible, nondegenerate curves $C \subset \mathbb{P}^r$ that are complete intersections of $r-1$ hypersurfaces of degree m.

1. Show that $\mathcal{H}^\circ_{ci}$ is open in $\mathcal{H}^\circ$;
2. Calculate the dimension of $\mathcal{H}^\circ_{ci}$ (and observe that it is irreducible); and
3. Show that the dimension of the image of $\mathcal{H}^\circ_{ci}$ in M_g is asymptotically $2g/r!$ as $m \to \infty$

The question is, can we do better? For example, if we fix r, can we find a sequence of components $\mathcal{H}_n$ of restricted Hilbert schemes $\mathcal{H}^\circ_{g_n,r,d_n}$ of curves in $\mathbb{P}^r$ such that

$$\lim \frac{\dim \mathcal{H}_n}{g_n} = 0?$$

8.5.3 Rigid curves?

In the last section, we considered components of the restricted Hilbert scheme whose image in M_g was "as small as possible." Let's go now all the way to the extreme, and ask: is there a component of the restricted Hilbert scheme $\mathcal{H}^\circ_{g,r,d}$ whose image in M_g is a single point? Of course M_0 itself is a single point, so we exclude genus 0! We can give three flavors of this question, in order of ascending preposterousness.

1. First, we'll say a smooth, irreducible and nondegenerate curve $C \subset \mathbb{P}^r$ is *moduli rigid* if it lies in a component of the restricted Hilbert scheme whose image in M_g is just the point $[C] \in M_g$—in other words, if the linear series $|\mathcal{O}_C(1)|$ does not deform to any nearby curves.

2. Second, we say that such a curve is *rigid* if it lies in a component $\mathcal{H}^\circ$ of the restricted Hilbert scheme such that PGL_{r+1} acts transitively on $\mathcal{H}^\circ$. This is saying that C is moduli rigid, plus the line bundle $\mathcal{O}_C(1)$ does not deform to any other g^r_d on C.

3. Finally, we say that such a curve is *deformation rigid* if the curve $C \subset \mathbb{P}^r$ has no nontrivial infinitesimal deformations other than those induced by PGL_{r+1}—in other words, every global section of the normal bundle $\mathcal{N}_{C/\mathbb{P}^r}$ is the image of the restriction of a vector field on $\mathbb{P}^r$.

In truth, these are not so much questions as howls of frustration. The existence of irrational rigid curves seems outlandish; we don't know anyone who thinks there are such things. But then *why can't we prove that they don't exist?*

Chapter 9

Scrolls and their divisors

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On Varieties of Minimal Degree (A Centennial Account)

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Abstract. This note contains a short tour through the folklore surrounding the rational normal scrolls, a general technique for finding such scrolls containing a given projective variety, and a new proof of the Del Pezzo–Bertini theorem classifying the varieties of minimal degree, which relies on a general description of the divisors on scrolls rather than on the usual enumeration of low-dimensional special cases and which works smoothly in all characteristics.

Introduction. Throughout, we work over an algebraically closed field k of arbitrary characteristic with subschemes $X \subset \mathbf{P}^r_k$. We say that X is a *variety* if it is reduced and irreducible, and that it is *nondegenerate* if it is not contained in a hyperplane. There is an elementary lower bound for the degree of such a variety:

PROPOSITION 0. *If $X \subset \mathbf{P}^r$ is a nondegenerate variety, then* $\deg X \geq 1 + \operatorname{codim} X$.

(PROOF. If $\operatorname{codim} X = 1$ the result is trivial. Else we project to $\mathbf{P}^{r-1}$ from a general point of X, reducing the degree by at least 1 and the codimension by 1, and are done by induction. □)

We say that $X \subset \mathbf{P}^r$ is a *variety of minimal degree* if X is nondegenerate and $\deg X = 1+\operatorname{codim} X$. One hundred years ago Del Pezzo (1886) gave a remarkable classification for surfaces of minimal degree, and Bertini (1907) showed how to deduce a similar classification for varieties of any dimension. Of course the case of codimension 1 is trivial, X being then a quadric hypersurface, classified by its dimension and that of its singular locus. In other cases we may phrase the result as:

THEOREM 1. *If $X \subset \mathbf{P}^r$ is a variety of minimal degree, then X is a cone over a smooth such variety. If X is smooth and* $\operatorname{codim} X > 1$, *then $X \subset \mathbf{P}^r$ is either a rational normal scroll or the Veronese surface $\mathbf{P}^2 \subset \mathbf{P}^5$.*

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(See §1 for the definition and some properties of rational normal scrolls.)

The purpose of this note is to give a short and direct proof of the Del Pezzo–Bertini theorem, valid in any characteristic. The proofs (Bertini (1907), Harris (1981), and Xambò (1981)) are all essentially similar: they treat first the cases of surfaces in general (which is also done in Nagata (1960) and Griffiths-Harris (1978)), and finally they reduce the case of arbitrary varieties to the case of surfaces, distinguishing according to whether the general 2-dimensional plane section of the given variety is a scroll or the Veronese surface. Instead, we base our discussion on the following general result (§2), which is useful in many other circumstances:

THEOREM 2. *Let $X \subset \mathbf{P}^r$ be a linearly normal variety, and $D \subset X$ a divisor. If D moves in a pencil $\{D_\lambda | \lambda \in \mathbf{P}^1\}$ of linearly equivalent divisors, then writing $\overline{D}_\lambda$ for the linear span of D_λ in $\mathbf{P}^r$, the variety*

$$S = \bigcup_\lambda \overline{D}_\lambda$$

is a rational normal scroll.

This allows us (in §3) to write an arbitrary variety X of minimal degree as a divisor on a scroll, and simple considerations of the geometry of scrolls then lead to the result.

1. Description of the varieties of minimal degree. We first explain some of the terms used in Theorems 1 and 2 above:

If $L \subset \mathbf{P}^{r+s+1}$ is a linear space of dimension s, $p_L: \mathbf{P}^{r+s+1} \to \mathbf{P}^r$ is the projection from L, and X is a variety in $\mathbf{P}^r$, then the cone over X is the closure of $p_L^{-1}X$. In equations, the cone is simply given by the same equations as X, written in the appropriate subset of the coordinates on $\mathbf{P}^{r+s+1}$. Thus a *cone in* $\mathbf{P}^r$ *over the Veronese surface* $\mathbf{P}^2 \hookrightarrow \mathbf{P}^5$ may be defined as a variety given, with respect to suitable coordinates $x_0, \ldots, x_r$, by the (prime) ideal of 2×2 minors of the generic symmetric matrix:

$$\begin{pmatrix} X_0 & X_1 & X_2 \\ X_1 & X_3 & X_4 \\ X_2 & X_4 & X_5 \end{pmatrix}$$

(It is easy to see that a cone over any variety of minimal degree has minimal degree; our definition of rational normal scroll is such that the cone over a rational normal scroll is another rational normal scroll.)

Note that the Veronese surface contains no lines—indeed, any curve that lies on it must have even degree, as one sees by pulling back to $\mathbf{P}^2$—and thus a cone over the Veronese surface cannot contain a linear space of codimension 1. We shall see that this property separates the varieties of minimal degree which are cones over the Veronese surface from those that are scrolls.

We now describe rational normal scrolls in the terms necessary for Theorem 1. In our proof of the theorem we reduce rapidly to the case where X is a divisor on a scroll, and we shall describe these as well.

A rational normal scroll is a cone over a smooth linearly normal variety fibered over $\mathbf{P}^1$ by linear spaces; in particular, a rational normal scroll contains a pencil of linear spaces of codimension 1 (and these are the only linearly normal varieties with this property, as will follow from Proposition 2.1, below).

To be more explicit, think of $\mathbf{P}^r$ as the space of 1-quotients of k^{r+1}, so that a d-plane in $\mathbf{P}^r$ corresponds to a $d+1$-quotient of k^{r+1}. A variety $X \subset \mathbf{P}^r$ with a map $\pi\colon X \to \mathbf{P}^1$ whose fibers are d-planes is thus the projectivization of a rank $d+1$ vector bundle on $\mathbf{P}^1$ which is a quotient of

$$k^{r+1} \otimes_k \mathcal{O}_{\mathbf{P}^1} = \mathcal{O}_{\mathbf{P}^1}^{r+1}.$$

Slightly more generally, let

$$\mathcal{E} = \bigoplus_0^d \mathcal{O}_{\mathbf{P}^1}(a_i)$$

be a vector bundle on $\mathbf{P}^1$, and assume

$$0 \le a_0 \le \cdots \le a_d, \quad \text{with } a_d > 0,$$

so that $\mathcal{E}$ is generated by $\sum a_i + d + 1$ global sections. Write $\mathbf{P}(\mathcal{E})$, or alternately $\mathbf{P}(a_0, \ldots, a_d)$, for the projectivized vector bundle

$$\mathbf{P}(\mathcal{E}) = \operatorname{Proj} \operatorname{Sym} \mathcal{E} \to \mathbf{P}^1$$

(whose points over $\lambda \in \mathbf{P}^1$ are quotients $\mathcal{E}_\lambda \to k(\lambda)$), and let $\mathcal{O}_{\mathbf{P}(\mathcal{E})}(1)$ be the tautological line bundle. Because the a_i are ≥ 0, $\mathcal{O}_{\mathbf{P}(\mathcal{E})}(1)$ is generated by its global sections (see the computation below) and defines a "tautological" map

$$\mathbf{P}(\mathcal{E}) \to \mathbf{P}^{\sum a_i + d}.$$

This map is birational because $a_d > 0$. We write $S(\mathcal{E})$ or $S(a_0, \ldots, a_d)$ for the image of this map, which, as we shall see, is a variety of dimension $d+1$ and degree $\sum a_i$, so that it is a variety of minimal degree. A *rational normal scroll* is simply one of the varieties $S(\mathcal{E})$. Note that $\mathcal{O}_{\mathbf{P}(\mathcal{E})}(1)$ induces $\mathcal{O}_{\mathbf{P}^d}(1)$ on each fiber $F \cong \mathbf{P}^d$ of $\mathbf{P}(\mathcal{E}) \to P^1$, so F is mapped isomorphically to a d-plane in $S(\mathcal{E})$.

The most familiar examples of rational normal scrolls are probably

(o) $\mathbf{P}^d$, which is $S(0, \ldots, 0, 1)$,

(i) the rational normal curve of degree a in $\mathbf{P}^a$, which is $S(a)$,

(ii) the cone over a plane conic, $S(0,2) \subset \mathbf{P}^3$,

(iii) the nonsingular quadric in $\mathbf{P}^3$, $S(1,1)$,

(iv) the projective plane blown up at one point, embedded as a surface of degree 3 in $\mathbf{P}^4$ by the series of conics in the plane passing through the point; this is $S(1,2)$.

There is a pretty geometric description of $S(a_0, \ldots, a_d)$ from which the name "scroll" derives, and from which the equivalence of the two definitions above may be deduced:

The projection

$$\mathcal{E} = \bigoplus_0^d \mathcal{O}(a_i) \to \mathcal{O}(a_i)$$

defines a section $\mathbf{P}^1 \cong \mathbf{P}(\mathcal{O}(a_i)) \hookrightarrow \mathbf{P}(\mathcal{E})$, and

$$\mathcal{O}_{\mathbf{P}(\mathcal{E})}(1)|_{\mathbf{P}(\mathcal{O}(a_i))} = \mathcal{O}_{\mathbf{P}(\mathcal{O}(a_i))}(1) = \mathcal{O}_{\mathbf{P}^1}(a_i),$$

so this section is mapped to a rational normal curve of degree a_i in the $\mathbf{P}^{a_i} \subset \mathbf{P}^{\sum a_i + d}$ corresponding to the quotient $H^0(\mathcal{E}) \to H^0(\mathcal{O}_{\mathbf{P}^1}(a_i))$. (Of course if $a_i = 0$, the "rational normal curve of degree a_i" is a point $\subset \mathbf{P}^0$!) Thus we may construct the rational normal scroll $S(a_0, \dots, a_d) \subset \mathbf{P}^{\sum a_i + d}$ by considering the parametrized rational normal curves

$$\mathbf{P}^1 \xrightarrow{\phi_i} C_{a_i} \subset \mathbf{P}^{a_i} \subset \mathbf{P}^{\sum a_i + d}$$

corresponding to the decomposition

$$k^{\sum a_i + d + 1} = \bigoplus_0^d k^{a_i + 1},$$

and letting $S(a_0, \dots, a_d)$ be the union over $\lambda \subset \mathbf{P}^1$ of the d-planes spanned by $\phi_0(\lambda), \dots, \phi_d(\lambda)$. In particular, we see that the cone in $\mathbf{P}^{\sum a_i + d + s}$ over $S(a_0, \dots, a_d)$ is

$$S(\underbrace{0, \dots, 0}_{s}, a_0, \dots, a_d).$$

Also, $S(a_0, \dots, a_d)$ is nonsingular iff $(a_0, \dots, a_d) = (0, \dots, 0, 1)$ or $a_i > 0$ for all i.

We note that this description is convenient for giving the homogeneous ideal of $S(a_0, \dots, a_d)$. As is well known, the homogeneous ideal of a rational normal curve $S(a) \subset \mathbf{P}^a$ may be written as the ideal of 2×2 minors

$$\det{}_2 \begin{pmatrix} X_0, X_1, \dots, X_{a-1} \\ X_1, X_2, \dots, X_a \end{pmatrix},$$

and this expression gives the parametrization sending $(s, t) \in \mathbf{P}^1$ to the point of $\mathbf{P}^a$ where the linear forms

$$sX_0 + tX_1, \dots, sX_{a-1} + tX_a$$

all vanish. (This is s times the first row of the given matrix plus t times the second row.) It follows at once that $S(a_0, \dots, a_d)$ is at least set-theoretically the locus where the minors of a matrix of the form

$$\left(\begin{array}{c|c|c} X_{0,0} X_{0,1}, \dots, X_{0,a_0-1} & X_{1,0}, \dots, X_{1,a_1-1} & \dots X_{d,a_d-1} \\ & & \dots \\ X_{0,1}, X_{0,2}, \dots, X_{0,a_0} & X_{1,1}, \dots, X_{1,a_1} & \dots X_{d,a_d} \end{array}\right)$$

all vanish. That these minors generate the whole homogeneous ideal follows easily as in the proof of Lemma 2.1 below.

The divisor class group of a projectivized vector bundle $\mathbf{P}(\mathcal{E})$ over $\mathbf{P}^1$ is easy to describe (Hartshorne (1977), Chapter II, exc. 7.9): Writing H for a divisor in the class determined by $\mathcal{O}_{\mathbf{P}(\mathcal{E})}(1)$, and F for the fiber of $\mathbf{P}(\mathcal{E}) \to \mathbf{P}^1$, the divisor class group may be written (confusing divisors and their classes systematically)

$$\mathbf{Z}H + \mathbf{Z}F.$$

Moreover, the chow ring is given by

$$\mathbf{Z}[F, H] / \left(F^2, H^{d+2}, H^{d+1}F, H^{d+1} - \left(\sum a_i\right) H^d F\right).$$

We shall only need a numerical part of this, giving the degree of a scroll:

$$\text{degree}\, S(a_0, \ldots, a_d) = H^{d+1} = \sum_0^d a_i.$$

The simplest way to understand this is perhaps from the geometric description given above: In $\mathbf{P}^{\sum_0^d a_i + d}$ we may take a hyperplane containing the natural copy of $\mathbf{P}^{\sum_1^d a_i + d - 1}$ and meeting $C_{a_0} \subset \mathbf{P}^{a_0}$ transversely. The hyperplane section is then the union of $S(a_1, \ldots, a_d)$ with a_0 copies of F (which is embedded as a d-plane).

It is also easy to compute the cohomology of the line bundles on $\mathbf{P}(\mathcal{E})$. In particular, the tautological map

$$\pi^* \mathcal{E} \to \mathcal{O}_{\mathbf{P}(\mathcal{E})}(1)$$

induces for any integer a a map

$$\operatorname{Sym}_a \mathcal{E} = \pi_* \operatorname{Sym}_a \pi^* \mathcal{E} \to \pi_* \mathcal{O}_{\mathbf{P}\mathcal{E}}(a)$$

and thus for every a, b a map

$$\mathcal{O}_{\mathbf{P}^1}(b) \otimes \operatorname{Sym}_a \mathcal{E} \to \mathcal{O}_{\mathbf{P}^1}(b) \otimes \pi_* \mathcal{O}_{\mathbf{P}\mathcal{E}}(a) \cong \pi_*(\pi^* \mathcal{O}_{\mathbf{P}^1}(b) \otimes \mathcal{O}_{\mathbf{P}\mathcal{E}}(a)),$$

which is an isomorphism, as one easily checks locally. Since π is surjective, π_* induces an isomorphism on global sections, and we see that an element

$$\sigma \in H^0(\pi^* \mathcal{O}_{\mathbf{P}^1}(b) \otimes \mathcal{O}_{\mathbf{P}\mathcal{E}}(a))$$

may be represented as an element of

$$\begin{aligned} H^0(\mathcal{O}_{\mathbf{P}^1}(b) \otimes \operatorname{Sym}_a \mathcal{E}) &= H^0\left(\mathcal{O}_{\mathbf{P}^1}(b) \otimes \sum_{|I|=a} \mathcal{O}_{\mathbf{P}^1}\left(\sum_{i \in I} a_i\right)\right) \\ &= \sum_{|I|=a} H^0\left(\mathcal{O}_{\mathbf{P}}\left(b + \sum_{i \in I} a_i\right)\right), \end{aligned}$$

where the notation $\sum_{|I|=a}$ indicates summation over all collections I consisting of a elements (with repetitions) from $\{0, \ldots, d\}$.

From this we may derive a useful representation of divisors in $\mathbf{P}\mathcal{E}$, generalizing the idea of "bihomogeneous forms" in the case of $\mathbf{P}(\mathcal{O}_{\mathbf{P}^1}^{d+1}) = \mathbf{P}^1 \times \mathbf{P}^d$. If we let

$$x_i \in H^0(\pi^* \mathcal{O}_{\mathbf{P}^1}(-a_i) \otimes \mathcal{O}_{\mathbf{P}\mathcal{E}}(1)) = H^0 \mathcal{E}(-a_i)$$

be an element corresponding to a generator of the ith summand

$$\mathcal{O}_{\mathbf{P}^1}(a_i - a_i) = \mathcal{O}_{\mathbf{P}^1} \subset \mathcal{E}(-a_i),$$

and write

$$x^I := \prod_{i\in I} x_i \in H^0\left[(\operatorname{Sym}_a \mathcal{E})\Big(-\sum_{i\in I} a_i\Big)\right]$$
$$= H^0\left(\pi^*\mathcal{O}_{\mathbf{P}^1}\Big(-\sum_{i\in I} a_i\Big)\otimes \mathcal{O}_{\mathbf{P}\mathcal{E}}(a)\right)$$

for the product, then we may represent σ conveniently as a "polynomial":

$$\sigma = \sum_{|I|=a} \alpha_I(s,t)x^I,$$

where s, t are homogeneous coordinates on $\mathbf{P}^1$ and where $\alpha_I(s,t)$ is a homogeneous form of degree

$$\deg \alpha_I(s,t) = b + \sum_{i\in|I|} a_i.$$

This representation is convenient because the "variables"

$$x_i \in H^0(\pi^*\mathcal{O}_{\mathbf{P}^1}(-a_i)\otimes \mathcal{O}_{\mathbf{P}\mathcal{E}}(1))$$

restrict to a basis of the linear forms on each fiber of $\mathbf{P}\mathcal{E} \to \mathbf{P}^1$, and the divisor D of σ meets the $\mathbf{P}^d \cong F_{(u,v)}$ over $(u,v)\in \mathbf{P}^1$ in the hypersurface with equation $\sum_{|I|=a} \alpha_I(u,v)x^I$.

In practice, we wish to use this idea on a Weil divisor X of a scroll $S(\mathcal{E})$. Since $S(\mathcal{E})$ is normal and $\mathbf{P}\mathcal{E} \to S\mathcal{E}$ is birational, we may do this by defining $\tilde{X} \subset \mathbf{P}\mathcal{E}$ to be the "strict transform" of X—that is, for an irreducible subvariety X of codimension 1, $\tilde{X}$ is the closure of the image in $\mathbf{P}\mathcal{E}$ of the complement, in X, of the fundamental locus of the inverse rational map, $S(\mathcal{E}) \to \mathbf{P}\mathcal{E}$. Then $\tilde{X}$ occupies a well-defined divisor class on $\mathbf{P}\mathcal{E}$, and we may apply the above technique to it.

2. Rational normal scrolls in the wild. The proof of Theorem 2 rests on a technique of constructing scrolls from their determinantal equations, as follows:

We say that a map of k-vector spaces

$$\phi: U\otimes V \to W$$

is *nondegenerate* if $\phi(u\otimes v)\neq 0$ whenever $u, v\neq 0$, or equivalently if each map $\phi_u: u\otimes V \to W$ is a monomorphism. The typical example, for our purposes, comes from a (reduced, irreducible) variety X and a pair of line bundles $\mathcal{L}$, $\mathcal{M}$; if $U = H^0(\mathcal{L})$, $V = H^0(\mathcal{M})$, and $W = H^0(\mathcal{L}\otimes\mathcal{M})$, then the multiplication map is obviously nondegenerate in the above sense. In our application, X will be embedded linearly normally in $\mathbf{P}^r$ by $\mathcal{L}\otimes\mathcal{M}$, so we may identify $H^0(\mathcal{L}\otimes\mathcal{M})$ with $H^0(\mathcal{O}_{\mathbf{P}^r}(1))$.

In general, given any map

$$k^\gamma \otimes k^\delta \to H^0\mathcal{O}_{\mathbf{P}^r}(1),$$

we define an associated map of sheaves

$$A_\phi: \mathcal{O}_{\mathbf{P}^r}^\delta(-1) \to \mathcal{O}_{\mathbf{P}^r}^\gamma$$

by twisting the obvious map

$$k^{\delta} \otimes \mathcal{O}_{\mathbf{P}^r} \to k^{\gamma^*} \otimes \mathcal{O}_{\mathbf{P}^r}(1)$$

by $\mathcal{O}_{\mathbf{P}^r}(-1)$. Taking $\gamma = 2$, we have

LEMMA 2.1. *If $\phi: k^2 \otimes k^{\delta} \to H^0\mathcal{O}_{\mathbf{P}^r}(1)$ is a nondegenerate pairing, then the ideal of 2×2 minors $\det_2 A_{\phi}$ is prime, and $V(\det_2 A_{\phi})$ is a rational normal scroll of degree δ.*

PROOF. If the image of ϕ is a proper subspace of $H^0\mathcal{O}_{\mathbf{P}^r}(1)$, then $V(\det_2 A_{\phi})$ is a cone. Since the cone over a scroll is a scroll, we may by reducing r assume that ϕ is an epimorphism, so that the rank of A_{ϕ} never drops to 0 on $\mathbf{P}^r$. It follows that $\mathcal{L} = \operatorname{Coker} A_{\phi}$ is a line bundle on $S = V(\det_2 A_{\phi})$, generated by the image of $V = k^{2^*}$. The linear series $(\mathcal{L}, V)$ defines a map $\pi: S \to \mathbf{P}^1$. If $(s,t) \in \mathbf{P}^1$, then the fiber F of π over (s,t) is the scheme defined by the vanishing of the composite map

$$\mathcal{O}_{\mathbf{P}^r}^{\delta}(-1) \to \mathcal{O}_{\mathbf{P}^r}^{2} \xrightarrow{(s,t)} \mathcal{O}_{\mathbf{P}^r};$$

and this scheme is, by our nondegeneracy hypothesis, given by the vanishing of δ linearly independent linear forms, so F is a plane of codimension δ. By the general formula for the maximum codimension of (any component of) a determinantal variety we have $\operatorname{codim} S \le \delta - 1$, so the map $S \to \mathbf{P}^1$ is onto, and since the fibers are smooth and irreducible, and the map is proper, S is smooth and irreducible of codimension $\delta - 1$.

Since $\det_2 A_{\phi}$ thus has height $\delta - 1$ in the homogeneous coordinate ring of $\mathbf{P}^r$, it is perfect, and in particular unmixed (Arbarello et al. (1984), Chapter II, 4.1; note that the characteristic 0 hypothesis there is irrelevant). Thus $\det_2 A_{\phi}$ is the entire homogeneous ideal of S, and since $\det_2 A_{\phi}$ is perfect, S is arithmetically Cohen-Macaulay, so in particular S is linearly normal.

The fibers of π, being linear spaces in $\mathbf{P}^r$, correspond to quotients of k^{r+1}, and this defines a vector bundle on $\mathbf{P}^1$ of rank $r - \delta + 1$ such that $S \to \mathbf{P}^1$ is the associated projective space bundle; thus S is a rational normal scroll as claimed. $\square$

REMARK. Using the same ideas, one sees that the height of $\det_2(A_{\phi})$ is $\delta - 1$ iff the rank of ϕ_u never drops by more than 1; then $X = V(\det_2 A_{\phi})$ is a "crown", that is, the union of a scroll of codimension $\delta - 1$ and some linear spaces of codimension $\delta - 1$ which intersect the scroll along linear spaces of codimension δ (fibers of π)—see Xambò (1981).

With this result in hand, it is easy to complete the proof of Theorem 2:

PROOF OF THEOREM 2. Let $k^2 \cong V \subset H^0\mathcal{O}_X(D)$ be the vector space of sections corresponding to the pencil D_{λ}, and let H be the hyperplane section of X. The natural multiplication map

$$V \otimes H^0\mathcal{O}_X(H - D) \to H^0\mathcal{O}_X(H) = H^0\mathcal{O}_{\mathbf{P}^r}(1)$$

is nondegenerate, and thus gives rise to a scroll S containing all the D_{λ}, and thus X. The linear space $\overline{D}_{\lambda}$ is the intersection of all the hyperplanes containing

D_λ, which correspond to elements of $H^0\mathcal{O}_X(H-D)$, so $\overline{D}_\lambda$ is the fiber over λ of $S \to \mathbf{P}^1$, as desired. $\square$

EXAMPLES. (i) Let C be a hyperelliptic (or elliptic) curve, $C \subset \mathbf{P}^r$ an embedding by a complete series of degree d. C is a divisor on the variety S which is the union of the secants corresponding to the $\mathfrak{g}^1_2$ on C (or, if C is elliptic, any $\mathfrak{g}^1_2$ on C). This variety is a rational normal scroll $S(\mathcal{E})$ and $\tilde{C} \sim 2H + (d-2r+2)F$ on $\mathbf{P}(\mathcal{E})$. More generally, a linearly normal curve $C \subset \mathbf{P}^r$ which possesses a $\mathfrak{g}^1_d$ lies on a scroll of dimension $\leq d$; if $C \subset \mathbf{P}^r$ is the canonical embedding, then this scroll is of dimension $\leq d-1$, so in particular the canonical image of any trigonal curve is a divisor on a 2-dimensional scroll $S(\mathcal{E})$, and $\tilde{C} \sim 3H + (4-g)F$ on $\mathbf{P}(\mathcal{E})$. See Schreyer (1986) for a study of canonical curves using this idea.

(ii) A $K3$ surface, embedded linearly normally in any projective space, is a divisor on a 3-dimensional scroll if it contains an elliptic cubic (which then moves in a nontrivial linear series). See for example Saint-Donat (1974).

3. The classification theorem. Before giving our proof of the Del Pezzo–Bertini Theorem, we record three elementary observations about projections:

(1) If X is a variety of minimal degree, then X is linearly normal. (*Proof*: If X were the isomorphic projection of a nondegenerate variety X' in $\mathbf{P}^{r+1}$, then X' would have degree less than that allowed by Proposition 0.)

(2) If $X \subset \mathbf{P}^r$ is a variety of minimal degree and $p \in X$, then the projection $\pi_p X \subset \mathbf{P}^{r-1}$ is a variety of minimal degree, the map $X - p \to \pi_p X$ is separable, and if p is singular then X is a cone with vertex p. (*Proof*: Indeed, $\pi_p X$ is obviously nondegenerate. If X is a cone with vertex p, the result is obvious. Else $\dim \pi_p X = \dim X$ but $\deg \pi_p X \leq \deg X - 1$. The inequality must actually be an equality by Proposition 0, which shows in particular that p is a nonsingular point, and $\pi_p : X - p \to \pi_p X$ is birational.)

(3) If $p \in X \subset \mathbf{P}^r$ is any point on any variety, E_X the exceptional fiber of the blow-up of p in X, and $E_{\mathbf{P}^r} \cong \mathbf{P}^{r-1}$ the exceptional fiber of the blow-up of $\mathbf{P}^r$ at p, then E_X is naturally embedded in $E_{\mathbf{P}^r}$, which is mapped isomorphically to $\mathbf{P}^{r-1}$ by the map induced by π_p. Thus $E_X \subset \pi_p(X) \subset \mathbf{P}^{r-1}$. In particular, if p is a nonsingular point on X, so that E_X is a linear subspace of $\mathbf{P}^{r-1}$, then the "image of p" under $\pi_p : X \to \pi_p(X) \subset \mathbf{P}^{r-1}$ is a linear subspace of $\mathbf{P}^{r-1}$ which is a divisor on $\pi_p(X)$. More naively, this is the image of the tangent plane to X at p.

In view of observation (3) it will be useful to begin with the following result, which "recognizes" scrolls:

PROPOSITION 3.1. *If $X \subset \mathbf{P}^r$ is a variety of minimal degree, and X contains a linear subspace of $\mathbf{P}^r$ as a subspace of codimension 1, then X is a scroll.*

PROOF. By Proposition 2.1 it suffices to show that X contains a pencil of linear divisors, though the given subspace itself may not move.

Let $F \subset X$ be the given linear subspace. We may assume (by projecting, if necessary) that X is smooth along F. Let $H \subset \mathbf{P}^r$ be a general hyperplane containing F, and let $S = H \cap X - F$. Let π_F be projection from F. We distinguish two cases:

Case 1. $\dim \pi_F(X) \geq 2$. By Bertini's Theorem and observation (2) above, S is then a reduced and irreducible variety, of degree and dimension one less than that of X. Thus by Proposition 0, S is degenerate in H, so $F = H \cdot X - S$ moves in (at least) a pencil of linear spaces, and we are done.

Case 2. $\dim \pi_F(X) = 1$. By observation (1), $\pi_F(X)$ is a curve of minimal degree, say of degree s in $\mathbf{P}^s$. Projecting $\pi_F(X)$ from $s-1$ general points on it gives a birational map to $\mathbf{P}^1$, so $\pi_F(X) \cong \mathbf{P}^1$. Further, the cone on S with vertex p is a union of s planes, the spans of F with the points of a general hyperplane section of $\pi_F(X)$, so S has s components. But $s = r - \dim F - 1 = \operatorname{codim} X = \deg X - 1 = \deg S$, so S is the union of s planes, and these are linearly equivalent to each other since the points of $\pi_F(X)$ are. Thus a component of S is a linear space moving in a pencil as desired. $\square$

Proof of Theorem 1. Let $X \subset \mathbf{P}^r$ be a variety of minimal degree. We may assume that the codimension c of X is ≥ 2 and that X is not a cone. By Proposition 3.1 we may as well also assume that X contains no linear space of codimension 1, so that in particular the dimension d of X is ≥ 2, and we must prove that under these hypotheses X is the Veronese surface $\mathbf{P}^2 \subset \mathbf{P}^5$. In fact, it suffices to prove that $X \cong \mathbf{P}^2$; for the embedding of $\mathbf{P}^2$ by the complete series of curves of degree d gives a surface of degree d^2 and codimension

$$\binom{2+d}{2} - 3,$$

which is $< d^2 - 1$ for $d \geq 3$.

Let $p \in X$ be any point. By observation (3) and Proposition 3.1, $\pi_p(X)$ is a scroll, so the cone $S \subset \mathbf{P}^r$ with vertex p over $\pi_p(X)$ (or over X) is a scroll, say $S = S(\mathcal{E})$, with $\mathcal{E} = \bigoplus_0^d \mathcal{O}_{\mathbf{P}^1}(a_i)$ and $0 \leq a_0 \leq \cdots \leq a_d$. X is a divisor on S.

Consider the strict transform $\tilde{X} \subset \mathbf{P}(\mathcal{E})$ of X under the desingularization $\mathbf{P}(\mathcal{E}) \to S(\mathcal{E}) = S$, and let its divisor class be $aH - bF$. We will prove under the hypotheses above that $a = 2$ and X is a surface. (Along the way we will see numerically that $b = 4$, $(a_0, a_1, a_2) = (0, 1, 2)$, so $c = 3$ and $X \subset \mathbf{P}^5$ as befits the Veronese, but we will not use this directly.)

First, because the degree $c+1$ of X is 1 more than that of S, and on the other hand is $H^{d-1} \cdot (aH - bF)$, we get $b = (a-1)c - 1$.

To bound a, first note that X must meet every fiber of $\mathbf{P}\mathcal{E} \to \mathbf{P}^1$, so $aH - bF|_F = aH|_F > 0$, and $a \geq 1$. If a were 1, then $\tilde{X}$ would meet each fiber F in a linear space of dimension $d-1$. Since each fiber F is mapped isomorphically to a d-plane in $\mathbf{P}^r$ under $\mathbf{P}(\mathcal{E}) \to S(\mathcal{E})$, X would contain linear spaces of dimension $d-1$, contrary to our hypothesis. Thus $a \geq 2$.

As in §2, $\tilde{X}$ may be represented by an equation $g = 0$ with g of the form:

$$g = \sum_{|I|=a} \alpha_I(s,t) x^I,$$

with

$$\deg \alpha_I = \left(\sum_{i\in I} a_i\right) - b = \sum_{i\in I} a_i - (a-1)c + 1.$$

If the variable x_0 did not occur in g, then $\tilde{X}$ would meet each fiber F in a cone over the preimage of p, and X itself would be a cone contrary to hypothesis. But for x_0 to occur we must have

$$0 \leq \deg \alpha_{0,d,\dots,d} = a_0 + (a-1)a_d - (a-1)c + 1.$$

Since S is a cone we have $a_0 = 0$, and we derive

$$(*) \qquad a_d \geq c - 1/(a-1).$$

If x_d occurred in every nonzero term of g, then for every fiber F, $\tilde{X} \cap F$ would contain the $(d-1)$-plane $x_d = 0$, and again X would contain a $(d-1)$-plane, contradicting our hypotheses. Thus

$$(**) \qquad 0 \leq \deg \alpha_{d-1,d-1,\dots,d-1} = a a_{d-1} - (a-1)c + 1.$$

Now if $a \geq 3$, then $a_d = c$ by $(*)$; but $c = \deg X - 1 = \deg S = \sum_{i=0}^{d} a_i$, so this implies $a_{d-1} = 0$, and $(**)$ gives a contradiction. Thus $a = 2$ as claimed, and $a_d \leq c-1$. Condition $(*)$ now gives $a_d = c-1$, so $a_{d-1} = 1$ and $a_0 = \cdots = a_{d-2} = 0$. Applying $(**)$ again we get $a_d = 1$ or $a_d = 2$.

In the first case $(a_0, \dots, a_d) = (0, \dots, 0, 1, 1)$, so S is a cone over $\mathbf{P}^1 \times \mathbf{P}^1 \subset \mathbf{P}^3$. A suitable hypersurface section of S will consist of the union of two planes F_1 and F_2, the cones over the rulings of $\mathbf{P}^1 \times \mathbf{P}^1$. Since each of these rulings sweeps out all of $\mathbf{P}^1 \times \mathbf{P}^1$, X must meet each of F_1 and F_2 in codimension 1. Because $c = 2$ we have $\deg X = 3$, so either $X \cap F_1$ or $X \cap F_2$ must be a linear space, contradicting our assumption on X.

We thus see that $a = 2$, $c = 3$, and $(a_0, \dots, a_d) = (0, \dots, 0, 1, 2)$. Under these circumstances the sum of the terms of g involving $x_0, \dots, x_{d-2}$ may be written

$$\left(\sum_0^{d-2} \alpha_{i,d} x_i\right) x_d,$$

with $\alpha_{i,d}$ constant. Thus if $d \geq 3$ the locus $g = 0$ in each fiber F is a cone with vertex the $(d-3)$-dimensional linear space given by

$$x_d = x_{d-1} = \sum_0^{d-2} \alpha_{i,d} x_i = 0.$$

Of course S is itself a cone with $(d-2)$-dimensional vertex L, say. The $(d-2)$-dimensional subspaces of the fibers F given by $x_d = x_{d-1} = 0$ are all mapped isomorphically to L under $\mathbf{P}(\mathcal{E}) \to S$, and the restrictions of the coordinates $x_0, \dots, x_{d-2}$ are all identified, and become coordinates on L. Thus X meets the image of each fiber in a cone with vertex given in L by $\sum_0^{d-2} \alpha_{i,d} x_i = 0$, so X is a cone, contradicting our assumption. This shows $d = 2$.

We have now shown that $a = 2$ and X is a surface. In this case, for every fiber $F \cong \mathbf{P}^2$ of $\mathbf{P}(\mathcal{E})$, $F \cap \tilde{X}$ is a conic, necessarily nonsingular since else X would contain a line. Thus $\tilde{X}$ is a rational ruled surface. But the preimage in X of p is a line, so $\tilde{X}$ is the blow-up of X at p, and is not a minimal surface. This is only possible if $\tilde{X} \cong \mathbf{P}(\mathcal{O}_{\mathbf{P}^1} \oplus \mathcal{O}_{\mathbf{P}^1}(1))$ and $X \cong \mathbf{P}^2$, as required. $\square$

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