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Chapter 18

Free resolutions and canonical curves

In Chapter 16 we related the resolutions of curves in $\mathbb{P}^3$ to their Hartshorne–Rao modules. The simplest case was that in which the Hartshorne–Rao module vanishes, that is, the case of arithmetically Cohen–Macaulay curves. The first 3 sections of this chapter constitue a quick review of some necessary parts of homological commutative algebra; see [Eisenbud 1995, Part III] for a more complete exposition. In particular, we explain how the condition that a curve in $\mathbb{P}^r$ is arithmetically Cohen–Macaulay manifests itself in the free resolution of the homogeneous ideal of the curve. We then introduce the Eagon–Northcott complexes, and explain their relation to the free resolutions of canonical curves. We close the chapter with an explanation of Green's conjecture, which proposes a way in which the intrinsic geometry of a curve may be connected to the shape of its minimal free resolution of its ideal in the canonical embedding.

18.1. Free resolutions

The basic results described here depend on Nakayama's lemma and hold in two parallel contexts: modules over a local ring, and graded modules over a polynomial ring whose variables have positive degree. We will work both with local rings and with polynomial rings whose variables have degree 1, as appropriate, and leave the translations to the reader.

Let M be a finitely generated graded module over $S := \mathbb{C}[x_0, \dots x_r]$. A *free resolution* of M is an exact complex of graded free modules, with maps of degree 0:

$$(\mathbb{F}, \phi): \quad 0 \leftarrow M \xleftarrow{\epsilon} \bigoplus_j S(-j)^{\beta_{0,j}} \leftarrow \cdots \xleftarrow{\phi_t} \bigoplus_j S(-j)^{\beta_{t,j}} \leftarrow 0.$$

Here $S(-j)$ denotes the graded free module of rank 1 with generator in degree j. The map to M is not considered part of the free resolution. The resolution is called *minimal* if a minimal set of generators of $F_i \coloneqq \bigoplus_j S(-j)^{\beta_{t,j}}$ maps to a minimal set of generators of the kernel of the following map, or equivalently (by Nakayama's lemma) the maps in $\mathbb{F} \otimes_S \mathbb{C}$ are all 0. The numbers $\beta_{i,j}$ are called the *Betti numbers* of M.

The most basic examples of minimal free resolutions are the Koszul complexes (first defined, despite the name, in [Cayley 1848], and found as examples, in [Hilbert 1890]), which resolve S/I when $I = (f_1, \dots, f_t)$ is a complete intersection, that is, $f_1, \dots, f_t$ is a regular sequence. For $t = 2, 3$, if $\deg f_i = d_i$, these have the form

$$S \xleftarrow{(f_1\ f_2)} S(-d_1) \oplus S(-d_2) \xleftarrow{\left(\begin{smallmatrix} -f_2 \\ f_1 \end{smallmatrix}\right)} S(-d_1 - d_2) \longleftarrow 0$$

and

$$S \xleftarrow{(f_1\ f_2\ f_3)} F_1 \xleftarrow{\begin{pmatrix} 0 & f_3 & -f_2 \\ -f_3 & 0 & f_1 \\ f_2 & -f_1 & 0 \end{pmatrix}} F_2 \xleftarrow{\begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix}} F_3 \longleftarrow 0$$

where

$$F_1 = \bigoplus_{j=1}^{3} S(-d_j), \quad F_2 = \bigoplus_{1 \le i < j \le 3} S(-d_i - d_j), \quad F_3 = S(-d_1 - d_2 - d_3).$$

Minimal free resolutions of a given module M are all isomorphic, and thus provide interesting invariants of M.

To construct the minimal free resolution of a finitely generated graded module M, suppose that a minimal homogeneous set of generators of M contains $\beta_{0,j}$ generators of degree j for each j; the choice of generators defines a degree 0 map ϵ from $\oplus_j S(-j)^{\beta_{0,j}}$ onto M. We proceed to do the same with the kernel of ϵ to construct ϕ_1, and continue similarly to construct $\phi_2 \dots$. Hilbert's basis theorem and syzygy theorem together imply that that for some $t \le r+1$ we will find that ϕ_t has kernel equal to 0; that is, every finitely generated S-module has a finite free resolution of length $t \le r + 1$ [Eisenbud 1995, Corollary 19.7]. The minimal such t is called the *projective dimension* of M written $\operatorname{pd} M$. Computing $\operatorname{Ext}(M, -)$ from such a resolution, Nakayama's lemma implies that,

$$t = \max\{s \mid \operatorname{Ext}^s_S(M, k) \neq 0\}.$$

It follows from the Auslander–Buchsbaum theorem [Eisenbud 1995, Theorem 19.9] that $t \ge \operatorname{codim} \operatorname{ann}_S(M)$, the codimension of the support of M.

Cheerful Fact 18.1. Fundamental results of Auslander, Buchsbaum and Serre say that a local ring R is *regular* — that is, the Krull dimension of R is equal to the minimal number of generators of is maximal ideal — if and only if the minimal free resolution of the residue field is finite, in which case the minimal free resolution of every module is finite.

18.2. Classification of 1-generic $2 \times f$ matrices

We can use the uniqueness of minimal free resolutions to give a simple proof of Kronecker's classification of 1-generic $2 \times f$ matrices, which was announced in Chapter 17.

Theorem 18.2. *Every* 1*-generic* $2 \times f$ *matrix of linear forms can be transformed by row and column operations and a linear change of variables to one of the type shown in Corollary 17.13, and thus the minors of any 1-generic matrix define a scroll.*

To prove this result we first reinterpret the 1-generic condition: We have observed that an $a \times b$ matrix of linear forms in c variables is the same as a $\mathbb{C}$-linear map of vector spaces $A \otimes B \to C$, where A, B and C have dimensions a, b and c respectively. Such a map can be viewed in several ways, for example as a map $C^* \otimes B \to A^*$ — in other words, a $c \times b$ matrix in a variables — or equivalently an a-dimensional family of $c \times b$ matrices (and similarly for other permutations of A, B, C). This bit of trivial formalism pays off in the following observation:

Proposition 18.3. *An* $a \times b$ *matrix of linear forms in* c *variables* M *corresponding to* $A \otimes B \to C$ *is 1-generic if and only if the* $c \times b$ *matrix* N *of linear forms in* a *variables corresponding to* $C^* \otimes B \to A$ *has constant rank* b*; that is, for any point* x *in* $\mathbb{P}^{a-1}$*, the rank of* N *evaluated at* x *is* b*.*

Proof. The b columns of N correspond to the b columns of M, while the rows of N are indexed by the c variables in M and the variables in N are indexed by the a rows of M. Thus the evaluation of N at a point x corresponds to a generalized row of M; and the image of $N(x)$ is the span of the variables in that generalized row. The matrix M is 1-generic if the dimension of that span is b for every generalized row. □

Thus the classification of 1-generic $2 \times f$ matrices of linear forms in $r + 1$ variables of Theorem 1.2 is equivalent to the classification of *matrix pencils* of constant maximal rank — that is $f \times (r+1)$ matrices of linear forms over $\mathbb{P}^1$ with constant rank f. Such "matrix pencils" were first classified by Kronecker; see [Gantmacher 1959, Chapter 12] for an exposition, and [Eisenbud and Harris 1987] for a geometric approach.

Proof of Theorem 1.2. Let M be a 1-generic $2 \times b$ matrix in $r+1$ variables. We may assume that the span of the entries is equal to the vector space of linear forms in $\mathbb{P}^r$. The associated $(r+1) \times b$ matrix

$$N : \mathcal{O}_{\mathbb{P}^1}(-1)^b \to \mathcal{O}_{\mathbb{P}^1}^{r+1}$$

of linear forms over $\mathbb{P}^1$ has constant rank b. For every point $x \in \mathbb{P}^1$ the scalar matrix $N(x)$ is a split inclusion, and thus coker N is a vector bundle on $\mathbb{P}^1$ that is generated by its global sections, and thus, necessarily of the form $\sum_{i=1}^d \mathcal{O}_{\mathbb{P}^1}(a_i)$ for some integers $a_i \geq 0$.

Regarding N as a matrix over the homogeneous coordinate ring $S \coloneqq \mathbb{C}[s,t]$ of $\mathbb{P}^1$, and taking global sections of all non-negative twists, we get

$$0 \to S(-1)^b \xrightarrow{N} S^{r+1} \to \bigoplus_{i=1}^{d} \Big(\bigoplus_{j=0}^{\infty} H^0(\mathcal{O}_{\mathbb{P}^1}(j + a_i)) \Big) \to 0,$$

which is a minimal free resolution because $H^1(\mathcal{O}_{\mathbb{P}^1}(j)) = 0$ for all $j \geq -1$. It follows that the map N is the direct sum of the minimal S-free resolutions of the modules

$$\bigoplus_{j=0}^{\infty} H^0(\mathcal{O}_{\mathbb{P}^1}(j + a_i)) = (s,t)^{a_i},$$

where we take the generators to be in degree 0. As we will explain in Example 1.11, the minimal presentation of $(s,t)^{a_i}$ is the $(a_i+1) \times a_i$ matrix

$$\phi_a = \begin{pmatrix} s & 0 & 0 & \dots & 0 \\ -t & s & 0 & \dots & 0 \\ 0 & -t & s & \dots & 0 \\ 0 & 0 & -t & \dots & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & \ddots & s \\ 0 & 0 & 0 & \cdots & -t \end{pmatrix}.$$

Thus, by a change of bases and variables, the matrix N can be transformed into the direct sum of such matrices. The direct sum decomposition of N corresponds to a block decomposition of M. Translating this back to a $2 \times (r+d-1)$ matrix M, we have transformed M into a matrix of the desired form. $\square$

How to look at a resolution. As is apparent even in the example $t = 3$ above, free resolutions can be bulky to describe; the *Betti table* is a compact representation of the numerical information in the resolution. Suppose that F is a minimal free resolution of a graded module M as illustrated in the beginning of the previous section. Since we choose a minimal set of generators at each stage, the matrices of the ϕ_i have entries in the maximal ideal $(x_0, \dots, x_r)$, and thus each $\beta_{i+1,j}$ must be strictly greater than some $\beta_{i,j}$. For this reason it is convenient to tabulate the Betti numbers so that $\beta_{i,j}$ is in the i-th column and $(j-i)$-th row:

j	$i=0$	1	$\cdots$	n
$\vdots$	$\vdots$	$\vdots$		$\vdots$
0	$\beta_{0,0}$	$\beta_{1,1}$	$\cdots$	$\beta_{n,n}$
1	$\beta_{0,1}$	$\beta_{1,2}$	$\cdots$	$\beta_{n,n+1}$
$\vdots$	$\vdots$	$\vdots$		$\vdots$

Example 18.4. The Koszul complex that resolves the homogeneous coordinate ring $S/(Q, F_1, F_2)$ of the complete intersection of 2 quadrics and a cubic in $\mathbb{P}^3$ has the form

$$S \leftarrow S(-2) \oplus S(-3)^2 \leftarrow S(-5)^2 \oplus S(-6) \leftarrow S(-8) \leftarrow 0,$$

which has Betti table

j	$i=0$	1	2	3
0	1	–	–	–
1	–	1	–	–
2	–	2	–	–
3	–	–	2	–
4	–	–	1	–
5	–	–	–	1

where a dash represents 0.

To simplify our language, we will speak of the *Betti table of a scheme X* rather than the "Betti table of the minimal free resolution of the ideal of X".

When is a finite free complex a resolution? How does a free resolution over $S := \mathbb{C}[x_0, \dots x_r]$ "know" to end no later than the $(r+1)$-st step? The main theorem of [Buchsbaum and Eisenbud 1973], describes a sense in which the maps in the resolution change as the resolution continues.See [Eisenbud 1995, Theorem 20.9] for further exposition.

A central role in the theorem is played by the ideals of minors of the differentials in the complex: If $\phi : F \to G$ is a map of finitely generated free R-modules with cokernel M, then $I_t(\phi)$ denotes the ideal generated by all the $t \times t$ minors (subdeterminants) of a matrix representing ϕ; this is independent of the choice of bases used to represent ϕ as a matrix. The ideal $I_{\operatorname{rank} G - j}(\phi)$ depends only on M; it called the j-th *Fitting ideal* of coker ϕ and usually written $\operatorname{Fitt}_j M$. Some basic properties of these ideals are given in Exercise 1.12, where the reader may show that the annihilator of M has the same radical as $\operatorname{Fitt}_0 M$. Actually more is true:

removed equals sign in parenthesis

Cheerful Fact 18.5. If $F \to G \to M \to 0$ is an exact sequence of finitely generated R-modules, then

$$(\operatorname{ann}_R(\operatorname{coker}\phi))^{\operatorname{rank} G} \subset \operatorname{Fitt}_0 \phi \subset \operatorname{ann}_R(\operatorname{coker}\phi).$$

For this and other such inequalities, see [Buchsbaum and Eisenbud 1977].

The *rank* of $\phi : F \to G$ is the largest size of a nonvanishing minor of a matrix for ϕ, or equivalently the largest k such that the exterior power

$$\bigwedge^k \phi : \bigwedge^k F \to \bigwedge^k G$$

is nonzero. If $r := \operatorname{rank}\phi = \operatorname{rank} G$, then the ideal $I(\phi) := I_{\operatorname{rank}\phi}(\phi)$ plays a special role: the cokernel of ϕ is projective (= locally free) if and only if $I(\phi) = R$. If $I(\phi)$ contains a nonzerodivisor, then the construction localizes, and we see that $I(\phi)$ defines the locus $P \in \operatorname{Spec} R$ where $(\operatorname{coker}\phi)_P$ is not free.

The *grade* of an ideal I is defined to be the length of a maximal regular sequence contained I, or ∞ if $I = R$. If R is a Cohen–Macaulay ring such as $\mathbb{C}[x_0, \dots, x_r]$ then the grade of any proper ideal is equal to its codimension, so grade becomes a geometric notion. Readers less familiar with commutative algebra will lose little if they stick with the case when R is regular, or even the case when R is $\mathbb{C}[x_0, \dots, x_r]$. This suffices, for example, for the applications of the Eagon–Northcott complex described below.

Theorem 18.6. ([*Buchsbaum and Eisenbud 1973*])

Let R be a Noetherian ring and let

$$\mathbb{F} : \quad F_0 \xleftarrow{\phi_1} F_1 \leftarrow \cdots \leftarrow F_{n-1} \xleftarrow{\phi_n} F_n \leftarrow 0$$

be a finite complex of free S-modules. Set $r_i := \operatorname{rank}\phi_i$. The complex $\mathbb{F}$ is acyclic (that is, $H_i(\mathbb{F}) = 0$ for all $i > 0$) if and only if, for all i,

$$\operatorname{rank} F_i = r_i + r_{i+1} \quad \textit{and} \quad \operatorname{grade} I_{r_i}(\phi_i) \geq i. \qquad \square$$

A familiar case occurs when $r = 1$ and R is a domain. In this case the theorem says that a map $F_1 \to F_0$ is a monomorphism if and only if it becomes a monomorphism after tensoring with the field of rational functions K, which follows from the flatness of localization and the fact that F_1 is torsion-free, so that $F_1 \subset F_1 \otimes K$. More generally, if R is Noetherian, then a map of finitely generated free modules $\phi : F \to G$ is injective if and only if $I_{\operatorname{rank} F}(\phi)$ contains a nonzerodivisor.

To see the relevance of the grade hypothesis to the conclusion, suppose for a moment that R is a regular local ring of dimension r, and suppose that the complex $\mathbb{F}$ is acyclic. The hypothesis grade $I(\phi_{d+1}) \geq d+1$ can only be satisfied if $I(\phi_{d+1}) = R$ (so that its grade is ∞ by convention). This is equivalent to the cokernel of ϕ_{d+1} being free. Thus the theorem "explains" why a minimal free resolution has length $\leq r+1$.

18.3. Depth and the Cohen–Macaulay property

If M is a graded $\mathbb{C}[x_0, \dots x_r]$-module then an *$M$-regular sequence* is a sequence of homogeneous polynomials $f_1, \dots, f_m \in (x_0, \dots, x_r)$ such that f_1 is a nonzerodivisor on M, f_2 is a nonzerodivisor on $M/(f_1 M)$, and so on. The maximal length of such a sequence is called the *depth* of M, or more properly the depth of $(x_0, \dots, x_r)$ on M. The lengths of all maximal M-regular sequences are the same:

Theorem 18.7 (Auslander–Buchsbaum). *If M is a finitely generated graded module over $S := \mathbb{C}[x_0, \dots x_r]$, then the length of every M-regular sequence is*

$$m = r + 1 - \mathrm{pd}\ M,$$

and m is the smallest integer m such that $\mathrm{Ext}^m_S(S/(x_0, \dots x_r), M) \neq 0$. □

The depth of a module M is bounded above by $\dim M$, the Krull dimension. The reason is that if the dimension of M is d, and $f_1 \in (x_0, \dots x_r)$ is a nonzerodivisor on M, then $\dim M/(f_1)M = \dim M - 1$. Thus by induction, if $f_1, \dots, f_d$ is M-regular then $M/(f_1, \dots, f_d)M$ has dimension 0, which is equivalent to its being Artinian. Thus any $f_{d+1} \in (x_0, \dots x_r)$ acts as a nilpotent endomorphism of $M/(f_1, \dots, f_d)M$.

It follows from these facts that the depth of an S-module M is equal to the dimension of M if and only if the projective dimension of M is equal to the codimension of M; in this case we say that M is a *Cohen–Macaulay module*.

As we showed in Chapter 16, a curve $C \subset \mathbb{P}^3$ is linked to a complete intersection if and only if $H^1_*(\mathcal{I}_C) := \bigoplus_{m \in \mathbb{Z}} H^1(\mathcal{I}_C(m)) = 0$, in which case we said that C was arithmetically Cohen–Macaulay. This is equivalent to the condition that the homogeneous coordinate ring R_C of C is Cohen-Macaulay. In general, we say that a projective scheme $X \subset \mathbb{P}^r$ is arithmetically Cohen-Macaulay if its homogeneous coordinate ring R_X is Cohen-Macaulay, and this is true if and only if $\mathrm{pd}\, R_X = \mathrm{codim}\, X$.

The Gorenstein property. An important homological condition on a scheme X is the condition that ω_X is an invertible sheaf; when this holds, we say that X is *quasi-Gorenstein*. When, in addition, X is Cohen–Macaulay we say that X is *Gorenstein*. As with the Cohen–Macaulay property, the Gorenstein property is interpreted locally on a scheme. Any scheme that is locally a complete intersection, such as any smooth scheme, is Gorenstein. Since the restriction of $\mathcal{O}_{\mathbb{P}^r}(1)$ to a subvariety is always invertible, saying that a scheme X is canonically embedded implies that X is at least quasi-Gorenstein. We say that a projective scheme X is *arithmetically Gorenstein* if its homogeneous coordinate ring is Gorenstein, and it follows that $\omega_{S/I} \cong S/I(a)$ for some integer $a = a(X)$.

In Chapter 16, we expressed ω_X for a subscheme $X \subset Y$ of a scheme Y as $\mathrm{Ext}^{\operatorname{codim} X}_{\mathcal{O}_Y}(\mathcal{O}_X, \omega_Y)$. Slightly extending this idea, if $C \subset \mathbb{P}^r$ is a curve with homogeneous coordinate ring R_C, we define ω_{R_C} to be $\mathrm{Ext}^{r-1}_S(R_C, S(-r-1))$, where S is the homogeneous coordinate ring of $\mathbb{P}^r$. Since $\omega_{\mathbb{P}^r} = \mathcal{O}_{\mathbb{P}^r}(-r-1)$, the sheafification of this module is ω_C.

If C is arithmetically Cohen–Macaulay, so that $\mathrm{pd}(C) = \operatorname{codim}(C) = r-1$, then by computing $\mathrm{Ext}^{r-1}_S(R_C, S(-r-1))$ from the minimal free resolution

$$(\mathbb{F}, \phi): \quad 0 \leftarrow R_C \leftarrow S \xleftarrow{\phi_1} F_1 \leftarrow \cdots \leftarrow F_{r-2} \xleftarrow{\phi_{r-1}} F_r \leftarrow 0$$

we see that $\omega_{R_C} = \operatorname{coker} \phi^*_{r-1}$. Theorem 1.6 implies that the complex $(\mathbb{F}^*, \phi^*)$ which is the dual of the resolution $(\mathbb{F}, \phi)$ is again acyclic, so it is the minimal free resolution of ω_{R_C}. Thus ω_{R_C} is a Cohen–Macaulay module. Just as the Cohen–Macaulay property of R_C implies that $R_C = H^0_*(\mathcal{O}_C)$, it follows that $\omega_{R_C} = H^0_*(\omega_C)$.

If, in addition, C is a canonical curve, so that $\omega_C = \mathcal{O}_C(1)$, then we derive:

$$\omega_{R_C} = \operatorname{coker}(\phi^*_{r-2})(-r-1) = R_C(1)$$

so $\operatorname{coker} \phi^*_{r-2} = R_C(r)$. Thus $(\mathbb{F}^*(-r), \phi^*)$ is a minimal free resolution of R_C, and is therefore isomorphic to $(\mathbb{F}^*, \phi^*)$; that is, $(\mathbb{F}, \phi)$ is self-dual. We have seen an example already in the Koszul complex (a complete intersection is arithmetically Gorenstein).

Taking into account that in a resolution $(\mathbb{F}, \phi)$ each summand $S(-j)$ of F_{i+1} can only map to summands $S(-l)$ of F_i with $\ell < j$, and similarly for the dual, we see that the Betti table of the minimal free resolution of a canonical curve of genus $g \geq 4$ must have the form

j	$i=0$	1	2	$\cdots$	$g-3$	$g-2$
0	1	–	–	$\cdots$	–	–
1	–	b_1	b_2	$\cdots$	b_{g-3}	–
2	–	b_{g-3}	b_{g-4}	$\cdots$	b_1	–
3	–	–	–	$\cdots$	–	1

It turns out that the b_i depend on the particular canonical curve, but since the Hilbert function of the curve is the alternating sum of the Hilbert functions in the resolution, the differences $b_i - b_{g-i-1}$ are independent of the curve. This together with the self-duality implies that the whole Betti table of a canonical curve is determined by the numbers $b_1, \ldots, b_{g-2}$.

The geometry of linear series on C influences these Betti numbers. If $|\mathcal{L}|$ is a g^r_d on C then the multiplication map $H^0(\mathcal{L}) \otimes H^0(\mathcal{L}^{-1} \otimes \omega_C) \to H^0(\omega_C) = H^0(\mathcal{O}_C(1))$ corresponds to a $2 \times h^1(\mathcal{L})$ matrix of linear forms on $\mathbb{P}^{g-1}$ as in Chapter 17. and the ideal I generated by the minors of this matrix is contained in the homogeneous ideal of C. It turns out that the whole resolution of I is a subcomplex of the resolution of R_C. Resolutions of such determinantal ideals can be described explicitly, and we turn now to this description.

18.4. The Eagon–Northcott complex

The Eagon–Northcott complex $EN(\phi)$ [Eagon and Northcott 1962] associated with a matrix, or a map of free modules $\phi : F \to G$, is a generalization of the Koszul complex, which is the case $\operatorname{rank} G = 1$. Like the Koszul complex, it is tautological: its existence depends only on the properties of commutative rings; and like the Koszul complex it is exact or not depending on a property of the matrix ϕ related to regular sequences. It is part of a family of complexes described in [Eisenbud 1995, Appendix A2], and, from a more conceptual and general point of view, in [Weyman 2003].

We are interested in $EN(\phi)$ because its shape influences the shape of the free resolutions of canonical curves in an interesting way, leading to Green's conjecture. This conjecture, one of the central open problems in the theory of algebraic curves, is described in the last section of this chapter. We will also use the the Eagon–Northcott complex, in a special case, to give a proof of the classification of matrix pencils and an analysis of the ideals of ACM curves in $\mathbb{P}^3$.

To prepare for the description of the Eagon–Northcott complex we will examing the cases $\operatorname{rank} G = 1$ (the Koszul complex), and the case $\operatorname{rank} F = \operatorname{rank} G + 1$.

<u>The case $\operatorname{rank} G = 1$</u>

Let $\phi : F = R^f \to R$ be a homomorphism from a free module to a ring R. In this case $EN(\phi) = K(\phi)$, the Koszul complex, and we may write it in the form

$$S \xleftarrow{\delta_1} F \xleftarrow{\delta_2} \bigwedge^2 F \xleftarrow{\delta_3} \cdots \xleftarrow{\delta_f} \bigwedge^f F \leftarrow 0,$$

where $\delta_1 = \phi$.

To define the complex, we must construct the differentials δ_i and prove that $\delta_i \delta_{i+1} = 0$. Since the modules are free, it suffices to do this for the dual maps

$$\partial_i : \bigwedge^i F^* \to \bigwedge^{i+1} F^*,$$

and it turns out that this is in a sense even more natural.

It is convenient to think of R as an $S := \mathbb{Z}[x_1, \dots, x_f]$-algebra by the map sending x_i to ϕ_i; we define the Koszul complex of ϕ over R by tensoring the Koszul complex of $(x_1, \dots, x_f)$ with R.

Thus for the definition we take the map ϕ to be

$$\phi : S^f \xrightarrow{(x_1 \dots x_f)} S.$$

First of all, the map ∂_i (like the map δ_i) is *linear*: the image of a basis vector of $\bigwedge^i F^*$ is a sum of variables times basis vectors of $\bigwedge^{i+1} F^*$. We may write S as $\mathrm{Sym}(V)$, where V is the free $\mathbb{Z}$-module generated by $x_1, \dots, x_f$, and we may think of F as the module $V \otimes S$ with the map ϕ sending $V \otimes 1 \subset F$ by the identity to $V = S_1 \subset S$ — the *tautological map.* Let $t \in V \otimes V^* \subset S \otimes \bigwedge V^*$ be the *trace element* represented in terms of any basis $\{x_i\}$ of V and dual basis $\{\hat{e}_i\}$ of V^* as $t = \sum x_i \otimes \hat{e}_i$. Because $\bigwedge V^*$ is an anti-commutative algebra, we have $t^2 = 0$.

We define the map

$$\partial_i : S \otimes_{\mathbb{C}} \bigwedge^i V^* = \bigwedge^i F^* \to \bigwedge^{i+1} F^* = S \otimes_{\mathbb{C}} \bigwedge^{i+1} V^*$$

to be multiplication by t, and thus $\partial_{i+1}\partial_i$ is multiplication by $t^2 = 0$.

Having defined the complex $K(\phi) = EN(\phi)$ in the case $\mathrm{rank}\, G = 1$, we next ask what conditions on ϕ make it acyclic (that is, a free resolution of $\mathrm{coker}\, \delta_1$).

Theorem 18.8. *Suppose that R is a ring and $\phi : F \to R$ is a map from a free R-module of rank f. The complex $K(\phi)$ is acyclic if and only if the ideal $I \coloneqq I_1(\phi)$ has grade $\geq f$.*

Proof. Theorem 1.6 directly implies that if $K(\phi)$ is acyclic then $\mathrm{rank}\, \delta_f = 1$ and $\mathrm{grade}\, I(\delta_f) \geq f$. Since $I(\delta_f) = I$, this proves one implication.

Now assume that $\mathrm{grade}\, I \geq f$. We first prove that $K(\phi)$ is split exact when $I = R$; that is, $\bigwedge^i F = \ker \delta_i \oplus \mathrm{im}\, \delta_i$ for every i, or equivalently $\bigwedge^i F^* = \mathrm{im}\, \partial_i \oplus \mathrm{coker}\, \partial_i$ for every i. The condition $I = R$ implies that δ_1 is s split surjection, or equivalently that ∂_1 is a split injection. In this case we may write $F^* = R \oplus F'^*$ in such a way that ∂_1 is the injection into the first summand, and we may choose a basis $\{\hat{e}_i\}$ of F^* so that the last $f - 1$ basis elements are a basis for F'^*. Specializing the sequence $x_1, x_2, \dots, x_f$ to the sequence $1, 0, \dots, 0$, the differential of $K(\phi)^*$ becomes the multiplication by $1 \otimes e_1$.

The module $\bigwedge^i F^*$ decomposes as

$$\bigwedge^i F^* = \left(Re_1 \otimes_R \bigwedge^{i-1} F^*\right) \oplus \bigwedge^i F'^*.$$

Because $e_1 \wedge e_1 = 0$ the differential $\partial_i : \wedge^i F^* \to \wedge^{i+1} F^*$ has the form

$$\begin{array}{ccccccc}
 & & Re_1 \otimes \wedge^{i-1} F'^* & \xrightarrow{0} & Re_1 \otimes \wedge^{i} F'^* & & \\
\wedge^i F^* & = & \bigoplus & \nearrow_{\cong} & \bigoplus & = & \wedge^{i+1} F^*. \\
 & & \wedge^i F'^* & \xrightarrow{0} & \wedge^{i+1} F'^* & &
\end{array}$$

Thus we see that $K(\phi)$ is split exact when ϕ is a split surjection.

We now assume only that $\operatorname{grade} I \geq f$. From what we just proved we see that if we localize R by inverting any element of I the complex $K(\phi)$ becomes split exact. Since $\operatorname{grade} f \geq 1$, we can find such a nonzerodivisor in I, and inverting it does not change the ranks of the maps ϕ_i. Because ranks of free modules are additive in direct sums, it is obvious that in the split exact case the condition on the ranks of the ϕ_i is satisfied; more precisely, $\operatorname{rank}(\delta_i) = \binom{i-1}{f-1}$. We also see that after inverting a nonzerodivisor in $I(\delta_1)$ we have $I(\delta_i) = R$; equivalently,

$$I \subset \sqrt{I(\delta_i)}.$$

(In fact $I(\delta_i) = I^{\binom{i-1}{f-1}}$, though this requires a separate argument.) Thus if $\operatorname{grade} I = f$ then $\operatorname{grade} I(\delta_i) \geq i$ for all i, so $K(\phi)$ is acyclic. $\square$

<u>The case $\operatorname{rank} F = \operatorname{rank} G + 1$</u>

We set $g = \operatorname{rank} G$ and $f = \operatorname{rank} F = g + 1$. In this case the Eagon–Northcott complex has the form

$$EN(\phi): \quad 0 \to G^* \otimes \bigwedge^f F \xrightarrow{\delta_2} \bigwedge^{f-1} F \xrightarrow{\delta_1} \bigwedge^g G.$$

Here $\delta_1 = \bigwedge^g \phi$, so that the entries of a matrix for δ_1 are the $g \times g$ minors of ϕ.

We choose an identification $\bigwedge^f F = S$, called an *orientation* of F, and get a perfect pairing

$$\bigwedge^g F \times F \to \bigwedge^f F = S$$

so that we may identify $\bigwedge^g F$ with F^*. With this identification, we define δ_2 as

$$\delta_2 : G^* \xrightarrow{\phi^*} F^* = \bigwedge^g F.$$

We also choose an orientation $\bigwedge^g G = S$, in terms of which the image of δ_1 is the ideal generated by the $(f-1) \times (f-1)$ minors of ϕ.

We first claim that $EN(\phi)$ is a complex; that is, $\delta_1\delta_2 = 0$. As with the Koszul complex, it is convenient to dualize and consider the maps

$$EN(\phi)^*: \quad 0 \to \bigwedge^g G \to \bigwedge^g F^* = F \xrightarrow{\phi} G.$$

The fact that this composition is 0 is often taught as Cramer's rule for solving a system of homogeneous equations represented by a $g \times (g+1)$ matrix of rank

g. The solutions — that is, the elements of $\ker \phi$ — are multiples of the column $\Delta_1, \dots, \Delta_g$ where the Δ_j is $(-1)^j$ times the determinant of the matrix obtained from ϕ by leaving out the j-th column. This works because the composition of the two maps is a column matrix whose i-th entry is the expansion of the $(g+1) \times (g+1)$ determinant of the matrix obtained from ϕ by repeating the i-th row.

Theorem 18.9. *Suppose that R is a ring and $\phi : F \to G$ is a map of free R-modules, where G has rank g and F has rank $f = g+1$. The complex $EN(\phi)$ is acyclic if and only if the ideal $I := I_g(\phi)$ has grade ≥ 2.*

Example 18.10. We have seen that the ideal of the twisted cubic is generated by the 2×2 minors of the matrix

$$\phi := \begin{pmatrix} x_0 & x_1 & x_2 \\ x_1 & x_2 & x_3 \end{pmatrix}$$

and it follows that the free resolution of its homogeneous coordinate ring is the Eagon–Northcott complex

$$0 \to S^2(-3) \xrightarrow{\begin{pmatrix} x_0 & x_1 \\ x_1 & x_2 \\ x_2 & x_3 \end{pmatrix}} S^3(-2) \xrightarrow{\phi \wedge \phi} S$$

used $\phi \wedge \phi$ instead of big wedge, ok?

Example 18.11. In Section 1.2 we asserted that the $(a+1) \times a$ matrix ϕ_a given on page 6 is the minimal presentation of the ideal $(s^a, s^{a-1}t, \dots, t^a) \subset R := \mathbb{C}[s,t]$. It is not hard to check this directly, but in any case it's easy to see that its $a \times a$ minors of ϕ_a generate this ideal, which has grade 2, so the Eagon–Northcott resolution $EN(\phi)$ has the form

$$R \xleftarrow{\bigwedge^a \phi_a} R^{a+1} \xleftarrow{\phi_a} R^a \leftarrow 0,$$

verifying the assertion.

Proof. Once having shown that $EN(\phi)$ is a complex, as we did above, the proof of the equivalence in the theorem follows the same pattern as the proof given above for the Koszul complex.

If $EN(\phi)$ is acyclic, then by Theorem 1.6 the $g \times g$ minors of $\phi = \delta_2$ must have grade ≥ 2. For the converse, suppose first that $I_g(\phi)$ is the unit ideal. We may split F as $S \oplus G$ with $\Delta_1 = 1$ and $\Delta_j = 0$ for $j > 1$, and then $EN(\phi)^*$ has the form

$$\begin{tikzcd}
 & 0 \arrow[r] & G \arrow[r, "0"] & S \\
0 \arrow[r] & \bigoplus & \bigoplus & \bigoplus. \\
 & G \arrow[r, "0"] \arrow[uur, "\cong"] & S \arrow[r] \arrow[uur, "\cong"] & 0
\end{tikzcd}$$

Thus $EN(\phi)$ is split exact in this case.

We now apply Theorem 1.6: From what we just proved we see that if we localize S by inverting any element of I then the complex $EN(\phi)$ becomes split exact, and therefore, before localizing, $\operatorname{rank}(\delta_1) = 1$ and $\operatorname{rank} \delta_2 = g$. In this case it follows from the definition that $I_1(\delta_1) = I_g(\phi) = I_g(\delta_2)$ so if $I_g(\phi)$ has grade 2 then both conditions of Theorem 1.6 are satisfied. □

The Hilbert–Burch theorem. In a regular local ring any ideal of whose primary components all have codimension 1 is principal (divisors are all Cartier). What about ideals of codimension 2? The answer is the content of the *Hilbert–Burch* theorem, proven in 1890 by David Hilbert in the case of homogeneous ideals in $\mathbb{C}[x_0, x_1]$ and in general by Lindsay Burch [1967]. We can deduce it as an application of the Eagon–Northcott complex in the case $f = g + 1$:

Corollary 18.12 (Hilbert–Burch theorem). *Suppose that R is a local ring. Any ideal $I \subset R$ of projective dimension* 1 *has the form aI' where I' is an ideal of grade 2 generated by the $g \times g$ minors of a $g \times (g+1)$ matrix and a is a nonzerodivisor of R; and conversely any ideal of this form has projective dimension* 1.

In particular, if $C \subset \mathbb{P}^3$ is an ACM curve whose homogeneous ideal I is generated by f elements, then I is minimally generated by the $(f-1) \times (f-1)$ *minors of the syzygy matrix of I.*

Proof. If $C \subset \mathbb{P}^3$ is ACM, then the projective dimension of the homogeneous coordinate ring R_C is 2 by the Auslander–Buchsbaum theorem, and thus the ideal of C has projective dimension 1.

Now suppose that $I \subset R$ is an ideal of projective dimension 1 in any local ring, and suppose that I is generated by f elements, so that we have a surjection $F := R^f \to I$. The module R/I has free resolution of the form

$$\mathbb{F}: \quad R \xleftarrow{\alpha} R^f \xleftarrow{\phi} G \leftarrow 0,$$

where $I = I_1(\alpha)$, so by Theorem 1.6 the free module G has rank $g = f - 1$, the $g \times g$ minors of ϕ generate an ideal I' of grade ≥ 2, and the ideal I has grade at least 1. Theorem 1.6 implies that both the Eagon–Northcott complex $EN(\phi)$ and its dual are acyclic.

The dual of the complex $\mathbb{F}$ will not be acyclic unless grade $I = 2$, but there is at least a comparison map

$$\begin{array}{rccccccc}
\mathbb{F}^*: & G^* & \xleftarrow{\phi^*} & F^* & \xleftarrow{\alpha^*} & R & \longleftarrow & 0 \\
& \big\downarrow = & & \big\downarrow = & & \big\downarrow a & & \\
EN(\phi)^*: & G^* & \xleftarrow{\phi^*} & F^* & \xleftarrow{\wedge^g \phi^*} & R & \longleftarrow & 0
\end{array}$$

It follows that $I = aI'$, and since I has grade at least 1, a must be a nonzero-divisor.

Conversely, if $\phi : R^f \to R^g$ is a map with $f = g + 1$ and grade $I_g(\phi) \geq 2$, then the acyclicity of $EN(\phi)$ shows that $I_g(\phi)$ has projective dimension 1; and if a is a nonzerodivisor, then $I \coloneqq aI_g(\phi) \cong I_g(\phi)$ as R-modules, so I has projective dimension 1 as well. □

This argument applies, for example, to the case of a nonhyperelliptic curve of genus 3 and degree 6 in $\mathbb{P}^3$, discussed on page 121.

The general case of the Eagon–Northcott complex. With these two special cases in mind, we are ready for the general case.

Definition 18.13. If $\phi : F \to G$ is a map of free S-modules with $f \coloneqq \operatorname{rank} F \geq g \coloneqq \operatorname{rank} G$, then the *Eagon–Northcott complex of* ϕ is the complex of free S-modules

$$EN(\phi) : \quad S \xleftarrow{\delta_1 := \bigwedge^g \phi} \bigwedge^g F \xleftarrow{\delta_2} G^* \otimes \bigwedge^{g+1} F \xleftarrow{\delta_3} (\operatorname{Sym}^2 G)^* \otimes \bigwedge^{g+2} F$$
$$\xleftarrow{\delta_4} \cdots \xleftarrow{\delta_{f-g+1}} (\operatorname{Sym}^{f-g} G)^* \otimes \bigwedge^f F \leftarrow 0.$$

It has the following properties:

(1) After identifying $\bigwedge^g G$ with S, the map δ_1 is identified with $\bigwedge^g \phi$.

(2) It is convenient to give a formula for $\partial_i = \delta_i^*$ by taking advantage of the algebra structures of $\operatorname{Sym}(G)$ and $\bigwedge F^*$. To do this, choose dual bases $\{e_i\}$ and $\{\hat{e}_i\}$ for F and F^*. In these terms

$$\delta_i^* = \partial_i : \operatorname{Sym}^{i-2} G \otimes \bigwedge^{g+i-2} F^* \to \operatorname{Sym}^{i-1} G \otimes \bigwedge^{g+i-1} F^*$$

is multiplication by the element $\sum_{i=1}^f \phi(e_i) \otimes \hat{e}_i$.

To show that $\delta_1\delta_2 = 0$ is almost the same as in the case $f = g + 1$ because a basis element

$$b := e_{i_1} \wedge \cdots \wedge e_{i_{g+1}} \in \bigwedge^{g+1} F$$

can be thought of as coming from a rank $g + 1$ summand of F, and the value of $\delta_2\delta_1 b$ is the same as it would be if F were replaced by this summand. Thus from the case $f = g + 1$ we see that $\delta_1\delta_2 b = 0$, and thus $\delta_2\delta_1 = 0$.

On the other hand, for $i \geq 1$ the map $\partial_{i+1}\partial_i$ is multiplication by

$$\left(\sum_{i=1}^f \phi(e_i) \otimes \hat{e}_i\right)^2,$$

which we may think of as the square of an element of degree 1 in the exterior algebra of the free $\operatorname{Sym}(G)$-module $\bigwedge(\operatorname{Sym}(G) \otimes F^*) = \operatorname{Sym}(G) \otimes \bigwedge F^*$, and hence this square is 0. Thus the given maps do define a complex.

Theorem 18.14. *Suppose that R is a ring and $\phi : F \to G$ is a map of free R-modules, where G has rank g and F has rank $f \geq g$. The complex $EN(\phi)$ is acyclic if and only if the ideal $I \coloneqq I_g(\phi)$ has grade $\geq f - g + 1$.*

Example 18.15. If ϕ is a matrix of linear forms, then the first map of $EN(\phi)$ is represented by the row of $g \times g$ minors of ϕ, which are forms of degree g, but all the rest of the maps are represented by matrices of linear forms. Thus, for example, the Betti table of the Eagon–Northcott complex of a $2 \times f$ matrix of linear forms is

j	$i = 0$	1	2	$\cdots$	$f-1$
0	1	–	–	$\cdots$	–
1	–	$\binom{f}{2}$	$2\binom{f}{3}$	$\cdots$	$(f-1)\binom{f}{f}$

Proof. The dual of the last differential of $EN(\phi)$ is

$$\partial_{f-g+1} : \mathrm{Sym}^{f-g-1} G \otimes \bigwedge^{f-1} F^* \to \mathrm{Sym}^{f-g}(G) \otimes \bigwedge^{f} F^*.$$

With our usual identifications $\bigwedge^f F^* = S$ and $\bigwedge^{f-1} F^* = F$ this becomes the map

$$\mathrm{Sym}^{f-g-1} G \otimes F \xrightarrow{1\cdot\phi} \mathrm{Sym}^{f-g} G,$$

whose cokernel is $\mathrm{Sym}^{f-g}(\operatorname{coker}\phi)$ by the right exactness of the symmetric algebra functor [Eisenbud 1995, Proposition A2.2]. Because Sym is a multilinear functor, the support of $\mathrm{Sym}^{f-g}(\operatorname{coker}\phi)$ is contained in the support of $\operatorname{coker}\phi$, but in fact they are equal, because if a localization of $\operatorname{coker}\phi$, over a local ring S_P, is nonzero, then by Nakayama's lemma it surjects onto $S_P/P_P = \kappa(P)$, and again by the right exactness of the symmetric algebra functor $\mathrm{Sym}^{f-g}(\operatorname{coker}\phi)_P$ surjects onto $\mathrm{Sym}^{f-g}(\kappa(P)) = \kappa(P)$.

By Theorem 1.6 we see from this that if $EN(\phi)$ is acyclic, then the support of $\operatorname{coker}\phi$ has grade $\geq f - g + 1$. This support is defined by the radical of $I_g(\phi)$, so $\operatorname{codim} I_g(\phi) \geq f - g + 1$ as required.

Conversely, to show that $EN(\phi)$ is acyclic under the given hypothesis we first treat the case $I_g(\phi) = S$, and prove that $EN(\phi)^*$ is split exact. This is the most complicated part of the proof, but it is purely formal:

As before we may split F and assume that $F = G \oplus F'$, the map ϕ being the projection onto the first summand. Assuming that the first summand corresponds to the basis elements $e_1, \dots, e_g \in G \subset F$ the dual differential ∂_i takes the form $\sum_{i=1}^{g} e_i \otimes \hat{e}_i$.

The map

$$\bigwedge^g G \xleftarrow{\delta_1} \bigwedge^g F = \bigoplus_{j=0}^{g} \bigwedge^j G \otimes \bigwedge^{g-j} F'$$

is the projection onto the $j = g$ summand, so we must show that the rest of $EN(\phi)$ is a split surjection ending with the terms of the source of δ_1 other than $\bigwedge^g G$.

Once again, it will be convenient to treat the dual complex. Using the splitting we may write the terms of the dual as

$$EN_i(\phi)^* = \mathrm{Sym}^{i-2} G \otimes \bigwedge^{g+i-2} F^* = \bigoplus_j \mathrm{Sym}^{i-2} G \otimes \bigwedge^j G^* \otimes \bigwedge^{g+i-2-j} F'^*$$

for $i \geq 1$. The map $\partial_i = \delta_i^*$ is a direct sum from $j = 0$ to g of the maps

$$\mathrm{Sym}^{i-2} G \otimes \bigwedge^j G^* \otimes \bigwedge^{g+i-2-j} F'^* \to \mathrm{Sym}^{i-1} G \otimes \bigwedge^{j+1} G^* \otimes \bigwedge^{g+i-2-j} F'^*$$

that are all equal to multiplication by $\sum_{k=1}^{g} e_k \otimes \hat{e}_k \otimes 1$ where the last tensor factor is the identity map of $\bigwedge^{g+i-2-j} F'^*$.

Thus it suffices to show that the complexes

$$(*_j) \qquad \mathrm{Sym}^0 G \otimes \bigwedge^j G^* \to \cdots \to \mathrm{Sym}^i G \otimes \bigwedge^{i+j} G^* \to \cdots$$

are split exact for $0 \leq j < g$.

Let $R = \mathrm{Sym}\, G = S[e_1, \dots, e_g]$. The Koszul complex over R of the sequence $\phi_i = e_i$ may be written as

$$R \otimes_S \bigwedge G^* = \mathrm{Sym}(G) \otimes_S \bigwedge G^* = \bigoplus_{p,q} \mathrm{Sym}^p G \otimes \bigwedge^q G^*$$

and we proved in Theorem 1.8 that it is a free resolution of $R/(e_1, \dots, e_g) = S$, which appears as $\mathrm{Sym}^0 G \otimes \bigwedge^0 G^*$. This complex is $(\mathrm{Sym}\, G) \otimes \bigwedge G^*$ which is the direct sum of the the complexes $(*_j)$ above. It follows that these finite complexes have no homology at all, and since the modules are free over S, they are split exact.

Since the complex $EN(\phi)$ is split exact after inverting any element of $I = (\phi_1, \dots, \phi_f)$, it follows that the rank condition of Theorem 1.6 is satisfied, and

$$I \subset \sqrt{I_{\mathrm{rank}\,\delta_i}(\delta_i)}.$$

Since the length of $EN(\phi)$ is $f - g + 1$, Theorem 1.6 implies that it is acyclic when grade $I \geq f - g + 1$, completing the proof. □

Corollary 18.16. *With notation as in Theorem 1.14, if the ideal $I_g(\phi)$ has codimension $\geq f - g + 1$ then it has codimension exactly $f - g + 1$, the ring $S/I_g(M)$ is Cohen–Macaulay, and the $\binom{f}{g}$ forms that are the $g \times g$ minors of a matrix for ϕ are linearly independent over $\mathbb{C}$.*

Proof. From the resolution $EN(M)$ we see that the projective dimension of $S/I_g(M)$ is $f - g + 1$. Since the projective dimension of a module is at least the codimension of its annihilator, the equality follows, and the Auslander–Buchsbaum formula implies that $S/I_2(M)$ is Cohen–Macaulay. The linear independence of the minors of M follows because $EN(M)$ is a resolution and there $EN(M)_2$ is generated in degree 3, so all the relations on the minors have coefficients of degree 1. □

In general when $X \subset Y \subset \mathbb{P}^r$, so that $I_X \supset I_Y$, it may be hard to see which syzygies of X come from syzygies of Y. But when the degrees of the syzygies of Y are smaller than those from X, the situation is simpler. Here is the special case we will use:

Proposition 18.17. *Let $C \subset \mathbb{P}^r$ be a nondegenerate curve. If $C \subset X \subset \mathbb{P}^r$, where X is a rational normal scroll, then the Eagon–Northcott complex that is the minimal free resolution of I_X is termwise a direct summand of the minimal free resolution of I_C. Thus each numbers in the Betti table of the resolution of I_C is no less than the corresponding number in the Betti table of the resoluiont of I_X.*

Proof. Let EN be the minimal resolution of I_X, and let $\mathbb{F}$ be the minimal resolution of I_C. The inclusion $I_X \subset I_C$ induces a map $\phi : EN \to \mathbb{F}$, unique up to homotopy. Since the minimal generators of I_X are quadratic, and I_C contains no linear forms, $\phi_0 : EN_0 \to \mathbb{F}_0$ is a split monomorphism.

By induction, we may assume that ϕ_{i-1} is a split monomorphism. The free module EN_i is generated in degree $i+1$, while the free module $\mathbb{F}_i$ is generated in degrees $\geq i+1$. It follows that the relations represented by EN_i, extended by 0, are among the minimal generators of the relations represented by $\mathbb{F}_i$, completing the proof. □

18.5. Green's Conjecture

Corollary 10.9 implies that the dimension of the vector space of forms of degree d vanishing on a canonical curve is independent of the curve; for example, for $d = 2$ we get $\dim(I_C)_2 = \binom{g-2}{2}$. The Hilbert function of I_C is determined by the Betti table of its resolution, so that the table generally has more information.

For example, when C is trigonal then by the geometric Riemann–Roch theorem, C has a 1-dimensional family of trisecant lines, and any quadric containing C must contain all these. As we have seen in Chapter 17, these lines sweep out the 2-dimensional rational normal scroll defined by the 1-generic $2\times(g-2)$ matrix M corresponding to the decomposition of $\mathcal{O}_C(1)$ into a tensor product of the line bundle $\mathcal{L}$ associated to the g^1_3 and the residual line bundle $\omega_C \otimes \mathcal{L}^{-1}$. The latter has $g-2$ sections, and we see from Section 16.2 that the scroll itself lies on the $\binom{g-2}{2}$ quadrics defined by the minors of M. The exactness of the Eagon–Northcott complex associated to this matrix shows that there are no relations of degree 0 on these minors — that is, they are linearly independent over the ground field. It follows that they generate the vector space of all quadrics containing C.

Furthermore, if $g = 6$ and C is isomorphic to a plane quintic curve, then the canonical series of the plane quintic is $5-3=2$ times the hyperplane series, and it follows that the canonical image of C lies on the Veronese surface in $\mathbb{P}^5$. Thus the Veronese surface is contained in (in fact, equal to) the intersection of the quadrics defined by the 2×2 minors of a generic symmetric matrix, coming from the multiplication map

$$H^0(\mathcal{O}_{\mathbb{P}^2}(1)) \otimes H^0(\mathcal{O}_{\mathbb{P}^2}(1)) \to H^0(\mathcal{O}_{\mathbb{P}^2}(2)) = H^0(\mathcal{O}_{\mathbb{P}^5}(1))$$

and there are $6 = \binom{g-2}{2}$ independent quadrics in this ideal. Again in this case, they cannot generate the ideal of the curve.

One might fear that this is the beginning of some long series of examples, but in fact it is not:

Theorem 18.18 (Petri). *The ideal of a canonical curve of genus ≥ 5 is generated by the $\binom{g-2}{2}$-dimensional space of quadrics it contains unless the curve is either trigonal or isomorphic to a plane quintic; in the latter cases, the ideal of the curve is generated by quadrics and cubics.*

For a modern treatment of Petri's theorem in this level of generality see [Schreyer 1991]; for a different treatment see [Arbarello and Sernesi 1978].

The two exceptions can be described simultaneously by using the Clifford index:

Definition 18.19. The *Clifford index* $\operatorname{Cliff}\mathcal{L}$ of a line bundle $\mathcal{L}$ on a curve C is $d-2r$, where $d := \deg\mathcal{L}$ and $r := h^0(\mathcal{L})-1$. The Clifford index $\operatorname{Cliff} C$ of a curve C of genus ≥ 2 is the minimum of the Clifford indices of special line bundles with at least 2 sections.

Clifford's theorem (Corollaries 2.33 and 10.14) says that $\operatorname{Cliff} C \geq 0$, and that $\operatorname{Cliff} C = 0$ if and only if C is hyperelliptic. If C is not hyperelliptic, then it turns out that $\operatorname{Cliff} C = 1$ if and only if C is either trigonal or isomorphic to a plane quintic. The Clifford index of any smooth curve of genus $g \geq 2$ is $\leq \lceil g/2\rceil + 1$, with equality for a general curve, as one sees from the Brill–Noether Theorem 12.2, and for "most" curves the line bundle $\mathcal{L}$ of maximal Clifford index has only 2 sections, though there is an infinite sequence of examples where this "Clifford dimension" is greater.

Moving to cubic forms, we see that $\dim(I_C)_3 = \binom{g+2}{3} - (5g-5)$. Comparing this number with the number of (possibly linearly dependent) cubics obtained by multiplying g linear forms and $\binom{g-2}{2}$ quadrics, we see that the ideal of the curve has at least $\binom{g-2}{2} - \binom{g+2}{3} - (5g-5)$ independent syzygies of total degree 3 (that is, linear syzygies on the quadrics). For example when $g = 4$ so that $C \subset \mathbb{P}^3$ there is one quadric and 5 independent cubics, at most 4 of which are multiples of the quadric. Since the curve has degree $6 = 2\times 3$, the ideal of the curve must be generated by the quadric and one cubic. When $g = 5$ there are genuinely two possibilities: the three quadrics in the ideal might be a complete intersection (then they generate the ideal), so the Betti table would be

j	$i=0$	1	2
0	1	–	–
1	–	2	–
2	–	–	1

— or the curve could be trigonal, in which case the 3 quadrics generate the ideal of a surface scroll F. In the latter case, the Eagon–Northcott complex resolves the homogeneous coordinate ring S_F of the scroll,

$$0 \to S^2(-3) \to S^3(-2) \to S \to S_F \to 0,$$

which has Betti table

j	$i=0$	1	2
0	1	–	–
1	–	3	2

and we see that there are 2 linear relations among the quadrics. Thus the minimal generators of I_C must include exactly 2 cubics as well as the 3 quadrics. Since the homogeneous ring of a canonical curve is Gorenstein, its minimal free resolution is symmetric, and this is enough for us to fill in its Betti table:

j	$i=0$	1	2	3
0	1	–	–	–
1	–	3	2	–
2	–	2	3	–
3	–	–	–	1

Note that we can see the scroll reflected in the top two lines of the table.

From the analogue of the Hilbert–Burch theorem for Gorenstein rings of codimension 3 one can show that the 5 generators can be written as the Pfaffians of a skew symmetric 5×5 matrix whose entries are of degrees 1 and 2, in the following pattern (we give just the degrees, and use – for 0):

$$\begin{pmatrix} - & - & 1 & 1 & 1 \\ - & - & 1 & 1 & 1 \\ 1 & 1 & - & 2 & 2 \\ 1 & 1 & 2 & - & 2 \\ 1 & 1 & 2 & 2 & - \end{pmatrix}.$$

Here the 2×2 minors of the upper 2×3 block of linear forms generate the ideal of the scroll.

Applying this logic more generally we get the following result about the canonical embedding of curves with low degree maps to $\mathbb{P}^1$:

Theorem 18.20. *Let $C \subset \mathbb{P}^{g-1}$ be a reduced, irreducible canonical curve. If C has a line bundle $\mathcal{L}$ of degree $d \le g-1$ with $h^0(\mathcal{L}) = 2$ then there is a 1-generic $2\times(g+1-d)$ matrix of linear forms whose minors define a scroll of codimension*

$g-d$ containing C; and thus an Eagon–Northcott complex of length $g-d$ is a subcomplex of the minimal free resolution of R_C. In particular, the Betti table of R_C is termwise $\geq$ that of the homogeneous coordinate ring of the scroll.

(We have stated this theorem for canonical curves, but in fact the construction applies much more generally to a linearly normal variety $X \subset \mathbb{P}^n$ of any dimension: if X has a divisor D that moves in a pencil and is contained in a subspace $\mathbb{P}^k$ with $k \leq n-2$, the planes spanned by the divisors of the pencil $|D|$ sweep out a rational normal scroll.)

Thus the existence of the g^1_d on C, together with the symmetry of the resolution of the Gorenstein ring R_C, implies that the Betti table of R_C has the form

j	$i=0$	1	2	…	$d-3$	$d-2$	…	$g-d-1$	$g-d$	…	$g-3$	$g-2$
0	1	–	–	⋯	–	–	–	–	–	–	–	–
1	–	*	*	⋯	*	*	⋯	*	?	⋯	?	?
2	–	?	?	⋯	?	*	⋯	*	*	⋯	*	*
3	–	–	–	⋯	–	–	–	–	–	–	–	1

where we have assumed for illustration that $d-2 < g-d-1$. As before, a dash indicates a place that is definitely 0; asterisks indicate some that are definitely nonzero. The entries of the rows marked 0 and 1 are greater than or equal to the corresponding entries of the Betti table of the scroll.

We can summarize this by saying that if the curve C has a line bundle $\mathcal{L}$ of degree d with exactly 2 sections (which is thus of Clifford index $c = d-2$) the row labeled 2 in the Betti diagram definitely has $\beta_{c,c+2} \neq 0$. As with the case of the plane quintics above, one can make a similar argument for *any* line bundle of Clifford index c. Thus:

Corollary 18.21. *If* Cliff $C \leq c$ *then* $\beta_{c,c+2}(S/I_C)) \neq 0$.

Starting from examples such as the case of genus 6, Mark Green made a bold conjecture that is still open as of this writing:

Conjecture 18.22 (Green's conjecture). *If C is a smooth canonical curve of genus g and S/I_C is the homogeneous coordinate ring of C in its canonical embedding, then the Clifford index of C is $\leq c$ if and only if $\beta_{c,c+2}(S/I_C) \neq 0$.*

The conjecture was made for curves over a field of characteristic 0, and is known in many cases, though it is also known to fail in small finite characteristics (see [Bopp and Schreyer 2021] for an amended conjecture that may hold in all characteristics.) Claire Voisin [2002; 2005] proved the conjecture for generic curves of each Clifford index; simpler proofs then appeared in [Aprodu et al. 2019; Kemeny 2021; Rathmann 2022]. As of this writing the full conjecture is known up to genus 9, for plane curves, and in a number of other special cases. See [Farkas 2017] for a survey on this and related topics.

Low-genus canonical embeddings. Frank-Olaf Schreyer, in his 1983 Brandeis thesis (reworked as [Schreyer 1986]), analyzed the possibilities for resolutions of smooth canonical curves up to genus 8. The project was carried further to genus 9 by Schreyer's student Michael Sagraloff in his Saarbrücken thesis [2018].

18.6. Exercises

Exercise 18.1. Let $S = k[x_0, \dots, x_r]$, and let

$$\mathbb{F}: \quad F_0 \xleftarrow{\phi_1} F_1 \leftarrow \cdots \leftarrow F_{n-1} \xleftarrow{\phi_n} F_n \leftarrow 0$$

be a finite complex of free S-modules. Set

$$X_i = \{p \in \mathbb{A}^{n+1} \mid H_i(\mathbb{F} \otimes \kappa(p)) \neq 0\}.$$

Use Theorem 1.6 to prove that $\mathbb{F}$ is acyclic if and only if $\operatorname{codim} X_i \geq i$ for all $i > 0$. Moreover, $X_0 \supseteq X_1 \supseteq \cdots \supseteq X_n$.

Hint: Elementary linear algebra shows that, if k is a field, then a complex

$$k^p \xleftarrow{\phi} k^q \xleftarrow{\psi} k^r$$

is exact at k^q if and only if $\operatorname{rank} \phi + \operatorname{rank} \psi = q$.

Exercise 18.2. Prove that if $X \subset \mathbb{P}^r$ is arithmetically Cohen–Macaulay then the dual of the minimal free resolution of S/I_X is the minimal free resolution of ω_X.

Hint: Use Theorem 1.6 and the characterization of ω_{S/I_X} as an Ext module.

Exercise 18.3. The *depth lemma* states that if

$$0 \to A \to B \to C \to 0$$

is an exact sequence of nonzero finitely generated modules over a local ring R then

$$\begin{aligned} \operatorname{depth} C &\geq \min\{\operatorname{depth} B, \operatorname{depth} A - 1\}, \\ \operatorname{depth} A &\geq \min\{\operatorname{depth} B, \operatorname{depth} C + 1\}. \end{aligned}$$

Prove this in the special case when R is regular using the characterization of depth via projective dimension.

Exercise 18.4. (1) Prove that if $\phi : F \to G$ is a free presentation of a finitely generated module M then

$$\operatorname{ann}_R(\operatorname{coker} \phi)^{\operatorname{rank} G} \subset \operatorname{Fitt}_0 \phi \subset \operatorname{ann}_R(\operatorname{coker} \phi).$$

(2) Prove that over a local ring every projective module is free; and show that $I(\phi)$ defines the nonfree locus of $\operatorname{coker} \phi$.

Exercise 18.5. In dealing with arithmetically Gorenstein schemes, we used the fact that arithmetically Gorenstein if $\omega_{S/I}$ is an invertible sheaf (over $\operatorname{Spec}(S/I)$) then it is isomorphic to $S/I(a)$ for some a. Why is this true?

Hint: Nakayama's lemma can be used to prove that projectives are free in some cases.

Exercise 18.6. Find a degree 6 embedding of a curve of genus 3 that is not arithmetically Cohen–Macaulay, and another that is.

Hint: Show that the 3×3 minors of a general 4×3 matrix of linear forms defines a Cohen–Macaulay curve of genus 3. Show that a curve of type $(2, 4)$ on a smooth quadric in $\mathbb{P}^3$ is not arithmetically Cohen–Macaulay.

Exercise 18.7. Referring to the Betti table of a canonical curve just before Section 1.4, give a formula for the differences $b_i - b_{g-i-1}$ that depends only on i and g.

Exercise 18.8. Show that if $I \subset S := \mathbb{C}[x_0, \dots, x_r]$ is a codimension 2 ideal, then S/I is Cohen–Macaulay if and only if the minimal S-free resolution of S/I has the form

$$0 \to S^{n-1} \to S^n \to S$$

for some n. Show that S/I is Gorenstein if and only if I is a complete intersection.

Hint: for the first part, tensor with the field of rational functions.

Exercise 18.9. Which sets of 4 distinct points in $\mathbb{P}^2$ are arithmetically Gorenstein? Which sets of 5 points? Which sets of 6 points?

Exercise 18.10. Give an example of a set of points in $\mathbb{P}^3$ that is arithmetically Gorenstein but not a complete intersection.

Hint: take the hyperplane section of a trigonal canonical curve of genus 5.

Exercise 18.11. Let $q : F \to G$, with $F = S^2(-1)$ and $G = S^2$, be described by the matrix $\left(\begin{smallmatrix} x_0 & x_1 \\ x_2 & x_3 \end{smallmatrix}\right)$, and let $Q \subset \mathbb{P}^3$ be the quadric defined by the determinant of q.

(1) Show that the sheafification of the graded module $M := \operatorname{coker} q$ is $\mathcal{O}_Q(1, 0)$ and the sheafification of $\operatorname{coker} q^*$ is $\mathcal{O}_Q(0, 1)$ by computing the vanishing locus of the two sections corresponding to the generators of the module.

(2) Show that the sheafification of $\operatorname{Sym}^a(M)$ is $\mathcal{O}_Q(a, 0)$. Conclude that if $a \le b$ then the relative ideal sheaf $\mathcal{I}_{C/Q} = \mathcal{O}(-a, -b)$ of a curve C of type (a, b) is the sheafification of the module $\operatorname{Sym}^{b-a}(M)(-b)$.

(3) Let $S = \mathbb{C}[x_0, \dots, x_3]$. Show that the minimal S-free resolution of $\operatorname{Sym}^a(M)$ has the form

$$0 \to \textstyle\bigwedge^2 F \otimes \operatorname{Sym}^{a-2} G(-2) \to F \otimes \operatorname{Sym}^{a-1} G(-1) \to \operatorname{Sym}^a G.$$

Hint: Use multilinear algebra (as in [Eisenbud 1995]) to define the maps, and use Theorem 1.6 to prove that this is a resolution.

(4) Show that if $C \subset \mathbb{P}^3$ is a curve of type (a, b) with $a \leq b$ then there is a free resolution of a module whose sheafification is $\mathcal{I}_{C/\mathbb{P}^3}$ that is a mapping cone of the map of complexes

$$\begin{tikzcd}
0 \arrow[r] & 0 \arrow[r] & \bigwedge^2 F \arrow[r, "\bigwedge^2 q"] & \bigwedge^2 G \\
0 \arrow[r] \arrow[u] & \bigwedge^2 F \otimes \mathrm{Sym}^{b-a-2} G(-b-2) \arrow[r] \arrow[u] & F \otimes \mathrm{Sym}^{b-a-1} G(-b-1) \arrow[r] \arrow[u] & \mathrm{Sym}^{b-a} G(-b) \arrow[u]
\end{tikzcd}$$

(5) Conclude that the deficiency module of a curve of type (a, b) with $a \leq b$ is the cokernel of a map

$$\bigwedge^2 F^* \otimes \mathrm{Sym}^{b-a-2}(G^*)(b+2) \xleftarrow{F^*} \quad \otimes \mathrm{Sym}^{b-a-1} G^*(b+1)$$

and thus, after choosing a basis of $F = S^2$, may be identified as the sheafification of the cokernel of

$$\mathrm{Sym}^{b-a-2}(G^*)(b+2) \xleftarrow{F} \quad \otimes \mathrm{Sym}^{b-a-1} G^*(b+1)$$

where the map is the action of F on $\bigwedge G^*$ via the map $q : F \to G$.

Hint: Imitate the proof of Theorem 1.14 to prove the exactness of the given resolution of $\mathrm{Sym}^a(M)$. See also [Eisenbud 1995, Appendix A2.6].

Exercise 18.12. Let $\phi : F \to G \to M \to 0$ be an exact sequence of finitely generated R-modules, with F and G free.

(1) Show that the annihilator of $\operatorname{coker} \phi$ has the same radical as the ideal of minors $I_{\operatorname{rank} G}(\phi)$.

(2) Show that the cokernel of ϕ is locally free if and only if $I_{\operatorname{rank} \phi}(\phi) = R$.