CHAPTER VI.

QUARTICS.

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positions which the intersections of the four lines with the conic can have with respect to the quadrilateral found by them. Thus when V meets all the lines in real points, he enumerates the following cases: (1), annular quartic, quadrifolium and internal oval; (2), quadrifolium and 2 ovals; (3), 4 unifolia; (4), trifolium, unifolium and oval; (5), bifolium, 2 unifolia and oval; (6), 2 bifolia and oval; (7), 2 bifolia and 2 ovals; (8), bifolium and 2 unifolia; (9), trifolium, unifolium and 2 ovals. He enumerates thus 36 cases in all, but the figures which he gives for the nine cases just mentioned sufficiently illustrate the rest, a very slight modification being enough to turn a unifolium into an oval, &c. It will be observed that the classification just made rests solely on projective properties and has no reference to the line infinity. In Art. 249 we state the principles on which these classes may be subdivided into species when the nature of the infinite branches is taken account of.

248 (*d*). Zeuthen also applies his method of classification to nodal quartics considered as limiting cases of non-singular quartics. He enumerates and discusses the following cases: (*a*), conjugate points considered as limiting cases of ovals; (*b*), nodes which arise when in limiting cases of annular quartics the inner branch comes to meet the outer;—in neither of these cases are the Zeuthen bitangents affected; (*c*), nodes which arise when two mutually external branches come to meet; (*d*), which arise when a branch of even order breaks up into the intersection of two of odd order; (*e*), the case of two imaginary double points. In the cases where the Zeuthen bitangents are affected, the investigation is carried on by considering the forms represented by the equation $wxyz = V^2$, when V passes through the intersection of two of the lines, or when two of the lines coincide with each other.

249. In order to see how quartics might be classified in respect of their infinite branches, we observe that the line infinity may meet a quartic, (*a*) in four real points, (*b*) in two real and two imaginary, (*c*) in four imaginary points, (*d*) in two coincident and two real points, (*e*) in two coincident and two imaginary points, (*f*) twice in two coincident points, these points being real, or (*g*) these points being imaginary, (*h*) in three coincident and one real point, (*i*) in four coincident points. Again, the cases (*d*), (*e*), (*f*), (*g*) would have to be further distinguished according as the line infinity when meeting the curve in two coincident points is simply a tangent or a line passing through a double point, which double point may be either crunode or acnode, cusp, or one of the special kinds above mentioned. Similarly in the case (*h*), the line infinity may be either an ordinary stationary tangent, or a tangent at a double point or cusp, or it may pass through a triple point, and in the case (*i*) it may be either a tangent at a point of undulation, a tangent at a double point of the special kind, or a tangent at a triple point. Lastly, any of the points which count only as single intersections of the line infinity with the curve may be on the curve a point of inflexion or undulation, and where this happens a difference in the figure will result which would have to be taken into account in a complete classification of quartics.

250. We have already shown (Art. 70) how to form the equation of the Hessian of a quartic, which is a curve of the sixth degree, intersecting the quartic in the twenty-four points of inflexion. We have also seen (Art. 92) that the equation of the reciprocal of a quartic is of the form $S^2 = T^3$, where S represents a curve of the fourth and T of the sixth class, and the form of the equation shows that both are touched by the twenty-four stationary tangents. We have postponed to another chapter the solution of the problem to form the equation of a curve passing through the points of contact of double tangents of a given curve. It will there be shown that, in the case of the quartic, the equation of such a bitangential curve may be written in the form $\Theta = 3H\Phi$, where Θ is the covariant $AL'^2 + \&c.$, as in Art. 231; that is to say, L' &c. represent the first differential coefficients of the Hessian, and A denotes $bc - f^2$, where a, b, &c. are the second differential coefficients of U. In like manner Φ denotes $Aa' + \&c.$, as in Ex. 1, Art. 230.

THE BITANGENTS.

251. It is convenient to commence by studying a more general theory in which that of the bitangent is included. Let us then consider first the form $UW = V^2$, where U, V, W

$\lambda^2 U + 2\lambda V + W$ have this higher contact with the quartic, namely, the twelve passing each through one of the intersections of the Jacobian with the quartic.

254. Six conics of the system $\lambda^2 U + 2\lambda V + W$ reduce to a pair of right lines; for the discriminant of this form being a function of the third degree in its coefficients will be one of the sixth degree in λ, and therefore six values of λ can be found for which it vanishes. When an enveloping conic reduces to a pair of right lines, the four points of contact lie two on each line, and each line is therefore a double tangent to the quartic. It appears from Art. 249, that if ab, cd be any two of these six pairs of bitangents, the equation of the quartic may be transformed to $abcd = V^2$, the eight points of contact lying on a conic V. Thus we see that the form $\lambda^2 U + 2\lambda V + W$ includes six pairs of the bitangents of the quartic, these twelve bitangents all touching a curve of the third class, viz. the Cayleyan of the system, and the intersections of each pair lying on the Jacobian. So again, if the points of contact of any of these pairs of bitangents be joined directly or transversely, the joining lines also touch the Cayleyan, and the intersection of each pair lies on the Jacobian. This may be stated in a slightly different form by considering the cubic S, of which U, V, W are polar conics. Then if the equation of a quartic is a function of the second degree in U, V, W, since the vanishing of such a function expresses the condition that the line $xU + yV + zW = 0$ should touch a fixed conic, it is easy to see that the quartic may be defined as the locus of a point whose polar with respect to S touches a fixed conic, or, in other words, the locus of the poles with respect to S of the tangents of that fixed conic; or, it will come to the same thing if it be defined as the envelope of the polar conics of the points of that conic. The double tangents of the quartic correspond to the points where the conic meets the Hessian of S.

255. Let us now consider any two of the bitangents of a quartic, which we take for the lines x, y; then if we make $x = 0$, the equation of the quartic is to reduce to a perfect

square, say $(z^2 + ayz + by^2)^2$, and if we make $y = 0$, the equation is to reduce to, say $(z^2 + cxz + dx^2)^2$. Hence, evidently the equation of the quartic must be of the form

$$xyU = (z^2 + ayz + by^2 + cxz + dx^2)^2;$$

that is to say, of the form $xyU = V^2$, which we have just discussed; an equation which may also be written

$$xy(\lambda^2 U + 2\lambda V + xy) = (xy + \lambda V)^2.$$

There are, as we have seen, beside the value $\lambda = 0$, corresponding to the pair of lines xy, five other values of λ for which $\lambda^2 U + 2\lambda V + xy$ will represent a pair of lines; and thus in five different ways the equation can be reduced to the form $wxyz = V^2$. Hence, *through the four points of contact of any two bitangents we can describe five conics, each of which passes through the four points of contact of two other bitangents.*

A non-singular quartic has 28 bitangents; and there are therefore $\frac{1}{2}(28.27)$, or 378 pairs of bitangents; each of these pairs gives rise to five different conics, but each conic may arise from any one of the six different pairs formed by the four bitangents which correspond to that conic, hence *there are in all* $\frac{5}{6}(378)$ *or* 315 *conics, each of which passes through the points of contact of four bitangents of a quartic.**

256. We have seen that each pair of bitangents combines with five other pairs to form a group of six pairs, the points of contact of any two of which pairs lie on a conic. It follows that the 378 pairs may be distributed into 63 such groups of six. The twelve bitangents of each group touch the same curve of the third class; and this is touched also by the lines joining directly and transversely the points of contact of each pair. The intersections of each pair of bitangents, and also those of each pair of joining lines, lie on a cubic. Corresponding to each group there are twelve conics, each of which touches the quartic twice with ordinary contact, and once so as to meet it in four

* Plücker first noticed the possibility of bringing the equation of any quartic to the form $wxyz = V^2$, but he hastily inferred that the six points of contact of *any* three bitangents lie on a conic, and thence drew an erroneous conclusion as to the total number of conics passing through eight points of contact of bitangents (see the *Theorie der Algebraischen Curven*, p. 246).

consecutive points, the twelve points of higher contact lying on the cubic last mentioned. There being 63 groups, 756 such conics may in all be drawn.

257. We shall show how to form a scheme of the 315 conics, and for that purpose we denote provisionally the first 26 bitangents by the letters of the alphabet, adding the symbols ϕ and ψ to denote the other two. We denote by *abcd* the conic passing through the eight points of contact of the bitangents a, b, c, d. If now *abcd*, *abef*, be two of the 315 conics, the pairs *ab*, *cd*, *ef* belong to the same group, and from what we have seen, *cdef* will be another of the conics. This may also be shown directly as follows. Let the equation of the quartic be $abcd = V^2$, or

$$ab\,(cd + 2\lambda V + \lambda^2 ab) = (V + \lambda ab)^2,$$

and we can determine λ so that $cd + 2\lambda V + \lambda^2 ab = ef$. Solve for V from this equation, and substitute in the equation of the quartic, when it becomes

$$\lambda^4 a^2 b^2 + c^2 d^2 + e^2 f^2 - 2\lambda^2 abcd - 2\lambda^2 abef - 2cdef = 0,$$

or

$$4cdef = (cd + ef - \lambda^2 ab)^2,$$

a form which proves the theorem stated. It appears thus, that given three pairs of lines which are to be pairs of bitangents of the same group of a quartic, the equation of the quartic will be of the form $l\sqrt{(ab)} + m\sqrt{(cd)} + n\sqrt{(ef)} = 0$, so that if two points were given in addition, a single quartic could be found satisfying the prescribed conditions. Corresponding to any group there are 15 conics, passing respectively through the points of contact of each two of the six pairs of which the group consists. There would thus seem to be $63 \times 15 = 945$ conics; but then every conic *abcd* is counted three times over, as belonging to the three groups *ab*, *cd*, &c., *ac*, *bd*, &c., *ad*, *bc*, &c.; the total number is therefore 315 as before.

258. Consider any conic *abcd*, then the group *ab*, *cd*, &c., and the group *ac*, *bd*, &c., can have no other bitangent common, the quartic being supposed to be non-singular. For example,

if *abef* be a conic of the first group, *aceg* cannot be a conic of the second. For (Art. 257) the equation of the conic through the points of contact of a, b, c, d may be written in the form

$$\lambda ab + \frac{1}{\lambda}(cd - ef) = 0,$$

and if *aceg* be another conic, this must be identical with the form

$$\mu ac + \frac{1}{\mu}(bd - eg) = 0.$$

From this identity we at once infer

$$(\lambda b - \mu c)\left(a - \frac{1}{\lambda\mu}\,d\right) = e\left(\frac{1}{\lambda}f - \frac{1}{\mu}\,g\right).$$

It follows that e, being identical with one of the factors into which the left-hand side breaks up, passes through the intersection either of b and c or of a and d. But in either case the point through which e is thus proved to pass will be a double point on

$$4\lambda^2 abcd = (\lambda^2 ab + cd - ef)^2,$$

and therefore the quartic could not be non-singular.

In precisely the same way we see that if *abef*, *acmn* be two conics, there is an identity

$$(\lambda b - \mu c)\left(a - \frac{1}{\lambda\mu}\,d\right) = \frac{1}{\lambda}\,ef - \frac{1}{\lambda}\,mn,$$

and hence the diagonals of the quadrilateral *efmn* pass one through *ad*, the other through *bc*; or, in other words, the intersections of each pair of bitangents lie, according to a certain rule, three by three on right lines. When once a scheme of the 315 conics has been made, there is no difficulty in discriminating which diagonal passes through *ad* and which through *bc*. For example, if it appears that *aemu*, *afnv*, *aduv* are conics of the system, we infer in like manner that the diagonals of the quadrilateral *emfn* pass through *ad* and *uv*; and thence we infer that *ad* lies on the line joining *en*, *fm*. Thus then consider any conic *abcd*, this belongs to the three groups *ab*, *cd*, &c., *ac*, *bd*, &c., and *ad*, *bc*, &c., and it appears now that each of the sixteen quadrilaterals formed by combining one of the four other pairs belonging to the group *ac*, *bd* with a pair from the group ad, bc, will have a diagonal passing through ab. Now the pair ab belongs to five different conics, and therefore there are eighty quadrilaterals having a diagonal passing through ab. But it will be found, as we have intimated, that these quadrilaterals may be distributed into pairs having a common diagonal; hence, through each of the 378 points ab can be drawn 40 lines, each passing through two others of these points, and there are in all 5040 such lines.

259. We are now in a position to form a scheme of the 315 tangents, in which nothing but the notation shall be arbitrary. Commence by writing down the group ab, cd, ef, gh, ij, kl; then since the groups ac, bd; ad, bc can have no bitangent common with the preceding nor with each other, these groups may be written, ac, bd, mn, op, qr, st; ad, bc, uv, wx, yz, $\phi\psi$. Proceed now to write down the group ae, bf; this must include no bitangent from the group ab; but in each term one of the bitangents from the group ac will be combined with one from the group ad. Now since it was free to us to write down the pairs of each group in any order we pleased, it is a mere matter of notation, and does not introduce any geometrical condition, if we take this group to be ae, bf, mu, ow, qy, $s\phi$. In like manner, it is a mere matter of notation to suppose that the bitangents have been so lettered, that ag and mx, ai and mz, ak and $m\psi$ shall respectively belong to the same group. This being assumed, it will be found that the group af, be is necessarily nv, px, rz, $t\psi$, and we can thus proceed, step by step, to write out the whole system. A table of the 315 conics was accordingly given in the first edition, but I do not occupy space with it now, because an algorithm has been given by Hesse (*Crelle*, 1855, XLIX, 243), and more minutely discussed by Professor Cayley (*Crelle*, 1868, LXVIII, 176), which exhibits in an easily recognizable form the mutual relations of the 28 tangents. Hesse's method introduces considerations from the geometry of three dimensions. He equates to zero the discriminant of $\alpha U+\beta V+\gamma W$ where U, V, W denote quadric surfaces. This discriminant being a function of the fourth degree in α, β, γ, if these quantities be regarded as variables, the equation denotes a plane quartic. But for any value of α, β, γ for which the discriminant vanishes, $\alpha U+\beta V+\gamma W$ denotes a cone, so that to every point on the plane quartic corresponds a point in space, namely, the vertex of this cone; and Hesse's method connects the double tangents of the plane quartic with the lines connecting each pair of 8 points in space which are the intersections of three quadric surfaces. We make no use here of any principles of solid geometry, but merely borrow the notation which Hesse's method suggests.*

260. Take then eight symbols 1, 2, 3, 4, 5, 6, 7, 8. Their combination in pairs gives us 28 symbols 12, 13...78, which we use to denote the 28 bitangents. This notation, the symbols being properly applied to the 28 bitangents, enables us correctly to represent their geometrical relations, though it fails completely to exhibit the symmetry of the system. In fact, the notation might suggest that the bitangent 12 was related in a different manner to the bitangents 13, 14, &c., and to the bitangents 34, 56, &c., whereas actually there is no geometric difference between the relations of any pair of bitangents. So again we suppose the symbols so applied, that 12, 34, 56, 78 shall denote bitangents whose 8 points of contact lie on a conic. The same property will then belong to every tetrad of bitangents represented by a like set of duads; that is, by any four duads containing all the eight symbols. But if we count, we shall find that we can only make 105 arrangements of the 8 symbols into sets, such as 12, 34, 56, 78. The remaining 210 conics correspond to four bitangents, whose symbols are such as 12, 23, 34, 41; that is to say, the duads are formed cyclically from any arrangement of four of the eight symbols, and it will be found that we

* Another mode of connecting the theory of 28 bitangents with Solid Geometry is used by Geiser, *Mathematische Annalen* I. 129, as follows: From any point on a cubic surface can be drawn a quartic cone touching the surface. This will be non-singular, its bitangent planes being the tangent plane to the cubic at the vertex, and the planes joining the vertex to the 27 lines on the surface. Zeuthen shows that his classification of quartics with regard to the reality of their bitangents leads by a different process to the results obtained by Schläfli in classifying cubic surfaces with respect to the reality of their right lines.

can have 210 such tetrads. Thus then the group belonging to the pair 12, 34, consists of 56, 78; 57, 68; 58, 67; 13, 24; 14, 23; and the group belonging to a pair such as 12, 13, is 24, 34; 25, 35; 26, 36; 27, 37; 28, 38. Thus the notation shows completely how the bitangents are to be combined in groups. It suggests, however, that the 105 conics of the form 12, 34, 56, 78 differ in their properties from the 210 of the form 12, 23, 34, 41. This is not the case, the whole 315 tetrads forming an indissoluble system.

261. Professor Cayley remarks that Hesse's researches establish the following general rule: *A bifid substitution makes no alteration in the geometrical relations of the bitangents denoted by any set of symbols.* What is meant by a bifid substitution is, that writing down such a symbol of substitution as 1234·5678, we interchange everywhere the duads 12, 34; 13, 24; 14, 23; and again, 56, 78; 57, 68; 58, 67; but leave unchanged such duads as 15, 36, where one of the first set of symbols is combined with one of the second. The number of possible bifid substitutions is 35, or, if we add unity (viz. no alteration of any duad) the number is 36.

For example, now if we apply the bifid substitution 1234·5678 to the pair 12, 34, we get the same pair in opposite order; if we apply it to 12, 13, we get 34, 24, a pair of the same type as 12, 13; if we apply it to 12, 15, we get 34, 15, a pair of apparently a different type, but not different in geometrical relations. Thus, then, if we apply the same bifid substitution as before to the tetrad 15, 67, 28, 34, which is one of the set of 105 already referred to, we get 15, 58, 82, 21, which is one of the set of 210, and which, according to the rule, possesses the same geometrical properties.

262. Professor Cayley has exhibited in the following table the geometrical relations of the bitangents, taken singly in twos, threes, or fours, and the number of terms belonging to each type of arrangement of the symbols.

	Representative term.	*No. of terms.*		*Geometrical character.*
|	12	28	28	Bitangents.
V	12.13	168	378	Pairs of bitangents.
||	12.34	210		
⊔	12.23.34	420	1260	Triads of bitangents such that 6 points of contact are on conic.
|||	12.34.56	840		
△	12.23.31	56		Triads such that 6 points of contact are not on conic.
V|	12.23.45	1680	2016	
Ѱ	12.13.14	280	3276	
||||	12.34.56.78	105	315	Tetrads of bitangents such that the 8 points of contact are on conic.
□	12.23.34.41	210		
||V	12.34.56.67	2520		Tetrads such that 6 out of the 8 points of contact are on conic.
⌈⊔	12.34.45.56	5040		
⊔⌈	12.23.34.45	3360	15120	
△/	12.23.31.14	840		
VΛ	12.13.14.45	3360		
|△	12.34.45.53	560		Tetrads such that no 6 points of contact are on conic.
V∠	12.13.14.15	280	5040	
|Ѱ	12.34.35.36	1680		
VV	12.13.45.46	2520	20475	

In the above, for greater clearness, a geometrical symbol has been attached to each term, viz. the symbols 1, 2, 3, 4, 5, 6, 7, 8 being regarded as points, when any two of these are combined into a duad, this is indicated by a line being drawn to join the two points; thus △ is the symbol of the term 12.23.31. This is very convenient; we can for instance, by mere inspection, see that the symbol of any partial set in the set of 15120 terms, contains as part of itself one of the symbols |||, ⊔, viz. that there are among the 8 bitangents six such that their points of contact lie in a conic; whereas, contrariwise in the symbols of the partial sets belonging to the set of 5040, no one of these symbols contains as part of itself either of the symbols |||, ⊔.

To the foregoing may be joined the following two groups of hexads of bitangents:

	Representative term	*No. of terms*	
	12.23.31.45.56.64	280	
	12.34.35.36.37.38	168	
	12.13.14.56.57.58	560	
	12.23.31.14.45.51	140	
	12.23.34.45.56.61	1680	5040
	12 34 35 36 67.68	2520	

These 1008 and 5040 hexads have been studied by Hesse and Steiner as bitangents whose twelve points of contact lie on a proper cubic, the former set having no six contacts on a conic, but the twelve points of contact in the latter case being divisible into two sets of six lying each on a conic. It may be added, that the six tangents of each of the 1008 hexads all touch the same conic, as will appear from Aronhold's investigations, which will be presently given. The six tangents of each of the 5040 hexads may be distributed into three pairs, whose points of intersection lie on a right line (see Art. 258).

263. We conclude this discussion of the bitangents with an account of the method by which Aronhold has shewn (see Berlin *Manatsberichte*, 1864, p. 499), that when seven arbitrary lines are given, a quartic can be found having these lines as bitangents, and of which the other bitangents can be found by linear constructions. The method depends on properties of a system of curves of the third class having seven common tangents, but it seems convenient to state them first in the reciprocal form with which the reader is more familiar, viz. as properties of a system of cubics passing through seven given points. (1) Consider any one cubic of the system, then if the eighth and ninth points in which it is intersected by any other cubic of the system be joined, the joining line passes through a fixed point on the assumed cubic, viz. the coresidual of the seven given points (Art. 160). (2) Through any assumed point 8 can be described one and but one cubic on which

this point shall be the coresidual of the seven given points. For all cubics of the system through the point 8 pass through another fixed point 9, and, by definition, the coresidual is the point where the line joining these points meets the curve again. If, therefore, the coresidual is to coincide with the point 8, the cubic must be that one which is determined by having the line 89 as its tangent at the point 8. (3) Four cubics of the system can be described to touch a given cubic of the system, the points of contact being obviously the points of contact of tangents drawn to the given cubic from the coresidual point on it. (4) If the points 8, 9 coincide, that is to say, if cubics of the system touch, the envelope of the common tangent 89 is a curve of the fourth class. For consider how many such lines can pass through any assumed point P. Suppose a cubic described through P, and through the points 8, 9, then, by definition, P is the coresidual point on that cubic, and by (2) this cubic having P for the coresidual is a determinate known cubic. We see then, from (3), that the envelope in question is of the fourth class, the four tangents from any point P being constructed by finding the cubic which has P for its coresidual, and drawing the four tangents from P to that cubic. (5) The point P will be a point on the envelope curve, if two of the tangents drawn from it coincide; but from the construction just given, it appears that this can only happen when the curve having P for its coresidual has a node; for in this case two tangents coincide with the line joining P to the node. Hence the envelope we are considering may also be defined as the locus of the coresidual of the given system of points on all the nodal cubics of the system. (6) If the cubic through the seven points break up into a conic through five of them, and a line joining the other two, it has two nodes, namely, the intersection of the line and conic. Any other cubic of the system meets this complex cubic in two other points, one on the line, one on the conic, and the coresidual is the point P where the line joining these two meets the conic again. In this case, then, P is a double point, the two tangents at it being the lines joining it to the intersections of line and conic. Now seven points can be divided in 21 different ways into a system of two and of five. The curve we are considering has, therefore,