Geometry of Algebraic Curves

Fall 2011

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Contents

LECTURE 1

September 2, 2011

The text for this class is ACGH, Geometry of Algebraic Curves, Volume I. There will be weekly homeworks and no final exam. Anand Deopurkar will hold a weekly section.

We are going to talk about compact Riemann surfaces, which is the same thing as a smooth projective algebraic curve over $\mathbb{C}$. Most of what we say will hold over an algebraically closed base field of characteristic zero.

We are going to assume something very special about curves – there exist non-constant meromorphic functions.

We are not going to cover systematically:

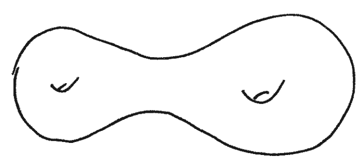
- singular algebraic curves,
- open Riemann surfaces,
- families of curves,
- curves over non-algebraically closed fields (e.g., $\mathbb{R}$, $\mathbb{Q}$, number fields).

Singular curves will inevitably enter our discussion.

First question: How can me map a curve into projective space?
For singular $X \subset \mathbb{P}^r$, we will treat X via its normalization (that is as an image of a smooth curve).

Today we will go through the basics in order to establish a common language and notation.

Let X be a smooth projective algebraic curves over $\mathbb{C}$. Naively, $g = \text{genus}(X)$ is the topological genus.



Riemann surface of genus 2.

There are other possible definitions of the genus:

$$1 - \chi(\mathcal{O}_X), \quad 1 - \frac{1}{2}\chi_{\text{top}}(X), \quad \frac{1}{2}\deg(K_X) + 1, \quad 1 - \text{constant term in the Hilbert polynomial of } X.$$

Remark 1.1. By the maximum principle, every holomorphic function on X is constant. We need to allow poles, so we will keep track of them.

Definition 1.2. A *divisor* D is a formal finite linear combination of points

$$D = \sum_i n_i \cdot p_i$$

for $p_i \in X$.

Definition 1.3. We call $D = \sum n_i \cdot p_i$ *effective* if $n_i \geq 0$ for all i. We will use $D_1 \geq D_2$ to denote that $D_1 - D_2$ is effective.

Definition 1.4. The degree of $D = \sum n_i \cdot p_i$ is given by

$$\deg(D) = \sum n_i \in \mathbb{Z}.$$

For each $d \geq 1$, we have a bijection:

$$\left\{\begin{matrix}\text{effective divisors} \\ \text{of degree } d\end{matrix}\right\} \longleftrightarrow X_d = \operatorname{Sym}^d X = \underbrace{X \times \cdots \times X}_{d} / S_d.$$

Given a meromorphic function f on X, we can associate

$$(f) = \sum_{p \in X} \operatorname{ord}_p(f) \cdot p.$$

Positive terms indicate zeros, and negative ones poles. Singularities and zeros are isolated so (f) is well-defined.

Given a divisor $D = \sum n_i \cdot p_i$, we can look at functions with poles only at p_i with order bounded by n_i:

$$\mathcal{L}(D) = \{f \text{ rational on } X \mid \operatorname{ord}_p(f) \geq -n_p \forall p \in X\} = \{f \mid (f) + D \geq 0\}.$$

This is a vector space over $\mathcal{C}$. Set

$$\ell(D) = \dim \mathcal{L}(D), \qquad r(D) = \ell(D) - 1.$$

Basic problem: Given D, find $\mathcal{L}(D)$, and, in particular, find $\ell(D)$.
There is a simple and well-understood algorithm for doing this.

As stated the problem has some redundancy. If E is another divisor and $D - E = (g)$ for a global rational function g, then

$$\mathcal{L}(D) \xrightarrow{\times g} \mathcal{L}(E)$$

is an isomorphism. We do not need to consider all divisors, but only up to equivalence.

Definition 1.5. We say two divisors D and E are *linearly equivalent*, denoted $D \sim E$, if $D - E = (g)$ for a global rational function g.

Remark 1.6. For each g as above, $\deg(g) = 0$. To see this complex analytically, observe that

$$\deg(g) = \int_X \frac{g'}{g} dz = 0.$$

Therefore $D \sim E$ implies $\deg(D) = \deg(E)$.

We are lead to define

$$\operatorname{Pic}^d(X) = \{\text{divisors of degree } d \text{ on } X\}/\text{linear equivalence}.$$

There is a divisor which plays a special role in the theory. Let ω be a meromorphic 1-form (locally, $\omega = f(z)dz$ for a meromorphic function f). Define

$$\operatorname{ord}_p(\omega) = \operatorname{ord}_p(f),$$

and

$$(\omega) = \sum_{p \in X} \operatorname{ord}_p(\omega) \cdot p.$$

If ω and η are meromorphic/rational 1-forms, then their ratio ω/η is a meromorphic/rational function (locally, if $\omega = f(z)dz$ and $\eta = g(z)dz$, then $\omega/\eta = f(z)/g(z)$). It follows that $(\omega) \sim (\eta)$. We define the *canonical class*, denoted K_X, to be (ω) for some meromorphic 1-form ω.

Now we would like to translate this to a slightly more abstract point of view. Namely, it is convenient to use a notation which avoids the language of divisors. For this, one can employ line bundles.

Consider a divisor $D = \sum_p n_p \cdot p$ on X. Let $\mathcal{O}_X$ be the sheaf of regular functions on X. Similarly, let $\mathcal{O}_X(D)$ be the sheaf of functions with zeros and poles prescribed by D. Concretely,

$$\mathcal{O}_X(D)(U) = \{f \text{ meromorphic on } U \mid \operatorname{ord}_p(f) \geq -n_p \forall p \in U\}$$

for any open $U \subset X$. The point is $L = \mathcal{O}_X(D)$ is locally free of rank 1, i.e., a line bundle.

There is already some ambiguity between locally free sheaves and vector bundles, but we will ignore this.

If $D \sim E$, then $\mathcal{O}_X(D) \cong \mathcal{O}_X(E)$ via multiplication by a rational function. The converse is also true, so we arrive at the following correspondence.

$$\{\text{divisors on } X\} / \sim \longleftrightarrow \{\text{line bundles on } X\} / \cong$$

For example, $L = T_X^\vee$ corresponds to the canonical divisor class.

For $L = \mathcal{O}_X(D)$, we will adopt the notation

$$\mathcal{L}(D) = \mathrm{H}^0(X, \mathcal{O}_X(D)) = \mathrm{H}^0(X, L).$$

Question: Given an abstract Riemann surface, how do we describe maps into projective space?

Let L be a line bundle on X, and $\sigma_0, \ldots, \sigma_r \in \mathrm{H}^0(L)$ be sections with no common zeros. We get a map

$$\begin{aligned} f\colon X &\longrightarrow \mathbb{P}^r, \\ p &\longmapsto [\sigma_0(p), \ldots, \sigma_r(p)]. \end{aligned}$$

Note that $\sigma_i(p)$ are not numbers but elements of the fiber L_p, so only their ratio is well-defined. Equivalently, L is locally-trivial, so we get a local definition of f, which turns out to be independent of the chosen trivialization. If $\{\sigma_i\}$ are not linearly independent, then the image of f would lie in a linear subspace of $\mathbb{P}^r$. A map $f\colon X \to \mathbb{P}^r$ whose image is not contained in a linear subspace is called *nondegenerate*.

If we replace the sections $\{\sigma_i\}$ with a linear combination, we get the same map up to linear isomorphism of $\mathbb{P}^r$. Therefore, modulo $\mathrm{Aut}(\mathbb{P}^r) = \mathrm{PGL}(r+1)$, the map f depends only on $V = \langle \sigma_0, \ldots, \sigma_r \rangle \subset \mathrm{H}^0(L)$. We arrived at the correspondence

$$\left\{ (L, V) \,\middle|\, \begin{array}{c} L \text{ line bundle over } X, \\ V \subset \mathrm{H}^0(L) \text{ has no common zeros} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{nondegenerate maps} \\ X \to \mathbb{P}^r \end{array} \right\} / \mathrm{Aut}(\mathbb{P}^r).$$

The degree of the line bundle L is the degree of the map it corresponds to.

Definition 1.7. A *linear series* of degree d and dimension r on X is a pair (L, V), where L is a line bundle on X of degree d, and $V \subset \mathrm{H}^0(L)$ is a subspace of dimension $r+1$.

Remark 1.8. Note that we dropped the hypothesis that there are no common zeros. If such exist, the induced f is a rational map defined away from the common zeros. We allow such behavior in order to ensure the space of linear series is compact.

A linear series of degree d and dimension r is called a g^r_d. Here g does not refer to the genus of the curve X. The notation comes from the Italian school which referred to a linear series as a "groups of points".

A section $\sigma \in \mathrm{H}^0(L)$ corresponds to an effective divisor on X. When necessary, we will denote the family of effective divisors $\mathbb{P}V$ corresponding to $V \subset \mathrm{H}^0(L)$ by $\mathcal{D}$.

Given a linear series (L, V) with no common zeros, we can alternatively describe the associated map as

$$\begin{aligned} f\colon X &\longrightarrow \mathbb{P}(V^\vee) \cong \mathbb{P}^r, \\ p &\longmapsto H = \{\sigma \in V \mid \sigma(p) = 0\}. \end{aligned}$$

We are utilizing the fact that points of $\mathbb{P}(V^\vee)$ correspond to hyperplanes $H \subset V$.

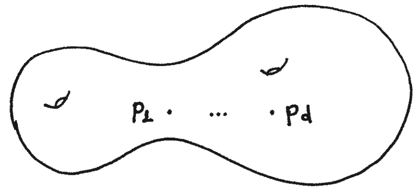
When $r \geq 1$, we have enough functions to get a map to projective space.

Fact 1.9.

$$\ell(K_X) = \dim\{\text{global holomorphic 1-forms}\} = g.$$

Therefore, the linear series $(K_X, \mathrm{H}^0(K_X))$ produces a map $\varphi = \varphi_X\colon X \to \mathbb{P}^{g-1}$.

Now, let us give a quick "proof" of Riemann-Roch. To some extent, this answers the basic question we previously proposed.



Consider a divisor $D = p_1 + \cdots + p_d$ where p_i are assumed to be distinct for convenience. We want to describe the set of all meromorphic functions with at worst simple poles at p_i. Given a local coordinate z_i around each p_i, a function $f \in \mathcal{L}(D)$ can locally be written as $f = a_i/z_i + f_i$ where f_i is holomorphic. Since there are no global holomorphic sections, f is specifies up to an additive constant by the constants a_i. Succinctly put, there is an exact sequence

$$0 \longrightarrow \{\text{constant functions on } X\} \longrightarrow \mathcal{L}(D) \longrightarrow \mathbb{C}^d$$

To find $\mathcal{L}(D)$, we want to find the image of $\mathcal{L}(D)$ in $\mathbb{C}^d$. In other words, we would like to answer the question: Given $a_1, \ldots, a_d \in \mathbb{C}$, does there exist f with polar part a_i/z_i at p_i and holomorphic elsewhere.

A holomorphic 1-form $\omega \in \mathrm{H}^0(K_X)$ gives us a necessary condition. Consider the global meromorphic differential $f\omega$ (potentially, only with simple poles at p_i) must satisfy

$$\sum_i \mathrm{Res}_{p_i}(f\omega) = 0.$$

Suppose $\omega = g_i(z_i)dz_i$ around p_i. Then

$$\mathrm{Res}_{p_i}(f\omega) = a_i g_i(p_i),$$

which implies

$$\sum_i a_i g_i(p_i) = 0.$$

We can phrase this condition by saying that the image of $\mathcal{L}(D)$ in $\mathbb{C}^d$ lies in the orthogonal complement of the values of the holomorphic differential ω. We are lead to the bound $\ell(D) \leq 1 + d$. If the holomorphic differentials impose linearly independent conditions, we would have $\ell(D) \leq 1 + d - g$ since $\dim \mathrm{H}^0(K_X) = g$. But we also need to account for the possibility that a holomorphic differential vanishes at all p_i. In the end, we are lead to the bound

$$\ell(D) \leq 1 + d - g + \ell(K_X - D).$$

Consider the canonical divisor class K_X. The Hopf index theorem implies

$$\deg(K_X) = -\chi_{\text{top}}(X) = 2g - 2.$$

The above inequality applied to $K_X - D$ yields

$$\ell(K - D) \leq 1 + (2g - 2 - d) - g + \ell(D).$$

Adding these up, we obtain

$$\ell(D) + \ell(K_X - D) \leq \ell(D) + \ell(K_X - D),$$

so both inequalities above are equalities. We have proved the following classical statement.

Theorem 1.10 (Riemann-Roch). *Any divisor D on a curve X satisfies*

$$\ell(D) = 1 + d - g + \ell(K_X - D).$$

It is worth noting that we cheated at several points.

- We assumed all points in the divisor D are distinct. The present proof can handle this at the cost of a little more notation.
- The entire theory derived from a basic fact $\sum_p \mathrm{Res}_p\, \omega = 0$ for a meromorphic 1-form ω.

LECTURE 2

September 7, 2011

2.1. Riemann surfaces associated to a polynomial

Consider $f(x,y) \in \mathbb{C}[x,y]$ and write it as

$$f(x,y) = a_d(x)y^d + \cdots + a_0(x).$$

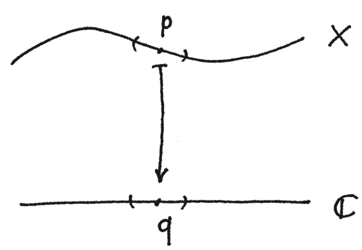
Setting $f = 0$, for each value of x, there are generically d values of y. Therefore, we can think of y as an (implicitly defined) multi-valued function of x. This was one of the motivations to study Riemann surfaces. We will start by doing things complex-analytically.

Set

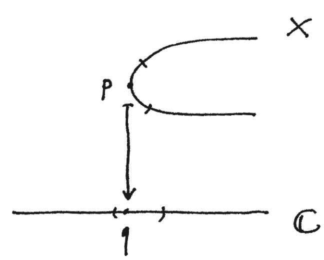
$$\begin{aligned} X &= \{(x,y) \mid f(x,y) = 0\} \subset \mathbb{C}^2, \\ X^0 &= \text{smooth locus of } X \text{ (complement of finitely many points in } X). \end{aligned}$$

There is a projection map on the first coordinate $\pi \colon X \to \mathbb{C}$ sending (x,y) to x. Take $p \in X$ and $q = \pi(p) \in \mathbb{C}$. There are several possibilities.

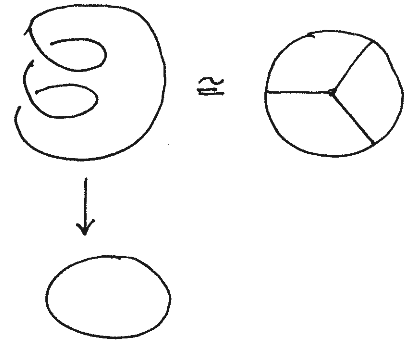
(i) $\frac{\partial f}{\partial y}(p) \neq 0$. This means X is smooth at p and π is a local isomorphism around p.



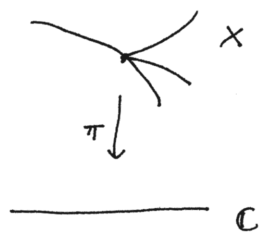
(ii) $\frac{\partial f}{\partial y}(p) = 0$, $\frac{\partial f}{\partial x}(p) \neq 0$. In this case X is smooth around p, but the map π is locally m-to-1 around the point p. We can choose a local coordinate z on X such that $\pi(z) = z^m$. We call p a *branch point* of π.



Remark 2.1. It looks like the image of a neighborhood of p under π is not open, but complex analytically this is the case.



(iii) $\frac{\partial f}{\partial x}(p) = \frac{\partial f}{\partial y}(p) = 0$. In this case, p is a singular point of X.



We can find a disc Δ around q such that the map

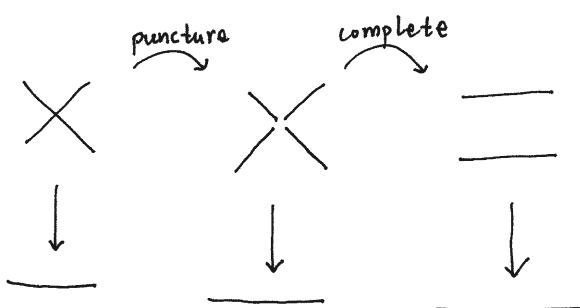
$$\pi|_{\pi^{-1}(\Delta^*)} \colon \pi^{-1}(\Delta^*) \longrightarrow \Delta^*$$

is a covering space. We denoted $\Delta^* = \Delta \setminus \{q\}$. Fortunately, the equations $\partial f/\partial x = \partial f/\partial y = 0$ define a set of isolated points in X. We are left to analyze covering spaces of a punctured disc Δ^*. Since $\pi_1(\Delta^*) \cong \pi_1(S^1) \cong \mathbb{Z}$, connected covers of Δ^* are described by the positive integers. For each $m \geq 1$, the corresponding cover is

$$\begin{array}{ccc} \Delta^* & \longrightarrow & \Delta^*, \\ z & \longmapsto & z^m. \end{array}$$

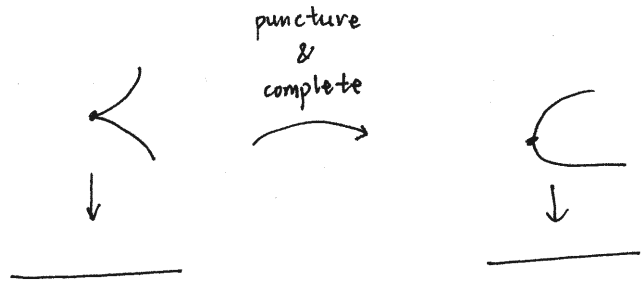
No matter how bad the singularity at p looks like, around it we have a disjoint union of punctured discs. Filling each connected component of $\pi^{-1}(\Delta^*)$ to a disc, we have modified X to a smooth complex manifold mapping down to $\mathbb{C}$.

Example 2.2. Consider $f(x, y) = y^2 - x^2$.



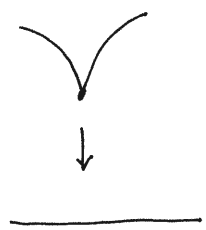
The curve $X = \{f = 0\}$ has a unique singularity at $(0, 0)$. Its complement is a disjoint union of two punctured lines, and the completion is comprised of two copies of $\mathbb{C}$.

Example 2.3. Consider $f(x, y) = y^2 - x^3$.



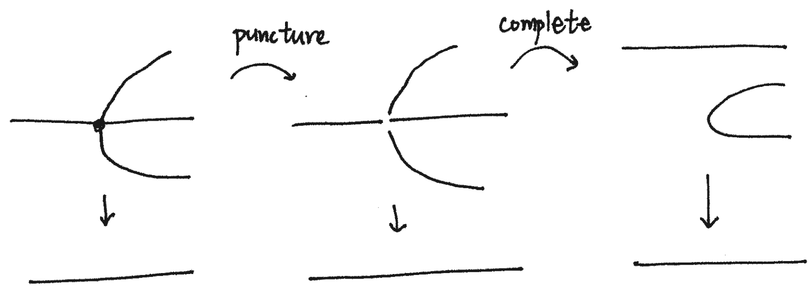
The completion is a 2-fold branched cover of $\mathbb{C}$ with a unique branch point over 0.

Example 2.4. Consider $f(x, y) = y^3 - x^2$.



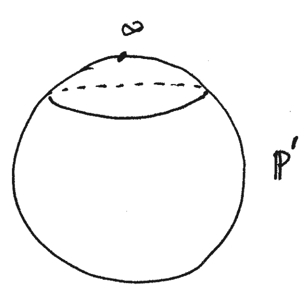
The map π is locally 3-to-1. We can complete it by adding a point to get a triple branch point.

Example 2.5. Consider $f(x, y) = y(x - y^2)$.



The completion is a disjoint union of a line and a conic as indicated by the factorization of f.

It is much more convenient to work with compact objects. Let us think of the target of the projection π as $\mathbb{C} \subset \mathbb{P}^1$. The complement of a disc around $\infty \in \mathbb{P}^1$ looks like a disc in $\mathbb{C}$.



Let Δ_R denote the disc of radius R around $0 \in \mathbb{C}$. For $R \gg 0$,

$$\pi^{-1}\left(\mathbb{C} \setminus \overline{\Delta}_R\right) \longrightarrow \mathbb{C} \setminus \overline{\Delta}_R$$

is a covering map and we can complete analogously. In this way each $X = \{f = 0\}$ can be modified to a smooth compact Riemann surface admitting a finite-to-one map to $\mathbb{P}^1$.

Example 2.6. The branch points of $y^2 = x^3 - 1$ occur over the cube roots of unity. As a point p goes around the origin in a large circle, $\arg(x)$ increases by 6π and $\arg(y)$ by 3π. This implies that $\pi^{-1}(\mathbb{C} \setminus \overline{\Delta}_R)$ is connected, hence we need only one point to complete at ∞.

Example 2.7. Consider $y^2 = x^4 - 1$. Similar reasoning implied that a neighborhood of ∞ is disconnected, so we need to add two points.

We can arrive at a compact desingularization of X purely algebraically. The homogenization of the defining polynomial defined the closure of $X \subset \mathbb{C}^2$ in $\mathbb{P}^2$. Then we normalize by blowing up singular points. It may, however, take multiple blow-ups to resolve singularities. In conclusion, the complex analytic procedure we described is much simpler to carry out, and should be preferred whenever possible.

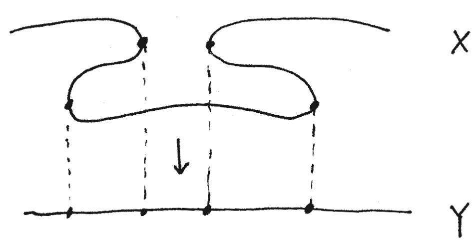
Example 2.8. Let a and b be positive coprime integers. The curve $x^a = y^b$ requires multiple blow-ups to desingularize but complex-analytically we complete with only one point.

In summary, the Riemann surface associated to a polynomial can be detected complex-analytically more easily.

2.2. The degree of K_X and Riemann-Hurwitz

Last lecture we assumed the fact $\deg(K_X) = 2g - 2$ for a compact Riemann surface X of genus g. We will deduce this from the Riemann-Hurwitz formula which we will discuss first.

Let X and Y be compact Riemann surfaces of genera g and h respectively, and $f\colon X \to Y$ be a non-constant finite map of degree d (complex-analytically, generically d-to-1).



At each $p \in X$ and $q = f(p) \in Y$ we can choose local coordinates on X and Y so that $f\colon z \mapsto z^m$ for an integer $m \geq 1$. If $m \geq 2$, we say p is a *ramification point* of order $\nu_p(f) = m - 1$. The *ramification divisor* is

$$R = \sum_{p \in X} \nu_p(f) \cdot p.$$

The images in Y of ramification points are called *branch points*. The *branch divisor* is defined as

$$B = \sum_{q \in Y} \left(\sum_{p \in f^{-1}(p)} \nu_p(f) \right) \cdot q = \sum_{q \in Y} n_q \cdot q.$$

It is easy to see that

$$\left| f^{-1}(q) \right| = d - n_q.$$

for all $q \in Y$. Let $q_1, \ldots, q_\delta$ be the points appearing in the branch divisor. The restriction

$$f|_{X \setminus f^{-1}(\{q_1, \ldots, q_\delta\})} \colon X \setminus f^{-1}(\{q_1, \ldots, q_\delta\}) \longrightarrow Y \setminus \{q_1, \ldots, q_\delta\}$$

is a d-sheeted covering map. We deduce

$$\begin{aligned}\chi(X \setminus f^{-1}(\{q_1, \dots, q_\delta\})) &= d\chi(Y \setminus \{q_1, \dots, q_\delta\}) \\ &= d(\chi(Y) - \delta) \\ &= d(2 - 2h - \delta).\end{aligned}$$

We are also removing finitely many points from X, so

$$\begin{aligned}\chi(X \setminus f^{-1}(\{q_1, \dots, q_\delta\}) &= \chi(X) - \sum_{i=1}^{\delta}(d - n_{q_i}) \\ &= 2 - 2g - \sum_{i=1}^{\delta}(d - n_{q_i}).\end{aligned}$$

Combining these we obtain the following.

Theorem 2.9 (Riemann-Hurwitz). *Let X and Y be Riemann surfaces of genera g and h respectively. For any degree d branched cover $f\colon X \to Y$, we have*

$$2 - 2g = d(2 - 2h) - \sum_{q \in Y} n_q.$$

It is convenient to note that

$$\deg B = \deg R = \sum_{q \in Y} n_q.$$

Then the Riemann-Hurwitz formula can be stated as

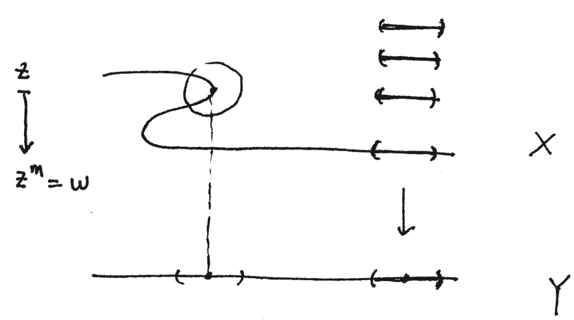
$$g - 1 = d(h - 1) + \frac{b}{2}$$

for $b = \deg B = \deg R$.

Remark 2.10. It is possible to generalize this considerably to varieties of arbitrary dimensions but we will not cover it here.

Onto the promised application. Suppose we have $f\colon X \to Y$ and ω is a meromorphic 1-form on Y. For simplicity assume $\mathrm{Supp}(\omega) \cap \mathrm{Supp}(B) = \emptyset$, i.e., ω has no zeros or poles at branch points. It is not necessary to make this assumption in general, but it makes our arguments easier. Analyzing the poles zeros and poles of ω, it is easy to deduce we have an equality of divisors

$$(f^*\omega) = f^*(\omega) + R.$$



Comparing this formula with the Riemann-Hurwitz formula, we see that $\deg(K_Y) = 2h - 2$ implies $\deg(K_X) = 2g - 2$. Since every compact Riemann surface admits a non-constant map to $\mathbb{P}^1$, it suffices to prove $K_{\mathbb{P}^1}$ has degree -2. This can be easily shown by considering a specific meromorphic differential on $\mathbb{P}^1$. We have shown that

$$\deg(K_X) = 2\,\mathrm{genus}(X) - 2$$

for all Riemann surfaces X.

2.3. Maps into projective space

We would like to give a criterion when a map of a Riemann surface into projective space in an embedding. Before doing so, let us introduce some notation.

A linear series on a Riemann surface X is a pair (L, V) where L is a line bundle on X and $V \subset \mathrm{H}^0(L)$ is a subspace of dimension $r+1$. We say (L, V) has degree d and dimension r which is often encoded by calling it a g_d^r.

A common zero of all $v \in V$ is called a *basepoint* of (L, V). If no such exist, we call the linear series *basepoint-free*.

A global section $\sigma \in \mathrm{H}^0(L)$ is determined up to scalars by its divisor of zeros. To see this consider two sections σ and σ' with the same divisor of zeros. It follows that the meromorphic function σ/σ' is actually holomorphic. On the other hand all holomorphic functions on X are constant, hence σ and σ' are the same up to scaling.

Given a linear series (L, V) we associate to it a family of effective divisors of degree f parametrized by $\mathbb{P}V$. The divisor corresponding to $\sigma \in V \setminus \{0\}$ is $D = (\sigma)$. In other words, a g_d^r corresponds to a family of effective divisors of degree d and parametrized by $\mathbb{P}^r$.

For a linear series (L, V) and an effective divisor E, both on X, define

$$V(-E) = \{\sigma \in V \mid (\sigma) \geq E\} \subset V.$$

If (L, V) is a basepoint-free linear series on X, we get a map

$$\begin{aligned} \varphi\colon X &\longrightarrow \mathbb{P}^r = \mathbb{P}V^\vee, \\ p &\longmapsto V(-p) \subset V. \end{aligned}$$

Since there are no basepoints, the space $V(-p)$ is guaranteed to be a hyperplane in V and φ is well-defined. More concretely, set $L = \mathcal{O}_X(D)$, $V \subset \mathcal{L}(D) = \mathrm{H}^0(L)$ and choose a basis $f_0, \ldots, f_r$ for V. Then in a local coordinate z on X, we can express

$$\varphi(z) = [f_0(z), \ldots, f_r(z)].$$

A priori this expression is not well-defined if some f_i has a pole or all f_i vanish. Take m to be the maximal order of pole or zero and scale throughout by z^m to make sense of the expression.

As constructed, the map φ is uniquely characterized (up to $\mathrm{Aut}(\mathbb{P}^r)$) by the family of divisors

$$\mathcal{D} = \{\varphi^{-1}(H) \mid H \subset \mathbb{P}^r \text{ hyperplane}\}.$$

This agrees with the family of divisors we associated to the linear system (L, V).

We are now ready to state the condition for embedding we alluded to earlier.

Proposition 2.11. *Let (L, V) be a g_d^r on X and $\varphi\colon X \to \mathbb{P}V^\vee$ the associated map.*

(i) *φ is injective if and only if $V(-p) \neq V(-q)$ for all distinct pairs $p, q \in X$.*
(ii) *φ is an immersion if and only if $V(-2p) \subsetneq V(-p)$ for all $p \in X$.*
(iii) *φ is an embedding if and only if $V(-p-q)$ has codimension 2 in V for all $p, q \in X$.*

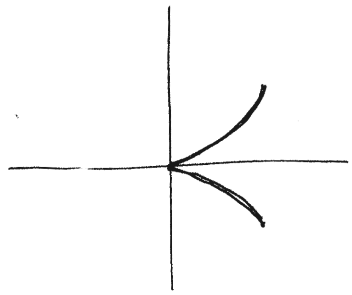
Together with Riemann-Roch, we deduce the following.

Corollary 2.12. *Let L be a linear bundle on X of degree $d \geq 2g+1$. Then the linear system $(L, \mathrm{H}^0(L))$ corresponds to an embedding of X in projective space.*

We proved that every compact Riemann surface can be realized inside projective space. We would like to do better, that is, obtain the smallest possible degree and dimension of ambient space. For example, give a curve X, what is the smallest d such that X is a degree d branched cover of $\mathbb{P}^1$?

2.4. An amusing fact

Consider the curve $y^2 = x^3$ in $\mathbb{C}^2$.

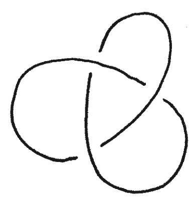


Take a ball $B_\varepsilon(0)$ of radius $\varepsilon > 0$ around the point $0 \in \mathbb{C}^2$. Its boundary $\partial B_\varepsilon(0)$ is a 3-sphere S^3. If ε is small enough, then $\partial B_\varepsilon(0)$ will intersect X in a 1-manifold (real curve) which is a link in S^3.

In the specific case we are considering, the equations

$$|y|^2 = |x|^3, \qquad |x|^2 + |y|^2 = \varepsilon$$

determine $|x|$ and $|y|$ uniquely. It follows that the knot $X \cap \partial B_\varepsilon(0)$ lies in a torus in S^3. We are left to find the possible arguments of x and y. Since $y^2 = x^3$, we have $2\arg(y) = 3\arg(x)$. hence we get a $(2,3)$-torus knot.



LECTURE 3

September 9, 2011

3.1. Embedding Riemann surfaces in projective space

We will start by completing our discussion when a map into $\mathbb{P}^n$ is an embedding. Recall the following from last time.

Proposition 3.1. *Let (L, V) be a linear series on a Riemann surface X. Then the associated map $\varphi_V \colon X \to \mathbb{P}V^\vee$ is am embedding if and only if $\dim V(-p-q) = \dim V - 2$ for all pairs $p, q \in X$.*

Corollary 3.2. *If $d = \deg L \geq 2g+1$ and $V = \mathrm{H}^0(L)$, then φ_V is am embedding.*

Proof. Let D be a divisor associated to L. Then Riemann-Roch implies

$$\ell(D) = d - g + 2 + \ell(D-K), \qquad \ell(D-p-q) = d - 2 - g + 1 + \ell(K - D + p + q)$$

for all $p, q \in X$. Since $\deg D \geq 2g+1$ and $\deg K = 2g-2$, we conclude

$$\ell(K-D) = 0, \qquad \ell(K - D + p + q) = 0. \qquad \square$$

In conclusion, given a line bundle L with $\deg L \gg 0$, we obtain an embedding. Let us analyze what happens in degrees which barely miss the hypothesis of the Corollary. For example, if $\deg L = 2g$, the argument goes through unless $D \sim K + p + q$ for some $p, q \in X$. Then either p and q are identified (if $p \neq q$), or we attain a cusp (if $p = q$).

Recall the canonical divisor K has degree $2g-2$ and its space of global sections has dimension g. Let us analyze the associated map $\varphi_L \colon X \to \mathbb{P}^{g-1}$. We have

$$\ell(K) = g, \qquad \ell(K-p-q) = 2g - 4 - g + 1 + \ell(p+q) = g - 3 + \ell(p+q).$$

Because $p + q \geq 0$, it follows $\ell(p+q) \geq \ell(0) = 1$. We would like to know whether $\ell(p+q) \geq 2$, or, equivalently, whether there exists a non-constant meromorphic function on X with at worst simple poles at p and q (double pole if $p = q$). Then φ_K is an embedding unless there exists a g^1_2 on X, that is, a divisor D of degree 2 with $\ell(D) \geq 2$, or, a non-constant meromorphic function of degree 2 on X.

Definition 3.3. We say a curve X is *hyperelliptic* if any of the following equivalent conditions hold:

(i) X has a meromorphic function of degree 2,
(ii) X is expressible as a 2-sheeted cover of $\mathbb{P}^1$.

We can then restate the conclusion we arrived at.

Corollary 3.4. *The map φ_K is an embedding if and only if X is not hyperelliptic.*

In this case, we call $\varphi_K(X) \subset \mathbb{P}^{g-1}$ the *canonical model* of X.

3.2. Geometric Riemann-Roch

For simplicity, assume X is not hyperelliptic so $\varphi_K \colon X \to \mathbb{P}^{g-1}$ is an embedding (this hypothesis can be disposed of). Let $D = p_1 + \cdots + p_d$ be a divisor consisting of d distinct points (another hypothesis made for convenience). Riemann-Roch states

$$r(D) = \ell(D) - 1 = d - g + \ell(K-D),$$

where $\ell(K-D)$ is the space of holomorphic differentials on $X \cong \varphi_K(X) \subset \mathbb{P}^{g-1}$ vanishing at $p_1, \ldots, p_d$. Identifying X with its canonical model, holomorphic 1-forms on X correspond to linear forms on $\mathbb{P}^{g-1}$. There are g linearly independent such corresponding to hyperplanes in $\mathbb{P}^{g-1}$. Actually,

$$\ell(K-D) = g - d + \dim\{\text{linear relations satisfied by } p_1, \ldots, p_d \text{ in the canonical model of } X\},$$

and

$$r(D) = \dim\{\text{linear relations satisfied by } p_1, \ldots, p_d \text{ in the canonical model of } X\}.$$

This statement is called the *geometric Riemann-Roch*.

Definition 3.5. We say a curve is *trigonal* if X is a 3-sheeted cover of $\mathbb{P}^1$.

Equivalently, there exist 3 collinear points in the canonical model. The divisor $p+q+r$ moves in a pencil if p, q and r are collinear in $\varphi_K(X)$.

3.3. Adjunction

Since we will deal both with curves in the abstract and curves in projective space, we need to develop some tools applying to varieties in projective space.

Let X be a smooth projective variety of dimension n. A *divisor* on X is a formal linear combination of irreducible subvarieties of dimension $n-1$. We write $D = \sum n_i Y_i$ for $n_i \in \mathbb{Z}$. Two divisors D and E are called *linearly equivalent*, denoted $D \sim E$, if there exists f in $K(X)$, the function field of X, such that $(f) = D - E$. Then

$$\mathrm{Pic}(X) = \{\text{line bundles on } X\}/\cong = \{\text{divisors on } X\}/\sim.$$

The canonical line bundle on X is

$$K_X = \bigwedge^n T_X^\vee.$$

Its sections are holomorphic forms of top degree, i.e., locally of the form

$$f(z_1, \ldots, z_n) dz_1 \wedge \cdots \wedge dz_n.$$

For example, if ω is a meromorphic n-form, then the divisor (ω) represents K_X. Unfortunately, there is no well-defined notion of degree in dimensions $n \geq 2$.

Example 3.6. In $X = \mathbb{P}^n$, any irreducible subvariety $Y \subset \mathbb{P}^n$ of dimension $n-1$ is the zero locus of a homogeneous polynomial F of degree d. Consider $Y = V(F)$ and $Y' = V(F')$ of degrees d and d' respectively. The divisors Y and Y' are linearly equivalent if and only if $d = d'$. To see this note that $Y - Y' = (F/F')$. We conclude $\mathrm{Pic}(\mathbb{P}^n) \cong \mathbb{Z}$ and $h = \mathcal{O}_{\mathbb{P}^n}(1)$ is a generator.

To find $K_{\mathbb{P}^n}$ we can write a meromorphic differential and analyze its zeros and poles. Consider

$$\omega = dz_1 \wedge \cdots \wedge dz_n$$

on an affine open $\mathbb{A}^n \subset \mathbb{P}^n$. When we go to the hyperplane at infinity ω attains a pole of order $n+1$ along the entire hyperplane. We conclude

$$K_{\mathbb{P}^n} \cong \mathcal{O}_{\mathbb{P}^n}(-n-1) \cong -(n+1)h.$$

Proposition 3.7 (Adjunction formula)**.** *Let X be a smooth variety and $Y \subset X$ a smooth subvariety of codimension 1. Then,*

$$K_Y = (K_X \otimes \mathcal{O}_X(Y))|_Y.$$

Proof. We have an exact sequence

$$0 \longrightarrow T_Y \longrightarrow T_X|_Y \longrightarrow N_{Y/X} \longrightarrow 0,$$

and, dually,

$$0 \longrightarrow N_{Y/X}^\vee \longrightarrow T_X^\vee|_Y \longrightarrow T_Y^\vee \longrightarrow 0.$$

Multilinear algebra implies

$$\bigwedge^n T_X^\vee|_Y \cong \bigwedge^{n-1} T_Y^\vee \otimes N_{Y/X}^\vee,$$

and, equivalently,

$$K_Y = K_X|_Y \otimes N_{Y/X}.$$

Combining this with the standard fact $N_{Y/X} \cong \mathcal{O}_X(Y)$ produces the desired result. $\square$

Example 3.8. Consider a smooth plane curve $X \subset \mathbb{P}^2$ of degree d, i.e.,

$$\mathcal{O}_{\mathbb{P}^2}(X) \cong \mathcal{O}_{\mathbb{P}^2}(d), \qquad K_{\mathbb{P}^2} \cong \mathcal{O}_{\mathbb{P}^2}(-3).$$

The adjunction formula implies

$$K_X = \mathcal{O}_{\mathbb{P}^2}(d-3)|_X = \mathcal{O}_X(d-3).$$

Since $\deg \mathcal{O}_X(1) = d$, we obtain

$$2g - 2 = \deg K_X = d(d-3).$$

Rearranging, we obtain the standard formula

$$g = \binom{d-1}{2}$$

relating the genus and the degree of a smooth plane curve.

Example 3.9. Consider a smooth quadric $Q \subset \mathbb{P}^3$. Since $Q \cong \mathbb{P}^1 \times \mathbb{P}^1$, we know that $\operatorname{Pic}(X) \cong \mathbb{Z}\langle e, f\rangle$ where e and f are lines on each of the two rulings on Q. Using similar reasoning, $K_X \sim -2e - 2f$. Note that $\mathcal{O}_Q(1) = \mathcal{O}_{\mathbb{P}^3}(1)|_Q$ is of class $e+f$ (a general intersection of the quadric with a hyperplane is the union of two lines, one from each ruling). Alternatively, we can apply adjunction to $Q \subset \mathbb{P}^3$ and compute

$$K_Q \cong (K_{\mathbb{P}^3} \otimes \mathcal{O}_{\mathbb{P}^3}(1))|_Q \cong \mathcal{O}_{\mathbb{P}^3}(-2)|_Q.$$

Consider a smooth curve $X \subset Q \cong \mathbb{P}^1 \times \mathbb{P}^1$. We can write $X \sim ae + bf$ for integers $a, b > 0$, that is, X meets a line of the first ruling in a points and a line from the second in b points, or, $X = V(F)$ where F is a bihomogeneous polynomial of bidegree (a, b). By adjunction,

$$K_X \cong (K_Q \otimes \mathcal{O}_Q(X))|_X \sim (a-2)e + (b-2)f.$$

We can relate this to the genus by

$$2g - 2 = \deg K_X = (a-2)b + (b-2)a,$$

or, equivalently,

$$g = (a-1)(b-1).$$

Example 3.10. Let X be a smooth projective curve of genus 0. Consider a point $p \in X$ and the divisor $D = p$ with the associated line bundle $L = \mathcal{O}_X(D)$. By Riemann-Roch, $h^0(L) = 2$, so we get a degree 1 map $X \to \mathbb{P}^1$ which is an isomorphism. We have shown there is a unique genus 0 curve up to isomorphism.

We know that $\operatorname{Pic} X \cong \operatorname{Pic} \mathbb{P}^1 \cong \mathbb{Z}$, so up to linear equivalence all divisors are of the form $D = d \cdot p$ for some $p \in X$ and $d \in \mathbb{Z}$. Consider a coordinate z in which $p = \infty$. Assume $d \geq 0$, so

$$\mathcal{L}(D) = \mathbb{C}\langle 1, z, \ldots, z^d\rangle$$

has dimension $d+1$. We get a map

$$\begin{aligned} \nu_d \colon \mathbb{P}^1 &\longrightarrow \mathbb{P}^d, \\ z &\longmapsto [1, z, \ldots, z^d]. \end{aligned}$$

(1) For $d = 1$, we obtain the isomorphism $X \cong \mathbb{P}^1$ that we just investigated.
(2) For $d = 2$, we get a smooth plane conic. The image of $z \mapsto [1, z, z^2]$ is the locus $AC - B^2 = 0$ for coordinates A, B, C on $\mathbb{P}^2$.

(3) For $d = 3$, we get the twisted cubic $X \subset \mathbb{P}^3$. If A, B, C, D are coordinates on $\mathbb{P}^3$ we can describe it as the zero locus of

$$AC - B^2, \qquad AD - BC, \qquad BD - C^2.$$

These equations are the minors of a 2×3 matrix

$$\begin{pmatrix} A & B & C \\ B & C & D \end{pmatrix},$$

so the twisted cubic is a *determinental variety.*

If X were a complete intersection, then, since it has degree 3, it has to be the intersection of a degree 3 surface an a plane in $\mathbb{P}^3$. But the twisted cubic is nondegenerate so it is not contained in any linear subspace. We conclude that X is not a complete intersection.

Disposing of any of the three defining equations we obtain the union of X and a line. It is also worth noting that any of the three equations defined a smooth quadric in $\mathbb{P}^3$ on which X is of type $(2, 1)$.

Open problem. Are all curves in space complete intersection set-theoretically. For the twisted cubic, there is a ribbon which is the complete intersection two hypersurfaces of degrees 2 and 3 respectively.

For a general d, the image of the degree d embedding $\nu_d \colon \mathbb{P}^1 \hookrightarrow \mathbb{P}^d$ is called the *rational normal curve* of degree d. Projecting down to a linear subspace yields maps $\mathbb{P}^1 \to \mathbb{P}^r$ of degree d for any $1 \le r \le d$.

$$\begin{tikzcd} & \mathbb{P}^d \arrow[d, "\text{linear projection}"] \\ \mathbb{P}^1 \arrow[ur, "\nu_d"] \arrow[r] & \mathbb{P}^r \end{tikzcd}$$

Example 3.11. Consider a quartic curve $\varphi \colon \mathbb{P}^1 \to \mathbb{P}^3$. We would like to know what surfaces it lies on. For example, does it lie on a quadric? Note that

$$\varphi^* \mathcal{O}_{\mathbb{P}^3}(1) \cong \mathcal{O}_{\mathbb{P}^1}(4), \qquad \varphi^* \mathcal{O}_{\mathbb{P}^3}(2) \cong \mathcal{O}_{\mathbb{P}^1}(8).$$

By restricting global sections, we get a map

$$\left\{ \begin{matrix} \text{homogeneous quadratic} \\ \text{polynomials on } \mathbb{P}^3 \end{matrix} \right\} = \mathrm{H}^0(\mathcal{O}_{\mathbb{P}^3}(2)) \longrightarrow \mathrm{H}^0(\mathcal{O}_{\mathbb{P}^1}(8)) = \left\{ \begin{matrix} \text{homogeneous octic} \\ \text{polynomials on } \mathbb{P}^1 \end{matrix} \right\}.$$

The dimensions of these vector spaces are 10 and 9 respectively, hence the map must have a non-trivial kernel. We conclude the image of $\mathbb{P}^1$ lies on the quadric surface defined by this kernel. Furthermore, one can show this quadric cannot be singular. A smooth quadric surface is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ and the quartic on it can be proved to be of type $(1, 3)$ (the remaining possibilities $(2, 2)$ and $(2, 1)$ can be eliminated).

LECTURE 4

September 12, 2011

4.1. A change of viewpoint

Looking at the historical development of mathematics, there seems to be a period between the 19th and 20th centuries when the definitions of many basic objects were changes. For example, during the 19th century a group was a subset of $\mathrm{GL}(n)$ closed under multiplication and inverse. This was later replaced by the modern definition – a set G equipped with a suitable binary operation $G \times G \to G$. In other words, the point of view has been shifted from subobjects of well-understood ones to objects in the abstract. As a second example, a manifold during the 19th century was a subset of $\mathbb{R}^n$, that is, it came with an embedding. Later on, an intrinsic point of view was formulated in which a manifold is a set with certain additional structure. The situation in algebraic geometry is quite similar – there was a transition of subsets of $\mathbb{P}^n$ cut out by polynomials to abstract varieties.

What is more important is that such a shift in the point of view lead to a restructuring of the subject. For example, group theory has been broken into (1) the study of abstract groups, and (2) representation theory (the study of maps from groups to $\mathrm{GL}(n)$). Similarly, the study of curves in algebraic geometry can be split into (1) the classification of abstract (smooth, projective) curves, and (2) describing the space of maps $X \to \mathbb{P}^r$ for given X and $r \geq 1$. Today we will talk about the second half, i.e., mapping curves in projective space.

The way we pose question (2) is interrelated with the answer to question (1). Namely, there is a space M_g parametrizing smooth curves of genus g for any $g \geq 0$. We can stratify M_g according to the answer to part (2). For example, the locus of curves that admit a degree d map into $\mathbb{P}^r$ is a locally closed subset of M_g. If the answer stabilizes on locally closed subsets, then there should be a generic answer. This is the topic of *Brill-Noether theory*.

4.2. The Brill-Noether problem

For a general curve X of genus g, we would like to describe the space of (nondegenerate) maps $X \to \mathbb{P}^r$ of degree d. In particular, are there any such? More specifically, we can pose the following questions for a general curve X of genus g.

(i) What is the smallest degree of a non-constant map $X \to \mathbb{P}^1$?
(ii) What is the smallest degree of a plane curve birational to X?
(iii) What is the smallest degree of an actual embedding in $\mathbb{P}^3$?

Remark 4.1. Any curve can be embedded into $\mathbb{P}^3$. To do so, first embed in $\mathbb{P}^n$ for some n and then project linearly down to $\mathbb{P}^3$. The point from which we project has to be in the complement of the secant variety of the curve, which has dimension 3.

Consider the following naive approach.

$$\begin{tikzcd}[column sep=small] & H_{g,d} = \left\{ (X, f\colon X \to \mathbb{P}^1) \;\middle|\; \begin{array}{c} f \text{ has degree } d \text{ and is} \\ \text{simply branched over } b \text{ points} \end{array} \right\} \arrow[dl] \arrow[dr] & \\ M_g & & \operatorname{Sym}^b \mathbb{P}^1 \setminus \Delta = \mathbb{P}^b \setminus \Delta. \end{tikzcd}$$

The space $H_{d,g}$ is called the *small Hurwitz scheme*. The right map is described by mapping a branched cover to its branch divisor which consists of an unordered b-tuple of distinct points. Given such a b-tuple, recovering the curve amounts to adding monodromy data, i.e., information how the sheets of the covering glue as we go around the branch points. It follows that $H_{g,d} \to \mathbb{P}^b \setminus \Delta$ is a finite covering, hence

$$\dim H_{g,d} = \dim(\mathbb{P}^b \setminus \Delta) = b.$$

The Riemann-Hurwitz formula allows us to compute

$$\dim H_{g,d} = b = 2g + 2d - 2.$$

Next, we would like to describe the fibers of the first projection $H_{d,g} \to M_g$. Given an abstract X, for $d \gg 0$ (actually $d > 2g+1$ suffices), a cover $X \to \mathbb{P}^1$ amounts to a degree d meromorphic function. Look at the polar divisor of such a function. For a divisor D of degree d on X, the space

$$\mathcal{L}(D) = \{\text{meromorphic functions on } X \text{ with poles along } D\}$$

has dimension $g - g + 1$ by Riemann-Roch (this is where we use $d > 2g+1$). We conclude

$$\dim M_g = b + (d + d - g + 1) = 2g - 3.$$

Note that this argument worked under the hypothesis d is large enough. It is an interesting question to find the smallest d such that $H_{d,g}$ dominates M_g. Since

$$\dim \mathrm{Aut}(\mathbb{P}^1) = \dim \mathrm{PGL}(2) = 3,$$

the fibers of the map $H_{d,g} \to M_g$ have dimension 3 or greater. We are lead to the inequality

$$b = 2g + 2d - 2 \geq 3g - 3 + 3 = 2g,$$

or, equivalently,

$$d \geq \frac{g}{2} + 1.$$

It turns out this is the correct answer to our question, namely, $H_{d,g}$ dominates M_g is and only if $d \geq g/2+1$. This can be verified by concrete considerations in low genera.

Let us consider another specific question. When can we represent a general abstract genus g curve as a plane curve? Again, take the naive approach illustrated by the following diagram.

$$\begin{tikzcd}[column sep=small] & V_{d,g} = \left\{ (X, f\colon X \to \mathbb{P}^2) \;\middle|\; \begin{array}{l} f \text{ has degree } d \text{ and is birational} \\ \text{onto a plane curve with } \delta \text{ nodes} \end{array} \right\} \arrow[dl] \arrow[dr] & \\ M_g & & \mathrm{Sym}^\delta \mathbb{P}^2 \setminus \Delta \end{tikzcd}$$

We can compute

$$\delta = \binom{d-1}{2} - g$$

as the difference of the expected genus of a smooth degree d plane curve and the actual genus. The fibers of the right map have dimension

$$\frac{d(d+3)}{2} - 3\delta.$$

To see this note that the space of all degree d plane curves has dimension $d(d+3)/2$ and each of the δ nodes imposes 3 linear conditions. We conclude

$$\dim V_{d,g} = \frac{d(d+3)}{2} - 3\delta + 2\delta = 3d + g - 1.$$

Remark 4.2. There is a serious problem with this argument if $3\delta > d(d+3)/2$ but this can be fixed using deformation theory.

The fibers on the left side have dimension at least

$$\dim \mathrm{Aut}(\mathbb{P}^2) = \dim \mathrm{PGL}(3) = 8.$$

Then $V_{d,g}$ dominated M_g only if

$$3d + g - 9 \geq 3g - 3,$$

or, equivalently

$$3d \geq 2g + 6.$$

In summary:

$$\begin{aligned}&\text{there exists a degree } d \text{ map to } \mathbb{P}^1 \text{ only if } d \geq \tfrac{1}{2}g + 1;\\ &\text{there exists a degree } d \text{ map to } \mathbb{P}^2 \text{ only if } d \geq \tfrac{2}{3}g + 2.\end{aligned}$$

This information is sufficient to guess the pattern. In general, for a general genus g curve, there exists a nondegenerate degree d map $X \to \mathbb{P}^r$ if and only if

$$d \geq \frac{r}{r+1}g + r.$$

Remark 4.3. The cases $r = 1, 2$ were known long before the general case. The latter was settled only in the late 1970s.

Open problem. Are all *rigid curves* rational normal curves? (A curve is called rigid if it admits no non-trivial deformations.)

Given g, d and r, the *Brill-Noether number* is defined as

$$\rho = g - (r+1)(g - d + r).$$

Theorem 4.4 (Brill-Noether Theorem)**.** *A general genus g curve admits a nondegenerate map to $\mathbb{P}^r$ of degree d if and only if $\rho \geq 0$. The dimension of the space of such maps is ρ.*

Moreover, for a general such map f:

(i) *f is an embedding if $r \geq 3$,*
(ii) *f is a birational embedding if $r \geq 2$;*
(iii) *f is a simply branched cover if $r = 1$.*

An abstract curve does not possess a lot of structure. On the other hand, once embedded in projective space, we get a great deal of structure:

(i) geometric – e.g., tangent and secant lines, inflection points,
(ii) algebraic – e.g., the homogeneous ideal $I_C \subset k[z_0, \ldots, z_r]$ of the image in $\mathbb{P}^r$.

It is possible to pose many questions about the ideal I_C, for example, about its generators, their degrees, smallest possible degrees. There can be bundled up by asking for the minimal resolution (syzygies) of I_C. We have arrived at the following.

[**.** Problem] For a general X, and a general map $f\colon X \to \mathbb{P}^r$ of degree d ($r \geq 3$), describe the minimal set of generators of the ideal I_C of the image $C = f(X) \subset \mathbb{P}^r$.

Example 4.5. Let X be a genus 2 curve. Either by Brill-Noether, or via general examination, we can embed X in $\mathbb{P}^3$ with degree 5. Pick a general effective divisor D of degree 5. Riemann-Roch implies

$$\ell(D) = 5 - 2 + 1 = 4,$$

so we can write

$$\mathcal{L}(D) = \mathbb{C}\langle 1, f_1, f_2, f_3, f_4 \rangle.$$

We get an embedding

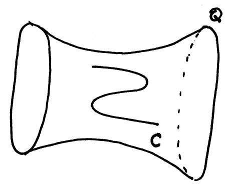
$$X \xrightarrow[{[1, f_1, f_2, f_3, f_4]}]{\cong} C \subset \mathbb{P}^3.$$

We would like to find the lowest degree of a generator of I_C. Since the map is nondegenerate, it cannot be 1.

Consider $d = 2$. Restriction gives us a map

$$\left\{\begin{matrix}\text{homogeneous quadratic} \\ \text{polynomials on } \mathbb{P}^3\end{matrix}\right\} = \mathrm{H}^0(\mathcal{O}_{\mathbb{P}^3}(2)) \longrightarrow \mathcal{L}(2D).$$

The domain has dimension 10 and the codomain 9, so X must lie on a quadric surface. If X lies on two distinct such, then Bezout's Theorem leads to a contradiction. We conclude the restriction map has full rank and X lies on a unique quadric Q. It is possible that Q is singular, but it is smooth in general. When this happens, C is a curve of type $(2, 3)$.



Next, look at cubics. The restriction map reads

$$\left\{\begin{matrix}\text{homogeneous cubic} \\ \text{polynomials on } \mathbb{P}^3\end{matrix}\right\} = \mathrm{H}^0(\mathcal{O}_{\mathbb{P}^3}(3)) \longrightarrow \mathcal{L}(3D).$$

These spaces have dimensions 20 and 15-2+1 = 14 respectively, so the kernel has dimension at least 6. Modulo Q, there are only two cubics left.

Exercise 4.6. We have an exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^3}(-4)^{\oplus 4} \longrightarrow \mathcal{O}_{\mathbb{P}^3}(-3)^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^3}(-2) \longrightarrow \mathcal{I}_C = \widetilde{I}_C \longrightarrow 0.$$

The first middle term represents the generators we found above and the first one two linear relations between them.

Remark 4.7. This procedure is much harder to carry out if X is embedded in a higher dimensional projective space.

Conjecture 4.8 (Maximal rank conjecture)**.** *Let X be a general curve of genus g and $f\colon X \hookrightarrow \mathbb{P}^r$ a general map of degree d with $r \geq 3$. Then the restriction map*

$$\left\{\begin{matrix}\textit{homogeneous polynomials} \\ \textit{of degree } m \textit{ on } \mathbb{P}^r\end{matrix}\right\} = \mathrm{H}^0(\mathcal{O}_{\mathbb{P}^r}(m)) \longrightarrow \mathcal{L}(mD)$$

has maximal rank. Here D stands for the divisor of a general hyperplane section of C.

LECTURE 5

September 16, 2011

5.1. Remark on a homework problem

Let $C \hookrightarrow \mathbb{P}^d$ be a rational normal curve. One of the homework problems asked to show the normal bundle is

$$N_{C/\mathbb{P}^d} \cong \mathcal{O}_{\mathbb{P}^1}(d+2)^{\oplus(d-1)}.$$

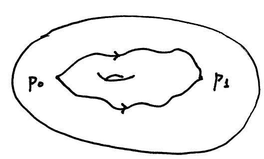
Note that we are utilizing an automorphism $C \cong \mathbb{P}^1$ above. More generally, one can ask what can be said about smooth rational curves $\mathbb{P}^1 \hookrightarrow \mathbb{P}^r$ of degree d in general, namely, what normal bundles occur? The answer is known for $r = 3$ but not in general.

5.2. Abel's Theorem

One of the original motivations for studying algebraic curves in the 19th century was to find indefinite integrals of algebraic functions. For example, there was an explicit answer for $\int dx/\sqrt{x^2+1}$ but not for $\int dx/\sqrt{x^3+1}$. One can interpret the latter as

$$\int \frac{dx}{\sqrt{x^3+1}} = \int \frac{dx}{y}$$

for a curve $y^2 = x^3+1$. This lead to the realization that indefinite integrals of the above form are well-defined only up to periods, i.e., up to integrals along loops in the curve.



Alternatively, we can say the inverse function of the antiderivative is doubly periodic. At the time, however, there were no known non-trivial doubly periodic functions.

Let C be a Riemann surface of genus g. The space of loops in C, ignoring basepoints, is $\pi_1(C)$. Integration of a 1-form ω along loops gives a map

$$\int_{-} \omega \colon \pi_1(C) \longrightarrow \mathbb{C}.$$

Since $\mathbb{C}$ is abelian, the kernel of $\int_{-} \omega$ contains the commutator subgroup $[\pi_1(C), \pi_1(C)]$, hence it induces a map

$$\int_{-} \omega \colon \pi_1(C)/[\pi_1(C), \pi_1(C)] \cong \mathrm{H}_1(C, \mathbb{Z}) \longrightarrow \mathbb{C}.$$

It is easy to see these maps vary linearly in ω, so we get

$$\int \colon \ \mathrm{H}^0(C, K_C) \times H_1(C, \mathbb{Z}) \longrightarrow \mathbb{C},$$

or, equivalently,

$$\int\colon \ \mathrm{H}_1(C,\mathbb{Z}) \to \mathrm{H}^0(C,K_C)^\vee.$$

One can use the exponential sequence to show this map is injective and the image of $\mathrm{H}_1(C,\mathbb{Z})$ is a lattice of rank $2g$ in $H^0(C,K_C)^\vee$. The quotient

$$\mathrm{Jac}(C) = \mathrm{H}^0(C,K_C)^\vee / \mathrm{H}_1(C,\mathbb{Z}),$$

called the *Jacobian* of C, is a torus of dimension $2g$. While non-trivial, it is possible to show $\mathrm{Jac}(C)$ it is a complex algebraic variety of dimension g.

Fixing a basepoint $p_0 \in C$, we can define a map

$$\begin{aligned} \int_{p_0}\colon \quad C &\longrightarrow \mathrm{Jac}(C), \\ p &\longmapsto \int_{p_0}^{p}. \end{aligned}$$

Remark 5.1. The map $\int = \int_{p_0}$ depends on a basepoint but we will suppress this detail for now.

It is possible to extend $\int$ linearly to all divisors by defining

$$D = \textstyle\sum_i n_i \cdot p_i \longmapsto \sum_i n_i \displaystyle\int_{p_0}^{p}.$$

Fixing an integer $d \geq 1$, and restricting to effective divisors of degree d, we get a map

$$u_d\colon \ C_d = \mathrm{Sym}^d\, C \longrightarrow \mathrm{Jac}(C).$$

The points of the image are called *abelian integrals*.

Abel considered two linearly equivalent divisors $D \sim E$, where $D - E = (f)$. We get a family of divisors

$$D_t = (f - t) + E$$

interpolating between D and E, more specifically, $D_0 = D$ and $D_\infty = E$. Therefore, we get a map

$$\begin{aligned} \mathbb{P}^1 &\longrightarrow \mathrm{Jac}(C), \\ t &\longmapsto u_d(D_t) = \sum_i \int_{p_0}^{p_i(t)}. \end{aligned}$$

Since $\mathrm{Jac}(C)$ is a complex torus, it has a lot of holomorphic 1-forms. In particular, the cotangent space at each point is generated by global holomorphic 1-forms. On the other hand, there are no such forms on $\mathbb{P}^1$, hence the differential of the map above is 0, and it must be constant.

Remark 5.2. Another way to see this, note that the universal cover of $\mathrm{Jac}(C)$ is $\mathbb{C}^g$. Since $\mathbb{P}^1$ is homeomorphic to S^2, it has trivial fundamental group, and we can find a lift as in the diagram below.

$$\begin{tikzcd} & \mathbb{C}^g \arrow[d] \\ \mathbb{P}^1 \arrow[r] \arrow[ur, dashed] & \mathrm{Jac}(C) \end{tikzcd}$$

The covering map $\mathbb{C}^g \to \mathrm{Jac}(C)$ is locally an isomorphism of complex manifolds, hence the lift $\mathbb{P}^1 \to \mathbb{C}^g$ is also holomorphic. On the other hand, the only such map is the constant.

In conclusion, the map $u_d\colon C_d \to \mathrm{Jac}(C)$ depends only on divisor classes. Clebsch extended this statement by showing this is the only way two images can coincide.

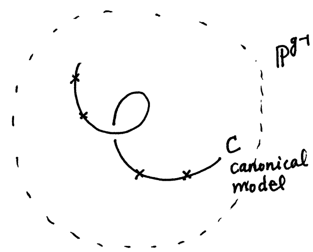
Theorem 5.3 (Abel's Theorem, due to Clebsch)**.** *For two effective degree d divisors D and E, $u_d(D) = u_d(E)$ if and only if $D \sim E$. In other words, u_d is injective on divisor classes.*

As a restatement, the non-empty fibers of the map $u_d \colon C_d \to \mathrm{Jac}(C)$ are complete linear systems

$$|D| = \{\text{effective } E \text{ such that } E \sim D\} = u_d^{-1}(u_d(D)),$$

which are projective spaces.

(1) By the geometric statement of Riemann-Roch, for $d \le g$, the general fiber of the map $u_d \colon C_d \to \mathrm{Jac}(C)$ is $\mathbb{P}^0$, i.e., a point. It follows that the map is birational onto its image.



(2) For $d = g$, we get a birational isomorphism

$$u_d \colon \; C_d \xrightarrow[\text{birational}]{\cong} \mathrm{Jac}(C).$$

This statement is also known as *Jacobi inversion*. Alternatively, we can always express a sum

$$\sum^g \int_{p_0}^{p_i} + \sum \int_{p_0}^{q_i}$$

as $\sum \int_{p_0}^{r_i}$ where the symmetric functions of r_i are algebraic in p_i and q_i.

(3) When $d \ge 2g-1$, then $u_d \colon C_d \to \mathrm{Jac}(C)$ is a $\mathbb{P}^{d-g}$-bundle, which is a nice handle on C_d.

5.3. Examples and applications

When $g = 1$, then $\mathrm{Jac}(C) \cong C$.

When $g = 2$, then

$$u_2 \colon \; C_2 \longrightarrow \mathrm{Jac}(C)$$

is generically 1-1 except on the locus of linear systems of degree 2. But there exists a unique such – the canonical one $|K_C| \cong \mathbb{P}^1$. We infer that C_2 is the blow-up of $\mathrm{Jac}(C)$ at a point.

Remark 5.4. We have given a somewhat antiquated definition of the Jacobian. For example, the Jacobian is only a torus, and not necessarily algebraic. This can be shown to be the case. Furthermore, our construction works only over $\mathbb{C}$, in particular, not in finite characteristic. Even over another field of characteristic 0 contained in $\mathbb{C}$, the Jacobian is only defined over $\mathbb{C}$. Andre Weil fixed this by reinventing the birational isomorphism $C_g \cong \mathrm{Jac}(C)$. He constructed $\mathrm{Jac}(C)$ by producing a cover by open affines in C_g. He developed the theory of algebraic varieties for this purpose!

Last time we discussed rational curves. A natural place to continue is curves C of genus $g = 1$. Let D be a divisor of degree $d \ge 1$. Then $\ell(D) = d$ and $r(D) = d - 1$ by Riemann-Roch.

When $d = 2$, we get an expression of C as a double cover of $\mathbb{P}^1$ branched at 4 points. Writing $y^2 = x(x-1)(x-\lambda)$, the branch points are $0, 1, \lambda$, and ∞.

When $d = 3$, $r(D) = 2$ and we get an embedding of C as a smooth plane cubic.

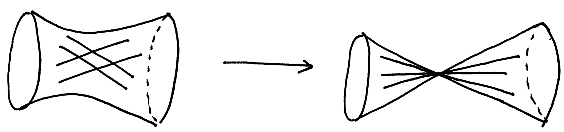
When $d = 4$, we get an embedding $C \hookrightarrow \mathbb{P}^3$ as a quartic curve. It would be interesting to know something about the equations which define it. For example, in degree 2, there is a map

$$\mathrm{H}^0(\mathcal{O}_{\mathbb{P}^3}(2)) \longrightarrow \mathrm{H}^0(\mathcal{O}_C(2D)).$$

The domain, comprised of homogeneous quadratic polynomials on $\mathbb{P}^3$, has dimension 10. The codomain has dimension 8 by Riemann-Roch. It follows that the kernel is at least of dimension 2, or, in other words, C lies on at least 2 quadrics, call them Q and Q'. Each of these is irreducible since the curve does lie in the union of two planes. Therefore, $C = Q \cap Q'$. Recall that a smooth quadric hypersurface is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. We actually, have a whole pencil of quadrics

$$\{Q_t = t_0 Q + t_1 Q'\}_{t=[t_0,t_1]\in\mathbb{P}^1}.$$

Exactly 4 of these will be singular. Each non-singular quadric has two rulings, while a singular one, being a cone, has only one.



Around a singular quadric, each of the rulings of the nearby non-singular quadrics has as a limit the unique ruling on the cone. We are lead to construct

$$E = \{(t, \text{ruling on } Q_t)\} \longrightarrow \mathbb{P}^1,$$

where the map is given by projecting on the first component. Studying the monodromy around a singular quadric, we can find that the two rulings can be exchanged. It is also possible to show that $E \to \mathbb{P}^1$ is a 2-to-1 branched cover at 4 points, and, in fact, $E \cong C$. This statement is false over fields other than $\mathbb{C}$, but it is a good question to think about.

Next, consider curves of genus $g = 2$. For the first time, we will see an example where the geometry of a map depends on the choice of line bundle. Consider a divisor D of degree d on a curve C of genus 2.

When $d = 2$, we can compute

$$\ell(D) = \begin{cases} 2 & \text{if } D = K, \\ 1 & \text{otherwise.} \end{cases}$$

We already observed this when we stated that $C_2 \to \mathrm{Jac}(C)$ is the blow-down of a line. The map $\varphi_K \colon C \to \mathbb{P}^1$ expresses C as a double cover branched at 6 points. If we write C as $y^2 = \prod(x - \lambda_i)$, there is an involution $i \colon C \to C$ exchanging the sheets given by $(x, y) \mapsto (x, -y)$.

When $d = 3$, $\ell(D) = 2$ for all divisors D. It seems that we get a family of maps $C \to \mathbb{P}^1$. If $D = K + p$, then p is a basepoint of $|D|$, and

$$\ell(D) = \ell(D - p) = \ell(K) = 2.$$

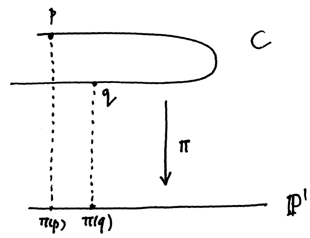
If $D \neq K + p$, then we get a degree 3 map $C \to \mathbb{P}^1$. Note that the latter type of divisors, that is $D \neq K + p$, exist for dimension reasons. The locus of divisors of the form $K + p$ is a curve in $\mathrm{Jac}(C)$, but $\dim \mathrm{Jac}(C) = g = 2$, so the curve cannot fill the entire Jacobian.

Next, consider $d = 4$. It is possible to show that $\ell(D - p) = \ell(D) - 1$ for all $p \in C$, and there is no problem with basepoints. For each such D, we get a degree 4 map $\varphi_D \colon C \to \mathbb{P}^2$. Note that $D - K$ has degree 2, and each such divisor is a sum of two points (Jacobi inversion). It follows that we can write $D = K + p + q$ for $p, q \in C$. There are three possibilities to consider.

(i) Consider $p \neq q$ and $p \neq i(q)$, i.e., p and q do no lie in the same fiber of the 2-sheeted cover. We can compute

$$\ell(D - p - q) = \ell(K) = \ell(D) - 1,$$

so $\varphi_D(p) = \varphi_D(q)$, and φ_D is singular. These are the only two points which are identified in the image of φ_D. In fact, it is a bijective immersion with the exception of $\varphi_D(p) = \varphi_D(q)$. In fact, $\varphi_D(C) \subset \mathbb{P}^2$ is a quartic curve with a node.



(ii) Consider $D = K + 2p$ and $p \neq i(p)$. Then the map φ_D fails to be an immersion at p, and $\varphi_D(C)$ is a quartic curve with a node.

(iii) Finally, consider $D = 2K$, that is, $D = K + p + q$ and $p = i(q)$. In this case, any conjugate pair of points under i are identified, i.e., $\varphi_D(r) = \varphi_D(i(r))$ for all $r \in C$. Then φ_D factors as

$$C \xrightarrow[2:1]{\varphi_K} \mathbb{P}^1 \xrightarrow{\text{conic}} \mathbb{P}^2.$$

The map φ_D is 2-to-1 onto a conic in $\mathbb{P}^2$.

Looking at the Jacobian of C, one can show that all of the three possibilities we listed above occur.

LECTURE 6

September 21, 2011

6.1. The canonical divisor on a smooth plane curve

In the following two lectures we will pay an outstanding debt, namely, the proof of Riemann-Roch and also of the fact that $H^0(K_C) = g$ where g is the topological genus of C.

Problem 1 Given a smooth projective curve C and a divisor F, we would like to find

$$|D| = \{E \text{ effective} \mid D \sim E\},$$

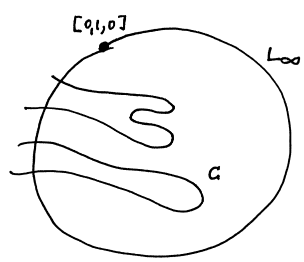
or, equivalently, $\mathcal{L}(D)$. In particular, we want to find

$$|K| = \{\text{effective canonical divisors}\},$$

or, equivalently, write down all holomorphic 1-forms on C.

We will start with a simple case. Let C be a smooth plane curve of degree d. We already showed that $\text{genus}(C) = \binom{d-1}{2}$, but this fact will also come out from our discussion. Start by choosing an affine open $\mathbb{A}^2 = \mathbb{P}^2 \setminus L_\infty$ where L_∞ is the line at infinity of $\mathbb{P}^2$. Let x and y be affine coordinates on $\mathbb{A}^2$. Let the corresponding coordinates on $\mathbb{P}^2$ be $[x, y, z]$ such that $L_\infty = \{z = 0\}$. Single out the point $[0, 1, 0] \in L_\infty \setminus C$ (if $[0, 1, 0] \in C$, we can pick slightly different coordinates). Adjunction computes $K_c \sim (d-3)L_\infty$, so $\deg K_C = d(d-3)$, which implies $g = \binom{d-1}{2}$. We will demonstrate the fact $K_c \sim (d-3)L_\infty$ directly.

Assume $C \subset \mathbb{P}^2$ has no vertical asymptotes and cuts L_∞ transversely (again, this can be arranged after a suitable linear change of coordinates). Say $C = V(f)$ for f a homogeneous polynomial of degree d. We want to write down a holomorphic differential ω_0 on C, but before that let us start with a meromorphic one. Say $\omega_0 = dx$. This is holomorphic on $\mathbb{A}^2 \subset \mathbb{P}^2$ but may have poles along $\mathbb{P}^2 \setminus \mathbb{A}^2 = L_\infty$. Consider the projection $\pi_{[0,1,0]} : C \to \mathbb{P}^1$ away from $[0, 1, 0] \notin C$ (in the affine place, this is projection on the x-axis). This expresses C as a degree d cover of $\mathbb{P}^1$ unramified at $\infty \in \mathbb{P}^1$.



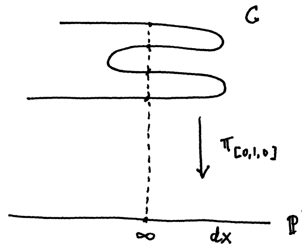
Note that on C the differential

$$\frac{\partial f}{\partial x} d + \frac{\partial f}{\partial y} dy = f_x dx + f_y dy$$

vanishes identically, hence, f_x and f_y do not vanish simultaneously. In other words, if dx vanishes, then so does f_y and the orders agree. Similarly for dy and f_x. This means

$$(dx) = (f_y), \qquad (dy) = (f_x).$$

Call $D_\infty = L_\infty \cdot C$. Note that dx has a double pole on $\mathbb{P}^1$ at ∞, and, since $\pi_{[0,1,0]} \colon C \to \mathbb{P}^1$ is unramified at ∞, $\pi_{[0,1,0]}^* dx$ has a pole of order 2 at each of the preimages of ∞, or, equivalently, it has double poles along D_∞.



To kill the poles, we have to divide dx by a polynomial h of degree at least 2. The problem is we can introduce new poles at points where the polynomials h vanishes. Recall that $(dx) = (f_y)$, so why not consider $h = f_y$. This way we have introduced no poles, but there are also no zeros in $\mathbb{A}^2$. Therefore, dx/f_y is holomorphic and has no zeros in $\mathbb{A}^2$. It has zeros of order $-2 + d - 1 = d - 3$ along D_∞. We showed that $(dx/f_y) = (d-3)D_\infty$, which proves one of our claims.

Assume $d \geq 3$ (the cases $d = 0, 1, 2$ are easy to handle separately). We still have the freedom to multiply dx/f_y by a polynomial of degree $\leq d-3$, and this will keep the 1-form holomorphic. We constructed the vector space

$$\left\{ g\frac{dx}{f_y} \;\middle|\; g \text{ polynomial of degree } \leq d-3 \right\}$$

of dimension $\binom{d-1}{2}$ consisting of holomorphic 1-forms on C.

6.2. More general divisors on smooth plane curves

Onto the original problem, consider an effective divisor D on C for which we want to find $|D|$, or, equivalently, $\mathcal{L}(D)$. We can treat sections of $\mathcal{O}_C(D)$ as rational function on C with poles along D.

To start, choose $g(x,y)$ of degree m vanishing along D, or, equivalently, a plane curve $G = V(g)$ of degree m such that $G \supset D$. (We are thinking of D as a subscheme of C, hence of $\mathbb{P}^2$.) Note that g may vanish outside $G \cap C$. Write $G \cdot C = D + E$ for E effective. Consider $1/g$ with poles along D and E.

We want to choose $h(x,y)$ of the same degree m as g such that h vanishes along E. Equivalently, we are looking for $H = V(h)$ a plane curve of degree m containing E. Once that is done, we can look at the divisor (h/g). We express $H \cdot C = E = F$ for F effective, so

$$H \cdot C \sim mD_\infty \sim G \cdot C = D + E.$$

Then $F \sim D$, so $(h/g) = F - D \sim 0$.

We claim that for any g as above,

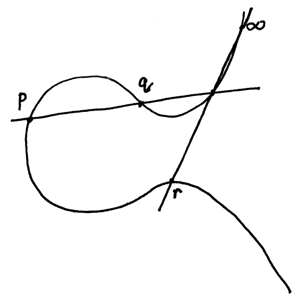
$$|D| = \{(h/g) + D \mid h \text{ is a polynomial of degree } m \text{ and } h(E) = 0\}$$

is the complete linear system of D. Alternatively,

$$\mathcal{L}(D) = \{h/g \mid h \text{ is a polynomial of degree } m \text{ and } h(E) = 0\}.$$

We will prove this claim in a broader context later on.

Example 6.1. We have already observed this behavior for elliptic curves.



Under the group law with identity at ∞, the equality $p+q=r$ means $p+q \sim r+\infty$ as divisors.

Remark 6.2. We assumed $D \geq 0$ above. For general D, write $D = D' - D''$ for $D', D'' \geq 0$. Take $G = V(g) \supset D'$ and express $G \cdot C = D' + E$. Then choose $H = V(h)$ such that $H \supset E + D''$ and express $H \cdot C = E + D'' + F$. Then $F \sim D' - D''$ which allows us to recover $\mathcal{L}(D)$ as the set of quotients h/g.

Back to our discussion above, say $\deg D = n$. Then

$$\deg E = \deg(G \cdot E - D) = md - n.$$

By looking at the construction, we can estimate

$$\ell(D) \geq \binom{m+2}{2} - \binom{m-d+2}{2} - (md-n) = n - \binom{d-1}{2} + 1.$$

In first term in the middle expression is the dimension of the space of degree m polynomials, the second, the dimension of the degree m polynomials vanishing along C, and the last, a correction. The given inequality is a statement of Riemann-Roch leaving our $\ell(K-D)$.

Historical note. Our notion of genus came about only after Riemann and the classification of surfaces in the second half of the 19th century. Originally, mathematicians say this as $\binom{d-1}{2}$. For them the genus represented a lack of rational functions, that is, a deficiency. This is where p came from, "deficiency" in Italian.

6.3. The canonical divisor on a nodal plane curve

Let C be a smooth curve of genus g. Consider a map

$$\nu\colon\ C \to C_0 \subset \mathbb{P}^2,$$

where $C \to C_0$ is birational, C_0 has degree d, and it has at worst poles in the plane.

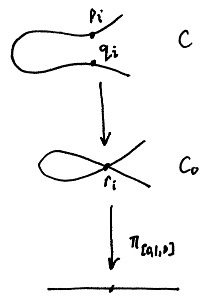
Remark 6.3. Such a map exists for any smooth abstract genus g curve. Details will appear in the homework.

Again, we would like to find all holomorphic differentials on C. For this purpose, start with a meromorphic one $\nu^* dx$. Last time we had a basic relation

$$f_y dx + f_x dy = 0$$

on C. Now this is true away from nodes, but not necessarily at them. For the sake of simplifying notation, assume there are no vertical tangents at the nodes. (This condition can be relaxed eventually.)

Denote the nodes of C_0 by $r_1, \ldots, r_\delta$, and the points of C above r_i by p_i, q_i. Consider the divisor $\Delta = \sum_i p_i + q_i$ of degree 2δ on C.



Then

$$\nu^*\left(\frac{dx}{f_y}\right) = (d-3)D_\infty - \Delta.$$

We still need to cancel the poles at Δ. For this purpose, we are allowed to multiply by any polynomial of degree $\leq d-2$. In other words, we want to study

$$\left\{ \nu^*\left(\frac{gdx}{f_y}\right) \;\middle|\; \begin{array}{c} g \text{ is a polynomial of degree } \leq d-3 \\ \text{and } g(r_i) = 0 \text{ for all } i \end{array} \right\}$$

Note that

$$K_C = (d-3)D_\infty - \Delta$$

is a generalization of a previous formula. We can estimate the dimension of the space above to be at least

$$\binom{d-1}{2} - \delta.$$

Note that this is the well-known expression for the genus of a nodal plane curve. We claim there is equality, that is, the space above is contains all holomorphic 1-forms.

Remark 6.4. Not only do we obtain all holomorphic 1-forms, but we also deduce that the points $r_1, \dots, r_\delta$ impose independent conditions on polynomials of degree $d-2$.

6.4. More general divisors on nodal plane curves

Again, start with an effective divisor D of degree n on C. For notational convenience, assume $\operatorname{Supp} D \cap \Delta = \emptyset$. Let $g(x,y)$ be a polynomial of degree m vanishing along D and $g(r_i) = 0$ for all i. Set $G = V(g)$, and express $G \cap C = D + \cdots$, i.e.,

$$\nu^*(g) = D + \Delta + E - mD_\infty.$$

Let h be a polynomial of degree m vanishing on E and also $h(r_i) = 0$. Again, write

$$\nu^*(h) = E + F + \Delta - mD_\infty.$$

Then $F - D = \nu^*(h/g)$. We claim

$$\mathcal{L}(D) = \left\{ \frac{h}{g} \;\middle|\; \begin{array}{l} h \text{ a polynomial of degree } m \\ \text{such that } h(E) = h(r_i) = 0 \end{array} \right\}.$$

Counting degrees, we get

$$\deg E = md - n - 2\delta.$$

Again, $\ell(D)$ can be estimated from below as

$$\binom{m+2}{2} - \binom{m-d}{2} - (md - n - 2\delta) + \delta \geq n - \left(\binom{d-1}{2} - \delta\right) + 1.$$

The difference from last time is the appearance of $+\delta$. This verifies the Riemann-Roch inequality in some sense. As before, the assumption D effective can be relaxed with a little more work.

We will use this setup next time as a basic tool in the proof of the Riemann-Roch Theorem.

Note that what we have is still inconvenient, since we cannot deal with arbitrary singular curves directly. It is possible to improve our discussion to arbitrary birational $\nu\colon C \to C_0 \subset \mathbb{P}^2$ which is carried out in an Appendix to Chapter 1 in ACGH. The bottom line is we can find complete linear series in the same way, that is, by replacing the condition $g(r_i) = 0$ by the more complicated "g satisfies the adjoint conditions at r_i". In modern language, it means that g lies in the *conductor ideal* at r_i, that is,

$$\operatorname{Ann}\left(\frac{\nu_*\mathcal{O}_C}{\mathcal{O}_{C_0}}\right)_{r_i}.$$

LECTURE 7

September 23, 2011

7.1. More on divisors

Recall the setup from last time. We took a smooth projective curve C of genus g, and a birational map

$$\nu\colon\ C \longrightarrow C_0 \subset \mathbb{P}^2,$$

where $C_0 = V(f)$ has degree d and only nodes as singularities at $r_1, \ldots, r_\delta$. Set the singular divisor to be $\Delta = r_1 + \cdots + r_\delta$. We saw that $\delta = \binom{d-1}{2} - g$, and then we constructed the space

$$\left\{ \omega = \frac{g\,dx}{f_y} \;\middle|\; \begin{array}{c} g \text{ polynomial of degree} \le d-3, \\ g(r_i) = 0 \text{ for all } i \end{array} \right\}$$

consisting of holomorphic differentials on C.

Claim. This is the space of all holomorphic differentials on C.

The dimension of this space is at least $\binom{d-1}{2} - \delta = g$.

Given a divisor D (for simplicity, say $D \geq 0$) of degree n, we choose a polynomial $g(x, y)$ of degree m such that $g(D) = g(r_i) = 0$ (i.e., g vanishes along D and the nodes). Then we write

$$(g) = D + \Delta + E - mD_\infty,$$

after which we considered

$$\left\{ \frac{h}{g} \;\middle|\; \begin{array}{c} h \text{ polynomial of degree} \le m, \\ h(E) = h(r_i) = 0 \end{array} \right\} \subset \mathcal{L}(D).$$

Again, the claim is we obtain all of $\mathcal{L}(D)$ in this way. We have

$$\deg E = md - 2\delta - n,$$

and the dimension of the vector space is

$$\binom{m+2}{2} - \binom{m-d+2}{2} - \delta - \deg E = n - g + 1.$$

Let us back up a little and look at the abstract curve C. First, we want to show $h^0(K_C) \le g$, which will complete our first claim. To see this, note that we have a map

$$\mathrm{H}^0(K_C) \longrightarrow \mathrm{H}^1_{\mathrm{dR}}(C) \cong \mathbb{C}^{2g}.$$

This is an embedding since a non-zero holomorphic differential cannot be exact. We can also think of the conjugate embedding

$$\overline{\mathrm{H}^0(K_C)} \longrightarrow \mathrm{H}^1_{\mathrm{dR}}(C),$$

where the domain is the space of anti-holomorphic differentials. These are locally of the form $\overline{f}(z)d\overline{z}$. We claim the two spaces of differentials intersect trivially in $\mathrm{H}^1_{\mathrm{dR}}(C)$.

To see this, construct a positive definite Hermitian form on $\mathrm{H}^0(K_C)$ given by

$$H(\omega) = i \int_C \omega \wedge \overline{\omega}.$$

One can show that H is positive definite by a local computation. Say $\omega = f(z)dz$, so $\overline{\omega} = \overline{f}(z)d\overline{z}$. Recall that

$$dz = dx + idy, \qquad d\overline{z} = dx - idy,$$

and write

$$\omega \wedge \overline{\omega} = |f(z)|^2(dx + idy) \wedge (dx - idy) = -2i|f(z)|^2 dx \wedge dy.$$

Multiplying by i clears the constant $-i$, hence H is positive definite. Next, consider a cohomology class lying in the intersection of the spaces of holomorphic and anti-holomorphic forms. Pick a holomorphic 1-form ω representing it. We also know that its conjugate $\overline{\omega}$ is homologous to another holomorphic 1-form, say η. Since integrals are well-defined on de Rham classes, we have

$$H(\omega) = i\int_C \omega \wedge \overline{\omega} = i\int_C \omega \wedge \eta.$$

The form $\omega \wedge \eta$ can verified to vanish locally, hence $H(\omega) = 0$, and $\omega = 0$ as necessary. Then we observe that

$$\dim \mathrm{H}^0(K_C) = \dim \overline{\mathrm{H}^0(K_C)},$$

so

$$\dim \mathrm{H}^0(K_C) \leq \frac{\dim \mathrm{H}^1_{\mathrm{dR}}(C)}{2} = g,$$

which proves the first claim.

7.2. Riemann-Roch, finally

We are left to prove our second claim. Let us start by recalling the bogus proof of Riemann-Roch we gave previously. For an effective divisor $D = p_1 + \cdots + p_d$ consider some $f \in \mathcal{L}(D)$. Observe that for any holomorphic differential ω, we can construct a meromorphic differential $f\omega$, with simple poles along D, hence

$$\sum_i \mathrm{Res}_{p_i}(f\omega) = 0.$$

From here, we deduce that

$$\ell(D) \leq d + 1 - (g - \ell(K - D)).$$

Then, we applied the same formula to $K - D$ which reads

$$\ell(K - D) \leq 2g - 2 - d + 1 - (g - \ell(D)).$$

The last step is wrong in general since it assumes $K - D$ is effective. If $K - D \geq 0$, or at least, there is a linearly equivalent effective divisor, then the argument goes through. If $\ell(K - D) = 0$, we still need to prove Riemann-Roch. More precisely, we need to prove the opposite inequality $\ell(D) \geq d + 1 - g$, which was carried out last time.

If the both divisor classes of D and $K - D$ do not contain effective representatives, then $\deg D < g$ and $\deg(K - D) < g$. To see this, think about Jacobi inversion. But then, we get $\deg D = \deg K - D = g - 1$ in which case Riemann-Roch is automatically satisfied.

While it seems we have completed the proof of Riemann-Roch, there still is a small hiccup. Recall the construction we already discussed. We have a curve C and a birational map $\nu\colon C \to C_0 \subset \mathbb{P}^2$, and a divisor D on C. Then, we constructed g such that $g(D) = g(r_i) = 0$ where the r_i denote the nodes of C_0. Then we write $(g) = D + \Delta + E + mD_\infty$, and consider

$$\left\{ \frac{h}{g} \;\middle|\; \begin{array}{c} h \text{ has degree } m, \\ h(E) = h(r_i) = 0 \end{array} \right\} \subset \mathcal{L}(D).$$

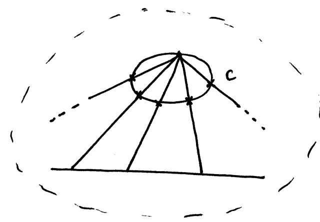
Exercise 7.1. We have equality if $m \geq d - 3$.

If $m > d - 3$, then every divisor D on C and $D \sim mD - \Delta$ is the intersection of C with a curve of degree r_i. The case $m = d - 3$ has to be handled separately.

Exercise 7.2. (harder) We have equality for all m. This is called *completeness of the adjoint series* in classical language.

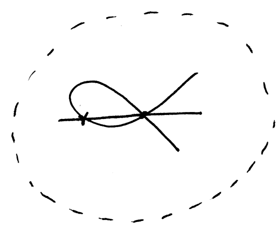
7.3. Fun applications

Let C be a smooth conic plane curve. To see this is isomorphic to $\mathbb{P}^1$, consider the intersection map, i.e., project from a point onto a line.

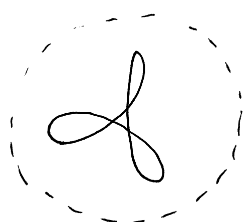


Alternatively, the images of the intersection cut a g^1_1 and hence produce an isomorphism with $\mathbb{P}^1$.

For C a nodal plane cubic, we can project from the node to obtain a birational isomorphism to $\mathbb{P}^1$.



Next, consider a plane quartic C with 3 nodes. The claim is C is still rational.



If we take a single point p and show that it moves in a pencil, then we are done. Let G be a conic plane curve (e.g., two lines) containing p and the three nodes r_1, r_2 and r_3. There should be 8 intersections by Bezout, and we have accounted for 7 so far (the nodes count as double intersections). There is a pencil of such conics, and they parametrize the points of C.

Alternatively, take two conics $G = V(g)$ and $G' = V(g')$. Then g/g' is a rational function on C as necessary and it produces a birational isomorphism $g/g' \colon C \to \mathbb{P}^1$.

7.4. Sheaf cohomology

Let us explain some notation we have already been using in some form. Consider a set of δ points in the plane $\Gamma = \{p_1, \dots, p_\delta\} \subset \mathbb{P}^2$. There is an inclusion map on degree m polynomials given as follows.

$$\begin{tikzcd}
\left\{\begin{matrix}\text{polynomials of degree } m \\ \text{vanishing at } p_1, \dots, p_\delta\end{matrix}\right\} \arrow[r, hook] \arrow[d, equal] & \left\{\begin{matrix}\text{polynomials} \\ \text{of degree } m\end{matrix}\right\} \arrow[d, equal] \\
\mathrm{H}^0(\mathcal{I}_\Gamma(m)) \arrow[r, hook] & \mathrm{H}^0(\mathcal{O}_{\mathbb{P}^2}(m))
\end{tikzcd}$$

We say that Γ *imposes independent conditions* on polynomials/curves of degree m if the codimension of the given vector spaces is δ. Equivalently, the restriction map $\mathcal{O}_{\mathbb{P}^2}(m) \to \mathcal{O}_\Gamma(m)$ is surjective on global sections.

There is a short exact sequence of sheaves

$$0 \longrightarrow \mathcal{I}_\Gamma(m) \longrightarrow \mathcal{O}_{\mathbb{P}^2}(m) \longrightarrow \mathcal{O}_\Gamma(m) \longrightarrow 0$$

with associated long exact sequence

$$0 \longrightarrow \mathrm{H}^0(\mathcal{I}_\Gamma(m)) \longrightarrow \mathrm{H}^0(\mathcal{O}_{\mathbb{P}^2}(m)) \longrightarrow \mathrm{H}^0(\mathcal{O}_\Gamma(m)) \longrightarrow \mathrm{H}^1(\mathcal{I}_\Gamma(m)) \longrightarrow \mathrm{H}^1(\mathcal{O}_{\mathbb{P}^2}(m)) = 0.$$

It follows that the condition $h^1(\mathcal{I}_\Gamma(m)) = 1$ is equivalent to the two we just provided.

Example 7.3. Consider the space of quartic space curves $C \subset \mathbb{P}^3$. If C has genus 1, then we can realize it as the intersection of two quadratic surfaces. It is possible that C has genus 0 and it is a rational normal curve in $\mathbb{P}^3$.

We would like to know how to distinguish these two cases. For a hyperplane $H \subset \mathbb{P}^3$, the intersection $H \cap C$ will generically consist of 4 points $\Gamma \subset H$. There exist two quadrics in H which vanish on Γ, but then we ask whether these can be extended to quadrics in $\mathbb{P}^3$ which contain C. There is a surjective map of sheaves $\mathcal{I}_C(2) \to \mathcal{I}_\Gamma(2)$ given by restriction. Its kernel can be recognized to be $\mathcal{I}_C(1)$, hence we have a short exact sequence

$$0 \longrightarrow \mathcal{I}_C(1) \longrightarrow \mathcal{I}_C(2) \longrightarrow \mathcal{I}_\Gamma(2) \longrightarrow 0.$$

Equivalently, our question is whether the last map is surjective on global sections. Looking at H^1 we recognize a situation which we already discussed.

The point of the previous example was to show that sheaf cohomology is a great tool for transferring information between different contexts. Namely, if we can reduce our questions to studying a given sheaf, and then recognize the same sheaf in a different problem, we have a good chance of applying our geometric intuition in the latter context.

Here is a basic result about finite sets of points.

Theorem 7.4.

(a) *If $\delta \leq m+1$, then any configuration Γ of degree δ imposes independent linear conditions on polynomials of degree m.*

(b) *If $\delta = m+2$, then a configuration Γ of degree δ fails to impose independent conditions on degree m polynomials if and only if Γ is contained in a linear subspace.*

Proof. To prove part (a) it suffices to construct a polynomial of degree $m+1$ which vanishes at all but one of the points of Γ. To do so, pick linear spaces through the m points which do not go through the last one, then take their product.

When $\delta = m+2$ the argument is a very similar investigation which we will omit. □

Example 7.5. Consider a smooth plane curve $C \subset \mathbb{P}^2$ of degree d. We would like to find the smallest degree of a non-constant meromorphic function on C. Equivalently, what is the smallest m such that there exists a degree m branched cover $C \to \mathbb{P}^1$? We can certainly do this for $m = d$ by taking the ratio of two linear polynomials on $\mathbb{P}^2$. We can improve the argument to $m = d-1$ by taking both linear polynomials to vanish on a fixed point on C.

More generally, consider a divisor $D = p_1 + \cdots + p_m$ of degree m on C. The question we posed is equivalent to the assertion there exist such D moving in a pencil. By the geometric form of Riemann-Roch $r(D) \geq 1$ occurs if and only if $p_1, \ldots, p_m$ fail to impose independent conditions on the complete canonical series $|K|$. But we already know that

$$\mathrm{H}^0(K) = \left\{ \frac{g\,dx}{f_y} \;\middle|\; \begin{matrix} g \text{ is a polynomial} \\ \text{of degree} \leq d-3 \end{matrix} \right\},$$

so the above statement is equivalent to $p_1, \ldots, p_m$ failing to impose independent conditions on polynomials of degree $d-3$ in $\mathbb{P}^2$. Any $d-1$ points impose independent conditions, so our bound $m \geq d-1$ is correct. When $m = d-1$, this D moves in a pencil if and only if $p_1, \ldots, p_m$ are collinear.

In the direction of more general geometry, we can assume C is smooth with the exception of a unique node r. We can address the same question for the normalization of C. We are looking for $D = p_1 + \cdots + p_m$ that fail to impose independent conditions on $|K|$, i.e., $p_1, \ldots, p_m$ such that $\Gamma = \{p_1, \ldots, p_m, r\}$ fail to impose independent conditions of polynomials of degree $d-3$. This can happen only if $m \geq d-2$. When $m = d-2$, we have to require that $p_1, \ldots, p_m$ are collinear with r.

LECTURE 8

September 28, 2011

8.1. Examples of low genus

Today we will talk about several examples of curves. We will look at curves of low genus and see why some behave differently than others. Next week, we will start on Castelnuovo theory (Chapter 3 in ACGH).

Let C be a curve of genus g and D a divisor of degree d on C with corresponding line bundle L. Provided D is basepoint free, there is an associated map $\varphi\colon C \to \mathbb{P}^r$.

When $g = 0$ and 1, the behavior of φ depends only on d (that is, neither on the specific curve C, nor on the line bundle L).

When $g = 2$, φ depends on the choice of line bundle L. Note that C is hyperelliptic, so there is an associated involution $i\colon C \to C$. For example, take $g = 2$ and $d = 4$. There are three cases for the map $\varphi\colon C \to \mathbb{P}^2$.

(i) If $D = K + p + q$ and $p \neq q$, $p \neq i(q)$, then φ is birational onto its image which has a single node.
(ii) If $D = K + 2p$ and $p \neq i(p)$, then φ is again birational but its image has a cusp.
(iii) If $D = 2K$, then φ is 2-to-1 onto a conic in $\mathbb{P}^2$.

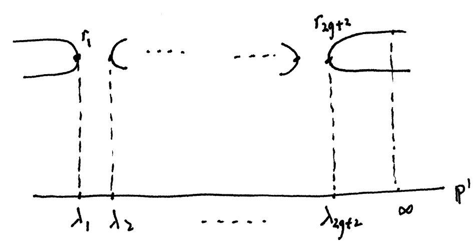
When $g \geq 3$, the map φ depends on both C and the line bundle L. Take for example $g = 3$ and $d = 4$. If the canonical map $\varphi_K\colon C \to \mathbb{P}^2$ is an embedding, then C has a smooth quartic plane model. This is the case unless C is hyperelliptic.

8.2. Hyperelliptic curves

Consider a hyperelliptic curve C of arbitrary genus g. We can express C as

$$y^2 = \prod_{i=1}^{2g+2} (x - \lambda_i)$$

for some $\lambda_i \in \mathbb{C}$. Let the projection down to the x-axis be denoted by π. Its branch points are given by $r_i = (\lambda_i, 0)$ for $1 \leq i \leq 2g+2$. The projection is unramified over ∞, and let p and q denote the points of C lying above ∞.



The first thing we want to do it write the holomorphic differentials on C. For this purpose, let us start with a meromorphic one, say dx. It is easy to compute the associated divisor

$$(dx) = \sum_i r_i - 2(p+q).$$

We would like to find a rational function vanishing at the r_i. For example, dx/y is holomorphic in the finite chart. We need to study its behavior over ∞. This can be carried out locally and we conclude

$(dx/y) = (g-1)(p+q)$. We can now multiply by any polynomials in x of degree $\leq g-1$. The differentials

$$\frac{dx}{y}, x\frac{dx}{y}, \dots, x^{g-1}\frac{dx}{y}$$

for a basis for the space of holomorphic differentials. The canonical map can then be expressed as

$$\varphi_K\colon\ (x,y) \longmapsto [1, x, \dots, x^{g-1}] \in \mathbb{P}^{g-1}.$$

It is 2-to-1 onto a rational normal curve $C_0 \subset \mathbb{P}^{g-1}$ (assuming $g \geq 2$). We have a degree 2 class that moves in a pencil, i.e., it is a g^1_2. In concrete terms, for any points in $\mathbb{P}^1$, we have

$$\pi^*(\text{point}) = \pi^*\mathcal{O}_{\mathbb{P}^1}(1) = g^1_2.$$

Observe that by Riemann-Roch, for any $s, t \in C$

$$r(s+t) \geq 1 \quad \Longleftrightarrow \quad |K(-s)| = |K(-t)| \quad \Longleftrightarrow \quad \varphi_K(s) = \varphi_K(t).$$

This tells us that the g^1_2 we described is unique.

Recall that the geometric version of Riemann-Roch applied for non-hyperelliptic curves C. Let $\varphi_K : C \to \mathbb{P}^{g-1}$ denote the canonical embedding. For a divisor $D = p_1 + \cdots + p_d$, we can compute $r(D)$ as the number of linear relations on the p_i. Let us relax the hypothesis C is not hyperelliptic. Then $\varphi_K : C \to \mathbb{P}^{g-1}$ is either an embedding or a 2-sheeted branched cover of a rational normal curve. Let D be a degree d effective divisor on C. Define

$$\overline{D} = \bigcap_{\substack{H\subset\mathbb{P}^{g-1} \text{ a hyperplane} \\ D\subset\varphi_K^{-1}(H)}} H.$$

Remark 8.1. The inclusion $D \subset \varphi_K^{-1}(H)$ must be taken scheme theoretically. For example, if D contains $2p$ then H needs to be tangent to $\varphi_K(p)$.

Note that the construction of $\overline{D}$ does not require C to be hyperelliptic. The geometric Riemann-Roch still applies stating that

$$r(D) = d - 1 - \dim \overline{D},$$

where we compute the dimension of $\overline{D}$ as a projective space.

Let us return to our discussion of hyperelliptic curves. Let C be such and $\varphi_K : C \to C_0 \subset \mathbb{P}^2$ be the canonical map. We know that $C_0 \subset \mathbb{P}^{g-1}$ is a rational normal curve, hence any effective divisor of degree $d \leq g$ is linearly independent, i.e., the points of D span a $\mathbb{P}^{d-1} \subset \mathbb{P}^{g-1}$.

Definition 8.2. A divisor D is called *special* if any of the following equivalent conditions holds:

(i) $\overline{D}$ is a proper subspace of $\mathbb{P}^{g-1}$,
(ii) $h^0(D-K) > 0$,
(iii) $K - D$ is effective.

Back to our discussion of the hyperelliptic case, the geometric Riemann-Roch implies that any special D can be expressed as

$$|D| = |mg^1_2| + D_0,$$

where D_0 is a fixed divisor of degree $d - 2m$. In other words, if D is special then φ_D factors through $\pi\colon C \to \mathbb{P}^1$. It follows that the only way to get a birational embedding of a hyperelliptic curve C of genus g is to take $d \geq g+2$ and $r = d-g$. To get an actual embedding, we have to take $d \geq g+3$ and $r = d-g$. These observations can be summarized by saying that hyperelliptic curves are the most resistant to embedding in projective space.

8.3. Low genus examples

We proceed to investigate curves of genera 3, 4 and 5.

Example 8.3. Let C be a hyperelliptic curve of genus $g=3$. We can take $d=6$ and $D \neq K+p+q$ for all $p,q \in C$ in order to get an embedding $C \hookrightarrow \mathbb{P}^3$ as a sextic curve. The image of C is of type $(2,4)$ on a smooth quadric surface. This statement is equivalent to saying that $D-g^1_2$ does not contain a g^1_2, i.e., $D-g^1_2$ is a non-special divisor of degree 4. Note that $r(D-g^1_2)=1$, so we get a 4-sheeted branched cover

$$\varphi = \varphi_{D-g^1_2}\colon\ C \longrightarrow \mathbb{P}^1.$$

We can realize C as sitting on the quadric in question as follows.

$$\begin{tikzcd}
 & \mathbb{P}^1 & \\
C \arrow[ur, "\varphi"] \arrow[r] \arrow[dr, "\pi"'] & \mathbb{P}^1 \times \mathbb{P}^1 \arrow[u, "\mathrm{pr}_1"'] \arrow[d, "\mathrm{pr}_2"] \arrow[r, hook, "\text{Segre}"] & \mathbb{P}^3 \\
 & \mathbb{P}^1 &
\end{tikzcd}$$

If C is not hyperelliptic, then $C \hookrightarrow \mathbb{P}^2$ is a smooth quartic. In this case, we can express C as a 3-sheeted branched cover of $\mathbb{P}^1$ in a 1-dimensional family of ways. To do so construct these projections we project from a point on C, hence there is a 1-dimensional family of projections.

Example 8.4. Let C be a non-hyperelliptic curve of genus $g=4$ (we already know about the hyperelliptic case). We would like to know whether C can be expressed as a 3-sheeted branched cover of $\mathbb{P}^1$.

Consider the canonical model $C_0 = \varphi_K(C) \subset \mathbb{P}^3$ of degree 6. Let us start by investigating the surfaces on which C_0 lies. In degree 2, we have a restriction map

$$\varphi_K^*\colon\ \mathrm{H}^0(\mathcal{O}_{\mathbb{P}^3}(2)) \longrightarrow \mathrm{H}^0(K^2).$$

The dimensions of the given spaces are 10 and $12-4+1=9$ respectively, so C_0 lies on at least one quadric surface Q. It is possible that Q is smooth or a cone. In the latter case C_0 is contained in the smooth locus of Q, that is, it does not pass through the vertex. In degree 3, the restriction map reads

$$\varphi_K^*\colon\ \mathrm{H}^0(\mathcal{O}_{\mathbb{P}^3}(3)) \longrightarrow \mathrm{H}^0(K^3),$$

and the given spaces have dimensions 20 and $18-4+1=15$ respectively. It follows that C_0 lies on at least 5 linearly independent cubics, four of them being products of Q with linear forms, hence there is a fifth one which we will call S. By Bezout, we have $C_0 = Q \cap S$. By Noether's $AF+BG$ Theorem, the degree 3 restriction map above is surjective.

Remark 8.5. Conversely if C is the smooth complete intersection of a quadric and a cubic in $\mathbb{P}^3$, adjunction implies $K_C = \mathcal{O}_C(1)$ and C has genus 4. Due to this correspondence, we will identify C with its image C_0 under φ_K.

Geometric Riemann-Roch says that a degree 3 divisor D on C satisfies $r(D) \geq 1$ if and only if D is contained in a line $L \subset \mathbb{P}^3$. The lines in question are the rulings on the quadric Q, namely, $L \subset Q$. As a conclusion C can be expressed as a degree 3 cover of $\mathbb{P}^1$. There are two such expressions if Q is smooth and only one if Q is singular (a cone). This is an example of the phenomenon we mentioned in the beginning of this lecture – different curves behave differently.

Example 8.6. Let C be a non-hyperelliptic curve of genus $g=5$. The canonical map $\varphi_K\colon C \to \mathbb{P}^4$ has degree 8. We identify C with its image under φ_K.

Let us look at the quadrics which contain C. The relevant restriction map reads

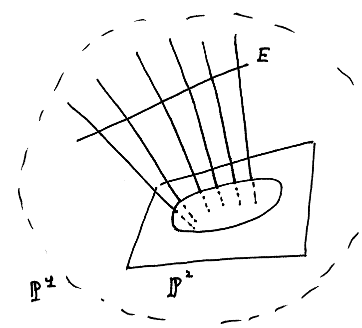
$$\varphi_K^*\colon\ \mathrm{H}^0(\mathcal{O}_{\mathbb{P}^4}(2)) \longrightarrow \mathrm{H}^0(K^2),$$

and the two vector spaces have dimensions 15 and $16 - 5 + 1 = 12$. It follows that C lies on at least three quadrics which we will call Q_1, Q_2 and Q_3.The curve may be the complete intersection of the three quadrics but this does not happen always. (Conversely, if we start with $C = Q_1 \cap Q_2 \cap Q_3$, adjunction implies $K_C = K_{\mathbb{P}^4}(2+2+2)|_C = \mathcal{O}_C(1)$.) This happens in general but now always. It is possible that the Q_i intersect in a surface. There is a unique such – the *cubic scroll*. For this to happen $Q_1 \cap Q_2$ gas to be reducible, the union of a cubic surface contained in Q_3 and a plane. We conclude there are two possibilities:

$$C = Q_1 \cap Q_2 \cap Q_3, \qquad C \subsetneq S = Q_1 \cap Q_2 \cap Q_3.$$

Next we would like to investigate whether C is *trigonal*, i.e., if it can be presented as a degree 3 cover of $\mathbb{P}^1$. As before, geometric Riemann-Roch implies that a degree 3 divisor D satisfies $r(D) \geq 1$ if and only if D is contained in a line L. We are looking for triples of collinear points on C, and the line L they line on is contained in all quadrics Q_i. This cannot happen if $C = Q_1 \cap Q_2 \cap Q_3$, hence the general genus 5 curve is not trigonal. When C is trigonal, it lies on a *rational normal scroll*, also called a cubic scroll for short.

Here is a brief construction of the rational normal scroll. Consider a line E and a planar conic, both in $\mathbb{P}^4$.



Identify the conic and the line and join corresponding points with lines. The result is a surface isomorphic to $\mathbb{P}^2$ blown-up at a point embedded in $\mathbb{P}^4$ via the divisor $2H - E$ (here H denotes the divisor of lines in $\mathbb{P}^2$). This also corresponds to the space of conics in $\mathbb{P}^2$ passing through a points. Equivalently, embed $\mathbb{P}^2$ in $\mathbb{P}^5$ via a Veronese map, and then project away from a point p on the image.

When the curve C lies on a cubic scroll, it meets the ruling 3 times, hence we get a unique g^1_3.

LECTURE 9

September 30, 2011

This lecture was prepared and delivered by Anand Deopurkar.

9.1. Automorphisms of genus 0 an 1 curves

Let C be a smooth projective curve of genus g. In what follows, we would like to investigate the group of automorphisms of C, denoted $\mathrm{Aut}(C)$.

When $g = 0$, we can identify $C = \mathbb{P}^1$. All automorphisms of $\mathbb{P}^1$ are linear, that is, they are given by coordinate transformations

$$[x, y] \longmapsto [ax + by, cx + dy],$$

for some $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}(2)$ unique up to scaling. Therefore

$$\mathrm{Aut}(\mathbb{P}^1) = \mathrm{GL}(2)/\mathbb{C}^\times = \mathrm{PGL}(2)$$

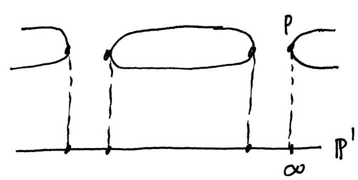
which is 3-dimensional. Additionally, it is not hard to show the action is 3-transitive, that is, it acts transitively on ordered triples of distinct points in $\mathbb{P}^1$.

Next, consider the case $g = 1$ and fix a point $p \in C$. There is a group law on C with p as the identity element. For all $q \in C$, there is an automorphism

$$\tau_q\colon\ x \longmapsto x + q,$$

and these furnish a 1-dimensional family of automorphisms. What can be say about other automorphisms? Equivalently, we would like to study automorphisms $\varphi\colon C \to C$ satisfying $\varphi(p) = p$. There is an obvious one given by $\varphi(x) = -x$, which can also be identified as the hyperelliptic involution of C. What about other ones?

Consider the linear series $|2p|$. This is the unique g^1_2 so it gives a map $\pi\colon C \to \mathbb{P}^1$ ramified at 4 points. One of these points is p, but there are three more.



Any automorphism $\varphi\colon C \to C$ satisfying $\varphi(p) = p$ must fit into a diagram

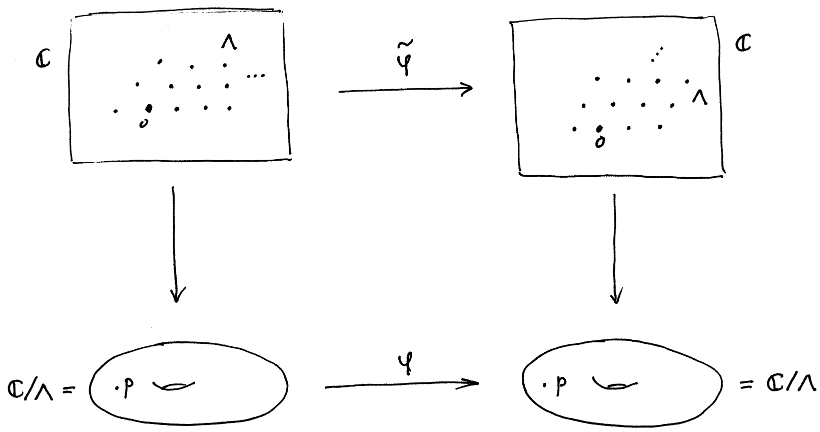
$$\begin{tikzcd} C \arrow[r, "\varphi"] \arrow[d, "\pi"'] & C \arrow[d, "\pi"] \\ \mathbb{P}^1 \arrow[r, "\widetilde{\varphi}"] & \mathbb{P}^1 \end{tikzcd}$$

for some $\widetilde{\varphi}\colon \mathbb{P}^1 \to \mathbb{P}^1$. It follows that $\widetilde{\varphi}$ permutes the branch points fixing $\infty = \pi(p)$. Conversely, any automorphism of $\mathbb{P}^1$ permuting the branch points and fixing ∞ lifts to an automorphism φ of C, unique up to the hyperelliptic involution. Up to coordinate transformations, there are only two choices of 3 points in $\mathbb{P}^1 \setminus \{\infty\}$ which admit a non-trivial automorphism. These are:

(i) $\{\infty, 1, 0, -1\}$ with automorphism $x \mapsto -x$ of order 2, and
(ii) $\{\infty, 1, \omega, \omega^2\}$ for ω a primitive cube root of unity with automorphism $x \mapsto \omega x$ of order 3.

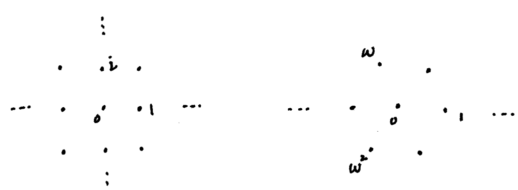
In conclusion, for $g = 1$, there is one curve with 4 automorphisms – the double cover of $\mathbb{P}^1$ branched over $\{\infty, 1, 0, -1\}$. There is another curve with 6 automorphisms 0 – the double cover of $\mathbb{P}^1$ branched over $\{\infty, 1, \omega, \omega^2\}$ where ω stands for a primitive cube root of unity. All other curves have only the hyperelliptic involution.

By realizing C as a quotient $\mathbb{C}/\Lambda$ for a rank 2 lattice $\Lambda \subset \mathbb{C}$, we can also deduce the statements above topologically. An automorphism $\varphi\colon (C, p) \to (C, p)$ lifts to $\widetilde{\varphi}\colon \mathbb{C} \to \mathbb{C}$ satisfying $\widetilde{\varphi}(0) = 0$ and $\widetilde{\varphi}(\Lambda) = \Lambda$. These facts imply that $\widetilde{\varphi}$ is linear. Conversely, any such linear $\widetilde{\varphi}$ descends to an automorphism of $\mathbb{C}/\Lambda = C$.



There is always one such $\widetilde{\varphi}$, namely, $x \mapsto -x$. This is the unique automorphism for a general lattice. There are two special ones which admit more automorphisms.

(i) $\Lambda = \mathbb{Z}[i] \subset \mathbb{C}$ with additional automorphism $x \mapsto ix$, and
(ii) $\Lambda = \mathbb{Z}[\omega] \subset \mathbb{C}$ with additional automorphism $x \mapsto \omega x$.



9.2. Automorphisms of higher genus curves

Curves of genus $g \geq 2$ have only finitely many automorphisms. We proceed to discuss the following general bound on the number of automorphisms.

Theorem 9.1 (Hurewicz)**.** *Let C be a smooth projective curve of genus g. If $g \geq 2$, then*

$$|\operatorname{Aut} C| \leq 84(g-1) = -42\chi(C) = 42\deg(K_C).$$

We will prove it as follows:

Step 1. $\operatorname{Aut} C$ is finite;
Step 2. assuming $\operatorname{Aut} C$ is finite, then $|\operatorname{Aut} C| \leq 84(g-1)$.

Proof of Step 1. Denote $G = \operatorname{Aut} C$ and $V = \mathrm{H}^0(K_C)$. The group G acts on V by pulling-back holomorphic differentials, so we get a map $G \to \mathrm{GL}(V)$.

Claim. The image of G in $\mathrm{GL}(V)$ is finite.

Recall that V has a positive definite form given by

$$\langle \omega, \eta \rangle \longmapsto i \int_C \omega \wedge \overline{\eta}.$$

It is not hard to see that G preserves this form, so its image lands in the unitary group with respect to the given form, which is compact. Next, observe that there is a distinguished lattice

$$\begin{tikzcd} \mathrm{H}^1(C, \mathbb{Z}) \arrow[r, hook] \arrow[d, "\cong"'] & \mathrm{H}^0(K_C) \arrow[d, "\cong"] \\ \mathbb{Z}^{2g} \arrow[r, hook] & \mathbb{C}^g \end{tikzcd}$$

which is preserved by G, hence the image of G lands in $\mathrm{GL}(2g, \mathbb{Z})$ which is discrete. In conclusion, the image of G lands in a compact discrete group, hence is finite.

Claim. The map $G \to \mathrm{GL}(V)$ is injective.

At this point we will need the hypothesis $g \geq 2$. Consider the de Rham cohomology $\mathrm{H}^*_{\mathrm{dR}}(C, \mathbb{C})$ which consists of

$$\mathrm{H}^0(C, \mathbb{C}), \qquad \mathrm{H}^1(C, \mathbb{C}) = \mathrm{H}^0(K_C) \oplus \overline{\mathrm{H}^0(K_C)}, \qquad \mathrm{H}^2(C, \mathbb{C}).$$

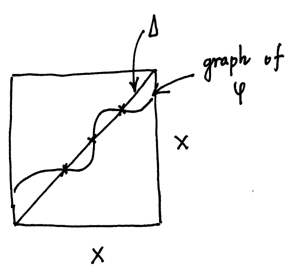
Let $g \in G$ be an element acting trivially on all topological cohomology groups (de Rham of singular). Let us recall the *Lefschetz Fixed Point Theorem.*

Theorem 9.2. *Let $\varphi \colon X \to X$ be a continuous map of finite CW complexes. We can associate to it the* Lefschetz number

$$\Lambda_\varphi = \sum_i (-1)^i \operatorname{Tr}(\varphi^* \colon \mathrm{H}^i(X) \longrightarrow \mathrm{H}^i(X)).$$

If the number of fixed points is finite, then Λ_φ is reflects the number of fixed points counted with the right multiplicity (the sign depends on orientation).

Remark 9.3. The fixed points of φ can be identified with the intersections of the diagonal $\Delta \subset X \times X$ and the graph of φ in $X \times X$. While we will not go into details, this setting makes it simpler to define multiplicities.



In our case, we are dealing with complex manifolds, so all intersection multiplicities are positive. If g acts trivially on $\mathrm{H}^*(C)$, then $\Lambda_\varphi = 2 - 2g < 0$ if $g \geq 2$. By the Lefschetz Fixed Point Theorem, the number of fixed points cannot be finite. But if an automorphism of compact complex manifolds has infinitely many fixed points, then there is an accumulation point, hence g is the identity. $\square$

Sketch of another proof. The idea this time is to use the following fact: if $\varphi \colon C \to C$ fixes more than $2g + 2$ points, then it must be the identity. To show this, one can use the Lefschetz fixed point theorem and consider the eigenvalues of the action on H^* which are algebraic integers.

Secondly, given a Riemann surface C, we associate a finite set of points $W \subset C$ such that any automorphism permutes W (these are called *Weierstrass points*. Then we can show that if $g \geq 2$ there are enough Weierstrass points, and there is an injection $\operatorname{Aut} C \to S_W$. The codomain S_W is the symmetric group on W which has finite size $|W|!$. □

Remark 9.4. Both proofs we presented so far use the Lefschetz fixed point theorem in a crucial way. This tool is available only when working over $\mathbb{C}$.

Sketch of a purely algebraic proof. The group $\operatorname{Aut} C$ can be given the structure of a quasi-projective scheme. To show that it is finite, it suffices to demonstrate that $\dim \operatorname{Aut} C = 0$. Actually, we show that all tangent spaces are 0. In fact, it suffices to consider the tangent space at the identity element. But deformation theory dictates that

$$T_{\mathrm{id}_C} \operatorname{Aut} C = \mathrm{H}^0(TC)$$

which is 0 when $g \geq 2$. □

Proof of Step 2. We know that $\operatorname{Aut} C$ is finite. Form $C' = C/G$ with quotient map $\pi \colon C \to C'$.

Claim. The space C' can be given the structure of a Riemann surface such that π is holomorphic.
(For more details, see Miranda's book *Algebraic curves and Riemann surfaces.*)

We can always form C' as a topological space and make $\pi \colon C \to C'$ continuous and open. If there are no fixed points, then π is a covering space and we are done. We only need to focus on fixed points of $\operatorname{Aut} C$.

Observation 9.5. *All non-trivial stabilizers are finite.*

Observation 9.6. *All stabilizers are cyclic groups. (This fact depends on the holomorphicity of the action in a crucial way.)*

Let us sketch a few points crucial for the proof of these facts. Let z be a local coordinate around a point p with non-trivial stabilizer such that p corresponds to $z = 0$. Given an element g, we can write $g(z) = a_1(g)z + a_2(g)z^2 + \cdots$. The map $G_p \to \mathbb{C}^\times$ given by $g \mapsto a_1(g)$ is a homomorphism. We claim it is injective. Consider g non-trivial in the kernel of $G_p \to \mathbb{C}^\times$, and write

$$\begin{aligned} g(z) &= z + az^m + \cdots, \\ g^2(z) &= z + 2az^m + \cdots, \\ &\vdots \\ g^k(z) &= z + kaz^m + \cdots, \end{aligned}$$

where $a \neq 0$ and $m > 1$. Provided $g^k = \mathrm{id}$ for some k, then $ka = 0$, hence $a = 0$, and $g = \mathrm{id}$.

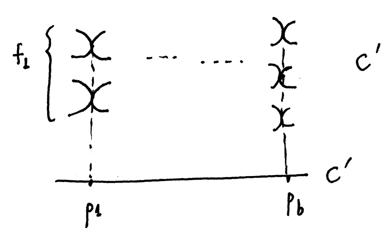
Back to our bound, we have constructed a map

$$\pi \colon C \longrightarrow C'$$

of degree $|G|$. Let the genera of C and C' be g and h respectively. Applying Riemann-Hurewicz, we can write

$$2g - 2 = |G|(2h - 2) + R,$$

where R denotes the degree of the ramification divisor. Let $p_1, \ldots, p_b$ denote the branch points of π.



Let f_i be the number of points in C above p_i. Around each of the preimages, the ramification index is the same, say e_i. Then $e_i f_i = |G|$ for all $1 \leq i \leq b$, so we can write

$$R = \sum_i f_i(e_i - 1) = \sum_i (|G| - f_i) = |G| \sum_i \left(1 - \frac{1}{e_i}\right).$$

We can then rewrite the statement of Riemann-Hurewicz as $2g - 2 = |G| \cdot Q$, where

$$Q = 2h - 2 + \sum_i \left(1 - \frac{1}{e_i}\right).$$

The assumption $g \geq 2$ forces $Q > 0$, so we need to find the minimal possible value of Q. The constraints are $h \geq 0$, $b \geq 0$, $e_i \geq 2$ and all are integers. At this point, we break into several cases.

(i) If $h \geq 2$, then $Q \geq 2$.
(ii) If $h = 1$, then $Q \geq 1/2$.
(iii) If $h = 0$, then $b \geq 3$.
  (a) If $b \geq 4$, then
$$Q \geq -2 + \left(1 - \frac{1}{2}\right) + \left(1 - \frac{1}{2}\right) + \left(1 - \frac{1}{2}\right) + \left(1 - \frac{1}{3}\right) = \frac{1}{6}.$$
  (b) If $b = 3$, then
$$Q \geq -2 + \left(1 - \frac{1}{2}\right) + \left(1 - \frac{1}{3}\right) + \left(1 - \frac{1}{7}\right) = 1 - \frac{41}{42} = \frac{1}{42}.$$

The minimal value which Q may take is $1/42$. We conclude that

$$|G| \leq \frac{1}{Q}(2g - 2) = 84(g - 1).$$ □

What can we say about the sharpness of the bound we obtained. When $g = 2$, the bound is not sharp. When $g = 3$, there is a sharp curve, the *Klein quartic*

$$\{x^3y + y^3z + z^3x = 0\} \subset \mathbb{P}^2.$$

All automorphisms come from projective automorphisms

(i) Cyclically permuting variables $[x, y, z] \mapsto [y, z, x]$,
(ii) Order 7 map $[x, y, z] \mapsto [\zeta^4 x, \zeta^2 y, \zeta z]$ where $\zeta^7 = 1$.

There are also infinitely many genera where the bound is not achieved. More generally, people study

$$N(g) = \text{size of the largest automorphism group of a curve of genus } g.$$

We just showed that $N(g) \leq 84(g - 1)$. Equality holds infinitely often, but so does the strict inequality. There are also ways to produce lower bounds. For example, we can show $N(g) \geq 8(g + 1)$ by studying the curve $y^2 = x^{2g+2} - 1$. This is hyperelliptic branched over ∞ and the vertices of a $(2g + 2)$-gon.

Remark 9.7. In positive characteristic, there are curves with many more automorphisms, i.e., the Hurewicz bound does not hold.

LECTURE 10

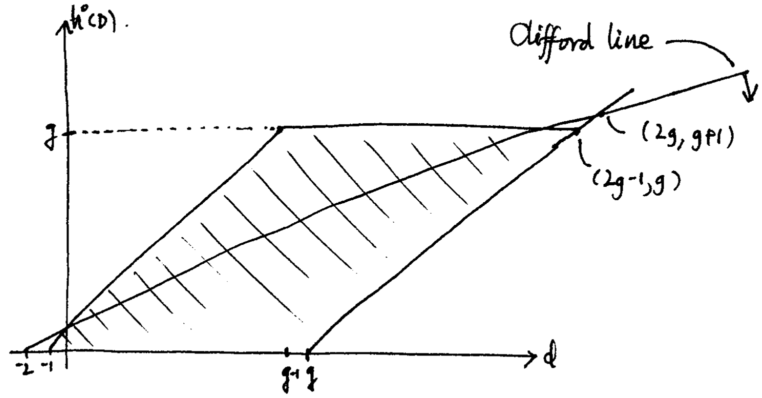
October 7, 2011

10.1. Clifford's Theorem

Let C be a smooth projective curve of genus g and D a divisor (class) of degree d. We would like to study the possible values of $h^0(D) = \ell(D)$, or, equivalently, $r(D) = \ell(D) - 1$.

(i) If $d < 0$, then $h^0(D) = 0$.
(ii) If $d > 2g - 2$, then $h^0(D) = d - g + 1$ by Riemann-Roch.
(iii) In general, Riemann-Roch tells us $h^0(D) \leq d + 1$ and $h^0(D) \geq d - g + 1$.

The question is what happens in the range $0 \leq d \leq 2g-1$. The following diagram illustrates the parallelogram enclosing all possible values in this range.



Even so, only about half of these occur.

Theorem 10.1 (Clifford's Theorem)**.** *Let $0 \leq d \leq 2g - 2$. Then $h^0(D) \leq 1 + d/2$, or, equivalently, $r(D) \leq d/2$. Furthermore, equality holds if and only if one of three things happen:*

(i) *$D = 0$,*
(ii) *$D = K$, or*
(iii) *C is hyperelliptic and $D = mg^1_2$.*

The line in the diagram above illustrates the result we stated. For proofs of the boundary cases we refer to ACGH.

Before proving Clifford's Theore, let us introduce some notation.

Definition 10.2. Consider $Dc \subset |D|$ a g^r_d and $\mathcal{E} \subset |E|$ a g^s_e. Let $\mathcal{D} + \mathcal{E}$ denote the subspace of $|D + E|$ spanned by divisors of the form $D' + E'$ for $D' \in \mathcal{D}$ and $E' \in \mathcal{E}$.

For any two line bundles L and M there is a map

$$\mathrm{H}^0(L) \otimes \mathrm{H}^0(M) \longrightarrow \mathrm{H}^0(L \otimes M).$$

In the language of line bundles, given $V \subset \mathrm{H}^0(L)$, $W \subset \mathrm{H}^0(M)$, we are looking at the image of $V \otimes W$ in $\mathrm{H}^0(L \otimes M)$.

Here is a general fact about bilinear maps of vector spaces over an algebraically closed field (e.g., $\mathbb{C}$). Note that this statement is not true over $\mathbb{R}$.

Lemma 10.3. *Let $\varphi\colon V \otimes W \to U$ be a linear map. If there exist $v \in V$ an $w \in W$ such that $\varphi|_{\langle v\rangle \otimes W}$ and $\varphi|_{V\otimes\langle w\rangle}$ are injective, then*

$$\dim \operatorname{Im}\varphi \geq \dim V + \dim W - 1.$$

Then we deduce the following.

Lemma 10.4. *Any two linear series $\mathcal{D}$ and $\mathcal{E}$ satisfy*

$$r(\mathcal{D}+\mathcal{E}) \geq r(\mathcal{D}) + r(\mathcal{E}).$$

Remark 10.5. The statement $r(\mathcal{D}) \geq r$ is equivalent to saying that for all $p_1, \ldots, p_r \in C$, there exists $D \in \mathcal{D}$ such that $p_1, \ldots, p_r \in D$. We can put equality $r(\mathcal{D}) = r$ if we take $p_1, \ldots, p_r$ general.

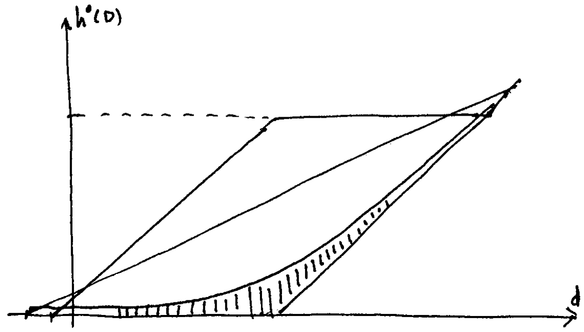
Now Clifford's Theorem follows easily.

Proof. Apply to $\mathcal{D} = |D|$ and $\mathcal{E} = |K-D|$. We get $h^0(D) + h^0(K-D) \leq g-1$. But Riemann Rock says $h^0(D) - h^0(K-D) = d-g+1$. Adding up, we conclude $h^0(D) \leq 1 + d/2$. $\square$

Remark 10.6. Points near Clifford's line occur only on hyperelliptic curves.

10.2. Various questions

While we were able to give a nice characterization, in some sense this was the wrong question as ask. We should have really asked what linear series exist on a general curve. In other words, for which r and d does every curve of genus g possess a g^r_d. This is the subject matter of *Brill-Noether theory.* The answer looks like a quadric.



Even further recall that very ample linear series are of greatest interest.

As a different question, we can ask what linear series occur that embed $C \hookrightarrow \mathbb{P}^r$ either regularly or biregularly. To be a bit more specific, let C be a smooth projective irreducible curve of genus g and D a divisor of degree d. If φ_D embeds C in $\mathbb{P}^r$, then what can we say about the possible values of (g,r,d)? So far we have been fixing g and asking about r, but the order can be changed. This leads us to a restatement of the previous question.

Question 10.7. If $C \subset \mathbb{P}^r$ is a smooth, irreducible, non-degenerate curve if degree d, how large can g be?

For example, if $r=2$, then the usual genus formula reads $g = \binom{d-1}{2}$.

If $r=3$, then we can list the following possibilities.

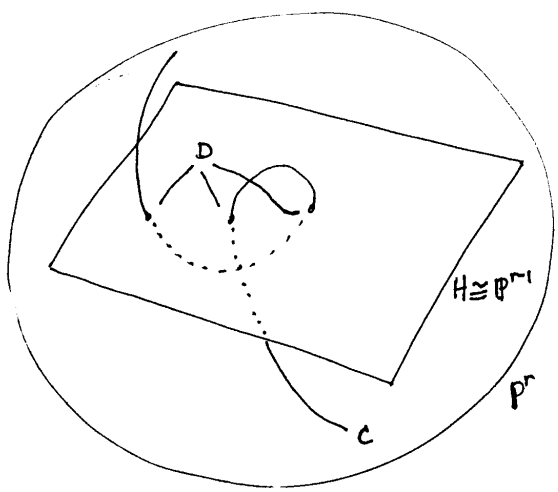
d	Possible values of g
3	0
4	0, 1
5	0, 1, 2
6	0, 1, 2, 3, 4
7	...

The cases $d = 4, 5$ are restricted by Clifford's Theorem. Actually, if $d \leq 5$, then D is non-special and $g = d - h^0(D) + 1$. The case $d = 6$ again can be handled using Clifford's Theorem. However, this is not helpful for $d \geq 7$.

10.3. Castelnuovo's approach

Let $C \hookrightarrow \mathbb{P}^r$ be a smooth irreducible non-degenerate curve, and D a hyperplane section of C. We will think of D as a divisor on C but also as a configuration of points in $\mathbb{P}^r$. Our aim is to bound $h^0(mD)$ from below. The point is that for $m \gg 0$, the divisor mD is non-special so Riemann-Roch applies, namely,

$$g = md - h^0(mD) + 1.$$



Onto the estimate, consider the restriction map

$$\mathrm{H}^0(\mathcal{O}_C(mD)) \xrightarrow{\rho_m} \mathrm{H}^0(\mathcal{O}_D(mD)) \cong \mathbb{C}^d.$$

We can identify the kernel of this map as $\mathrm{H}^0(\mathcal{O}_C((m-1)D))$, so the first row in the following diagram is exact.

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{H}^0(\mathcal{O}_C((m-1)D)) \arrow[r] & \mathrm{H}^0(\mathcal{O}_C(mD)) \arrow[r, "\rho_m"] & \mathrm{H}^0(\mathcal{O}_D(mD)) \\
 & & \mathrm{H}^0(\mathcal{O}_{\mathbb{P}^r}(m)) \arrow[u] \arrow[ur] &
\end{tikzcd}$$

We would like to study

$$\operatorname{rank} \rho_m = \#\{\text{linear conditions on a section of } \mathcal{O}_C(mD) \text{ to vanish at } p_1, \ldots, p_d\},$$

where $D = C \cap H = p_1 + \cdots + p_d$. It is easier to estimate the rank of the composite map

$$\mathrm{H}^0(\mathcal{O}_{\mathbb{P}^r}(m)) \longrightarrow \mathrm{H}^0(\mathcal{O}_D(mD)).$$

Therefore,

$$\operatorname{rank}\rho_m \geq \#\{\text{linear conditions on polynomials of degree } m \text{ in } \mathbb{P}^r \text{ to vanish at } p_1, \dots, p_d\}.$$

Note that this is independent of C. The simple estimate is $\min\{d, m+1\}$ but we can do better. The crucial result reads as follows.

Lemma 10.8 (General position Lemma)**.** *Let C be a smooth, irreducible, and non-degenerate curve in $\mathbb{P}^r$, $H \subset \mathbb{P}^r$ a hyperplane section, and $D = C \cap H = p_1 + \cdots + p_d$. If H is general, then the points of D are in* linear general position *in $H \cong \mathbb{P}^{r-1}$.*

Being in linear general position means that no r points are linearly dependent. For example, when $r = 3$, this means that not 3 are collinear.

Question 10.9. Say $p_1, \dots, p_d \in \mathbb{P}^n$ span $\mathbb{P}^n$ and they are in linear general position. How many conditions do they impose on hypersurfaces of degree m?

By definition, the number of conditions is $h_D(m)$ where $D = \sum_i p_i$. We claim

$$h_D(m) \geq \min\{d, mn+1\}.$$

Proof. Start by assuming $d \geq mn+1$, that is, $\min\{d, mn+1\} = mn+1$. Given $mn+1$ points of D, we claim we can find a hypersurface of degree m that contains all points with the exception of any given one q. For any q, group the remaining points in m sets of n. Each set of n spans a hyperplane, and their union is a hypersurface of degree m. By linear general position, none of the hyperplanes contains q, hence we get the desired lower bound. The same argument works if $d < mn+1$, we just need to add a few points in a smart way. $\square$

As surprising as it is, this bound happens to be sharp. For example, we get equality for any configuration D lying on a rational normal curve.

We have all ingredients to recover Castelnuovo's bound. Apply the Lemma to the points of a general hyperplane section. Then $n = r-1$, and

$$h^0(\mathcal{O}_C(mD) \geq h^0(\mathcal{O}_C((m-1)D)) + \min\{d, m(r-1)+1\}.$$

Assuming $d \gg 0$, we obtain the following inequalities for the first few values of m.

$$h^0(\mathcal{O}_C(D)) \geq r+1 = r-1+2$$

$$h^0(\mathcal{O}_C(2D)) \geq r+1+(2r-1) = 3r = 3(r-1)+3h^0(\mathcal{O}_C(D)) \quad \geq 3r+3(r-1)+1 = 6r-1 = 6(r-1)+4$$

Express $d = m_0(r-1)+\varepsilon+1$ where $0 \leq \varepsilon \leq r-2$. The pattern above continues until we get to

$$h^0(\mathcal{O}_C(m_0D)) \geq \binom{m_0}{2}(r-1)+m_0+1.$$

From there on the $\min\{d, m(r-1)+1\} = d$, so we get

$$h^0(\mathcal{O}_C((m_0+k)D)) \geq \binom{m_0}{2}(r-1)+m_0+1+kd.$$

Riemann-Roch applies for $k \gg 0$, so we deduce

$$\begin{aligned} g &= \deg((m_0+k)D) + h^0((m_0+k)D)+1 \\ &\leq (m_0+k)d - \left(\binom{m_0}{2}(r-1)+m_0+1+kd\right)+1 \\ &= \binom{m_0+1}{2}(r-1)+m_0\varepsilon. \end{aligned}$$

We will denote the bottom expression by

$$\pi(d,r) = \binom{m_0+1}{2}(r-1)+m_0\varepsilon,$$

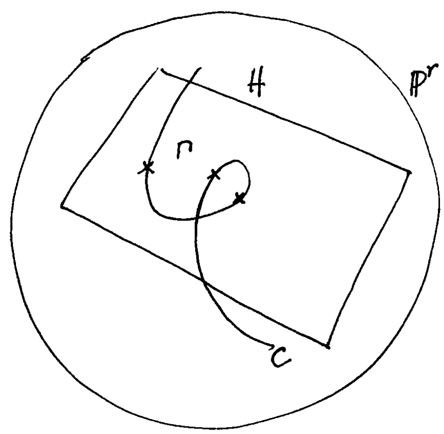
and this is an upper bound for the genus of C.

LECTURE 11

October 12, 2011

11.1. Castelnuovo's estimate

Let us recall the computation we started last time. We consider $C \hookrightarrow \mathbb{P}^r$ a smooth, non-degenerate, irreducible curve of degree d. The assumption C is embedded in $\mathbb{P}^r$ can be relaxed to a birational embedding but for our purposes this is more convenient. Let $H \subset \mathbb{P}^r$ be a general hyperplane, and denote $\Gamma = H \cap C$.



The basic estimate goes as follows.

$$\begin{aligned} r(k\Gamma) - r((k-1)\Gamma) &= \#\{\text{conditions imposed by } \Gamma \text{ on } \mathrm{H}^0(k\Gamma)\} \\ &\geq \#\{\text{conditions imposed by } \Gamma \text{ on polynomials of degree } k\} \\ &\geq \min\{d, k(r-1)+1\}. \end{aligned}$$

The only non-trivial step is the last one which relies on the fact Γ is in general linear position for general H. For small k the minimum above is achieved by $k(r-1)+1$, so we get the following.

$$\begin{aligned} h^0(\Gamma) &\geq r+1 \\ h^0(2\Gamma) &\geq r+1+2r-1 = 3r \\ h^0(3\Gamma) &\geq 3r+3r-2 = 6r-2 \\ &\vdots \end{aligned}$$

This behavior stops around $m = \lfloor (d-1)/(r-1) \rfloor$. Set $d - 1 = m(r-1) + \varepsilon$ for $0 \le \varepsilon \le r-2$. Then the sequence about continues as follows.

$$h^0(m\Gamma) \ge \binom{m+1}{2}(r-1) + m + 1$$
$$\vdots$$
$$h^0((m+n)\Gamma) \ge \binom{m+1}{2}(r-1) + m + 1 + nd.$$

For $n \gg 0$, we can apply Riemann-Roch to estimate the genus g.

$$\begin{aligned}
g &= (m+n)d - \mathrm{h}^0((m+n)\Gamma) + 1 \\
&\le (m+n)d - \binom{m+1}{2}(r-1) - m - 1 - nd + 1 \\
&= md - \binom{m+1}{2}(r-1) - m \\
&= m^2(r-1) + m\varepsilon + n - \binom{m+1}{2}(r-1) - m \\
&= \binom{m}{2}(r-1) + m\varepsilon.
\end{aligned}$$

For convenience, we will denote

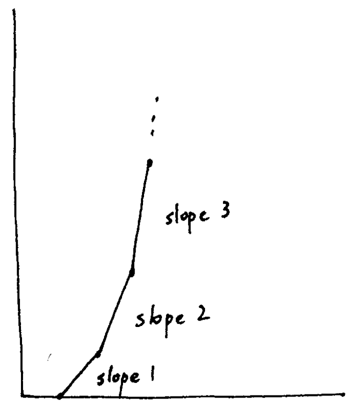
$$\pi(d,r) = \binom{m}{2}(r-1) + m\varepsilon.$$

There are several things to carry out at this point. First we need to study the given estimate. We also need to return and prove the crucial bound we used. Let us start with tabulating the behavior of π.

d	m	ε	π
r	1	0	0
$r+1$	1	1	1
$\vdots$			
$2r-2$	1	$r-2$	$r-2$
$2r-1$	2	0	$r-1$
$2r$	2	1	$r+1$
$2r+1$	2	2	$r+3$
$\vdots$			
$3r-3$	2	$r-2$	$3r-5$
$3r-2$	3	0	$3r-3$
$3r-1$	3	1	$3r$
$3r$	3	2	$3r+2$
$\vdots$			

In the range $r \le d \le 2r-1$, Clifford's Theorem applies implying Γ is non-special on C, so $g \le d - h^0(\Gamma) + 1 \le d - r$. The next case $d = 2r$ yields *canonical curves.*

Note that π starts by increasing in steps of 1 until $r = 2r-1$. After that it increases in steps of 2, and then in 3. We can plot its values as follows.



Asymptotically,

$$\pi \sim \frac{d^2}{2(r-1)}$$

for fixed r as $d \to \infty$.

Now that we have investigated the behavior of π, we can proceed and check the basic estimate

$$h_\Gamma(l) \geq \min\{d, k(r-1)+1\}.$$

Once that is done, we can also show the given bound is sharp. After that, we can provide a criterion telling us when curves have maximal genus. As a consequence of our investigations, we can prove several useful results such as Noether's Theorem and Enriques' Theorem.

As we described last time, Castelnuovo's bound depends on the following result.

Lemma 11.1. *Let $C \subset \mathbb{P}^r$ be a irreducible non-degenerate curve, $H \subset \mathbb{P}^r$ a general hyperplane, and $\Gamma = H \cap C$. Then Γ is in general linear position in $H \cong \mathbb{P}^{r-1}$.*

This depends on a general result on monodromy groups.

11.2. Monodromy in general

Let $\pi \colon X \to Y$ be surjective and generically finite of degree d where X and Y are smooth projective varieties. Then there exists a Zariski open $U \subset Y$ such that for all $p \in U$, the fiber $\pi^{-1}(p)$ consists of d points exactly. We can then replace X with $\pi^{-1}(U)$ and study an honest degree d map $X \to U$. Then $\pi_1(U, p)$ acts on $\pi^{-1}(p)$. The subgroup $G \subset S_{\pi^{-1}(p)}$ which is realized is called the *monodromy group* of the cover $X \to U$, denoted $G(X/U)$. It coincides the group of deck transformations if and only if the covering is normal (Galois).

We can also recover this group algebraically. Consider the following inclusion of fields.

$$\begin{tikzcd}
\overline{K(X)} \arrow[d, dash] \arrow[dr, dash] & \\
K(X) \arrow[d, dash] \arrow[r, dash] & L \\
K(Y) &
\end{tikzcd}$$

Here $K(X)$ and $K(Y)$ denote the function fields of X and Y respectively. Let L be the Galois normalization of $K(X)$ in $\overline{K(X)}$. We can then identify the monodromy group as

$$G(X/U) = \mathrm{Gal}(L/K(X)).$$

There are also some geometric characterizations of the monodromy group G. For example, the action of the monodromy group G is transitive on the elements of a fiber if and only if X is connected. Since X is smooth, it suffices to show it is irreducible. The action of G is *twice-transitive* (acts transitively on pairs of distinct elements of a fiber) if and only if the monodromy group of $X \times_U X \setminus \Delta \to U$ is transitive, that is, if and only if $X \times_U X \setminus \Delta$ is irreducible. Note that it is crucial to remove the diagonal Δ since it is an separate irreducible component consisting of pairs of repeating points. In general, define

$$X_U^{(r)} = X \times_U \cdots \times_U X \setminus \Delta = \left\{ (q, p_1, \dots, p_r) \;\middle|\; \begin{matrix} q \in U \\ p_i \in X \text{ distinct} \end{matrix} \right\}.$$

Then the action of G is r-transitive if and only if $X_U^{(r)}$ is irreducible.

11.3. The Uniform Position Theorem

Theorem 11.2 (Uniform Position Theorem)**.** *Let $C \subset \mathbb{P}^r$ be degree d irreducible non-degenerate curve, and $C^* \subset \mathbb{P}^{r*}$ the dual hypersurface (the set of tangent hyperplanes). Set $U = \mathbb{P}^{r*} \setminus C^*$ and construct*

$$X = \{(H, p) \mid p \in H \cap C\} \subset U \times C.$$

Then the projection map $X \to U$ is a covering with monodromy group $G = G(X/U)$ the complete symmetric group on d elements S_d.

Remark 11.3. This is true only in characteristic 0.

Proof. We start by showing that G is twice-transitive. Consider

$$X_U^{(2)} = X \times_U X \setminus \Delta = \{(H, p, q) \mid p, q, \in H \cap C, p \neq q\} \subset U \times C \times C.$$

To see it is irreducible, consider the projection

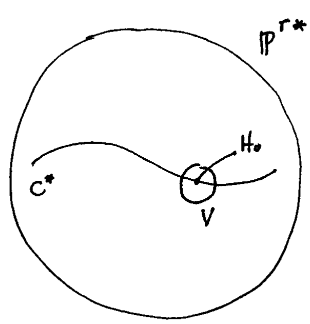
$$X_U^{(2)} = X \times_U X \setminus \Delta \longrightarrow C \times C \setminus \Delta.$$

The non-empty fibers are projective spaces of dimension $r-2$ and $C \times C \setminus \Delta$ is irreducible, hence $X_U^{(2)}$ is irreducible and G is twice-transitive.

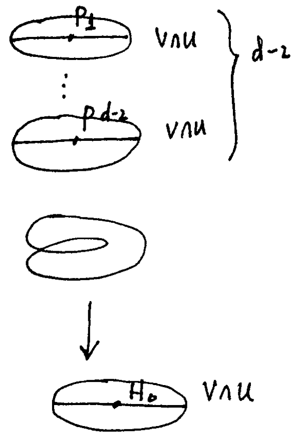
Next, we would like to show that G contains a transposition. Choose a hyperplane $H_0 \in \mathbb{P}^{r*}$ simply tangent to C (this step requires characteristic 0). This means that

$$H_0 \cap C = 2p + p_1 + \cdots + p_{d-2}.$$

Choose a small analytic neighborhood V of H_0 in $\mathbb{P}^{r*}$.



Then $X_{V \cap U}$ consists of $d-2$ copies of $U \cap V$ and one component which is connected but maps 2-to-1 onto $V \cap U$.



The projection

$$\begin{array}{c} \overline{X} = \{(H,p) \mid p \in H \cap C\} \subset \mathbb{P}^{r*} \times C \\ \Big\downarrow \\ \mathbb{P}^{r*} \end{array}$$

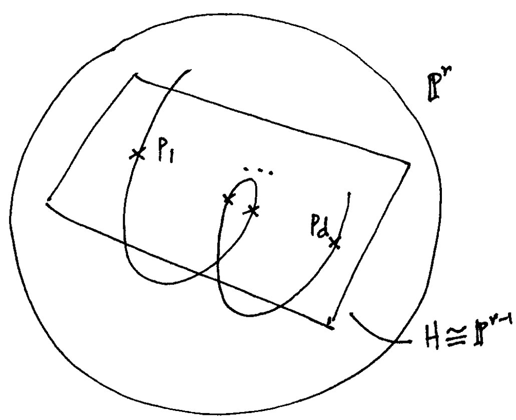
is smooth over H_0, so we can induce a transposition on the two points which lie in the same connected component of $X_{V\cap U}$. This concludes the proof of the Uniform Position Theorem. $\square$

October 14, 2011

12.1. General linear position

The following is the only standing claim from Castelnuovo's Theorem we have not proved.

Proposition 12.1. *Let $C \subset \mathbb{P}^r$ be an irreducible, non-degenerate curve and $H \subset \mathbb{P}^r$ a general hyperplane. Then $H \cap C$ is a collection of points in general linear position, i.e., there are no r linearly dependent points.*



Proof. Consider

$$U = \mathbb{P}^{r*} \setminus C^* = \{\text{hyperplane } H \subset \mathbb{P}^r \text{ such that } H \text{ intersects } C \text{ transversely}\}.$$

We can then study the following d-sheeted cover.

$$\begin{array}{c} X = \{(H,p) \mid p \in H \cap C\} \subset U \times C \\ \downarrow \\ U \end{array}$$

Last time we proved the monodromy group of this cover is S_d, the complete symmetric group on the points of a given fiber. Equivalently,

$$X_U^{(r)} = \{(H, p_1, \ldots, p_r) \mid p_i \text{ distinct in } H \cap C\} \subset U \times C^r$$

is irreducible. Define

$$Z = \{(H, p_1, \ldots, p_r) \mid p_1, \ldots, p_r \text{ are linearly dependent}\} \subset X_U^{(r)}.$$

Since being linearly dependent is a determinental condition, it follows that Z is closed in $X_U^{(r)}$. The point it Z is proper in $X_U^{(r)}$ since we can exhibit a point in the complement of Z. Then $\dim Z < \dim X_U^{(r)}$, and $\operatorname{Im} Z \subset U$ is contained in a proper subvariety which concludes our claim. $\square$

Remark 12.2. This statement holds more generally for projective varieties of arbitrary dimension.

12.2. Application to projective normality

Let $C \subset \mathbb{P}^r$ be a smooth curve.

Definition 12.3. We say that C is *projectively normal* if hypersurfaces of degree m in $\mathbb{P}^r$ cut out a complete linear series on C for all $m > 0$.

There are several equivalent formulations of this statement.

(i) If $D \subset C$ is a divisor of degree m satisfying $D \sim mH$ for a hyperplane section H, then there exists a degree m hypersurface $Z \subset \mathbb{P}^r$ such that $Z \cap C = D$.

(ii) The restriction map
$$\mathrm{H}^0(\mathcal{O}_{\mathbb{P}^r}(m)) \to \mathrm{H}^0(\mathcal{O}_C(m))$$
is surjective for all $m > 0$.

(iii) For all $m > 0$, we have $\mathrm{H}^0(\mathcal{I}_{C,\mathbb{P}^r}(m)) = 0$.

(iv) If $S = \overline{p, C}$ denotes the cover over C in $\mathbb{P}^{r+1}$, then the local ring $\mathcal{O}_{S,p}$ is Cohen-Macaulay.

Let us assume C is smooth, irreducible and non-degenerate. Back to Castelnuovo's bound, denote
$$E_K = \mathrm{H}^0(\mathcal{O}_{\mathbb{P}^r}(k))|_C \subset \mathrm{H}^0(\mathcal{O}_C(k)).$$
We showed that
$$\dim E_k - \dim E_{k-1} \geq \min\{d, k(r-1)+1\},$$
and then we used this to find a lowed bound for $\dim E_k$. Eventually, this implies $g \leq \pi(d,r)$. If $g = \pi(d,r)$, that is C has maximal genus, then we must have equalities at all positions above. We can restate this by saying that a curve of maximal genus is projectively normal.

For example, if $C \subset \mathbb{P}^{g-1}$ is a canonical curve (of degree $d = 2g-2$), then it has maximal genus, hence is projectively normal. It follows that every quadratic differential on C is a quadratic polynomial in the holomorphic differentials $\omega_1, \ldots, \omega_g$.

12.3. Sharpness of Castelnuovo's bound

To show Castelnuovo's bound is sharp, we would like to construct curves of degree d in $\mathbb{P}^r$ having maximal genus $\pi(d,r)$. If H is a general hyperplane, then $\Gamma = H \cap C$ should satisfy $h_\Gamma(k) = \min\{d, k(r-1)+1\}$. For example, this condition is always satisfied if $\Gamma \subset H \cong \mathbb{P}^{r-1}$ lies on a rational normal curve $B \subset \mathbb{P}^{r-1}$.

When $r = 3$, one way to achieve this is by picking C on a quadratic surface Q. For simplicity, let us assume Q is smooth and C is of type (a,b) where $d = a+b$. Then
$$g(C) = (a-1)(b-1) \leq \begin{cases} (k-1)^2 & \text{if } d = 2k, \\ k(k-1) & \text{if } d = 2k+1. \end{cases}$$
These bounds are respectively achieved by curves of type (k,k) and $(k,k+1)$ respectively. It also happens that
$$\pi(d,r) = \begin{cases} (k-1)^2 & \text{if } d = 2k, \\ k(k-1) & \text{if } d = 2k+1. \end{cases}$$
This argument shows the bound is sharp when working in $\mathbb{P}^3$.

In general, we are looking for $C \subset \mathbb{P}^r$ such that $H \cap C$ lies on a rational normal curve. We would like to find a surface $S \subset \mathbb{P}^r$ such that a general hyperplane section is a rational normal curve.

12.4. Varieties of minimal degree

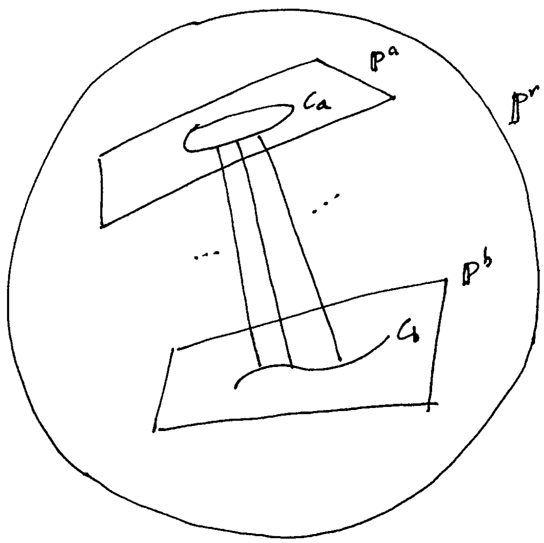
Before attacking the specific problem we encountered above, let us consider a slightly more general situation. We would like to know what is the smallest possible degree of an irreducible, non-degenerate variety $X \subset \mathbb{P}^r$ of dimension k. When $k = 1$, the answer is r (e.g., the rational normal curve can be shown to achieve the minimum). One step further, we would like to study the case of surfaces ($k = 2$).

Before we proceed, let us recall a general statement whose proof will be omitted. Let $S \subset \mathbb{P}^r$ be a irreducible non-degenerate surface, and consider a hyperplane section $C = S \cap H$ for $H \subset \mathbb{P}^r$. Then for general H, the C is an irreducible non-degenerate curve and $\deg S = \deg S \geq r-1$. For varieties X of higher dimension k, we get a bound $\deg X \geq r-k+1$.

The nature of the problem we are studying is such that if we can solve it for surfaces, then we should be able to solve it in higher-dimensions too. The generalization mainly consists of improving the notation so it can handle a broader case. For this reason, we will start by focusing on surfaces.

Let us start by choosing two complimentary linear subspaces $\mathbb{P}^a, \mathbb{P}^b \subset \mathbb{P}^r$. A simple dimension count tells is $a+b = r-1$. Next, choose a rational normal curve in each: $C_a \subset \mathbb{P}^a$ and $C_b \subset \mathbb{P}^b$. Note that each of these curves is abstractly isomorphic to $\mathbb{P}^1$, so we can choose an isomorphism between them $\varphi\colon C_a \to C_b$. Consider the surface

$$S = \bigcup_{p \in C_a} \overline{p, \varphi(p)}.$$



Let us calculate the degree of S. Start by choosing a general hyperplane H which contains $\mathbb{P}^a$, and look at the intersection $H \cap S$. By construction $C_a \subset H \cap S$. Note that $H \cap \mathbb{P}^b$ is a general hyperplane in $\mathbb{P}^b$, so it intersects C_b in b points, say $p_1, \ldots, p_b$. It follows that $H \cap S$ contains the lines of S through $p_1, \ldots, p_b$. The converse is also true. If $H \cap S$ contains another point of $S \setminus C_a$, then it contains the entire line passing through this point. It follows that $H \cap S$ is the union of C_a with b lines, so $\deg H \cap S = a+b = r-1$. (One also needs to check each of these components occur with multiplicity 1, but that is left as an exercise.)

We can modify our construction slightly so it accounts for cones over a rational normal curve too. Choose parametrizations $\varphi_a\colon \mathbb{P}^1 \to C_a$ and $\varphi_b\colon \mathbb{P}^1 \to C_b$. Then

$$S = \bigcup_{p \in \mathbb{P}^1} \overline{\varphi_a(p), \varphi_b(p)}.$$

When $a = 0$, take $C_a = \mathbb{P}^0 = \{\text{point}\}$ to get a cone over the curve C_b, and similarly for $b = 0$. The surface S, denoted $X_{a,b}$, is called a *rational normal (surface) scroll*.

Remark 12.4. The projective equivalence class of $X_{a,b}$ depends only on a and b. This is to say that all choices we made in the construction above can be adjusted for via a linear isomorphism of the ambient projective space $\mathbb{P}^r = \mathbb{P}^{a+b+1}$.

There are two other descriptions of $X_{a,b}$ worth mentioning.

(i) We can construct $X_{a,b} = S$ as the projectivization of the rank 2 bundle $\mathcal{O}(a) \oplus \mathcal{O}(b)$ on $\mathbb{P}^1$. Note we are using the post-Grothendieckian projectivization; otherwise $X \cong \mathbb{P}(\mathcal{O}(-a) \oplus \mathcal{O}(-b))$ which is less aesthetically pleasant. The embedding $X_{a,b} \hookrightarrow \mathbb{P}^{a+b+1}$ corresponds to the complete linear series $|\mathcal{O}_{X_{a,b}}(1)|$. As a corollary of this construction, we conclude that $X_{a,b} \cong X_{a',b'}$ if and only if $|a-b| = |a'-b'|$.

(ii) Recall that rational normal curves are determinenral varieties. Pick a general $2 \times r$ matrix

$$A = \begin{pmatrix} L_1 & \cdots & L_r \\ M_1 & \cdots & M_r \end{pmatrix}$$

of linear forms on $\mathbb{P}^r$. Then

$$\Phi = \{p \in \mathbb{P}^r \mid \operatorname{rank} A(p) \leq 1\}$$

is a rational normal curve. Note that the condition $\operatorname{rank} A(p) = 1$ is equivalent since rank 0 cannot occur generally.

Next, consider a similar matrix

$$A = \begin{pmatrix} L_1 & \cdots & L_{r-1} \\ M_1 & \cdots & M_{r-1} \end{pmatrix}$$

of size $2 \times (r-1)$. Then

$$S = \{p \in \mathbb{P}^r \mid \operatorname{rank} A(p) = 1\}$$

is a rational normal scroll. To see this note that for all $[\alpha, \beta] \in \mathbb{P}^1$, the equations

$$\alpha L_1 + \beta M_1 = \cdots = \alpha L_{r-1} + \beta M_{r-1} = 0$$

define a line $L_{[\alpha,\beta]} \subset S$. Varying $[\alpha, \beta] \in \mathbb{P}^1$, we cover S with lines. There seems to be no easy way to read off a and b from the linear forms.

Remark 12.5. There is a different notion called *1-generic* which is more suitable for our choice of linear forms above.

Here is the reason why we are interested in rational normal scrolls. A proof of this statement will be posted on the course website.

Theorem 12.6. *Let $S \subset \mathbb{P}^r$ be an irreducible non-degenerate surface of minimal degree $r-1$. Then S is*

(i) *a rational normal scroll (in the extended definition including cones), or*

(ii) *the quadratic Veronese surface $\nu_2(\mathbb{P}^2) \subset \mathbb{P}^5$.*

Our next goal is to learn about the theory of divisors on scrolls so we can apply it in the case we are interested in.

Before we close, let us briefly make a note about quadric hypersurfaces. Given an irreducible non-degenerate variety $X \subset \mathbb{P}^r$ of dimension k, we would like to know how many independent quadrics can contain X.

Remark 12.7. As a form of motivation, we are trying to study the "size" of varieties. Degree is a form of size measurement, so it is naturally interesting to ask about varieties of minimal degree. Similarly, the more quadrics which contain a variety, the smaller its size. Again, we are interested in the largest number of such independent quadrics, hence, this is again a question about varieties of "minimal size".

The answer to the question we posed above is

$$\binom{r+1-k}{2}.$$

To see this, consider a general hyperplane $H \subset \mathbb{P}^r$ and set $Y = H \cap X$. Then there is a short exact sequence

$$0 \longrightarrow \mathcal{I}_{X,\mathbb{P}^r}(1) \longrightarrow \mathcal{I}_{X,\mathbb{P}^r}(2) \longrightarrow \mathcal{I}_{Y,H \cong \mathbb{P}^{r-1}}(2) \longrightarrow 0.$$

Since X is non-degenerate $\mathrm{H}^0(\mathcal{I}_{X,\mathbb{P}^r}(1)) = 0$, so the natural map

$$\mathrm{H}^0(\mathcal{I}_{X,\mathbb{P}^r}(2)) \longrightarrow \mathrm{H}^0(\mathcal{I}_{Y,\mathbb{P}^{r-1}}(2))$$

is an injection. This allows us to reduce to the case $r = 1$ which is easy to handle.

It is possible to characterize the list of varieties which achieve this bound as we did above, and the list is actually almost identical.

LECTURE 13

October 17, 2011

13.1. Scrolls

Given a smooth, irreducible, non-degenerate curve $C \subset \mathbb{P}^r$, we know that

$$g(C) \leq \pi(d,r) = \binom{m}{2}(r-1) + m\varepsilon,$$

where $d-1 = m(r-1)+\varepsilon$ for $0 \leq \varepsilon \leq r-2$. To find curves of maximal genus $\pi(d,r)$, we would like to look for curves $C \subset \mathbb{P}^r$ whose hyperplane section Γ lies on a rational normal curve. If $d > 2r-2$, then $Q \subset \mathbb{P}^{r-1}$ containing Γ cut our a rational normal curve. It follows that the quadrics in $\mathbb{P}^r$ containing C cut out a surface $S \subset \mathbb{P}^r$ of degree $d-1$. We are then looking for curves C on a surface $S \subset \mathbb{P}^r$ of minimal degree; with one exception, all such surfaces are scrolls.

Start with two integers $0 \leq a \leq b$, and consider two complimentary subspaces $\mathbb{P}^a, \mathbb{P}^b \subset \mathbb{P}^r$ where $r = a+b+1$. Pick rational normal curves $C_a \subset \mathbb{P}^a$, $C_b \subset \mathbb{P}^b$, and an isomorphism between them. Joining the corresponding points of C_a and C_b with lines, we get a surface $X_{a,b}$. The curve C_a is called the *directrix* of $X_{a,b}$. Let us start by discussing a few facts about scrolls.

Proposition 13.1. *For $a > 1$, there is an isomorphism $X_{a,b} \cong X_{a-1,b-1}$.*

This fact follows immediately by projecting from a line in the ruling. In particular,

$$X_{a,a} \cong X_{1,1} \cong \mathbb{P}^1 \times \mathbb{P}^1$$

because $X_{1,1}$ is a quadric smooth surface in $\mathbb{P}^3$. Similarly,

$$X_{a,a+1} \cong X_{1,2} \cong \operatorname{Bl}_p \mathbb{P}^2.$$

Here is a way to argue this. By blowing up $\mathbb{P}^2$ at a point p, we replace it with the exceptional divisor $E \cong \mathbb{P}^1$. The lines passing through p in $\mathbb{P}^2$ become disjoint; their proper transform in $\operatorname{Bl}_p \mathbb{P}^2$ records the slope at which the line approaches p. Choose a second line L in $\mathbb{P}^2$ not passing through p. Lines joining p with a point of L cover $\mathbb{P}^2$ and they are mutually disjoint, with the exception of p. The proper transform of L in $\operatorname{Bl}_p \mathbb{P}^2$ is not modified. The transforms of lines joining p and a point in L become disjoint relating $\operatorname{Bl}_p \mathbb{P}^2$ to the construction of $X_{a,a+1}$. It is possible to make this argument formal by considering the space of conics in $\mathbb{P}^2$ passing through p.

Our next goal is to understand the intersection theory of scrolls. Thinking of $X_{a,b}$ as a $\mathbb{P}^1$-bundle over $\mathbb{P}^1$, each fiber defines a Cartier divisor, and these are linearly equivalent by the construction of $X_{a,b}$. We will denote the *fiber class* by f. We constructed $X_{a,b}$ as a surface embedded in projective space, so we will also consider the associated *hyperplane class* h.

Proposition 13.2. *The Picard group $\operatorname{Pic}(X_{a,b})$ is freely generated by f and h.*

Proof. Start by removing a representative of the hyperplane class h from $X_{a,b}$. What is left is an $\mathbb{A}^1$-bundle over $\mathbb{P}^1$. Removing a fiber produces an $\mathbb{A}^1$-bundle over $\mathbb{A}^1$ which is trivial, hence isomorphic to $\mathbb{A}^2$. Since $\operatorname{Pic}(\mathbb{A}^2) = 1$, we conclude that h and f generate $\operatorname{Pic}(X_{a,b})$.

Let us compute the intersection pairing on $\operatorname{Pic}(X_{a,b})$. Since all fibers are linearly equivalent and disjoint, it follows that $h \cdot h = 0$. A similar observation implies that $h \cdot f = 1$. Finally, the product $h \cdot h$ is the number of points of the intersection between $X_{a,b}$ and two hyperplanes; this can be interpreted as the degree of the surface $r-1$. These computations imply that h and f are linearly independent and not torsion, hence $\mathrm{Pic}(X_{a,b}) = \mathbb{Z}\langle h, f\rangle$.

Alternatively, one can perform this computation by appealing to general facts about Chow rings of projective bundles. $\square$

Recall that we have two curves C_a and C_b sitting in $X_{a,b}$ and their classes are also of interest. Pick a general hyperplane H containing C_a. Its intersection with $X_{a,b}$ is the union of C_a and b lines of the ruling, hence $C_a \cdot f = 1$ and $C_a \cdot h = a$. We can them compute $C_a \sim h-(r-1-a)f = h-bf$. Similarly, $C_b = g-af$. Note that $C_a \cdot C_a = a-b \leq 0$ and $C_b \cdot C_b = b-a \geq 0$, so the two curves have fundamentally different behavior when $a<b$. Actually, if $a<b$, other than lines, C_a is the unique curve on $X_{a,b}$ of degree $\leq a$.

Let us compute the canonical class K of $X_{a,b}$. Start by writing $K = \alpha h + \beta f$. We will use adjunction applied to representatives of f and h. Each fiber is isomorphic to $\mathbb{P}^1$, so its canonical class has degree -2. Adjunction then tells us

$$-2 = (K+f)|_f = (K+f)\cdot f = (\alpha h + \beta f)\cdot f + f \cdot f = \alpha.$$

Similarly, a hyperplane section is a rational normal curve, again isomorphic to $\mathbb{P}^1$. We get

$$-2 = (K+h)|_h = (K+h)\cdot h = (\alpha h + \beta f)\cdot h + h\cdot h = \beta - (r-1),$$

hence $\beta = r-3$. We just proved that

$$K = -2h + (r-3)h.$$

It is possible to derive the same result applying adjunction to C_a and C_b.

Remark 13.3. If $r=3$, then $X_{a,b}$ is a smooth quadric surface whose canonical class is $-2h$. This agrees with the computation above.

Our next goal is to derive the degree and genus of a curve $C \subset X_{a,b}$ from its class. Assume $C = \alpha h + \beta f$. The degree is given by the intersection of C with a hyperplane, that is,

$$\deg C = C \cdot h = \alpha + (r-1)\beta.$$

To find the genus, start by writing the statement of adjunction:

$$\begin{aligned}(K+C)|_C &= (K+C)\cdot C = ((\alpha-2)h + (\beta+r-3)f)\cdot(\alpha h + \beta f)\\ &= (\alpha-2)\alpha(r-1) + (\alpha-2)\beta + (\beta+r-3)\alpha\\ &= \alpha(\alpha-1)(r-1) + 2\alpha\beta - 2\alpha - 2\beta\\ &= 2\binom{\alpha}{2}(r-1) + 2(\alpha-1)(\beta-1).\end{aligned}$$

The genus is half this number, hence

$$\mathrm{genus}(C) = \binom{\alpha}{2}(r-1) + (\alpha-1)(\beta-1).$$

13.2. Curves on scrolls

Our next goal is to fix a given degree d of C and maximize its genus. As before, let us write $d = m(r-1)+\varepsilon+1$. Take

$$\alpha = m+1, \qquad \beta = \varepsilon - r + 2.$$

In this case the degree turns out to be $m(r-1)+\varepsilon+1 = d$, and the genus is $\binom{m}{2}(r-1)+m\varepsilon$. It is easy to show this is indeed the maximum possible genus.

It remains to answer whether this class can be represented by a smooth curve. While one can investigate in detail all possibilities for α and β under which $\alpha h + \beta f$ is smoothly represented, we will only list a few of the cases here:

(i) $\alpha = 0$ and $\beta \geq 0$ (a collection of β disjoint lines),
(ii) $\alpha > 0$ and $\beta \geq -a\alpha$,

(iii) $\alpha = 1$ and $\beta = -b$ (the curve C_b is a representative).

For our purposes it suffices to state a weaker result.

Lemma 13.4. *If $\alpha > 0$ and $\beta \geq -\alpha a$, then $\alpha h + \beta f$ is represented by a smooth curve (not necessarily irreducible).*

PROOF. We will use Bertini's Theorem, namely, if a linear series has no basepoints, then the general member is that linear series is smooth. Write $\alpha h + \beta f = \alpha C_b + (\beta + a\alpha) f$. It is possible to check that if the coefficients $\beta + a\alpha$ is non-negative, then the associated linear series has no basepoints. This follows from observing that C_b moves in a linear series without basepoints. We omit the remaining details. □

It is possible to deduce there are irreducible smooth representatives under the hypotheses of the result above. To show this one can assume there are two smooth curves whose classes add up to $\alpha h + \beta f$, and then demonstrate they have to intersect.

Remark 13.5. Bertini's Theorem is commonly used in this setting. We start by exhibiting a singular or irreducible member of a basepoint-free linear series, and then conclude there is a smooth linearly equivalent representative.

Remark 13.6. When $r = 5$ and $d = 2k$ is even, then the curve C is the image of a degree k plane curve under the Veronese embedding.

Having exhibited extremal curves, our next goal is to show these are all such.

Theorem 13.7 (Castelnuovo's Lemma)**.** *Let $\Gamma \subset \mathbb{P}^n$ be a configuration of $d \geq 2n+3$ points in linear general position. If Γ imposes only $2n+1$ conditions on quadrics, then Γ lies on a rational normal curve.*

We will start by giving an indication of what goes into the proof of this result. One of the main ideas is the so called *Steiner construction.* Consider a sequence of n codimension 2 linear subspaces $\Lambda_1, \ldots, \Lambda_n \subset \mathbb{P}^n$. For each $1 \leq i \leq n$, let $\{H_t^i\}_{t \in \mathbb{P}^1}$ be a pencil of hyperplanes containing Λ_i. For general Λ_i, the intersection

$$H_t^1 \cap \cdots \cap H_t^n$$

should be a point for all $t \in \mathbb{P}^1$. It is an exercise that these points trace out a rational normal curve $\bigcup_{t \in \mathbb{P}^1} H_t^1 \cap \cdots \cap H_t^n$. More generally, consider $\ell \leq n$ pencils of hyperplanes $\{H_t^i\}_{t \in \mathbb{P}^1}$ where $1 \leq i \leq \ell$. If we assume that all intersections

$$H_t^1 \cap \cdots \cap H_t^\ell$$

are linear spaces of dimension $n - \ell$, then their union is a rational normal scroll of dimension $n - \ell + 1$. Let us give an indication how to show this.

PROOF. Define a map

$$\begin{array}{rccl} \varphi\colon & \mathbb{P}^1 & \longrightarrow & \mathbb{G}(n-\ell, n), \\ & t & \longmapsto & \bigcap_{i=1}^{\ell} H_t^i. \end{array}$$

If we consider the target Grassmannian sitting in a projective space via the Plücker embedding, this map has degree ℓ. In general, when we have a map to a Grassmannian, the degree of the variety swept out is the same as the degree of the map. It follows that the variety swept out by these linear spaces has degree ℓ which is minimal, hence it has to be a scroll. □

October 19, 2011

14.1. Castelnuovo's Lemma

Lemma 14.1. *Let $\Lambda^1, \ldots, \Lambda^k \subset \mathbb{P}^n$ be codimension 2 linear spaces and $k \leq n$. For each i, let $\{H_t^i\}_{t\in\mathbb{P}^1}$ denote the pencil of hyperplanes containing Λ^i. Assume that*

$$H_t^1 \cap \cdots \cap H_t^k \cong \mathbb{P}^{n-k}$$

for all $t \in \mathbb{P}^1$, i.e., the hyperplanes H_t^i intersect transversely. Then

$$X = \bigcup_{t\in\mathbb{P}^1} H_t^1 \cap \cdots H_t^k$$

is a rational normal scroll of dimension $n-k+1$.

Proof. Consider the map

$$\begin{array}{rcl} \mathbb{P}^1 & \longrightarrow & \mathbb{G}(n-k,n), \\ t & \longmapsto & H_t^1 \cap \cdots \cap H_t^k. \end{array}$$

Under the Plücker embedding, this gives a rational curve of degree k. It follows that X has degree k in $\mathbb{P}^n$, hence X is a scroll.

As an alternative approach, assume that

$$H^i_{t=[t_0,t_1]} = V(t_0 L^i + t_1 M^i).$$

Then

$$X = \left\{ \operatorname{rank} \begin{pmatrix} L^1 & \cdots & L^k \\ M^1 & \cdots & M^k \end{pmatrix} \leq 1 \right\},$$

and this is a scroll. $\square$

Lemma 14.2. *If $p_1, \ldots, p_{n+3} \in \mathbb{P}^n$ are points in linear general position, then $p_1, \ldots, p_{n+3}$ lie on a rational normal curve.*

Example 14.3. In $\mathbb{P}^2$, any 5 points such that no 3 are collinear lie on a conic. In $\mathbb{P}^3$, any 6 points in general linear position lie on a twisted cubic.

Proof. Let Λ^i be the span of $p_1, \ldots, \widehat{p}_i, \ldots, p_n$ which is a linear subspace of codimension 2. Choose a parametrization for each pencil H_t^i such that

$$p_{n+1} \in H_0^i, \qquad p_{n+2} \in H_\infty^i, \qquad p_{n+3} \in H_1^i,$$

and take

$$X = \bigcup_{t\in\mathbb{P}^1} H_t^1 \cap \cdots \cap H_t^n.$$

It is easy to see that $p_1, \ldots, p_n$ lie in X. For example, $p_1 \in H_t^2 \cap \cdots \cap H_t^n$ for all t, and there exists one value of t such that $p_1 \in H_t^1$. Finally, by the choice of parametrizations $p_{n+1}, p_{n+2}, p_{n+3} \in X$. $\square$

Lemma 14.4 (Castelnuovo's Lemma)**.** *Let $\Gamma = \{p_1, \ldots, p_d\} \subset \mathbb{P}^n$ be in linear general position and $d \geq 2n+3$. If Γ imposes only $2n+1$ conditions on quadrics (i.e., $h_\Gamma(2) = 2n+1$), then Γ is contained in a rational normal curve.*

Example 14.5. Let us see what $d \geq 2n+3$ is necessary. Take $n = 3$. Can we find 8 points on a net of quadrics? Yes. Take $\Gamma = Q_1 \cap Q_2 \cap Q_3$ for general Q_i. Then Γ imposes only 7 conditions on quadrics.

PROOF. Start by taking

$$\Lambda^i = \overline{p_1, \dots, \widehat{p_i}, \dots, p_n}.$$

Choose parametrizations of the pencils $\{H_t^i\}$ such that

$$p_{n+1} \in H_0^i, \qquad p_{n+2} \in H_\infty^i, \qquad p_{n+3} \in H_1^i$$

for all i. Then the corresponding rational normal curve X contains the first $n+3$ points. Next, take

$$\Lambda = \overline{p_{n+4}, \dots, p_{2n+2}}$$

and H_t the pencil of hyperplanes containing Λ normalized as before. Consider

$$Q_i = \bigcup_{t \in \mathbb{P}^1} H_t^i \cap H^i.$$

This is a scroll of codimension 1, hence a quadric (of rank 4). It is easy to see that Q_i contains

$$p_1, \dots, \widehat{p_i}, \dots, p_n, \dots, p_{2n+2}.$$

These are a total of $2n+1$ points from Γ. But $h_\Gamma(2) = 2n+1$, so $\Gamma \subset Q_i$. Note that for all $p \in \{p_{2n+3}, \dots, p_d\}$, the value of t such that $p \in H_t^i$ is the same t as $p \in H_t$. Therefore $p_{2n+3}, \dots, p_d \in X$. We still need to show that $p_{n+4}, \dots, p_{2n+2} \in C$. We can swap one point at a time in the choices we made above and the curve X does not change, hence proving all points lie on it. □

14.2. Curves of maximal genus

Let $C \subset \mathbb{P}^r$ be an irreducible, non-degenerate, smooth degree d curve. We know that $g(C) \leq \pi(d,r)$. If $g(C) = \pi(d,r)$ and $d \geq 2r+1$, then $\Gamma = C \cap H$ lies on a rational normal curve $B \subset \mathbb{P}^{r-1}$. It follows that

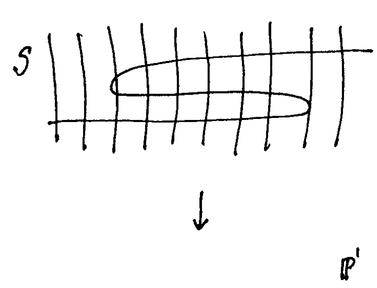
$$\bigcap_{\substack{\Gamma \subset Q \subset \mathbb{P}^{r-1},\\ Q \text{ a quadric}}} Q = B,$$

so

$$\bigcap_{\substack{C \subset Q \subset \mathbb{P}^{r},\\ Q \text{ a quadric}}} Q = X,$$

where X is a surface of minimal degree.

Say $C \subset \mathbb{P}^r$ is a canonical curve of genus $g = r+1$ and degree $d = 2g-2 = 2r$. Consider the X as above containing C. Suppose there exists $p \in X \setminus C$. Take a general hyperplane $H \subset \mathbb{P}^r$ containing p. In this circumstance, $p_1, \dots, p_{2r}, p$ are in linear general position. (This requires a monodromy argument.) Therefore these points impose only $2r-1$ conditions on quadrics. Castelnuovo's Lemma implies that $p_1, \dots, p_{2r}, p$ lie on a rational normal curve. Then C lies on a scroll $S \subset \mathbb{P}^{r-1}$ of degree $r-1$ or a Veronese surface in $\mathbb{P}^5$.



If C lies on a scroll, it is possible to show the class of C is $C \sim 3h-(r-3)f$ where h and f respectively denote the hyperplane and fiber classes on S. It follows that C is trigonal. In the case of a Veronese surface, C is the image of a smooth plane quintic. We arrived at the following result.

Theorem 14.6 (Enriques)**.** *Let $C \subset \mathbb{P}^{g-1}$ be a canonical curve. Then C is cut out by quadrics unless*

(i) *C is trigonal, or*
(ii) *C is a plane quintic.*

So far we have been studying curves of maximal genus, or, equivalently, of smallest Hilbert polynomial. It would be interesting to investigate curves of almost maximal genus, say $\pi(d,r)-1$ or $\pi(d,r)-2$. Recall that we considered $h_\Gamma(m)=\min\{d, m(r-1)+1\}$ which is achieved by Γ lying on a rational normal curve. The second best case Γ lies on a elliptic normal curve. Then $h_\Gamma(m) \sim \min\{d, mr\}$. In this case the conclusion is

$$g \leq \pi_1(d,r) \sim \frac{d^2}{2r}.$$

We can contrast this with

$$\pi(d,r) \sim \frac{d^2}{2r-2}.$$

The gap between $\pi(d,r)$ and $\pi_1(d,r)$ is quite significant and curves in it lie on scrolls.

LECTURE 15

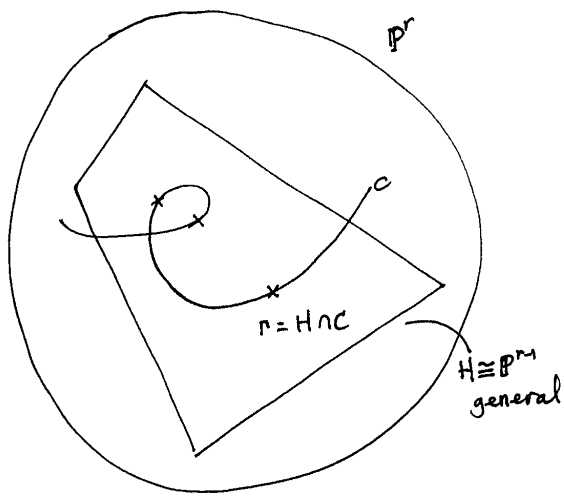
October 21, 2011

15.1. Beyond Castelnuovo

Today we will talk about extensions of Castelnuovo's theory. It is also a great opportunity to get acquainted with an interesting open problem.

In a 19th century mindset, all curves exist in projective space. Classifying curves starts with addressing their discrete invariants – genus and degree. Today we will focus on curves of large degree. In other words, we would like to follow Castelnuovo's argument and modify it accordingly.

Let $C \subset \mathbb{P}^r$ be a smooth, irreducible, non-degenerate curve of degree d.



We showed that the monodromy on $H \cap C = \Gamma$ as H varies is S_d. It follows that for any two subsets $\Gamma', \Gamma'' \subset \Gamma$ of equal cardinality satisfy

$$h_{\Gamma'}(m) = h_{\Gamma''}(m)$$

for all m. Collections of points Γ satisfying this condition are said to be in *uniform position*. Recall that our arguments used

$$X = \{(H, p) \mid p \in H \cap C\} \subset U \times C,$$

and we know that $X_U^{(k)}$ is irreducible for all $k \leq d$. The case $m = 1$ precisely means that Γ is in linear general position.

Castelnuovo's estimate says that if $\Gamma \subset \mathbb{P}^r$ are in linear general position (even better, in uniform position), then

$$h_\Gamma(m) \geq h_0(m) = \min\{d, mn - 1\},$$

and this bound is achieved for Γ lying on a rational normal curve B. Note that $h_B(m) = mn + 1$ for all m. Last time we shows that h_0 is the smallest Hilbert function a configuration of points may possess. What is the second smallest?

Before moving on, let us clarify some language. A curve $C \subset \mathbb{P}^r$ is called *k-normal* if the restriction map

$$\mathrm{H}^0(\mathcal{O}_{\mathbb{P}^r}(k)) \longrightarrow \mathrm{H}^0(\mathcal{O}_C(k))$$

is surjective. We call a curve *linearly normal* if it is 1-normal. Equivalently, C is not the projection of a curve in $\mathbb{P}^{r+f}$ for $f \geq 1$. We can see this by consulting the following diagram.

$$\begin{tikzcd} & \mathbb{P}^{r+f} \arrow[d, "\pi"] \\ C \arrow[ur, "|\mathcal{O}_C(1)|"] \arrow[r] & \mathbb{P}^r \end{tikzcd}$$

Note that a curve is always k-normal for $k \gg 0$. For curves up to canonical degree linear normality implies projective normality.

The second smallest Hilbert polynomial for an irreducible, non-degenerate curve $B \subset \mathbb{P}^n$ occurs for an elliptic normal curve of degree $n+1$. This is an elliptic curve B embedded via the complete linear series $|L|$ for some $L \in \operatorname{Pic}^{n+1}(B)$. This means that

$$h_B(m) = h^0(\mathcal{O}_B(L^m)) = m(n+1).$$

If Γ lies on an elliptic normal curve, then

$$h_\Gamma(m) = \begin{cases} m(n+1) & \text{if } d > m(n+1), \\ d-1 & \text{if } d = m(n+1), \\ d & \text{if } d < m(n+1). \end{cases}$$

Let us call this function $h_1(m)$. We conclude that if Γ does not lie on a rational normal curve then $h_\Gamma \geq h_1$. Then $g(C) \leq \pi_1(d,r)$ where $\pi_1(d,r)$ is begotten from h_1 by summing it up. Asymptotically $pi_1(d,r) \sim d^2/2r$. The conclusion, contrapositively, is that is $g(C) > \pi_1(d,r)$, then C lies on a rational normal surface scroll.

Example 15.1. In $\mathbb{P}^3$, any curve C of degree d with genus $g(C) > \pi_1(d,r) \sim d^2/2r$ must lie on a quadric. Then $g = (a-1)(d-a-1)$ for some a. There are fairly few such integers in the given range.

15.2. Analogues of Castelnuovo

If $\Gamma \subset \mathbb{P}^n$ is a configuration of points in uniform position of degree $d \geq 2n+5$ and $h_\Gamma(2) = 2n+2$, then Γ lies on an elliptic normal curve. For more details on this refer to the Montreal notes.

Conjecture 15.2. *Let $0 \leq \alpha \leq n-2$ be an integer and $\Gamma \subset \mathbb{P}^n$ a configuration of points in uniform position of degree $d \geq 2n+3+2\alpha$. If $h_\Gamma(2) \geq 2n+1+\alpha$, then Γ is contained in a curve $B \subset \mathbb{P}^n$ of degree $\leq n+\alpha$.*

Assuming this, let h_α be the minimal Hilbert function of Γ contained in a curve of degree $n+\alpha$ in $\mathbb{P}^n$. Then we derive a corresponding bound on the genus $g(C) \leq \pi_\alpha(d,r)$ for a curve $C \subset \mathbb{P}^r$ such that the general hyperplane section of C does not lie on a curve of degree $\leq r-2+\alpha$. In this way, we arrive at a sequence of bounds:

$$\pi(d,r) \sim \frac{d^2}{2(r-1)}, \quad \pi_1(d,r) \sim \frac{d^2}{2r}, \quad \cdots, \quad \pi_\alpha(d,r) \sim \frac{d^2}{2(r-1+\alpha)}, \quad \cdots, \quad \pi_{r-2}(d,r).$$

If $g(C) > \pi_\alpha(d,r)$, then C lies on a surface of degree $\leq r-2+\alpha$. For $g \leq \pi_{r-2}(d,r)$, every possible genus occurs.

Let us make a few remarks about such surfaces $S \subset \mathbb{P}^r$. Assume the degree of S is $r-1+\alpha < 2r-1$. For example, if $\alpha = 0$, this is a scroll. If $B = S \cap H$ is a general hyperplane section, then Clifford's Theorem implies that B is non-special, hence $g(B) \leq d-(r-1) = \alpha$. We have $B \cdot B = \deg S = r-1+\alpha$, so by adjunction

$$K_S \cdot B = 2g(B) - 2 - (r-1+\alpha) \leq 2\alpha - 2 - (r-1+\alpha) \leq \alpha - (r-1) < 0.$$

Therefore, no positive multiple of K_S admits sections. By the classification of surfaces, S is birational to $\mathbb{P}^1$ bundle over a curve. We can then classify the curves on such surfaces.

Example 15.3. Take $\alpha = 1$, so $S \subset \mathbb{P}^r$ is a surface of degree r. Then $S \cap H$ is either rational or elliptic. In the former case, S lies on the projection of a scroll $\widetilde{S} \subset \mathbb{P}^{r+1}$. In the latter case, one possibility is S is the cone over an elliptic normal curve in $\mathbb{P}^{r-1}$ (birational to a rules surface over an elliptic curve). The remaining possibility is S is a *del Pezzo surface*.

A del Pezzo surface S is birational to $\mathbb{P}^2$ blown up at δ points $p_1, \ldots, p_\delta$. It is embedded in $\mathbb{P}^{9-\delta}$ via the linear system of cubics through $p_1, \ldots, p_\delta$. For $\delta = 0$ we get the cubic Veronese surface. For $\delta = 6$, we get a cubic surface in $\mathbb{P}^3$. Note that the last possibility in the example above occurs only if $r \leq 9$.

LECTURE 16

October 26, 2011

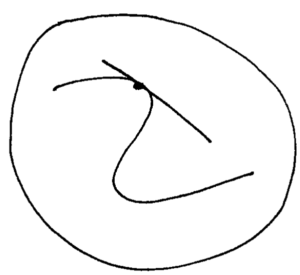
16.1. Inflectionary points

By way of introduction, consider the following general remark.

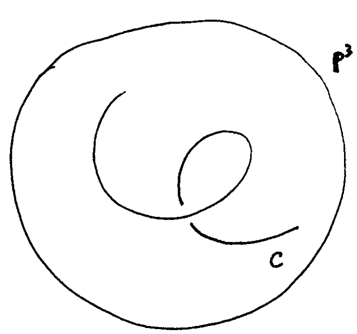
Question 16.1. What cab we say about the degree d and dimensions r of linear series on a curve of genus g?

(i) If $d > 2g-2$, then for complete linear series $r = d - g$ by Riemann-Roch.
(ii) If $d < 2g-2$, then $r \le d/2$ by Clifford and this bound is sharp.
(iii) How does the question change if we require the associated map to the linear series to be a (birational) embedding? We proved Castelnuovo's bound $g \le \pi(d,r)$.
(iv) What can we say if C is a general curve? We will not answer this immediately. It is the subject of *Brill-Noether theory* which we will discuss later on.

Let $C \subset \mathbb{P}^2$ be a smooth, non-degenerate curve. At a general point $p \in C$, the we have $m_p(C \cdot \mathbb{T}_p C) = 2$, that is, the multiplicity of intersection of C and the tangent line at p is 2. This is true only in characteristic 0. We say p is a *flax point* if $m_p(C \cdot \mathbb{T}_p C) \ge 3$.



We would like to generalize this notion to curves embedded in higher dimensional projective spaces, i.e., $C \subset \mathbb{P}^r$ for $r \ge 3$. Let us start with $r = 3$.



It is interesting to consider how the curve meets both lines and planes.

Definition 16.2. For $\Lambda \subset \mathbb{P}^n$ a k-plane, the *order of contact* of C and Λ at p is

$$\mathrm{ord}_p(\Lambda \cdot C) = \min\{m_p(C \cdot H) \mid H \supset \Lambda \text{ a hyperplane}\}.$$

In higher dimensions, flex lines are atypical, i.e., a general curve $C \subset \mathbb{P}^r$ for $r > 2$ does not have any flex points. This is unlike in $\mathbb{P}^2$ where every curve of degree > 2 has flex points.

For now, assume C meets its tangent line $\mathbb{T}_p C$ with order 2. We would like to consider planes $H \supset \mathbb{T}_p C$. We claim one of them will have an order of intersection > 2. In $\mathbb{P}^3$, if the order of contact with the tangent line is 2, then there exists a unique plane $H \supset \mathbb{T}_p C$ meeting C with multiplicity ≥ 3. This is called the *osculating plane* to C at p. It agrees with the span of $3p$. For some $p \in C$, we will have $m_p(C \cdot H) > 3$, and these are called *stalls*.

Let us introduce some of the modern notation. Throughout, we will asume C is a smooth curve, $\mathcal{D} = (L, V)$ is a linear system, where L is a line bundle, and $V \subset \mathrm{H}^0(L)$ is a subspace of dimension $r+1$. In other words, $\mathcal{D}$ is a g^r_d. For $\sigma \in V \setminus \{0\}$, let $\operatorname{ord}_p \sigma$ denote the order of vanishing of σ at p.

Proposition 16.3. *For all $p \in C$, we have*

$$\#\{\operatorname{ord}_p \sigma \mid \sigma \in V \setminus \{0\}\} = r + 1.$$

Proof. We will prove the statement by demonstrating inequalities in both directions. It is immediate to see there is an inequality $\leq$. For the other direction, start by choosing a basis $\sigma_1, \ldots, \sigma_r$ of V. If $\operatorname{ord}_p \sigma_i = \operatorname{ord}_p \sigma_j$ for some $i \neq j$, then replace σ_j with a linear combination of σ_i and σ_j which vanishes to higher order. In this way, we arrive at a basis with distinct orders of vanishing. $\square$

Let us write

$$\{\operatorname{ord}_p \sigma \mid \sigma \in V \setminus \{0\}\} = \{a_0, \ldots, a_r\}$$

where

$$a_0 < \cdots < a_r.$$

This is called the *vanishing sequence* of $\mathcal{D}$ at p. If we want to stress the dependent on V and p, we will write $a_i(V, p)$. It is clear the minimal series is $0, 1, \ldots, r$. Construct a second sequence

$$\alpha_i = a_i - i$$

called the *ramification sequence*. It is immediate that

$$0 \leq \alpha_0 \leq \cdots \leq \alpha_r.$$

The *total ramification* is

$$\alpha = \alpha(V, p) = \sum_{i=0}^{r} \alpha_i.$$

We say p is an *inflectionary point* of V if $\alpha > 0$. Equivalently $\alpha_r > 0$, $a_r > r$, or there exists $\sigma \in V$ satisfying $\operatorname{ord}_p \sigma > r$.

Observe that $a_0 = \alpha_0 > 0$ if and only if p is a basepoint of V. Assume $a_0 = \alpha_0 = 0$. Then $\alpha_1 > 0$ ($\alpha_1 > 1$) means that $d\varphi_V = 0$ at p, that is, φ_V is not an immersion at p.

Lemma 16.4. *Let (L, V) be a linear system on a smooth curve C. Then for a general $p \in C$, $\alpha(V, p) = 0$.*

Proof. Say the map $\varphi_V \colon C \to \mathbb{P}^r$ is given locally by the vector valued function

$$v(z) = (\sigma_0(z), \ldots, \sigma_r(z)),$$

where V has been trivialized around p. To say p is inflectionary is equivalent to

$$v(p) \wedge v'(p) \wedge \cdots \wedge v^{(r)}(p) = 0.$$

If p is not an isolated point, so $v \wedge v' \wedge \cdots \wedge v^{(r)} = 0$ identically around p. Let k be the smallest integer such that $v \wedge v' \wedge \cdots \wedge v^{(k)} = 0$, i.e., we can assume that $v \wedge v' \wedge \cdots \wedge v^{(k-1)} \neq 0$. In other terms $v^{(k)} \in \langle v, v', \ldots, v^{(k-1)} \rangle$ at a general p. Taking the derivative of $v \wedge v' \wedge \cdots \wedge v^{(k)}$, we get

$$\begin{aligned} 0 &= v' \wedge v' \wedge v^{(2)} \wedge \cdots \wedge v^{(k)} \pm \cdots \pm v \wedge \cdots \wedge v^{(k-1)} \wedge v^{(k-1)} \pm v \wedge \cdots \wedge v^{(k-1)} \wedge v^{(k+1)} \\ &= v \wedge \cdots \wedge v^{(k-1)} \wedge v^{(k+1)}, \end{aligned}$$

so

$$v^{(k+1)} \in \langle v, \ldots, v^{(k-1)} \rangle.$$

Taking the derivative again of $v \wedge \cdots \wedge v^{(k-1)} \wedge v^{(k+1)} = 0$, the only non-zero terms are

$$v \wedge v' \wedge \cdots \wedge v^{(k-1)} \wedge v^{(k+2)} \pm v \wedge \cdots \wedge v^{(k-2)} \wedge v^{(k)} \wedge v^{(k+1)} = 0.$$

The latter one vanishes by the previous observation, so

$$v^{(k+2)} \in \langle v, \ldots, v^{(k-1)} \rangle.$$

Continuing this argument implies that all derivatives $v^{(\ell)}$ lie in the span $\langle v, \ldots, v^{(k-1)} \rangle \subsetneq \mathbb{C}^{r+1}$. It follows that $\varphi_V(C)$ lies in a proper subspace of $\mathbb{P}^r$ which is a contradiction of non-degeneracy. $\square$

Note that in characteristic p only the last step of this argument fails.

Example 16.5. Consider the curve $xy - z^2 = 0$ in characteristic 2. It is smooth but each tangent line contains $[0, 0, 1]$.

16.2. Plücker formula

It is possible to push the observations in the proof of the proposition above a little further. We used the fact that

$$v \wedge v' \wedge \cdots \wedge v^{(r)} \in \Lambda^{r+1}\mathbb{C}^{r+1} \cong \mathbb{C}$$

is a function which is not identically zero. This is the *Wronskian determinant*

$$\begin{vmatrix} \sigma_0 & \cdots & \sigma_r \\ \sigma_0' & \cdots & \sigma_r' \\ \vdots & \ddots & \vdots \\ \sigma_0^{(r)} & \cdots & \sigma_r^{(r)} \end{vmatrix}.$$

Note that this happens in a local coordinate and the content of the result above is the function does not vanish identically. We can improve the observation by saying that

$$\mathrm{ord}_p(v \wedge v' \wedge \cdots \wedge v^{(r)}) = \alpha(V, p) = \sum_{i=0}^{r} \alpha_i(V, p).$$

Proof. Choose σ_o to vanish to minimal α_0 order at p and write

$$\sigma_0(z) = z^{\alpha_0} + \text{higher order terms}.$$

Proceed by choosing σ_i with increasing orders of vanishing

$$\begin{aligned} \sigma_1(z) &= z^{\alpha_1+1} + \text{higher order terms}, \\ &\vdots \\ \sigma_r(z) &= z^{\alpha_r+r} + \text{higher order terms}. \end{aligned}$$

The first non-zero derivative of the matrix is of order

$$\alpha_0 + \cdots + \alpha_r = \alpha(V, p),$$

that is, the first non-zero term in the derivative of $v \wedge \cdots \wedge v^{(r)}$ is $v^{(\alpha_0)} \wedge \cdots \wedge v^{(\alpha_r+r)}$. $\square$

Choose local coordinates z_α in U_α, and let L be a line bundle with transition functions $f_{\alpha\beta}$. Changing the local coordinate, the derivatives of functions are multiplied by $dz_\beta/dz_\alpha = g_{\alpha\beta}$. Note that these are the transition functions for the canonical line bundle. We can treat the Wronskian

$$\begin{vmatrix} \sigma_0 & \cdots & \sigma_r \\ \sigma_0' & \cdots & \sigma_r' \\ \vdots & \ddots & \vdots \\ \sigma_0^{(r)} & \cdots & \sigma_r^{(r)} \end{vmatrix}$$

as a function on U_α. Changing coordinates to U_β the derivative is multiplies by $f_{\alpha\beta}^{r+1} g_{\alpha\beta}^{\binom{r+1}{2}}$. It other words, the Wronskian is a section of $L^{r+1} \otimes K^{\binom{r+1}{2}}$. So the total number of zeros is

$$\deg\left(L^{r+1} \otimes K^{\binom{r+1}{2}}\right) = d(r+1) + \binom{r+1}{2}(2g-2) = (r+1)(d + r(g-1)).$$

We have arrived at the following result.

Proposition 16.6 (Plücker forula). *For all linear series (L, V) on a smooth curve C, we have*

$$\sum_{p \in C} \alpha(V, p) = (r+1)(d + r(g-1)).$$

Example 16.7. Assume $g = 0$ and take $V = \mathrm{H}^0(\mathcal{O}_{\mathbb{P}^1}(d))$. This complete linear series yields an embedding $\varphi_V : \mathbb{P}^1 \hookrightarrow \mathbb{P}^d$ whose image is a rational normal curve. There are no inflectionary points since all points are alike (the automorphisms of $\mathbb{P}^r$ which fix the rational normal curve act transitively on its points). By the Plücker formula $d = r$, so the result is 0.

It is more interesting to consider non-complete series $V \subsetneq \mathrm{H}^0(\mathcal{O}_{\mathbb{P}^1}(d))$. For example, for $d = 4$ and $r = 3$, we get a rational quartic $\mathbb{P}^3$. It has

$$(3+1)(4 + 3 \cdot (-1)) = 4$$

flex points. It is possible to see this directly. The osculating planes to the rational normal curve in $\mathbb{P}^4$ fill up $\mathbb{P}^4$ 4-times, i.e., the general point in $\mathbb{P}^4$ lies on 4 such planes. If we project away from such a point $p \in \mathbb{P}^4$, these are the planes we are seeking.

Example 16.8. Consider $g = 1$. A complete linear series V satisfied $d = r + 1$ so it corresponds to an embedding $\varphi_V : C \hookrightarrow \mathbb{P}^r$. To say there is an inflectionary plane means it meets the curve in exactly one point. Two such hyperplanes have ratio a torsion point of order $r + 1$. Plücker formula predicts the number of inflectionary points is

$$(r + 1)(d + r(g - 1)) = (r + 1)d = (r + 1)^2$$

which is precisely the number of $(r+1)$-torsion points once an origin is chosen. In other words, the inflection points are a coset of $(r + 1)$-torsion points once an origin is chosen.

Observation 16.9. *When $r = 1$, the discussion above applies even if the map φ_V is not an embedding, i.e., basepoints do not affect the computation. Following this logic, we can recover the Riemann-Hurwitz formula.*

LECTURE 17

November 2, 2011

17.1. Osculating linear spaces

Let C be a smooth projective genus g curve, L a line bundle of degree d on C, $V \subset \mathrm{H}^0(L)$ a subspace, and $p \in C$ a point. We will write

$$\{\mathrm{ord}_p(\sigma) \mid \sigma \in V \setminus \{0\}\} = \{a_0, \dots, a_r\}$$

where $a_0 < \cdots < a_r$. Set

$$\alpha_i = \alpha_i(V, p) = a_i - i,$$
$$\alpha = \alpha(V, p) = \sum_{i=0}^{r} \alpha_i.$$

The integer α is called the *total ramification index* of V at p. The Plücker formula reads

$$\sum_{p \in C} \alpha(p, V) = (r+1)d + \binom{r+1}{2}(2g-2) = (r+1)(d + r(g-1)).$$

Let us provide an alternative perspective. Consider $\varphi\colon C \hookrightarrow \mathbb{P}^r$ (or more generally, a map $\varphi\colon C \to C_0 \subset \mathbb{P}^r$ birational onto its image). We have the *Gauss map*

$$\begin{aligned} \varphi^{(1)}\colon C &\longrightarrow \mathbb{G}(1, r), \\ p &\longmapsto \mathbb{T}_p C. \end{aligned}$$

If $\varphi\colon C \to \mathbb{P}^r$ is given by $z \mapsto [v(z)]$, then $\varphi^{(1)}: z \mapsto [v(z) \wedge v'(z)]$. As presented $\varphi^{(1)}$ is only a rational map, but it extends to a regular one since both C and $\mathbb{G}(1, r)$ are projective. For $k \leq r-1$, there is a natural generalization

$$\begin{aligned} \varphi^{(k)}\colon C &\longrightarrow \mathbb{G}(k, r), \\ z &\longmapsto [v(z) \wedge \cdots \wedge v^{(k)}(z)] \end{aligned}$$

called the *k-th associated map*. As before, we first define it as a rational map and then extend. The *osculating k-plane* to C at p is defined as $\varphi^{(k)}(p) \in \mathbb{G}(k, r)$. Alternatively, the osculating k-plane is the k-plane through p with maximal order of contact with C (at least $k+1$).

In the special case $k = r-1$, we call $\varphi^{(r-1)}(p)$ the *osculating hyperplane* at p. The image of $\varphi^{(r-1)}$ is called the *dual curve* to C.

Remark 17.1. For $X \subset \mathbb{P}^r$, we associate the *dual variety*

$$X^* = \{\text{tangent hyperplanes to } X\} \subset \mathbb{P}^{r*}.$$

Note that when X is a curve this is different from its dual curve.

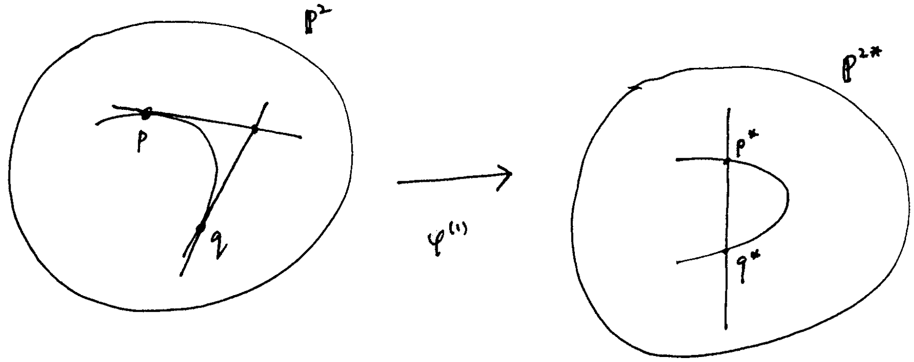
Question 17.2. What is the dual curve of a twisted cubic $C \subset \mathbb{P}^3$?

Interestingly enough, the answer is another twisted cubic. To see this note that the twisted cubic is homogeneous, i.e., projective automorphisms fixing it act transitively on its points. These induce automorphisms on the dial projective space and on the dual curve, whose action, again, is transitive. These it suffices to observe that there is a unique irreducible non-degenerate homogeneous curve in projective space.

17.2. Plane curves

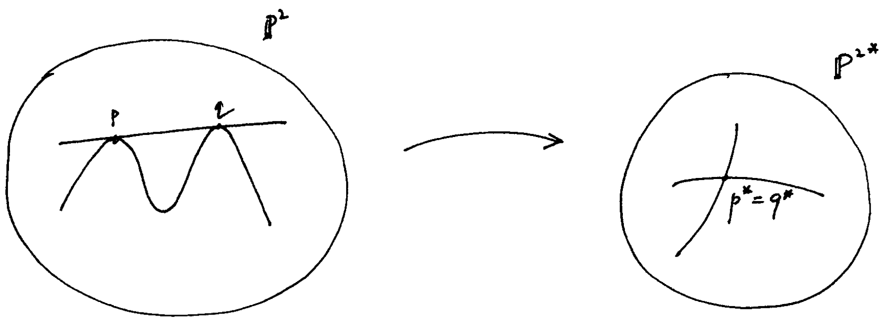
Consider a plane curve $C \subset \mathbb{P}^2$. For the sake of notation think of an embedded curve and not a morphism into $\mathbb{P}^2$. Assume C is not a line. Let C^* be the dual curve in $\mathbb{P}^{2*}$, the image of C under the Gauss map $\varphi^{(1)}: C \to \mathbb{P}^{2*}$, $p \mapsto \mathbb{T}_p C$. We know that in characteristic 0, the map $\varphi^{(1)}$ is birational onto its image. It follows that $\mathrm{genus}(C) = \mathrm{genus}(C^*)$.

It is a basic fact that $C = (C^*)^*$. To see this consider the following diagram.



The intersection point of the tangent lines to p and 1 of C corresponds to the secant line $\overline{p^*, q^*}$ in $\mathbb{P}^{2*}$. But as p is fixed and q approaches it, the intersection point approaches p. The secant line approaches the tangent line $\mathbb{T}_{\mathrm{p}^*} C^*$. Therefore, the double application of the Gauss map sends p to p.

People in the 19-th century thought there should be a theory of curves symmetric with respect to dualization of curves. We would like to restrict the type of curves we are studying so their class is closed under dualization. For example, the dual of a smooth curve may not be smooth. Take for example a curve with a bitangent. It follows that there is a corresponding node in the dual curve.



In fact, there is a unique smooth curve with smooth dual – a smooth conic. Similarly, a flex point yields a cusp in the dual. This class of singularities is closed under dualization (if imposed on both C and C^*).

Definition 17.3. We sat that $C \subset \mathbb{P}^2$ has *traditional singularities* if both C and C^* have only nodes and cusps.

Equivalently, C has only nodes and cusps as singularities, only simple flexes, and only bitangents, or any line $L \subset \mathbb{P}^2$ can meet C only in

(i) d simple points,
(ii) 1 double point and $d-2$ simple ones,
(iii) 1 triple point and $d-3$ simple ones,
(iv) 2 double points and $d-4$ simple ones.

Let C be a curve with traditional singularities. Let

$$d = \deg(C), \qquad d^* = \deg(C^*), \qquad g = \mathrm{genus}(C), \qquad g^* = \mathrm{genus}(C^*),$$

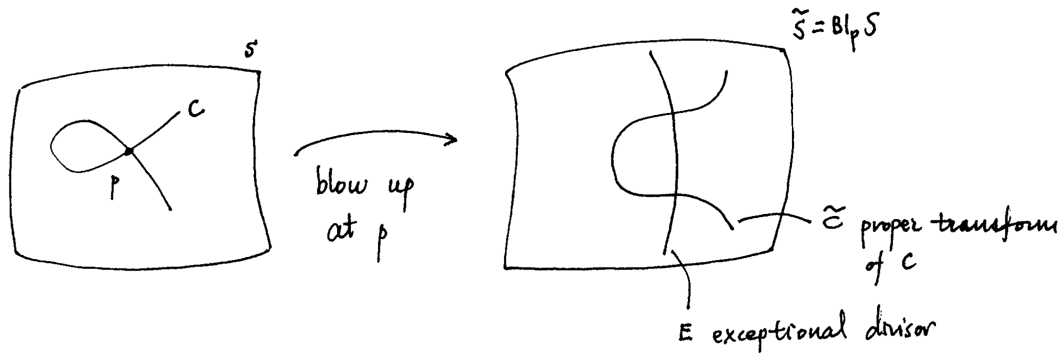
where genus$(-)$ measures the geometric genus. Further, denote

$$\begin{aligned} \delta &= \#\{\text{nodes of } C\} = b^*, \\ \kappa &= \#\{\text{cusps of } C\} = f^*, \\ b &= \#\{\text{bitangents of } C\} = \delta^*, \\ f &= \#\{\text{flexes of } C\} = \kappa^*. \end{aligned}$$

We would like to derive relations between these invariants. These were classically known as the *Plücker relations*. Recall that nodes and cusps drop the geometric genus by 1, that is,

$$g = \binom{d-1}{2} - \delta - \kappa = g^* = \binom{d^*-1}{2} - b - f.$$

As a slight digression, consider a curve C on a surface S, and assume $p \in C$ is a node point. Blowing up p in S resolves the node in the proper transform $\widetilde{C}$ of C.



If $\pi\colon \widetilde{S} \to S$ is the blow-down projection, then $K_{\widetilde{S}} = \pi^* K_S + E$ where $E = \pi^{-1}(p)$ is the exceptional fiber. Similarly, $\widetilde{C} = \pi^* C - 2E$. Then

$$K_{\widetilde{S}} + \widetilde{C} = \pi^*(K_S + C) - E,$$

and

$$(K_{\widetilde{S}} + \widetilde{C}) \cdot \widetilde{C} = (K_S + C) \cdot C - 2.$$

A similar argument applies to a curve with a cusp. This gives us another way to derive the equality

$$g = \binom{d-1}{2} - \delta - \kappa = g^* = \binom{d^*-1}{2} - b - f,$$

which allows us to relate d and d^*. There is however an easier approach. If C is smooth and $C = V(F)$, note that the degree d^* of C^* is the number of points in a general intersection $C^* \cap L$ where L is a line. But this is the same as

$$\begin{aligned} d^* &= \deg(C^*) = \#(C^* \cap L) \\ &= \#\{\text{tangent lines to } C \text{ that pass through } L^* \subset \mathbb{P}^{2*}\} \\ &= V\left(F, \frac{\partial F}{\partial x}\right) \\ &= d(d-1). \end{aligned}$$

Above, $\partial F/\partial x$ denoted a general first order derivative of F. The above argument used the fact C is smooth. In the presence of nodes, a similar argument yields $d^* = d(d-1) - 2\delta$. More generally,

$$d^* = d(d-1) - 2\delta - 3\kappa.$$

Dually, we get

$$d = d^*(d^* - 1) - 2b - 3f.$$

Remark 17.4. An alternative way to derive these formulas is by applying Riemann-Hurwitz to a general projection of C.

Example 17.5. If C is smooth of degree d, then $\delta = \kappa = 0$. We compute

$$g = \binom{d-1}{2}, \qquad d^* = d(d-1).$$

From $d = d^*(d^* - 1) - 2b - 3f$, we get

$$2b + 3f = d^*(d^* - 1) - d = (d^2 - d)(d^2 - d - 1) - d = d^4 - 2d^3.$$

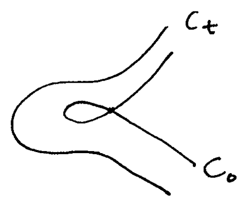
The other relation comes from $g = \binom{d^*-1}{2} - b - f$. We double to avoid denominators and write

$$\begin{aligned} 2b + 2f = 2\left(\binom{d^* - 1}{2} - \binom{d-1}{2}\right) &= (d^* - 1)(d^* - 2) - (d-1)(d-2) \\ = (d^2 - d - 1)(d^2 - d - 2) - (d-1)(d-2) &= d^4 - 2d^3 - 3d^2 + 6d. \end{aligned}$$

Solving, we get

$$\begin{aligned} f &= 3d^2 - 6d = 3d(d-2), \\ b &= \frac{1}{2}(d^4 - 2d^3 - 9d^2 + 18d) = \frac{d(d-2)(d-3)(d+3)}{2}. \end{aligned}$$

Finally, it is interesting to make a few remarks about the behavior of these invariants in families. Suppose we have a family of smooth curves specializing to a curve with a single node.

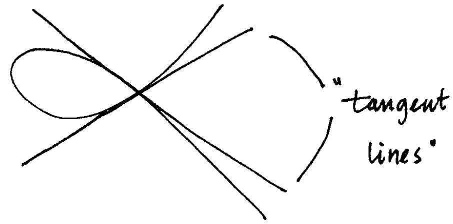


Then

$$\begin{aligned} f_0 &= f - 6, \\ b_0 &= b - 2(d^2 - d - 6) = b - 2(d-3)(d+2). \end{aligned}$$

Question 17.6. What happens to the missing bitangents and flexes in C_0?

For example, in a nodal curve the two tangent lines to the node qualify as flexes in some sense.



Each of these is approaches by 3 flexes, hence the 6 missing ones. The bitangents are tangent lines passing through the node; 2 approach each such.

Question 17.7. What happens if a family of smooth curve degenerates to a curve with a single cusp?

LECTURE 18

November 4, 2011

18.1. A remark about inflectionary points

Let $C \subset \mathbb{P}^r$ be a smooth curve. We have a distinguished finite subset $\Gamma = \{\text{inflectionary points}\} \subset C$. Note that Γ is not intrinsic to C, i.e., it depends on the embedding $C \subset \mathbb{P}^r$. There is an exception, namely, the inflectionary points for a canonical curve are intrinsic. This holds even for curves which are not embedded via the canonical series.

Example 18.1. A genus 2 curve is hyperelliptic and maps 2-to-1 to a rational curve with 6 ramification points. These are intrinsic.

This idea can be repeated for pluricanonical divisors. In fact, these are all intrinsic invariants one can get.

Theorem 18.2. *Consider the following morphism.*

$$\begin{tikzcd} \mathcal{P}_{d,g} = \{(C, L) \mid C \in \mathcal{M}_g, L \in \operatorname{Pic}^d(C) \arrow[d] \\ \mathcal{M}_g = \{\text{smooth curves of genus } g\}/\text{isomorphism} \arrow[u, bend left, dashed] \end{tikzcd}$$

The only rational sections are given by powers of t he canonical bundle.

We can repeat this question for embedded curves.

Conjecture 18.3. *Consider the following morphism.*

$$\begin{tikzcd} \mathcal{P}_{n,g,r,d} = \{(C, L) \mid C \in \mathcal{H}_{n,g,r}, L \in \operatorname{Pic}^d(C)\} \arrow[d] \\ \mathcal{H}_{n,g,r} = \{\text{smooth curve of genus } g \text{ and degree } n \text{ in } \mathbb{P}^r\} \subset \text{Hilbert scheme} \arrow[u, bend left, dashed] \end{tikzcd}$$

For $r = 1, 2$, the only rational sections are linear combinations of the canonical class and the hyperplane class.

Remark 18.4. For $r \geq 3$, this statement is false.

Remark 18.5. The space $\mathrm{H}_{n,g,r}$ need not be irreducible. The make the conjecture meaningful, we need to look at one irreducible component at a time.

18.2. Weierstrass points

Let C be a smooth projective curve of genus g and $p \in C$ a point. We would like to relate the inflectionary behavior of p with respect to the canonical series and the meromorphic functions on C with poles only at p. First, Riemann-Roch predicts $h^0(mp)$ for $m \gg 0$.

Let D be an effective divisor on C of degree d. Riemann-Roch states

$$r(D) = d - g + h^0(K - D),$$
$$r(D + p) = d + 1 - g + h^0(K - D - p),$$

so

$$r(D+p) = \begin{cases} r(D) & \text{if } p \text{ is not a basepoint of } |K-D|, \\ r(D)+1 & \text{if } p \text{ is a basepoint of } |K-D|. \end{cases}$$

Applying this to the sequence of divisors $0, p, 2p, \dots$, we would like to make inferences about the sequence

$$h^0(K) = g, h^0(K-p), \dots, h^0(K-mp), \dots.$$

For $m \gg 0$, $h^0(K-mp) = 0$, so there are exactly g values $m_1, \dots, m_g$ such that $h^0(K-mp) < h^0(K-(m-1)p)$. On the other hand, $h^0(K-mp) = h^0(K-(m-1)p)$ if and only if there exists a meromorphic function f on C with polar divisor $(f)_\infty = mp$. In conclusion, there exists exactly g natural numbers m such that there exists no meromorphic function f with $(f)_\infty = mp$. Consider the semigroup

$$H = \{m \mid \text{there exists } f \text{ meromorphic with } (f)_\infty = mp\} \subset \mathbb{Z}_{>0}.$$

Then the statement above is $|\mathbb{Z}_{>0} \setminus H| = g$. The sequence $m_1, \dots, m_g$ is called the sequence of *Weierstrass gaps*. Note that $m_i \leq 2g-1$ for all i.

Fix an arbitrary curve C. For a general $p \in C$, the associated semigroup is $H = \{g+1, g+2, \dots\}$.

Definition 18.6. We say $p \in C$ is a *Weierstrass point* if $H \neq \{g+1, g+2, \dots\}$. Equivalently, the set of gaps G satisfies $G \neq \{1, \dots, g\}$, or $r(gp) > 0$.

Definition 18.7. The *weight of a (Weierstrass) point* p is

$$w(p) = \sum_{m \in G} m - \binom{g+1}{2} = \sum_i (m_i - i).$$

Then being a Weierstrass point means $w(p) > 0$.

Theorem 18.8. *For any smooth curve C,*

$$\sum_{p \in C} w(p) = g^3 - g.$$

PROOF. Recall that $m \in G$ if and only if $h^0(K-mp) < h^0(K-(m-1)p)$, or, equivalently, there exists a global holomorphic differential ω on C such that $\operatorname{ord}_p \omega = m-1$. This implies that

$$G = \{a_i(\mathrm{H}^0(K), p) + 1\},$$

i.e., the sequence of gaps is a shift of the vanishing sequence of K. In other words, p is not a Weierstrass point if and only if $G = \{1, \dots, g\}$, if and only if $a(\mathrm{H}^0(K), p) = \{0, 1, \dots, g-1\}$. The first part of the theorem follows since a general points is not inflectionary (in characteristic 0). The second part is the Plücker formula applied to thee canonical series ($d = 2g-2$, $r = g-1$):

$$\sum_p w(p) = (r+1)(d + r(g-1)) = g(g-1)(g+1) = g^3 - g. \qquad \square$$

It is possible to talk about inflectionary points for pluricanonical series but these do not have a convenient interpretation in terms of meromorphic functions.

Example 18.9. For $g = 2$, the possible semigroups are:

$\{3, 4, 5, \dots\}$ not a Weierstrass point,

$\{2, 4, 5, \dots\}$ a Weierstrass point which occurs at the 6 ramification points of $\varphi_K \colon C \xrightarrow{\text{2-to-1}} \mathbb{P}^1$.

Example 18.10. For $g = 3$, the possible semigroups are

$\{4, 5, 6, 7, \dots\}$ not a Weierstrass point,

$\{3, 5, 6, 7, \dots\}$ a Weierstrass point of weight 1,

$\{3, 4, 6, 7, \dots\}$ a Weierstrass point of weight 2,

$\{2, 4, 6, 7, \dots\}$ a Weierstrass point of weight 3 which occurs only on hyperelliptic curves.

Note that the total weight is $g^3 - g = 24$. In the hyperelliptic case, we get a 2-fold cover of $\mathbb{P}^1$ with 8 ramification points, i.e., 8 Weierstrass points of weight 3. If C is not hyperelliptic, the canonical curve if a smooth plane quartic. A Weierstrass point of weight 1 is an ordinary flex, and a point of weight 2 is a hyperflex. These are the only two possibilities. If α is the number of ordinary flexes and β the number of hyperflexes, then we know that $\alpha + 2\beta = 24$, so $0 \leq \beta \leq 12$. It is interesting to ask which values of β occur? Note that on a general curve all Weierstrass points have weight 1; these are called *normal Weierstrass points*.

Example 18.11. Consider a curve C of genus $g = 4$. There is a unique semigroup of weight 1:

$$\{g, g+2, g+3, \dots\}.$$

There are two semigroups of weight 2:

$$\{g-1, g+2, g+3, \dots\}, \qquad \{g, g+1, g+3, g+4, \dots\}.$$

A fundamental question is: Which semigroups occur? The answer is no but the lowest example occurs in $g = 16$. We do not know which occur.

LECTURE 19

November 9, 2011

19.1. Real curves

We would like to briefly study real algebraic curves. More specifically, consider a curve $C = V(F)$ where F is a homogeneous polynomial in $\mathbb{R}[x, y, z]$ of degree d. We can then study both its real and complex points:

$$C_{\mathbb{R}} \subset \mathbb{RP}^2, \qquad C_{\mathbb{C}} \subset \mathbb{CP}^2.$$

Consider the space of all degree d polynomials in the variables x, y, z with complex coefficients, and the subspace of polynomials which cut out singular curves.

$$\begin{array}{l} \mathbb{CP}^N = \{\text{curves } C \subset \mathbb{P}^2 \text{ of degree } d \text{ over } \mathbb{C}\} \\ \uparrow \\ \Delta = \{\text{singular curves}\} \end{array}$$

Since Δ has real codimension 2, it follows that its complement is connected. This does not follow in the real case.

$$\begin{array}{l} \mathbb{RP}^N = \{\text{curves } C \subset \mathbb{P}^2 \text{ of degree } d \text{ over } \mathbb{R}\} \\ \uparrow \\ \Delta = \{\text{singular curves}\} \end{array}$$

In fact, the complement may have many connected components.

If C is smooth, then so is $C_{\mathbb{R}}$. This can be seen by observing that singular points satisfy

$$F = \frac{\partial F}{\partial x} = \frac{\partial F}{\partial y} = \frac{\partial F}{\partial z} = 0.$$

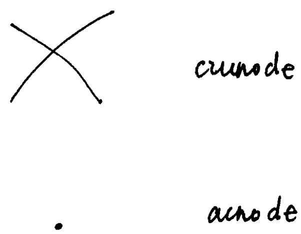
Then $C_{\mathbb{R}}$ is a compact smooth manifold of dimension 1, hence $C_{\mathbb{R}} = \bigsqcup S^1$. The connected components are called *ovals* or *circuits*. There are two possibilities for a component $\S^1 \cong \gamma \subset \mathbb{RP}^2$.

(i) If $\gamma \sim 0$ (homologous) in $\mathrm{H}_1(\mathbb{RP}^2, \mathbb{Z})$, then $\mathbb{RP}^2 \setminus \gamma$ is disconnected. One component is homeomorphic to a disc, called the *interior* of γ, and the other, called the *exterior*, is homeomorphic to a Möbius band.

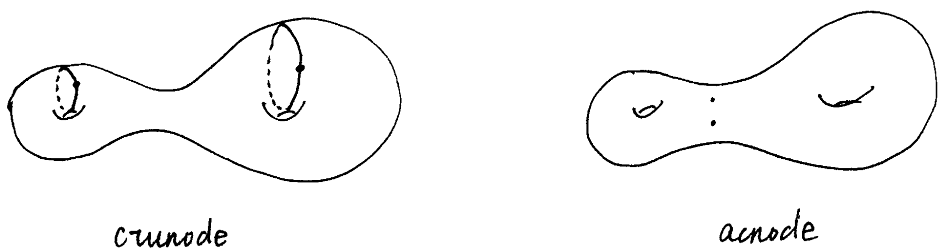
(ii) If $\gamma \not\sim 0$ in $\mathrm{H}_1(\mathbb{RP}^2, \mathbb{Z})$, then the complement is connected.

These two types of components are referred to as *even* and *odd* ovals respectively. It follows from the intersection pairing on $\mathrm{H}_1(\mathbb{RP}^2, \mathbb{Z})$ that any two off ovals must meet. In particular, if C is smooth, then it has at most one odd oval. We can push this analysis further. If $d = \deg C$ is even then C has only even ovals. If d is odd, then C has one odd oval.

It is interesting to related the singularities of C and $C_{\mathbb{R}}$. For example, a node of C may manifest itself in two ways in $C_{\mathbb{R}}$.



The former case, called a *crunode*, corresponds to local rings satisfying $\widehat{\mathcal{O}}_{C,p} \cong \mathbb{R}[[x,y]]/(x^2 - y^2)$. The latter case, called an *acnode*, corresponds to $\widehat{\mathcal{O}}_{C,p} \cong \mathbb{R}[[x,y]]/(x^2 + y^2)$. If we think of the normalization of C, a crunode arises from the identification of two real points. An acnode is the result of the identification of two complex conjugate points.



19.2. Harnack's Theorem

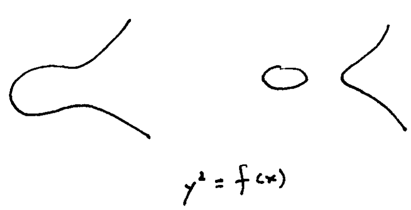
Theorem 19.1. *Let $C \subset \mathbb{P}^2$ be a smooth plane curve of degree d. Then the number of ovals of $C_{\mathbb{R}}$ is at most*

$$\binom{d-1}{2} + 1,$$

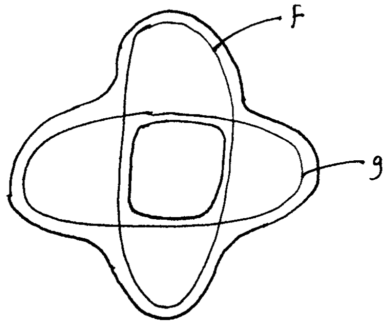
and this bound is sharp.

Example 19.2. If $d = 2$, then the number of ovals is 0 or 1. This many not be apparent from the planar description of parabolas, but they are connected in $\mathbb{RP}^2$.

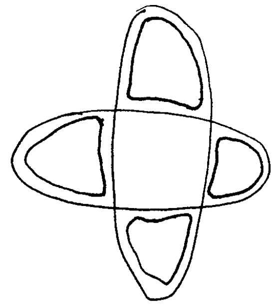
Example 19.3. If $d = 3$, then the number of ovals is either 1 or 2.



Example 19.4. If $d = 4$, then the number of ovals may vary between 0 and 4. It is easy to produce a quadric with no real points, for example, take $x^4 + y^4 + z^4 = 0$. It is also not hard to find one with 4 connected components. Start with two ellipses $V(f)$ and $V(g)$, then add or subtract a small constant ε to fg.



2 components



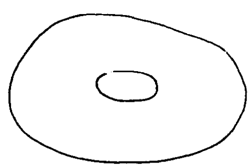
4 components

Similar arguments produce 1 or 3 components.

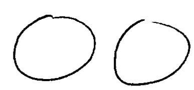
We can also ask whether ovals are nested.

Definition 19.5. Given two ovals γ and γ', we sat they are *nested* if one lies in the interior of the other.

Example 19.6. The first pair of ovals are nested, while the latter one is not.



nested

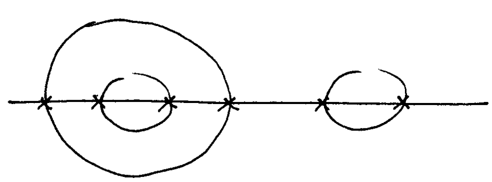


non-nested

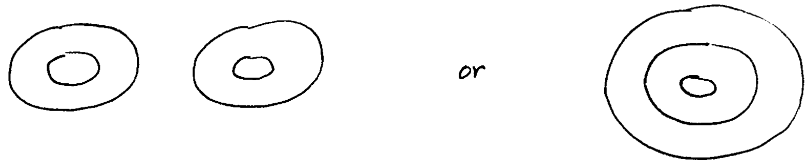
A further question is to describe the possible nesting the ovals.

Example 19.7. Consider a real curve of degree $d = 4$.

(i) If the number of ovals is 0 or 1, there is no ambiguity regarding nesting.
(ii) If there are 2 ovals, both nested and non-nested configurations occur.
(iii) If there are 3 ovals, no two of them are nested. To see this apply Bezout's Theorem. The given curve meets a suitably chosen line in at least 6 points leading to a contradiction.



Example 19.8. If $d = 6$, there are up to 11 ovals. It is not hard to see the following two configurations do not occur.



It follows that there exists an ovals, with $k = 0, \dots, 10$ simply nested ovals inside, and $10 - k$ outside. It is possible to show that not all values of k occur. In fact, the difference between k and $10 - k$ is divisible by 4.

Back to Harnack's Theorem, we will only prove the bound. The sharpness argument is a construction similar to the degree 4 case presented above, but it can be very tricky to set up (see fold-out in Coolidge for a proof).

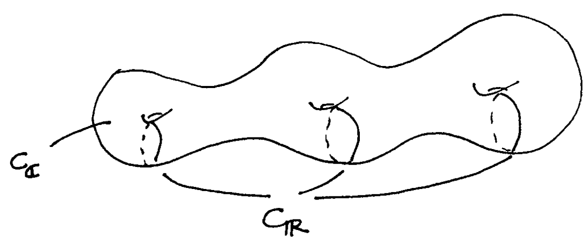
PROOF. Let C be a smooth plane curve of degree d with m ovals, $\gamma_1, \dots, \gamma_m$. For contradiction assume $m > \binom{d-1}{2} + 1$. At least $m - 1$ of the ovals are even, say $\gamma_1, \dots, \gamma_{m-1}$ are such. Choose a point $p_i \in \gamma_i$ for each i. Then take a curve $Y \subset \mathbb{P}^2$ of degree $d - 2$ passing through $p_1, \dots, p_{\binom{d-1}{2}+1}$. To see this is possible, note that $\mathrm{h}^0(\mathcal{O}_{\mathbb{P}_2}(d-2)) = \binom{d}{2}$. In fact, there are still $d-3$ degrees of freedom left. Choose $q_1, \dots, q_{d-3} \in \gamma_m$ and arrange so that Y passes through all of them. Note that if a curve goes in an even oval, it mist come out. The case when Y is singular or tangent do not pose a problem, since we get even higher multiplicities of intersection. Then the total number of intersections counted with multiplicities is

$$\#(Y \cap C) \geq (d-1)(d-2) + 2 + d - 3 = d(d-2) + 1,$$

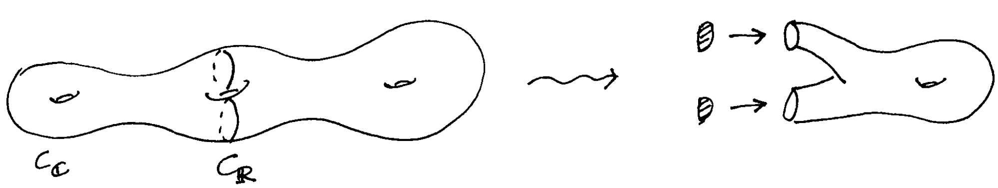
which violates Bezout's Theorem. $\square$

We will give a second simpler proof of Harnack's Theorem, one that is not biased towards the plane.

PROOF. Let C be a smooth projective curve of genus g over $\mathbb{R}$. We picture the set of real points $C_{\mathbb{R}} \subset C_{\mathbb{C}}$ as follows.



Let us investigate the complement $C_{\mathbb{C}} \setminus C_{\mathbb{R}}$. Complex conjugation σ acts on it freely, so we can take the quotient $(C_{\mathbb{C}} \setminus C_{\mathbb{R}})/\sigma$ which is a manifold. If we did not remove $C_{\mathbb{R}}$, then $C_{\mathbb{C}}/\sigma$ is a manifold with boundary. It is possible to show both of these are connected . These surfaces are orientable if $C_{\mathbb{R}}$ disconnects $C_{\mathbb{C}}$, and non-orientable otherwise. Either way, we complete it to a compact surface X by attaching disks. Say $C_{\mathbb{R}}$ has δ ovals, then the number of disks necessary is also δ.



We can now carry a simple Euler characteristic argument.

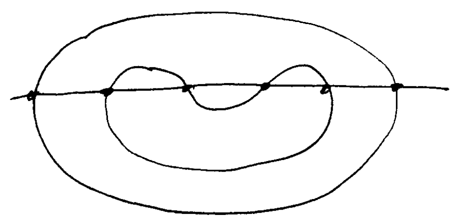
$$\begin{aligned} \chi(C_{\mathbb{C}}) &= 2 - 2g \\ \chi(C_{\mathbb{C}} \setminus C_{\mathbb{R}}) &= 2 - 2g \\ \chi((C_{\mathbb{C}} \setminus C_{\mathbb{R}})/\sigma) &= 1 - g \\ \chi(X) &= 1 - g + \delta \geq 2 \end{aligned}$$

We conclude $\delta \leq g + 1$ which is the statement of Harnack's Theorem for plane curves. □

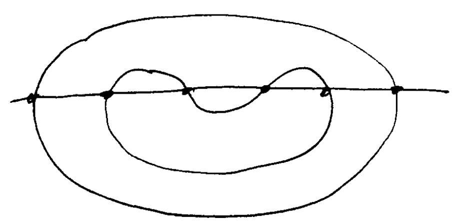
19.3. Other questions

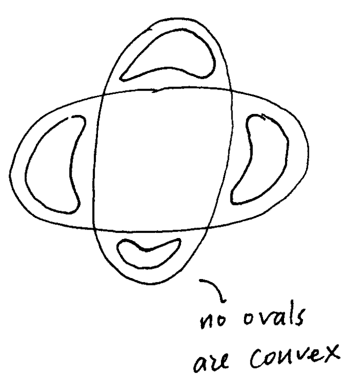
There are a few other interesting questions pertaining to real algebraic curves. One such is convexity. In other words, once over $\mathbb{R}$, we can decide whether an oval is convex or not.

Example 19.9. In a quartic curve, an interior oval always has to be convex so it does not violate Bezout's Theorem.

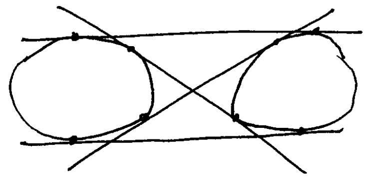


We can also ask about bitangents and flexes. For example, a quartic has 28 bitangents, and they are all real. Consider the 4 oval quartic we constructed earlier.

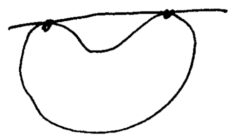




Each pair of ovals give rise to 4 bitangents as shown on the following diagram.



Furthermore, no oval is convex, so it gives rise to one more bitangent.



The total number then is

$$4\binom{4}{2} + 4 = 28.$$

In the case of flexes, there is a theorem saying at most 1/3 of them can be real.

LECTURE 20

November 16, 2011

20.1. Setting up Brill-Noether theory

Let us start by recalling a simple problem about matrices. Consider the space of all matrices up to scaling

$$M_{a,b} = \{a \times b \text{ matrices}\}/\text{scalars} \cong \mathbb{P}^{ab-1},$$

and the subspace

$$M^k_{a,b} = \{A \mid \operatorname{rank}(A) \leq k\}$$

where $k < \min\{a, b\}$.

(i) $\operatorname{codim}(M^k_{a,b} \subset M_{a,b}) = (a-k)(b-k)$
(ii) $(M^k_{a,b})_{\text{sing}} = M^{k-1}_{a,b}$
(iii) If $A \in M^k_{a,b} \setminus M^{k-1}_{a,b}$, then

$$\mathbb{T}_A M^k_{a,b} = \{\varphi \mid \varphi(\operatorname{Ker} A) \subset \operatorname{Im} A\}.$$

Alternatively,

$$(N_{M^k_{a,b}/M_{a,b}})_A = \operatorname{Hom}(\operatorname{Ker} A, \operatorname{Coker} A).$$

While this remark will become useful later, we can now return to the description of Brill-Noether theory. There are two basic object we can associate to a smooth projective curve of genus g:

- $C_g = \operatorname{Sym}^d C = C^d/S_d = \{\text{effective divisors of degree } d \text{ on } C\}$,
- $\operatorname{Pic}^d(C) = \{\text{line bundles of degree } d \text{ on } C\} \cong \operatorname{Jac}(C) = \frac{\mathrm{H}^0(K)^*}{\mathrm{H}_1(C,\mathbb{Z})}$.

The isomorphisms $\operatorname{Pic}^d(C) \cong \operatorname{Jac}(C)$ depends on a choice of basepoint $p_p \in C$. There is a map

$$\begin{aligned} u\colon C_d &\longrightarrow \operatorname{Jac}(C), \\ \sum_i p_i &\longmapsto \sum_i \int_{p_0}^{p_i}. \end{aligned}$$

This is the precomposition of the isomorphism $\operatorname{Pic}^d(C) \cong \operatorname{Jac}(C)$ with $C_d \to \operatorname{Pic}^d(C)$ given by $\sum p_i \mapsto \mathcal{O}_C(\sum p_i)$. We introduce the following two subvarieties.

$$\begin{array}{c} C^r_d = \{D \mid r(D) \geq r\} \subset C_d \\ \Big\downarrow u \\ W^r_d = \{L \mid h^0(L) \geq r+1\} \subset \operatorname{Pic}^d(C) \end{array}$$

Alternatively,

$$W^r_d = \{L \mid u^{-1}(L) \text{ has dimension } \geq r\}$$

which is closed in $\operatorname{Pic}^d(C)$. Many of the questions we addressed so far in this course can be phrased in terms of C^r_d or W^r_d being or not being empty. Now we would like to touch on issues such as irreducibility, smoothness, dimension, and other. We introduced C^r_d and W^r_d as closed subvarieties, but they admit natural scheme structures (other than the reduced one) which is a fine moduli space for certain functors.

Consider the map u as given above. We would like to understand its differential. Suppose $p_1, \ldots, p_d$ are distinct points in C. Then étale locally C_d are C^d coincide around $D = \sum p_i$ and (p_i) respectively. In particular, $T_D C_d = \bigoplus T_{p_i} C$. Note that $T_{u(D)} \operatorname{Jac}(C) \cong \mathrm{H}^0(K)^*$. Then, we would like to understand the map

$$du\colon \ \bigoplus_i T_{p_i} C \longrightarrow \mathrm{H}^0(K)^*$$

Given a tangent vector (v_i) in the domain of du, we claim that

$$du(v_i) = \left(\begin{array}{rccc} \ell\colon & \mathrm{H}^0(K) & \longrightarrow & \mathbb{C} \\ & \omega & \longmapsto & \langle \omega(p_i), v_i \rangle \end{array} \right)$$

As an alternative point of view, we can look at the transpose (dual) of du. Namely, the map

$$du^*\colon \ \mathrm{H}^0(K) \longrightarrow \bigoplus_i T^*_{p_i} C = \bigoplus_i K_{p_i} = \mathrm{H}^0(K/K(-D)),$$

called the *codifferential map*, is given by evaluation. Let us briefly comment on the case when the points p_i are not necessarily distinct. For $p \in C$, we have $T^*_p C = \mathfrak{m}_p/\mathfrak{m}_p^2$. If we treat $\mathfrak{m}_p$ as an ideal sheaf inside $\mathcal{O}_C$, then the above is a skyscraper sheaf whose stalk is $T^*_p C = \mathrm{H}^0(\mathfrak{m}_p/\mathfrak{m}_p^2)$. Dualizing we get $T_p C = \mathrm{H}^0(\mathcal{O}_C(p)/\mathcal{O})$. The elements of $\mathrm{H}^0(\mathfrak{m}_p/\mathfrak{m}_p^2)$ are classes of global functions vanishing at p, while these of $\mathrm{H}^0(\mathcal{O}_C(p)/\mathcal{O}_C)$ are represented by global meromorphic functions which are allowed a unique simple pole at p. Then $\mathrm{H}^0(\mathfrak{m}_p/\mathfrak{m}_p^2)$ and $\mathrm{H}^0(\mathcal{O}_C(p)/\mathcal{O}_C)$ are dual via the pairing $\langle f, g \rangle = (fg)(p)$. Then

$$T_D C = \mathrm{H}^0(\mathcal{O}_c(D)/\mathcal{O}_C) \cong \bigoplus \mathrm{H}^0(\mathcal{O}(p_i)/\mathcal{O}_C),$$

where the second isomorphisms makes sense in the case of distinct points. We will focus on this case, but everything we say extends appropriately once we interpret the (co)tangent space as suggested above, that is,

$$T^*_D C_d = \mathrm{H}^0(K/K(-D)), \qquad T_D C_d = \mathrm{H}^0(\mathcal{O}_C(D)/\mathcal{O}_C).$$

The pairing is given by

$$\langle \omega, f \rangle = \sum \operatorname{Res}(f\omega).$$

Assume the points p_i are distinct. It is easy to see that

$$\operatorname{Im}(du_D) = \operatorname{Ann}(\mathrm{H}^0(K - D)) \subset \mathrm{H}^0(D)^*.$$

Then

$$\begin{gathered} \operatorname{rank}(du_D) = g - h^0(K - D), \\ \dim(\operatorname{Ker}(du_D)) = d - (h^0(K) - h^1(K - D)). \end{gathered}$$

By Geometric Riemnan-Roch, the last number agrees with the dimension of the fiber $u^{-1}(u(D))$, hence

$$C^r_d = \{D \mid \dim u^{-1}(u(D)) \geq r\} = \{D \mid \operatorname{rank}(du_D) \leq d - r\}.$$

This is interesting since we can interpret du as a map of vector bundles

$$du\colon \ TC_d \longrightarrow u^* T \operatorname{Jac}(C),$$

hence the induced scheme structure on C^r_d.

20.2. Marten's Theorem

Theorem 20.1 (Marten's Theorem). *We have*

$$\dim W^r_d \leq d - 2r$$

with equality holding only if $r = 0$ or C is hyperelliptic.

This statement looks like Clifford's Theorem, and in some sense it is a version of Clifford's Theorem for families. We will deduce it from more general observations later, but for now it is possible to give a proof from basic facts.

Proof. Assume the curve C is not hyperelliptic. If we embed C canonically in $\mathbb{P}^{g-1}$, them by Geometric Riemann-Roch

$$C_d^r = \{\text{locus of divisors } D \subset C \subset \mathbb{P}^{g-1} \text{ lying on } \mathbb{P}^{d-r-1}\}.$$

Consider the following incidence correspondence.

$$\begin{tikzcd}
 & \Sigma = \{(H, D) \mid D \subset H \cap C\} \subset (\mathbb{P}^{g-1})^* \times C_d^r \arrow[dl] \arrow[dr] & \\
(\mathbb{P}^{g-1})^* & & C_d^r \arrow[r] & W_d^r
\end{tikzcd}$$

The general fiber of $C_d^r \to W_d^r$ is isomorphic to $\mathbb{P}^r$.

Claim. $W_d^{r+1} \subsetneq W_d^r$. Even better, $W_d^r \setminus W_d^{r+1}$ is dense in W_d^r.

Proof. If D is an effective divisor, then for all but finitely many points $p \in C$, we have $r(D-p) = r(D) - 1$. Dually, assuming $K - D$ is effective, we have $r(K - D - p) = r(K - D) - 1$. By Riemann-Roch $r(D + p) = r(D)$ for all but finitely many points. We conclude that given $D \in C_d^r$ such that $0 \leq r < d - g$, for general $p, q \in C$, we have $r(D + p - q) = r(D) - 1$. □

The fibers of $\Sigma \to C_d^r$ are generically $\mathbb{P}^{g-d+r-1}$. The conclusion is

$$\dim \Sigma = \dim W_d^r + r + g - d + r - 1.$$

Let us look at the projection on the other side. The map $\Sigma \to (\mathbb{P}^{g-1})^*$ is finite and not surjective (dominant). We obtain $\dim \Sigma \leq g - 2$, so $\dim W_d^r \leq d - 2r - 1$. If C is hyperelliptic, then

$$W_d^r = \{r \cdot g_2^1 + W_{d-2r}^0\}$$

which has dimension $d - 2r$. □

LECTURE 21

November 18, 2011

21.1. Setting up Brill-Noether theory even further

Recall the following setting.

$$\begin{tikzcd} & C_d^r \arrow[r, hook] \arrow[d] & C_d \arrow[d, "u"] \\ G_d^r \arrow[r] & W_d^r \arrow[r, hook] & \operatorname{Pic}^d(C) = \operatorname{Jac}(C) \end{tikzcd}$$

Here

$$G_d^r = \{(L, V) \mid L \in \operatorname{Pic}^d(C), V^{r+1} \subset \mathrm{H}^0(L)\}.$$

We also have description of several tangent and cotangent spaces:

$$\begin{aligned} T_D C_d &= \mathrm{H}^0(\mathcal{O}_C(D)/\mathcal{O}_C), \\ T_D^* C_d &= \mathrm{H}^0(K/K(-D)), \\ T_L \operatorname{Jac}(C) &= \mathrm{H}^0(K)^*, \\ T_L^* \operatorname{Jac}(C) &= \mathrm{H}^0(K). \end{aligned}$$

Out goal is to study the codifferential map du^*.

$$\begin{tikzcd} du^* \colon T_L^* \operatorname{Jac}(C) \arrow[r] \arrow[d, equal] & T_D^* C_d \arrow[d, equal] \\ \mathrm{H}^0(K) \arrow[r] & \mathrm{H}^0(K/K(-D)) \end{tikzcd}$$

The map in the bottom row corresponds to the quotient map $K \to K/K(-D)$. Let $D = p_1 + \cdots + p_d \in C_d$ be a divisor of d distinct points. In a neighborhood of D, we can write

$$C_d^r = \left\{ \operatorname{rank} \begin{pmatrix} \omega_1(p_1) & \ddots & \omega_g(p_1) \\ \vdots & \ddots & \vdots \\ \omega_1(p_d) & \cdots & \omega_g(p_d) \end{pmatrix} \leq d - r \right\},$$

where $\omega_1, \ldots, \omega_g$ is a basis for $\mathrm{H}^0(K)$. More generally, this is a matrix representative of the map $\mathrm{H}^0(K) \to \mathrm{H}^0(K/K(-D))$. These vector spaces can be interpreted as fibers of vector bundles, the above morphism is induced by a morphism of vector bundles. Comparing to the problem about matrices we mentioned discussed earlier, we get the following local models.

$$\begin{tikzcd} C_d \supset U \arrow[r] & M_{d,g} \\ U \cap C_d^r \arrow[r] \arrow[u] & M_{d,g}^{d-r} \end{tikzcd}$$

Both vertical inclusions are of codimension $r(g-d+r)$. The conclusion is

$$\dim C_d^r \geq d - r(g-d+r)$$

everywhere. Then

$$\dim W_d^r = \dim C_d^r - r \geq d - r(g-d+r) - r = g - (r+1)(g-d+r).$$

We call this the *Brill-Noether number*

$$\rho = g - (r+1)(g-d+r).$$

There is second consequence worth mentioning. Consider the map

$$\begin{aligned} \alpha\colon \ \mathrm{H}^0(K) &\longrightarrow \mathrm{H}^0(K/K(-D)) = \mathrm{H}^0(K|_D), \\ \mu\colon \ \mathrm{H}^0(D) \otimes \mathrm{H}^0(K-D) &\longrightarrow \mathrm{H}^0(K). \end{aligned}$$

Then if $D \in C_d^r \setminus C_d^{r+1}$ and $L = \mathcal{O}_C(D)$, then

$$\begin{aligned} T_D C_d^r &= \mathrm{Ann}(\mathrm{Im}(\alpha\mu)), \\ T_L W_d^r &= \mathrm{Ann}(\mathrm{Im}(\mu)). \end{aligned}$$

Note that if μ is injective at a point L, then

$$\dim \mathrm{Im}(\mu) = (r+1)(g-d+r),$$

and W_d^r is smooth of dimension ρ at L.

21.2. Results

Theorem 21.1 (Brill-Noether existence)**.** *If $\rho \geq 0$, then $W_d^r \neq \emptyset$.*

Theorem 21.2. *If C is general, then $\rho < 0$ implies $W_d^r = \emptyset$. If $\rho \geq 0$, then $\dim W_d^r = \rho$.*

Corollary 21.3. *Assume C and $L \in W_d^r$ are general.*
- (i) *If $r \geq 3$, then φ_L is an embedding.*
- (ii) *If $r = 2$, then φ_L is birational onto a plane curve C_0 with only nodes as singularities.*
- (iii) *If $r = 1$, then $\varphi_L\colon C \to \mathbb{P}^1$ is simply branched.*

Example 21.4. Let C be a general curve of genus g. Then

$$\begin{aligned} \text{the smallest degree if a non-constant meromorphic function on } C &= \text{the smallest degree of a map } C \to \mathbb{P}^1 \\ &= \left\lceil \frac{g}{2} \right\rceil + 1, \end{aligned}$$

$$\text{the smallest degree of a plane curve birational to } C = \left\lceil \frac{2g}{3} \right\rceil + 2,$$

$$\text{the smallest degree of an embedding of } C = \left\lceil \frac{3g}{4} \right\rceil + 3.$$

The last of these equalities holds only for $g \geq 3$.

Theorem 21.5 (Gieseker-Petri)**.** *For a general curve C and L an arbitrary line bundle, the map*

$$\mu\colon \ \mathrm{H}^0(L) \otimes \mathrm{H}^0(K-L) \longrightarrow \mathrm{H}^0(K)$$

is injective. In particular,

$$(W_d^r)_{\mathrm{sing}} = W_d^{r+1}, \qquad (C_d^r)_{\mathrm{sing}} = C_d^{r+1},$$

and G_d^r is smooth.

Remark 21.6. Gieseker-Petri implies Theorem 2 above.

Remark 21.7. When $\rho = 0$, Gieseker-Petri says G^r_d consists of a finite set of reduced points. Therefore, we can compute this number using the Porteus formula:

$$\#\{g^r_d\text{-s on } C\} = g! \prod_{\alpha=0}^{r} \frac{\alpha!}{(g-d+r+\alpha)!} = g! \prod_{\alpha=0}^{g-d+r-1} \frac{\alpha!}{(r+1-\alpha)!}.$$

The last equality follows uses that

$$W^r_d = K - W^{g-d+r-1}_{2g-2-d},$$

that is, the map $\mathrm{Pic}^d(C) \to \mathrm{Pic}^{2g-2-d}(C)$ given by $L \mapsto K - L$ carries W^r_d isomorphically to $W^{g-d+r-1}_{2g-2-d}$.

Example 21.8.

(i) If $g = 4, r = 1, d = 3$, the corresponding number is 2, i.e., a general curve of genus 4 has 2 g^1_3-s.
(ii) If $g = 6, r = 1, d = 4$, the corresponding number is 5. We can see this by looking at a model for such a curve.
(iii) The next case is $g = 8, r = 1, d = 5$ when the count is 14, but there is no known way of deriving this directly.

Theorem 21.9 (Fulton-Lazarsfeld)**.** *If C is general and $\rho > 0$, then W^r_d is irreducible.*

Let C be a curve, $p_1, \dots, p_\delta \in C$ a sequence of points, and $\alpha^1, \dots, \alpha^\delta$ a list of ramification sequences (i.e., non-decreasing sequences of integers). Introduce

$$G^r_d(p, \alpha) = \{(L, V) \mid \alpha_k(p_i, V) \geq \alpha^i_k\}.$$

Theorem 21.10 (Brill-Noether with inflection)**.** *For C, $p_1, \dots, p_\delta$ general, the dimension of $G^4_d(p, \alpha)$ is the expected one, that is,*

$$\dim G^r_d(p, \alpha) = \rho - \sum_{i,k} \alpha^i_k.$$

Example 21.11. Let C be a non-hyperelliptic curve of genus 4. (The curve C is hyperelliptic if and only if $\dim W^1_3 = 1$.) Look at W^1_3, $\rho = 0$. Such a curve is the intersection of a quadric and a cubic surface in $\mathbb{P}^3$, say, $C = Q_2 \cap S_3$. Then the g^1_3-s correspond to triples of collinear points on Q_2.

Assume Q_2 is smooth and let L and M denote the two rulings on Q_2; then we get 2 g^1_3-s on C. Note that $L + M = K_C$ and consider the map

$$\mu\colon\ \mathrm{H}^0(L) \otimes \mathrm{H}^0(M) \longrightarrow \mathrm{H}^0(K_C).$$

The vector spaces in question have dimensions 2, 2, and 4. While the general machinery implies μ is an isomorphism, we can check this directly. If we think pass to a bilinear map $\mathrm{H}^0(L) \times \mathrm{H}^0(M) \to \mathrm{H}^0(K)$ and projectivize, we get a morphism $\mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^3$. Its image can be shown to be a smooth quadric, hence it spans all of $\mathbb{P}^3$. In other words, μ is surjective and this suffices for dimension reasons. A similar analysis can be carried out in the case Q_2 is a cone.

November 30, 2011

22.1. Towards the a Brill-Noether Theorem

Our aim is to prove a version of the Brill-Noether Theorem. The standard prove uses limit linear series. We will not need these objects in their complete generality, so we will introduce them only to the extent necessary.

We start by setting some notation. Let C be a smooth projective curve. Pick a g^r_d (L, V) on C, that is, $L \in \mathrm{Pic}^d(C)$ and $V^{r+1} \subset \mathrm{H}^0(L)$. For the sake of convenience, we will abbreviate this object by V. For $p \in C$, we define the vanishing sequence

$$\{a_0(V,p) < \cdots < a_r(V,p)\} = \{\mathrm{ord}_p(\sigma) \mid \sigma \in V \setminus \{0\}\}.$$

The ramification sequence is given by

$$\alpha_i(V,p) = a_i(V,p) - i,$$

and the total ramification is defined as

$$\alpha(V,p) = \sum_i \alpha_i(V,p).$$

The Plücker relation says

$$\sum_{p \in C} \alpha(V,p) = (r+1)(d + r(g-1)) = (r+1)d + \binom{r+1}{2}(2g-2).$$

The Brill-Noether number of the linear series V is

$$\rho(V) = g - (r+1)(g-d+r).$$

We plan to prove the following.

Theorem 22.1 (Brill-Noether, non-existence half)**.** *If C is general, then any linear series V on C satisfies $\rho(V) \geq 0$.*

Remark 22.2. The condition "any linear series V on C satisfies $\rho(V) \geq 0$" is open on the moduli space of curves. We will assume the existence of this space, and, in some sense, its irreducibility. Then, to show the result above, it suffices to exhibit a single curve. This will be our strategy.

Let C be a smooth projective curve and $p_1, \ldots, p_m$ a set of points on C. If C is any g^r_d on c, we can define the *adjusted Brill-Noether number* with respect to $p_1, \ldots, p_m$ as

$$\rho(V, p_1, \ldots, p_m) = \rho(V) - \sum_{k=1}^{m} \alpha(V, p_k).$$

We are now ready to state a more general result, which, it turns out, is easier to prove.

Theorem 22.3 (Adjusted Brill-Noether)**.** *Let C be a general curve, and $p_1, \ldots, p_m$ a sequence of general points on it. Then any linear series V on C satisfies*

$$\rho(V; p_1, \ldots, p_m) \geq 0.$$

The previous version is the specialization $m = 0$.

Remark 22.4. This is a statement with a variational components (i.e., it concerns general curves), so to prove it we have to talk about families of curves (even if we do not talk about the entire moduli space).

22.2. Families of curves, line bundles, and linear series

Definition 22.5. A *family of curves* over (a curve) B is a map $\pi\colon \mathcal{C} \to B$ such that for all $t \in B$, the fiber $C_t = \pi^{-1}(t)$ is a smooth projective curve of genus g. A *family of line bundles* is a collection $\{L_t \in \mathrm{Pic}^d(C_t)\}$. We can also think of these as a single line bundle $\mathcal{L}$ on $\mathcal{C}$ and then take $L_t = \mathcal{L}|_{C_t}$. Finally, a *family of linear series* on $\pi\colon \mathcal{C} \to B$ is a collection $\{V_t \subset \mathrm{H}^0(L_t)\}$ which can be interpreted as a rank $r+1$ vector subbundle $\mathcal{V}$ of $\pi_*\mathcal{L}$. Then

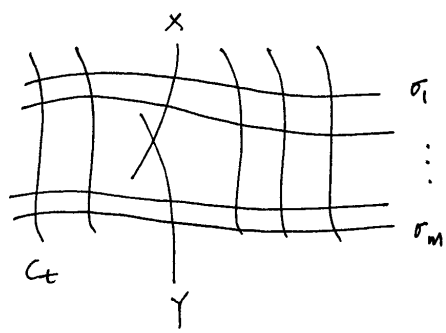
$$V_t = \mathcal{V}_t \subset (\pi_*\mathcal{L})_t = \mathrm{H}^0(C_t, \mathcal{L}|_{C_t}) = \mathrm{H}^0(C_t, L_t).$$

Remark 22.6. The bundle $\pi_*\mathcal{L}$ is locally-free. To see this start by noting it is a pushforward of a torsion free sheaf, hence torsion free. Assuming the base B is a curve, this suffices to conclude $\pi_*\mathcal{L}$ is locally-free.

Consider a family of curves $\pi\colon \mathcal{C} \to B$ with marking, i.e., these are sections $\sigma_1, \ldots, \sigma_m \colon B \to \mathcal{C}$ of π.

Claim. The function $\rho(V_t, \sigma_1(t), \ldots, \sigma_m(t))$ is lower semi-continuous. Alternatively, $\alpha(V_t, \sigma_1(t), \ldots, \sigma_m(t))$ is upper semi-continuous.

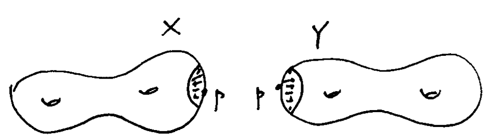
We will often use a disk Δ for a base. Consider a family $\pi\colon \mathcal{C} \to \Delta$. Then $\mathcal{C}$ is a smooth surface such that C_t is a smooth projective curve for $t \neq 0$. Furthermore, $C_0 = X \cup Y$ where both X and Y are smooth projective curves, and they intersect in a unique point p which is a node of C_0.



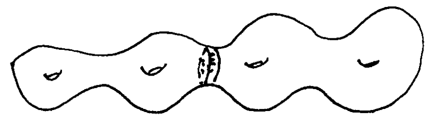
We would like to relate the Brill-Noether theory of the general and special fibers. Then, we would like to show that if Brill-Noether holds separately on the components X and Y, then it holds on the general fiber. Assume $\sigma_1, \ldots, \sigma_m$ are sections of this family such that $\sigma_k(0) \in X$ for $k = 1, \ldots, \delta$ and $\sigma_k(0) \in Y$ for $k = \delta+1, \ldots, m$. We aim to show the following statement.

Theorem 22.7. *If the adjusted Brill-Noether statement holds for $(X, \sigma_1(0), \ldots, \sigma_\delta(0), p)$ and for $(Y, \sigma_{\delta+1}(0), \ldots, \sigma_m(0), p)$, then it also holds for $(C_t, \sigma_1(t), \ldots, \sigma_m(t))$ for $|t|$ sufficiently small.*

After this, it suffices to demonstrate adjusted Brill-Noether for curves of genus 0 with arbitrary number of markings, and for curves of genus 1 with a unique marking. We start by describing how to construct families with the desired properties. Start with two smooth curves X and Y, and points $p \in X$, $p \in Y$ which we conveniently give the same name.



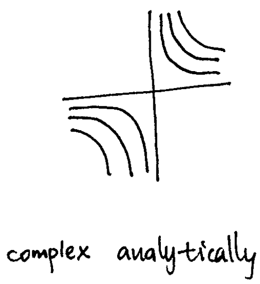
Choose a neighborhood U of p in each, and consider $(X \cup Y \setminus U) \times \Delta$. The fibers over Δ look like a single Riemann surface of genus $\mathrm{genus}(X) + \mathrm{genus}(Y)$ with a bank removed.

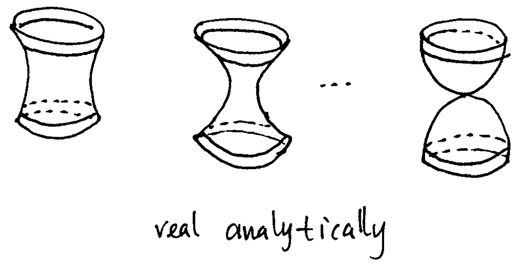


Next, consider

$$N = \{(z, w, t) \mid zw - t = 0, |z| < 1, |w| < 1, |t| < 1\} \subset \mathbb{C}^3$$

which looks as follows.





Remove a pair of annuli on each end and glue in the family we just considered. The result if a family of Riemann surfaces with the desired properties. To make our description complete, one needs to fill a number of analytic and topological details. Alternatively, this can be done purely algebraically via deformation theory.

Given a collection of markings

$$\sigma_1(0), \ldots, \sigma_\delta(0) \in X, \qquad \sigma_{\delta+1}(0), \ldots, \sigma_m(0) \in Y,$$

it is clear from the above construction how to extend these over the whole family. Denote $\Delta^* = \Delta \setminus \{0\}$ and $\mathcal{C}^* = \pi^{-1}(\Delta^*)$, so we have a punctured family $\mathcal{C}^* \to \Delta^*$. Consider a line bundle $\mathcal{L}^*$ on $\mathcal{C}^*$ and a vector subbundle $\mathcal{V}^* \subset \pi_*\mathcal{L}^*$.

Question 22.8. Can we extend $\mathcal{L}^*$ to a line bundle $\mathcal{L}$ on all of $\mathcal{C}$?

Question 22.9. If so, is this extension unique?

The answer to the first question is positive. Suppose $\mathcal{L}^* = \mathcal{O}_{\mathcal{C}^*}(D^*)$ and take the closure D of D^* (take closure component-wise and then copy multiplicities). The answer to the second question is no; rather than a complication, this turns out to be a useful feature. Let us investigate this in greater detail. Given two extensions $\mathcal{L}_1$, $\mathcal{L}_2$, then $\mathcal{L}_1 \otimes \mathcal{L}_2^{-1}$ is trivial on $\mathcal{C}^*$. Therefore, it suffices to study how $\mathcal{O}_{\mathcal{C}^*}$ extends over $\mathcal{C}$. Give $\mathcal{L}^* \cong \mathcal{O}_{\mathcal{C}^*}$, take a section σ of $\mathcal{L}^*$ corresponding to 1 and extend to a rational section σ of $\mathcal{L}$. Then the divisor (σ) is supported on $\mathcal{C} \setminus \mathcal{C}^* = X \cup Y$. Therefore, $(\sigma) = \alpha X + \beta Y$ for some $\alpha, \beta \in \mathbb{Z}$. In other words, we conclude that $\mathcal{L} = \mathcal{O}_{\mathcal{C}}(\alpha X + \beta_Y)$ which is trivial on $\mathcal{C}^*$ but not on $\mathcal{C}$. To see the latter, note that $\mathcal{O}_{\mathcal{C}}(X + Y) \cong \mathcal{O}_{\mathcal{C}}$, so $\mathcal{O}_{\mathcal{C}}(\alpha X + \beta Y) = \mathcal{O}_{\mathcal{C}}(mX)$ for $m = \alpha - \beta$. In conclusion, given $\mathcal{L}^*$, there exist $\mathcal{L}$ on $\mathcal{C}$ extending $\mathcal{L}^*$ and any other such is of the form $\mathcal{L} \otimes \mathcal{O}_{\mathcal{C}}(mX)$ for some $m \in \mathbb{Z}$. We should say why $\mathcal{O}_{\mathcal{C}}(mX)$ is non-trivial for $m \neq 0$. Note that $\mathcal{O}_{\mathcal{C}}(mX)|_Y = \mathcal{O}_Y(m \cdot p)$ which is non-trivial for $m \neq 0$. Dually,

$$\mathcal{O}_{\mathcal{C}}(mX)|_X = \mathcal{O}_{\mathcal{C}}(-mY)|_X = \mathcal{O}_X(-m \cdot p).$$

In particular, given $\mathcal{L}^*$ and any $\alpha \in \mathbb{Z}$, there exists a unique extension $\mathcal{L}_\alpha$ such that

$$\deg(\mathcal{L}_\alpha|_X) = \alpha, \qquad \deg(\mathcal{L}_\alpha|_Y) = d - \alpha.$$

The key idea is to get the full picture we need to look at all α. Actually, two extensions, $\mathcal{L}_d$ and $\mathcal{L}_0$, suffice. These have degrees concentrated on X and Y respectively. Note that $\mathcal{L}_d = \mathcal{L}_0(dY)$.

Next, look at a family of linear series on C_t for $t \neq 0$, that is, a line bundle $\mathcal{L}^*$ on $\mathcal{C}^*$, and a vector bundle $\mathcal{V}^* \subset \pi_*\mathcal{L}^*$. Choose an extension $\mathcal{L}_\alpha$ on $\mathcal{C}$ and $\mathcal{V}_\alpha \subset \pi_*\mathcal{L}_\alpha$. We have $(\mathcal{V}_\alpha)_0 \subset \mathrm{H}^0(\mathcal{L}_\alpha|_{C_0})$. Let us focus on $\mathcal{L}_d$ and $\mathcal{L}_0$. We get an inclusion

$$(\mathcal{V}_d)_0 \subset \mathrm{H}^0(\mathcal{L}_d|_{C_0}) \hookrightarrow \mathrm{H}^0(\mathcal{L}_d|_X)$$

since the values of a section over Y is determined by its value at p. This is a g^r_d on X. Focusing on $\mathcal{V}_0$, we get a g^r_d on Y. The crux of the matter is to understand the relative relation between these two g^r_d's on X and Y. We will continue this investigation next time.

December 2, 2011

23.1. The setup

Previously, we considered a smooth projective curve C, markings $p_1, \ldots, p_m \in C$, and V a g^r_d on C. We define

$$V(-ap) = \{\sigma \in V \mid \mathrm{ord}_p(\sigma) \geq a\} \subset V,$$

which can also be naturally viewed as a subspace of $\mathrm{H}^0(L(-ap))$. Define the adjusted Brill-Noether number

$$\rho(V; p_1, \ldots, p_m) = g - (r+1)(g-d+r) - \sum_{k=1}^{m} \alpha(V, p_k).$$

Theorem 23.1 (Adjusted Brill-Noether)**.** *If C, $p_1, \ldots, p_m$ are general, then any linear series V on C satisfies*

$$\rho(V; p_1, \ldots, p_m) \geq 0.$$

There are several cases in which we can already resolve this claim.

(a) Let $g = 0$, $C = \mathbb{P}^1$, and let $p_1, \ldots, p_m \in \mathbb{P}^1$ be general. Then theorem above reads

$$(r+1)(d-r) - \sum_{k=1}^{m} \alpha(V, p_k) \geq 0$$

for any V. Compare this to the Plücker formula

$$\sum_{p \in \mathbb{P}^1} \alpha(V, p) = (r+1)(d-r).$$

Therefore, the claim above holds even without the genericity hypothesis on the points p_i.

(b) Take $g = 1$ and $m = 1$. Then $C = E$ is an elliptic curve, $p_1 = p \in E$. Suppose $V \subset \mathrm{H}^0(L)$, and L has degree d. For of all, it is clear that $a_1(V, p) \leq d$. The key is to notice $a_{r-1}(V, p) \leq d - 2$. It is clear that $a_{r-1}(V, p) \leq d - 1$. If this number is $d - 1$, then $\dim V(-(d-1)p) \geq 0$ contradicting the fact $L(-(d-1)p)$ has degree 1. We can infer the following inequalities:

$$\begin{aligned}
a_r(V,p) &\leq d, & \alpha_r(V,p) &\leq d-r,\\
a_{r-1}(V,p) &\leq d-2, & \alpha_{r-1}(V,p) &\leq d-r-1,\\
a_{r-2}(V,p) &\leq d-3, & \alpha_{r-2}(V,p) &\leq d-r-1,\\
&\vdots & &\vdots\\
a_0(V,p) &\leq d-r-1, & \alpha_0(V,p) &\leq d-r-1.
\end{aligned}$$

Adding the right hand sides of the second column of inequalities we compute

$$\alpha(V, p) \leq (r+1)(d-r-1) + 1,$$

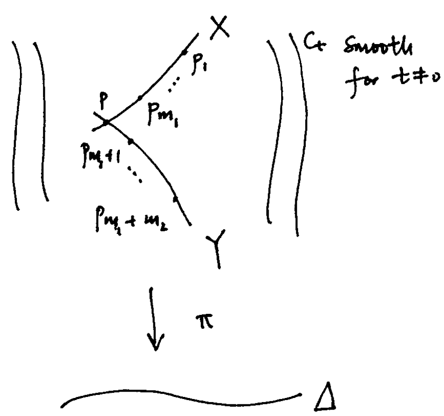
which is equivalent to the adjuster Brill-Noether statement in this case.

23.2. The proof

As discussed previously, we plan to give an inductive proof of the adjusted Brill-Noether statement, denoted by $\mathrm{ABN}_{g,m}$. We just tackles the base case by demonstrating $\mathrm{ABN}_{0,m}$ for all $m \geq 0$ and $\mathrm{ABN}_{1,1}$. For the inductive step, we will prove

$$\mathrm{ABN}_{g,m_1+1} \quad \text{and} \quad \mathrm{ABN}_{h,m_2+1} \quad \Longrightarrow \quad \mathrm{ABN}_{g+h,m_1+m_2}.$$

Start with general smooth curves X, Y of genera g, h respectively, and general points $p, p_1, \ldots, p_{m_1} \in X$, $p, p_{m_1+1}, \ldots, p_{m_1+m_2} \in Y$. Form a reducible nodal curve $C_0 = (X \cup Y)/p$, and deform it to a family over the disk Δ.



Consider a family $V_t \subset \mathrm{H}^0(L_t)$ of g^r_d's on C_t for $t \neq 0$. This is equivalent to a rank $r+1$ subbundle $\mathcal{V}^* \subset \pi_*\mathcal{L}^*$ for a line bundle $\mathcal{L}^*$ on $\mathcal{C}^*$. For each $\alpha \in \mathbb{Z}$, there exists a unique extension $\mathcal{L}_\alpha$ of $\mathcal{L}^*$ to $\mathcal{C}$ such that $\deg(\mathcal{L}_\alpha|_X) = \alpha$ and $\deg(\mathcal{L}_\alpha|_Y) = d - \alpha$. For example, if we take $\mathcal{L}_d$ to have degree d on X and 0 on Y, then

$$\mathcal{L}_\alpha = \mathcal{L}_d(-(d-\alpha)Y) \cong \mathcal{L}_d((d-\alpha)X).$$

For any α, let $\mathcal{V}_\alpha$ be the saturation/closure of $\mathcal{V}^*$ in $\pi_*\mathcal{L}_\alpha$. Set $V_\alpha = (\mathcal{V}_\alpha)_0 \subset \mathrm{H}^0(\mathcal{L}_\alpha|_{C_0})$. We will focus on the two extremes $\mathcal{L}_0$ and $\mathcal{L}_d$. There is an inclusion

$$V_d \subset \mathrm{H}^0(\mathcal{L}_d|_{C_0}) \qquad \mathrm{H}^0(\mathcal{L}_d|_X).$$

Set $L = \mathcal{L}_d|X$, $W = \operatorname{Im} V_d \subset \mathrm{H}^0(L)$, a g^r_d on X. Similarly, there is an inclusion

$$V_0 \subset \mathrm{H}^0(\mathcal{L}_0|_{C_0}) \hookrightarrow \mathrm{H}^0(\mathcal{L}_0|_Y),$$

and set $M = \mathcal{L}_0|_Y$, $U = \operatorname{Im} V_0 \subset \mathrm{H}^0(M)$, a g^r_d on Y. Choose sections $\sigma_1, \ldots, \sigma_{m_1+m_2}$ such that $\sigma_k(0) = p_k$ for all $1 \leq k \leq m_1 + m_2$, i.e., extend the markings of C_0 over all of $\mathcal{C}$. This might require shrinking the base disk Δ but that is not a problem. Note that by upper semi-continuity

$$\alpha_i(V_t, \sigma_k(t)) \leq \alpha_i(W, p_k) \quad \text{for} \quad k = 1, \ldots, m_1,$$
$$\alpha_i(V_t, \sigma_k(t)) \leq \alpha_i(U, p_k) \quad \text{for} \quad k = m_1+1, \ldots, m_1+m_2.$$

For each $\alpha \in \mathbb{Z}$, we have $\mathcal{L}_\alpha = \mathcal{L}_d(-(d-\alpha)Y) = \mathcal{L}_0(-\alpha X)$ and $V_\alpha \subset \mathrm{H}^0(\mathcal{L}_\alpha|_{C_0})$. Note that the natural map

$$\mathrm{H}^0(\mathcal{L}_\alpha|_{C_0}) \qquad \mathrm{H}^0(\mathcal{L}_\alpha|_X) \oplus \mathrm{H}^0(\mathcal{L}_\alpha|_Y)$$

is an inclusion. The image is either the entire space (if p is a basepoint of both W and U, that is, all sections over X and Y vanish at p) of codimension 1. It follows there are inclusions

$$V_\alpha|_X \hookrightarrow W(-(d-\alpha)p) \qquad \text{and} \qquad V_\alpha|_Y \hookrightarrow W(-\alpha p).$$

It follows that

$$(*)\qquad \dim W(-(d-a)p) + \dim U(-ap) \geq r+1$$

for all $\alpha \in \mathbb{Z}$, and equality holds only of p is a basepoint of both linear series. Observe that

$$\dim W(-(d-a)p) = r+1 - \#\{i \mid a_i(W,p) < d-a\},$$
$$\dim U(-ap) = r+1 - \#\{i \mid a_i(U,p) < a\}.$$

Adding these inequalities we obtain

$$a_i(W,p) + a_{r-i}(U,p) \geq d, \qquad \alpha_i(W,p) + \alpha_{r-i}(U,p) \geq d-r,$$

hence

$$\alpha(W,p) + \alpha(U,p) \geq (r+1)(d-r).$$

There is an alternative way to say this. Assume $a_i(W,p) = b$, $a_{r-i}(U,p) = a$, and $a+b < d$. It follows that $a_0(U,p), \dots, a_{r-i-1}(U,p)$ are all strictly less than a. When we require vanishing to order a, we reduce the number of sections by this precise number, hence $\dim U(-ap) = i+1$. Furthermore, there are no basepoints since the hypothesis is vanishing to order a precisely. Using analogous logic

$$\dim W(-(d-a)p) \leq \dim W(-(b+1)p) \leq r-i.$$

Adding these inequalities, we contradict $(*)$.

Finally, let us compare the statements of adjusted Brill-Noether in the cases we are interested in. Assume equalities

$$\rho(W; p_1, \dots, p_{m_1}, p) = g - (r+1)(g-d+r) - \sum_{k=1}^{m_1} \alpha(W, p_k) - \alpha(W,p),$$
$$\rho(U; p_{m_1+1}, \dots, p_{m_1+m_2}, p) = h - (r+1)(h-d+r) - \sum_{k=m_1+1}^{m_1+m_2} \alpha(U, p_k) - \alpha(U,p).$$

Adding these up and using upper semi-continuity, we get

$$\begin{aligned}
\rho(W; p_1, \dots, p_{m_1}, p) + \rho(U; p_{m_1+1}, \dots, p_{m_1+m_2}, p) &= g+h+(r+1)(g+h-d+r) + (r+1)(d-r) \\
&\quad - \sum_{k=1}^{m_1+m_2} \alpha(W \text{ or } U, \sigma_k(t)) - (r+1)(d-r) \\
&\leq g+h+(r+1)(g+h-d+r) + (r+1)(d-r) \\
&\quad - \sum_{k=1}^{m_1+m_2} \alpha(V_k, \sigma_k(t)) - (r+1)(d-r) \\
&= \rho(V_t; \sigma_1(t), \dots, \sigma_{m_1+m_2}(t)).
\end{aligned}$$

This completes our claim.