Contents

Chapter 15

Using a singular plane model

In the first part of Chapter 4 we showed how to use an embedding of a smooth curve C in $\mathbb{P}^2$ to understand differentials and linear series on C. In that form, the technique has limited applicability, since most smooth curves cannot be embedded in $\mathbb{P}^2$. However, by Proposition 10.11, any smooth curve C can be projected birationally to a curve $C_0 \subset \mathbb{P}^2$ with only nodes. We open this chapter by showing how to use such a nodal model C_0 to describe differentials and linear series on C, a theory well-understood by Brill and Noether; and then explain what is necessary to adapt the technique to birational images of C with arbitrary singularities.

15.1. Nodal plane curves

The methods of Chapter 4 can be applied, with one change, when $C_0 \subset \mathbb{P}^2$ is a nodal curve.

Let $\nu : C \to C_0 \subset \mathbb{P}^2$ be the normalization morphism from a smooth curve, and let $B \subset C_0$ be a subscheme. It will be convenient to speak of *the linear series cut out on C* by curves of degree m containing B: though B is only a subscheme, $\nu^{-1}(B)$ may be considered as a Cartier divisor because C is smooth, and we define the linear series cut out on C by curves of degree m containing B to be the linear series $\mathcal{V}$ of divisors in C that are residual to $\nu^{-1}(B)$ in the pullbacks of the intersections of C_0 with curves of degree m. Formally, since $B' := \nu^{-1}(B)$ is a Cartier divisor on C, we can say that if L is the divisor of a line in $\mathbb{P}^2$ and L' its pullback to C then $\mathcal{V}$ is a space of divisors corresponding to sections of $\mathcal{O}_C(mL' - B')$; more precisely, it is the space of divisors corresponding to

sections of $\mathcal{O}_C(mL' - B')$ in the image of the restriction/pullback map
$$H^0(\mathcal{I}_{B/\mathbb{P}^2}(m)) \to H^0(\mathcal{O}_C(mL' - B')).$$

15.1.1. Differentials on a nodal plane curve. Let $C_0 \subset \mathbb{P}^2$ be a curve of degree d with δ nodes and no other singularities. By the adjunction formula (Proposition 2.8), Proposition 2.25, and the first example that follows it, the genus g of the normalization C of C_0 is the arithmetic genus $p_a(C_0) = \binom{d-1}{2}$ of C_0 minus δ, that is,
$$g = \binom{d-1}{2} - \delta.$$
We will make this explicit by exhibiting a vector space of g regular differential forms on C.

Choose homogeneous coordinates $[X, Y, Z]$ on $\mathbb{P}^2$ so that C_0 intersects the line $L = V(Z)$ in a divisor D consisting only of smooth points of C_0 other than $[0, 1, 0]$, and so that at each node of C_0 (necessarily contained in the affine plane $U = \mathbb{P}^2 \setminus L$) the tangents to C_0 have finite slope. Let the nodes of C_0 be $q_1, \dots, q_\delta$, with $r_i, s_i \in C$ lying over q_i; we'll denote by Δ the divisor $\sum r_i + \sum s_i$ on C.

Let $F(X, Y, Z)$ be the homogeneous polynomial of degree d defining the curve C_0, and let $f(x, y) = F(x, y, 1)$ be the defining equation of the affine part $C_0^\circ := C_0 \cap U$ of C_0. Let $\nu : C \to C_0$ be the normalization map. We start by considering the rational differential $\nu^*(dx)$ on $C^\circ := \nu^{-1}(C_0^\circ)$.

In the smooth case where $C_0 = C$ we saw that this differential was regular and nonzero on C°; this followed from the fact that f_x and f_y had no common zeroes on C_0. But now f_x and f_y have common zeroes: they both vanish to order 1 at the points q_i and thus $\nu^*(f_x)$ and $\nu^*(f_y)$ have simple zeroes at the points r_i and s_i.

As before, the differential $\nu^*(dx)$ has double poles along the divisor D on C_0 lying over the point at infinity in $\mathbb{P}^1$ and we see that for a polynomial $e(x, y)$ of degree $\leq d - 3$, the differential
$$\nu^*\left(\frac{e(x, y)\, dx}{f_y}\right)$$
is regular except for simple poles at the points r_i and s_i.

We can get rid of these poles by requiring that e vanishes at the points q_i. We say in this case that e (and the curve defined by e) . *satisfies the conditions of adjunction*.

Theorem 15.1. *If C_0 is a nodal plane curve of degree d with normalization ν : $C \to C_0$ then the regular differentials on C, in terms of the notation above, are precisely those of the form*
$$\nu^*\left(\frac{e(x, y)\, dx}{f_y}\right),$$

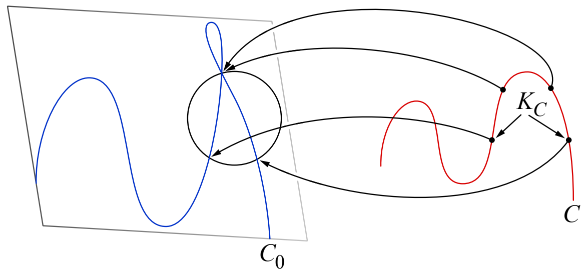

Figure 15.1. A curve C of geometric genus 2 represented as the normalization of a plane curve C_0 of degree 4 with a node, and a canonical divisor, represented by a conic containing the node.

where $e(x, y)$ ranges over the polynomials of degree $\leq d-3$ vanishing at the nodes of C_0.

Thus if $\mathfrak{F}(C_0) \subset C_0$ denotes the union of the reduced points at the nodes of C, then $|\omega_C|$ is the linear series cut out on C by forms of degree $d-3$ containing $\mathfrak{F}(C_0)$.

See Figure 15.1 for a picture in a case where C_0 has a single node.

Proof. The dimension of the space of polynomials $e(x, y)$ of degree at most $d-3$ is $\binom{d-1}{2}$, and vanishing at δ nodes imposes at most δ linear conditions on e. The linear map sending $e \mapsto \nu^*(e\,dx/f_y)$ is injective, and the target has dimension $\binom{d-1}{2} - \delta$, so this must be an isomorphism. □

We will give a more conceptual proof of this theorem in Section 15.2.

In particular, Theorem 15.1 shows that the linear series cut out on C by forms of degree $d-3$ containing $\mathfrak{F}(C_0)$ is complete. (We will soon see that the linear series cut out on C by forms of degree m containing $\mathfrak{F}(C_0)$ is complete for every m.)

This gives another proof of Lemma 8.16.

Corollary 15.2. *If C is a nodal plane curve of degree d, then the nodes of C_0 impose independent conditions on forms of degree $d-3$.*

Proof. Otherwise the space of differential forms on the normalization of C_0 would be too large. □

One can use Theorem 15.1 to re-embed a plane curve of geometric genus g as a canonical curve in $\mathbb{P}^{g-1}$:

Corollary 15.3. *The canonical ideal of the normalization of a nodal plane curve of degree d is the ideal of polynomial relations among the forms of degree $d-3$ that vanish at the nodes of the curve.* □

15.1.2. Linear series on a nodal plane curve. Since C_0 is singular, not every effective divisor on C is the preimage of an effective Cartier divisor on C_0. As an example, one may take a single point lying over a node as in Figure 15.2.

However, we can still represent every divisor on C as the preimage of a divisor on C_0 up to linear equivalence, and the same goes for any reduced curve:

Lemma 15.4. *Let $\nu : C \to C_0$ be the normalization of any reduced projective curve. If D is any divisor on C, then D is linearly equivalent to the pullback of a divisor supported on the smooth locus of C_0. More precisely, every effective divisor on C containing Δ can be written as the pullback of a Cartier divisor on C_0.*

We will see a more general version in Theorem 15.12.

Proof. It suffices to prove the result locally on C_0, where it is geometrically obvious: if the node $p \in C_0$ has preimages q, r corresponding to branches Q and R of C_0 at p, then a divisor $aq + br$ with both a, b strictly positive, is locally the pullback of the intersection of C_0 with a curve C' meeting Q with multiplicity a at p and meeting R with multiplicity b at p. For example, assuming that $a \le b$, we could take C' to be the union of $a - 1$ general lines through p and a smooth plane curve meeting R with multiplicity $b - a + 1$ at p. □

Returning to the case of a nodal plane curve C_0 and its normalization $\nu : C \to C_0$, suppose that D is a divisor on C that is the pullback of a difference of Cartier divisors $D_+ - D_-$ on C_0. We will compute the complete linear series $|D|$. Let $\mathfrak{F}(C_0)$ be the set of nodes in C and let Δ be the preimage of $\mathfrak{F}(C_0)$ in C.

Theorem 15.5. *Let $D = D_+ - D_-$ be a divisor on C, and let G be a form on $\mathbb{P}^2$ that vanishes on $D_+ + \mathfrak{F}(C_0)$ but not identically on C_0.*

If G has degree m and $A = (\nu^* G) - D_+ - \Delta$, then every effective divisor on C linearly equivalent to D (if any) has the form $(\nu^* H) - D_- - A - \Delta$ for some H of degree m that vanishes on $\mathfrak{F}(C_0) + A$ but not identically on C_0.

The reason for including $\mathfrak{F}(C_0)$ is that an arbitrary linear series on C cannot be represented as the pullback of a linear series on C_0 without this. We will see a generalization in Theorem 15.13 and the surrounding discussion.

Corollary 15.6. *With notation as in Theorem 15.5, the ideal of the image of C with respect to the linear series $|D|$ is the ideal of polynomial relations among the forms of degree m in $\mathfrak{f}_{C/C_0}$ that vanish on $D_- + A$.* □

Proof of Theorem 15.5. Choose an integer m big enough that there is a form G vanishing on D_+ and $\mathfrak{F}(C_0)$ so that $\nu^*(G)$ vanishes on $D_+ + \Delta$, but not everywhere on C. (If D_+ contains some positive multiple of a point p of Δ this means that G defines a curve sufficiently tangent to the corresponding branch of C_0.) Then, as before, we can write the zero locus of G pulled back to C as

$$(\nu^* G) = D_+ + \Delta + A.$$

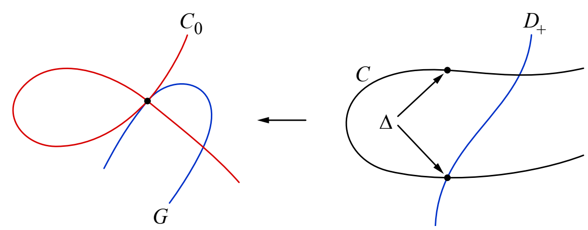

Figure 15.2. If G is tangent to a branch of C_0 at the node, then D_+ contains a point of Δ on C.

Next, we look for forms H of the same degree m, vanishing at $A + D_-$ and on $\mathfrak{F}(C_0)$ but not on all of C_0. If there are no such polynomials H then, as we shall show, there are no effective divisors equivalent to D. Supposing that there is such a form H, let D' be the divisor

$$D' = (\nu^* H) - (D_+ + \Delta),$$

that is, D' is residual to $(D_+ + \Delta)$ in $(\nu^* H)$.

Since $\nu^*(G/H)$ is a rational function on C we have

$$D_- + \Delta + A + D' = (\nu^* H) \sim (\nu^* G) = D_+ + \Delta + A,$$

and thus D' is an effective divisor linearly equivalent to $D = D_+ - D_-$ on C.

To complete the argument we must show that we get *all* divisors D' in this way. In this case the curve C can be desingularized by blowing up the plane once at each node, and we can give a proof based on the resulting surface S. The same technique would work for any curve with only ordinary multiple points, in which case the total transform of C_0 on S has normal crossings. We will give a different proof, extending this theorem to curves with arbitrary singularities, in Section 15.2.

Proposition 15.7. *If C_0 is a reduced irreducible plane curve all of whose singularities are ordinary nodes, then for each integer m, the linear series cut out on the normalization C of C_0 by forms of degree m containing the nodes is complete.*

Proof. To prove Proposition 15.7, we work on the blow-up $\pi : S \to \mathbb{P}^2$ of $\mathbb{P}^2$ at the nodes q_i of C_0. The proper transform of $C_0 \subset \mathbb{P}^2$ in S is the normalization of C_0, which we will again call C.

Let L be the class on S of the pullback of a line in $\mathbb{P}^2$ and let E be the sum of the exceptional divisors, the preimage of $\mathfrak{F}(C_0)$. We write $h = L \cap C$ and $e = E \cap C = \sum(p_i + q_i)$ for the corresponding divisors on C. Because C has double points at each q_i we have $C \sim dL - 2E$ and by Theorem 2.39 we have $K_S \sim -3L + E$.

The proper transform of a degree m curve $A \subset \mathbb{P}^2$ passing simply through the points q_i is $\pi^*A - E$; this gives an isomorphism

$$H^0(\mathcal{I}_{\{q_1,\dots,q_\delta\}/\mathbb{P}^2}(m)) \cong H^0(\mathcal{O}_S(mL - E)).$$

In these terms we can describe the linear series cut on C by plane curves of degree m passing through the nodes of C_0 as the image of the map

$$H^0(\mathcal{O}_S(mL - E)) \to H^0(\mathcal{O}_C(mL - E)),$$

and we must show that this map is surjective.

From the long exact cohomology sequence associated to the exact sequence of sheaves

$$0 \to \mathcal{O}_S((m-d)L + E) \to \mathcal{O}_S(mL - E) \to \mathcal{O}_C(mL - E) \to 0,$$

we see that it will suffice to prove that $H^1(\mathcal{O}_S((m-d)L + E)) = 0$.

By Serre duality on S,

$$H^1(\mathcal{O}_S((m-d)L + E)) \cong H^1(\mathcal{O}_S((d-m-3)L))^*.$$

The line bundle $\mathcal{O}_S((d-m-3)L)$ is the pullback to S of the bundle $\mathcal{O}_{\mathbb{P}^2}(d-m-3)$, which has vanishing H^1. Lemma 15.8 completes the proof. □

Lemma 15.8. *Let X be a smooth projective surface, and $\pi : S \to X$ the blow-up of a finite set of reduced points. If $\mathcal{L}$ is any line bundle on X, then*

$$H^1(S, \pi^*\mathcal{L}) = H^1(X, \mathcal{L}).$$

Proof. Because $\mathbb{P}^2$ is normal, and $\pi_*(\mathcal{O}_S)$ is a finite birational algebra over $\mathcal{O}_{\mathbb{P}^2}$, we have $\pi_*(\mathcal{O}_S) = \mathcal{O}_{\mathbb{P}^2}$. Since any invertible sheaf $\mathcal{L}$ on $\mathbb{P}^2$ is locally isomorphic to $\mathcal{O}_{\mathbb{P}^2}$, is also an isomorphism.

The Leray spectral sequence (Theorem 2.26) gives an exact sequence

$$0 \to H^1(\pi_*(\mathcal{L})) \to H^1(\mathcal{L}) \to H^0(R^1(\pi_*(\mathcal{L})) \to 0$$

The restriction of $\pi^*(\mathcal{L})$ to any fiber of π is trivial and has vanishing H^1, so $H^0(R^1(\pi_*(\mathcal{L}))) = 0$, and $\pi_*\pi^*\mathcal{O}_{\mathbb{P}^2}(1)$ is an invertible sheaf. The natural map $\pi_*\pi^*(\mathcal{L}) \to \mathcal{L}$ is an isomorphism away from the codimension 2 set of points blown up. Thus these two sheaves are isomorphic, and

$$H^1(S, \pi^*\mathcal{L}) = H^1(\pi_*\pi^*\mathcal{L}) = H^1(\mathcal{L}),$$

completing the proof. □

This concludes the proof of Theorem 15.5 □

Proposition 15.9. *Let C be a curve on a smooth surface S, and let $\nu : S' \to S$ be the blowup of S at p. If C' is the strict transform of C, then*

$$p_a(C') = p_a(C) - \binom{m}{2},$$

where m is the multiplicity of $p \in C$.

Proof. This follows from comparing the adjunction formulas on S and S'. To start, we have

$$p_a(C) = \frac{C^2 + K_S \cdot C}{2} + 1.$$

On S' let E be the divisor class of the exceptional divisor. As we've seen,

$$K_{S'} = \nu^* K_S + E,$$

while the class of C' is given by

$$C' \sim \nu^* C - mE.$$

It follows that

$$(C')^2 = C^2 + m^2E^2 = C^2 - m^2 \quad \text{and} \quad K_{S'} \cdot C' = K_S \cdot C + m.$$

Thus, applying adjunction on S', we find the desired equality:

$$p_a(C') = \frac{C'^2 + K_{S'} \cdot C'}{2} + 1 = \frac{C^2 + K_S \cdot C - m(m-1)}{2} + 1 = p_a(C) - \binom{m}{2}. \qquad \square$$

Cheerful Fact 15.10. Any plane curve can be desingularized by iteratively blowing up of singular points of C, then of the strict transform, and so on. See for example [?] or [?]. This is in fact the same sequence of transforms as the one given in Exercise 2.15. The points on the various blowups that map to the original singular point are called *infinitely near points*.

This gives a nice formula for the δ invariant of any singularity:

Corollary 15.11. *The δ invariant of any singularity of a plane curve C_0 at a point p can be computed as the sum of the numbers $\binom{m_q}{2}$ over all infinitely near singular points q, where m_q denotes the multiplicity of the pullback of C_0 at q.* $\square$

15.2. Arbitrary plane curves

Throughout this section $C_0 \subset \mathbb{P}^2$ denotes a reduced and irreducible plane curve with arbitrary singularities. Let $\nu : C \to C_0$ be its normalization, and write L' for the pullback to C of the class L of a line in $\mathbb{P}^2$.

It is possible to carry out an analysis of linear series on the normalization of an arbitrary plane curve in a manner analogous to what we did in the preceding section for nodal curves, replacing the set of nodes by the *adjoint scheme* $\mathfrak{F}(C_0) \subset C_0$, which in the case of a plane curve is the scheme defined by the *conductor ideal* $\mathfrak{f}_{C/C_0}$, the annihilator in $\mathcal{O}_{C_0}$ of $\nu_*(\mathcal{O}_C)/\mathcal{O}_{C_0}$ (see Theorem 1.17).

The conductor ideal and linear series on the normalization. Let $\Delta \subset C$ be the divisor defined by the pullback of $\mathfrak{f}_{C/C_0}$ to C. The following result gives a simple way of expressing any divisor on C in terms of Δ and a Cartier divisor on C_0:

Theorem 15.12. *If D is an effective divisor on C that contains Δ, then D is the pullback of a Cartier divisor on C_0.*

Proof. The result is local, so it suffices to treat the affine case of an affine curve C_0 with coordinate ring $\mathcal{O}'$ contained in its integral closure $\mathcal{O}$, the coordinate ring of its normalization C. The ideal of Δ in $\mathcal{O}$ is the conductor $\mathfrak{f}_{\mathcal{O}/\mathcal{O}'}$ (which is contained in $\mathcal{O}'$, but stable under multiplication by elements of $\mathcal{O}$, and thus also an ideal of $\mathcal{O}$). Thus $\mathcal{I}_D$ may be regarded as an ideal — though not necessarily a principal ideal — of $\mathcal{O}'$. Since the ground field $\mathbb{C}$ is infinite, $\mathcal{I}_D \subset \mathcal{O}'$ is the integral closure of a principal ideal (x) of $\mathcal{O}'$. (See [?, Chapter 8], for example.) This means that for sufficiently large n we have $x\mathcal{I}_D^n = \mathcal{I}_D^{n+1}$.

Let D_0 be the Cartier divisor on C_0 corresponding to x. Pulling everything back to C we have $\nu^*(D_0) + nD = (n+1)D$, whence $\nu^*(D_0) = D$. □

In view Theorem 15.12 we can specify an arbitrary linear series on C as the difference $\mathcal{E} - \Delta$, where $\mathcal{E}$ is a linear series with Δ in its base locus, by specifying a linear series on C_0 containing $\mathfrak{F}(C_0) \subset C_0$. The following theorem makes this algorithmic.

Theorem 15.13. *Let $D = D_+ - D_-$ be a divisor on C, and let G be a form on $\mathbb{P}^2$, vanishing on $\mathfrak{F}(C_0)$, whose pullback to C vanishes on D_+ but not on all of C.*

If G has degree m and $(\nu^*G) = D_+ + A + \Delta$, then every effective divisor on C linearly equivalent to D (if any) has the form $(\nu^*H) - D_- - A - \Delta$ for some H of degree m vanishing on $\mathfrak{F}(C_0) + A$. Thus, the global sections of $H^0(\mathcal{O}_C(D))$ can be written as divisors cut by forms of degree m in $\mathfrak{f}_{C/C_0}$ with basepoints at $D_- + A + \Delta$.

It follows that one can compute the image of C under the map given by $|D|$ directly from C_0 and the conductor:

Corollary 15.14. *With notation as in Theorem 15.13, the ideal of the image of C with respect to the linear series $|D|$ is the ideal of polynomial relations among the forms of degree m in $\mathfrak{f}_{C/C_0}$ that vanish on $D_- + A$.* □

Proof of Theorem 15.13. If $H \in \mathfrak{f}(C_0)$ vanishes on A but not on all of C_0, then as before

$$D_- + \Delta + A + D' = (\nu^*H) \sim (\nu^*G) = D_+ + \Delta + A,$$

So $D' = (\nu^*H) - D_- - A - \Delta$ is linearly equivalent to D. The proof that every divisor D' linearly equivalent to D has this form is the content of Theorem 15.15 below, which was known classically as the *completeness of the adjoint series*. □

Theorem 15.15. *For every integer $m \geq 0$ the series cut out on C by forms of degree m on $\mathbb{P}^2$ containing $\mathfrak{F}(C_0)$ is complete.*

Proof. If $R_0 \subset R$ is an inclusion of commutative rings, then the ideal

$$\mathfrak{f}_{R/R_0} := \operatorname{ann}_{R_0}(R/R_0) \subset R_0$$

is called the *conductor of* $R_0 \subset R$. It is by definition an ideal of R_0, but is also an ideal of R; this follows because if $f \in \mathfrak{f}_{R/R_0}$ and $r \in R$, then $fR \subset R_0$ so $(rf)R = frR \subset fR \subset R_0$.

If R_0 is a domain and R is a subring of the quotient field $Q(R)$ of R, then $\mathfrak{f}_{R/R_0} \cong \operatorname{Hom}_{R_0}(R, R_0)$. To see this, note that R_0 and R become equal after tensoring with $Q(R_0)$ and thus $\operatorname{Hom}_{R_0}(R, R_0) \subset \operatorname{Hom}_Q(Q, Q) = Q$ may be identified with the set of elements $\{\alpha \in Q \mid \alpha R \subset R_0\}$. If α is in this set, then $\alpha \cdot 1 = \alpha \in R_0$, as required.

Returning to the case of the curve C_0, it follows that the global sections of $(\nu^*(\mathcal{O}_{C_0}(m))(-\Delta)$ on C are, on each affine open set U, represented by the elements of $\mathcal{O}_{C_0}(U)$ that are restrictions to U of forms of degree m contained in the ideal $\mathfrak{f}_{C/C_0}$. Thus the global sections of the sheaf $\widetilde{\mathfrak{f}_{C/C_0}}(m)$ cut out a complete linear series on C.

Write $S = \mathbb{C}[x_0, x_1, x_2]$ for the homogeneous coordinate ring of $\mathbb{P}^2$. It remains to prove that the homogeneous ideal $\mathfrak{f}_{C/C_0}$ maps surjectively to $H^0_*(\widetilde{\mathfrak{f}_{C/C_0}})$, and this amounts to the statement that the depth of $\mathfrak{f}_{C/C_0}$ as an S-module is at least 2. Set $R_0 = H^0_*(\mathcal{O}_{C_0})$ and $R = H^0_*(\nu_*(\mathcal{O}_C))$. We see from the general considerations above that

$$\mathfrak{f}_{R/R_0} = \operatorname{Hom}_{R_0}(R, R_0).$$

Any nonzerodivisor on a module M is a nonzerodivisor on $\operatorname{Hom}(P, M)$ for any module P since $(a\phi)(p) = a(\phi(p))$ by definition. Since $R_0 = S/(F)$, it is a module of depth 2, and we may choose a regular sequence a, b of elements in R_0. From the short exact sequence

$$0 \to R_0 \xrightarrow{a} R_0 \to R_0/(a) \to 0$$

we get a left exact sequence

$$0 \to \operatorname{Hom}_{R_0}(R, R_0) \xrightarrow{a} \operatorname{Hom}_{R_0}(R, R_0) \to \operatorname{Hom}_{R_0}(R, R_0/(a)).$$

Thus

$$\operatorname{Hom}_{R_0}(R, R_0)/a \operatorname{Hom}_{R_0}(R, R_0) \subset \operatorname{Hom}_{R_0}(R, R_0/(a))$$

and since b is a nonzerodivisor on $\operatorname{Hom}_{R_0}(R, R_0/(a))$, it is a nonzerodivisor on $\operatorname{Hom}_{R_0}(R, R_0)/(a \operatorname{Hom}_{R_0}(R, R_0))$ as well. □

Since we saw directly that the adjoint ideal was equal to the conductor ideal in the case of a nodal curve, this result gives another, less ad hoc, proof that the effective divisors equivalent to D are all defined by pullbacks of forms of degree m that contain Δ as constructed in Proposition 15.7.

Differentials. Let C_0° be the intersection of C_0 with the open set $\mathbb{A}^2 \cong U \subset \mathbb{P}^2$ where $Z \neq 0$, and let $C^\circ \subset C$ be the preimage of C_0°.

Theorem 15.16. *If C_0 meets the line L at infinity only in smooth points of C_0 other than $(0,1,0)$, then the complete canonical series on the normalization $\nu : C \to C_0$ is cut out by differentials of the form*

$$\frac{e(x,y)\,dx}{f_y}$$

where $e(x,y)$ is a polynomial of degree $\leq d-3$ contained in the conductor ideal $\mathfrak{f}_{C^\circ/C_0^\circ}$.

As in the case of nodal curves, one can use Theorem 15.16 to re-embed a plane curve of geometric genus g as a canonical curve in $\mathbb{P}^{g-1}$:

Corollary 15.17. *The canonical ideal of the normalization C of a plane curve C_0 of degree d is the ideal of polynomial relations among the forms of degree $d-3$ in the conductor ideal $\mathfrak{f}_{C/C_0}$.* □

Proof of Theorem 15.16. We proceed in four steps. First, because $(0,1,0)$ does not lie on C, the function x defines a ramified d-sheeted cover of C to $\mathbb{P}^1$. Because C_0 meets L only in smooth points and the differential dx has a pole of order 2 at the point at infinity in $\mathbb{P}^1$, dx has polar locus twice the divisor $\nu^{-1}(C_0 \cap L)$. It follows that the differential $\varphi_0 := dx/f_y$ is regular, with a zero of order $d-3$, along the divisor of C lying over $C_0 \cap L$.

Second, the function on C_0° defined by x is a finite map to $\mathbb{A}^1$, and thus the field of rational functions $\kappa(C) = \kappa(C_0^\circ)$ is a finite separable extension of $\mathbb{C}(x)$. By [?, Section 16.5], the module of differentials $\omega_{\kappa(C)/\mathbb{C}}$ is generated over $\kappa(C_0^\circ)$ by dx. Thus every rational differential form on C can be expressed as a rational function times dx. Since $\varphi_0 := dx/f_y$ vanishes to order $d-3$ along $C_0 \cap L$, the regular differential forms on C must be of the form $e(x,y)\varphi_0$ where $e(x,y)$ is a rational function of degree $\leq d-3$. (The set of rational forms that occur in this way is called the *Dedekind complementary module.*)

Third, a sophisticated form of Hurwitz's theorem, to be explained in Chapter 1, shows that the sheaf ω_C of regular differential forms on C can be expressed as $\nu^!\pi^!\,\mathcal{H}om_{\mathbb{P}^1}(\pi_*\nu_*(\mathcal{O}_C)\omega_{\mathbb{P}^1})$, where π is the map $C_0 \to \mathbb{P}^1$ defined by x. Locally, this is expressed more simply as

$$\mathcal{H}om_{\mathbb{P}^1}(\nu_*(\mathcal{O}_C), \omega_{\mathbb{P}^1}),$$

where we use the action of $\mathcal{O}_C$ on $\nu_*(\mathcal{O}_C)$. Since the maps involved are finite, we will identify $\mathcal{O}_C$ with $\nu_*(\mathcal{O}_C)$ and write

$$\omega_C = \mathcal{H}om_{\mathbb{P}^1}(\mathcal{O}_C, \omega_{\mathbb{P}^1}),$$

Since $\mathcal{O}_{C_0} \subset \mathcal{O}_C$, this sheaf is naturally contained in

$$\mathcal{H}om_{\mathbb{P}^1}(\mathcal{O}_{C_0}, \omega_{\mathbb{P}^1}).$$

As will be explained in Chapter 1,

$$\omega_{C_0} := \mathcal{H}om_{\mathbb{P}^1}(\mathcal{O}_{C_0}, \omega_{\mathbb{P}^1}) = \mathcal{H}om_{\mathbb{P}^1}(\mathcal{O}_{C_0}, \mathcal{O}_{\mathbb{P}^1})(-2),$$

is properly called the dualizing module of the singular curve C_0. We will show in Theorem 1.17 that $\mathfrak{f}_{C/C_0}$ is the annihilator of the quotient of two locally cyclic modules

$$\mathfrak{f}_{C/C_0} = \operatorname{ann}_{C_0} \frac{\mathcal{H}om_{\mathbb{P}^1}(\mathcal{O}_{C_0}, \omega_{\mathbb{P}^1})}{\mathcal{H}om_{\mathbb{P}^1}(\mathcal{O}_C, \omega_{\mathbb{P}^1})}.$$

Thus every regular differential form on C can be expressed as an element of the conductor times some element of ω_{C_0}.

To prove the theorem, we must show that $\varphi_0 = dx/f_y$ generates ω_{C_0} as a module over $\mathcal{O}_{C_0}$.

Passing to the field of rational functions $\kappa := \kappa(C)$, and noting that $\kappa(C) = \kappa(C_0)$ we use the well-known result from Galois theory that

$$\operatorname{Hom}_{\kappa(\mathbb{P}^1)}(\kappa, \kappa(\mathbb{P}^1))$$

is a 1-dimensional vector space over κ, generated by the trace map T. Moreover, because $\mathcal{O}_{C_0}$ is integral over $\mathcal{O}_{\mathbb{P}^1}$, which is normal,

$$T(\mathcal{O}_{C_0}) \subset \mathcal{O}_{\mathbb{P}^1}.$$

For the fourth and last step we need a result from commutative algebra:

Theorem 15.18. *If C_0° is an affine plane curve defined by the equation $f(x, y) = 0$ and such that $\mathbb{C}[x, y]/(f)$ is finite over $\mathbb{C}[x]$, then $\operatorname{Hom}_{\mathbb{C}[x]}(\mathbb{C}[x, y]/(f), \mathbb{C}[x])$ is generated by $(1/f_y)T$.*

See [?, Theorem 15.1] for a proof using valuations, and [?, Theorem A.1] for a proof in a more general context.

Given Theorem 15.18, we can complete the proof of Theorem 15.16. Via the isomorphism $\mathbb{C}(x) \cong \mathbb{C}(x)\,dx$ sending 1 to dx, the trace map is identified (up to scalar) with the map $(1 \mapsto dx) \in \operatorname{Hom}_{\mathbb{C}(x)}(\kappa, \mathbb{C}(x)\,dx)$. Thus by Theorem 15.18 the canonical module of C_0° is identified with $\mathcal{O}_{C_0^\circ}\phi_0$ as required. □

Example 15.19 (nodes and cusps). We have already seen that in case q is a node of C_0, there are two points of C lying over it, and the multiplicities of φ_0 at these two points are $m_1 = m_2 = 1$; the adjoint ideal is thus the maximal ideal $\mathcal{I}_q$ at q. When q is a cusp, analytically isomorphic to the zero locus of $y^2 - x^3$, there is only one point r of C lying over q. The cusp can be parametrized, locally analytically, by $x = t^2$, $y = t^3$ and it follows that the differential

dropped “the cusp point”

removed extra right parens

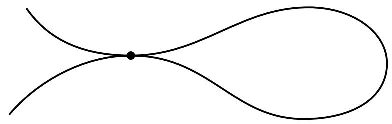

Figure 15.3. A tacnode: two smooth branches tangent to one another.

$$\varphi_0 = \frac{dx}{f_y} = \frac{2t\,dt}{2t^3} = \frac{dt}{t^2}$$

has a pole of order 2. Since the pullback to C of any polynomial g vanishing dropped repeated "of"
at the cusp q vanishes to order at least two at r, the adjoint ideal is again the maximal ideal at q. We can also see this by computing the conductor ideal as the annihilator of $\mathbb{C}[[t]]/\mathbb{C}[[t^2, t^3]]$.

Example 15.20 (tacnodes). Next, consider the case of a *tacnode* (Figure 15.3); that is, a plane-curve singularity with two smooth branches simply tangent to one another, analytically isomorphic to the zero locus of $y^2 - x^4$, parametrized locally analytically with two branches $x = t$, $y = t^2$ and $x = t$, $y = -t^2$.

At the two points of C lying over q, we have

$$\varphi_0 = \pm\frac{dt}{2t}$$

and each of these differentials has a simple pole there. The adjoint ideal is thus the ideal of functions vanishing at q and having derivative 0 in the direction of the common tangent line to the branches.

Example 15.21 (ordinary n-fold points). In the case of an ordinary n-fold point of a plane curve — where n smooth branches pairwise transverse to one another meet — there are n points r_i of C lying over q. The polynomial f_y vanishes to order $n-1$ at q, so dx/f_y has a pole of order $n-1$ at each r_i. It follows that for $e(x, y)\,dx/f_y$ to be regular, e must vanish to order $n-1$ at each r_i.

We can see that the conductor ideal is the full $(n-1)$-rst power of (x, y) by using the normalization map

$$\nu^* : \frac{\mathbb{C}[x, y]}{\prod_{i=1}^{n}(x - \alpha_i y)} \to \prod_{i=1}^{n} \frac{\mathbb{C}[x, y]e_i}{(x - \alpha_i y)}$$

where the α_i are distinct elements of $\mathbb{C}$ and the e_i are orthogonal idempotents. The element $1 = \sum_i e_i$ goes to 0 in the quotient, and e_i is annihilated by $x - \alpha_i y$, so the quotient is annihilated by each of the n elements $g_j \coloneqq \prod_{i \neq j}(x - \alpha_i y)$. As forms on $\mathbb{P}^1$, all the g_i except g_j vanish at the point $(\alpha_j, 1)$, so the g_i are linearly independent. Since $(x, y)^{n-1}$ is minimally generated by n elements, $(x, y)^{n-1} = (g_1, \ldots, g_n)$.

A consequence of this computation is that the δ invariant of the ordinary multiple point — that is, the difference in arithmetic genus between a plane curve that has just one such singular point and its normalization — is $\binom{n}{2}$, the dimension of $k[x,y]/(x,y)^{n-1}$, as we proved before in Proposition 15.9.

Example 15.22 (spatial triple points). Spatial triple points provide a contrast to the last example. A spatial triple point is a singularity consisting of three smooth branches, with linearly independent tangent lines, meeting in a point p so that its Zariski tangent space is 3-dimensional; the simplest example is the origin as a point on the union of the three coordinate axes in $\mathbb{A}^3$.

In this case the conductor is the annihilator of the cokernel of

$$\nu^* : R := \frac{\mathbb{C}[x,y,z]}{(xy,xz,yz)} \to \frac{\mathbb{C}[x,y,z]}{(x,y)} \times \frac{\mathbb{C}[x,y,z]}{(x,z)} \times \frac{\mathbb{C}[x,y,z]}{(y,z)} =: \overline{R}.$$

Since $x\overline{R} = x\mathbb{C}[x,y,z]/(y,z)$ is in the image of $\mathbb{C}[x,y,z]/(xy,xz,yz)$, and similarly with y and z, we see that the conductor is the maximal ideal (x,y,z). However, the δ invariant, the length of the quotient $(\overline{R})/R$, is 2: for a function f in $\overline{R}$ to be in R, it is necessary and sufficient that f take the same value at the three points above the singular point, and this gives two linear conditions.

15.3. Exercises

In Exercise 4.1, we saw how to use the description of the canonical series on a smooth plane curve to determine its gonality. Now that we have an analogous description of the canonical series on (the normalization of) a nodal plane curve, we can deduce a similar statement about the gonality of such a curve. Here are the first two cases:

Exercise 15.1. Let C_0 be a plane curve of degree $d \geq 4$ with one node p and no other singularities, and let C be its normalization. Show that C admits a unique map $C \to \mathbb{P}^1$ of degree $d-2$, but does not admit a map $C \to \mathbb{P}^1$ of degree $d-3$ or less.

Hint: If $D = q_1 + \cdots + q_{d-2}$ had $r(D) \geq 1$, the points $q_1 + \cdots + q_{d-2}$ and p would fail to impose independent conditions on plane curves of degree $d-3$ and hence lie on a line.

Exercise 15.2. Let C_0 be a plane curve of degree $d \geq 5$ with two nodes p and p' and no other singularities, and let C be its normalization. Show that C admits two maps $C \to \mathbb{P}^1$ of degree $d-2$, but does not admit a map $C \to \mathbb{P}^1$ of degree $d-3$ or less.

Hint: If $D = q_1 + \cdots + q_{d-2}$ had $r(D) \geq 1$, the points $q_1 + \cdots + q_{d-2}$ and p, p' would fail to impose independent conditions on plane curves of degree $d-3$ and hence by Proposition 15.23 below $d-1$ of them would lie on a line.

Exercise 15.3. Generalizing the examples above, show that if a nodal plane curve of degree d has $\delta \leq d+3$ nodes, then its gonality is $d-2$, and moreover every g^1_d on the curve is given by projection from one of the nodes. You may use the following result:

Proposition 15.23. *A set of $n \leq 2d+2$ distinct points in the plane fails to impose independent conditions on curves of degree d if and only if either $d+2$ of the points are collinear or $n = 2d+2$ and all the points lie on a conic.*

See [?, p. 302] for a proof of this.

Hint: A g^1_{d-2} is a set of points that, together with the nodes, impose dependent conditions on forms of degree $d-3$.

Exercise 15.4. Suppose that C is a smooth curve and $\nu : C \to C_0$ is a map to a plane curve with only nodes as singularities. Let $D = D_+ - D_-$ be a divisor on C. Modify the technique of Section 15.1.2 to compute the complete linear series $|D|$ without assuming that D is disjoint from the preimages of the singular points.

Hint: Be careful to subtract the right multiples of the points that are preimages of the singular points.

Exercise 15.5. Let C_0 be a plane quartic curve with two nodes q_1, q_2 and let $\nu : C \to C_0$ be its normalization. By the adjunction formula, C has genus 1. For an arbitrary point $o \in C$ not lying over a node of C_0, give a geometric description of the group law on C with o as origin.

Hint: To add two points s and $t \in C$, choose a conic curve D passing though s, t, q_1 and q_2, and let u and v be the remaining points of $C_0 \cap D$; then take the conic D' passing though u, v, q_1, q_2 and o. The sum $s + t$ will then be the remaining point of $D' \cap C_0$.

Exercise 15.6. Let p be a point in $\mathbb{P}^2$ and let C_0 be a curve that, in a neighborhood of p, consists of 3 smooth branches that are pairwise simply tangent (for example, C_0 could be given locally analytically by the equation $y(y-x^2)(y+x^2) = 0$.) Use Corollary 15.11 to show that the δ invariant of C_0 at p is 6.

Hint: Show that in addition to the triple point at p, the curve C_0 has one infinitely near point of multiplicity 3.

Exercise 15.7. Find the adjoint ideals of the following plane curve singularities:

(1) a *triple tacnode*, also known in classical language as a *triple point with an infinitely near triple point*: three smooth branches, pairwise simply tangent; removed from parentheses in view of the next entry where the same language is used

(2) a triple point with an infinitely near double point: three smooth branches, two of which are simply tangent, with the third transverse; dropped "to both" (redundant, right?)

(3) a unibranch triple point, such as the zero locus of $y^3 - x^4$.

Hint: For the first, the adjoint ideal is the ideal of functions vanishing to order 4 on each branch (so that the general member of the ideal with have zero locus consisting of two smooth branches simply tangent to the branches of the triple point). For the last, the adjoint is simply the square of the maximal ideal.

Here is a general description in case the individual branches of C_0 at p are each smooth:

Exercise 15.8. Let $\nu : C \to C_0$ be the normalization of a plane curve C_0 and $p \in C_0$ a singular point. Denote the branches of C_0 at p by $B_1, \dots, B_k$, and let r_i be the point in B_i lying over p. If the individual branches B_i of C_0 at p are each smooth, and we set

$$m_i = \sum_{j \neq i} \mathrm{mult}_p(B_i \cdot B_j)$$

does mult here have the meaning defined in chapter 10?

then the adjoint ideal of C_0 at p is the ideal of functions g such that $\mathrm{ord}_{r_i}(\nu^* g) \geq m_i$.

Hint: The computation can be done locally analytically. Let $R = \widehat{\mathcal{O}_{C_0,p}}$ be the completion of the local ring of C_0 at r_i. The integral closure is then the product of rings $R_i = \widehat{\mathcal{O}_{B_i}} \cong k[[t_i]]$, with $R_i = R/P_i$ as P_i runs over the minimal primes of R. The multiplicity m_i is the colength of the ideal $\sum_{j \neq i} P_j \subset R$.