# Personalities of Curves

©David Eisenbud and Joe Harris

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# Contents

# Chapter 1

# Rational Normal Scrolls

ScrollsChapter

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The naming of cats is a difficult matter,
It isn't just one of your everyday games.
You may think that I am as mad as a hatter,
When I tell you each cat must have three different names.
The first is the name that the family use daily ...
But I tell you, a cat needs a name that's particular ...
But above and beyond there's still one name left over,  ...
[his] deep and inscrutable, singular name.


--T.S.Eliot, Practical Cats
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**(( Notation in this chapter the field often has to be algebraically closed. I am writing it as $\mathbb{C}$, but this will be easy to replace globally with $k$ if this seems desirable – in which case we should add a blanket assumption. Also we write $\mathbb{P}$ instead of $\mathbb{P}_{\mathbb{C}}$. ))**

Some of the simplest subvarieties in projective space are the *rational normal scrolls*. They appear in many contexts in algebraic geometry, and are useful, in particular, for describing the embeddings of curves of low degree and genus.

We begin this chapter by giving three different characterizations of these varieties, each useful in a different context: First a classical geometric construction that gives a good picture, then an algebraic description that allows one to "find" the scrolls containing a given variety, and then a more modern geometric definition that makes it easy to understand the divisors on a scroll. Finally, we turn to some of the applications to the embeddings of curves.

In each section we will focus on the 2-dimensional case, both because this is the case that occurs in our applications, and to simplify the discussion. We will also indicate the surprisingly simple extensions to higher dimensions.

In this chapter we will refer to rational normal scrolls simply as scrolls. The third characterization we will give lends itself to a natural generalization to families of irrational ruled varieties, which we'll briefly mention, and the reader should be aware that in the literature the word "scroll" is used for this wider class.

[Notation: $\mathbb{P}^N$ and $\mathbb{P}^1$ mean $\mathbb{P}^N_{\mathbb{C}}$ and $\mathbb{P}^1_{\mathbb{C}}$. ]

## 1.1 Some classical geometry (the name the family use daily)

daily name

Recall that the image of the map

$$\mathbb{P}^1 \to \mathbb{P}^a : (s,t) \mapsto (s^a, s^{a-1}t, \dots, t^a),$$

corresponding to the complete linear series $|\mathcal{O}_{\mathbb{P}^1}(a)|$, is called a *rational normal curve*; it is, up to linear transformations, the unique nondegenerate curve of degree $a$ in $\mathbb{P}^a$.

Scrolls are one answer to the question: "What are the higher-dimensional analogues of rational normal curves?" To construct a scroll of dimension 2, we choose integers $0 < a_1, a_2$ and consider a projective space

$$\mathbb{P}^{a_1+a_2+1} = \mathbb{P}_B(\mathbb{C}^{a_1+1} \oplus \mathbb{C}^{a_2+1})$$

with its two subspaces $\mathbb{P}^{a_i} = \mathbb{P}_B(\mathbb{C}^{a_i+1})$. We choose a rational normal curve $C_i \subset \mathbb{P}^{a_i}$, and we choose an isomorphism $\phi : C_1 \to C_2$. We define the scroll $S(a_1, a_2)$ to be the union of the lines

$$S(a_1,a_2) := \bigcup_{p \in C_1} \overline{p, \phi(p)}.$$

We call the curves $C_{a_1}$ and $C_{a_2}$ the *directrices* (singular: directrix) of the scroll, and we call the lines $\overline{p, \phi(p)}$ the *rulings* of the scroll.

It is not hard to prove directly that $S(a_1, a_2)$ is an algebraic variety, but as we shall soon write down its defining equations we will not bother to do so.

From this description we can immediately deduce the dimension and degree of the scroll:

**Proposition 1.1.1.** *1. $S(a_1, a_2)$ is a nondegenerate surface.*

deg and codim

*2. $S(a_1, a_2)$ has degree $a_1 + a_2$, and codimension $a_1 + a_2 - 1$.*

*Proof.* The rational normal curves separately span the spaces $\mathbb{P}^{a_i}$, so a hyperplane containing both of them would contain $\overline{\mathbb{P}^{a_1}, \mathbb{P}^{a_2+1}} = \mathbb{P}$, proving nondegeneracy.

It is clear from our description that $S$ is 2-dimensional, and thus of codimension $a_1 + a_2 + 1 - 2 = a_1 + a_2 - 1$.

To compute the degree, we choose a general hyperplane $H$ containing $\mathbb{P}^{a_1}$. The intersection $H \cap C_2$ consists of $a_2$ reduced points. Thus the intersection $H \cap S$ consists of $C_1$ and the $a_2$ reduced lines connecting the points of $H \cap C_2$ with their corresponding points on $C_1$; this union has degree $a_1 + a_2$. $\square$

A completely parallel construction creates rational normal scrolls of dimension $r$. Set $N = \sum_{i=1}^{r}(a_i + 1)$, where each $a_i > 0$ and decompose $\mathbb{C}^N$ as

$$\mathbb{C}^{\sum_{i=1}^{r}(a_i+1)} = \oplus_{i=1}^{r} \mathbb{C}^{a_i+1}.$$

Let $\mathbb{P}^{a_i} \subset \mathbb{P}^N$ be the subspaces corresponding to the summands, choose rational normal curves $C_i \subset \mathbb{P}^{a_i}$ and choose isomorphisms $C_1 \to C_i$. Set

$$S := S(a_1, \ldots, a_r) = \bigcup_{p \in C_1} \overline{p, \phi_2(p), \ldots, \phi_r(p)}.$$

The variety $S$ is nondegenerate of codimension $N - r$ and degree $\sum_i a_i = N - r + 1$. The proof is similar to the one we gave for $r = 2$. Note the case $r = 1$, in which $S(a)$ is simply the rational normal curve of degree $a$.

To put this result in context, we recall an elementary fact of projective geometry:

minimal degree **Proposition 1.1.2.** *Any nondegenerate variety of codimension $c$ in $\mathbb{P}^N$ has degree $\geq c+1$.*

*Proof.* We do induction on $c$.The case $c = 0$ being trivial, we may assume that $c \geq 1$. A general plane $L \subset \mathbb{P}^N$ meets $X$ in $\deg X$ distinct general points, which must be nonsingular points of $X$.

Let $p \in L \cap X$ be a point. If every secant to $X$ through $p$ lies entirely in $X$, then $X$ is a cone over $p$; but since $p$ was a general point, this would imply that $X$ is a plane, contradicting non-degeneracy.

It follows that the projection $\pi_p : X \to \mathbb{P}^{N-1}$ is a generically finite (rational) map from $X$ to $X' := \pi_p(X)$, and thus $\dim X' = \dim X$ and $\operatorname{codim} X' = \operatorname{codim} X + 1$. The plane $\pi_p(L)$ meets $X'$ in the images of the points of $L \cap X$ other than $p$, so $\deg X \geq \deg X' + 1$. By induction, $\deg X' \geq \operatorname{codim} X' + 1 = \operatorname{codim} X$, completing the argument. $\square$

Thus scrolls are *varieties of minimal degree*. The reader already knows that the rational normal curves of degree $a$ in $\mathbb{P}^a$ are the only curves of degree $a$ and codimension $a-1$. A celebrated Theorem of Del Pezzo (for surfaces) and Bertini (in general) generalizes this statement:

ation of scrolls **Theorem 1.1.3.** *Any nondegenerate variety $X \subset \mathbb{P}^N$ with with $\deg X = \operatorname{codim} X + 1$, is either a scroll or the Veronese surface in $\mathbb{P}^5$ or a cone over one of these.*

A proof is given in the appendix to this chapter.

One interesting way to view the construction above is that we chose subvarieties $C_i \subset \mathbb{P}^{a_i}$ and a one-to-one correspondence between them, that is, a subscheme $\Gamma \subset \prod_i C_i$ that projects isomorphically onto each $C_i$; the scroll is then the union of the planes spanned by sets of points $p_i \in C_i$ that are "in correspondence". There are other interesting varieties constructed starting with other choices of subvarieties $C_i$ and subschemes—not necessarily reduced—of $\prod_i C_i$. See [**?**] for an exploration of this idea.

We tend to speak of "the" rational normal scroll rather than "a" rational normal scroll, despite the choices made in the definition, for the following reason:

eness of scrolls **Proposition 1.1.4.** *The scroll $S(a_1, a_2)$ is, up to a linear automorphism of $\mathbb{P}^{a_1+a_2+1}$, independent of the choices made in its definition.*

*Proof.* To simplify the notation, set $S := S(a_1, a_2)$ and $\mathbb{P} := \mathbb{P}^{a_1+a_2+1}$. To construct $S$ we chose

1. disjoint subspaces $\mathbb{P}^{a_i} \subset \mathbb{P}$;
2. a rational normal curve in each subspace; and
3. an isomorphism between these curves.

Elementary linear algebra shows that there are automorphisms of $\mathbb{P}$ carrying any choice of disjoint subspaces to any other choice. Further, since the rational normal curve of degree $a$ is unique up to an automorphism of $\mathbb{P}^a$, the choice in (2) can be undone by a linear automorphism. Finally, any automorphism of $C_{a_2}$ extends to an automorphism of $\mathbb{P}^{a_2}$, and this extends to an automorphism of $\mathbb{P}$ fixing $\mathbb{P}^{a_1}$ pointwise, showing that $S(a_1, a_2)$ is independent, up to an automorphism of the ambient space, of the choice in (3) as well. $\square$

## 1.2 1-generic matrices and the equations of scrolls (the name that's particular)

particular name

From the definition above it is easy to find the equations of a rational normal scroll. To warm up, we look at the case of the rational normal curve, $S(a) \subset \mathbb{P}^a$. For $a = 1$ there is of course no problem, so we assume $a > 1$. TConsider the expression of the hyperplane bundle $\mathcal{O}_{\mathbb{P}^1}(a)$ as the product

$$\mathcal{O}_{\mathbb{P}^1}(1) \otimes_{\mathbb{P}^1} \mathcal{O}_{\mathbb{P}^1}(a-1) = \mathcal{O}_{\mathbb{P}^1}(a).$$

This leads to the map of vector spaces

$$\mu : H^0\mathcal{O}_{\mathbb{P}^1}(1) \otimes_{\mathbb{C}} H^0\mathcal{O}_{\mathbb{P}^1}(a-1) \to H^0\mathcal{O}_{\mathbb{P}^1}(a),$$

which, in coordinates $s, t$ on $\mathbb{P}^1$, is just the multiplication map

$$< s, t > \otimes_{\mathbb{C}} < s^{a-1}, s^{a-2}t, \ldots, t^{a-1}.$$

We may represent this map as a multiplication table, with matrix

$$\begin{array}{cc} & \begin{array}{cccc} s^{a-1} & s^{a-2}t & \ldots & t^{a-1} \end{array} \\ \begin{array}{c} s \\ t \end{array} & \begin{pmatrix} s^a & s^{a-1}t & \ldots & st^{a-1} \\ s^{a-1}t & s^{a-2}t^2 & \ldots & t^a \end{pmatrix} \end{array}$$

More abstractly, what we have done is to use the equivalence between maps of vector spaces $U \otimes_{\mathbb{C}} V \to W$ and maps $W^* \to Hom_{\mathbb{C}}(V, U^*)$. Because this is a multiplication table, the $2 \times 2$ minors of any of the $2 \times 2$ submatrices

$$\begin{pmatrix} s^{a-i}t^i & s^{a-j}t^j \\ s^{a-i-1}t^{i+1} & s^{a-j-1}t^{j+1} \end{pmatrix}$$

vanish on $\mathbb{P}^1$. Thus, if we give the parametrization $\mathbb{P}^1 \to \mathbb{P}^a$ of the rational normal curve by

$$x_i = s^{a-i}t^i$$

we see that $2 \times 2$ minors of the matrix

$$(*) \qquad M_a = \begin{pmatrix} x_0 & x_1 & \dots & x_{a-1} \\ x_1 & x_2 & \dots & x_a \end{pmatrix}$$

vanish on the curve. In fact:

`RNC generators` **Proposition 1.2.1.** *The ideal of forms vanishing on the rational normal curve in $\mathbb{P}^a$ given parametrically by $x_i = s^{a-i}t^i$ is generated by the $2 \times 2$ minors of $M_a$.*

*Proof.* The ideal of forms vanishing on the rational normal curve is the kernel of the map of graded rings

$$\mathbb{C}[x_0, \dots, x_a] \twoheadrightarrow \mathbb{C}[s^a, s^{a-1}t, \dots, t^a],$$

and the map is homogeneous of degree 0 if we take the monomials $s^{a-i}t^i$ in the target to have degree 1. Write $I_2(M_a)$ for the ideal generated by the $2 \times 2$ minors of $M_a$. By what we have seen above, this map factors through $\mathbb{C}[x_0, \dots, x_a]/I_2(M_a)$.

We claim that, modulo the $2 \times 2$ minors, any monomial in the $x_i$ of degree $d$ can be reduced to the form

$$x_0^m x_i^\epsilon x_a^{m-d-\epsilon}$$

for some $i$, with $\epsilon \in \{0, 1\}$ and $0 \le m \le d + \epsilon$. To see this, Suppose that $m$ is a monomial that cannot be so expressed, so that $m$ contains $z_i z_j$ as a factor, with $1 \le i \le j \le a - 1$. We do induction on $i$. Since $M_a$ contains the submatrix

$$\begin{pmatrix} z_{i-1} & z_j \\ z_i & z_{j+1} \end{pmatrix}$$

we see that $z_i z_j \equiv z_{i-1}z_{j+1} \mod I$, proving the claim.

Thus the dimension of the $d$-th graded component of $\mathbb{C}[x_0, \ldots, x_a]/I_2(M_a)$ is at most $ad+1$. The $d$-graded component of $\mathbb{C}[s^a, s^{a-1}t, \ldots, t^a]$ is the $ad$-th graded component of $\mathbb{C}[s,t]$, which has dimension exactly $ad+1$. Thus the map

$$\mathbb{C}[x_0, \ldots, x_a]/I_2(M_a) \cong \mathbb{C}[s^a, s^{a-1}t, \ldots, t^a]$$

is an isomorphism, as required. $\square$

By a *generalized row* of $M_a$, we mean a $\mathbb{C}$-linear combination of the given rows of $M_a$. Note that the points at which the $2\times 2$ minors of $M_a$ vanish are the points at which the evaluations of the two rows are linearly dependent; that is, the points at which some generalized row of $M_a$ vanishes identically. Given Proposition 1.2.1, we see that the points of the rational normal curve are exactly the points where all the linear forms in some generalized row of $M_a$ vanish.

The matrix $M_a$ has a special property: it is 1-generic in the sense below:

**Definition 1.2.2.** A matrix of linear forms $M$ is said to be *1-generic* if every generalized row of $M$ consists of $\mathbb{C}$-linearly independent forms..

**Exercise 1.2.3.** Show that a matrix $M$ of linear form is 1-generic iff, even after arbitrary row and column transformations, it's entries are all non-zero.

For example, the matrix

$$M = \begin{pmatrix} x & y \\ z & x \end{pmatrix}$$

over $\mathbb{C}[x,y,z]$ is 1-generic, since if a row and column transformation produced a 0 the determinant would be a product of linear forms, whereas $\det M = x^2 - yz$ is irreducible.

On the other hand, the matrix

$$M' = \begin{pmatrix} x & y \\ -y & x \end{pmatrix}$$

over $\mathbb{C}[x,y]$ is not 1-generic, since

$$\begin{pmatrix} 1 & 0 \\ -i & 1 \end{pmatrix} M' \begin{pmatrix} 1 & 0 \\ i & 1 \end{pmatrix} = \begin{pmatrix} x+iy & 0 \\ 0 & x-iy \end{pmatrix}$$

(but note that it would be 1-generic if we restricted scalars to $\mathbb{R}$—thus the definition depends on the field).

The fact that $M_a$ is 1-generic is most conveniently seen as a special case of the next proposition:

Let $X$ be an irreducible, reduced variety, and suppose that $(\mathcal{L}, V)$ is a linear series. Suppose that there are two linear series $(\mathcal{L}_1, V_1), (\mathcal{L}_2, V_2)$ on $X$ such that $\mathcal{L} = \mathcal{L}_1 \otimes \mathcal{L}_2$ and $V_1 \otimes V_2 \subset V$. If we choose bases $\{u_i\}$ of $V_1$ and $\{v_j\}$ of $V_2$ then we may regard $\ell_{i,j} := u_i v_j \in V$ as a linear form on $\mathbb{P}(V)$.

some generators

**Proposition 1.2.4.** *With notation above, the* $\dim V_1 \times \dim V_2$ *matrix* $M := (\ell_{i,j})$ *is 1-generic. Moreover, its* $2 \times 2$ *minors are contained in the image of* $X$ *under the linear series* $(cL, V)$.

*Proof.* The entries of $M$, after any row and column operations, have the form $uv$, where $u$ and $v$ are nonzero sections of $\mathcal{L}_1$ and $\mathcal{L}_2$, respectively. Since $X$ is irreducible and reduced, $u$ and $v$ can vanish only on nowhere dense subsets of $X$, so $uv$ is nonzero. This proves the first statement.

To see that the $2 \times 2$ minors of $M$ vanish on $X$ we interpret all the sections of the line bundles $\mathcal{L}_1, \mathcal{L}_2$ and $\mathcal{L}$ as elements of the field of rational functions on $X$, so a $2 \times 2$ minor

$$\det \begin{pmatrix} \ell_{i,j} & \ell_{i,j'} \\ \ell_{i',j} & \ell_{i',j'} \end{pmatrix} = (u_i v_j)(u_{i'} v_{j'}) - (u_i v_{j'})(u_{i'} v_j)$$

vanishes on $X$ by the associativity of multiplication. $\square$

The fact that the matrix $M'$ above is *not* 1-generic can be seen from a more general point of view as well:

ize of 1-generic

**Lemma 1.2.5.** *There exist 1-generic* $p \times q$ *matrices of linear forms in* $n+1$ *variables over* $\mathbb{C}$ *if and only if* $n \geq p+q$*; In particular, the dimension of the space of linear forms spanned by the* $1 \times 1$ *minors of a 1-generic matrix* $M$ *of size* $p \times q$ *is at least* $p+q-1$. *Moreover, if this space of linear forms has dimension* $> p+q-1$, *then the restriction of* $M$ *to a general hyperplane is still 1-generic.*

*Proof.* If we think of a polynomial ring $\mathbb{C}[z_0, \ldots, z_n]$ as the symmetric algebra of a vector space $V$ of rank $n+1$, then we may regard a $p \times q$ matrix of linear forms $M$ as coming from a map $m : \mathbb{C}^p \otimes \mathbb{C}^q \to V$. The matrix is 1-generic if and only if no "pure" tensor $r \otimes s$ goes to zero, that is, iff the kernel $K$ of $m$ intersects the cone of pure tensors only in 0. The cone of pure tensors is the cone over the Segre embedding of $\mathbb{P}^{p-1} \times \mathbb{P}^{q-1}$, an thus has dimension $(p-1)+(q-1)+1$. Thus a general subspace $K$ of codimension $\geq p+q-1$ will intersect the cone only in 0, but any larger subspace $K$ will intersect the cone non-trivially, and the first two statements follow.

Moreover, if $K$ is any space of codimension $> p+q-1$ that intersects the cone only in 0, then the general subspace $K' \supset K$ of dimension one larger still intersects the cone only in 0, proving the last statement. $\square$

Note that the idea behind this argument is present in the usual proof of the bound in Clifford's Theorem: if $\mathcal{L}$ is a special line bundle on a curve $C$ then the map

$$\mathrm{H}^0(\mathcal{L}) \otimes \mathrm{H}^0(\mathcal{L}^{-1} \otimes \omega_C) \to \mathrm{H}^0(\omega_C)$$

is 1-generic by Proposition some generators 1.2.4, and thus

$$h^0(\mathcal{L}) + h^1(\mathcal{L}) - 1 \leq g.$$

By Riemann-Roch, $h^1(\mathcal{L}) = h^0(\mathcal{L}) - d + g - 1$, so this last relation becomes $h^0(\mathcal{L}) + (h^0(\mathcal{L}) - d + g - 1) - 1 \leq g$, or $2(h^0(\mathcal{L}) - 1) \leq d$.

We have already seen that the ideal of minors of the 1-generic $2 \times a$ matrix $M_a$ associated to the rational normal curve is a prime ideal of codimension $a-1$ and degree $a$. This too is part of a more general pattern:

-generic basics **Theorem 1.2.6.** *Let $M$ be a 1-generic $2 \times a$ matrix of linear forms on $\mathbb{P}^n$. Let $I = I_2(M)$ be the ideal generated by the $2 \times 2$ minors of $M$.*

1. *The ideal $I$ is prime, and the variety $V = V(I) \subset \mathbb{P}^n$ has degree $a$ and codimension $a-1$.*

2. *We have $n \geq a$. If $n = a$ then the 1-generic matrix $M$ is equivalent up to row and column transformations to the matrix $M_a$ of Proposition RNC generators 1.2.1; in particular, $I$ is prime and $V(I)$ is a rational normal curve of degree $a$.*

*Proof.* 1) The inequality $a \leq n$ was established in Lemma size of 1-generic 1.2.5. We will reduce the case $a < n$ to the case $a = n$. Thus we suppose $a < n$, and let $\ell$ be a general linear form. By induction on $n$, the image of $I_2(M)$ in $S/\ell \cong \mathbb{C}[x_0, \dots, x_{n-1}$ is prime of codimension $a-1$, and degree $a$ so $V(I_2(M))$ is irreducible of degree $a$ and codimension $a-1$. It remains to show that $I_2(M)$ is prime.

It follows from the induction that the image $\overline{\ell}$ of $\ell$ in $R := S/I_2(M)$ generates a prime ideal containing the unique minimal prime $P$ of $R$. Every element $f \in P$ is divisible by $\ell$. If $f \in P$ is a minimal generator, then $f = \ell f'$, and since $\ell \notin P$ we must have $f' \in P$, a contradiction. Thus $P = 0$; that is, $I_2(M)$ is prime, as required.

(2) The points of $X = V(I_2(M))$ are the points where some generalized row of $M$ vanish. Since the family of generalized rows is $\mathbb{P}^1$, this variety is at most one-dimensional. If it were 0-dimensional then, since $\mathbb{P}^1$ is irreducible it would have to be a single point—that is, all the generalized rows would be contained in a single vector space of linear forms of dimension $n$, contradicting Lemma 1.2.5. Thus $X$ is a curve in $\mathbb{P}^N$, and $I_2(M)$ has codimension $n-1 = a-1$.

We now use an important general result from commutative algebra, Theorem 1.2.3, from which we see that $I_2(M)$ is unmixed, and that, even after factoring out two (= dimension $I_2(M)$) general linear forms $\ell_1, \ell_2$, the $2 \times 2$ minors of $M$ remain linearly indedpendent. Since the dimension of the space of quadratic forms in $T := S/(\ell_1, \ell_2)$ in $n-1$ variables is $\binom{n}{2}$, the same as the number of minors,we see that the vector space dimension of $\overline{S}/(I_2(M) + (\ell_1, \ell_2)$ is $1 + (n-1) = n$; thus the curve defined (in the scheme-theoretic sense) by $I_2(M)$ has degree $n$. As it is nondegenerate in $\mathbb{P}^N$, $X_{\text{red}}$ must be the rational normal curve, and since $I_2(M)$ is unmixed it follows that $I_2(M)$ is the whole homogeneous ideal defining the rational normal curve.

Let $\phi : \mathbb{P}^1 \to \mathbb{P}^N$ be the parametrization of this rational normal curve, and let

$$\begin{pmatrix} \ell_{0,0}, \dots, \ell_{0,a-1} \\ \ell_{1,0}, \dots, \ell_{1,a-1} \end{pmatrix}$$

Write $\overline{\ell_{i,j}}$ for the restriction of $\ell_{i,j}$ to $X \cong \mathbb{P}^1$; we thus consider $\overline{\ell_{i,j}}$ as a form of degree $a$ in 2 variables. Rechoosing coordinates on $\mathbb{P}^1$, we may assume that the first row vanishes at the point $\phi(0,1)$, and the second row vanishes at $\phi(1,0)$ so that each $\overline{\ell_{0,j}}$ is divisible by $s$ and each $\overline{\ell_{1,j}}$ is divisible by $t$. Since the vector space of forms of degree $a$ divisible by $s$ has dimension $a$, we may, rechoosing coordinates on $\mathbb{P}^n$, assume that $\overline{\ell_{0,i}} = s^{a-i}t^i$. It follows that each $\overline{\ell_{1,i}}$ is divisible by $t$, and that the restriction to $\mathbb{P}^1$ of the second row is proportional to $(t/s)(s^a, \ldots, st^{a-1}$; thus, after multiplying by a scalar, it will become equal to $s^{a-1}t, \ldots, t^a$; that is, the matrix $M$ is equivalent under row and column operations to the matrix $M_a$ defined above.

$\square$

ions of scrolls **Corollary 1.2.7.** *Let $a_1, \ldots a_r$ be positive integers, and let $N = r - 1 + sum_{i=1}^r a_i$. The ideal of $S(a_1, \ldots, a_r) \subset \mathbb{P}^N$ is generated by the $2 \times 2$ minors of the matrix*

$$M = \begin{pmatrix} x_{1,0} & x_{1,1} & \ldots & x_{1,a_1-1} & | & x_{2,0} & \ldots & x_{2,a_1-1} & | & \ldots & | & x_{r,0} & \ldots & x_{r,a_1-1} \\ x_{1,1} & x_{1,2} & \ldots & x_{1,a_1}. & | & x_{2,1} & \ldots & x_{2,a_1} & | & \ldots & | & x_{r,1} & \ldots & x_{r,a_1} \end{pmatrix}$$

*Proof.* We may think of the matrix $M$ as consisting of $r$ blocks, $M_{a_i}$ of the form (*). These blocks are 1-generic by Proposition some generators 1.2.4. Since they involve distinct variables, it follows that $M$ is 1-generic. Thus by Theorem **??**, the ideal $I_2(M)$ is prime and of codimension $\sum a_i - 1$, as is the ideal of the scroll. Thus it suffices to show that the minors of $M$ vanish on the scroll.

Let $C_i$ be the rational normal curve in the subspace $\mathbb{P}^{a_i} \subset \mathbb{P}^N$. As always, the set $V(M)$ is the union of the linear spaces on which generalized rows of $M$ vanish; and each such space is the space spanned by the points in the curves $C_{a_i}$ corresponding to the part of that row in the block $M_{a_i}$—that is, $V(I_2(M))$ is the union of the spans of sets of corresponding points on the $C_{a_i}$, as required. $\square$

More is true:

**Cheerful Fact 1.2.1.** Every 1-generic matrix of linear forms is equivalent to one of the type shown in Corollary equations of scrolls 1.2.7, and thus the minors of any 1-generic matrix defines a scroll or the cone over a scroll.

*References.* A $2 \times a$ matrix of linear forms in $N+1$ variables may be though of as a tensor in $\mathbb{C}^2 \otimes \mathbb{C}^a \otimes \mathbb{C}^{N+1}$, or, equivalently, as an $a \times N+1$ matrix of linear forms in 2 variables. This, in turn is equivalent to a *pencil* (that is, a projective line) in the vector space of scalar $a \times N+1$ matrices. These were first classified by Kronecker; see [**?**, Theorems *** and ***] for a modern exposition. $\square$

**(( mention $2 \times n$ matrices in general; and the Kac classification of matrix formats with finite classification problems ))**

In fact it will be convenient to widen the definition of scrolls to allow cones over scrolls as well: indeed, if we think of a *rational normal curve of degree 0*, then the cone over $S(a_1, \ldots, a_r)$with $b$-dimensional vertex is $S(0, \ldots, 0, a_1, \ldots a_r)$, where there are $b$ zeros in the sequence. With this in mind, we can say that giving a scroll of codimension $a - 1$ is "the same" as giving a 1-generic $2 \times a$ matrix of linear forms.

There are not many types of varieties of minimal degree:

**Cheerful Fact 1.2.2.** Suppose that $X \subset \mathbb{P}^N$ is a nondegenerate variety of minimal degree; that is, $\deg X - 1 = \operatorname{codim} X$. Then $X$ is one of the following:

- a quadric hypersurface;
- a (cone over the) Veronese surface $\mathbb{P}^2 \subset \mathbb{P}^5$; or
- a (cone over a) rational normal scroll.

We have used an important result from commutative algebra. We say that a matrix of forms is *homogeneous* if the entries $f_{i,j}$ satisfy

$$\deg f_{i,j} + \deg f_{k,l} = \deg f_{i,l} + \deg f_{k,j} \text{ for all } i, j, k, l \text{ where this makes sense;}$$

that is, if the determinant of each $2 \times 2$ submatrix is "naturally" homogeneous.

`caulay's Theorem`

**Cheerful Fact 1.2.3.** Every minimal prime over the ideal $I$ of $p \times p$ minors of a homogeneous $p \times q$ matrix forms has codimension $\leq q - p + 1$. If $I$ has codimension $q - p + 1$, then it is unmixed—that is, there are no embedded primes—and the $p \times p$ minors are linearly independent over the ground field.

*References.*

**(( I think we're going to prove this in the free res chapter ))**

[**?**, Theorem *** ] $\square$

## 1.3 Scrolls as Images of Projective Bundles (the deep and inscrutable name)

nscrutable name

Our third description of scrolls is that they are projective space bundles on $\mathbb{P}^1$, embedded by the complete series associated to the tautological line bundle. Before we review the general theory, we establish the facts about scrolls that characterize such bundles. For simplicity we focus on the 2-dimensional case; the case of a higher dimensional scroll is similar. We start from the description of $X := S(a_1, a_2)$ as the vanishing locus of the minors of the matrix

$$M := M_{a_1,a_2} = \begin{pmatrix} x_{1,0} & x_{1,1} & \dots & x_{1,a_1-1} & | & x_{2,0} & \dots & x_{2,a_1-1} \\ x_{1,1} & x_{1,2} & \dots & x_{1,a_1}. & | & x_{2,1} & \dots & x_{2,a_1} \end{pmatrix}$$

particular name

of Section 1.2. For $p = (s, t) \in \mathbb{P}^1$ we write $R_p$ for the locus where the linear forms

$$sx_{1,0} + tx_{1,1}, \dots, sx_{2,a_1-1} + tx_{2,a_1}$$

all vanish, so that $R_p$ is a ruling of $X$ in $\mathbb{P}^N$

**Proposition 1.3.1.** *Let $X = S(a_1, a_2) \subset \mathbb{P}^N$, with $N = a_1 + a_2 + 1$, be a non-singular rational normal scroll. The rullings $R_p$ of $X$ are the preimages of points under a morphism $\pi : X \to \mathbb{P}^1$. Furthermore, the line bundle*

$$\mathcal{L} := \mathcal{O}_{\mathbb{P}^N}(1)|_X$$

*restricts to $\mathcal{O}_{\mathbb{P}^1}(1)$ on each $R_p \cong \mathbb{P}^1$, and the pushforward $\mathcal{E} := \pi_*\mathcal{L}$ is isomorphic to $\mathcal{O}_{\mathbb{P}^1}(a_1) \oplus \mathcal{O}_{\mathbb{P}^1}(a_2)$.*

*Proof.* We write $M$ for the matrix $M_{a_1,a_2}$. Since there are $N+1$ independent entries of $M$ the intersection of $R_p$ with $R_q$ is empty when $p \neq q$, so there is at least a set-theoretic map $X \to \mathbb{P}^1$ sending the points of $R_p$ to $p$. To see that this is really a morphism, consider the sheaf

$$\mathcal{L} = \operatorname{coker} \phi : \mathcal{O}_{\mathbb{P}^N}(-1)^{a+b} \to \mathcal{O}_{\mathbb{P}^N}^2$$

given by the matrix $M$. Let $p, q$ be distinct points of $\mathbb{P}^1$ and let $\tilde{p}$ be a point in the ruling $L_p$. Since $L_q$ is disjoint from $L_p$, some linear form in the generalized row corresponding to $q$ does not vanish at $p$. Thus the restriction of $M$ to the point $p$ is equivalent to the matrix

$$\begin{pmatrix} 0 & 0 & \dots & 0 \\ 1 & 0 & \dots & 0 \end{pmatrix}$$

and we see that the fiber of $\mathcal{L}$ at $p$ is $\mathbb{C}$. It follows that $\mathcal{L}$ is a line bundle on $X$.

The images of the two basis vectors of $\mathcal{O}^2_{\mathbb{P}^N}$ map to two global sections $\sigma_1, \sigma_2$ of $\mathcal{L}$. By the argument above these two sections generated $\mathcal{L}$ locally everywhere on $X$, and indeed $\sigma_1$ fails to generate $\mathcal{L}$ locally precisely at the points where the second row of $M$ vanishes. Thus the linear series defined by these sections corresponds to a morphism to $\mathbb{P}^1$ whose fibers are exactly the rulings of $X$.

Because $L_p$ is a linear space, the general hyperplane in $\mathbb{P}^N$ meets $L_p$ in a point; that is $\mathcal{O}_{\mathbb{P}^N}(1)$ restricts to $\mathcal{O}_{\mathbb{P}^1}(1)$ as claimed.

Since $\mathcal{L}$ is a line bundle and $X$ is a variety, $\mathcal{L}$ is flat over $\mathbb{P}^1$ and $\mathcal{E}$ is a vector bundle on $\mathbb{P}^1$. Since the restriction of $\mathcal{L}$ to each fiber is $\mathcal{O}_{\mathbb{P}^1}(1)$, which has two global sections, we see that $\mathcal{E}$ has rank 2 Moreover, since $X \subset \mathbb{P}^N$ is non-degenerate, we see that $\mathcal{E}$ has at least $N+1 = a_1 + a_2 + 2$ independent global sections.

Now consider the directrices $C_i := C_{a_i} \subset X$. The restriction $\pi|_{C_i}$ is an isomorphism inverse to the parametrization $\mathbb{P}^1 \to C_i$, and $\mathcal{O}_{\mathbb{P}^N}|_{C_i}$ pulls back to $\mathcal{O}_{\mathbb{P}^1}(a_i)$, so $\pi_*(\mathcal{L}|_{C_i}) = \mathcal{O}_{\mathbb{P}^1}(a_i)$. Thus the maps $\mathcal{L} \to \mathcal{L}|_{C_{a_i}}$ induce maps $\mathcal{E} = \pi_*\mathcal{L} \to \pi_*(\mathcal{L}|_{C_i})$. Putting this together, we get a map

$$\mathcal{E} \to \mathcal{O}_{\mathbb{P}^1}(a_1) \oplus \mathcal{O}_{\mathbb{P}^1}(a_2) =: \mathcal{E}'.$$

Since the spaces spanned by $C_1$ and $C_2$ are complementary, this map of rank 2 vector bundles is an inclusion. Since $\mathcal{E}'$ has only $a_1+a_2+2$ independent global sections, and is generated by them, the map is an isomorphism, completing the argument. $\square$

This is a special case of a very general situation, where, among other things, the Picard group is easy to compute, and which we now explain.

Recall that the projective space $\mathbb{P}^n$ may be defined as $\operatorname{Proj}\operatorname{Sym}_{\mathbb{C}}(\mathbb{C}^{n+1})$. The inclusion of rings $\mathbb{C} = \operatorname{Sym}_{\mathbb{C}}(\mathbb{C}^{n+1})_0 \subset \operatorname{Sym}_{\mathbb{C}}(\mathbb{C}^{n+1})$ induces a structure map $\pi : \mathbb{P}^n \to \operatorname{Spec}\mathbb{C}$. The variety $\mathbb{P}^n$ comes equipped with a tautological line bundle $\mathcal{O}_{\mathbb{P}^N}(1)$, which is associated to the graded module $(\operatorname{Sym}_{\mathbb{C}} \mathbb{C}^{n+1})(1)$, and a tautological map

$$\mathbb{C}^{n+1} \otimes \mathcal{O}_{\mathbb{P}^N} = \pi^*(\mathbb{C}^{n+1}) \to \mathcal{O}_{\mathbb{P}^N}(1)$$

that induces an isomorphism on global sections.

e space bundles

**Cheerful Fact 1.3.1.** In an exactly parallel way, we may make a projective space bundle $\mathbb{P}_B(\mathcal{E})$ over a variety $B$ from a vector bundle $\mathcal{E}$ on $B$ by taking $\mathbb{P}_B(\mathcal{E}) = \operatorname{Proj}\operatorname{Sym}_{\mathcal{O}_B}(\mathcal{E})$. The inclusion of sheaves of rings $\mathcal{O}_B = (\operatorname{Sym}_{\mathcal{O}_B}(\mathcal{E}))_0 \hookrightarrow \operatorname{Sym}_{\mathcal{O}_B}(\mathcal{E})$ induces a structure map $\pi : \mathbb{P}_B(\mathcal{E}) \to B$. If $\mathcal{E}$ has rank $n+1$, then over any closed point $b \in B$ we have $\mathcal{E}_b \cong \mathbb{C}^{n+1}$, and so the fiber $\pi^{-1}(b)$ is $\mathbb{P}^N$. The restriction of $\mathcal{O}_{\mathbb{P}_B(\mathcal{E})}(1)$ to $\pi^{-1}(b)$ is $\mathbb{P}^N$ is $\mathcal{O}_{\mathbb{P}^N}(1)$.

The variety $\mathbb{P}_B(\mathcal{E})$ comes equipped with a tautological line bundle $\mathcal{O}_{\mathbb{P}_B}(\mathcal{E})(1)$, which is associated to the graded module $(\operatorname{Sym}_{\mathcal{O}_B(\mathcal{E})}(1)$, and a tautological map

$$\pi^*(\mathcal{E}) \to \mathcal{O}_{\mathbb{P}_B(\mathcal{E})}(1)$$

that induces an isomorphism on global sections. Furthermore,

$$\pi_*\mathcal{O}_{\mathbb{P}_B(\mathcal{E})}(p)) = \operatorname{Sym}^p(\mathcal{E})$$

for every $p$.

Thus the pair $(\mathbb{P}_B(\mathcal{E}), \mathcal{O}_{\mathbb{P}_B(\mathcal{E})}(1))$ determines $\mathcal{E}$; but $\mathbb{P}_B(\mathcal{E})$ alone determines $\mathcal{E}$ only up to twisting with a line bundle on $B$. For example, if $\mathcal{E}$ is itself a line bundle on $B$, then $\mathbb{P}_B(\mathcal{E}) \cong B$, but $\mathcal{O}_{\mathbb{P}_B(\mathcal{E})}(1)) \cong \mathcal{E}$.

Conversely, if $\pi : X \to B$ is a map whose fibers are isomorphic to $\mathbb{P}^N$, and if $X$ carries a line bundle $\mathcal{L}$ whose restriction to each fiber of $\pi$ is $\mathcal{O}_{\mathbb{P}^N}(1)$, then $X \cong \mathbb{P}_B(\mathcal{E})$ and $\mathcal{L} \cong \mathcal{O}_{\mathbb{P}_B(\mathcal{E})}(1)$, where $\mathcal{E} = \pi_*(\mathcal{L})$.

Finally, the Picard group, of line bundles on $X$ is $\operatorname{Pic} X \cong \operatorname{Pic} B \oplus \mathbb{Z}h$, where $h$ is the class of the tautological bundle, and the map $\operatorname{Pic} B \to \operatorname{Pic} X$ is pull-back by $\pi$.

In particular, since $\operatorname{Pic}(\mathbb{P}^1) = \mathbb{Z}$, we see that the divisor class group of a scroll $S(a_1, a_2)$ is freely generated by the class $H$ of a hyperplane section and the class $F$ of a ruling. The intersection form, is now easy to compute. If $C, D$ are divisor classes on the scroll, we write $C \cdot D \in \mathbb{Z}$ for their intersection number.

**Theorem 1.3.2.** *Let $X = S(a_1, a_s) \subset \mathbb{P}^N$ be a scroll and let $C_{a_i}$, for $i = 1, 2$ be the directrices. The divisor class group of $X$ is $\mathbb{Z}F \oplus \mathbb{Z}H$, where $F$ is the class of a fiber of the structure map and $H$ is the hyperplane section. The intersection form is given by*

$$\begin{array}{cc} \cdot & \begin{array}{cc} F & H \end{array} \\ \begin{array}{c} F \\ H \end{array} & \begin{pmatrix} 0 & 1 \\ 1 & a_1 + a_2 \end{pmatrix} \end{array}$$

*The canonical class of $X$ is $K_X = -2H + (a + b - 2)F$.*

*Moreover $C_{a_i} = H - a_j F$, where $\{i, j\} = \{1, 2\}$, so that $F \cdot C_{a_i} = 1, H \cdot C_{a_i} = a_i$, and $C_{a_i}^2 = a_i - a_j$.*

*Proof.* The values in the intersection matrix follow at once because any two fibers are disjoint straight lines, meeting a general hyperplane transversely in single points, and $H^2$ is the degree of $X \subset \mathbb{P}^N$.

Finally, if we choose a general hyperplane containing $C_{a_1}$ then it meets $C_{a_2}$ in $a_2$ points, so $H \cap X$ consists of $C_{a_1}$ plus $a_2$ fibers, proving the formula $C_{a_i} = H - a_j F$. The self-intersection formulas follow.

Finally, the canonical class $K_X$ must have the form $pH + qF$ for some integers $p, q$. By the adjunction formula, $(F + K_X).F = -2$, whence $p = -2$. But also $(C_a + K_X)C_a = -2$, yielding $q = a + b - 2$. □



Note that if $a_1 < a_2$, then $C_{a_1}^2 = a_1 - a_2$ is negative. We shall see that this is the only curve of negative self-intersection on $X$.

Our interest in scrolls in this book is primarily for the curves that lie on them. The following result tells us where to look:

are the curves?

**Theorem 1.3.3.** *Let $X = S(a_1, a_2)$ be a scroll of dimension 2 with $a_1 \leq a_2$, and let $F, H$ denote the class of the ruling and the hyperplane section, respectively. There are reduced irreducible curves in the class $D = pH + qF$ if and only if one of the following holds:*

1. *$D \sim F$; that is, $p = 0, q = 1$; or*

2. *$D \sim C_{a_1}$; that is, $p = 1, q = -a_2$; or*

3. *$p \geq 1$ and $D \cdot C_{a_1} \geq 0$; that is, $q \geq -pa_1$.*

*In case (3) the linear series $|D|$ is basepoint free, and thus in each case the class contains smooth curves.*

Note that in case (1) we have $D^2 = 0$; in case (2) we have $D^2 = a_1 - a_2 \leq 0$ and in case (3) we have $D^2 > 0$. This result follows by identifying the global sections of line bundle $\mathcal{O}_X(D)$:

global sections

**Theorem 1.3.4.** *Suppose that $D$ is a divisor on the scroll $X = S(a_1, a_2)$ with $a_1 \leq a_2$, and set $\mathcal{E} = \mathcal{O}_{\mathbb{P}^1}(a_1) \oplus \mathcal{O}_{\mathbb{P}^1}(a_1)$. If $D \sim pH + qF$, then*

$$\begin{aligned} H^0(\mathcal{O}_X(D)) &= H^0(\mathcal{O}_{\mathbb{P}^1}(q) \otimes \operatorname{Sym}^p \mathcal{E}) \\ &= \oplus_{0 \leq i \leq p} H^0\big(\mathcal{O}_{\mathbb{P}^1}(q + (p-i)a_1 + ia_2)\big). \end{aligned}$$

*and $|D|$ is basepoint free iff every summand is nonzero. Thus, numerically,*

$$h^0(\mathcal{O}_X(D)) = \sum_{\{i \ | \ q + (p-i)a_1 + ia_2 \geq 0\}} 1 + (q + (p-i)a_1 + ia_2),$$

*and $|D|$ is base point free iff $p \geq 0$ and $q \geq -pa_1$.*

*Proof.* Let $\pi : X \to \mathbb{P}^1$ be the structure map of the projective bundle $X = \mathbb{P}_{\mathbb{P}^1}(\mathcal{E})$. We have $H^0(\mathcal{O}_X(pH + qF)) = H^0(\pi_*(\mathcal{O}_X(pH + qF)))$. Also, We may write $\mathcal{O}_X(pH + qF)$ as $\mathcal{O}_X(p) \otimes \pi^* \mathcal{O}_{\mathbb{P}^1}(q)$, so by the push-pull formula and Fact projective space bundles 1.3.1,

$$\begin{aligned} \pi_*(\mathcal{O}_X(pH + qF)) &= \pi_*(\mathcal{O}_X(p) \otimes \pi^* \mathcal{O}_{\mathbb{P}^1}(q)) \\ &= \pi_*(\mathcal{O}_X(p)) \otimes \mathcal{O}_{\mathbb{P}^1}(q) \\ &= \operatorname{Sym}^p(\mathcal{E}) \otimes \mathcal{O}_{\mathbb{P}^1}(q) \\ &= \big(\oplus_{0 \leq i \leq p} \mathcal{O}_{\mathbb{P}^1}((p-i)a_1 + ia_2)\big) \otimes \mathcal{O}_{\mathbb{P}^1}(q), \end{aligned}$$

and the first formula follows, and we see that every term $H^0(\big(\mathcal{O}_{\mathbb{P}^1}(q + (p-i)a_1 + ia_2)\big))$ is nonzero iff and only if $H^0(\mathcal{O}_{\mathbb{P}^1}(q + pa_1)$ is nonzero iff $q \geq -pa_1$.

To establish the condition for base-point freeness, note that if all the summands are nonzero then there are sections vanishing on $C_{a_1}$ but not $C_{a_2}$, and vice versa, so the system is base point free. Conversely, if $q < -pa_1$, then

$$D \cdot C_{a_1} = (pH + qF) \cdot (H - a_2 F) = p(a_1 + a_2) - pa_1 + q = pa_1 + q < 0.$$

so any effective divisor in the class of $D$ must have a component in common with $C_{a_1}$. □

**Exercise 1.3.5.**
**(( keep this? sketch proof! ))**
Minimal degree varieties; as the varieties of given degree lying on the maximal number of quadrics.

We can easily compute the degrees and genera of curves that lie on scrolls:

**Proposition 1.3.6.** *Suppose that $D \sim pH + qF$ is a smooth irreducible curve on $S(a_1, a_2)$ as in Theorem* where are the curves? *1.3.3.*

- *The degree of $D$ is $p(a_1 + a_2) + q$.*
- *The genus of $D$ is $\binom{p}{2}(a_1 + a_2) + (p - 1)(q - 1)$.*

*Proof.* The degree of $D$ is $H \cdot D$, yielding the given formula. Let $g$ be the genus of $D$. By the adjunction formula

$$\begin{aligned} 2g - 2 = \big((p-2)H + &(q + a_1 + a_2 - 2)F\big) \cdot (pH + qF) \\ &= (p^2 - p)(a_1 + a_2) + 2(pq - p - q) \end{aligned}$$

so $g = \binom{p}{2}(a_1 + a_2) + (p - 1)(q - 1)$ as required. □

**Cheerful Fact 1.3.2.** A general curve $C$ of genus $\geq 22$ does not lie on any 2-dimensional scroll.

*Proof.* Except when $D \sim C_{a_1}$, a rational curve of negative self-intersection, every nonsingular curve on $X$ moves in a non-trivial linear series. However the moduli space of curves of genus $\geq 22$ is of general type, and this implies in particular that there there is no nontrivial rational family of curves containing a general curve of genus $\geq 22$. But if the linear series containing $C$ had all nonsingular fibers isomorphic to $C$, then $X$ would be birationally isomorphic to $C \times \mathbb{P}^1$, and thus not rational, a contradiction. □

## 1.4 Smooth curves on a 2-dimensional scroll

Prove there is one in a given class iff it meets the directrix non-negatively.

## 1.5 Automorphisms

(first compute the intersection form...)
**(( probably drop this section – seems we don't use it elsewhere. ))**

## 1.6 Appendix: Varieties of minimal degree

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# On Varieties of Minimal Degree (A Centennial Account)

DAVID EISENBUD AND JOE HARRIS



Abstract. This note contains a short tour through the folklore surrounding the rational normal scrolls, a general technique for finding such scrolls containing a given projective variety, and a new proof of the Del Pezzo–Bertini theorem classifying the varieties of minimal degree, which relies on a general description of the divisors on scrolls rather than on the usual enumeration of low-dimensional special cases and which works smoothly in all characteristics.

**Introduction.** Throughout, we work over an algebraically closed field $k$ of arbitrary characteristic with subschemes $X \subset \mathbf{P}^r_k$. We say that $X$ is a *variety* if it is reduced and irreducible, and that it is *nondegenerate* if it is not contained in a hyperplane. There is an elementary lower bound for the degree of such a variety:

PROPOSITION 0. *If $X \subset \mathbf{P}^r$ is a nondegenerate variety, then* $\deg X \geq 1 + \operatorname{codim} X$.

(PROOF. If $\operatorname{codim} X = 1$ the result is trivial. Else we project to $\mathbf{P}^{r-1}$ from a general point of $X$, reducing the degree by at least 1 and the codimension by 1, and are done by induction. $\square$)

We say that $X \subset \mathbf{P}^r$ is a *variety of minimal degree* if $X$ is nondegenerate and $\deg X = 1 + \operatorname{codim} X$. One hundred years ago Del Pezzo (1886) gave a remarkable classification for surfaces of minimal degree, and Bertini (1907) showed how to deduce a similar classification for varieties of any dimension. Of course the case of codimension 1 is trivial, $X$ being then a quadric hypersurface, classified by its dimension and that of its singular locus. In other cases we may phrase the result as:

THEOREM 1. *If $X \subset \mathbf{P}^r$ is a variety of minimal degree, then $X$ is a cone over a smooth such variety. If $X$ is smooth and* $\operatorname{codim} X > 1$, *then $X \subset \mathbf{P}^r$ is either a rational normal scroll or the Veronese surface $\mathbf{P}^2 \subset \mathbf{P}^5$.*

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(See §1 for the definition and some properties of rational normal scrolls.)

The purpose of this note is to give a short and direct proof of the Del Pezzo–Bertini theorem, valid in any characteristic. The proofs (Bertini (1907), Harris (1981), and Xambò (1981)) are all essentially similar: they treat first the cases of surfaces in general (which is also done in Nagata (1960) and Griffiths-Harris (1978)), and finally they reduce the case of arbitrary varieties to the case of surfaces, distinguishing according to whether the general 2-dimensional plane section of the given variety is a scroll or the Veronese surface. Instead, we base our discussion on the following general result (§2), which is useful in many other circumstances:

THEOREM 2. *Let $X \subset \mathbf{P}^r$ be a linearly normal variety, and $D \subset X$ a divisor. If $D$ moves in a pencil $\{D_\lambda | \lambda \in \mathbf{P}^1\}$ of linearly equivalent divisors, then writing $\overline{D}_\lambda$ for the linear span of $D_\lambda$ in $\mathbf{P}^r$, the variety*

$$S = \bigcup_\lambda \overline{D}_\lambda$$

*is a rational normal scroll.*

This allows us (in §3) to write an arbitrary variety $X$ of minimal degree as a divisor on a scroll, and simple considerations of the geometry of scrolls then lead to the result.

**1. Description of the varieties of minimal degree.** We first explain some of the terms used in Theorems 1 and 2 above:

If $L \subset \mathbf{P}^{r+s+1}$ is a linear space of dimension $s$, $p_L: \mathbf{P}^{r+s+1} \to \mathbf{P}^r$ is the projection from $L$, and $X$ is a variety in $\mathbf{P}^r$, then the cone over $X$ is the closure of $p_L^{-1}X$. In equations, the cone is simply given by the same equations as $X$, written in the appropriate subset of the coordinates on $\mathbf{P}^{r+s+1}$. Thus a *cone in* $\mathbf{P}^r$ *over the Veronese surface* $\mathbf{P}^2 \hookrightarrow \mathbf{P}^5$ may be defined as a variety given, with respect to suitable coordinates $x_0, \ldots, x_r$, by the (prime) ideal of $2 \times 2$ minors of the generic symmetric matrix:

$$\begin{pmatrix} X_0 & X_1 & X_2 \\ X_1 & X_3 & X_4 \\ X_2 & X_4 & X_5 \end{pmatrix}$$

(It is easy to see that a cone over any variety of minimal degree has minimal degree; our definition of rational normal scroll is such that the cone over a rational normal scroll is another rational normal scroll.)

Note that the Veronese surface contains no lines—indeed, any curve that lies on it must have even degree, as one sees by pulling back to $\mathbf{P}^2$—and thus a cone over the Veronese surface cannot contain a linear space of codimension 1. We shall see that this property separates the varieties of minimal degree which are cones over the Veronese surface from those that are scrolls.

We now describe rational normal scrolls in the terms necessary for Theorem 1. In our proof of the theorem we reduce rapidly to the case where $X$ is a divisor on a scroll, and we shall describe these as well.

A rational normal scroll is a cone over a smooth linearly normal variety fibered over $\mathbf{P}^1$ by linear spaces; in particular, a rational normal scroll contains a pencil of linear spaces of codimension 1 (and these are the only linearly normal varieties with this property, as will follow from Proposition 2.1, below).

To be more explicit, think of $\mathbf{P}^r$ as the space of 1-quotients of $k^{r+1}$, so that a $d$-plane in $\mathbf{P}^r$ corresponds to a $d+1$-quotient of $k^{r+1}$. A variety $X \subset \mathbf{P}^r$ with a map $\pi: X \to \mathbf{P}^1$ whose fibers are $d$-planes is thus the projectivization of a rank $d+1$ vector bundle on $\mathbf{P}^1$ which is a quotient of

$$k^{r+1} \otimes_k \mathcal{O}_{\mathbf{P}^1} = \mathcal{O}_{\mathbf{P}^1}^{r+1}.$$

Slightly more generally, let

$$\mathcal{E} = \bigoplus_0^d \mathcal{O}_{\mathbf{P}^1}(a_i)$$

be a vector bundle on $\mathbf{P}^1$, and assume

$$0 \le a_0 \le \cdots \le a_d, \quad \text{with } a_d > 0,$$

so that $\mathcal{E}$ is generated by $\sum a_i + d + 1$ global sections. Write $\mathbf{P}(\mathcal{E})$, or alternately $\mathbf{P}(a_0, \dots, a_d)$, for the projectivized vector bundle

$$\mathbf{P}(\mathcal{E}) = \operatorname{Proj} \operatorname{Sym} \mathcal{E} \xrightarrow{\pi} \mathbf{P}^1$$

(whose points over $\lambda \in \mathbf{P}^1$ are quotients $\mathcal{E}_\lambda \to k(\lambda)$), and let $\mathcal{O}_{\mathbf{P}(\mathcal{E})}(1)$ be the tautological line bundle. Because the $a_i$ are $\ge 0$, $\mathcal{O}_{\mathbf{P}(\mathcal{E})}(1)$ is generated by its global sections (see the computation below) and defines a "tautological" map

$$\mathbf{P}(\mathcal{E}) \to \mathbf{P}^{\sum a_i + d}.$$

This map is birational because $a_d > 0$. We write $S(\mathcal{E})$ or $S(a_0, \dots, a_d)$ for the image of this map, which, as we shall see, is a variety of dimension $d+1$ and degree $\sum a_i$, so that it is a variety of minimal degree. A *rational normal scroll* is simply one of the varieties $S(\mathcal{E})$. Note that $\mathcal{O}_{\mathbf{P}(\mathcal{E})}(1)$ induces $\mathcal{O}_{\mathbf{P}^d}(1)$ on each fiber $F \cong \mathbf{P}^d$ of $\mathbf{P}(\mathcal{E}) \to P^1$, so $F$ is mapped isomorphically to a $d$-plane in $S(\mathcal{E})$.

The most familiar examples of rational normal scrolls are probably

(o) $\mathbf{P}^d$, which is $S(0, \dots, 0, 1)$,

(i) the rational normal curve of degree $a$ in $\mathbf{P}^a$, which is $S(a)$,

(ii) the cone over a plane conic, $S(0,2) \subset \mathbf{P}^3$,

(iii) the nonsingular quadric in $\mathbf{P}^3$, $S(1,1)$,

(iv) the projective plane blown up at one point, embedded as a surface of degree 3 in $\mathbf{P}^4$ by the series of conics in the plane passing through the point; this is $S(1,2)$.

There is a pretty geometric description of $S(a_0, \dots, a_d)$ from which the name "scroll" derives, and from which the equivalence of the two definitions above may be deduced:

The projection

$$\mathcal{E} = \bigoplus_0^d \mathcal{O}(a_i) \to \mathcal{O}(a_i)$$

defines a section $\mathbf{P}^1 \cong \mathbf{P}(\mathcal{O}(a_i)) \hookrightarrow \mathbf{P}(\mathcal{E})$, and

$$\mathcal{O}_{\mathbf{P}(\mathcal{E})}(1)|_{\mathbf{P}(\mathcal{O}(a_i))} = \mathcal{O}_{\mathbf{P}(\mathcal{O}(a_i))}(1) = \mathcal{O}_{\mathbf{P}^1}(a_i),$$

so this section is mapped to a rational normal curve of degree $a_i$ in the $\mathbf{P}^{a_i} \subset \mathbf{P}^{\sum a_i+d}$ corresponding to the quotient $H^0(\mathcal{E}) \to H^0(\mathcal{O}_{\mathbf{P}^1}(a_i))$. (Of course if $a_i = 0$, the "rational normal curve of degree $a_i$" is a point $\subset \mathbf{P}^0$!) Thus we may construct the rational normal scroll $S(a_0, \ldots, a_d) \subset \mathbf{P}^{\sum a_i+d}$ by considering the parametrized rational normal curves

$$\mathbf{P}^1 \xrightarrow{\phi_i} C_{a_i} \subset \mathbf{P}^{a_i} \subset \mathbf{P}^{\sum a_i+d}$$

corresponding to the decomposition

$$k^{\sum a_i+d+1} = \bigoplus_0^d k^{a_i+1},$$

and letting $S(a_0, \ldots, a_d)$ be the union over $\lambda \subset \mathbf{P}^1$ of the $d$-planes spanned by $\phi_0(\lambda), \ldots, \phi_d(\lambda)$. In particular, we see that the cone in $\mathbf{P}^{\sum a_i+d+s}$ over $S(a_0, \ldots, a_d)$ is

$$S(\underbrace{0, \ldots, 0}_{s}, a_0, \ldots, a_d).$$

Also, $S(a_0, \ldots, a_d)$ is nonsingular iff $(a_0, \ldots, a_d) = (0, \ldots, 0, 1)$ or $a_i > 0$ for all $i$.

We note that this description is convenient for giving the homogeneous ideal of $S(a_0, \ldots, a_d)$. As is well known, the homogeneous ideal of a rational normal curve $S(a) \subset \mathbf{P}^a$ may be written as the ideal of $2 \times 2$ minors

$$\det{}_2 \begin{pmatrix} X_0, X_1, \ldots, X_{a-1} \\ X_1, X_2, \ldots, X_a \end{pmatrix},$$

and this expression gives the parametrization sending $(s,t) \in \mathbf{P}^1$ to the point of $\mathbf{P}^a$ where the linear forms

$$sX_0 + tX_1, \ldots, sX_{a-1} + tX_a$$

all vanish. (This is $s$ times the first row of the given matrix plus $t$ times the second row.) It follows at once that $S(a_0, \ldots, a_d)$ is at least set-theoretically the locus where the minors of a matrix of the form

$$\left(\begin{array}{c|c|c} X_{0,0}X_{0,1}, \ldots, X_{0,a_0-1} & X_{1,0}, \ldots, X_{1,a_1-1} & \ldots X_{d,a_d-1} \\ & & \ldots \\ X_{0,1}, X_{0,2}, \ldots, X_{0,a_0} & X_{1,1}, \ldots, X_{1,a_1} & \ldots X_{d,a_d} \end{array}\right)$$

all vanish. That these minors generate the whole homogeneous ideal follows easily as in the proof of Lemma 2.1 below.

The divisor class group of a projectivized vector bundle $\mathbf{P}(\mathcal{E})$ over $\mathbf{P}^1$ is easy to describe (Hartshorne (1977), Chapter II, exc. 7.9): Writing $H$ for a divisor in the class determined by $\mathcal{O}_{\mathbf{P}(\mathcal{E})}(1)$, and $F$ for the fiber of $\mathbf{P}(\mathcal{E}) \to \mathbf{P}^1$, the divisor class group may be written (confusing divisors and their classes systematically)

$$\mathbf{Z}H + \mathbf{Z}F.$$

Moreover, the chow ring is given by

$$\mathbf{Z}[F,H]/\left(F^2, H^{d+2}, H^{d+1}F, H^{d+1} - \left(\sum a_i\right) H^d F\right).$$

We shall only need a numerical part of this, giving the degree of a scroll:

$$\text{degree}\, S(a_0,\dots,a_d) = H^{d+1} = \sum_0^d a_i.$$

The simplest way to understand this is perhaps from the geometric description given above: In $\mathbf{P}^{\sum_0^d a_i + d}$ we may take a hyperplane containing the natural copy of $\mathbf{P}^{\sum_1^d a_i + d - 1}$ and meeting $C_{a_0} \subset \mathbf{P}^{a_0}$ transversely. The hyperplane section is then the union of $S(a_1,\dots,a_d)$ with $a_0$ copies of $F$ (which is embedded as a $d$-plane).

It is also easy to compute the cohomology of the line bundles on $\mathbf{P}(\mathcal{E})$. In particular, the tautological map

$$\pi^*\mathcal{E} \to \mathcal{O}_{\mathbf{P}(\mathcal{E})}(1)$$

induces for any integer $a$ a map

$$\operatorname{Sym}_a \mathcal{E} = \pi_* \operatorname{Sym}_a \pi^*\mathcal{E} \to \pi_*\mathcal{O}_{\mathbf{P}\mathcal{E}}(a)$$

and thus for every $a$, $b$ a map

$$\mathcal{O}_{\mathbf{P}^1}(b) \otimes \operatorname{Sym}_a \mathcal{E} \to \mathcal{O}_{\mathbf{P}^1}(b) \otimes \pi_*\mathcal{O}_{\mathbf{P}\mathcal{E}}(a) \cong \pi_*(\pi^*\mathcal{O}_{\mathbf{P}^1}(b) \otimes \mathcal{O}_{\mathbf{P}\mathcal{E}}(a)),$$

which is an isomorphism, as one easily checks locally. Since $\pi$ is surjective, $\pi_*$ induces an isomorphism on global sections, and we see that an element

$$\sigma \in H^0(\pi^*\mathcal{O}_{\mathbf{P}^1}(b) \otimes \mathcal{O}_{\mathbf{P}\mathcal{E}}(a))$$

may be represented as an element of

$$\begin{aligned} H^0(\mathcal{O}_{\mathbf{P}^1}(b) \otimes \operatorname{Sym}_a \mathcal{E}) &= H^0\left(\mathcal{O}_{\mathbf{P}^1}(b) \otimes \sum_{|I|=a} \mathcal{O}_{\mathbf{P}^1}\left(\sum_{i\in I} a_i\right)\right) \\ &= \sum_{|I|=a} H^0\left(\mathcal{O}_{\mathbf{P}}\left(b + \sum_{i\in I} a_i\right)\right), \end{aligned}$$

where the notation $\sum_{|I|=a}$ indicates summation over all collections $I$ consisting of $a$ elements (with repetitions) from $\{0,\dots,d\}$.

From this we may derive a useful representation of divisors in $\mathbf{P}\mathcal{E}$, generalizing the idea of "bihomogeneous forms" in the case of $\mathbf{P}(\mathcal{O}_{\mathbf{P}^1}^{d+1}) = \mathbf{P}^1 \times \mathbf{P}^d$. If we let

$$x_i \in H^0(\pi^*\mathcal{O}_{\mathbf{P}^1}(-a_i) \otimes \mathcal{O}_{\mathbf{P}\mathcal{E}}(1)) = H^0\mathcal{E}(-a_i)$$

be an element corresponding to a generator of the $i$th summand

$$\mathcal{O}_{\mathbf{P}^1}(a_i - a_i) = \mathcal{O}_{\mathbf{P}^1} \subset \mathcal{E}(-a_i),$$

and write

$$x^I := \prod_{i\in I} x_i \in H^0\Big[(\mathrm{Sym}_a\,\mathcal{E})\Big(-\sum_{i\in I} a_i\Big)\Big]$$
$$= H^0\Big(\pi^*\mathcal{O}_{\mathbf{P}^1}\Big(-\sum_{i\in I} a_i\Big)\otimes \mathcal{O}_{\mathbf{P}\mathcal{E}}(a)\Big)$$

for the product, then we may represent $\sigma$ conveniently as a "polynomial":

$$\sigma = \sum_{|I|=a} \alpha_I(s,t)x^I,$$

where $s$, $t$ are homogeneous coordinates on $\mathbf{P}^1$ and where $\alpha_I(s,t)$ is a homogeneous form of degree

$$\deg \alpha_I(s,t) = b + \sum_{i\in|I|} a_i.$$

This representation is convenient because the "variables"

$$x_i \in H^0(\pi^*\mathcal{O}_{\mathbf{P}^1}(-a_i)\otimes \mathcal{O}_{\mathbf{P}\mathcal{E}}(1))$$

restrict to a basis of the linear forms on each fiber of $\mathbf{P}\mathcal{E}\to\mathbf{P}^1$, and the divisor $D$ of $\sigma$ meets the $\mathbf{P}^d \cong F_{(u,v)}$ over $(u,v)\in\mathbf{P}^1$ in the hypersurface with equation $\sum_{|I|=a}\alpha_I(u,v)x^I$.

In practice, we wish to use this idea for a Weil divisor $X$ of a scroll $S(\mathcal{E})$. Since $S(\mathcal{E})$ is normal and $\mathbf{P}\mathcal{E}\to S\mathcal{E}$ is birational, we may do this by defining $\tilde{X}\subset\mathbf{P}\mathcal{E}$ to be the "strict transform" of $X$—that is, for an irreducible subvariety $X$ of codimension 1, $\tilde{X}$ is the closure of the image in $\mathbf{P}\mathcal{E}$ of the complement, in $X$, of the fundamental locus of the inverse rational map, $S(\mathcal{E})\to\mathbf{P}\mathcal{E}$. Then $\tilde{X}$ occupies a well-defined divisor class on $\mathbf{P}\mathcal{E}$ and we may apply the above technique to it.

**2. Rational normal scrolls in the wild.** The proof of Theorem 2 rests on a technique of constructing scrolls from their determinantal equations, as follows:

We say that a map of $k$-vector spaces

$$\phi: U\otimes V\to W$$

is *nondegenerate* if $\phi(u\otimes v)\neq 0$ whenever $u,v\neq 0$, or equivalently if each map $\phi_u: u\otimes V\to W$ is a monomorphism. The typical example, for our purposes, comes from a (reduced, irreducible) variety $X$ and a pair of line bundles $\mathcal{L}$, $\mathcal{M}$; if $U = H^0(\mathcal{L})$, $V = H^0(\mathcal{M})$, and $W = H^0(\mathcal{L}\otimes\mathcal{M})$, then the multiplication map is obviously nondegenerate in the above sense. In our application, $X$ will be embedded linearly normally in $\mathbf{P}^r$ by $\mathcal{L}\otimes\mathcal{M}$, so we may identify $H^0(\mathcal{L}\otimes\mathcal{M})$ with $H^0(\mathcal{O}_{\mathbf{P}^r}(1))$.

In general, given any map

$$k^\gamma\otimes k^\delta \to H^0\mathcal{O}_{\mathbf{P}^r}(1),$$

we define an associated map of sheaves

$$A_\phi: \mathcal{O}^\delta_{\mathbf{P}^r}(-1)\to\mathcal{O}^\gamma_{\mathbf{P}^r}$$

by twisting the obvious map

$$k^{\delta} \otimes \mathcal{O}_{\mathbf{P}^r} \to k^{\gamma^*} \otimes \mathcal{O}_{\mathbf{P}^r}(1)$$

by $\mathcal{O}_{\mathbf{P}^r}(-1)$. Taking $\gamma = 2$, we have

LEMMA 2.1. *If $\phi: k^2 \otimes k^{\delta} \to H^0\mathcal{O}_{\mathbf{P}^r}(1)$ is a nondegenerate pairing, then the ideal of $2 \times 2$ minors $\det_2 A_\phi$ is prime, and $V(\det_2 A_\phi)$ is a rational normal scroll of degree $\delta$.*

PROOF. If the image of $\phi$ is a proper subspace of $H^0\mathcal{O}_{\mathbf{P}^r}(1)$, then $V(\det_2 A_\phi)$ is a cone. Since the cone over a scroll is a scroll, we may by reducing $r$ assume that $\phi$ is an epimorphism, so that the rank of $A_\phi$ never drops to 0 on $\mathbf{P}^r$. It follows that $\mathcal{L} = \operatorname{Coker} A_\phi$ is a line bundle on $S = V(\det_2 A_\phi)$, generated by the image of $V = k^{2^*}$. The linear series $(\mathcal{L}, V)$ defines a map $\pi: S \to \mathbf{P}^1$. If $(s,t) \in \mathbf{P}^1$, then the fiber $F$ of $\pi$ over $(s,t)$ is the scheme defined by the vanishing of the composite map

$$\mathcal{O}^{\delta}_{\mathbf{P}^r}(-1) \to \mathcal{O}^2_{\mathbf{P}^r} \xrightarrow{(s,t)} \mathcal{O}_{\mathbf{P}^r};$$

and this scheme is, by our nondegeneracy hypothesis, given by the vanishing of $\delta$ linearly independent linear forms, so $F$ is a plane of codimension $\delta$. By the general formula for the maximum codimension of (any component of) a determinantal variety we have $\operatorname{codim} S \leq \delta - 1$, so the map $S \to \mathbf{P}^1$ is onto, and since the fibers are smooth and irreducible and the map is proper, $S$ is smooth and irreducible of codimension $\delta - 1$.

Since $\det_2 A_\phi$ thus has height $\delta - 1$ in the homogeneous coordinate ring of $\mathbf{P}^r$, it is perfect, and in particular unmixed (Arbarello et al. (1984), Chapter II, 4.1; note that the characteristic 0 hypothesis there is irrelevant). Thus $\det_2 A_\phi$ is the entire homogeneous ideal of $S$, and since $\det_2 A_\phi$ is perfect, $S$ is arithmetically Cohen-Macaulay, so in particular $S$ is linearly normal.

The fibers of $\pi$ being linear spaces in $\mathbf{P}^r$, correspond to quotients of $k^{r+1}$, and this defines a vector bundle on $\mathbf{P}^1$ of rank $r - \delta + 1$ such that $S \to \mathbf{P}^1$ is the associated projective space bundle; thus $S$ is a rational normal scroll as claimed. $\square$

REMARK. Using the same ideas, one sees that the height of $\det_2(A_\phi)$ is $\delta - 1$ iff the rank of $\phi_u$ never drops by more than 1; then $X = V(\det_2 A_\phi)$ is a "crown", that is, the union of a scroll of codimension $\delta - 1$ and some linear spaces of codimension $\delta - 1$ which intersect the scroll along linear spaces of codimension $\delta$ (fibers of $\pi$)—see Xambò (1981).

With this result in hand, it is easy to complete the proof of Theorem 2:

PROOF OF THEOREM 2. Let $k^2 \cong V \subset H^0\mathcal{O}_X(D)$ be the vector space of sections corresponding to the pencil $D_\lambda$, and let $H$ be the hyperplane section of $X$. The natural multiplication map

$$V \otimes H^0\mathcal{O}_X(H - D) \to H^0\mathcal{O}_X(H) = H^0\mathcal{O}_{\mathbf{P}^r}(1)$$

is nondegenerate, and thus gives rise to a scroll $S$ containing all the $D_\lambda$, and thus $X$. The linear space $\overline{D}_\lambda$ is the intersection of all the hyperplanes containing

$D_\lambda$, which correspond to elements of $H^0\mathcal{O}_X(H-D)$, so $\overline{D}_\lambda$ is the fiber over $\lambda$ of $S \to \mathbf{P}^1$, as desired. $\square$

EXAMPLES. (i) Let $C$ be a hyperelliptic (or elliptic) curve, $C \subset \mathbf{P}^r$ an embedding by a complete series of degree $d$. $C$ is a divisor on the variety $S$ which is the union of the secants corresponding to the $\mathfrak{g}^1_2$ on $C$ (or, if $C$ is elliptic, any $\mathfrak{g}^1_2$ on $C$). This variety is a rational normal scroll $S(\mathcal{E})$ and $\tilde{C} \sim 2H + (d-2r+2)F$ on $\mathbf{P}(\mathcal{E})$. More generally, a linearly normal curve $C \subset \mathbf{P}^r$ which possesses a $\mathfrak{g}^1_d$ lies on a scroll of dimension $\leq d$; if $C \subset \mathbf{P}^r$ is the canonical embedding, then this scroll is of dimension $\leq d-1$, so in particular the canonical image of any trigonal curve is a divisor on a 2-dimensional scroll $S(\mathcal{E})$, and $\tilde{C} \sim 3H + (4-g)F$ on $\mathbf{P}(\mathcal{E})$. See Schreyer (1986) for a study of canonical curves using this idea.

(ii) A $K3$ surface, embedded linearly normally in any projective space, is a divisor on a 3-dimensional scroll if it contains an elliptic cubic (which then moves in a nontrivial linear series). See for example Saint-Donat (1974).

**3. The classification theorem.** Before giving our proof of the Del Pezzo–Bertini Theorem, we record three elementary observations about projections:

(1) If $X$ is a variety of minimal degree, then $X$ is linearly normal. (*Proof*: If $X$ were the isomorphic projection of a nondegenerate variety $X'$ in $\mathbf{P}^{r+1}$, then $X'$ would have degree less than that allowed by Proposition 0.)

(2) If $X \subset \mathbf{P}^r$ is a variety of minimal degree and $p \in X$, then the projection $\pi_p X \subset \mathbf{P}^{r-1}$ is a variety of minimal degree, the map $X - p \to \pi_p X$ is separable, and if $p$ is singular then $X$ is a cone with vertex $p$. (*Proof*: Indeed, $\pi_p X$ is obviously nondegenerate. If $X$ is a cone with vertex $p$, the result is obvious. Else $\dim \pi_p X = \dim X$ but $\deg \pi_p X \leq \deg X - 1$. The inequality must actually be an equality by Proposition 0 which shows in particular that $p$ is a nonsingular point, and $\pi_p: X - p \to \pi_p X$ is birational.)

(3) If $p \in X \subset \mathbf{P}^r$ is any point on any variety, $E_X$ the exceptional fiber of the blow-up of $p$ in $X$, and $E_{\mathbf{P}^r} \cong \mathbf{P}^{r-1}$ the exceptional fiber of the blow-up of $\mathbf{P}^r$ at $p$, then $E_X$ is naturally embedded in $E_{\mathbf{P}^r}$, which is mapped isomorphically to $\mathbf{P}^{r-1}$ by the map induced by $\pi_p$. Thus $E_X \subset \pi_p(X) \subset \mathbf{P}^{r-1}$. In particular, if $p$ is a nonsingular point on $X$, so that $E_X$ is a linear subspace of $\mathbf{P}^{r-1}$, then the "image of $p$" under $\pi_p: X \to \pi_p(X) \subset \mathbf{P}^{r-1}$ is a linear subspace of $\mathbf{P}^{r-1}$ which is a divisor on $\pi_p(X)$. More naively, this is the image of the tangent plane to $X$ at $p$.

In view of observation (3) it will be useful to begin with the following result, which "recognizes" scrolls:

PROPOSITION 3.1. *If $X \subset \mathbf{P}^r$ is a variety of minimal degree, and $X$ contains a linear subspace of $\mathbf{P}^r$ as a subspace of codimension 1, then $X$ is a scroll.*

PROOF. By Proposition 2.1 it suffices to show that $X$ contains a pencil of linear divisors, though the given subspace itself may not move.

Let $F \subset X$ be the given linear subspace. We may assume (by projecting, if necessary) that $X$ is smooth along $F$. Let $H \subset \mathbf{P}^r$ be a general hyperplane containing $F$, and let $S = H \cap X - F$. Let $\pi_F$ be projection from $F$. We distinguish two cases:

*Case* 1. $\dim \pi_F(X) \geq 2$. By Bertini's Theorem and observation (2) above, $S$ is then a reduced and irreducible variety, of degree and dimension one less than that of $X$. Thus by Proposition 0, $S$ is degenerate in $H$, so $F = H \cdot X - S$ moves in (at least) a pencil of linear spaces, and we are done.

*Case* 2. $\dim \pi_F(X) = 1$. By observation (1), $\pi_F(X)$ is a curve of minimal degree, say of degree $s$ in $\mathbf{P}^s$. Projecting $\pi_F(X)$ from $s-1$ general points on it gives a birational map to $\mathbf{P}^1$, so $\pi_F(X) \cong \mathbf{P}^1$. Further, the cone on $S$ with vertex $p$ is a union of $s$ planes, the spans of $F$ with the points of a general hyperplane section of $\pi_F(X)$, so $S$ has $s$ components. But $s = r - \dim F - 1 = \operatorname{codim} X = \deg X - 1 = \deg S$, so $S$ is the union of $s$ planes, and these are linearly equivalent to each other since the points of $\pi_F(X)$ are. Thus a component of $S$ is a linear space moving in a pencil as desired. □

Proof of Theorem 1. Let $X \subset \mathbf{P}^r$ be a variety of minimal degree. We may assume that the codimension $c$ of $X$ is $\geq 2$ and that $X$ is not a cone. By Proposition 3.1 we may as well also assume that $X$ contains no linear space of codimension 1, so that in particular the dimension $d$ of $X$ is $\geq 2$, and we must prove that under these hypotheses $X$ is the Veronese surface $\mathbf{P}^2 \subset \mathbf{P}^5$. In fact, it suffices to prove that $X \cong \mathbf{P}^2$; for the embedding of $\mathbf{P}^2$ by the complete series of curves of degree $d$ gives a surface of degree $d^2$ and codimension

$$\binom{2+d}{2} - 3,$$

which is $< d^2 - 1$ for $d \geq 3$.

Let $p \in X$ be any point. By observation (3) and Proposition 3.1, $\pi_p(X)$ is a scroll, so the cone $S \subset \mathbf{P}^r$ with vertex $p$ over $\pi_p(X)$ (or over $X$) is a scroll, say $S = S(\mathcal{E})$, with $\mathcal{E} = \bigoplus_0^d \mathcal{O}_{\mathbf{P}^1}(a_i)$ and $0 \leq a_0 \leq \cdots \leq a_d$. $X$ is a divisor on $S$.

Consider the strict transform $\tilde{X} \subset \mathbf{P}(\mathcal{E})$ of $X$ under the desingularization $\mathbf{P}(\mathcal{E}) \to S(\mathcal{E}) = S$, and let its divisor class be $aH - bF$. We will prove under the hypotheses above that $a = 2$ and $X$ is a surface. (Along the way we will see numerically that $b = 4$, $(a_0, a_1, a_2) = (0, 1, 2)$, so $c = 3$ and $X \subset \mathbf{P}^5$ as befits the Veronese, but we will not use this directly.)

First, because the degree $c+1$ of $X$ is 1 more than that of $S$, and on the other hand is $H^{d-1} \cdot (aH - bF)$, we get $b = (a-1)c - 1$.

To bound $a$, first note that $X$ must meet every fiber of $\mathbf{P}\mathcal{E} \to \mathbf{P}^1$, so $aH - bF|_F = aH|_F > 0$, and $a \geq 1$. If $a$ were 1, then $\tilde{X}$ would meet each fiber $F$ in a linear space of dimension $d-1$. Since each fiber $F$ is mapped isomorphically to a $d$-plane in $\mathbf{P}^r$ under $\mathbf{P}(\mathcal{E}) \to S(\mathcal{E})$, $X$ would contain linear spaces of dimension $d-1$, contrary to our hypothesis. Thus $a \geq 2$.

As in §2, $\tilde{X}$ may be represented by an equation $g = 0$ with $g$ of the form:

$$g = \sum_{|I|=a} \alpha_I(s,t) x^I,$$

with

$$\deg \alpha_I = \left( \sum_{i \in I} a_i \right) - b = \sum_{i \in I} a_i - (a-1)c + 1.$$

If the variable $x_0$ did not occur in $g$, then $\tilde{X}$ would meet each fiber $F$ in a cone over the preimage of $p$, and $X$ itself would be a cone contrary to hypothesis. But for $x_0$ to occur we must have

$$0 \leq \deg \alpha_{0,d,\ldots,d} = a_0 + (a-1)a_d - (a-1)c + 1.$$

Since $S$ is a cone we have $a_0 = 0$, and we derive

$$(*) \qquad a_d \geq c - 1/(a-1).$$

If $x_d$ occurred in every nonzero term of $g$, then for every fiber $F$, $\tilde{X} \cap F$ would contain the $(d-1)$-plane $x_d = 0$, and again $X$ would contain a $(d-1)$-plane, contradicting our hypotheses. Thus

$$(**) \qquad 0 \leq \deg \alpha_{d-1,d-1,\ldots,d-1} = a a_{d-1} - (a-1)c + 1.$$

Now if $a \geq 3$, then $a_d = c$ by $(*)$; but $c = \deg X - 1 = \deg S = \sum_{i=0}^{d} a_i$, so this implies $a_{d-1} = 0$, and $(**)$ gives a contradiction. Thus $a = 2$ as claimed, and $a_d \leq c - 1$. Condition $(*)$ now gives $a_d = c - 1$, so $a_{d-1} = 1$ and $a_0 = \cdots = a_{d-2} = 0$. Applying $(**)$ again we get $a_d = 1$ or $a_d = 2$.

In the first case $(a_0, \ldots, a_d) = (0, \ldots, 0, 1, 1)$, so $S$ is a cone over $\mathbf{P}^1 \times \mathbf{P}^1 \subset \mathbf{P}^3$. A suitable hypersurface section of $S$ will consist of the union of two planes $F_1$ and $F_2$, the cones over the rulings of $\mathbf{P}^1 \times \mathbf{P}^1$. Since each of these rulings sweeps out all of $\mathbf{P}^1 \times \mathbf{P}^1$, $X$ must meet each of $F_1$ and $F_2$ in codimension 1. Because $c = 2$ we have $\deg X = 3$, so either $X \cap F_1$ or $X \cap F_2$ must be a linear space, contradicting our assumption on $X$.

We thus see that $a = 2$, $c = 3$, and $(a_0, \ldots, a_d) = (0, \ldots, 0, 1, 2)$. Under these circumstances the sum of the terms of $g$ involving $x_0, \ldots, x_{d-2}$ may be written

$$\left( \sum_0^{d-2} \alpha_{i,d} x_i \right) x_d,$$

with $\alpha_{i,d}$ constant. Thus if $d \geq 3$ the locus $g = 0$ in each fiber $F$ is a cone with vertex the $(d-3)$-dimensional linear space given by

$$x_d = x_{d-1} = \sum_0^{d-2} \alpha_{i,d} x_i = 0.$$

Of course $S$ is itself a cone with $(d-2)$-dimensional vertex $L$, say. The $(d-2)$-dimensional subspaces of the fibers $F$ given by $x_d = x_{d-1} = 0$ are all mapped isomorphically to $L$ under $\mathbf{P}(\mathcal{E}) \to S$, and the restrictions of the coordinates $x_0, \ldots, x_{d-2}$ are all identified, and become coordinates on $L$. Thus $X$ meets the image of each fiber in a cone with vertex given in $L$ by $\sum_0^{d-2} \alpha_{i,d} x_i = 0$, so $X$ is a cone, contradicting our assumption. This shows $d = 2$.

We have now shown that $a = 2$ and $X$ is a surface. In this case, for every fiber $F \cong \mathbf{P}^2$ of $\mathbf{P}(\mathcal{E})$, $F \cap \tilde{X}$ is a conic, necessarily nonsingular since else $X$ would contain a line. Thus $\tilde{X}$ is a rational ruled surface. But the preimage in $X$ of $p$ is a line, so $\tilde{X}$ is the blow-up of $X$ at $p$, and is not a minimal surface. This is only possible if $\tilde{X} \cong \mathbf{P}(\mathcal{O}_{\mathbf{P}^1} \oplus \mathcal{O}_{\mathbf{P}^1}(1))$ and $X \cong \mathbf{P}^2$, as required. $\square$

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