Contents

Chapter 16

Linkage and the canonical sheaves of singular curves

16.1. Introduction

In this chapter, curves are purely 1*-dimensional projective schemes, not always reduced or irreducible.*

Linkage is an equivalence relation on varieties and schemes of a given dimension embedded in a common space. It was a key element in the classification of curves in $\mathbb{P}^3$ for which Max Noether and Georges-Henri Halphen received the Steiner prize of the Prussian Academy of Sciences in 1880, and it was a necessary ingredient in the work of Clebsch, Brill, Noether and Macaulay toward a version of the Riemann–Roch theorem couched in terms of the algebra of plane curves near the end of the nineteenth century. It was put on a firm modern footing in [Peskine and Szpiro 1974], and this foundation was used for further progress in projective geometry by Hartshorne, Rao and others. In this chapter we will explain some of these developments, starting with a simple example, and including the algebra necessary for a formulation in the natural generality of purely 1-dimensional schemes.

As we have seen, any plane curve is arithmetically Cohen–Macaulay, and its arithmetic genus is determined by its degree. Similarly, a curve in $\mathbb{P}^3$ that is a complete intersection of surfaces of degrees d, e is arithmetically Cohen–Macaulay (Theorem 0.1) and has arithmetic genus determined by d, e. Next simplest, perhaps, is a curve C that is *directly linked* to a complete intersection, which means roughly that its union $X = C \cup D$ with a complete intersection curve D is again a complete intersection (see Definition 16.3 for the general definition). We will see that, once again, such a curve C is arithmetically Cohen–Macaulay, and its genus is determined by the degrees of the equations of X and the degree and genus of D. Allowing sequences of direct links we define an equivalence relation called *linkage* or *liaison*, and curves in the linkage class of a complete intersection are said to be *licci*.

A famous theorem of Hartshorne and Rao [Prabhakar Rao 1978/79] shows that the linkage class of a curve $C \subset \mathbb{P}^3$ is classified by the finite-dimensional graded module

$$D(C) \coloneqq H^1_*(\mathcal{I}_C) \coloneqq \bigoplus_{m\in\mathbb{Z}} H^1(\mathcal{I}_C(m)),$$

called the *deficiency module* or *Hartshorne–Rao module* of C. (This is a module over the homogeneous coordinate ring $S = H^0_*(\mathcal{O}_{\mathbb{P}^3})$.) The correspondence is explicit: from a finite-dimensional graded module over S one can actually construct curves.

In the first sections of this chapter we will examine the equivalence relation on curves in $\mathbb{P}^3$ that is defined by linkage. Much of the story extends to the case of singular curves. This extension requires an understanding of the dualizing sheaves of singular curves, to which we turn in Section 16.5. We conclude the chapter with an analysis of the adjoint ideal, completing a result from Chapter 15, and allowingl us to formulate the Riemann–Roch theorem for general curves and coherent sheaves.

Aside from the classification result above, linkage is useful in analyzing Hilbert schemes. We will exploit this systematically in cases of low degree and genera in Chapter 19, and we begin this chapter with what is perhaps the simplest example, computing the dimension of the component of the Hilbert scheme $\mathrm{Hilb}_{3m+1}(\mathbb{P}^3)$ that is the closure of the open subset $\mathcal{H}^\circ$ parametrizing twisted cubics (see Proposition 7.11 for another proof).

16.2. Linkage of twisted cubics

The simplest example of linkage is that of the union of a twisted cubic and one of its secant lines, pictured in Figure 16.1, and we will start with that.

Any twisted cubic curve $C \subset \mathbb{P}^3$ lies on a nonsingular quadric in class $(1,2)$. Adding a line L of class $(1,0)$ we get a divisor of class $(2,2)$, the class of the complete intersection of two quadrics. Since L is also a complete intersection, C is licci.

We can make the relation of L and C explicit as follows: The ideal of C is minimally generated by the three 2×2 minors of the matrix

$$\begin{pmatrix} x_0 & x_1 & x_2 \\ x_1 & x_2 & x_3 \end{pmatrix}.$$

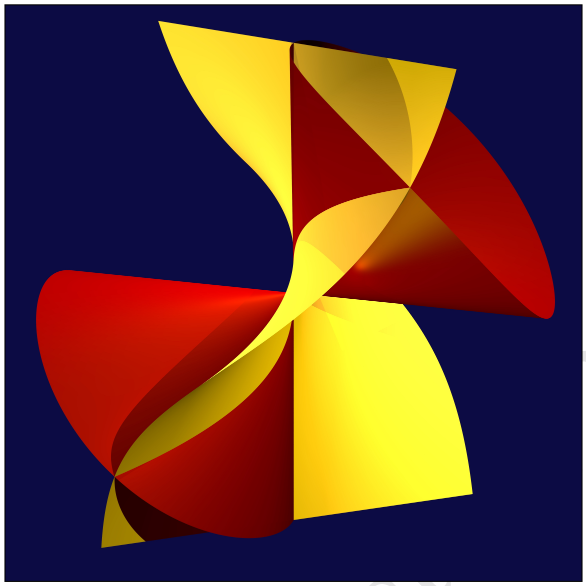

Figure 16.1. A quadratic cone (red) intersecting a smooth quadric (yellow) in the union of a vertical line and a twisted cubic (credit: Herwig Hauser).

The minor $Q_{1,2}$ involving the first two columns and the minor $Q_{2,3}$ involving the last two columns both vanish on the line $L : x_1 = x_2 = 0$, which meets the twisted cubic in the two points $x_0 = x_1 = x_2 = 0$ and $x_1 = x_2 = x_3 = 0$. Thus L is a secant line to C. A general linear combination Q of $Q_{1,2}$ and $Q_{2,3}$ defines a smooth quadric, which is thus isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. The curve C necessarily lies in the divisor class $(1, 2)$ (or, symmetrically, $(2, 1)$), and the line in class $(1, 0)$ (respectively, $(0, 1)$), summing to the complete intersection $(2, 2)$ of Q with (say) $Q_{1,2}$. See Figure 16.1.

Conversely, if two irreducible quadrics Q_1, Q_2 both contain a twisted cubic C then, by Bézout's theorem, $Q_1 \cap Q_2$ is the union of C with a line. If at least one of the quadrics is smooth, we are in the situation above.

This suggests that we set up an incidence correspondence between twisted cubics and their secant lines. Let $\mathbb{P}^9$ denote the projective space of quadrics in $\mathbb{P}^3$, and consider

$$\Phi = \{(C, L, Q, Q') \in \mathcal{H}^\circ \times \mathbb{G}(1,3) \times \mathbb{P}^9 \times \mathbb{P}^9 \mid Q \cap Q' = C \cup L\}.$$

We'll analyze Φ by considering the projection maps to $\mathcal{H}^\circ$ and $\mathbb{G}(1,3)$; that is, by looking at the diagram

$$\begin{tikzcd} & \Phi \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ \mathcal{H}^\circ & & \mathbb{G}(1,3) \end{tikzcd}$$

Consider the projection map $\pi_2 : \Phi \to \mathbb{G}(1,3)$ on the second factor. By what we just said, the fiber over any point $L \in \mathbb{G}(1,3)$ is an open subset of $\mathbb{P}^6 \times \mathbb{P}^6$, where $\mathbb{P}^6$ is the space of quadrics containing L. Since $\dim \mathbb{G}(1,3) = 4$ we see that Φ is irreducible of dimension $4 + 2 \times 6 = 16$. On the other hand, the map $\pi_1 : \Phi \to \mathcal{H}^\circ$ is surjective, with fiber over a curve C an open subset of $\mathbb{P}^2 \times \mathbb{P}^2$, where $\mathbb{P}^2$ is the projective space of quadrics containing C; we conclude that $\mathcal{H}^\circ$ is irreducible of dimension 12, in accord with our computation in Proposition 7.11 of the space of twisted cubics as $\mathrm{PGL}_4 / \mathrm{PGL}_2$.

16.3. Linkage of smooth curves in $\mathbb{P}^3$

If the union of two smooth curves in $\mathbb{P}^3$ is a complete intersection then the degrees and genera of the curves are related:

Theorem 16.1. *Let $C_1, C_2 \subset \mathbb{P}^3$ be distinct smooth irreducible curves whose union is the complete intersection of two surfaces S, T of degrees s, t, with S smooth. Then* $\deg C_1 + \deg C_2 = st$ *and*

$$g(C_1) - g(C_2) = \frac{s+t-4}{2}(\deg C_1 - \deg C_2).$$

In words, the difference between the genera of C_1 and C_2 is proportional to the difference in their degrees, with constant of proportionality $(s+t-4)/2$. In the example of a complete intersection of two quadrics described above, the multiplier $(s+t-4)/2$ is zero, and indeed the line and the twisted cubic have the same genus. The relation of degrees and genera is true more generally, as we shall see in the next section, but the special case is already useful.

Proof. The sum of degrees of C_1 and C_2 is the degree of $C_1 \cup C_2 = S \cap T$ which, by Bézout's theorem, is st.

By the adjunction formula in $\mathbb{P}^3$ the canonical divisor of S has class $K_S = (s-4)H$. Thus, from the adjunction formula on the surface S we get

$$g(C_i) = \frac{C_i^2 + C_i \cdot K_S}{2} + 1 = \frac{C_i^2 + (s-4)\deg C_i}{2} + 1.$$

Subtracting,

$$g(C_1) - g(C_2) = \frac{C_1^2 - C_2^2 + (s-4)(\deg C_1 - \deg C_2)}{2}.$$

Because $C_1 + C_2$ is in the class tH on S we have

$$C_1^2 - C_2^2 = (C_1 + C_2)(C_1 - C_2) = t(\deg C_1 - \deg C_2).$$

Combining the last two displays yields the second formula of the theorem. □

Remark 16.2. Linkage is closely related to linear equivalence. Here is a special case: suppose that S is a smooth surface in $\mathbb{P}^3$, and $C \subset S$ is a curve. If T is a sufficiently general surface of degree t containing C then the curve C' that is the link of C with respect to S, T lies in the class $tH - C$. If we link again with respect to another surface T' of degree t' we thus arrive at $C'' = C + (t - t')H$. Thus if $t = t'$ we get a curve in the same linear equivalence class as C. Moreover, since every rational function on S is the restriction to S of the ratio of two forms of the same degree on $\mathbb{P}^3$, the set of curves on S that can be obtained from C by two linkages with surfaces T, T' of the same degree is exactly the linear series $|C|$ on S. This idea is generalized in the notion of a *basic double link*; see Exercise 16.8.

Exercise 16.1. Let C_1, C_2 be irreducible curves on a sm

16.4. Linkage of purely 1-dimensional schemes in $\mathbb{P}^3$

To say that the unionX of distinct reduced irreducible curves $C \cup C'$ is a complete intersection means that the ideal I_X equals $I_C \cap I_{C'}$. Since the latter contains $I_C I_{C'}$, the ideal quotient $(I_X : I_C) := \{F \mid FI_C \subset I_X\}$ contains $I_{C'}$.

On the other hand, if $F \notin I_{C'}$ and we choose $G \in I_C \setminus I_{C'}$, then $FG \notin I_{C'}$, so $F \notin (I_X : I_C)$, and thus $(I_X : I_C) = I_{C'}$. This relationship underlies the formulas connecting the degrees and genera of C, C', which hold for arbitrary purely 1-dimensional subschemes of $\mathbb{P}^3$, as we shall see in Theorem 16.5.

Definition 16.3. Let C, C' be purely 1-dimensional subschemes of $\mathbb{P}^3$. We say that C' is *directly linked* to C if there is a complete intersection X containing C, C' and satisfying $(I_X : I_C) = I_{C'}$. We say that C' is *linked* to C if they are connected by a chain of such direct linkages, and we say that C' is *evenly linked* to C if the chain involves an even number of direct linkages.

Note that in this setting, C and C' can have components in common. For example the subscheme $C \subset \mathbb{P}^3$ defined by the square of the ideal $\mathcal{I}_{L/\mathbb{P}^3}$ of a line $L \subset \mathbb{P}^3$ is linked to the reduced line L in the complete intersection of two quadrics. This is an example of a rope as discussed in Exercise 16.12. (This makes sense, since this scheme is a flat limit of twisted cubics.)

As in the smooth case treated above, direct linkage is a symmetric relation:

Proposition 16.4. *Let $C_1 \subset \mathbb{P}^3$ be a purely 1-dimensional subscheme with saturated homogeneous ideal I_1 and suppose that C_1 is contained in a complete intersection of hypersurfaces $X := S \cap T$. The ideal $I_2 = (I_X : I_1)$ is a saturated ideal, defining a purely 1-dimensional subscheme and $I_1 = (I_X : I_2)$ as well.*

Proof. The ideal I_X is unmixed of codimension 2, since X is a complete intersection [Eisenbud 1995a, Proposition 18.13]. It follows that $I_2 = (I_X : I_1)$ is also unmixed of codimension 2, and therefore saturated. Thus it suffices to prove that $I_1 = (I_X : I_2)$ after localizing at a codimension 2 prime P that contains I_X.

Write R for the localization at P of the homogeneous coordinate ring of X. Because I_X is a complete intersection, the ring R is zero-dimensional and Gorenstein. By [Eisenbud 1995a, Propositions 21.1 and 21.5], every finitely generated R-module is reflexive. Since $I_{C_2}R = \operatorname{ann}_R(I_{C_1}R) \cong \operatorname{Hom}_R(R/I_{C_1}R, R)$, the proposition follows. $\square$

16.5. Degree and genus of linked curves

The degrees and arithmetic genera of directly linked schemes are related exactly as in the pilot case of Section 16.3.

Theorem 16.5. *If $C_1, C_2 \subset \mathbb{P}^3$ are purely 1-dimensional schemes that are directly linked by surfaces S, T of degrees s, t, then* $\deg C_1 + \deg C_2 = st$ *and*

$$p_a(C_1) - p_a(C_2) = \frac{s+t-4}{2}(\deg C_1 - \deg C_2).$$

Since we have left the realm of smooth curves and surfaces, we will need a more sophisticated duality theory, and we postpone the proof to explain the necessary ideas.

Dualizing sheaves for singular curves. In Chapter 2 we said that the canonical sheaf of a smooth curve — the sheaf of differential forms — was the most important invertible sheaf after the structure sheaf. In the general setting of Cohen–Macaulay schemes, the analogue of the canonical sheaf is known as the dualizing sheaf. The general definition of the dualizing sheaf of a pure dimensional projective scheme this notion is not very illuminating; what is useful is how it is constructed and its cohomological properties relating to duality. However, having a definition may be comforting.

Definition 16.6. Let X be a projective scheme of pure dimension d over $\mathbb{C}$. The *dualizing sheaf* for X is a coherent sheaf ω_X together with a *residue map* $\eta : H^d(\omega_X) \to \mathbb{C}$ such that for every coherent sheaf $\mathcal{F}$ the composite map

$$H^d(\mathcal{F}) \times \operatorname{Hom}(\mathcal{F}, \omega_X) \to H^d(\omega_X) \xrightarrow{\eta} \mathbb{C}$$

is a perfect pairing.

If $\mathcal{F} = \mathcal{L}$ is an invertible sheaf on a projective curve C then $\operatorname{Hom}(\mathcal{L}, \omega_X) = \mathcal{L}^{-1} \otimes \omega$, so we recover Serre duality: $H^1(\mathcal{L})$ is the dual of $H^0(\mathcal{L}^{-1} \otimes \omega_C)$.

It follows from the definition that the pair (ω_X, η) is unique up to canonical isomorphism if it exists: The module $H^0_*(\omega_X) = \bigoplus_{n\in\mathbb{Z}}(\operatorname{Hom}_X(\mathcal{O}_X(n), \omega_X))$ is determined as the graded vector space dual of $H^d_*(\mathcal{O}_X)$, and the choice of η simply fixes the isomorphism. It may not be apparent that such a sheaf exists, but we will give a construction in Section 16.6.

Several properties of the dualizing sheaf on a purely 1-dimensional scheme are the same as in the smooth case, and follow easily from the definition:

Proposition 16.7. *Let C be a purely 1-dimensional projective scheme.*

(1) *$\mathcal{H}om_C(\omega_C, \omega_C) = \mathcal{O}_C$; thus if C is integral then the generic rank of ω_C is 1.*

(2) *For any invertible sheaf $\mathcal{L}$ on C we have*

$$H^1(\mathcal{L}^{-1}) = H^0(\mathcal{L} \otimes \omega_C) \quad \text{and} \quad H^0(\mathcal{L}^{-1}) = H^1(\mathcal{L} \otimes \omega_C).$$

In particular, $\chi(\mathcal{L}^{-1}) = -\chi(\mathcal{L} \otimes \omega_C)$.

Proof. (1) We will show that the natural map $\mathcal{O}_C \to \mathcal{H}om_C(\omega_C, \omega_C)$ is an isomorphism. Since the map is globally defined, it suffices to prove that it is an isomorphism locally.

Choose a Noether normalization of C, that is, a finite map $f : C \to \mathbb{P}^1$. We shall see in Theorem 16.8 below that $\omega_C \cong \mathcal{H}om_{\mathbb{P}^1}(\mathcal{O}_C, \omega_{\mathbb{P}^1})$, regarded as a sheaf on C (in Theorem 16.8 this is the sheaf $f^! \omega_{\mathbb{P}^1}$). Since C is purely 1-dimensional, $\mathcal{O}_C$ is torsion-free as an $\mathcal{O}_{\mathbb{P}^1}$-module, and is thus locally free. Also, since $\mathbb{P}^1$ is smooth, $\omega_{\mathbb{P}^1}$ is locally isomorphic to $\mathcal{O}_{\mathbb{P}^1}$. But if B is any commutative ring and A is a B-algebra that is finitely generated and free as a B-module, then the composite map

$$A \to \mathrm{Hom}_A(\mathrm{Hom}_B(A, B), \mathrm{Hom}_B(A, B)) \cong \mathrm{Hom}_B(\mathrm{Hom}_B(A, B), B)$$

sending an element a to the multiplication by a and thence to the map $f \mapsto f(a)$ may be identified with the isomorphism of A to its double dual as a B-module this completes the argument. (See Exercise 16.9 for a generalization of this last step.)

(2) The definition of ω_C shows that, if $\mathcal{L}$ is an invertible sheaf, then

$$H^1(\mathcal{L}^{-1}) = \mathrm{Hom}_C(\mathcal{L}^{-1}, \omega_C)) = H^0(\mathcal{H}om_C(\mathcal{L}^{-1}, \omega_C)) = H^0(\mathcal{L} \otimes_C \omega_C).$$

Using part (1), we have

$$\begin{aligned} H^0(\mathcal{L}^{-1}) &= H^0(\mathcal{L}^{-1} \otimes_C \mathcal{O}_C) \\ &= H^0\big(\mathcal{L}^{-1} \otimes_C \mathcal{H}om_C(\omega_C, \omega_C)\big) \\ &= \mathrm{Hom}_C(\mathcal{L} \otimes_C \omega_C, \omega_C)) \\ &= H^1(\mathcal{L} \otimes_C \omega_C) \end{aligned}$$

□

16.6. The construction of dualizing sheaves

Dualizing sheaves do exist on any purely 1-dimensional projective scheme, and more generally on any projective Cohen–Macaulay scheme. We have already seen constructions in three cases:

- If X is a smooth scheme of dimension d over $\mathbb{C}$ then $\omega_X = \bigwedge^d \Omega_{X/\mathbb{C}}$ is a dualizing sheaf [Hartshorne 1977, Section III.7; [1978, p. 648, 708]].
- If $f : X \to Y$ is a map of smooth curves, then $\omega_X = f^*(\omega_Y)(\mathrm{ram}_{X/Y})$, where ram denotes the ramification divisor.

- If $X \subset Y$ is a Cartier divisor on a surface, then $\omega_X = \omega_Y(X)|_X$.

How can such different looking formulas all be correct? Grothendieck provided a general scheme that unifies them and gives many more. To understand what is needed for the general case, we first consider a setting generalizing Hurwitz's theorem. Suppose that $X \to Y$ is a finite map of projective schemes, and that $\mathcal{F}$ is a coherent sheaf on X.

If we restrict ourselves to open affine subsets $U := \operatorname{Spec} A \subset X$ mapping to $V := \operatorname{Spec} B \subset Y$ via the map of rings $f^* : B \to A$, then $F := \mathcal{F}_U$ is an A-module. Moreover, $f_* F := f_*(\mathcal{F})(V)$ is just F regarded as a B-module via the map f^*.

For any B-module M the module $\operatorname{Hom}_B(A, M)$ has a natural A-module structure, where $(a\phi)(m)$ is defined to be $\phi(am)$. The functor

$$f^!(-) := \operatorname{Hom}_B(A, -) : \operatorname{mod}_B \to \operatorname{mod}_A$$

defined in this way is the right adjoint of the functor $f_*(-) : \operatorname{mod}_A \to \operatorname{mod}_B$, which means that there is a natural isomorphism of functors

$$\operatorname{Hom}_B(f_* F, -) \cong \operatorname{Hom}_A(F, \operatorname{Hom}_B(A, -)) = \operatorname{Hom}_A(F, f^!(-)).$$

Thus if Y has dualizing sheaf ω_Y and we set $o_Y := \omega_Y(V)$, then

$$\operatorname{Hom}_B(f_* F, o_Y) \cong \operatorname{Hom}_A(F, f^! o_Y).$$

Also, there is a natural transformation η from $f_* f^!$ to the identity functor given by the formula

$$\eta : f_* f^!(M) = f_*(\operatorname{Hom}_B, (A, M)) = \operatorname{Hom}_B, (A, M) \to M, \quad \eta(\phi) = \phi(1),$$

for any A-module M. The transformation η called the *counit* of the adjoint pair $(f_*, f^!)$.

We also write $f^!(-)$ for the sheafification of the functor $\operatorname{Hom}_B(A, -)$. Again $f^!$ is right adjoint to f_* on coherent sheaves, and again there are natural maps $\eta : f_* f^!(\mathcal{F}) \to \mathcal{F}$.

Theorem 16.8. *Let $f : X \to Y$ be a finite map of d-dimensional projective schemes. If Y has a dualizing sheaf ω_Y, with residue map $\eta_Y : H^d(\omega_Y) \to \mathbb{C}$, then*

$$\omega_X := f^! \omega_Y,$$

with residue map

$$\rho_X : H^d(f^! \omega_X) = H^d(f_* f^! \omega_X) \xrightarrow{H^d(\eta)} H^d(\omega_Y) \xrightarrow{\rho_Y} \mathbb{C},$$

where η is the counit of the adjoint pair $(f_*, f^!)$, is a dualizing sheaf on X.

Proof. Let $\mathcal{F}$ be a coherent sheaf on X. Since f is finite, $H^d(\mathcal{F}) = H^d(f_*(\mathcal{F}))$. dropped "we have"
Thus, since $f^!(-)$ is a right adjoint of f_* there are natural isomorphisms

$$H^d(\mathcal{F}) = H^d(f_* \mathcal{F}) \cong \operatorname{Hom}_Y(f_* \mathcal{F}, \omega_Y)^\vee \cong \operatorname{Hom}_X(\mathcal{F}, f^! \omega_Y)^\vee,$$

the first being induced by ρ_Y. One can check that the composite isomorphism is the one induced by ρ_X, so $\omega_X = f^!(\omega_Y)$, completing the proof. □

Cheerful Fact 16.9. Given this theorem, it seems natural to look for an adjoint functor $f^!$ for a wider class of morphisms f, but... in most cases, for example when f is the inclusion of a divisor on a smooth surface, no such functor exists on the category of coherent sheaves! An adjoint functor $f^!$ does exist on the derived category, where it is the right adjoint to Rf_*, leading to a theory of dualizing complexes.

Fortunately for the reader who is mostly interested in curves, this level of complication is unnecessary, and there is an intermediate level of generality that suffices for all the purposes of this book and more:

Theorem 16.10. *Suppose that $f : X \to Y$ is a finite map of projective schemes. If Y is Gorenstein with dualizing module ω_Y, then*

$$f^!(\omega_Y) \cong \mathcal{E}xt_Y^{\dim Y - \dim X}(\mathcal{O}_X, \omega_Y)$$

is a dualizing module for X.

The hypothesis is satisfied for any Y that is smooth, or even locally a complete intersection. The reason this works is that the complex $f^!\omega_Y$ can be identified with its one nonvanishing cohomology module, $\mathcal{E}xt_Y^{\dim Y - \dim X}(\mathcal{O}_X, \omega_Y)$. See for example [Altman and Kleiman 1970] for a thorough and accessible exposition and the Wikipedia page [?].

Proof of Theorem 16.5. Let X be the complete intersection of surfaces of degrees s, t containing C, and let $R_X = S/(F, G)$ be its homogeneous coordinate ring, where $S = \mathbb{C}[x_0, \dots, x_3]$ is the homogeneous coordinate ring of $\mathbb{P}^3$. From the free resolution

$$0 \to S(-s-t) \xrightarrow{\begin{pmatrix} G \\ -F \end{pmatrix}} S(-s) \oplus S(-t) \xrightarrow{(F\ G)} S \to R_X \to 0$$

and Theorem 16.10 we see that

$$\omega_X = \mathcal{E}xt_C^2(\mathcal{O}_X, \omega_{\mathbb{P}^3}) = \mathcal{E}xt^2(\mathcal{O}_X, \mathcal{O}_{\mathbb{P}^3}(-4)) = \mathcal{O}_X(s+t-4).$$

Note that for any ideals $J \subset I$ in a ring A we have $\mathrm{Hom}_A(A/I, A/J) \cong (J : I)/J$, where the isomorphism sends a homomorphism ϕ to the element $\phi(1)$. From Theorem 16.10 we have

$$\omega_C = \mathcal{H}om_X(\mathcal{O}_C, \omega_X) = \mathcal{H}om_X(\mathcal{O}_C, \mathcal{O}_X)(s+t-4) = \frac{\mathcal{I}_X : \mathcal{I}_C}{\mathcal{I}_X}(s+t-4),$$

where we have identified $\mathcal{O}_C$ with its pushforward under the inclusion map $C \to X$.

By Proposition 16.7 we have $\chi(\omega_C(m)) = -\chi(\mathcal{O}_C(-m))$. It follows that the leading coefficient of the Hilbert polynomial of ω_C is equal to $\deg C$, and thus

$$st = \deg \mathcal{O}_X = \deg \mathcal{O}_{C'} + \deg \mathcal{O}_C,$$

as required by the formula for the sum of the degrees.

From Theorem 16.8 (or Theorem 16.10) we see that $\chi(\mathcal{O}_X) = st(4-s-t)/2$. Since $\mathcal{O}_{C'} = \mathcal{O}_{\mathbb{P}^3}/(\mathcal{I}_X : \mathcal{I}_C)$ and $(\mathcal{I}_X : \mathcal{I}_C)/(\mathcal{I}_X) = \omega_C(4-s-t)$ we have

$$\begin{aligned}
\frac{4-s-t}{2}(\deg C + \deg C') &= \frac{4-s-t}{2} st \\
&= \chi(\mathcal{O}_X) \\
&= \chi(\mathcal{O}_{C'}) + \chi(\omega_C(4-s-t)) \\
&= \chi(\mathcal{O}_{C'}) - \chi(\mathcal{O}_C(s+t-4)) \\
&= \chi(\mathcal{O}_{C'}) - (s+t-4)\deg C - \chi(\mathcal{O}_C) \\
&= (1-p_a(\mathcal{O}_{C'})) - (1-p_a(\mathcal{O}_C)) - (s+t-4)\deg C,
\end{aligned}$$

whence

$$p_a(\mathcal{O}_C) - p_a(\mathcal{O}_{C'}) = \frac{s+t-4}{2}(\deg C - \deg C'). \qquad \square$$

Linkage also behaves in a simple way with respect to deficiency modules:

Theorem 16.11. *If C, C' are purely 1-dimensional subschemes of $\mathbb{P}^3$ that are directly linked by a complete intersection of degrees s, t then*

$$D(C') = \operatorname{Hom}_{\mathbb{C}}(D(C), \mathbb{C})(-s-t+4)$$

as graded modules over the homogeneous coordinate ring of $\mathbb{P}^3$.

A more general form of this result appears as Proposition 2.5 in [Peskine and Szpiro 1974], with an attribution to Daniel Ferrand.

Proof. Suppose that the homogeneous ideal of C is generated by forms of degree a_i, $i = 1, \dots, s$. Since C is locally Cohen–Macaulay, the local rings $\mathcal{O}_{C,p}$ have projective dimension 2 as modules over $\mathcal{O}_{\mathbb{P}^3,p}$, and $\mathcal{I}_{C,p}$ has projective dimension 1. Thus we have an exact sequence

$$0 \to \mathcal{E} \to \bigoplus_i \mathcal{O}_{\mathbb{P}^3}(-a_i) \to \mathcal{I}_C \to 0.$$

Since the first and second cohomology groups of the twists of $\mathcal{O}_{\mathbb{P}^3}$ vanish, we deduce an isomorphism

$$D(C) := \bigoplus_{m\in\mathbb{Z}} H^1(\mathcal{I}_C(m)) \cong \bigoplus_{m\in\mathbb{Z}} H^2(\mathcal{E}(m)).$$

Let X be the complete intersection of two hypersurfaces, of degrees s, t, containing C. From the inclusion we deduce a map of resolutions

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{E} \arrow[r] & \bigoplus_i \mathcal{O}_{\mathbb{P}^3}(-a_i) \arrow[r] & \mathcal{O}_{\mathbb{P}^3} \arrow[r] & \mathcal{O}_C \arrow[r] & 0 \\
0 \arrow[r] \arrow[u] & \mathcal{O}_{\mathbb{P}^3}(-s-t) \arrow[r] \arrow[u] & \mathcal{O}_{\mathbb{P}^3}(-s) \oplus \mathcal{O}_{\mathbb{P}^3}(-t) \arrow[r] & \mathcal{O}_{\mathbb{P}^3} \arrow[r] \arrow[u, "="'] & \mathcal{O}_X \arrow[r] \arrow[u] & 0
\end{tikzcd}$$

We dualize this diagram, form the mapping cone, and twist by $-s-t$. Note that $\mathrm{Hom}_{\mathbb{P}^3}(\mathcal{O}_C, \mathcal{O}_{\mathbb{P}^3}) = 0$. Also, since the vertical map $\mathcal{O}_{\mathbb{P}^3} \to \mathcal{O}_{\mathbb{P}^3}$ on the right is the identity we may cancel these terms in the mapping cone. Noting that $\omega_C = \mathcal{E}xt^2(\mathcal{O}_C, \mathcal{O}_{\mathbb{P}^3}(-4))$ the result is a diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[d] & \omega_C(-s-t+4) \arrow[l] \arrow[d] & \mathcal{E}^*(-s-t) \arrow[l] \arrow[d] & \bigoplus_i \mathcal{O}_{\mathbb{P}^3}(a_i-s-t) \arrow[l] \arrow[d] & 0 \arrow[l] \\
0 & \mathcal{O}_X \arrow[l] \arrow[d] & \mathcal{O}_{\mathbb{P}^3} \arrow[l] & \mathcal{O}_{\mathbb{P}^3}(-t) \oplus \mathcal{O}_{\mathbb{P}^3}(-s) \arrow[l] & \\
 & \mathcal{O}_{C'} \arrow[d] & & & \\
 & 0 & & &
\end{tikzcd}$$

The map ϕ is a monomorphism because $(\mathcal{I}_X : \mathcal{I}_C)/\mathcal{I}_X \cong \omega_C(-s-t+4)$, as explained above, so the column on the left is a short exact sequence. We can now write a resolution of $\mathcal{I}_{C'}$ as the mapping cone:

$$0 \leftarrow \mathcal{I}_{C'} \leftarrow \mathcal{O}_{\mathbb{P}^3}(-t) \oplus \mathcal{O}_{\mathbb{P}^3}(-s) \oplus \mathcal{E}^*(-s-t) \leftarrow \bigoplus_i \mathcal{O}_{\mathbb{P}^3}(a_i - s - t) \leftarrow 0.$$

From this we see that

$$H^1(\mathcal{I}_{C'}(m)) \cong H^1(\mathcal{E}^*(-s-t+m)) \cong \mathrm{Hom}_{\mathbb{C}}(H^2(\mathcal{E}(s+t-m-4)), \mathbb{C}),$$

where the last equality is from Serre duality on $\mathbb{P}^3$. Summing over m we see that $D(C') \cong \mathrm{Hom}_{\mathbb{C}}(D(C)(s+t-4), \mathbb{C})$, and since Serre duality is functorial, the isomorphism holds not only as graded vector spaces, but as graded S-modules. □

Sometimes the following consequence is a useful way to compute the deficiency module:

Proposition 16.12. *If C is a purely 1-dimensional subscheme of $\mathbb{P}^3$ with homogeneous ideal $I = I_C$ then*

$$D(C) \cong \mathrm{Hom}_{\mathbb{C}}(\mathrm{Ext}^3(S/I, S), \mathbb{C})(-4), \mathbb{C})$$

as graded modules over the homogeneous coordinate ring S of $\mathbb{P}^3$.

Proof. We may choose a surjection $\psi : \oplus_i S(-a_i) \to I$, and choose the map $\phi : \oplus_i \mathcal{O}_{\mathbb{P}^3}(-a_i) \to \mathcal{I}_C$ in the proof of Theorem 16.11 to be the corresponding map of sheaves, so that $\mathcal{E}$ is the sheafification of the graded module $E = \ker \psi$.

Since I is a saturated ideal, the depth of S/I is at least 1, so pd $S/I \leq 3$, and I has a free resolution of the form

$$0 \to G \to F \to \oplus_i S(-a_i) \to S \to S/I \to 0.$$

where $G \to F$ is a free presentation of E. and there is an exact sequence

$$0 \to E^* \to F^* \to G^* \to \mathrm{Ext}^3_S(S/I, S) \to 0.$$

By Theorem 18.7, the fact that C is Cohen–Macaulay implies that the projective dimension of each of its local rings is ≤ 2, and it follows that $\mathrm{Ext}^3_S(S/I, S)$ has finite length. Writing $\widetilde{(\ )}$ for the sheafification functor, we have a short exact sequence of sheaves

$$0 \to \mathcal{E}^* \to \widetilde{F^*} \to \widetilde{G^*} \to 0.$$

From this we see that

$$\mathrm{Ext}^3_S(S/I, S) = H^1_*(\mathcal{E}^*) = \mathrm{Hom}_{\mathbb{C}}(H^2_*(\mathcal{E}(-4)), \mathbb{C}) = H^1_*(\mathcal{I})(-4),$$

proving the assertion. □

Proposition 16.12 is actually a special case of the local duality isomorphism between local cohomology and the dual of Ext; see for example [Eisenbud 2005, Theorem A.1.9].

16.7. The linkage equivalence relation

As an immediate consequence of Theorem 16.11 we have:

Corollary 16.13 (Hartshorne). *If two curves C, C' are linked by an even length chain of direct linkages, then $D(C)$ and $D(C')$ are isomorphic up to a shift in grading.* □

As we mentioned at the beginning of this chapter, the converse is also true: the Hartshorne–Rao modules, up to shift in grading, provide a complete invariant of linkage.

Cheerful Fact 16.14. Even more precise results are known (and the characteristic 0 hypothesis is largely unnecessary); here is a sample:

Theorem 16.15. *Let $S = \mathbb{C}[x_0, \dots, x_3]$ be the homogeneous coordinate ring of $\mathbb{P}^3$, and let M be a graded S-module of finite length.*

(1) *There is a smooth curve C with $D(C) = M(m)$ for some integer m.*
(2) *There is a minimum value of m such that $M(m) = D(C_0)$ for some purely one-dimensional scheme C_0.*

Moreover, each linkage class has a relatively simple structure, known as the *Lazarsfeld-Rao property*. We say that C' is obtained from C by an *ascending double link* if $I_{C'} = fI_C + (g)$ for some regular sequence contained in I_C — see Exercise 16.8.

Theorem 16.16 [Ballico et al. 1991]. *Let $M = D(C_0)$ the Hartshorne–Rao module of a purely 1-dimensional subscheme of $\mathbb{P}^3$, and suppose that M is minimal in the sense that no $M(m)$ with $m > 0$ is the invariant of a purely 1-dimensional scheme.*

(1) *Any curve in $\mathbb{P}^3$ with $D(C) = M$ is a deformation of C_0 through curves with invariant M.*

(2) *Every curve in the even linkage class of C_0 is the result of a series of ascending double links followed by a deformation.*

In [Lazarsfeld and Rao 1983] it is shown that general curves in $\mathbb{P}^3$ that have reasonably large degree compared to their genus are minimal in the sense of Theorem 16.16.

16.8. Comparing the canonical sheaf with that of the normalization

In Chapter 15 we boasted in that we could effectively compute linear series on a smooth curve C given any plane curve C_0 with normalization C, and we showed how to do this when the plane curve has only nodes. To complete the discussion we need to compare the canonical sheaf ω_C of C with the dualizing sheaf ω_{C_0} of C_0; that is, we need a formula for the adjoint ideal of any curve singularity. A simplification occurs when the dualizing sheaf is invertible, that is, when the curve is Gorenstein (as is every plane curve).

Theorem 16.17. *If $\nu : C \to C_0$ is the normalization of a reduced connected projective curve then the adjoint ideal*

$$\mathfrak{A}_{C/C_0} \coloneqq \operatorname{ann}_{\mathcal{O}_{C_0}} \frac{\omega_{C_0}}{\nu_*\omega_C}$$

is equal to the conductor ideal

$$\mathfrak{f}_{C/C_0} := \operatorname{ann}_{\mathcal{O}_{C_0}} \frac{\nu_*\mathcal{O}_C}{\mathcal{O}_{C_0}}.$$

Moreover, if C_0 is Gorenstein, then

$$\delta(C_0) = \operatorname{length} \frac{\nu_*\mathcal{O}_C}{\mathcal{O}_{C_0}} = \operatorname{length} \frac{\mathcal{O}_{C_0}}{\mathfrak{f}_{C/C_0}}.$$

As explained in Chapter 2, we can think of $\delta(C_0)$ as the number of nodes equivalent to the singularities of C_0. The formula for $\delta(C_0)$ in the theorem was first noted in Daniel Gorenstein's thesis [1] under Oscar Zariski. Hyman Bass [1963] explains that this is why Grothendieck named Cohen–Macaulay rings that have cyclic canonical modules after Gorenstein.

In Examples 15.19–15.21 we used the second equality in the formula for $\delta(C_0)$ in Theorem 16.17 to compute the δ invariant for several plane curve singularities. It fails for many space curve singularities; see Example 16.19 for a singularity that is not Gorenstein and behaves differently.

Proof of Theorem 16.17. Let $\rho : C_0 \to \mathbb{P}^1$ be a finite morphism. Both $\rho_* \nu_* \mathcal{O}_C$ and $\rho_* \mathcal{O}_{C_0}$ are torsion free coherent sheaves over $\mathcal{O}_{\mathbb{P}^1}$, and are thus locally free. Since ρ_* is left-exact, the inclusion $\mathcal{O}_{C_0} \subset \nu_* \mathcal{O}_C$ pushes forward to an inclusion

$$\alpha : \rho_* \mathcal{O}_{C_0} \hookrightarrow \rho_* \nu_* \mathcal{O}_C$$

and since $\mathcal{O}_C$ is equal to $\mathcal{O}_{C_0}$ generically on $\mathbb{P}^1$, the cokernel $\operatorname{coker} \alpha$ has finite length; indeed, it is supported on the image in $\mathbb{P}^1$ of the singular locus of C_0. Since the maps ν and ρ are finite, we may harmlessly think of both $\mathcal{O}_{C_0}$ and $\mathcal{O}_C$ as coherent sheaves on $\mathbb{P}^1$, and we will simplify the notation by dropping ν_* and ρ_*. Taking duals into $\omega_{\mathbb{P}^1} = \mathcal{O}_{\mathbb{P}^1}(-2)$ and defining $\alpha^\vee := \operatorname{Hom}_{\mathbb{P}^1}(\alpha, \mathcal{O}_{\mathbb{P}^1})$ we get a map that fits into the long exact sequence of $\mathcal{E}xt_{\mathbb{P}^1}(-, \omega_{\mathbb{P}^1})$:

$$\begin{aligned} 0 \to \operatorname{Hom}_{\mathbb{P}^1}(\operatorname{coker} \alpha, \omega_{\mathbb{P}^1}) \to \omega_C \xrightarrow{\alpha^\vee} \omega_{C_0} & \\ \to \mathcal{E}xt^1_{\mathbb{P}^1}(\operatorname{coker} \alpha, \omega_{\mathbb{P}^1}) &\to \mathcal{E}xt^1_{\mathbb{P}^1}(\mathcal{O}_C, \omega_{\mathbb{P}^1}) \to \cdots \end{aligned}$$

We know that $\operatorname{coker} \alpha$ has finite support, so $\operatorname{Hom}_{\mathbb{P}^1}(\operatorname{coker} \alpha, \omega_{\mathbb{P}^1})$ is trivial and $\mathcal{E}xt^1_{\mathbb{P}^1}(\operatorname{coker} \alpha, \omega_{\mathbb{P}^1})$ has the same length and the same annihilator as $\operatorname{coker} \alpha$. Because $\mathcal{O}_C$ is locally free as an $\mathcal{O}_{\mathbb{P}^1}$-module, the term $\mathcal{E}xt^1_{\mathbb{P}^1}(\mathcal{O}_C, \omega_{\mathbb{P}^1})$ vanishes, and we get the more manageable exact sequence

$$0 \to \omega_C \xrightarrow{\alpha^\vee} \omega_{C_0} \to \mathcal{E}xt^1_{\mathbb{P}^1}(\operatorname{coker} \alpha, \mathcal{O}_{\mathbb{P}^1}) \to 0.$$

It follows that the sheaves $\nu_* \mathcal{O}_C / \mathcal{O}_{C_0}$ and $\omega_{C_0} / \nu_* \omega_C$ have the same length $\delta(C_0)$. Note that the conductor $\mathfrak{f}_{C/C_0}$ (the annihilator of $\mathcal{O}_C / \mathcal{O}_{C_0}$ in $\mathcal{O}_{C_0}$) is at the same time an ideal sheaf of $\mathcal{O}_{C_0}$ and an ideal sheaf of $\mathcal{O}_C$ via the inclusion $\mathcal{O}_{C_0} \subset \mathcal{O}_C$. The argument above shows that $\mathfrak{f}_{C/C_0}$ is also the annihilator ideal of ω_{C_0}/ω_C. By definition, this is the adjoint ideal of C_0, proving the first statement of the theorem.

A further simplification occurs when $C_0 \subset \mathbb{P}^2$ is a plane curve, or more generally any Gorenstein curve. If the defining equation of C_0 is the form F of degree d, then there is a locally free resolution of $\mathcal{O}_{C_0}$ of the form

$$0 \to \mathcal{O}_{\mathbb{P}^2}(-d) \xrightarrow{F} \mathcal{O}_{\mathbb{P}^2} \to \mathcal{O}_{C_0} \to 0.$$

[1] Gorenstein is better remembered for his work on the classification of finite simple groups.

Thus $\omega_{C_0} \cong \mathcal{E}xt^1_{\mathbb{P}^2}(\mathcal{O}_{C_0}, \omega_{\mathbb{P}^2})$ is the cokernel of the map $\mathcal{O}_{\mathbb{P}^2}(-3) \to \mathcal{O}_{\mathbb{P}^2}(d-3)$ given by multiplication by F. It follows that ω_{C_0} is locally cyclic, and since ω_{C_0}/ω_C has finite support, ω_{C_0}/ω_C is globally cyclic, so $\omega_{C_0}/\omega_C \cong \mathcal{O}_{C_0}/\mathfrak{f}_{C/C_0}$. Since the length of ω_{C_0}/ω_C is equal to the length of $\mathcal{O}_C/\mathcal{O}_{C_0}$, the last statement of the theorem follows. □

Example 16.18. Working locally, consider the germ of a node, represented by the ring $R_0 := k[\![x, y]\!]/(xy)$, and the projection to a line represented by the inclusion

$$P := k[\![t]\!] \subset R_0 \, : \, t \mapsto x + y.$$

The normalization of R_0 is the map $R_0 \to R := k[\![x]\!]e_1 \times k[\![y]\!]e_2$, where $e_1 = x/t$ and $e_2 = y/t$ are orthogonal idempotents. Writing

$$Q_1 = k(\!(t)\!) \subset Q := k(\!(x)\!) \times k(\!(y)\!)$$

for the map of total quotient rings, we know that, because the extension is separable, the trace map $\mathrm{Tr} := \mathrm{Tr}_{Q/Q_1} : Q \to Q_1$ generates $\mathrm{Hom}_{Q_1}(Q, Q_1)$ as a Q-vector space. Thus we may write the elements of $\omega_{R_0} = \mathrm{Hom}_P(R_0, P)$ and $\omega_R = \mathrm{Hom}_P(R, P)$ as multiples of Tr by elements of Q.

Since $R \cong Pe_1 \oplus Pe_2$ as a P-module, the module $\mathrm{Hom}_P(R, P)$ is generated by the two projections, and it is easy to check that these are the maps $(x/t)\,\mathrm{Tr}$ and $(y/t)\,\mathrm{Tr}$. One can also check easily that

$$g := \frac{x-y}{t^2}\,\mathrm{Tr} \in \mathrm{Hom}_P(R_0, R).$$

Since $xg = x/t$ and $yg = y/t$ in Q we have $xg = x/t\,\mathrm{Tr}$ and $yg = y/t\,\mathrm{Tr}$, the generators of $\mathrm{Hom}_P(R, P)$. The ring R_0, regarded as a P-module, is freely generated by 1 and x. Immediate computation shows that $g\,\mathrm{Tr}(1) = 0$ while $g\,\mathrm{Tr}(x) = 1$. Furthermore $(x-y)g = 1$ in Q, and thus $\frac{1}{2}(x-y)g\,\mathrm{Tr}$ takes 1 to 1 and x to t, proving that $g\,\mathrm{Tr}$ generates $\mathrm{Hom}_P(R_0, P)$. We also see directly from this that the adjoint ideal of ω_{R_0}/ω_R is the conductor ideal $(x, y)R_0 = (x, y)R = \mathfrak{f}_{R/R_0}$, as shown by Theorem 16.17.

Example 16.19. In Example ?? we showed that if C has a spatial triple point at 0, so that the completion of its local ring has the form $R_0 := k[\![x, y, z]\!]/(xy, xz, yz)$, with normalization R, then it has δ invariant 2 but the conductor is $\mathfrak{f}_{R/R_0} = (x, y, z)R = (x, y, z)R_0$, and thus

$$\delta(C_0) \neq \mathrm{length}\, \frac{\mathcal{O}_{C_0}}{\mathfrak{f}_{C/C_0}}.$$

The S-module R_0, on the other hand, has free resolution

$$0 \longrightarrow S^2 \xrightarrow{\begin{pmatrix} 0 & z \\ y & -y \\ -x & 0 \end{pmatrix}} S^3 \xrightarrow{(xy\ xz\ yz)} S \longrightarrow R_0 \longrightarrow 0.$$

The canonical module of R_0 (which would be the germ of the canonical module of a global curve with such a singularity) is thus

$$\omega_{R_0} = \mathrm{Ext}^2(R_0, S) = \mathrm{coker}\begin{pmatrix} 0 & y & -x \\ z & -y & 0 \end{pmatrix},$$

a module requiring 2 generators. This shows that R_0 is not Gorenstein.

16.9. A general Riemann–Roch theorem

Using dualizing sheaves we can state a more version of the Riemann–Roch theorem, applicable to any reduced, irreducible projective curve and any coherent sheaf thereupon. We will not make use of this generality later, so we only sketch the argument. We first need to extend the notion of degree of a sheaf:

Definition 16.20. If $\mathcal{F}$ is a sheaf of generic rank r on a projective reduced and irreducible curve C we define the *degree* of $\mathcal{F}$ as $\deg \mathcal{F} := \chi(\mathcal{F}) - r\chi(\mathcal{O}_C)$. Keeping in mind the definition of the arithmetic genus, we can write

$$(*) \qquad \chi(\mathcal{F}) = \deg \mathcal{F} + r\chi(\mathcal{O}_C) = \deg \mathcal{F} + r(1 - p_a(C)).$$

This formula would be pointless if there were no other way to compute $\deg \mathcal{F}$, but that is not the case:

Cheerful Fact 16.21. The degree of $\mathcal{F}$ coincides with that of a certain divisor class, the *first Chern class* of $\mathcal{F}$. See [Eisenbud and Harris 2016a, Chapter 5] for more information.

More to the point, if $\mathcal{F}$ is a coherent sheaf on the reduced, irreducible projective curve C of generic rank r, and if $\mathcal{L}$ is an invertible sheaf on C, then the following assertions can be proved using nothing more than the additivity of the Euler characteristic:

(1) If $\mathcal{F}$ is generated by its global sections and $\sigma_1, \dots, \sigma_r$ is a maximal generically independent collection of global sections of $\mathcal{F}$, then

$$M = \mathrm{coker}\,(\mathcal{O}_C^r \xrightarrow{(\sigma_1,\dots,\sigma_r)} \mathcal{F})$$

has finite support, and

$$\deg \mathcal{F} = r\chi(\mathcal{O}_C) + \dim_{\mathbb{C}} H^0(M).$$

(2) $\deg(\mathcal{L} \otimes \mathcal{F}) = \deg \mathcal{F} + r \deg \mathcal{L}$.

Thus if $\mathcal{O}_C(1)$ is a very ample invertible sheaf on C and m is an integer that is large enough so that $\mathcal{F}(m)$ is generated by global sections, then the degree of $\mathcal{F}(m)$ and the degree of $\mathcal{O}_C(1)$ are computed by the formula in item (1) and $\deg \mathcal{F} = \deg \mathcal{F}(m) - m\,\mathrm{rank}(\mathcal{F}) \deg \mathcal{O}_C(1)$.

Using these facts and the dualizing property of ω_C we can reexpress the equality ($*$) in the definition of $\deg \mathcal{F}$ as follows:

Theorem 16.22. *If C is a reduced and irreducible curve and $\mathcal{F}$ is a coherent sheaf on C, then*

$$h^0(\mathcal{F}) = \deg \mathcal{F} + \operatorname{rank}(\mathcal{F})(1 - p_a(C)) + h^0(\mathrm{Hom}_C(\mathcal{F}, \omega_C)).$$

16.10. Exercises

Exercise 16.2. Verify that the genus formula in Theorem 16.5 agrees with the usual calculation of degrees and genera for divisors on a quadric of classes (a, b) and $(d-a, d-b)$.

Exercise 16.3. Let $C_1, C_2 \subset \mathbb{P}^3$ be distinct smooth irreducible curves whose union is the complete intersection of two surfaces S, T of degrees s, t, with S smooth. Compute the intersection number $(C_1 \cdot C_2)$ in terms of the degrees and genera of C_1 and C_2.

Exercise 16.4. Let C be a reduced and irreducible projective curve, and let $\mathcal{E}$ be a locally free sheaf of rank r on C. Show that $\deg \mathcal{E} = \deg \bigwedge^r(\mathcal{E})$.

Hint: First show that any locally free sheaf on C is an iterated extension of invertible sheaves.

Exercise 16.5. Let C be the disjoint union of 3 skew lines (see Figure 16.2).

(1) Prove that C lies on a unique quadric, and that $H^2(\mathcal{I}_C) = 0$.

(2) Compute the Hartshorne–Rao module $D(C)$.

(3) Show that if Γ is the union of 3 points in $\mathbb{P}^3$ then $H^1\mathcal{I}(\Gamma) = 0$ if and only if the three points are collinear.

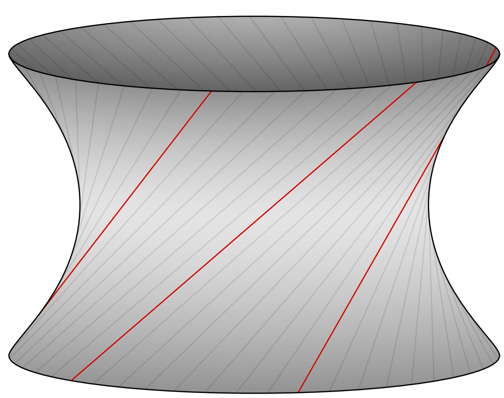

Figure 16.2. Any three skew lines in space lie on a unique (and necessarily smooth) quadric surface, and all belong to the same ruling.

(4) Using the exact sequence in cohomology coming from the short exact sequence

$$0 \to \mathcal{I}_C \xrightarrow{\ell} \mathcal{I}_C(1) \to \mathcal{I}_\Gamma(1) \to 0,$$

where ℓ is a linear form, show that the map of vector spaces

$$H^1(\mathcal{I}_C) \xrightarrow{\ell} H^1(\mathcal{I}_C(1))$$

has rank < 2 if and only if ℓ vanishes on 3 collinear points on the three lines (including the case when ℓ vanishes identically on one of the lines). Conclude that if a different union C' of 3 skew lines is linked to C, then C' lies on the same quadric as C.

See [Migliore 1986] for more examples of this type.

Exercise 16.6. Compute the Hilbert function of the Hartshorne–Rao module of a curve of type (a, b) on a smooth quadric surface.

Hint: The ideal sheaf of the curve on the quadric Q is an extension of the ideal sheaf of the quadric in $\mathbb{P}^3$ with the ideal sheaf of the curve on the quadric, which is

$$\mathcal{O}_Q(-a, -b) = \pi_1^*(\mathcal{O}_{\mathbb{P}^1}(-a)) \otimes \pi_2^*(\mathcal{O}_{\mathbb{P}^1}(-b)),$$

where π_1, π_2 are the projections to $\mathbb{P}^1$. Use the Künneth formula

$$H^1(\mathcal{O}_Q(p, q)) = H^1(\mathcal{O}_{\mathbb{P}^1}(p)) \otimes H^0(\mathcal{O}_{\mathbb{P}^1}(q)) \oplus H^0(\mathcal{O}_{\mathbb{P}^1}(p)) \otimes H^1(\mathcal{O}_{\mathbb{P}^1}(q))$$

to compute the necessary cohomology.

Exercise 16.7 (linkage addition [Schwartau 1982]). Suppose that I, J are saturated ideals defining purely 1-dimensional subschemes of $\mathbb{P}^3$ and that f, g is a regular sequence with $f \in I$ and $g \in J$. Prove that $gI \cap fJ = (fg)$, and conclude that if I, J are saturated codimension 2 ideals defining purely 1-dimensional schemes C, C' in $\mathbb{P}^3$ then $(gI + fJ)$ is a saturated ideal defining a purely 1-dimensional scheme C'' with $D(C'') = D(C)(-\deg g) \oplus D(C')(-\deg f)$.

Hint: Use the exact sequence

$$0 \to (fg) \to gI \oplus fJ \to gI + fJ \to 0$$

and the corresponding exact sequence of quotients by these ideals.

Exercise 16.8 (basic double links). The special case of the construction in Exercise 16.7 in which C' is trivial is already interesting.

(1) Show that if I is a saturated ideal of codimension 2 defining a purely 1-dimensional scheme C in $\mathbb{P}^3$ and (f, g) is a regular sequence with $g \in I$, then $fI + (g)$ defines a scheme C' with $D(C') = D(C)(-\deg f)$.

(2) Show directly that, with notation as above, C' is directly linked to C in two steps. Since the degrees of the generators of $D(C')$ are more positive, this is sometimes called an *ascending double link*. Geometrically it amounts to

taking the union of C with some components that are complete intersections.

Exercise 16.9. Here is a more general form of the last step in the proof of Proposition 16.7(1). Suppose that $B \to A$ is a homomorphism of rings, X is an A-module and Y is a B-module. Show that there is a natural transformation

$$\phi : \mathrm{Hom}_A(X, \mathrm{Hom}_B(A, Y)) \cong \mathrm{Hom}_B(X, Y)$$

and that if $X = \mathrm{Hom}_B(A, Y)$, then the map

$$A \to \mathrm{Hom}_A(\mathrm{Hom}_B(A, Y), \mathrm{Hom}_B(A, Y))$$

taking an element $a \in A$ to multiplication by a on the A-module $\mathrm{Hom}_B(A, Y)$ is sent by ϕ to the evaluation map $\alpha \mapsto \alpha(a)$ for $\alpha \in \mathrm{Hom}_B(A, Y)$.

Ropes and ribbons. The simplest way to construct well-behaved nonreduced curves is to take neighborhoods of smooth ones. Ropes and ribbons are examples of this sort:

Definition 16.23. The *rope defined from a curve* $C \subset \mathbb{P}^n$ is the scheme $V(I_C^2)$ defined by the square of the ideal C.

Exercise 16.10. If C is the rope defined from a line $L \subset \mathbb{P}^3$ then the Hilbert function $h_C(m)$ and Hilbert polynomial $p_C(m)$ are both equal to $3m+1$. Thus C has degree 3 and arithmetic genus 0. Note that the degree can also be computed as the degree of a general hyperplane section, since this is defined by the square of the ideal of a point in $\mathbb{P}^2$.

Hint: Count the monomials of each degree in square of the ideal of a line.

Exercise 16.11. To see why the rope in Exercise 16.10 should look like a twisted cubic, show that it is the flat limit of a twisted cubic as follows: Let $X \subset \mathbb{P}^3$ be the twisted cubic with parametrization $x_i = s^i t^{3-i}$. Consider the one-parameter subgroup of PGL_4 given in homogeneous coordinates $x_0, \dots, x_3$ on $\mathbb{P}^3$ by

$$A_t : (x_0, \dots, x_3) \mapsto (tX_0, X_1, X_2, tX_3).$$

Show that the flat limit, as $t \to 0$, of the twisted cubics $A_t(C)$ is the rope $V(X_0^2, X_0X_1, X_1^2)$.

Hint: Use the description of $I(X)$ as the ideal of 2×2 minors of

$$\begin{pmatrix} x_0 & x_1 & x_2 \\ x_1 & x_2 & x_3 \end{pmatrix}.$$

Exercise 16.12. We saw in Section 16.1 that a twisted cubic curve is linked to a line by the complete intersection of two quadrics. Show that the same is true for the rope of Exercise 16.10.

If C is the rope defined from a line in $\mathbb{P}^2$, then the Zariski tangent space to C at any point is 2-dimensional; that is, it looks like a ribbon. More generally:

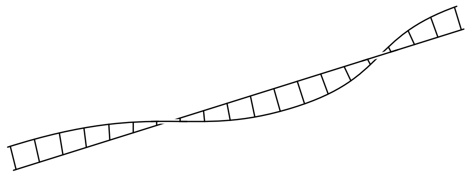

Figure 16.3. A ribbon supported on a line.

Definition 16.24. By a *ribbon* $X \subset \mathbb{P}^n$ we mean a scheme of pure dimension 1 and multiplicity 2 whose support is a smooth, irreducible curve $C \subset \mathbb{P}^n$ and whose Zariski tangent space at every point is 2-dimensional (Figure 16.3).

Exercise 16.13. Suppose that $C \subset \mathbb{P}^n$ is a ribbon. Show that C is contained in the rope defined from C_{red}, and show that the degree of C is twice that of C_{red}.

Hint: Look at hyperplane sections of C.

Unlike ropes, there are many different ribbons C with the same smooth curve C_{red}, and they can have different arithmetic genera. Suppose that $C \subset \mathbb{P}^3$ is a ribbon such that $X = C_{red}$ is the line $V(x_0, x_1)$. Since C is contained in the rope defined from X we must have $(x_0^2, x_0x_1, x_1^2) \subset I(C)$. The tangent space to C at a point $(0, 0, s, t)$ meets the line $X' = V(x_2, x_3)$ at some point $(F(s,t), G(s,t), 0, 0)$, so F and G define a morphism $\mathbb{P}^1 \to \mathbb{P}^1$; thus they are homogeneous polynomials of the same degree d. It follows that $I(C)$ also contains the element $x_0G(x_2, x_3) - x_1F(x_2, x_3)$. Show that the ideal of C is obtained by adding this form to the ideal of the rope, that is,

$$I_C = \left(X_0^2, X_0X_1, X_1^2, F(X_3, X_4)X_0 + G(X_3, X_4)X_1\right).$$

In case $d = 1$, show that C lies on a smooth quadric.

General adjunction. The next two exercises illustrate Theorem 16.10:

Exercise 16.14. Show that if $C \to D$ is a map of smooth curves with ramification index e at $p \in C$, and t is a local analytic parameter at p, then locally analytically at p the sheaf $\mathcal{H}om_C(\mathcal{O}_C, \omega_D)$ is $\mathcal{O}_C(e)$.

Exercise 16.15. Show that if $C \subset S$ is a Cartier divisor on a surface S with canonical sheaf ω_S, then $\mathcal{E}xt^1(\mathcal{O}_C, \omega_S) \cong \mathcal{O}_C \otimes \mathcal{O}_S(C)$, and thus

$$K_C = (K_S + C) \cap C.$$