



*Dedicated to...*



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## Contents



# Introduction

## 1. Why you want to read this book

Algebraic geometry is an old subject. Descartes' introduction in 1637 of co-ordinates in the plane and in space made it possible to relate the algebra of polynomials to the geometry of their zero loci. Gauss' proof of the Fundamental Theorem of Algebra showed that the degree of a polynomial, an algebraic invariant, was equal to the number of roots counted with multiplicity, a geometric invariant. Mathematicians have been doing what is recognizably algebraic geometry for more than two centuries.

Within algebraic geometry, the study of algebraic curves is the oldest topic. Newton already classified all the possible types of real affine cubics. By the middle of the 19th century a rich theory of curves in the complex projective plane was a central topic, overturned by Riemann's work in mid-century—what became the theory of Riemann surfaces, which introduced new techniques of complex analysis into the field. This was taken up and made algebraic as a theory of plane curves by Alexander Brill, Max Noether, F. S. Macaulay and many others. By the end of the century Halphen and others were interested in the classification of space curves as well. The interested reader will find more about the early work on algebraic curves in the Appendix to this book, written by historian Jeremy Gray.

The development continues unabated today: in the second half of the 20th century Grothendieck's foundations led to the solution of many classical problems and, in particular, to a firm foundation for the theory of moduli, allowing mathematicians to exploit the fact that algebraic varieties generally come in families parameterized by other algebraic varieties—something that we will

return to often in this book. Algebraic geometry has merged more and more with number theory, and the moduli space of curves plays an important role in string theory, responding to the needs of physicists.

For these reasons, the subject of algebraic curves is one of the richest in algebraic geometry, if not in all of mathematics. If you want to know whether a conjecture is plausible, you can generally find well-understood special cases on which to test it. Some of the fundamental constructions of algebraic geometry, like the construction of moduli spaces and their description, can be carried out in the setting of algebraic curves with a degree of precision and detail far beyond what has been possible in higher dimensions.

Already the geometry of plane curves of degrees 3 and 4 shows some of the promise of the subject:

Every smooth plane cubic has exactly 9 flexes—points where the tangent line has contact of order 3 with the curve—and these points form a remarkable configuration; they’re the only known example of a finite, non-colinear set of points in the plane such that the line joining any two contains a third. Generalizing the notion of the flexes of a plane cubic leads us to the subject of inflectionary behavior of linear series on curves in general, which arose separately in the theory of Weierstrass points on Riemann surfaces, and has become a powerful tool.

Every smooth quartic curve has exactly 28 bitangents, also forming a beautiful and mysterious configuration; see Figure ???. In [Salmon 1960], first published in 1837, Salmon computed the number (315) of 4-tuples of bitangents whose eight points of tangency lie on a conic. The extension of these ideas to curves of higher genus leads us to the rich theory of theta-characteristics, bringing together algebra (in the form of a bilinear pairing on the points of order 2 in the Jacobian of a curve), analysis (in the form of theta-functions) and of course projective geometry.

The richest subject in what is arguably the richest branch of mathematics<sup>1</sup>—of course you want to read this book!

## 2. Why we wrote this book

The wealth of beauty, both in theory and in examples, certainly makes the study of algebraic curves an attractive prospect. But it comes at a price: to absorb in detail all the things we’ve learned over the centuries about algebraic curves would take years, if not decades. This is, in essence, the conundrum facing anyone who undertakes to write a book on the subject: how to convey the wealth of information (and the many many ways in which our knowledge is incomplete)

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<sup>1</sup>Number theorists may quibble

without writing an encyclopedia. We have chosen to try to be useful and broad but not necessarily complete.

When we introduce a technique or a construction without full proofs, we do so as a “cheerful fact”, in a different font, and we try to make clear which category is which.<sup>2</sup>”

*I'm very well acquainted, too, with matters mathematical,  
I understand equations, both the simple and quadratical,  
About binomial theorem I am teeming with a lot o' news,  
With many cheerful facts about the square of the hypotenuse.*

—Gilbert and Sullivan, Pirates of Penzance, Major General’s Song

Our intended audience is a graduate student considering working in the field of algebraic curves, or a researcher in a related field whose work has led them to questions about algebraic curves. Our goal is to equip the reader with the understanding of both the techniques and the state of our knowledge necessary to read the current literature and work on open problems.

### 3. What's with Practice?

Be simple by being concrete. Listeners are prepared to accept unstated (but hinted) generalizations much more than they are able, on the spur of the moment, to decode a precisely stated abstraction and to re-invent the special cases that motivated it in the first place.

—Paul Halmos, How to Talk Mathematics

This book aims to present those ideas and methods from the theory of algebraic curves that used *in practice* by mathematicians working in a variety of fields of mathematical research.

Although mathematicians aspire to understand their subjects deeply, we feel that we learn in stages: in early stages we accept large and difficult results as black boxes and explore the rich examples that they yield. That is how we have tried to organize this book: We begin with two chapters that we hope will bridge the gap between first courses in algebraic geometry/commutative algebra at the level of Fulton’s or Reid’s well-known books [Fulton 1989, 2008], [Reid 1988] and the professional language of invertible sheaves, cohomology and linear series. An ideal background would be Hartshorne’s book [Hartshorne 1977] or Vakil’s notes [Vaki 2023] but very much less will suffice if the reader is willing to accept some advanced ideas or look them up at leisure—we have

<sup>2</sup>Hats off to the Berkeley “Many cheerful facts” seminar run by the graduate students, which gave us this idea!

tried to give precise references where this might be required. Subsequent chapters roughly alternate between expositions of basic techniques (partly without proofs) and families of examples, treated in detail.

#### 4. What's in this book

In organizing this book we faced a common problem of mathematical exposition:

We are dealing here with a fundamental and almost paradoxical difficulty. Stated briefly, it is that learning is sequential but knowledge is not. A branch of mathematics... consists of an intricate network of interrelated facts, each of which contributes to the understanding of those around it. When confronted with this network for the first time, we are forced to follow a particular path, which involves a somewhat arbitrary ordering of the facts.

—Robert Osserman [[Osserman 2011](#)]

In Chapters 1 and 2 we lay out the central objects of algebraic curve theory: invertible sheaves, linear systems, canonical sheaves and the Riemann-Roch theorem, as well as Hurwitz' theorem, the adjunction formula and some elementary facts about the geometry of surfaces. We prove or sketch the proofs of many of these basic results. These chapters may serve as review for someone who has been exposed already to material roughly equivalent to Chapter IV of [[Hartshorne 1977](#)].

Thereafter, we alternate between chapters focussed on special cases and chapters developing more of the theory. Chapter 3 describes the geometry of curves of genus 0 and 1 in projective space. We emphasize some of the things that make rational normal curves so special, and take the opportunity to introduce the conditions implying that a curve is *arithmetically Cohen-Macaulay*. We describe the low-degree linear series on curves of genus 1, and demonstrate how counts of parameters suggest the presence of a moduli space.

In genus  $\geq 2$  there is a fundamental shift: not all invertible sheaves of a given degree on a curve  $C$  of genus  $g \geq 2$  are congruent modulo the automorphism group of the curve. It is a salient feature of algebraic geometry that families of similar algebro-geometric objects are often naturally parameterized by algebraic varieties. This idea goes back to the beginning of algebraic geometry and the family of plane curves of degree  $d$ , but was only fully clarified in the work of Grothendieck, Deligne and Mumford,

Chapter 5 introduces our first examples of this phenomenon: the *fine moduli spaces* of divisors on a curve, as well as the Jacobian, for which we give the classic analytic construction, and the Picard varieties  $\text{Pic}_d(C)$  parametrizing

isomorphism classes of invertible sheaves. We also introduce the subvarieties  $W_d^r(C)$  parametrizing invertible sheaves with many sections.

Since the Jacobian is irreducible, we can speak meaningfully of a *general* invertible sheaf, and of the dimension of various families of special invertible sheaves. With this in hand we prove that every curve of genus  $g > 1$  can be embedded in  $\mathbb{P}^3$  as a curve of degree  $g + 3$ .

Equipped with information about the Jacobian, we proceed in Chapter 6 to study curves of genus 2 and 3, describing in particular how the geometry of a map of the curve to projective space, given by sections of an invertible sheaf, depends on the sheaf. We show how a hyperelliptic curve of genus  $g$  can be described by a set of  $2g + 2$  points in  $\mathbb{P}^1$ , and we describe the canonical maps. Again, this suggests the existence of moduli spaces.

We spend Chapters 7 and 8 on other moduli spaces as they appear in the world of curves, though we do not give complete proofs of all the assertions made. We start in Chapter 7 with some central examples of fine moduli spaces, primarily the Hilbert scheme. Using properties of the Hilbert scheme we show that curves of genus  $\geq 2$  can have only finitely many automorphisms, and more generally that there can only be finitely many morphisms from one such curve to another.

The most interesting example of a moduli space, the moduli of smooth (or stable) curves of genus  $g$ , is not a fine moduli space, and we spend Chapter 8 describing what it is and isn't. In that chapter we also describe the Hurwitz space of coverings and the Severi variety of nodal plane curves.

After this, it's back to examples: in Chapter 9 we analyze aspects of the geometry of curves of genus 4 and 5.

In the following two chapters we take up the properties of the points of a general hyperplane sections of a curve in projective space. In Chapter 10 we give Rathmann's proof that the points of a general hyperplane section of a reduced irreducible curve are in linearly general position (independent of the characteristic). Consequences include Castelnuovo's bound on the maximal genus of curves of a given degree. We show that the curves that achieve the maximum genus, called Castelnuovo curves, are arithmetically Cohen-Macaulay. These include all plane curves, as well as canonical curves and linearly normal curves of high degree compared to their genera.

We also use the linear general position result to prove the strong forms of Clifford's and Martens' theorems on special linear series; to show that every curve has a nodal plane model; and to show that a general invertible sheaf of

degree  $g+2$  on a curve of genus  $g$  maps the curve to a nodal curve in the plane—unless the curve is hyperelliptic, in which case the image has a single multiple point of multiplicity  $g$ .

In characteristic 0 an even stronger statement than linearly general position is true: the monodromy of the hyperplane divisors is the full symmetric group. We take up this and some other monodromy questions in Chapter 11 and prove consequences for secant planes and for sums of linear series.

In Chapter 12 we return to the general question: “What linear series exist on curves of genus  $g$ ?” We give three possible interpretations of this question, and the answers to each: Clifford’s theorem, the Castelnuovo bound, and the Brill-Noether theorem, postponing the proof of the latter. We apply the Brill-Noether result to explore various special classes of curves of genus 6.

Chapter 13 prepares for the proof of the Brill-Noether theorem with a discussion of inflection points of linear series, generalizing the flexes of plane curves. The Plücker formula counts the number of inflection points, with appropriate weights. In this chapter we explain the remarkable connection between the study of inflections on a rational curve and the Schubert calculus of cycles in the Grassmannian. In Chapter 14 we use this material, together with a degeneration to cuspidal rational curves, to give a relatively short proof of the Brill-Noether theorem.

In Chapter 15 we take the classical point of view of Brill and Noether, and describe an explicit algorithm for finding the complete linear series on a smooth curve, using a singular plane model. This involves the result classically known as the completeness of the adjoint linear series. For the general case we use a simple case of the theory of dualizing sheaves, introduced in more generality in the next chapter.

Returning to curves in  $\mathbb{P}^3$  in Chapter 16 we explain the theory of Hartshorne and Rao classifying curves up to linkage. This theory applies to all purely 1-dimensional subschemes of  $\mathbb{P}^3$ , and to explain this we detour to discuss dualizing sheaves and Grothendieck’s  $f^!$  operation.

Curves that lie on a quadric in  $\mathbb{P}^3$  are easy to understand. In Chapter 17 we systematically describe a natural generalization: rational normal scrolls and the curves that lie on them. This includes all the Castelnuovo curves of high degree.

Chapter 18 presents some aspects of the theory of syzygies of the homogeneous ideals of curves and the famous—and, as of this writing, open—conjecture of Mark Green that connects syzygies of canonical curves to the Clifford index and the theory of rational normal scrolls. A table reproduced from

work of Frank-Olaf Schreyer shows the sensitivity of the numerical information in the free resolution of a canonical curve to questions of the existence of special linear series on the curve.

Chapter 19 is in some ways the culmination of the book. It is concerned with *Hilbert schemes*, schemes  $H_{g,3,d}$  that parametrize smooth curves of genus  $g$  and degree  $d$  in  $\mathbb{P}^3$ . We work out examples up to degree 7, define the “principal component”—the only one dominating the moduli space of curves—and derive dimension estimates from deformation theory and from the Brill-Noether theorems using most of the ideas we have introduced.

The historical appendix written by Jeremy Gray, Chapter 20, is a survey of some of the work on algebraic curves up to that of Brill and Noether, and we are grateful to him for allowing us to include it.

**4.1. Exercises and hints.** There are many exercises, gathered at the end of the chapters. There is a separate document, available at [\\*\\*\\*.ams.org](http://www.ams.org), that collects all the exercises and includes hints to many of them.

**4.2. Relation of this book to other texts.** Chapter IV of [Hartshorne 1977] has a similar flavor to that of this book, and contains details of most of the results in our Chapters 1 and 2. A beautiful (and brief) account of a number of topics in a style we particularly admire is found in [Mumford 1975].

A far more extensive treatment, partially overlapping that of the present book and containing many other topics, can be found in the important and encyclopaedic [Arbarello et al. 1985] and [Arbarello et al. 2011], and yet even these works do not cover all the major topics in the field. One of the topics we do not cover is the construction of the moduli space of curves, which can be found in [Arbarello et al. 2011]. Though not a complete account, [Harris and Morrison 1998a] deals directly with this topic.

There are more elementary accounts of some of our material, in [Fulton 1989, 2008] and [Walker 1950] (who goes farther than we do into local resolution of singularities) as well as [Griffiths 1989]. The book [Kunz 2005] treats plane curves and their normalizations via the theory of valuations, and contains a detailed account of the ideas we treat in Chapter 15 from this more algebraic point of view. A comprehensive treatment of the local topological theory of plane curves and their singularities in [Brieskorn and Knörrer 1986]. The topological questions there are developed in different directions in [Milnor 1968] and [Eisenbud and Neumann 1985]. An idiosyncratic collection of interesting topics is presented in [Clemens 2003].

The Riemann surface point of view is well represented in the books [Forster 1991], [Gunning 1966], [Hulek 1995], [Kirwan 1992], and [Miranda 1995].

## 5. Prerequisites, notation and conventions

The reader should be familiar with the (Krull) dimension of rings and varieties, and their primary decomposition at the level of [Atiyah and Macdonald 1969]. Ideally the reader will already have some familiarity with the geometry of curves and surfaces at the level of [Hartshorne 1977, Chapter IV] and the beginning of Chapter V, there, though our summary of the necessary material in the first two chapters of this book may suffice for the intrepid.

Unless otherwise mentioned, we assume that the ground field is the field of complex numbers  $\mathbb{C}$ , though much of what we do could be done over any algebraically closed field.

**5.1. Commutative Algebra.** All the rings we consider are commutative with unit and Noetherian.

Since we are working over a field of characteristic 0, we use the terms smooth and nonsingular interchangeably when referring to a point on a scheme ■

Some results that we use:

**Theorem 5.1** (Lasker's Theorem). *If  $f_1, \dots, f_c \subset \mathbb{C}[x_0, \dots, x_n]$  generates an ideal of codimension  $c$ , then the ideal  $(f_1, \dots, f_c)$  is unmixed (all its primary components have codimension  $c$ ).*

**Theorem 5.2.** *If  $R$  is a domain that is a finitely generated algebra over a field or a localization of such an algebra, then the normalization (= integral closure) of  $R$  is a finitely generated  $R$ -module. If  $R$  is 1-dimensional, then its normalization is nonsingular.*

**5.2. Projective geometry.** Schemes are assumed quasi-projective, and varieties (including curves) are reduced and irreducible schemes unless otherwise stated. “Points” will always be closed points unless we explicitly say otherwise. **Note to the patient reader:** We usually use the term curve to refer to a smooth irreducible projective purely 1-dimensional scheme, but the usage in this DRAFT is not completely consistent.

Though we occasionally use the classical topology, the term “open set” refers to the Zariski topology unless otherwise stated.

We adopt the Grothendieck convention that points of projective space  $\mathbb{P}(V)$  are 1-dimensional quotients of  $V$ , or hyperplanes in  $V$ , so that  $\text{Sym}(V)$  is the homogeneous coordinate ring of  $\mathbb{P}(V)$  and lines in  $V$  correspond to points of  $\mathbb{P}(V^*)$ .

However we write  $G(k, V)$  for the variety of  $k$ -dimensional linear subspaces of  $V$ , so that in particular  $G(1, V) = \mathbb{P}(V^*)$ . We also write  $\mathbb{G}(k, r)$  for  $k$ -dimensional projective subspaces of  $\mathbb{P}^r$ . ■

The results we assume are well-represented by the following classical theorems:

**Theorem 5.3** (Bézout's Theorem). *If  $X, Y \subset \mathbb{P}^r$  are subvarieties, and if  $\text{codim}(X \cap Y) = \text{codim } X + \text{codim } Y$ , then  $\deg(X \cap Y) = \deg(X) \deg(Y)$ .*

**Theorem 5.4** (Bertini's Theorem). *If  $X \subset \mathbb{P}^r$  is a nonsingular quasi-projective variety, and  $\{H_\lambda \mid \lambda \in \Lambda\}$  is a linear family of hyperplanes of  $\mathbb{P}^r$ , then for an open subset of  $\lambda \in \Lambda$  the scheme  $H_\lambda \cap X$  is nonsingular away from the union of the base locus  $\cap_{\lambda \in \Lambda} H_\lambda$  and the singular locus of  $X$ .*

**Theorem 5.5** (Main theorem of elimination theory). *Any morphism  $\phi : X \rightarrow Y$  of projective varieties (or schemes) is closed: if  $X' \subset X$  is a Zariski closed subset, then  $\phi(X') \subset Y$  is also closed.*

**Corollary 5.6.** *If  $\phi : C \rightarrow D$  is a non-constant morphism of (projective) curves, then  $\phi$  is finite and surjective.*

If  $D$  is a smooth curve then the local ring of  $D$  at any point is a discrete valuation ring, so any torsion free module is flat. Thus:

**Proposition 5.7.** *If  $\phi : C \rightarrow D$  is a non-constant morphism of smooth curves, then  $\phi$  is finite, surjective, and flat.*

If  $X \subset \mathbb{P}^r$  is any scheme, we define the *homogeneous coordinate ring* of  $X \subset \mathbb{P}^r$  to be  $R_X = R_{X/\mathbb{P}^r} := S/I(X)$ , where  $S = \mathbb{C}[x_0, \dots, x_r]$  is the homogeneous coordinate ring of  $\mathbb{P}^r$ . We emphasize that, unlike the coordinate ring of an affine variety, this is not an intrinsic invariant of  $X$ , but depends on the embedding in  $\mathbb{P}^r$ .

When  $X$  is reduced and irreducible we write  $\kappa(X)$  for the field of rational functions on  $X$ . In particular, if  $p$  is a closed point we write  $\kappa(p) \cong \mathbb{C}$  for the residue class field at  $p$ .

**5.3. Sheaves and cohomology.** Some familiarity with coherent sheaves is recommended; a possible source is the first chapter of our book [[Eisenbud and Harris 2000](#)]. As for cohomology, it is probably enough if the reader can write down  $H^i$  and exact sequences without blushing. In any case we review some of the theory of coherent sheaves and their cohomology theory, and that of divisors on projective varieties, in the first two Chapters.

We occasionally use the bijection between algebraic and analytic sheaves on smooth projective curves, which preserves cohomology and exact sequences. This is a special case of the results in [[Serre 1955/56](#)].

If  $\mathcal{F}$  is a sheaf on  $X$  and  $X \subset Y$  then we will identify  $\mathcal{F}$  with the sheaf usually written  $\iota_* \mathcal{F}$ , where  $\iota : X \rightarrow Y$  is the inclusion map, and thus regard  $\mathcal{F}$

as a sheaf on  $Y$  as well. The cohomology  $H^i(X, \mathcal{F}) = H^i(Y, \iota_* \mathcal{F})$  canonically, so we will simply and unambiguously write  $H^i(\mathcal{F})$  for either of these.

We write  $h^i(\mathcal{F})$  or (if  $D$  is a divisor)  $h^i(D)$  for  $\dim_{\mathbb{C}} H^i(\mathcal{F})$  or  $\dim_{\mathbb{C}} H^i(\mathcal{O}_X(D))$  ■

If  $\mathcal{F}$  is a sheaf on projective space (perhaps supported on a subvariety) we write  $H_*^i(\mathcal{F})$  for  $\bigoplus_{m \in \mathbb{Z}} H^i(\mathcal{F}(m))$ .

# Linear series and morphisms to projective space

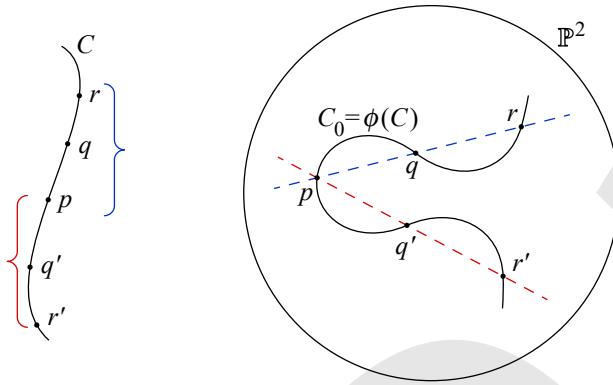
In the first two sections of this chapter we lay out four major definitions: divisors, linear series, invertible sheaves, and maps to projective space; and we explain the close relationships among them. We will use these ideas throughout this book. In the last section we will explore some special cases, culminating with the condition for an invertible sheaf to provide an embedding of a curve in projective space.

We will prove only some of our assertions; a reader who wants to see all the proofs should keep a copy of [Hartshorne 1977] or the equivalent handy. On the other hand, a more experienced reader could skip ahead to Chapter 3.

As an analytic space, a complex projective smooth curve is a compact Riemann surface. This is a compact, orientable 2-manifold with a complex structure but it's hard to "see" the global complex structure. Any two Riemann surfaces with the same genus<sup>1</sup> are  $C^\infty$  isomorphic, but except for genus 0, where the sphere has a unique complex structure, there are continuous families of non-isomorphic global structures. The differences among Riemann surfaces or algebraic curves is revealed in geometry of their maps to projective spaces.

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<sup>1</sup>The notion of genus was introduced by Riemann in his great paper [Riemann 1857], though he never used the term. Instead, he defined the *connectivity* (*Zusammenhang*) of a Riemann surface  $X$ —the number of disjoint closed curves  $C_i$  (1-dimensional real submanifolds) that can be drawn on  $X$  such that  $X \setminus UC_i$  is connected. The term genus (*Geschlect*) was first used by Clebsch. Both uses were subtly different than the modern one: where we speak of genus as a property of a given surface, Riemann and Clebsch used this idea as a way of partitioning the family of all Riemann surfaces into groups [Lê 2020].



**Figure 1.** Divisors of the linear system corresponding to the map  $\phi : C \rightarrow \mathbb{P}^2$  are preimages of the hyperplanes—in this case the dashed lines—in the plane. The image of  $p$  is determined by the intersection of the two lines. Silvio: please show the two divisors, perhaps by color or bracket :  $p,q,r$  is one group and  $p, q', r'$  the other. The dashed lines could be the corresponding colors too.

Throughout this book we will study curves in this way. The algebraic and complex analytic theories are equivalent, and in this book we generally take the algebraic path.

Since the only globally defined regular functions (that is, functions locally equal to a polynomial in affine coordinates) on a projective variety are the constant functions, we must take a different approach to describing the maps we will study, and a large part of this chapter is devoted to the necessary machinery for doing this. The idea is simple: a point in projective space is the intersection of the hyperplanes containing it, so a map  $\phi : C \rightarrow \mathbb{P}^r$  from a curve to projective space can be described set-theoretically by the set of preimages of these hyperplanes: a point  $p \in C$  is sent to the point that is the intersection of those hyperplanes whose preimages contain  $p$ .

Some hyperplanes in  $\mathbb{P}^r$  will be tangent to the image  $\phi(C)$  at a point  $\phi(p)$ , and in that case the point  $p$  should be counted with higher multiplicity in the preimage of that hyperplane; in this way we arrive at a notion of divisor as a sum of points with multiplicities.

## 1. Divisors

We begin again, more formally:

On a smooth projective curve  $C$ , we define a *divisor* to be a finite formal sum of points of  $C$  with integer coefficients, which we can write compactly as  $\sum_{p \in C} m_p \cdot p$  with all but finitely many  $m_p = 0$ . The coefficient  $m_p$  is called the *multiplicity* of the point  $p$  in the divisor  $D$ ; if all coefficients  $m_p$  are nonnegative

we say that  $D$  is *effective*. Thus the group of divisors  $\text{Div}(C)$  is the free abelian group whose generators are the points of  $C$ .

The *degree* of  $D = \sum_{p \in C} m_p \cdot p$  is by definition the sum  $\sum m_p$  of its coefficients. Because the curve is smooth each local ring is a discrete valuation ring, so two elements of  $\mathcal{O}_{C,p}$  with the same order of vanishing at  $p$  generate the same ideal in the local ring. Thus an effective divisor can be identified with a finite subscheme of  $C$ , and vice versa.

## 2. Divisors and rational functions

Let  $C$  be a smooth projective curve,  $U \subset C$  an open subset and  $f \in \mathcal{O}_C(U)$  a regular function, not identically zero. At every point  $p \in U$ , we define the *order* of  $f$  at  $p$ , denoted  $\text{ord}_p(f)$ , to be the highest power of the maximal ideal  $m_p \subset \mathcal{O}_p$  containing  $f$ , and we define the divisor  $(f)$  associated to the function  $f$  by

$$(f) := \sum_{p \in U} \text{ord}_p(f) \cdot p.$$

More generally, if  $h$  is any rational function on  $C$ , given locally as a quotient  $f/g$  of regular functions  $f$  and  $g$ , we define the divisor of  $h$  to be

$$(h) := (f) - (g);$$

this is independent of the choice of  $f$  and  $g$ . The divisor of a rational function  $h$  is called a *principal divisor*; principal divisors form a subgroup  $\text{Div}_0(C)$  of the group  $\text{Div}(C)$  of all divisors. We say that two divisors  $D$  and  $E$  on  $C$  are *linearly equivalent* if their difference  $D - E$  is a principal divisor. The group  $\text{Div}(C)/\text{Div}_0(C)$  of linear equivalence classes of divisors is called the *Picard group* of  $C$ , denoted  $\text{Pic}(C)$ .

Linear equivalence is important in the description of a map  $C \rightarrow \mathbb{P}^r$  by divisors because in  $\mathbb{P}^r$  any two hyperplanes, defined by the vanishing of linear forms  $\ell_1 = 0$  and  $\ell_2 = 0$ , differ by the globally defined rational function  $\ell_1/\ell_2$ . The preimages of these hyperplanes differ by the pull-back of this rational function, and are thus linearly equivalent divisors.

**2.1. Generalizations.** We can generalize the notion of a divisor on a smooth curve in two respects.

First, we can extend the notion to higher dimensional smooth varieties  $X$ , defining a divisor to be a finite formal linear combination of irreducible subvarieties of codimension 1 in  $X$ . In this setting, most of the notions introduced above have natural analogues; for example, a divisor is called *effective* if all the irreducible varieties in its expression as a sum appear with nonnegative coefficients. We can similarly define the divisor of a rational function on  $X$ , and

the notion of linear equivalence of divisors. (One notion that does not have an obvious analog is the degree of a divisor.)

We can also extend the notion to possibly singular varieties  $X$ , though this requires more care. We can mimic the definition above, defining a divisor to be a finite formal linear combination of irreducible subvarieties of codimension 1 in  $X$ ; this is called a *Weil divisor* on  $X$ . But for most purposes we have to introduce a slightly different notion of divisor, called a *Cartier divisor*. We first define an *effective Cartier divisor* to be a subscheme locally defined on some open covering  $U_i$  by the vanishing of one non-zerodivisor in  $\mathcal{O}_X(U_i)$ ; and a *Cartier divisor* to be a difference of effective Cartier divisors

A simple example will serve to illustrate this. Suppose  $C \subset \mathbb{A}^2$  is a plane curve with a node at a point  $p = (0, 0) \in C$ . The point  $p$  itself (that is, the reduced subscheme) is *not* a Cartier divisor, since its ideal  $(x, y)$  is not generated by a single element. Likewise, the *fat point*—that is, the subscheme  $\Gamma = V(x^2, xy, y^2) \subset C$ —is not Cartier. But a subscheme of degree 2 supported at  $p$ —that is, a scheme of the form

$$\Gamma_{\alpha, \beta} := V(\alpha x + \beta y, x^2, xy, y^2) \subset C -$$

is in general Cartier, as it is defined—globally as well as locally—by the equation  $\alpha x + \beta y = 0$ . For distinct lines  $L$  the schemes  $\Gamma_L$  are also distinct, since  $\Gamma_L \subset L$ . The two exceptions, when  $\Gamma_L$  is not Cartier, occur when the line  $L$  is tangent to either branch of  $C$  at the node; in these cases the scheme  $\Gamma_{\alpha, \beta}$  is not a Cartier divisor on  $C$  (the intersection  $L \cap C$  has multiplicity 3 at  $p$ , and is not equal to  $\Gamma_{\alpha, \beta}$ ). All of these assertions are the content of Exercise 5.7 below; the reader who is not already familiar with the notion of Cartier divisor is encouraged to carry out the verifications.

Here is another way to characterize Cartier divisors on a scheme  $X$ : for each open affine set  $U \subset X$  we define  $K_X(U)$  to be the field of fractions of  $\mathcal{O}_X(U)$ , and write  $K_X$  for the associated sheaf. We write  $\mathcal{O}_X^*$  and  $K_X^*$  for the sheaves of groups of units in  $\mathcal{O}_X(U)$  and  $K_X(U)$ . A Cartier divisor on  $X$  is then by definition a global section of  $K_X^*/\mathcal{O}_X^*$ . Moreover, if  $X$  is a *normal* scheme (so that its local rings at the generic points of codimension 1 subvarieties are discrete valuation rings) then the Cartier divisors are the same as locally principal Weil divisors. See [Hartshorne 1977, Section II.6] for more information.

In this book, we will be dealing for the most part with smooth curves; even when singular curves  $C$  arise, they will be viewed as images of smooth curves  $\tilde{C}$  (their normalizations) and their geometry analyzed in terms of that of  $\tilde{C}$ . Thus for the most part we'll be dealing with divisors in the simplest setting of smooth curves. But there will be occasions when we want to extend our analysis to singular curves and/or higher-dimensional varieties, and in those cases the notion of Cartier divisor is called for.

**2.2. Divisors of functions.** Returning to the case of smooth curves, we have:

**Theorem 2.1.** *Let  $C$  be a smooth projective curve. If  $f \in K(C)$ , then the degree of the divisor  $(f)$  is 0. Thus any two linearly equivalent divisors on  $C$  have the same degree.*

**Proof.** The result is evident on  $\mathbb{P}^1$ , where a rational function is the ratio of two forms of the same degree. In general, a rational function  $\phi$  on a smooth curve  $C$  defines a map  $\pi : C \rightarrow \mathbb{P}^1$ , such that

$$(\phi) = \pi^{-1}(0) - \pi^{-1}(\infty).$$

If  $C \rightarrow D$  is any map of smooth projective curves, then restricting to an affine open subset  $D'$  of  $D$  and its preimage  $C' \subset C$  the map is represented by a homomorphism of rings  $\mathcal{O}_D(D') \rightarrow \mathcal{O}_C(C')$  in such a way that  $\mathcal{O}_C(C')$  becomes a finitely generated module over  $\mathcal{O}_D(D')$ , which is torsion-free because  $C$  is reduced and irreducible. Since  $D$  is smooth,  $\mathcal{O}_D(D')$  is a Dedekind domain, so  $\mathcal{O}_C(C')$  is free of constant rank equal to the degree of the extension of the corresponding fields of rational functions  $\kappa(D) \subset \kappa(C)$ . This implies that the degree of the fibers is constant.

Returning to the case of the map  $C \rightarrow \mathbb{P}^1$  defined by a rational function, we see that  $\pi^{-1}(0)$  and  $\pi^{-1}(\infty)$  have the same degree, so  $(\phi)$  has degree 0 as required.  $\square$

In particular, we see that if  $C \rightarrow \mathbb{P}^r$  is a morphism, then the pullbacks of different hyperplanes have the same degree, and we define this to be the *degree of the morphism*.

It follows from Theorem 2.1 that we can write  $\text{Pic}(C)$  as a disjoint union

$$\text{Pic}(C) = \bigsqcup_{d \in \mathbb{Z}} \text{Pic}_d(C),$$

where  $\text{Pic}_d(C)$  is the set of linear equivalence classes of divisors of degree  $d$ .

**2.3. Invertible sheaves.** To deal with families of linearly equivalent divisors we will use the language of invertible sheaves.

Recall first that a *coherent sheaf*  $\mathcal{L}$  on  $X$  may be defined by giving

- An open affine cover  $\{U_i\}$  of  $X$ ;
- For each  $i$ , a finitely generated  $\mathcal{O}_X(U_i)$ -module  $L_i$ ;
- For each  $i, j$ , an isomorphism  $\sigma_{i,j} : L_i|_{U_i \cap U_j} \rightarrow L_j|_{U_i \cap U_j}$  satisfying the compatibility conditions  $\sigma_{j,k}\sigma_{i,j} = \sigma_{i,k}$ .

A *global section* of  $\mathcal{L}$  is a family of elements  $t_i \in L_i$  such that  $\sigma_{i,j}t_i = t_j$ . Such a section may be realized as the image of the constant function 1 under a homomorphism of sheaves  $\mathcal{O}_X \rightarrow \mathcal{L}$ .

A coherent sheaf  $\mathcal{L}$  is said to be *locally free* if the modules  $L_i$  are all free; when  $X$  is irreducible, the ranks of the free modules  $L_i$  are all the same, and this is called the *rank* of  $\mathcal{L}$ . An *invertible sheaf* is a locally free coherent sheaf of rank 1; that is,  $L|_U \cong \mathcal{O}_U$ , for every open set in some covering of  $X$ .

Invertible sheaves form a group: the tensor product  $\mathcal{L}_1 \otimes_{\mathcal{O}_X} \mathcal{L}_2$  of two invertible sheaves is again an invertible sheaf. The inverse is given by the dual invertible sheaf  $\mathcal{L}^{-1} := \mathcal{H}\text{om}_{\mathcal{O}_X}(\mathcal{L}, \mathcal{O}_X)$  (Proof: Evaluation of functions defines a natural map  $\mathcal{F} \otimes \mathcal{H}\text{om}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X)$ , and when  $\mathcal{F} = \mathcal{L}$  is locally free of rank 1 this is locally an isomorphism).

Invertible sheaves can be defined from divisors on a smooth scheme (or more generally from Cartier divisors on any scheme). In case  $C$  is a smooth projective curve, this is concrete: if  $D = \sum m_p \cdot p$ , then we define the sections of  $\mathcal{O}_C(D)$  on any open set  $U$  by

$$\mathcal{O}_C(D)(U) := \{\text{rational functions } f \mid \text{ord}_p(f) + m_p \geq 0 \text{ for all } p \in U\}.$$

Thus, at a point  $p$  with  $m_p > 0$ , we are allowing  $f$  to have a pole of order at most  $m_p$ ; if  $m_p \leq 0$  then  $f$  must be regular at  $p$ , with a zero of order at least  $-m_p$ .

More generally, we can associate an invertible sheaf to any Cartier divisor on a scheme  $X$ . First, if  $D \subset X$  is an effective Cartier divisor (and thus locally defined by the vanishing of a non-zerodivisor) then  $D$  corresponds to a subscheme of  $X$  whose ideal sheaf  $\mathcal{I}_{D/X}$  is invertible. We define  $\mathcal{O}_X(-D)$  to be the ideal sheaf  $\mathcal{I}_{D/X}$ , and define  $\mathcal{O}_X(D)$  to be its inverse. The dual of the inclusion of  $\mathcal{O}_X(-D) \subset \mathcal{O}_X$  is a homomorphism  $\sigma := \mathcal{O}_X \rightarrow \mathcal{O}_X(D)$ , that we regard as a global section. It vanishes precisely along  $D$ .

For a general (Cartier) divisor  $D = E - F$ , we define

$$\mathcal{O}_X(D) := \mathcal{O}_X(E) \otimes \mathcal{O}_X(F)^{-1}.$$

If  $D$  and  $E$  are linearly equivalent divisors on  $C$ —that is, there is a rational function  $f$  on  $C$  with  $(f) = D - E$ —then multiplication by  $f$  defines an isomorphism  $\mathcal{O}_C(E) \xrightarrow{\cong} \mathcal{O}_C(D)$ ; thus the isomorphism class of the invertible sheaf  $\mathcal{O}_C(D)$  corresponds to the linear equivalence class of  $D$ . We will soon see that every invertible sheaf on a projective scheme has this form (Corollary 1.2).

If  $\sigma \in H^0(\mathcal{L})$  is a global section of an invertible sheaf on  $X$  and  $p \in X$  is a point, then the *value*  $\sigma(p)$  of  $\sigma$  at  $p$  is the image of  $\sigma$  under the natural map  $H^0(\mathcal{L})$  to the fiber  $\kappa(p) \otimes \mathcal{L}_p \cong \mathbb{C}$ . Since the isomorphism is not canonical,

$\sigma$  does not define a function on  $X$  at  $p$ ; but since any two isomorphisms differ by a unit in  $\mathcal{O}_{X,p}$ , the vanishing locus of  $\sigma$ , denoted  $(\sigma)_0$ , is a well-defined subscheme of  $X$ .<sup>2</sup> Moreover, if  $X$  is reduced and irreducible, then the ratio  $\sigma/\tau$  of two global sections of the same invertible sheaf is a well-defined rational function  $\sigma(p)/\tau(p)$  at all the points where the denominator  $\tau(p)$  is not 0, so the divisor class of  $(\sigma)_0$  is independent of the choice of  $\sigma \in H^0(\mathcal{L})$ .

The tensor product of (rational) sections  $\sigma$  of  $\mathcal{L}$  and  $\sigma'$  of  $\mathcal{L}'$  is a rational section of  $\mathcal{L} \otimes \mathcal{L}'$  that has divisor the sum of the divisors of  $\sigma$  and  $\sigma'$ . Thus the group of divisor classes  $\text{Pic}(X)$  is naturally isomorphic to the group of invertible sheaves under  $\otimes$ , and we will identify the two.

**2.4. Invertible sheaves and line bundles.** If  $\mathcal{L}$  is an invertible sheaf on a variety  $X$ , and  $p \in X$  is a point then the *stalk*  $\mathcal{L}_p$  of  $\mathcal{L}$  is isomorphic to the local ring  $\mathcal{O}_{X,p}$ . We write  $\mathfrak{m}_{X,p}$  for the maximal ideal of  $\mathcal{O}_{X,p}$ . By definition the *fiber* of  $\mathcal{L}$  is

$$\kappa(p) \otimes \mathcal{L} = \mathcal{L}_p / \mathfrak{m}_{X,p} \mathcal{L}_p \cong \mathbb{C}$$

where  $\kappa(p) := \mathcal{O}_{X,p}/\mathfrak{m}_{X,p}$  is the residue field of  $X$  at  $p$ . It is not hard to prove that the collection of these fibers forms a line bundle on  $X$ ; that is, a morphism of schemes  $L \rightarrow X$  whose fibers have the structure of 1-dimensional vector spaces such that  $L|_{U_i} \cong U_i \times \mathbb{C}^1$  for some open covering  $\{U_i\}$  of  $X$ .

Moreover, given a line bundle  $L$  on  $X$ , we can recover an invertible sheaf  $\mathcal{L}$  associated to  $L$  by defining  $\mathcal{L}(U)$  to be the set of sections of  $L$  defined over  $U$ . These two processes are inverse to one another, and allow us to think of invertible sheaves and line bundles interchangeably.

Though we will systematically prefer the invertible sheaf terminology, there are at least two points in which the line bundle approach is more natural. First, the vanishing of a section of an invertible sheaf at a point  $p$  is genuinely the vanishing of the section of the line bundle as a function. Second, and more serious, given a morphism  $f : Y \rightarrow X$  of schemes, the pullback  $f^*(\mathcal{L})$  of an invertible sheaf  $\mathcal{L}$  on  $X$  is defined as the tensor product of  $\mathcal{O}_Y$  with a sort of naive pull-back; whereas the pullback of a line bundle is a straightforward set-theoretic operation.

**Example 2.2** (Invertible sheaves on  $\mathbb{P}^r$ ). Since  $\mathbb{C}[x_0, \dots, x_r]$  is a unique factorization domain, the ideal of any codimension 1 subvariety of  $\mathbb{P}^r$  is generated by one nonzero element, which is thus a non-zerodivisor—it is a hypersurface. As we explained above, any two hypersurfaces of degree  $d$  differ by the divisor of a rational function, so the group of divisor classes on  $\mathbb{P}^r$  is  $\mathbb{Z}$ —that is, the class

<sup>2</sup>The terminology of “vanishing” is slightly confusing: when we say that  $\sigma$  vanishes at  $p$ , or  $\sigma(p) = 0$ , we mean that the image of  $\sigma$  in the stalk  $\mathcal{L}_p = \mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{O}_{X,p}$  is in the maximal ideal  $\mathfrak{m}_{X,p}$  of  $\mathcal{O}_{X,p}$  times  $\mathcal{L}_p$ —and *not* that  $\sigma$  is zero as an element of the stalk  $\mathcal{L}_p$  itself. Similarly, we say that  $\sigma$  vanishes to order  $m$  at  $p$  if  $\sigma(p)$  lies in  $\mathfrak{m}_{X,p}^m \mathcal{L}_p$ .

of a divisor is defined by its degree. Thus if  $D = V(F) \subset \mathbb{P}^r$  is a hypersurface defined by the vanishing of a form  $F$  of degree  $d$ , it is natural to use the name  $\mathcal{O}_{\mathbb{P}^r}(d)$  for  $\mathcal{O}_{\mathbb{P}^r}(D)$ .

If  $D$  is an effective divisor other than 0, then  $H^0(\mathcal{O}_{\mathbb{P}^r}(-D)) = 0$ , since there are no globally defined functions vanishing on  $D$  except 0.

To compute  $H^0(\mathcal{O}_{\mathbb{P}^1}(d))$  directly, let  $D = z_1 + z_2 + \cdots + z_d$  be a divisor of degree  $d$  and suppose that the coordinates are chosen so that none of the  $z_i$  are at infinity. The sections of  $\mathcal{O}_{\mathbb{P}^1}(D)$  are the rational functions with poles in  $\mathbb{P}^1$  only at the  $z_i$ . Identifying  $\mathbb{P}^1 \setminus \{\infty\} = \mathbb{A}^1$  with  $\mathbb{C}$  these can each be written

$$\frac{g(z)}{(z - z_1)(z - z_2) \cdots (z - z_d)}$$

where  $g$  is a polynomial. The condition that the point at infinity is not a pole is the condition  $\deg(g) \leq d$ . With this condition, these rational functions form a vector space of dimension  $d + 1$ .

More generally, because every rational function on  $\mathbb{P}^r$  has degree 0, and any two global sections differ by a rational function, it follows that every global section of  $\mathcal{O}_{\mathbb{P}^r}(d)$  vanishes on a divisor of degree  $d$ . Thus we may identify  $H^0(\mathcal{O}_{\mathbb{P}^r}(d))$  with the  $\binom{r+d}{r}$ -dimensional vector space of forms of degree  $d$  on  $\mathbb{P}^r$ .

Putting this together for future reference we have:

**Proposition 2.3.** *Every invertible sheaf  $\mathcal{L}$  on  $\mathbb{P}^r$  has the form  $\mathcal{L} \cong \mathcal{O}_{\mathbb{P}^r}(m)$  for a unique  $m = \deg \mathcal{L} \in \mathbb{Z}$ ; and*

$$H^0(\mathcal{O}_{\mathbb{P}^r}(m)) = \mathbb{C}[x_0, \dots, x_r]_m$$

*the space of forms of degree  $m$  in  $r + 1$  variables.*

### 3. Linear series and maps to projective space

We will use invertible sheaves to describe maps of a given variety  $X$  to projective space. For this we add the notion of linear series (sometimes called linear system).

**Definition 3.1.** A *linear series* on a scheme  $X$  is a pair  $\mathcal{V} = (\mathcal{L}, V)$  where  $\mathcal{L}$  is an invertible sheaf on  $X$  and  $V$  is a nonzero vector space of global sections of  $\mathcal{L}$ . We define the *dimension* of the linear series to be

$$\dim \mathcal{V} := \dim_{\mathbb{C}} V - 1.$$

To every global section  $\sigma$  of an invertible sheaf  $\mathcal{L}$  on a variety  $X$  we can associate an effective divisor  $(\sigma) = (\sigma)_0$  defined by the vanishing of  $\sigma$ . If  $\tau$  is a scalar multiple of  $\sigma$ , it has the same divisor; and if  $H^0(\mathcal{O}_X) = \mathbb{C}$  (for example if

$X$  is reduced, connected, and projective) then the converse is true: two sections of  $\mathcal{L}$  with the same divisor differ by multiplication by a scalar.

We sometimes write  $|\mathcal{L}|$  for the complete linear series  $(\mathcal{L}, H^0(\mathcal{L}))$ . If  $\mathcal{L} = \mathcal{O}(D)$  we often write this as  $|D|$ .

Thus a linear series  $\mathcal{V} = (\mathcal{L}, V)$  gives rise to a family of effective divisors on  $X$ , all in the same linear equivalence class, parametrized by the projective space  $\mathbb{P}V^*$  of nonzero  $\sigma \in V$  mod scalars. This is indeed the way we think of a linear series: as a family of divisors parametrized by a projective space. This is reflected in the definition of the dimension of a linear series above: it's not the dimension of  $V$  as a vector space, but the dimension of the corresponding projective space. It is also reflected in the terminology: we often speak of “the divisors of a linear series  $\mathcal{V}$ ” or as divisors “moving” in a linear series. The intersection of the vanishing loci of all the sections in  $V$  is called the *base locus* of  $\mathcal{V}$ . It is in general a subscheme of  $C$ . The points in its support are called *base points* of  $\mathcal{V}$ . If the vector space  $V = H^0(\mathcal{L})$ , the linear system is said to be *complete*; in terms of the *family of divisors* point of view, this is the same as saying the linear system includes every effective divisor in the linear equivalence class.

Suppose that  $C$  is a smooth projective curve. Since the divisors in a linear series are linearly equivalent, they all have the same degree, called the degree of the linear series. If  $\mathcal{V}$  is a linear series of degree  $d$  and dimension  $r$  we say that  $\mathcal{V}$  is a  $g_d^r$ —this classical language means that  $\mathcal{V}$  represents a group of  $d$  points “moving” within a linear equivalence class with  $r$  degrees of freedom.<sup>3</sup>

The next result is fundamental:

**Theorem 3.2.** *For any scheme  $X$  there is a natural bijection between the set of nondegenerate morphisms  $\phi : X \rightarrow \mathbb{P}^r$  modulo  $PGL_{r+1}$ , and base-point free linear series of dimension  $r$  on  $X$ , up to isomorphism.*

Here *nondegenerate* means the image of the morphism  $\phi$  is not contained in any hyperplane. The phrase “modulo  $PGL_{r+1}$ ” is needed because the notation  $\mathbb{P}^r$  supposes a choice of projective coordinates, and  $PGL_{r+1}$  is the group of linear coordinate transformations (actually all automorphisms, by Exercise 5.4). To get a correspondence without the dependence on a basis we could think of morphisms to  $\mathbb{P}V$ , the set of 1-quotients of  $V$ .

Suppose that  $(\mathcal{L}, V)$  is a linear series on a smooth projective curve  $C$  that does have a base locus  $D_0$ . We can then subtract  $D_0$  from all the divisors of the linear series, replacing  $\mathcal{L}$  by  $\mathcal{L}(-D_0)$  and dividing each section in  $V$  by the

<sup>3</sup>The “ $g$ ” in the symbol  $g_d^r$  has nothing to do with the genus of the curve  $C$ ; it's short for *gruppi di punti*, Italian for “divisors.”

section  $\sigma$  of  $\mathcal{O}_C(D_0)$  vanishing on  $D_0$  to get a new, base-point free linear series of the same dimension (but of lower degree).

We will prove Theorem 3.2 by describing the correspondence in both directions:

**3.0.1. The linear series coming from a morphism to projective space.** Let  $f : X \rightarrow \mathbb{P}^r$  be any nondegenerate morphism. The associated linear series  $\mathcal{V} = (\mathcal{L}, V)$  on  $X$  has  $\mathcal{L} = f^*\mathcal{O}_{\mathbb{P}^r}(1)$ , the pullback of the invertible sheaf  $\mathcal{O}_{\mathbb{P}^r}(1)$ , and

$$V = f^*H^0(\mathcal{O}_{\mathbb{P}^r}(1)) \subset H^0(\mathcal{L}).$$

In geometric terms, if we think of a linear series as a family of effective divisors, this is the linear series on  $X$  consisting of preimages of hyperplanes in  $\mathbb{P}^r$ . The condition of nondegeneracy assures us that the preimage of a hyperplane in  $\mathbb{P}^r$  is indeed a divisor on  $X$ .

**3.0.2. The morphism to projective space coming from a linear series.** Suppose that  $X$  is any scheme, and  $\mathcal{V} = (\mathcal{L}, V)$  is a base-point free linear series of dimension  $r$  on  $X$ ; we want to describe a corresponding morphism  $f : X \rightarrow \mathbb{P}^r$ . Choose a basis  $\sigma_0, \dots, \sigma_r$  for  $V$ . If we let  $D_i = (\sigma_i) \subset X$  be the divisor of zeroes of  $\sigma_i$  and  $U_i := X \setminus D_i$ , then the ratio  $\sigma_j/\sigma_i$  is a regular function on  $U_i$ , and we can define a map  $f_i : U_i \rightarrow \mathbb{P}^r$  by

$$f_i : p \mapsto \left( \frac{\sigma_0}{\sigma_i}(p), \dots, \frac{\sigma_r}{\sigma_i}(p) \right),$$

where the component  $\sigma_i/\sigma_i = 1$  ensures that not all the components are 0, so that the image point is well-defined. The maps  $f_i$  and  $f_j$  agree on the overlap  $U_i \cap U_j$ , and by the hypothesis that  $\mathcal{V}$  is base-point free the  $U_i$  cover  $X$ ; so together they define a regular map  $f$  from  $X$  to  $\mathbb{P}^r$ .

We can describe this map set-theoretically without having to choose a basis: since  $\mathcal{V}$  is assumed base-point free, for any point  $p \in X$  the subspace  $H_p := \{\sigma \in V \mid \sigma(p) = 0\}$  is a hyperplane in  $V$ ; thus we get a map  $f : X \rightarrow \mathbb{P}V$ .

**Example 3.3.** The morphism from  $\mathbb{P}^r$  defined by the complete linear series  $(\mathcal{O}_{\mathbb{P}^r}(d), H^0(\mathcal{O}_{\mathbb{P}^r}(d)))$  has target  $\mathbb{P}^{\binom{r+d}{r}-1}$ , and takes a point  $(a_0, \dots, a_r)$  to the point whose coordinates are all the monomials of degree  $d$  in  $x_0, \dots, x_r$ . It is called the  $d$ -th Veronese morphism from  $\mathbb{P}^r$ . For example on  $\mathbb{P}^1$ , this has the form

$$(x_0, x_1) \mapsto (x_0^d, x_0^{d-1}x_1, \dots, x_1^d).$$

The image of  $\mathbb{P}^1$  under this morphism is called the *rational normal curve* of degree  $d$ ; in the case  $d = 2$  is the *plane conic*, and in the case  $d = 3$  it is called the *twisted cubic*. Veronese himself studied the image of  $\mathbb{P}^2$  by the Veronese morphism of degree 2 now simply called the *Veronese surface*.

## 4. The geometry of linear series

**4.1. An upper bound on  $h^0(\mathcal{L})$ .** We will develop sophisticated ways of estimating the dimensions of linear series. We begin with an elementary bound:

**Theorem 4.1.** *Let  $C$  be a smooth projective curve. If  $\mathcal{L}$  is an invertible sheaf of degree  $d \geq 0$  on  $C$ , then  $h^0(\mathcal{L}) \leq d + 1$ ; and equality holds if and only if  $C \cong \mathbb{P}^1$ .*

**Proof.** First, let  $p_1, \dots, p_{d+1}$  be any points of  $C$ . If  $h^0(\mathcal{L}) \geq d + 2$ , there would be a nonzero section  $\sigma \in H^0(\mathcal{L})$  vanishing at  $p_1, \dots, p_{d+1}$ ; the divisor  $(\sigma)$  would then have degree  $\geq d + 1$ , contradicting the hypothesis that  $\deg(\mathcal{L}) = d$ .

For the second part, suppose that  $h^0(\mathcal{L}) = d + 1$ . If  $p_1, \dots, p_d$  are any points of  $C$ , there is a nonzero section  $\sigma \in H^0(\mathcal{L})$  vanishing at  $p_1, \dots, p_d$ ; by degree considerations, it cannot vanish anywhere else. It follows that any two divisors of degree  $d$  on  $C$  are linearly equivalent, hence that any two points  $p, q \in C$  are linearly equivalent. Thus there is a rational function  $f$  on  $C$  with exactly one zero and one pole, giving an isomorphism  $C \cong \mathbb{P}^1$ .  $\square$

From the correspondence between invertible sheaves and maps to projective space, we now get:

**Corollary 4.2.** *If  $C \subset \mathbb{P}^d$  is a nondegenerate curve, then the degree of  $C$  is  $\geq d$  with equality only in the case that  $C$  is a rational normal curve.*  $\square$

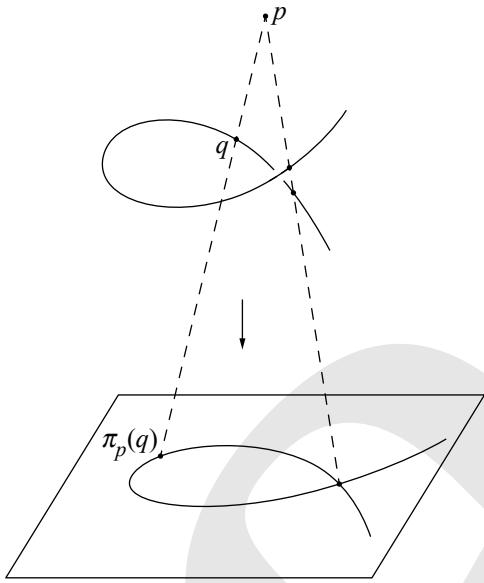
Rational normal curves arise often in the literature because they have many extremal properties, such as those of Corollaries 4.2 and 4.3. For a related result see Corollary 1.3. Since there is only one invertible sheaf of degree  $d$  on  $\mathbb{P}^1$  any two rational normal curves of degree  $d$  differ by a transformation in  $\mathrm{PGL}_{d+1}$ , and we will therefore often speak of *the* rational normal curve.

**Corollary 4.3.** *Let  $C \subset \mathbb{P}^d$  be the rational normal curve of degree  $d$ . If  $E$  is an effective divisor on  $C$  of degree  $e \leq d + 1$ , then the span of  $E$  (that is the smallest linear space containing the subscheme  $E$ ) has dimension  $e - 1$ .*

Less formally: any finite set or subscheme of  $C$  is as linearly independent as possible.

**Proof.** Let  $L$  be the span of  $E$ . Since  $E$  imposes at most  $e$  conditions on hyperplanes, it follows that the dimension of the span of  $E$  is at most  $e - 1$ .

On the other hand, the hyperplanes containing  $E$  meet  $C$  in a divisor of the form  $E + E'$ , where  $\deg E' = d - e$ . Thus the projection of  $C$  from  $E$  is a nondegenerate curve of degree  $d - e$  in  $\mathbb{P}^{d-(\dim \mathrm{span}(E))-1}$  so from Corollary 4.2 we get  $d - e \geq d - (\dim \mathrm{span}(E) - 1)$ , as required.  $\square$



**Figure 2.** Projection of a space curve from a general point  $p$  to  $\mathbb{P}^2$  Silvio:  $p$  needs a dot. perhaps add a line from  $p$  to the double point, meeting the space curve in two points? figure out a standard way to show projections

In the case of distinct points on a rational normal curve it is easy to make a direct argument why they are as independent as possible: Choose coordinates so that none of the points are at infinity. We can identify the points  $\lambda_1, \dots, \lambda_{d+1} \in C \cong \mathbb{P}^1$  with distinct complex numbers, and the independence (for  $\ell = d + 1$ ) is equivalent to the nonvanishing of the Vandermonde determinant

$$\begin{vmatrix} 1 & \lambda_1 & \lambda_1^2 & \dots & \lambda_1^d \\ 1 & \lambda_2 & \lambda_2^2 & \dots & \lambda_2^d \\ \vdots & & \vdots & & \vdots \\ 1 & \lambda_{d+1} & \lambda_{d+1}^2 & \dots & \lambda_{d+1}^d \end{vmatrix} = \prod_{1 \leq i < j \leq d+1} (\lambda_j - \lambda_i)$$

**4.2. Incomplete linear series.** In classical algebraic geometry a linear series of dimension 1 is called a *pencil*, a linear series of dimension 2 is called a *net* and, less commonly, a three-dimensional linear series is called a *web*. We will use only the first of these.

The morphism associated to an incomplete linear series  $V \subset H^0(\mathcal{L})$  is the composition of the morphism associated to the complete linear series  $|\mathcal{L}|$  with a linear projection. In general, if  $V \subset W \subset H^0(\mathcal{L})$  are a pair of nested linear series, then a 1-dimensional quotient of  $W$  restricts to a 1-dimensional quotient of  $V$  unless it vanishes on  $V$ . Thus we have a partially defined linear morphism

$\pi : \mathbb{P}W \rightarrow \mathbb{P}V$ . The *indeterminacy locus* of the map consists of the set of 1-quotients vanishing on  $V$ , that is, to  $\mathbb{P}(W/V) \subset \mathbb{P}W$ ; we will call it the *center of the projection*  $\pi$ . (It is sometimes useful to think of the dual picture: lines in  $W^*$  map to lines in  $V^*$  except when they lie in the subspace  $(W/V)^* = \text{Ann}(V) \subset W^*$ .) Thus there is a commutative diagram

$$\begin{array}{ccc} & \mathbb{P}W^* & \\ \phi_W \nearrow & \downarrow \pi & \\ C & \xrightarrow{\phi_V} & \mathbb{P}V^*. \end{array}$$

If  $W$  is base-point free, then  $V$  is base-point free if and only if the center of the projection  $\pi$  is disjoint from  $\phi_W(C)$ . In this case we say that  $\pi$  is *regular* on  $C$ .

By way of language, we will say that a curve  $C \subset \mathbb{P}^r$  embedded by a complete linear series  $|\mathcal{L}|$  is *linearly normal*; this is equivalent to saying that the pullback map

$$H^0(\mathcal{O}_{\mathbb{P}^r}(1)) \rightarrow H^0(\mathcal{L})$$

is surjective. Since regular projections of a curve correspond to subseries, this is equivalent to saying that  $C$  is *not* the regular projection of a nondegenerate curve  $\tilde{C} \subset \mathbb{P}^{r+1}$ .

**4.3. Sums of linear series.** If  $D = (\mathcal{L}, V)$  and  $E = (\mathcal{M}, W)$  be two linear series on a curve  $C$ . By the *sum*  $D + E$  of  $D$  and  $E$ , we will mean the pair

$$D + E = (\mathcal{L} \otimes \mathcal{M}, U)$$

where  $U \subset H^0(\mathcal{L} \otimes \mathcal{M})$  is the subspace generated by the image of  $V \otimes W$ , under the multiplication/cup product map  $H^0(\mathcal{L}) \otimes H^0(\mathcal{M}) \rightarrow H^0(\mathcal{L} \otimes \mathcal{M})$ —in other words, it's the subspace of the complete linear series  $|\mathcal{L} \otimes \mathcal{M}|$  spanned by divisors of the form  $D + E$ , with effective divisors  $D \in D$  and  $E \in E$ .

The sum of two base-point free linear series is clearly base-point free; if the two series correspond to maps  $C \xrightarrow{\phi} \mathbb{P}^r$  and  $C \xrightarrow{\psi} \mathbb{P}^s$  then the sum corresponds to the composition of the map  $(\phi, \psi) : C \rightarrow \mathbb{P}^r \times \mathbb{P}^s$  with the *Segre* embedding  $\sigma : \mathbb{P}^r \times \mathbb{P}^s \rightarrow \mathbb{P}^{(r+1)(s+1)-1}$  given in coordinates by

$$((x_0, \dots, x_r), (y_0, \dots, y_s)) \mapsto (x_0 y_0, \dots, x_i y_j, \dots, x_r y_s).$$

Note that this says nothing about the dimension of  $D + E$ , since the composite map  $\sigma \circ (\phi, \psi)$  will typically not be nondegenerate. In fact, the best we can do in general is the following proposition.

**Proposition 4.4.** *If  $D$  and  $E$  are two linear series that contain effective divisors on a curve  $C$ , then*

$$\dim(D + E) \geq \dim D + \dim E.$$

**Proof.** Saying  $\dim D \geq m$  is equivalent to saying that we can find a divisor  $D \in D$  containing any given  $m$  points of  $C$ ; since  $D + E$  contains all pairwise sums  $D + E$  with  $D \in D$  and  $E \in E$ , we can certainly find a divisor  $F \in D + E$  containing any given  $\dim D + \dim E$  points of  $C$ .  $\square$

**4.4. Which linear series define embeddings?** A linear series  $\mathcal{V} = (\mathcal{L}, V)$  is called *very ample* if it is base-point free and defines an embedding (what Hartshorne calls a “closed immersion”). If  $D$  is a Cartier divisor on  $X$ , then we say that  $D$  is *very ample* if the complete linear series  $|D|$  is very ample, and we say that  $D$  is *ample* if  $mD$  is very ample for some integer  $m > 0$ .

Similarly,  $\mathcal{V}$  or  $D$  are called *birationally very ample* if  $V$  or  $D$  are base-point free and define a map that is generically one-to-one or, equivalently, an embedding when restricted to an open set. This arises frequently when speaking of a map from a smooth curve  $C$  onto a curve  $C_0$  in the plane, since the latter frequently has singularities.

Given a linear series  $\mathcal{V} = (\mathcal{L}, V)$  and an effective divisor  $D$  on  $C$ , we set  $\mathcal{V}(-D) = (\mathcal{L}(-D), V(-D))$  where

$$\mathcal{L}(-D) := \mathcal{L} \otimes \mathcal{O}(-D) \text{ and } V(-D) := \{\sigma \in V \mid \sigma(D) = 0\}.$$

The difference  $\dim \mathcal{V} - \dim \mathcal{V}(-D)$  is called the *number of conditions imposed by  $D$  on the linear series  $\mathcal{V}$* ; we say that  $D$  *imposes independent conditions* on  $\mathcal{V}$  if  $\dim \mathcal{V} - \dim \mathcal{V}(-D) = \deg(D)$ .

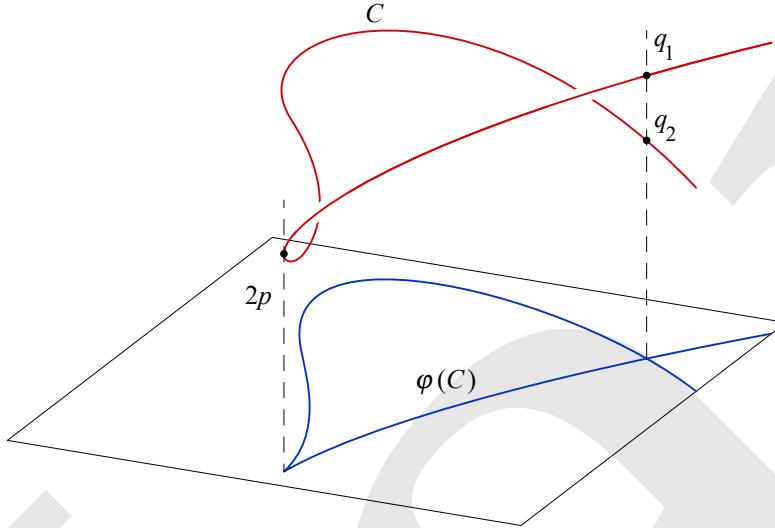
Via the correspondence of Theorem 3.2, statements about the geometry of a morphism  $\phi : C \rightarrow \mathbb{P}^r$  can be formulated as statements about the relevant linear series. In the case of complete series, these are statements about the vector space  $H^0(\mathcal{L})$  of global sections of  $\mathcal{L}$ . We write  $h^0(\mathcal{L})$  for the vector space dimension of  $H^0(\mathcal{L})$  (and similarly for other cohomology groups). It is useful to have criteria in these terms for when a linear series defines an embedding, or even to be base-point free so that it defines a morphism:

**Proposition 4.5.** [Hartshorne 1977, Thm. IV.3.1] *Let  $\mathcal{L}$  be an invertible sheaf on a smooth curve  $C$ . The complete linear series  $|\mathcal{L}|$  is base-point free if and only if*

$$h^0(\mathcal{L}(-p)) = h^0(\mathcal{L}) - 1 \quad \forall p \in C;$$

and  $\mathcal{L}$  is very ample if and only if

$$h^0(\mathcal{L}(-p - q)) = h^0(\mathcal{L}) - 2 \quad \forall p, q \in C.$$



**Figure 3.** Two distinct points or one double point might impose just one condition on the linear series defined by projection from a point  $a$  (here illustrated by vertical projection).

**Proof.** First, if  $\mathcal{L}$  is base-point free, then vanishing at a point imposes one linear condition on sections of  $\mathcal{L}$ , so  $h^0(\mathcal{L}(-D)) \geq h^0(\mathcal{L}) - \deg D$  for any effective divisor  $D$ .

To say that  $|\mathcal{L}|$  is base-point free means that for every point  $p \in C$  there is a section of  $\mathcal{L}$  that does not vanish at  $p$ ; thus vanishing at  $p$  is a nontrivial linear condition on  $H^0(\mathcal{L})$ . Conversely, if  $h^0(\mathcal{L}(-p)) = h^0(\mathcal{L}) - 1$  then  $p$  imposes a nontrivial condition, so some section of  $\mathcal{L}$  does not vanish at  $p$ .

Since a divisor of degree  $d$  cannot impose more than  $d$  conditions on a linear series, the statement  $h^0(\mathcal{L}(-p-q)) = h^0(\mathcal{L}) - 2$  for all  $p, q$  implies the condition for base-point freeness; and saying that  $\phi_{\mathcal{L}}(p) \neq \phi_{\mathcal{L}}(q)$  implies that the linear series defines a set-theoretic injection.

Let  $\phi : C \rightarrow \mathbb{P}^r$  be the map defined by  $\mathcal{L}$ . To say that  $\phi$  is an embedding locally at a point  $p$ , we need to know that the map of local rings

$$\phi^* : \mathcal{O}_{\mathbb{P}^r, \phi(p)} \rightarrow \mathcal{O}_{C,p}$$

is surjective.

Since  $C$  is projective, the map  $C \rightarrow \phi(C)$  is finite, so  $\phi^*$  makes  $\mathcal{O}_{C,p}$  into a finitely generated  $\mathcal{O}_{\mathbb{P}^r, \phi(p)}$ -module. By Nakayama's lemma it suffices to show that  $\mathcal{O}_{C,p}/\phi^*(\mathfrak{m}_{\mathbb{P}^r, \phi(p)})$  is generated by the image of  $\mathcal{O}_{\mathbb{P}^r, \phi(p)}/\mathfrak{m}_{\mathbb{P}^r, \phi(p)} = \mathbb{C}$ .

Since the constants  $\mathbb{C} = \mathcal{O}_{\mathbb{P}^r, \phi(p)}/\mathfrak{m}_{\mathbb{P}^r, \phi(p)}$  pull back to the constants in  $\mathcal{O}_{C,p}/\mathfrak{m}_{C,p}$ , surjectivity will follow if

$$\frac{\mathcal{O}_{C,p}}{\phi^*(\mathfrak{m}_{\mathbb{P}^r, \phi(p)})\mathcal{O}_{C,p}}$$

is 1-dimensional; that is, if the linear series contains a section that vanishes to order exactly 1 at  $p$ , which is equivalent to the condition that  $h^0(\mathcal{L}(-2p)) \neq h^0(\mathcal{L}(-p))$ .  $\square$

A more geometric version of the last part of the proof would be to say that the condition of the existence of a section vanishing to order exactly 1 implies that  $\phi$  is an injection on the tangent space to  $C$  at  $p$ . This implies that the map is analytically an isomorphism onto its image, and together with the finiteness of the map, this suffices.

If  $\phi : X \rightarrow \mathbb{P}^r$  is a generically finite morphism, then the *degree* of  $\phi$  is the number of points in the preimage of a general point of  $\phi(X)$ . It follows that if  $D := \sum_{p \in C} n_p p$  is a divisor on a smooth curve, and the linear series  $|D|$  is base-point free, then the degree of the morphism associated to  $|D|$  is  $\deg D := \sum_{p \in C} n_p$ .

## 5. Exercises

**Exercise 5.1.** Show that there is no non-constant morphism  $\mathbb{P}^r \rightarrow \mathbb{P}^s$  when  $s < r$  by showing that any nontrivial linear series of dimension  $< r$  on  $\mathbb{P}^r$  has a non-empty base locus. Hint: use Bézout's theorem.

**Exercise 5.2.** Let

$$V = \langle s^{a_0} t^{d-a_0}, \dots, s^{a_r} t^{d-a_r} \rangle \subset \mathbb{C}[s, t]_d,$$

and let  $\mathcal{V} = (\mathcal{O}_{\mathbb{P}^1}(d), V)$ . Determine the conditions for:

- (1)  $\mathcal{V}$  is base-point free.
- (2)  $\mathcal{V}$  is ample.
- (3)  $\mathcal{V}$  is very ample.
- (4)  $\mathcal{V}$  is complete.

**Exercise 5.3.** Extend the statement of Proposition 4.5 to incomplete linear series; that is, prove that the morphism associated to a linear series  $(\mathcal{L}, V)$  on a smooth curve is an embedding iff

$$\dim(V \cap H^0(\mathcal{L}(-p - q))) = \dim V - 2 \quad \forall p, q \in C.$$

**Exercise 5.4.** Show that an automorphism of  $\mathbb{P}^r$  takes hyperplanes to hyperplanes. Deduce that it is given by the linear series  $\mathcal{V} = (\mathcal{O}_{\mathbb{P}^r}(1), H^0(\mathcal{O}_{\mathbb{P}^r}(1)))$ , and use this to show that  $\text{Aut } \mathbb{P}^r = PGL(r+1)$ . Hint: for the first part, use Bézout's theorem.

**Exercise 5.5.** Let  $C \subset \mathbb{P}^r$  be any linearly normal curve and  $\phi : C \rightarrow C$  an automorphism. Show that  $\phi$  is induced by an automorphism of  $\mathbb{P}^r$  if and only if  $\phi$  carries the invertible sheaf  $\mathcal{O}_C(1)$  to itself; that is,  $\phi^*(\mathcal{O}_C(1)) \cong \mathcal{O}_C(1)$ . In this case we say that the automorphism is projective. Show that every automorphism of a rational normal curve  $C \subset \mathbb{P}^d$  extends to  $\mathbb{P}^d$ . Since the automorphism group  $PGL_2$  of  $\mathbb{P}^1$  acts transitively on  $\mathbb{P}^1$ , we say that  $C$  is *projectively homogeneous*.

**Exercise 5.6.** Show that the ring  $\mathbb{C}[s^d, s^{d-1}t, \dots, t^d]$  is normal (= integrally closed) by noting that its integral closure must be contained in  $\mathbb{C}[s, t]$  and then showing that if  $f$  is any polynomial in the integral closure then the homogeneous components of  $f$  are also in the integral closure.

**Exercise 5.7.** Suppose  $C = V(y^2 - x^2 - x^3) \subset \mathbb{A}^2$  (so that  $C$  is a plane curve with a node at the point  $p = (0, 0) \in C$ ).

- (1) Show that the point  $p$  itself (that is, the reduced subscheme) is not a Cartier divisor, since its ideal  $(x, y)$  is not generated by a single element.
- (2) Show that the fat point,—that is, the subscheme  $\Gamma = V(x^2, xy, y^2)$ —is contained in  $C$  but is not a Cartier divisor on  $C$ .
- (3) Let  $\Gamma_{\alpha, \beta} := V(\alpha x + \beta y, x^2, xy, y^2) \subset C$ . Show that if  $\beta \neq \pm\alpha$  then  $\Gamma_{\alpha, \beta}$  is a Cartier divisor on  $C$ ; and
- (4) With  $\Gamma_{\alpha, \beta}$  as above, show that if  $\beta = \pm\alpha$  then  $\Gamma_{\alpha, \beta}$  is not a Cartier divisor on  $C$ .

Hint: for the last two parts, observe that if  $\beta \neq \pm\alpha$  then  $\Gamma_{\alpha, \beta}$  is locally the intersection of  $C$  with the line  $L \subset \mathbb{A}^2$  given by  $\alpha x + \beta y = 0$ , but when  $\beta = \pm\alpha$  the intersection  $L \cap C$  has multiplicity 3 at  $p$ , and is not equal to  $\Gamma_{\alpha, \beta}$ .



# The Riemann-Roch Theorem

## 1. How many sections?

To study curves via their maps to projective spaces, we want to estimate the dimension of the space of global sections of an invertible sheaf  $\mathcal{L}$ . The beginning of the story is the Riemann-Roch theorem.

Though we would like to be able to compute  $h^0(\mathcal{L})$ , it is much easier to compute the Euler characteristic

$$\chi(\mathcal{L}) := \sum_{i \geq 0} (-1)^i h^i(\mathcal{L}).$$

This computes  $h^0(\mathcal{L})$  itself in many cases, by virtue of the following result:

**Theorem 1.1.** (*Serre-Grothendieck vanishing theorem*) *If  $\mathcal{F}$  is a coherent sheaf on a projective scheme  $X$  of dimension  $n$ , then for any  $i$ , the vector space  $H^i(\mathcal{F})$  is finite dimensional, and is 0 if  $i > n$ . Moreover, if  $X \subset \mathbb{P}^m$  then for  $d \gg 0$ ,  $\mathcal{F}(d)$  is generated by its global sections and  $H^i(\mathcal{F}(d)) = 0$  for all  $i > 0$ .*  $\square$

A deficit of this vanishing theorem is the lack of a bound on the number  $d$  needed to achieve the second assertion. For smooth curves and invertible sheaves this is corrected by Theorem 3.2, which gives a bound in terms of the genus and the degree.

**Proof.** This is a combination of Theorems [Hartshorne 1977, Theorems III.2.7 and III.5.2], due to Grothendieck and Serre; see also [Serre 1955] for a relatively concrete proof.  $\square$

One immediate consequence of Theorem 1.1 is that on a smooth variety the groups of invertible sheaves and divisor classes are the same:

**Corollary 1.2.** *If  $X$  is a projective variety that is nonsingular in codimension 1, then every invertible sheaf  $\mathcal{L}$  on  $X$  is of the form  $\mathcal{L} = \mathcal{O}_C(D)$  for some Cartier divisor  $D$  on  $X$ . Thus if  $X$  is a smooth projective variety the map  $\div$  is an isomorphism from the group of invertible sheaves to the group of divisor classes.*

**Proof.** Let  $H \subset \mathbb{P}^r$  be a general hyperplane, and  $E$  the divisor of intersection of  $C$  with  $H$ . We know that for  $n \gg 0$ ,  $\mathcal{L}(n)$  has sections; and if  $F$  is the divisor of zeroes of one such section, we have

$$\mathcal{L} = \mathcal{O}_C(F - nE).$$

If  $X$  is smooth, then, since a regular local ring is a unique factorization domain, every codimension 1 subvariety is defined locally by a single non-zerodivisor, and thus corresponds to a Cartier divisor. This implies that  $\div$  is surjective. Furthermore any isomorphism between invertible sheaves is defined by multiplication with a global rational function, so that invertible sheaves defining linearly equivalent divisors are isomorphic. Thus  $\div$  is injective as well.  $\square$

**1.1. Riemann-Roch without duality.** It follows from Theorem 1.1 that on any scheme  $X \subset \mathbb{P}^r$  we have  $\chi(\mathcal{L}(d)) = h^0(\mathcal{L}(d))$  for large  $d$ , and in the case of a curve  $C \subset \mathbb{P}^r$ , we have  $\chi(\mathcal{L}) = h^0(\mathcal{L}) - h^1(\mathcal{L})$ .

**Theorem 1.3.** *(Easy Riemann-Roch) If  $C$  is a smooth curve, and  $\mathcal{L}$  is an invertible sheaf on  $C$ , then  $\chi(\mathcal{L}) = \deg \mathcal{L} + \chi(\mathcal{O}_C)$ .*

**Proof.** The result is tautological if  $\mathcal{L} = \mathcal{O}_C$ . Every invertible sheaf on  $C$  has the form  $\mathcal{L} = \mathcal{O}_C(D)$  for some divisor  $D$ . If  $p \in C$ , then writing  $\kappa(p)$  for the 1-dimensional skyscraper sheaf at  $p$ , the long exact sequence in cohomology associated to the short exact sequence

$$0 \rightarrow \mathcal{L}(-p) \rightarrow \mathcal{L} \rightarrow \mathcal{L} \otimes \kappa(p) \rightarrow 0$$

together with the isomorphism  $\mathcal{L} \otimes \kappa(p) \cong \kappa(p)$  and the vanishing of higher cohomology of a sheaf with zero-dimensional support allows us to compute

$$\chi(\mathcal{L}) = \chi(\mathcal{L}(-p)) + \chi(\kappa(p)) = \chi(\mathcal{L}(-p)) + 1.$$

Since every divisor on  $C$  can be reached by adding and subtracting points, this suffices.  $\square$

Since the Euler characteristic of a sheaf is well-behaved, we can extend the result of Theorem 1.3 to invertible sheaves on any 1-dimensional scheme  $C$ , by

defining  $\deg \mathcal{L} := \chi(\mathcal{L}) - \chi(\mathcal{O}_C)$ . We will use this definition to express the self-intersection of a divisor on a surface in Section 6.

We can make the Riemann-Roch theorem still more useful by understanding the error term  $h^1(\mathcal{L})$ . This requires the canonical divisor and Serre duality, to which we now turn.

## 2. The most interesting linear series

The most important vector bundles on a manifold are the tangent and cotangent bundles. For reasons that will become clear, the focus in algebraic geometry is on the cotangent bundle or, equivalently, the sheaf of differential 1-forms. On a smooth curve  $C$  the *canonical sheaf* is the sheaf of differentials, which is an invertible sheaf; on a smooth variety of dimension  $n$  we define the canonical sheaf to be the  $n$ th exterior power of the sheaf of differentials. A section of  $\omega_C$  is thus a differential form, and the class of the divisor of such a form is usually denoted  $K_C$ .

**Cheerful Fact 2.1.** *Canonical sheaves are defined for any projective scheme; see Section 5.1. They are usually called “dualizing sheaves” in that generality. The condition for the dualizing sheaf to be an invertible sheaf is that the scheme is (locally) Gorenstein, something that is true, for example, for any subscheme of  $\mathbb{P}^r$  that is locally a complete intersection (see Section 6).*

On projective space we can compute the canonical sheaf directly; other computations of the canonical sheaf will usually reduce to this central case.

**Theorem 2.2.** *The canonical sheaf of  $\mathbb{P}^r$  is  $\mathcal{O}_{\mathbb{P}^r}(-r - 1)$ .*

**Proof.** Let  $x_0, \dots, x_r$  be the projective coordinates on  $\mathbb{P}^r$  and let  $U = \mathbb{P}^r \setminus H$  be the affine open set where  $x_0 \neq 0$ . Thus  $U \cong \mathbb{A}^r$  with coordinates  $z_1 := x_1/x_0, \dots, z_r := x_r/x_0$ . The space of  $r$ -dimensional differential forms on  $U$  is spanned by  $d(x_1/x_0) \wedge \dots \wedge d(x_r/x_0)$ , which is regular everywhere in  $U$ . In view of the formula

$$d \frac{x_i}{x_0} = \frac{x_0 dx_i - x_i dx_0}{x_0^2}$$

we get

$$d(x_1/x_0) \wedge \dots \wedge d(x_r/x_0) = \frac{dx_1 \wedge \dots \wedge dx_r}{x_0^r} - \sum_{i=1}^r x_i \frac{dx_1 \wedge \dots \wedge \widehat{dx_i} \wedge \dots \wedge dx_r}{x_0^{r+1}}$$

which has a pole of order  $r + 1$  along the locus  $H$  defined by  $x_0$ . Thus the divisor of this differential form is  $-(r + 1)H$ , and this is the canonical class.  $\square$

**Cheerful Fact 2.3.** *A different derivation: there is a short exact sequence of sheaves, called the Euler sequence ([Hartshorne 1977, Ch. II.8])*

$$0 \rightarrow \Omega_{\mathbb{P}^r} \rightarrow \mathcal{O}_{\mathbb{P}^r}^{r+1}(-1) \rightarrow \mathcal{O}_{\mathbb{P}^r} \rightarrow 0.$$

*Summing over all twists, and taking global sections, that is, applying  $H_*^0$ , we see that  $H_*^0(\Omega_{\mathbb{P}^r})$  fits into an exact sequence:*

$$0 \rightarrow H_*^0(\Omega_{\mathbb{P}^r}) \longrightarrow S^{r+1}(-1) \xrightarrow{\delta_1} S \longrightarrow \mathbb{C} \rightarrow 0,$$

*where  $S$  is the homogeneous coordinate ring of  $\mathbb{P}^r$  and  $\delta_1$  sends the  $i$ -th basis vector of  $S^{r+1}(-1)$  to the  $i$ -th variable of  $S$ ; that is,  $H_*^0(\Omega_{\mathbb{P}^r})$  is the second syzygy of the residue field  $\mathbb{C}$  of  $S$ . We can extend this sequence to the free resolution of  $\mathbb{C}$ , the Koszul complex ([Eisenbud 1995a, Ch. 17.5]):*

$$0 \rightarrow S(-r-1) \xrightarrow{\delta_{r+1}} \bigwedge^r S^{r+1}(-r) \xrightarrow{\delta_r} \dots \longrightarrow S^{r+1}(-1) \longrightarrow S \rightarrow \mathbb{C} \rightarrow 0.$$

*For each  $i$ , the  $i$ -th exterior power of the map  $H_*^0(\Omega_{\mathbb{P}^r}) \longrightarrow S^{r+1}(-1)$  is an inclusion, and represents  $\bigwedge^i(\Omega_{\mathbb{P}^r})$  as the sheaf associated to the graded module that is the  $(i+1)$ -st syzygy of  $\mathbb{C}$ . In particular, the canonical module  $\omega_{\mathbb{P}^r} = \bigwedge^r(\Omega_{\mathbb{P}^r})$  is the sheaf associated to the  $(r+1)$ -st syzygy,  $S(-r-1)$ .*

*For more on syzygies, see Chapter 18.*

The most important invariant of a smooth curve can be defined in terms of the canonical sheaf:

**Definition 2.4.** If  $C$  is an irreducible smooth curve we define the genus  $g(C)$  to be the dimension of  $H^0(\omega_C)$ .

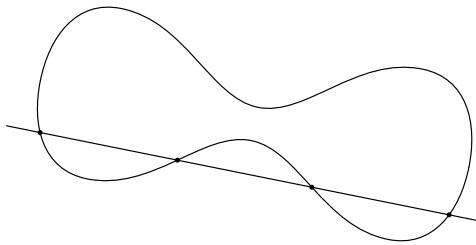
Computations of the canonical sheaf on a variety usually involve comparing the variety to a variety where the canonical sheaf is already known. The most useful results of this type are the *adjunction formula* and *Hurwitz' theorem*.

**2.1. The adjunction formula.** The adjunction formula describes the difference between the canonical divisor of a subscheme and the restriction of the canonical divisor from the ambient variety. If  $X \subset Y$  we define the *conormal sheaf* of  $X$  in  $Y$  to be  $\mathcal{I}_{X/Y}/\mathcal{I}_{X/Y}^2$ , and the *normal sheaf* of  $X$  in  $Y$  to be its dual,

$$\mathcal{N}_{X/Y} = \mathcal{H}\text{om}(\mathcal{I}_{X/Y}/\mathcal{I}_{X/Y}^2, \mathcal{O}_Y).$$

If  $X$  and  $Y$  are smooth then  $X$  is locally a complete intersection in  $Y$ , so  $\mathcal{I}_{X/Y}/\mathcal{I}_{X/Y}^2$  is a vector bundle on  $X$  of rank equal to the codimension,  $\dim Y - \dim X$ . When, in addition, the codimension is 1, so that  $X$  is a divisor and  $\mathcal{I}_X = \mathcal{O}_Y(-X)$ , we get

$$\mathcal{N}_{X/Y} = \mathcal{O}_X(X).$$



**Figure 1.** On a smooth plane quartic, the canonical divisors are its intersections with lines.

**Proposition 2.5.** (adjunction formula) Let  $X \subset Y$  a smooth subscheme of codimension  $c$  in a smooth variety  $Y$ , and let  $K_Y$  be the canonical class of  $Y$ . The canonical class  $K_X$  of  $X$  is

$$\omega_X = \bigwedge^c \mathcal{N}_{X/Y} \otimes \omega_Y.$$

In particular, when  $X$  is a divisor,  $K_X$  is the restriction to  $X$  of the divisor  $K_Y + X$  on  $Y$ .

**Proof.** Because  $X$  is locally a complete intersection in  $Y$  there is an exact sequence of sheaves

$$0 \rightarrow \mathcal{I}_{X/Y}/\mathcal{I}_{X/Y}^2 \rightarrow \Omega_Y|_X \rightarrow \Omega_X \rightarrow 0$$

where  $\Omega_X$  is the sheaf of differential forms on  $X$  (see [Eisenbud 1995b, Proposition 16.3]), and  $\mathcal{I}_{X/Y}|_X = \mathcal{O}_Y(-X)|_X = \mathcal{O}_X(-X)$ . The proposition follows by taking top exterior powers, as in Lemma 2.6.  $\square$

**Lemma 2.6.** If

$$0 \rightarrow \mathcal{E} \rightarrow \mathcal{F} \rightarrow \mathcal{G} \rightarrow 0$$

is a short exact sequence of locally free sheaves of ranks  $e, f, g$  respectively on a scheme  $X$  then there is a natural isomorphism

$$\wedge^e \mathcal{E} \otimes \wedge^g \mathcal{G} \rightarrow \wedge^f \mathcal{F}.$$

**Proof of Lemma 2.6.** We may define a map  $\wedge^e \mathcal{E} \otimes \wedge^g \mathcal{G} \rightarrow \wedge^f \mathcal{F}$  in terms of local sections as

$$(\epsilon_1 \wedge \cdots \wedge \epsilon_e) \otimes (\gamma_1 \wedge \cdots \wedge \gamma_g) \mapsto \epsilon_1 \wedge \cdots \wedge \epsilon_e \wedge \gamma_1 \wedge \cdots \wedge \gamma_g.$$

This is globally well-defined because changing one of the  $\gamma_i$  by a local section of  $\mathcal{E}$  would not change the exterior product. To check that the map is an isomorphism, it is enough to show that this is true locally.

Because  $\mathcal{G}$  is locally free, there is a covering of  $X$  by open sets  $U$  so that the sequence

$$0 \rightarrow \mathcal{E}|_U \rightarrow \mathcal{F}|_U \rightarrow \mathcal{G}|_U \rightarrow 0$$

is a split exact sequence of free modules,  $\mathcal{F}|_U = \mathcal{E}|_U \oplus \mathcal{G}|_U$ . It follows that

$$\bigwedge^f \mathcal{F}|_U = \bigoplus_{i+j=f} \bigwedge^i \mathcal{E}|_U \otimes \bigwedge^j \mathcal{G}|_U.$$

In our case all the exterior powers of  $\mathcal{E}$  vanish above the  $e$ -th, and all the exterior powers of  $\mathcal{G}$  vanish above the  $g$ -th, so

$$\bigwedge^f \mathcal{F}|_U = \wedge^e \mathcal{E}|_U \otimes \bigwedge^g \mathcal{G}|_U,$$

with isomorphism given as above.  $\square$

**Corollary 2.7.** *If  $C \subset \mathbb{P}^2$  is a smooth plane curve of degree  $d$ , then  $\omega_C = \mathcal{O}_C(d-3)$ ; more generally, if  $X \subset \mathbb{P}^r$  is a smooth complete intersection of hypersurfaces of degrees  $d_1, \dots, d_c$  in  $\mathbb{P}^r$  then  $\omega_X = \mathcal{O}_X(\sum d_i - r - 1)$ .*

**Proof.** Since  $\mathcal{N}_{X/Y} = \bigoplus_{i=1}^c \mathcal{O}_X(d_i)$ , the result follows from Theorem 2.5.  $\square$

**2.2. Hurwitz' theorem.** Given a (nonconstant) morphism  $f : C \rightarrow X$  of smooth projective curves, the Riemann-Hurwitz formula computes the canonical sheaf  $C$  in terms of that of  $X$  and the local geometry of  $f$ . To do this we define the *ramification index* of  $f$  at  $p \in C$ , denoted  $\text{ram}(f, p)$ , by the formula

$$f^{-1}(q) = \sum_{p \in C | f(p)=q} (\text{ram}(f, p) + 1) \cdot p$$

for any point  $q \in X$ .

**Proposition 2.8.** *If  $f : C \rightarrow X$  is a (nonconstant) morphism of smooth projective curves then there are only finitely many points  $p \in C$  such that  $\text{ram}(f, p) > 0$ .*

In light of this result we define the *ramification divisor* of  $f$  to be the divisor

$$R = \sum_{p \in C} \text{ram}(f, p) \cdot p \in \text{Div}(C).$$

and the *branch divisor* to be

$$B = \sum_{q \in X} \left( \sum_{p \in f^{-1}(q)} \text{ram}(f, p) \right) \cdot q \in \text{Div}(X).$$

Note that  $R$  and  $B$  have the same degree  $\sum_{p \in C} \text{ram}(f, p)$ .

**Proof.** The result follows from the separability of the map of fields of rational functions,  $K(X) \rightarrow K(C)$ , which holds because we are in characteristic 0 (in characteristic  $p$  the Frobenius map provides a counterexample). A proof using separability is given in [Hartshorne 1977, Section IV.2]. Here is an analytic version:

In terms of local parameters  $z$  on  $C$  around  $p$  and  $w$  on  $X$  around  $f(p)$ , we can write the morphism as  $z \mapsto w = z^m$  for some integer  $m > 0$ ; that is, if  $w$  is a local parameter on  $X$  and  $z$  is a local parameter in the source, then the map of convergent power series rings induced by  $f^*$

$$\mathbb{C}\{\{w\}\} \cong \widehat{\mathcal{O}}_{X,f(p)} \xrightarrow{\widehat{f}^*} \widehat{\mathcal{O}}_{C,p} \cong \mathbb{C}\{\{z\}\}$$

sends  $w$  to  $uz^m$ , where  $u$  is a power series with nonvanishing constant term. In this case  $\text{ram}(f, p) = m - 1$ . These power series expansions are valid in a neighborhood of  $p$ , and the derivative of  $f$  vanishes at the ramification points in this neighborhood. Since the zeros of a nonconstant analytic function are isolated, the ramification points are isolated. Since  $C$  is compact in the classical topology, there are only finitely many.  $\square$

Hurwitz' theorem describes the difference between the canonical divisor of  $C$  and the pullback of the canonical divisor of  $X$ .

**Theorem 2.9.** (*Hurwitz' theorem*) [Hartshorne 1977, Proposition IV.2.3] *If  $f : C \rightarrow X$  is a nonconstant morphism of smooth curves, with ramification divisor  $R$ , then*

$$K_C = f^*(K_X) + R,$$

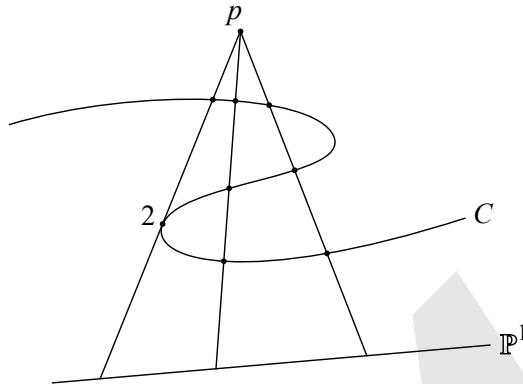
or equivalently  $\omega_C = (f^*\omega_X)(R)$ .

**Proof.** Choose a rational 1-form  $\omega$  on  $X$ , and let  $\eta = f^*(\omega)$  be its pullback to  $C$ . For simplicity, we will assume that the zeroes and poles of  $\omega$  lie outside the branch divisor  $B$ , so that  $\omega$  will be regular and nonzero at each branch point. (Since we have the freedom to multiply by any rational function on  $X$  we can certainly find such a form, and in any event the calculation goes through without this assumption, albeit with more complicated notation.)

Since the zeroes of  $\omega$  lie outside the branch divisor  $B$ , for every zero of  $\omega$  of multiplicity  $m$  we have exactly  $d$  zeroes of  $\eta$ , each with multiplicity  $m$ ; and likewise for the poles of  $\omega$ . Meanwhile, at every point of  $B$ , the form  $\omega$  is regular and nonzero. At a point  $p$  where (locally)  $f$  has the form  $z \mapsto w = z^e$  and  $\omega = dw$ , we have  $\eta = z^{e-1}dz$ ; that is  $\eta$  has a zero of multiplicity  $\text{ram}(f, p)$  at  $p$ . Thus the divisor  $K_C$  of  $\eta$  is  $K_C = f^*(K_X) + R$ .  $\square$

**Example 2.10.** Let  $C \subset \mathbb{P}^2$  be a smooth plane curve and let  $p$  be a point of  $\mathbb{P}^2$  not on  $C$ . Suppose that the coordinates on  $\mathbb{P}^2$  are chosen so that the ideal sheaf of  $p$  is generated by the vector space of linear forms  $W = \langle x_0, x_1 \rangle$ . The linear series  $(\mathcal{O}_C(1), W)$  defines the projection of  $C$  from  $p$  to  $\mathbb{P}^1$ , a map of degree  $d = \deg C$ .

The canonical sheaf of  $\mathbb{P}^1$  has degree  $-2$ , so by Hurwitz' theorem  $K_C$  has degree  $-2d + \deg R$ , where  $R$  is the branch locus. We may choose coordinates so that none of the branch points lie on the line  $x_0 = 0$ . Taking this to be the



**Figure 2.** Projection of a plane cubic from a general point  $p$  to  $\mathbb{P}^1$  is a three-to-one map. Silvio: put in correct style. also  $\mathbb{P}'$  should be  $\mathbb{P}^1$

line at infinity, we may compute the branch locus after passing to the affine open set  $x_0 \neq 0$ , where the projection map is given by the function  $z = x_1/x_0$ . Suppose that  $C$  is defined, in this open set, by the equation  $f(x, y) = 0$ . A point  $q \in C$  is a branch point if the tangent line to  $C$  at  $q$  passes through  $p$ , that is, if  $dx$  and

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$$

are linearly dependent, that is, if  $\partial f / \partial y$  vanishes at  $q$ . The intersection of  $C$  with the curve defined by  $\partial f / \partial y = 0$  has degree  $d(d - 1)$  by Bézout's theorem, so the degree of the ramification divisor  $R$  is  $d(d - 1)$  whence the degree of the canonical divisor on  $C$  is  $\deg K_C = -2d + d(d - 1) = d(d - 3)$ , which is in accord with Corollary 2.7.

**Example 2.11.** Let  $V$  be the vector space of homogeneous polynomials of degree  $d$  in two variables; that is,  $V = H^0(\mathcal{O}_{\mathbb{P}^1}(d))$ . In the projectivization  $\mathbb{P}(V^*) \cong \mathbb{P}^d$ , let  $\Delta$  be the locus of polynomials with a repeated factor. Since  $\Delta$  is defined by the vanishing of the discriminant, it is a hypersurface. What is its degree?

To answer this, we intersect  $\Delta$  with a general line; the degree of  $\Delta$  is the degree of the intersection. Let  $W \subset V$  be a general 2-dimensional linear subspace—that is, a general pencil of forms of degree  $d$  on  $\mathbb{P}^1$ . The linear series  $\mathcal{W} = (\mathcal{O}_{\mathbb{P}^1}, W)$  defines a morphism  $\phi_{\mathcal{W}} : \mathbb{P}^1 \rightarrow \mathbb{P}(W) \cong \mathbb{P}^1$  and the fiber over the point of  $\mathbb{P}(W)$  corresponding to a form  $f$  of degree  $d$  is the divisor  $\{f = 0\} \subset \mathbb{P}^1$ . Thus the intersection of  $\Delta$  with the line is the locus of polynomials in  $W$  with a multiple root; that is, the branch locus of  $\phi_{\mathcal{W}}$ , where we would count an  $m$ -fold root  $m - 1$  times if there were multiple roots. By Hurwitz' formula, the degree of the branch locus  $B$  of a degree  $d$  morphism from  $\mathbb{P}^1$  to  $\mathbb{P}^1$  is

$$\deg B = \deg \omega_{\mathbb{P}^1} - d \deg \omega_{\mathbb{P}^1} = 2d - 2.$$

Thus  $\deg \Delta = 2d - 2$ .

### 3. Riemann-Roch with duality

We now return to the task of understanding  $h^0(\mathcal{L})$  for an invertible sheaf  $\mathcal{L}$  on a smooth curve. Since  $\chi(\mathcal{L}) = h^0(\mathcal{L}) - h^1(\mathcal{L})$  is easier to compute, we would like to understand  $h^1(\mathcal{L})$  in a more concrete way. The key is duality:

**Theorem 3.1** (Serre duality). *If  $C$  is a smooth curve and  $D$  is a divisor on  $C$ , then*

$$H^1(D) = H^0(K_C - D)^* := \text{Hom}_{\mathbb{C}}(H^0(K_C - D), \mathbb{C}),$$

and thus  $h^1(D) = h^0(K_C - D)$ .

For proofs see [Hartshorne 1977, Theorem III.5.2 and III.7.6].

For example we see that if  $C$  is a smooth connected curve then  $h^1(\mathcal{O}_C) = h^0(K_C) = g(C)$  and thus  $\chi(\mathcal{O}_C) = 1 - g(C)$ . Using this we can recast the Easy Riemann-Roch theorem above in a more useful form.

**Theorem 3.2.** (Riemann-Roch) *If  $D$  is any divisor on  $C$ , then*

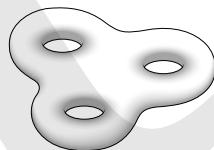
$$h^0(D) - h^0(K_C - D) = \deg D - g(C) + 1.$$

See Sections 5.1 and Theorem 9.5 for the corresponding results on singular curves.

**Proof.** Combine Theorem 1.3 with Theorem 3.1. □

Applying the formula with  $D = K_C$ , we get  $\deg K_C = 2g(C) - 2$ .

We can now explain the relationship between the genus of a smooth curve, as we have defined it and the topological genus, the “number of holes” in the Riemann surface:



**Figure 3.** Silvio: An image that fits the style of other images would be nicer...A Riemann surface of genus 3 (from Wikimedia).

**Cheerful Fact 3.3.** (Hodge theory) *The sole topological invariant of a smooth projective curve  $C$ , viewed as an analytic space, is its genus. As a manifold it is a compact, oriented surface, and its genus is half the rank of its first singular cohomology,  $H^1(C; \mathbb{C})$ , which is equal to its first deRham cohomology. Breaking up the deRham cohomology of any smooth projective complex variety  $X$  in terms of*

holomorphic and antiholomorphic differential forms we get the Hodge decomposition

$$H^i(X, \mathbb{C}) = H_{deRham}^i(X) = \bigoplus_{j=0}^i H^j(\bigwedge^{i-j} \Omega_X).$$

For a smooth curve  $C$ , this says in particular that

$$H^1(C; \mathbb{C}) = H^0(\omega_C) \oplus H^1(\mathcal{O}_C) = H^0(\omega_C) \oplus (H^0(\omega_C))^\vee,$$

so  $h^0(\omega_C)$  is half the rank of the middle singular cohomology group, justifying the name “genus”. For details, see [Griffiths and Harris 1978, p. 116]

If  $E$  is a divisor of negative degree then  $H^0(E) = 0$  so we get the form of the Riemann-Roch theorem originally proved by Riemann:

**Corollary 3.4.** *For any divisor  $D$  of degree  $d$  we have*

$$h^0(D) \geq d - g + 1,$$

with equality if  $d > 2g - 2$ .

It was Riemann’s student Roch who supplied the correction term  $h^0(K_C - D)$  for divisors of lower degree. The dimension  $h^0(K_C - D) = h^1(D)$  was called the *superabundance* of  $D$ : the “expected” number of sections was  $d - g + 1$ , and  $h^1(\mathcal{L})$  reflected how much larger the actual number was.

Corollary 3.4 and Proposition 4.5 together show that all high degree divisors come from hyperplane sections in suitable embeddings; and unlike the general vanishing theorems, they give a bound on the degree necessary for vanishing of cohomology and for global generation:

**Corollary 3.5.** *Let  $D$  be a divisor of degree  $d$  on a smooth, connected projective curve of genus  $g$ .*

- (1) *If  $d > 2g - 2$  then  $H^1(\mathcal{O}_C(D)) = 0$ .*
- (2) *If  $d \geq 2g$  then  $\mathcal{O}_C(D)$  is generated by global sections; that is, the complete linear series  $|D|$  is basepoint free; and  $\mathcal{O}_C(D)$  is very ample unless  $D = K_C + E$  for some divisor  $E$  of degree 2.*
- (3) *If  $d \geq 2g + 1$  then  $\mathcal{O}_C(D)$  is very ample; that is, the associated morphism  $\phi_D : C \rightarrow \mathbb{P}^{d-g}$  is an embedding, and  $D$  is the preimage of the intersection of  $C$  with a hyperplane in  $\mathbb{P}^{d-g}$ .*

**Proof.** If  $d > 2g - 2$  then  $K - D$  has negative degree, and thus  $h^1(D) = h^0(K - D) = 0$ . The last two parts follow from Proposition 4.5 and the Riemann-Roch theorem.  $\square$

Since the complement of a hyperplane in projective space is an affine space, we get an affine embedding result too:

**Corollary 3.6.** *If  $C$  is any smooth projective curve and  $\Gamma \subset C$  a nonempty finite subset then  $C \setminus \Gamma$  is affine (that is, isomorphic to a closed subscheme of an affine space.)*

**Proof.** Let  $D$  be the divisor defined by  $\Gamma$ . By Corollary 3.5 a high multiple of  $D$  is very ample, and gives an embedding  $\phi : C \rightarrow \mathbb{P}^n$  such that the preimage of the intersection of  $C$  with some hyperplane  $H$  is a multiple of  $D$ . It follows that  $C \setminus \Gamma$  is embedded in  $\mathbb{A}^n = \mathbb{P}^n \setminus H$ .  $\square$

We can use Corollary 3.4 to determine the Hilbert polynomial of a projective curve. To do this, let  $C \subset \mathbb{P}^r$  be a smooth curve of degree  $d$  and genus  $g$ , and consider the exact sequence of sheaves

$$0 \longrightarrow \mathcal{I}_{C/\mathbb{P}^r}(m) \longrightarrow \mathcal{O}_{\mathbb{P}^r}(m) \longrightarrow \mathcal{O}_C(m) \longrightarrow 0$$

and the corresponding exact sequence

$$H^0(\mathcal{O}_{\mathbb{P}^r}(m)) \xrightarrow{\rho_m} H^0(\mathcal{O}_C(m)) \longrightarrow H^1(\mathcal{I}_{C/\mathbb{P}^r}(m)) \longrightarrow 0.$$

The *Hilbert function*  $h_C$  of  $C$  is defined in terms of the homogeneous coordinate ring  $R_C$  of  $C$  by:

$$h_C(m) = \dim_{\mathbb{C}}(R_C)_m = \text{rank}(\rho_m),$$

where  $(R_C)_m$  is the degree  $m$  component of the homogeneous coordinate ring of  $C$  in  $\mathbb{P}^n$ .

By Theorem 1.1 we have  $H^1(\mathcal{I}_{C/\mathbb{P}^r}(m)) = 0$  for large  $m$ , so  $h_C(m) = h^0(\mathcal{O}_C(m))$  for large  $m$ , which, by the Riemann-Roch theorem, equals  $dm - g + 1$ , again for large  $m$ . Thus, the Hilbert polynomial of  $C \subset \mathbb{P}^r$  is  $p_C(m) = dm - g + 1$ . More generally, we define what is called the *arithmetic genus*:

**Definition 3.7.** If  $C \subset \mathbb{P}^n$  is a 1-dimensional projective scheme with Hilbert polynomial  $p_C(m) = \chi(\mathcal{O}_C(m))$ , the *arithmetic genus* of  $p_a(C)$  is  $1 - \chi(\mathcal{O}_C) = 1 - p_C(0)$ . If  $C$  is reduced and irreducible, then the *geometric genus*  $g(C)$  is the genus of the normalization of  $C$ .

We see from the Riemann-Roch theorem that if  $C$  is smooth and connected, then  $p_a(C) = g(C) = h^0(\omega_C)$ , the genus of  $C$ . We will see that for reduced and irreducible curves  $p_a(C) \geq g(C)$ , with equality only when  $C$  is smooth. For some examples with curves that are not reduced and irreducible, see Exercise 7.8.

The Riemann-Roch theorem and Serre duality have extensions to arbitrary coherent sheaves in place of invertible sheaves and to singular curves, which we will explain in Chapter 16.

Divisors  $D$  for which  $h^0(K_C - D) > 0$  are called *special divisors*. The existence or nonexistence of divisors  $D$  with given  $h^0(D)$  and  $h^1(D)$  often serves to distinguish one curve from another, and will be an important part of our study.

**3.1. Residues, and a complex-analytic argument for the Riemann-Roch theorem.** The Riemann-Roch theorem is so central to the study of curves that it is worth understanding from another point of view as well. We remarked at the beginning of Chapter 1 that a smooth projective curve over  $\mathbb{C}$  is the same thing as a compact Riemann surface. In this section, we'll adopt the complex analytic viewpoint, and give an explanation of the Riemann-Roch theorem (Theorem 3.2) in a special case. If  $D = \sum a_i p_i$  is an effective divisor on a compact Riemann surface  $X$  then we write  $L(D)$  for the vector space of meromorphic functions on  $X$  with poles of order  $\leq a_i$  at  $p_i$ .

**Theorem 3.8.** *Let  $X$  be a compact Riemann surface of genus  $g$ , and let  $D$  be an effective divisor of degree  $d$  on  $X$ . Suppose that  $K - D$  is also effective for some canonical divisor  $K$ . The dimension of  $L(D)$  is  $d - g + 1 + \dim_{\mathbb{C}} L(K - D)$ .*

Because meromorphic functions on a Riemann surface are rational functions on the corresponding algebraic curve, and  $L(D) = H^0(\mathcal{O}_C(D))$ , the assertion is equivalent to Theorem 3.2.

**Proof.** Recall that the *residue* of a meromorphic 1-form  $\phi$  at a point  $p$  on  $X$  is defined by an integral: choose a disc  $\Delta \subset X$  containing  $p$  and in which  $\phi$  is holomorphic except for its pole at  $p$ . The residue  $\text{Res}_p(\phi)$  is  $\frac{1}{2\pi i}$  times the integral of  $\phi$  along the boundary of  $\Delta$ . If  $z$  is a local coordinate on  $\Delta$  zero at  $p$  and we write the differential  $\phi$  as

$$\phi = \sum_{i=-n}^{\infty} a_i z^i dz$$

then by Cauchy's formula, the residue of  $\phi$  at  $p$  is the coefficient  $a_{-1}$ .

**Proposition 3.9.** *If  $\phi$  is a meromorphic differential on a compact Riemann surface  $X$ , then the sum of the residues of  $\phi$  at all the poles of  $\phi$  is 0.*

**Proof.** Apply Stokes' theorem to the complement of the union of small discs around each of the poles of  $\phi$ .  $\square$

Let  $D = \sum a_i p_i$  be an effective divisor on  $X$ , and set  $d = \sum a_i = \deg D$ . Locally, a function with a pole of order at most  $n$  at  $p$  may be written in terms of a local coordinate  $z$  at  $p$  as  $\sum_{i=-n}^{\infty} a_i z^i$ ; the sum  $\sum_{i=-n}^{-1} a_i z^i$  is called its *polar part*. By the maximum principle, a meromorphic function in  $L(D)$  is determined, up to the addition of a constant, by its polar parts at the points  $p_i$ . Thus we have  $\dim L(D) \leq d + 1$ .

When is a given collection  $c_1, \dots, c_d \in \mathbb{C}[z^{-1}]$  the polar parts of a global meromorphic function  $f$  on  $X$ ? A necessary condition is that if  $\phi \in L(K)$  is a holomorphic differential on  $X$ , then

$$\sum Res_{p_i}(f \cdot \phi) = 0.$$

This gives  $g$  linear relations on the  $c_i$ . However, if  $\phi$  is a holomorphic differential vanishing at all the points  $p_i$  then the corresponding relation is trivial. Thus the number of linearly independent relations on the polar parts of  $f$  is actually  $g - \dim L(K - D)$ ; and we arrive at an inequality

$$\dim L(D) \leq d + 1 - g + \dim L(K - D).$$

This is a priori only an inequality. But we can apply the same logic to an effective divisor  $K - D$ , and we see that

$$\begin{aligned} \dim L(K - D) &\leq \deg(K - D) + 1 - g + L(K - (K - D)) \\ &= 2g - 2 - d + 1 - g + L(D) \\ &= g - d - 1 + L(D) \end{aligned}$$

Adding the two inequalities we have

$$L(D) + L(K - D) \leq L(D) + L(K - D).$$

Since the sum of the inequalities is an equality, we conclude that each inequality is also an equality; this is the Riemann-Roch formula in our special case.  $\square$

**3.2. Arithmetic genus and geometric genus.** Throughout this book, we will be primarily concerned with the geometry of smooth curves. Of course singular curves will arise—for example, as images of smooth curves under morphisms that are not embeddings. At least when  $C$  is a 1-dimensional variety (that is, a reduced irreducible 1-dimensional scheme) we can regard  $C$  as the image of a smooth curve in an optimal way (Proposition 3.10).

**Proposition 3.10.** *If  $C_0$  is a projective variety of dimension 1 then the normalization  $\phi : C \rightarrow C_0$  of  $C_0$  is a birational morphism from a smooth curve  $C$ . The curve  $C$  is again projective, and the pair  $(C, \phi)$  is unique up to isomorphism. In particular, every birational map of smooth curves is an isomorphism.*

**Proof.** We use the result that the normalization (= integral closure) of a domain finitely generated over a field is again finitely generated, and nonsingular in codimension 1 [Eisenbud 1995a, Theorem 4.14 and 11.5]. Thus, starting with a projective embedding of  $C$  we normalize the homogeneous coordinate ring of  $C$ . The resulting ring may have generators in many degrees, but a suitable Veronese subring will be generated in a single degree, and thus is the homogeneous coordinate ring of a smooth projective curve.

Since localization commutes with normalization, and any map from a normal ring to a domain factors uniquely through the normalization, the normalization is unique.  $\square$

A different, less canonical procedure for finding a smooth curve birational to a given curve is explained in Exercise ??.

Still assuming that  $C_0$  is reduced and irreducible, we can relate the arithmetic and geometric genera of  $C_0$  using the map of sheaves

$$\mathcal{O}_{C_0} \rightarrow \nu_* \mathcal{O}_C.$$

Since the normalization map of rings is injective and finite, this map is injective. The cokernel  $\mathcal{F}$  is a coherent sheaf supported on the singular points of  $C_0$ , and is thus finite over  $\mathbb{C}$ . The definition implies that there is a short exact sequence

$$0 \rightarrow \mathcal{O}_{C_0} \rightarrow \nu_* \mathcal{O}_C \rightarrow \mathcal{F} \rightarrow 0.$$

**Proposition 3.11.** *Suppose that  $C_0$  is a reduced, irreducible curve and let  $\nu : C \rightarrow C_0$  be its normalization. Let  $\mathcal{F} = \nu_* \mathcal{O}_C / \mathcal{O}_{C_0}$ . If we set  $\delta(C_0) := h^0(\mathcal{F})$ , we have*

$$p_a(C_0) - g(C) = \delta(C_0)$$

**Proof.** Since the normalization map  $\nu : C \rightarrow C_0$  is finite, the direct images  $R^i \nu_* \mathcal{O}_C$  vanish for  $i > 0$ , so the Leray spectral sequence 3.12 computes  $\chi(\nu_* \mathcal{O}_C) = \chi(\mathcal{O}_C)$ . Thus

$$p_a(C_0) - g(C) = \chi(\mathcal{O}_C) - \chi(\mathcal{O}_{C_0}) = \chi(\mathcal{F}) = h^0(\mathcal{F})$$

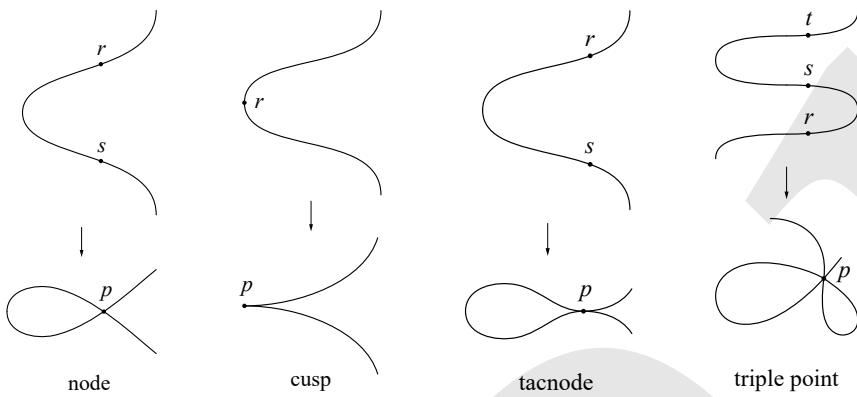
since the support of  $\mathcal{F}$  is finite.  $\square$

The operation of normalization localizes, so we can understand  $\mathcal{F}$  by looking at it locally. Writing  $R$  for the affine ring of some affine subset  $U \subset C_0$  we see that  $\mathcal{O}_C|_U$  is the integral closure  $\bar{R}$  of  $R$ , so  $\mathcal{F}|_U$  is  $\bar{R}/R$ . Thus the annihilator of  $\mathcal{F}|_U$ , called the *conductor*  $\mathfrak{f}_{\bar{R}/R}$  is both an ideal of  $\bar{R}$  and of  $R$ , and correspondingly  $\nu^{-1}(\mathfrak{f}_{C/C_0})$  is also an ideal sheaf in  $\mathcal{O}_C$ .

Informally, we may say that  $\delta(C_0)$  is the number of linear conditions a locally defined function  $f$  on  $C$  has to satisfy to be the pullback of a function from  $C_0$ . The length of the stalk of  $\mathcal{F}$  at a particular singular point  $p \in C_0$  is called the  $\delta$ -*invariant*  $\delta_p$  of the singularity; Thus

$$p_a(C_0) - g(C) = \sum_{p \in (C_0)^{\text{sing}}} \delta_p$$

**Cheerful Fact 3.12.** *In general, the Leray spectral sequence addresses the situation of a morphism  $f : X \rightarrow Y$  of varieties or schemes, and a coherent sheaf  $\mathcal{G}$*



**Figure 4.** Simple planar curve singularities Silvio: easy to make these mathematical?

on the source  $X$ ; it says in this circumstance that there is a spectral sequence

$$H^p(R^q f_*(\mathcal{G})) \Longrightarrow H^{p+q}(\mathcal{G}).$$

(This is a special case of the spectral sequence for the derived functors of a composite functor ( $H^0$  composed with  $f_*$ ); see [[Godement 1973, II.4.17.1](#)] or [[Gelfand and Manin 2003, Section III.7](#)] for proofs.)

In the simplest cases the  $\delta$  invariant is easy to compute:

- (1) (nodes) A *node* of a curve  $C_0$  is a point  $p$  such that an analytic neighborhood of  $p$  in  $C_0$  consists of two smooth arcs intersecting transversely at  $p$ . Equivalently, the completion of the local ring  $\mathcal{O}_{C_0, p}$  is isomorphic to  $k[[x, y]]/(xy)$ . If  $p \in C_0$  is a node, its preimage in the normalization  $C$  of  $C_0$  consists of two points  $r, s \in C$ . The condition for a function  $f$  on  $C$  to descend to  $C_0$ —that is, to be the pullback of a function on  $C_0$ —is that  $f(r) = f(s)$ ; this is one linear condition so  $\delta_p = 1$ .
- (2) (cusps) An *ordinary cusp* of a curve  $C_0$  is a point  $p$  such that an analytic neighborhood of  $p$  in  $C_0$  is given by the equation  $y^2 = x^3$ . If  $p \in C_0$  is an ordinary cusp then its preimage in the normalization  $C$  of  $C_0$  will consist of one point  $r \in C$ . The condition for a function  $f$  on  $C$  is that the derivative  $f'(r) = 0$ ; again, this is one linear condition so  $\delta_p = 1$ .

We will give more examples at the end of Chapter 15.

## 4. The canonical morphism

We now return to the world of smooth curves.

The canonical sheaf on  $\mathbb{P}^1$  has negative degree, so  $|K_{\mathbb{P}^1}| = \emptyset$ . If  $C$  is a curve of genus 1 then the canonical sheaf has a nonzero global section, and since

the sheaf has degree  $2g - 2 = 0$  this is nowhere vanishing, whence  $\omega_C = \mathcal{O}_C$ , and  $K_C = 0$ . Thus in studying the canonical series we restrict our attention to curves  $C$  of genus  $g \geq 2$ .

**Theorem 4.1.** *Let  $C$  be a smooth curve of genus  $\geq 2$ .*

- (1)  $|K_C|$  is basepoint free.
- (2)  $|K_C|$  is very ample if and only if  $C$  admits no map of degree 2 to  $\mathbb{P}^1$ .

A curve of genus  $\geq 2$  is said to be *hyperelliptic* if there exists a morphism  $f : C \rightarrow \mathbb{P}^1$  of degree 2. It is easy to describe this in terms of linear series:

**Lemma 4.2.** *Let  $C$  be a smooth, projective curve of genus  $g \geq 2$ . If  $C$  has an invertible sheaf  $\mathcal{L}$  of degree  $\leq 2$  with two independent sections, then  $\mathcal{L}$  has degree 2 and  $|\mathcal{L}|$  defines a morphism of degree 2 to  $\mathbb{P}^1$ , so  $C$  is hyperelliptic. In particular, if  $g(C) = 2$  then the canonical series  $|K_C|$  defines a 2 to 1 morphism to  $\mathbb{P}^1$ , so  $C$  is hyperelliptic.*

**Proof.** Since  $C \not\cong \mathbb{P}^1$ , Theorem 4.1 shows that  $C$  cannot have a 1-dimensional linear series of degree  $< 2$ . Thus  $\mathcal{L}$  has degree exactly 2, and no basepoints, so it defines a morphism of degree 2 to  $\mathbb{P}^1$  as claimed.  $\square$

**Proof of Theorem 4.1.** A point  $p$  is a basepoint of  $|K_C|$  if and only if  $h^0(K_C - p) = h^0(K_C) = g$ . By the Riemann-Roch theorem, this is equivalent to  $h^0(p) = 2$ , which would imply that  $C \cong \mathbb{P}^1$  by Theorem 4.1. Thus  $K_C$  has no basepoints.

By Proposition 4.5 we have to show that for any pair of points  $p, q \in C$  we have

$$h^0(K_C(-p - q)) = h^0(K_C) - 2 = g - 2.$$

Applying the Riemann-Roch Theorem we see this fails if and only if  $h^0(\mathcal{O}_C(p + q)) \geq 2$  for some  $p, q \in C$ . By Lemma 4.2, this implies that  $C$  is hyperelliptic. Conversely, if  $C$  is hyperelliptic then for some divisor  $D = p + q$  we have  $h^0(D) = 2$ , whence  $h^0(K - p - q) = h^0(K) - 1$  by the Riemann-Roch formula.  $\square$

The image of the canonical morphism of a nonhyperelliptic curve of genus  $g > 2$  is called a *canonical curve*, and (for a non-hyperelliptic curve) we speak of the image as being *canonically embedded*.

**4.1. Geometric Riemann-Roch.** There is a useful way of expressing the Riemann-Roch formula in terms of the geometry of the canonical map. Suppose  $C \subset \mathbb{P}^r$  is a smooth curve and  $D$  an effective divisor on  $C$ , thought of as a subscheme of  $C$ . We define the *span*  $\overline{D}$  of  $D$  to be the intersection of the hyperplanes  $H \subset \mathbb{P}^r$

containing  $D$ . More generally, if  $\phi : C \rightarrow \mathbb{P}^r$  is a map and  $D$  an effective divisor on  $C$ , we define the span  $\overline{\phi(D)} \subset \mathbb{P}^r$  to be the intersection

$$\overline{\phi(D)} := \bigcap_{\{H \subset \mathbb{P}^r \text{ a hyperplane } | D \subset \phi^{-1}(H)\}} H.$$

Now suppose  $C$  is a smooth curve of genus  $g \geq 2$  and  $\phi_K : C \rightarrow \mathbb{P}^{g-1}$  is the canonical morphism. If  $D$  is an effective divisor on  $C$  of degree  $d$ , then since the hyperplanes in  $\mathbb{P}^{g-1}$  containing  $\phi_K(D)$  correspond (up to scalars) to sections of  $K_C$  vanishing on  $D$ , we see that

$$h^0(K_C - D) = \text{codim } \overline{\phi_K(D)} \subset \mathbb{P}^{g-1} = g - 1 - \dim \overline{\phi_K(D)}.$$

Applying the Riemann-Roch theorem we obtain the *geometric Riemann-Roch theorem*:

**Corollary 4.3.** *If  $D$  is a divisor on a smooth curve  $C$  of genus  $\geq 2$  then*

$$r(D) = d - 1 - \dim \overline{\phi_K(D)}.$$

In particular, if  $D = \sum_{i=1}^d p_i$  is a sum of distinct points, then the dimension of the linear series  $|D|$  is equal to the number of linear relations among the images of the  $p_i$  on the canonical image of  $C$ . Thus, for example, a divisor given as the sum  $D = p + q + r$  of three points will move in a pencil if and only if the three points are colinear on the canonical model.

In general, if  $p_a \in \mathbb{P}^r$  is a point then the condition on forms of degree  $d$  to vanish at  $p$  is given by one homogeneous linear equation on the coefficients of the form, obtained by evaluating the variables at the coordinates of  $p$ , so vanishing on a set  $\Gamma$  of  $\gamma$  points is given by  $\gamma$  linear equations and the same is true when  $\Gamma$  is a finite subscheme of length  $\gamma$ , as one sees from the exact sequence

$$0 \longrightarrow H^0(\mathcal{I}_\Gamma(d)) \longrightarrow H^0(\mathcal{O}_{\mathbb{P}^r}(d)) \xrightarrow{ev} H^0(\mathcal{O}_\Gamma(d)) \longrightarrow 0,$$

bearing in mind that

$$H^0(\mathcal{O}_\Gamma(d)) \cong H^0(\mathcal{O}_\Gamma) \cong \mathbb{C}^\gamma.$$

However, the linear equations may be linearly dependent; that is, the map marked  $ev$  above may have rank  $< \gamma$ . We will say that the points of  $\Gamma$  *fail to impose independent conditions on hypersurfaces of degree  $d$*  by the amount equal to  $\gamma - \text{rank}(ev)$ .

With this terminology, the geometric Riemann-Roch theorem says that the dimension of the complete linear series  $|D|$  on a smooth curve  $C$  is the amount by which the points of  $D$  fail to impose independent conditions on canonical divisors, and the geometric version simply translates this into hyperplanes in

the canonical embedding of  $C$ . The Cayley-Bacharach-Macaulay Theorem 2.5 and the result of Exercise ?? are other useful special cases.

**Remark 4.4.** For a singular curve  $C_0$  to have a canonical morphism it is necessary first of all that its canonical sheaf  $\omega_{C_0}$  be invertible; when this is the case, we say that  $C_0$  is (locally) Gorenstein, and then the theory above can be applied. See for example [Bayer and Eisenbud 1991] and [Bayer and Eisenbud 1995] for examples.

**4.2. Linear series on a hyperelliptic curve.** We can describe the canonical map of a hyperelliptic curve—and indeed all its special linear series—quite precisely:

**Corollary 4.5.** *Let  $C$  be a smooth hyperelliptic curve of genus  $g \geq 2$ . The curve  $C$  admits a unique degree 2 morphism  $\pi : C \rightarrow \mathbb{P}^1$ , and the canonical map  $\phi_K : C \rightarrow \mathbb{P}^{g-1}$  is the composition of  $\pi$  with the Veronese map of degree  $g - 1$  from  $\mathbb{P}^1$  to the rational normal curve of degree  $g - 1$ ; in particular, every canonical divisor is a sum of  $g - 1$  fibers of  $\pi$  and vice versa.*

**Proof.** If  $D$  is a general fiber of a degree 2 map  $\pi : C \rightarrow \mathbb{P}^1$  so that  $D = p + q$  and  $r(D) = 1$ , then by the Riemann-Roch theorem or its geometric version,  $D$  imposes only one condition on sections of  $K_C$ ; that is,  $\phi_K(p) = \phi_K(q)$ . Consequently the degree of the canonical morphism  $\phi_K$  is at least 2, and the image is thus a nondegenerate curve in  $\mathbb{P}^{g-1}$  of degree  $\leq (2g - 2)/2 = g - 1$ . By Theorem 4.1, the image of  $\phi_K$  is the rational normal curve of degree  $g - 1$  and every fiber of  $\phi_K$  is a divisor linearly equivalent to  $D$ . It follows that  $\pi$  is determined by  $K$ , so it is unique. Since every canonical divisor is the pullback of a hyperplane section of the rational normal curve, every canonical divisor is a sum of  $g - 1$  fibers of  $\pi$ .  $\square$

We can use these ideas to analyze all special divisors.

**Corollary 4.6.** *Let  $C$  be a smooth hyperelliptic curve with map  $\pi : C \rightarrow \mathbb{P}^1$  of degree 2. Every special divisor  $D$  of degree  $d$  on a hyperelliptic curve is a sum of  $s$  fibers of  $\pi$  and  $d - 2s$  points  $p_1, \dots, p_{d-2s}$  with distinct images under the canonical map; and in this case we have  $r(D) = s$  and the points  $p_i$  are base points of the linear series  $|D|$ .*

In particular, we see that *no divisor on a hyperelliptic curve is both special and very ample*.

**Proof.** Suppose that  $D$  contains the sum  $E$  of  $s$  fibers of  $\pi$  and no more; and write  $D = E + D'$ , with  $\deg D' = d - 2s$ .

If  $\phi_K : C \rightarrow \mathbb{P}^{g-1}$  is the canonical map, then  $\phi_K(D')$  consists of  $d - 2s$  distinct points, while  $\phi_K(E)$  consists of  $s$  points. By Corollary 4.3 the span of

$\phi_K(D)$  has dimension  $\min\{g-1, d-s-1\}$ , so  $r(D) = d-1-\min\{g-1, d-s-1\} = \max\{s, d-g\}$ . Since  $D$  is a special divisor,  $r(D) > d-g$ , so  $r(D) = s$ . Since  $r(E) = s$ , we see that  $E$  is basepoint free, so  $D'$  is the base locus of  $D$ .  $\square$

## 5. Clifford's theorem

While the Riemann-Roch theorem gives a lower bound for the dimension of a linear series,  $r(\mathcal{L}) := h^0(\mathcal{L}) - 1 \geq \deg \mathcal{L} - g$ , Clifford's theorem gives an upper bound. If  $\deg \mathcal{L} > 2g-2$ , then the Riemann-Roch inequality becomes an equality, so it is enough to treat the case  $\deg \mathcal{L} \leq 2g-2$ . The bound is actually a corollary of the Riemann-Roch theorem.

**Corollary 5.1.** *Let  $C$  be a curve of genus  $g$  and  $\mathcal{L}$  a line bundle of degree  $d \leq 2g-2$ . Then*

$$r(\mathcal{L}) \leq \frac{d}{2}.$$

**Proof.** If  $\mathcal{L}$  is nonspecial then, since  $g \geq d/2+1$ , we have  $r(\mathcal{L}) = d-g+1 \leq d/2$ . Otherwise we have

$$r(K_C \otimes \mathcal{L}^{-1}) = r(\mathcal{L}) + g - d - 1$$

by the Riemann-Roch Theorem, and so by Proposition 4.4

$$g = r(K_C) + 1 \geq r(\mathcal{L}) + r(K_C \otimes \mathcal{L}^{-1}) + 1 = 2r(\mathcal{L}) + g - d;$$

hence  $r(\mathcal{L}) \leq d/2$ .  $\square$

The usual statement of Clifford's theorem includes a description of when equality can occur:

**Theorem 5.2.** *Let  $C$  be a curve of genus  $g$  and  $\mathcal{L}$  a line bundle of degree  $d \leq 2g-2$ . If*

$$r(\mathcal{L}) = \frac{d}{2},$$

*the largest possible value, then either*

- (1)  $d = 0$  and  $\mathcal{L} = \mathcal{O}_C$ ;
- (2)  $d = 2g-2$  and  $\mathcal{L} = K_C$ ; or
- (3)  $C$  is hyperelliptic, and  $|\mathcal{L}|$  is a multiple of the  $g_2^1$  on  $C$ .

From Corollary 4.5 and Corollary 4.3 we see that the equality does indeed hold in each of the cases enumerated. See Corollary 3.4 for the converse and [Hartshorne 1977, IV.5.4] for a different proof.

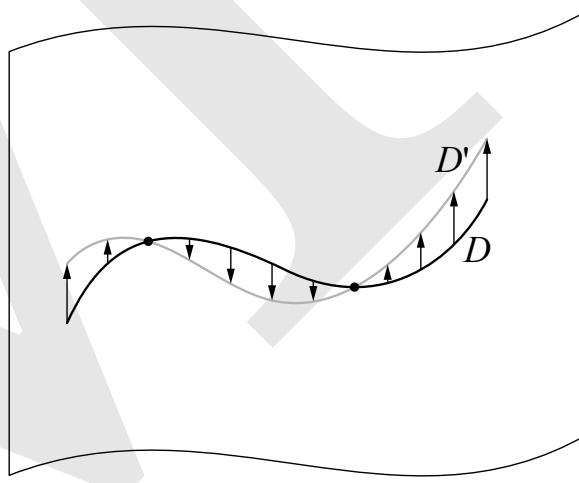
## 6. Curves on surfaces

We will often analyze curves on a smooth surface. Compared to the theory of linear series on curves, there is a new element: the intersection pairing. We refer to [Hartshorne 1977, Chapter V] and [Beauville 1996, Chapter I] for proof of the unproven statements in this section. We suppose for this section that  $S$  is a smooth projective surface, and write  $\text{Pic}(S)$  for the group of invertible sheaves on  $S$ .

**6.0.1. The intersection pairing.** When two codimension 1 subschemes  $D, E$  on  $S$  meet transversely we define  $D \cdot E$  to be the number of points in which they meet. It is less obvious how one might define an intersection number  $D \cdot E$  in more general cases, including the case  $D = E$ . However a codimension 1 subvariety of  $S$  has a fundamental class in  $H^2(X, \mathbb{Z})$ , and the product just defined agrees with the cup product pairing

$$H^2(S, \mathbb{Z}) \times H^2(S, \mathbb{Z}) \rightarrow H^4(S, \mathbb{Z}) \cong \mathbb{Z},$$

showing that it is possible. For example, the self-intersection  $D \cdot D$  can be realized geometrically by choosing a vector field  $\sigma$  on  $D$  and using it to push a copy  $D'$  of  $D$  infinitesimally away from  $D$ ; then one can count the points where  $\sigma$  becomes tangent to  $D$ . Here one must take orientations into account, and the result is that  $D \cdot D$  may be negative. In the previous case (and when  $\sigma$  is complex-analytic) the multiplicities are all positive because a complex subvariety is canonically oriented using the orientation of  $\mathbb{C}$  itself.



**Figure 5.** The result,  $D'$ , of moving  $D$  infinitesimally along a normal vector field meets  $D$  twice.

(( Silvio Let's consider differentiating the “real” curve from the ghost infinitesimally close one ))

This pairing can be realized algebraically as follows:

First, if  $D$  and  $E$  have no common components, then the *intersection multiplicity*  $m_S(D, E, p)$  at  $p \in S$  is defined as the vector space dimension of the local ring  $\mathcal{O}_{S,p}/(\mathcal{I}_{D,p} + \mathcal{I}_{E,p})$ , and

$$D \cdot E = \sum_p m_S(D, E, p).$$

Quite generally, the intersection product can be computed using the (algebraic) Euler characteristic, and this works even for Cartier divisors on a singular surface: Setting  $\mathcal{L} := \mathcal{O}_S(D)$  and  $\mathcal{M} := \mathcal{O}_S(E)$  to simplify the notation, we have

$$D \cdot E = \chi(\mathcal{O}_S) - \chi(\mathcal{L}^{-1}) - \chi(\mathcal{M}^{-1}) + \chi(\mathcal{L}^{-1} \otimes \mathcal{M}^{-1})$$

**Theorem 6.1.** *The pairing  $(D, E) \mapsto (D \cdot E)$  is the unique bilinear map  $\text{Pic}(S) \times \text{Pic}(S) \rightarrow \mathbb{Z}$  extending the case of intersections of two curves meeting transversely at smooth points of  $S$ .*

An important special case is that of the self-intersection  $D \cdot D = D^2$ . The general formula reduces immediately to the case of an effective divisor, and in that case it resembles the  $C^\infty$  construction above:

**Corollary 6.2.** *If  $D$  is a codimension 1 subscheme of  $S$ , then the self-intersection  $D \cdot D$  is the degree of the normal bundle  $\mathcal{N}_{D/S} := \mathcal{O}_D(D)$ .*

**Proof.** Substituting  $\mathcal{O}_S(-D)$  for both  $\mathcal{L}^{-1}$  and  $\mathcal{M}^{-1}$  we compute from the short exact sequences

$$\begin{aligned} 0 &\rightarrow \mathcal{O}_S(-D) \rightarrow \mathcal{O}_S \rightarrow \mathcal{O}_D \rightarrow 0, \\ 0 &\rightarrow \mathcal{O}_S(-2D) \rightarrow \mathcal{O}_S(-D) \rightarrow \mathcal{O}_D(-D) \rightarrow 0, \end{aligned}$$

and the Riemann-Roch theorem that

$$\begin{aligned} \chi(\mathcal{O}_S) - \chi(\mathcal{O}_S(-D)) &= \chi(\mathcal{O}_D) = -p_a(D) + 1, \\ \chi(\mathcal{O}_S(-D)) - \chi(\mathcal{O}_S(-2D)) &= \chi(\mathcal{O}_D(-D)) = -\deg \mathcal{O}_D(D) - p_a(D) + 1. \end{aligned}$$

Subtracting the second of these equations from the first we see that  $D \cdot D = \deg \mathcal{O}_D(D)$ .  $\square$

Using the intersection pairing we can turn the adjunction formula of Proposition 2.5 into a numerical formula:

**Theorem 6.3** (Adjunction formula). *If  $C$  is a Cartier divisor on a smooth surface  $S$  then*

$$p_a(C) = \frac{(K_X + C) \cdot C}{2} + 1.$$

**6.1. Quadrics in  $\mathbb{P}^3$  and the curves they contain.** We will frequently be interested in curves on a quadric surface in  $\mathbb{P}^3$ , and we can describe these curves and their intersections very concretely.

6.1.1. *The classification of quadrics.* The form defining a quadric  $Q \subset \mathbb{P}^3$  can be written in suitable coordinates  $x_i$  as  $q = \sum_{i=0}^3 x_i^2$ , where  $r \leq 4$  is the *rank* of  $Q$ . (The corresponding statement is also true in  $\mathbb{P}^r$ , with  $m \leq r+1$ .) Equivalently, this is the rank of the symmetric matrix  $M$  representing the bilinear form  $(q(x+y) - q(x) - q(y))/2$ .

- A quadric of rank 1 is a double plane.
- A quadric of rank 2 is the union of two distinct planes.
- A quadric  $Q$  of rank 3 is a cone over a smooth plane conic. The cone point is called the vertex. Every line on such a quadric passes through the vertex. The Picard group of  $Q$  is  $\mathbb{Z}$  ([Hartshorne 1977, Exercise II.6.5]).
- A quadric  $Q$  of rank 4 is smooth and is isomorphic to  $\mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^3$ , embedded by the Segre embedding. Every line on such a quadric has the form  $\mathbb{P}^1 \times \{p\}$  or  $\{p\} \times \mathbb{P}^1$ ; the two families of such lines are called *rulings*. The Picard group of  $Q$  is  $\mathbb{Z} \times \mathbb{Z}$ , generated by the lines of the rulings. [Hartshorne 1977, Example II.6.1].
- Quadrics in  $\mathbb{P}^r$ : Inside the space  $\mathbb{P}(\mathcal{O}_{\mathbb{P}^r}(2)) = \mathbb{P}^{\binom{r+2}{2}-1}$  the space of singular quadrics is a hypersurface of degree  $r+1$ , called the *discriminant*, which is defined by the determinant of a generic symmetric  $(r+1) \times (r+1)$  matrix  $M$ . More generally, the rank  $k$  locus is defined by the  $(k+1) \times (k+1)$  minors of  $M$ .

#### 6.1.2. Curves on quadrics.

- An irreducible nondegenerate curve cannot be contained in a quadric of rank 1 or 2.
- Let  $C$  be a reduced curve on a quadric  $Q$  of rank 3 in  $\mathbb{P}^3$ . If  $C$  has even degree  $2m$  then  $C$  is a complete intersection of the quadric with a hypersurface of degree  $m$ . Therefore if the curve is smooth it cannot contain the vertex of the quadric. The genus of  $C$  is  $(m-1)^2$ .

If  $C$  has odd degree  $2m + 1$ , then the union of  $C$  with a line on the quadric has even degree, so if  $L_1, L_2$  are lines on the quadric, then

$$C \cup L_1 = Q \cap F_1, \quad C \cup L_2 = Q \cap F_2$$

are complete intersections of  $Q$  with forms of degree  $m + 1$ , and the homogeneous ideal of  $C$  is generated by the equations of  $Q, F_1$  and  $F_2$ . The genus of  $C$  is  $m(m - 1)$ . The curve  $C$  contains the vertex, and is a Weil divisor, not a Cartier divisor. These things follow from the description of the desingularization of  $Q$  in Corollary 4.9.

- If  $C$  is a reduced curve on a quadric  $Q$  of rank 4, then in terms of the isomorphism  $Q \cong \mathbb{P}^1 \times \mathbb{P}^1$  we can express the divisor class of  $C$  as  $(a, b) \in \text{Pic}(Q) = \mathbb{Z} \oplus \mathbb{Z}$ . The intersection form on  $Q$  is determined by the fact that the lines of a ruling are all linearly equivalent, so that  $(1, 0)^2 = (0, 1)^2 = 0$  and  $(1, 0) \cdot (0, 1) = 1$ ; thus  $(a, b) \cdot (d, e) = ae + bd$ . The hyperplane section has class  $(1, 1)$ , and the canonical class is  $(K_{\mathbb{P}^3} + Q)|_Q = (-2, -2)$  by the adjunction formula.

Thus if  $C$  has class  $(a, b)$  then the degree of  $C$  is  $a + b$ , while the genus of  $C$  is  $(a - 1)(b - 1)$ . Note that if  $a = b$ , the curve is a complete intersection, and if  $b = a + 1$  then the curve is residual to a line in a complete intersection of the quadric with a surface of degree  $b$ . In these cases we get the same genus and degree as we would for a curve on a rank 3 quadric.

The cohomology of  $\mathcal{O}_Q(a, b)$  is determined by the Künneth formula (or the Leray spectral sequence) as

$$H^i(\mathcal{O}_Q(a, b)) = \bigoplus_{i=s+t} H^s(\mathcal{O}_{\mathbb{P}^1}(a)) \otimes H^t(\mathcal{O}_{\mathbb{P}^1}(b)).$$

The degrees of the generators of the homogeneous ideal of  $C$  can be determined from the exact sequence

$$0 \rightarrow \mathcal{I}_{Q/\mathbb{P}^3} \rightarrow \mathcal{I}_{C/\mathbb{P}^3} \rightarrow \mathcal{I}_{C/Q} \rightarrow 0.$$

Since  $\mathcal{I}_{C/Q}(m) = (m - a, m - b)$  we see that, supposing  $0 < a \leq b$  and  $1 < b$  there are no generators other than the equation  $q$  of  $Q$  having degree  $< b$ , and there are  $b - a + 1$  generators of degree exactly  $b$ ; with  $q$  these are the minimal generators of the homogeneous ideal (Exercise 7.12).

**6.2. The Riemann-Roch theorem for smooth surfaces.** Often we wish to compute the dimension of the space of sections of an invertible sheaf, but as with the case of curves, the Euler characteristic is more accessible:

**Theorem 6.4** (Riemann-Roch for surfaces). *Let  $\mathcal{L}$  be an invertible sheaf on a smooth surface  $S$ . The Euler characteristic  $\chi(D) := h^0(\mathcal{L}) - h^1(\mathcal{L}) + h^2(\mathcal{L})$ ,*

where  $\mathcal{L} = \mathcal{O}_S(D)$ , is given by

$$\chi(D) = \chi(\mathcal{O}_S) + \frac{(D - K_S) \cdot D}{2} + 1$$

**6.3. Blowups of smooth surfaces.** It is useful to know what happens under mappings of surfaces, particularly the case of the mapping corresponding to blowing up a point.

**Theorem 6.5.** *If  $\pi : X \rightarrow Y$  is a birational map of smooth surfaces, then the pullback map on divisors preserves the intersection pairing. If  $X$  is the blowup of  $Y$  at a point  $p$  with exceptional divisor  $E = \pi^{-1}(p)$ , then:*

- (1)  $\text{Pic } X = \pi^*(\text{Pic } Y) \oplus \mathbb{Z}E$ .
- (2) The canonical class on  $X$  is given by  $K_X = \pi^*(K_Y) + E$ .
- (3) The intersection pairing on  $\text{Pic } X$  is given by
  - $\pi^*(D) \cdot \pi^*(D') = D \cdot D'$  for all  $D, D' \in \text{Pic } Y$ .
  - $\pi^*(D) \cdot E = 0$  for all  $D \in \text{Pic } Y$ .
  - $E \cdot E = -1$ .
  - $K_X \cdot E = -1$ .
  - If  $C$  is a curve that has an  $m$ -fold point at  $p$  then  $\pi^{-1}(C)$  contains  $E$  with multiplicity  $m$ , so that the proper transform  $\tilde{C}$ —that is, the closure in  $X$  of the preimage of  $C \setminus \{p\}$ —has class

$$\tilde{C} \sim \pi^*C - mE.$$

Using these facts, we can compare the adjunction formula applied to a curve  $C$  on a smooth surface  $S$  to the formula applied to the proper transform  $\tilde{C} \subset \tilde{S}$ : we have

$$\tilde{C} \cdot \tilde{C} = C \cdot C - m^2 \quad \text{and} \quad \tilde{C} \cdot K_{\tilde{S}} = C \cdot K_S + m.$$

Combining these, we arrive at the formula

$$p_a(\tilde{C}) = p_a(C) - \binom{m}{2},$$

and thus the  $\delta$ -invariant of  $p \in C$  is  $\binom{m}{2}$  plus the sum of the  $\delta$ -invariants of all the points of  $\tilde{C}$  in the preimage of  $p$ . One can resolve the curve singularity by repeatedly blowing up in this way and thus compute the  $\delta$ -invariant of any singularity as a sum of binomial coefficients.

In the special case where  $\tilde{C}$  is smooth—for example, if the point  $p \in C$  is an *ordinary  $m$ -fold point*, consisting of  $m$  smooth branches with distinct tangent lines—we conclude that the  $\delta$ -invariant of the point  $p$  is  $\binom{m}{2}$ .

Blowups occur frequently in the theory of surfaces, and are characterized by *Castelnuovo's Theorem* [Hartshorne 1977, Theorem V.5.7].

**Theorem 6.6.** *If  $E \subset X$  is a curve on a smooth projective surface  $X$  and that  $E^2 = E \cdot K_X = -1$  then  $E$  can be “blown down” in the sense that  $X$  is the blowup of a smooth surface  $Y$  at a point  $p \in Y$ , and  $E$  is the exceptional divisor.*

## 7. Exercises

**Exercise 7.1.** (1) The degree of a zero-dimensional subscheme  $X \subset \mathbb{P}^r$  is by definition the value of the Hilbert polynomial of  $X$ , which is a constant. Show that this is the sum of the lengths of the components of  $X$ . (Hint: use the Serre-Grothendieck vanishing theorem.)

- (2) The degree of a projective subscheme  $X \subset \mathbb{P}^r$  of dimension  $n$  is defined to be the degree of the 0-dimensional scheme that is the intersection of  $X$  with a general plane of degree  $r - n$ . Prove that this is the leading coefficient of the Hilbert polynomial, multiplied by  $n!$ . (Hint: compare the Hilbert polynomial of  $X$  with that of a hyperplane section of  $X$ .)
- (3) If  $C \subset \mathbb{P}^r$  is a smooth curve of genus  $g$  and degree  $d$ , show that the Hilbert polynomial of  $C$  is  $H_C(m) = dm - g + 1$ . (Hint: use the Riemann-Roch formula together with the Serre-Grothendieck vanishing theorem.)

**Exercise 7.2.** If  $C \subset \mathbb{P}^r$  is a 1-dimensional variety with normalization  $\phi : \tilde{C} \rightarrow C$ , and  $X \subset \mathbb{P}^r$  is any subscheme that does not contain  $C$  we define the multiplicity of intersection  $\text{mult}(C, X; p)$  at the point  $p \in X \cap C$  to be the sum of the lengths of the finite scheme  $\phi^{-1}(X)$  at the points of  $\phi^{-1}(p)$ .

- (1) Show that  $C$  is singular at  $p$  if and only if  $\text{mult}(C, X; p) \geq 2$  for all  $X$  containing  $p$ . Show further that if  $H \subset \mathbb{P}^r$  is a hyperplane then  $\deg C = \sum_{p \in C} \text{mult}(C, H; p)$ .

- (2) Show that the degree of the image of  $C$  under projection from  $p$  is  $\deg C - \text{mult}(C, H; p)$  for a general hyperplane  $H$ .

Hint: For the degree formula note that  $\sum_{p \in C} \text{mult}(C, H; p)p$  is the divisor associated to the morphism  $\tilde{C} \rightarrow \mathbb{P}^r$  with image  $C$

In the following series of exercises, we will work with the smooth projective curve birational to a possibly singular affine curve  $C^\circ := V(f(x, y)) \subset \mathbb{A}^2$ ; this is the unique smooth projective curve containing the normalization of  $C^\circ$  as a Zariski dense open subset.

**Exercise 7.3.** Let  $C$  be the smooth projective curve birational to the affine plane curve  $C^\circ := V(y^3 + x^3 - 1)$ , and let  $\pi : C \rightarrow \mathbb{P}^1$  be the map given by the rational function  $x$ .

- (1) Show that the closure  $\overline{C^\circ}$  of  $C^\circ$  in  $\mathbb{P}^2$  is smooth (i.e.,  $C = \overline{C^\circ}$ ).
- (2) Find the branch points and ramification points of  $\pi$ , and deduce that the genus of  $C$  is 1.
- (3) For any two points  $p, q \in C$  find the complete linear series  $|p + q|$ .  
(Hint: let  $r$  be the third point of intersection of the line  $\overline{pq}$  with the closure of  $C^\circ$ , and consider the pencil of lines through  $r$ .)
- (4) Find the (unique) map  $\eta : C \rightarrow \mathbb{P}^1$  of degree 2 such that  $\eta((1, 0)) = \eta((0, 1))$ , and determine the ramification points of  $\eta$ .
- (5) Show that  $C$  is isomorphic to the smooth projective curve associated to the affine plane curve  $y^2 + x^3 = 1$ .

For the next three exercises, let  $C^\circ$  be the affine plane curve given as the zero locus of  $y^2 - x^6 + 1$ , and let  $C$  be the corresponding smooth projective curve. Note that the map  $C^\circ \rightarrow \mathbb{A}^1$  given by the projection  $(x, y) \mapsto x$  extends to a map  $\pi : C \rightarrow \mathbb{P}^1$ , expressing  $C$  as a 2-sheeted cover of  $\mathbb{P}^1$  branched over the points  $1, \zeta, \dots, \zeta^5$ , where  $\zeta$  is any primitive 6th root of unity. In addition, let  $p$  and  $q \in C$  be the two points lying over the point  $\infty \in \mathbb{P}^1$ .

**Exercise 7.4.** With  $C^\circ$  and  $C$  as above show that the map  $C^\circ \rightarrow \mathbb{A}^1$  given by the projection  $(x, y) \mapsto x$  extends to a map  $\pi : C \rightarrow \mathbb{P}^1$ , expressing  $C$  as a 2-sheeted cover of  $\mathbb{P}^1$  branched over the points  $1, \zeta, \dots, \zeta^5$ , where  $\zeta$  is any primitive 6th root of unity. Show that there are two distinct points  $p$  and  $q \in C$  lying over the point  $\infty \in \mathbb{P}^1$ , so that  $C$  is unramified over  $\infty$ .

What is the genus of  $C$ ?

**Exercise 7.5.** With  $C$  as in Exercise 7.4:

- (1) Let  $r_\alpha$  be the ramification point over  $\zeta^\alpha$ . Show that

$$p + q \sim 2r_\alpha \quad \text{and} \quad \sum_{\alpha=0}^5 r_\alpha \sim 3p + 3q.$$

- (2) Find the vector space  $H^0(\mathcal{O}_C(D))$  where  $D = r_0 + r_2 + r_4$ , and find the (unique) divisor  $E$  on  $C$  such that  $E + r_1 \sim r_0 + r_2 + r_4$ .

**Exercise 7.6.** With  $C$  as in Exercise 7.4: Let  $D$  be the divisor  $D = p + q + r_0 + r_3$

- (1) Find the vector space  $H^0(\mathcal{O}_C(D))$ .
- (2) Describe the map  $\phi_{|D|} : C \rightarrow \mathbb{P}^2$ .

- (3) Find the equation of the image curve  $\phi_{|D|}(C) \subset \mathbb{P}^2$ , and describe its singularities.

Hint: for the first part, find rational functions in  $x$  and  $y$  whose pullback to  $C$  has poles along  $D = p + q + r_0 + r_3$  but nowhere else. For the last part, observe that the equation of the image corresponds to the kernel of the map  $\text{Sym}^v H^0(\mathcal{O}_C(D)) \rightarrow H^0(\mathcal{O}_C(4D))$ .

**Exercise 7.7.** Let  $C$  be the smooth projective curve associated to the affine curve  $y^3 = x^5 - 1$ . Note that the map  $\pi : C \rightarrow \mathbb{P}^1$  given by the function  $x$  expresses  $C$  as a cyclic, 3-sheeted cover of  $\mathbb{P}^1$ , branched over the 5th roots of unity and the point at  $\infty$ . By way of notation, if we take  $\eta = e^{2\pi i/5}$  a primitive 5th root of unity, we'll denote by  $r_\alpha$  the point  $(\eta^\alpha, 0) \in C$  lying over  $\eta^\alpha$ , and by  $p$  the point lying over  $\infty \in \mathbb{P}^1$ .

- (1) Verify that there is indeed a unique point  $p \in C$  lying over  $\infty \in \mathbb{P}^1$ , and the map has ramification index 2 at  $p$ .

- (2) Show that the genus of  $C$  is 4.

- (3) Establish the linear equivalences

$$3p \sim 3r_\alpha \quad \text{and} \quad r_1 + \cdots + r_5 \sim 5p.$$

- (4) Find a basis for the space  $H^0(K_C)$  of regular differentials on  $C$ .

- (5) Show that  $C$  is not hyperelliptic.

- (6) Describe the canonical map  $\phi_K : C \rightarrow \mathbb{P}^3$  and find the equations of the image.

- (7) Let  $D$  be the divisor  $D = r_1 + \cdots + r_5$ . Show that  $h^0(K_C(-D)) = 1$ ; deduce that  $r(D) = 2$ , and find a basis for  $H^0(\mathcal{O}_C(D))$ .

- (8) If  $E = 3p$ , show that  $r(E) = 1$ ; that  $|E|$  is the unique  $g_3^1$  on  $C$  and that  $2E \sim K$ .

**Exercise 7.8.** (1) Show that the arithmetic genus of the disjoint union of two lines is  $-1$ .

- (2) Let  $L \subset Q \subset \mathbb{P}^3$  be a line on a smooth quadric surface in  $\mathbb{P}^3$ . Show that the divisor  $3L$ , regarded as a 1-dimensional scheme, has  $p_a(3L) = -2$ .

- (3) Compare these results with the result of simply applying the adjunction formula.

**Exercise 7.9.** Show that a curve of genus  $g \geq 3$  cannot be simultaneously hyperelliptic and a three-sheeted cover of  $\mathbb{P}^1$ . See Proposition ?? for a generalization.

**Exercise 7.10.** Show that normalization of the affine curve  $C = V(xy(x - y)) \subset \mathbb{A}^2$  is the disjoint union of three affine lines (that is,  $\text{Spec}(\mathbb{C}[x] \times \mathbb{C}[y]) \times$

$\mathbb{C}[z]$ ). Compute the linear conditions on the values and derivatives of three polynomial functions  $f, g, h$  defined on these three lines that they “descend” to give a regular function on the planar triple point.

**Exercise 7.11.** Let  $p \in C$  be a singular point of a reduced curve  $C$ . Show that if  $\delta_p = 1$ , then  $p$  must be either a node or an ordinary cusp.

Hint: first show that there are at most two points in the normalization of  $C$  lying over  $p$ .

**Exercise 7.12.** Suppose that  $C \subset Q \subset \mathbb{P}^3$  is a curve on a smooth quadric  $Q$ , and that  $C$  lies in the class  $(a, b)$  with  $0 \leq a \leq b$

- (1) Compute the generators of the homogeneous ideal of  $C$  in the case  $a = 0$ ; and in the case  $(a, b) = (1, 1)$  not treated in Example 6.1.2.
- (2) Prove the assertion about the generation of the homogeneous ideal by forms of degrees 2 and  $b$  that was made at the end of Example 6.1.2.

The following three exercises give a proof (in arbitrary characteristic) of resolution of singularities for curves; that is, every reduced curve is birational to a smooth curve.

**Exercise 7.13.** Let  $C_0 \subset \mathbb{P}^2$  be an irreducible and reduced plane curve of degree  $d$ . Show that for large  $m$ , the  $m$ -th Veronese map  $\mathbb{P}^2 \hookrightarrow \mathbb{P}^{\binom{m+2}{2}-1}$  embeds  $C_0$  as a curve  $C$  of degree  $md$  spanning a linear space  $\mathbb{P}^N$  of dimension

$$N = md - \binom{d-1}{2}.$$

Hint: observe that the linear forms in  $\mathbb{P}^{\binom{m+2}{2}-1}$  vanishing on the image of  $C_0$  correspond to elements of the kernel of the map  $H^0(\mathcal{O}_{\mathbb{P}^2}(m)) \rightarrow H^0(\mathcal{O}_C(m))$ .

**Exercise 7.14.** We define the *multiplicity*  $m_p(C)$  of a point  $p \in C$  on a curve  $C \subset \mathbb{P}^r$  to be the degree of the component of the intersection  $H \cap C$ , where  $H \subset \mathbb{P}^r$  is a general hyperplane through  $p$ . (Note that the definition of multiplicity is more tricky for varieties of dimension greater than 1; see [Eisenbud and Harris 2016a] for more.) Show that if the projection  $\pi_p : C \rightarrow \mathbb{P}^{r-1}$  from  $p$  is birational onto its image, then the degree of the image curve  $C_0 = \pi_p(C) \subset \mathbb{P}^{r-1}$  is  $\deg(C_0) = \deg(C) - m_p(C)$ , and more generally if  $\pi_p : C \rightarrow \mathbb{P}^{r-1}$  has degree  $k$  onto its image, then

$$\deg(C_0) = \frac{\deg(C) - m_p(C)}{k}.$$

**Exercise 7.15.** Returning to the situation of Exercise 7.13, suppose we take the curve  $C \subset \mathbb{P}^N$  and project it from a singular point, then do the same to its image  $C_1 \subset \mathbb{P}^{N-1}$  and continue in this fashion to produce a sequence of

curves  $C_n \subset \mathbb{P}^{N-n}$ . Combining the last two exercises with the statement of Corollary 4.2, show that

- (1) All the projection maps are birational onto their images; and
- (2) For some  $n$ , the process terminates; that is, the image curve  $C_n$  is smooth.



# Curves of genus 0

In this chapter, we'll begin our project of describing curves in projective space with the simplest case, curves of genus 0. (The theory is “simple” because we are operating over  $\mathbb{C}$ ; the study of curves of genus 0 over  $\mathbb{Q}$  was a major part of Gauss' work.) Even here, there are interesting statements to make about the geometry of their embeddings in  $\mathbb{P}^r$ , and there are many open problems.

Using Theorem 4.1 and the Riemann-Roch theorem, we can show that (over an algebraically closed field)  $\mathbb{P}^1$  is the only curve of arithmetic genus 0:

**Corollary 0.1.** *Every reduced irreducible projective curve  $C$  of arithmetic genus 0 over an algebraically closed field is isomorphic to  $\mathbb{P}^1$ .*

**Proof.** The curve  $C$  is smooth since otherwise its normalization would have negative genus. By the Riemann-Roch theorem, any linear series  $\mathcal{L}$  of degree  $d$  on  $C$  has  $h^0(\mathcal{L}) \geq d + 1$ , so we may use Theorem 4.1 to conclude that  $C \cong \mathbb{P}^1$ .  $\square$

Even images of genus 0 curves must have genus 0. The following proof works as well if we replace  $\mathbb{C}$  by any algebraically closed field.

**Theorem 0.2.** ([*Lüroth 1875*])

- (1) *If  $C \rightarrow D$  is a non-constant map of reduced, irreducible projective curves, then the geometric genus of  $C$  must be at least that of  $D$ . In particular, if  $C$  has geometric genus 0, then  $D$  has geometric genus 0.*
- (2) *If  $K$  is a field with  $\mathbb{C} \subsetneq K \subset \mathbb{C}(x)$  then  $K = \mathbb{C}(y)$  for some  $y \in \mathbb{C}(x)$ .*

The second part is called *Lüroth's theorem*

**Proof.** 1. Normalizing, we get a map  $\tilde{C} \rightarrow \tilde{D}$ , and the first statement follows from Hurwitz' theorem.

2. Since  $\mathbb{C}$  is algebraically closed,  $K$  is a transcendental extension. Since  $\mathbb{C}(x)$  has transcendence degree 1, if  $z \in K \setminus \mathbb{C}$  then  $x$  is algebraic over  $\mathbb{C}(z)$ . Thus  $\mathbb{C}(x)$  is finite over  $\mathbb{C}(z)$ , so  $K$  is finite over  $\mathbb{C}(z)$  as well. In particular,  $K$  is finitely generated over  $\mathbb{C}$ . It follows that  $K$  is the field of rational functions on a curve  $D$ , and the inclusion  $K \subset \mathbb{C}(x)$  corresponds to a finite map  $\mathbb{P}^1 \rightarrow D$ . Applying the first part of the theorem, we see that  $D \cong \mathbb{P}^1$  so  $K \cong \mathbb{C}(y)$  for some  $y$ .  $\square$

**Cheerful Fact 0.3.** *The last item of the theorem remains true if we replace  $\mathbb{C}$  by any algebraically closed field; see for example [Jacobson 1989, Section 8.13] for an elementary proof. There is an analogue for rational surfaces proven by Castelnuovo: every complex surface admitting a dominant rational map from  $\mathbb{P}^2$  is rational (see for example [Beauville 1996, Corollary V.5]). However, the analogue of Lüroth's theorem in higher dimension is false; for example, ([Clemens and Griffiths 1972]) shows that a smooth cubic threefold  $X$  admits a dominant rational map  $\mathbb{P}^3 \rightarrow X$  but is not rational.*

**Cheerful Fact 0.4.** *Over a non-algebraically closed field, a curve  $C$  of genus 0 need not have any points, or any line bundles of odd degree. Since the canonical bundle  $K_C$  has degree  $-2$ , there do necessarily exist line bundles of every even degree; thus an arbitrary curve of genus 0 is isomorphic to a plane conic. The classification of curves of genus 0 over non-algebraically closed fields is a subject that goes back to Gauss.*

A projective curve of genus 0 over a field  $k$  is called a form of  $\mathbb{P}^1$  if it becomes isomorphic to  $\mathbb{P}^1$  after extension of scalars to the algebraic closure of  $k$ . The unique example with  $k = \mathbb{R}$  that is not isomorphic over  $\mathbb{R}$  to  $\mathbb{P}^1$  is the conic with no  $\mathbb{R}$ -rational points,  $x^2 + y^2 + z^2 = 0$ .

Noncommutative algebras enter the subject of forms of  $\mathbb{P}^1$  (and  $\mathbb{P}^n$  more generally) in a surprising way: The curve  $\mathbb{P}_k^1$  itself may be described as the scheme of left ideals of  $k$ -vector-space dimension 2 in the ring of  $2 \times 2$  matrices over  $k$  (such an ideal can be embedded in the matrix ring as a linear combination of the 2 columns in an appropriate sense). More generally, any scheme that is a form of  $\mathbb{P}^1$  over  $k$  may be described as the scheme of 2-dimensional left ideals in a 4-dimensional central simple (= Azumaya) algebra over  $k$ . For example, the conic  $x^2 + y^2 + z^2 = 0$  with no points over  $\mathbb{R}$  is the scheme of left ideals in the algebra of quaternions. See [Serre 1979, Section X.6].

## 1. Rational normal curves

1.0.1. *The homogeneous coordinate ring of a rational normal curve.* If  $\mathcal{V} = (V, \mathcal{L})$  is a linear series on a scheme  $X$ , then the inclusion  $V \subset H^0(\mathcal{L})$  induces by multiplication a map from the symmetric algebra

$$\rho_{\mathcal{V}} : \text{Sym}(V) \rightarrow \bigoplus_{n \in \mathbb{Z}} H^0(\mathcal{L}^n).$$

When  $\mathcal{V}$  embeds  $X$  in  $\mathbb{P}^r = \mathbb{P}V$ , the image  $R_X$  of this map is called the *homogeneous coordinate ring* of  $X$ .

The rational normal curve  $C \subset \mathbb{P}^d$  of degree  $d$  is embedded by the complete linear series  $\mathcal{V} = (\mathcal{O}_{\mathbb{P}^1}(d), \mathbb{C}[s, t]_d)$ . Since any form of degree  $nd$  in  $\mathbb{C}[s, t]$  is a sum of products of  $n$  forms of degree  $d$ , the corresponding homogeneous coordinate ring of the rational normal curve is

$$\mathbb{C}[s, t]_{(d)} := \bigoplus_n (\mathbb{C}[s, t]_{nd}),$$

and the map  $\rho_{\mathcal{V}}$  is surjective. This is expressed by saying that  $C$  is *arithmetically Cohen-Macaulay*; more generally, if  $C \subset \mathbb{P}^r$  is a one-dimensional scheme, we say that  $C$  is linearly (respectively, quadratically, ...,  $n$ -ically) normal if the natural maps

$$\rho_m : H^0(\mathcal{O}_{\mathbb{P}^r}(m)) \rightarrow H^0(\mathcal{O}_C(m))$$

are surjective for  $m = 1$  (respectively  $m = 2, \dots, m = n$ ). We say that  $C$  is arithmetically Cohen-Macaulay (usually abbreviated ACM) when  $\rho_m$  is surjective for all  $m$ . We'll discuss the significance of this condition in Section 3 of this chapter, and we will prove in particular that it is equivalent to the condition that  $R_C$  is a Cohen-Macaulay ring, justifying the name.

1.0.2. *The equations defining a rational normal curve.* Choosing a basis  $s, t$  for the linear forms on  $\mathbb{P}^1$ , we can write the  $d$ -th Veronese map  $\mathbb{P}^1 \rightarrow \mathbb{P}^d$  as

$$\phi_d : (s, t) \mapsto (s^d, s^{d-1}t, \dots, t^d)$$

from which we see that the image  $C$  of  $\phi_d$  lies in the zero locus of the homogeneous quadratic polynomial  $x_i x_j - x_{i+1} x_{j-1}$  for every  $i, j$ . We can realize these quadratic forms as the  $2 \times 2$  minors of the matrix

$$M = \begin{pmatrix} x_0 & x_1 & \dots & x_{d-1} \\ x_1 & x_2 & \dots & x_d \end{pmatrix}.$$

Note that if we substitute  $s^i t^{(d-i)}$  for  $x_i$  and identify  $H^0(\mathcal{O}_{\mathbb{P}^1}(i))$  with  $\mathbb{C}[s, t]_i$ , this becomes the multiplication table

$$H^0(\mathcal{O}_{\mathbb{P}^1}(1)) \times H^0(\mathcal{O}_{\mathbb{P}^1}(d-1)) \rightarrow H^0(\mathcal{O}_{\mathbb{P}^1}(d)).$$

It is easy to see that  $C$  is set-theoretically defined by the  $2 \times 2$  minors of  $M$ : the affine set  $s = 1$  in  $\mathbb{P}^1$  maps to the affine set  $x_0 = 1$  in  $\mathbb{P}^d$ , and the affine

form of the map is  $t \mapsto (t, t^2, \dots, t^d)$ . But if  $x_1 = t$  then from the equations  $x_0x_i = x_1x_{i-1}$  we see successively that  $x_i = t^i$ ; that is, the vanishing of the minors of  $M$  at a point  $p$  implies that  $p$  lies on  $C$ .

A much stronger statement is true:

**Proposition 1.1.** *The homogeneous ideal of the rational normal curve*

$$\mathbb{P}^1 \longrightarrow C \subset \mathbb{P}^d \quad (s, t) \mapsto (s^d, s^{d-1}t, \dots, t^d)$$

*is generated by the  $2 \times 2$  minors of  $M$ .*

**Proof.** Let  $I \subset \mathbb{C}[x_0, \dots, x_d]$  be the ideal generated by the  $2 \times 2$  minors of  $M$ . As we have seen, the map

$$\phi : \mathbb{C}[x_0, \dots, x_d]/I \longrightarrow \mathbb{C}[s, t]_{(d)} : \quad x_i \mapsto s^{d-i}t^i$$

is surjective, and we must show that this is a monomorphism, or equivalently that the source of  $\phi$  in degree  $n$  has (at most) the same dimension  $nd + 1$  as the target of  $\phi$  in degree  $nd$ .

If  $0 < i \leq j < d$  then

$$x_i x_j \equiv x_{i-1} x_{j+1} \pmod{I}.$$

Thus every monomial in the  $x_i$  of degree  $t$  is equivalent, modulo  $I$ , to a monomial of the form

$$(*) \quad x_0^a x_1^{\epsilon_1} \cdots x_{d-1}^{\epsilon_{d-1}} x_d^b$$

where at most one  $\epsilon_i$  is 1, and the rest are 0. There are  $n + 1$  such elements of degree  $n$  with all the  $\epsilon_i = 0$  and  $n(d - 1)$  elements with one of the  $\epsilon_i = 1$ . Thus there are  $nd + 1$  such elements in all, proving that  $\phi$  is an isomorphism.  $\square$

**Cheerful Fact 1.2.** *The description of the equations above can be considerably extended; see Proposition 2.3. For example, the equations of the Veronese surface, which is the image of  $\mathbb{P}^2$  under the complete linear system of quadrics, is defined by the  $2 \times 2$  minors of the generic symmetric matrix*

$$\begin{pmatrix} x_0 & x_1 & x_2 \\ x_1 & x_3 & x_4 \\ x_2 & x_4 & x_5 \end{pmatrix}$$

*which arises from the multiplication table of*

$$H^0(\mathcal{O}_{\mathbb{P}^2}(1)) \otimes H^0(\mathcal{O}_{\mathbb{P}^2}(1)) \rightarrow H^0(\mathcal{O}_{\mathbb{P}^2}(2)).$$

**Corollary 1.3.** *The dimension of the degree  $n$  part of the homogeneous ideal of the rational normal curve of degree  $d$  is*

$$H^0(\mathcal{I}_{C/\mathbb{P}^d}(n)) = \binom{n+d}{n} - (nd + 1).$$

and in particular the  $\binom{d}{2}$  minors of the matrix  $M$  are linearly independent.

**Proof.** The homogeneous coordinate ring of the rational normal curve is

$$\mathbb{C}[s, t]_{(d)} \subset \mathbb{C}[s, t].$$

Comparing the dimension of  $\mathbb{C}[x_0, \dots, x_d]_n$  with the dimension of  $\mathbb{C}[s, t]_{nd}$  gives the result.  $\square$

The number of quadrics containing the rational normal curve is extremal:

**Proposition 1.4.** *If  $C \subset \mathbb{P}^d$  is any irreducible, nondegenerate curve, then*

$$h^0(\mathcal{I}_{C/\mathbb{P}^d}(2)) \leq \binom{d}{2};$$

and if equality holds then  $C$  is a rational normal curve

In the proof we will use a general fact about hyperplane sections:

**Proposition 1.5.** *If  $X \subset \mathbb{P}^r$  is a nondegenerate, reduced, irreducible variety, then  $H^0(\mathcal{I}_X(1)) = H^1(\mathcal{I}_X) = 0$ . Moreover if  $H$  is any hyperplane of  $\mathbb{P}^r$ , defined by a linear form  $h$ , then the natural sequence*

$$0 \rightarrow \mathcal{I}_{X/\mathbb{P}^r} \xrightarrow{h} \mathcal{I}_{X/\mathbb{P}^r}(1) \rightarrow \mathcal{I}_{(H \cap X)/H}(1) \rightarrow 0$$

arising from the restriction of the ideal sheaf  $\mathcal{I}_{X/\mathbb{P}^r}$  to  $H$  is exact, and the subscheme  $H \cap X$  spans  $H$ ; that is,  $H^0(\mathcal{I}_{(H \cap X)/H}(1)) = 0$ .

See Exercise 4.5 for the necessity of the hypotheses and Exercise 4.6 for a generalization.

**Corollary 1.6.** *If  $X \subset \mathbb{P}^r$  is a nondegenerate, reduced, irreducible variety of codimension  $c$ , then  $\deg X \geq c + 1$ .*

**Proof.** The intersection of  $X$  with a general plane  $\Lambda$  of dimension  $c$  is a finite scheme spanning  $\Lambda$ .  $\square$

**Proof of Proposition 1.5.** Because  $X$  is reduced irreducible and nondegenerate,  $h$  is a non-zerodivisor modulo  $\mathcal{I}_{X/\mathbb{P}^r}$  so  $h\mathcal{I}_{X/\mathbb{P}^r} = (h) \cap \mathcal{I}_{X/\mathbb{P}^r}$  where  $(h) = h\mathcal{O}_X$ . Thus, setting  $\Gamma = H \cap X$  we have

$$\begin{aligned} \mathcal{I}_{X/\mathbb{P}^r}(1)/h\mathcal{I}_{X/\mathbb{P}^r} &= \mathcal{I}_{X/\mathbb{P}^r}(1)/((h) \cap \mathcal{I}_{X/\mathbb{P}^r}) \\ &= (\mathcal{I}_{X/\mathbb{P}^r}(1) + (h))/(h) \\ &= \mathcal{I}_{\Gamma/H}(1). \end{aligned}$$

From which the exactness assertion follows.

Again because  $X$  is reduced and irreducible,  $H^0(\mathcal{O}_X)$  contains only the constant function, so the map  $H^0(\mathcal{O}_{\mathbb{P}^r}) \rightarrow H^0(\mathcal{O}_X)$  is surjective, from which it

follows that  $H^1(\mathcal{I}_{X/\mathbb{P}^r}) = 0$ . From the long exact sequence in cohomology it follows that the restriction map  $H^0(\mathcal{I}_{X/\mathbb{P}^r}(1)) \rightarrow H^0(\mathcal{I}_{\Gamma/H}(1))$  is surjective. Since  $X$  is nondegenerate,  $H^0(\mathcal{I}_{X/\mathbb{P}^r}(1)) = 0$ , and the desired result follows.  $\square$

**Proof of Proposition 1.4.** Consider the restriction of the quadrics containing  $C$  to a general hyperplane  $H \cong \mathbb{P}^{d-1} \subset \mathbb{P}^d$ , and let  $\Gamma = H \cap C$ . There is an exact sequence:

$$0 \rightarrow \mathcal{I}_{C/\mathbb{P}^d}(1) \rightarrow \mathcal{I}_{C/\mathbb{P}^d}(2) \rightarrow \mathcal{I}_{\Gamma/\mathbb{P}^{d-1}}(2) \rightarrow 0.$$

Proposition 1.5 shows that  $h^0(\mathcal{I}_{C/\mathbb{P}^d}(1)) = h^1(\mathcal{I}_{C/\mathbb{P}^d}(1)) = 0$ , so  $\Gamma$  imposes the same number of conditions on quadrics as  $C$  does.

Proposition 1.5 also shows that  $H$  is the linear span of  $\Gamma$ . In particular, the hyperplane section  $\Gamma$  of  $C$  must contain at least  $d$  linearly independent points (this gives another proof that such a curve must have degree  $\geq d$ .) Any set of linearly independent points imposes independent conditions on linear forms, and thus also on quadrics (see Proposition 2.3 for a stronger result). Thus

$$h^0(\mathcal{I}_{\Gamma/\mathbb{P}^{d-1}}(2)) \leq h^0(\mathcal{O}_{\mathbb{P}^{d-1}}(2)) - d = \binom{d+1}{2} - d = \binom{d}{2}$$

establishing the desired inequality.

To prove that the equality can be achieved only with the rational normal curve, suppose that  $C \subset \mathbb{P}^d$  is not a rational normal curve, so that  $\deg C > d$ . Let

$$\Gamma' \subset H \cong \mathbb{P}^{d-1}$$

be a subset of  $d$  linearly independent points of a general hyperplane section  $H \cap C$  and let  $\Gamma = \Gamma' \cup \{p\} \subset H$  be a subset containing one more point.

It suffices to show that  $p$  imposes a nontrivial vanishing condition on quadrics containing  $\Gamma'$ , since then  $C$  lies on fewer quadrics than the rational normal curve. Since the points of  $\Gamma'$  are linearly independent, each subset of  $d-1$  of them spans a  $(d-2)$ -plane inside  $H$  and the intersection of these spans is empty. Thus one of the spans  $\Lambda$  does not contain  $p$ . The union of a general hyperplane  $H$  containing  $\Lambda$  and a general hyperplane containing the point  $\Gamma' \setminus \Gamma' \cap \Lambda$  is a quadric containing  $\Gamma'$  but not  $p$ , as required.  $\square$

In fact, for each  $m$ , rational normal curves lie on more hypersurfaces of degree  $m$  than any other irreducible, nondegenerate curve in  $\mathbb{P}^d$ ; see Exercise 4.6.

1.0.3. *Rational normal curves are projectively homogeneous.* Another important property of rational normal curves  $C \subset \mathbb{P}^d$  is that they are *projectively homogeneous* in the sense that the subgroup  $G \subset PGL_{d+1}$  of automorphisms of  $\mathbb{P}^d$  that carry  $C$  to itself acts transitively on  $C$ . More generally,

**Proposition 1.7.** *If  $X \subset \mathbb{P}^n$  is the image of  $\mathbb{P}^r$  by the Veronese map corresponding to  $|\mathcal{O}_{\mathbb{P}^r}(d)|$ , then  $X$  is projectively homogeneous.*

**Proof.** First,  $\mathbb{P}^r$  is a homogeneous variety in the sense that  $\text{Aut } \mathbb{P}^r$  acts transitively. If  $\sigma$  is an automorphism then, because  $\mathcal{O}_{\mathbb{P}^r}(d)$  is the unique invertible sheaf of degree  $d$  on  $\mathbb{P}^r$ , we have  $\sigma^* \mathcal{O}_{\mathbb{P}^r}(d) = \mathcal{O}_{\mathbb{P}^r}(d)$  so  $\sigma$  induces an automorphism  $\phi$  on  $H^0(\mathcal{O}_{\mathbb{P}^r}(d))$ , and an automorphism  $\bar{\phi}$  on the ambient space  $\mathbb{P}^N := \mathbb{P}H^0(\mathcal{O}_{\mathbb{P}^r}(d))$  of the target of the  $d$ -th Veronese map. If  $\ell \in H^0(\mathcal{O}_{\mathbb{P}^r}(d))$  with divisor  $D \subset X$ , then  $\phi(\ell) = \ell \circ \sigma$  has divisor  $\sigma^{-1}(D)$ , so  $\bar{\phi}^{-1}$  induces  $\sigma$  on  $\mathbb{P}^r$ .  $\square$

The rational normal curve  $C \subset \mathbb{P}^d$  can be characterized among irreducible, nondegenerate curves as the unique projectively homogeneous curve in  $\mathbb{P}^d$  (Corollary 1.3).

1.0.4. *Interpolation for rational normal curves.* Another remarkable property of rational normal curves is expressed in the following Proposition. Recall that a collection of points (or a finite subscheme) of  $\mathbb{P}^d$  is said to be in *linearly general position* if no  $k + 1$  of them lie in a  $(k - 1)$ -plane with  $k \leq n$ .

**Proposition 1.8.** *If  $p_1, \dots, p_{d+3} \in \mathbb{P}^d$  are any  $d + 3$  points in  $\mathbb{P}^d$  in linearly general position, then there exists a unique rational normal curve  $C \subset \mathbb{P}^d$  containing them.*

**Proof.** There is an automorphism  $\Phi : \mathbb{P}^d \rightarrow \mathbb{P}^d$  carrying the points  $p_1, \dots, p_{n+1}$  to the coordinate points  $(0, \dots, 0, 1, 0, \dots, 0) \in \mathbb{P}^d$  and the point  $p_{d+2}$  to the point  $(1, 1, \dots, 1)$ . Denote the image of the remaining point  $p_{d+3}$  by  $[\alpha_0, \dots, \alpha_n]$ . We consider maps  $\mathbb{P}^1 \rightarrow \mathbb{P}^d$  given in terms of an inhomogeneous coordinate  $z$  on  $\mathbb{P}^1$  by

$$z \mapsto \left( \frac{1}{z - \nu_0}, \frac{1}{z - \nu_1}, \dots, \frac{1}{z - \nu_d} \right)$$

with  $\nu_0, \dots, \nu_d$  any distinct scalars. Clearing denominators, we see that the image of such a map is a rational normal curve, and it passes through the  $d + 1$  coordinate points of  $\mathbb{P}^d$ , which are the images of the points  $z = \nu_0, \dots, \nu_d \in \mathbb{P}^1$ . Moreover, the image of the point  $z = \infty$  at infinity is the point  $(1, 1, \dots, 1)$ . Because the points are assumed linearly independent, the  $\alpha_i$  must all be nonzero, and thus if we take  $\nu_i = -1/\alpha_i$ , the image of the point  $z = 0$  is  $(\alpha_0, \dots, \alpha_d)$ . This proves existence; we leave uniqueness to Exercise 4.11.  $\square$

**Cheerful Fact 1.9.** *A generalization of Proposition 1.8 is given, with a proof that is different even in our case, in [Harris 1982a, Proposition 3.19]. There is also a version replacing the set of  $d + 3$  distinct points by a finite scheme of degree  $d + 3$  satisfying necessary conditions, in [Eisenbud and Harris 1992].*

We remark that for any natural family of curves in projective space  $\mathbb{P}^r$  such as an open set of smooth curves in a component of the Hilbert scheme that we will describe in Chapter 7 and Chapter 19 and any integer  $m$ , we can ask whether there exists a curve in the family passing through  $m$  given general points of  $\mathbb{P}^r$ . This is the only example we know of curves other than complete intersections for which there is a *unique* such curve.

## 2. Other rational curves

What about other rational curves in projective space?

Any linear series  $D$  of degree  $d$  on  $\mathbb{P}^1$  is a subseries of the complete series  $|\mathcal{O}_{\mathbb{P}^1}(d)|$ , so any map  $\phi : \mathbb{P}^1 \rightarrow \mathbb{P}^r$  of degree  $d$  may be given as the composition of the embedding  $\phi_d : \mathbb{P}^1 \rightarrow \mathbb{P}^d$  of  $\mathbb{P}^1$  as a rational normal curve with a linear projection  $\pi : \mathbb{P}^d \rightarrow \mathbb{P}^r$ . Since the natural degree  $d$  Veronese embedding of  $\mathbb{P}^1 = \mathbb{P}(V)$  as the rational normal curve is the map  $\mathbb{P}V \rightarrow \mathbb{P}(\text{Sym}^d(V))$ , the projection is a map into  $\mathbb{P}W$ , where  $W \subset \text{Sym}^d(V)$ .

We can make this more explicit in several ways: first, choosing a basis  $s, t$  for  $V$  and a basis of forms  $F_i$  of degree  $d$  on  $\mathbb{P}^1$  for  $W$ , we may write the map as

$$(s, t) \mapsto (F_0(s, t), \dots, F_r(s, t)).$$

If the  $F_i$  have a greatest common factor  $G(s, t)$  then the set  $G = 0$  will be the base locus of the linear series, and the map

$$(s, t) \mapsto \left( \frac{F_0(s, t)}{G(s, t)}, \dots, \frac{F_r(s, t)}{G(s, t)} \right).$$

gives the same map, represented as a linear series of lower degree  $d - \deg G$ . Thus we will now assume that the forms in  $W$  have no common factor.

Perhaps even simpler, we can pass to an open affine set  $\mathbb{A}^1$  given by  $t = 1$  in  $\mathbb{P}^1$ , with coordinate function  $z = s/t$  and dehomogenize the  $F_i$  to get a vector space of polynomials  $f_i = F_i(s, 1)$  of degree  $\leq d$ . Then we may write the map as

$$z \mapsto (f_0(z), \dots, f_r(z)).$$

Since the  $F_i$  have no common factor, and in particular are not all divisible by  $t$ , at least one of the  $f_i$  will have degree exactly  $d$ . For example, the twisted cubic itself can be represented by the map  $z \mapsto (1, z, z^2, z^3)$ .

Here, we think of a polynomial  $f(z) = f(s/t)$  of degree  $\leq d$  as a rational function on  $\mathbb{P}^1$  having a pole of order at most  $d$  at the point at infinity  $(1, 0) \in \mathbb{P}^1$ ; but we could also take rational functions  $F_i(s, t)/G_i(s, t)$  of total degree  $\deg F_i - \deg G_i = d$  or, dehomogenizing,  $\phi_i(z) = f_i(z)/g_i(z)$ .

**Smooth rational quartics.** Given how easy it is to describe rational curves in projective space in this way, it is surprising how many open questions there are. We begin with one of the simplest cases: a smooth nondegenerate rational curve  $C$  of degree 4 in  $\mathbb{P}^3$ . As described above, such a curve is the image of  $\mathbb{P}^1$  under a map given by 4 quartic forms, or, in the most coordinate-free formulation, a codimension 1 subspace of  $\text{Sym}^4(V)$ , where  $V \cong \mathbb{C}^2$ .

**Example 2.1.** In Exercise 4.10 the reader may prove that there is a 1-parameter family of  $PGL_4$  orbits of smooth rational quartic curves in  $\mathbb{P}^3$ . Perhaps the simplest is given by the parameterization

$$\mathbb{P}^1 \ni (s, t) \mapsto (s^4, s^3t, st^3, t^4) \in \mathbb{P}^3,$$

or, more simply,  $t \mapsto (t, t^3, t^4)$ .

Proposition 2.2 shows that the homogeneous ideal of this curve requires 4 generators, and it is not hard to show that the ideal is generated scheme theoretically (that is, up to saturation) by 3 elements. Nevertheless it is one of the most famous open problems in the theory of curves to determine whether or not the ideal is generated up to radical by just 2 elements—that is, whether it can be written *set theoretically* as the intersection of two surfaces. For a sample of the recent work on this, see [Hartshorne and Polini 2015].

**Proposition 2.2.** *If  $C \subset \mathbb{P}^3$  is a smooth rational quartic curve, then  $C$  lies on a smooth quadric surface  $S$  in the divisor class  $(1, 3)$ , and the homogenous ideal of  $C$  is minimally generated by the quadric defining  $S$  and three cubic forms.*

**Proof.** We first consider the restriction maps

$$\rho_e : H^0(\mathcal{O}_{\mathbb{P}^3}(e)) \rightarrow H^0(\mathcal{O}_C(e)).$$

Since  $C \cong \mathbb{P}^1$ , we may identify  $H^0(\mathcal{O}_C(e))$  with  $H^0(\mathcal{O}_{\mathbb{P}^1}(4e))$ . Since we assume that  $C$  is nondegenerate, it does not lie on a hyperplane, so  $\rho_1$  is a monomorphism.

The source  $H^0(\mathcal{O}_{\mathbb{P}^3}(2))$  of  $\rho_2$  is 10-dimensional, and the target  $H^0(\mathcal{O}_{\mathbb{P}^1}(8))$  is 9-dimensional, so  $\rho_2$  has a kernel of dimension at least 1; that is,  $C$  lies on a quadric surface, and since  $C$  is irreducible and nondegenerate, any quadric surface containing  $C$  is irreducible. This quadric is smooth, as we have established in Section 6.1.2.

Given that  $C$  lies on a smooth quadric surface  $Q$ , we consider its divisor class  $(a, b)$  in the Picard group  $\text{Pic}(Q) = \mathbb{Z} \oplus \mathbb{Z}$ . As we saw in Section 6.1.2, a smooth curve in the class  $(a, b)$  has degree  $a + b$  and genus  $(a - 1)(b - 1)$ ; solving the equations  $a + b = 4, (a - 1)(b - 1) = 0$  we see that  $C \sim (1, 3)$  or  $C \sim (3, 1)$ . Since the cases are symmetric we assume  $C \sim (1, 3)$ .

The source of  $\rho_3$  is 20 dimensional and the target is 13 dimensional so there are at least 7 cubics in the ideal of  $C \subset \mathbb{P}^3$ . Four of these come from multiplying

the quadric by the 4 variables on  $\mathbb{P}^3$ , so there are at least 3 more cubic generators in  $I_{C/\mathbb{P}^3}$ , the homogeneous ideal of  $C$ .

From the exact sequence

$$0 \rightarrow I_{Q/\mathbb{P}^3} \rightarrow I_{C/\mathbb{P}^3} \rightarrow I_{C/Q} \rightarrow 0.$$

we see that  $I_{C/\mathbb{P}^3}$  is generated by the form defining the quadric together with generators for  $I_{C/Q}$ . Moreover, the natural inclusion  $I_{C/Q} \subset H_*^0(I_{C/Q})$  is an equality; this follows from the exact sequence of sheaves

$$0 \rightarrow \mathcal{I}_{Q/\mathbb{P}^3} \rightarrow \mathcal{I}_{C/\mathbb{P}^3} \rightarrow \mathcal{I}_{C/Q} \rightarrow 0$$

because  $H_*^1(\mathcal{I}_{Q/\mathbb{P}^3}) = H_*^1(\mathcal{O}_{\mathbb{P}^3}(-2)) = 0$ . Thus we need only compute generators for  $H_*^0(I_{C/Q})$ .

Since  $C$  lies in class  $(1, 3)$  we see that

$$\mathcal{I}_{C/Q}(d) = \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(d-1, d-3)$$

Thus by the Künneth formula,

$$h^0(\mathcal{I}_{C/Q}(3)) = h^0(\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(2, 0)) = h^0(\mathcal{O}_{\mathbb{P}^1}(2)) \cdot h^0(\mathcal{O}_{\mathbb{P}^1}(0)) = 3$$

and we see that  $I(C)$  contains just 3 cubic generators modulo the quadric defining  $Q$ .

We claim that the three generators of  $h^0(\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(2, 0))$  generate all of

$$H_*^0(\mathcal{I}_{C/Q}(3)) = H_*^0(\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(2, 0)) = \bigoplus_{m \geq 0} H^0(\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(m+2, m)).$$

For this we must show that the maps

$$\rho_m : H^0(\mathcal{O}_{\mathbb{P}^3}(1)) \otimes H^0(\mathcal{I}_{C/Q}(m+3)) \rightarrow H^0(\mathcal{I}_{C/Q}(m+1+3))$$

are surjective for  $m \geq 0$ .

Since the restriction of  $H^0(\mathcal{O}_{\mathbb{P}^3}(1))$  to the quadric is

$$H^0(\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1, 1)) = H^0(\mathcal{O}_{\mathbb{P}^1}(1)) \otimes H^0(\mathcal{O}_{\mathbb{P}^1}(1)),$$

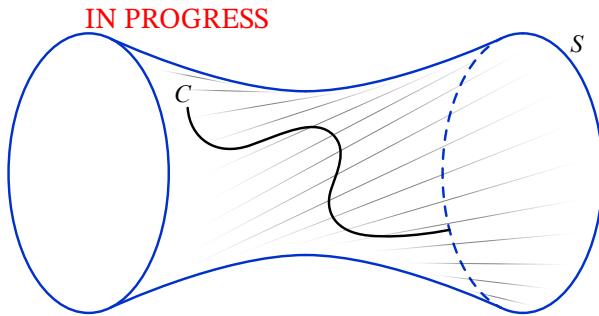
the map  $\rho_m$  is the tensor product of the two maps

$$H^0(\mathcal{O}_{\mathbb{P}^1}(1)) \otimes H^0(\mathcal{O}_{\mathbb{P}^1}(m+2)) \rightarrow H^0(\mathcal{O}_{\mathbb{P}^1}(m+1+2))$$

and

$$H^0(\mathcal{O}_{\mathbb{P}^1}(1)) \otimes H^0(\mathcal{O}_{\mathbb{P}^1}(m)) \rightarrow H^0(\mathcal{O}_{\mathbb{P}^1}(m+1))$$

both of which are surjective for  $m \geq 0$ . This completes the proof.  $\square$



**Figure 1.** A rational quartic space curve has type  $(1, 3)$  on the unique quadric surface containing it.

**2.1. Some open problems about rational curves.** We can say far less about rational curves of higher degree, even in  $\mathbb{P}^3$ . For example, when  $d$  is large, we don't know the possible Hilbert functions for curves of degree  $d$  in  $\mathbb{P}^3$ , and the situation in  $\mathbb{P}^r$  for  $r > 3$  is even worse. However, we do know the Hilbert function of a *general* rational curve  $C \subset \mathbb{P}^r$  of degree  $d$ .

Since such statements will come up often, we pause to explain the meaning of the statement, “A general  $X$  has property  $Y$ .” This statement presupposes a choice of a parameter space for objects of type  $X$ , that is to say an algebraic variety  $Z$  whose points each correspond to an object of type  $X$ . Thus, to be precise, the statement should be, “An object of type  $X$  that is general with respect to parameter space  $Z$  has property  $Y$ .” The statement then means: inside  $Z$  there is a dense open set whose elements correspond to objects of type  $X$  that have property  $Y$ . Often  $Z$  is irreducible, and then it is enough to have a nonempty open set, since every such set is Zariski dense.

In the case of nondegenerate rational curves of degree  $d$  in  $\mathbb{P}^r$  we could take as parameter spaces the Grassmannian of  $(r + 1)$ -dimensional planes in  $H^0(\mathcal{O}_{\mathbb{P}^1}(d))$  or the  $(r + 1)$ -tuples of independent elements of  $H^0(\mathcal{O}_{\mathbb{P}^1}(d))$ ; it would not make a difference since the first is a topological quotient of the latter. We will later make statements about general curves of genus  $g$ , referring either to a component of a Hilbert scheme (discussed in Chapter 6) or to the moduli space of curves that is discussed at length in Chapter 7.

As in Proposition 2.2, knowing the Hilbert function of a curve  $C \subset \mathbb{P}^r$  is tantamount to knowing the ranks of the restriction maps

$$\rho_m : H^0(\mathcal{O}_{\mathbb{P}^r}(m)) \rightarrow H^0(\mathcal{O}_C(m)) = H^0(\mathcal{O}_{\mathbb{P}^1}(md)).$$

Equivalently we ask: if  $V$  is a general  $(r + 1)$ -dimensional vector space of homogeneous polynomials of degree  $d$ , what is the dimension of the space of polynomials spanned by  $m$ -fold products of polynomials in  $V$ ?

We might guess that the answer is, “as large as possible,” meaning that the rank of  $\rho_m$  is  $\binom{m+r}{r}$  when that number is less than  $md + 1$ , and equal to  $md + 1$  when it is greater—in other words, the map  $\rho_m$  is either injective or surjective for each  $m$ . This was proven in [Ballico and Ellia 1983], and is a special case of Larson’s Maximal Rank Theorem, [Larson 2017].

As we will see in subsequent chapters it is possible to speak of a general curve of genus  $g$  and a general invertible sheaf of degree  $d$  on such a curve; and the analogous statement about Hilbert functions was proven in [Larson 2018]; see Chapter 12.

Nevertheless, even the degrees of the generators of the homogeneous ideal of a general rational curve of degree  $d$  in  $\mathbb{P}^3$  is unknown for larger  $d$ .

**2.1.1. The secant plane conjecture.** If  $C \subset \mathbb{P}^r$ , then an  $e$ -secant  $s$ -plane to  $C$  is an  $s$ -plane  $\Lambda \cong \mathbb{P}^s \subset \mathbb{P}^r$  such that the intersection  $\Lambda \cap C$  has degree  $\geq e$ ; if we exclude degenerate cases (for example, where  $\Lambda \cap C$  fails to span  $\Lambda$ ), an  $e$ -secant  $s$ -plane to  $C$  exists if there is a divisor  $D \subset C$  of degree  $e$  contained in an  $s$ -plane.

Should you expect a curve  $C \subset \mathbb{P}^r$  to have any  $e$ -secant  $s$ -planes? The set of  $s$ -planes in  $\mathbb{P}^r$  is parametrized by the Grassmannian  $\mathbb{G} = \mathbb{G}(s, r)$ , which has dimension  $(s+1)(r-s)$ . Inside  $\mathbb{G}$ , the locus of planes that meet  $C$  has codimension  $r-s-1$  (reason: the locus of planes containing a given point  $p$  is isomorphic by projection from  $p$  to  $\mathbb{G}(s-1, r-1)$ , and thus has codimension  $r-s$ ). Thus one might conjecture that a curve  $C \subset \mathbb{P}^r$  to have  $e$ -secant  $s$ -planes when

$$e \leq (s+1) \frac{r-s}{r-s-1},$$

perhaps with a few low degree exceptions. Is this true of a general rational curve? For most  $e, r$  and  $s$ , we don’t know.

### 3. The Cohen-Macaulay property

In Section 1 we defined a curve  $C \subset \mathbb{P}^r$  to be Arithmetically Cohen-Macaulay (ACM) if the natural maps

$$\rho_m : H^0(\mathcal{O}_{\mathbb{P}^r}(m)) \rightarrow H^0(\mathcal{O}_C(m))$$

are surjective for all  $m$ . When this is the case we can accurately predict the number of independent forms of each degree vanishing on the curve, and the condition has other important consequences that we will explore here and in Chapters 16 and 18. For general treatments of Cohen-Macaulay rings see [Eisenbud 1995a, Chapter 18] or the book [Bruns and Herzog 1993]. One of the characterizations of the condition is in terms of regular sequences:

**Definition 3.1.** A sequence of elements  $f_1, \dots, f_c$  in a ring  $R$  is a *regular sequence* if  $f_{i+1}$  is a non-zerodivisor modulo  $(f_1, \dots, f_i)$  for  $i = 0, \dots, c - 1$  and the ideal  $(f_1, \dots, f_c)$  is not the unit ideal.

The *grade* of an ideal  $I$  in a ring  $R$  (sometimes called the depth of  $I$  on  $R$ ) is the maximal length of a regular sequence contained in  $I$ , or  $\infty$  if  $I = R$ .

A local ring  $(R, \mathfrak{m})$  is *Cohen-Macaulay* if  $\text{grade } \mathfrak{m} = \dim R$ . If  $R$  is not necessarily local, then  $R$  is called Cohen-Macaulay if every localization is Cohen-Macaulay. Similarly, a scheme is Cohen-Macaulay if all its local rings are Cohen-Macaulay.

**Example 3.2.**

- The sequence of elements  $x_0, \dots, x_r \subset S := \mathbb{C}[x_0, \dots, x_r]$  is regular, proving from the definition that the localization of  $S$  at the homogeneous maximal ideal is Cohen-Macaulay.
- Regular local rings are Cohen-Macaulay. Thus every smooth scheme is Cohen-Macaulay, and in particular  $S := \mathbb{C}[x_0, \dots, x_r]$  is Cohen-Macaulay.
- It follows from the definition that any complete intersection in a Cohen-Macaulay scheme is again Cohen-Macaulay. Thus every plane curve, and more generally every complete intersection curve, is ACM; and we shall show in Section 2 that every canonically embedded curve and every curve embedded by a complete linear series of sufficiently high degree is also ACM.
- Zero-dimensional rings are all Cohen-Macaulay. A 1-dimensional ring is Cohen-Macaulay if and only if it is pure-dimensional (sometimes called “unmixed”—that is, every associated prime ideal is 1-dimensional) (Proof: the zerodivisors in  $R$  are the elements of the associated primes of 0.) Thus any purely 1-dimensional scheme is Cohen-Macaulay.

**Cheerful Fact 3.3.**

- We defined the grade as the maximal length of a regular sequence in  $I$ ; but in fact all maximal regular sequences have the same length, equal to the smallest integer  $k$  such that  $\text{Ext}^k(R/I, R) \neq 0$  [Eisenbud 1995a, Theorem 17.4 and Proposition 18.4].

- For any proper ideal  $I \subsetneq R$  we have  $\text{grade } I \leq \text{codim } I$ . On the other hand, if  $R$  is Cohen-Macaulay, then  $\text{grade } I = \text{codim } I$  for every ideal  $I$  of  $R$ . In this sense the grade is an arithmetic approximation to the codimension.
- Every localization of a Cohen-Macaulay ring is Cohen-Macaulay. If  $X \subset \mathbb{P}^r$  has homogeneous coordinate ring  $R_X$ , then we say that  $X \subset \mathbb{P}^r$  is arithmetically Cohen-Macaulay if  $R_X$  is Cohen-Macaulay, a property that depends on the embedding, and implies the intrinsic property that

$X$  is Cohen-Macaulay (since the local rings of  $X$  are essentially localizations of  $R_X$ ). Previously we defined a curve  $C \subset \mathbb{P}^r$  to be arithmetically Cohen-Macaulay if the natural map  $R_C \rightarrow H_*^0(\mathcal{O}_C)$  is surjective. We will prove that this is equivalent to  $R_C$  being Cohen-Macaulay in Proposition 3.4. The same definitions and theorem apply to a positively graded ring, and homogeneous ideals.

- By Serre's Criterion [Eisenbud 1995a, Section 11.2] the homogeneous coordinate ring  $R_C$  of a curve  $C$  is normal (that is, integrally closed) iff  $C$  is both nonsingular and ACM, and sometimes  $C$  is said to be projectively normal in this case. (This is also the excuse for the terminology "linearly normal" for a curve embedded by a complete linear series, quadratically normal if  $\rho_2$  is surjective, and so on.)

**Proposition 3.4.** Suppose that  $C \subset \mathbb{P}^r$  is a 1-dimensional subscheme. Let  $S = H_*^0(\mathcal{O}_{\mathbb{P}^r})$  be the homogeneous coordinate ring of  $\mathbb{P}^r$ , and let  $R_C = S/I_C$  be the homogeneous coordinate ring of  $C$ . The following conditions are equivalent:

- (1) The natural injective map  $R_C \rightarrow H_*^0(\mathcal{O}_C(1))$  is surjective;
- (2)  $H_*^1(\mathcal{I}_{C/\mathbb{P}^r}) = 0$ .
- (3) The homogeneous coordinate ring  $R_C$  of  $C$  is a Cohen-Macaulay ring; that is, there are linear forms  $h, h'$  on  $\mathbb{P}^r$  whose images in  $R_C$  form a regular sequence. In particular,  $R_C$  has no 0-dimensional primary components so  $C$  is purely 1-dimensional and thus Cohen-Macaulay as a scheme.
- (4) For every hyperplane  $H \subset \mathbb{P}^r$  that does not contain any component of  $C$ , the homogeneous ideal of  $H \cap C$  is equal to  $I_C + (h)$ , where  $h$  is a linear form defining  $H$ .

**Proof.**  $1 \Leftrightarrow 2$ : We may assume that  $r \geq 2$ , so  $H_*^1(\mathcal{O}_{\mathbb{P}^r}(n)) = 0$  for all  $n$ . Using this and the exact sequence

$$0 \longrightarrow \mathcal{I}_{C/\mathbb{P}^r} \longrightarrow \mathcal{O}_{\mathbb{P}^r} \longrightarrow \mathcal{O}_C \longrightarrow 0$$

we see that  $H^0(\mathcal{O}_{\mathbb{P}^r}(n)) \rightarrow H^0(\mathcal{O}_C(n))$  is surjective for all  $n$  if and only if  $H^1(\mathcal{I}_{C/\mathbb{P}^r}(n)) = 0$  for all  $n$ , proving the equivalence of 1 and 2.

$1 \Leftrightarrow 3$ : First let  $C \subset \mathbb{P}^r$  be an arbitrary 1-dimensional subscheme, and let  $R = H_*^0(\mathcal{O}_C) := \bigoplus_{n \in \mathbb{Z}} H^0(\mathcal{O}_C(n))$ . If  $H$  is a general hyperplane, with equation  $h = 0$ , then  $h$  does not vanish on any primary component of  $C$ , and thus the sequence

$$0 \longrightarrow \mathcal{O}_C(-1) \xrightarrow{h} \mathcal{O}_C \longrightarrow \mathcal{O}_{C \cap H} \longrightarrow 0$$

is exact. Applying  $H_*^0$  and using item 2, we see that  $h$  is a non-zerodivisor on  $R$ , and that  $R/hR$  is a subring of  $H_*^0(\mathcal{O}_{C \cap H})$ . A general linear form  $h'$  doesn't

vanish on any point of  $C \cap H$ , so  $h'$  is a unit on  $H_*^0(\mathcal{O}_{C \cap H})$  and thus a non-zerodivisor on  $R/hR$ .

The ring  $R_C$  is the image of the natural map  $S \rightarrow R$ , and by definition  $C$  is ACM if and only if this map is surjective, so that  $R_C = R$ . This shows that if  $C$  is arithmetically Cohen-Macaulay then  $R_C$  is a Cohen-Macaulay ring, and is in particular unmixed; that is,  $C$  has no 0-dimensional primary components (see for example [Eisenbud 1995a, Chapter 18] for general information about Cohen-Macaulay rings). This proves the equivalence of conditions 1 and 3.

$3 \Leftrightarrow 4$ : If  $h$  does not vanish on any component of  $C$  then  $h$  is a non-zerodivisor on  $R_C$ . The ideal  $I_{C \cap H}$  is in any case the saturation of  $I_C + (h)$ . If  $I_{C \cap H} = I_C + (h)$ , then any linear form  $h'$  not containing a point of  $C \cap H$  is a non-zerodivisor on  $I_C + (h)$ , so  $C$  satisfies condition 2. Conversely, in a 2-dimensional positively graded Cohen-Macaulay ring, any system of parameters is a regular sequence [Eisenbud 1995a, Section 18.2], so  $I_C + (h)$  is unmixed, and in particular, saturated.  $\square$

**Cheerful Fact 3.5.** *The Cohen-Macaulay property is hard to interpret geometrically; the definition is justified by its usefulness. Here are two results that help our intuition:*

- (1) *A scheme  $X$  is Cohen-Macaulay if some (equivalently every) finite map  $f : X \rightarrow P$  to a smooth scheme  $P$  of the same dimension is flat, or equivalently the pushforward  $f_*(\mathcal{O}_X)$  is locally free. (This follows from the Auslander-Buchsbaum formula [Eisenbud 1995a, Section 19.3].)*
- (2) *(Hartshorne) If a scheme  $X$  is Cohen-Macaulay then  $X$  is connected in codimension 1 (that is,  $X$  remains connected after removing any closed subset of codimension  $\geq 2$ ). See for example [Eisenbud 1995a, Theorem 18.12] for a proof.*

## 4. Exercises

**Exercise 4.1.** Let  $\mathcal{V} = (\mathcal{O}_{\mathbb{P}^1}(d), V)$  be the linear series of degree  $d$  on  $\mathbb{P}^1$  defined by the vector space  $V = \langle s^d, s^{d-1}t, st^{d-1}, t^d \rangle$ , where  $s, t$  are coordinates on  $\mathbb{P}^1$ . Show that  $\mathcal{V}$  defines an isomorphism from  $\mathbb{P}^1$  onto a smooth curve of type  $(1, d - 1)$  on a quadric surface.

Hint: use Proposition 4.5.

**Exercise 4.2.** With notation as in Section 1, show that the sheaf associated to the graded module  $\text{coker } M$ , that is, the cokernel of the map  $\mathcal{O}_{\mathbb{P}^d}^d(-1) \rightarrow \mathcal{O}_{\mathbb{P}^d}^2$  defined by  $M$ , is the unique invertible sheaf of degree 1 on the rational normal curve  $C$ , and that thus the associated complete linear series defines the isomorphism  $C \rightarrow \mathbb{P}^1$  inverse to the Veronese map.

**Exercise 4.3.** Considering  $\mathbb{P}^n$  as  $\text{Proj } \mathbb{C}[x_0, \dots, x_n]$ , we may index the variables  $z_p$  of  $\mathbb{P}^{\binom{n+d}{d}-1}$  by monomials  $p$  of degree  $d$  in the  $x_i$ . Let  $M_{n,d}$  be an  $(n+1) \times \binom{n+d-1}{n}$  matrix of linear forms on  $\mathbb{P}^{\binom{n+d}{d}-1}$  whose rows are indexed by the variables  $x_i$ , whose columns are indexed by the monomials  $m$  of degree  $d-1$  in the  $x_i$  and whose  $(i, m)$  entry is  $z_{x_i m}$ . (For example the matrix  $M$  of Section 1 is the matrix  $M_{1,d}$ .) Show that the  $2 \times 2$  minors of  $M_{n,d}$  generate the ideal of the image  $V_{n,d}$  of the Veronese map  $\mathbb{P}^n \rightarrow \mathbb{P}^{\binom{n+d}{d}-1}$ , and that the cokernel of  $M_{n,d}$  is the unique invertible sheaf of degree 1 supported on  $V_{n,d}$ .

**Exercise 4.4.** Let  $\nu_d : \mathbb{P}^r \rightarrow \mathbb{P}^{\binom{r+d}{r}-1}$  be the  $d$ -Veronese map, and let  $C \subset \mathbb{P}^r$  be the rational normal curve of degree  $r$ . Is  $\nu_d(C)$  nondegenerate? If not, what is the dimension of its linear span (that is, of the smallest linear space that contains it)?

Hint: find the rank of the restriction map  $H^0(\mathcal{O}_{\mathbb{P}^N}(1)) \rightarrow H^0(\mathcal{O}_{\nu_d(C)}(1))$ .

**Exercise 4.5.** Let  $C_1$  be the union of two skew lines in  $\mathbb{P}^3$ , and let  $C_2$  be the double line on a smooth quadric in  $\mathbb{P}^3$ . Show that a general hyperplane section of  $C_1$  and of  $C_2$  violates the conclusion of Proposition 1.5.

**Exercise 4.6.** Suppose that  $X \subset \mathbb{P}^r$  is a subscheme and  $Z$  is the hypersurface in  $\mathbb{P}^r$  defined by a polynomial  $F \in H^0(\mathcal{O}_{\mathbb{P}^r}(m))$ . If the restriction of  $F$  is a non-zerodivisor in  $\mathcal{O}_X(m)$  then there is a short exact sequence

$$\mathcal{I}_X(d-m) \xrightarrow{F} \mathcal{I}_X(m) \longrightarrow \mathcal{I}_{X \cap Z}(m) \rightarrow 0.$$

**Exercise 4.7.** Let  $C \subset \mathbb{P}^3$  be the smooth rational quartic (or any smooth curve embedded by an incomplete linear series), and let  $h$  be a linear form defining a hyperplane  $H$ . Show that the irrelevant ideal is associated to the homogeneous ideal  $I_C + (h)$ , and thus  $I_C(1)/hI_C$  is not the saturated homogeneous ideal of the finite set  $C \cap H$ .

**Exercise 4.8.** Show that the twisted cubic is the unique irreducible, nondegenerate space curve lying on three quadrics by considering the possible intersections of two of the quadrics. Hint: use Bézout's theorem.

**Exercise 4.9.** As a consequence of our description of rational quartic curves on a smooth quadric in Proposition 2.2, show that a general  $g_4^3$  on  $\mathbb{P}^1$  is uniquely expressible as a sum of the  $g_1^1$  and a  $g_3^1$  (in other words, a general 4-dimensional vector space of quartic polynomials on  $\mathbb{P}^1$  is uniquely expressible as the product of a 2-dimensional vector space of cubics and the 2-dimensional space of linear forms).

Hint: look at the linear series cut on  $C$  by the two rulings of the quadric.

**Exercise 4.10.** Show that, up to projective equivalence, there is a 1-parameter family of embeddings of  $\mathbb{P}^1$  as a smooth quartic curve in  $\mathbb{P}^3$  by constructing an invariant that distinguishes them.

Hint: To any such embedding we can associate the  $g_3^1$  given by one of the rulings of the quadric surface containing the image curve. Modulo  $PGL_2$ , there is a one-parameter family of  $g_3^1$ s on  $\mathbb{P}^1$ .

**Exercise 4.11.** Complete the proof of Proposition 1.8 by showing that if  $C, C' \subset \mathbb{P}^n$  are two rational normal curves meeting in at least  $n + 3$  distinct points, then  $C = C'$ .

Hint: use induction on  $n$ , starting with the case  $n = 2$ .

**Exercise 4.12.** Let  $V = \mathbb{C} \cdot e_1 \oplus \mathbb{C} \cdot e_2$  be a 2-dimensional vector space.

The group  $SL_2 = SL(V)$  acts on the rational normal curve of degree  $d$  through automorphisms induced from its action on the ambient space  $\mathbb{P}^d$  of the rational normal curve, which may be identified with  $\mathbb{P}(\text{Sym}^d(V))$ .

In [Fulton and Harris 1991, pp. 146–150] it is shown that every finite dimensional rational representation of  $SL(V)$  is a direct sum of representations of the form  $\text{Sym}^e(V)$  for various  $e \geq 0$ . Moreover, it is often easy to understand how a given representation decomposes by looking at the action of

$$\alpha := \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \in SL(V).$$

Note that  $\text{Sym}^e(V)$  is spanned by “weight vectors” ( $\equiv$  eigenvectors of  $\alpha$ )  $w_s := e_1^{e-s} e_2^s$  which satisfy  $\alpha w_s = t^{e-2s} w_s$  for  $s = 0, \dots, e$ . To decompose an arbitrary representation  $W$ , knowing that  $W$  is a direct sum of  $\text{Sym}^{e_i} V$ , it is enough to know the eigenvalues for the action of  $\alpha$ : We begin by finding an element  $w \in W$  that is an eigenvector of  $\alpha$  and transforms by  $\alpha$  as  $\alpha w = t^m w$  with the highest possible  $m$  (this is called a “highest weight vector”). This element  $w$  must be contained in a summand  $\text{Sym}^m(V)$ , and after removing the eigenvalues of the action of  $SL_2$  on  $\text{Sym}^m(V)$ , we continue.

(1) Use this method to show that

$$\text{Sym}^d(V) \otimes \text{Sym}^d(V) = \text{Sym}^{2d}(V) \oplus \text{Sym}^{2d-2}(V) \oplus \text{Sym}^{2d-4}(V) \dots$$

$$\text{Sym}^2(\text{Sym}^d(V)) = \text{Sym}^{2d}(V) \oplus \text{Sym}^{2d-4}(V) \oplus \text{Sym}^{2d-8}(V) \dots$$

$$\bigwedge^2(\text{Sym}^d(V)) = \text{Sym}^{2d-2}(V) \oplus \text{Sym}^{2d-6}(V) \oplus \text{Sym}^{2d-10}(V) \dots$$

(where we take  $\text{Sym}^m(V) = 0$  when  $m < 0$ ).

- (2) Show that the space of quadrics containing the rational normal curve is a representation of  $SL_2$  of the form

$$\text{Sym}^{2d-4}(V) \oplus \text{Sym}^{2d-8}(V) \dots$$

- (3) Show there is a distinguished nonsingular skew-symmetric form (up to scalars) on the ambient space of the twisted cubic; in particular is, given a twisted cubic in  $\mathbb{P}^3$  there is a distinguished plane containing each point of  $\mathbb{P}^3$ .
- (4) Show that if  $d$  is divisible by 4 there is a distinguished quadric in the ideal of the rational normal curve.

**Exercise 4.13.** Let  $\mathbb{P}^1 \hookrightarrow C \subset \mathbb{P}^3$  be a twisted cubic. Show that the normal bundle  $\mathcal{N}_{C/\mathbb{P}^3}$  (defined to be the quotient of the restriction  $T_{\mathbb{P}^3}|_C$  to  $C$  of the tangent bundle of  $\mathbb{P}^3$  by the tangent bundle  $T_C$ ) is

$$\mathcal{N}_{C/\mathbb{P}^3} \cong \mathcal{O}_{\mathbb{P}^1}(5) \oplus \mathcal{O}_{\mathbb{P}^1}(5).$$

Hint: for any point  $p \in C$ , let  $L_p \subset \mathcal{N}_{C/\mathbb{P}^3}$  be the sub-line bundle of  $\mathcal{N}_{C/\mathbb{P}^3}$  whose fiber over any point  $q \neq p \in C$  is the one-dimensional subspace of  $(\mathcal{N}_{C/\mathbb{P}^3})_q$  spanned by the line  $\overline{p, q}$ . (This of course only defines a sub-line bundle of  $\mathcal{N}_{C/\mathbb{P}^3}$  over  $C \setminus \{p\}$ , but there is a unique extension to a sub-line bundle of  $\mathcal{N}_{C/\mathbb{P}^3}$  over all of  $C$ .) Show that for  $p \neq p'$  we have

$$\mathcal{N}_{C/\mathbb{P}^3} = L_p \oplus L_{p'}.$$

**Exercise 4.14.** Let  $\mathbb{P}^1 \hookrightarrow C \subset \mathbb{P}^d$  be a rational normal curve. Show that the normal bundle  $\mathcal{N}_{C/\mathbb{P}^d}$  is

$$\mathcal{N}_{C/\mathbb{P}^d} \cong \bigoplus_{i=1}^{d-1} \mathcal{O}_{\mathbb{P}^1}(d+2).$$

Hint: For  $p \in C$ , define a sub-line bundle  $L_p \subset \mathcal{N}_{C/\mathbb{P}^d}$  as in the preceding problem, and show that for  $p_1, \dots, p_{d-1}$  distinct points the bundles  $L_{p_i}$  are independent.

**Exercise 4.15.** In the situation of the preceding problem, the set of direct summands of  $\mathcal{N}_{C/\mathbb{P}^d}$  is a projective space  $\mathbb{P}^{d-2}$ . How does the group of automorphisms of  $\mathbb{P}^d$  carrying  $C$  to itself act on this  $\mathbb{P}^{d-2}$ ? (For more on normal bundles of rational curves, see for example [Coskun and Riedl 2018].)

Hint: It is the projectivization of the action of  $SL(V)$  on  $\text{Sym}^{d-2} V$ .

**Exercise 4.16.** If  $C = \bigcap_{i=1}^{r-1} X_i \subset \mathbb{P}^r$  is a complete intersection of hypersurfaces, then  $C$  is arithmetically Cohen-Macaulay.

Hint: Since the  $r - 1$  surfaces intersect only in codimension  $r - 1$ , they form a regular sequence; and the length of any maximal regular sequence in  $\mathbb{C}[x_0, \dots, x_r]$  is  $r + 1$ .

# Smooth plane curves and curves of genus 1

If  $C$  is a curve of genus 1, then by Corollary 3.5, any complete linear series  $D$  of degree  $3 = 2g + 1$  on  $C$  is very ample. By the Riemann-Roch theorem any invertible sheaf of degree  $d$  on  $C$  has  $d + 1$  sections, so the morphism associated to  $D$  embeds  $C$  as a plane curve of degree 3, on which  $D$  is the intersection of  $C$  with a line.

We therefore begin this chapter with a general explanation of sheaves of differentials and linear series on smooth plane curves, and use that theory to say a little about other embeddings of curves of genus 1. Though most curves cannot be realized as smooth plane curves, we shall see in Proposition 3.1 that every curve can be projected birationally to a plane curve whose only singularities are ordinary nodes, and in Chapter 15 we will explain the analogous treatment of differentials and linear series on such nodal curves, and more generally—but with less specificity—on all reduced plane curves.

Using the plane model of a curve of genus 1, we shall see that the theory is only a little more complicated than for genus 0 (but curves of genus 1 over number fields occupies a major part of modern number theory!)

## 1. Riemann, Clebsch, Brill and Noether

For a long time, plane curves were the only algebraic curves that were studied. Originally these were curves in the affine plane over the real numbers, but by the second half of the 19th century the complex projective plane was well understood, and curves in  $\mathbb{P}^2 = \mathbb{P}_{\mathbb{C}}^2$ , corresponding to irreducible forms in 3

variables, were recognized as the natural objects of study—see the historical appendix to this book for more details.

The work of Bernhard Riemann dramatically changed the focus of the theory to branched coverings of the “Riemann Sphere” ( $\mathbb{P}_{\mathbb{C}}^1$ ). The Riemann-Roch theorem, in particular, gave information about the existence of meromorphic functions on such coverings, well beyond what could be done in the earlier theory. However, Riemann’s work, depending as it did on the then-obscure “Dirichlet principle”, was not universally accepted. In the 1860s Alfred Clebsch and, after the death of Clebsch in 1872, Alexander Brill and Max Noether (Emmy Noether’s father), undertook the ambitious program of redoing the Riemann-Roch theorem entirely in terms of plane curves. They went beyond Riemann in certain directions, too: the Brill-Noether theorem treated in our Chapter 12 was formulated by Brill and Noether, and “proved” by them through an unsupported general position assumption.

A central difficulty in the Brill-Noether attempt on the Riemann-Roch theorem was that, although any smooth curve can be embedded in  $\mathbb{P}^r$  for any  $r \geq 3$ , most curves cannot be embedded in the plane. However, as is shown in Section 3.1, we can embed  $C$  as a curve  $C \subset \mathbb{P}^r$  in a higher-dimensional projective space and find a projection  $\mathbb{P}^r \rightarrow \mathbb{P}^2$  that carries  $C$  birationally onto its image  $C_0$ , called a plane model of  $C$ . The curve  $C_0$  typically has singularities, and  $C$  is the normalization of  $C_0$ . Brill and Noether wanted to prove the Riemann-Roch theorem for  $C$  by formulating and proving a related theorem for  $C_0$ . In particular, they tried to characterize linear equivalence of divisors on  $C$  in terms of certain “clusters” of points—we would say 0-dimensional subschemes—of  $C_0$ .

To carry out this program, a key step was to show that if  $D \subset C_0$  is contained in the intersection  $D'$  of  $C_0$  and some other plane curve  $C'_0$ , then a divisor  $E := D' - D$  can be defined with properties such as that  $D' - E = D$ ; Brill and Noether seem simply to have assumed that this is so. A bit later Frances Sowerby Macaulay proved that this is in fact possible and also understood that it would not generally be possible if  $D'$  were the intersection of three or more curves.<sup>1</sup>

Macaulay exploited this theory of residuation to prove what he called the “Generalized Riemann-Roch theorem” Theorem 2.5. This early work of Macaulay led directly to his definitions of “perfection” (a homogeneous ideal  $I \subset S := k[x_0, \dots, x_n]$  is perfect if  $S/I$  is Cohen-Macaulay) and “super-perfection” (the case when  $S/I$  is Gorenstein). For all this, see [Eisenbud and Gray 2023].

In this section we will take the point of view of Clebsch, Brill and Noether, and explain how to understand the differential forms and, given a (possibly ineffective) divisor  $D$  on  $C$ , how to find all the effective divisors on  $C$  that are

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<sup>1</sup>In modern terms, the intersection of two curves is Gorenstein; the intersection of 3 is generally not.

linearly equivalent to  $D$ , in terms of a smooth plane curve. In Chapter 15 we will return to this point of view and treat the case of nodal curves and the case of arbitrary reduced plane curves. We will give effective algorithms for two constructions:

- (1) Given the equation  $F(X, Y, Z)$  of a smooth  $d$  plane curve  $C$  of degree  $d$ , we can construct a basis for  $H^0(K_C)$ ; and
- (2) Given a possibly ineffective divisor  $D = D_+ - D_-$  on  $C$  we can construct the complete linear series  $|D|$  on  $C$ . In particular we can:
  - (a) determine whether  $D$  is equivalent to any effective divisor on  $C$ ; and if so,
  - (b) find all effective divisors  $E$  on  $C$  with  $E \sim D$ ; also,
  - (c) find a basis of  $H^0(\mathcal{O}_C(D))$ , expressed in terms of curves of high degree with base locus.
  - (d) find the homogeneous coordinate ring of the morphism defined by  $|D|$  or a subseries.

## 2. Smooth plane curves

**2.1. Differentials on a smooth plane curve.** Let  $C \subset \mathbb{P}^2$  be a smooth plane curve, given as the zero locus of a homogeneous polynomial  $F(X, Y, Z)$  of degree  $d$ . By the adjunction formula (Proposition 2.5) the canonical divisors on  $C$  are the intersections of  $C$  with curves of degree  $d - 3$ . In the spirit of Brill and Noether we will make this explicit by constructing all the regular differential forms on  $C$  in terms of forms of degree  $d - 3$ .

For this purpose we introduce coordinates  $x = X/Z$  and  $y = Y/Z$  on the affine open subset  $U \cong \mathbb{A}^2$  given by  $Z \neq 0$ , and let  $f(x, y) = F(x, y, 1)$  be the inhomogeneous form of  $F$ , so that  $C^\circ = C \cap U$  is the zero locus  $V(f) \subset \mathbb{A}^2$ .

Since an automorphism of  $\mathbb{P}^2$  can carry any point in  $\mathbb{P}^2$  to any other point, we may assume that the point  $[0, 1, 0]$  (that is, the point at infinity in the vertical direction) does not lie on  $C$  so that the projection  $\pi : C \rightarrow \mathbb{P}^1$  from  $(0, 1, 0)$ , which is given by  $[X, Y, Z] \mapsto [X, Z]$  (or, in affine coordinates,  $(x, y) \mapsto x$ ) has degree  $d$ . Let  $D$  be the divisor defined by the intersection of  $C$  with the line  $Z = 0$  at infinity.

Consider the regular 1-form  $dx$  on  $\mathbb{A}^2$ , which we may regard as the pull-back of the form  $dx$  on  $\mathbb{A}^1$ . Since the form  $dx$  on  $\mathbb{P}^1$  has a double pole at infinity the form  $dx|_C$  has polar locus  $2D$ .

How do we get rid of the poles of  $dx$ ? The extension to  $\mathbb{P}^2$  of a polynomial  $h(x, y)$  of degree  $m$  on  $\mathbb{A}^2$  has a pole of order  $m$  along the line  $L$  at infinity. Thus if  $h$  has degree at least 2 then  $dx/h$  has no poles at infinity. However,  $h(x, y)$  may well vanish at points of  $C^\circ$ , and this may create new poles of  $dx/h$ . Of

course if  $h$  vanishes only at points of  $C^\circ$  where  $dx$  has a zero, the zeroes of  $h$  may cancel the zeroes of  $dx$  rather than creating new poles.

To avoid producing new poles in this way we may take

$$h(x, y) = f_y := \frac{\partial f}{\partial y}(x, y).$$

We claim that

$$\varphi_0 = \frac{dx}{f_y}$$

is everywhere regular and nowhere 0 in  $C^\circ$ .

Note that  $df$  vanishes identically when restricted to  $C^\circ$ , so

$$0 \equiv df|_{C^\circ} = f_x dx|_{C^\circ} + f_y dy|_{C^\circ}.$$

Clearly  $\varphi_0$  is regular at points  $p$  where  $f_y(p) \neq 0$ . At such a point, if  $dx|_{C^\circ}$  were 0, then since  $C^\circ$  is smooth we would have  $dy|_{C^\circ} \neq 0$ , contradicting the equation above. Thus  $\varphi_0$  is both regular and nonzero at such points. On the other hand, if  $f_y(p) = 0$ , then since  $C^\circ$  is smooth  $f_x(p) \neq 0$ , so  $dx|_{C^\circ}$  and  $f_y$  vanish to the same order, whence, again,  $\varphi_0 = (dx/f_y)|_{C^\circ}$  is regular and nonzero, proving the claim.

Put differently, if  $L$  is the line at infinity, so that  $U = \mathbb{P}^2 \setminus L \cong \mathbb{A}^2$ , then the cotangent bundle on  $U$  is  $\Omega_U = \mathcal{O}_U dx \oplus \mathcal{O}_U dy$ , so the cotangent bundle  $\Omega_{C^\circ}$  on  $C^\circ$ , which is the canonical bundle  $\omega_{C^\circ}$ , is the cokernel of the map from the normal sheaf  $\mathcal{O}_C(-d)|_U$  to the restriction of  $\Omega_U$  to  $C^\circ$ . This map sends the local generator  $f$  of the normal sheaf to  $f_x dx + f_y dy \in \Omega_U$ , and because  $f_x$  and  $f_y$  have no common zeros, the generator  $d_y \in \Omega_{C^\circ}$  is a multiple of  $dx/f_y$ . Thus the free  $\mathcal{O}_{C^\circ}$ -module  $\omega_{C^\circ}|_{C^\circ}$  is generated by  $dx/f_y$ .

Since  $f_y$  has degree  $d - 1$ , the rational function  $1/f_y$  vanishes to order  $d - 1$  on the line at infinity, and thus in particular on the divisor  $D$ . Thus  $\varphi_0$  vanishes to order  $d - 3$  on  $D$ ; in other words, as divisors,

$$(\varphi_0) = (d - 3)D.$$

In particular, if  $d \geq 3$  then  $\varphi_0$  is a globally regular differential on  $C$ . The divisor of zeros of this differential has degree  $d(d - 3)$ . If  $g$  is the genus of  $C$ , then we must have  $2g - 2 = d(d - 3)$ , whence

$$g = \frac{d(d - 3)}{2} + 1 = \binom{d - 1}{2}.$$

We can produce a vector space of  $\binom{d-1}{2}$  regular differentials by multiplying  $\varphi_0$  by polynomials  $e(x, y)$  of degree  $d - 3$ , since this does not introduce any poles. This proves:

**Theorem 2.1.** *The space of regular differentials on a smooth plane curve  $C$  with affine equation  $f = 0$  is*

$$\left\{ \frac{e(x,y)dx}{f_y} \mid e(x,y) \text{ is a polynomial degree } \leq d - 3 \right\}. \quad \square$$

**2.2. Linear series on a smooth plane curve.** Any divisor on a smooth plane curve  $C$  may be expressed as the difference of two effective divisors,  $D = D_+ - D_-$ . We would like to find all the *effective* divisors linearly equivalent to  $D$ , that is, of the form  $D + (H/G)$ , where  $G, H$  are forms of the same degree  $m$ . We begin by choosing an integer  $m$ , large enough so there is a form  $G$  of degree  $m$  that vanishes on  $D_+$  plus some divisor  $A$  (but not on all of  $C$ ).

**Theorem 2.2.** *Let  $D = D_+ - D_-$  be a divisor on the smooth plane curve  $C$ . If there is a form  $G$  of degree  $m$  vanishing on  $D_+$  then we may write  $(G) = D_+ + A$ , and then the effective divisors equivalent to  $D$ , if any, are precisely those of the form  $(H) - A - D_-$  where  $H$  is a form of degree  $m$  vanishing on  $D_- + A$ , but not on all of  $C$ .*

In particular, if no homogeneous polynomial  $H$  of degree  $m$  vanishes on  $A + D_-$  but not on  $C$ , then  $D$  is not linearly equivalent to any effective divisor. The existence of such an  $H$  is thus independent of the choices of  $m$  and  $G$ , as we shall see in the proof.

The simplest special case of the theorem is the completeness of the linear series defined by intersections of  $C$  with curves of a given degree, or, equivalently, the fact that the restriction maps

$$H^0(\mathcal{O}_{\mathbb{P}^n}(m)) \rightarrow H^0(\mathcal{O}_C(m))$$

are surjective for all  $m$ .

**Proposition 2.3.** *If  $C$  is a smooth plane curve, then any Cartier divisor on  $D$  that is linearly equivalent to the divisor of a form of degree  $m$  is itself the divisor of a form of degree  $m$ ; that is, plane curves are arithmetically Cohen-Macaulay.*

The results for singular curves explained in Chapter 15 depend on strengthenings of this condition.

**Proof.** Let  $C$  be the plane curve defined by  $F = 0$ , and let  $D$  be the divisor on  $C$  defined by a form  $L$  of degree  $m$ , not vanishing on (any component of) the curve  $C$ . If  $G$  and  $H$  are forms of the same degree  $t$ , not vanishing on any component of  $C$ , and  $D + (G/H)$  is effective, then the divisor  $(LG)$  on  $C$  must contain the divisor  $(H)$  on  $C$ . This means that the subscheme of  $\mathbb{P}^2$  defined by  $(LG, F)$  contains the scheme defined by  $(H, F)$ . Since  $H$  and  $F$  have no components in common, Theorem 5.1 implies that  $LG = AH + BF$  for some forms  $A, B$  with  $\deg A = \deg LG - \deg H = m$  whence  $D + (G/H) = (A)$  as required.  $\square$

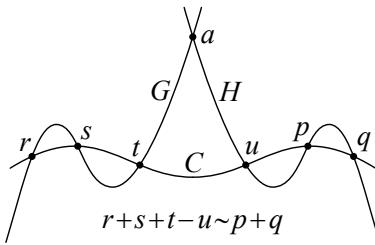


Figure 1. If  $G$  and  $H$  have the same degree, then  $r + s + t - u \sim p + q$ .

**Example 2.4.** Suppose that  $C$  has degree 3 and thus genus 1. If we choose as origin on the curve  $C$  a point  $o$ , then to add two points  $p$  and  $q \in C$  means to find the (unique) effective divisor of degree 1 linearly equivalent to  $p + q - o$ . In this situation, Theorem 2.2 applies with  $m = 1$ : there is a line  $L$  containing  $p + q$  defined by a linear form  $G$ . If  $r \in C$  be the remaining point of intersection of  $L$  with  $C$  we can choose a linear form  $H$  vanishing on  $o + r$ , and the line it defines meets  $C$  in one additional point  $s$ . This is the classical construction of the group law on the points of  $C$  (or, for curves over a field that is not algebraically closed, on the rational points of  $C$ ). See Figure 3

**Proof of Theorem 2.2.** First, suppose that we can find a form  $H$  of degree  $m$  as in the theorem. Setting  $D' = (H) - (D_- + A)$  we have

$$D' = D + (H/G) = D_+ - D_- - (D_+ + A) + (D_- + A + D')$$

so  $D'$  is linearly equivalent to  $D$ .

We claim that we find in this way all effective divisors  $D' \sim D$ . To see this, suppose  $D'$  is any effective divisor with  $D' \sim D$ , so that

$$\mathcal{O}_C(A + D_- + D') = \mathcal{O}_C(A + D_- + D) = \mathcal{O}_C(m),$$

that is,  $A + D_- + D' \equiv (G)$  for some form  $G$  of degree  $m$ . By Proposition 2.3 this implies that  $A + D_+ = (H)$  for some form of degree  $m$  as required.  $\square$

The argument given in Proposition 2.2 can be stated more generally thus: if curves  $F(X, Y, Z) = 0$  and  $Q(X, Y, Z) = 0$  meet only in a finite set of points  $\Gamma$  in  $\mathbb{P}^2$ , and  $E(X, Y, Z) = 0$  is a curve containing the intersection in an appropriate sense, then  $E = QH + LF$  for some forms  $H$  and  $L$ , and this statement applies to arbitrarily singular curves. Recognizing its importance for the argument above and the generalizations to come, Max Noether in [Noether 1873] dubbed it the *Fundamental theorem*, noting that it had often been used by geometers but not proven. After successive attempts and criticisms involving many mathematicians, he and Brill gave a complete proof in [Brill and Noether 1874]. For more of this story see the account in [Eisenbud and Gray 2023].

**2.3. The Cayley-Bacharach-Macaulay theorem.** The following result was proven by Macaulay [Macaulay 1900, p. 424], as a version of the Riemann-Roch theorem. It is now widely referred to as the Cayley-Bacharach theorem, named for an incorrect version asserted by Cayley and a correct special case later proven by Bacharach [Bacharach 1886]; see [Eisenbud and Gray 2023, Section 2.3] for more on this history, and [Eisenbud et al. 1996] (where the result is incorrectly attributed to Bacharach) for generalizations and related conjectures. Here is the version for divisors on a smooth plane curve:

**Theorem 2.5** (Cayley-Bacharach-Macaulay). *Let  $C$  be a smooth plane curve of degree  $d$ , suppose that  $E', E''$  are effective divisors on  $C$  such that  $E := E' + E'' = C \cap C'$ , the complete intersection of  $C$  with a curve  $C'$  of degree  $d'$ . For any integer  $0 \leq k \leq d + d' - 3$ , the difference in the number of conditions imposed on forms of degree  $k$  by  $E''$  and by  $E$  is equal to the degree of  $E'$  minus the number of conditions  $E'$  imposes on forms of degree  $d'' := d + d' - 3 - k$ —that is, the “failure of  $E'$  to impose independent conditions on forms of degree  $d''$ ”.*

Writing  $H$  for the divisor class on  $C$  of the intersection of  $C$  with a line, and setting  $s := d + d' - 3 - k$  this is the equality:

$$h^0(kH - E) - h^0(kH - E') = \deg E'' - (h^0(sH) - h^0(sH - E''))$$

**Proof.** Set  $e' := \deg E'$ ,  $e'' := \deg E''$  and  $e = e' + e'' = \deg E$ . By the adjunction formula, the divisor class of the canonical bundle on  $C$  is  $K := (d - 3)H$ . Using the Riemann–Roch theorem, the left hand side of the equality is

$$kd - e - h^0(K - (kH - E)) - (kd - e' - h^0(K - (kH - E'))).$$

Since  $K - (kH - E) = K - (kH - d'H) = sH$  and  $K - (kH - E') = sH + E''$ , we see that the left hand side is equal to  $e'' - h^0(sH) + h^0(sH + E'')$  as required.  $\square$

As noted in [Eisenbud and Gray 2023], the converse, proving the Riemann–Roch theorem for  $C$  from Theorem 2.5, is also easy.

**Corollary 2.6.** *A divisor  $E'$  on a smooth plane curve  $C \subset \mathbb{P}^2$  of degree  $d$  moves in a linear series of dimension  $r$  if and only if  $E'$  fails by  $r$  to impose independent conditions on curves of degree  $d - 3$ .*

**Proof.** For sufficiently large  $d'$  we can choose a curve  $C'$  of degree  $d'$  containing  $E$  and meeting  $C$  in  $E = E' + E''$ , where  $E''$  is disjoint from  $E'$ . Since  $C \cap C'$  is a complete intersection, every form vanishing on  $E + E'$  is a linear combination of the forms defining  $C$  and  $C'$ , and thus the dimension of the space of forms of degree  $d'$  vanishing on  $E$  modulo those vanishing on  $C$  is 1. By Theorem 2.2 the dimension  $r$  of the linear series  $|E'|$  is the dimension of the space of forms of degree  $d'$  modulo those vanishing on  $E + E'$ , and by Theorem 2.5

this is the failure of  $E'$  to impose independent conditions on forms of degree  $d'' = d + d' - d' - 3 = d - 3$ .  $\square$

To apply Theorem 2.5 it is helpful to know when points impose independent conditions on forms of a certain degree. Here is a first result of this kind:

**Proposition 2.7.** *Suppose that  $1 \leq n$ . Any set  $\Gamma$  of  $k \leq n$  distinct points in  $\mathbb{P}^2$  imposes independent conditions on forms of degree  $n-1$ ; and if  $\Gamma$  is not contained in a line then  $\Gamma$  imposes independent conditions on forms of degree  $n-2$ .*

**Proof.** To show that  $\Gamma$  imposes independent conditions on forms of degree  $d$  we must produce, for each  $p \in \Gamma$ , a form of degree  $d$  vanishing on

$$\Gamma_p := \Gamma \setminus \{p\}$$

but not  $p$ . If  $\Gamma$  imposes independent conditions on forms of degree  $d$  then  $\Gamma$  automatically imposes independent conditions on forms of degree  $d+1$ , so we may assume that  $k = n$ .

The product of general linear forms vanishing on general lines through the points of  $\Gamma_p$  does not vanish at  $p$ , proving the first statement.

Now assume that  $\Gamma$  is not contained in a line and  $p \in \Gamma$ . It suffices to show that for each  $p \in \Gamma$  there is a form of degree  $n-2$  vanishing on  $\Gamma_p$  but not on  $p$ . Since  $\Gamma$  spans  $\mathbb{P}^2$ , there is a spanning set of three points  $p, q, r$  of  $\Gamma$ . The union of the line spanned by  $q, r$  with general lines through the  $n-3$  points of  $\Gamma_p \setminus \{q_1, q_2\}$  is defined by a form of degree  $n-2$  containing  $\Gamma_p$  but not  $p$ , as required.  $\square$

**Corollary 2.8.** *Suppose that  $C \subset \mathbb{P}^2$  is a smooth plane curve of degree  $d \geq 3$ .*

- (1) *If  $\mathcal{V}$  is a  $g_e^1$  on  $C$  with  $e \leq d-1$  then  $e = d-1$  and  $\mathcal{V}$  corresponds to projection from a point of  $C$ .*
- (2) *If  $\mathcal{V}$  is a  $g_e^2$  on  $C$  with  $e \leq d \geq 4$  then  $e = d$ . Furthermore, the embedding of  $C$  in  $\mathbb{P}^2$  is unique up to automorphisms of  $\mathbb{P}^2$ .*

**Proof.** 1. If  $E$  is a divisor in the linear series  $\mathcal{V}$  then by Corollary 2.6 the points of  $E$  fail to impose independent conditions on forms of degree  $d-3$ . By Proposition 2.7 the degree of  $E$  is  $d-1$  and the points lie in a line  $L$ , which must meet  $C$  in an additional point  $p$ . By Theorem 2.2 the linear series  $|E|$  is residual to  $p$  in curves of degree 1; that is, it is the linear series corresponding to projection from  $p$ .

2. In this case a divisor in the series  $\mathcal{V}$  fails by 2 to impose independent conditions on forms of degree  $d-3$ , and thus fails (by at least 1) to impose independent conditions on forms of degree  $d-2$ . By Proposition 2.7 the degree

of  $E$  is  $d$ . Moreover, the points lie in a line  $L$  thus  $\mathcal{V}$  is a subseries of the series by which  $C$  is embedded.  $\square$

### 3. Curves of genus 1 and the group law of an elliptic curve

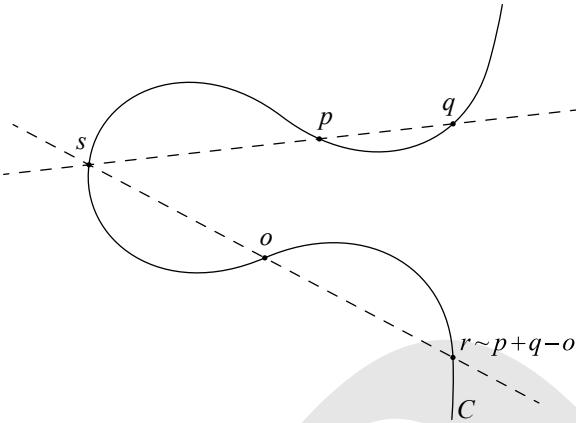
We will describe the maps of a curve of genus 1 given by the complete linear series in the lowest degree cases of interest:  $d = 2, 3, 4$  and  $5$ . Along the way we will see several ways of parametrizing the family of curves of genus 1 by one-dimensional varieties, forerunners of the moduli spaces that we will introduce in Chapters 7 and 8.

On a smooth, irreducible curve  $E$  of genus 1 the canonical sheaf has degree 0; and since it has a global section, it must be  $\mathcal{O}_C$ . Since invertible sheaves of negative degree cannot have any sections, the Riemann-Roch theorem shows that  $h^0(\mathcal{L}) = \deg \mathcal{L}$  for any  $\mathcal{L}$  of positive degree. Among the surprising consequences is that, given a point  $o \in E$ , there is a natural structure of abelian algebraic group on the points of  $E$  for which  $o$  is the zero element. A curve of genus 1 with a chosen point  $o$  is called an *elliptic curve*.

**Proposition 3.1.** *Let  $E$  be a curve of genus 1 and let  $o$  be a chosen point  $o \in E$ . If we set  $p + q = r$  where  $r$  is the unique effective divisor linearly equivalent to  $p + q - o$ , then  $E$  becomes a commutative algebraic group. Moreover, the group of divisor classes is divisible, in the sense that for any divisor  $D$  of degree  $n > 0$  there is a point  $p$  such that  $D \sim np$ .*

**Proof.** The group operation is easy to describe: Let  $E$  be an elliptic curve with the point  $o \in E$  chosen arbitrarily. If  $p, q$  are points of  $E$  then  $\mathcal{O}_E(p + q - o)$  has degree 1, and thus has a unique global section. This vanishes at a unique point  $r$ , which may also be described as the unique effective divisor linearly equivalent to  $p + q - o$ , and thus  $p + q = r$  in the group operation, which is thus obviously commutative. For the inverse, if  $r$  is the unique point linearly equivalent to  $2o - p$  then  $p + r - o \sim o$ , so that  $r = -p$ . For any divisor  $D$  we define  $\Sigma D$  to be the unique point linearly equivalent to  $\Sigma D - (\deg D - 1)o$ . Note that in this group operation any two linearly equivalent divisors have the same sum  $\Sigma D = \Sigma D'$ .

We can determine the point  $r \sim p + q - o$  and show that the group law is given by regular functions as follows. By Corollary 3.5 we may embed  $E$  in  $\mathbb{P}^2$  by any linear series of degree 3. Let  $L$  be the line through  $p + q$ , and suppose that  $L$  meets  $C$  in an additional point  $s$ . Since  $s$  is unique, its coordinates are polynomial functions (on a suitable affine chart) of the coordinates of  $p$  and  $q$ . The line through  $s$  and  $o$  meets  $C$  additionally in a point  $r$ , and by Theorem 2.2 we have  $r \sim p + q - o$ , as in Figure 3. and similarly the point  $r$  is a polynomial function of the coordinates of  $s$  and  $o$ .



**Figure 2.** Adding points  $p, q$  on a plane cubic with origin  $o$ : Silivo: please make the picture less symmetric:  $o$  should not be a flex point and  $p$  plays a role different from  $o$ .

To see that  $E$  is a divisible group, consider the maps  $n : E \rightarrow E$  given by multiplication by integers  $n$ . Since the group law is given by polynomial functions, the number of solutions  $p$  of the equation  $np = q$  is finite, and thus for  $n \neq 0$  this map is non-constant. This implies that it is surjective.  $\square$

It is convenient for some purposes to suppose that the linear series used to embed  $C$  in  $\mathbb{P}^2$  is  $|3o|$ ; this has the advantage that  $p, q, r \in C$  are collinear if and only if  $p + q + r = o$  in the group law.

**Remark 3.2.** From the definition it is obvious that the map  $E \rightarrow \text{Pic}_0(E)$  sending  $p$  to  $\mathcal{O}_E(p - o)$  is an isomorphism of groups, and adding multiples of  $o$  induces an isomorphism with each  $\text{Pic}_d(E)$  as well. This provides a natural sense in which the family of invertible sheaves on  $C$  can be treated as a smooth curve. In Chapter 5 we will see a general construction: the Picard group  $\text{Pic}_0(C)$  can be made into a variety, and for a curve  $C$  of genus  $g$  the effective divisors of degree  $g$  form a variety that surjects birationally to  $\text{Pic}_g(C)$ .

**Corollary 3.3.** *Given two invertible sheaves  $\mathcal{L}, \mathcal{L}'$  of the same degree on a curve  $E$  genus 1, there is an automorphism  $\sigma : E \rightarrow E$  such that  $\sigma^*\mathcal{L} = \mathcal{L}'$ .*

**Proof.** By Proposition 3.1 we may write  $\mathcal{L} \cong \mathcal{O}_E(np)$  and  $\mathcal{L}' \cong \mathcal{O}_E(np')$  for some points  $p, p'$ ; and translation by  $p - p'$  is an automorphism of  $E$  carrying one into the other.  $\square$

#### 4. Low degree divisors on curves of genus 1

In this section we will describe the complete linear series of degrees 2,3,4, and 5 on a curve of genus 1, and say something about the geometry of each case.

Several of these descriptions lead to natural guesses answering the apparently silly question: How many different curves of genus 1 are there?

**4.1. The dimension of families.** To make sense of this question, we observe a fundamental fact of algebraic curve theory that the set of isomorphism classes of smooth, projective curves of a given genus  $g$  is naturally parametrized by the points of a an irreducible quasiprojective variety  $M_g$ , called the *moduli space* of curves of genus  $g$ . We will have a great deal more to say about moduli spaces in general, and  $M_g$  in particular, in Chapters 7 and 8.

We will use the low degree embeddings to describe the family of isomorphism classes of curves of genus 1 in several ways, giving what seem to be natural estimates of its dimension. We leave to Chapter 8 the reasoning that leads to a map from the base of such a family to  $M_1$ , validating the estimates we give here.

**4.2. Double covers of  $\mathbb{P}^1$ .** Let  $E$  be a smooth projective curve of genus 1 and let  $\mathcal{L}$  be an invertible sheaf of degree 2 on  $E$ . By the Riemann-Roch theorem,  $h^0(\mathcal{L}) = 2$  and the linear series  $|\mathcal{L}|$  is base-point free, so we get a map  $\phi : E \rightarrow \mathbb{P}^1$  of degree 2. By the Riemann-Hurwitz theorem the map  $\phi$  will have 4 branch points, which must be distinct because in a degree 2 map only simple branching is possible. By Corollary 3.3, this set of four points are determined, up to automorphisms of  $\mathbb{P}^1$ , by the curve  $E$ , and are independent of the choice of  $\mathcal{L}$ .

In terms of a given embedding of  $E$  as a smooth cubic in  $\mathbb{P}^3$ , Corollary 2.8 shows that the linear series corresponding to a map to  $\mathbb{P}^1$  of degree 2 are projections from points of  $E$ , suggesting again that  $M_1$  has dimension 1.

We will see in Chapter 6 that a double cover of  $\mathbb{P}^1$  is determined by its branch points, so the family  $H$  of double covers of  $\mathbb{P}^1$  having genus 1 is four-dimensional. There is a map  $H \rightarrow M_1$ , sending such a double cover to the isomorphism class of the cover, and by what we have said, the fibers of this map are isomorphic to the automorphism group  $PGL_2$  of  $\mathbb{P}^1$ ; thus we expect that  $M_1$  has dimension  $4 - 3 = 1$ .

**4.3. Plane cubics.** We can also represent an arbitrary smooth curve of genus 1 as a plane cubic: Let  $\mathcal{L}$  be an invertible sheaf of degree 3 on  $E$ . As in the proof of Proposition 3.1 the linear series  $|\mathcal{L}|$  gives an embedding of  $E$  as a smooth plane cubic curve of degree 3; conversely, the adjunction formula implies that any smooth plane cubic curve has genus 1.

The space of plane cubic curves is parametrized by the space of cubic forms in 3 variables up to scalars, a  $\mathbb{P}^9$ . The locus of forms defining smooth curves is a Zariski open subset. If two plane cubics are abstractly isomorphic, that is

if we have two different degree 3 linear series  $|\mathcal{L}|, |\mathcal{L}'|$  mapping a given genus 1 curve  $E$  to the plane, then by Proposition 3.3 we may precompose one of the maps with an automorphism of  $E$  and suppose that  $\mathcal{L} = \mathcal{L}'$ . Thus the two curves differ by an element of  $PGL_3$  of automorphisms of  $\mathbb{P}^2$ . Since the group  $PGL_3$  has dimension 8, one should expect that the family of such curves up to isomorphism has dimension 1, which accords with the dimension computed in the previous section.

Corollary 2.8 does not determine the degree 3 maps of  $E$  to  $\mathbb{P}^2$ , but Theorem 2.2 can be applied directly, given one embedding  $E \subset \mathbb{P}^2$ . If  $D$  is a divisor of degree 3 on  $E \subset \mathbb{P}^2$ , we can find a conic containing  $D$ . The conic will meet  $E$  in  $D$  plus another divisor  $D'$  of degree 3, and the complete linear series  $D$  is then cut out by conics containing  $D'$ —that is, the divisors equivalent to  $D$  are those residual to  $D'$  in the intersection of  $E$  with conics containing  $D'$ . Since  $D$  fails by 2 to impose independent conditions on constants, the linear series has dimension 2 as expected. In fact, since any  $n$  points of  $E$  fail by  $n - 1$  to impose independent conditions on constants, this shows again that  $\dim |D| = \deg D - 1$ , and we can find the divisors in  $D$  using Theorem 2.2

**4.4. Genus 1 quartics in  $\mathbb{P}^3$ .** By Corollary 3.5, any divisor of degree 4 on  $E$  embeds  $E$  in  $\mathbb{P}^3$ . Representing  $E$  as a plane cubic, we see that a conic containing a divisor  $D$  of degree 4 on  $E$  meets  $E$  in  $D + D'$ , where  $D'$  has degree 2. Following the prescription of Theorem 2.2, we see that the linear series  $|D|$  may be represented as the residual to  $D'$  in the series of conics containing  $D'$ . We can also regard this space of conics as a linear series on  $\mathbb{P}^2$ , with base locus  $D'$ . As such it maps  $\mathbb{P}^2$  rationally to  $\mathbb{P}^3$ . The pullbacks of planes in  $\mathbb{P}^3$  are the conics through  $D'$ , and the fact that a general pair of them meet, away from  $D'$ , in 2 points, means that the intersection of the image surface with a line in  $\mathbb{P}^3$  consists of two points—the surface is a quadric. Assuming for simplicity that  $D' = p + q$ , the sum of two distinct points, the linear series is well-defined on the blowup of the plane at  $p$  and  $q$ . However, a conic meeting the line  $L$  spanned by  $p$  and  $q$  in any additional point  $r$  must contain  $L$ , and thus  $L$  is contracted by the linear series: the smooth quadric in  $\mathbb{P}^3$  can be described in this way.

The quadric we have constructed is not distinguished in the ideal of  $E$ :

**Proposition 4.1.** *The image of a genus 1 curve  $E$  by a linear series of degree 4 is the complete intersection of two quadrics in  $\mathbb{P}^3$ , and conversely any smooth complete intersection of two quadrics in  $\mathbb{P}^3$  is a curve of genus 1.*

**Proof.** Consider the restriction map

$$\rho_2 : H^0(\mathcal{O}_{\mathbb{P}^3}(2)) \rightarrow H^0(\mathcal{O}_E(2)) = H^0(\mathcal{L}^2).$$

The space on the left—the space of homogeneous polynomials of degree 2 in four variables—has dimension 10, while by the Riemann-Roch theorem the

space  $H^0(\mathcal{L}^2)$  has dimension 8. It follows that  $E$  lies on at least two linearly independent quadrics  $Q$  and  $Q'$ . Since  $E$  does not lie in any plane, neither  $Q$  nor  $Q'$  can be reducible, so they have no component in common. Thus by Bezout's Theorem  $Q \cap Q'$  has degree 4, so

$$E = Q \cap Q'.$$

Moreover,  $Q, Q'$  form a regular sequence, so the ideal  $(Q, Q')$  is unmixed, and thus the homogeneous ideal  $I(E)$  is generated by these two quadrics.

Conversely, if  $E := Q \cap Q'$  is a smooth complete intersection of two quadrics then every quadric in the pencil of quadrics spanned by  $Q$  and  $Q'$  is nonsingular along the base locus  $E$ , and thus by Bertini's theorem the general member  $Q_0$  of this pencil is nonsingular. Since  $E$  is the intersection of  $Q_0$  with another quadric, we see that  $E$  has class  $(2, 2)$  on  $Q_0$ , and thus has genus 1 by the adjunction formula. (In fact the same argument works for complete intersection of any two quadrics, showing that it has arithmetic genus 1, as we will see in Chapter 16.)  $\square$

From Proposition 4.1 we see that an elliptic quartic in  $\mathbb{P}^3$  determines a point in the Grassmannian  $G(2, H^0(\mathcal{O}_{\mathbb{P}^3}(2))) = G(2, 10)$  of pencils of quadrics; and by Bertini's theorem, a Zariski open subset of that Grassmannian corresponds to smooth quartic curves of genus 1. The Grassmannian  $G(2, 10)$  has dimension 16, while the group  $PGL_4$  of automorphisms of  $\mathbb{P}^3$  has dimension 15, so this suggests again that the family of curves of genus 1 up to isomorphism has dimension 1.

There is a direct way to go back and forth between the representation of the smooth genus 1 curve  $E$  as the intersection of two quadrics in  $\mathbb{P}^3$  and the representation of  $E$  as a double cover of  $\mathbb{P}^1$  branched at 4 distinct points. First, by Bertini's theorem, we may take the two quadrics to be nonsingular, since they must meet transversely along  $E$ , and elsewhere the pencil of quadrics they span has no base points. Representing the quadrics as symmetric matrices  $A, B$ , the pencil of all quadrics containing  $E$  can be written as  $sA + tB$ . A quadric in the pencil is singular at the points  $(s, t)$  such that the quartic polynomial  $\det(sA + tB)$  vanishes; thus at 4 points.

A smooth quadric has two rulings by lines; a cone has one. Thus the family

$$\Phi := \{(\lambda, \mathcal{L}) \mid \mathcal{L} \in \text{Pic}(Q_\lambda) \text{ is the class of a ruling of } Q_\lambda\}$$

is—at least set-theoretically—a 2-sheeted cover of  $\mathbb{P}^1$ , branched over the four values of  $\lambda$  corresponding to singular quadrics in the pencil. In fact, we claim

**Proposition 4.2.** *There is an isomorphism of  $\Phi$  with  $E$ , and thus the branch points of  $\Phi$  over  $\mathbb{P}^1$ —that is, the set of singular elements of the pencil of quadrics—are the same, up to automorphisms of  $\mathbb{P}^1$  as the four points over which a double cover of  $\mathbb{P}^1$  by  $E$  are ramified.*

**Proof.** First, choose a base point  $o \in E$ . We will construct inverse maps  $E \rightarrow \Phi$  and  $\Phi \rightarrow E$  as follows:

- (1) Suppose  $q \in E$  is any point other than  $o$ , and let  $M \subset \mathbb{P}^3$  be the line  $\overline{o, q}$  spanned by  $o, q$ . Every quadric  $Q_\lambda$  contains the two points  $o, q \in M$ . It is one linear condition for a quadric  $Q_\lambda$  to contain a given point so if  $r \in M$  is any third point, there will be a unique  $\lambda$  such that  $r$ , and hence all of  $M$ , lies in  $Q_\lambda$ . Thus the choice of  $q$  determines both one of the quadrics  $Q_\lambda$  of the pencil, and a ruling of that quadric, giving us a map  $E \rightarrow \Phi$ .
- (2) Given a quadric  $Q_\lambda$  and a choice of ruling of  $Q_\lambda$ , there is a unique line  $M \subset Q_\lambda$  of that ruling passing through  $o$ , and that line  $M$  will meet the curve  $E$  in one other point  $q$ ; this gives us the inverse map  $\Phi \rightarrow E$ .

□

**Cheerful Fact 4.3.** *There is a beautiful extension of this result to pencils of quadrics in any odd-dimensional projective space. Briefly: a smooth quadric  $Q \subset \mathbb{P}^{2g+1}$  has two rulings by  $g$ -planes, which merge into one family when the quadric specializes to quadric of rank  $2g + 1$ , that is, a cone over a smooth quadric in  $\mathbb{P}^{2g}$ . If  $\{Q_\lambda\}_{\lambda \in \mathbb{P}^1}$  is a pencil with smooth base locus  $X = \cap_{\lambda \in \mathbb{P}^1} Q_\lambda$ , then exactly  $2g + 2$  of the quadrics will be singular, and they will all be of rank  $2g + 1$ . The space  $\Phi$  of rulings of the quadrics  $Q_\lambda$  is thus a double cover of  $\mathbb{P}^1$  branched at  $2g + 2$  points. We shall see in Chapter 6 that this double cover is a hyperelliptic curve of genus  $g$ . For a proof see for example [Harris 1995, Proposition 22.34]. This shows in particular that the polynomial  $\det(sA + tB)$  has  $2g + 2$  distinct roots.*

*There is also a remarkable analogue of Proposition 4.2 for any  $g$ : the variety  $F_{g-1}(X)$  of  $(g - 1)$ -planes in the base locus  $X$  of the pencil is isomorphic to the Jacobian of the curve  $\Phi$ . (We will discuss Jacobians in Chapter 5.) A proof of this in case  $g = 2$  can be found in [Griffiths and Harris 1978]; for all  $g$  it is done in [Donagi 1980]. For a further study of the equivalence, see [Eisenbud and Schreyer 2022].*

**4.5. Genus 1 quintics in  $\mathbb{P}^4$ .** Now suppose that  $D$  is a divisor of degree 5 on the smooth cubic  $E \subset \mathbb{P}^2$ , and consider the embedding  $\phi_D : E \hookrightarrow \mathbb{P}^4$ , the embedding by the complete linear series  $|D|$ . We first observe that since  $D$  is contained in a cubic, at most 3 points of  $D$  can lie on a line. Thus  $D$  imposes independent conditions on conics: that is, there is a unique conic  $C$  containing

$D$ , and thus we may write the divisor  $C \cap E$  as  $D + p$  for some point  $p \in E$ . By Theorem 2.2 the linear series  $|D|$  consists of the divisors residual to  $p$  in the intersection of  $E$  with conics through  $p$ .

This description again determines an interesting surface containing  $\phi_D(E)$ : the image  $X$  of  $\mathbb{P}^2$  by the rational map defined by the conics containing  $p$ . This map is well-defined on the blowup of  $\mathbb{P}^2$  at  $p$ , and it follows that the degree of  $X$  is the number of intersections of two conics away from  $p$ , that is  $\deg X = 3$ . We can also see from this that  $X$  is ruled by lines, since the images of lines in  $\mathbb{P}^2$  through  $p$  become disjoint in the blowup. The image of a line through  $p$  meets  $\phi_D(E)$  in the images of the two points of intersection  $E \cap L$  away from  $p$ ; that is, the lines are the spans of the points in the  $g_2^1$  on  $E$  given by projection from  $p$ . We shall study this construction in a much more general setting in Chapter 17.

Another interesting surface containing  $\phi_D(D)$  can be constructed in a similar way. Since  $D$  imposes independent conditions on conics, it also imposes independent conditions on cubics in  $\mathbb{P}^2$ . If we choose a cubic  $E'$  containing  $D$  and distinct from  $E$ , then we may write  $E \cap E' = D + D'$  where  $D' \subset E$  is a divisor of degree  $9 - 5 = 4$ . By Theorem 2.2 the series  $|D|$  is also the linear series of intersections of  $E$  with cubics containing  $D'$ .

The linear series of cubics in  $\mathbb{P}^2$  containing  $D'$ , without restricting it to  $E$ , is a series of dimension 6, and maps  $\mathbb{P}^2$  rationally to a surface  $Y \subset \mathbb{P}^5$ , called a *del Pezzo surface of degree 5* (see Section 3.2 for more on del Pezzo surfaces). Since  $E$  itself is one of the cubics containing  $D'$ , the embedded curve  $\phi_D(E)$  is a hyperplane section of  $Y$ , which must have degree 5; the quadrics containing  $Y$  in  $\mathbb{P}^5$  pull back to sextic curves in  $\mathbb{P}^2$  vanishing doubly on  $D'$ . Vanishing doubly at a point imposes 3 linear conditions on a form (one to vanish, and two more for the derivatives to vanish), so the linear series of quadrics in  $\mathbb{P}^5$ , restricted to  $Y$ , has dimension at least  $\binom{6+2}{2} - 3 * 4 = 16$  and one can show (Exercise 5.2) that in this case the conditions are independent. Thus the family of quadrics containing  $Y$ , modulo the ideal of  $Y$ , has vector space dimension 16, and we see that  $Y$  lies on  $5 = \binom{5+2}{2} - 16$  quadrics.

On the other hand, we can determine the space of quadrics containing the quintic curve  $\phi_D(E)$  as the kernel of the map

$$H^0(\mathcal{O}_{\mathbb{P}^4}(2)) \rightarrow H^0(\mathcal{O}_{\phi_D(E)}(2)) = H^0(\mathcal{O}_E(5)).$$

The left hand term has dimension 15, and the right hand term dimension 5. As we shall see in Corollary 4.5, the curve  $\phi_D(E)$  is arithmetically Cohen-Macaulay, so the map above is surjective and the kernel has dimension exactly 5; it is thus the restriction to the hyperplane section of the family of quadrics containing the del Pezzo surface  $Y$ .

Much more can be said about the surface  $Y$  and the equations of  $\phi_D(E)$ , which we will only summarize:

**Cheerful Fact 4.4** (Quintic curves of genus 1 in  $\mathbb{P}^4$ ). *Recall that if  $A$  is a skew-symmetric matrix of even size, then the determinant of  $A$  is the square of a polynomial in the entries of  $A$  called the Pfaffian of  $A$ . For example, if*

$$M = \begin{pmatrix} 0 & x_{1,1} & x_{1,2} & x_{1,3} \\ -x_{1,1} & 0 & x_{2,2} & x_{2,3} \\ -x_{1,2} & -x_{2,2} & 0 & x_{3,3} \\ -x_{1,3} & -x_{2,3} & -x_{3,3} & 0 \end{pmatrix}$$

*then the Pfaffian of  $M$  is by definition the quadric  $x_{1,1}x_{3,3} - x_{1,2}x_{2,3} + x_{1,3}x_{2,2}$ . Let  $\tilde{A}$  be a  $5 \times 5$  generic alternating matrix (that is, a skew-symmetric matrix with 0 on the diagonal and independent variables in the 10 entries above the diagonal), and let  $I$  be the ideal generated by the  $4 \times 4$  Pfaffians of the submatrices of  $A$  leaving out one row and the corresponding column.*

*As a consequence of the main theorem of [Buchsbaum and Eisenbud 1977a] (see also [Eisenbud 1995a, Theorem 11]) we have:*

**Proposition 4.5.** *The scheme defined by  $I$  is a smooth irreducible Cohen-Macaulay variety  $\tilde{Y}$  of codimension 3 in  $\mathbb{P}^9$ , and both the quintic del Pezzo surface  $Y$  and the quintic curve  $\phi_D(E)$  are plane sections of  $\tilde{Y}$ . Moreover, with a suitable choice of bases and variables, the presentation matrix of  $I(\phi_D(E))$  is a  $5 \times 5$  matrix of linear forms*

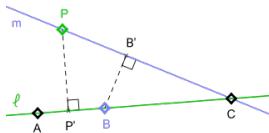
$$B = \begin{pmatrix} 0 & 0 & x_0 & x_1 & x_2 \\ 0 & 0 & x_1 & x_2 & x_3 \\ -x_0 & -x_1 & 0 & \ell_1 & \ell_2 \\ -x_1 & -x_2 & -\ell_1 & 0 & \ell_3 \\ -x_2 & -x_3 & -\ell_2 & -\ell_3 & 0 \end{pmatrix}$$

*and the homogeneous ideal of  $\phi_D(E)$  is generated by the  $4 \times 4$  Pfaffians of this matrix. The ideal of the scroll  $X$  is defined by the  $2 \times 2$  minors of the  $2 \times 3$  matrix in the upper right corner of  $A$ , which are equal to the Pfaffians of the matrices leaving out the 3-rd, 4-th and 5-th row and corresponding columns of  $B$*

## 5. Exercises

**Exercise 5.1.** Let  $C$  be a smooth plane curve of degree  $d$ . Show that  $C$  admits a one-parameter family of maps  $C \rightarrow \mathbb{P}^1$  of degree  $d - 1$ . Using the Riemann-Roch theorem, and Proposition 2.7, show that  $C$  does not admit a map  $C \rightarrow \mathbb{P}^1$  of degree  $d - 2$  or less.

Hint: show that any  $d - 2$  points impose independent conditions on forms of degree  $d - 3$ , and use Bertini to reduce to this case.



**Exercise 5.2.** Show that the vector space of sextic curves in  $\mathbb{P}^2$  vanishing doubly at 4 points, no three of which are colinear, has dimension exactly 16.

Hint: First, the space of sextics double at three non-colinear points visibly has dimension 19 (take the points to be the coordinate points and count monomials). Then, since this includes the triangle with vertices at the three points plus arbitrary cubics, the subspace of those double at the fourth point will have codimension 3.

**Exercise 5.3.** Let  $p \in C$  be a smooth point of a plane curve with equation  $F(x_0, x_1, x_2) = 0$  of degree  $d > 1$ . Show that the tangent line to  $C$  at  $p$  meets  $C$  in a scheme of order  $m + 2$  if and only if the determinant of the Hessian matrix vanishes along  $C$  to order  $m$  as is done in [Kunz 2005, pp. 84–85]:

- (1) Assume that  $p = (0, 0) \in \mathbb{A}^2 \subset \mathbb{P}^2$ , where  $\mathbb{A}^2$  is the locus  $x_0 \neq 0$ , and suppose that  $f(x, y) = 0$  is the affine equation of  $C$ , with  $x = x_1/x_0, y = x_2/x_0$ . Reduce to the affine case by showing that

$$\det \text{Hess}(C) = x_0^2 \det \begin{pmatrix} d(d-1)f & (d-1)\partial f/\partial x_1 & (d-1)\partial f/\partial x_2 \\ (d-1)\partial f/\partial x_1 & \partial^2 f/\partial x_1 \partial x_1 & \partial^2 f/\partial x_1 \partial x_2 \\ (d-1)\partial f/\partial x_2 & \partial^2 f/\partial x_2 \partial x_1 & \partial^2 f/\partial x_2 \partial x_2 \end{pmatrix}.$$

Writing the partial derivatives as subscripts, this becomes

$$\det \begin{pmatrix} d(d-1)f & (d-1)f_x & (d-1)f_y \\ (d-1)f_x & f_{xx} & f_{xy} \\ (d-1)f_y & f_{xy} & f_{yy} \end{pmatrix}$$

when restricted to  $\mathbb{A}^2$ .

- (2) Assume that the tangent line to  $C$  at  $p$  is  $y = 0$ . Show that  $f$  can be written as

$$f = x^{m+2}\phi(x) + y\psi(x, y)$$

where  $\phi(0) \neq 0$  and  $\psi(0, 0) \neq 0$ , and thus, modulo  $f$ , the Hessian determinant, up to a constant factor, has the form

$$f_x^2 f_{yy} + f_y^2 f_{xx} - 2f_x f_y f_{xy}.$$

Note that  $x$  is a local parameter at  $p$ . Using the form of  $f$  above, show that there is a unique term vanishing to order  $m$  at  $p$ , and no term vanishing to lower order there.

**Exercise 5.4.** As we have seen, each line through two flexes of a smooth genus 1 curve of degree 3 in  $\mathbb{P}_{\mathbb{C}}^2$  passes through a third flex; but this configuration is not possible in  $\mathbb{P}_{\mathbb{R}}^2$ . Prove

**Theorem 5.5** (Sylvester-Gallai theorem). *In any finite set  $\Gamma \subset \mathbb{P}_{\mathbb{R}}^2$  there are 3 non-colinear points unless  $\Gamma$  is contained in a line.*

Hint: (Following Leroy Milton Kelly): We may assume that  $\Gamma \subset \mathbb{R}^2$ . Choose a pair  $P, L$  consisting of a point  $P \in \Gamma$  and line  $\ell$  containing at least 2 of the points of  $\Gamma$ , but not  $P$ , such that the distance from  $L$  to  $P$  is minimal among such pairs. If there were at least 3 points of  $\Gamma$  on  $L$  then, considering diagram 5, show that there is a point of  $\Gamma$  and a line  $L'$  violating the minimal distance hypothesis.

## Jacobians

We have seen that the points of an elliptic curve naturally form an algebraic group. This is not true for curves of higher genus, but there is a substitute, of which Barry Mazur wrote:

The useful generalization [of the group structure on the points of an elliptic curve] ...has the advantage that by its very definition it is seen to be intrinsic (it depends only on the birational equivalence class of the curve and not upon its imbedding in the projective plane) and, moreover, three felicitous miracles occur (analytic, algebraic, and arithmetic) which make it an extraordinarily powerful tool. — [Mazur 1986]

An essential construction in studying a curve  $C$  is the association of an invertible sheaf to a divisor—in other words, the map

$$\{\text{effective divisors of degree } d \text{ on } C\} \xrightarrow{\mu} \{\text{invertible sheaves of degree } d \text{ on } C\}.$$

sending  $D$  to  $\mathcal{O}_C(D)$ .

A priori, this is a map of sets. But it is a fundamental fact that each set may be given the structure of an algebraic variety in a natural way, and that in terms of this structure the map between them is a morphism. In many ways the geometry of the map governs the geometry of the curve. In this chapter we will describe the source and target of  $\mu$ , and give references to proofs of their properties.

We start with the effective divisors. Since  $C$  is smooth, an effective divisor of degree  $d$  on  $C$  is the same thing as a subscheme  $D \subset C$  of dimension 0 and degree  $d$ , and thus the family  $C_d$  of effective divisors of degree  $d$  on  $C$  is a

Hilbert scheme; see Section 3. This Hilbert scheme may be identified with the  $d$ -th symmetric power  $C_d$  of  $C$ , described in Section 1.

The parametrization of the set of invertible sheaves on  $C$  of a given degree  $d$  by the variety  $\text{Pic}_d(C)$ , called the *Picard variety* of  $C$ , requires different techniques. We will define it by a universal property in the category of schemes, and exhibit its construction as an analytic variety, actually a complex torus  $\text{Jac}(C)$ , whose group structure reflects the tensor product of sheaves in  $\text{Pic}_0(C)$ . Historically, the algebraic construction was a major milestone, first reached in the work of Andre Weil in the middle of the 20th century, and then reshaped by Grothendieck and his school. The interested reader will find a beautiful, detailed account both of the history and the modern theory of the scheme of divisors and the Picard scheme in the exposition [Kleiman 2005], which has extensive references to the original literature. We will give more specific references to this text as we go.

A consequence of the modern theory is that we can extend the construction of the Picard scheme to singular curves as well as smooth ones. Also, given a family  $\pi : C \rightarrow B$  of curves there is a corresponding family of Picard schemes  $\text{Pic}_d(C/B) \rightarrow B$ , whose fiber over a closed point  $b \in B$  is the Picard scheme  $\text{Pic}_d(C_b)$  of the corresponding curve  $C_b$  in our family. In Chapter 14, we'll have occasion to describe the Picard schemes of  $g$ -nodal and  $g$ -cuspidal curves, and to see how these fit into families with Picard schemes of smooth curves.

As an application of the existence of the spaces  $C_d$  and  $\text{Pic}_d$ , we show in Section 5 that a general divisor of degree  $g + 3$  on any curve of genus  $g$  gives rise to an embedding in  $\mathbb{P}^3$  as a curve of degree  $g + 3$ , and there are related results for general divisors of degree  $g + 2$  and  $g + 1$ .

## 1. Symmetric products and the universal divisor

Let  $C$  be a smooth curve. The space of effective divisors on  $C$  can be characterized by a universal property:

**Definition 1.1.** Let  $B$  be any scheme. A *family* of effective divisors of degree  $d$  on  $C$ , parametrized by the scheme  $B$ , is a Cartier divisor  $X \subset B \times C$  whose intersections with fibers  $\{b\} \times C \cong C$  over points of  $B$  are divisors of degree  $d$  on  $C$ .

Given this, we have a contravariant functor

$$F : (\text{schemes}) \rightarrow (\text{sets}),$$

taking a scheme  $B$  to the set of families of divisors of degree  $d$  on  $C$  over  $B$ ; if  $\pi : B' \rightarrow B$  is any morphism, the induced map  $F(B) \rightarrow F(B')$  is defined by taking a family  $D \subset B \times C$  to the fiber product  $D' := B' \times_B D \subset B' \times C$ . We say

that a scheme  $C_d$  is a fine moduli space for divisors of degree  $d$  on  $C$  if there is an isomorphism of functors

$$F \cong \text{Hom}_{\text{Schemes}}(-, C_d).$$

By Yoneda's lemma (Exercise 9.4), this is equivalent to the existence of a *universal family*  $D \subset C_d \times C$ , with the property that for any family  $X \subset B \times C$  of divisors on  $C$  over any scheme  $B$ , there is a unique map  $\phi : B \rightarrow C_d$  such that  $X = (\phi \times id_C)^{-1}(D)$ .

From the universal property it is clear that a fine moduli space for divisors of degree  $d$  on  $C$  is unique if it exists. Indeed, it does exist, and we'll sketch the construction, using symmetric products. This construction relies on the existence of quotients of schemes by finite groups, and we pause to discuss such quotients.

**1.1. Finite group quotients.** If  $G$  is a finite group acting by automorphisms on an affine scheme  $X := \text{Spec } A$  then  $X/G$  is by definition  $\text{Spec}(A^G)$ , the spectrum of the ring  $A^G$  of invariant elements of  $A$ . Since the functions in  $A^G$  are constant on orbits, every fiber is a union of orbits, but something much better is true:

**Theorem 1.2.** *The map  $\pi : X \rightarrow X/G$  induced by the inclusion of rings is finite. If  $X$  is a normal variety, then each fiber of  $\pi$  is a single orbit of  $G$ .*

For a proof, see for example [Eisenbud 1995a, Proposition 13.10]).

Since the map  $X \rightarrow X/G$  is finite, we have  $\dim X/G = \dim X$ .

The construction commutes with the passage to  $G$ -invariant open affine sets, and thus passes to quasi projective schemes as well (Exercise 9.1), though not to arbitrary schemes (see [Olsson 2016, Example 5.3.2]).

When the group  $G$  is positive-dimensional, the situation becomes much more complex, and is the subject of Geometric Invariant Theory—see Chapter 8 for some idea of what can and cannot be done in a special case.

If  $X$  is any quasi-projective scheme we define the  $d$ -th symmetric power  $X^{(d)}$  of  $X$  to be the quotient of the Cartesian product  $X^d$  of  $d$  copies of  $X$  by the action of the symmetric group of all permutations of the factors, and let  $\pi_d : X^d \rightarrow X^{(d)}$  be the projection. There is a natural subvariety

$$D = \{(x, y) \in X^{(d)} \times X \mid y \text{ is in the projection of } \pi_d^{-1}(x) \text{ to the first factor}\};$$

in other words, if we think of a point  $x \in X^{(d)}$  as an unordered  $d$ -tuple of points on  $X$ , the fiber of  $D$  over  $x$  is the collection of points in  $x$ .

If  $X = \mathbb{A}^1$  then  $X^d = \mathbb{A}^d = \text{Spec } \mathbb{C}[x_1, \dots, x_d]$ . The ring of invariants of the symmetric group acting on  $\mathbb{C}[x_1, \dots, x_d]$  by permuting the variables is generated by the  $d$  elementary symmetric functions, which generate a polynomial subring, and  $X^{(d)}$  is isomorphic to  $\mathbb{A}^d$ . Since the symmetric functions of the roots of a polynomial in one variable are the coefficients of that polynomial, we may identify  $X^{(d)}$  with the affine space of monic polynomials of degree  $d$  in  $\mathbb{C}[z]$ . ([Eisenbud 1995a, Exercises 1.6, 13.2-13.4])

Next consider  $X = \mathbb{P}^1$  and the product  $(\mathbb{P}^1)^d$ . Taking the homogeneous coordinates of the  $i$ -th copy of  $\mathbb{P}^1$  to be  $(s_i, t_i)$ , the  $d+1$  multilinear symmetric functions of degree  $d$ ,

$$t_1 t_2 \cdots t_d, \sum_i s_i t_1 \cdots \hat{t}_i \cdots t_d, \dots, s_1 \cdots s_d$$

are the coefficients of the polynomial

$$(s_1\lambda + t_1\mu)(s_2\lambda + t_2\mu) \cdots (s_d\lambda + t_d\mu)$$

defining a general subscheme of  $\mathbb{P}^1$  of degree  $d$ . These forms define an isomorphism  $(\mathbb{P}^1)^{(d)} \rightarrow \mathbb{P}^d$ . Again, we may think of this map as taking a  $d$ -tuple of points to the unique-up-to-scalars homogeneous form of degree  $d$  vanishing on it. Identifying  $(\mathbb{P}^1)^{(d)}$  with the space of forms  $F$  of degree  $d$ , the fiber of the subscheme  $D \subset \mathbb{P}^1 \times \mathbb{P}^d \rightarrow \mathbb{P}^d$  over a form  $F$  is the subscheme defined by  $F$ .

We shall see that when  $C$  is a smooth curve of higher genus, the global geometry of  $C^{(d)}$  is nontrivial, but at least the local geometry is simple:

**Proposition 1.3.** *If  $C$  is a smooth curve then each symmetric power  $C^{(d)}$  is smooth, and the subscheme  $D \subset C \times C^{(d)}$  is a family of divisors.*

**Proof.** It suffices to show that the quotient of an invariant formal neighborhood of the preimage  $\{p_1, \dots, p_s\}$  of a point  $\bar{p} \in C^{(d)}$  in  $C^d$  is smooth. After completing the local rings, we get an action of the symmetric group  $G$  on the product of the completions of  $X$  at the  $p_i$ , and this depends only on the orbit structure of  $G$  acting on  $\{p_1, \dots, p_s\}$ . Thus it would be the same for some orbit of points on  $\mathbb{A}^1$ , so the general case follows from the smoothness of  $(\mathbb{A}^1)^{(d)}$ .  $\square$

By contrast, if  $\dim X \geq 2$ , the symmetric powers  $X^{(d)}$  are singular for all  $d \geq 2$ . See Exercise 9.3 for the case of  $(\mathbb{A}^2)^{(2)}$  and Exercise 9.2 for a well-behaved case.

It is clear from Theorem 1.2 that the points of  $C^{(d)}$  are in one-to-one correspondence with the effective divisors of degree  $d$  on  $C$ , but much more is true.

**Cheerful Fact 1.4.**

**Theorem 1.5.** *If  $C$  is a smooth projective curve, then the  $d$ th symmetric power  $C^{(d)}$  of  $C$  is the fine moduli space  $C_d$  of effective divisors of degree  $d$  on  $C$  with universal family  $D \subset C \times C_d \rightarrow C_d$ .*

*For a proof in the analytic category, see [Arbarello et al. 1985, p. 164]; for the algebraic fact, see [Kleiman 2005, Remark 9.3.9]. The result remains true for families of curves, that is, smooth curves in the category of schemes over a base scheme  $B$ .*

Henceforward we will write  $C_d$  in place of  $C^{(d)}$ .

## 2. The Picard varieties

As with  $C_d$ , we may define  $\text{Pic}_d(C)$  by a universal property. We start by saying what we mean by a family of invertible sheaves on a smooth curve  $C$ :

**Definition 2.1.** For any scheme  $B$ , a *family of invertible sheaves on  $C$  over  $B$*  is an equivalence class of invertible sheaves  $\mathcal{L}$  on  $B \times C$ , where two such families  $\mathcal{L}$  and  $\mathcal{L}'$  are equivalent if they differ by an invertible sheaf pulled back along the projection  $\pi_1 : B \times C \rightarrow B$ , that is, if

$$\mathcal{L}' \cong \mathcal{L} \otimes \pi_1^* F$$

for some invertible sheaf  $F$  on  $B$ . The family  $\mathcal{L}$  is a *family of invertible sheaves of degree  $d$*  if, moreover, the restriction of  $\mathcal{L}$  to  $\{b\} \times C$  has degree  $d$  for each point  $b \in B$ .

If  $p \in C$  and  $F$  is a sheaf on  $B$  then  $\pi_1^*(F)|_{B \times p} = F$ , so we could have eliminated the equivalence relation at the expense of choosing a point by insisting that the restriction of  $\mathcal{L}$  to  $B \times \{p\}$  be trivial.

We define the functor

$$\text{Pic}_d : (\text{schemes}) \rightarrow (\text{sets})$$

by associating to any scheme  $B$  the set of invertible sheaves of degree  $d$  on  $B \times C$ , modulo tensoring with pullbacks of invertible sheaves on  $B$ . Since the tensor product and inverse of an invertible sheaf of degree 0 again has degree 0,  $\text{Pic}_0$  factors through the category of abelian groups. These functors too are representable by schemes:

**Cheerful Fact 2.2.** [Kleiman 2005, Theorem 9.4.8] *There exists a fine moduli space  $\text{Pic}_d(C)$  for invertible sheaves of degree  $d$  on  $C$ ; that is, a scheme  $\text{Pic}_d(C)$  and an invertible sheaf  $P$  of degree  $d$  on  $\text{Pic}_d(C) \times C$ , such that for any scheme  $B$  there is a natural bijection between families of invertible sheaves of degree  $d$  over  $B$ , modulo invertible sheaves on  $B$ , and morphisms  $B \rightarrow \text{Pic}_d(C)$ , defined by pulling back  $P$ . The tensor product of invertible sheaves makes  $\text{Pic}_0(C)$  an*

algebraic group, which acts on each  $\text{Pic}_d(C)$ . The sheaf  $P$  is called a Poincaré sheaf for  $C$ .

The choice of  $P$  makes the variety  $\text{Pic}_0(C)$  into a projective algebraic group with group law given by tensor product of invertible sheaves of degree 0, and origin the point of  $\text{Pic}_0(C)$  that is the image of  $\text{Spec } \mathbb{C}$  under the map corresponding to the structure sheaf of  $C$  regarded as family of sheaves over  $\text{Spec } \mathbb{C}$ ; and for each  $d$ ,  $\text{Pic}_d(C)$  is a principal homogeneous space for  $\text{Pic}_0(C)$ .

The construction works for families of curves as long as there is a section (the Poincaré sheaf on  $\text{Pic}_d(C) \times C$  relies on the existence of a point on  $C$ , so that it does not apply to families of curves unless the family has a section.) That is, given a family of curves  $\mathcal{C} \rightarrow S$  with a section  $\sigma : S \rightarrow \mathcal{C}$  there is a scheme  $\text{Pic}_d(\mathcal{C}/S) \rightarrow S$  that represents the functor of families of invertible sheaves of degree  $d$  over  $S$ -schemes  $B \rightarrow S$ . We will use this important refinement in the proof of the Brill-Noether theorem given in Chapter 14.

If  $\mathcal{L}$  is any invertible sheaf of degree  $e$  on  $C$ , we can define a bijection between families of invertible sheaves of degree  $d$  over  $B$  and families of invertible sheaves of degree  $d + e$  over  $B$ , uniformly for all  $B$ , by tensoring with the pullback  $\pi_2^*\mathcal{L}$ . Thus  $\text{Pic}_d(C) \cong \text{Pic}_{d+e}(C)$  (but not canonically, since the isomorphism depends on the choice of  $\mathcal{L}$ ).

The description of the functors represented by the symmetric product  $C_d$  and the Picard scheme  $\text{Pic}_d$  implies that the set-theoretic map

$$C_d \rightarrow \text{Pic}_d \quad D \mapsto \mathcal{O}_c(D)$$

is the underlying set map of a morphism of schemes: on the level of functors the map takes a family of divisors  $D \subset B \times C$  to the family of sheaves  $\mathcal{O}_{B \times C}(D)$ .

The characterization of  $\text{Pic}_d(C)$  above does not shed much light on the geometry of  $\text{Pic}_d(C)$ : whether it's irreducible, for example, or what its dimension is. But we can get a good picture from the classical construction of the Jacobian,  $\text{Jac}(C)$ , which has roots in the 19th century.

### 3. Jacobians

The history leading to the analytic construction of the Jacobian starts from a calculus problem. A goal of the 19th century mathematicians was to make sense of integrals of algebraic functions. In the early development of calculus, mathematicians figured out how to evaluate integrals such as

$$\int_{t_0}^t \frac{dx}{\sqrt{x^2 + 1}}.$$

From a modern point of view such integrals can be thought of as path integrals of meromorphic differentials on the Riemann surface associated to the equation  $y^2 = x^2 + 1$ . This surface is isomorphic to  $\mathbb{P}^1$ , meaning that  $x$  and  $y$  can be expressed as rational functions of a single variable  $z$ ; making the corresponding change of variables transformed the integral into one of the form

$$\int_{s_0}^s R(z)dz,$$

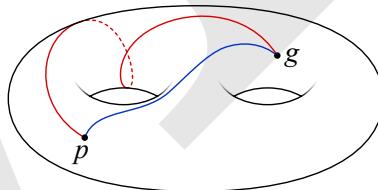
with  $R$  a rational function, and such integrals can be evaluated by the technique of partial fractions.

When they tried to extend this to similar-looking integrals, such as

$$\int_{t_0}^t \frac{dx}{\sqrt{x^3 + 1}},$$

which arises when one studies the length of an arc of an ellipse, they were stymied. The reason gradually emerged: the problem is that the Riemann surface associated to the equation  $y^2 = x^3 + 1$  is not  $\mathbb{P}^1$ , but rather a curve of genus 1, and so has nontrivial homology group  $H_1(C, \mathbb{Z}) \cong \mathbb{Z}^2$ . In particular, if one expresses this “function” of  $t$  as a path integral, then the value depends on a choice of path; it is defined only modulo a lattice  $\mathbb{Z}^2 \subset \mathbb{C}$ . This implies that the inverse function is a doubly periodic meromorphic function on  $\mathbb{C}$ , and not an elementary function. Many new special functions, such as the Weierstrass  $P$ -function, were studied as a result. The name “elliptic curve” arose from these considerations.

Once this case was understood, the next step was to extend the theory to path integrals of holomorphic differentials on curves of arbitrary genus. One problem is that the dependence of the integral on the choice of path is much worse; the set of homology classes of paths between two points  $p_0, p \in C$  is identified with  $H_1(C, \mathbb{Z}) \cong \mathbb{Z}^{2g}$  rather than  $\mathbb{Z}^2$ , rendering the expression  $\int_p^q \omega$  for a given 1-form  $\omega$  virtually meaningless.



**Figure 1.** Two paths from  $p$  to  $g$  might give different values of a path integral.

(( Silvio: the straight line doe not look like it's on the surface of a donut.

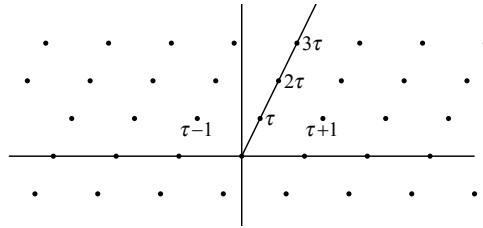


Figure 2. An elliptic curve is the quotient of  $\mathbb{C}$  by a lattice  $\mathbb{Z} \oplus \mathbb{Z}\tau$ .

Also, could be a different color than the outlines of the torus ))

To express the solution to this problems in relatively modern terms, let  $C$  be a smooth projective curve of genus  $g$  over  $\mathbb{C}$ , and let  $\omega_C$  be the sheaf of differential forms on  $C$ . We will consider  $C$  as a complex manifold. Every meromorphic differential form is in fact algebraic [Serre 1955–1956], and we consider  $\omega_C$  as a sheaf in the analytic topology.

Integration over a closed loop in  $C$  defines a linear function on 1-forms, so that we have a map

$$\iota : \mathbb{Z}^{2g} = H_1(C, \mathbb{Z}) \rightarrow H^0(\omega_C)^* \cong H^1(\mathcal{O}_C) = \mathbb{C}^g.$$

By Hodge theory,

$$H^1(C, \mathbb{C}) \cong H^1(C, \mathcal{O}_C) \oplus \overline{H^1(C, \mathcal{O}_C)}$$

where the bar denotes complex conjugation in  $H^1(C, \mathbb{C})$ , and the map  $\iota$  is the composition of the natural inclusion with the projection to the first summand. Now  $H_1(C, \mathbb{C}) = \mathbb{C} \otimes_{\mathbb{Z}} H_1(C, \mathbb{Z})$ , so any basis of  $H_1(C, \mathbb{Z})$  maps to a basis of  $H^1(C, \mathbb{C})$  invariant under conjugation in  $H^1(C, \mathbb{C})$ —See [Voisin 2007] or [Griffiths and Harris 1978, p. 116].

One can show that the image of  $\iota$  is a lattice in  $H^0(\omega_C)^*$ , and thus the quotient is a torus of real dimension  $2g$ . Moreover, the complex structure on  $H^0(\omega_C)^*$  yields a complex analytic structure on the quotient  $\mathbb{C}^g/\iota(\mathbb{Z}^{2g})$ , which is thus a complex torus of dimension  $g$ .

**Definition 3.1.** The complex torus  $J(C) := \mathbb{C}^g/\iota(\mathbb{Z}^{2g})$  is called the *Jacobian* of  $C$ .

The point of this construction is that for any pair of points  $p, q \in C$ , the expression  $\int_q^p$  describes a linear functional on  $H^0(\omega_C)$ , defined up to functionals obtained by integration over closed loops, and thus a point of  $J(C)$ . We can think of  $p - q$  as a divisor of degree 0, and the map  $p - q \mapsto \int_q^p$  extends to a well defined map  $\mu_0$  from divisors of degree 0 to  $J(C)$  because if  $p', q'$  are two

more points, then

$$\int_q^p + \int_{q'}^{p'} - (\int_q^{p'} + \int_{q'}^p)$$

is the integral around a closed path  $q \rightarrow p \rightarrow q' \rightarrow p' \rightarrow q$ .

Further, if we choose a “base point”  $p \in C$ , we can define the holomorphic map

$$\mu : C \rightarrow J(C); \quad q \mapsto \int_p^q$$

and more generally maps

$$\mu_d : C_d \rightarrow J(C) : \quad (q_1, \dots, q_d) \mapsto \sum_i \int_p^{q_i}.$$

These maps are called the *Abel-Jacobi* maps. When there is no ambiguity about  $d$ , we will denote all these maps by  $\mu$ , and we extend the definition by defining  $\mu(-D)$  to be  $-\mu(D)$ . The map  $\mu$  is a group homomorphism in the sense that if  $D, E$  are divisors, then  $\mu(D + E) = \mu(D) + \mu(E)$ ; this is immediate when the divisors are effective, and follows in general because the group of divisors is a free abelian group.

#### 4. Abel's theorem

The connection between the discussion above and the geometry of linear series is made by one of the landmark theorems of the 19th century:

**Theorem 4.1** (Abel's theorem). *Two effective divisors  $D, D'$  on  $C$  of degree  $d$  are linearly equivalent if and only if their images under the Abel-Jacobi map are equal,  $\mu(D) = \mu(D')$ ; in other words, the fibers of  $\mu_d$  are the complete linear systems of degree  $d$  on  $C$ . Thus  $\mu_0$  factors through a canonical isomorphism  $\text{Pic}_0(C) \rightarrow J(C)$  and, after choosing a base point  $p$ , an isomorphism  $\text{Pic}_d(C) \rightarrow \text{Jac}(C)$ , factoring through the isomorphism*

$$\text{Pic}_d(C) \xrightarrow{- \otimes \mathcal{O}_C(-dp)} \text{Pic}_0(C) \rightarrow J(C).$$

We will prove only one implication; see [Griffiths and Harris 1978, Section 2.2] for a complete proof.

**Proof of “only if”.** This was the part proved by Abel; the converse, which is substantially more subtle, was proved by Clebsch. Suppose that  $D$  and  $D'$  are linearly equivalent; that is,  $\mathcal{O}_C(D) \cong \mathcal{O}_C(D')$ . Call this invertible sheaf  $\mathcal{L}$ , and suppose that  $D$  and  $D'$  are the zero divisors of sections  $\sigma, \sigma' \in H^0(\mathcal{L})$ . Taking linear combinations of  $\sigma$  and  $\sigma'$ , we get a pencil  $\{D_\lambda\}_{\lambda \in \mathbb{P}^1}$  of divisors on  $C$ , with

$$D_\lambda = V(\lambda_0 \sigma + \lambda_1 \sigma'),$$

and by the universal property of the symmetric product, this corresponds to a regular map  $\alpha : \mathbb{P}^1 \rightarrow C_d$ .

Consider the composition

$$\phi = \mu \circ \alpha : \mathbb{P}^1 \rightarrow J(C).$$

If  $z$  is any linear functional on  $V = H^0(\omega_C)^*$  then the differential  $dz$  on  $V$  descends to a global holomorphic 1-form on  $J(C)$ , since this is the quotient of  $V$  by a discrete lattice. Thus the regular one-forms on  $J(C)$  generate the cotangent space to  $J(C)$  at every point. But for any 1-form  $\omega$  on  $J(C)$ , the pullback  $\phi^*\omega$  is a global holomorphic 1-form on  $\mathbb{P}^1$ , and hence identically zero. It follows that the differential  $d\phi$  vanishes identically, and hence that  $\phi$  is constant, proving that  $\mu(D) = \mu(D')$ .  $\square$

**Corollary 4.2.** *Let  $C$  be a curve of genus  $g$ . The number of isomorphism classes of invertible sheaves whose  $n$ -th power is  $\mathcal{O}_C$  is  $n^{2g}$ .*

**Proof.** This is the number of  $n$ -torsion points in the group  $\mathbb{C}^g/\mathbb{Z}^{2g} \cong (S^1)^{2g}$ .  $\square$

André Weil applied Abel's theorem to describe the structure of the Jacobian algebraically, using the following fact (often called the Jacobi inversion theorem):

**Corollary 4.3.** *If  $C$  is a smooth curve of genus  $g$  then the Abel-Jacobi map  $\mu_g : C_d \rightarrow J(C)$  is surjective for  $d \geq g$  and generically injective for  $d \leq g$ . In particular,  $\mu_g : C_g \rightarrow J(C)$  is birational.*

**Proof.** For  $d \leq g = \dim H^0(\omega_C)$ , a divisor  $D$  that is the sum of  $d$  general points  $p_1, \dots, p_d \in C$  will impose independent vanishing conditions on the sections of  $\omega_C$ , and thus

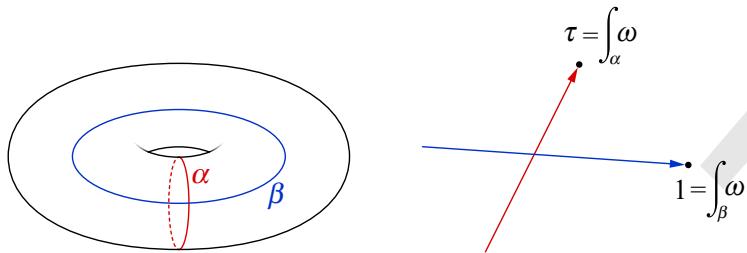
$$h^0(\omega_C(-D)) = g - d.$$

Using this, the Riemann-Roch formula gives  $h^0(\mathcal{O}_C(D)) = 1$ , so the fiber of  $\mu_d$  consists of a single point, proving generic injectivity.

If  $d \geq g$  then  $h^0(\omega_C(-D)) = 0$  and hence  $r(D) = d - g = \dim C_d - \dim J(C)$ , and it follows that the image of  $C_d$  has dimension  $g$ . Since  $C_d$  is compact and  $J(C)$  is irreducible,  $\mu_d$  is surjective.  $\square$

#### 4.0.1. The differential of the Abel-Jacobi map.

**Theorem 4.4.** *The Abel-Jacobi map  $\mu_1 : C \rightarrow \text{Pic}_1(C)$  is a closed embedding. Moreover, if  $p \in C$  is a point on a smooth projective curve of genus  $g > 0$  then the cotangent space of  $C$  at  $p$  is  $\kappa(p) \otimes \omega_C$  and the cotangent space to  $\text{Pic}_1$  at  $\mu_1(p)$  is*



**Figure 3.** The map  $\mu$  maps a curve of genus 1 to the quotient of the plane by the lattice  $\Lambda = \mathbb{Z}\langle w_1, w_2 \rangle$ .

$H^0(\omega_C)$  where  $\kappa(p) \cong \mathbb{C}$  is the residue class field at  $p$ . With these identifications the cotangent map

$$T^*(\mu_1) : H^0(\omega_C) \rightarrow \kappa(p) \otimes \omega_C$$

is evaluation at  $p$ , which is surjective, with kernel  $H^0(\omega_C(-p))$ .

**Proof.** Since  $g(C) > 0$  distinct points  $p, q \in C$  cannot be linearly equivalent, so the Abel-Jacobi map  $\mu_1$  is set-theoretically one-to-one. To prove that it is an embedding, we must show that its differential is everywhere nonzero.

To compute the differential of  $\mu_1$  at a point  $p \in C$  we choose a local parameter  $z$  at  $p$ , so that any differential form on a neighborhood of  $p$  has the form  $f(z)dz$ . We may thus write the cotangent space to  $C$  at  $p$  as  $\kappa(p) \otimes \omega_C \cong \mathbb{C}dz$ . The tangent space to  $J(C) = H^0(\omega_C)^*/H_1(C, \mathbb{Z})$  at any point is  $H^0(\omega_C)^*$ , so the cotangent space is  $H^0(\omega_C)$ .

The map  $\mu_1$  sends each point  $q$  to

$$\begin{aligned} \mu_1 : C &\rightarrow J(C) := H^0(\omega_C)^*/H_1(C, \mathbb{Z}) \\ q &\mapsto \int_p^q . \end{aligned}$$

The fundamental theorem of calculus tells us the derivative of an integral: for any  $q \in C$  close to  $p$  we have

$$\frac{\partial}{\partial q} \int_p^q f(z)dz = f(q).$$

Thus the co-differential  $T^*(\mu_1)$  of  $\mu_1$ , is the natural restriction map  $H^0(\omega_C) \rightarrow \kappa(p) \otimes \omega_C$  sending a differential form to its value at  $p$ . Thus the kernel of  $T^*(\mu_1)$  evaluated at  $p$  is the space of differential forms vanishing at  $p$ , that is  $H^0(\omega_C(-p))$ . Since  $\omega_C$  is basepoint free, there is a differential form not vanishing at  $p$ , so  $T^*(\mu_1)$  is surjective, proving that the differential of  $\mu_1$  is injective as required.  $\square$

**Corollary 4.5.** *The composition of the map  $\mu_1 : C \rightarrow \text{Pic}_1(C)$  with the Gauss map that sends a point  $\mu_1(p)$  to the tangent line to  $\mu_1(C)$  at  $p$  is the canonical embedding.*

**Proof.** Since the kernel of  $T^*(\mu_1)(p)$  is the hyperplane  $H^0(\omega_C(-p)) \subset H^0(\omega_C)$ , the differential of  $\mu_1$  carries the tangent space to  $C$  at  $p$  to the line in  $H^0(\omega_C)^*$  corresponding to the hyperplane  $H^0(\omega_C(-p)) \subset H^0(\omega_C)$ . This is the image of  $p$  in  $\mathbb{P}(H^0(\omega_C)) = \mathbb{P}^{g-1}$  under the canonical map.  $\square$

## 5. The $g + 3$ theorem

The Picard varieties are mysterious objects. But even the facts that there exists a moduli space for invertible sheaves of degree  $d$  on  $C$ , and that this space is an irreducible algebraic variety of dimension  $g$ , suffice to prove non-trivial results.

We proved in Corollary 3.5 that every divisor of degree  $\geq 2g + 1$  is very ample, and this is sharp as stated: on any curve  $C$  there exist divisors of every degree  $d \leq 2g$  that are not very ample. But if we look at a *general* divisor  $D$  of degree  $d$  on  $C$ , we can do much better: we have the

**Theorem 5.1** ( $g + 3$  theorem). *Let  $C$  be a smooth projective curve of genus  $g$ . If  $D \in C_{g+3}$  is a general effective divisor of degree  $g + 3$  on  $C$ , then  $D$  is very ample. In particular, every curve of genus  $g$  may be embedded in  $\mathbb{P}^3$  as a curve of degree  $g + 3$ .*

Moreover, this is sharp: if  $g \geq 2$  and  $C$  is hyperelliptic, then no invertible sheaf of degree  $\leq g + 2$  is very ample on  $C$ .

This is not the final word on low-degree embeddings of curves: if we look at a general curve  $C$ , Theorem 2.6 says that  $C$  can in fact be embedded in  $\mathbb{P}^3$  as a curve of degree  $d = \lceil 3g/4 \rceil + 3$ , and can be birationally embedded in  $\mathbb{P}^2$  as a nodal curve of degree  $d = \lceil 2g/3 \rceil + 2$ .

**Proof.** If  $D$  is general of degree  $g + 3$  then  $D$  is nonspecial, so  $h^0(\mathcal{O}_C(D)) = 4$ . By Proposition 4.5 the divisor  $D$  is very ample iff and only if  $h^0(\mathcal{O}_C(D - F)) = 2$  for every effective divisor  $F = p + q$  of degree 2 on  $C$ .

If, on the contrary,  $h^0(\mathcal{O}_C(D - F)) \geq 3$  then by the Riemann-Roch theorem  $h^0(\omega_C(-D + F)) \geq 1$  so  $K_C - D + F$  is linearly equivalent to an effective divisor  $E$  of degree  $g - 3$ .

Now, consider the map

$$\nu : C_{g-3} \times C_2 \rightarrow J(C)$$

given by

$$\nu : (E, F) \mapsto \mu_{g-3}(K_C - E + F) = \mu_{2g-2}(K_C) - \mu_{g-3}(E) + \mu_2(F),$$

where the + and – on the right refer to the group law on  $J(C)$ .

By what we have just said and Abel's theorem, the divisor  $D$  is very ample if and only if  $\mu(D) \notin \text{Im}(\nu)$ . However the source of  $\nu$  has dimension  $g - 3 + 2 = g - 1$ , so its image in  $J(C)$  is a proper subvariety. Since  $\mu_{g+3}$  is surjective, the image of a general divisor  $D \in C_{g-3}$  is a general point of  $J(C)$  and thus will not lie in  $\text{Im}(\nu)$ .

To prove that  $g + 3$  is sharp for hyperelliptic curves, recall that a smooth plane curve  $C$  of genus  $\geq 2$  has degree  $d \geq 4$  and canonical class  $\omega_C = \mathcal{O}_C(d - 3)$ , which is very ample. Thus no hyperelliptic curve can be embedded in the plane.

Thus every very ample divisor  $D$  on a hyperelliptic curve has  $h^0(D) \geq 3$ . If also  $\deg D \leq g + 2$  then  $D$  is a special divisor. By Corollary 4.6,  $D$  would have the form  $rD_0$ , where  $D_0$  is the  $g_2^1$ , the unique divisor of degree 2 with 2 independent sections, and  $r \leq (g + 2)/2$ . But the smallest multiple of the  $D_0$  that is very ample is  $(g + 1)D_0$ , and since  $g \geq 2$  we have  $(g + 2)/2 < g + 1$  so  $D$  is not very ample.  $\square$

It would be natural to speculate that a general divisor of degree  $g + 2$  defines a map to  $\mathbb{P}^2$  whose image has only nodes as singularities, but this is not quite true. Here is the correct result, whose proof we postpone until Section 3.3 because it depends on a general position result that we will prove in Section 1.

**Theorem 5.2** ( $g + 2$  theorem). *Let  $C$  be any smooth projective curve of genus  $g$ , and let  $D$  be a general divisor of degree  $g + 2$  on  $C$ .*

- (1) *If  $C$  is non-hyperelliptic, the map  $\phi_D : C \rightarrow \mathbb{P}^2$  is birational onto its image  $C_0$ , and  $C_0$  is a plane curve of degree  $g + 2$  with exactly  $\binom{g}{2}$  nodes and no other singularities; and*
- (2) *If  $C$  is hyperelliptic, the map  $\phi_D : C \rightarrow \mathbb{P}^2$  is birational onto its image  $C_0$ , and  $C_0$  is a plane curve of degree  $g + 2$  with one ordinary  $g$ -fold point and no other singularities.*

## 6. The schemes $W_d^r(C)$

Now that we have a parameter space  $\text{Pic}_d(C)$  for invertible sheaves of degree  $d$ , we can ask about the geometry of the set of invertible sheaves defining linear series of dimension  $\geq r$ , that is,

$$W_d^r(C) := \{\mathcal{L} \in \text{Pic}_d(C) \mid h^0(\mathcal{L}) \geq r + 1\}.$$

Thus, for example,  $W_d^0(C)$  is the locus of effective divisor classes, the image of the map  $\mu : C_d \rightarrow \text{Pic}_d(C)$ . We often omit the “0” in this case and write this simply as  $W_d(C)$ .

$W_d^r(C)$  can also be characterized as the locus

$$W_d^r(C) = \{\mathcal{L} \in \text{Pic}_d(C) \mid \dim(\mu^{-1}(\mathcal{L})) \geq r\}.$$

where  $\mu : C_d \rightarrow \text{Pic}_d(C)$  is the Abel-Jacobi map. By the upper-semicontinuity of fiber dimension,  $W_d^r(C)$  is a Zariski-closed subset. Thus  $W_d^r(C)$  is an algebraic variety, so we can talk about its dimension, irreducibility, smoothness or singularity and so on. The preimage  $\mu^{-1}(W_d^r) \subset C_d$  parametrizes effective divisors  $D$  with  $r(D) \geq r$ , and is denoted  $C_d^r$ .

The set  $W_d^r(C)$  can be given the structure of a scheme, with which it is a fine moduli space representing the functor of families of invertible sheaves with  $r+1$  or more sections (that is, the functor that associates to a scheme  $B$  the set of invertible sheaves  $\mathcal{L}$  on  $B \times C$  such that the pushforward  $(\pi_2)_*\mathcal{L}$  has a locally free subsheaf of rank  $r+1$  over every curvilinear subscheme of  $B$ , as always modulo tensoring with pullbacks of invertible sheaves on  $B$ ). In Chapter 9 we will see examples of curves  $C$  of genus 4 for which the scheme  $W_3^1(C)$  is reducible, and in Chapter 12 we will see curves  $C$  of genus 6 for which the scheme  $W_4^1(C)$  is non-reduced (in fact, in one case isomorphic to  $\text{Spec } \mathbb{C}[\epsilon]/(\epsilon^5)$ ). For a definition of the scheme structure, see [Arbarello et al. 1985, Section IV.3].

## 7. Examples in low genus

**7.1. Genus 1.** If  $C$  is a curve of genus 1, then since the Abel-Jacobi map  $\mu : C \rightarrow \text{Pic}_1(C)$  is always an embedding, it is an isomorphism, and also  $\text{Pic}_d(C) \cong C$  for all  $d$ . If we fix any point  $p \in C$ , we get a map  $C \rightarrow \text{Pic}_0(C)$  sending  $q \in C$  to the invertible sheaf  $\mathcal{O}_C(q - p)$ , which is again an isomorphism.

The fact that a curve of genus 1 over  $\mathbb{C}$  is the quotient of the complex plane by a lattice  $\Lambda \subset \mathbb{C}$  has an immediate consequence:

**Corollary 7.1.** *If  $E$  is a smooth projective curve of genus 1 and  $p \in E$  any point, there are only finitely many automorphisms  $\phi : E \rightarrow E$  with  $\phi(p) = p$ .*

**Proof.** The choice of  $p$  allows us to identify  $C$  with  $\mathbb{C}/\Lambda$  for some lattice  $\Lambda$ . Any automorphism  $\phi$  lifts to a linear map  $\tilde{\phi} : \mathbb{C} \rightarrow \mathbb{C}$  carrying  $\Lambda$  to itself.  $\square$

**7.2. Genus 2.** The map  $\mu_2 : C_2 \rightarrow J(C)$  is an isomorphism except along the locus  $\Gamma \subset C_2$  of divisors of the unique  $g_2^1$  on  $C$ . In fact, the symmetric square  $C_2$  of  $C$  is the blow-up of  $J(C)$  at the point corresponding to the invertible sheaf corresponding to the  $g_2^1$ ; see Exercise 9.10.

### 7.3. Genus 3.

**Cheerful Fact 7.2.** *Let  $C$  be a curve of genus 3. The geometry of the map  $\mu_2$  depends on whether or not  $C$  is hyperelliptic. If it is,  $\mu_2$  will collapse the locus in*

$C_2$  of divisors of the hyperelliptic  $g_2^1$  to a point, but is otherwise one-to-one, and  $W_2(C)$  is singular. If  $C$  is not hyperelliptic, by contrast,  $\mu_2$  is an embedding, and  $W_2(C)$  is smooth.

Furthermore, the birational surjection  $\mu_3 = \mu_g$  is the blow-up of  $J(C) \cong \text{Pic}_3(C)$  along the locus  $W_3^1(C)$ . At the same time, we know that

$$W_3^1(C) = K - W_1(C),$$

in the sense that any invertible sheaf  $\mathcal{L}$  defining a  $g_3^1$  has, by the Riemann-Roch theorem, the form  $\omega_C \otimes \mathcal{L}'$ , where  $\mathcal{L}'$  has degree 1 and  $h^0(\mathcal{L}') = 1$ , so  $W_3^1$  is isomorphic to  $C$ . Thus  $C_3$  is the blow-up of  $J(C)$  along the curve  $C$ . (see [Mumford 1975, pp. 53–4] for a discussion).

## 8. Martens' theorem

If  $C$  is a smooth projective curve of genus  $g$  and  $r \leq d - g$ , then by the Riemann-Roch theorem  $W_d^r = \text{Pic}_d$ . On the other hand Clifford's theorem says that, if  $C$  is a smooth projective curve of genus  $g$ , and  $d \leq 2g - 2$  then the set  $W_d^r(C)$  is empty when  $d - 2r < 0$ . Martens' theorem extends Clifford's theorem to the statement  $\dim W_d^r \leq d - 2r$  whenever  $r > d - g$ .

**Theorem 8.1** (Martens' theorem). [Martens 1967] Let  $C$  be a smooth projective curve of genus  $g$ . If  $r > d - g$ , then

$$\dim(W_d^r(C)) \leq d - 2r;$$

moreover, equality holds for some  $r$  in this range if and only if  $d = 2g - 2$  and  $r = g - 1$  (in which case  $W_d^r = \{\omega_C\}$ ) or  $d = r = 0$  (in which case  $W_d^r = \{\mathcal{O}_C\}$ ) or  $C$  is hyperelliptic. In the last case, equality holds for all  $d$  and  $r$  with  $d \geq 2r > 2(d - g)$ .

Given a smooth projective curve  $C \subset \mathbb{P}^r$ , we write  $\Sigma_d^r \subset C_d$  for the locus of effective divisors  $D$  of degree  $d$  on  $C$  with  $\dim \overline{D} \leq d - r - 1$ . This is a closed condition, so  $\Sigma_d^r$  is an algebraic subset. Using the Geometric Riemann-Roch theorem we will reduce the inequality in Martens' theorem to the following.

**Lemma 8.2** (elementary secant plane lemma). Let  $C \subset \mathbb{P}^r$  be a smooth, nondegenerate curve. If  $d \leq r$  and  $r \geq 0$  then

$$\dim \Sigma_d^r \leq d - r.$$

**Proof.** Since  $\Sigma_d^0 = C^d$ , the conclusion is immediate for  $r = 0$ .

Suppose that  $D \in \Sigma_d^r$  with  $r > 0$ . Since  $\dim \overline{D} \leq d - r - 1 \leq d - 2$  the plane  $\Lambda$  spanned by  $D$  is already spanned by some divisor  $D' := D - p$  of degree  $d - 1$ , obtained by subtracting one point in the support of  $D$ . It follows that for

points  $q \notin \Lambda \cap C$ , the divisor  $q + D'$  spans a plane of dimension  $\leq d - r$ , so  $q + D' \in \Sigma_d^{r-1}$ . Furthermore,  $D$  is in the closure of the set of divisors

$$\{q + D' \mid q \notin \Lambda \cap C\}.$$

From this we see that  $\Sigma_d^r \setminus \Sigma_d^{r-1}$  is in the closure of  $\Sigma_d^{r-1}$ . It follows that

$$\dim \Sigma_d^0 > \dim \Sigma_d^1 > \cdots > \dim \Sigma_d^r,$$

and thus  $\dim \Sigma_d^r \leq d - r$  as claimed.  $\square$

**Proof of Martens' inequality.** The listed cases of equality are easy to verify: for example on a hyperelliptic curve Corollary 4.6 shows that a special invertible sheaf with  $r(\mathcal{L}) = s$  can be written  $\mathcal{L}_0^{\otimes s}(p_1 + \cdots + p_{d-2s})$ , where  $\mathcal{L}_0$  is the unique  $g_2^1$  on  $C$  and the  $p_i$  are base points—a family of dimension  $d - 2s$ . We postpone the proof that these are the only cases of equality to Theorem 3.2, which we prove using a stronger version of Lemma 8.2 that is a corollary of Theorem 1.3.

Since we have assumed  $r > d - g$ , the set  $W_d^r$  contains only sheaves  $\mathcal{L}$  with  $h^1(\mathcal{L}) \neq 0$ . For such sheaves there is a one-to-one correspondence  $\mathcal{L} \rightarrow \omega_C \otimes \mathcal{L}^{-1}$  inducing an isomorphism

$$\{\mathcal{L} \in W_d^r(C) \mid h^1(\mathcal{L}) \neq 0\} = \{\mathcal{L} \in W_{2g-2-d}^{g-1-d+r} \mid h^1(\mathcal{L}) \neq 0\}$$

Since

$$2g - 2 - d - 2(g - 1 - d + r) = d - 2r,$$

it will suffice to prove the theorem in case  $d \leq g - 1$ .

Let  $C_d^r \subset C_d$  be the preimage of  $W_d^r(C)$ . Since the fibers of  $C_d^r$  over points of  $W_d^r$  have dimension at least  $r$ , Marten's inequality will follow from the inequality

$$\dim C_d^r \leq d - r.$$

Let  $\phi_K : C \rightarrow \mathbb{P}^g - 1$  be the canonical map. By the Geometric Riemann-Roch theorem 4.3,  $r(D) = \deg D - 1 - \dim \overline{\phi_K(D)}$ , so  $D \in C_d^r$  if and only if  $\dim \overline{\phi_K(D)} \leq d - r - 1$ . Since  $d \leq g - 1$  we may apply Lemma 8.2, showing that the set of such  $D$  has dimension  $\leq d - r$ , as required.  $\square$

If  $C$  is hyperelliptic with  $g_2^1 = |D|$  then for  $r \leq g - 1$  we have  $\mu(rD) \in W_{2r}^r$  and

$$W_d^r(C) \supset W_{d-2r}(C) + \mu(rD).$$

In fact this is an equality by 4.6. Since  $W_{d-2r}(C)$  has dimension  $d - 2r$ , we see that Martens' theorem is sharp.

There are extensions of Martens' theorem showing that  $\dim(W_d^r(C)) < d - 2r$  in certain cases; see for example [Mumford 1974] [Keem 1990], and [Coppens 1983].

## 9. Exercises

**Exercise 9.1.** Let  $G$  be a finite group acting on a quasi-projective scheme  $X$ . Show that there is a finite covering of  $X$  by invariant open affine sets.

Hint: use the fact that the intersection of two affine open subsets of a quasi-projective scheme is again affine.

**Exercise 9.2.** We say that a group  $G$  acts freely on  $X$  if  $gx = gy$  only when  $g$  is the identity or  $x = y$ . Show that if  $G$  is a finite group acting freely on a smooth affine variety  $X$  then the quotient  $X/G$  is smooth.

**Exercise 9.3.** Let  $X = (\mathbb{A}^2)^2$  and let  $G := \mathbb{Z}/2$  act on  $X$  by permuting the two copies of  $\mathbb{A}^2$ ; algebraically,  $(\mathbb{A}^2)^2 = \text{Spec } S$ , with  $S = k[x_1, x_2, y_1, y_2]$  and the nontrivial element  $\sigma \in G$  acts by  $\sigma(x_i) = y_i$ .

- (1) Show that  $G$  acts freely on the complement of the diagonal, but fixes the diagonal pointwise.
- (2) Supposing that the ground field has characteristic  $\neq 2$ , show that the algebra  $S^G$  is the affine coordinate ring of the product of  $A^2$  with the affine cone over a conic in the projective plane. (Hint: change coordinates to  $z_i := x_i + y_i, w_i := x_i - y_i$ .)
- (3) Conclude that the symmetric square has singular locus equal to the image of the diagonal in  $(\mathbb{A}^2)^2$ .

**Exercise 9.4.** It is not an accident that we can characterize a fine moduli space  $M$  in terms of the maps into it. Let  $X$  be a category, and  $F, G$  two functors from  $X$  to the category of sets. A morphism  $\eta : F \rightarrow G$  in the category of functors is called a *natural transformation*: for every object  $a \in X$  there is a morphism  $\eta_a : F(a) \rightarrow G(a)$  such that for every morphism  $f : a \rightarrow b$  in  $X$  the compositions  $G(f) \circ \eta_a$  and  $\eta_b \circ F(f)$  are equal.

- (1) Prove Yoneda's lemma: If  $X$  is any category, and  $F$  is a contravariant functor from  $X$  to the category of sets, then

$$\text{Hom}_{\text{Functors on } X}(\text{Hom}_X(-, Z), F) = F(Z)$$

- (2) Conclude that if the functors  $\text{Hom}_X(-, Z)$  and  $\text{Hom}_X(-, Z')$  are isomorphic in the functor category, then  $Z \cong Z'$  in  $X$ ; that is, the functor  $\text{Hom}_X(-, Z)$  determines the object  $Z$  up to isomorphism.

**Exercise 9.5.** Show that if  $r \geq d - g$ , then  $W_d^r(C) \setminus W_d^{r+1}(C)$  is dense in  $W_d^r(C)$  (that is,  $W_d^{r+1}(C)$  does not contain any irreducible component of  $W_d^r(C)$ ).

Hint: if  $\mathcal{L}$  is an invertible sheaf of degree  $d$  with  $h^0(\mathcal{L}) > d - g + 1$ , show that for general  $p, q \in C$  we have  $h^0(\mathcal{L}(p - q)) = h^0(\mathcal{L}) - 1$ .

**Exercise 9.6.** Let  $C$  be a curve of genus 2, and let  $C \subset J(C)$  be the image of the Abel-Jacobi map  $\mu_1$ . Show that the self-intersection of the curve  $C$  is 2,

- (1) by applying the adjunction formula to  $C \subset J(C)$ ; and
- (2) by calculating the self-intersection of its preimage  $C + p \subset C_2$  and using the geometry of the map  $\mu_2$ .

Hint: for the second part, use the fact that the Abel-Jacobi map  $\mu_2 : C_2 \rightarrow J(C)$  is the blow-up of  $J(C)$  at a point.

**Exercise 9.7.** Let  $C$  be a curve of genus 2, and consider the map  $\nu : C \times C \rightarrow \text{Pic}_0(C)$  defined by sending  $(p, q) \in C \times C$  to the invertible sheaf  $\mathcal{O}_C(p - q)$ . Show that this map is generically finite, and compute its degree.

Hint: show that for general points  $p, q \in C$ , there exists a unique pair  $(p', q') \neq (p, q)$  with  $p - q \sim p' - q'$ .

**Exercise 9.8.** Let  $C_d^r$  be the preimage of  $W_d^r$ , the set of divisors moving in at least an  $r$ -dimensional linear series. Show that the image of the differential of the Abel-Jacobi map  $C_d \rightarrow J(C)$  at a point of  $C_d \setminus C_d^1$  corresponding to a reduced divisor is the plane in  $\mathbb{P}^{g-1}$  spanned by the divisor  $D$  on the canonical curve.

**Exercise 9.9** ( $g+1$  theorem). Let  $C$  be a smooth projective curve of genus  $g$ . If  $D \in C_{g+1}$  is a general effective divisor of degree  $g+1$  on  $C$ , then  $D$  is base-point free and defines a map to  $\mathbb{P}^1$  that has only simple ramification, with  $2g+2$  distinct branch points.

Hint: A general divisor of degree  $\geq g+1$  is nonspecial, so  $h^0(\mathcal{O}_C(D)) = (g+1) - g+1 = 2$ . To say that the linear series  $|D|$  has no base points means that  $D$  is not equivalent to a divisor of the form  $D' + p$ , where  $h^0(D')$  has degree  $g$  and two independent sections. By the Riemann-Roch theorem,  $h^0(D') = 1 + h^0(K - D')$ , so the condition on  $D'$  means that  $K - D'$  is effective. But  $K - D'$  is a general divisor of degree  $g-2$ , and the set of classes of effective divisors, the image of  $C_{g-2}$ , has dimension only  $g-2$ . Thus the set of divisors of the form  $D' + p$  with  $h^0(D') \geq 2$  has dimension only  $g-2+1$  so a general divisor  $D$  of degree  $g+1$  is base-point-free and defines a map  $C \rightarrow \mathbb{P}^1$ .

By Hurwitz' theorem the total ramification of such a map is  $2g+2$ . The divisors with points of ramification index 2 or more are linearly equivalent to divisors of the form  $D' + 3p$ , and the divisors that have two ramification points mapping to the same point are equivalent to divisors of the form  $D'' + 2p + 2q$ . Each of these sets of divisors fills only a  $g-1$ -dimensional family of equivalence classes, the images of  $C_{g-2} \times C$  or  $C_{g-3} \times C_2$  respectively; so not every divisor  $D$  is of this form.

**Exercise 9.10.** Let  $C$  be a curve of genus 2. The canonical map  $\phi_K : C \rightarrow \mathbb{P}^1$  expresses  $C$  as a 2-sheeted cover of  $\mathbb{P}^1$ , and we have correspondingly an

involution  $\tau : C \rightarrow C$  exchanging points in the fibers of  $\phi_K$  (equivalently, for any  $p \in C$ , we have  $h^0(K_C(-p)) = 1$ ;  $\tau$  will send  $p$  to the unique zero of the unique section  $\sigma \in H^0(K_C(-p))$ ). Let  $\Gamma \subset C \times C$  be the graph of  $\tau$ .

- (1) Using the fact that a birational morphism of smooth surfaces must be the inverse of a sequence of blow-ups of reduced points ([[Hartshorne 1977](#), V.5.5]) show the self-intersection of the image  $C_2^1$  of  $\Gamma$  in  $C_2$  is  $-1$ .
- (2) Find the self-intersection of  $\Gamma$  in  $C \times C$ .



# Hyperelliptic curves and curves of genus 2 and 3

## 1. Hyperelliptic curves

We met hyperelliptic curves in Chapter 2 and proved there that, in genus  $g \geq 2$ , the canonical map of a hyperelliptic curve is two to one onto the rational normal curve of degree  $g - 1$  in  $\mathbb{P}^{g-1}$ . We used this to show that every special linear series on a hyperelliptic curve is a sum of a multiple of the unique  $g^1_2$  plus base points. We will begin this chapter with an explicit construction of hyperelliptic curves and use it to give a very concrete computation of the canonical series, reproving what we did in Chapter 2. Then we will consider the projective embeddings of curves of genus 2 (which are all hyperelliptic) and genus 3.

There will be a further discussion of hyperelliptic curves in Chapter 17.

**1.1. The equation of a hyperelliptic curve.** Recall that a hyperelliptic curve  $C$  is a curve of genus  $\geq 2$  admitting a degree two map  $\pi : C \rightarrow \mathbb{P}^1$ , and we have seen that such a map is unique. Because the degree is only 2, each point in  $\mathbb{P}^1$  has either two distinct preimages, or only one; in the latter case, this point is a ramification point with ramification index 1, where the map is given in terms of local analytic coordinates on  $C$  and  $\mathbb{P}^1$  by  $z \mapsto z^2$ . In particular, both the ramification divisor and the branch divisor (as defined in Chapter 2) are reduced. By Hurwitz' formula there are exactly  $2g + 2$  branch points in  $\mathbb{P}^1$ . These points determine the curve:

**Theorem 1.1.** *There is a unique smooth projective hyperelliptic curve  $C$  expressible as a 2-sheeted cover of  $\mathbb{P}^1$  branched over any given set of  $2g + 2$  distinct points  $\{q_1, \dots, q_{2g+2}\}$ .*

**Proof.** We can easily construct such a curve, postponing for a moment the uniqueness: If the coordinate of the point  $q_i \in \mathbb{P}^1$  is  $\lambda_i$ , then  $C$  is the smooth projective model of the affine curve

$$C^\circ = \{(x, y) \in \mathbb{A}^2 \mid y^2 = \prod_{i=1}^{2g+2} (x - \lambda_i)\}.$$

Note that we're choosing a coordinate  $x$  on  $\mathbb{P}^1$  with the point  $x = \infty$  at infinity not among the  $q_i$ , so that the pre-image of  $\infty \in \mathbb{P}^1$  is two points  $r, s \in C$ . Concretely, we see that as  $x \rightarrow \infty$ , the ratio  $y^2/x^{2g+2} \rightarrow 1$ , so that

$$\lim_{x \rightarrow \infty} \frac{y}{x^{g+1}} = \pm 1.$$

The two possible values of this limit correspond to the two points  $r, s \in C$ .

The curve  $C$  is *not* simply the closure of the affine curve  $C^\circ \subset \mathbb{A}^2$  in either  $\mathbb{P}^2$  or  $\mathbb{P}^1 \times \mathbb{P}^1$ : as you can see from a direct examination of the equation, each of these closures will be singular at the (unique) point at infinity.

The uniqueness statement of Theorem 1.1 will be proven in Section 2 below. □

To give a smooth projective model of the hyperelliptic curve  $C$  with given branch divisor, we divide the  $2g + 2$  branch points into two sets of  $g + 1$ —say, for example,  $q_1, \dots, q_{g+1}$  and  $q_{g+2}, \dots, q_{2g+2}$ . We can then take  $C$  to be the closure in  $\mathbb{P}^1 \times \mathbb{P}^1$  of the locus

$$\{(x, y) \in \mathbb{A}^2 \mid y^2 \prod_{i=1}^{g+1} (x - \lambda_i) = \prod_{i=g+2}^{2g+2} (x - \lambda_i)\};$$

in projective coordinates, this is

$$C = \{((X_0, X_1), (Y_0, Y_1)) \in \mathbb{P}^1 \times \mathbb{P}^1 \mid Y_1^{g+1} \prod_{i=1}^{g+1} (X_1 - \lambda_i X_0) = Y_0^{2g+2} \prod_{i=g+2}^{2g+2} (X_1 - \lambda_i X_0)\}.$$

To see that  $C \subset \mathbb{P}^1 \times \mathbb{P}^1$  is smooth we note that it is a curve of bidegree  $(2, g + 1)$  in  $\mathbb{P}^1 \times \mathbb{P}^1$ , and the formula for the genus of a curve in  $\mathbb{P}^1 \times \mathbb{P}^1$  derived in Example 6.1 tells us that such a curve has arithmetic genus  $g$ , and thus no singular points.

From this model, we deduce:

**Corollary 1.2.** *If  $C$  is a hyperelliptic curve and  $p_1, \dots, p_{2g+2} \in C$  are the ramification points of the unique degree 2 map  $C \rightarrow \mathbb{P}^1$ , then for any division of  $\{1, \dots, 2g+2\}$  into two sets  $A, B$  of cardinality  $g+1$ ,*

$$\sum_{i \in A} p_i \sim \sum_{i \in B} p_i.$$

**Proof.** The abstract curve  $C \subset \mathbb{P}^1 \times \mathbb{P}^1$  above is independent of the way in which the points  $p_i$  are divided into two sets of size  $g+1$  since, in any case the projection to the first factor is ramified over the same set  $p_1, \dots, p_{2g+2}$ . Given the representation above, the sets  $\{p_1, \dots, p_{g+1}\}$  and  $\{p_{g+2}, \dots, p_{2g+2}\}$  are preimages of  $(0, 1)$  and  $(1, 0)$  in the second factor.  $\square$

The map  $\iota : C \rightarrow C$  that exchanges the two points in each reduced fiber of the map  $C \rightarrow \mathbb{P}^1$  and fixes the ramification points is algebraic: in terms of the last representation of  $C$ , it is given by  $((X_0, X_1), (Y_0, Y_1)) \mapsto ((X_0, X_1), (Y_0, -Y_1))$ . The map  $\iota$  is called the *hyperelliptic involution* on  $C$ .

**1.2. Differentials on a hyperelliptic curve.** We can give a pleasantly concrete description of the differentials, and thus the canonical linear system, on a hyperelliptic curve  $C$  by working with the affine model  $C^\circ = V(f) \subset \mathbb{A}^2$ , where

$$f(x, y) = y^2 - \prod_{i=1}^{2g+2} (x - \lambda_i).$$

We will again denote the two points at infinity—that is, the two points of  $C \setminus C^\circ$ —by  $r$  and  $s$ ; for convenience, we'll denote the divisor  $r + s$  by  $D$ . We write  $\pi : C \rightarrow \mathbb{P}^1$  for the morphism that, on  $C^\circ$ , sends  $(x, y) \in C$  to  $x$ .

We can construct a differential form on  $C$  by following the proof of Hurwitz' theorem of Chapter 2. Let  $dx$  denote the usual differential on  $\mathbb{P}^1$  having a double pole at infinity, and consider  $\pi^*dx$  on  $C$ . The function  $x$  is regular on  $C^\circ$ , and is a local parameter over points other than the  $\lambda_i$ ; from the local description of the map  $\pi$ , we see that  $\pi^*dx$  is regular on  $C^\circ$  with simple zeros at the ramification points  $q_i = (\lambda_i, 0)$ . Since  $dx$  has a double pole at the point at  $\infty \in \mathbb{P}^1$  and  $\pi$  is a local isomorphism near  $r$  and  $s$ , the differential  $\pi^*dx$  has double poles at the points  $r$  and  $s$ . Thus the canonical divisor of  $C$  is

$$(*) \quad K_C \sim (dx) \sim R - 2D,$$

where  $R$  denotes the ramification divisor, in this case the sum of the ramification points.

How can we find differentials that are regular everywhere on  $C$ ? If we divide  $dx$  by  $x^2$  (or any quadratic polynomial in  $x$ ) to kill the poles we introduce new poles in the finite part  $C^\circ$  of  $C$ .

Instead, we want to multiply  $dx$  by a rational function with zeros at  $r$  and  $s$ , but whose poles occur only at the points where  $dx$  has zeroes—that is, the points  $\lambda_i$ . A natural choice is the reciprocal of the partial derivative  $f_y := \partial f / \partial y = 2y$ , which vanishes at the points  $q_i$ , and has a pole of order  $g + 1$  at each of the points  $r$  and  $s$  (reason:  $y/x^{g+1}$  approaches  $\pm 1$  as  $x$  goes to infinity, and  $x$  has a pole of order 1 at  $\infty \in \mathbb{P}^1$  and thus also at each of  $r, s$ ). In other words, as long as  $g \geq 1$ , the differential

$$\omega = \pi^* \left( \frac{dx}{f_y} \right)$$

is regular, with divisor

$$(\omega) = (g - 1)r + (g - 1)s = (g - 1)D.$$

The remaining regular differentials on  $C$  are now easy to find: Since  $x$  has only a simple pole at the two points at infinity we can multiply  $\omega$  by any  $x^k$  with  $k = 0, 1, \dots, g - 1$ . Since this gives us  $g$  independent differentials, we see that the differentials

$$\omega, x\omega, \dots, x^{g-1}\omega$$

form a basis for  $H^0(K_C)$ .

With this description of the differentials, we can see clearly why the canonical map of a hyperelliptic curves is degree 2 onto a rational normal curve, as proven in Chapter 2: the relations on

$$\omega, x\omega, \dots, x^{g-1}\omega$$

are the relations on  $x^i$ , and we see that the canonical image is the rational normal curve of degree  $g - 1$ .

## 2. Branched covers with specified branching

Given a curve  $B$  and a collection  $\Delta = \{p_1, \dots, p_b\} \subset B$  of points in  $B$ , what are the branched covers  $\pi : C \rightarrow B$  of degree  $d$  with specified branching over each of the points  $p_i$ , up to isomorphism over  $B$ ? The number of these is called the *Hurwitz number* of the configuration, and its computation in general is the subject of a large and active literature; see for example [Ekedahl et al. 2001].

First, we will reduce the problem to the classification of topological covering spaces of the complement  $U = B \setminus \Delta$ ; we will then use our knowledge of the fundamental group of  $U$  to enumerate such covering spaces.

**Theorem 2.1.** *Let  $B$  be a smooth curve, let  $\Delta \subset B$  be a finite set of points, and let  $U := B \setminus \Delta$ . If  $\pi^\circ : V \rightarrow U$  is a topological covering space then  $V$  may be given the structure of a Riemann surface in a unique way so that the map  $\pi^\circ$  is holomorphic; and  $V$  may be compactified to a compact Riemann surface  $C$  in a unique way such that the map  $\pi^\circ$  extends to a holomorphic map  $\pi : C \rightarrow B$ .*

**Proof.** The space  $V$  inherits the structure of a complex manifold from  $U$  because if  $D \subset U$  is any simply connected coordinate chart, then the preimage  $(\pi^\circ)^{-1}(D)$  is a disjoint union of  $d$  copies of  $D$ , and we may use them as coordinate charts on  $V$ .

To compactify  $V$  we observe that if  $D^* = \{z \in \mathbb{C} \mid 0 < |z| < 1\}$  is a punctured disc, then the map  $D \rightarrow D : z \mapsto z^n$  restricts to a connected  $n$ -fold covering space  $D^* \rightarrow D^*$ . Since  $\pi_1(D^*) = \mathbb{Z}$ , any connected covering space  $E$  of degree  $n$  is homeomorphic to this one by a homeomorphism inducing the identity on the target of  $\pi$ . If we define a holomorphic structure on  $E$  by pulling back the one on  $D$ , then this homeomorphism is biholomorphic.

Thus if  $D_i$  is a small neighborhood of the point  $p_i \in B$  biholomorphic to a disc, then the preimage in  $V$  of the punctured disc  $D_i^* := D_i \cap U$  is a disjoint union of punctured discs  $E_{i,j}^*$ . The maps  $E_{i,j}^* \rightarrow D_i^*$  are bihomeomorphic to the maps  $z \mapsto z^{n_{i,j}}$  for some  $n_{i,j}$ . Because of the way the holomorphic structure of  $V$  is defined, the maps  $z \mapsto z^{n_{i,j}}$  are actually holomorphic. Thus they extend holomorphically to maps of the full disks  $E_{i,j} \rightarrow D_i$  and  $V \cup \bigcup E_{i,j}$  is a compact Riemann surface in a unique way.  $\square$

The problem of classifying smooth curves  $C$  that have a map  $\pi : C \rightarrow B$  of degree  $d$  thus becomes one of classifying covering spaces of  $U$ .

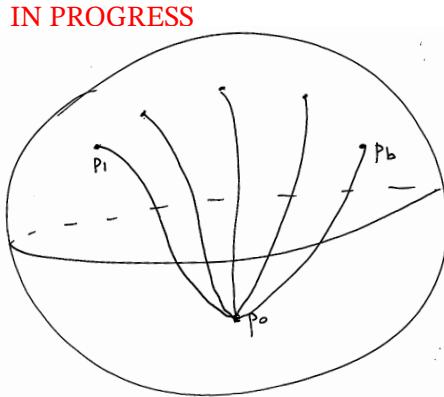
**2.1. Branched covers of  $\mathbb{P}^1$ .** We continue with the notation  $U = B \setminus \Delta$ , now supposing that  $B = \mathbb{P}_{\mathbb{C}}^1$ , the Riemann sphere. Again, let  $\pi : V \rightarrow U$  be a covering space.

Choose a base point  $p_0 \in U$ , and draw simple, non-intersecting arcs  $\gamma_i$  joining  $p_0$  to  $p_i$  in  $U$ . If  $\Sigma$  is the complement of the union of these arcs in the sphere, then the preimage of  $\Sigma$  in  $V$  will be the disjoint union of  $d$  copies of  $\Sigma$ , called the *sheets* of the cover; label these  $\Sigma_1, \dots, \Sigma_d$ .

Given  $U$ , elementary homotopy theory asserts the existence of a bijection between coverings  $V \rightarrow U$  of  $U$  of degree  $d$  (up to homeomorphisms of  $V$  fixing  $U$ ) and group homomorphisms

$$M : \pi_1(U, p_0) \rightarrow S_d,$$

to the symmetric group on  $d$  letters, up to inner automorphisms of  $S_d$ . The map  $M$  is called the *monodromy* of the covering: given  $V \rightarrow U$ , and a labeling of the  $d$  sheets of  $V$  over the point  $p$ , the value of  $M$  at a loop  $\beta$  in  $U$  based at  $p_0$  to the permutation of the points of  $\pi^{-1}(p_0)$  given by sending a point  $q \in \pi^{-1}(p_0)$  to the endpoint of the unique lift of  $\beta$  starting at  $q$ . A permutation  $\sigma$  of the labels of the sheets leads to a map  $M'$  equal to the composition of  $M$  with conjugation by  $\sigma$ .



**Figure 1.** Generators for the fundamental group of a multiply punctured sphere

A convenient set of generators of  $\pi_1(U, p_0)$  is the set of paths  $\beta_i$  indicated in Figure 2.1: starting at  $p_0$ , going out along the arc  $\gamma_i$  until just short of  $p_i$ , going once around  $p_i$  and then going back to  $p_0$  along the same path  $\gamma_i$ . The fundamental group of  $U$  is the free group generated by the paths  $\beta_1, \dots, \beta_b$  modulo the relation  $\prod_{i=1}^b \beta_i = 1$  which comes from the fact that the sphere minus the part enclosed by the paths  $\beta_i$  is contractible.

Given a degree  $d$  covering space  $V$  and a labeling of the  $d$  sheets over the point  $p_0$ , let  $\tau_i$  be the permutation of  $\{1, 2, \dots, d\}$  corresponding to the path  $\beta_i$ . The space  $V$  is connected if and only if the  $\tau_i$  generate a transitive subgroup of  $S_d$ .

If we start from a map of Riemann surfaces  $C \rightarrow \mathbb{P}^1$  that is simply branched, and take  $V$  to be the complement of the set of branch points, then each  $\tau_i$  is a transposition.

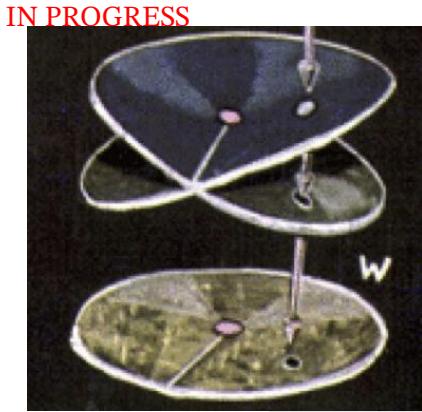
Summarizing we have proven:

**Lemma 2.2.** *Let  $p_1, \dots, p_b \in \mathbb{P}^1$  be any  $b$  given points. There is a natural bijection between*

- (1) *the set of simply branched covers  $\pi : C \rightarrow \mathbb{P}^1$  of degree  $d$ , branched over the points  $p_i$ , up to isomorphism over  $\mathbb{P}^1$ ; and*
- (2) *the set of  $b$ -tuples of transpositions  $\tau_1, \dots, \tau_b \in S_d$  such that  $\prod \tau_i = id$  and such that  $\tau_1, \dots, \tau_b$  generate a transitive subgroup of  $S_d$ , modulo simultaneous conjugation by  $S_d$ .*

□

**Proof of the uniqueness statement in Theorem 1.1.** In the case  $d = 2$  that is relevant to hyperelliptic curves, we note that there is only one transposition



**Figure 2.** Local picture of a simple branch point  $z \mapsto z^2$ . [Drawing by George Francis to be replaced after discussion]

in  $S_2$ . Thus there is a unique double cover of  $\mathbb{P}^1$  with given branch points  $p_1, \dots, p_b$ . (The product condition shows again that the number of branch points must be even.) This completes the proof of Theorem 1.1.  $\square$

**Example 2.3.** We can use these ideas to count the number of degree 3 branched covers  $C \rightarrow \mathbb{P}^1$  with given simple branch points, using that fact that every odd permutation  $\tau \in S_3$  is a transposition. Thus if  $b$  is even and  $\tau_1, \dots, \tau_{b-1} \in S_3$  are arbitrary transpositions, then the product  $\tau_1 \cdots \tau_{b-1}$  is also a transposition. It follows that the number of ordered  $b$ -tuples of transpositions  $\tau_1, \dots, \tau_b \in S_3$  with  $\prod \tau_i$  equal to the identity is  $3^{b-1}$ . The requirement that the group generated by the  $\tau_i$  is transitive eliminates just the three cases where all the  $\tau_i$  are equal. The group  $S_3$  acts on the set of  $b$ -tuples of permutations without stabilizing any  $b$ -tuple, so every cover corresponds to exactly 6 sequences  $\tau_1, \dots, \tau_b$ . In sum, the number of simply branched three-sheeted covers of  $\mathbb{P}^1$  with specified branch points  $q_1, \dots, q_b \in \mathbb{P}^1$  is

$$\frac{3^{b-1} - 3}{6} = \frac{3^{b-2} - 1}{2}.$$

One can use a similar strategy to count covers in other cases, when the target has higher genus and/or the degree of the covering is larger, but the combinatorics becomes more complicated.

### 3. Curves of genus 2

Since curves of genus 2 are hyperelliptic, everything we said above applies to them; in particular, the canonical map  $\phi_K : C \rightarrow \mathbb{P}^1$  on a curve of genus 2 is the expression of  $C$  as a double cover of  $\mathbb{P}^1$ , simply branched over 6 points in  $\mathbb{P}^1$ , which are unique up to automorphisms of  $\mathbb{P}^1$ .

In this section, we'll consider other maps from hyperelliptic curves  $C$  to projective space, starting with maps  $C \rightarrow \mathbb{P}^1$ . See for example [Eisenbud 1980] for a treatment of certain embeddings of hyperelliptic curves of all genera.

**3.1. Maps of  $C$  to  $\mathbb{P}^1$ .** The curve  $C$  has a unique degree 2 morphism to  $\mathbb{P}^1$  associated to the canonical system  $|K_C|$ . But there are many other morphisms to  $\mathbb{P}^1$ . For example, there is a 2-parameter family of maps of degree 3:

Let  $L$  be an invertible sheaf of degree 3 on  $C$ . Since  $3 > 2g - 2$ , the Riemann-Roch theorem tells us immediately that  $h^0(L) = 2$ , and there are two possibilities:

- (1) If the linear system  $|L|$  has a base point  $p \in C$ , then  $h^0(L(-p)) = 2$ , and hence  $L$  must be of the form  $L = K_C(p)$ . Conversely, if  $L = K_C(p)$ , then  $h^0(L(-p)) = h^0(L)$ , which is to say  $p$  is a base point of  $|L|$ . There is a 1-parameter family of such  $L$ .
- (2) If  $L$  is not of the form  $L = K_C(p)$ , then  $|L|$  does not have a basepoint, and so defines a degree 3 map  $\phi_L : C \rightarrow \mathbb{P}^1$ .

Since the variety  $\text{Pic}_3(C)$  has dimension  $g = 2$  the general invertible sheaf of degree 3 is of the second kind, and this gives a 2-parameter family of such maps.

There are plenty of higher-degree maps as well: an invertible sheaf of degree  $d \geq 4 = 2g$  is basepoint free, and gives a map to  $\mathbb{P}^{d-2}$ , from which we can project in many ways to  $\mathbb{P}^1$ .

**3.2. Maps of  $C$  to  $\mathbb{P}^2$ .** Next consider maps of  $C$  of genus 2 to the plane. By the Riemann-Roch theorem, an invertible sheaf  $L$  of degree 4 on  $C$  will have  $h^0(L) = 3$  and is basepoint free by Corollary 3.5 so the linear system  $|L|$  gives a morphism  $\phi_L : C \rightarrow \mathbb{P}^2$ . The invertible sheaf  $\mathcal{L} \otimes \omega_C^{-1}$  is either  $\omega_C$  or nonspecial; in either case, by the Riemann-Roch theorem, it has at least one section, so we may write  $\mathcal{L} \otimes \omega_C^{-1} = \mathcal{O}_C(p + q)$  for some points  $p, q$ . There are two possibilities:

- (1) If  $p + q = K_C$ , then  $\mathcal{L} = \omega_C^2$ . Since the elements of  $H^0(\omega_C)$  may be written as  $\omega, x\omega$ , the map

$$\text{Sym}^2 H^0(\omega_C) \rightarrow H^0(\mathcal{L}).$$

is injective, and since both sides are 3-dimensional vector spaces, they are equal. In other words, every divisor  $D \sim 2K_C$  is the sum of two divisors  $D_1, D_2 \in |K_C|$ . We conclude that the map  $\phi_L$  is the composition of the canonical map  $\phi_K : C \rightarrow \mathbb{P}^1$  with the Veronese embedding  $\nu_2 : \mathbb{P}^1 \rightarrow \mathbb{P}^2$  of  $\mathbb{P}^1$  as a conic in the plane and the map  $\phi_L$  is generically 2-to-1 onto the conic.

- (2) If  $p + q \neq K_C$  then  $h^0(p + q) = 1$ , so the pair  $p, q$  is unique. Furthermore,  $h^0(\mathcal{L} - p) = 2 = h^0(\mathcal{L}(-p - q))$  so  $H^0(\mathcal{L}(-p)) = H^0(\mathcal{L}(-q))$  and  $\phi_{\mathcal{L}}(p) = \phi_{\mathcal{L}}(q)$ . By the genus formula, the  $\delta$  invariant of this point must be 1. By Exercise 7.11 this is a node (if  $p \neq q$ ) or cusp (if  $p = q$ ).

Thus for  $\mathcal{L}$  in an open subset of  $\text{Pic}_4(C)$  the image is a quartic with a node; for a one-dimensional locus in  $\text{Pic}_4(C)$ , the image is a quartic with a cusp; and for one point in  $\text{Pic}_4(C)$  the image is a conic.

**3.3. Embeddings in  $\mathbb{P}^3$ .** By Corollary 3.5 any invertible sheaf  $L$  of degree 5 is very ample. Write  $\phi_{\mathcal{L}} : C \rightarrow \mathbb{P}^3$  for the map given by the complete linear system  $|L|$ . Since  $\phi_{\mathcal{L}}$  is an embedding, we'll also denote the image  $\phi_{\mathcal{L}}(C) \subset \mathbb{P}^3$  by  $C$  and write  $\mathcal{O}_C(1)$  for  $L$ .

What degree surfaces in  $\mathbb{P}^3$  contain the curve  $C$ ? We start with degree 2, and consider the restriction map

$$H^0(\mathcal{O}_{\mathbb{P}^3}(2)) \rightarrow H^0(\mathcal{O}_C(2)) = H^0(L^2).$$

The space on the left has dimension 10; by the Riemann-Roch theorem we have  $h^0(L^2) = 2 \cdot 5 - 2 + 1 = 9$ . It follows that  $C$  lies on a quadric surface  $Q$ . Since  $C$  is not contained in a plane or a union of planes, any quadric containing  $C$  is irreducible; if there were more than one such, Bézout's theorem would imply that  $\deg(C) \leq 4$ . Thus  $Q$  is unique.

We might ask at this point: is  $Q$  smooth or a quadric cone? The answer depends on the choice of invertible sheaf  $L$ .

**Proposition 3.1.** *Let  $C \subset \mathbb{P}^3$  be a smooth curve of degree 5 and genus 2 and  $Q \subset \mathbb{P}^3$  the unique quadric containing  $C$ . If  $L = \mathcal{O}_C(1) \in \text{Pic}_5(C)$ , then  $Q$  is singular if and only if we have*

$$L \cong K^2(p)$$

for some point  $p \in C$ ; in this case, the point  $p$  maps to the vertex of  $Q$ .

Note that the variety  $\text{Pic}_5(C)$  has dimension 2, while the sheaves  $K^2(p)$  form a one-dimensional subfamily. Thus in general  $Q$  is smooth; the set of invertible sheaves  $L$  for which the quadric is singular is 1-dimensional, and thus of codimension one.

**Proof.** First, suppose that the invertible sheaf  $L \cong K^2(p)$  for some  $p \in C$ . Then  $L(-p) \cong K^2$ , so that the map  $\pi : C \rightarrow \mathbb{P}^2$  given by projection from  $p$  is the map  $\phi_{K^2} : C \rightarrow \mathbb{P}^2$  given by the square of the canonical sheaf. As we've seen, the map  $\phi_{K^2}$  is two-to-one onto a conic  $E \subset \mathbb{P}^2$ , so that the curve  $C$  lies on the cone  $Q$  over  $E$  with vertex  $p$ , and this is the unique quadric surface containing  $C$ .

If  $L$  is not of the form  $K^2(p)$ , then we can write

$$L = K \otimes M,$$

where by hypothesis  $M$  is not of the form  $K(p)$ . As we saw in Section 3.1, this means that the pencil  $|M|$  gives a degree 3 map  $C \rightarrow \mathbb{P}^1$ .

This gives us a way of factoring the map  $\phi_L : C \rightarrow \mathbb{P}^3$ : we have maps  $\phi_K : C \rightarrow \mathbb{P}^1$  of degree 2 and  $\phi_M : C \rightarrow \mathbb{P}^1$  of degree 3, and we can compose their product with the Segre embedding  $\sigma : \mathbb{P}^1 \times \mathbb{P}^1 \rightarrow \mathbb{P}^3$ :

$$C \xrightarrow{\phi_K \times \phi_M} \mathbb{P}^1 \times \mathbb{P}^1 \xrightarrow{\sigma} \mathbb{P}^3$$

This description of the map  $\phi_L$  shows that  $C$  is a curve of type  $(2, 3)$  on a smooth quadric  $Q \subset \mathbb{P}^3$ , completing the proof of Proposition 3.1.  $\square$

**3.3.1. The ideal of a quintic space curve of genus 2.** To describe a minimal set of generators of the homogeneous ideal  $I(C) \subset \mathbb{C}[x_0, x_1, x_2, x_3]$  in either case, we look at the restriction map

$$H^0(\mathcal{O}_{\mathbb{P}^3}(3)) \rightarrow H^0(\mathcal{O}_C(3)).$$

Since the dimensions of these spaces are 20 and  $15 - 2 + 1 = 14$  respectively, we see that the vector space of cubics vanishing on  $C$  has dimension at least 6. The subspace of cubics divisible by  $Q$  has dimension 4. It follows that there are at least two cubics vanishing on  $C$  that are linearly independent modulo those vanishing on  $Q$ .

We can identify these cubics geometrically. Suppose first that  $Q$  is smooth, so that  $C$  is a curve of type  $(2, 3)$  on  $Q$ . In that case, if  $L \subset Q$  is any line of the first ruling, the sum  $C + L$  is the complete intersection of  $Q$  with a cubic  $S_L$ , unique modulo the ideal of  $Q$ ; conversely, if  $S$  is any cubic containing  $C$  but not containing  $Q$ , the intersection  $S \cap Q$  will be the union of  $C$  and a line  $L$  of the first ruling; thus, mod  $I(Q)$ ,  $S = S_L$ . A similar argument applies in case  $Q$  is a cone, and  $L$  is any line of the (unique) ruling of  $Q$ . In Exercise 6.6 you may show that there are no more cubics containing  $C$ .

**3.4. The dimension of the family of genus 2 curves.** Each of the types of maps that we described from a curve  $C$  of genus 2 to projective space suggests a way to compute the dimension of the family of genus 2 curves, and indeed, as we will explain in Chapter 8, there is a moduli space of this dimension.

First, every curve of genus 2 is uniquely expressible as a double cover of  $\mathbb{P}^1$  branched at six points, modulo the group  $PGL_2$  of automorphisms of  $\mathbb{P}^1$ . The space of such double covers has dimension 6, and  $\dim(PGL_2) = 3$ , and since the group acts with finite stabilizers this gives a family of dimension  $6 - 3 = 3$ .

Also, each curve of genus 2 is expressible as a 3-sheeted cover of  $\mathbb{P}^1$  (with eight branch points) in a 2-dimensional family of ways. As we saw in Section 2, such a triple cover is determined up to a finite number of choices by its branch divisor, so the space of such triple covers has dimension 8; modulo  $PGL_2$  it has dimension 5, and since every curve is expressible as a triple cover in a two-dimensional family of ways, we arrive again at a family of dimension  $5 - 2 = 3$ .

We've also seen that each curve of genus 2 can be realized as (the normalization of) a plane quartic with a node in a 2-dimensional family of ways. The space of plane quartics has dimension 14; the family of those with a node has codimension one (Section 6.1) and hence dimension 13. Since the automorphism group  $PGL_3$  of  $\mathbb{P}^2$  has dimension 8, this suggests that that the family of nodal plane quartics modulo  $PGL_3$  has dimension 5. Finally, since every curve of genus 2 corresponds to a 2-parameter family of such curves, this again suggests a family of dimension  $5 - 2 = 3$ .

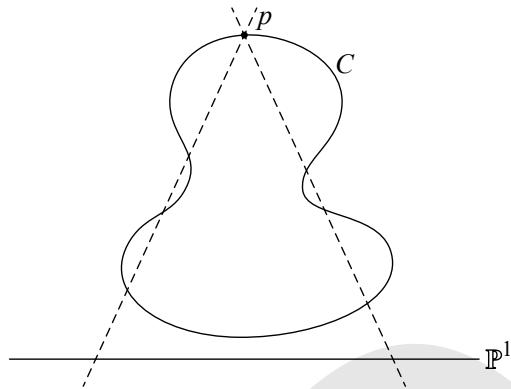
Finally, a curve of genus 2 may be realized as a quintic curve in  $\mathbb{P}^3$  in a two-parameter family of ways. To count the dimension of the family of such curves, note that each one lies on a unique quadric  $Q$ . We can assume for this purpose that  $Q$  is smooth, since the singular quadrics and curves on them occur in codimension 1. The curve  $C$  is of type  $(2, 3)$  on  $Q$ . Thus to specify such a curve we have to specify  $Q$  (9 parameters) and then a bihomogeneous polynomial of bidegree  $(2, 3)$  on  $Q \cong \mathbb{P}^1 \times \mathbb{P}^1$  up to scalars; these have  $3 \cdot 4 - 1 = 11$  parameters. Thus there is a 20-dimensional family of such divisors; modulo the automorphism group  $PGL_4$  of  $\mathbb{P}^3$ , this is a 5-dimensional family. Again, every abstract curve  $C$  of genus 2 corresponds to a 2-parameter family of these curves modulo  $PGL_4$ , so once more this suggests a family of dimension  $5 - 2 = 3$ .

## 4. Curves of genus 3

Let  $C$  be a smooth projective curve of genus 3. Since we have already discussed hyperelliptic curves, we will assume that  $C$  is not hyperelliptic. By Theorem 4.1, the canonical map  $\phi_K : C \rightarrow \mathbb{P}^2$  embeds  $C$  as a smooth plane quartic curve. Conversely, by Proposition 2.1 any smooth plane curve of degree 4 has genus 3 and is embedded by the complete canonical series.

Since the space of plane quartic curves is 14-dimensional, and  $\dim PGL(3) = 8$ , this suggests that there is a 6-dimensional family of curves of genus 3, and in Chapter 8 we will see that there is indeed a 6-dimensional moduli space.

**4.1. Other representations of a curve of genus 3.** Since we have assumed that  $C$  is not hyperelliptic there is no degree 2 cover of  $\mathbb{P}^1$ . On the other hand, there are degree 3 covers: if  $L \in \text{Pic}_3(C)$  is an invertible sheaf of degree 3 then,



**Figure 3.** Expression of a plane quartic  $C$  of genus 3 as a 3-sheeted cover of  $\mathbb{P}^1$  by projecting the canonical model from a point on it.

(( Silvio: sometimes points on curves are lowercase, sometimes uppercase. Standardize? ))

by the Riemann-Roch theorem, we have

$$h^0(L) = \begin{cases} 2, & \text{if } L \cong K - p \text{ for some point } p \in C; \text{ and} \\ 1 & \text{otherwise.} \end{cases}$$

There is thus a 1-dimensional family of representations of  $C$  as a 3-sheeted cover of  $\mathbb{P}^1$ . These are visible directly from the canonical model: a degree 3 map  $\phi_{K-p} : C \rightarrow \mathbb{P}^1$  is the composition of the canonical embedding  $\phi_K : C \rightarrow \mathbb{P}^2$  with a projection from  $p$ .

There are other representations of  $C$  as the normalization of a plane curve. By Clifford's theorem  $C$  has no  $g_3^2$ , and the canonical system is the only  $g_4^2$ , but there are plenty of models as plane quintic curves: by Proposition 4.5, if  $L$  is any invertible sheaf of degree 5, the linear system  $|L|$  will be a basepoint free  $g_5^2$  as long as  $L$  is not of the form  $K + p$ , so that  $\phi_L$  maps  $C$  birationally onto a plane quintic curve  $C_0 \subset \mathbb{P}^2$ . These can also be described geometrically in terms of the canonical model: any such invertible sheaf  $L$  is of the form  $2K - p - q - r$  for some trio of points  $p, q, r \in C$  that are not collinear in the canonical model, and we see that  $C_0$  is obtained from the canonical model of  $C$  by applying a Cremona transform with respect to the points  $p, q$  and  $r$ , that is, by applying the birational transformation of the plane defined by the linear series of conics through  $p, q, r$ .

Proposition 4.5 implies that a divisor  $D$  of degree 6 is very ample if and only if it is not of the form  $K + p + q$  for any  $p, q \in C$  and since the family of invertible

sheaves on  $C$  has dimension 3, we see that a general invertible sheaf of degree 6 is very ample (indeed, this is a simple case of Theorem 5.1).

If  $C \subset \mathbb{P}^3$  is a curve of genus 3 embedded as a curve of degree 6, then  $C$  cannot lie on a singular quadric since by Example 6.1 it would have to be a complete intersection of the quadric with a cubic, and then such a curve has genus 4. If  $C$  lies on a smooth quadric in class  $(a, b)$  then  $a$  or  $b$  would be 2, so  $C$  would be hyperelliptic, and conversely any curve in class  $(2, 4)$  is a hyperelliptic curve of genus 3, degree 6.

Thus if  $C$  is not hyperelliptic, then  $C$  does not lie on a quadric surface. We have  $h^0(\mathcal{O}_{\mathbb{P}^3}(3)) = 20$  while, by the Riemann-Roch formula,  $h^0(\mathcal{O}_C(3)) = 18 - 3 + 1 = 16$ , so  $C$  lies on (at least) 4 independent cubics. Each of these cubics must be irreducible, so any two of them intersect in a curve of degree 9 containing  $C$  and another component or components  $D$  of degree totaling 3. By Bertini's theorem if we choose two *general* cubics containing  $C$ , then each of the components of  $D$  will be smooth. We shall see in Theorem 3.1 that the arithmetic genus of  $D$  must be 0; thus  $D$  must be a twisted cubic curve. The ideal of the twisted cubic is generated by the  $2 \times 2$  minors of a matrix of the form

$$\begin{pmatrix} \ell_0 & \ell_1 & \ell_2 \\ \ell_1 & \ell_2 & \ell_3 \end{pmatrix}$$

where the  $\ell_i$  are linear forms, and it follows that the two cubics can be written as the two  $3 \times 3$  minors involving the first two rows of a matrix of the form

$$\begin{pmatrix} \ell_0 & \ell_1 & \ell_2 \\ \ell_1 & \ell_2 & \ell_3 \\ \ell_4 & \ell_5 & \ell_6 \\ \ell_7 & \ell_8 & \ell_9 \end{pmatrix}$$

where  $\ell_4, \dots, \ell_9$  are linear forms as well. From the Hilbert-Burch theorem (Corollary 4.5) one can show that the ideal of  $C$  is generated by the four  $3 \times 3$  minors of this matrix, whose columns generate the syzygies of the ideal of the curve.

## 5. Theta characteristics

In this section we sketch the algebraic theory of theta-characteristics, starting with the case of curves of genus 3.

Suppose that  $C \subset \mathbb{P}^2$  is a smooth plane curve. A *bitangent* to  $C$  is a line  $L \subset \mathbb{P}^2$  that is either tangent to  $C$  at two distinct points, or has contact of order  $\geq 4$  with  $C$  at a point. Alternatively, we can say that a bitangent corresponds to an effective divisor of degree 2 on  $C$  such that  $2D$  is contained in the intersection of  $C$  with a line  $L \subset \mathbb{P}^2$ .

A naive dimension count suggests that a smooth plane curve should have a finite number of bitangents (it's one condition on a line  $L \in \mathbb{P}^2^*$  to be tangent to  $C$ , so it should be two conditions for it to be bitangent). Indeed, this is the case; by Bézout's theorem a conic or cubic curve cannot have any bitangents, but as we will show in Section 1.3 every smooth curve of degree  $d \geq 4$  has

$$12\binom{d+1}{4} - 4d(d-2),$$

counted with appropriate multiplicities (for example, a line simply tangent to  $C$  at 3 points counts as three bitangents). Accordingly, a smooth plane quartic has 28 bitangents (see the drawing by Plücker in Figure 5).

The bitangents to a smooth plane quartic  $C$  (a canonical curve of genus 3) have a special significance: since  $4 = 2 \times 2$ , if  $D = p + q$  is a bitangent, then the divisor  $2D$  comprises the complete intersection of  $C$  with a line; in other words, we have a linear equivalence

$$2D \sim K_C$$

or equivalently the invertible sheaf  $\mathcal{O}_C(D)$  is a square root of the canonical sheaf of  $C$ . Because of their appearance in the theory of theta functions, Riemann named the square roots of the canonical sheaf *theta-characteristics*.

How many such square roots are there? If  $\mathcal{L}$  and  $\mathcal{M}$  are invertible sheaves with  $\mathcal{L}^2 = \mathcal{M}^2 = K$ , then  $\mathcal{L}$  and  $\mathcal{M}$  differ by an invertible sheaf of order 2; that is,

$$\mathcal{M} = \mathcal{L} \otimes F, \quad \text{where} \quad F \otimes F \sim \mathcal{O}_C.$$

In other words,  $F$  is an invertible sheaf of degree 0 and, having fixed  $\mathcal{L}$ , the other sheaf,  $\mathcal{M}$ , corresponds to a point of order 2 in the Picard group  $\text{Pic}_0(C)$ . Since we've seen that  $\text{Pic}_0(C) = \text{Jac}(C)$  is a complex torus of dimension  $g = 3$ —the quotient of  $\mathbb{C}^3$  by a lattice  $\Lambda \cong \mathbb{Z}^6$ —we see that there are  $2^6 = 64$  such invertible sheaves, and thus, given that there is some invertible sheaf  $\mathcal{L}$  satisfying  $\mathcal{L}^2 \cong K_C$ , there are exactly  $64 = 2^{2g}$  of them.

The reader will have noticed that the number 64 of theta-characteristics does not agree with the number 28 of bitangents. The reason is easy to see: bitangents correspond to *effective* divisors  $D$  with  $2D \sim K$ , while a theta characteristic  $\mathcal{L}$  may have  $h^0(\mathcal{L}) = 0$ , that is, may not correspond to an effective divisor. This situation also occurs in other genera. What can we say about the dimensions  $h^0(\mathcal{L})$  of the space of sections of the theta-characteristics on  $C$ ?

There is a beautiful partial answer to this question, which can be deduced from a remarkable fact: the dimension  $h^0(\mathcal{L})$  of the space of sections of a theta characteristic mod 2 is invariant in families. We will now sketch the necessary results; see [Mumford 1971] and [Harris 1982c] for a full treatment.

**Theorem 5.1.** *Let  $C \rightarrow B$  be a family of smooth curves, and  $\mathcal{L}_b$  a family of theta characteristics on the curves in this family—in other words, an invertible sheaf  $\mathcal{L}$  on  $C$  such that  $(\mathcal{L}|_{C_b})^2 \cong K_{C_b}$  for each  $b \in B$ . If we define a function  $f : B \rightarrow \mathbb{Z}/2$  by*

$$f(b) = h^0(\mathcal{L}|_{C_b}) \pmod{2}$$

*then  $f$  is locally constant.*

We say that a theta-characteristic  $\mathcal{L}$  is *even* or *odd* according to the parity of  $h^0(\mathcal{L})$ . Given the irreducibility of the space of smooth irreducible curves of genus  $g$  (which we'll discuss in Chapter 8), Theorem 5.1 suggests that all curves of genus  $g$  have the same number of even (equivalently, of odd) theta characteristics, and this is in fact the case.

**Theorem 5.2.** *If  $C$  is a curve of genus  $g$ , then of the  $2^{2g}$  theta-characteristics on  $C$  there are  $2^{g-1}(2^g + 1)$  even theta characteristics and  $2^{g-1}(2^g - 1)$  odd theta characteristics.*

Using Theorem 5.1 and the connectedness of the moduli space of curves, Theorem 5.2 is reduced to the case when  $C$  is hyperelliptic. We will carry out the computation for hyperelliptic curves in Section 5.1 below.

Note that in the case of a nonhyperelliptic curve  $C$  of genus 3, the dimension  $h^0(\mathcal{L})$  of a theta characteristic  $\mathcal{L}$  cannot be  $\geq 2$ , so the odd theta characteristics are exactly the effective theta characteristics, and this says exactly that there are  $2^{g-1}(2^g - 1) = 28$  effective theta-characteristics corresponding to the 28 bitangents.

The proof of Theorem 5.1 follows, via an ingenious construction of Mumford's, from a fact about quadratic forms in an even number of variables. To state this, recall that if  $Q$  is a bilinear form on  $V$  then an *isotropic plane* is a subspace  $\Lambda \subset V$  such that  $Q(\Lambda, \Lambda) = 0$ .

**Cheerful Fact 5.3.** *Suppose that  $V$  is a  $2n$ -dimensional complex vector space with a nondegenerate bilinear form  $Q$ .*

- (1) *The maximal isotropic subspaces for  $Q$  have dimension  $n$ ;*
  - (2) *The set of maximal isotropic subspaces for  $Q$  is a subvariety of the Grassmannian  $G(n, V)$ , of dimension  $\binom{n}{2}$ , that has exactly two connected components (the “rulings” of  $Q$ ); and*
  - (3) *If  $\Lambda, \Lambda' \subset V$  are any two maximal isotropic subspaces, then*
- $$\dim(\Lambda \cap \Lambda') \equiv n \pmod{2} \iff \Lambda, \Lambda' \text{ belong to the same ruling.}$$

*A proof is given in [Griffiths and Harris 1978, pp. 735–740].*

The first of these assertions is completely elementary: since the map  $\tilde{Q} : V \xrightarrow{\cong} V^*$  associated to the form  $Q$  carries an isotropic subspace to its annihilator, there can't be an isotropic plane of dimension  $> n$ ; and similarly if  $\Lambda \subset V$  is any isotropic plane of dimension  $< n$  we can include  $\Lambda$  in a larger isotropic plane by adding any vector  $v$  with  $\tilde{Q}(v, v) = 0$  for the induced bilinear form  $\tilde{Q}$  on  $\text{ann}(\Lambda)/\Lambda$ .

The second and third assertions are less elementary, but the reader may already have seen the first two nontrivial cases of each:

**Example 5.4.**

- When  $n = 2$  the form  $Q$  corresponds to a smooth quadric surface in  $\mathbb{P}^3$ , and the lines on this surface correspond to the isotropic 2-planes in  $\mathbb{C}^4$ . There are two rulings by lines, and lines of opposite rulings meet in a point, while lines of the same ruling are either disjoint or equal.
- When  $n = 3$ , the Grassmannian  $\mathbb{G}(1, 3)$ , in its Plücker embedding, is a smooth quadric in  $\mathbb{P}^5$ . The isotropic planes in the two distinct components are easy to describe: in one component they are the projective 2-plane of lines containing a given point  $p \in \mathbb{P}^3$ . In the other component they are the planes corresponding to the lines contained in a given plane  $H \subset \mathbb{P}^3$ . Two of these families of lines intersect in either a point, or coincide; that is, the linear subspaces either intersect in a line or a 3-plane. See Exercise 6.9.

Now suppose that  $C$  is a smooth curve of genus  $g$ , and  $\mathcal{L}$  an invertible sheaf on  $C$  with  $\mathcal{L}^2 \cong K_C$ ; that is, a theta characteristic. Choose a divisor  $D = p_1 + \dots + p_n$  of degree  $n > g - 1$  consisting of distinct points, and let  $V$  be the  $2n$ -dimensional vector space

$$V := H^0(\mathcal{L}(D)/\mathcal{L}(-D)).$$

From the exact sequence

$$0 \rightarrow \mathcal{L}(-D) \rightarrow \mathcal{L}(D) \rightarrow \mathcal{L}(D)/\mathcal{L}(-D) \rightarrow 0$$

we see that The sheaf  $\mathcal{L}(D)/\mathcal{L}(-D)$  is supported on  $D$ , with stalk isomorphic to  $\mathcal{O}_p/\mathfrak{m}_{C,p}^2$  of dimension 2 at each  $p \in D$ . We can define a bilinear form on  $V$  by setting

$$Q(\sigma, \tau) := \sum_i \text{Res}_{p_i}(\sigma \cdot \tau)$$

where we use the isomorphism  $\mathcal{L}^2 \cong K_C$  to identify the product  $\sigma \tau$  with a rational differential.

We now introduce two isotropic subspaces for  $Q$ : first, we set

$$\Lambda := H^0(\mathcal{L}/\mathcal{L}(-D)).$$

This is isotropic because the product  $\sigma\tau$  of two elements of  $\Lambda$  corresponds to a regular differential, and so has no residues. Second, we set

$$\Lambda' := \text{im}(H^0(\mathcal{L}(D)) \rightarrow H^0(\mathcal{L}(D)/\mathcal{L}(-D)))$$

Since  $H^0(\mathcal{L}(-D)) = 0$ , the map is injective and according to Riemann-Roch,  $h^0(\mathcal{L}(D)) = n$ , so this is again an  $n$ -dimensional subspace of  $V$ ; it's isotropic because the sum of the residues of a global rational differential on  $C$  is 0. Finally,

$$H^0(\mathcal{L}) \cong \Lambda \cap \Lambda',$$

and Theorem 5.1 follows.

As for proving Theorem 5.2, it is possible to describe the configurations of odd and even theta characteristics as subsets of the set  $S$  of all theta characteristics, which as we've seen is a principle homogeneous space for the group  $\text{Jac}(C)_2 \cong (\mathbb{Z}/2\mathbb{Z})^{2g}$  of points of order 2 on the Jacobian [Harris 1982c].

There is more to say about the configuration of theta characteristics. For example:

**Cheerful Fact 5.5.** *As noted, if we choose any theta characteristic on a curve  $C$ , we may identify the set  $S^-$  of odd theta-characteristics with a subset of the group  $\text{Jac}(C)_2$  of points of order 2 on the Jacobian of  $C$ . We might expect that some 4-tuples of these points will add up to 0 in  $\text{Jac}(C)$ ; in other words, there should exist some 4-tuples  $\mathcal{L}_1, \dots, \mathcal{L}_4 \in S^-$  such that*

$$\mathcal{L}_1 + \cdots + \mathcal{L}_4 = 2K_C.$$

*What this means in the case of genus  $g = 3$  is that among the 28 bitangents to a smooth plane quartic curve  $C$ , there are some subsets of 4 whose eight points of tangency form the intersection of  $C$  with a plane conic. From the more detailed knowledge of the configuration  $S^-$  we can say how many. Indeed, the number was first found by Salmon; it is 315 ([Salmon 1960]).*

**5.1. Counting theta-characteristics.** One way to count the number of odd and even theta characteristics on a curve of genus  $g$  is to describe these explicitly in the case of a hyperelliptic curve and then use Theorem 5.1 to deduce the corresponding statements for any smooth curve of genus  $g$ . The reader who wishes may try a relatively simple case in Exercise 6.7 before looking at the general case below. We start with some preliminary calculations:

**Lemma 5.6.** *For any positive integer  $n$ ,*

(1)

$$\sum_{k=0}^n \binom{2n}{2k} = \sum_{k=0}^{n-1} \binom{2n}{2k+1} = 2^{2n-1}$$

(2)

$$\sum_{k=0}^n \binom{4n}{4k} = 2^{4n-2} + (-1)^n 2^{2n-1} \quad \text{and} \quad \sum_{k=0}^{n-1} \binom{4n}{4k+2} = 2^{4n-2} - (-1)^n 2^{2n-1}$$

(3)

$$\sum_{k=0}^n \binom{4n+2}{4k+1} = 2^{4n} + (-1)^n 2^{2n} \quad \text{and} \quad \sum_{k=0}^{n-1} \binom{4n}{4k+3} = 2^{4n} - (-1)^n 2^{2n}$$

**Proof.** The first is completely elementary; by the binomial theorem, we have

$$2^{2n} = (1+1)^{2n} = \sum_{l=0}^{2n} \binom{2n}{l} \quad \text{and} \quad 0 = (1-1)^{2n} = \sum_{l=0}^{2n} (-1)^l \binom{2n}{l}$$

and taking the sum and the difference of these two equations yields the desired result.

The second follows similarly by applying the binomial theorem to the expression  $(1+i)^{4n} = (-1)^n 2^{2n}$ . Equating the real parts, we have

$$\sum_{k=0}^n \binom{4n}{4k} - \sum_{k=0}^{n-1} \binom{4n}{4k+2} = (-1)^n 2^{2n}$$

while by the first part of the lemma we have

$$\sum_{k=0}^n \binom{4n}{4k} + \sum_{k=0}^{n-1} \binom{4n}{4k+2} = 2^{4n-1};$$

taking the sum and difference of these two equations yields the desired formula.

Finally, for the third we apply the binomial theorem to the expression  $(1+i)^{4n+2} = (-1)^n 2^{2n+1}i$ . Equating the imaginary parts, this gives

$$\sum_{k=0}^n \binom{4n+2}{4k+1} - \sum_{k=0}^{n-1} \binom{4n+2}{4k+3} = (-1)^n 2^{2n+1}$$

whereas by the first part of the lemma,

$$\sum_{k=0}^n \binom{4n+2}{4k+1} + \sum_{k=0}^{n-1} \binom{4n+2}{4k+3} = (-1)^n 2^{4n+1}$$

and as before taking the sum and difference of these two yields the result.  $\square$

We will count the number of theta characteristics on a hyperelliptic curve in terms of sums of subsets of the ramification points, so we need to know what linear equivalences exist among sums of these subsets:

**Lemma 5.7.** *Let  $C$  be the hyperelliptic curve of genus  $g$  expressed as a 2-sheeted cover of  $\mathbb{P}^1$  with ramification points  $p_1, \dots, p_{2g+2}$ . The divisor class of any half of the ramification points is equal to the divisor class of the other half, but there are no smaller relations. More precisely, let  $I_1, I_2$  be subsets of  $\{1, \dots, 2g+2\}$  and set*

$$D_i = \sum_{j \in I_i} p_j.$$

*The divisors  $D_1, D_2$  are linearly equivalent if and only if  $I_1 = I_2$  or they have the same cardinality  $g+1$  and  $I_1 \cup I_2 = \{1, \dots, 2g+2\}$ .*

**Proof.** Note that the “if” part is simply Corollary 1.2 above.

For the “only if” part, subtracting whatever points  $D_1$  and  $D_2$  have in common we may suppose that  $I_1 \cap I_2 = \emptyset$ . If  $D_1 \sim D_2$ , it follows at once that they have the same degree,  $d \leq g+1$ , and we must show that either  $d = 0$  or  $d = g+1$ .

We have  $D_1 \sim D_2$  iff  $D_1 + D_2 \equiv 2D_1$ . If  $d \leq g$  we have  $r(2D_1) = d$ : for  $d < g$  this is the extremal case of Clifford’s theorem, while for  $d = g$  this follows simply from the Riemann-Roch formula. Thus in case  $d \leq g$  every divisor in  $|2D_1|$  is a sum of  $d$  fibers of the 2 to 1 map of  $C$  to  $\mathbb{P}^1$ , and for such a divisor to be a sum of distinct points  $p_i$  the degree  $d$  must be 0, concluding the argument.  $\square$

Returning to the counting, let  $C$  be the hyperelliptic curve of genus  $g$  expressed as a 2-sheeted cover of  $\mathbb{P}^1$ , with ramification points  $p_1, \dots, p_{2g+2}$ .

First of all, if we denote the class of the unique  $g_2^1$  on  $C$  by  $E$ , and  $D$  is any theta characteristic, then  $D + E$  will be effective, and so we can write

$$D \sim mE + F$$

with  $-1 \leq m \leq \frac{g-1}{2}$  and  $F$  the sum of  $g-1-2m$  distinct points  $p_i$ . Moreover, this representation is unique, except in case  $m = -1$ ; in that case, we note that the sum of  $g+1$  of the branch points of  $C$  is linearly equivalent to the sum of the other  $g+1$  by Corollary 1.2. Thus the total number of theta characteristics is a sum of binomial coefficients; if  $g$  is odd, it is

$$\binom{2g+2}{0} + \binom{2g+2}{2} + \binom{2g+2}{4} + \dots + \binom{2g+2}{g-1} + \frac{1}{2} \binom{2g+2}{g+1}$$

and similarly if  $g$  is even it is

$$\binom{2g+2}{1} + \binom{2g+2}{3} + \binom{2g+2}{5} + \dots + \binom{2g+2}{g-1} + \frac{1}{2} \binom{2g+2}{g+1}.$$

In either case, we are adding up every other entry in the  $(2g+2)$ -nd row of Pascal’s triangle, starting from the left and ending up with one half of the middle

term. This sum is exactly one half of the sum of every other entry in the whole row; by the first part of Lemma 5.6 this is exactly  $\frac{1}{4} \cdot 2^{2g+2} = 2^{2g}$ .

Finally, we can add up the number of even and odd theta characteristics separately simply by taking every other term in the sums above; using the second and third parts of Lemma 5.6 (in case  $g$  is odd and even, respectively) we can conclude that  $C$  has  $2^{g-1}(2^g - 1)$  odd theta characteristics and  $2^{g-1}(2^g + 1)$  even theta characteristics. By Theorem 5.1 (and the connectedness of the space of smooth irreducible curves of genus  $g$ ), this count then holds for all curves of genus  $g$ , establishing Theorem 5.2.

## 6. Exercises

**Exercise 6.1.** We have seen that a curve  $C$  of genus  $g = 1$  is expressible as a 2-sheeted cover of  $\mathbb{P}^1$  branched over four points; that is, as the smooth projective curve associated to the affine curve  $C^\circ \subset \mathbb{A}^2$  given by  $y^2 - \prod_{i=1}^4 (x - \lambda_i)$ . Show that the closure  $\overline{C^\circ}$  of  $C^\circ \subset \mathbb{A}^2$  in either  $\mathbb{P}^2$  or  $\mathbb{P}^1 \times \mathbb{P}^1$  consists of the union of  $C^\circ$  with one additional point, with that point a tacnode of  $\overline{C^\circ}$  in either case.

Hint: Observe that in either case the complement  $\overline{C^\circ} \setminus C^\circ$  consists of a single point, with two points of  $C$  mapping to it; now use the genus formula in either  $\mathbb{P}^2$  or  $\mathbb{P}^1 \times \mathbb{P}^1$ .

**Exercise 6.2.** Find the number of 3-sheeted covers  $C \rightarrow \mathbb{P}^1$  of genus  $g$  with simple branching except for one point of total ramification (that is, one point with just a single preimage point.)

Hint: such a cover is specified by giving  $2g + 2$  transpositions, not all equal, whose product is a nontrivial 3-cycle, modulo simultaneous conjugation. We have already worked out the number of such tuples whose product is the identity; just subtract.

**Exercise 6.3.** Let  $B$  be a curve of genus  $h$ . How many unramified double covers of  $B$  are there?

Hint: Topologically, such covers are in 1-1 correspondence with subgroups of index 2 in  $\pi_1(C)$ ; and such a subgroup is necessarily the preimage of a subgroup of index 2 in the abelianization  $H_1(C, \mathbb{Z}) \cong \mathbb{Z}^{2g}$ .

**Exercise 6.4.** Show that unramified double covers of a smooth curve  $C$  are in one-to-one correspondence with invertible sheaves  $\mathcal{L}$  on  $C$  such that  $\mathcal{L}^2 \cong \mathcal{O}_C$ , that is with the 2-torsion points of  $\text{Jac}(C)$ .

Hint: If  $f : X \rightarrow C$  is an unramified double cover, consider the direct image  $f_*(\mathcal{O}_X)$ . This is a locally free sheaf of rank 2 on  $C$ , on which the group  $\mathbb{Z}/2$  acts; the  $+1$ -eigenspace is the structure sheaf  $\mathcal{O}_C$ , and the  $-1$ -eigenspace is an invertible sheaf  $\mathcal{L}$  on  $C$  such that  $\mathcal{L}^2 \cong \mathcal{O}_C$ .

**Exercise 6.5.** Let  $E$  be a curve of genus 1, and  $q_1, \dots, q_b \in E$ . How many double covers  $C \rightarrow E$  are there branched over the  $q_i$ ?

Hint: By our analysis, to specify such a cover, we have to specify the monodromy around representative loops generating  $H_1(E, \mathbb{Z}) \cong \mathbb{Z}^2$ ; thus there are four possibilities.

In the next exercise, we ask you to complete the description in Section 3.3.1 of the ideal of a quintic space curve of genus 2.

**Exercise 6.6.** Show that for any pair of lines  $L, L'$  of the appropriate ruling of  $Q$ , the three polynomials  $Q, S_L$  and  $S_{L'}$  generate the homogeneous ideal  $I(C)$ . Find relations among them. Write out the minimal resolution of  $I(C)$ .

Hint: choose any line  $M \subset Q$  of the opposite ruling, and look at the linear forms  $H, H'$  on  $\mathbb{P}^3$  vanishing on  $L \cup M$  and  $L' \cup M$ .

**Exercise 6.7.** Let  $C$  be a curve of genus 2, expressed as a 2-sheeted cover of  $\mathbb{P}^1$  with ramification points  $p_1, \dots, p_6$ . In this exercise we will count the number of even and odd theta characteristics. The text contains the count for a hyperelliptic curve of any genus; we offer the case of genus 2 as a warmup.

- (1) Show that the theta-characteristics on  $C$  are either of the form  $\mathcal{L} = \mathcal{O}_C(p_i)$  or of the form  $\mathcal{L} = \mathcal{O}_C(p_i + p_j - p_k)$  with  $i, j, k$  distinct.
- (2) Show that in the first case we have  $h^0(\mathcal{L}) = 1$ , and in the second case we have  $h^0(\mathcal{L}) = 0$ .
- (3) Finally, show that there are six of the former kind, and 10 of the latter, making  $2^4 = 16$  in all.

Hint: if  $h^0(\mathcal{L}) = 0$ , pick any ramification point  $p_k$  and observe that  $h^0(\mathcal{L}(p_k)) = 1$ ; show that the unique effective divisor in  $|\mathcal{L}(p_k)|$  must be the sum of two ramification points.

**Exercise 6.8.** Let  $C$  be a curve of genus 2 and let  $\mathcal{L} \in \text{Pic}_4(C)$  be an invertible sheaf of the form  $L = K_C(p + q)$  with  $p \neq q$  and  $p + q \sim K_C$  as in 2. Show that

- (1)  $h^0(L(-2r)) = 1$  for any point  $r \in C$ , and
- (2)  $h^0(L(-2p - 2q)) = 0$ .

Deduce from this that the map  $\phi_L$  is an immersion, and that the tangent lines to the two branches of  $\phi_L(C)$  at the point  $\phi_L(p) = \phi_L(q)$  are distinct, meaning the point  $\phi_L(p) = \phi_L(q)$  is a node of  $\phi_L(C)$ .

Hint: For the first part (which implies that the map  $\phi_L$  is an immersion), observe that  $h^0(LK_C^{-1}) = 1$ , meaning  $p$  and  $q$  are unique. The second part says that the images of the differential  $d\phi_L$  at  $p$  and  $q$  are distinct.

**Exercise 6.9.** We can represent any line in  $\mathbb{P}^3$  by 2 points on it, and using their coordinates as the two rows of a  $2 \times 4$  matrix. The *Plücker coordinates* of the line are the six  $2 \times 2$  minors

$$\{p_{i,j}\}_{0 \leq i < j \leq 3}$$

of this matrix. They are independent, up to a common scalar multiple, of the two points chosen, and define the *Plücker embedding* of the Grassmannian  $\mathbb{G}(1, 3)$  in  $\mathbb{P}^5$ .

The minors  $p_{i,j}$  satisfy a nonsingular quadratic equation: if we stack two copies of the  $2 \times 2$  matrix to produce a  $4 \times 4$  matrix, its determinant is zero, and the Laplace expansion of this determinant is the *Plücker equation*

$$p_{0,1}p_{2,3} - p_{0,2}p_{1,3} + p_{0,3}p_{1,2} = 0.$$

- (1) Show that the quadratic form  $Q = p_{0,1}p_{2,3} - p_{0,2}p_{1,3} + p_{0,3}p_{1,2}$  is nonsingular, and deduce that it generates the ideal of  $\mathbb{G}(1, 3)$  in  $\mathbb{P}^5$ .
- (2) Write the bilinear form corresponding to  $Q$  as the determinant of a matrix, and deduce that two points in  $\mathbb{G}(1, 3)$  correspond to vectors that pair to 0 if and only if they correspond to lines that intersect.
- (3) Deduce that a maximal isotropic plane for  $Q$  corresponds either to the set of lines containing a given point or the set of lines contained in a given plane; and that two such sets of lines meet in a single point or coincide.

# Fine moduli spaces

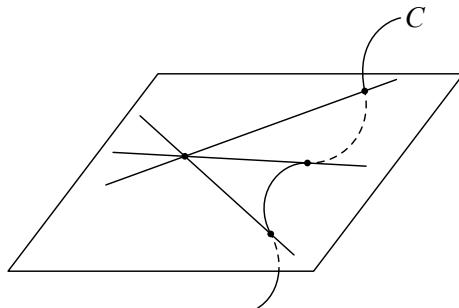
## 1. What is a moduli problem?

Algebraic geometry is almost unique among geometric theories in that the objects involved—varieties, schemes or maps between them—can be parametrized by other varieties or schemes. The set of submanifolds of a given manifold, or more generally of maps between two given manifolds, seems too large to be given the structure of a finite-dimensional manifold itself. By contrast, any algebraic variety is specified by a finite collection of polynomials, which in turn have a finite number of coefficients, so it's not too far-fetched that the collection of all varieties with specified numerical invariants, or morphisms between two given varieties, could be given the structure of a “moduli space” that is a variety (or scheme or...) in its own right.

For an example—perhaps the original one—projective plane curves of degree  $d$  are in natural one-to-one correspondence with the forms of degree  $d$  modulo the group of nonzero scalars—that is, with the points of the dual of the projective space  $\mathbb{P}(H^0(\mathcal{O}_{\mathbb{P}^2}(d))) = \mathbb{P}^{\binom{d+2}{2}-1}$ . Thus, for example, plane cubics are parametrized by  $\mathbb{P}^9$ , and a family  $C \rightarrow B$  of cubics corresponds to a map  $\phi : B \rightarrow \mathbb{P}^9$ . In this chapter, we'll give a general framework for the notion of moduli space, introducing the main examples that we will treat in this book.



Figure 1. A one-parameter family of plane cubics.



**Figure 2.** A general one-parameter linear family of lines in  $\mathbb{P}^3$ —that is, the family of lines contained in a general plane and passing through a general point in that plane—meets a space curve  $C$  in  $\deg C$  points.

There are several ways in which the possibility of making moduli spaces has been useful in algebraic geometry. First, the existence of a moduli space that parameterizes objects of a certain type allows us to speak of “the general object,” meaning that we allow ourselves to avoid the “special” properties of objects parameterized by closed subvarieties of the moduli space. We have already used this possibility in many places in this book.

Second, it allows us to speak coherently about families of objects. Some moduli spaces carry *universal families*, and every nice family of the sort of objects they parameterize is pulled back from this one by a unique map.

This idea was already exploited informally in the nineteenth century in the guise of “preservation of number,” used to count configurations of points or curves with a given property by specializing the data, and we have also exploited this idea in Chapter 5 to explain the count of odd and even theta characteristics on a general curve by appealing to the existence of a specialization to a hyperelliptic curve. In a related manner, we have already seen how the fact that invertible sheaves of degree  $d$  on a curve  $C$  are parametrized by a  $g$ -dimensional variety allows us to prove the “ $g + 3$ ” theorem (Theorem 5.1).

Third, to the extent that we can describe the intersection theory of a moduli space, it opens up the possibility of doing enumerative geometry on it to count solutions of geometric problems—and in particular to prove the existence of solutions. For example, knowing that the parameter space for lines in  $\mathbb{P}^3$  is the Grassmannian  $\mathbb{G}(1, 3)$ , a projective variety of dimension 4, and that the condition that the set of lines meeting a given curve of degree  $d$  is a divisor linearly equivalent to  $d$  times the hyperplane section, we can conclude that there exists a line in  $\mathbb{P}^3$  meeting any four given curves [Eisenbud and Harris 2016a, Section 3.4.1]. We will use the same idea to prove the much deeper existence of certain linear series on all curves (the “existence” half of the Brill-Noether theorem, discussed in Chapter 12).

In modern terms, a *moduli problem* consists of a class of objects in algebraic geometry—schemes, subschemes of a given scheme, sheaves on schemes, maps of schemes, typically defined by some common attributes—and a notion of what it means to have a *family* of these objects parametrized by a scheme  $B$ . The notion is formalized in the idea of a *moduli functor*, which associates to each scheme  $B$  the set of families over  $B$  of the given sort. Examples will make this vague notion more concrete.

### 1.1. Some moduli problems.

- (1) *Effective divisors on a given curve.* The objects are effective divisors of given degree on a given smooth, projective curve  $C$ . A family of such divisors is a subscheme  $D \subset B \times C$ , flat of degree  $d$  over  $B$ . Here we are using the equivalence between divisors of degree  $d$  on a smooth curve and degree  $d$  subschemes of the curve. The moduli space is the  $d$ -th symmetric power  $C_d$  of  $C$ , discussed in Section 1.
- (2) *Invertible sheaves on a given curve.* The objects are invertible sheaves on a given smooth projective curve  $C$ . A family over a scheme  $B$  is an equivalence class of invertible sheaves  $\mathcal{L}$  on  $B \times C$  whose restriction to each fiber of  $B \times C$  over  $B$  has degree  $d$ , where two families  $\mathcal{L}$  and  $\mathcal{L}'$  on  $B \times C$  are equivalent if  $\mathcal{L} \cong \mathcal{L}' \otimes \mathcal{M}$ , where  $\mathcal{M}$  is an invertible sheaf pulled back from  $B$ . Usually one restricts attention to invertible sheaves whose restrictions to each fiber  $b \times C$  have a given degree  $d$ . The moduli spaces are the Jacobian and Picard varieties, discussed in Section 2.
- (3) *Moduli of smooth curves.* The objects are isomorphism classes of smooth projective curves of genus  $g$ . A family over  $B$  is an equivalence class of smooth, projective morphisms  $f : C \rightarrow B$  whose fibers are curves of genus  $g$ , where two such families  $f, f'$  are equivalent if there is an isomorphism from the source of  $f$  to the source of  $f'$  making the diagram

$$\begin{array}{ccc} C & \xrightarrow{\cong} & C' \\ \searrow f & & \swarrow f' \\ & B & \end{array}$$

commute.

The moduli spaces  $M_g$  of curves are harder to construct, and we will have a separate discussion of them in the following chapter. Nonetheless, it will be useful at several points in this chapter to assume their existence.

- (4) *Hurwitz spaces.* An object is a smooth projective curve  $C$  of a given genus, together with a map  $f : C \rightarrow \mathbb{P}^1$  of given degree, up to isomorphisms of the curves that commute with the map, as in the diagram

$$\begin{array}{ccc} C & \xrightarrow{\cong} & C' \\ \searrow & & \swarrow \\ & \mathbb{P}^1 & \end{array} .$$

A family over  $B$  is a family  $C \rightarrow B$  of smooth projective curves of a given genus, together with flat map  $C \rightarrow B \times \mathbb{P}^1$  of degree  $d$ . Often the allowable ramification indices are specified. We will discuss the simplest Hurwitz scheme in Chapter 8.

- (5) *Severi varieties.* The objects are plane curves of given degree  $d$  and geometric genus  $g$ . If  $g \neq \binom{d-1}{2}$  the curves will necessarily be singular but are often constrained to have only mild singularities, usually only nodes. We will discuss the Severi variety in Chapter 8.
- (6) *Hilbert schemes.* The objects are subschemes of a given projective space; a family over  $B$  is a subscheme  $C \subset B \times \mathbb{P}^r$ , flat over  $B$ . Since the Hilbert polynomials of the fibers of a flat family are all equal ([Hartshorne 1977, Section III.9], the Hilbert scheme is the disjoint union of subschemes corresponding to particular Hilbert polynomials, and these subschemes are of finite type. If  $B$  is reduced then the flatness condition is equivalent to the constancy of the Hilbert polynomial [Hartshorne 1977, ].

In Chapter 19 we will study the open set consisting of smooth curves  $C \subset \mathbb{P}^r$  of given degree and genus.

## 2. What is a solution to a moduli problem?

Typically what we want from a solution to a moduli problem is to understand all possible families of the objects in question. In particular, the individual objects should be in natural one-to-one correspondence with the closed points of the moduli space. The word *natural* is the key. Most of the time, the set of objects we are interested in has cardinality  $2^{\aleph_0}$ , as do all positive-dimensional varieties  $M$  over  $\mathbb{C}$ , so a mere bijection between the points of  $M$  and the objects to be parametrized is meaningless.

In the nicest situations there is a *universal* family  $\phi : \mathcal{X} \rightarrow M$  of these objects over  $M$ , such that any family over a scheme  $B$  is pulled back from the one on  $M$  via a unique morphism  $B \rightarrow M$ . Such a space  $M$  with its universal family  $\phi$ , if it exists, is called a *fine moduli space*. This can be expressed more abstractly but more succinctly by saying that the moduli scheme *represents the*

*moduli functor*, which means there is an isomorphism of functors:

$$\{B \mapsto \text{families of } X \text{ over } B\} \cong \{B \mapsto \text{Mor}_{\text{Schemes}}(B, M)\}.$$

If  $M$  is a fine moduli space then the identity map  $M \rightarrow M$  corresponds to the “universal family”  $\phi : \mathcal{X} \rightarrow M$ . The Hilbert scheme, as well as  $\text{Div}_d(C)$  and  $\text{Pic}_d(C)$  are fine moduli spaces but the moduli space of curves and the Hurwitz schemes are not.

If a fine moduli space and its universal family exist, then it is unique up to unique isomorphism: given two avatars  $M$  and  $M'$  the universal family on  $M$  corresponds to a map  $M \rightarrow M'$ , and we similarly produce a map  $M' \rightarrow M$ . The pullback of the universal family on  $M$  by the composition of these two maps is again the universal family, so the composition is the identity map.

Although the moduli space of curves and the Hurwitz spaces are not fine moduli spaces, they are still defined in a way that makes them unique, as we shall see below.

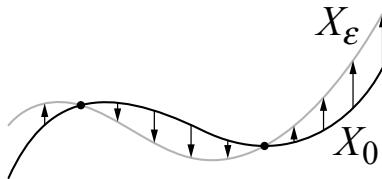
One of the useful features of a fine moduli space is that it makes the computation of tangent spaces relatively easy. Recall that if  $(R, \mathfrak{m})$  is the local ring at a point  $m$  on a variety  $\mathcal{M}$  then the (Zariski) tangent space to  $M$  at  $m$  is the vector space of linear functionals  $\text{Hom}_{R/\mathfrak{m}}(\mathfrak{m}/\mathfrak{m}^2, R/\mathfrak{m})$ . Assuming that  $R$  contains its residue field  $k := R/\mathfrak{m}$ , such functionals are precisely the restrictions to  $\mathfrak{m}/\mathfrak{m}^2$  of the ring homomorphisms  $R \rightarrow k[\epsilon]/(\epsilon^2)$  inducing the identity on  $k$ . If  $\mathcal{M}$  is a fine moduli space for some functor  $F$ , then such maps are in one-to-one correspondence with the set  $F(E)$  of families over  $E := \text{Spec}(k[\epsilon]/(\epsilon^2))$ .

The vector space structure on the tangent space is also accessible from this description: the sum of tangent vectors corresponding to families  $X_i \rightarrow E$  is the restriction to the diagonal  $E \subset E \times E$  of the product family  $X_1 \times X_2 \rightarrow E \times E$ . Often one can compute the tangent space to a moduli space before even knowing that the moduli space exists!

### 3. Hilbert schemes

The Hilbert scheme is a fine moduli space representing the functor of flat families of subschemes of  $\mathbb{P}^r$ , that is, the functor that takes a scheme  $B$  over  $\mathbb{C}$  to the set of subschemes  $\mathcal{X} \subset B \times \mathbb{P}^r$  that are flat over  $B$ ; a morphism  $f : B' \rightarrow B$  induces a map carrying a family to the pullback of the family by  $f$ .

**Example 3.1** (The Hilbert scheme of plane curves). Let  $p(m) := dm + 1 - \binom{d-1}{2}$  be the Hilbert polynomial of a plane curve of degree  $d$ , and consider the family  $\text{Hilb}_{p(m)}(\mathbb{P}^2)$  of schemes  $X \subset \mathbb{P}^2$  having Hilbert polynomial  $p$ . Since  $\dim X = \deg p = 1$ ,  $X$  has at least one component that is 1-dimensional. The union of such components must have degree equal to  $d$ , the leading coefficient of  $p(m)$ .



**Figure 3.** Infinitesimal perturbation of a curve by a normal vector field.

Since the Hilbert polynomial of this union is also equal to  $p(m)$  we see that  $X$  is in fact a (possibly nonreduced and/or reducible) plane curve of degree  $d$ . The space  $Hilb_{p(m)}(\mathbb{P}^2)$  is the projective space  $\mathbb{P}^{\binom{d+2}{2}-1}$  of forms of degree  $d$ , and the universal family is the projection

$$\{(x, F) \in \mathbb{P}^2 \times \mathbb{P}^{\binom{d+2}{2}-1} \mid F(x) = 0\} \rightarrow \mathbb{P}^{\binom{d+2}{2}-1}.$$

More typically, the set of curves  $C \subset \mathbb{P}^r$  of degree  $d$  and genus  $g$  corresponds to a subset of the Hilbert scheme parametrizing subschemes of  $\mathbb{P}^r$  with Hilbert polynomial  $p(m) = dm - g + 1$ , though not all schemes with this Hilbert polynomial are purely one-dimensional subschemes, as we shall see.

In this section, we'll compute the Zariski tangent space to the Hilbert scheme at a point, and we'll sketch the construction of the scheme itself. For a rigorous treatment including many generalizations, see [Álvarez et al. 2008] or [Nitsure 2005]. In Chapter 19, we'll describe the open subsets of smooth curves in the Hilbert schemes of curves of low degree and genus in  $\mathbb{P}^3$  in more detail.

**3.1. The tangent space to the Hilbert scheme.** Following the general recipe for tangent spaces to fine moduli spaces, we need to understand flat families of projective schemes over  $E := \text{Spec } \mathbb{C}[\epsilon]/(\epsilon^2)$ . Recall ([Hartshorne 1977, p.182]) that if  $X \subset \mathbb{P}^r$  is a smooth subscheme then the normal bundle  $\mathcal{N}_{X/\mathbb{P}^r}$  of  $X$  in  $\mathbb{P}^r$  is defined in terms of the tangent bundles of  $X$  and  $\mathbb{P}^r$  by the exact sequence:

$$0 \longrightarrow \mathcal{I}_X \xrightarrow{\phi} \mathcal{I}_{\mathbb{P}^r}|_X \longrightarrow \mathcal{N}_{X/\mathbb{P}^r} \longrightarrow 0.$$

Thus a global section of  $\mathcal{N}_{X/\mathbb{P}^r}$  can be thought of as an infinitesimal motion of  $X$ .

To define the normal sheaf for locally complete intersection subschemes  $X$  we first define the *conormal sheaf* to be the kernel of the dual of  $\phi$ , which is the natural map  $\Omega_{\mathbb{P}^r}|_X \rightarrow \Omega_X$ , and since  $X$  is locally a complete intersection, this corresponds to the exact sequence:

$$0 \rightarrow \mathcal{I}_{X/\mathbb{P}^r}/\mathcal{I}_{X/\mathbb{P}^r}^2 \rightarrow \Omega_{\mathbb{P}^r}|_X \rightarrow \Omega_X \rightarrow 0.$$

We thus define  $\mathcal{N}_{X/\mathbb{P}^r} := \mathcal{H}\text{om}(\mathcal{I}_{X/\mathbb{P}^r}/\mathcal{I}_{X/\mathbb{P}^r}^2, \mathcal{O}_X)$ .

By analogy we define the *normal sheaf* of any subscheme  $X \subset \mathbb{P}^r$

$$\mathcal{N}_{X/\mathbb{P}^r} := \mathcal{H}\text{om}(\mathcal{I}_{X/\mathbb{P}^r}/\mathcal{I}_{X/\mathbb{P}^r}^2, \mathcal{O}_X).$$

The following theorem shows that this has the same connection with infinitesimal motions for arbitrary subschemes  $X$ .

**Theorem 3.2.** *Let  $X \subset \mathbb{P}^r$  be a subscheme with Hilbert polynomial  $p(m)$  and let  $E = \text{Spec } \mathbb{C}[\epsilon]/(\epsilon)^2$ . The flat families  $\mathcal{X} \subset \mathbb{P}^r \times E$  specializing to  $X$  at the closed point defined by  $(\epsilon)$  are in natural one-to-one correspondence with the vector space  $\text{Hom}_{\mathbb{P}^r}(\mathcal{I}_{X/\mathbb{P}^r}, \mathcal{O}_X) = H^0(\mathcal{N}_{X/\mathbb{P}^r})$ , which is thus the tangent space to the Hilbert scheme  $\text{Hilb}_p(\mathbb{P}^r)$  at  $[X]$ .*

**Proof.** We will actually treat the analogous result for an affine subscheme  $X \subset \mathbb{A}^r = \text{Spec } S$ , where  $S = \mathbb{C}[x_1, \dots, x_r]$ ; since our construction is natural, it will patch on an affine cover to give the projective case stated above.

An  $E$ -module  $M$  is flat over  $E$  if and only if  $\text{Tor}_1^E(\mathbb{C}, M) = 0$ . Since the free resolution of  $\mathbb{C}$  as an  $E$ -module has the form

$$\dots \xrightarrow{\epsilon} E \xrightarrow{\epsilon} E \xrightarrow{\epsilon} E \longrightarrow \mathbb{C} \longrightarrow 0$$

we have  $\text{Tor}_1^E(\mathbb{C}, M) = 0$  if and only if the submodule of  $M$  annihilated by  $\epsilon$  is  $\epsilon M$ .

We first construct a flat family from a homomorphism: Let  $I = (g_1, \dots, g_t) \subset S = \mathbb{C}[x_1, \dots, x_r]$  be the ideal defining  $X$ . Note that  $\text{Hom}_S(I/I^2, S/I) = \text{Hom}_S(I, S/I)$ . Let  $\phi : I \rightarrow S/I$  be a homomorphism, and let  $h_i \in S$  be any element reducing to  $\phi(g_i)$  modulo  $I$ . The ideal

$$I' := (g_1 + \epsilon h_1, \dots, g_t + \epsilon h_t) \subset S[\epsilon]/(\epsilon^2) =: S'$$

defines a scheme  $\text{Spec } S'/I'$  over  $E$  which restricts to  $X$  modulo  $(\epsilon)$ ; that is,  $I' + (\epsilon) = I$ .

To see that  $I'$  is independent of the lifting chosen, suppose that  $k_1, \dots, k_t \in I$  so that the elements  $g_i + \epsilon(h_i + k_i)$  are a different lifting, generating an ideal  $I''$ . Writing  $k_i = \sum r_{i,j} g_j$  we have

$$g_i + \epsilon(h_i + k_i) = g_i + \epsilon h_i + \sum \epsilon r_{i,j} (g_j + \epsilon h_j)$$

so  $I'' \subset I$ , and symmetrically  $I' \subset I''$ .

To prove flatness, suppose that  $a \in S'$  and  $\epsilon a \in I'$ , so that we can write  $\epsilon a = \sum (r_i + \epsilon s_i)(g_i + \epsilon h_i)$  for some  $r_i, s_i \in S$ . It follows that  $\sum r_i g_i = 0$ . Since  $\phi$  is a homomorphism this implies  $\sum r_i h_i \in I$ . Thus  $\epsilon a \in \epsilon I \subset S'$ , whence  $a \in I + (\epsilon)$ . Writing  $a = \sum p_i g_i + \epsilon b'$  and using the relations  $g_i \equiv -\epsilon h_i \pmod{I'}$  we get  $a \equiv \epsilon(-\sum p_i h_i + b') \pmod{I'}$ , as required.

Finally, starting from a flat family  $S'/I'$  over  $E$  with  $I' + (\epsilon) = I + \epsilon$ , let  $J$  be the image of  $I'$  in  $S'/(I') = S \oplus (\epsilon S)/(I)$ . We claim that  $J$  is the graph of a homomorphism  $\phi : I \rightarrow (\epsilon S)/(I) \cong S/I$ .

Since  $I' + \epsilon S = I + \epsilon S$ , the projection to the summand  $S \subset S'$  maps  $J$  onto  $I$ . To prove that  $J$  is the graph of a homomorphism, it suffices to show that this projection is an isomorphism. The kernel is the intersection of  $J$  with  $(\epsilon S)/(I)$ , so we must show that if  $r \in S$  and  $\epsilon r \in I'$  then  $\epsilon r \in I$ .

Since  $\epsilon r \in I'$ , the condition of flatness implies that the image of  $r$  in  $S'/I'$  is in  $\epsilon(S'/I')$ , which is to say that  $r \in I' + \epsilon S = I + \epsilon S$ . Thus  $r \in I + \epsilon S$ , whence  $\epsilon r \in I$  as required.

This shows that  $J$  is the graph of a homomorphism  $\phi$  such that  $I'$  is generated by  $\{g + \phi g \mid g \in I\}$ , so the two constructions are inverse to one another.  $\square$

**Example 3.3.** If  $C$  is the plane curve of degree  $d$  defined by a form  $F$ , then the ideal sheaf of  $C$  is  $\mathcal{O}_{\mathbb{P}^2}(-d)$ , and thus

$$\mathcal{H}\text{om}(\mathcal{I}_{C/\mathbb{P}^2}/\mathcal{I}_{C/\mathbb{P}^2}^2, \mathcal{O}_C) = \mathcal{O}_{\mathbb{P}^2}(d)|_C = \mathcal{O}_C(d).$$

From the exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^2} \xrightarrow{F} \mathcal{O}_{\mathbb{P}^2}(d) \longrightarrow \mathcal{O}_C(d) \longrightarrow 0$$

we deduce that the dimension of the tangent space to  $Hilb_{p(m)}(\mathbb{P}^2) = \mathbb{P}^{\binom{d+2}{2}-1}$  at  $C$  with  $p(m) = dm + 1 - \binom{d-1}{2}$ , is  $h^0(\mathcal{O}_C(d)) = h^0(\mathcal{O}_{\mathbb{P}^2}(d)) - 1 = \dim \mathbb{P}^{\binom{d+2}{2}-1}$ , as expected.

**3.2. Parametrizing twisted cubics.** By Lemma 3.4 below, the set of twisted cubics—that is, smooth, irreducible, nondegenerate curves of degree 3 in  $\mathbb{P}^3$ —is an open subset of the Hilbert scheme  $Hilb_{3m+1}(\mathbb{P}^3)$ . As we've seen, a twisted cubic curve  $C \subset \mathbb{P}^3$  can be described as the zero locus of three homogeneous quadratic polynomials  $Q_1, Q_2$  and  $Q_3$  in the homogeneous coordinates on  $\mathbb{P}^3$ ; to specify the twisted cubic we could just list the  $3 \times 10 = 30$  coefficients of these. But of course we could replace the three quadrics  $Q_i$  with any three independent linear combinations of them; what matters—and what is naturally associated to  $C$ —is the vector space  $V = \langle Q_1, Q_2, Q_3 \rangle \subset H^0(\mathcal{O}_{\mathbb{P}^3}(2))$ . This suggests that we consider the map of sets

$$h : \{\text{twisted cubic curves } C \subset \mathbb{P}^3\} \rightarrow G = G(3, H^0(\mathcal{O}_{\mathbb{P}^3}(2)))$$

obtained by associating to a twisted cubic  $C$  the second graded piece of its homogeneous ideal.

This differs significantly from the example of plane curves given at the beginning of this chapter: there, the objects to be parametrized were the zero locus of a single polynomial, and we could vary those coefficients arbitrarily and still have a plane curve; thus, the image of the analogous map was open in the projective space  $\mathbb{P}^{\binom{d+2}{2}-1}$ . But if we generically perturb the coefficients of the three quadratic polynomials  $Q_i$  defining the twisted cubic the resulting quadrics will be a complete intersection, generating the ideal of a set of eight points. Thus the image of the map  $h$  does not contain an open set of  $G$ . We will give equations defining the image in Section 3.3, and we can consider it the Hilbert scheme of twisted cubics.

As in this case, we will mainly be interested in the subsets of the Hilbert schemes corresponding to smooth irreducible curves:

**Lemma 3.4.** *Suppose that  $X \rightarrow B$  is a flat family of projective schemes (that is, the projection to the second factor of  $X \subset B \times \mathbb{P}^n$  is flat). The points  $b \in B$  such that the fiber  $X_b$  is smooth and irreducible form an open set.*

**Proof.** To prove the Lemma we may assume that  $B = \text{Spec } A$  is affine and irreducible. Let  $x_0, \dots, x_r$  be homogeneous coordinates on  $\mathbb{P}^r$ , and suppose that  $X \subset \mathbb{P}_{\mathbb{C}}^r$  is defined by the ideal  $I_X = (F_1, \dots, F_n) \in A[x_0, \dots, x_r]$ . Since  $X$  is flat over  $B$  the fiber dimension  $d$  is constant, and the singular locus of a given fiber  $X_b$  is the subscheme of  $X_b$  defined by the ideal  $J$  of  $(r-d) \times (r-d)$  minors of the Jacobian matrix  $(\partial F_i / \partial x_j)$ .

Let  $Y \subset B \times \mathbb{P}^r$  be the closed subscheme defined by  $J$  alone. It follows that  $Y \cap X \subset X$  defines a scheme of singular points of fibers of the family, so this set is closed in  $X$ . Since the map  $X \rightarrow B$  is projective, the image of  $X \cap Y$  is closed, and its complement is open. Finally, a smooth fiber  $X_b$  is irreducible if and only if it is connected. This holds if and only if  $h^0(\mathcal{O}_{X_b}) < 2$  and by the flat base change theorem [Hartshorne 1977, Theorem 12.11] this set is open as well.  $\square$

**Proposition 3.5.** *The open subset  $H^\circ$  of the Hilbert scheme  $\text{Hilb}_{3m+1}(\mathbb{P}^3)$  parametrizing twisted cubics is irreducible of dimension 12.*  $\blacksquare$

**Proof.** Let  $C_0 \subset \mathbb{P}^3$  be a twisted cubic, and consider the family of translates of  $C_0$  by automorphisms  $A \in \text{PGL}_4$  of  $\mathbb{P}^3$ : that is, the family

$$C = \{(A, p) \in \text{PGL}_4 \times \mathbb{P}^3 \mid p \in A(C_0)\}.$$

Via the projection  $\pi : C \rightarrow \text{PGL}_4$ , this is a family of twisted cubics, and so it induces a map

$$\phi : \text{PGL}_4 \rightarrow H^\circ.$$

Since every twisted cubic is a translate of  $C_0$ , this is surjective, with fibers isomorphic to the stabilizer of  $C_0$ , that is, the subgroup of  $\text{PGL}_4$  of automorphisms

of  $\mathbb{P}^3$  carrying  $C_0$  to itself. By Exercise 5.5, every automorphism of  $C_0$  is induced by an automorphism of  $\mathbb{P}^3$ , so the stabilizer is isomorphic to  $\mathrm{PGL}_2$  and thus has dimension 3. Since  $\mathrm{PGL}_4$  is irreducible of dimension 15, we conclude that  $H^\circ$  is irreducible of dimension 12.  $\square$

**3.3. Construction of the Hilbert scheme in general.** The Hilbert scheme is more complicated than would appear from the examples above, and this is even true for the Hilbert polynomial  $3m + 1$ . There are many subschemes of  $\mathbb{P}^3$  that have the same Hilbert polynomial  $3m + 1$  as a twisted cubic—for example, the union of a plane cubic and a point—and are not the intersection of the quadrics containing them. (See Exercises 7.1 and 5.1). In Chapter 19 we will discuss many more components of Hilbert schemes.

A fundamental result of [Matsusaka 1972] provides a place to start:

**Lemma 3.6.** *Let  $p(m) \in \mathbb{Q}[m]$  be a polynomial. There exists an integer  $m_0$  such that*

- (1) *For any subscheme  $X \subset \mathbb{P}^r$  with Hilbert polynomial  $p_X = p$  we have*

$$h^0(\mathcal{I}_{X/\mathbb{P}^r}(m)) = \binom{m+r}{r} - p(m) \quad \text{for all } m \geq m_0$$

*or in other words the Hilbert function of  $X$  agrees with the Hilbert polynomial  $p_X = p$  for all  $m \geq m_0$ ; and*

- (2) *For any subscheme  $X \subset \mathbb{P}^r$  with Hilbert polynomial  $p_X = p$  and the saturated ideal of  $X$  is defined by forms of degree  $\leq m$ .*

Note that for any given  $X$  the existence of an  $m_0$  satisfying the statement of the lemma is immediate by Theorem 1.1. The point of the lemma is that we can find one value of  $m_0$  that works for all  $X$  with Hilbert polynomial  $p$ . The following result of [Gotzmann 1978] provides a method for determining  $m_0$ .

**Theorem 3.7.** *The Hilbert polynomial of the homogeneous coordinate ring of any scheme  $X \subset \mathbb{P}^r$  can be written uniquely in the form*

$$\chi(\mathcal{O}_X(m)) = \binom{m+a_1}{a_1} + \binom{m+a_2-1}{a_2} + \cdots + \binom{m+a_s-(s-1)}{a_s},$$

*with*

$$a_1 \geq \cdots \geq a_s \geq 0$$

*where the binomial coefficients are interpreted as polynomials in  $m$ . Moreover, the saturated homogeneous ideal of  $X$  is generated in degrees  $\leq s$ , and one can take  $m_0 = s$  in the construction of the Hilbert Scheme, above.*

See [Green 1989] for an exposition and a proof. From the coefficients  $a_j$  one can read off uniform vanishing theorems for  $H^i(\mathcal{I}_X(m))$  as well.

For example, the Hilbert polynomial  $3m + 1$  of the twisted cubic may be written as

$$3m + 1 = \binom{m+1}{1} + \binom{m+1-1}{1} + \binom{m+1-2}{1} + \binom{m+0-(3)}{0},$$

Here  $s = 4$ , and indeed the homogeneous ideal of the union of a plane cubic with a point, also in the plane, requires equations of degree 4.

**3.4. Grassmannians.** The most fundamental (and simple) of Hilbert schemes are the Grassmannians (including the projective spaces themselves), which parameterize the families of linear subspaces of given dimension in vector spaces or in projective spaces. For  $0 \leq k \leq r$  we write  $\mathbb{G}(k, r)$  for the set of  $k$ -planes in  $\mathbb{P}^r$  and identify it with  $G(k+1, r+1)$ , the set of  $(k+1)$ -dimensional vector subspaces of an  $(r+1)$ -dimensional vector space. When we want to make the  $(r+1)$ -dimensional vector space  $V$  explicit, we write  $G(k+1, V)$  instead.

We embed  $G(k+1, V)$  in  $\mathbb{P}(\wedge^{r-k} V) = \mathbb{P}^{\binom{r+1}{r-k}-1}$  by sending a subspace  $W \subset V$  to the 1-quotient  $\wedge^{r-k} V \rightarrow \wedge^{r-k}(V/W)$ . This map is a monomorphism called the *Plücker embedding*, and its image is an algebraic subvariety, which we take to be the algebraic structure of the Grassmannian.

Concretely, if we choose a basis of  $V/W$  so that the projection map  $V \rightarrow V/W$  is given by an  $(r-k) \times (r+1)$  matrix  $A$  then the coordinates of the image of  $W$  are the  $(r-k) \times (r-k)$  minors of  $A$ , called *Plücker coordinates* of  $W$ . On the open set where the first  $(r-k) \times (r-k)$  minor is nonzero, we may multiply by its inverse, and it is not hard to check that the minors become the entries of the complementary  $k \times (r-k)$  submatrix; thus  $G(k, V)$  is covered by open sets isomorphic to affine  $k(r-k)$ -space, and thus  $G(k+1, r+1) = \mathbb{G}(k, r)$  is smooth and irreducible of dimension  $k(r-k)$ .

**Example 3.8.** The Grassmannian  $\mathbb{G}(1, 3)$  of lines in  $\mathbb{P}^3$  has Plücker coordinates of a line  $L$  that is the span of points  $q, r \in \mathbb{P}^3$  the  $2 \times 2$  minors of the  $2 \times 4$  matrix whose rows are the coordinates of the points:

$$\begin{pmatrix} q_0 & q_1 & q_2 & q_3 \\ r_0 & r_1 & r_2 & r_3 \end{pmatrix}.$$

Indexing the minors  $p_{i,j}$  by pairs of distinct column indices one can easily prove the *Plücker relation*

$$p_{0,1}p_{2,3} - p_{0,2}p_{1,3} + p_{0,3}p_{1,2} = 0,$$

and this defines  $\mathbb{G}(1, 3) \subset \mathbb{P}^5$  as a quadric hypersurface.

The *universal sub-bundle*  $\mathcal{W} \subset V \times G(k+1, r+1)$ , also called the *tautological sub-bundle*, is the vector bundle with fiber  $W \subset V$  over the point of  $G(k+1, V)$

corresponding to  $W$ . It is universal in the sense that given any scheme  $X$  and a  $(k+1)$ -dimensional sub-bundle  $\mathcal{W}'$  of the trivial bundle  $V \times X$  there is a unique morphism  $X \rightarrow G(k+1, V)$  such that the pullback of  $\mathcal{W}$  is  $\mathcal{W}'$ .

For a thorough introduction to the Grassmannian, see for example [Eisenbud and Harris 2016a, Chapters 3–5].

**3.5. Equations defining the Hilbert scheme.** Matsusaka's theorem allows us to define an injective map of sets

$$h : \{\text{subschemes } X \subset \mathbb{P}^r \text{ with } p_X = p\} \rightarrow G\left(\binom{m_0 + r}{r} - p(m_0), \binom{m_0 + r}{r}\right)$$

by sending  $X$  to  $H^0(\mathcal{I}_{X/\mathbb{P}^r}(m_0))$ , and its image is the set of closed points of the Hilbert scheme. It remains to describe the scheme structure.

We observed above that though there are vector spaces  $V$  of 3 quadrics in  $\mathbb{P}^3$  that define twisted cubics, a general such vector space would generate the ideal of 8 points, not a twisted cubic. What we want to know is how to tell these cases apart algebraically. Consider the multiplication map

$$V \otimes H^0(\mathcal{O}_{\mathbb{P}^3}(1)) \rightarrow H^0(\mathcal{O}_{\mathbb{P}^3}(3)).$$

We saw in Chapter 3 that the cokernel of this map is the 10-dimensional space  $H^0(\mathcal{O}_{\mathbb{P}^1}(9))$ , so the image of this map is 10-dimensional, whereas 3 general quadrics form a complete intersection and would have only Koszul syzygies, so in the case of general quadrics this map would have 12-dimensional image. This is a map from a 12-dimensional vector space to a 20-dimensional one, and what we've seen is that if  $V$  is the net of quadrics containing a twisted cubic, it has a 2-dimensional kernel; that is, it has rank 10.

Thus if  $E$  is the universal subbundle on  $G = G(3, H^0(\mathcal{O}_{\mathbb{P}^3}(2)))$ , and  $H^0(\mathcal{O}_{\mathbb{P}^3}(d)) \otimes \mathcal{O}_G$  is the trivial bundle, then the multiplication map above gives a map of vector bundles

$$\mu : E \otimes H^0(\mathcal{O}_{\mathbb{P}^3}(1)) \rightarrow H^0(\mathcal{O}_{\mathbb{P}^3}(3)).$$

We can represent this locally as a matrix of functions, and the  $11 \times 11$  minors of this matrix define the rank 10 locus, and thus vanish on the points of the Hilbert scheme: in a neighborhood of a point in  $G$  corresponding to a twisted cubic, the common zero locus of these minors is the locus of nets of quadrics containing a twisted cubic

In fact, the construction of the Hilbert scheme in general is no more structurally complicated than this special case. Given a polynomial  $p(m)$ , we find a value of  $m_0$  that satisfies the statement of Lemma 3.6; we let

$$G = G\left(\binom{m_0 + r}{r} - p(m_0), \binom{m_0 + r}{r}\right)$$

be the Grassmannian, and let  $h$  be the map from the set of subschemes of  $\mathbb{P}^r$  with Hilbert polynomial  $p$  to  $G$  sending  $X$  to  $H^0(\mathcal{I}_{X/\mathbb{P}^r}(m_0))$ . We then get a map of vector bundles on  $G$

$$\mathcal{E} \otimes H^0(\mathcal{O}_{\mathbb{P}^r}(1)) \rightarrow H^0(\mathcal{O}_{\mathbb{P}^r}(m_0 + 1)).$$

In a neighborhood of a point of  $G$  in the image of  $h$ , the common zero locus of the minors of size  $\binom{r+m_0+1}{r} - p(m_0 + 1)$  of a matrix representative of this map is the image of  $h$ . Thus these functions define the Hilbert scheme.

#### 4. Bounding the number of maps between curves

A priori, the Hilbert scheme parametrizes subschemes of projective space. But the construction is adaptable to many other situations. In this section we'll sketch a proof of such an application. As we have seen, there can be infinitely many maps of given degree from a curve to  $\mathbb{P}^1$ , and this is also the case for maps to a curve of genus 1, even modulo the automorphisms of the target. But this is not the case in higher genus: given two smooth projective curves  $C$  and  $D$  of genera  $g, h \geq 2$ , we'll show that there are at most a finite number of nonconstant morphisms  $C \rightarrow D$ . In fact, the number is bounded purely in terms of  $g$  and  $h$ :

**Theorem 4.1.** *Given integers  $g, h \geq 2$  there is a bound  $N(g, h)$  on the number of distinct non-constant morphisms from a smooth projective curve  $C$  of genus  $g$  to a smooth projective curve  $D$  of genus  $h$ .*

A special case of this result is a bound on the size of the group of automorphisms of a curve of genus  $g \geq 2$ . We will give a second proof of that result, which doesn't rely on the Hilbert scheme, in Theorem 2.1.

**Proof.** Hurwitz' Theorem 2.9 implies a bound on the degree  $d$  of a morphism  $f : C \rightarrow D$  so it suffices to bound the number of morphisms of a fixed degree  $d$ .

We will use the Hilbert scheme in the relative setting, as Grothendieck originally defined it: Given a base scheme  $S$ , the scheme  $Hilb_{p(m)}(\mathbb{P}_S^r)$  represents the functor on  $S$ -schemes  $X \rightarrow S$  that associates to  $X$  the set of subschemes of  $\mathbb{P}_X^r$ , flat over  $X$ , whose fibers have Hilbert polynomial  $p$ .

We can construct a family of products of pairs of curves of genera  $g$  and  $h$  embedded in projective space as follows (the details don't matter—only the existence): Let  $S_g \rightarrow S$  be the Hilbert scheme of smooth curves of genus  $g$  embedded by invertible sheaves of degree  $2g + 1$  in  $\mathbb{P}_S^{g+1}$  and similarly for  $S_h$  and curves of genus  $h$ , and write  $([C], [D])$  for the corresponding point in the fiber of  $S_g \times_S S_h$ . From the universal families of curves over  $S_g$  and  $S_h$  we may

construct a family  $\mathcal{C} \rightarrow S = S_g \times_S S_h$  whose fiber over  $s$  includes all products of pairs of smooth curves of genera  $g$  and  $h$ , embedded in  $\mathbb{P}_s^{g+1} \times_S \mathbb{P}_s^{h+1}$ . Finally we embed this product of projective spaces by the Segre embedding in  $\mathbb{P}_s^N$ , with  $N = (g+2)(h+2) - 1$ .

If  $\Gamma \subset C \times D \subset \mathbb{P}^N$  is the graph of a morphism  $f : C \rightarrow D$  of degree  $d$ , then  $\Gamma$  has genus  $g$  and degree  $2g + 1 + d(2h + 1)$ , so we know its Hilbert polynomial  $p$ . The set of subschemes in this Hilbert scheme that project isomorphically to  $C$  and  $d$ -to-1 to  $D$  correspond to the points of a locally closed subset, of this Hilbert scheme, and therefore are a scheme of finite type. The fibers of this scheme over  $S$  are the sets of morphisms of degree  $d$  from curves of genus  $g$  to curves of genus  $h$ .

We first prove that each fiber is finite—that is, there can only be finitely many maps of degree  $d$  from one fixed curve to another. By Theorem 3.2 it suffices for this to show that if we fix the curves  $C, D$  and a map  $\phi$  with graph  $\Gamma_\phi$  then the normal bundle

$$\mathcal{N}_{\Gamma_\phi/(C \times D)} = \mathcal{T}_{C \times D}/\mathcal{T}_{\Gamma_\phi}$$

has no sections. The projection onto the first factor identifies  $\mathcal{T}_{\Gamma_\phi}$  with  $\mathcal{T}_C$ , and the quotient is thus  $\phi^*\mathcal{T}_D$ , which has degree  $2 - 2h < 0$ . Thus, fiber by fiber, the scheme of morphisms is finite.

Since the base  $S$  of the family is also a scheme of finite type and the degree of the fiber is semicontinuous, there is an absolute bound  $N(g, h)$  as claimed.  $\square$

A small variation of this construction, essentially the case  $g = h$  and  $d = 1$ , parametrizes families of isomorphisms of curves: given two families of projective curves  $\mathcal{X} \subset B \times \mathbb{P}_S^r$  and  $\mathcal{Y} \subset B \times \mathbb{P}_S^s$ , the result is a scheme  $\text{Isom}(\mathcal{X}, \mathcal{Y}) \rightarrow S$  whose fiber over a point of  $S$  is the set of isomorphisms of  $X_b \rightarrow Y_b$  for  $b$  in the fiber over  $s$ . This turns out to be useful in describing the properties of the moduli space  $M_g$  treated in the next chapter.

## 5. Exercises

**Exercise 5.1.** Suppose that a scheme  $X \subset \mathbb{P}^n$  is the disjoint union of subschemes  $Y, Z$ . Show that the Hilbert polynomial of  $X$  is the sum of the Hilbert polynomials of  $Y$  and  $Z$ . What statement can you make about the Hilbert functions?

Hint: the Hilbert polynomials satisfy  $p_X = p_Y + p_Z$ , which follows from the vanishing of  $h^1(\mathcal{I}_{Y \cup Z}(m))$  for large  $m$ ; the Hilbert functions satisfy  $h_X \leq h_Y + h_Z$ . (When  $h_X(m) = h_Y(m) + h_Z(m)$ , we say that  $Y$  and  $Z$  impose independent conditions on  $|\mathcal{O}_{\mathbb{P}^n}(m)|$ .)

**Exercise 5.2.** More generally, suppose that a scheme  $X \subset \mathbb{P}^n$  is the union of subschemes  $Y, Z$ . Show that the Hilbert polynomial of  $X$  is the sum of the Hilbert polynomials of  $Y$  and  $Z$  minus the Hilbert polynomial of  $Y \cap Z$ .

Hint: use the exact sequence

$$0 \rightarrow \mathcal{I}_{Y \cup Z} \rightarrow \mathcal{I}_Y \oplus \mathcal{I}_Z \rightarrow \mathcal{I}_{Y \cap Z} \rightarrow 0$$

**Exercise 5.3.** Let  $H \subset \mathbb{P}^3$  be a 2-plane; let  $C \subset H$  be a plane cubic curve and  $p \in H \setminus C$  a point in  $H$  not on  $C$ ; let  $X = C \cup \{p\}$ .

- (1) Show that the Hilbert polynomial of  $X$  is  $p_X(m) = 3m + 1$ .
- (2) Show that the smallest value of  $m_0$  satisfying the statement of Lemma 3.6 is 4.

Hint: any cubic vanishing on  $X$  vanishes identically on  $H$ .

**Exercise 5.4.** Use an argument like that of Proposition 3.5 to show that the restricted Hilbert scheme  $H^\circ$  of rational normal curves  $C \subset \mathbb{P}^r$  is irreducible of dimension  $r^2 + 2r - 3$ .

Hint: As in the twisted cubic case, the group  $PGL_{r+1}$  acts transitively on  $H^\circ$  with stabilizer  $PGL_2$

**Exercise 5.5.** If  $C = X \cap Y \subset \mathbb{P}^3$  is a complete intersection of surfaces of degrees  $d, e$ , then  $Hilb$  is smooth at the point  $[C]$ , of dimension  $2\binom{3+d}{3} - 4$  if  $d = e$  or  $\binom{3+d}{3} + \binom{3+e}{3} - \binom{3+e-d}{3} - 2$  if  $d < e$ .

Hint: The normal bundle is  $\mathcal{N} = \mathcal{O}_C(d) + \mathcal{O}_C(e)$ . To prove smoothness, use Exercise 4.16 to compute  $H^0(\mathcal{N})$ .



# Moduli of curves

In the preceding chapter, we described the *Hilbert scheme*, a fine moduli space for curves in projective space. In this chapter we will discuss the second moduli space central to the theory of algebraic curves:  $M_g$ , which parametrizes isomorphism classes of smooth projective curves of genus  $g$ . As we'll see,  $M_g$  is not a fine moduli space, but it comes close.

To describe the situation, we will start with the case of curves of genus 1, where everything can be made explicit.

## 1. Curves of genus 1

Let  $C$  be a smooth curve of genus 1. Any invertible sheaf of degree 2 on  $C$  can be written as  $\mathcal{O}_C(2p)$ , and defines a morphism to  $\mathbb{P}^1$  with 4 distinct branch points. Since the automorphism group of  $C$  is transitive, these 4 points in  $\mathbb{P}^1$  are independent of the choice of  $p$ , and are well-defined up to an automorphism of  $\mathbb{P}^1$ . As explained in Section 2, this means that every such curve  $C$  can be realized as the completion of an affine curve

$$y^2 = f(x)$$

where  $f$  is a quartic polynomial with distinct roots:

$$f(x) = \prod_{i=1}^4 (x - \lambda_i).$$

Thus we would like to define  $M_1$  to be the set of 4-tuples of distinct points  $\{\lambda_1, \dots, \lambda_4\}$  of  $\mathbb{P}^1$  modulo the action of  $\text{Aut}(\mathbb{P}^1) = PGL_2$ .

As we will explain in the next sections, quotients by infinite groups can behave badly, but in this case we can compute the quotient in a much simpler way: There is a unique automorphism of  $\mathbb{P}^1$  carrying the three points  $\lambda_1, \lambda_2, \lambda_3$  to the points 0, 1 and  $\infty \in \mathbb{P}^1$  respectively, so that we can write  $C$  as the zero locus of

$$y^2 = x(x - 1)(x - \lambda)$$

for some complex number  $\lambda \in \mathbb{P}^1 \setminus \{0, 1, \infty\}$ ; we'll call this curve  $C_\lambda$ . This expression is not unique, since if we reordered the original four points  $\lambda_i$ , we might arrive at a different value of  $\lambda$ ; for example, if we exchanged 0 and  $\infty$  and fixed 1,  $\lambda$  would be replaced by  $1/\lambda$ . Thus the symmetric group  $S_4$  acts on the set  $\mathbb{A}^1 \setminus \{0, 1, \infty\}$  and one can show that the orbit of  $\lambda$  under this action is

$$\left\{ \lambda, 1 - \lambda, \frac{1}{\lambda}, \frac{1}{1 - \lambda}, \frac{\lambda - 1}{\lambda}, \frac{\lambda}{\lambda - 1} \right\}.$$

There are 6 points in the orbit rather than 24 because the Klein 4-group  $K = \mathbb{Z}/2 \times \mathbb{Z}/2 \subset S_4$  of fixed-point free involutions acts trivially, so what we really have is an action of  $S_4/K \cong S_3$ .

Since  $S_3$  is finite and  $\mathbb{P}^1 \setminus \{0, 1, \infty\}$  is a normal affine curve, the quotient space by the action is again a normal affine curve whose points are in one-to-one correspondence with the orbits, and thus with the set of curves of genus 1. Moreover, by Lüroth's theorem, Theorem 0.2, the quotient is rational, meaning that the field of rational functions on the quotient—that is, the subfield of  $\mathbb{C}(\lambda)$  invariant under the action of  $S_3$ —is of the form  $\mathbb{C}(j)$  for some rational function  $j(\lambda)$  of degree 6. Of course, there are many possible generators of the field of rational functions on the quotient; one that works is

$$(1) \quad j(\lambda) := 256 \frac{(\lambda^2 - \lambda + 1)^3}{\lambda^2(\lambda - 1)^2},$$

known as the *j-function*. As  $\lambda$  varies in  $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ ,  $j(\lambda)$  can assume any value in  $\mathbb{A}^1$ .

Summarizing, we have proven:

**Theorem 1.1.** *The set of isomorphism classes of smooth projective curves of genus 1 is in bijection with the points of the affine line  $M_1 \cong \mathbb{A}^1$ . The bijection maps the curve defined by  $y^2 = x(x - 1)(x - \lambda)$  to  $j(\lambda) \in \mathbb{A}^1$ .*

**1.1.  $M_1$  is a coarse moduli space.** As we will see in Exercises 7.3 and 7.4,  $M_1$  is not a fine moduli space, but it comes close in two senses.

**Proposition 1.2.** *To any family  $\pi : C \rightarrow B$  of smooth projective curves of genus 1 over a reduced base  $B$  we can associate a natural morphism of schemes  $\phi : B \rightarrow M_1$  whose value at any point  $b \in B$  is the *j*-invariant of the corresponding fiber  $C_b$ .*

This is slightly weaker than saying that  $M_1$  is a coarse moduli space, because we are restricting ourselves to families with reduced bases; in fact, the statement is true in general, but requires more machinery to prove.

**Proof.** To start, we will work locally in  $B$ : for a given  $b_0 \in B$ , we will choose a suitably small neighborhood  $U$  of  $b_0 \in B$  and restrict ourselves to the preimage  $C_U = \pi^{-1}(U)$ . The first thing to do is to express the curves  $C_b$  in our family as 2-sheeted covers of  $\mathbb{P}^1$ , which is to say we want to choose an invertible sheaf on  $C_U$  having degree 2 on each fiber  $C_b$ . Since we're working locally in  $B$ , we can find a section  $\rho : U \rightarrow C_U$  of  $\pi : C \rightarrow B$ . If we let  $D = \rho(U) \subset C_U$  be the image, then we can take our invertible sheaf to be  $\mathcal{L} := \mathcal{O}_{C_U}(2D)$ .

Next, we use the following result, which is a special case of the theorem on “cohomology and base change” (see for example [Eisenbud and Harris 2016a, Appendix, Theorems B.5 and B.9] or, for a treatment that some of our students find harder to understand, [Hartshorne 1977, Theorem 12.11].)

**Theorem 1.3. (Cohomology and base change)** *If  $f : X \rightarrow Y$  is a morphism and  $\mathcal{F}$  is a coherent sheaf on  $X$  such that  $H^1(\mathcal{F}|_{f^{-1}(y)}) = 0$  for all  $y \in Y$ , then  $h^0(\mathcal{F}|_{f^{-1}(y)})$  is a constant function of  $y$ , and  $f_*(\mathcal{F})$  is a vector bundle of this rank.*

□

This result implies that the direct image  $E := \pi_*(\mathcal{O}_{C_U}(2D))$  is locally free of rank 2, and we get a morphism  $C_U \rightarrow \mathbb{P}(E)$  expressing each curve  $C_b$  as a two-sheeted cover of the corresponding fiber  $\mathbb{P}(E_b)$ . Again, since we are working locally in  $B$ , we can trivialize the bundle  $E$ , so that we get a diagram

$$\begin{array}{ccc} C_U & \xrightarrow{\quad} & U \times \mathbb{P}^1 \\ & \searrow & \swarrow \\ & U & \end{array}$$

Once more restricting to a smaller neighborhood  $U$  if necessary, we can write the family  $C_U \rightarrow U$  as the locus

$$y^2 = \prod_1^4 (x - \lambda_i),$$

where the  $\lambda_i$  are regular functions on  $U$ . The  $j$ -function of the  $\lambda_i$  yields a map  $U \rightarrow M_1$ ; since the value of the  $j$ -function at a point is determined by the isomorphism type of the fiber over this point, these maps agree on overlaps to give the desired morphism  $B \rightarrow M_1$ . □

**1.2. The good news.** However,  $M_1$  comes close to being a fine moduli space in the following sense:

**Proposition 1.4.** *Let  $B$  be a reduced scheme.*

- (1) *If  $j : B \rightarrow \mathbb{A}^1$  is any regular function on  $B$ , then there exists a finite cover  $\alpha : B' \rightarrow B$  such that  $j \circ \alpha$  is the  $j$ -function of a family of curves of genus 1 on  $B'$ ; and*
- (2) *If  $\pi : C \rightarrow B$  and  $\eta : D \rightarrow B$  are two families of curves of genus 1 with the same associated  $j$ -function, then there exists a finite cover  $\alpha : B' \rightarrow B$  and an isomorphism  $C \times_B B' \cong D \times_B B'$  such that the diagram*

$$\begin{array}{ccc} C \times_B B' & \xrightarrow{\quad} & D \times_B B' \\ & \searrow & \swarrow \\ & B' & \end{array}$$

commutes.

**Proof.** For the first of these assertions, let

$$B' := \{(b, \lambda) \in B \times (\mathbb{A}^1 \setminus \{0, 1\}) \mid j(b) = j(\lambda)\}$$

where  $j(\lambda)$  is as given in formula (1). We have already described a family of curves of genus 1 over the “ $\lambda$ -line”  $\mathbb{A}^1 \setminus \{0, 1\}$ ; the pullback to  $B'$  is the desired family.

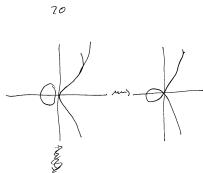
For the second half, we want to do something similar. Specifically, we want to choose sections  $\sigma : B \rightarrow C$  and  $\tau : B \rightarrow D$  and take

$$B' := \{(b, \phi) \mid b \in B, \phi : C_b \xrightarrow{\cong} D_b \text{ and } \phi(\sigma(b)) = \tau(b)\};$$

by Corollary 7.1,  $B'$  is a finite cover of  $B$  and when we pull back the two families to  $B'$  we have a tautological isomorphism between them. The only issue is how to give  $B'$  an appropriate scheme structure, and for this we can use the Isom scheme described at the end of Section 4.  $\square$

Thus,  $M_1$  is not a fine moduli space for smooth curves of genus 1, but it is the next best thing: we don’t get a bijection between families of curves of genus 1 over a given base  $B$  and maps  $j : B \rightarrow M_1$ ; but we do get a map from the former to the latter with “finite kernel and cokernel”.

**1.3. Compactifying  $M_1$ .** A natural question to ask is, if every value of  $j \in \mathbb{A}^1$  corresponds to an isomorphism class of curves  $C_j$  of genus 1, what happens to



**Figure 1.** A curve of genus 1 degenerating to a rational curve with a node in the family  $y^2 = x(x - 1)(x - \lambda)$ .

the curves  $C_j$  as  $j$  goes to  $\infty$ ? Or, equivalently, what happens to the curve  $C_\lambda$  given as the double cover

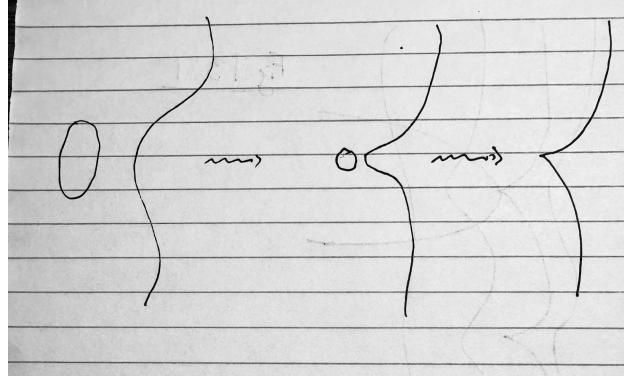
$$y^2 = x(x - 1)(x - \lambda)$$

as  $\lambda$  approaches 0, 1 or  $\infty$ —the other branch points of the double cover? The answer is clear from the equation: when two branch points of a double cover of smooth curves come together, the limiting curve has a node. In fact, there is a unique isomorphism class of irreducible curves of arithmetic genus 1 having a node; it's represented by the curve with equation  $y^2 = x^2(x - 1)$ .

The upshot of this is that if we enlarge the original class of curves parametrized by  $M_1$ —“smooth projective curves of genus 1”—to the slightly larger class, “irreducible nodal projective curves of arithmetic genus 1,” we still have a coarse moduli space  $\overline{M}_1$  for this slightly larger class of objects. This enlarged moduli space is obtained by adding one point “at  $\infty$ ” to the existing space  $M_1 \cong \mathbb{A}^1$  to form  $\overline{M}_1 \cong \mathbb{P}^1$ .

This is an example of what is called a *modular compactification*. There is no precise definition, but if we have a class of objects parametrized by a (non-compact) moduli space  $M$  we may be able enlarge the class of objects to be parametrized, with the result that the moduli space  $\overline{M}$  of the larger class is compact.

Modular compactifications of a given moduli problem may or may not exist. It's sometimes a tricky problem to find a suitable class of objects to parametrize: if we don't add enough additional isomorphism classes, not every 1-parameter family of objects in our original class will have a limit in the larger class, meaning the enlarged moduli space will still not be compact; if we add too many,



**Figure 2.** A curve of genus 1 degenerating to a cuspidal curve in the family  $C_t = V(y^2 - x^3 - t^2x - t^3)$ .  
(( Silvio: still to do, I guess. ))

1-parameter families may have more than one possible limit, meaning the enlarged space won't be separated. For example in the family of curves  $C_t$  given as

$$C_t = V(y^2 - x^3 - t^2x - t^3)$$

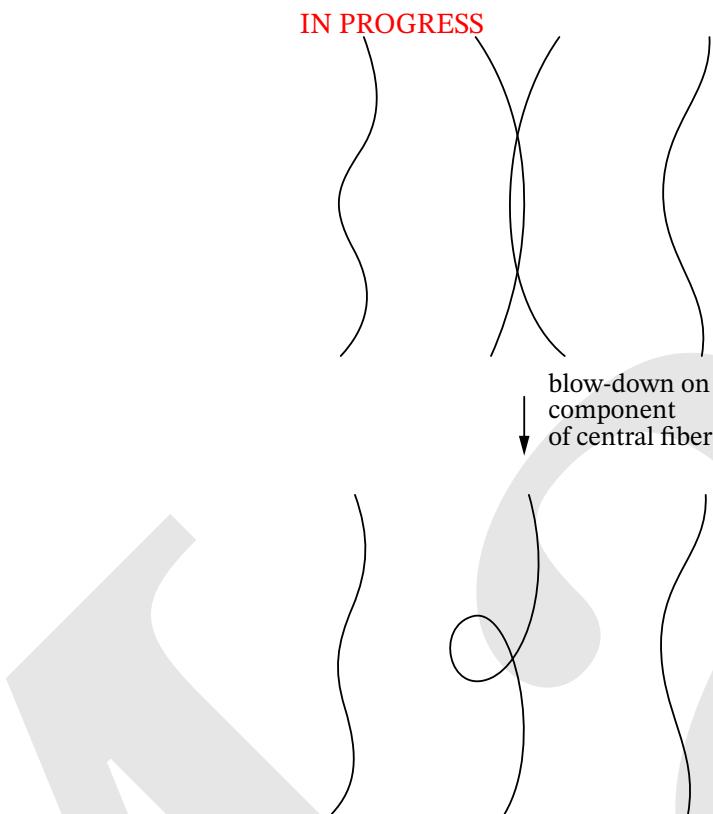
the  $j$  function is constant when  $t \neq 0$ , but the limiting curve  $C_0$  has a cusp. This shows that we could not have added cuspidal curves to  $M_1$ .

When modular compactifications do exist, they are extremely valuable for the study of both the space  $M$  and of the objects parametrized by  $M$ : compactness allows us to apply the techniques of modern algebraic geometry to the space  $\overline{M}$ , while the fact that it is still a moduli space gives us a handle on its geometry. In the following section, we will describe a modular compactification of  $M_g$ . The objects parametrized are called “stable curves.”

Getting back to the moduli space  $\overline{M}_1$ , if we have a family where  $j(\lambda)$  has a pole, we would like to say that the limit of the curves in the family is an irreducible nodal curve, but this is not necessarily true! For example, the limit of the curves

$$y^2 = x(x - t)(tx - 1)$$

as  $t \rightarrow 0$  is reducible, with two components meeting in two points, 0 and  $\infty$ . What is true is that a process called *semistable reduction* shows that after a base change and a birational modification of the family around the pole we can replace the family with one where the singular fiber is indeed an irreducible



**Figure 3.** In this case a birational modification of the total space of the family changes the unstable reducible curve to a stable curve.

nodal curve. (Figure 7.3) See [Harris and Morrison 1998b] for a description of this process in general, and several explicit examples.

(( Silvio: beware: this is the wrong sketch. ))

## 2. Higher genus

The idea is analogous to the one used for genus 1 curves: to construct a moduli space, first parametrize curves with a choice of some additional structure, such as a map to projective space, and then mod out by the choices made. For any smooth projective curve  $C$  of genus  $g \geq 2$ , the tricanonical linear series  $|3K_C|$  is very ample; it embeds  $C$  as a curve of degree  $6g - 6$  in  $\mathbb{P}^{5g-6}$ . Thus we have a way of realizing a given abstract curve  $C$  as a curve in projective space, unique up to automorphisms of  $\mathbb{P}^{5g-6}$ .

We claim next that the set of smooth, tricanonically embedded curves is a locally closed subset  $X$  of the Hilbert scheme  $Hilb_{(6g-6)m+1-g}(\mathbb{P}^{5g-6})$  parametrizing curves of genus  $g$  and degree  $6g-6$  in  $\mathbb{P}^{5g-6}$ . By Lemma 3.4, the set of points in the base over which the curves are smooth is open. Let

$$Hilb^\circ = Hilb_{(6g-6)m+1-g}^\circ(\mathbb{P}^{5g-6}) \subset Hilb_{(6g-6)m+1-g}(\mathbb{P}^{5g-6})$$

be this open set.

Next, on the universal family  $C \subset Hilb^\circ \times \mathbb{P}^{5g-6}$ , we have two families of invertible sheaves: we have the pullback of  $\mathcal{O}_{\mathbb{P}^{5g-6}}(1)$ ; and we have the cube  $K^3$  of the dualizing sheaf. Each gives rise to a section of the relative Picard variety over  $Hilb^\circ$ , and the locus where they agree is thus a closed subset  $X \subset Hilb^\circ$ .

The group  $PGL_{5g-5}$  of automorphisms of  $\mathbb{P}^{5g-6}$  acts on the variety  $X$  and its orbits are the isomorphism classes of smooth curves of genus  $g$ ; thus, we might hope to realize the moduli space  $M_g$  as the quotient of  $X$  by  $PGL_{5g-5}$ . But here things go awry in a hurry: unlike the case of an action of a finite group on a variety, the orbit spaces of infinite groups are often not algebraic varieties. (Think of the action of  $\mathbb{C}^*$  on  $\mathbb{C}$  by multiplication.) What is needed is a tool to extract the “best possible approximation” to a quotient. Happily, David Mumford created a tool that does exactly this: *geometric invariant theory* (GIT). To see how GIT can be used in this setting to produce the space  $M_g$ , see the wonderful introduction in [Mumford and Suominen 1972] (linked from the book’s AMS website) or the more technical version in [Mumford 1977].

**Theorem 2.1.** (Mumford) *The space of orbits of  $PGL_{5g-5}$  acting on the subset of the Hilbert scheme representing tricanonical curves has the structure of an algebraic variety  $M_g$  which is a coarse moduli space in the sense that*

- (1) *Given any flat family  $Y \rightarrow B$  of smooth curves of genus  $g$  there is a morphism of schemes  $B \rightarrow M_g$  sending each closed point  $p \in B$  to the point of  $M_g$  representing the fiber  $Y_p$ ;*
- (2) *These maps form a natural transformation from the functor  $G(-)$  of families of smooth curves to the functor  $Mor_{schemes}(-, M_g)$  through which any natural transformation  $G \rightarrow Mor_{schemes}(-, M')$  factors;*

The power of the theory of the moduli space of curves was greatly increased when compactifications of the space (there are many interesting ones) were understood. One of these, the compactification of  $M_1 = \mathbb{A}^1$  to  $\overline{M}_1 = \mathbb{P}^1$  by adding a nodal curve, has already been mentioned. This has the desirable properties that the subset added to  $M_1$  is a divisor; and the compactification is *modular* in the sense that the point added corresponds to a curve almost of the same type as the curves in  $M_1$ .

There are two reasons why a compactification is important:

First, the great majority of the techniques that algebraic geometers have developed for dealing with varieties apply directly only to projective varieties. For example, the Satake compactification is a projective variety containing  $M_g$  in such a way that the complement—usually referred to as the “boundary”—has codimension 2. Taking successive hyperplane sections that pass through a given point but don’t meet the boundary, we see that for  $g \geq 2$  there is a complete one-dimensional family of *smooth* curves containing any smooth curve of genus  $\geq 2$ .

Often, though, we can learn the most from a compactification where the “boundary”—the part that is added—is a divisor, and this is the case for the Deligne-Mumford compactification  $\overline{M}_g$  introduced by Deligne and Mumford in their groundbreaking 1969 paper [[Deligne and Mumford 1969](#)], described below. A central example of how this is used is given in Section 4, where we take up the question, “can we write down a general curve of genus  $g$ ?”

To describe this compactification, we first explain some of the language and results of geometric invariant theory.

**2.1. Stable, semistable, unstable.** Given a quasi projective variety  $X \subset \mathbb{P}^N$  and a group  $G \subset PGL_{N+1}$  that carries  $X$  into itself, we wish to construct as good a map as possible from the set of orbits to a projective space. Whatever map we take, the closure of the image will correspond to a graded ring. We want to preserve as much of the structure of the orbit space as possible, and on an open affine cover this means finding as many functions as possible that are invariant on the orbits. Thus it is natural to take the ring of invariants of the homogeneous coordinate ring  $A$  of the closure of  $X$  as the homogeneous coordinate ring of the closure of the image of  $X$ .

The first difficulty is that the elements of  $A$  are not functions on  $X$ , so  $G$  may not even act on  $A$ . However, it is possible to lift the action of  $G$  to an action on  $A$  of the slightly larger group,  $SL_{N+1}$ , a process called *linearization*. The kernel of the map  $SL_{N+1} \rightarrow PGL_{N+1}$  consists of diagonal matrices of finite order dividing  $N + 1$ , and the choice of a linearization amounts to a choice of a character of this abelian group. However, the choice doesn’t matter: since the kernel acts trivially on forms of degree a multiple of  $N + 1$ , and thus the action of  $PGL_{N+1}$  itself extends to an action on the homogeneous coordinate ring of the  $(N + 1)$ -st Veronese embedding.

The second difficulty in this program is that the ring of invariants of an infinite group may not even be finitely generated, so it may not correspond to a projective variety. Hilbert showed that if  $G = SL_{N+1}$ , then the ring of invariants is finitely generated. Since Hilbert’s time this result has been extended to the class of *linearly reductive* groups—see [[Haboush 1975](#)]. Thus the subring  $A^G \subset A$  of invariant elements is finitely generated over the ground field.

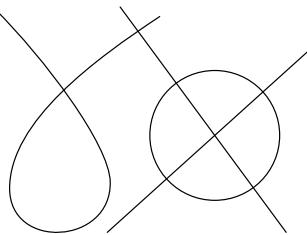
The third difficulty is that the points of  $\text{Proj}(A^G)$ , usually denoted  $X//G$ , are generally not in one-to-one correspondence with the orbits of  $G$  on  $X$ ! is  $\text{Proj}(A^G)$ , usually denoted  $X//G$ .

Geometric invariant theory explains the relationship of  $X//G$  to the set of orbits. To do this, it performs a sort of triage on the points of  $X$  (or their orbits), dividing them into three classes: stable, semistable and unstable. The theory also provides tools for determining this stratification.

- (1) *Stable* points. These are the points whose orbits are closed. They comprise an open subset  $X^{stable} \subset X$ , and the points of an open subset of  $X//G$  correspond one-to-one to the stable orbits, that is, an open subset that is set-theoretically  $X^{stable}/G$ . In general, this set may be empty, but in the case of the action of  $PGL_3$  on the  $\mathbb{P}^9$  of plane cubics, the stable points are the smooth plane cubics, and the quotient is the affine  $j$ -line.
- (2) *Strictly semistable* points. These are the points  $p$  such that there exists an invariant form not vanishing at  $p$ . Together with the stable points, comprise a larger open subset  $X^{semistable} \subset X$ , called the *semistable* locus. Two semistable points  $p, q$  map to the same point in  $X//G$  if and only if  $\overline{Gp} \cap \overline{Gq} \cap X^{semistable} \neq \emptyset$ . In the example of the action of  $PGL_3$  on  $\mathbb{P}^9$ , the semistable locus contains the orbits of smooth and nodal plane cubics; that is, smooth cubics together with the three orbits consisting of irreducible cubics with a node, unions of lines and conics meeting transversely, and triangles. In the quotient, these last three orbits correspond to just one additional point, and this quotient is the compactification of the affine line to the projective line obtained by adding one point.
- (3) *Unstable* orbits. These are the points  $p$  on which all invariant polynomials vanish, so that the induced map  $\text{Proj } A \rightarrow \text{Proj}(A^G)$  is not even defined at  $p$ . Thus unstable points do not correspond to any points of  $X//G$ ; in fact, they cannot be included in any topologically separated quotient of an open subset of  $X$  defined in this way (but there may well be other compactifications, coming from other representations of  $M_g$  as  $X'//G'$ ; see [Smyth 2013]).

### 3. Stable curves

The compactification  $\overline{M}_g$  is also a *modular* compactification in the sense that the points of the boundary correspond to slightly more general objects of the same type as the points of  $M_g$ .



**Figure 4.** A stable curve.

**Definition 3.1.** A reduced irreducible connected curve is *stable* if it has at most nodes as singularities, and if every smooth rational component meets the other components at least three times.

The last phrase of the definition could be replaced by the equivalent condition that the automorphism group of  $C$  is finite.

These are stable points in the Hilbert scheme of tri-canonical embeddings in the sense of geometric invariant theory, and the result is that  $M_g$  has a modular compactification that is a projective variety:

**Theorem 3.2.** (Deligne, Mumford, Knudsen) [Deligne and Mumford 1969], [Knudsen 1983]  $\overline{M}_g$  is a projective variety such that

- (1) The points of  $\overline{M}_g$  correspond one-to-one to isomorphism classes of smooth curves.
- (2) For every family  $C \rightarrow B$  of stable curves there is a morphism of schemes  $B \rightarrow \overline{M}_g$  carrying each closed point  $b \in B$  to the point representing the isomorphism class of the fiber of  $C$  over  $b$ . These maps form a natural transformation from the functor  $G(-)$  of families of stable curves to the functor  $\text{Mor}_{\text{schemes}}(-, \overline{M}_g)$  through which any natural transformation  $G \rightarrow \text{Mor}_{\text{schemes}}(-, M')$  factors.

The deepest theorems about  $M_g$  have been proven using the divisor class group of  $\overline{M}_g$ , and many of the divisors that play a role are actually supported on the complement  $\overline{M}_g \setminus M_g$ , often called the *boundary*.

**Cheerful Fact 3.3.**

**Proposition 3.4.** For  $g \geq 1$  the boundary  $\overline{M}_g \setminus M_g$  is the union of  $1 + \lfloor g - 1/2 \rfloor$  divisors whose generic points are of the form:

- (1) irreducible nodal curves of geometric genus  $g - 1$

- (2) for  $i = 1, \dots, \lfloor (g-1)/2 \rfloor$  the union of two smooth curves  $C_i \cup C_{g-i-1}$  of genera  $i$  and  $g-1-i$  meeting in a point.

We will not prove either of Theorems 2.1 and 3.2. For an introduction to the actual proofs, with references, see [[Harris and Morrison 1998b](#)].

**3.1. How we deal with the fact that  $\overline{M}_g$  is not fine.** The fact that  $\overline{M}_g$  is not a fine moduli space—and that correspondingly there does not exist a universal family of curves over it—is unquestionably a nuisance. Nonetheless, there are ways of dealing with the situation. The first step is to identify the cause of the problem, which is that some curves have nontrivial automorphisms. There are three ways to proceed:

- (1) *Kill the automorphisms.* The idea here is to add additional structure to the objects parametrized, so as to eliminate automorphisms. Here is an example of such a construction. We saw in Chapter 5 that on a smooth projective curve  $C$  of genus  $g$ , the collection of invertible sheaves  $\mathcal{L}$  with  $\mathcal{L}^m \cong \mathcal{O}_C$  forms a group isomorphic to  $(\mathbb{Z}/m)^{2g}$ . We define a *curve with level  $m$  structure* to be such a curve, together with a choice of  $2g$  generators  $\mathcal{L}_1, \dots, \mathcal{L}_{2g}$  for this group. On every curve  $C$  of genus  $\geq 2$  an automorphism fixing all line bundles of order  $m$  is trivial, and there does exist a fine moduli space  $M_g[m]$  for curves with level  $m$  structure; this space is a finite cover of  $M_g$ . Thus, while a universal family does not exist over  $M_g$ , one does exist over a finite cover of  $M_g$ , and this is sufficient for many purposes.
- (2) *Ignore the automorphisms.* Here we use a basic fact, which we'll establish in Section 3: in  $M_g$ , the locus  $A \subset M_g$  of curves that do have automorphisms other than the identity has codimension  $g-2$ . If we restrict to the complement  $M_g^\circ = M_g \setminus A$ , accordingly, there does exist a universal family, and again this is sufficient for many purposes; for example, if  $g \geq 4$  then a divisor class on  $M_g$  is determined by its restriction to  $M_g^\circ$ , so we can just work over that open set.
- (3) *Embrace the automorphisms.* We mentioned above that there does not exist a fine moduli space for curves of genus  $g$  in the category of schemes. But there is a larger category, called *stacks*, in which a fine moduli space does exist. This solution to the problem, pioneered by Deligne and Mumford, has many advantages but involves a substantial investment in mastering the technical issues; readers who wish to pursue this avenue may consult [[Deligne and Mumford 1969](#)] or [[Olsson 2016](#)].

#### 4. Can one write down a general curve of genus $g$ ?

We have made a fuss over the value of compactifying  $M_g$  to a projective variety. To see an example of the usefulness of  $\overline{M}_g$ , we'll take up a question we've raised before: Can one write down a general curve of genus  $g$ ? More precisely, does there exist a family of curves depending freely on parameters that includes all the curves in an open subset of  $M_g$ —for example, as the equation  $y^2 = x(x - 1)(x - \lambda)$  represents general curves of genus 1? Still more precisely, we say that a variety is *unirational* if it admits a dominant morphism from an open subset of  $\mathbb{A}^n$ , and our question is: “Is  $M_g$  unirational?”.

We have produced families with free parameters in genera 2 and 3. Essentially the same approach works in genera 4 and 5; in each case a general canonical curve is a complete intersection, so that if we take the coefficients of its defining polynomials to be general scalars we have a general curve.

This method breaks down when we get to genus 6, where a canonical curve is not a complete intersection. But it's close enough: as discussed in Chapter 12, a general canonical curve of genus 6 is the intersection of a smooth del Pezzo surface  $S \subset \mathbb{P}^5$  with a quadric hypersurface  $Q$ ; since all smooth del Pezzo surfaces in  $\mathbb{P}^5$  are isomorphic, we can just fix one such surface  $S$  and let  $Q$  be a general quadric.

It gets harder as the genus increases. Already genus 7 calls for a different approach. Here we want to argue that, by Brill-Noether theory, a general curve of genus 7 can be realized as (the normalization of) a plane septic curve with 8 nodes  $p_1, \dots, p_8 \in \mathbb{P}^2$ . Conversely, if  $p_1, \dots, p_8 \in \mathbb{P}^2$  are general points then having nodes at the points  $p_i$  imposes  $3 \times 8 = 24$  independent conditions on the  $\mathbb{P}^{35}$  of curves of degree 7, so that we would expect that the septic curves double at the  $p_i$  form a linear series, parametrized by a projective space  $\mathbb{P}^{11}$ .

This suggests that we consider the space

$$\Sigma := \{(p_1, \dots, p_8, C) \in (\mathbb{P}^2)^8 \times \mathbb{P}^{35} \mid C \text{ is double at } p_1, \dots, p_8\}$$

With a little work, we can see that there is a unique component  $\Sigma^\circ$  of  $\Sigma$  dominating  $(\mathbb{P}^2)^8$ , which is a  $\mathbb{P}^{11}$ -bundle over an open subset of  $(\mathbb{P}^2)^8$  and hence rational; this component dominates  $M_7$ , showing that  $M_7$  is unirational.

A similar approach works through genus 10, and Severi conjectured that it would be possible to do something similar for all genera. The approach through plane curves, however, fails in genus 11: by the Brill-Noether theorem, the smallest degree of a planar embedding of a general curve of genus 11 is 10; by our  $g + 2$  theorem, such a curve has  $\binom{9}{2} - 11 = 25$  nodes. But  $3 \times 25 > 65$ , the dimension of the space of plane curves of degree 10. Thus, if we introduce the analog of the incidence correspondence we used in the case of genus 7—that

is,

$$\Sigma := \{(p_1, \dots, p_{25}, C) \in (\mathbb{P}^2)^{25} \times \mathbb{P}^{65} \mid C \text{ is double at } p_1, \dots, p_{25}\}$$

then the projection  $\Sigma \rightarrow (\mathbb{P}^2)^{25}$  is not dominant, and we have no idea if  $\Sigma$  is rational. Ad hoc (and much more difficult) arguments have been given in genera 11, 12 13 and 14, but so far no-one can go further in producing general curves; in genus 15 it is only known that any two general curves can be connected by a chain of rational curves that passes through the locus of irreducible nodal curves in  $\overline{M}_g$  ([Bruno and Verra 2005]). In genera 15 and 16 Chang and Ran showed that  $\overline{M}_g$  has no pluricanonical divisors a (possibly) weaker statement.

However the issue is resolved for all genera  $\geq 22$ . Surprisingly, this depends (in the current state of our knowledge) on an understanding of the complement  $\overline{M}_g \setminus M_g$  and its image in the divisor class group of  $\overline{M}_g$ . The starting point is the fact that a smooth  $n$ -dimensional projective variety  $X$  with an effective pluricanonical canonical divisor—that is, a nonzero section of the sheaf  $\omega_X^{\otimes p}$  for some  $p > 0$ —cannot be unirational: if there were a dominant rational map  $\mathbb{P}^n \rightarrow X$ , we could pull this section back to get an effective pluricanonical divisor on  $\mathbb{P}^n$ , which doesn't exist because the canonical divisor on  $\mathbb{P}^n$  has negative degree. At the same time, we can analyze the divisor class theory of the space  $\overline{M}_g$  and for large  $g$  exhibit an effective pluricanonical divisor on  $M_g$  by using components of  $\overline{M}_g \setminus M_g$ . The upshot is

**Theorem 4.1.** ([Harris and Mumford 1982], [Harris 1984], [Eisenbud and Harris 1987b], [Farkas et al. 2023]) For all  $g \geq 22$ ,  $M_g$  is not unirational.

In each case, what is actually proven is the stronger but more technical statement that  $\overline{M}_g$  has *general type*. This line of argument requires a great deal of work—the interested reader can find more details, plus a guide to the literature, in [Harris and Morrison 1998b].

## 5. Hurwitz spaces

Hurwitz spaces are spaces parametrizing branched covers. They are fascinating objects; we know quite a bit about their geometry but there is much that is unknown as well. In this discussion, we'll stick to the simplest case, that of the *small Hurwitz spaces*, parametrizing simply branched covers of  $\mathbb{P}^1$ .

To start with the definition: the small Hurwitz space  $Hur_{g,d}^\circ$  parametrizes pairs  $(C, f)$  where  $C$  is a smooth curve of genus  $g$  and  $f : C \rightarrow \mathbb{P}^1$  a map of degree  $d$  with simple branching; that is,

$$Hur_{g,d}^\circ = \{(C, f) \mid C \in M_g \text{ and } f : C \rightarrow \mathbb{P}^1 \text{ simply branched of degree } d\}.$$

There are two natural maps from the Hurwitz space to other spaces. First, we can “project on the first factor;” that is, simply forget the map  $f$  to arrive at a map  $\pi : \text{Hur}_{g,d}^\circ \rightarrow M_g$ . Secondly, we can associate to a point  $(C, f) \in \text{Hur}_{g,d}^\circ$  the branch divisor  $B \subset \mathbb{P}^1$ , which is an unordered  $b$ -tuple of distinct points in  $\mathbb{P}^1$ , which we can think of as a point in the  $b$ th symmetric product  $(\mathbb{P}^1)_b \cong \mathbb{P}^b$ . We thus have a diagram

$$\begin{array}{ccc} & \text{Hur}_{g,d}^\circ & \\ \pi \swarrow & & \searrow \beta \\ M_g & & U \subset \mathbb{P}^b \end{array}$$

where  $U \subset \mathbb{P}^b$  is the complement of the hypersurface in  $\mathbb{P}^b$  where at least 2 of the  $b$  points are equal, called the discriminant hypersurface. Thus the Hurwitz space is positioned between an object  $U$  we understand relatively well, and an object  $M_g$  about which we would like to know more; this accounts for the historical importance of Hurwitz spaces. We’ll now illustrate how this can be exploited.

To begin with, by the analysis in Section 2, we see that *the map  $\beta$  is a covering space*: for any reduced divisor  $B \subset \mathbb{P}^1$  there are a finite number of simply branched covers of  $\mathbb{P}^1$  with branch divisor  $B$ ; and as we vary the points of  $B$  locally we can deform the cover along with them. This allows us to give the Hurwitz space  $\text{Hur}_{g,d}^\circ$  the structure of a smooth variety, and also tells us that

$$\dim(\text{Hur}_{g,d}^\circ) = b = 2d + 2g - 2$$

**5.1. The dimension of  $M_g$ .** Next, we look at the projection  $\pi : \text{Hur}_{g,d}^\circ \rightarrow M_g$ . To start, let’s assume  $d$  is large relative to  $g$ ;  $d \geq g + 1$  suffices, but you can take  $d$  as large as you like; taking  $d > 2g$  may make the argument simpler. We have then the

**Proposition 5.1.** *If  $d \geq g + 1$ , the map  $\pi : \text{Hur}_{g,d}^\circ \rightarrow M_g$  is surjective, with fibers of dimension  $2d - g + 1$ .*

**Proof.** The question is, given a curve  $C$ , how many simply branched maps  $f : C \rightarrow \mathbb{P}^1$  of degree  $d$  are there? To begin with, the  $g + 1$  theorem (9.9) tells us that there is one, whence we see that  $\pi$  is surjective.

We can compute the dimension of the fibers, too: To specify a map  $f : C \rightarrow \mathbb{P}^1$ , we can start by choosing a divisor  $D \in C_d$ , which will be the divisor  $f^{-1}(\infty)$ ; this can be a general divisor of degree  $d$  on  $C$ . Second, we choose a divisor  $E$  which will be  $f^{-1}(0)$ ; this can be a general member of the linear system  $|D|$ , which has dimension  $d - g$ . Finally, specifying  $f^{-1}(\infty)$  and  $f^{-1}(0)$  determines

the map  $f$  up to scalar multiplication on  $\mathbb{P}^1$ ; adding up the degrees of freedom, we see that the fibers of  $\pi$  have dimension

$$d + (d - g) + 1 = 2d - g + 1.$$

□

Finally, we conclude that if  $g \geq 2$  then

$$\dim(M_g) = (2d + 2g - 2) - (2d - g + 1) = 3g - 3.$$

We can use this in turn to analyze the cases of smaller  $d$ . As a basic application, note that the group  $PGL_2$  of automorphisms of  $\mathbb{P}^1$  acts on the Hurwitz space: given  $\varphi \in PGL_2$ , we can send  $(C, f)$  to  $(C, \varphi \circ f)$ . Moreover, the orbits of this action lie in fibers of the projection  $\pi : Hur_{g,d}^\circ \rightarrow M_g$ , meaning that the fibers of  $\pi$  have dimension at least 3.

**Corollary 5.2.** *If  $d < \lceil \frac{g}{2} \rceil + 1$ , then a general curve  $C$  of genus  $g$  does not admit a map of degree  $d$  to  $\mathbb{P}^1$ .*

This is one-half of the case  $r = 1$  of the Brill-Noether theorem, about which we will say much more later.

**5.2. Irreducibility of  $M_g$ .** Another important application is the original proof of the irreducibility of  $M_g$ : in [Hurwitz 1891], Hurwitz analyzed the monodromy of the map  $\beta : Hur_{g,d}^\circ \rightarrow U \subset \mathbb{P}^b$ —what happens, in other words, when you let the branch points of a cover wander around in  $U$  before coming back to their original locations. He proved that the monodromy is transitive, and hence that the Hurwitz space  $Hur_{g,d}^\circ$  is irreducible; since  $Hur_{g,d}^\circ$  dominates  $M_g$  for  $d$  large, he deduced that  $M_g$  must be irreducible as well.

Hurwitz' argument illustrates a fundamental point: in practice, moduli spaces of curves “with extra structure,” such as a map to projective space, are often easier to work with, and provide a useful tool for understanding the geometry of abstract moduli spaces. For example, if we're given an abstract curve  $C$  of genus  $g$ , it's hard—without developing a fair amount of deformation theory—to show that  $C$  varies in a nontrivial family. But if  $C$  is expressed as a branched cover, we can find such families just by varying the branch points.

There are many open problems connected with the Hurwitz scheme; here are a few:

- (1) A compactification of the Hurwitz scheme by *admissible covers* (allowing both source and target of the covering to be reducible in a controlled way) is known [Harris and Morrison 1998b], but the boundary is very complicated, and it would be interesting to find a simpler one.

- (2) It is conjectured that Picard group of the Hurwitz scheme is torsion; see [Deopurkar and Patel 2015], who prove the conjecture for  $g \leq 5$ , and [Mullane 2023] for the case  $d > g - 1$ .
- (3) There is active work and many open problems around computing the *Hurwitz numbers*, that is, the number of curves having maps to  $\mathbb{P}^1$  with specified degree and branching; see for example [Hurwitz 1901] and [Ekedahl et al. 2001].

## 6. The Severi variety

Despite having been studied for so long, many questions about plane curves remain open—for example: which ones degenerate into which others, and in what way. All plane curves of degree  $d$  have the same Hilbert function, and thus the same arithmetic genus  $\binom{d-1}{2}$ , but since curves of degree  $d$  can have different sorts and numbers of singularities, they can have geometric genera from 0 to  $\binom{d-1}{2}$ . In this section we will explore the subset of (reduced, irreducible) curves of degree  $d$  with a fixed geometric genus. We will focus on the open set consisting of curves with only nodes as singularities, which we call *nodal curves*, and compute its dimension.

Let  $\mathbb{P}^N := \mathbb{P}^{\binom{d+2}{2}-1}$  be the projective space parametrizing plane curves of degree  $d$ . Within  $\mathbb{P}^N$  the set of reduced irreducible curves is open—it is the complement of the union of the images of the maps

$$\mathbb{P}^{\binom{d_1+2}{2}-1} \times \mathbb{P}^{\binom{d_2+2}{2}-1} \rightarrow \mathbb{P}^N$$

with  $d_1 + d_2 = d$  given by multiplication of forms.

**Proposition-Definition 6.1.** The *Severi variety*  $V_{g,d} \subset \overline{V}_{g,d}$  is the locus of irreducible plane curves of degree  $d$  with  $\delta = \binom{d-1}{2} - g$  nodes and no other singularities. This is a locally closed subset of  $\mathbb{P}^N$ . (Reason: having only nodes as singularities is an open condition; having at least a certain number of them is a closed condition.) It is sometimes called the *small Severi variety*, since we are excluding curves with more complicated singularities.

We will see that in a neighborhood of  $V_{g,d}$ , the closure  $\overline{V}_{g,d}$  is well behaved; but away from this, even the singularities of  $\overline{V}_{g,d}$  are not well understood. It is an interesting open problem to find a simpler partial compactification of  $V_{g,d}$ .

**Cheerful Fact 6.2.** As we shall see, the variety  $V_{g,d}$  is smooth. In 1921 F. Severi gave an incorrect proof that  $V_{g,d}$  was connected, and thus irreducible. This was finally proven in [Harris 1986].

**6.1. Local geometry of the Severi variety.** We first consider the *universal singular point*

$$\Phi := \{(C, p) \in \mathbb{P}^N \times \mathbb{P}^2 \mid p \in C_{\text{sing}}\}$$

and its image  $\Delta \subset \mathbb{P}^N$ , the *discriminant variety*.

**Proposition 6.3.**  $\Phi$  is smooth and irreducible of dimension  $N - 1$ , and the discriminant  $\Delta$  is a hypersurface in  $\mathbb{P}^N$ .

**Proof.** Projection on the second factor expresses  $\Phi$  as a  $\mathbb{P}^{N-3}$ -bundle over  $\mathbb{P}^2$ . Explicitly, if  $[X, Y, Z]$  are homogeneous coordinates on  $\mathbb{P}^2$ , and  $\{a_{i,j,k} \mid i + j + k = d\}$  are homogeneous coordinates on  $\mathbb{P}^N$ , then the universal curve

$$\mathbb{C} := \{(C, p) \in \mathbb{P}^N \times \mathbb{P}^2 \mid p \in C\}$$

is given as the zero locus of the single bihomogenous polynomial of bidegree  $(1, d)$

$$F([a_{i,j,k}], [X, Y, Z]) = \sum a_{i,j,k} X^i Y^j Z^k;$$

and the universal singular point is the common zero locus of the three partial derivatives  $\partial F/\partial X$ ,  $\partial F/\partial Y$  and  $\partial F/\partial Z$ .

The set of forms  $F$  that define curves singular at a given point is defined by 3 independent linear conditions, and since the set of points is 2-dimensional, the set  $\Delta$  of singular forms has dimension  $N - 1$ .  $\square$

We next compute the differential of the map  $\pi : \Phi \rightarrow \mathbb{P}^N$ :

**Lemma 6.4.** Suppose that  $(C, p) \in \Phi$ , with  $p$  a node of  $C$ . The differential

$$d\pi : T_{(C,p)}\Phi \rightarrow T_C\mathbb{P}^N$$

is injective, with image the hyperplane  $H_p \subset \mathbb{P}^N$  of plane curves containing the point  $p$ .

Thus, if  $p$  is a node of  $C$  and the only singularity of  $C$ , then  $\Delta$  is smooth at  $C$ ; and more generally the image of a small analytic neighborhood of  $(C, p) \in \Phi$  is smooth, and we can identify its tangent space at  $p$  with the hyperplane  $H_p$ .

**Proof.** We will prove this using affine coordinates on  $\mathbb{P}^2$  and  $\mathbb{P}^N$ . Changing coordinates if necessary, we may assume that the point  $[1, 0, 0] \notin C$ , and that the point  $p$  is  $[0, 0, 1]$ . let  $x = X/Z$  and  $y = Y/Z$  be coordinates on the affine plane  $Z \neq 0$  and write the polynomial  $F(x, y, 1)$  above as

$$f(x, y) = \sum_{i+j \leq d} a_{i,j} x^i y^j$$

with  $a_{d,0}$  normalized to 1.

Let  $g, h$  be the two partial derivatives of  $f$ , that is:

$$g(x, y) := \frac{\partial f}{\partial x} = \sum_{i+j \leq d} ia_{i,j}x^{i-1}y^j$$

$$h(x, y) := \frac{\partial f}{\partial y} = \sum_{i+j \leq d} ja_{i,j}ix^iy^{j-1}.$$

The functions  $f, g$  and  $h$  are local defining equations for  $\Phi$ ; we consider their partial derivatives with respect to  $x, y$  and  $a_{0,0}$ , evaluated at the point  $(C, p)$ , as in Table 6.1.

	$f$	$g$	$h$
$\frac{\partial}{\partial x}$	0	$a_{2,0}$	$a_{1,1}$
$\frac{\partial}{\partial y}$	0	$a_{1,1}$	$a_{0,2}$
$\frac{\partial}{\partial a_{0,0}}$	1	0	0

The fact that  $p$  is a node of  $C$  (and not a more complicated singularity) implies that the upper right  $2 \times 2$  submatrix is nonsingular, which shows that the differential  $d\pi$  is injective, and its image is the hyperplane  $a_{0,0} = 0$  in  $\mathbb{P}^N$ , which is exactly the hyperplane of curves containing  $p$ .  $\square$

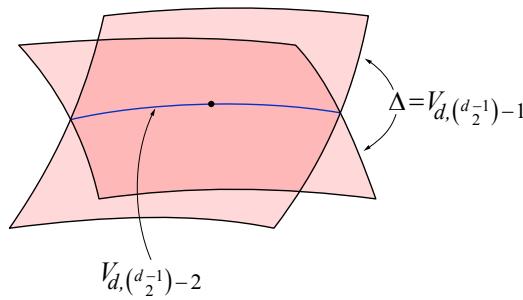
**Lemma 6.5.** *The nodes  $q_i$  of an irreducible nodal plane curve  $C$  of degree  $d$  impose independent conditions on curves of degree  $d - 3$ , and hence on curves of any degree  $m \geq d - 3$ .*

**Proof.** We will prove in Chapter 15 that the  $g$  sections of the canonical sheaf on the normalization  $\tilde{C}$  of  $C$  are the preimages of the sections of  $\mathcal{O}_C(d - 3)$  that vanish at the nodes. On the other hand,  $h^0(\mathcal{O}_C(d - 3)) = \binom{d-1}{2}$ , and the difference is exactly the number of nodes.  $\square$

**Corollary 6.6.** *If  $C$  is a nodal curve of degree  $d$  with geometric genus  $g = \binom{d-1}{2} - \delta$ , then in a neighborhood of  $C \in \mathbb{P}^N$  the discriminant hypersurface of all singular curves consists of  $\delta$  smooth sheets, meeting transversely, and hence  $V_{g,d}$  is smooth.*

Moreover, in a neighborhood  $C \in \mathbb{P}^N$  the variety  $\bar{V}_{g,d'}$  with  $g' = \binom{d-1}{2} - \delta' > g$  is the union of  $\binom{\delta}{\delta'}$  smooth branches, each of dimension  $N - \delta'$ , corresponding bijectively with subsets of  $\{p_1, \dots, p_{\delta'}\}$  of cardinality  $\delta'$ .

**Proof.** Lemma 6.4 shows that in an analytic neighborhood of  $C \in \mathbb{P}^N$  the discriminant hypersurface  $\Delta$  consists of  $\delta$  smooth sheets, each corresponding to one node, and Lemma 6.5 implies that the tangent spaces to these sheets are linearly independent.  $\square$



**Figure 5.** In a neighborhood of the point corresponding to a plane curve with 2 nodes,  $V_{d, \binom{d-1}{2}-2}$  is the intersection of two smooth hypersurfaces intersecting transversely.

(( Silvio, in Fig 6.6 the two "α"s in the image should both be d. Also, replace Δ with  $\Delta = V_{d, \binom{d-1}{2}-1}$  . ))

**Corollary 6.7.** *The Severi variety  $V_{g,d}$  has pure dimension  $N - \delta$ , where  $\delta = \binom{d-1}{2} - g$ .*

In Section 11, we give a heuristic calculation of the “expected dimension”  $h(d, g, r)$  of the variety parametrizing curves of degree  $d$  and genus  $g$  in  $\mathbb{P}^r$ :

$$h(g, r, d) := 4g - 3 + (r + 1)(d - g + 1) - 1.$$

The actual dimension of the restricted Hilbert scheme may be quite different. But Corollary 6.7 shows that in case  $r = 2$  (as in the case of  $r = 1$ ), the actual dimension is always the expected.

## 7. Exercises

**Exercise 7.1.** Consider the action of  $G_m$  on  $\mathbb{P}^3$  given in coordinates by

$$t : (x_0, x_1, x_2, x_3) \mapsto (tx_0, tx_1, t^{-1}x_2, t^{-1}x_3)$$

for  $t \in G_m = \mathbb{C}^*$ .

- (1) Show that the ring of forms in  $\mathbb{C}[x_0, \dots, x_3]$  that are invariant is generated by

$$x_0x_3, x_0x_2, x_1x_3, x_1x_2$$

and thus  $\mathbb{P}^3/G_m \cong \mathbb{P}^1 \times \mathbb{P}^1$ .

- (2) Show that the unstable locus for this action is the union of the two lines  $x_0 = x_1 = 0$  and  $x_2 = x_3 = 0$ .

- (3) Show that the orbits of  $G_m$  are the points on the unstable lines and, for each point  $p$  not on an unstable line, a copy of  $\mathbb{P}^1 \setminus \{0, \infty\} \cong G_m$  whose closure is the unique line containing  $p$  and meeting both unstable lines.

**Exercise 7.2.** Consider the action of  $G_m$  on  $\mathbb{P}^3$  given in coordinates by

$$t : (x_0, x_1, x_2, x_3) \mapsto (tx_0, tx_1, tx_2, t^{-1}x_3)$$

for  $t \in G_m = \mathbb{C}^*$ .

- (1) Show that the ring of forms in  $\mathbb{C}[x_0, \dots, x_3]$  that are invariant is generated by forms

$$F(x_0, x_1, x_2)x_3$$

where  $F$  is a cubic form on  $\mathbb{P}^2$ , and thus  $\mathbb{P}^3 // G_m \cong \mathbb{P}^2$ , with the embedding given by the third Veronese map.

- (2) Show that the unstable locus for this action is the union of the point  $x_0 = x_1 = x_2 = 0$  and the plane  $x_3 = 0$ .
- (3) Show that the orbits of  $G_m$  are the points on the components of the unstable locus and, for each point  $p$  that is not unstable, a copy of  $\mathbb{P}^1 \setminus \{0, \infty\} \cong G_m$  whose closure is the unique line containing  $p$  and the unstable point. Thus the quotient map is the composition of the linear projection from the unstable point with the 3-uple embedding.

**Exercise 7.3.** Show from the explicit formula 1 for the  $j$ -function that if  $j : B \rightarrow M_1 = \mathbb{A}^1$  is a map associated to a family  $C \rightarrow B$  of curves of genus 1, then every zero of the  $j$ -function has multiplicity divisible by 3, and conclude that some maps  $B \rightarrow M_1$  do not correspond to families of curves; in particular there is no universal family over  $M_1$ , and thus  $M_1$  is not a fine moduli space for curves of genus 1. There is a similar problem at  $j(\lambda) = 1728$ .

**Exercise 7.4.** In Exercise 7.3 we saw a local obstruction to the existence of a universal family over  $M_1$ . There is also a global obstruction, coming from the fact that some genus 1 curves have extra automorphisms. Show that there is a “tautological” family over the punctured  $j$ -line  $L := \mathbb{A}^1 \setminus \{0, 1728\}$ —that is, a family  $\mathcal{X} \rightarrow L$  whose fiber over  $t$  has  $j$ -invariant  $t$ ; but show that this family is not universal as follows: Let  $B$  be any curve of genus 1 and  $\tau : B \rightarrow B$  a translation of order 2, and let  $E$  be a fixed elliptic curve (that is, curve of genus 1 with a chosen point, so that we may identify the points of  $E$  with an abelian group.) Let  $\mathcal{X} \rightarrow L$  be the family  $E \times B$  modulo the equivalence relation  $(e, b) \sim (-e, \tau(b))$ . The projection to  $B/\tau$  has all fibers isomorphic to  $E/(\pm) \cong E$ . But the family is not isomorphic to the trivial family  $E \times B/\tau \rightarrow B/\tau$

Hint: show that the canonical bundle  $K_{\mathcal{X}}$  is nontrivial.



# Curves of genus 4 and 5

In this chapter we focus on the linear systems that exist on curves of genus 4 and 5, and what this says about maps to  $\mathbb{P}^r$ , focusing as usual on the nonhyperelliptic case.

## 1. Curves of genus 4

In genus 4 we have a question that the elementary theory based on the Riemann-Roch formula cannot answer: are nonhyperelliptic curves of genus 4 *trigonal*, that is, expressible as three-sheeted covers of  $\mathbb{P}^1$ ? The answer will emerge from our analysis in Proposition 1.2 below.

**1.1. The canonical model.** Let  $C$  be a nonhyperelliptic curve of genus 4. The canonical map  $\phi_K : C \hookrightarrow \mathbb{P}^3$  embeds  $C$  as a curve of degree 6 in  $\mathbb{P}^3$ , and we identify  $C$  with the image. As in previous cases we may describe the homogeneous ideal  $I$  of  $C$  by considering the restriction maps

$$\rho_m : H^0(\mathcal{O}_{\mathbb{P}^3}(m)) \rightarrow H^0(\mathcal{O}_C(m)) = H^0(mK_C).$$

In case  $m = 2$  we have  $h^0(\mathcal{O}_{\mathbb{P}^3}(2)) = \binom{5}{3} = 10$ , while by the Riemann-Roch theorem we have

$$h^0(\mathcal{O}_C(2)) = 12 - 4 + 1 = 9.$$

Thus  $C \subset \mathbb{P}^3$  lies on at least one quadric surface  $Q$ . The quadric  $Q$  must be irreducible, since any any reducible and/or nonreduced quadric must be a union of planes, and thus cannot contain an irreducible nondegenerate curve. If  $Q \neq Q' \subset \mathbb{P}^3$  is another quadric then, by Bézout's theorem,  $Q \cap Q'$  is a curve of degree 4 and thus could not contain  $C$ . From this we see that  $Q$  is unique, and it follows that  $\rho_2$  is surjective.

What about cubics? Again we consider the restriction map

$$\rho_3 : H^0(\mathcal{O}_{\mathbb{P}^3}(3)) \rightarrow H^0(\mathcal{O}_C(3)) = H^0(3K_C).$$

The space  $H^0(\mathcal{O}_{\mathbb{P}^3}(3))$  has dimension  $\binom{6}{3} = 20$ , while the Riemann-Roch formula shows that

$$h^0(\mathcal{O}_C(3)) = 18 - 4 + 1 = 15.$$

It follows that the ideal of  $C$  contains at least a 5-dimensional vector space of cubic polynomials. We can get a 4-dimensional subspace as products of the unique quadratic polynomial  $F$  vanishing on  $C$  with linear forms—these define the cubic surfaces containing  $Q$ . Since  $5 > 4$  we conclude that the curve  $C$  lies on at least one cubic surface  $S$  not containing  $Q$ . Bézout's theorem shows that the curve  $Q \cap S$  has degree 6; thus it must be equal to  $C$ .

Let  $G = 0$  be the cubic form defining the surface  $S$ . By Lasker's theorem the ideal  $(F, G)$  is unmixed, and thus is equal to the homogeneous ideal of  $C$ .

Conversely, Let  $C = Q \cap S$  with  $Q$  a quadric and  $S$  a cubic. By Corollary 2.7 the canonical sheaf of  $C$  is

$$\omega_C = ((\omega_Q \otimes \mathcal{O}_Q(3))|_C = \mathcal{O}_C(-2 + 3) = \mathcal{O}_C(1)$$

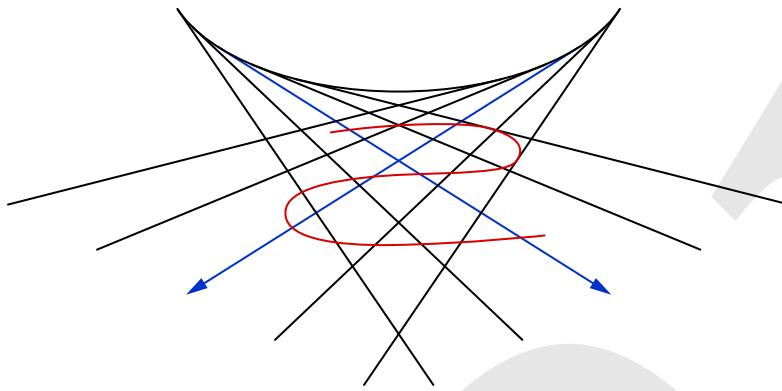
so  $C$  is a canonical curve. Since  $C$  is smooth, the quadric and the cubic must meet transversely at points that are nonsingular on each of them. Thus:

**Theorem 1.1.** *The canonical model of a nonhyperelliptic curve of genus 4 is a complete intersection of a quadric  $Q = V(F)$  and a cubic surface  $S = V(G)$  meeting transversely along nonsingular points of each. Conversely, any smooth curve that is the complete intersection of a quadric and a cubic surface in  $\mathbb{P}^3$  is the canonical model of a nonhyperelliptic curve of genus 4.*  $\square$

Since the quadric surface  $Q$  containing the canonical curve  $C$  is unique, its rank is an invariant of  $C$ . Since  $C$  is irreducible and nondegenerate, the quadric cannot be a double plane or the union of two planes, but it can be singular (rank 3) or smooth (rank 4). On the other hand, the singularities of a cubic  $S$  such that  $S \cap Q = C$  play no role. Of course  $S$  must be nonsingular along  $C$ , since else  $C$  would be singular. We can vary  $S$  by adding a multiple of the equation of  $Q$  to the equation of  $S$ , and since this linear system of cubics has base locus only along  $C$ , Bertini's theorem shows that the general such cubic is nonsingular everywhere.

## 1.2. Maps to projective space.

1.2.1. *Maps to  $\mathbb{P}^1$ .* We can now answer the question we asked at the outset, whether a nonhyperelliptic curve of genus 4 can be expressed as a three-sheeted cover of  $\mathbb{P}^1$ . This amounts to asking if there are any divisors  $D$  on  $C$  of degree



**Figure 1.** Canonical curve of genus 4 on a smooth quadric, showing two  $g_3^1$ s.

3 with  $r(D) \geq 1$ ; since we can take  $D$  to be a general fiber of a map  $\pi : C \rightarrow \mathbb{P}^1$ , we can for simplicity assume  $D = p + q + r$  is the sum of three distinct points.

By the geometric Riemann-Roch theorem, a divisor  $D = p + q + r$  on a canonical curve  $C \subset \mathbb{P}^{g-1}$  has  $r(D) \geq 1$  if and only if the three points  $p, q, r \in C$  are collinear. If three points  $p, q, r \in C \subset Q$  lie on a line  $L \subset \mathbb{P}^3$  then the quadric  $Q$  will meet  $L$  in at least three points, and hence will contain  $L$ . Conversely, if  $L$  is a line contained in  $Q$ , then the divisor  $D = C \cap L = S \cap L$  on  $C$  has degree 3. Thus we can answer our question in terms of the family of lines contained in  $Q$ .

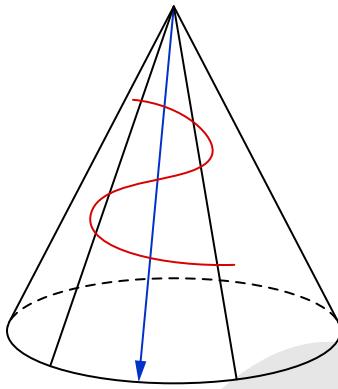
Any smooth quadric is isomorphic to  $\mathbb{P}^1 \times \mathbb{P}^1$ , and contains two families of lines, or *rulings*. On the other hand, any quadric of rank 3 is a cone over a smooth plane conic, and thus has just one ruling. By the argument above, the pencils of divisors on  $C$  cut out by the lines of these rulings are the  $g_3^1$ s on  $C$ . This proves:

**Proposition 1.2.** *A nonhyperelliptic curve of genus 4 may be expressed as a 3-sheeted cover of  $\mathbb{P}^1$  in either one or two ways, depending on whether the unique quadric containing the canonical model of the curve is singular or smooth.*

See Figures 1 and 2 for an idea how this might look in the case of a smooth quadric.

A curve expressible as a 3-sheeted cover of  $\mathbb{P}^1$  is called *trigonal*; by the analyses of the preceding sections, we have shown that *every curve of genus  $g \leq 4$  is either hyperelliptic or trigonal*.

**1.2.2. Maps to  $\mathbb{P}^2$ .** We can also describe the lowest degree plane models of nonhyperelliptic curves  $C$  of genus 4. We can get a plane model of degree 5 by projecting  $C$  from a point  $p$  of the canonical model of  $C$ . Moreover, the Riemann-Roch theorem shows that if  $D$  is a divisor of degree 5 with  $r(D) = 2$



**Figure 2.** Canonical curve of genus 4 on a quadric cone, with just one  $g_3^1$ .  
 (( Silvio: Nice picture!. Maybe make the two  $g_3^1$  arrows different colors?  
 and add their target lines ))

then,  $h^0(K - D) = 1$ . Thus  $D$  is of the form  $K - p$  for some point  $p \in C$ , and the map to  $\mathbb{P}^2$  corresponding to  $D$  is  $\pi_p$ . These maps  $\pi_p : C \rightarrow \mathbb{P}^2$  have the lowest possible degree (except for those whose image is contained in a line) because by Clifford's theorem a nonhyperelliptic curve of genus 4 cannot have a  $g_4^2$ .

We now consider the singularities of the plane quintic  $\pi_p(C)$ . Suppose as above that  $C = Q \cap S$ , with  $Q$  a quadric. If a line  $L$  through  $p$  meets  $C$  in  $p$  plus a divisor of degree  $\geq 2$  then, as we have seen,  $L$  must lie in  $Q$ . Any line through  $p$  that is not contained in  $Q$  meets  $C$  in at most a single reduced point, whose image is thus a nonsingular point of  $\pi(C)$ . Moreover, a line that met  $C$  in  $> 3$  points would have to lie in both the quadric and the cubic containing  $C$ , and therefore would be contained in  $C$ . Since  $C$  is irreducible there can be no such line; thus the image  $\pi_p(C)$  has at most double points.

We will distinguish two cases, depending on whether the quadric  $Q$  is smooth or singular. We will make use of the Gauss map of the quadric, described by the next lemma.

**Lemma 1.3.** *Let  $L \subset S \subset \mathbb{P}^3$  be a line on a surface  $S \subset \mathbb{P}^3$  of degree  $d \geq 2$ , and write  $S_{\text{sing}}$  for the singular locus of  $S$ . The Gauss map  $G : S \rightarrow \mathbb{P}^{3*}$  sending each point  $p \in S \setminus S_{\text{sing}}$  to the tangent plane  $\mathbb{T}_p(S)$  maps  $L$  into the dual line in  $\mathbb{P}^3$  (that is, the locus of planes containing  $L$ ); if  $S$  is smooth along  $L$  then  $G$  has degree  $d - 1$ , and if  $S$  is singular anywhere along  $L$  it has strictly lower degree.*

The geometric idea behind this result is easy to understand: If  $S$  is smooth along  $L$  and  $H \cong \mathbb{P}^2$  is a plane containing  $L$ , then  $H \cap S = L \cup D$  for a plane curve  $D$  of degree  $d - 1$ . The plane  $H$  is tangent to  $S$  at a point  $p$  if and only if  $H \cap S$  is singular at  $p$ ; and this occurs at each of the  $d - 1$  points (counted with

multiplicity) at which  $D$  meets  $L$ , suggesting that the restriction of the Gauss map is  $(d - 1)$ -to-one in this case.

If  $S$  is singular somewhere along  $L$ , then every plane through that point would be among the tangent planes in the sense above, and the image of  $L$  would be a component of a curve of degree  $d - 1$  containing a line; thus of lower degree.

To see that the multiplicities count correctly in this argument, we will give an algebraic version:

**Proof.** Suppose that in terms of homogeneous coordinates  $[X, Y, Z, W]$  on  $\mathbb{P}^3$  the line  $L$  is given by  $X = Y = 0$ . Then the defining equation  $F$  of  $S$  can be written

$$F(X, Y, Z, W) = X \cdot G(Z, W) + Y \cdot H(Z, W) + J(X, Y, Z, W)$$

where  $J$  vanishes to order 2 along  $L$ ; that is,  $J \in (X, Y)^2$ . The Gauss map  $G|_L$  restricted to  $L$  is then given by

$$[0, 0, Z, W] \mapsto [G(Z, W), H(Z, W), 0, 0].$$

The polynomials  $G$  and  $H$  have degree  $d - 1$ , and have a common zero if and only if  $S$  is singular somewhere along  $L$ ; the lemma follows.  $\square$

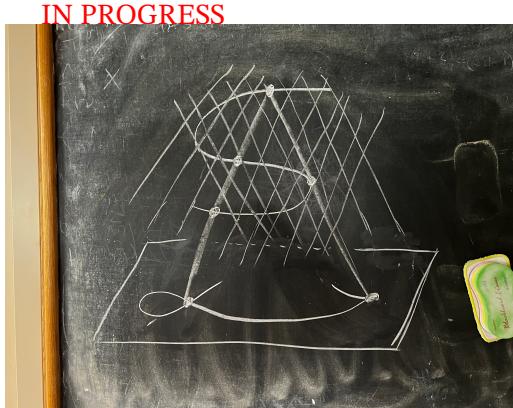
**Example 1.4.** (Gauss map of a quadric) Let  $Q \subset \mathbb{P}^3$  be a smooth quadric, and let  $L \subset Q$  be the line  $X = Y = 0$ . Since we may write the equation of  $Q$  as  $XZ + YW = 0$ , the Gauss map of  $Q$ , restricted to  $L$ , maps  $L$  one-to-one onto the dual line. Indeed, the Gauss map takes  $Q$  isomorphically onto its dual, which is also a smooth quadric.

We can also see this geometrically: if  $H \subset \mathbb{P}^3$  is any plane containing the line  $L \subset Q$ , then  $H$  intersects  $Q$  in the union of  $L$  and a line  $M$ ; the hyperplane section  $Q \cap H = L \cup M$  is then singular at a unique point  $p \in L$ . Thus the Gauss map gives a bijection between points on  $L$  and planes containing  $L$ .

Given this, we can analyze the geometry of projections  $\pi_p(C)$  of our canonical curve  $C = Q \cap S$  as follows:

- (1)  $Q$  is nonsingular: In this case there are two lines  $L_1, L_2$  on  $Q$  that pass through  $p$ ; they meet  $C$  in  $p$  plus divisors  $E_1$  and  $E_2$  of degree 2. If each  $E_i$  consists of distinct points then, since the tangent planes to the quadric along  $L_i$  are all distinct by Example 1.4 the plane curve  $\pi(C)$  has two nodes, one at the image of each  $E_i$ .

On the other hand, if  $E_i$  consists of a double point  $2q$  (that is,  $L_i$  is tangent to  $C$  at  $q \neq p$ , or meets  $C$  three times at  $q = p$ ), then  $\pi(C)$  has a cusp at the corresponding image point.



**Figure 3.** If a genus 4 canonical curve  $C$  lies on a smooth quadric, the image of the projection of  $C$  from a point  $p \in C$  is a plane quintic with two singular points.

In either case,  $\pi(C)$  has two distinct singular points, each either a node or a cusp. The two  $g_3^1$ s on  $C$  correspond to the projections from these singular points. These possibilities are illustrated in Figure 3

- (2)  $Q$  is a cone: In this case, since the curve cannot pass through the singular point of  $Q$  there is a unique line  $L \subset Q$  that passes through  $p$ . Let  $p + E$  be the divisor on  $C$  in which this line meets  $C$ . The tangent planes to  $Q$  along  $L$  are all the same. Thus if  $E = q_1 + q_2$  consists of two distinct points, the image  $\pi_p(C)$  has two smooth branches sharing a common tangent line at  $\pi_p(q_1) = \pi_p(q_2)$ . Such a point is called a *tacnode* of  $\pi_p(C)$ . On the other hand, if  $E = 2q$ , that is, if  $L$  meets  $C$  tangentially at one point  $q \neq p$  (or meets  $C$  3 times at  $p$ ) then the image curve has a higher order cusp, called a *ramphoid cusp*. In either case, the one  $g_3^1$  on  $C$  is the projection from the unique singular point of  $\pi(C)$ .

## 2. Curves of genus 5

We next consider a nonhyperelliptic curve  $C$  of genus 5. There are now two questions that cannot be answered by simple application of the Riemann-Roch /:

- (1) Is  $C$  expressible as a 3-sheeted cover of  $\mathbb{P}^1$ ? In other words, does  $C$  have a  $g_3^1$ ?
- (2) Is  $C$  expressible as a 4-sheeted cover of  $\mathbb{P}^1$ ? In other words, does  $C$  have a base-point-free  $g_4^1$ ?

Note that the first question is answered in the negative by the dimension count of Section 5, but the analysis below will give us a way of characterizing those curves of genus 5 that are trigonal. Note also that if a curve  $C$  of genus 5 is trigonal, the  $g_3^1$  on  $C$  is unique: if there were two distinct  $g_3^1$ s on  $C$ , with associated maps  $\alpha, \beta : C \rightarrow \mathbb{P}^1$ , the product map

$$\alpha \times \beta : C \rightarrow \mathbb{P}^1 \times \mathbb{P}^1$$

would give a birational embedding of  $C$  as a curve of bidegree  $(3, 3)$  in  $\mathbb{P}^1 \times \mathbb{P}^1$ , and it would follow from the genus formula for curves on  $\mathbb{P}^1 \times \mathbb{P}^1$  that the genus of  $C$  would be at most 4.

The argument of the preceding paragraph can be applied more broadly, and it's worth stating the results here:

**Proposition 2.1.** *Let  $d$  and  $e$  be relatively prime integers. If  $C$  is a curve of genus  $g$  with a basepoint free  $g_d^1$  and a basepoint free  $g_e^1$ , then*

$$g \leq (d-1)(e-1).$$

*The same conclusion holds when  $C$  has two distinct  $g_d^1$ s with  $d$  prime.*

**Proof.** As before, we let  $\alpha$  and  $\beta : C \rightarrow \mathbb{P}^1$  be the maps associated to the  $g_d^1$  and the  $g_e^1$ . In either case product map  $\alpha \times \beta : C \rightarrow \mathbb{P}^1 \times \mathbb{P}^1$  is then a birational embedding of  $C$  as a curve of bidegree  $(d, e)$  in  $\mathbb{P}^1 \times \mathbb{P}^1$ , and the result follows.  $\square$

The condition of (relative) primeness is necessary: if  $d = ma$  and  $e = mb$ , the map  $\alpha \times \beta : C \rightarrow \mathbb{P}^1 \times \mathbb{P}^1$  could express  $C$  as an  $m$ -sheeted cover of a curve  $C_0 \subset \mathbb{P}^1 \times \mathbb{P}^1$  of bidegree  $(a, b)$ , and the genus of  $C$  could be arbitrarily large.

### 3. Canonical curves of genus 5

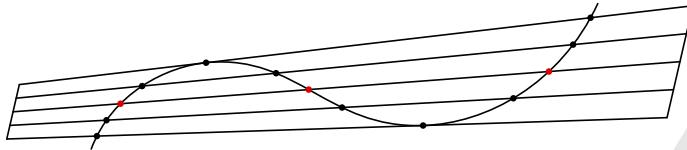
As in the case of genus 4, the answers to the basic questions above about linear series on a curve  $C$  of genus 5 can be found through an investigation of the geometry of the canonical model  $C \subset \mathbb{P}^4$  of  $C$ . This is an octic curve in  $\mathbb{P}^4$ , and as before the first question to ask is what sort of polynomial equations define  $C$ . We start with quadrics, by considering the restriction map

$$\rho_2 : H^0(\mathcal{O}_{\mathbb{P}^4}(2)) \rightarrow H^0(\mathcal{O}_C(2)).$$

On the left, we have the space of homogeneous quadratic polynomials on  $\mathbb{P}^4$ , which has dimension  $\binom{6}{4} = 15$ , while by the Riemann-Roch theorem the target is a vector space of dimension

$$2 \cdot 8 - 5 + 1 = 12.$$

We deduce that  $C$  lies on at least 3 independent quadrics. (We will see in the course of the following analysis that it is exactly 3; that is,  $\rho_2$  is surjective.) Since



**Figure 4.** If  $C$  has a  $g_3^1$  then, in the canonical embedding, the three points of each divisor are collinear, and the lines they span sweep out a surface. Silvio: Not clear why one triple of points is distinguished—red. The lines could be a different color than the curve. The surface looks unreasonably flat, too.

$C$  is irreducible and, by construction, does not lie on a hyperplane, each of the quadrics containing  $C$  is irreducible, and thus the intersection of any two is a surface of degree 4. There are now two possibilities: The intersection of (some) three independent quadrics  $Q_1 \cap Q_2 \cap Q_3$  containing the curve is 1-dimensional; or every such intersection is two dimensional.

**3.1. First case: the intersection of the quadrics is a curve.** We first consider the case where  $Q_1 \cap Q_2 \cap Q_3$  is 1-dimensional. By Bézout's theorem the intersection is a curve of degree 8, and since  $C$  also has degree 8 we must have  $C = Q_1 \cap Q_2 \cap Q_3$ ; that is, the canonical curve is a complete intersection. Lasker's theorem then shows that the three quadrics  $Q_i$  generate the whole homogeneous ideal of  $C$ ; in particular, there are no additional quadrics containing  $C$ .

We can now answer the first of our two questions for curves of this type. As in the genus 4 case the geometric Riemann-Roch theorem implies that  $C$  has a  $g_3^1$  if and only if the canonical model of  $C$  contains 3 colinear points or, more generally, meets a line  $L$  in a divisor of degree 3 (not necessarily 3 distinct points). When  $C$  is the intersection of quadrics, this cannot happen, since the line  $L$  would have to be contained in all the quadrics that contain  $C$ . Thus, in this case,  $C$  has no  $g_3^1$ .

What about  $g_4^1$ s? Again invoking the geometric Riemann-Roch theorem, a divisor of degree 4 moving in a pencil lies in a 2-plane; so the question is, does  $C \subset \mathbb{P}^4$  contain a divisor of degree 4, say  $D = p_1 + \dots + p_4 \subset C$ , that lies in a plane  $\Lambda$ ? Supposing this is so, we consider the restriction map

$$H^0(\mathcal{I}_{C/\mathbb{P}^4}(2)) \rightarrow H^0(\mathcal{I}_{D/\Lambda}(2)).$$

By what we have said, the left hand space is 3-dimensional. We will show that the right-hand space is 2-dimensional, so that one of the quadrics vanishes identically on  $\Lambda$ .

**Lemma 3.1.** *Let  $\Gamma \subset \mathbb{P}^2$  be any scheme of dimension 0 and degree 4. Either  $\Gamma$  is contained in a line  $L \subset \mathbb{P}^2$ , or  $\Gamma$  imposes independent conditions on quadrics, that is,  $h^0(\mathcal{I}_{\Gamma/\mathbb{P}^2}(2)) = 2$ .*

**Proof.** We will do this in case  $\Gamma$  is reduced, that is, consists of four distinct points; the reader is asked to supply the analogous argument in the general case in Exercise 4.8. Suppose to begin with that  $\Gamma$  fails to impose independent conditions on quadrics, and let  $q \in \mathbb{P}^2$  be a general point. Since we are assuming that  $h^0(\mathcal{I}_{\Gamma/\mathbb{P}^2}(2)) \geq 3$ , we see that there are at least two conics  $C', C'' \subset \mathbb{P}^2$  containing  $\Gamma \cup \{q\}$ . By Bézout's theorem, these two conics have a component in common, which can only be a line  $L$ ; thus we can write  $C' = L \cup L'$  and  $C'' = L \cup L''$  for some pair of distinct lines  $L', L'' \subset \mathbb{P}^2$ . The intersection  $C' \cap C''$  thus consists of the line  $L$  and the single point  $L' \cap L''$ . Since this must contain  $\Gamma \cup \{q\}$ , and  $q$  does not lie on the line joining any two points of  $\Gamma$ , we conclude that  $L' \cap L'' = \{q\}$  and hence  $\Gamma \subset L$ .  $\square$

**Cheerful Fact 3.2.** *Lemma 3.1 is the first case of a more general statement: If  $n \leq 2d + 1$  points in the plane fail to impose independent conditions on forms of degree  $d$ , then  $d + 2$  of the points lie on a line. See [Eisenbud et al. 1996, p. 302] for a proof.*

It follows that the 2-plane  $\Lambda$  spanned by  $D$  must be contained in one of the quadrics  $Q \subset \mathbb{P}^4$  containing  $C$ . This implies in particular that the quadric is singular: If  $V = \mathbb{C}^3 \subset \mathbb{C}^5$  is a 3-dimensional subspace of a 5-dimensional inner-product space, then  $V$  meets its orthogonal space in a line, which is a singular point of the corresponding quadric.

Thus  $Q$  is a cone over a quadric in  $\mathbb{P}^3$ , and it is ruled by the (one or two) families of 2-planes it contains, which are the cones over the (one or two) rulings of the quadric in  $\mathbb{P}^3$ . The argument above shows that the existence of a  $g_4^1$ s on  $C$  in this case implies the existence of a singular quadric containing  $C$ .

Conversely, suppose that  $Q \subset \mathbb{P}^4$  is a singular quadric containing  $C = Q_1 \cap Q_2 \cap Q_3$ . Now say  $\Lambda \subset Q$  is a 2-plane. If  $Q'$  and  $Q''$  are quadrics that, with  $Q$  span the space of quadrics containing  $C$ , then we can write

$$\Lambda \cap C = \Lambda \cap Q' \cap Q'',$$

from which we see that  $D = \Lambda \cap C$  is a divisor of degree 4 on  $C$ , and so has  $r(D) = 1$  by the geometric Riemann-Roch theorem. Thus, the rulings of singular quadrics containing  $C$  cut out on  $C$  pencils of degree 4; and every pencil of degree 4 on  $C$  arises in this way.

Does  $C$  lie on singular quadrics? There is a  $\mathbb{P}^2$  of quadrics containing  $C$ —a 2-plane in the space  $\mathbb{P}^{14}$  of quadrics in  $\mathbb{P}^4$ —and the family of singular quadrics consists of a hypersurface of degree 5 in  $\mathbb{P}^{14}$ , called the *discriminant* hypersurface. By Bertini's theorem, not every quadric containing  $C$  is singular. Thus the set of singular quadrics containing  $C$  is a plane curve  $B$  cut out by a quintic equation. So  $C$  does indeed have a  $g_4^1$ , and is expressible as a 4-sheeted cover of

$\mathbb{P}^1$ . Moreover, each singular quadric contributes either 1 or 2  $g_4^1$ s, depending on whether it has rank 3 or 4. In sum, we have proven:

**Proposition 3.3.** *Let  $C \subset \mathbb{P}^4$  be a canonical curve, and assume  $C$  is the complete intersection of three quadrics in  $\mathbb{P}^4$ . Then  $C$  may be expressed as a 4-sheeted cover of  $\mathbb{P}^1$  in a one-dimensional family of ways, and there is a map from the set  $W_4^1(C)$  of  $g_4^1$ s on  $C$  to a plane quintic curve  $B$ , whose fibers have cardinality 1 or 2.*

One can go further and ask about the geometry of the plane curve  $B$  and how it relates to the geometry of  $C$ . The list of possibilities is given in [Arbarello et al. 1985, p. 274].

**3.2. Second case: the intersection of the quadrics is a surface.** We will show that the intersection must contain an irreducible, nondegenerate surface. This follows from Fulton's Elementary Bézout theorem:

**Theorem 3.4.** [Fulton 1984] *Let  $Z_1, \dots, Z_k \subset \mathbb{P}^n$  be hypersurfaces of degrees  $d_1, \dots, d_k$ . If  $\Gamma_1, \dots, \Gamma_m$  are the irreducible components of the intersection  $\bigcap_1^k Z_j$ , then*

$$\sum_{\alpha=1}^m \deg(\Gamma_\alpha) \leq \prod_{i=1}^k d_i.$$

**Proof.** We do induction on  $k$ , the result being trivial for  $k = 1$ . Assuming that the result is true for  $k - 1$ , we consider the irreducible components  $V_i$  of  $\bigcap_1^{k-1} Z_j$ . If  $Z_k$  contains  $V_i$ , then  $V_i$  is again a component of  $\bigcap_1^k Z_j$ . Otherwise,  $V_i \cap Z_k$  is a union of components whose degrees sum to  $d_k \deg V_i$ . Thus the sum of the degrees of the components of  $\bigcap_1^k Z_j$  is at most  $d_i$  times the sum of the degrees of components of  $\bigcap_1^{k-1} Z_j$ , as required.  $\square$

Returning to the canonical curve  $C \subset \mathbb{P}^4$ , suppose that the intersection  $X = Q_1 \cap Q_2 \cap Q_3$  of the three quadrics containing  $C$  has dimension 2. If  $C$  were a component of  $X$ , then the sum of the degrees of the irreducible components of  $X$  would be strictly greater than 8, which Fulton's theorem doesn't allow. Thus  $C$  must be contained in a 2-dimensional irreducible component  $S$  of  $X$ , and this surface  $S$  is necessarily nondegenerate.

We will return to this surface in Chapter 17, where we develop the theory of rational normal scrolls. Here is the result:

**Theorem 3.5.** *Let  $C \subset \mathbb{P}^4$  be a canonical curve of genus 5. Then  $C$  lies on exactly three quadrics, and either*

- (1)  *$C$  is the intersection of these quadrics; in which case  $C$  is not trigonal, and the variety  $W_4^1(C)$  of expressions of  $C$  as a 4-sheeted cover of  $\mathbb{P}^1$  is a two-sheeted cover of a plane quintic curve; or*

- (2) The quadrics containing  $C$  intersect in a cubic surface that is a nonsingular rational normal scroll; in this case,  $C$  has a unique  $g_3^1$ , and the variety  $W_4^1(C)$  consists of the union of two copies of  $C$  meeting at two points.

**Cheerful Fact 3.6.** In the second case of the theorem, the ideal of the curve has the form  $I_C = (Q_1, Q_2, Q_3, F_1, F_2)$  where the  $Q_i$  are quadrics and the  $F_i$  are cubic forms. The quadrics  $Q_i$  cut out a surface scroll, which must be smooth by Corollary 4.10).

Moreover, there are 2 linear relations among the  $Q_i$ , and we shall see in Chapter 18 that this is a special case of the Eagon-Northcott complex. The generators of  $I_C$  above may be written as the  $4 \times 4$  Pfaffians of a skew-symmetric  $5 \times 5$  matrix of the form

$$\begin{pmatrix} 0 & g_1 & g_2 & \ell_0 & \ell_1 \\ -g_1 & 0 & g_3 & \ell_2 & \ell_3 \\ -g_2 & -g_3 & 0 & \ell_3 & \ell_4 \\ -\ell_0 & -\ell_2 & -\ell_3 & 0 & 0 \\ -\ell_1 & -\ell_3 & -\ell_4 & 0 & 0 \end{pmatrix}$$

where  $\ell_0, \dots, \ell_3$  are linear forms and  $g_1, g_2, g_3$  are quadrics. The  $2 \times 2$  minors of the matrix of linear forms in the last two columns are the three quadrics  $Q_i$  contained in the ideal of  $C$ , and those two columns are the linear relations on the  $Q_i$  mentioned above. The columns of the whole  $5 \times 5$  matrix generate the syzygies of  $I_C$ . Moreover, the  $4 \times 4$  Pfaffians of any sufficiently general matrix of this form define a trigonal canonical curve (see [Buchsbaum and Eisenbud 1977a]).

## 4. Exercises

**Exercise 4.1.** Let  $C$  be a smooth projective nonhyperelliptic curve of genus 4, and  $|D|$  a  $g_3^1$  on  $C$ . Show that the following are equivalent:

- (1)  $D \sim K - D$ .
- (2) the multiplication map  $\mu : H^0(D) \otimes H^0(K - D) \rightarrow H^0(K)$  fails to be surjective
- (3) the unique quadric  $Q$  containing the canonical curve of  $C$  is singular; and
- (4)  $|D|$  is the unique  $g_3^1$  on  $C$ .

Hint: recall that the  $g_3^1$ s on  $C$  are cut by the rulings of the quadric  $Q$ .

**Exercise 4.2.** Let  $C$  again be a smooth projective nonhyperelliptic curve of genus 4 whose canonical model lies on a smooth quadric. We have seen that  $C$  is birational to a quintic plane curve  $C_0$  with two nodes  $p, q \in C_0$ . Show that the canonical series of  $C$  is cut out by the system of plane conic curves passing

through  $p$  and  $q$  in the sense that if  $D$  is a curve meeting  $C$  at each node and transverse to each branch at the nodes, then the sum of the points of  $D \cap C$  other than the nodes is a canonical divisor.

Hint: Try blowing up the plane at the nodes. Look at Section 1.1 if you get stuck.

**Exercise 4.3.** Let  $C$  be a smooth projective nonhyperelliptic curve of genus 4 and let  $D$  be a general divisor of degree 7 on  $C$ . By the  $g+3$  theorem (Theorem 5.1),  $h^0(D) = 4$  and the map  $\phi_D : C \rightarrow \mathbb{P}^3$  is an embedding. Show that the image  $C \subset \mathbb{P}^3$  does not lie on any quadric surfaces, but does lie on two cubic surfaces  $S$  and  $T$ ; describe the intersection  $S \cap T$ .

Hint: For the first part: if the curve  $C \subset \mathbb{P}^3$  lay on a quadric, what would be its class? For the second, if  $S \cap T = C \cup D$ , calculate the degree and genus of  $D$  (see Chapter 16 if you get stuck).

**Exercise 4.4.** In the setting of the preceding exercise, suppose now that  $C$  is hyperelliptic. Show that in this case the image of  $C$  under the map  $\phi_D : C \rightarrow \mathbb{P}^3$  does lie on a quadric surface  $Q$ , and in fact is a curve of type  $(2, 5)$  on  $Q$ . Show also that if  $D$  is of the form  $D \sim 2g_2^1 + p + q + r$  then the quadric surface  $Q$  is singular, and the image curve  $\phi_D(C)$  has a triple point at the vertex of  $Q$ .

Hint: Show that each of the divisors  $E$  of the  $g_2^1$  span a line in  $\mathbb{P}^3$ .

**Exercise 4.5.** Consider a space  $M_{d,g}^r$  parametrizing  $g_d^r$ s on curves of genus  $g$ ; that is,

$$M_{d,g}^r = \{(C, L) \mid C \text{ a smooth curve of genus } g, L \in \text{Pic}^d(C) \text{ and } h^0(L) \geq r+1\}.$$

The analysis of this chapter shows that  $M_{3,4}^1$  is a two-sheeted cover of  $M_4$ . Show that it is in fact irreducible.

Hint: Consider the incidence correspondence

$$\Gamma := \{(Q, L) \in \mathbb{P}^9 \times \mathbb{G}(1, 3) \mid L \subset Q\}$$

where  $\mathbb{P}^9$  is the space of quadrics  $Q \subset \mathbb{P}^3$ .

**Exercise 4.6.** The arguments in the chapter show that the canonical model of a nonhyperelliptic trigonal curve of genus 5 lies on an irreducible, nondegenerate cubic surface  $S \subset \mathbb{P}^4$ . In Chapter 17, we'll see that such a surface is either smooth or a cone over a twisted cubic curve. Show that the latter case cannot occur.

Hint: by projection, show that  $C$  would have to be double at the vertex of the cone.

**Exercise 4.7.** Let  $C$  be a smooth projective curve of genus 5. The  $g+3$  theorem (Theorem 5.1) says that  $C$  admits an embedding in  $\mathbb{P}^3$  as a curve of degree 8. Does it admit an embedding of degree 7?

Hint: show that the linear series  $|K_C - p|$  is very ample if and only if  $C$  is not trigonal.

**Exercise 4.8.** Complete the proof of Lemma 3.1.

Hint: consider separately the cases where  $\Gamma$  contains a fat point (that is, the scheme defined by the square of the maximal ideal at a point) or is curvilinear (that is, has Zariski tangent space of dimension at most 1 everywhere).

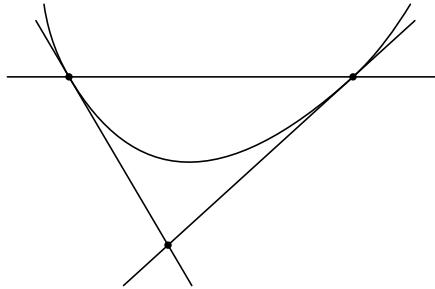


## Hyperplane sections of a curve

One way to study a curve in  $\mathbb{P}^r$  is to use properties of its hyperplane sections. In this chapter, we will take up the question: if  $C \subset \mathbb{P}^r$  is a reduced, irreducible and nondegenerate curve, what can we say about the geometry of the points in the hyperplane section  $\Gamma = C \cap H$  of  $C$ , and what more can we say about a general hyperplane section?

In this chapter we prove and apply a result originally due to Castelnuovo, that every subset of  $r$  points in a general hyperplane section of  $C$  spans the hyperplane; that is, any  $r$  points of  $\Gamma$  are linearly independent. In this situation, we say that the points are in *linearly general position* as in Section 1.0.4. This holds for smooth curves in arbitrary characteristic (but see Exercise 4.4 for interesting counterexamples involving singular curves in positive characteristic). This is followed by a series of applications, including the famous theorem of Castelnuovo bounding the genus of a curve in terms of its degree, and, through that result, a proof that canonical curves and linearly normal curves “of high degree” are arithmetically Cohen-Macaulay (Theorem 2.4).

In Chapter 11, we will explain and prove a still stronger general position result, which requires characteristic 0. We give applications to the irreducibility of fiber products, to the Hilbert functions of subsets of the general hyperplane section, and to sums of linear series.



**Figure 1.** A stationary secant.Silvio: the horizontal line plays a different role from the other lines; should be red—it's the stationary secant.

## 1. Linearly general position

In this section we will allow our algebraically closed ground field to have arbitrary characteristic.

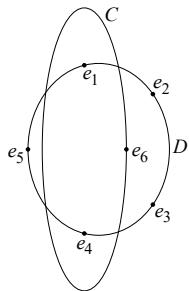
An irreducible and reduced curve  $C \subset \mathbb{P}^r$  over an algebraically closed field  $k$ , other than a line, is called *strange* if all the tangent lines at smooth points of  $C$  meet in a point  $p \in \mathbb{P}^r$ . If this condition seems somewhat...strange, that may be because no such curves exist in characteristic 0: if it were the case that all the tangent lines at smooth points of  $C$  met in a point, then the projection of  $C$  from that point would have derivative 0 everywhere and hence be constant. But strange curves do exist in characteristic  $p > 0$ , albeit rarely: Pierre Samuel showed that the only *smooth* curve that is strange (in any characteristic) is the conic in characteristic 2; a proof is given in [Hartshorne 1977, Theorem IV.3.9]. Without the smoothness hypothesis there are more examples; see Exercise 4.4.

Given this language, we can state our first result. Recall that if  $C \subset \mathbb{P}^n$  is a reduced curve then, by Bertini's theorem, a general hyperplane meets  $C$  transversely.

**Theorem 1.1.** ([Rathmann 1987, Lemma 1.1]) Suppose that  $n \geq 3$ . If  $C \subset \mathbb{P}^r$  is a nondegenerate irreducible, reduced curve, that is not strange and  $H$  is a general hyperplane, then any  $r$  points of  $H \cap C$  are linearly independent.

We postpone the proof to introduce some related definitions.

For example, the theorem says that for  $C \subset \mathbb{P}^3$ , the general hyperplane section does not contain a line that meets  $C$  three or more times—a *multiplican*. Another special configuration which is not present in a general hyperplane section, as it turns out, is any pair of points  $p \neq q$  on  $C$  such that the tangent lines  $\mathbb{T}_p(C), \mathbb{T}_q(C)$  to  $C$  at  $p$  and  $q$  intersect—what was classically called a stationary secant (Figure 4)).



**Figure 2.** A smooth rational quintic in  $\mathbb{P}^3$  with a unique 4-fold secant line can be obtained as the image of a plane conic  $C$  by the rational map of  $\mathbb{P}^2$  to  $\mathbb{P}^3$  induced by the 4 cubics through 6 points  $e_1, \dots, e_6$ , exactly one of which lies on  $C$ ; the image of the unique conic  $D$  through the other 5 points becomes the 4-secant line to the image of  $C$ . Silvio: make the two conics different colors

1

More formally we define two subsets of the symmetric square  $C_2$  of  $C$ :

$$\text{mult}(C) := \{(p, q) \in C_2 \mid \overline{p, q} \text{ meets } C \text{ in a scheme of length } \geq 3\};$$

$$\text{stat}(C) := \{(p, q) \in C_2 \setminus \Delta \mid \text{the tangent lines to } C \text{ at } p, q \text{ have nonempty intersection}\},$$

where  $\overline{p, q}$  denotes the secant line through  $p, q$  (or the tangent line if  $p = q$ ) and  $\Delta \subset C_2$  is the diagonal. Clearly if  $C$  is strange or planar then  $\text{stat}(C) = C_2$ . If  $C$  is planar with  $\deg C > 2$  then  $\text{mult}(C) = C_2$  as well. It turns out that these are the only cases:

**Proposition 1.2.** *If  $C \subset \mathbb{P}^r$  is a reduced, irreducible, curve that is neither planar nor strange, then  $\text{mult}(C) \subset C_2$  and  $\text{stat}(C) \subset C_2 \setminus \Delta$  are closed subsets of dimension  $\leq 1$ .*

**Proof.** To show that  $\text{mult}(C)$  is closed, consider the projection map,

$$\{((p, q), r) \in C_2 \times \mathbb{P}^r \mid r \in \overline{p, q}\} \rightarrow C_2$$

sending  $((p, q), r)$  to  $(p, q)$ . By the semicontinuity of the degree in proper families of constant fiber dimension, the set of points lying in fibers of length  $\geq 3$  is closed, and  $\text{mult}(C)$  is the image of this set. Because the map is finite,  $\text{mult}(C)$  is closed as well.

<sup>1</sup>After [Kronecker 1882] proved that every algebraic set in  $\mathbb{P}^n$  is set-theoretically the intersection of  $n + 1$  hypersurfaces, [Vahlen 1891] claimed that Kronecker's result was sharp because the rational quintic space-curve described in Figure ?? could not be the intersection of 3 hypersurfaces. Vahlen played a role in the rise of the Nazi party, and perhaps for that reason [Perron 1941] produced 3 hypersurfaces intersecting precisely in this curve. It is now known that every algebraic set in  $\mathbb{P}^n$  over an infinite field can be written set-theoretically as the intersection of just  $n$  hypersurfaces [Eisenbud and Evans 1973].

Similarly since

$$\{((p, q), r) \in (C_2 \setminus \Delta) \times \mathbb{P}^r \mid r \in T_p(C) \cup T_q(C)\}$$

is closed in  $(C_2 \setminus \Delta) \times \mathbb{P}^r$ , its image under the projection to the first factor, which is  $\text{stat}(C)$ , is closed in  $C_2 \setminus \Delta$

Suppose, contrary to the Proposition, that  $\dim \text{stat}(C) > 1$ . Since  $\text{stat}(C)$  is a closed subset of the irreducible variety  $C_2 \setminus \Delta$  it follows that  $\text{stat}(C) = C_2 \setminus \Delta$ , so every pair of tangent lines meet. Any three lines in projective space that meet pairwise must be either coplanar or meet in a point. Thus if  $C$  is not strange, then all the tangents to  $C$  lie in a common plane, and  $C$  is planar, contradicting our hypothesis and proving that  $\dim \text{stat}(C) \leq 1$ .

Finally suppose, contrary to the Proposition, that  $\dim \text{mult}(C) > 1$ . As in the previous argument, this implies that every secant is a multisecant. We will show in this case that  $\dim \text{stat}(C) = 2$ , a contradiction.

To this end, consider the projections  $\pi_p : C \rightarrow \mathbb{P}^{r-1}$  and the set

$$I = \{(p, r, r') \in C^3 \mid p, r, r' \text{ are distinct and } \pi_p(r) = \pi_p(r'), \text{ is a smooth point of } \pi_p(C)\}. \blacksquare$$

The tangent lines  $T_r(C)$  and  $T_{r'}(C)$  both project from  $p$  to the tangent line  $T_{\pi_p(r)}(\pi_C)$ , and thus they lie in the 2-plane spanned by  $p$  and this line; it follows that  $(r, r') \in \text{stat}(C)$ . The projection  $I \rightarrow \overline{C_2} : (p, r, r') \mapsto (r, r')$  is finite since the secant line  $r, r'$  meets  $C$  in only finitely many points, so  $\dim I \leq \dim \text{stat}(C)$ .

If the projection from some point  $p$  were everywhere ramified, then all tangents to  $C$  would pass through  $p$ , and  $C$  would be strange. Since also every secant to  $C$  is a multisecant it follows that for every smooth point  $p \in C$  there is an open set of points  $q \in \pi_p(C)$  such that the fiber  $\pi_p^{-1}(q)$  contains two distinct points of  $C$ . Thus the projection  $I \rightarrow C : (p, r, r') \mapsto p$  maps  $I$  onto an open set of  $C$ , with 1-dimensional fibers, so  $2 = \dim I \leq \dim \text{stat}(C)$  as claimed.  $\square$

**Proof of Theorem 1.1.** First suppose that  $n = 3$ , and consider the incidence correspondence

$$I := \{((p, q), H) \in \text{mult}(C) \times \mathbb{P}^{3*} \mid \overline{p, q} \in H\}.$$

Since  $\dim \overline{p, q} = 1$ , the fibers of the projection of  $I$  to  $\text{mult}(C)$  have dimension 1, so  $\dim I = 1 + \dim \text{mult}(C) \leq 2$ . Consequently the image of the projection of  $I$  to  $\mathbb{P}^{3*}$  has dimension  $\leq 2$ , so there is an open set of planes that contain no multisecants, proving the theorem in this case.

We next do induction on  $n \geq 4$ . We will show first that the image  $C'$  of  $C$  under the projection  $\pi_p : C \rightarrow C'$  from a general point of  $C$  is not strange. Since  $C \subset \mathbb{P}^r$  is neither planar nor strange Proposition 1.2 shows that we may

choose two smooth points  $r, r' \in C \subset \mathbb{P}^r$  such that  $T_r(C) \cap T_{r'}(C) = \emptyset$ , and thus  $T_r(C)$  and  $T_{r'}(C)$  together span a 3-plane  $L$ . Since  $C$  is nondegenerate we may choose a point  $p \in C$  outside  $L$ , and it follows that  $\pi_p$  restricted to  $L$  is an isomorphism. Thus  $T_{\pi_p(r)}(C') \cap T_{\pi_p(r')}(C') = \emptyset$ , so  $C'$  is not strange, and since  $n \geq 4$  the curve  $C'$  is not planar either.

Arguing by contradiction, suppose that the general hyperplane section of  $C$  contains a set of  $r$  linearly dependent points. Consider the closed subsets

$$I_1 \subset I \subset \{(p, H) \in C \times \mathbb{P}^{n*}\}$$

where  $I$  is defined by the conditions that  $p \in H$  and  $I_1$  is defined by the condition that, in addition, there exist  $p_2, \dots, p_r \in H$  such that  $p, p_2, \dots, p_r$  are dependent. Both  $I_1$  and  $I$  are closed subsets.

By hypothesis, both  $I_1$  and  $I$  project onto open sets of  $\mathbb{P}^{n*}$ , so they have the same dimension. But  $I$  is irreducible: it projects onto an open subset of  $C$  with fibers isomorphic to  $\mathbb{P}^{r-1}$ . Thus  $I_1 = I$ , and  $p$  is part of a dependent set  $\{p, p_2, \dots, p_r\}$  in a general hyperplane containing  $p$ .

Let  $H'$  be a general hyperplane in  $\mathbb{P}^{n-1}$  and let  $H = \pi_p^{-1}(H')$ , which is a general hyperplane containing  $p$ . The intersection  $H \cap C$  contains  $r$  dependent points  $\{p, p_2, \dots, p_r\}$ , and it follows that  $\pi_p(p_2), \dots, \pi_p(p_r)$  are dependent points of  $H' \cap C'$ . This contradicts the induction hypothesis, and proves that the general hyperplane section of  $C$  is in linearly general position.  $\square$

## 2. Castelnuovo's theorem

Clifford's theorem gives a complete and sharp answer to the question, “what linear series can exist on a curve of genus  $g$ ?”. But maybe that wasn't the question we meant to ask! After all, we're interested in describing curves in projective space as images of abstract curves  $C$  under maps given by linear systems on  $C$ . Observing that the linear series that achieve equality in Clifford's theorem give maps to  $\mathbb{P}^r$  that are 2 to 1 onto a rational curve, we might hope that we would have a different—and more meaningful—answer if we restrict our attention to linear series  $D = (\mathcal{L}, V)$  for which the associated map  $\phi_D$  is at least birational onto its image.

A classical result of Castelnuovo does exactly that: it gives a sharp bound on the arithmetic genus of a reduced, irreducible, nondegenerate curve of degree  $d$  in  $\mathbb{P}^r$ . To state it, for positive integers  $d$  and  $r$ , let

$$M := M(d, r) := \lfloor (d - 1)/(r - 1) \rfloor,$$

so that

$$d - 1 = M(r - 1) + \epsilon \quad \text{for some } \epsilon = \epsilon(d, r) \text{ with } 0 \leq \epsilon \leq r - 2.$$

**Theorem 2.1** (Castelnuovo's bound). *Let  $C \subset \mathbb{P}^r$  be a reduced, irreducible, non-degenerate curve of degree  $d$ . With  $M$  and  $\epsilon$  defined as above,*

$$p_a(C) \leq \pi(d, r) := \frac{M(M-1)}{2}(r-1) + M\epsilon.$$

*Moreover, if  $p_a(C) = \pi(d, r)$  then  $C$  is arithmetically Cohen-Macaulay and every hyperplane section  $H \cap C$  has Hilbert function  $\dim(R_{H \cap C})_m = \min\{m(r-1), d\}$ .*

We will say that a curve achieving the bound is a *Castelnuovo curve*.

**2.1. Proof of Castelnuovo's bound.** The idea of Castelnuovo's proof is simple and beautiful. To start with, the hypothesis that  $C$  is nondegenerate in  $\mathbb{P}^r$  says that  $h^0(\mathcal{O}_C(1)) \geq r+1$ . We'd like to use this and the Riemann-Roch formula, in the form

$$g = d + 1 - h^0(\mathcal{O}_C(1)) + h^1(\mathcal{O}_C(1)),$$

but this doesn't work because we have a priori no way to estimate  $h^1(\mathcal{O}_C(1))$ .

Castelnuovo's solution is to derive lower bounds not just on  $h^0(\mathcal{O}_C(1))$  but on  $h^0(\mathcal{O}_C(m))$  for all  $m$ ; since we know that  $h^1(\mathcal{O}_C(m)) = 0$  for large  $m$ , a lower bound on  $h^0(\mathcal{O}_C(m))$  for large  $m$  will translate directly into an upper bound on  $g$ .

How can we go about estimating  $h^0(\mathcal{O}_C(m))$ ? The answer is to derive lower bounds on the successive differences  $h^0(\mathcal{O}_C(m)) - h^0(\mathcal{O}_C(m-1))$ , by letting  $\Gamma = C \cap H$  be a general hyperplane section of  $C$ , and viewing  $H^0(\mathcal{O}_C(m-1))$  as the subspace of  $H^0(\mathcal{O}_C(m))$  consisting of sections vanishing on  $\Gamma$ . The estimates we get may seem crude—not surprisingly, considering how little we know about  $\Gamma$ —but remarkably, the bound we ultimately derive on the genus of  $C$  turns out to be sharp!

For the proof, the following definition will be convenient:

**Definition 2.2.** If  $\mathcal{V} = (V, \mathcal{L})$  is a linear system on a variety  $X$  and  $\Gamma$  is a subscheme then the *number of conditions imposed by  $\Gamma$  on  $\mathcal{V}$*  (that is, the number of independent linear conditions for a section in  $V$  to vanish identically on  $\Gamma$ ) is the dimension of the image of  $V$  in  $H^0(\mathcal{L}|_\Gamma) = H^0(\mathcal{L} \otimes \mathcal{O}_\Gamma)$ ; or, numerically,

$$\dim(V) - \dim(V \cap H^0(\mathcal{L} \otimes \mathcal{I}_{\Gamma/X})).$$

Thus, for example, if  $\Gamma \subset \mathbb{P}^r$ , then the number of conditions imposed by  $\Gamma$  on  $H^0(\mathcal{O}_{\mathbb{P}^r}(m))$  is the value  $h_\Gamma(m)$  of the Hilbert function of  $\Gamma$  at  $m$ . Note that in case  $\Gamma$  is zero-dimensional, the number of conditions imposed by  $\Gamma$  on a linear system  $V$  is necessarily less than or equal to the degree  $d$  of  $\Gamma$ ; if it is equal we say that  $\Gamma$  *imposes independent conditions on  $V$* .

**Proof of Theorem 2.1.** Suppose that  $C \subset \mathbb{P}^r$  is an irreducible, nondegenerate curve. For large  $m$  we have  $h^0(\mathcal{O}_C(m)) = md - p_a(C) + 1$ , so to bound the genus from above we must bound  $h^0(\mathcal{O}_C(m))$  from below.

Let  $\Gamma = C \cap H$  be a general hyperplane section of  $C$ , and let  $V_m \subset H^0(\mathcal{O}_C(m))$  be the linear series cut on  $C$  by hypersurfaces of degree  $m$  in  $\mathbb{P}^r$ , that is, the image of the restriction map

$$H^0(\mathcal{O}_{\mathbb{P}^r}(m)) \rightarrow H^0(\mathcal{O}_C(m)).$$

The number of conditions imposed by  $\Gamma$  on  $V_m$  is the rank of the restriction map  $\rho_m : V_m \rightarrow \mathcal{O}_\Gamma(m)$ . The number of conditions imposed by  $\Gamma$  on  $H^0(\mathcal{O}_C(m))$  is the rank of the restriction map from this potentially larger space, and thus is at least as large. This accounts for the inequality in the following:

$$\begin{aligned} h^0(\mathcal{O}_C(m)) - h^0(\mathcal{O}_C(m-1)) &= \# \text{ of conditions imposed by } \Gamma \text{ on } H^0(\mathcal{O}_C(m)) \\ &\geq \# \text{ of conditions imposed by } \Gamma \text{ on } V_m \\ &= \# \text{ of conditions imposed by } \Gamma \text{ on } H^0(\mathcal{O}_{\mathbb{P}^r}(m)) \\ &= h_\Gamma(m). \end{aligned}$$

Summing these relations, we see that the dimension  $h^0(\mathcal{O}_C(m))$  is bounded below by

$$h^0(\mathcal{O}_C(m)) \geq \sum_{k=0}^m h_\Gamma(k).$$

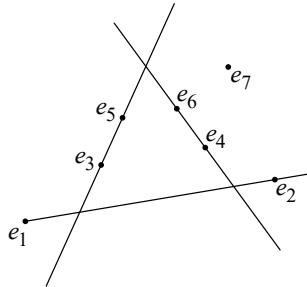
In order to bound the genus  $C$  from above, we will bound the Hilbert function of its hyperplane section  $\Gamma$  from below; and for this, we need to know something about the geometry of  $\Gamma$ . In fact, all we need to know is Theorem 1.1, which says that the points of  $\Gamma$  are in linearly general position!

**Proposition 2.3.** *If  $\Gamma \subset \mathbb{P}^r$  is a collection of  $d$  points in linearly general position that span  $\mathbb{P}^r$ , then*

$$h_\Gamma(m) \geq \begin{cases} mr + 1 & \text{if } m \leq M(d, r+1) \\ d & \text{otherwise.} \end{cases}$$

One way to understand the bound  $mr + 1$  is to realize that if  $\Gamma$  is any finite subscheme of a rational normal curve  $C \subset \mathbb{P}^r$  of degree  $r$ , then  $h_\Gamma(m) = \min\{\deg \Gamma, mr+1\}$  for every  $m$  (Exercise 4.7). Thus the bound in Proposition 2.3 is best possible. On the other hand, sets of points on a rational normal curve are almost the only sets for which the bound is sharp.

**Proof.** Suppose first that  $d \geq mr + 1$ , and let  $p_1, \dots, p_{mr+1} \in \Gamma$  be any subset of  $mr+1$  points. It suffices to show that  $\Gamma' = \{p_1, \dots, p_{mr+1}\}$  imposes independent conditions on  $H^0(\mathcal{O}_{\mathbb{P}^r}(m))$ , that is, for any  $p_i \in \Gamma'$  there is a hypersurface  $X \subset$



**Figure 3.** The case  $d = 3*2+1$ : the point  $e_7$  imposes an additional condition on the cubics (such as the union of the three lines shown) passing through  $e_1, \dots, e_6$ .

$\mathbb{P}^r$  of degree  $m$  containing all the points  $p_1, \dots, \hat{p}_i, \dots, p_{mr+1}$  but not containing  $p_i$ .

To construct such an  $X$ , group the  $mr$  points of  $\Gamma' \setminus \{p_i\}$  into  $m$  subsets  $\Gamma_k$  of cardinality  $r$ ; each set  $\Gamma_k$  will span a hyperplane  $H_k \subset \mathbb{P}^r$ , and we can take  $X = H_1 \cup \dots \cup H_m$ .

In the case where  $d < mr + 1$ , we add  $mr + 1 - d$  general points; each one imposes exactly one additional condition on hypersurfaces of degree  $m$ .  $\square$

To complete the proof of Theorem 2.1 we add up the lower bounds in the proposition. To this end, let  $C \subset \mathbb{P}^r$  be as above an irreducible, nondegenerate curve of degree  $d$ , and again set  $M = M(d, r) = \lfloor \frac{d-1}{r-1} \rfloor$ . We have

$$\begin{aligned} h^0(\mathcal{O}_C(M)) &= \sum_{k=0}^M h^0(\mathcal{O}_C(k)) - h^0(\mathcal{O}_C(k-1)) \\ &\geq \sum_{k=0}^M (k(r-1) + 1) \\ &= \frac{M(M+1)}{2}(r-1) + M + 1 \end{aligned}$$

and similarly

$$h^0(\mathcal{O}_C(M+m)) \geq \frac{M(M+1)}{2}(r-1) + M + md + 1$$

For sufficiently large  $m$ , the line bundle  $\mathcal{O}_C(M + m)$  is nonspecial, so by the Riemann-Roch theorem,

$$\begin{aligned} g &= (M + m)d - h^0(\mathcal{O}_C(M + m)) + 1 \\ &\leq (M + m)d - \left( \frac{M(M + 1)}{2}(r - 1) + M + 1 + md \right) + 1 \\ &= M(M(r - 1) + 1 + \epsilon) - \left( \frac{M(M + 1)}{2}(r - 1) + M \right) \\ &= \frac{M(M - 1)}{2}(r - 1) + M\epsilon. \end{aligned}$$

In the case of equality, to show that  $C$  is arithmetically Cohen-Macaulay we must show that  $V_m = H^0(\mathcal{O}_C(m))$  for all  $m$ . Consider the diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & H^0(\mathcal{O}_{\mathbb{P}^r}(m-1)) & \longrightarrow & H^0(\mathcal{O}_{\mathbb{P}^r}(m)) & \longrightarrow & H^0(\mathcal{O}_{\Gamma}(m)) \\ & & \text{surjection} \downarrow & & \text{surjection} \downarrow & & = \downarrow \\ 0 & \longrightarrow & V_{m-1} & \longrightarrow & V_m & \longrightarrow & H^0(\mathcal{O}_{\Gamma}(m)). \\ & & \text{injection} \downarrow & & \text{injection} \downarrow & & = \downarrow \\ 0 & \longrightarrow & H^0(\mathcal{O}_C(m-1)) & \longrightarrow & H^0(\mathcal{O}_C(m)) & \longrightarrow & H^0(\mathcal{O}_{\Gamma}(m)) \end{array}$$

The top and bottom rows are left exact. From the left exactness of the bottom row it follows that the map  $V_{m-1} \rightarrow V_m$  is an injection. From the left exactness of the top row it follows that this map is the kernel of the restriction map  $V_m \rightarrow H^0(\mathcal{O}_{\Gamma}(m))$ . Thus the middle row is also left exact, and we see that the number of conditions that  $\Gamma$  imposes on  $V_m$  is equal to  $\dim V_m - \dim V_{m-1}$ .

If  $g = \pi(d, r)$ , then the inequalities

$$\begin{aligned} &\# \text{ of conditions imposed by } \Gamma \text{ on } H^0(\mathcal{O}_C(m)) \\ &\geq \# \text{ of conditions imposed by } \Gamma \text{ on } V_m, \end{aligned}$$

that is,

$$h^0(\mathcal{O}_C(m)) - h^0(\mathcal{O}_C(m-1)) \geq \dim V_m - \dim V_{m-1}$$

from the first part of the proof must be equalities for all  $m$ . For large  $m$  the restriction map  $H^0(\mathcal{O}_{\mathbb{P}^r}(m)) \rightarrow H^0(\mathcal{O}_C(m))$  is surjective, so

$$\begin{aligned} & \sum_{k=0}^m (h^0(\mathcal{O}_C(k)) - h^0(\mathcal{O}_C(k-1))) \\ &= h^0(\mathcal{O}_C(m)) = \dim V_m \\ &= \sum_{k=0}^m (\dim V_k - \dim V_{k-1}). \end{aligned}$$

However, for each  $k$  we have

$$\dim V_k - \dim V_{k-1} \geq h^0(\mathcal{O}_C(k)) - h^0(\mathcal{O}_C(k-1)),$$

and thus we must have equality for all  $k$ , whence  $\dim V_k = h^0(\mathcal{O}_C(k))$  for all  $k$ ; that is,  $C$  is arithmetically Cohen-Macaulay.

Always supposing that  $p_a(C) = \pi(d, r)$  the inequalities in Proposition 2.3 must be equalities, so for a general hyperplane section  $\Gamma = H \cap C$  we have  $h_\Gamma(m) = \min\{m(r-1)+1, d\}$  for all  $m \geq 0$ . However, since  $C$  is arithmetically Cohen-Macaulay,  $h_\Gamma(m) = h^0(\mathcal{O}_C(m)) - h^0(\mathcal{O}_C(m-1))$  for every hyperplane section  $\Gamma$ , completing the proof.  $\square$

**2.2. Consequences and special cases.** First of all, in the case  $r = 2$  we have  $\pi(d, r) = \binom{d-1}{2}$ , so every plane curve (smooth or not) is a Castelnuovo curve.

In case  $r = 3$ , we have

$$\pi(d, r) = \begin{cases} \left(\frac{d-2}{2}\right)^2, & \text{if } d \text{ is even; and} \\ \left(\frac{d-1}{2}\right)\left(\frac{d-3}{2}\right), & \text{if } d \text{ is odd} \end{cases}$$

which we can recognize as the genus of curves of type  $(\frac{d}{2}, \frac{d}{2})$  on a quadric surface in case  $d$  is even, and the genus of curves of type  $(\frac{d+1}{2}, \frac{d-1}{2})$  on a quadric when  $d$  is odd. Thus we see that the Castelnuovo bound is sharp when  $r = 3$  as well.

Indeed, the Castelnuovo bound is sharp for all  $d$  and  $r$ ; we'll prove this, and describe explicitly the curves that achieve it, in Chapter 17.

**Corollary 2.4.** *Suppose that  $C \subset \mathbb{P}^r$  is a smooth curve of arithmetic genus  $p_a$  and degree  $d$  embedded by a complete linear series. The curve  $C$  is Castelnuovo (and thus arithmetically Cohen-Macaulay) if and only if one of the following holds:*

- (1)  $d < 2r$  and  $d \geq 2p_a + 1$ ;
- (2)  $d = 2r$ : The curve  $C$  is a canonical curve;

- (3)  $r = 3, d > 6$ : The curve  $C$  is a divisor on a (smooth or singular) irreducible quadric in the class  $mH$  or  $mH + L$ , where  $H$  is the class of a hyperplane and  $L$  is the class of a line;
- (4) More generally, if  $d > 2r$  then  $C$  lies on a rational normal scroll in a certain range of divisor classes (Theorem 4.5).

**Cheerful Fact 2.5.** Corollary 2.4 remains true with appropriate definitions also for reduced irreducible curves. This can be proven with the techniques developed in Chapters 16 and 17.

**Proof.** Simple arithmetic shows that if  $d \geq 2p_a + 1$  then  $d < 2r$  and  $p_a = \pi(d, r)$ . Moreover if  $p_a = \pi(d, r)$  and  $d < 2r$ , then indeed  $d \geq 2p_a + 1$ , proving item 1.

If  $d = 2r$  then  $\pi(d, r) = r + 1$ , and Clifford's theorem shows that  $C$  is a canonical curve, proving the second item.

Now suppose that  $r = 3, d > 6$  and let  $\Gamma$  be a hyperplane section of  $C$ . Since  $h_\Gamma(2) = \min(2 \cdot 2 + 1, d) = 5$ ,  $\Gamma$  lies on a conic, and since  $C$  is arithmetically Cohen-Macaulay,  $C$  must lie on a quadric. If  $C$  is smooth then arithmetic plus the result of Section 6.1 establishes the result; for the more general case, and the case  $r > 3$ , see Theorem 4.5.  $\square$

The most important special case is that of canonical curves:

**Corollary 2.6.** If  $C \subset \mathbb{P}^{g-1}$  is a canonical curve then the Hilbert function of the homogeneous coordinate ring  $S_C$  of  $C$  depends only on  $g$ , and is given by:

$$\dim(S_C)_n = h^0(\mathcal{O}_C(n)) = \begin{cases} 0 & \text{if } d < 0 \\ 1 & \text{if } d = 0 \\ g & \text{if } d = 1 \\ (2g - 2)n + 1 - g & \text{if } d > 1 \end{cases}$$

**Proof.** Theorem 2.1 implies that the homogeneous coordinate ring of  $C$  can be identified with  $\bigoplus_{n \in \mathbb{Z}} H^0(\mathcal{O}_C(n))$ , and the dimensions of these spaces are determined by the Riemann-Roch theorem.  $\square$

**Cheerful Fact 2.7.** A famous theorem of Gruson and Peskine [Gruson and Peskine 1982] (see [Hartshorne 1982] for an exposition and also the cases of characteristic  $> 0$  and small fields) completes the picture of the possibilities for the degree  $d$  and genus  $g$  of a smooth curve in  $\mathbb{P}^3$ . If the curve does not lie on a plane or a quadric, then the genus satisfies the stronger inequality

$$g \leq \pi_1(d, 3) := \frac{d^2 - 3d}{6} + 1$$

and smooth curves with all such degree and genus exist; they can all be realized as curves on cubic or quartic surfaces.

Note that there can be gaps in the possible genera of curves a given degree: for example  $\pi_1(9, 3) = 10 < \pi(9, 3) = 12$  but there is no curve of degree 9 and genus 11. The full range of possible degrees and genera for curves in  $\mathbb{P}^r$  remains open for larger  $r$ . See for example [Hartshorne 1980] for a summary and a number of conjectures.

### 3. Other applications of linearly general position

**3.1. Existence of good projections.** We can use Theorem 1.1 to show that every smooth curve  $C$  is birational to a nodal plane curve  $C_0 \subset \mathbb{P}^2$ , in many ways.

**Proposition 3.1.** *If  $C \subset \mathbb{P}^r$  is a smooth nondegenerate curve in projective space, let  $\Lambda \cong \mathbb{P}^k \subset \mathbb{P}^r$  be a general  $k$ -plane, and let  $\pi_\Lambda : C \rightarrow \mathbb{P}^{n-k-1}$  be the projection from  $\Lambda$ , restricted to  $C$ . If  $n - k - 1 \geq 3$ —that is,  $k \leq n - 4$ —then  $\pi_\Lambda : C \rightarrow \mathbb{P}^{n-k-1}$  defines an isomorphism of  $C$  onto its image, while if  $n - k - 1 = 2$  (i.e.,  $k = n - 3$ ) then  $\pi_\Lambda$  is birational onto its image, which is a curve with only nodes.*

**Proof.** Recall that the secant variety of  $C$  consists of the union of the lines  $\overline{q, r}$  joining pairs of distinct points  $q, r \in C$ , plus the tangent lines  $\mathbb{T}_q(C)$ ; altogether, these lines form a family, parametrized by the symmetric square  $C_2$  of  $C$ . More precisely every subscheme  $\lambda$  of length 2 in  $\mathbb{P}^r$  spans a line  $\bar{\lambda}$ . The incidence variety

$$I := \{(\lambda, p) \mid \lambda \in C_2 \text{ is a divisor of degree 2 on } C, p \in \bar{\lambda} \subset \mathbb{P}^r\}$$

projects to  $C_2$  with 1-dimensional fibers isomorphic to  $\mathbb{P}^1$ , and thus is irreducible of dimension 3. Its image in  $\mathbb{P}^r$  under the second projection is the secant variety of  $C$ , which is thus irreducible of dimension  $\leq 3$ . It follows that a general  $(n-4)$ -plane does not meet the secant variety, and the first statement of the proposition follows.

If  $n > 3$  then by first projecting from a general  $(n-4)$ -plane inside  $\Lambda$  we may reduce to the case  $n = 3$ , and assume that  $\Lambda$  is a general point of  $\mathbb{P}^3$ . By a variant of the argument above, the union of the tangent lines to  $C$  is a surface, and thus does not contain  $\Lambda$ . It follows that  $\pi_\Lambda$  is locally in the source an analytic isomorphism.

To show that the fibers of  $\pi_\Lambda$  are subschemes of length at most 2, we need to show that  $\Lambda$  does not lie on any multisecant line.

By Theorem 1.1 the family of multisecant lines to  $C$  is a proper subscheme of the irreducible two-dimensional family of secant lines, so the union of the

trisecant lines is at most 2 dimensional, and we see that the fibers of  $\pi_\Lambda$  all have degree  $\leq 2$ . Furthermore, the general fiber of the projection from the incidence correspondence  $I$  to  $\mathbb{P}^3$  is empty or finite, so only a finite number of secant lines contain  $\Lambda$ , and we see that  $\pi_\Lambda$  is birational.

We have shown that the map  $\pi_\Lambda$  is an immersion, and at most two-to-one everywhere; thus the image curve  $C_0 \subset \mathbb{P}^2$  has at most double points, and an analytic neighborhood of each double point consists of two smooth branches. To complete the proof of Proposition 3.1 we have to show that those two branches have distinct tangent lines; that is, that if  $q, r \in C$  are any two points collinear with  $\Lambda$ , then the images of the tangent lines  $\mathbb{T}_q(C)$  and  $\mathbb{T}_r(C)$  in  $\mathbb{P}^2$  are distinct. But if  $\pi_p(\mathbb{T}_q(C)) = \pi_p(\mathbb{T}_r(C))$  then  $\mathbb{T}_q(C)$  and  $\mathbb{T}_r(C)$  lie in a plane, and thus intersect.

Since the family of all secant lines is irreducible of dimension 2, it will suffice to show that not every secant line to  $C$  is a stationary secant or, equivalently, that not every pair of tangent lines to  $C$  meet. We saw in Proposition 1.2 that in the contrary case the curve  $C$  would be either strange or planar, a contradiction.  $\square$

**3.2. The case of equality in Martens' theorem.** Theorem 1.1 allows us to analyze the case of equality in Martens' theorem bounding the dimension of the variety  $W_d^r(C)$  parametrizing divisor classes of degree  $d$  on a curve  $C$  with  $r(D) \geq r$  (Theorem 8.1). To start, recall the statement:

**Theorem 3.2. (Martens)** *If  $C$  is any smooth projective curve of genus  $g$ , then for any  $r > d - g$  we have*

$$\dim W_d^r(C) \leq d - 2r.$$

*Equality holds iff either  $C$  is hyperelliptic or  $d = r = 0$  or  $d = 2g - 2, r = g - 1$ ; in either of the last two cases  $W_d^r$  is a single point.*

Note that the inequality  $\dim W_d^r(C) \leq d - 2r$  is equivalent to the inequality  $\dim C_d^r \leq d - r$ , which is what we actually showed in Chapter 5. This in turn followed by combining the geometric form of Riemann-Roch with an elementary bound on the dimension of the variety of secant planes to a curve in projective space.

Theorem 1.1 allows us to do exactly one better:

**Lemma 3.3. (strong secant plane lemma)** *Let  $C \subset \mathbb{P}^n$  be a smooth, irreducible and nondegenerate curve. If we denote by  $\Sigma_d^r \subset C_d$  the locus of effective divisors  $D$  of degree  $d$  on  $C$  with  $\dim \overline{D} \leq d - r - 1$ , then for any  $d \leq r$  and  $r > 0$ ,*

$$\dim \Sigma_d^r \leq d - r - 1.$$

**Proof.** Consider the incidence correspondence:

$$\Gamma := \{(D, H) \in \Sigma_d^r \times \mathbb{P}^{r^*} \mid \overline{D} \subset H\}.$$

The curve being nondegenerate, the projection map  $\Gamma \rightarrow \mathbb{P}^{r^*}$  is finite. But the fibers of  $\Gamma$  over  $\Sigma_d^r$  have dimension at least  $n - d + r$ ; if we had  $\dim \Sigma_d^r \geq d - r$ , it would follow that  $\dim \Gamma \geq n$ , and hence that the projection map  $\Gamma \rightarrow \mathbb{P}^{r^*}$  is dominant—contradicting Theorem 1.1.  $\square$

**Proof of the case of equality in Martens theorem.** Now, if a curve  $C$  is non-hyperelliptic, we can apply the strong secant plane lemma to the canonical curve. Except for the trivial cases  $d = r = 0$  and  $d = 2g - 2, r = g - 1$ , we can apply Lemma 3.3 to conclude that  $\dim W_d^r(C) \leq d - 2r - 1$ ; it follows that if we have  $\dim W_d^r(C) = d - 2r$  the curve  $C$  in question must be hyperelliptic.  $\square$

Using the case of equality in Martens' theorem, we can analyze equality in Clifford's theorem:

**Corollary 3.4.** *If  $C$  is a smooth curve of genus  $g$  and  $D$  a divisor on  $C$  of degree  $\leq 2g - 2$ , such that  $\deg D = 2r(D)$ , then either  $D = 0$  or  $D = K_C$  or  $C$  is hyperelliptic.*

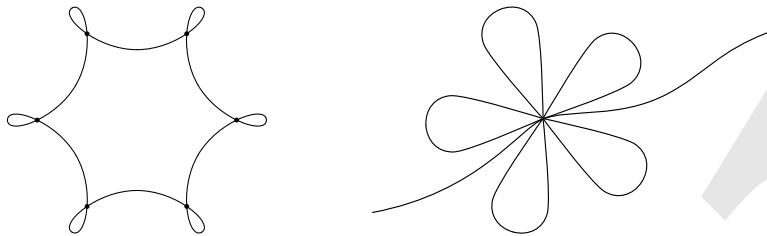
**Proof.** If  $C$  has a divisor  $D$  with  $\deg D = 2r(D)$  then by the Riemann-Roch theorem,  $\deg D = 2(\deg D - g + h^1(D))$ , so  $\deg D = 2g - 2h^1(D)$ . If also  $\deg D \leq 2g - 2$ , then  $h^1(D) \geq 1$  and thus  $r(D) > \deg D - g$ . Since  $W_{\deg D}^{r(D)}$  contains  $D$  its dimension is  $\geq 0$ , and we have a case of equality in Martens' theorem.  $\square$

**3.3. The  $g + 2$  theorem.** Now that we have the strong form of Martens' theorem, we can prove the analogue of Theorem 5.1 for general linear series of degree  $g + 2$ . This was stated in Section 5, but we'll reproduce the statement here.

**Theorem 3.5.** *Let  $C$  be any smooth projective curve of genus  $g$ , and let  $D$  be a general divisor of degree  $g + 2$  on  $C$ . The complete linear series  $|D|$  is base-point free of dimension 2, and the associated map  $\phi_D : C \rightarrow \mathbb{P}^2$  is birational onto its image  $C_0$ . Moreover,  $\phi_D$  is an immersion, and no two branches of  $C_0$  share a tangent line. Finally,*

- (1) *If  $C$  is not hyperelliptic, then  $C_0$  has only  $\binom{g}{2}$  nodes and no other singularities; and*
- (2) *If  $C$  is hyperelliptic,  $C_0$  has only one singular point, which is an ordinary  $g$ -fold point.*

**Proof.** • (Dimension) Since  $D$  is general and  $\deg D > h^0(K)$  we see that  $D$  is nonspecial, whence  $h^0(D) = (g + 2) - g + 1 = 3$ .



**Figure 4.** A general linear series of degree  $g + 2$  maps a non-hyperelliptic curve of genus  $g$  to a plane curve with  $\binom{g}{2}$  nodes, but the image of a hyperelliptic curve has just one singular point, where  $g$  smooth branches intersect with distinct tangents. I like the flower, on the right, but it doesn't look very algebraic. Maybe a triple-point ( $g=3$ ) would be enough

- (no base points) If  $|D|$  had a base point  $p \in C$ , then  $3 = h^0(D) = h^0(D - p) = (g + 1) - g + 1 + h^0(K_C - D + p)$ , so  $K_C - D$  is effective of degree  $2g - 2 - (g + 1) = g - 3$ ; thus the family of such  $D$  depends on only  $g - 3 + 1$  parameters, and  $\mu(D)$  would lie in a proper subvariety of  $\text{Pic}_{g+2}(C)$ .
- (birationality) To say that  $\phi_D(p) = \phi_D(q)$  for some pair of points  $p, q \in C$  means that  $h^0(D - p - q) = 2$ . By the Riemann-Roch theorem,  $D - p - q = K - E$  for some effective divisor  $E$  of degree  $g - 2$ ; Thus  $\mathcal{O}_C(D)$  is in the image of the map

$$\nu : C_2 \times C_{g-2} \rightarrow \text{Pic}_{g+2}(C)$$

sending  $(p + q, E)$  to  $K_C - E + p + q$ .

The map  $\nu$  is the composition of two surjective maps,  $C_2 \times C_{g-2} \rightarrow C_g$  and  $C_g \rightarrow \text{Pic}_g \rightarrow \text{Pic}_{g+2}$  by duality. Since the source and target of  $\nu$  have the same dimension, the general fiber of  $\nu$  is finite. Thus for general  $D$  there are only finitely many pairs  $p, q \in C$  such that  $\phi_D(p) = \phi_D(q)$ ; it follows that  $\phi_D$  is birational onto its image.

- (no tacnodes): To say that a pair of points  $p, q \in C$  map to a point of  $C_0$  such that the two branches have a common tangent means two things: that  $h^0(D - p - q) \geq 2$ ; and that  $h^0(D - 2p - 2q) \geq 1$ . If this is the case, set  $E = D - 2p - 2q$ . The condition  $h^0(D - 2p - 2q) \geq 1$  is equivalent to  $E$  being an effective divisor. Considering the canonical map  $\phi_K : C \rightarrow \mathbb{P}^{g-1}$  the geometric Riemann-Roch theorem  $r(E) = \deg E - 1 - \dim \overline{\phi_K(E)}$  implies that  $\overline{\phi_K(E)}$  has dimension  $g - 3$ . The additional condition  $h^0(D - p - q) \geq 2$  says that  $\dim \overline{\phi_K(E + p + q)} = g - 2$ , so the secant line  $\overline{p, q}$  meets the  $(g - 3)$ -plane  $\overline{\phi_K(E)}$ . Now, not every secant line to  $C$  can meet a given linear subspace  $\Lambda \subset \mathbb{P}^{g-1}$  of dimension  $g - 3$ —otherwise, the projection  $\pi_\Lambda : C \rightarrow \mathbb{P}^1$  would be constant—so we see that  $\mu(D)$  would have to lie on the image of a proper subvariety

of  $C_{g-2} \times C_2$  under the map  $(E, p+q) \mapsto \mathcal{O}(E+2p+2q)$  from  $C_{g-2} \times C_2$  to  $\text{Pic}_{g+2}(C)$ . Since the dimension of this subvariety is  $< g$  we see that if  $C'$  has a point at which two branches are tangent, then  $\mathcal{O}_C(D)$  lies in a proper subvariety of  $\text{Pic}_{g+2}(C)$ . Since the Abel-Jacobi map  $C_{g+2} \rightarrow \text{Pic}_{g+2}(C)$  is surjective,  $D$  would not be general.

Let's now suppose that  $C$  is non-hyperelliptic and  $D$  is a general divisor of degree  $g+2$  on  $C$ . To prove the theorem in this case, we have to show three things: that the image  $C_0 = \phi_D(C)$  does not have cusps or triple points.

- (no cusps) To say that a point  $p \in C$  maps to a cusp of  $C_0$  (that is, the differential  $d\phi_D$  is zero at  $p$ ) amounts to saying that  $h^0(D - 2p) \geq 2$ ; that is,  $D - 2p$  is a  $g^1_g$ . But by the Riemann-Roch theorem,  $W_g^1 = K_C - W_{g-2}$ ; so to say  $\phi_D$  has a cusp means that

$$\mu(D) \in 2W_1 + K_C - W_{g-2},$$

and since the locus on the right has dimension at most  $g-1$ , a general point of  $J(C)$  will not lie in it. Note that this subsumes the fact that  $|D|$  has no base points.

- (no triple points) To say that  $C_0$  has a triple point means that for some divisor  $E = p + q + r$  of degree 3,  $h^0(D - E) \geq 1$ ; thus we must have

$$\mu(D) \in W_3 + W_{g-1}^1$$

To argue that this is not the case, we need to know that  $\dim W_{g-1}^1 \leq g-4$ . Here we have to invoke Marten's theorem, which says that  $\dim W_{g-1}^1 \leq g-4$  if  $C$  is non-hyperelliptic; given this, we conclude that  $C_0$  has no triple points.

Thus, in the non-hyperelliptic case, the image curve  $C_0 = \phi_D(C)$  has only nodes as singularities; the fact that there are exactly  $\binom{g}{2}$  of them follows from the adjunction formula.

This concludes the proof in the non-hyperelliptic case. Now suppose that  $C$  is hyperelliptic, and let  $|E|$  be the  $g^1_2$  on  $C$ . If  $D$  is any divisor of degree  $g+2$ , the divisor  $D - E$  will have degree  $g$ , and so be effective; thus we can write

$$D \sim E + p_1 + \cdots + p_g$$

for some  $g$ -tuple of points  $p_i$ . If the divisor  $D$  is general, the points  $p_i$  will be general as well, and in particular distinct.

The fact that

$$h^0(D - p_1 - \cdots - p_g) = h^0(E) = 2 = h^0(D) - 1$$

implies that  $\phi_D$  maps all the points  $p_i$  to the same point. The image curve  $C_0$  thus has a point with at least  $g$  branches. Since these branches cannot share a

tangent line, they are smooth and their tangents are distinct; that is, the point is an ordinary multiple point. By the adjunction formula  $p_a(C) = \binom{g+1}{2}$  and since  $C$  has genus  $g$  the singularity must have  $\delta$  invariant  $\binom{g+1}{2} - g = \binom{g}{2}$ . Such a singularity cannot have multiplicity  $> g$ , and is thus an ordinary  $g$ -fold point. Since the  $\delta$ -invariant of such a point is  $\binom{g}{2}$  by Section 6.3 (see also Proposition 1.9), there can be no other singularities.  $\square$

## 4. Exercises

**Exercise 4.1.** Let  $C \subset \mathbb{P}^r$  be a smooth curve. If we re-embed  $C$  by a Veronese map of sufficiently high degree—that is, we let  $\nu_m : \mathbb{P}^r \rightarrow \mathbb{P}^N$  be the  $m$ th Veronese map, and let  $\tilde{C} = \nu_m(C)$  be image of  $C$ . Prove:

**Proposition 4.2** (Proposition 3.1 in characteristic  $p$ ). *The projection of  $\tilde{C}$  from a general  $\mathbb{P}^{N-3}$  is a nodal plane curve.*

Hint: show that the items of 3.5 are true in this situation.

**Exercise 4.3.** Let  $C_0$  be a plane quartic curve with two nodes  $q_1, q_2$ ; let  $\nu : C \rightarrow C_0$  be its normalization, and let  $o \in C$  be any point not lying over a node of  $C_0$ . By the adjunction formula,  $C$  has genus 1. Give a geometric description of the group law on  $C$  with  $o$  as origin.

Hint: To add two points  $s$  and  $t \in C$ , choose a conic curve  $D$  passing through  $s, t, q_1$  and  $q_2$ , and let  $u$  and  $v$  be the remaining points of  $C_0 \cap D$ ; then take  $D'$  the conic passing through  $u, v, q_1, q_2$  and  $o$ . The sum  $s+t$  will then be the remaining point of  $D' \cap C_0$ .

**Exercise 4.4.** [Rathmann 1987]. Let  $k$  be an algebraically closed field of characteristic  $p > 0$ , and let  $q = p^e$  for some  $e \geq 1$ . Let  $C \subset \mathbb{P}^r$  be the closure of the image  $C_0$  of the morphism

$$\mathbb{A}^1 \ni t \mapsto (t, t^q, t^{q^2}, \dots, t^{q^r}) \in \mathbb{A}^r$$

where  $\mathbb{A}^r \subset \mathbb{P}^r$  is the open set  $x_0 = 1$ .

- (1) Show that  $C$  is a complete intersection, defined by the equations

$$x_0^{q-1}x_2 - x_1^q, x_0^{q-1}x_3 - x_2^q, \dots, x_0^{q-1}x_r - x_{r-1}^q.$$

- (2) Show that  $C$  is singular unless  $q = r = 2$ .
- (3) Show every secant line to  $C_0$  contains  $q$  points of  $C_0$ ; more generally, if  $a_1, \dots, a_r$  are linearly independent points of  $C_0$ , show that the linear span of  $a_1, \dots, a_r$  contains  $q^{(r-1)}$  points of  $C_0$ . Compare this with the configuration of points in affine  $r$ -space over a field of  $q$  elements.

**Exercise 4.5.** Here is another approach to the  $g+2$  theorem in the hyperelliptic case: Let  $C$  be a hyperelliptic curve of genus  $g$  and  $D$  a general divisor of degree  $g+1$  on  $C$ ; let  $|E|$  be the  $g_2^1$  on  $C$ . Consider the map  $\phi : C \rightarrow \mathbb{P}^1 \times \mathbb{P}^1$  given as the product of the maps  $\phi_D : C \rightarrow \mathbb{P}^1$  and  $\phi_E : C \rightarrow \mathbb{P}^1$  given by the pencils  $|D|$  and  $|E|$ .

- (1) Show that  $\phi$  embeds the curve  $C$  as a curve of bidegree  $(g+1, 2)$  on  $\mathbb{P}^1 \times \mathbb{P}^1$ .
- (2) Now embed  $\mathbb{P}^1 \times \mathbb{P}^1$  into  $\mathbb{P}^3$  as a quadric surface  $Q$ ; pick a general point  $p \in C \subset Q$  and project  $C$  from the point  $p$ . Show that the image curve  $C_0$  is a plane curve of degree  $g+2$  with one ordinary  $g$ -fold point.

Hint: for the first part, show that  $h^0(D - K_C) = 0$ , and conclude that the map  $\phi_D \times \phi_E$  is birational onto its image; then use the genus formula to conclude the image curve is smooth. For the second, observe that if  $L$  is a line through  $p$  meeting  $C$  more than once elsewhere, the line  $L$  must be contained in  $Q$ .

**Exercise 4.6.** Establish the analog of Proposition 1.4 for hypersurfaces of any degree  $m$ , that is to say no irreducible, nondegenerate curve in  $\mathbb{P}^d$  lies on more hypersurfaces of degree  $m$  than the rational normal curve. To do this, let  $C \subset \mathbb{P}^d$  be any irreducible nondegenerate curve. Let  $\Gamma$  be a general hyperplane section of  $C$ , and use the exact sequences

$$0 \rightarrow \mathcal{I}_{C/\mathbb{P}^d}(l-1) \rightarrow \mathcal{I}_{C/\mathbb{P}^d}(l) \rightarrow \mathcal{I}_{\Gamma/\mathbb{P}^{d-1}}(l) \rightarrow 0.$$

with  $2 \leq l \leq m$  to show that

$$h^0(\mathcal{I}_{C/\mathbb{P}^d}(m)) \leq \binom{d+m}{m} - (md + 1)$$

with equality only if  $C$  is a rational normal curve.

**Exercise 4.7.** Let  $D \subset \mathbb{P}^r$  be a rational normal curve. If  $\Gamma \subset D$  is any collection of  $d$  points on  $D$  (or for that matter any subscheme of  $D$  of degree  $d$ ) then the Hilbert function of  $\Gamma$  is

$$h_{\Gamma}(m) = \min\{d, mr + 1\}$$

Hint: if  $d > mr$ , then the hypersurfaces of degree  $m$  containing  $\Gamma$  are exactly the hypersurfaces of degree  $m$  containing  $D$ , and we know how many there are. If  $d \leq mr$ , then the hypersurfaces of degree  $m$  containing  $\Gamma$  cut out on  $D$  the linear series  $|\mathcal{O}_D(m)(-\Gamma)| = |\mathcal{O}_{\mathbb{P}^1}(mr - d)|$  and again we know the dimension.

**Exercise 4.8.** Let  $C$  be a reduced irreducible curve with general hyperplane section  $\Gamma$ , and write  $d - 1 = M(r - 1) + \epsilon$  with  $\epsilon < r - 1$  as in Castelnuovo's theorem.

Prove that

- (1) if  $\epsilon > 0$ , then  $\mathcal{O}_C(M)$  is nonspecial, but  $\mathcal{O}_C(M - 1)$  is special; and
- (2) if  $\epsilon = 0$ , then  $\mathcal{O}_C(M - 1)$  is nonspecial, but  $\mathcal{O}_C(M - 2)$  is special.

Hint: Show that if  $h^0(\mathcal{O}_C(m+1)) - h^0(\mathcal{O}_C(m)) \leq \deg C$ , and that if equality holds for all  $m \geq m_0$ , then  $\mathcal{O}_C(m)$  is nonspecial for all  $m \geq m_0$ .

**Exercise 4.9.** If  $C \subset \mathbb{P}^r$  is a Castelnuovo curve of degree  $d \geq 2r$ , show that  $|D| = |\mathcal{O}_C(1)|$  is the unique  $g_d^r$  on  $C$ .

Hint: If  $|D'|$  were another  $g_d^r$  on  $C$ , show that  $h^0(mD + D') > h^0((m+1)D)$ .

**Exercise 4.10.** We have seen that complete intersections  $C = Q \cap S \subset \mathbb{P}^3$  of a quadric surface  $Q$  and a surface  $S$  of degree  $k$  achieve Castelnuovo's bound  $g = \pi(2k, 3)$  on the genus of curves of degree  $2k$  in  $\mathbb{P}^3$ . In fact, we will see in Chapter 17 that any curve  $C \subset \mathbb{P}^3$  of degree  $2k$  and genus  $g = \pi(2k, 3) = (k-1)^2$  is of this form.

- (1) Find the dimension of the subvariety  $\Gamma \subset M_g$  consisting of Castelnuovo curves.
- (2) Find the dimension of the subvariety  $H \subset M_g$  of hyperelliptic curves, and compare this to the result of the first part.

Hint: For the first part, we can calculate the dimension of the Hilbert scheme  $H^\circ$  of Castelnuovo curves by calculating the dimension of the incidence correspondence of pairs  $(Q, C)$  with  $Q$  a quadric surface in  $\mathbb{P}^3$  and  $C \subset Q$  a curve of type  $(k, k)$ ; then invoke Exercise 4.9 to say the fibers of  $H^\circ \rightarrow M_g$  are isomorphic to  $PGL_4$ . For the second, we already know the dimension of the locus of hyperelliptic curves is  $2g - 1$  ( $2g + 2$ -tuples of points in  $\mathbb{P}^1$  mod  $PGL_2$ ); the point is just that Castelnuovo curves are rarer than hyperelliptic ones.



# Monodromy of Hyperplane Sections

## 1. Uniform position and monodromy

We now return to the situation where the ground field  $k$  is algebraically closed of characteristic 0.

The central result of this chapter is the *uniform position lemma*, which deals with the monodromy group of the points of a general hyperplane section of a curve  $C \subset \mathbb{P}^r$ . To define the monodromy group and prove the uniform position lemma, we will use the classical topology. An equivalent algebraic definition is described in Cheerful Fact 1.1 below, but the strong form of uniform position can fail in positive characteristic, as shown by the examples in Exercise 4.4.

We may describe the monodromy group informally as follows: Suppose that  $C \subset \mathbb{P}^r$  is an irreducible curve of degree  $d$  over  $\mathbb{C}$ , and  $H_0 \subset \mathbb{P}^r$  a hyperplane transverse to  $C$ ; say  $C \cap H_0 = \{p_1, \dots, p_d\}$ . As we vary  $H_0$  continuously along a real arc  $\{H_t\}$ , staying within the open subset  $U \subset \mathbb{P}^{r*}$  of hyperplanes transverse to  $C$ , we can “follow” each of the points  $p_i(t)$  of intersection of  $C$  with the hyperplane  $H_t$ .

Now imagine that the hyperplanes  $H_t$  come back to the original  $H_0$  at time  $t = 1$ ; that is, we have a continuous family  $\{H_t\}_{0 \leq t \leq 1} \subset U$  with  $H_1 = H_0$ . Each of the points  $p_i$  then traces out a continuous real arc  $\{p_i(t) \in C \cap H_t\}_{0 \leq t \leq 1}$ . Since  $H_1 = H_0$ , the end point  $p_i(1)$  is one of the original points  $p_j \in C \cap H_0$ . In this way, we get a permutation of the set  $C \cap H_0$ ; the group of all permutations arrived at in this way is called the *monodromy group* of the points  $C \cap H_0$ .

We will now give a precise definition of the monodromy group in a more general setting, and prove that the monodromy group of the points of a general hyperplane section of an irreducible curve is the full symmetric group; this is the uniform position lemma. The rest of the chapter will be a series of applications.

**1.1. The monodromy group of a generically finite morphism.** Let  $f : Y \rightarrow X$  be a dominant map between varieties of the same dimension over  $\mathbb{C}$ , and suppose that  $X$  is irreducible. There is then a Zariski open subset  $U \subset X$  such that  $U$  and its preimage  $V = f^{-1}(U)$  are smooth, and the restriction of  $f$  to  $V$  is a covering space in the classical topology. Let  $d$  be the number of sheets. This is the degree of the extension  $K(Y)/K(X)$ .

Homotopy theory associates a monodromy group to any finite topological covering map  $f : V \rightarrow U$ , defined as follows: Choose a base point  $p_0 \in U$ , and suppose  $\Gamma := f^{-1}(p_0) = \{q_1, \dots, q_d\}$ . If  $\gamma$  is any loop in  $U$  with base point  $p_0$ , for any  $i = 1, \dots, d$  there is a unique lifting of  $\gamma$  to an arc  $\tilde{\gamma}_i$  in  $V$  with initial point  $\tilde{\gamma}_i(0) = q_i$  and end point  $\tilde{\gamma}_i(1) = q_j$  for some  $j \in \{1, 2, \dots, d\}$ . Since we could traverse the loop in the opposite direction, the index  $j$  determines  $i$ , and the map  $i \mapsto j$  is a permutation of  $\{1, 2, \dots, d\}$ . Since the set  $\Gamma$  is discrete, the permutation depends only on the class of  $\gamma$  in  $\pi_1(U, p_0)$  so we have defined a homomorphism to the symmetric group:

$$\pi_1(U, p_0) \rightarrow \text{Perm}(\Gamma) \cong S_d.$$

The image  $M$  of this map is called the *monodromy group* of the map  $f$ . It depends on the labeling of the points of  $\Gamma$ , but a change in labeling only changes the group by conjugation with the corresponding permutation, so the monodromy group is well defined up to conjugation in  $S_d$ .

**Cheerful Fact 1.1.** *In our setting the monodromy group is independent of the choice of open set  $U$ : if  $U' \subset U$  is a Zariski open subset, the complement of  $U \setminus U'$  has real codimension  $\geq 2$  so the map  $\pi_1(U', p_0) \rightarrow \pi_1(U, p_0)$  is surjective. Thus the image of  $\pi_1(U', p_0)$  in  $S_d$  is the same.*

*The theory of finite coverings of algebraic varieties is not only analogous to Galois theory, it is Galois theory: In the situation described above, if  $Y$  is irreducible, then the pullback map  $f^*$  expresses the function field  $K(Y)$  as a finite algebraic extension of  $K(X)$ . The monodromy group of  $f$  is the Galois group of the Galois normalization of  $K(Y)$  over  $K(X)$  (see [Harris 1979]). Indeed, in early treatments of Galois theory, such as the famous Traité des Substitutions [Jordan 1889], originally published in 1870, function fields played as large a role as number fields.*

Since we assumed that  $X$  is irreducible, the space  $U$  is (path) connected, and it follows that the monodromy group is transitive if and only if the space  $V$  is (path) connected. But  $V$  is a smooth variety, so this is the case if and only if

$V$  is irreducible. For example, the monodromy in the family of smooth quadric surfaces in  $\mathbb{P}^3$  interchanges the two rulings:

**Example 1.2.** Consider the family

$$X := \{(p, Q) \mid p \in Q \text{ a point on a smooth quadric surface in } \mathbb{P}^3\}$$

of pointed smooth quadric surfaces in  $\mathbb{P}^3$  and the double covering by

$$V := \{((p, Q), L) \mid (p, Q) \in X, L \text{ a line with } p \in L \subset Q\}.$$

The variety  $V$  is irreducible: if we project  $V$  to the (irreducible) variety of lines in  $\mathbb{P}^3$ , the fiber consists of a point on the line and quadric containing the line—that is, the product of  $\mathbb{P}^1$  and the (dual of the) projective space on the space of quadratic forms in the ideal of the line. Thus the monodromy of the family is transitive, and exchanges the two lines through the point.

**1.2. Uniform position.** We will next compute the monodromy group of the universal hyperplane section of a curve, constructed as follows: Let  $C \subset \mathbb{P}^r$  be an irreducible, nondegenerate curve of degree  $d$ , and let  $X = \mathbb{P}^{r*}$  be the space of hyperplanes in  $\mathbb{P}^r$ . We define the *universal hyperplane section* of  $C$  to be the projection  $f : Y \rightarrow \mathbb{P}^{r*}$  of the incidence variety

$$Y = \{(H, p) \in \mathbb{P}^{r*} \times C \mid p \in H\}.$$

The fibers of  $f$  are the hyperplane sections of  $C$ , so  $f$  is a finite surjective map. If we let  $U \subset \mathbb{P}^{r*}$  be the open subset of hyperplanes meeting  $C$  transversely, then the restriction of  $f$  to the preimage  $V$  of  $U$  is a covering space whose fibers each consist of  $d$  distinct points. The preimage in  $Y$  of a point  $p \in C$  is the set of hyperplanes containing  $p$ , a copy of  $\mathbb{P}^{r-1}$ , and thus  $Y$  is irreducible. Thus the monodromy group of  $f$  is transitive. But much more is true:

**Theorem 1.3** (Uniform Position Theorem). *The monodromy group of the universal hyperplane section of an irreducible curve  $C \subset \mathbb{P}^r$  is the full symmetric group  $S_d$ .*

We postpone the proof to develop some necessary tools.

Theorem 1.3 fails over fields of finite characteristic, though there is no known counterexample for smooth curves; see Exercise 4.4 for singular examples, and [Rathmann 1987] and [Kadets 2021] for what is known.

Theorem 1.3 implies that two subsets of the same cardinality in the general hyperplane section of  $C$  are indistinguishable from the point of view of any discrete invariant that is semicontinuous in the Zariski topology. To make this precise, we introduce a definition:

**Definition 1.4.** Let  $\phi : Y \rightarrow X$  be a finite morphism. By the *restricted fiber power*  $\tilde{Y}^n/X$  we will mean the complement of all diagonals in the ordinary fiber power; that is,

$$\tilde{Y}^n/X := \{(x, y_1, \dots, y_n) \in X \times Y^n \mid \phi(y_i) = x \text{ and } y_i \neq y_j \forall i \neq j\}$$

In down-to-earth terms, a point of  $\tilde{Y}^n/X$  is a set of  $n$  distinct points in a fiber of  $\phi$  together with the choice of a total order on these points.

**Lemma 1.5.** Let  $f : Y \rightarrow X$  be a generically finite cover of degree  $d$ , with monodromy group  $M \subset S_d$ .  $M$  is  $n$ -transitive if and only if the restricted fiber power  $\tilde{Y}^n/X$  is irreducible.

**Proof.** If we restrict ourselves to open subsets  $U \subset X$  and  $V = f^{-1}(U) \subset Y$  such that  $U$  and  $V$  are smooth and the restriction  $f|_V : V \rightarrow U$  is a covering in the classical topology, then the restricted fiber powers are unions of connected components of the usual fiber powers  $Y^n/X$ . The condition that the monodromy is  $n$ -transitive is equivalent to the condition that the restricted fiber power  $\tilde{Y}^n/X$  is connected; since the fiber powers are all smooth, this is equivalent to  $\tilde{Y}^n/X$  being irreducible.  $\square$

## 2. Flexes and bitangents are isolated

There are further general position results that we need for the proof of the uniform position lemma. We have separated them from the results on linear general position because these require characteristic 0; they are really local analytic statements.

**2.1. Not every tangent line is tangent at a flex.** Recall that a smooth point  $p$  on a curve  $C \subset \mathbb{P}^r$  is called a *flex point* if the tangent line to  $C$  at  $p$  has contact of order 3 or more with  $C$  at  $p$ .

**Lemma 2.1.** If  $r > 1$  and  $C \subset \mathbb{P}^r$  is a smooth, irreducible and nondegenerate curve, then not every point of  $C$  is a flex point.

The proof applies to any linear series on a curve.

**Proof.** We begin by lifting the inclusion  $C \hookrightarrow \mathbb{P}^r$  to an arc  $v(t)$  in  $\mathbb{C}^{r+1}$ , so that the tangent line at the point  $t = 0$  in projective space is represented by the span of  $v(0)$  and  $v'(0) \in \mathbb{C}^{r+1}$ . To say that  $v(t)$  is a flex point is to say that the vectors  $v(t), v'(t)$  and  $v''(t)$  are linearly dependent. If this holds for all  $t$  then

$$v(t) \wedge v'(t) \wedge v''(t) \equiv 0.$$

When we take the derivative of the wedge product  $v(t) \wedge v'(t) \wedge v''(t)$  by applying the product rule we see that the first two terms are zero because they contain a repeated factor; it follows that

$$v(t) \wedge v'(t) \wedge v'''(t) \equiv 0,$$

so that  $v'''(t)$  lies in the span of  $v(t)$  and  $v'(t)$  as well. Indeed, as we continue to take derivatives, we see in each case that all but one term is zero, and we deduce that  $v^{(l)}(p)$  lies in the span of  $v(t)$  and  $v'(t)$  for all  $l$ . This being characteristic 0, it follows that  $C$  lies in a line, contrary to our hypothesis  $\square$

**2.2. Not every tangent is bitangent.** This statement seems even more obvious than Lemma 2.1 above, but—like that lemma—is false in characteristic  $p$ —see Exercise 5.3

**Lemma 2.2.** *Let  $C \subset \mathbb{P}^r$  be a smooth, irreducible, nondegenerate curve, with  $r > 1$ . If  $p \in C$  is a general point, then the tangent line  $\mathbb{T}_p(C) \subset \mathbb{P}^r$  is not tangent to  $C$  at any other point.*

**Proof.** Again the result is local, this time with a pair of germs  $D \xrightarrow{v} \mathbb{P}^r$  and  $D \xrightarrow{w} \mathbb{P}^r$ . If every tangent line to the two curves is a bitangent, we will show that they are both contained in a line.

Let  $C_1, C_2$  be the images of  $v, w$  respectively, and let  $\tilde{v}, \tilde{w}$  be lifts to  $\mathbb{C}^{r+1}$ . Let

$$\Sigma := \{(p, q) \in D \times D \mid \text{and } \mathbb{T}_{v(p)}(C_1) = \mathbb{T}_{w(q)}(C_2)\}$$

be the variety parametrizing bitangents to the two germs.

The statement that the tangent lines to  $C$  at the points  $v(t)$  and  $w(t)$  are equal then says that the vectors  $\tilde{v}(t), \tilde{v}'(t), \tilde{w}(t)$  and  $\tilde{w}'(t)$  all lie in a 2-dimensional subspace  $\Lambda \subset \mathbb{C}^{r+1}$ ; in particular,

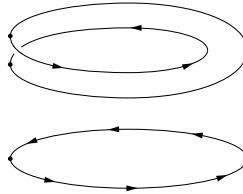
$$\tilde{v}(t) \wedge \tilde{v}'(t) \wedge \tilde{w}(t) \equiv 0 \quad \text{and likewise} \quad \tilde{v}(t) \wedge \tilde{w}(t) \wedge \tilde{w}'(t) \equiv 0$$

We proceed now exactly as in the proof of Lemma 2.1: taking derivatives, we see that all derivatives of  $\tilde{v}(t)$  and  $\tilde{w}$  at  $t = 0$  lie in  $\Lambda$  and hence  $C_1$  and  $C_2$  are both contained in the line in  $\mathbb{P}^r$  corresponding to  $\Lambda$ .  $\square$

In characteristic zero, [Kaji 1986, Theorem 3.1] shows that a general tangent line to a smooth projective curve  $C$  can intersect  $C$  in a point other than  $p$ , except of course in case  $r = 2$ , although this is possible in characteristic  $p > 0$ .

### 3. Proof of the uniform position theorem

Let  $C \subset \mathbb{P}^r$  be an irreducible, nondegenerate curve of degree  $d$  and  $f : Y \subset \mathbb{P}^{r*} \times C \rightarrow X = \mathbb{P}^{r*}$  its universal hyperplane section; let  $U \subset \mathbb{P}^{r*}$  be the open subset of hyperplanes transverse to  $C$  and  $V = f^{-1}(U)$ ; let  $M \subset S_d$  be the



**Figure 1.** Monodromy action around the ramification point of a double cover Silvio: better to have 6 arrows above corresponding to the 6 below. Maybe different colors on the arrows/points on the upper sheet (as drawn) than the lower sheet?

monodromy group of  $V$  over  $U$ . To show that  $M$  is the full symmetric group, it suffices to show that  $M$  contains all transpositions, and for this it is enough to show that  $M$  is doubly transitive and contains one transposition.

To prove that  $M$  is doubly transitive, we can give a concrete description of the restricted fiber power  $\tilde{V}^2/U$ : let

$$\Sigma := \{(H, p, q) \in \mathbb{P}^{r^*} \times C \times C \mid p, q \in H \text{ and } p \neq q\}.$$

Projection on the second and third factors expresses  $\Sigma$  as a  $\mathbb{P}^{r-2}$ -bundle over the complement  $C \times C \setminus \Delta$  of the diagonal in  $C \times C$ . Thus  $\Sigma$  is irreducible, and it follows that the restricted fiber square  $\tilde{V}^2/U$ , which is a Zariski open subset of  $\Sigma$ , is as well. Note that this part of the argument does not rely on any assumption about the characteristic.

Next we give a criterion for a monodromy group to contain a transposition:

**Lemma 3.1.** *Let  $f : Y \rightarrow X$  be a generically finite cover of degree  $d$  over an irreducible variety  $X$ , with monodromy group  $M \subset S_d$ . If, for some smooth point  $p \in X$  the fiber  $f^{-1}(p) \subset V$  consists of  $d-2$  reduced points  $p_1, \dots, p_{d-2}$  and one point  $q$  of multiplicity 2, where  $q$  is also a smooth point of  $Y$ , then  $M$  contains a transposition.*

**Proof.** Note that the hypothesis implies that  $Y$  is smooth locally near the fiber over  $p$ . Let  $U \subset X$  be a Zariski open subset of the smooth locus in  $X$ , as in the definition of the monodromy group, so that  $V := f^{-1}(U)$  is also smooth and the restriction  $f|_V : V \rightarrow U$  expresses  $V$  as a finite  $d$ -sheeted covering space of  $U$ , with  $U, V$  smooth and  $p \in X$ .

Let  $B_i \subset Y$  be disjoint small connected closed neighborhoods of the points in  $f^{-1}(p)$ . Because  $f|_V$  is finite, the image  $A := \cap f(B_i)$  is a locally closed subset of the same dimension as  $X$  and thus  $A$  contains a neighborhood of  $p$  in the classical topology.

Let  $p' \in A \cap U$ . Two of the  $d$  points of  $f^{-1}(p')$  lie in the component of  $B$  containing  $q$ ; call these  $q'$  and  $q''$ . Since  $B \cap V$  is the complement of a proper

subvariety in  $B$  it is connected, and we can draw a real arc  $\gamma : [0, 1] \rightarrow B \cap V$  joining  $q'$  to  $q''$ ; by construction, the permutation of  $f^{-1}(p')$  associated to the loop  $f \circ \gamma$  will exchange  $q'$  and  $q''$  and fixes each of the remaining  $d - 2$  points of  $f^{-1}(p')$ .  $\square$

**Completion of the proof of the uniform position lemma.** It remains to show that a reduced irreducible curve  $C$  of degree  $d$  (in characteristic 0) has a hyperplane section consisting of  $d - 2$  smooth points and one double point; that is, a hyperplane simply tangent to the curve at one point and transverse everywhere else. By the results of Section 2 there is a tangent line  $L$  to  $C$  at a smooth point that is not flex tangent, and is not tangent to  $C$  at any other point. A general hyperplane  $H$  containing  $L$  meets  $C$  doubly in the point of tangency. At any other point  $p$  of  $L \cap C$ , if any, the intersection  $C \cap H$  is transverse unless  $H$  contains the tangent line at  $p$ . Containing such a line is a proper codimension 1 condition on the hyperplanes containing  $L$ , and since there can only be finitely many such points, a general  $H$  will be transverse at all such  $p$ . On the other hand, at points not on  $L$  the intersection  $H \cap C$  is transverse by Bertini's theorem. This completes the proof.  $\square$

**3.1. Uniform position for higher-dimensional varieties.** We mention here a generalization of Theorem 1.3 for irreducible varieties  $X \subset \mathbb{P}^r$  of any dimension  $k$ . To set this up, let  $\mathbb{G}(r - k, r)$  be the Grassmannian parametrizing  $(r - k)$ -planes  $\Lambda \subset \mathbb{P}^r$ . We introduce the *universal  $(r - k)$ -plane section of  $X$* :

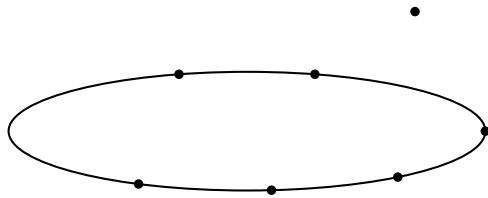
$$Y = \{(\Lambda, p) \in \mathbb{G}(r - k, r) \times X \mid p \in \Lambda\}.$$

Projection on the first factor expresses  $Y$  as a generically finite cover of  $\mathbb{G}(r - k, r)$ , and we can ask for its monodromy. The answer is the same as for curves:

**Theorem 3.2.** *The monodromy group of the universal  $(r - k)$ -plane section of an irreducible  $k$ -dimensional variety  $X \subset \mathbb{P}^r$  is the full symmetric group  $S_d$ .*

**Proof.** In fact, this follows from Theorem 1.3. To prove it, fix a general  $(r - k + 1)$ -plane  $\Gamma \subset \mathbb{P}^r$ ; since a general hyperplane section of an irreducible variety  $X \subset \mathbb{P}^r$  of dimension  $k \geq 2$  is again irreducible, we see that  $C := \Gamma \cap X$  is an irreducible curve. The restriction of the universal  $(r - k)$ -plane section of  $X$  to the sub-Grassmannian  $\mathbb{G}(r - k, \Gamma) \cong \mathbb{P}^{r-k}$  of  $(r - k)$ -planes contained in  $\Gamma$  is just the universal hyperplane section of the curve  $C$ , which we know has monodromy  $S_d$ ; since the monodromy group of a cover can only get smaller under restriction to a subvariety of the target, the result follows.  $\square$

We will see an application of this result in Proposition 4.4 below.



**Figure 2.** 7 points in the plane, 6 on a conic, in linearly general but not uniform position

#### 4. Applications of uniform position

**4.1. Irreducibility of fiber powers.** If we apply the uniform position lemma to the universal hyperplane section of a curve  $C \subset \mathbb{P}^r$  we get an irreducibility result:

**Corollary 4.1.** *If  $C \subset \mathbb{P}^r$  is a smooth curve of degree  $d$ , then for  $k \leq d$ , the restricted fibered powers  $\tilde{Y}^k/X$  of the universal hyperplane section of  $C$  are irreducible.*

#### 4.2. Numerical uniform position.

**Corollary 4.2** (numerical uniform position lemma). *Let  $C \subset \mathbb{P}^r$  be an irreducible curve, and let  $\Gamma = H \cap C$  be a general hyperplane section. Any two subsets of  $\Gamma$  with the same cardinality impose the same number of conditions on forms of any degree; that is, any two subsets of the same cardinality have the same Hilbert function.*

Silvio: three of these points could easily be collinear; I suggest moving the "floating" point off to the left (makes the figure less high, too.)

**Proof.** Let  $U = \mathbb{P}^{r^*} \setminus C^*$  be the open subset of hyperplanes transverse to  $C$ , and let  $Y \rightarrow U$  be the universal hyperplane section. Corollary 4.1 says that the restricted fiber powers  $\tilde{V}^n/U$  are irreducible.

Now, for each  $m$  the number of conditions that  $\Gamma$  imposes on forms of degree  $m$  is lower semicontinuous, so it achieves its maximum on a Zariski open subset of  $\tilde{V}^n/U$ . Since  $\tilde{V}^n/U$  is irreducible, the complement  $Z$  of this open set has dimension strictly less than  $\dim \tilde{V}^n/U = \dim U$ . Thus a general hyperplane  $H \in \mathbb{P}^{r^*}$  lies outside the image of  $Z$ , meaning that the number of conditions imposed by all the  $k$ -element subsets  $\Gamma \subset C \cap H$  have this maximal value.  $\square$

This result may be seen as an important strengthening of Theorem 1.1, since if  $C$  is a reduced, irreducible nondegenerate curve in  $\mathbb{P}^n$  then a general

subset of  $n$  points of  $C$  is linearly independent and spans a hyperplane; Corollary 4.2 says that this is true for every subset of  $n$  points of every general hyperplane section, which reproves Theorem 1.1, though only in characteristic 0.

**4.3. Sums of linear series.** Another consequence of the uniform position lemma is a result about sums of linear series. Recall that if  $D$  is a divisor on a curve  $C$  we write  $r(D) = \dim |D| = h^0(\mathcal{O}_C(D)) - 1$ .

**Corollary 4.3.** *If  $D, E$  are effective divisors on a curve  $C$  then*

$$r(D + E) \geq r(D) + r(E).$$

*If the genus of  $C$  is  $> 0$  and  $|D + E|$  is birationally very ample, then the inequality is strict.*

On  $\mathbb{P}^1$ , by contrast, any effective divisor  $D$  has  $r(D) = \deg D$ , so the inequality above is always an equality for  $C = \mathbb{P}^1$ .

**Proof.** The inequality follows in general because the sums of divisors in  $|D|$  and divisors in  $|E|$  already move in a family of dimension  $r(D) + r(E)$ ; the key point is the strict inequality in case  $D + E$  is birationally very ample.

If  $D + E$  is birationally very ample and  $r(D + E) = r(D) + r(E)$  then restricting to an open set we may identify  $C$  with its image under the complete linear series  $|D + E|$ , and we see that a general hyperplane section  $H \cap C$  contains a divisor equivalent to  $D$ .

Let  $Y$  be the  $\deg D + \deg E$  restricted fiber power of the universal hyperplane. A point  $y \in Y$  is a hyperplane section plus an ordering of its points. Let  $\phi : Y \rightarrow \text{Pic}_d(C)$  be the Abel-Jacobi map taking  $y$  to the class of the divisor that is the sum of first  $d$  points in this order. The preimage  $Y'$  of the point of  $\text{Pic}_d(C)$  corresponding to the class of  $D$  is a closed subset, and since every divisor in the class of a hyperplane section contains a divisor linearly equivalent to  $D$ , the subvariety  $Y'$  dominates  $\mathbb{P}^{n*}$ , and thus has the same dimension as  $Y$ . Consequently  $Y' = Y$ , and the sum of the first  $d$  points in any ordering of the general hyperplane section—that is, the sum of any  $d$  of the points—is equivalent to  $D$ .

Thus if  $p \in D$  and  $q \notin D$ , then  $D - p + q \equiv D$ , whence  $q \equiv p$ . Thus  $r(p) \geq 1$ , so  $C \cong \mathbb{P}^1$ .  $\square$

**4.4. Nodes of plane curves.** In Section 6, we introduced the *Severi variety*  $V_{d,g}$ ; this is the locally closed subset of the projective space  $\mathbb{P}^N$  of all plane curves of degree  $d$  parametrizing irreducible plane curves of degree  $d$  having

$\delta := \binom{d-1}{2} - g$  nodes and no other singularities. We proved there that  $V_{d,g}$  was smooth, and by Cheerful Fact 6.2 it is irreducible for all  $d$  and  $g$ .

To compute the monodromy we introduce the incidence correspondence

$$\Phi := \{(C, p) \in V_{d,g} \times \mathbb{P}^2 \mid p \in C_{\text{sing}}\}.$$

This is a  $\delta$ -sheeted covering space of  $V_{d,g}$ , and we compute its monodromy is. We start with the extremal case  $g = 0$ , where we can prove

**Proposition 4.4.** *The monodromy group of  $\Phi$  over  $V_{d,0}$  is the full symmetric group  $S_\delta$ , with  $\delta = \binom{d-1}{2}$ .*

**Proof.** Every rational nodal curve  $C \subset \mathbb{P}^2$  is the projection of a rational normal curve  $\tilde{C} \subset \mathbb{P}^d$  from a  $(d-3)$ -plane  $\Lambda \subset \mathbb{P}^d$ . Moreover, if  $\Lambda$  is general, the nodes of the projection correspond to the points of intersection of  $\Lambda$  with the secant variety  $X \subset \mathbb{P}^d$  of the rational normal curve  $\tilde{C}$ . Applying Theorem 3.2, the result follows.  $\square$

This result has an immediate consequence, which played a major role in the proof of Cheerful Fact 6.2: given the description in Section 6 of  $V_{d,g}$  in a neighborhood of a point in  $V_{d,0}$ , it follows that there is a unique irreducible component of  $V_{d,g}$  containing  $V_{d,0}$  in its closure. Thus, to prove the irreducibility of  $V_{d,g}$ , it is sufficient to show that every component of  $V_{d,g}$  contains  $V_{d,0}$  in its closure.

Going in the other direction, note also that if we assume Cheerful Fact 6.2, then we can deduce the analogous result for all  $g$ :

**Proposition 4.5.** *The monodromy group of  $\Phi$  over  $V_{d,g}$  is the full symmetric group  $S_\delta$ , with  $\delta = \binom{d-1}{2} - g$ .*

## 5. Exercises

As a consequence of Theorem 3.2, we can deduce the Bertini irreducibility theorem:

**Exercise 5.1.** Let  $X \subset \mathbb{P}^r$  be an irreducible variety of dimension  $k \geq 2$ . Show that a general hyperplane section of  $X$  is irreducible.

Hint: Say  $k = 2$ . If the general hyperplane section of  $X$  were reducible, there would be distinguished subsets of the intersection of  $X$  with a general  $(r-2)$ -plane  $\Lambda$  (the intersections of  $\Lambda$  with the components of  $H \cap X$ , for  $H$  a general hyperplane containing  $\Lambda$ ); but we know the monodromy on  $X \cap \Lambda$  is the full symmetric group.

**Exercise 5.2.** In Example 1.2, we gave a global argument to say that in the family of smooth quadric surfaces in  $\mathbb{P}^3$  the monodromy exchanges the two rulings of a quadric by lines. Prove this with a local calculation, analyzing the family

$$Q_t := V(X^2 + Y^2 + Z^2 + tW^2)$$

in a neighborhood of  $t = 0$ .

Hint: Choose a base point of the pencil, say  $p = [1, i, 0, 0]$ . The lines of the two rulings of  $Q_t$  passing through  $p$  are  $Y - iX = Z - \pm i\sqrt{t}W = 0$ , which are exchanged under the monodromy as  $t$  goes around 0.

**Exercise 5.3.** (Tangential degeneracy [Kaji 1986, Example 4.1]) Show that the linear series in Exercise 4.1 defines an isomorphism with a smooth curve  $C$  of type  $(1, d - 1)$  on a nonsingular quadric in  $\mathbb{P}_k^2$  no matter what field  $k$  is chosen. Now suppose that  $k$  has characteristic  $p > 0$  and  $d - 1 = p^\ell n$  with  $\ell \geq 2$  and  $n$  relatively prime to  $p$ . Show that the tangent lines to  $C$  are the rulings of the quadric in the family that meet  $C$  with multiplicity  $d - 1$ , and that the intersection of each tangent line meets  $C$  in  $n$  distinct points, each with multiplicity  $p^\ell$ . Thus Lemma 2.2 fails in positive characteristic.

**Exercise 5.4.** Let  $C \subset \mathbb{P}^r$  be a union of irreducible curves  $C_i$  of degrees  $d_i$ . Prove that the monodromy group of the points of a general hyperplane section of  $C$  is the product  $\prod S_{d_i}$ .

Hint: We need to know that the dual hypersurfaces  $C_i^* \subset \mathbb{P}^{r^*}$  are all distinct; given this, we can exhibit loops that induce a given permutation of the points of  $H \cap C_i$  while fixing the points of  $H \cap C_j$  for  $j \neq i$ . To see that the dual hypersurfaces  $C_i^* \subset \mathbb{P}^{r^*}$  are all distinct, invoke the duality theorem  $(C_i^*)^* = C_i$  (see for example [Eisenbud and Harris 2016a]).

**Exercise 5.5.** Let  $d$  and  $e$  be positive integers,  $\mathbb{P}^M$  the space of plane curves of degree  $d$  and  $\mathbb{P}^N$  the space of plane curves of degree  $e$ , and

$$\Phi := \{(D, E, p) \in \mathbb{P}^M \times \mathbb{P}^N \times \mathbb{P}^2 \mid p \in D \cap E\}.$$

Prove that the monodromy group of the projection  $\pi : \Phi \rightarrow \mathbb{P}^M \times \mathbb{P}^N$  is the symmetric group  $S_{de}$  on  $de$  letters

- (1) by applying Theorem 1.3; and
- (2) from scratch, using the method used in the proof of Theorem 1.3.

Hint: for the first part, we can fix the curve  $E$  and prove the a priori stronger statement that the monodromy on the points of  $D \cap E$  as  $D$  varies is the symmetric group; this follows from Theorem 1.3 applied to the Veronese embedding  $\nu_d(E)$ . Alternatively, we can exhibit a transposition by finding a pair  $(D, E)$

such that  $D \cap E$  consists of  $de - 2$  simple points and one double point, and prove double transitivity with the usual incidence correspondence argument.

**Exercise 5.6.** If  $E \subset \mathbb{P}^2$  is a smooth plane cubic curve, then a point  $p \in E$  is a *flex* if  $\mathcal{O}_E(3p) \cong \mathcal{O}_E(1)$ . By Corollary 4.2 in case  $g = 1$ , there are nine of them. Now let  $\mathbb{P}^9$  be the space of plane cubics, and let

$$\Phi := \{(E, p) \in \mathbb{P}^9 \times \mathbb{P}^2 \mid p \text{ is a flex of } E\}.$$

Show that the monodromy group of  $\Phi \rightarrow \mathbb{P}^9$  is a proper subgroup of  $S_9$ .

Hint: the line joining any two flexes of a plane cubic  $E$  contains a third.

**Exercise 5.7.** Let  $C$  be a smooth projective curve. Show that if  $\pi : C \rightarrow \mathbb{P}^1$  is a simply branched cover of degree  $n$ , then the monodromy of the map  $\pi$  is the full symmetric group  $S_n$ .

Hint: Show that a transitive subgroup of  $S_n$  generated by transpositions is all of  $S_n$ .

# Brill-Noether theory and applications to genus 6

## 1. What linear series exist?

Let's start with a naive question: when does there exist a curve  $C$  of genus  $g$  and a  $g_d^r$  on  $C$ —equivalently, a line bundle  $\mathcal{L}$  of degree  $d$  on  $C$  with  $h^0(\mathcal{L}) \geq r + 1$ ? The Riemann-Roch and Clifford theorems together provide a complete answer to this question:

**Theorem 1.1.** *There exists a curve  $C$  of genus  $g$  and a line bundle  $\mathcal{L}$  of degree  $d$  on  $C$  with  $h^0(\mathcal{L}) \geq r + 1$  if and only if*

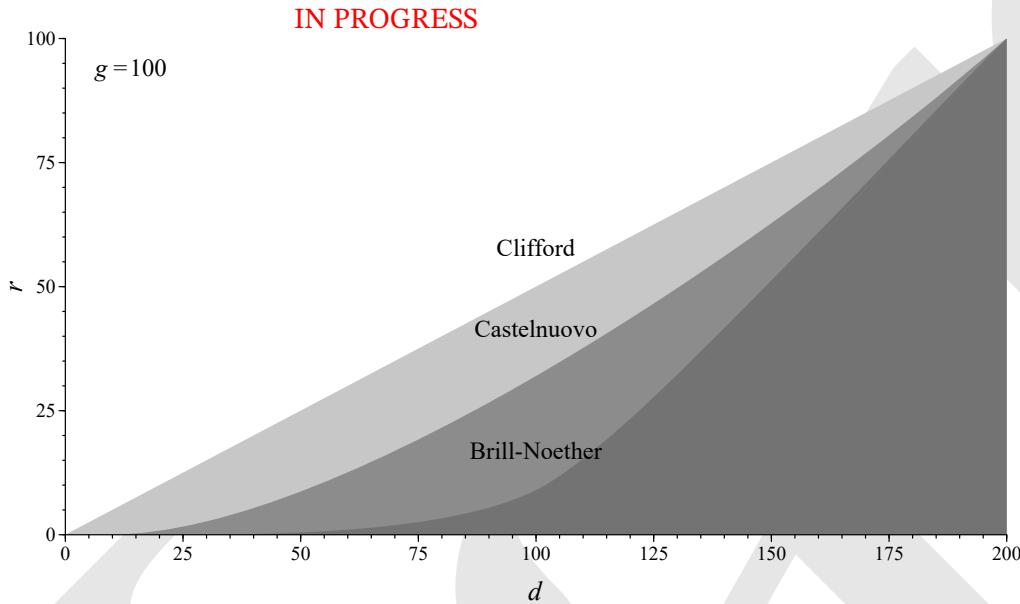
$$r \leq \begin{cases} d - g, & \text{if } d \geq 2g - 1; \text{ and} \\ d/2, & \text{if } 0 \leq d \leq 2g - 2. \end{cases}$$

For the—perhaps more interesting—question of when there exists a curve of genus  $g$  with a birationally very ample  $g_d^r$ , Castelnuovo's theorem gives a quadratic bound, roughly  $d \geq \sqrt{g(2r - 2)}$ .

In both these situations, the curves that achieve the bounds are quite special. Perhaps the most interesting question of all is, for which  $r, d$  do *all* curves of genus  $g$  have a  $g_d^r$ , and what is the behavior of these series on a general curve? Brill-Noether theory provides some answers to both these questions.

## 2. Brill-Noether theory

The following result was stated by Brill and Noether in 1874, and finally proven in a series of works by [Kempf 1971], [Kleiman and Laksov 1972], [Kleiman



**Figure 1.** For smooth curves of genus 100, these are bounds on  $(d, r)$  for all linear series (Clifford), birationally very ample series (Castelnuovo), and all linear series on general curves (Brill-Noether).

and Laksov 1974], [Kleiman 1976] culminating in a paper by Griffiths and the second author [Griffiths and Harris 1980].

**Theorem 2.1** (Basic Brill Noether). *If  $r \geq 0$  and*

$$\rho(g, r, d) := g - (r + 1)(g - d + r) \geq 0,$$

*then every smooth projective curve of genus  $g$  possesses a  $g_d^r$ . Conversely, if  $\rho < 0$  then a general curve  $C$  of genus  $g$  does not possess a  $g_d^r$ .*

It is interesting to compare the values of  $d, r$  that are possible on special and general curves; see Figure 1.

Gathering the inequalities, and putting them all in terms of lower bounds on  $d$  given  $g, r$ , we get

$$d \geq \min\{r + g, 2r\} \text{ by the Riemann-Roch and Clifford theorems}$$

$$d \geq \sqrt{(2r - 2)g} \text{ by an approximation to the Castelnuovo theorem}$$

$$d \geq r + g - \frac{g}{r + 1} \text{ for a general curve.}$$

In the following sections, we'll explain the heuristic argument that led Brill and Noether to the statement of Theorem 2.1 and discuss some refinements. In

Chapter 13 we'll give a proof based on the study of inflections and on families of Jacobians.

The case  $r = 1$  is already interesting:

**Corollary 2.2.** *If  $C$  is any curve of genus  $g$ , then  $C$  admits a map to  $\mathbb{P}^1$  of degree  $d$  for some  $d \leq \lceil \frac{g+2}{2} \rceil$ .*

Thus any curve of genus 2 is hyperelliptic, any curve of genus 3 or 4 is either hyperelliptic or trigonal (admits a 3-1 map to  $\mathbb{P}^1$ ), and so on. We have already verified this assertion in genus  $g \leq 5$  by analyzing the geometry of the canonical map; for higher genera, though, this is not feasible.

Note also that this is exactly the converse to Corollary 5.2 of Chapter 7.

**2.1. A Brill-Noether inequality.** The proof of the Brill-Noether theorem starts with a dimension estimate that was first carried out by Brill and Noether in 1874 [Brill and Nöther 1874]. The estimate provides an inequality on the dimension of the variety  $W_d^r$ , and the assertion of the theorem is that this is sharp for a general curve.

Let  $C$  be a smooth projective curve of genus  $g$ , and  $D = p_1 + \dots + p_d$  a divisor on  $C$ . Assume for simplicity that the points  $p_i$  are distinct; the same argument can be carried out in general, but requires more complicated notation.

When does the divisor  $D$  move in an  $r$ -dimensional linear series? By the Riemann-Roch theorem  $h^0(D) \geq r + 1$  if and only if the vector space  $H^0(K - D)$  of 1-forms vanishing on  $D$  has dimension at least  $g - d + r$ —that is, if and only if the evaluation map

$$H^0(K) \rightarrow H^0(K|_D) = \bigoplus k_{p_i}$$

has rank at most  $d - r$ .

We can represent this map by a  $g \times d$  matrix. Choose a basis  $\omega_1, \dots, \omega_g$  for the space  $H^0(K)$  of 1-forms on  $C$ ; choose an analytic open neighborhood  $U_j$  of each point  $p_j \in D$  and choose a local coordinate  $z_j$  in  $U_j$  around each point  $p_j$ , and write

$$\omega_i = f_{i,j}(z_j) dz_j$$

in  $U_j$ . We have  $r(D) \geq r$  if and only if the matrix-valued function

$$A(z_1, \dots, z_d) = \begin{pmatrix} f_{1,1}(z_1) & f_{2,1}(z_1) & \dots & f_{g,1}(z_1) \\ f_{1,2}(z_2) & f_{2,2}(z_2) & \dots & f_{g,2}(z_2) \\ \vdots & \vdots & & \vdots \\ f_{1,d}(z_d) & f_{2,d}(z_d) & \dots & f_{g,d}(z_d) \end{pmatrix}$$

has rank  $d - r$  or less at  $(z_1, \dots, z_d) = (0, \dots, 0)$ .

In the space  $M_{d,g}$  of  $d \times g$  matrices, the subset of matrices of rank  $d - r$  or less has codimension  $r(g - d + r)$  ([Eisenbud 1995a, Exercise 10.9]). It follows that if an effective divisor  $D$  of degree  $d$  with  $h^0(D) \geq r + 1$  exists, then in a neighborhood of the point  $D \in C_d$  the locus  $C_d^r$  of such divisors must have dimension at least  $d - r(g - d + r)$ , with equality if the map  $A$  is dimensionally transverse to the locus in  $M_{d,g}$  of matrices of rank at most  $d - r$ . Since a general fiber of the map  $\mu : C_d^r \rightarrow W_d^r(C)$  has dimension  $r$ , it follows that

$$\dim W_d^r(C) \geq d - r(g - d + r) - r = g - (r + 1)(g - d + r)$$

and this is exactly the Brill-Noether (in)equality. We will give a proof of the Brill-Noether theorem in Chapter 14.

**2.2. Refinements of the Brill-Noether theorem.** Theorem 2.1 suggests a slew of questions, both about the geometry of the schemes  $W_d^r(C)$  parametrizing linear series on a general curve  $C$  (are they irreducible? what are their singular loci,...), and about the geometry of the linear systems themselves (do they give embeddings? what's the Hilbert function of the image? ...). This is an active area of research. Here is some of what is currently known, starting with results about the geometry of  $W_d^r(C)$ :

**Theorem 2.3.** *Let  $C$  be a general curve of genus  $g$ . If we set  $\rho = g - (r + 1)(g - d + r)$ , then for  $d \leq g + r$ ,*

- (1)  $\dim(W_d^r(C)) = \rho$  ([Griffiths and Harris 1980]);
- (2) *the singular locus of  $W_d^r(C)$  is exactly  $W_d^{r+1}(C)$  ([Gieseker 1982], [Lazarsfeld 1986]);*
- (3) *if  $\rho > 0$  then  $W_d^r(C)$  is irreducible ([Fulton and Lazarsfeld 1981]);*
- (4) *if  $\rho = 0$  then  $W_d^r(C)$  consists of a finite set of points of cardinality*

$$\#W_d^r(C) = g! \prod_{\alpha=0}^r \frac{\alpha!}{(g - d + r + \alpha)!}$$

*and the monodromy of the generically finite covering of  $M_g$  by the universal family  $W_d^r$  of  $W_d^r$ 's is transitive. ([Eisenbud and Harris 1987a]).*

- (5) *if  $\mathcal{L}$  is an invertible sheaf on  $C$ , then the multiplication map*

$$m : H^0(L) \otimes H^0(\omega_C \otimes L^{-1}) \longrightarrow H^0(\omega_C)$$

*is injective, and the Zariski tangent space to the scheme  $W_d^r(C)$  at the point  $L$ , as a subspace of the tangent space  $T_L \text{Pic}_d(C) = H^0(\omega_C)^*$ , is the annihilator of the image of  $m$  or, equivalently, the kernel of the dual of  $m$  ([Gieseker 1982]).*

**Corollary 2.4.** *If  $C$  is a general curve and  $\mathcal{L}$  is a general point of  $W_d^r(C)$  with  $r \geq 2$ , then  $\mathcal{L}^m$  is nonspecial for all  $m \geq 2$ .*

**Proof.** If  $\mathcal{L}^m$  were special—that is, if  $\omega_C \otimes \mathcal{L}^{-m} = E$  were effective—then we would have an inclusion  $H^0(\mathcal{L}) = H^0(\omega_C \otimes \mathcal{L}^{-m+1}(-E)) \hookrightarrow H^0(\omega_C \otimes \mathcal{L}^{-m+1})$ . By Part 5 of Theorem 2.3, the map

$$m : H^0(\mathcal{L}^{m-1}) \otimes H^0(\omega_C \otimes \mathcal{L}^{-m+1}) \longrightarrow H^0(\omega_C)$$

is injective, so the map

$$H^0(\mathcal{L}^{m-1}) \otimes H^0(\mathcal{L}) \subset H^0(\mathcal{L}^{m-1}) \otimes H^0(\omega_C \otimes \mathcal{L}^{-m+1})$$

obtained by restriction would likewise be injective. However if  $\sigma, \tau \in H^0(\mathcal{L})$  are two linearly independent sections, then  $\sigma^{m-1} \otimes \tau - \sigma^{m-2}\tau \otimes \sigma$  lies in the kernel, contradicting the specialness of  $\mathcal{L}^m$ .  $\square$

**Remark 2.5.** (1) As a special case of Part 4 of the Theorem we see that the number of  $g_d^1$ s in the case  $\rho = 0$ , that is,  $g = 2d - 2$ , is the Catalan number  $C_{d-1} := \frac{1}{d} \binom{2d}{d}$ .

We have already seen this in the first two cases: in genus 2, it says the canonical series  $|K|$  is the unique  $g_2^1$  on a curve of genus 2, and in the case of genus 4 we have already seen that there are exactly two  $g_3^1$ s on a general curve of genus 4. In genus 6, it says that a general curve of genus 6 has 5  $g_4^1$ s; we'll describe these in Section 3 below.

- (2) Part 5 and Part 1 imply Part 2. [Arbarello et al. 1985, Section IV.4] shows that at a point  $\mathcal{L} \in W_d^r(C) \setminus W_d^{r+1}(C)$ , the tangent space to  $W_d^r$  at the point  $\mathcal{L}$  is the annihilator in  $(H^0(\omega_C))^*$  of the image of  $\mu$ ; given that  $\mu$  is injective, we can compare dimensions and deduce that  $W_d^r$  is smooth at  $\mathcal{L}$ .
- (3) For any curve  $C$ , there exists a scheme  $G_d^r(C)$  parametrizing linear series of degree  $d$  and dimension  $r$ ; that is, in set-theoretic terms,

$$G_d^r(C) = \{(\mathcal{L}, V) \mid \mathcal{L} \in \text{Pic}_d(C), \text{ and } V \subset H^0(\mathcal{L}) \text{ with } \dim V = r + 1\}.$$

$G_d^r(C)$  maps to  $W_d^r(C)$ ; the map is an isomorphism over the open subset  $W_d^r(C) \setminus W_d^{r+1}(C)$  and has positive-dimensional fibers over  $W_d^{r+1}(C)$ . It was conjectured by Petri and proven in [Gieseker 1982] that for a general curve the scheme  $G_d^r(C)$  is smooth for any  $d$  and  $r$ .

Recall that in theorems 9.9, 5.2, 5.1 we proved that general invertible sheaves of degrees  $g + 1, g + 2$  and  $g + 3$  on any curve give the nicest possible maps to (respectively)  $\mathbb{P}^1, \mathbb{P}^2$  and  $\mathbb{P}^3$ . These linear series, being general of degree  $\geq g$ , are nonspecial and have respectively 2, 3, or 4-dimensional spaces of sections. The following result shows that something similar is true on a general curve for general linear series with 2, 3, or 4-dimensional spaces of sections, though they may have degrees much less than  $g + 1, g + 2, g + 3$ :

**Theorem 2.6.** ([Eisenbud and Harris 1983, Proposition 5.4]) Let  $C$  be a general curve of genus  $g$ , and suppose that  $|D|$  is a general  $g_d^r$  on  $C$ .

- (1) If  $r \geq 3$  then  $D$  is very ample; that is, the map  $\phi_D : C \rightarrow \mathbb{P}^r$  embeds  $C$  in  $\mathbb{P}^r$ ;
- (2) If  $r = 2$  the map  $\phi_D : C \rightarrow \mathbb{P}^2$  gives a birational embedding of  $C$  as a nodal plane curve; and
- (3) If  $r = 1$ , the map  $\phi_D : C \rightarrow \mathbb{P}^1$  expresses  $C$  as a simply branched cover of  $\mathbb{P}^1$ .

In case  $\rho = 0$ —so that there are a finite number of  $g_d^r$ s on a general curve  $C$ —these statements hold for *all* the  $g_d^r$ s on  $C$ .

In the course of investigating embeddings of a curve  $C \subset \mathbb{P}^n$  we have again and again asked about the ranks of the maps  $H^0(\mathcal{O}_{\mathbb{P}^n}(d)) \rightarrow H^0(\mathcal{O}_C(d))$ . In the case of a general curve, the following theorem of [Larson 2017] gives a comprehensive answer; in particular, it gives the Hilbert function of any general embedding:

**Theorem 2.7** (E. Larson). (*Maximal Rank theorem*) If  $C$  is a general curve of genus  $g$  and  $\mathcal{L} \in W_d^r(C)$  is a general point, then for each  $m > 0$  the multiplication map

$$\rho_m : \text{Sym}^m H^0(\mathcal{L}) \rightarrow H^0(\mathcal{L}^m)$$

has maximal rank; that is, it is injective if  $\binom{m+r}{r} \leq h^0(\mathcal{L}^m)$  and surjective if  $\binom{m+r}{r} \geq h^0(\mathcal{L}^m)$ .

If  $\mathcal{L} \in W_d^r(C)$  is a general point, then Corollary 2.4 shows that  $h^0(\mathcal{L}^m) = md - g + 1$  for all  $m \geq 2$  and this allows us to compute the Hilbert function of a general embedding as a curve of degree  $d$  as

$$h_C(m) = \min\left(\binom{m+r}{r}, md - g + 1\right).$$

A key step in Larson's proof is an interpolation theorem [Atanasov et al. 2019], later extended in [Larson and Vogt 2023] to show:

**Theorem 2.8** ([Larson and Vogt 2023]). Let  $d, g$  and  $r$  be nonnegative integers with  $\rho(d, g, r) \geq 0$ . There is a general curve of degree  $d$  and genus  $g$  through  $n$  general points in  $\mathbb{P}^r$  if and only if

$$(r-1)n \leq (r+1)d - (r-3)(g-1)$$

except in the four cases  $(d, g, r) = (5, 2, 3), (6, 4, 3), (7, 2, 5)$  and  $(10, 6, 5)$ .

There is a possible extension of the maximal rank theorem. If  $C \subset \mathbb{P}^r$  is a general curve embedded by a general linear series, the maximal rank theorem

tells us the dimension of the  $m$ th graded piece of the ideal of  $C$ , for any  $m$ : this is just the dimension of the kernel of  $\rho_m$ . But it doesn't tell us the degrees of generators of the homogeneous ideal of  $C$ . For example, if  $m_0$  is the smallest  $m$  for which  $I(C)_m \neq 0$ , or numerically the smallest  $m$  such that  $\binom{m+r}{r} > md - g + 1$ , we can ask: is the homogeneous ideal  $I(C)$  generated by  $I(C)_{m_0}$ ? This can't always be the case, since there are examples where the smallest nonzero graded piece of  $I(C)$  has dimension 1. But one might conjecture that  $I(C)$  is always be generated by its graded pieces of degrees  $m$  and  $m+1$ ; this is an open problem.

To answer this—given that we know the dimensions of  $I(C)_m$  for every  $m$ —we would need to know the ranks of the multiplication maps

$$\sigma_m : I(C)_m \otimes H^0(\mathcal{O}_{\mathbb{P}^r}(1)) \rightarrow I(C)_{m+1}$$

for each  $m$ . In particular, we may conjecture that *the maps  $\sigma$  have maximal rank*; if this were true we could deduce the degrees of a minimal set of generators for the homogeneous ideal  $I(C)$ .

Another recent strand of work on Brill-Noether theory was developed in the thesis [Larson 2021] and in [Larson et al. 2008], providing analogues of many of the parts of the classical Brill-Noether theorem for general curves of given gonality in the cases  $\rho \geq 0$ .

There are many remaining questions! One is the question of *secant planes*: a naive dimension count would suggest that an irreducible, nondegenerate curve  $C \subset \mathbb{P}^r$  should have an  $s$ -secant  $t$ -plane if and only if  $s(r-t-1) \leq (t+1)(r-t)$  (for example a curve  $C \subset \mathbb{P}^3$  has 4-secant lines, but no 5-secant lines). Is this true for a general curve embedded in  $\mathbb{P}^r$  by a general linear series?

**Exercise 2.9.** We saw in Chapter 5 that if  $C$  is any curve of genus  $g$  and  $D$  a general divisor of degree  $g+3$  on  $C$ , then  $\phi_D : C \hookrightarrow \mathbb{P}^3$  is an embedding. Using the Brill-Noether theorem, show that if  $C$  is general then the image curve in  $\mathbb{P}^3$  has no 5-secant lines.

### 3. Linear series on curves of genus 6

We have seen in our analysis of curves of genus up to 5 that curves of the same genus can look quite different from the point of view of the linear series they possess: the existence of  $g_d^r$ 's, the geometry of the schemes  $W_d^r(C)$  parametrizing them, and the geometry of the associated maps to projective space, can look quite different on different curves.

The variety of possible behaviors has increased modestly with the genus. Genus 6 is a tipping point: we could still enumerate all the possible behaviors

of the schemes  $W_d^r(C)$ —as distinguished by the number of components, dimension and singularities of the various schemes  $W_d^r(C)$ , and the geometry of the associated maps—but it's quite a long list, and we will actually study just a few cases. For genus 7 and higher a full analysis has probably never been carried out.

In lower genus we tacitly verified the statements of the Brill-Noether theorem from our descriptions of the canonical models. In genus 6, by contrast, we cannot deduce the Brill-Noether theorem from studying the geometry of the canonical curve—though we can easily see that a canonical curve  $C \subset \mathbb{P}^5$  of genus 6 lies on a 6-dimensional vector space of quadrics, that doesn't tell us much about its geometry.

Instead will appeal directly to the Brill-Noether theorems. Here is a summary of what we will use:

**Theorem 3.1.** *Every smooth curve  $C$  of genus 6 has at least one  $g_6^2$ . If  $C$  is general, then  $W_6^2(C)$  and  $W_4^1(C)$  each consist of 5 reduced points, while  $W_5^2 = W_3^1 = \emptyset$ . Less formally,  $C$  has precisely 5  $g_6^2$ s and 5  $g_4^1$ s, but no  $g_5^2$ s and no  $g_3^1$ s. The image of the map associated to each  $g_6^2$  is a nodal plane curve and its nodes are in linearly general position, that is, no three are collinear.*

All these assertions except for the linear general position of the nodes follow immediately from Theorem 2.1 and Theorem 2.3; we will deduce the linear general position of the nodes from the relationship of the different linear series on a general curve that are given by these theorems.

**3.1. General curves of genus 6.** *We suppose for the rest of this section that  $C$  is a general smooth curve of genus 6.*

By Theorem 3.1 we can map  $C$  birationally to a plane sextic  $C_0$  with only nodes as singularities. Since a plane sextic has arithmetic genus  $\binom{6-1}{2} = 10$ , the curve  $C_0$  must have exactly 4 nodes.

Once we have exhibited one birational map of  $C$  to a plane sextic with 4 nodes, we can describe all five  $g_6^2$ s and all five  $g_4^1$ s in terms of this plane model. For example, composing a  $g_6^2$  corresponding to  $f : C \rightarrow \mathbb{P}^1$  with the projections from the 4 nodes gives four  $g_4^1$ s. To see the 5th  $g_4^1$  we introduce some terminology:

Suppose  $f : X \rightarrow S$  is a regular map from any smooth curve  $X$  to a surface  $S$ . If  $\mathcal{L}$  is a line bundle on  $S$  and  $V \subset H^0(\mathcal{L})$  a vector space of sections, we can associate to them a linear system on  $X$  by taking the pullback linear system  $f^*V \subset H^0(f^*\mathcal{L})$  on  $X$  and subtracting the base points; this is called the *linear series cut out on  $X$  by  $V$* .

To see the  $g_4^1$ s on  $C$  in this way, suppose again that  $C$  is a general curve of genus 6 as above and  $f : C \rightarrow \mathbb{P}^2$  is a birational map onto a sextic curve  $C_0$  with four nodes; let  $p \in C_0$  be one of the nodes and consider the linear system  $(\mathcal{O}_{\mathbb{P}^2}(1), V)$  of lines in  $\mathbb{P}^2$  through  $p$ . The pullback  $f^*\mathcal{O}_{\mathbb{P}^2}(1)$  of course has degree 6, but the pullback linear series  $f^*V$  has two base points, at the points  $q, r \in C$  lying over  $p$ . The linear series cut on  $C$  by  $V$  is thus a  $g_4^1$ .

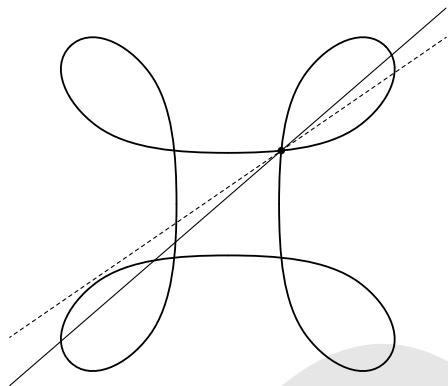
To produce the fifth  $g_4^1$ , consider the linear series cut on  $C$  by conic plane curves passing through all four nodes of  $C_0$ . There is a pencil of such conics, and the pullback  $f^*\mathcal{O}_{\mathbb{P}^2}(2)$  has degree 12. If the nodes are linearly independent then the pullback series has eight base points; thus we arrive at another  $g_4^1$  on  $C$ . Not all the nodes can be contained in a line, since then, by Bézout's theorem, the line would be a component of  $C_0$ . Thus if the nodes are linearly dependent, then exactly 3 lie on a line so the linear series cut by the conics containing the nodes coincides with the projection from the 4th node. This would represent a non-reduced point of the scheme  $W_4^1(C)$ , the subject of Exercise 5.8 below. Thus the nodes are independent.

For another example, consider the linear system cut on  $C$  by cubics passing through all four nodes. This has degree  $3 \cdot 6 - 8 = 10$  and dimension 6. It follows that this is the complete canonical series on  $C$ . (In Chapter 15 we will see directly that this is the case.)

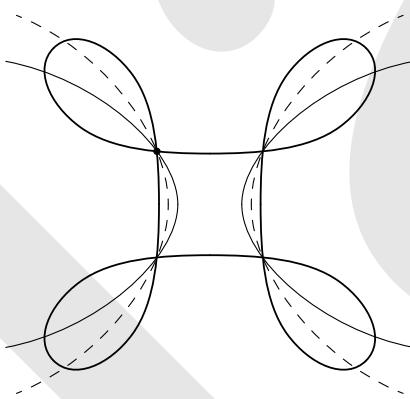
Given the degree six map  $f : C \rightarrow C_0 \subset \mathbb{P}^2$  corresponding to one  $g_6^2$  we can use the fact that the five  $g_6^2$ s on  $C$  are residual to the five  $g_4^1$ s in the canonical series to construct the other four  $g_6^2$ s: they are cut out on  $C$  by the linear system of plane conics passing through three of the four nodes of  $C_0$ . Equivalently, their images are the curves obtained from  $C_0$  by the quadratic transformation of  $\mathbb{P}^2$  centered at 3 of the 4 nodes, which blows up these 3 nodes and blows down the three lines joining them.

In previous chapters we have seen that in genus  $\leq 5$  a general canonical curve is a complete intersection, but this fails for a canonical curve  $C$  of genus 6. There is a 21-dimensional vector space of quadratic forms on  $\mathbb{P}^5$ , and  $h^0(\mathcal{O}_C(2)) = 2(2g - 2) - g + 1 = 15$ , so  $C$  lies on at least 6 quadrics, and we will show that its ideal sheaf is generated by exactly 6 quadrics. Since  $6 > \text{codim } C$ , the canonical curve of genus 6 is not a complete intersection. However, such curves lie on a quintic del Pezzo surface, which may be described as follows.

**3.2. Del Pezzo surfaces.** We met the del Pezzo surface of degree 5 in Section 4.5. We briefly sketch, without proofs, a little of the rich classical theory of del Pezzo surfaces in general. The basics are well treated in [Beauville 1996, pp. 45–50]; for more, see the beautiful book [Manin 1986], which also goes into some of the arithmetic theory. We will use only the case of the del Pezzo surface of degree 4, which lies in  $\mathbb{P}^5$



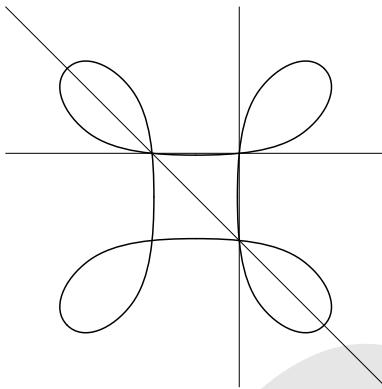
**Figure 2.** A  $g^1_4$  as the projection from a node of a plane sextic. Silvio: both lines of the projection should meet the sextic 4 times; the dashed line could be tangent. In general, perhaps our convention for showing a map by projection should be to show more than two projection lines (and note that they are all on the same footing).



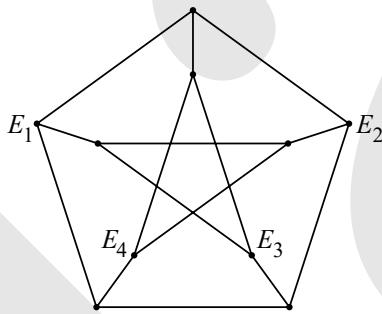
**Figure 3.** A  $g^1_4$  as a pencil of hyperbolas through the four nodes of a plane sextic.

By definition, a *del Pezzo* surface is a smooth surface embedded in  $\mathbb{P}^n$  by its complete anticanonical series  $-K_S$ . These exist only for  $3 \leq n \leq 9$ . The best-known example is a smooth cubic surface in  $\mathbb{P}^3$ . That it is a del Pezzo surface follows from the adjunction formula.

A del Pezzo surface in  $\mathbb{P}^n$  has degree  $n$ , and is isomorphic to the blow-up of  $\mathbb{P}^2$  at  $9 - n$  points of which no 3 lie on a line and no 6 lie on a conic, embedded by the linear series on  $\mathbb{P}^2$  consisting of the cubics passing through the  $9 - n$  points—except when  $n = 8$ , in which case the linear series of curves of type  $(2, 2)$  on  $\mathbb{P}^1 \times \mathbb{P}^1$  provides another example.



**Figure 4.** A sextic with 4 nodes and the fundamental triangle of the quadratic transformation giving a different  $g_6^2$ . Silvio: nice picture. Might be better to use color for the three lines of the distinguished triangle. Is the sextic mathematically correct? If not, it might be clearer if the lines of the triangle weren't so tight to the sextic—they are a little hard to distinguish.



**Figure 5.** Dual graph of the configuration of 10 lines on a quintic del Pezzo surface, the plane blown up at 4 points showing 4 pairwise disjoint exceptional divisors.

Comparing the linear series of cubic forms containing  $p_1, \dots, p_4$  with the linear series of sextic forms vanishing to order 2 at  $p_1, \dots, p_4$ , we see that a quintic del Pezzo surface  $S \subset \mathbb{P}^5$  lies on at least 5 quadrics. In fact, its homogeneous ideal is generated by exactly 5 quadrics.

A quintic del Pezzo surface  $S \subset \mathbb{P}^5$  contains exactly 10 lines, which (in terms of the description of  $S$  as the blow up of  $\mathbb{P}^2$  at four points  $p_1, \dots, p_4 \in \mathbb{P}^2$ ) are the 4 exceptional divisors and the 6 proper transforms of the lines joining the  $p_i$  pairwise. It is the intersection of a  $\mathbb{P}^5$  with the Grassmannian  $G(2, 5) \subset \mathbb{P}^9$ , and correspondingly the five quadrics containing  $S$  can be realized as the Pfaffians of a  $5 \times 5$  skew-symmetric matrix of linear forms on  $\mathbb{P}^5$ .

There is also a notion of a *weak del Pezzo* surface; this is a smooth surface whose anticanonical bundle is nef but not necessarily ample. We get such a

surface if we blow up  $\mathbb{P}^2$  at a configuration of points of which three are collinear; in this circumstance the anticanonical bundle on the blow-up  $S$  has degree 0 on the proper transform of the line containing the three points, and this proper transform is correspondingly collapsed to a rational double point of the image  $\phi_{-K}(S)$ . In general, the description of del Pezzo surfaces as blow-ups of the plane extends to the case of weak del Pezzos.

**3.3. The canonical image of a general curve of genus 6.** Using the Brill-Noether theorem, we have seen that a general curve  $C$  of genus 6 is the normalization of a plane sextic  $C_0$  with four nodes, and that the canonical series on  $C$  is cut out by cubics in the plane passing through the four nodes. Thus the canonical model lies on the surface  $S \subset \mathbb{P}^5$  that is the image of the plane under the (rational) map given by cubics through these four points, which we now recognize as a quintic del Pezzo surface.

**Theorem 3.2.** *A general canonical curve  $C$  of genus 6 is the intersection of a quintic del Pezzo surface and a quadric.*

**Proof.** We have seen that  $C \subset \mathbb{P}^5$  lies on a quintic del Pezzo surface  $S \subset \mathbb{P}^5$ . The surface is cut out by 5 quadrics, and we know that  $C$  lies on 6 independent quadrics, so  $C$  is contained in the complete intersection of  $S$  with a quadric. Since this scheme has degree 10, which is the degree of  $C$ , they are equal.  $\square$

For corresponding theorems for general curves with  $g = 7, 8, 9$  see [Mukai 1995a], [Mukai 2010], and [Mukai 1995b].

#### 4. Other curves of genus 6

From Theorem 3.1 we know that every curve  $C$  of genus 6 has a  $g_6^2$ . For a general curve the corresponding morphism  $\phi_D$  maps  $C$  birationally onto a plane sextic with 4 nodes and exactly 5  $g_4^1$ s. In this section we will analyze some of the other curves of genus 6, starting with the question “what could go wrong?” with the birational map to  $\mathbb{P}^2$ .

Since we already have a complete picture in the hyperelliptic case, we’ll assume that  $C$  is non-hyperelliptic. Clifford’s theorem then rules out the existence of a  $g_6^3$  so the  $g_6^2$  is a complete linear series  $|D|$  for some divisor  $D$  of degree 6.

Further analysis can be divided as follows:

- (1)  $C$  is not trigonal. Then either
  - (a)  $|D|$  has a base point; the canonical image of  $C$  lies on the Veronese surface.

- (b)  $\phi_D$  maps  $C$  two to one onto a cubic  $E \subset \mathbb{P}^2$  of genus 1; the canonical image is the complete intersection of a quadric and the cone over the elliptic quintic  $E \subset \mathbb{P}^4$ .
  - (c)  $\phi_D$  is birational onto a plane sextic with no triple point.
- (2)  $C$  is trigonal. The canonical image lies on a rational normal scroll  $S(a, 4 - a)$ ; see Chapter 17. In this case  $|D|$  is base-point free and  $\phi_D$  either maps  $C$  three to one onto a conic or birationally onto a sextic with a triple point.

In the rest of this section we will examine cases 1a and 1b. Case 2 can be further divided by the value of  $a$ . Case 1c may be divided into many parts according to the various configurations of double points of the plane sextic, which may be nodes, cusps, tacnodes or higher-order double points, corresponding to various possible schemes  $W_4^1$ .

The analysis of the other cases is lengthy, but largely accessible with the tools we've introduced.

**4.1.  $|D|$  has a base point.** Clifford's theorem shows that a nonhyperelliptic curve of genus 6 cannot have a  $g_3^2$ , so  $|D|$  has exactly one base point, and when we subtract the base point we get a base-point-free  $g_5^2$ .

If  $C \rightarrow C_0 \subset \mathbb{P}^r$  is the map given by a base-point free  $g_d^r$ , the degree  $d$  of the linear series is the degree of the image curve  $C_0$  times the degree of the map  $C \rightarrow C_0$ . Since 5 is prime, the associated map  $\phi_D : C \rightarrow \mathbb{P}^2$  is birational onto a quintic curve. Moreover, since plane quintic curves have arithmetic genus 6, the image  $\phi_D(C)$  is smooth; thus  $C$  is isomorphic to a smooth plane quintic.

This allows us to describe the other special linear series on  $C$ . By the adjunction formula, the canonical series  $|K_C|$  is cut on  $C$  by conics in the plane.

The plane  $\mathbb{P}^2$  is embedded by the complete linear series of quadrics as the Veronese surface in  $\mathbb{P}^5$ , and since the canonical series of  $C$  is cut out by quadrics, the canonical model of  $C$  lies on this surface, and the canonical ideal is generated by the 6-dimensional family of quadrics containing the Veronese surface—the  $2 \times 2$  minors of the generic symmetric  $3 \times 3$  matrix corresponding to the multiplication map

$$H^0(\mathcal{O}_{\mathbb{P}^2}(1)) \otimes H^0(\mathcal{O}_{\mathbb{P}^2}(1)) \rightarrow H^0(\mathcal{O}_{\mathbb{P}^2}(2))$$

as explained in Proposition 2.3—together with a 3-dimensional family of cubics, the image of the 3 dimensional family of forms of degree 6 that are multiples of the quintic form defining  $C$  in  $\mathbb{P}^2$ .

As we showed in Chapter 4, the  $g_4^1$ s on  $C$  are exactly the projections from points of  $C$ , so  $W_4^1(C) \cong C$ , and there are no  $g_3^1$ s.

**4.2.  $C$  is not trigonal and the image of  $\phi_D$  is two to one onto a plane curve of degree 3.** The cubic curve  $E$  is smooth since otherwise it would have geometric genus 0 and  $C$  would be hyperelliptic. Thus  $C$  is a double cover of a smooth curve of genus 1; we say that  $C$  is *bielliptic*.

In this case the canonical divisor class  $K_C$  is the pullback of an invertible sheaf  $\mathcal{O}_E(F)$  for some divisor class of degree 5 on  $E$ . But it is not the case that the canonical series  $|K_C|$  is the pullback of the linear series  $|\mathcal{O}_E(F)|$ : by the Riemann-Roch theorem, the latter has dimension 4, rather than 5. Indeed, if we recall that the target of the canonical map  $\phi_K : C \rightarrow \mathbb{P}^5$  is the projective space  $\mathbb{P}H^0(K_C)$ , there is a point  $X \in \mathbb{P}^5$  corresponding to the hyperplane  $\pi^*H^0(F) \hookrightarrow H^0(K_C)$ , and projection of the canonical curve from this point maps  $C$  2-to-1 onto the image  $\phi_F(E) \subset \mathbb{P}^4$ . In other words, the canonical model of  $C$  lies on a cone  $S = \overline{X, E}$  over an elliptic normal quintic curve  $E \subset \mathbb{P}^4$ .

As we saw in Chapter 4, the quintic curve  $\phi_F(E) \subset \mathbb{P}^4$  lies on 5 quadrics, as does the cone  $S \subset \mathbb{P}^5$  over it. Thus there is a quadric  $Q \subset \mathbb{P}^5$  containing  $C$  but not containing  $S$ . Bézout's theorem shows that in this case, the canonical model of a bielliptic curve of genus 6 is the intersection of the cone over an elliptic quintic curve with a quadric.

## 5. Exercises

**Exercise 5.1.** Let  $C$  be a bielliptic curve of genus 6; that is, a double cover of a smooth projective curve  $E$  of genus 1.

- (1) Show that  $C$  cannot be hyperelliptic (going forward, we will identify  $C$  with its canonical image in  $\mathbb{P}^5$ ).
- (2) Let  $F$  an invertible sheaf of degree 5 on  $E$  and  $\phi_F(E) \subset \mathbb{P}^4$  the corresponding elliptic normal quintic curve. Show that  $\phi_F(E)$  lies on 5 quadrics, as does the cone  $S \subset \mathbb{P}^5$  over it
- (3) Deduce that there is a quadric  $Q \subset \mathbb{P}^5$  containing  $C$  but not containing  $S$ . Now invoke Bézout's theorem to deduce that in this case, the canonical model of a bielliptic curve of genus 6 is the intersection of the cone over an elliptic quintic curve with a quadric.

Hint: For the second part, we know that  $\phi_F(E)$  lies on at least 5 quadrics by the usual restriction sequence; if it lay on 6 or more it would be a rational normal curve. (Alternatively, see Section 4.5.)

**Exercise 5.2.** Use the preceding exercise to show that if  $C$  is a bielliptic curve of genus 6, a 2-sheeted cover of an elliptic curve  $E$ , then every  $g_4^1$  on  $C$  is the pullback of a  $g_2^1$  on  $E$ , and likewise every  $g_6^2$  on  $C$  is the pullback of a  $g_3^2$  on  $E$ . Deduce that in this case,  $W_4^1(C)$  and  $W_6^2(C)$  are each isomorphic to  $E$ .

Hint: Use the description of the canonical model of  $C$  and the geometric Riemann-Roch theorem.

**Exercise 5.3.** Let  $C$  be a trigonal curve of genus 6 with  $g_3^1 |E|$ , and  $p \in C$  a general point. Show that the linear series  $|K_C - E - p|$  is a  $g_6^2$ , and that the corresponding map  $C \rightarrow \mathbb{P}^2$  maps  $C$  birationally onto a plane sextic curve with a triple point.

**Exercise 5.4.** Let  $C$  be the normalization of a plane sextic  $C_0$  with four nodes, three of which are collinear. Show that by choosing a different  $g_6^2$  on  $C$ , we can express it as the normalization of a plane sextic with two nodes and a tacnode.

Hint: Take the image of  $C$  under the map associated to the  $g_6^2$  cut out by conics through two of the three collinear nodes and the one remaining node.

**Exercise 5.5.** Show that if  $C$  is the normalization of a plane sextic  $C_0$  with only double points, then  $W_4^1(C) \cong W_6^2(C)$  is zero-dimensional (so in particular this case does not overlap with any of the previous cases)

**Exercise 5.6.** Find an example of a curve  $C$  of genus 6 such that  $W_4^1(C)$  consists of one point of degree 5. Bonus points for showing that in this case the scheme  $W_4^1(C) \cong \text{Spec } \mathbb{C}[\epsilon]/(\epsilon^5)$ .

Hint: the curve in question is the normalization of a plane sextic curve with one double point, consisting of two smooth branches with contact of order 4 with each other and contact of order 3 with their common tangent line. The exercise asks you to both prove that such a curve exists, and that the  $g_4^1$  cut out by lines through the double point is the unique  $g_4^1$  on  $C$ .

**Exercise 5.7.** Show that if  $C$  is a smooth plane quintic, then the  $g_6^2$ s on  $C$  all have a base point; that is, they are all of the form  $|K_C| + p$  for  $p \in C$ .

Furthermore, the canonical model of  $C$  will lie on a quadratic Veronese surface  $S$ ; and the six quadrics containing the canonical curve  $C$  are the six quadrics containing  $S$  (in particular, the intersection of the quadrics containing  $C$  will be  $S$ , so the ideal of  $C$  requires generators of degree  $> 2$ )

**Exercise 5.8.** Show that if the nodes of the curve  $C_0$  are in linear general position—■ that is, no three collinear—then indeed the map  $\mu : H^0(D) \otimes H^0(K - D) \rightarrow H^0(K)$  is an isomorphism for each of the five  $g_4^1$ s on  $C$ .



# Inflection points

Generalizing the ramification points of a map from a smooth curve  $C$  to  $\mathbb{P}^1$ , there are finitely many “special” points determined by any linear series on any curve, the *inflection points*. We love this topic for its echoes of classical algebraic geometry and it will provide the tools to give a proof of the Brill-Noether theorem in the following chapter. In characteristic 0 every linear series has finitely many inflection points, and the number of these, properly counted, depends only on the genus of the curve and the degree of the linear series.

## 1. Inflection points, Plücker formulas and Weierstrass points

**1.1. Definitions.** To characterize the inflection points of a linear series  $\mathcal{D} = (\mathcal{L}, V)$  on a curve  $C$ , we will use the following result:

**Proposition 1.1.** *Let  $V$  be a finite-dimensional vector space of global sections of an invertible sheaf  $\mathcal{L}$  on a smooth curve  $C$ , and let  $p \in C$  be a point. There exists a basis  $\sigma_0, \dots, \sigma_r$  of  $V$  consisting of sections vanishing to different orders at  $p$ . Thus the set*

$$\{\text{ord}_p(\sigma) \mid \sigma \neq 0 \in V\}$$

*has cardinality  $\dim V$ .*

**Proof.** Let  $\tau_0, \dots, \tau_r$  be any basis of  $V$ . If  $\tau_i$  and  $\tau_j$  vanish to the same order at  $p$ , then some nonzero linear combination  $\tau'_i := a\tau_i + b\tau_j$  vanishes to strictly higher order. Since the coefficients  $a$  and  $b$  are both necessarily nonzero we may modify the basis, replacing  $\tau_i$  with  $\tau'_i$ , strictly increasing the sum of the orders. The order of vanishing of each  $\sigma_i$  is bounded above by  $\deg \mathcal{L}$ , so the sum of the orders is bounded by  $(r+1)\deg \mathcal{L}$ . Thus the process must terminate, and when it does, the orders must be distinct.  $\square$

According to Proposition 1.1, if  $D = (\mathcal{L}, V)$  is any  $g_d^r$  on  $C$ , we may write

$$\{\text{ord}_p(\sigma) \mid \sigma \neq 0 \in V\} = \{a_0, \dots, a_r\} \text{ with } 0 \leq a_0 < a_1 < \dots < a_r.$$

The sequence  $a_i = a_i(D, p)$  is called the *vanishing sequence* of  $D$  at  $p$ . Since  $a_i \geq i$ , the numbers  $\alpha_i = \alpha_i(D, p) := a_i - i$  are often more interesting, and the sequence  $0 \leq \alpha_0 \leq \alpha_1 \leq \dots \leq \alpha_r$  is called the *ramification sequence* of  $D$  at  $p$ .

We say that  $p$  is an *inflection point* of the linear series  $D$  if  $(\alpha_0, \dots, \alpha_r) \neq (0, \dots, 0)$ —equivalently, if  $\alpha_r > 0$ —and we define the *weight* of  $p$  to be

$$w(D, p) = \sum_{i=0}^r \alpha_i(D, p).$$

If  $D$  is very ample, so that it may be viewed as the linear series cut on  $C$  by hyperplanes for some embedding  $C \subset \mathbb{P}^r$ , then  $p$  is an inflection point if  $a_r > r$ ; that is, if there is a hyperplane  $H \subset \mathbb{P}^r$  having contact of order  $r + 1$  or more with  $C$  at  $p$ .

The first two terms in the ramification sequence are particularly important:  $\alpha_0(D, p)$  is nonzero if and only if  $p$  is a base point of  $D$ ; and if  $\alpha_0(D, p) = 0$ , then  $\alpha_1(D, p) = 0$  if and only if, in addition, the map  $\phi_D$  is an immersion (that is, has nonzero derivative) at  $p$ .

**1.2. The Plücker formula.** In characteristic 0, a linear series on a smooth projective curve can have only finitely many inflection points (this can be proven directly just as in Lemma 2.1), and indeed the sum of the weights of all the inflection points depends only on the genus of the curve and the degree and dimension of the the linear series. This is the Plücker formula:

**Theorem 1.2.** *If  $C$  be a smooth curve of genus  $g$  and  $D$  is a linear series of degree  $d$  and dimension  $r$ , then*

$$(2) \quad \sum_{p \in C} w(D, p) = (r+1)d + r(r+1)(g-1).$$

For a proof, see for example [Eisenbud and Harris 2016b, Theorem 7.13].

Theorem 1.2 also holds in positive characteristic under the hypothesis that the number of inflection points is finite (equivalently, not every point is an inflection point). This may seem like an unnecessary hypothesis—it's hard to imagine a plane curve in which every point is a flex!—but in positive characteristic there are such curves; see Exercise 5.1.

As an immediate consequence of the Plücker formula, we have

**Corollary 1.3.** *If  $C \subset \mathbb{P}^r$  is a smooth nondegenerate curve with no inflection points, then  $C$  is the rational normal curve of degree  $r$ .*

**Proof.** Suppose  $C \subset \mathbb{P}^r$  is a curve of degree  $d$  and genus  $g$ . If  $C$  has no inflection points then, by the Plücker formula, we must have

$$(r+1)d + r(r+1)(g-1) = 0.$$

This immediately implies that  $g = 0$ , so that we must have  $(r+1)(d-r) = 0$  and hence  $d = r$ ; thus  $C$  is a rational normal curve.  $\square$

**Corollary 1.4.** *A nondegenerate smooth curve  $C \subset \mathbb{P}^r$  is the rational normal curve of degree  $r$  if and only if any of the following equivalent properties holds:*

- (1)  $C$  is projectively homogeneous.
- (2)  $C$  has no inflection points.
- (3) Every subscheme of length  $r+1$  contained in  $C$  spans  $\mathbb{P}^r$ .

**Proof.** Corollary 1.3 implies that the rational normal curves are the only ones without any inflection points.

By Proposition 1.7, the rational normal curve is projectively homogeneous. On the other hand, since not every point on  $C$  is an inflection point, a curve with an inflection point cannot be projectively homogeneous.

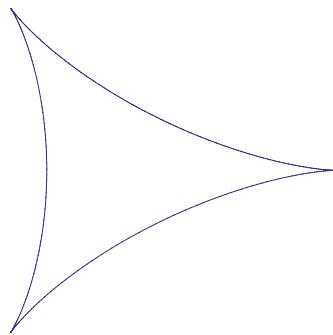
If  $p \in C$  is an inflection point, then  $(r+1)p$  is a subscheme that lies in a hyperplane whereas in Proposition 4.3 we showed that if  $C \subset \mathbb{P}^r$  is a rational normal curve, and  $\Gamma \subset C$  any proper subscheme of  $C$  of degree  $r+1$ , then  $\Gamma$  spans  $\mathbb{P}^r$ .  $\square$

The Plücker formula leaves many questions about the possible configurations of flex points unanswered. For example, how many cusps can there be on a plane curve of geometric genus  $g$  and degree  $d$ ? The answer is known only up to degree 8: [Zariski 1931] proves that a plane curve of degree 8 can have 15 but not 16 cusps. See [Calabri et al. 2014] and [Kulikov and Shustin 2015] for recent contributions and [Kharlamov and Sottile 2003] for a study of what is possible over the real numbers.

One thing we do know is the behavior of the inflection points for a general linear series on a general curve; we will state this here and prove it as a corollary to the proof of the Brill-Noether theorem in Chapter 14:

**Theorem 1.5.** *If  $C$  is a general curve of genus  $g$ ,  $\mathcal{L} \in W_d^r(C) \subset \text{Pic}_d(C)$  a general line bundle of degree  $d$  with  $h^0(\mathcal{L}) = r+1$  and  $V = H^0(\mathcal{L})$ , then every inflection point of the linear series  $D = (\mathcal{L}, V)$  has weight 1 and hence ramification sequence  $(0, \dots, 0, 1)$ .*

**1.3. Flexes of plane curves.** Specializing Theorem 1.2 to a smooth curve of degree  $d$  in the plane, and using the formula  $g = \binom{d-1}{2}$ , we see that the total number of flexes is  $3(d-2)d$ .



**Figure 1.** A quartic plane curve with 3 cusps. Silvio: doesn't need to be blue. could be reproduced. the credit could be less important (footnote?) (Image credit F. Sottile)

It turns out that the “flex divisor”  $\sum w(C, p)p$  is the intersection of  $C$  with a curve of degree  $3(d-2)$ , called the *Hessian*: If  $C$  is defined by a form  $F(x_0, x_1, x_2)$  of degree  $d$ , then the Hessian of  $C$  is the curve defined by the determinant of the *Hessian matrix* of partial derivatives

$$\text{Hess}(C) := \begin{pmatrix} \partial^2 F / \partial x_0 \partial x_0 & \partial^2 F / \partial x_0 \partial x_1 & \partial^2 F / \partial x_0 \partial x_2 \\ \partial^2 F / \partial x_1 \partial x_0 & \partial^2 F / \partial x_1 \partial x_1 & \partial^2 F / \partial x_1 \partial x_2 \\ \partial^2 F / \partial x_2 \partial x_0 & \partial^2 F / \partial x_2 \partial x_1 & \partial^2 F / \partial x_2 \partial x_2 \end{pmatrix}$$

**Theorem 1.6.** *If  $C$  is a smooth plane curve then the flex divisor of  $C$  is the intersection of  $C$  with the Hessian curve defined by  $\det \text{Hess}(C)$ .*

The proof is an exercise in Euler’s formula and matrix manipulation; see Exercise 5.3.

**1.4. Weierstrass points.** As with any extrinsic invariant of a curve in projective space, we can derive an intrinsic invariant of an abstract curve by applying the invariant to the canonical linear series. We define a *Weierstrass point* of a curve  $C$  to be an inflection point of the canonical linear series  $|K_C|$ .

Thus  $p$  is a Weierstrass point of  $C$  if there exists a differential form on  $C$  vanishing to order  $g$  or more at  $p$ . The *weight*  $w_p$  of a Weierstrass point  $p \in C$  is defined to be the weight  $w(|K_C|, p)$  of  $p$  as an inflection point of the canonical series.

The Plücker formula tells us the total weight of the Weierstrass points on a given curve  $C$ :

**Corollary 1.7.** *The sum of the weights of the Weierstrass points on a curve  $C$  of genus  $g$  is*

$$\sum_{p \in C} w_p = g^3 - g.$$

□

Note that Theorem 1.5 implies that on a general curve  $C$  of genus  $g$ , every Weierstrass point has weight 1; thus there are  $g^3 - g$  distinct Weierstrass points on  $C$ .

For example, suppose  $C$  is a curve of genus 2. The canonical series on  $C$  gives a map  $\phi_K : C \rightarrow \mathbb{P}^1$  of degree 2; the Weierstrass points of  $C$  are the 6 ramification points of this map.

In genus 3, if  $C$  is hyperelliptic then the Weierstrass points are exactly the 8 ramification points of the 2-sheeted cover  $C \rightarrow \mathbb{P}^1$ , and each has weight 3. If  $C$  is non-hyperelliptic, then it is a plane quartic curve. A general such curve has 24 ordinary flexes, which are Weierstrass points of weight 1; special quartics may have some number  $\alpha$  of *hyperflexes*—points where the tangent line has contact of order 4 with the curve—which are Weierstrass points of weight 2; in this case  $C$  has  $\alpha$  Weierstrass points of weight 2 and  $24 - 2\alpha$  Weierstrass points of weight 1. (It has been shown that  $\alpha$  cannot be 11, but all other values between 0 and 12 occur ([Vermeulen 1983]).)

**1.5. Another characterization of Weierstrass points.** The Riemann-Roch formula tells us that

$$h^0(\mathcal{O}_C(gp)) = g - g + 1 + h^0(K_C(-gp))$$

so the condition  $h^0(K_C(-gp)) \neq 0$  that  $p$  be a Weierstrass point is equivalent to the condition  $h^0(\mathcal{O}_C(gp)) > 1$ ; in other words, we have

**Proposition 1.8.** *A point  $p \in C$  is a Weierstrass point if and only if there exists a nonconstant rational function on  $C$ , regular on  $C \setminus \{p\}$  and having a pole of order at most  $g$  at  $p$ .*

Indeed, this is how Weierstrass points were characterized when they were first studied.

**1.5.1. The Weierstrass semigroup.** Proposition 1.8 suggests that we look at the set of all possible orders of pole at  $p$  of rational functions regular on  $C \setminus \{p\}$ ; that is,

$$W(C, p) := \{-\text{ord}_p(f) \mid f \in K(C) \text{ with } f \text{ regular on } C \setminus \{p\}\}.$$

This is clearly a sub-semigroup of the natural numbers  $\mathbb{N}$ ; it is called the *Weierstrass semigroup* of the point  $p$ .

Another way to characterize the condition that there exists a rational function on  $C$ , regular on  $C \setminus \{p\}$ , with a pole of order exactly  $k$  at  $p$  is to say that

$$h^0(\mathcal{O}_C(kp)) = h^0(\mathcal{O}_C((k-1)p)) + 1.$$

Applying Riemann-Roch to both sides of this equation, we see that it is equivalent to the condition

$$h^0(K_C(-kp)) = h^0(K_C((-k+1)p)).$$

In English: there exists a rational function on  $C$ , regular on  $C \setminus \{p\}$ , with a pole of order exactly  $k$  at  $p$ , if and only if there does *not* exist a regular differential on  $C$  with a zero of order exactly  $k - 1$  at  $p$ . In other words, the complement  $\mathbb{N} \setminus W(C, p)$  is exactly the vanishing sequence of the canonical series at  $p$ , shifted by 1; in particular, it has cardinality exactly  $g$ . This is called the *Weierstrass gap sequence* of the point  $p$ .

There is still much we don't know about Weierstrass points in general. Most notably, we don't know what semigroups of finite index in  $\mathbb{N}$  occur as Weierstrass semigroups; an example of Buchweitz shows that not all semigroups occur, but there are also positive results, such as the statement ([EH]) that every semigroup of weight  $w \leq g/2$  occurs, and its refinement and strengthening in [Pflueger 2018]).

## 2. Finiteness of the automorphism group

Because the Weierstrass points of a smooth projective curve  $C$  are defined intrinsically, any automorphism of  $C$  must carry Weierstrass points to Weierstrass points. We can use this observation to give a direct proof of the

**Theorem 2.1.** *If  $C$  is a smooth curve of genus  $\geq 2$  then the automorphism group  $\text{Aut}(C)$  is finite.*

We will actually prove a strong form of the assertion: the subgroup of  $\text{Aut}(C)$  of automorphisms that fix each of the finitely many Weierstrass points is either trivial or, in the case of hyperelliptic curves, just the  $\mathbb{Z}/2$  generated by the hyperelliptic involution.

We will study the fixed points of automorphisms by studying the intersection of the graph of an automorphism with the diagonal divisor in  $C \times C$  (an alternate proof uses the Lefschetz fixed point formula). For this we use the *Neron-Severi group*  $N(S)$  of a surface  $S$ . This consists of divisors modulo *numerical equivalence*—that is, in  $N(S)$  we identify divisors  $H, H'$  if for all divisors  $D$  on  $S$  we have  $H \cdot D = H' \cdot D$ . The group  $N(S)$  is a finitely generated free abelian group for any surface—in characteristic 0 it is a subgroup of the quotient of the second integral homology group by its torsion elements. The rank of  $N(S)$  is called the *Picard number* of  $S$ .

**Theorem 2.2** (Hodge index theorem). *If  $H \subset S$  is an ample divisor on a smooth projective surface, and  $D \neq 0 \in N(S)$  is a divisor class with  $D \cdot H = 0$ , then  $D^2 < 0$ ; that is, the intersection pairing is negative definite on the orthogonal complement of an ample divisor.*

**Proof.** The result follows easily from the Riemann-Roch theorem for surfaces; see [Hartshorne 1977, Theorem V.1.9].  $\square$

There is also a stronger form of the Hodge index theorem, in which the condition that  $D$  is ample is weakened to the requirement that  $D^2 > 0$ . The proof of this stronger form relies on Hodge theory; see for example [Voisin 2003] or [Griffiths and Harris 1978].

Together, the following two lemmas imply the stronger form of Theorem 2.1.

**Lemma 2.3.** *Let  $C$  be a smooth projective curve of genus  $g \geq 2$ , and  $f : C \rightarrow C$  an automorphism of  $C$ .*

- (1) *If  $f$  has  $2g + 3$  or more distinct fixed points, then  $f$  is the identity; and*
- (2) *If  $f$  has  $2g + 2$  distinct fixed points, then either  $f$  is the identity or  $C$  is hyperelliptic and  $f$  is the hyperelliptic involution.*

**Proof.** Let  $S = C \times C$ , and let  $\Delta$  and  $\Gamma \subset S$  be the diagonal and the graph of  $f$  respectively, and let  $\Phi_1$  and  $\Phi_2 \subset S$  be fibers of the two projection maps. Let  $\delta, \gamma, \varphi_1$  and  $\varphi_2$  be the classes of these curves in  $N(S)$ . The number of fixed points of  $f$  (counted with multiplicities) is the intersection number  $b = \delta \cdot \gamma$ .

We know all the other pairwise intersection numbers of these classes. To begin with, the ones involving  $\varphi_1$  or  $\varphi_2$  are obvious. From the sequence

$$0 \rightarrow \mathcal{T}_\Delta \rightarrow \mathcal{T}_{C \times C}|_\Delta \rightarrow \mathcal{N}_{\Delta/C \times C} \rightarrow 0$$

we see that the normal bundle of the diagonal is isomorphic to the tangent bundle of  $C$ , so

$$\delta^2 = 2 - 2g.$$

Since the automorphism  $id_C \times f : C \times C \rightarrow C \times C$  carries  $\Delta$  to  $\Gamma$ , we see that  $\gamma^2 = 2 - 2g$  as well.

We can now apply the index theorem for surfaces to deduce our inequality. To keep things relatively simple, let's introduce two new classes: set

$$\delta' = \delta - \varphi_1 - \varphi_2 \quad \text{and} \quad \gamma' = \gamma - \varphi_1 - \varphi_2,$$

so that  $\delta'$  and  $\gamma'$  are orthogonal to the class  $\varphi_1 + \varphi_2$ . Since  $\varphi_1 + \varphi_2$  has positive self-intersection, the index theorem tells us that the intersection pairing must be negative definite on the span  $\langle \delta', \gamma' \rangle \subset N(S)$ . In particular, the determinant of the intersection matrix

	$\delta'$	$\gamma'$
$\delta'$	$-2g$	$b - 2$
$\gamma'$	$b - 2$	$-2g$

(where again  $b = \gamma \cdot \delta$ ) is nonnegative, or equivalently,  $b \leq 2g + 2$ .  $\square$

Having established an upper bound on the number of fixed points an automorphism  $f$  of  $C$  (other than the identity) may have, it remains to find a lower

bound on the number of distinct Weierstrass points; this is the content of the next lemma.

**Lemma 2.4.** *If  $C$  is a smooth projective curve of genus  $g \geq 2$ , then  $C$  has at least  $2g + 2$  distinct Weierstrass points; and if it has exactly  $2g + 2$  Weierstrass points it is hyperelliptic.*

**Proof.** Let  $p \in C$  be any point, and  $w_1 = w_1(p), \dots, w_g = w_g(p)$  the ramification sequence of the canonical series  $|K_C|$  at  $p$ . By definition,

$$h^0(K_C(-(w_i + i)p)) = g - i.$$

Applying Clifford's theorem we have

$$g - i \leq \frac{2g - 2 - w_i - i}{2} + 1;$$

solving, we see that

$$w_i \leq i$$

and hence

$$w_p \leq \binom{g}{2}$$

where  $w_p$  is the total weight of  $p$  as a Weierstrass point. Since the total weight of the Weierstrass points on  $C$  is  $g^3 - g$  by the Plücker formula Theorem 1.2, the number of distinct Weierstrass points is at least

$$\frac{g^3 - g}{\binom{g}{2}} = 2g + 2.$$

Finally, by the strong form of Clifford's theorem (5.2), equality implies that the curve is hyperelliptic.  $\square$

### 3. Curves with automorphisms are special

In genus  $g > 2$ , curves with nontrivial automorphisms are rare. The general curve has none, and more is true:

**Lemma 3.1.** *In the moduli space  $M_g$  of smooth curves of genus  $g \geq 3$ , the locus of curves with nontrivial automorphisms has codimension  $g - 2$ ; and the only component of this codimension is the locus of hyperelliptic curves.*

**Proof.** Let  $C$  be a smooth curve of genus  $g$  admitting a nontrivial automorphism  $\phi : C \rightarrow C$ . Taking a power, we may assume  $\phi$  has prime order  $p$ . Let  $\langle \phi \rangle = \{1, \phi, \phi^2, \dots, \phi^{p-1}\}$  be the group of automorphisms of  $C$  generated by  $\phi$ , and let  $B = C/\langle \phi \rangle$  be the quotient of  $C$  by this group, so that the quotient map  $\pi : C \rightarrow B$  expresses  $C$  as a  $p$ -sheeted branched cover of  $B$ , having  $b$  branch points  $q_1, \dots, q_b$  of multiplicity  $p - 1$  and otherwise unramified. As we have

seen in Chapter 6, if we know  $B$  and the set of branch points on  $B$  then we know  $C$  up to a finite set of choices.

The point is, in a neighborhood of  $[C] \in M_g$ , the locus of curves with an automorphism of order  $p$  has dimension at most  $3h - 3 + b$ . Our claim is thus that

$$3g - 3 - (3h - 3 + b) \geq g - 2$$

or equivalently

$$2g - 1 - (3h - 3) - b \geq 0.$$

By applying the Riemann-Hurwitz formula to the map  $C \rightarrow B$ , we get

$$2g - 2 = p(2h - 2) + b(p - 1).$$

Plugging this in, our claim is equivalent to the assertion that

$$p(2h - 2) + b(p - 1) + 1 - (3h - 3) - b \geq 0$$

that is,

$$(2p - 3)h - 2p + b(p - 2) + 4 \geq 0.$$

Now, the expression on the left could be negative only if  $h = 0$ , in which case the expression reduces to

$$(b - 2)(p - 2) \geq 0;$$

and since  $h = 0$  the number  $b$  of branch points must be at least 2, so we're done.  $\square$

#### 4. Inflections of linear series on $\mathbb{P}^1$

We are in a position to describe all possible inflectionary behavior of linear series on  $\mathbb{P}^1$ . This will provide the essential ingredient in our proof of the Brill-Noether theorem in the next chapter.

Giving a  $g_d^r$  on  $\mathbb{P}^1$  amounts to choosing an  $(r+1)$ -dimensional subspace  $W$  of  $H^0(\mathcal{O}_{\mathbb{P}^1}(d))$  and thus the family of  $g_d^r$ 's is parametrized by the Grassmannian  $G(r+1, H^0(\mathcal{O}_{\mathbb{P}^1}(d))) = G(r+1, d+1)$ .

**Theorem 4.1.** *The set of  $g_d^r$ 's on  $\mathbb{P}^1$  with ramification sequence that is termwise at least as big as  $\alpha = (\alpha_0, \dots, \alpha_r)$  at a given point  $p \in \mathbb{P}^1$  is an irreducible subvariety of  $G(r+1, d+1)$  of codimension  $|\alpha| := \sum \alpha_i$ . The intersection of these subvarieties for any set of distinct points and any ramification sequences at those points is either empty or dimensionally transverse, depending only on the collection of ramification sequences.*

We will restate and prove this theorem, giving the combinatorial condition on the ramification indices, as Theorem 4.10 below. To simplify the notation, we will write  $\ell := r+1, e := d+1$ .

Let  $V = H^0(\mathcal{O}_{\mathbb{P}^1}(d))$  and, for  $p \in \mathbb{P}^1$ , write  $V_i(p)$  for the space of forms of degree  $d$  that vanish to order  $\geq e - i$  at  $p$ , and defined the *vanishing flag*  $\mathcal{V}(p)$  at  $p$  to be the chain of subspaces

$$0 \subset V_1(p) \subsetneq V_2(p) \subsetneq \cdots \subsetneq V_e(p) = V.$$

Note that  $\dim V_i(p) = i$ . A subspace  $W$  of dimension  $\ell = r + 1$  has vanishing sequence termwise  $\geq a = a_0, \dots, a_r$ , and thus ramification sequence termwise

$$\geq \alpha = (\alpha_0 = a_0 - 0, \alpha_1 = a_1 - 1, \dots, \alpha_r = a_r - r)$$

at  $p$  if and only if  $\dim W \cap V_{e-a_{\ell-i}}(p) \geq i$  for  $i = 0, \dots, \ell$ . Such a condition is called a Schubert condition on  $W$ , and we pause to describe the *Schubert varieties* in the Grassmannian  $G(\ell, e)$  that consist of subspaces satisfying such a condition. See for example [Eisenbud and Harris 2016a, Chapters 3 and 4] for a full exposition. To accord with the notation there, which indexes Schubert cycles by certain decreasing sequences, we define  $\beta_i = \alpha_{\ell-i}$  so that

$$d - r = e - \ell \geq \beta_1 \geq \cdots \geq \beta_\ell \geq 0,$$

and we write  $\beta$  for the sequence of  $\beta_i$ . Putting this together, we have:

**Proposition 4.2.** *The condition for an  $\ell$ -dimensional subspace  $W \subset H^0(\mathcal{O}_{\mathbb{P}^1}(d))$  to define a  $g_d^r$  with ramification sequence at least  $\alpha = (\alpha_0, \dots, \alpha_r)$  at  $p$  is*

$$\dim W \cap V_{e-\ell+i-\beta_i}(p) \geq i \text{ for } 0 \leq i \leq \ell.$$

where  $e := d + 1$  and  $\beta_i = \ell - \alpha_i$ .

#### 4.1. Schubert cycles.

**Definition 4.3.** A *complete flag*  $\mathcal{V}$  in an  $e$ -dimensional vector space  $V$  is a nested sequence of vector spaces

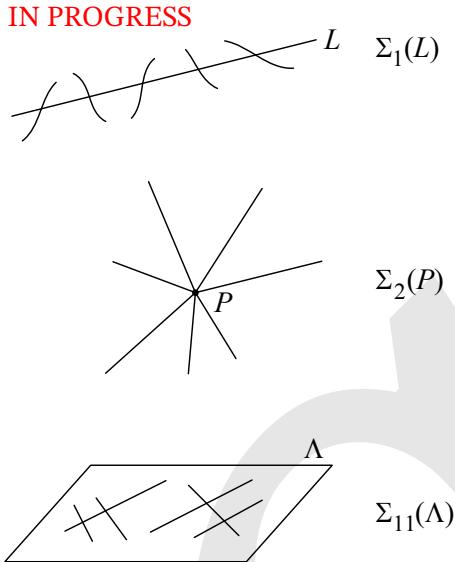
$$0 \subset V_1 \subset V_2 \subset \cdots \subset V_e = V.$$

with  $\dim V_i = i$ .

Given a flag in  $V$  and any  $\ell$ -dimensional subspace  $W \subset V$ , we can derive the nested sequence of  $e$  subspaces of  $W$ :

$$0 \subset W \cap V_1 \subset W \cap V_2 \subset \cdots \subset W \cap V_d \subset W \cap V_e = W$$

Each term in this sequence is either equal to the preceding one, or of dimension 1 greater; the former occurs  $e - \ell$  times, and the latter  $\ell$  times. For a general  $\ell$ -plane  $W$ , the jumps occur at the end; that is, we have  $W \cap V_{e-\ell} = 0$ , and thereafter the dimension of the intersection goes up by 1 each time. This makes it natural to describe the special position of a given  $\ell$ -plane by how early the  $i$ -th jump occurs:



caption  $\Sigma_1(L)$  = Lines meeting  $L \subset \mathbb{P}^3$ .  $\Sigma_2(p)$ : Lines passing through  $p \in \mathbb{P}^3$ .  
 $\Sigma_{1,1}(\Lambda)$ : Lines in the plane  $\Lambda \subset \mathbb{P}^3$ . Silvio: I realize that this is in process, but here are some notes: 1. the lines meeting  $L$  should look like lines 2. all the lines in  $\Lambda$  should meet each other 3. The point  $p$  is lowercase in the description, should be lowercase in the illustration 4. The lines through  $p$  should look as non-planar as possible. 5. Silvio: let's use color here

**Definition 4.4.** A *Schubert index* for  $G(\ell, e)$  is a sequence  $\beta = (\beta_1, \dots, \beta_\ell)$  of integers with  $e - \ell \geq \beta_1 \geq \beta_2 \geq \dots \geq \beta_\ell \geq 0$ . The *Schubert cycle*  $\Sigma_\beta(V) \subset G$  associated to a complete flat  $V$  in  $\mathbb{C}^e$  and Schubert index  $\beta$  is

$$\Sigma_\beta(V) := \{W \in G \mid \dim(W \cap V_{e-\ell+i-\beta_i}) \geq i \ \forall i\}$$

In particular, if  $V(p)$  is the vanishing flag for forms of degree  $d$  at a point  $p \in \mathbb{P}^1$ , then a  $g_d^r$  has ramification sequence (termwise)  $\geq \alpha$  at  $p$  if and only if  $W$  belongs to the Schubert variety  $\Sigma_\beta(V(p))$ , where  $\beta_i = \alpha_{\ell-i}$  as above.

For example, the Schubert cycle  $\Sigma_{0,\dots,0}$  is the whole Grassmannian, and  $\Sigma_{1,0,\dots,0}(V)$  is the set of  $\ell$ -planes that meet  $V_{e-\ell}$  nontrivially, which is a hyperplane section of the Grassmannian in its Plücker embedding. More generally, the *special Schubert cycle*  $\Sigma_\gamma(V) := \Sigma_{\gamma,0,\dots,0}(V)$  is the set of  $\ell$ -planes meeting  $V_{\gamma-\ell-\beta+1}$  nontrivially. Since this condition really involves only the single space  $U = V_{e-\ell-\gamma+1}$ , we sometimes write it as  $\Sigma_\gamma(U)$ .

For any Schubert index  $\beta$ , the codimension of  $\Sigma_\beta(V)$  in  $G(\ell, e)$  is  $|\beta| := \sum \beta_i$  (Exercise 5.2).

#### 4.2. Special Schubert cycles and Pieri's formula.

**Cheerful Fact 4.5.** *The variety of complete flags in  $\mathbb{C}^e$  is rational, and it follows that the class of  $\Sigma_\beta(V) \subset G$  in the Chow ring  $A^*(G(\ell, e))$  of the Grassmannian is independent of the flag  $V$ . It is typically denoted  $\sigma_\beta$ . Moreover, the classes  $\sigma_\beta$  form a basis for the Chow ring as a free abelian group.*

Thus the product  $\sigma_\beta \cdot \sigma_{\beta'}$  is a linear combination of Schubert classes, given combinatorially by the *Littlewood-Richardson rule*—see for example [Vakil 2006].  
*Pieri's formula* is the special case of the Littlewood-Richardson rule that expresses the product in the Chow ring  $A^*(G(\ell, e))$  of an arbitrary Schubert class with an arbitrary Schubert class. In Chapter 14 we will use it to describe the possible ramification behavior of rational curves with cusps.

**Cheerful Fact 4.6.**

**Proposition 4.7. (Pieri's Formula)** *If  $\sigma_\gamma$  is a special Schubert class and  $\sigma_\beta$  is an arbitrary Schubert class, then*

$$\sigma_\gamma \cdot \sigma_\beta = \sum \sigma_\delta$$

where the sum ranges over all Schubert indices  $\delta = (\delta_1, \dots, \delta_\ell)$  with

$$\sum \delta_i = \gamma + \sum \beta_i \quad \text{and} \quad \beta_i \leq \delta_i \leq \beta_{i-1} \text{ for all } i$$

For a proof, see for example [Eisenbud and Harris 2016a, Section 4.2.4].

Figure ?? illustrates the degeneration of  $\Sigma_1 \cdot \Sigma_1$  to  $\Sigma_2 \cup \Sigma_{1,1}$ .

To understand Proposition 4.7, represent the Schubert class  $\sigma_\beta$  by stacks of coins, with  $\beta_1$  coins in the first stack,  $\beta_2$  coins in the second stack, and so on. We now want to add a total of  $\gamma$  coins to the stacks; we can add any number of them to any stack (including a stack that was previously empty), with the one condition that the new height of each stack can't be larger than the previous height of the stack to its left. This interpretation makes the following corollary clear:

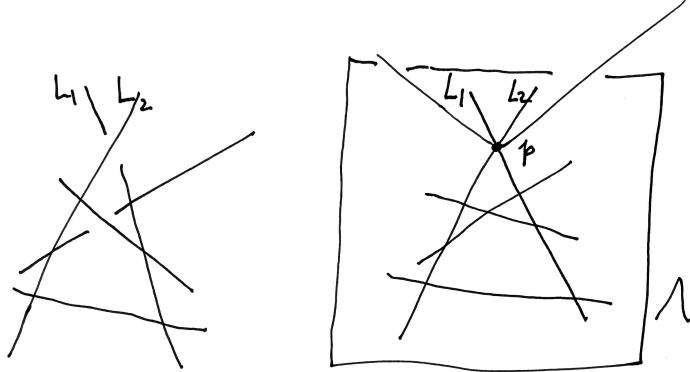
**Corollary 4.8.** *If  $\sigma_\gamma$  is any special Schubert class,  $\sigma_\beta$  is any Schubert class, and  $m \geq 0$  is an integer with  $m\gamma + \sum \beta_i \leq \dim G(\ell, e) = \ell(e - \ell)$ , then*

$$(\sigma_\gamma)^l \cdot \sigma_\beta \neq 0 \in A^*(G(\ell, e), \mathbb{Z}).$$

**Proposition 4.9.** *Let  $C \subset \mathbb{P}^d$  be a rational normal curve, and  $(V_W, \mathcal{O}_C(1))$  be the  $g_d^r$  on  $\mathbb{P}^1$  cut by hyperplanes in  $\mathbb{P}^d$  containing a plane  $W \subset \mathbb{P}^d$  of dimension  $d - r - 1$ , and write  $V = 0 \subset V_1 \subset \dots \subset V_d = \mathbb{P}^d$  for the osculating flag of  $C$  at  $p$ . The ramification sequence  $\alpha(V_W, p)$  is determined by the formula*

$$W \in \Sigma_a(V) \iff \alpha_i(V_W, p) \geq a_{r+1-i}^* = \#\{j \mid a_j \geq r + 1 - i\}.$$

IN PROGRESS



$$\sum_1(L_1) \cap \sum_1(L_2) \rightsquigarrow \sum_2(p) \cup \sum_{1,1}(1)$$

**Figure 2.** When the lines  $L_1$  and  $L_2$  move to meet each other at  $P$  they become coplanar in  $\Lambda$ , and the intersection  $\Sigma_1(L_1) \cap \Sigma_1(L_2)$  degenerates to the union  $\Sigma_2(P) \cup \Sigma_{1,1}(\Lambda)$ . Silvio:  $L_1$  red,  $L_2$  blue. The other lines black. You could add another line emanating from  $p$ , and maybe dotted extensions of the lines coming out of  $p$  to show that they continue behind the plane  $\Lambda$

In other words, the ramification sequence  $\alpha(V_W, p)$  of the linear series  $V_W$  at  $p$  is exactly the reverse of the transpose of the Schubert index of the smallest Schubert cycle containing  $\widetilde{W}$ , the vector space associated to  $W$ , in  $G(d-r, d+1)$ .

**Proof.** To keep the notation of Section 4.1, we work with the affine versions  $\widetilde{W}, \widetilde{\mathcal{V}} = (\widetilde{V}_1, \dots, \widetilde{V}_{d+1})$  of the projection center and the osculating flag at  $p$ , respectively, so that, for example  $\dim \widetilde{W} = d-r$ . (Note that the space  $\widetilde{V}_W$  of the linear series denoted  $(V_W, \mathcal{O}_{\mathbb{P}^1}(d))$  is already a vector space, but for consistency of notation we will write  $\widetilde{V}_{\widetilde{W}}$  anyway.)

The condition  $\widetilde{W} \in \Sigma_a(\widetilde{\mathcal{V}})$  means that

$$\dim \widetilde{W} \cap \widetilde{V}_{r+i-a_i} \geq i$$

for each  $i$ . It follows that the codimension of the space of hyperplanes containing  $\widetilde{W}$  has codimension  $\leq r - a_i + 1$ . since the projective space corresponding to  $\widetilde{V}_{r+i-a_i}$  is the linear span of the divisor  $(r+i-a_i+1)p \subset C$ , a codimension  $\leq r - a_i + 1$  space of sections of  $\widetilde{V}_{\widetilde{W}}$  vanishes to order  $\geq r - a_i + 1 + i$  at  $p$ .

If we write the distinct orders of vanishing of the sections in  $\widetilde{V}_{\widetilde{W}}$  as  $b_0 < b_1 < \dots < b_r$ , we see that  $\widetilde{W} \in \Sigma_a$  if and only if  $a_i$  of the  $b_j$  are  $\geq r + i + a_i + 1$ , that is,  $b_{r-a_i+1} \geq r - a_i + 1 + i$  or equivalently  $\alpha_{r-a_i+1} \geq i$  for each  $i$ . Since  $\alpha_i \leq \alpha_{i+1} \leq \alpha_r$ , this is equivalent to saying that the number of  $\{j \mid \alpha_j \geq i\}$  is at

least  $a_i$ , or, for the reverse sequence  $\alpha' = \alpha_r \geq \alpha_{r-1} \geq \dots \geq 0$ , that the  $a_i$ -th term is  $\geq i$ . On the other hand  $a'_{a_i} = \#\{j \mid a_j \geq a_i\} = i$  since the sequence  $a$  is weakly decreasing, so  $\alpha'$  is termwise  $\geq a'$ .  $\square$

**4.3. Conclusion.** Using these ideas we can rephrase and prove Theorem 4.1 in terms of Schubert cycles, adding the precise condition for the existence of a  $g_d^r$  with prescribed ramification sequences at an arbitrary collection of distinct points in  $\mathbb{P}^1$ :

**Theorem 4.10.** *Let  $p_1, \dots, p_\delta \in C$  be distinct points on a rational normal curve  $C \subset \mathbb{P}^d$ , and  $V^1, \dots, V^\delta$  the corresponding vanishing flags. If  $\beta^1, \dots, \beta^\delta$  are  $\delta$  Schubert indices for  $G(d-r, d+1)$ , the Schubert cycles  $\Sigma_{\beta^1}(V^1), \dots, \Sigma_{\beta^\delta}(V^\delta) \subset G(d-r, d+1)$  intersect properly; that is, the intersection is either empty or has codimension exactly  $\sum_{j=1}^\delta |\beta^j|$ , the sum of the codimensions of the cycles  $\Sigma_{\beta^j}$ . Moreover, the intersection is nonempty if and only if the intersection product of the classes  $[\Sigma_{\beta^j}]$  is nonzero in  $A^*(G(d-r, d+1))$ .*

**Proof.** If the intersection is empty then the Chow class is 0, so it suffices to show that the intersection is proper, and we may assume that it is non-empty. Because the Grassmannian is smooth, the codimension of the intersection of any subvarieties is at most the sum of their codimensions, so it is enough to show that the codimension of the intersection cannot be too small.

The Schubert cycle  $\Sigma_1$  is a hyperplane section of the Grassmannian  $G(d-r, r+1)$ , so that if  $\Phi \subset G$  is any subvariety of dimension  $m$ , its intersection with  $m$  Schubert cycles  $\Sigma_1$  is nonempty. Thus, if the intersection

$$X := \bigcap_{i=1}^\delta \Sigma_{\beta^i}(V^i)$$

had dimension strictly bigger than the expected

$$\rho := (r+1)(d-r) - \sum_{i=1}^\delta |\beta^i|,$$

we could choose  $\rho+1$  additional points  $q_1, \dots, q_{\rho+1}$  on  $C$ , with vanishing flags  $V^1, \dots, V^{\rho+1}$  and the intersection of  $X$  with the Schubert cycles  $\Sigma_1(V^i)$  would still be nonempty.

It thus suffices to show that the intersection is empty if

$$\sum_{i=1}^\delta |\beta^i| > (r+1)(d-r) = \dim G(d-r, d+1).$$

If on the contrary

$$W \in \bigcap_{i=1}^{\delta} \Sigma_{\beta^i}(V^i),$$

then, by Proposition 4.9, the linear series  $(W, \mathcal{O}_{\mathbb{P}^1}(d))$  would have ramification of weight  $|\beta_i|$  at  $p_i$ , and the sum of the weights would be strictly greater than  $(r+1)(d-r)$ . This contradicts the Plücker formula, Theorem 1.2.  $\square$

Using Proposition 4.9 this result becomes a characterization of the sets of possible ramification sequences for linear series at specified points on  $\mathbb{P}^1$ . Explicitly, suppose we are given a collection of distinct points  $p_1, \dots, p_\delta \in \mathbb{P}^1$ , and for each point  $p_i$  a ramification sequence

$$\alpha^i = (\alpha_0^i, \alpha_1^i, \dots, \alpha_r^i) \quad \text{with} \quad 0 \leq \alpha_0^i \leq \alpha_1^i \leq \dots \leq \alpha_r^i \leq d-r.$$

Let  $\beta^i$  be this sequence in reverse (and relabelled); that is

$$\beta^i = (\alpha_r^i, \alpha_{r-1}^i, \dots, \alpha_1^i, \alpha_0^i)$$

and finally let  $b^i$  be the transpose of the Schubert index  $\beta^i$ .

**Corollary 4.11.** *With  $\alpha^i$  and  $b^i$  related as above, there exists a  $g_d^r$  on  $\mathbb{P}^1$  with ramification sequence at  $p_i$  equal to  $(\alpha_0^i, \alpha_1^i, \dots, \alpha_r^i)$  if and only if*

$$\prod_{i=0}^{\delta} \sigma_{b^i} \neq 0 \quad \text{in} \quad A^*(G(d-r, r+1)).$$

**Proof.** If the product is nonzero, Proposition 4.9 and Theorem 4.10 immediately show the existence of a  $g_d^r$  on  $\mathbb{P}^1$  with ramification sequence greater than or equal to  $\alpha^i$  at  $p_i$ . But by Theorem 4.10, the ones with ramification strictly greater than the  $\alpha^i$  form a family of strictly smaller dimension. Thus a general  $g_d^r$  with ramification sequence greater than or equal to  $\alpha^i$  at  $p_i$  has ramification sequence exactly equal to  $\alpha^i$  at  $p_i$ .  $\square$

We can deduce a result about general secant loci strengthening one that was originally proven in [Griffiths and Harris 1980]: Given a curve  $C \subset \mathbb{P}^d$ , we say that a *secant flag* in  $\mathbb{P}^d$  is a flag

$$0 \subset V_1 \subset V_2 \subset \dots \subset V_{d-1} \subset V_d = \mathbb{P}^d$$

where each  $V_i$  is spanned by its (scheme-theoretic) intersection with  $C$ . In other words, there is a sequence of points  $p_1, p_2, \dots, p_{d+1} \in C$  such that

$$V_i = \overline{p_1 + p_2 + \dots + p_{i+1}}$$

(An osculating flag is just the special case where all  $p_i$  are equal.) Since any secant flag can specialize to an osculating flag, we can deduce the

**Corollary 4.12.** *Schubert cycles defined relative to general secant flags to a rational normal curve intersect properly.*

Unlike the case for osculating flags, the hypothesis of generality is necessary for secant flags; see Exercise 5.6

## 5. Exercises

**Exercise 5.1.** Let  $k$  be a field of characteristic  $p$ . Show that every point of the affine curve  $y = x^{p+1} + 1$  over  $k$  is a flex point.

**Exercise 5.2.** Show that the codimension of  $\Sigma_\beta(V)$  in the Grassmannian  $G$  is equal to  $\sum \beta_i$ .

Hint: for  $\Lambda \in \Sigma_\beta(V)$ , consider bases of  $\Lambda$  such that  $\Lambda \cap V_i$  is the span of basis vectors.

**Exercise 5.3.** If  $\beta^*$  is the transpose Schubert index to  $\beta$ ,

- (1) Show that  $|\beta^*| = |\beta|$
- (2) Show that  $(\beta^*)^* = \beta$
- (3) Show that the isomorphism  $\phi : G(k+1, V) \xrightarrow{\cong} G(n-k, V^*)$  carries the Schubert cycle  $\Sigma_\beta(V)$  to the Schubert cycle  $\Sigma_{\beta^*}(V^*)$ .

(Of course, the first two parts follow from the third; think of the first two as warmups.)

**Exercise 5.4.** (Ramification and osculating planes) Let  $p \in C \subset \mathbb{P}^d$  be a point on a rational normal curve of degree  $d$ , and let  $\Lambda \subset \mathbb{P}^d$  be a  $d-r-1$  plane, and let  $\mathcal{U} := (\mathcal{O}_{\mathbb{P}^1}(d), V)$  be the  $g_d^r$ , possibly with base points, cut out by hyperplanes containing  $\Lambda$ . Consider the complete flag

$$U(p) := \{U_1, \dots, U_d\}$$

of osculating spaces at  $p$ , where  $U_t$  is the linear span of the divisor  $tp$  considered as a subscheme of length  $t$  in  $C$ . Show that the ramification sequence of  $\mathcal{U}$  at  $p$  is determined by the formula

$$W \in \Sigma_a(U(p)) \iff \alpha_i(\mathcal{U}, p) \geq a_{r+1-i}^* = \#\{j \mid a_j \geq r+1-i\}.$$

In other words, the ramification sequence  $\alpha(\mathcal{U}, p)$  of the linear series  $\mathcal{U}$  at  $p$  is the reverse of the transpose of the Schubert index of the smallest Schubert cycle containing  $L$ , the vector space associated to  $\Lambda$ , in  $G(d-r, d+1)$ . Conclude from Theorem 4.10 that any collection of Schubert cycles associated to osculating flags of  $C$  at distinct points have intersection that is dimensionally transverse or empty.

**Exercise 5.5.** Let  $C$  be the rational normal curve in  $\mathbb{P}^r$ . And let  $(p_i, q_i) \in C^2$  be  $t$  pairs of points, all distinct. For each  $i$ , let  $S_i$  be the Schubert variety of  $k$ -planes in  $\mathbb{P}^r$  that meet the secant line  $\overline{p_i q_i}$ . Show that if the  $p_i, q_i$  are sufficiently general, then the intersection of the  $S_i$  is dimensionally transverse. Hint: Let each pair of points come together, and use the result for the tangent lines, a special case of Theorem 4.10.

**Exercise 5.6.** Show by example that Schubert cycles defined relative to arbitrary secant flags to a rational normal curve may fail to intersect properly. (Hint: it's enough to look at the case  $d = 2$  and  $r = 1$ .)

**Exercise 5.7.** We can define the osculating flag to any nondegenerate curve  $C \subset \mathbb{P}^d$  at any smooth point  $p$ . If  $p$  is an inflection point then  $\overline{mp}$  may not be an  $(m - 1)$ -plane, but we can look at the nested sequence of subspaces

$$\{p\} \subset \overline{2p} \subset \overline{3p} \subset \dots$$

and pick out exactly the terms whose dimension is strictly greater than the preceding term; this gives a complete flag

$$\{p\} \subset \overline{b_2 p} \subset \overline{b_3 p} \subset \dots \subset \overline{b_{d+1} p} = \mathbb{P}^d$$

Show that the sum  $\sum b_i - i$  is equal to the weight of the point  $p$  as an inflection point of the hyperplane series.

**Exercise 5.8.** The Plücker formula shows that an elliptic normal curve  $E \subset \mathbb{P}^{n-1}$  has  $n^2$  inflection points. Show that they are all simple, and that if you choose one as the origin for the group law on  $E$ , then they are the  $n$ -torsion points of  $E \cong \text{Jac}(E)$ .

Hint: observe that the inflection points are the points  $p \in E$  such that  $\mathcal{O}_E(np) \cong \mathcal{O}_E(1)$ .

**Exercise 5.9.** (Buchweitz) It was long believed that every semigroup  $H \subset \mathbb{N}$  of finite index  $g = \#(\mathbb{N} \setminus H)$  could occur as the Weierstrass semigroup of a Weierstrass point on a curve of genus  $g$ , but this is not the case. The following counterexample for a curve of genus 16 was found by Buchweitz (unpublished): Show that on a curve of genus 16 there can be no Weierstrass point  $p$  with ramification sequence  $(0^{12}, 6, 7, 9, 9)$  by showing that there would be too many vanishing orders at  $p$  of quadratic differentials.



# Proof of the Brill Noether Theorem

Our goal in this chapter is to give a proof of the Brill-Noether theorem based on the analysis of inflections of linear series on  $\mathbb{P}^1$  of Theorem 4.10. We will focus on proving what we call the “Basic Brill-Noether” theorem (Theorem 2.1), which we reproduce for convenience:

**Theorem 0.1** (Basic Brill Noether). *If*

$$\rho(g, r, d) := g - (r + 1)(g - d + r) \geq 0,$$

*then every smooth projective curve of genus  $g$  possesses a  $g_d^r$ ; and for a general curve  $C$ ,  $\dim W_d^r(C) = \rho$ . Conversely, if  $\rho < 0$  then a general curve  $C$  of genus  $g$  does not possess a  $g_d^r$ .*

In fact, a closer examination of our proof will yield some of the assertions of the stronger Theorems 2.3 and 2.6, as well as additional results on the existence of linear series with specified inflectionary behavior; we will discuss these at the end of the chapter.

## 1. Castelnuovo’s approach

The original argument of Brill and Noether reduced the proof to the assertion that a matrix depending on an invertible sheaf on a general curve behaves in a certain sense like a matrix of indeterminates. This was finally proven in [Griffiths and Harris 1980] using an idea that goes back to Castelnuovo in [Castelnovo 1889].

Interestingly, Castelnuovo's goal was not to prove the Brill-Noether theorem, which was considered established at the time (or at least not in need of further demonstration). Given that that a general curve of genus  $2d - 2$  has finitely many  $g_d^1$ s, Castelnuovo asked: how many? We have answered this question in cases  $g = 2, 4$  and  $6$  in earlier chapters, and these cases were certainly known to Castelnuovo, but the cases of higher  $g$  were unknown. His idea was to specialize to a general curve  $C_0$  with  $g$  nodes whose normalization is  $\mathbb{P}^1$ , and count the number of  $g_d^1$ s on that curve. Appealing to the (then) vague “principle of conservation of number”, Castelnuovo felt that the result probably reflected the number on a general smooth curve as well.<sup>1</sup>

By way of notation, we'll write  $r_1, \dots, r_g$  for the nodes of  $C_0$ , and let  $p_i, q_i \in \mathbb{P}^1$  be the two points lying over  $r_i$ . Castelnuovo counted the  $g_d^1$ s on  $C_0$  by observing that any pencil on  $C_0$  can be pulled back to a pencil on the normalization  $\mathbb{P}^1$ ; if we embed  $\mathbb{P}^1$  in  $\mathbb{P}^d$  as a rational normal curve of degree  $d$ , such a pencil is cut out by the hyperplanes containing a  $(d-2)$ -plane  $\Lambda \subset \mathbb{P}^d$ . Moreover, to say that such a pencil is a pullback from  $C_0$  means that every divisor of the pencil that contains  $p_i$  contains  $q_i$  and vice versa. This is equivalent to the condition that  $\Lambda \cap \overline{p_i, q_i} \neq \emptyset$ .

In the Grassmannian  $G(d-1, d+1)$  of  $(d-2)$ -planes in  $\mathbb{P}^d$ , the locus of those that meet the line  $L_i = \overline{p_i, q_i}$  is what we called in the last chapter the Schubert cycle  $\Sigma_1(L_i)$ . In these terms the set of  $g_d^1$ s on  $C_0$  is the intersection

$$W_d^1(C_0) = \bigcap_{i=1}^{2d-2} \Sigma_1(L_i).$$

Castelnuovo proposed that if the points  $p_i, q_i \in \mathbb{P}^1$  were chosen generally, then the Schubert cycles  $L_i$  would meet transversely, and thus that the cardinality of this intersection is the degree of the power  $\sigma_1^{2d-2}$  in the Chow ring  $A(G(d-1, d+1))$ . Castelnuovo evaluated this power, and came to the conclusion that a general curve  $C$  of genus  $g = 2d - 2$  has

$$\#W_{d+1}^1(C) = \frac{(2d-2)!}{(d-1)!d!}$$

pencils of degree  $d$ . Indeed, we see in Exercise 6.3 that if the points  $p_i, q_i$  are general, then the Schubert cycles  $\Sigma_1(L_i)$  at least intersect properly, so the given number is the number of  $g_d^1$ s counted with appropriate multiplicities.

<sup>1</sup>Castelnuovo presented his computation as heuristic, not claiming it was a full proof, and the reviewer of Castelnuovo's paper in the Zentralblatt wrote very politely: “Das Resultat, welches er bekommen hat, gibt mit grosser Wahrscheinlichkeit den wahren Wert von  $N \dots$ ; daher sind wir mit dem Verf. einverstanden, wenn er seinen Versuch nicht für wertlos hält” (The result that he obtained gives the true value of [the number of  $g_d^1$ s] with high probability, and thus we agree with the author that his work is not worthless...)

The papers [Kleiman and Laksov 1972], [Kleiman and Laksov 1974] and [Kempf 1971] use a different idea, which we'll describe in Section 1.1 below, to prove the “existence” part of Theorem 0.1—that is, that  $W_d^r(C)$  is nonempty when  $\rho \geq 0$ . In [Kleiman 1976], the general “non-existence” part is reduced to the proof of Exercise 6.3 and completed for  $r = 1$ . The general statement was finally proven in [Griffiths and Harris 1980].

We will also adopt this general approach, and we will prove the existence and non-existence parts together for all  $g, d, r$ . However, the strength of Theorem 4.10 compared to Exercise 6.3 suggests specializing to a  $g$ -cuspidal curve  $C_0$  rather than a  $g$ -nodal one, and we will use this refinement.

**1.1. Upper bound on the codimension of  $W_d^r(C)$ .** Let  $C$  be a reduced irreducible projective curve of arithmetic genus  $g$ . In Section 2.1 we showed how we might arrive at the “expected” dimension of the locus  $W_d^r(C)$  by estimating the dimension of the subvariety  $C_d^r \subset C_d$  of divisors moving in an  $r$ -dimensional linear series; here we'll give a similar argument using the Picard variety  $\text{Pic}(C)$ . Again the heart of the proof is the estimation of the codimension of an ideal of minors of a certain matrix of functions. Since the proof uses degeneration to a cuspidal curve, we will work with flat families of curves that may include singular fibers. For simplicity we will take the base of the family to be a complex disk  $\Delta$  centered at the origin in  $\mathbb{C}$ ; the more algebraically-minded reader can substitute  $\text{Spec}$  of a DVR.

**Theorem 1.1.** *Let  $\mathcal{C}/\Delta$  be a family of reduced irreducible projective curves of arithmetic genus  $g$ . If the fiber  $C_0$  of the family has  $\dim W_d^r(C_0) = \rho(g, r, d) \geq 0$  locally at a particular invertible sheaf  $\mathcal{L}_0$ , then  $\dim W_d^r(C_b) = \rho$  locally for all  $b$  in a neighborhood of  $0 \in \Delta$  and invertible sheaves in a neighborhood of  $\mathcal{L}_0$  in the relative Picard variety  $\text{Pic}_d(\mathcal{C}/\Delta)$ . In particular,  $W_d^r(C_b)$  is nonempty for all  $b$  in a neighborhood of  $0 \in \Delta$ .*

The idea of the following proof is a modern form of the discussion in the original paper of Brill and Noether.

**Proof.** Possibly after pulling back the family  $\mathcal{C}/\Delta$  along a ramified covering  $\Delta \rightarrow \Delta$  and restricting to a smaller disk around 0, we may choose  $m$  sections  $p_i(b)$  with distinct values in the smooth locus of each fiber, and  $m$  as large as we like. We define a family of divisors  $D_b = \sum_i p_i(b)$ . By the semicontinuity of fiber dimension, we may choose  $m$  so large that  $h^1(\mathcal{O}_{C_b}(D_b)) = 0$  and that for every invertible sheaf  $\mathcal{L}_b$  of degree  $d$  on  $C_b$  we have  $h^1(\mathcal{L}_b(D_b)) = 0$ . By Theorem 1.3 we have

$$h^0(\mathcal{L}_b(D_b)) = d + m - g + 1.$$

The pushforward of the Poincaré bundle  $P$  on  $\text{Pic}_{d+m}(\mathcal{C}/\Delta) \times \mathcal{C}$  to  $\text{Pic}_{d+m}(\mathcal{C}/\Delta)$  is thus a vector bundle  $E$  of rank  $d+m-g+1$  on  $\Delta$  whose fiber over a point representing  $\mathcal{L}_b$  is the vector space  $H^0(\mathcal{L}(D_b))$ . Similarly, the pushforward  $\pi_{1*}(P|_{D_b})$  is a trivial vector bundle  $F$  whose fiber at every point is the  $m$ -dimensional vector space  $\bigoplus_{i=1}^m \mathcal{L}(E)|_{p_i}$ . Regarding  $D$  as a family of finite schemes inside  $\mathcal{C}$ , the restriction map

$$P \longrightarrow P_D$$

pushes forward to give a map of vector bundles  $\phi : E \rightarrow F$  on  $\mathcal{C}$  which, on a fiber over  $b$ , is the evaluation of sections  $\sigma \in H^0(\mathcal{L}_b(D_b))$  at the points  $p_i$ .

If  $\mathcal{L}_b$  is an invertible sheaf of degree  $d$  on  $C_b$  then the space  $H^0(\mathcal{L}_b)$  is the kernel of the map  $\phi$  at the point corresponding to  $\mathcal{L}_b(D_b) \in \text{Pic}_{d+m}(\mathcal{C}/\Delta)$ . Locally on  $\mathcal{C}$  the map  $\phi$  can be defined by a  $(d+m) \times (d+m-g+1)$  matrix of regular functions. The locus  $W_d^r(C_b)$  is thus the translate by  $\otimes \mathcal{O}_{\mathcal{C}}(D)$  of the locus where  $\phi$  has rank  $d+m-g-r$  or less, and by [Eisenbud 1995a, Exercise 10.9] this locus is either empty or its components have codimension  $\leq (r+1)(g-d+r)$  in  $\text{Pic}_{d+m}(\mathcal{C})$ , which has dimension  $g$ . Consequently, every component of  $W_d^r(C)$  has dimension at least  $\rho$ .  $\square$

It was in fact this set-up that was used in the papers [Kleiman and Laksov 1972], [Kleiman and Laksov 1974] and [Kempf 1971] to prove the existence half of Brill-Noether: they determined the characteristic classes of the bundles involved and applied the *Thom-Porteous formula*, a formula for the class of the degeneracy locus of a map of vector bundles. An account of this proof is given in Appendix D of [Eisenbud and Harris 2016a].

We will see that if  $C_0$  is any rational curve with  $g$  cusps, and  $\rho(g, r, d) \geq 0$ , then  $\dim W_d^r(C_0) = \rho$  locally at every invertible sheaf on  $C_0$ .

## 2. Specializing to a $g$ -cuspidal curve

Our first goal is to find a family  $\{C_t\}$  of curves of arithmetic genus  $g$ , with  $C_t$  smooth for  $t \neq 0$  and  $C_0$  a rational curve with  $g$  cusps. To do this we show how to construct a rational curve  $C_0$  with  $g$  cusps and deform it to a smooth curve.

### 2.1. Constructing curves with cusps.

**Proposition 2.1.** *Let  $C$  be any curve and  $p \in C$  a smooth point. There exists a curve  $C_0$  and a bijective morphism  $f : C \rightarrow C_0$  such that  $f$  maps  $C \setminus \{p\}$  isomorphically to  $C_0 \setminus \{r\}$  and the image  $r = f(p) \in C_0$  is a cusp of  $C_0$ .*

We will say that we *crimp*  $C$  at  $p$  to obtain  $C_0$ . For the corresponding result with nodes instead of cusps, see Exercise 6.2.

**Proof.** We can construct  $C_0$  explicitly as a topological space homeomorphic to  $C$ , with structure sheaf  $\mathcal{O}_{C_0}$  that is the subsheaf of  $\mathcal{O}_C$  consisting of functions on  $C$  whose derivative at  $p$  is 0.  $\square$

If we start with  $\mathbb{P}^1$ , pick any  $g$  points  $p_1, \dots, p_g \in \mathbb{P}^1$  and crimp at each  $p_i$ , we arrive at a  $g$ -cuspidal curve  $C_0$ .

**2.2. Smoothing a cuspidal curve.** Given a curve  $C_0$  with a finite number of cusps and no other singularities, we can find a proper flat family  $C \rightarrow \Delta$  with special fiber  $C_0$  and all other fibers smooth; that is, we can smooth  $C_0$ .

To begin with, we can do this locally in the complex analytic setting: if  $p \in C_0$  is a cusp, we can find an analytic neighborhood of  $p$  in which  $C_0$  is given by the equation  $y^2 = x^3$ ; we can smooth this by taking the family

$$y^2 = x(x - t)(x - 2t)$$

for  $t \in \Delta$ . (See Figure ??.)

The next step is to argue that we can glue together these local smoothings to obtain a proper family  $C \rightarrow \Delta$ , and this is where we need to invoke a result from deformation theory ([Fantechi and Göttsche 2005]):

**Lemma 2.2.** *Let  $p_1, \dots, p_g \in \mathbb{P}^1$  be distinct points, and  $C_0$  the curve obtained by crimping  $\mathbb{P}^1$  at each  $p_i$ . There exists a family of curves  $\pi : C \rightarrow \Delta$ , where*

- (1)  *$\Delta$  is a disk centered at the origin in  $\mathbb{C}$ .*
- (2) *for all  $b \neq 0 \in \Delta$ , the fiber  $C_b = \pi^{-1}(b)$  is a smooth, projective curve of genus  $g$ ; and*
- (3) *the fiber over 0 is the curve  $C_0$ .*

**Proof Sketch.** The cuspidal curve  $C_0$  can be embedded in projective space  $\mathbb{P}^n$  and there it is locally a complete intersection so, as with the case of one cusp, above, the local obstructions to smoothing vanish. The global obstruction is the first cohomology of a sheaf supported at the cusps, and therefore vanishes as well.  $\square$

### 3. The family of Picard varieties

By Lemma 2.2 there is a flat family  $C \rightarrow \Delta$  of curves, specializing from a smooth curve of genus  $g$  to a  $g$ -cuspidal curve. The next step is to relate linear series on the general fiber of our family to their limits on  $C_0$ .

**3.1. The Picard variety of a cuspidal curve.** We can describe the invertible sheaves on a cuspidal curve  $C_0$  in terms of the invertible sheaves on its normalization:

**Proposition 3.1.** *Let  $p \in C_0$  be a cusp that is the result of crimping a reduced irreducible projective curve  $C$  at a smooth point  $q \in C$ , and let  $\nu : C \rightarrow C_0$  be the natural morphism. There is an exact sequence of groups*

$$0 \longrightarrow (\mathbb{C}, +) \longrightarrow \text{Pic}_0(C_0) \xrightarrow{\nu^*} \text{Pic}_0(C) \longrightarrow 0.$$

**Proof.** The preimage  $\nu^{-1}(p)$  is the nonreduced scheme  $2q \subset C$ . If  $\mathcal{L}$  is an invertible sheaf on  $C_0$  then  $\nu^*(\mathcal{L})$  is invertible on  $C$ , and thus locally trivial near  $q$  and, in particular, trivial when restricted to the subscheme  $2q$ . Conversely, if  $\mathcal{L}'$  is an invertible sheaf on  $C$  and we choose a trivialization of  $a : \mathcal{L}'|_U \cong \mathcal{O}_U$  on a neighborhood  $U \subset C$  of  $q$ , then there is an invertible sheaf  $\mathcal{L}$  on  $C_0$  together with a trivialization  $a : \mathcal{L}|_{\nu(U)} \cong \mathcal{O}_{\nu(U)}$  on  $\nu(U)$ , unique up to an automorphism of  $\mathcal{L}'$  (that is, up to multiplication by a scalar), that pulls back to  $\mathcal{L}'$  with the given trivialization  $a$ . Thus the data of an invertible sheaf  $\mathcal{L}$  on  $C_0$  is equivalent to the data of an invertible sheaf  $\mathcal{L}'$  on  $C$ , together with a trivialization of  $\widetilde{\mathcal{L}}$  on the preimage  $\nu^{-1}(p) = 2q$  up to multiplication by a nonzero scalar.

Of course every invertible sheaf on the zero-dimensional scheme  $2q$  is trivial, and a change of trivialization thus corresponds to an automorphism of the structure sheaf  $\mathcal{O}_{2q} \cong \mathbb{C}[\epsilon]/(\epsilon^2)$ —not as an algebra, but as a rank 1 free module over  $\mathbb{C}[\epsilon]/(\epsilon^2)$ . Such a map is determined by its effect on the generator 1, and can take this element to any other generator  $a + b\epsilon$  with  $a \neq 0$  and  $b \in \mathbb{C}$  arbitrary. After multiplying  $\mathcal{L}'$  by  $a^{-1}$  we may suppose that  $a = 1$  and we may take the automorphism to induce the identity on  $\mathbb{C}$ . In this case, the family of trivializations, modulo multiplication by a nonzero scalar, is determined by the element  $b \in \mathbb{C}$ . Moreover, composing the isomorphisms  $1 \mapsto 1 + b\epsilon$  and  $1 \mapsto 1 + b'\epsilon$  gives  $1 \mapsto 1 + (b + b')\epsilon$ , so the kernel of the map  $\nu^* : \text{Pic}_d(C_0) \rightarrow \text{Pic}_d(C)$  is the additive group  $(\mathbb{C}, +)$ .  $\square$

**Corollary 3.2.** *If  $C_0$  is a  $g$ -cuspidal rational curve, then  $\text{Pic}_d(C_0) \cong \mathbb{C}^g$ . More precisely,  $\text{Pic}_d(C_0)$  is a principal homogeneous space for the additive group  $\mathbb{C}^g$ , in the sense that this group acts faithfully and transitively.*

**Proof.** We can construct one invertible sheaf on  $C_0$  as the inverse of the ideal sheaf of a divisor of  $d$  smooth points, and any other will differ from this one by the choice of an element of  $\text{Pic}_0(C_0)$ , corresponding to a choice of  $g$  trivializations, as in the proof of the Proposition.  $\square$

Thus we see that  $\text{Pic}_d(C_0)$  has dimension  $g$  just as does the Picard variety of a smooth curve of genus  $g$ . An important difference is that  $\text{Pic}_d(C_0)$  is not compact.

**3.2. The relative Picard variety.** Returning to the family  $\pi : C \rightarrow \Delta$  at the beginning of Section 3, the Picard varieties  $\text{Pic}_d(C_t)$  form a family  $\text{Pic}_d(C/\Delta)$ , and the varieties  $W_d^r(C_t)$  form a subfamily  $W_d^r(C/\Delta)$ . In the argument of Theorem 1.1 bounding the dimensions of the  $W_d^r$  we may replace the points  $p_i$  by sections of the family, and thus the codimension of  $W_d^r(C/\Delta) \subset \text{Pic}_d(C/\Delta)$  is  $\leq (r+1)(g-d+r)$  locally at each point.

We will soon show that  $W_d^r(C_0)$ , the fiber of  $W_d^r(C/\Delta)$  over 0, is nonempty of dimension  $\rho(g, r, d)$  when  $\rho(g, r, d) \geq 0$  and otherwise empty. If the family of Picard varieties over  $\Delta$  were proper, this would prove Theorem 0.1; but it is not proper, and thus it is a priori possible that  $W_d^r(C_0) = \emptyset$  but  $W_d^r(C_t) \neq \emptyset$  for  $t \neq 0$ . This would mean that families of invertible sheaves  $\mathcal{L}_t$  on  $C_t$  simply don't have limits in  $\text{Pic}_d(C_0)$ . Indeed, the limit of a family of invertible sheaves need not be an invertible sheaf. Nevertheless we can describe these limits quite precisely.

**3.3. Limits of invertible sheaves.** Suppose that  $\pi : C \rightarrow \Delta$  is a family of smooth genus  $g$  curves specializing to a rational curve  $C_0$  with  $g$  cusps as in Lemma 2.2. Let

$$\pi^\circ : C^\circ := C \setminus C_0 \rightarrow \Delta^\circ := \Delta \setminus 0,$$

and suppose that there is an invertible sheaf  $\mathcal{L}^\circ$  on  $C^\circ$  such that  $h^0(\mathcal{L}^\circ|_{C_b}) \geq r+1$  for each  $b \neq 0 \in \Delta$  so that  $\mathcal{L}^\circ|_{C_b} \in W_d^r(C_b)$ . We would like to describe the “limit” of  $\mathcal{L}$  as  $b \rightarrow 0$ .

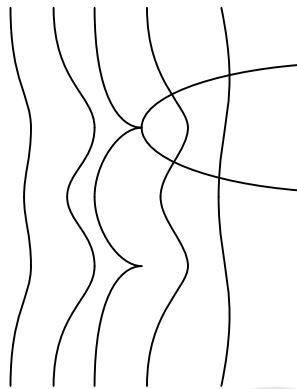
**Lemma 3.3.** *In the situation above, there exists a torsion-free sheaf  $\mathcal{L}$  of rank 1 on  $C$ , flat over  $\Delta$  and locally isomorphic to an ideal sheaf of  $C$ , such that  $\mathcal{L}|_{C^\circ} \cong \mathcal{L}^\circ$ .*

In fact, any torsion free sheaf of rank 1 on  $C_0$  is locally isomorphic to an ideal sheaf; this follows from the fact that  $C_0$  is generically Gorenstein. However, the argument we will give shows directly that the extension is isomorphic to an ideal sheaf tensored with an invertible sheaf, and the fact that it is locally isomorphic to an ideal sheaf follows at once.

**Proof.** Choose an auxiliary invertible sheaf  $\mathcal{M}$  on  $C$  with relative degree  $e > d + 2g$  and let  $\mathcal{M}^\circ$  be the restriction of  $\mathcal{M}$  to  $C^\circ$ . Consider the invertible sheaf

$$\mathcal{N}^\circ = (\mathcal{L}^\circ)^* \otimes \mathcal{M}^\circ.$$

Choose a section  $\sigma$  of  $\mathcal{N}^\circ$ ; let  $D^\circ \subset C^\circ$  be its divisor of zeros, and let  $D \subset C$  be the closure of  $D^\circ$  in  $C$ . Because it is the closure, the scheme  $D$  has no embedded 0-dimensional components, and thus  $\mathcal{O}_C/\mathcal{I}_{D/C}$  has no  $\mathcal{O}_\Delta$ -torsion.



**Figure 1.** Divisor on a family of curves degenerating to a cuspidal curve. Silvio: distinguish the divisor (parabola) with color.

Since  $\Delta$  is a smooth curve, this implies that  $\mathcal{O}_C/\mathcal{I}_{D/C}$  is flat over  $\Delta$ , and thus  $\mathcal{I}_{D/C}|_{C_0}$  is an ideal sheaf, the kernel of  $\mathcal{O}_C \rightarrow \mathcal{O}_C/\mathcal{I}_{D/C}$ . Thus

$$\mathcal{L} := \mathcal{I}_{D/C} \otimes \mathcal{M}$$

has the desired properties.  $\square$

Fortunately the ideal sheaves on cuspidal curves have a simple local structure. The reason lies in the relation of the local ring  $R_0$  of a cusp to its integral closure.

**Definition 3.4.** The *conductor* of an integral domain  $R_0$  is the annihilator of the  $R_0$ -module  $R/R_0$ , where  $R$  is the integral closure of  $R_0$ .

It follows at once from the definition that the conductor of  $R_0$  is also an ideal of  $R$ , and that it is the largest ideal of  $R_0$  that is also an ideal of  $R$ . The following result is a restatement of Example 2 after Proposition 3.12:

**Proposition 3.5.** *If  $R_0$  is the local ring of an ordinary cusp singularity of a curve, then  $R/R_0 \cong k$ , the residue field of  $R$ , and thus the conductor of  $R_0$  is the maximal ideal  $\mathfrak{m}_0$  of  $R_0$ .*  $\square$

**Proof.** These properties can be verified after completing at the maximal ideal of  $R_0$ . To say that  $R_0$  has an ordinary cusp singularity means that the completion of  $R_0 \subset R$  is  $k[[x^2, x^3]] \subset k[[x]]$ , and the quotient is  $kx \cong k$ .  $\square$

**Theorem 3.6.** *Let  $p$  be an ordinary cusp of a curve  $C_0$  with normalization  $\pi : C \rightarrow C_0$ . Let  $R_0 = \mathcal{O}_{C_0, p}$  be the local ring of the cusp, with maximal ideal  $\mathfrak{m}_0 = \mathcal{I}_p R_0$ , and let  $R_0 \xrightarrow{\pi^*} R$  be its normalization. If  $I_0 \neq 0$  is an ideal of  $R_0$  then either  $I_0 \cong R_0$  or  $I_0 \cong \mathfrak{m}_0$ .*

In the latter case,  $I_0 \cong Ra$ , and if  $R_0/I_0$  has length  $v$  then  $R/RI_0 = R/I_0$  has length  $v + 1$ . Moreover, there is a split exact sequence

$$0 \rightarrow R/\mathfrak{m}_0R \rightarrow R \otimes I_0 \rightarrow RI_0 \rightarrow 0$$

with  $RI_0 \cong R$ .

Interpreting Theorem 3.6 in the context of an ideal sheaf on a cuspidal curve we get:

**Corollary 3.7.** *Let  $p \in C_0$  be an ordinary cusp in a reduced irreducible curve and let  $\pi : C \rightarrow C_0$  be the partial normalization of  $C_0$  at  $p$ , so that  $p_a(C) = p_a(C_0) - 1$ . Let  $q \in C$  be the point lying over  $p \in C_0$ .*

*If  $\mathcal{F}_0$  is locally isomorphic to a nonzero ideal sheaf on  $C_0$  and  $\mathcal{F}_0$  is not locally free at  $p$ , then there is a unique locally free sheaf  $\mathcal{F}$  on  $C$  and a short exact sequence*

$$0 \rightarrow \mathcal{O}_{\pi^{-1}(p)} \rightarrow \pi^*\mathcal{F}_0 \rightarrow \mathcal{F} \rightarrow 0.$$

Thus  $\chi(\mathcal{F}) = \chi(\mathcal{F}_0) - 2$ , so

$$\deg \mathcal{F} = \chi(\mathcal{F}) - \chi(\widetilde{\mathcal{O}}_C) = \deg(\mathcal{F}_0) - 1.$$

Moreover the map  $H^0(\mathcal{F}_0) \xrightarrow{\pi^*} H^0(\mathcal{F}(-q))$  is a monomorphism.  $\square$

**Proof of Theorem 3.6.** The endomorphism ring of  $I_0$  is commutative, it contains  $R_0$ , and (since it stabilizes the finitely generated module) it is integral over  $R_0$ . Thus

$$R_0 \subset \text{End}(I_0) \subset R.$$

Since  $R/R_0 \cong k$ , the ring  $\text{End}(I_0)$  is equal to either  $R_0$  or  $R$ .

First, suppose  $\text{End}(I_0) = R$ , which is a discrete valuation ring. Every ideal of  $R$  is principal, so we may write  $I_0 = Ra$ . Since  $\mathfrak{m}_0$  is also an ideal of  $R$ , it is isomorphic to  $R$  as an  $R$ -module, and since  $R_0 \subset R$ ,  $I \cong \mathfrak{m}_0$  as  $R_0$ -modules. Moreover, after completing we have  $\widehat{R}_0 \cong \mathbb{C}[[t^2, t^3, \dots]]$  and it is evident that if  $a$  has valuation  $d$  in  $R$  then the length of  $R_0/(aR)$  is  $d - 1$ , proving the claims in this case.

On the other hand, suppose  $\text{End}(I) = R_0$  and consider the inclusions

$$\mathfrak{m}_0I \subset I \subset RI.$$

The left and right hand modules both have endomorphism ring  $R$ , so both containments must be strict. Since  $R/\mathfrak{m}_0$  has length 2, we see that  $I/\mathfrak{m}_0I$  is principal. By Nakayama's lemma  $I \cong R_0$ .  $\square$

Returning once again to the family  $C \rightarrow \Delta$  at the beginning of Section 3, assume that for general  $t$  the scheme  $W_d^r(C_t) \neq \emptyset$ . After replacing  $\Delta$  with a ramified covering we may assume that  $W_d^r(C/\Delta) \rightarrow \Delta$  has a section, that is, an invertible sheaf  $\mathcal{L}$  on  $C$  such that the restriction of  $\mathcal{L}$  to each fiber  $C_t$  with

$t \neq 0$  has at least  $r + 1$  sections, so the  $g$ -cuspidal curve  $C_0$  that is the limit of this family has a torsion-free sheaf  $\mathcal{L}_0$  of degree  $d$ . By the semi-continuity of cohomology,  $\mathcal{L}_0$  also has at least  $r + 1$  sections.

At each cusp  $r_i$  of  $C_0$ ,  $\mathcal{L}_0$  is either locally free or isomorphic to the maximal ideal of the cuspidal curve. Thus, if we denote by  $p_1, \dots, p_k$  the cusps of  $C_0$  at which  $\mathcal{L}_0$  is isomorphic to the maximal ideal, and let  $C'_0$  be the partial normalization of  $C_0$  at those cusps, then  $C'_0$  is a curve of arithmetic genus  $g - k$  having  $g - k$  cusps, and the pullback of  $\mathcal{L}_0$  to  $C'_0$  has at least  $r$  sections with base points at the points lying over  $p_1, \dots, p_k$ ; this is a  $g_{d-k}^r$  on the  $(g - k)$ -cuspidal curve  $D_0$ .

#### 4. Putting it all together

**4.1. Non-existence.** We can now prove that if  $\rho(g, r, d) < 0$  then  $W_d^r(C) = \emptyset$  for curves in an open dense subset of  $M_g$ —that is, a general curve of genus  $g$  does not possess a  $g_d^r$ . Here we use Theorem 4.10 from the last chapter, which implies exactly this statement for invertible sheaves on an arbitrary  $g$ -cuspidal curve  $C_0$ . If it were the case that a general curve possessed a  $g_d^r$  with  $\rho(g, r, d) < 0$ , then by the results of Section 3, there would be an invertible sheaf of degree  $d$  with  $\geq r + 1$  sections on a  $(g - k)$ -cuspidal curve  $C'_0$  for some  $k \geq 0$ . But since

$$\rho(g - k, r, d - k) = \rho(g, r, d) - k < 0$$

this is impossible by Theorem 4.10

**4.2. Existence.** Once more we let  $C \rightarrow \Delta$  be a family of smooth curves specializing to a  $g$ -cuspidal curve  $C_0$ . We can use Corollary 4.8 in combination with Theorem 4.10 to say that the variety  $W_d^r(C_0)$  is nonempty of dimension  $\rho(g, r, d)$ ; since the codimension of  $W_d^r(C/\Delta) \subset \text{Pic}_d(C/\Delta)$  is at most  $(r + 1)(g - d + r)$  locally everywhere, and we may conclude that likewise for general  $t$  the variety  $W_d^r(C_t)$  is nonempty of dimension exactly  $\rho(g, r, d)$ .

#### 5. Brill-Noether with inflection

The approach we've taken here to the proof of Brill-Noether is well-suited to analyzing the inflectionary behavior of linear series on a general curve; indeed, a small modification of the argument above allows us to prove a stronger form of the Brill-Noether statement, concerning the existence of  $g_d^r$ s on a general curve  $C$  that are required to have inflection points of given weights.

**Definition 5.1.** Let  $C$  be a smooth curve of genus  $g$  and  $p_1, \dots, p_n \in C$  distinct points of  $C$ . If  $D = (L, V)$  is a linear system on  $C$  of degree  $d$  and dimension  $r$ ,

we define the *adjusted Brill-Noether number* of  $D$  relative to the points  $p_k$  to be

$$\rho(D; p_1, \dots, p_k) := g - (r+1)(g-d+r) - \sum_{k=1}^n w(D, p_k).$$

In these terms, we can prove the following generalization of the “nonexistence” half of Brill-Noether:

**Theorem 5.2.** *Let  $(C; p_1, \dots, p_n)$  be a general  $n$ -pointed curve of genus  $g$  (that is, let  $C$  be a general curve and  $p_1, \dots, p_n \in C$  general points). If  $D$  is any linear system on  $C$ , then*

$$\rho(D; p_1, \dots, p_n) \geq 0.$$

**Proof.** To start, let  $C \rightarrow \Delta$  be a family of curves as in the proof of Lemma 2.2. As in the argument of Section 4.1, possibly after pulling back the family along a ramified map  $\Delta \rightarrow \Delta$ , we can choose sections  $\sigma_1, \dots, \sigma_n : \Delta \rightarrow C$  of  $C \rightarrow \Delta$  such that the  $\sigma_k(0)$  are distinct smooth points of  $C_0$ . If the general curve  $C_b$  in our family admits a  $g_d^r$   $D$  with

$$\rho(D; \sigma_1(b), \dots, \sigma_n(b)) < 0$$

then, possibly after pulling the family back along a ramified map  $\Delta \rightarrow \Delta$ , we can choose a family  $\{D_b\}$  of such linear series on the fibers  $C_b$  for  $b \neq 0$  and, taking limits, we arrive at a  $g_d^r D_0$  on  $\mathbb{P}^1$  with

$$w(D_0, q_i) \geq r$$

for each of the  $g$  points  $q_i \in \mathbb{P}^1$  lying over the cusps of  $C_0$ , and in addition

$$w(D_0, r_k) \geq w(D_b, \sigma_k(b))$$

where  $r_k \in \mathbb{P}^1$  is the point in  $\mathbb{P}^1$  lying over  $\sigma_k(0) \in C_0$ . Adding up, we have

$$\begin{aligned} \sum_{i=1}^g w(D_0, q_i) + \sum_{k=1}^n w(D_0, r_k) &\geq rg + \sum_{k=1}^n w(D_b, \sigma_k(b)) \\ &> rg + g - (r+1)(g-d+r) = (r+1)(d-r) \end{aligned}$$

since we assumed that

$$\rho(D_b; \sigma_1(b), \dots, \sigma_n(b)) = g - (r+1)(g-d+r) - \sum_{k=1}^n w(D_b, \sigma_k(b)) < 0.$$

But as before the Plücker formula for  $\mathbb{P}^1$  tells us that

$$\sum_{p \in \mathbb{P}^1} w(D_0, p) = (r+1)(d-r),$$

a contradiction.  $\square$

This extension of the “nonexistence” part of Brill-Noether raises the question of a converse: if  $(C; p_1, \dots, p_n)$  is a general  $n$ -pointed curve of genus  $g$ , and we specify ramification sequences  $\alpha^1, \dots, \alpha^n$ , can we say that there exists a  $g_d^r$   $D$  on  $C$  with  $\alpha_i(D, p_k) \geq \alpha_i^k$  for  $k = 1, \dots, n$  and  $i = 0, \dots, r$ ? If the product of the corresponding Schubert classes in  $G(d-r, d+1)$  is nonzero, then we can indeed deduce the existence of such a linear series; and if the product is 0, we can deduce that no such linear series exists.

Here is one way to state what we know without getting lost in a thicket of Schubert calculus:

**Theorem 5.3.** *Let  $C$  be a smooth curve of genus  $g$  and  $p_1, \dots, p_n \in C$  distinct points; for  $k = 1, \dots, n$  let  $\alpha^k = (\alpha_0^k, \dots, \alpha_r^k)$  be a nondecreasing sequence of non-negative integers, and let*

$$G_d^r(p_1, \dots, p_n; \alpha^1, \dots, \alpha^n) = \{D \in G_d^r(C) \mid \alpha_i(D, p_k) \geq \alpha_i^k\}.$$

*If  $(C, p_1, \dots, p_n)$  is a general  $n$ -pointed curve, then either  $G_d^r(p_1, \dots, p_n; \alpha^1, \dots, \alpha^n)$  is empty or*

$$\dim G_d^r(p_1, \dots, p_n; \alpha^1, \dots, \alpha^n) = \rho(g, r, d) - \sum_{k+1}^n \sum_{i=0}^r \alpha_i^k.$$

Finally, from the codimensions of the Schubert varieties we can deduce the ramification behavior of general linear series:

**Theorem 5.4.** *If  $D$  is a general  $g_d^r$  on a general curve, then  $D$  has only simple ramification; that is,*

$$w(D, p) \leq 1 \quad \text{for all } p \in C.$$

Applying this in case  $d = 2g - 2$  and  $r = g - 1$ , we arrive at the statement made in Section 1.4: that a general curve  $C$  of genus  $g$  has only Weierstrass points of weight 1.

## 6. Exercises

**Exercise 6.1.** By Theorem 5.3, there is a  $g_4^1$  on a general curve  $C$  of genus 3 ramified at any 2 general points  $p, q \in C$ . Construct it.

Hint: using the canonical embedding to realize  $C$  as a plane quartic and project from the point of intersection of the tangent lines at  $p$  and  $q$ .

The following three Exercises give the results necessary to prove Theorem 0.1 using degeneration to nodal curves rather than to cuspidal curves.

**Exercise 6.2.** (Construction of nodal curves) Let  $C$  be a smooth curve, and let  $\{(p_i, q_i) \mid i = 1 \dots h\}$  be  $2h$  distinct points of  $C$ .

Show that if  $C$  is embedded in  $\mathbb{P}^r$  by a complete linear series of sufficiently high degree, then the projection of  $C$  from a general  $(r - 4)$ -plane  $\Lambda$  that meets each secant  $\overline{p_i, q_i}$  maps  $C$  to a curve with  $h$  ordinary nodes in  $\mathbb{P}^3$ .

Hint: Use dimension counts to show that

- (1)  $\Lambda$  does not contain any tangent line to  $C$ ;
- (2)  $\Lambda$  does not meet any secant line to  $C$  other than the lines  $\overline{p_i, q_i}$ ; and
- (3)  $\Lambda$  does not meet the 3-plane  $\overline{\mathbb{T}_{p_i} C, \mathbb{T}_{q_i} C}$  in a line.

**Exercise 6.3.** Suppose that  $C \subset \mathbb{P}^d$  is the rational normal curve of degree  $d$ , and  $\{L_i = \overline{p_i, q_i}\}$  is a collection of secants of  $C$ . Show that if the points  $p_i, q_i \in C$  are chosen sufficiently generally, then any collection of Schubert cycles  $\Sigma_{a_i}(L_i)$  meet dimensionally transversely.

Hint: Let the  $p_i, q_i$  approach each other, reducing to the case of tangent lines. Then use Theorem 4.10. For a direct proof see [Griffiths and Harris 1980, Lemma, p. 259].

**Exercise 6.4.** Imitate the proof of Proposition 3.1 to show that if  $R_0$  is the local ring of a node—that is, the completion  $\widehat{R}$  is isomorphic to  $\mathbb{C}[x, y]/(xy)$ —then every nonzero ideal of  $R$  is isomorphic either to  $R$  or to the maximal ideal of  $R$ . Deduce that if  $C \rightarrow C_0$  is the partial normalization of a single node on a reduced irreducible projective curve, then there is an exact sequence  $0 \rightarrow \mathbb{C}^* \rightarrow \text{Pic}_0(C_0) \rightarrow \text{Pic}_0(C) \rightarrow 0$ .

Hint: Let  $C_0$  be a curve with a node  $p$ , and  $C \xrightarrow{\nu} C_0$  its partial normalization at  $p$ . Denote by  $q, r \in \widetilde{C}$  the points lying over  $p$ . If  $\mathcal{L}$  is an invertible sheaf on  $C$ , and  $\mathcal{M} := \nu^*(\mathcal{L})$  the pullback of  $\mathcal{L}$  to  $\widetilde{C}$ , then  $\mathcal{M}$  is an invertible sheaf on  $\widetilde{C}$ . Its fibers over  $q$  and  $r$  are both identified with the fiber  $\mathcal{L}_p$  of  $\mathcal{L}$  at  $p$ , and hence with each other. Conversely, given an invertible sheaf  $\mathcal{M}$  on  $\widetilde{C}$  and an identification of the fibers  $\mathcal{M}_q$  and  $\mathcal{M}_r$ , we can form an invertible sheaf  $\mathcal{L}$  on  $C$  by taking the subsheaf of  $\nu_* \mathcal{M}$  whose sections “agree” at  $q$  and  $r$ , in terms of the identification.



# Using a singular plane model

In the first part of Chapter 4 we showed how to use an embedding of a smooth curve  $C$  in  $\mathbb{P}^2$  to understand differentials and linear series on  $C$ . But most smooth curves cannot be embedded in  $\mathbb{P}^2$  so, as such, the technique is not generally applicable. However, by Proposition 3.1, any smooth curve  $C$  can be projected birationally to a curve  $C_0 \subset \mathbb{P}^2$  with only nodes. In the first part of this chapter we will show how to use such a nodal model  $C_0$  to describe differentials and linear series on  $C$ , a theory well-understood by Brill and Noether; and then explain what is necessary to adapt the technique to birational images of  $C$  with arbitrary singularities.

## 1. Nodal plane curves

The methods of Chapter 4 can be applied, with one change, when  $C_0 \subset \mathbb{P}^2$  is a nodal curve.

Let  $\nu : C \rightarrow C_0 \subset \mathbb{P}^2$  be the normalization morphism from a smooth curve, and let  $B \subset C_0$  be a subscheme. It will be convenient to speak of *the linear series cut out on  $C$*  by curves of degree  $m$  containing  $B$ : though  $B$  is only a subscheme,  $\nu^{-1}(B)$  may be considered as a Cartier divisor because  $C$  is smooth, and we define the linear series cut out on  $C$  by curves of degree  $m$  containing  $B$  to be the linear series  $\mathcal{V}$  of divisors in  $C$  that are residual to  $\nu^{-1}(B)$  in the pullbacks of the intersections of  $C_0$  with curves of degree  $m$ . More formally, since  $B' := \nu^{-1}(B)$  is a Cartier divisor on  $C$ , we can say that if  $L$  is the divisor of a line in  $\mathbb{P}^2$  and  $L'$  its pullback to  $C$  then  $\mathcal{V}$  is a space of divisors corresponding to sections

of  $\mathcal{O}_C(mL' - B')$ ; more precisely, it is the space of divisors corresponding to sections of  $\mathcal{O}_C(mL' - B')$  in the image of the restriction/pullback map

$$H^0(\mathcal{I}_{B/\mathbb{P}^2}(m)) \rightarrow H^0(\mathcal{O}_C(mL' - B')).$$

**1.1. Differentials on a nodal plane curve.** Let  $C_0 \subset \mathbb{P}^2$  be a curve of degree  $d$  with  $\delta$  nodes and no other singularities. By the adjunction formula (Proposition 2.5), Proposition 3.11, and the first example that follows it, the genus  $g$  of the normalization  $C$  of  $C_0$  is the arithmetic genus  $p_a(C_0) = \binom{d-1}{2}$  of  $C_0$  minus  $\delta$ , that is,

$$g = \binom{d-1}{2} - \delta.$$

We will make this explicit by exhibiting a vector space of  $g$  regular differential forms on  $C$ .

Choose homogeneous coordinates  $[X, Y, Z]$  on  $\mathbb{P}^2$  so that the curve  $C_0$  intersects the line  $L = V(Z)$  in a divisor  $D$  consisting only of smooth points of  $C_0$  other than  $[0, 1, 0]$ . In particular, all the nodes of  $C_0$  will lie in the affine plane  $U = \mathbb{P}^2 \setminus L$ . We may harmlessly assume that neither branch of  $C_0$  at a node has vertical tangent. (These conditions are satisfied by a general choice of coordinates.) Let the nodes of  $C_0$  be  $q_1, \dots, q_\delta$ , with  $r_i, s_i \in C$  lying over  $q_i$ ; we'll denote by  $\Delta$  the divisor  $\sum r_i + \sum s_i$  on  $C$ .

Let  $F(X, Y, Z)$  be the homogeneous polynomial of degree  $d$  defining the curve  $C_0$ , and let  $f(x, y) = F(x, y, 1)$  be the defining equation of the affine part  $C_0^\circ := C_0 \cap U$  of  $C_0$ . Let  $\nu : C \rightarrow C_0$  be the normalization map. We start by considering the rational differential  $\nu^*(dx)$  on  $C^\circ := \nu^{-1}(C_0^\circ)$ .

In the smooth case where  $C_0 = C$  we saw that this differential was regular and nonzero on  $C^\circ$ ; this followed from the fact that that  $f_x$  and  $f_y$  had no common zeroes on  $C_0$ . But now  $f_x$  and  $f_y$  have common zeroes: they both vanish to order 1 at the points  $q_i$  and thus  $\nu^*(f_x)$  and  $\nu^*(f_y)$  have simple zeroes at the points  $r_i$  and  $s_i$ .

As before, the differential  $\nu^*dx$  has double poles along the divisor  $D$  on  $C_0$  lying over the point at infinity in  $\mathbb{P}^1$  and we see that for a polynomial  $e(x, y)$  of degree  $\leq d - 3$ , the differential

$$\nu^*\left(\frac{e(x, y)dx}{f_y}\right)$$

is regular except for simple poles at the points  $r_i$  and  $s_i$ .

We can get rid of these poles by requiring that  $e$  vanishes at the points  $q_i$ . We say in this case that  $e$  (and the curve defined by  $e$ ) *satisfies the conditions of adjunction*.

**Theorem 1.1.** *If  $C_0$  is a nodal plane curve of degree  $d$  with normalization  $\nu : C \rightarrow C_0$  then the regular differentials on  $C$ , in terms of the notation above, are precisely those of the form*

$$\nu^*\left(\frac{e(x,y)dx}{f_y}\right)$$

where  $e(x,y)$  ranges over the polynomials of degree  $\leq d-3$  vanishing at the nodes of  $C_0$ .

Thus if  $\mathfrak{F}(C_0) \subset C_0$  denotes the union of the reduced points at the nodes of  $C$ , then  $|\omega_C|$  is the linear series cut out on  $C$  by forms of degree  $d-3$  containing  $\mathfrak{F}(C_0)$ .

**Proof.** The dimension of the space of polynomials  $e(x,y)$  of degree at most  $d-3$  is  $\binom{d-1}{2}$ , and vanishing at  $\delta$  nodes imposes at most  $\delta$  linear conditions on  $e$ . The linear map sending  $e \mapsto \nu^*(edx/f_y)$  is injective, and the target has dimension  $\binom{d-1}{2} - \delta$ , so this must be an isomorphism.  $\square$

We will give a more conceptual proof of this theorem in Section 2.

In particular, Theorem 1.1 shows that the linear series cut out on  $C$  by forms of degree  $d-3$  containing  $\mathfrak{F}(C_0)$  is complete. (We will soon see that the linear series cut out on  $C$  by forms of degree  $m$  containing  $\mathfrak{F}(C_0)$  is complete for every  $m$ .)

This gives another proof of Lemma 6.5.

**Corollary 1.2.** *If  $C$  is a nodal plane curve of degree  $d$ , then the nodes of  $C_0$  impose independent conditions on forms of degree  $d-3$ .*

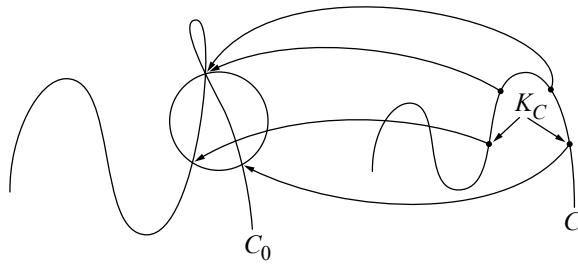
**Proof.** Otherwise the space of differential forms on the normalization of  $C_0$  would be too large.  $\square$

One can use Theorem 1.1 to re-embed a plane curve of geometric genus  $g$  as a canonical curve in  $\mathbb{P}^{g-1}$ :

**Corollary 1.3.** *The canonical ideal of the normalization of a nodal plane curve of degree  $d$  is the ideal of polynomial relations among the forms of degree  $d-3$  that vanish at the nodes of the curve.*  $\square$

**1.2. Linear series on a nodal plane curve.** Since  $C_0$  is singular, not every effective divisor on  $C$  is the preimage of an effective Cartier divisor on  $C_0$ . As an example, one may take a single point lying over a node.

However, we can still represent every divisor on  $C$  as the preimage of a divisor on  $C_0$  up to linear equivalence, and the same goes for any reduced curve:



**Figure 1.** A curve  $C$  of geometric genus 2 represented as a plane curve  $C_0$  of degree 4 with a node, and a canonical divisor, represented as a conic containing the node. Silvio: Nice picture. Would be clearer if the plane of  $C_0$  were shown. Maybe  $C, C_0$  different colors?

**Lemma 1.4.** *Let  $\nu : C \rightarrow C_0$  be the normalization of any reduced projective curve. If  $D$  is any divisor on  $C$ , then  $D$  is linearly equivalent to the pullback of a divisor supported on the smooth locus of  $C_0$ . More precisely, every effective divisor on  $C$  containing  $\Delta$  can be written as the pullback of a Cartier divisor on  $C_0$ .*

We will see a more general version in Theorem 2.1.

**Proof.** It suffices to prove the result locally on  $C_0$ , where it is geometrically obvious: if the node  $p \in C_0$  has preimages  $q, r$  corresponding to the branches of  $C_0$  at  $p$ , then a divisor  $aq + br$  with both  $a, b$  strictly positive, is locally the pullback of the intersection of  $C_0$  with a curve  $C'$  passing through  $p$  that is  $(a-1)$ -times tangent to one branch and  $(b-1)$ -times tangent to the other.  $\square$

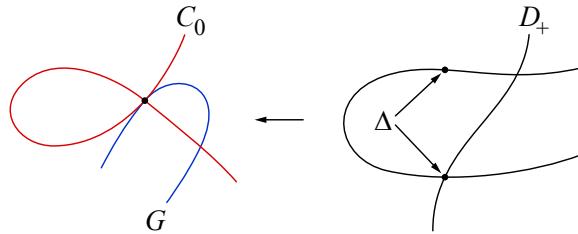
Returning to the case of a nodal plane curve  $C_0$  and its normalization  $\nu : C \rightarrow C_0$ , suppose that  $D$  is a divisor on  $C$  that is the pullback of a difference of Cartier divisors  $D_+ - D_-$  on  $C_0$ . We will compute the complete linear series  $|D|$ . Let  $\mathfrak{F}(C_0)$  be the set of nodes in  $C$  and let  $\Delta$  be the preimage of  $\mathfrak{F}(C_0)$  in  $C$ .

**Theorem 1.5.** *Let  $D = D_+ - D_-$  be a divisor on  $C$ , and let  $G$  be a form on  $\mathbb{P}^2$  that vanishes on  $D_+ + \mathfrak{F}(C_0)$  but not identically on  $C_0$ .*

*If  $G$  has degree  $m$  and  $A = (\nu^*G) - D_+ - \Delta$ , then every effective divisor on  $C$  linearly equivalent to  $D$  (if any) has the form  $(\nu^*H) - D_- - A - \Delta$  for some  $H$  of degree  $m$  that vanishes on  $\mathfrak{F}(C_0) + A$  but not identically on  $C_0$ .*

The reason for including  $\mathfrak{F}(C_0)$  is explained in Section 2.1.

**Corollary 1.6.** *With notation as in Theorem 1.5, the ideal of the image of  $C$  with respect to the linear series  $|D|$  is the ideal of polynomial relations among the forms of degree  $m$  in  $\mathfrak{f}_{C/C_0}$  that vanish on  $D_- + A$ .*  $\square$



**Figure 2.** If \$G\$ is tangent to a branch of \$C\_0\$ at the node, then \$D\_+\$ contains a point of \$\Delta\$ on \$C\$. Silvio: \$C\$ should be labeled. \$G\$ and \$D\_+\$ could be the same color

**Proof of Theorem 1.5.** Choose an integer \$m\$ sufficiently big that there is a form \$G\$ vanishing on \$D\_+\$ and \$\mathfrak{F}(C\_0)\$ so that \$\nu^\*(G)\$ vanishes on \$D\_+ + \Delta\$, but not everywhere on \$C\$. (If \$D\_+\$ contains some positive multiple of a point \$p\$ of \$\Delta\$ this means that \$G\$ defines a curve sufficiently tangent to the corresponding branch of \$C\_0\$.)

We can then write the zero locus of \$G\$ pulled back to \$C\$ as

$$(\nu^*G) = D_+ + \Delta + A,$$

as before.

Next, we look for forms \$H\$ of the same degree \$m\$, vanishing at \$A + D\_-\$ and on \$\mathfrak{F}(C\_0)\$ but not on all of \$C\_0\$. If there are no such polynomials \$H\$ then, as we shall show, there are no effective divisors equivalent to \$D\$. Supposing that there is such a form \$H\$, let \$D'\$ be the divisor

$$D' = (\nu^*H) - (D_+ + \Delta),$$

that is, \$D'\$ is residual to \$(D\_+ + \Delta)\$ in \$(\nu^\*H)\$.

Since \$\nu^\*(G/H)\$ is a rational function on \$C\$ we have

$$D_- + \Delta + A + D' = (\nu^*H) \sim (\nu^*G) = D_+ + \Delta + A,$$

and thus \$D'\$ is an effective divisor linearly equivalent to \$D = D\_+ - D\_-\$ on \$C\$.

To complete the argument we must show that we get *all* divisors \$D'\$ in this way. In this case the curve \$C\$ can be desingularized by blowing up the plane once at each node, and we can give a proof based on the resulting surface \$S\$. The same technique would work for any curve with only ordinary multiple points, in which case the total transform of \$C\_0\$ on \$S\$ has normal crossings. We will give a different proof, extending this theorem to curves with arbitrary singularities, in Section 2.

**Proposition 1.7.** *If \$C\_0\$ is a reduced irreducible plane curve all of whose singularities are ordinary nodes, then for each integer \$m\$, the linear series cut out on the normalization \$C\$ of \$C\_0\$ by forms of degree \$m\$ containing the nodes is complete.*

**Proof.** To prove Proposition 1.7, we work on the blow-up  $\pi : S \rightarrow \mathbb{P}^2$  of  $\mathbb{P}^2$  at the nodes  $q_i$  of  $C_0$ . The proper transform of  $C_0 \subset \mathbb{P}^2$  in  $S$  is the normalization of  $C_0$ , which we will again call  $C$ .

Let  $L$  be the class on  $S$  of the pullback of a line in  $\mathbb{P}^2$  and let  $E$  be the sum of the exceptional divisors, the preimage of  $\mathfrak{F}(C_0)$ . We write  $h = L \cap C$  and  $e = E \cap C = \sum(p_i + q_i)$  for the corresponding divisors on  $C$ . Because  $C$  has double points at each  $q_i$  we have  $C \sim dL - 2E$  and by Theorem 6.5 we have  $K_S \sim -3L + E$ .

The proper transform of a degree  $m$  curve  $A \subset \mathbb{P}^2$  passing simply through the points  $q_i$  is  $\pi^*A - E$ ; this gives an isomorphism

$$H^0(\mathcal{I}_{\{q_1, \dots, q_\delta\}/\mathbb{P}^2}(m)) \cong H^0(\mathcal{O}_S(mL - E)).$$

In these terms we can describe the linear series cut on  $C$  by plane curves of degree  $m$  passing through the nodes of  $C_0$  as the image of the map

$$H^0(\mathcal{O}_S(mL - E)) \rightarrow H^0(\mathcal{O}_C(mL - E)),$$

and we must show that this map is surjective.

From the long exact cohomology sequence associated to the exact sequence of sheaves

$$0 \rightarrow \mathcal{O}_S((m-d)L + E) \rightarrow \mathcal{O}_S(mL - E) \rightarrow \mathcal{O}_C(mL - E) \rightarrow 0,$$

we see that it will suffice to prove that  $H^1(\mathcal{O}_S((m-d)L + E)) = 0$ .

By Serre duality on  $S$ ,

$$H^1(\mathcal{O}_S((m-d)L + E)) \cong H^1(\mathcal{O}_S((d-m-3)L))^*.$$

The line bundle  $\mathcal{O}_S((d-m-3)L)$  is the pullback to  $S$  of the bundle  $\mathcal{O}_{\mathbb{P}^2}(d-m-3)$ , which has vanishing  $H^1$ . Lemma 1.8 completes the proof.  $\square$

**Lemma 1.8.** *Let  $X$  be a smooth projective surface, and  $\pi : S \rightarrow X$  the blow-up of a finite set of reduced points. If  $\mathcal{L}$  is any line bundle on  $X$ , then*

$$H^1(S, \pi^*\mathcal{L}) = H^1(X, \mathcal{L}).$$

**Proof.** Because  $\mathbb{P}^2$  is normal, and  $\pi_*(\mathcal{O}_S)$  is a finite birational algebra over  $\mathcal{O}_{\mathbb{P}^2}$ , we have  $\pi_*(\mathcal{O}_S) = \mathcal{O}_{\mathbb{P}^2}$ . Since any invertible sheaf  $\mathcal{L}$  on  $\mathbb{P}^2$  is locally isomorphic to  $\mathcal{O}_{\mathbb{P}^2}$ , is also an isomorphism.

The Leray spectral sequence (Theorem 3.12) gives an exact sequence

$$0 \rightarrow H^1(\pi_*(\mathcal{L})) \rightarrow H^1(\mathcal{L}) \rightarrow H^0(R^1(\pi_*(\mathcal{L}))) \rightarrow 0$$

The restriction of  $\pi^*(\mathcal{L})$  to any fiber of  $\pi$  is trivial and has vanishing  $H^1$ , so  $H^0(R^1(\pi_*(\mathcal{L}))) = 0$ , and  $\pi_*\pi^*\mathcal{O}_{\mathbb{P}^2}(1)$  is an invertible sheaf. The natural map

$\pi_*\pi^*(\mathcal{L}) \rightarrow \mathcal{L}$  is an isomorphism away from the codimension 2 set of points blown up. Thus these two sheaves are isomorphic, and

$$H^1(S, \pi^*\mathcal{L}) = H^1(\pi_*\pi^*\mathcal{L}) = H^1(\mathcal{L}),$$

completing the proof.  $\square$

This concludes the proof of Theorem 1.5  $\square$

**Proposition 1.9.** *Let  $C$  be a curve on a smooth surface  $S$ , and let  $\nu : S' \rightarrow S$  be the blowup of  $S$  at  $p$ . If  $C'$  is the strict transform of  $C$ , then*

$$p_a(C') = p_a(C) - \binom{m}{2},$$

where  $m$  is the multiplicity of  $p \in C$ .

**Proof.** This follows from comparing the adjunction formulas on  $S$  and  $S'$ . To start, we have

$$p_a(C) = \frac{C^2 + K_S \cdot C}{2} + 1.$$

On  $S'$  let  $E$  be the divisor class of the exceptional divisor. As we've seen,

$$K_{S'} = \nu^*K_S + E,$$

while the class of  $C'$  is given by

$$C' \sim \nu^*C - mE.$$

It follows that

$$(C')^2 = C^2 + m^2E^2 = C^2 - m^2 \quad \text{and} \quad K_{S'} \cdot C' = K_S \cdot C + m.$$

Thus, applying adjunction on  $S'$ , we find

$$p_a(C') = \frac{C'^2 + K_{S'} \cdot C'}{2} + 1 = \frac{C^2 + K_S \cdot C - m(m-1)}{2} + 1 = p_a(C) - \binom{m}{2}$$

as stated.  $\square$

**Cheerful Fact 1.10.** *Any plane curve can be desingularized by iteratively blowing up of singular points of  $C$ , then of the strict transform, and so on. See for example [Fulton 1989, 2008] or [Brieskorn and Knörrer 1986]. This is in fact the same sequence of transforms as the one given in Exercise 7.15. The points on the various blowups that map to the original singular point are called infinitely near points.*

This gives a nice formula for the delta-invariant of any singularity:

**Corollary 1.11.** *Thus the  $\delta$  invariant of any singularity of a plane curve  $C_0$  at a point  $p$  can be computed as the sum of the numbers  $\binom{m_q}{2}$  where the sum runs over all infinitely near singular points  $q$  and  $m_q$  denotes the multiplicity of the pullback of  $C_0$  at  $q$ .*  $\square$

## 2. Arbitrary plane curves

Throughout this section  $C_0 \subset \mathbb{P}^2$  denotes a reduced and irreducible plane curve with arbitrary singularities. Let  $\nu : C \rightarrow C_0$  be its normalization, and write  $L'$  for the pullback to  $C$  of the class  $L$  of a line in  $\mathbb{P}^2$ .

It is possible to carry out an analysis of linear series on the normalization of an arbitrary plane curve in a manner analogous to what we did in the preceding section for nodal curves, replacing the set of nodes by the scheme  $\mathfrak{F}(C_0) \subset C_0$  called the *adjoint scheme*, which in this case is the scheme defined by the *conductor ideal*  $\mathfrak{f}_{C/C_0}$ , the annihilator in  $\mathcal{O}_{C_0}$  of  $\nu_*(\mathcal{O}_C)/\mathcal{O}_{C_0}$ —see Theorem 8.1. Let  $\Delta \subset C$  be the Cartier divisor defined by the pullback of  $\mathfrak{f}_{C/C_0}$  to  $C$ .

**2.1. The conductor ideal and linear series on the normalization.** The following result gives a simple way of expressing any divisor on  $C$  in terms of  $\Delta$  and a Cartier divisor on  $C_0$ :

**Theorem 2.1.** *If  $D$  is an effective (Cartier) divisor on  $C$  that contains  $\Delta$ , then  $D$  is the pullback of a Cartier divisor on  $C_0$ .*

**Proof.** The result is local, so it suffices to treat the affine case of an affine curve  $C_0$  with coordinate ring  $\mathcal{O}'$  contained in its integral closure  $\mathcal{O}$ , the coordinate ring of its normalization  $C$ . The ideal of  $\Delta$  in  $\mathcal{O}$  is the conductor  $\mathfrak{f}_{\mathcal{O}/\mathcal{O}'}$  (which is contained in  $\mathcal{O}'$ , but stable under multiplication by elements of  $\mathcal{O}$ , and thus also an ideal of  $\mathcal{O}$ ). Thus  $\mathcal{I}_D$  may be regarded as an ideal—though not necessarily a principal ideal—of  $\mathcal{O}'$ . Since the ground field  $\mathbb{C}$  is infinite,  $\mathcal{I}_D \subset \mathcal{O}'$  is the integral closure of a principal ideal  $(x)$  of  $\mathcal{O}'$ . (see for example [Huneke and Swanson 2006, Chapter 8]). This means that for sufficiently large  $n$  we have  $x\mathcal{I}_D^n = \mathcal{I}_D^{n+1}$ .

Let  $D_0$  be the Cartier divisor on  $C_0$  corresponding to  $x$ . Pulling everything back to  $C$  we have

$$\nu^*(D_0) + nD = (n+1)D$$

whence  $\nu^*(D_0) = D$ . □

In view of the previous theorem, we can specify an arbitrary linear series on  $C$  as the difference  $E - \Delta$ , where  $E$  is a linear series with  $\Delta$  in its base locus, by specifying a linear series on  $C_0$  containing  $\mathfrak{F}(C_0) \subset C_0$ . The following theorem makes this algorithmic.

**Theorem 2.2.** *Let  $D = D_+ - D_-$  be a divisor on  $C$ , and let  $G$  be a form on  $\mathbb{P}^2$ , vanishing on  $\mathfrak{F}(C_0)$ , whose pullback to  $C$  vanishes on  $D_+$  but not on all of  $C$ .*

*If  $G$  has degree  $m$  and  $(\nu^*G) = D_+ + A + \Delta$ , then every effective divisor on  $C$  linearly equivalent to  $D$  (if any) has the form  $(\nu^*H) - D_- - A - \Delta$  for some  $H$  of*

degree  $m$  vanishing on  $\mathfrak{F}(C_0) + A$ . Thus, the global sections of  $H^0(\mathcal{O}_C(D))$  can be written as divisors cut by forms of degree  $m$  in  $\mathfrak{f}_{C/C_0}$  with base points at  $D_- + A + \Delta$ .

The following consequence allows one to embed  $C$  via the linear series  $|D|$ :

**Corollary 2.3.** *With notation as in Theorem 2.2, the ideal of the image of  $C$  with respect to the linear series  $|D|$  is the ideal of polynomial relations among the forms of degree  $m$  in  $\mathfrak{f}_{C/C_0}$  that vanish on  $D_- + A$ .  $\square$*

**Proof of Theorem 2.2.** If  $H \in \mathfrak{f}(C_0)$  vanishes on  $A$  but not on all of  $C_0$ , then as before

$$D_- + \Delta + A + D' = (\nu^*H) \sim (\nu^*G) = D_+ + \Delta + A,$$

So  $D' = (\nu^*H) - D_- - A - \Delta$  is linearly equivalent to  $D$ . The proof that every divisor  $D'$  linearly equivalent to  $D$  has this form is the content of Theorem 2.4 below.  $\square$

The following result was known classically as the *completeness of the adjoint series*.

**Theorem 2.4.** *For every integer  $m \geq 0$  the series cut out on  $C$  by forms of degree  $m$  on  $\mathbb{P}^2$  containing  $\mathfrak{F}(C_0)$  is complete.*

**Proof.** If  $R_0 \subset R$  is an inclusion of commutative rings, then the set  $\mathfrak{f}_{R/R_0} := \text{ann}_{R_0}(R/R_0) \subset R_0$  is by definition an ideal of  $R_0$ , but is also an ideal of  $R$ ; this follows because if  $f \in \mathfrak{f}_{R/R_0}$  and  $r \in R$ , then  $fR \subset R_0$  so  $(rf)R = frR \subset fR \subset R_0$ .

If  $R_0$  is a domain and  $R$  is a subring of the quotient field  $Q(R)$  of  $R$ , then  $\mathfrak{f}_{R/R_0} \cong \text{Hom}_{R_0}(R, R_0)$ . To see this, note that  $R_0$  and  $R$  become equal after tensoring with  $Q(R_0)$  and thus  $\text{Hom}_{R_0}(R, R_0) \subset \text{Hom}_Q(Q, Q) = Q$  may be identified with the set of elements  $\{\alpha \in Q \mid \alpha R \subset R_0\}$ . If  $\alpha$  is in this set, then  $\alpha \cdot 1 = \alpha \in R_0$ , as required.

Returning to the case of the curve  $C_0$ , it follows that the global sections of  $(\nu^*(\mathcal{O}_{C_0}(m))(-\Delta)$  on  $C$  are, on each affine open set  $U$ , represented by the elements of  $\mathcal{O}_{C_0}(U)$  that are restrictions to  $U$  of forms of degree  $m$  contained in the ideal  $\widetilde{\mathfrak{f}}_{C/C_0}$ . Thus the global sections of the sheaf  $\widetilde{\mathfrak{f}}_{C/C_0}(m)$  cut out a complete linear series on  $C$ .

Write  $S = \mathbb{C}[x_0, x_1, x_2]$  for the homogeneous coordinate ring of  $\mathbb{P}^2$ . It remains to prove that the homogeneous ideal  $\mathfrak{f}_{C/C_0}$  maps surjectively to  $H_*^0(\widetilde{\mathfrak{f}}_{C/C_0})$ , and this amounts to the statement that the depth of  $\mathfrak{f}_{C/C_0}$  as an  $S$ -module is (at least) 2. Set  $R_0 = H_*^0(\mathcal{O}_{C_0})$  and  $R = H_*^0(\nu_*(\mathcal{O}_C))$ . We see from the general considerations above that

$$\mathfrak{f}_{R/R_0} = \text{Hom}_{R_0}(R, R_0).$$

Any non-zerodivisor on a module  $M$  is a non-zerodivisor on  $\text{Hom}(P, M)$  for any module  $P$  since  $(a\phi)(p) = a(\phi(p))$  by definition. Since  $R_0 = S/(F)$ , it is a module of depth 2, and we may choose a regular sequence  $a, b$  of elements in  $R_0$ . From the short exact sequence

$$0 \longrightarrow R_0 \xrightarrow{a} R_0 \longrightarrow R_0/(a) \longrightarrow 0$$

we get a left exact sequence

$$0 \longrightarrow \text{Hom}_{R_0}(R, R_0) \xrightarrow{a} \text{Hom}_{R_0}(R, R_0) \longrightarrow \text{Hom}_{R_0}(R, R_0/(a)).$$

Thus

$$\text{Hom}_{R_0}(R, R_0)/a\text{Hom}_{R_0}(R, R_0) \subset \text{Hom}_{R_0}(R, R_0/(a))$$

and since  $b$  is a non-zerodivisor on  $\text{Hom}_{R_0}(R, R_0/(a))$ , it is a non-zerodivisor on  $\text{Hom}_{R_0}(R, R_0)/a\text{Hom}_{R_0}(R, R_0)$  as well.  $\square$

Since we saw directly that the adjoint ideal was equal to the conductor ideal in the case of a nodal curve, this result gives another, less ad hoc, proof that the effective divisors equivalent to  $D$  are all defined by pullbacks of forms of degree  $m$  that contain  $\Delta$  as constructed in Proposition 1.7.

**2.2. Differentials.** Let  $C_0^\circ$  be the intersection of  $C_0$  with the open set  $\mathbb{A}^2 \cong U \subset \mathbb{P}^2$  where  $Z \neq 0$ , and let  $C^\circ \subset C$  be the preimage of  $C_0^\circ$ .

**Theorem 2.5.** *If  $C_0$  meets the line  $L$  at infinity only in smooth points other than  $(0, 1, 0)$ , then the complete canonical series on the normalization  $\nu : C \rightarrow C_0$  is cut out by differentials of the form*

$$\frac{e(x, y)dx}{f_y}$$

where  $e(x, y)$  is a polynomial of degree  $\leq d - 3$  contained in the conductor ideal  $\mathfrak{f}_{C^\circ/C_0^\circ}$

As in the case of nodal curves, one can use Theorem 2.5 to re-embed a plane curve of geometric genus  $g$  as a canonical curve in  $\mathbb{P}^{g-1}$ :

**Corollary 2.6.** *The canonical ideal of the normalization  $C$  of a plane curve  $C_0$  of degree  $d$  is the ideal of polynomial relations among the forms of degree  $d - 3$  in the conductor ideal  $\mathfrak{f}_{C/C_0}$ .  $\square$*

**Proof.** The proof of Theorem 2.5 consists of four parts.

First, because we have assumed that  $(0, 1, 0)$  does not lie on  $C$ , the function  $x$  defines a ramified  $d$ -sheeted cover of  $C$  to  $\mathbb{P}^1$ . Because  $C_0$  meets  $L$  only in smooth points and the differential  $dx$  has a pole of order 2 at the point at infinity in  $\mathbb{P}^1$ ,  $dx$  has polar locus twice the divisor  $\nu^{-1}(C_0 \cap L)$ . It follows that the

differential  $\varphi_0 := dx/f_y$  is regular, with a zero of order  $d - 3$ , along the divisor of  $C$  lying over  $C_0 \cap L$ .

Second, the function on  $C_0^\circ$  defined by  $x$  is a finite map to  $\mathbb{A}^1$ , and thus the field of rational functions  $\kappa(C) = \kappa(C_0^\circ)$  is a finite separable extension of  $\mathbb{C}(x)$ . By [Eisenbud 1995a, Section 16.5], the module of differentials  $\omega_{\kappa(C)/\mathbb{C}}$  is generated over  $\kappa(C_0^\circ)$  by  $dx$ . Thus every rational differential form on  $C$  can be expressed as a rational function times  $dx$ . Since  $\varphi_0 := dx/f_y$  vanishes to order  $d - 3$  along  $C_0 \cap L$ , the regular differential forms on  $C$  must be of the form  $e(x, y)\varphi_0$  where  $e(x, y)$  is a rational function of degree  $\leq d - 3$ . (The set of rational forms that occur in this way is called the *Dedekind complementary module*.)

A sophisticated form of Hurwitz' theorem that will be explained in Chapter 16 shows that the sheaf  $\omega_C$  of regular differential forms on  $C$  can be expressed as

$$\mathcal{H}\text{om}_{\mathbb{P}^1}(\nu_*(\mathcal{O}_C), \omega_{\mathbb{P}^1}),$$

or more correctly  $\nu^!\pi^!\mathcal{H}\text{om}_{\mathbb{P}^1}(\pi_*\nu_*(\mathcal{O}_C), \omega_{\mathbb{P}^1})$  where  $\pi$  is the map  $C_0 \rightarrow \mathbb{P}^1$  defined by  $x$ —see Chapter 16. Since the maps involved are finite, we will ignore this refinement, and treat  $\mathcal{O}_{C_0}$  and  $\nu_*(\mathcal{O}_C)$  as sheaves on  $\mathbb{P}^1$ .

Since  $\mathcal{O}_{C_0} \subset \nu_*(\mathcal{O}_C)$ , this sheaf is naturally contained in

$$\mathcal{H}\text{om}_{\mathbb{P}^1}(\mathcal{O}_{C_0}, \omega_{\mathbb{P}^1}).$$

We will show in Theorem 8.1 that

$$\mathfrak{f}_{C/C_0} = \frac{\mathcal{H}\text{om}_{\mathbb{P}^1}(\mathcal{O}_{C_0}, \omega_{\mathbb{P}^1})}{\mathcal{H}\text{om}_{\mathbb{P}^1}(\nu_*(\mathcal{O}_C), \omega_{\mathbb{P}^1})}.$$

As will be explained in Chapter 16, the numerator of this quotient

$$\omega_{C_0} := \mathcal{H}\text{om}_{\mathbb{P}^1}(\mathcal{O}_{C_0}, \omega_{\mathbb{P}^1}) = \mathcal{H}\text{om}_{\mathbb{P}^1}(\mathcal{O}_{C_0}, \mathcal{O}_{\mathbb{P}^1})(-2)$$

is properly called the dualizing module of the singular curve  $C_0$ . Thus every regular differential form on  $C$  can be expressed as an element of the conductor times some element of  $\omega_{C_0}$ . To prove the Theorem, we must show that  $\varphi_0$  generates  $\omega_{C_0}$  as a module over  $\mathcal{O}_{C_0}$ .

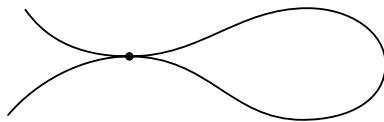
Passing to the field of rational functions  $\kappa := \kappa(C)$ , and noting that  $\kappa(C) = \kappa(C_0)$  we use the well-known result from Galois theory that

$$\text{Hom}_{\kappa(\mathbb{P}^1)}(\kappa, \kappa(\mathbb{P}^1))$$

is a 1-dimensional vector space over  $\kappa$ , generated by the trace map  $T$ . Moreover, because  $\mathcal{O}_{C_0}$  is integral over  $\mathcal{O}_{\mathbb{P}^1}$ , which is normal,

$$T(\mathcal{O}_{C_0}) \subset \mathcal{O}_{\mathbb{P}^1}.$$

Less well-known is the result from commutative algebra:



**Figure 3.** A tacnode: two smooth branches tangent to one another.

**Theorem 2.7.** *If  $C_0^\circ$  is an affine plane curve defined by the equation  $f(x, y) = 0$  and such that  $\mathbb{C}[x, y]/(f)$  is finite over  $\mathbb{C}[x]$ , then  $\text{Hom}_{\mathbb{C}[x]}(\mathbb{C}[x, y]/(f), \mathbb{C}[x])$  is generated by  $(1/f_y)T$ .*

See [Kunz 2005, Theorem 15.1] for a proof using valuations, and [Eisenbud and Ulrich 2019, Theorem A.1] for a proof in a more general context.

Given Theorem 2.7, we can complete the proof of Theorem 2.5. Via the isomorphism  $\mathbb{C}(x) \cong \mathbb{C}(x)dx$  sending 1 to  $dx$ , the trace map is identified (up to scalar) with the map  $(1 \mapsto dx) \in \text{Hom}_{\mathbb{C}(x)}(\kappa, \mathbb{C}(x)dx)$ . Thus by Theorem 2.7 the canonical module of  $C_0^\circ$  is identified with  $\mathcal{O}_{C_0^\circ}\phi_0$  as required.  $\square$

**Example 2.8** (nodes and cusps). We have already seen that in case  $q$  is a node of  $C_0$ , there are two points of  $C$  lying over it, and the multiplicities of  $\varphi_0$  at these two points are  $m_1 = m_2 = 1$ ; the adjoint ideal is thus the maximal ideal  $\mathcal{I}_q$  at  $q$ . In the case of a cusp, analytically isomorphic to the zero locus of  $y^2 - x^3$ , there is only one point  $r$  of  $C$  lying over the cusp point  $q$ . The cusp can be parameterized, locally analytically, by  $x = t^2, y = t^3$  and it follows that the differential

$$\varphi_0 = \frac{dx}{f_y} = \frac{2tdt}{2t^3} = \frac{dt}{t^2}$$

has a pole of order 2. since the pullback to  $C$  of any polynomial  $g$  vanishing at the cusp  $q$  vanishes to order at least two at  $r$ , the adjoint ideal is again the maximal ideal at  $q$ . We can also see this by computing the conductor ideal as the annihilator of  $\mathbb{C}[[t]]/\mathbb{C}[[t^2, t^3]]$ .

**Example 2.9** (tacnodes). Next, consider the case of a *tacnode*; that is, a plane-curve singularity with two smooth branches simply tangent to one another, analytically isomorphic to the zero locus of  $y^2 - x^4$ , parameterized locally analytically with two branches  $x = t, y = t^2$  and  $x = t, y = -t^2$ .

At the two points of  $C$  lying over  $q$ , we have

$$\varphi_0 = \frac{dt}{2t} \text{ and } \frac{-dt}{2t}$$

which has a simple pole at each. The adjoint ideal is thus the ideal of functions vanishing at  $q$  and having derivative 0 in the direction of the common tangent line to the branches.

**Example 2.10** (ordinary  $n$ -fold points). In the case of an ordinary  $n$ -fold point of a plane curve— $n$  smooth branches pairwise transverse to one another—there are  $n$  points  $r_i$  of  $C$  lying over  $q$ . The polynomial  $f_y$  vanishes to order  $n - 1$  at  $q$ , so  $dx/f_y$  has a pole of order  $n - 1$  at each  $r_i$ . It follows that for  $e(x, y)dx/f_y$  to be regular,  $e$  must vanish to order  $n - 1$  at each  $r_i$ .

We can see that the conductor ideal is the full  $(n - 1)$ -rst power of  $(x, y)$  by using the normalization map

$$\nu^* : \frac{\mathbb{C}[x, y]}{\prod_{i=1}^n (x - \alpha_i y)} \rightarrow \prod_{i=1}^n \frac{\mathbb{C}[x, y]e_i}{(x - \alpha_i y)}$$

where the  $\alpha_i$  are distinct elements of  $\mathbb{C}$  and the  $e_i$  are orthogonal idempotents. The element  $1 = \sum_i e_i$  goes to 0 in the quotient, and  $e_i$  is annihilated by  $x - \alpha_i y$ , so the quotient is annihilated by each of the  $n$  elements  $g_j := \prod_{i \neq j} (x - \alpha_i y)$ . As forms on  $\mathbb{P}^1$ , all the  $g_i$  except  $g_j$  vanish at the point  $(\alpha_j, 1)$ , so the  $g_i$  are linearly independent. Since  $(x, y)^{n-1}$  is minimally generated by  $n$  elements,  $(x, y)^{n-1} = (g_1, \dots, g_n)$ .

A consequence of this computation is that the  $\delta$ -invariant of the ordinary multiple point—that is, the difference in arithmetic genus between a plane curve that has just one such singular point and its normalization—is  $\binom{n}{2}$ , the dimension of  $k[x, y]/(x, y)^{n-1}$ , as we proved before in Proposition 1.9.

**Example 2.11.** (spatial triple points) Spatial triple points provide a contrast to the last example. A spatial triple point is a singularity consisting of three smooth branches, with linearly independent tangent lines, meeting in a point  $p$  so that its Zariski tangent space is 3-dimensional; the simplest example is the origin as a point on the union of the three coordinate axes in  $\mathbb{A}^3$ .

In this case the conductor is the annihilator of the cokernel of

$$\nu^* : R := \frac{\mathbb{C}[x, y, z]}{(xy, xz, yz)} \rightarrow \frac{\mathbb{C}[x, y, z]}{(x, y)} \times \frac{\mathbb{C}[x, y, z]}{(x, z)} \times \frac{\mathbb{C}[x, y, z]}{(y, z)} =: \bar{R}.$$

Since  $x\bar{R} = x\mathbb{C}[x, y, z]/(y, z)$  is in the image of  $\mathbb{C}[x, y, z]/(xy, xz, yz)$ , and similarly with  $y$  and  $z$ , we see that the conductor is the maximal ideal  $(x, y, z)$ . However, the  $\delta$ -invariant, the length of the quotient  $(\bar{R})/R$ , is 2: for a function  $f$  in  $\bar{R}$  to be in  $R$ , it is necessary and sufficient that  $f$  take the same value at the three points above the singular point, and this is 2 linear conditions.

### 3. Exercises

In Exercise 5.1, we saw how to use the description of the canonical series on a smooth plane curve to determine its gonality. Now that we have an analogous description of the canonical series on (the normalization of) a nodal plane

curve, we can deduce a similar statement about the gonality of such a curve. Here are the first two cases:

**Exercise 3.1.** Let  $C_0$  be a plane curve of degree  $d \geq 4$  with one node  $p$  and no other singularities, and let  $C$  be its normalization. Show that  $C$  admits a unique map  $C \rightarrow \mathbb{P}^1$  of degree  $d - 2$ , but does not admit a map  $C \rightarrow \mathbb{P}^1$  of degree  $d - 3$  or less.

Hint: If  $D = q_1 + \cdots + q_{d-2}$  had  $r(D) \geq 1$ , the points  $q_1 + \cdots + q_{d-2}$  and  $p$  would fail to impose independent conditions on plane curves of degree  $d - 3$  and hence lie on a line.

**Exercise 3.2.** Let  $C_0$  be a plane curve of degree  $d \geq 5$  with two nodes  $p$  and  $p'$  and no other singularities, and let  $C$  be its normalization. Show that  $C$  admits two maps  $C \rightarrow \mathbb{P}^1$  of degree  $d - 2$ , but does not admit a map  $C \rightarrow \mathbb{P}^1$  of degree  $d - 3$  or less.

Hint: If  $D = q_1 + \cdots + q_{d-2}$  had  $r(D) \geq 1$ , the points  $q_1 + \cdots + q_{d-2}$  and  $p, p'$  would fail to impose independent conditions on plane curves of degree  $d - 3$  and hence by Proposition 3.4 below  $d - 1$  of them would lie on a line.

**Exercise 3.3.** Generalizing the examples above, show that if a nodal plane curve of degree  $d$  has  $\delta \leq d + 3$  nodes, then its gonality is  $d - 2$ , and moreover every  $g_d^1$  on the curve is given by projection from one of the nodes. You may use the following result:

**Proposition 3.4.** *A set of  $n \leq 2d + 2$  distinct points in the plane fails to impose independent conditions on curves of degree  $d$  if and only if either  $d + 2$  of the points are collinear or  $n = 2d + 2$  and all the points lie on a conic.*

See [Eisenbud et al. 1996, p. 302] for a proof of this.

Hint: a  $g_{d-2}^1$  is a set of points that, together with the nodes, impose dependent conditions on forms of degree  $d - 3$ .

**Exercise 3.5.** Suppose that  $C$  is a smooth curve and  $\nu : C \rightarrow C_0$  is a map to a plane curve with only nodes as singularities. Let  $D = D_+ - D_-$  be a divisor on  $C$ . Modify the technique of Section 1.2 to compute the complete linear series  $|D|$  without assuming that  $D$  is disjoint from the preimages of the singular points.  
Hint: Be careful to subtract the right multiples of the points that are preimages of the singular points.

**Exercise 3.6.** Find the adjoint ideals of the following plane curve singularities:

- (1) a *triple tacnode*: three smooth branches, pairwise simply tangent (in classical language, this is a *triple point with an infinitely near double point*)

- (2) a triple point with an infinitely near double point: three smooth branches, two of which are simply tangent, with the third transverse to both
- (3) a unibranch triple point, such as the zero locus of  $y^3 - x^4$

**Hint:** For the first, the adjoint ideal is the ideal of functions vanishing to order 4 on each branch (so that the general member of the ideal will have zero locus consisting of two smooth branches simply tangent to the branches of the triple point). For the last, the adjoint is simply the square of the maximal ideal.

Here is a general description in case the individual branches of  $C_0$  at  $p$  are each smooth:

**Exercise 3.7.** Let  $\nu : C \rightarrow C_0$  be the normalization of a plane curve  $C_0$  and  $p \in C_0$  a singular point. Denote the branches of  $C_0$  at  $p$  by  $B_1, \dots, B_k$ , and let  $r_i$  be the point in  $B_i$  lying over  $p$ . If the individual branches  $B_i$  of  $C_0$  at  $p$  are each smooth, and we set

$$m_i = \sum_{j \neq i} \text{mult}_p(B_i \cdot B_j)$$

then the adjoint ideal of  $C_0$  at  $p$  is the ideal of functions  $g$  such that  $\text{ord}_{r_i}(\nu^* g) \geq m_i$ .

**Hint:** The computation can be done locally analytically. Let  $R = \widehat{\mathcal{O}_{C_0, p}}$  be the completion of the local ring of  $C_0$  at  $r_i$ . The integral closure is then the product of rings  $R_i = \widehat{\mathcal{O}_{B_i}} \cong k[[t_i]]$ , with  $R_i = R/P_i$  as  $P_i$  runs over the minimal primes of  $R$ . The multiplicity  $m_i$  is the colength of the ideal  $\sum_{j \neq i} P_j \subset R$ .



# Linkage and the canonical sheaves of singular curves

## 1. Introduction

*In this chapter, curves are purely 1-dimensional projective schemes—not necessarily reduced or irreducible.*

Linkage is an equivalence relation on varieties and schemes of a given dimension embedded in a common space. It was a key element in the classification of curves in  $\mathbb{P}^3$  for which Max Noether and Georges-Henri Halphen received in the Steiner prize of the Prussian Academy of Sciences in 1880, and it was a necessary ingredient in the work of Clebsch, Brill, Noether and Macaulay toward a version of the Riemann-Roch theorem couched in terms of the algebra of plane curves near the end of the 19-th century. It was put on a firm modern footing in [Peskine and Szpiro 1974], and this foundation was used for further progress in projective geometry by Hartshorne, Rao and others. In this Chapter we will explain some of these developments, starting with a simple example, and including the algebra necessary for a formulation in the natural generality of purely 1-dimensional schemes.

As we have seen, any plane curve is arithmetically Cohen-Macaulay, and its arithmetic genus is determined by its degree. Similarly, a curve in  $\mathbb{P}^3$  that is a complete intersection of surfaces of degrees  $d, e$  is arithmetically Cohen-Macaulay (Theorem 5.1) and has arithmetic genus determined by  $d, e$ . Next simplest, perhaps is a curve  $C$  that is *directly linked* to a complete intersection,

which means roughly that its union  $X = C \cup D$  with a complete intersection curve  $D$  is again a complete intersection (see Definition 4.1 for the general definition). We will see that, once again, such a curve  $C$  is arithmetically Cohen-Macaulay, and its genus is determined by the degrees of the equations of  $X$  and the degree and genus of  $D$ . Allowing sequences of direct links we define an equivalence relation called *linkage* or *liaison*, and curves in the linkage class of a complete intersection are often said to be “licci.”

A famous theorem of Hartshorne and Rao [Prabhakar Rao 1978/79] shows that the linkage class of a curve  $C \subset \mathbb{P}^3$  is classified by the finite dimensional graded module

$$D(C) := H_*^1(\mathcal{I}_C) := \bigoplus_{m \in \mathbb{Z}} H^1(\mathcal{I}_C(m)),$$

called the *deficiency module* of  $C$ . (This is a module over the homogeneous coordinate ring  $S = H_*^0(\mathcal{O}_{\mathbb{P}^3})$ .) The correspondence is explicit: from a finite-dimensional graded module over  $S$  one can actually construct curves.

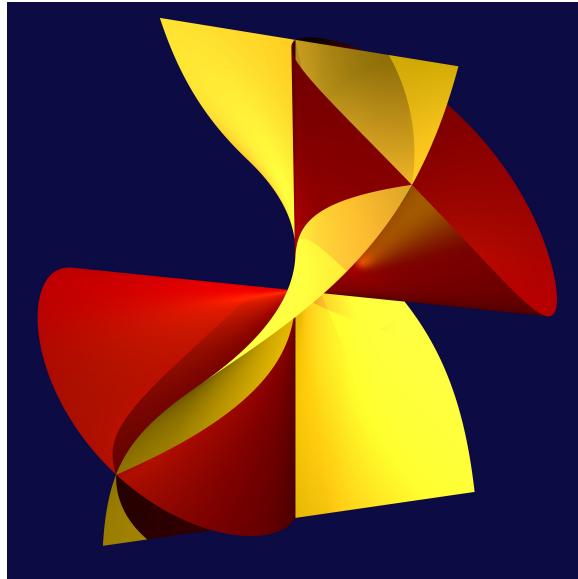
In the first sections of this chapter we will examine the equivalence relation on curves in  $\mathbb{P}^3$  that is defined by linkage. Much of the story extends to the case of singular curves. This extension requires an understanding of the dualizing sheaves of singular curves, to which we turn in Section 5.1. We conclude the chapter with an analysis of the adjoint ideal, completing a result from Chapter 15, and to the Riemann-Roch theorem for general curves and coherent sheaves.

Aside from the classification result above, linkage is useful in analyzing Hilbert schemes. We will exploit this systematically in cases of low degree and genera in Chapter 19, and we begin this chapter with what is perhaps the simplest example, computing the dimension of the component of the Hilbert scheme  $Hilb_{3m+1}(\mathbb{P}^3)$  that is the closure of the open subset  $H^\circ$  parametrizing twisted cubics (see Proposition 3.5 for another proof).

## 2. Linkage of twisted cubics

The simplest example of linkage is that of the union of a twisted cubic and one of its secant lines, pictured in Figure 2, and we will start with that.

Any twisted cubic curve  $C \subset \mathbb{P}^3$  lies on a nonsingular quadric in class  $(1, 2)$ . Adding a line  $L$  of class  $(1, 0)$  we get a divisor of class  $(2, 2)$ , the class of the complete intersection of two quadrics. Since  $L$  is also a complete intersection,  $C$  is licci.



**Figure 1.** A quadratic cone (red) intersecting a smooth quadric (yellow) in the union of a vertical line and a twisted cubic (credit: Herwig Hauser)

We can make the relation of  $L$  and  $C$  explicit as follows: The ideal of  $C$  is minimally generated by the three  $2 \times 2$  minors of the matrix

$$\begin{pmatrix} x_0 & x_1 & x_2 \\ x_1 & x_2 & x_3 \end{pmatrix}.$$

The minor  $Q_{1,2}$  involving the first two columns and the minor  $Q_{2,3}$  involving the last two columns both vanish on the line  $L : x_1 = x_2 = 0$ , which meets the twisted cubic in the two points  $x_0 = x_1 = x_2 = 0$  and  $x_1 = x_2 = x_3 = 0$ . Thus  $L$  is a secant line to  $C$ . A general linear combination  $Q$  of  $Q_{1,2}$  and  $Q_{2,3}$  defines a smooth quadric, which is thus isomorphic to  $\mathbb{P}^1 \times \mathbb{P}^1$ . The curve  $C$  necessarily lies in the divisor class  $(1, 2)$  (or, symmetrically,  $(2, 1)$ ), and the line in class  $(1, 0)$  (respectively,  $(0, 1)$ ), summing to the complete intersection  $(2, 2)$  of  $Q$  with (say)  $Q_{1,2}$ . See Figure 2.

Conversely, if two irreducible quadrics  $Q_1, Q_2$  both contain a twisted cubic  $C$  then, by Bézout's theorem,  $Q_1 \cap Q_2$  is the union of  $C$  with a line. If at least one of the quadrics is smooth, we are in the situation above.

This suggests that we set up an incidence correspondence between twisted cubics and their secant lines. Let  $\mathbb{P}^9$  denote the projective space of quadrics in  $\mathbb{P}^3$ , and consider

$$\Phi = \{(C, L, Q, Q') \in H^\circ \times G(1, 3) \times \mathbb{P}^9 \times \mathbb{P}^9 \mid Q \cap Q' = C \cup L\}.$$

We'll analyze  $\Phi$  by considering the projection maps to  $H^\circ$  and  $\mathbb{G}(1, 3)$ ; that is, by looking at the diagram

$$\begin{array}{ccc} & \Phi & \\ \pi_1 \swarrow & & \searrow \pi_2 \\ H^\circ & & \mathbb{G}(1, 3) \end{array}$$

Consider the projection map  $\pi_2 : \Phi \rightarrow \mathbb{G}(1, 3)$  on the second factor. By what we just said, the fiber over any point  $L \in \mathbb{G}(1, 3)$  is an open subset of  $\mathbb{P}^6 \times \mathbb{P}^6$ , where  $\mathbb{P}^6$  is the space of quadrics containing  $L$ . Since  $\dim \mathbb{G}(1, 3) = 4$  we see that  $\Phi$  is irreducible of dimension  $4 + 2 \times 6 = 16$ . On the other hand, the map  $\pi_1 : \Phi \rightarrow H^\circ$  is surjective, with fiber over a curve  $C$  an open subset of  $\mathbb{P}^2 \times \mathbb{P}^2$ , where  $\mathbb{P}^2$  is the projective space of quadrics containing  $C$ ; we conclude that  $H^\circ$  is irreducible of dimension 12, in accord with our computation in Proposition 3.5 of the space of twisted cubics as  $PGL_4/PGL_2$ .

### 3. Linkage of smooth curves in $\mathbb{P}^3$

If the union of two smooth curves in  $\mathbb{P}^3$  is a complete intersection of surfaces, then the degrees and genera of the curves are related:

**Theorem 3.1.** *Let  $C, C_2 \subset \mathbb{P}^3$  be distinct smooth irreducible curves of degrees  $c, d$  whose union is the complete intersection of two surfaces  $S, T$ , with  $S$  smooth. If the degrees of  $S, T$  are  $s, t$  respectively, then*

$$\begin{aligned} \deg C_1 + \deg C_2 &= st \\ g(C_1) - g(C_2) &= \frac{s+t-4}{2}(\deg C_1 - \deg C_2). \end{aligned}$$

In words, the difference between the genera of  $C_1$  and  $C_2$  is proportional to the difference in their degrees, with constant of proportionality  $(s+t-4)/2$ . For example, in the case of the complete intersection of two quadrics described above, the multiplier  $(s+t-4)/2 = 0$ , and indeed the line and the twisted cubic have the same genus. The relation of degrees and genera is true more generally, as we shall see in the next section, but the special case is already useful.

**Proof.** Bézout's theorem implies that the degree of  $S \cap T = C_1 \cup C_2$  is  $st$ , whence the first formula.

By the adjunction formula in  $\mathbb{P}^3$  the canonical divisor of  $S$  has class  $K_S = (s-4)H$ . Thus, from the adjunction formula on the surface  $S$  we get

$$g(C_i) = \frac{C_i^2 + C_i \cdot K_S}{2} + 1 = \frac{C_i^2 + (s-4)\deg C_i}{2} + 1.$$

Subtracting, we get

$$g(C_1) - g(C_2) = \frac{C_1^2 - C_2^2 + (s-4)(\deg C_1 - \deg C_2)}{2}.$$

Because  $C_1 + C_2$  is in the class  $tH$  on  $S$  we have

$$C_1^2 - C_2^2 = (C_1 - C_2)(C_1 + C_2) = t(\deg C_1 - \deg C_2)$$

and substituting this into the previous formula we get the second formula of the theorem.  $\square$

**Remark 3.2.** Linkage is closely related to linear equivalence; here is a special case: Suppose that  $S$  is a smooth surface in  $\mathbb{P}^3$ , and  $C \subset S$  is a curve. If  $T$  is a sufficiently general surface of degree  $t$  containing  $C$  then the curve  $C'$  that is the link of  $C$  with respect to  $S, T$  lies in the class  $tH - C$ . If we link again with respect to another surface  $T'$  of degree  $t'$  we thus arrive at  $C'' = C + (t - t')H$ . Thus if  $t = t'$  we get a curve in the same linear equivalence class as  $C$ . Moreover, since every rational function on  $S$  is the restriction to  $S$  of the ratio of two forms of the same degree on  $\mathbb{P}^3$ , the set of curves on  $S$  that can be obtained from  $C$  by two linkages with surfaces  $T, T'$  of the same degree is exactly the linear series  $|C|$  on  $S$ . This idea is generalized in the notion of a *basic double link*; see Exercise 10.6.

#### 4. Linkage of purely 1-dimensional schemes in $\mathbb{P}^3$

To say that the union of distinct reduced irreducible curves  $C \cup C'$  is a complete intersection  $X$  means that the ideal  $I_X = I_C \cap I_{C'}$ . Since  $I_C \cap I_{C'}$  contains  $I_C I_{C'}$ , the ideal quotient  $(I_X : I_C) := \{F \mid FI_C \subset I_X\}$  contains  $I_{C'}$ .

On the other hand, if  $F \notin I_{C'}$  and we choose  $G \in I_C \setminus I_{C'}$ , then  $FG \notin I_{C'}$ , so  $F \notin (I_X : I_C)$ , and thus  $(I_X : I_C) = I_{C'}$ . It turns out that this relationship is the key to the formulas connecting the degrees and genera of  $C, C'$ , which hold for arbitrary purely 1-dimensional subschemes of  $\mathbb{P}^3$ , as we shall see in Theorem 5.1.

**Definition 4.1.** Let  $C, C'$  be purely 1-dimensional subschemes of  $\mathbb{P}^3$ . We say that  $C'$  is *directly linked* to  $C$  if there is a complete intersection  $X$  containing  $C, C'$  and  $(I_X : I_C) = I_{C'}$ . We say that  $C'$  is *linked* to  $C$  if they are connected by a chain of such direct linkages, and we say that  $C'$  is *evenly linked* to  $C$  if the chain involves an even number of direct linkages.

Note that in this setting,  $C$  and  $C'$  can have components in common. For example, a “rope”—that is, the subscheme  $C \subset \mathbb{P}^3$  defined by the square of the ideal  $\mathcal{I}_{L/\mathbb{P}^3}$  of a line  $L \subset \mathbb{P}^3$ —is linked to the reduced line  $L$  in the complete intersection of two quadrics, as the reader will be invited to verify in Exercise 10.11. (This makes sense, since a rope is a flat limit of twisted cubics.)

As in the smooth case treated above, direct linkage is a symmetric relationship:

**Proposition 4.2.** *Let  $C_1 \subset \mathbb{P}^3$  be a purely 1-dimensional subscheme with saturated homogeneous ideal  $I_1$  and suppose that  $C_1$  is contained in a complete intersection of hypersurfaces  $X := S \cap T$ . The ideal  $I_2 = (I_X : I_1)$  is a saturated ideal, defining a purely 1-dimensional subscheme and  $I_1 = (I_X : I_2)$  as well.*

**Proof.** Since  $X$  is a complete intersection, the ideal  $I_X$  is unmixed of codimension 2 ([Eisenbud 1995a, Proposition 18.13]), and it follows that  $I_2 = (I_X : I_1)$  is unmixed of codimension 2 as well. In particular it is saturated. Thus it suffices to prove that  $I_1 = (I_X : I_2)$  after localizing at a codimension 2 prime  $P$  that contains  $I_X$ .

Write  $R$  for the localization at  $P$  of the homogeneous coordinate ring of  $X$ . Because  $I_X$  is a complete intersection, the ring  $R$  is zero-dimensional and Gorenstein. By [Eisenbud 1995a, Propositions 21.1 and 21.5], every finitely generated  $R$ -module is reflexive. Since

$$I_{C_2}R = \text{ann}_R(I_{C_1}R) \cong \text{Hom}_R(R/I_{C_1}R, R)$$

the proposition follows.  $\square$

We define the equivalence relation of *linkage* among purely 1-dimensional subschemes of  $\mathbb{P}^3$  as the closure of the relation of direct linkage, and we say that two such subschemes are *evenly linked* if they are connected by an even number of direct linkages.

## 5. Degree and genus of linked curves

The degrees and (arithmetic) genera of directly linked schemes are related exactly as in the simple case above:

**Theorem 5.1.** *If  $C_1, C_2 \subset \mathbb{P}^3$  are purely 1-dimensional schemes that are directly linked by surfaces  $S, T$  of degrees  $s, t$  then*

$$\begin{aligned} \deg C_1 + \deg C_2 &= st \\ p_a(C_1) - p_a(C_2) &= \frac{s+t-4}{2}(\deg C_1 - \deg C_2). \end{aligned}$$

Since we have left the realm of smooth curves and surfaces, we will need a more sophisticated duality theory, and we postpone the proof to explain the necessary ideas.

**5.1. Dualizing sheaves for singular curves.** Recall that in Chapter 2 we claimed that the canonical sheaf of a smooth curve—the sheaf of differential forms—was “the most important invertible sheaf” after the structure sheaf. In the general setting of Cohen-Macaulay schemes, the analogue of the canonical sheaf is called the dualizing sheaf. The general definition of the dualizing sheaf is not very illuminating; what is useful is how it is constructed and its cohomological properties relating to duality. However, having a definition may be comforting.

**Definition 5.2.** Let  $X$  be a projective scheme of pure dimension  $d$  over  $\mathbb{C}$ . The *dualizing sheaf* for  $X$  is a coherent sheaf  $\omega_X$  with a *residue map*  $\eta : H^d(\omega_X) \rightarrow \mathbb{C}$  such that for every coherent sheaf  $\mathcal{F}$  the composite map

$$H^d(\mathcal{F}) \times \text{Hom}(\mathcal{F}, \omega_X) \rightarrow H^d(\omega_X) \xrightarrow{\eta} \mathbb{C}$$

is a perfect pairing.

Thus if  $F = \mathcal{L}$  is an invertible sheaf on a projective curve  $C$  then  $\text{Hom}(\mathcal{L}, \omega_X) = \mathcal{L}^{-1} \otimes \omega_C$ , so this relation becomes Serre duality:  $H^1(\mathcal{L})$  is the dual of  $H^0(\mathcal{L}^{-1} \otimes \omega_C)$ .

It follows from the definition that the pair  $(\omega_X, \eta)$  is unique up to canonical isomorphism if it exists: The module  $H_*^0(\omega_X) = \bigoplus_{n \in \mathbb{Z}} (\text{Hom}_X(\mathcal{O}_X(n), \omega_X))$  is determined as the graded vector space dual of  $H_*^d(\mathcal{O}_X)$ , and the choice of  $\eta$  simply fixes the isomorphism. It may not be apparent that such a sheaf exists, but we will give a construction in Section 6 below.

Several properties of the dualizing sheaf on a purely 1-dimensional scheme are the same as in the smooth case, and follow easily from the definition:

**Proposition 5.3.** *If  $C$  is a purely 1-dimensional projective scheme over  $\mathbb{C}$ , then:*

- (1)  $\text{Hom}_C(\omega_C, \omega_C) = \mathcal{O}_C$ , and thus if  $C$  is integral then the generic rank of  $\omega_C$  is 1.
- (2) For any invertible sheaf  $\mathcal{L}$  on  $C$  we have:

$$H^1(\mathcal{L}^{-1}) = H^0(\mathcal{L} \otimes \omega_C); \quad H^0(\mathcal{L}^{-1}) = H^1(\mathcal{L} \otimes \omega_C).$$

In particular,

$$\chi(\mathcal{L}^{-1}) = -\chi(\mathcal{L} \otimes \omega_C).$$

**Proof.** 1: We claim that the natural map  $\mathcal{O}_C \rightarrow \text{Hom}_C(\omega_C, \omega_C)$  is an isomorphism. Since the map is globally defined, it suffices to prove that it is an isomorphism locally.

Choose a Noether normalization of  $C$ , that is, a finite map  $f : C \rightarrow \mathbb{P}^1$ . We shall see in Theorem 6.1 below that  $\omega_C \cong \text{Hom}_{\mathbb{P}^1}(\mathcal{O}_C, \omega_{\mathbb{P}^1})$ , regarded as a sheaf on  $C$  (in Theorem 6.1 this is the sheaf  $f^! \omega_{\mathbb{P}^1}$ ). Since  $C$  is purely 1-dimensional,  $\mathcal{O}_C$  is torsion-free as an  $\mathcal{O}_{\mathbb{P}^1}$ -module, and is thus locally free. Also, since  $\mathbb{P}^1$  is

smooth,  $\omega_{\mathbb{P}^1}$  is locally isomorphic to  $\mathcal{O}_{\mathbb{P}^1}$ . But if  $B$  is any commutative ring and  $A$  is a  $B$ -algebra that is finitely generated and free as a  $B$ -module, then the composite map  $A \rightarrow \text{Hom}_A(\text{Hom}_B(A, B), \text{Hom}_B(A, B)) \cong \text{Hom}_B(\text{Hom}_B(A, B), B)$  sending an element  $a$  to the multiplication by  $a$  and thence to the map  $f \mapsto f(a)$  may be identified with the isomorphism of  $A$  to its double dual as a  $B$ -module, completing the proof of 1. See Exercise 10.7 for a generalization of this last step.

2: The definition of  $\omega_C$  shows that, if  $\mathcal{L}$  is an invertible sheaf, then

$$\begin{aligned} H^1(\mathcal{L}^{-1}) &= \text{Hom}_C(\mathcal{L}^{-1}, \omega_C) \\ &= H^0(\mathcal{H}\text{om}_C(\mathcal{L}^{-1}, \omega_C)) \\ &= H^0(\mathcal{L} \otimes_C \omega_C). \end{aligned}$$

Using item 1, we have:

$$\begin{aligned} H^0(\mathcal{L}^{-1}) &= H^0(\mathcal{L}^{-1} \otimes_C \mathcal{O}_C) \\ &= H^0(\mathcal{L}^{-1} \otimes_C \mathcal{H}\text{om}_C(\omega_C, \omega_C)) \\ &= \text{Hom}_C(\mathcal{L} \otimes_C \omega_C, \omega_C) \\ &= H^1(\mathcal{L} \otimes_C \omega_C) \end{aligned}$$

□

## 6. The construction of dualizing sheaves

Dualizing sheaves do exist on any projective Cohen-Macaulay scheme. We have already seen constructions in three different cases:

- If  $X$  is a smooth scheme of dimension  $d$  over  $\mathbb{C}$  then  $\omega_X = \wedge^d \Omega_{X/\mathbb{C}}$  is a dualizing sheaf ([Hartshorne 1977, Hartshorne[Section III.7] or [Griffiths and Harris 1978, p. 648, 708]).
- If  $f : X \rightarrow Y$  is a map of smooth curves, then  $\omega_X = f^*(\omega_Y)(\text{ram}_{X/Y})$ , where  $\text{ram}$  denotes the ramification divisor.
- If  $X \subset Y$  is a Cartier divisor on a surface, then  $\omega_X = \omega_Y(X)|_X$ .

How can such different looking formulas all be correct? Grothendieck provided a general scheme that unifies them and gives many more. To understand what is needed for the general case, we first consider a setting generalizing Hurwitz' theorem. Suppose that  $X \rightarrow Y$  is a finite map of projective schemes, and that  $\mathcal{F}$  is a coherent sheaf on  $X$ .

If we restrict ourselves to open affine subsets  $U := \text{Spec } A \subset X$  mapping to  $V := \text{Spec } B \subset Y$  via the map of rings  $f^* : B \rightarrow A$ , then  $F := \mathcal{F}_U$  is an  $A$ -module. Moreover,  $f_* F := f_*(\mathcal{F})(V)$  is just  $F$  regarded as a  $B$ -module via the map  $f^*$ .

For any  $B$ -module  $M$  the module  $\text{Hom}_B(A, M)$  has a natural structure of  $A$ -module, where  $(a\phi)(m)$  is defined to be  $\phi(am)$ . The functor  $f^!(-) := \text{Hom}_B(A, -) : \text{mod}_B \rightarrow \text{mod}_A$  defined in this way is the right adjoint of the functor  $f_*(-) : \text{mod}_A \rightarrow \text{mod}_B$ , which means that there is a natural isomorphism of functors

$$\text{Hom}_B(f_*F, -) \cong \text{Hom}_A(F, \text{Hom}_B(A, -)) = \text{Hom}_A(F, f^!(-)).$$

In particular if  $Y$  has dualizing sheaf  $\omega_Y$  and we set  $\mathcal{O}_Y := \omega_Y(V)$ , then  $\text{Hom}_B(f_*F, \mathcal{O}_Y) \cong \text{Hom}_A(F, f^!\mathcal{O}_Y)$ . Also, there is a natural transformation  $\eta$  from  $f_*f^!$  to the identity functor given by the formula

$$\eta : f_*f^!(M) = f_*(\text{Hom}_B(A, M)) = \text{Hom}_B(A, M) \rightarrow M : \quad \eta(\phi) = \phi(1)$$

for any  $A$ -module  $M$ . The transformation  $\eta$  called the *counit* of the adjoint pair  $(f_*, f^!)$ .

We also write  $f^!(-)$  for the sheafification of the functor  $\text{Hom}_B(A, -)$ . Again  $f^!$  is right adjoint to  $f_*$  on coherent sheaves, and again there are natural maps  $\eta : f_*f^!(\mathcal{F}) \rightarrow \mathcal{F}$ .

**Theorem 6.1.** *Let  $f : X \rightarrow Y$  be a finite map of  $d$ -dimensional projective schemes. If  $Y$  has a dualizing sheaf  $\omega_Y$ , with residue map  $\eta_Y : H^d(\omega_Y) \rightarrow \mathbb{C}$ , then*

$$\omega_X := f^!\omega_Y,$$

with residue map

$$\rho_X : H^d(f^!\omega_X) = H^d(f_*f^!\omega_X) \xrightarrow{H^d(\eta)} H^d(\omega_Y) \xrightarrow{\rho_Y} \mathbb{C},$$

where  $\eta$  is the counit of the adjoint pair  $(f_*, f^!)$ , is a dualizing sheaf on  $X$ .

**Proof.** Let  $\mathcal{F}$  be a coherent sheaf on  $X$ . Since  $f$  is finite, we have  $H^d(\mathcal{F}) = H^d(f_*(\mathcal{F}))$ . Thus, since  $f^!(-)$  is a right adjoint of  $f_*$  there are natural isomorphisms

$$H^d(\mathcal{F}) = H^d(f_*\mathcal{F}) \cong^{\rho_Y} \text{Hom}_Y(f_*\mathcal{F}, \omega_Y)^\vee \cong \text{Hom}_X(\mathcal{F}, f^!\omega_Y)^\vee,$$

where  $\cong^{\rho_Y}$  is the isomorphism induced by  $\rho_Y$ . One can check that the composite isomorphism is the one induced by  $\rho_X$ , so  $\omega_X = f^!(\omega_Y)$ , completing the proof.  $\square$

**Cheerful Fact 6.2.** *Given this theorem, it seems natural to look for an adjoint functor  $f^!$  for a wider class of morphisms  $f$  but...in most cases, for example when  $f$  is the inclusion of a divisor on a smooth surface, no such functor exists on the category of coherent sheaves! However, an adjoint functor  $f^!$  does exist on the derived category, where it is the right adjoint to  $Rf_*$ , leading to a theory of dualizing complexes.*

Fortunately for the reader who is mostly interested in curves, this level of complication is unnecessary, and there is an intermediate level of generality that suffices for all the purposes of this book and more:

**Theorem 6.3.** Suppose that  $f : X \rightarrow Y$  is a finite map of projective schemes. If  $Y$  is Gorenstein with dualizing module  $\omega_Y$ , then

$$f^!(\omega_Y) \cong \mathcal{E}xt_Y^{\dim Y - \dim X}(\mathcal{O}_X, \omega_Y)$$

is a dualizing module for  $X$ .

The hypothesis is satisfied for any  $Y$  that is smooth, or even locally a complete intersection. The reason this works is that the complex  $f^!\omega_Y$  can be identified with its one nonvanishing cohomology module,  $\mathcal{E}xt_Y^{\dim Y - \dim X}(\mathcal{O}_X, \omega_Y)$ .

See for example [Altman and Kleiman 1970] for a thorough and accessible exposition.

(( check A-K for precise hypotheses ))

### 6.1. Proof of Theorem 5.1.

**Proof.** Let  $X$  be the complete intersection of surfaces of degrees  $s, t$  containing  $C$ , and let  $R_X = S/(F, G)$  be its homogeneous coordinate ring, where  $S = \mathbb{C}[x_0, \dots, x_3]$  is the homogeneous coordinate ring of  $\mathbb{P}^3$ . From the free resolution

$$0 \longrightarrow S(-s-t) \xrightarrow{\begin{pmatrix} G \\ -F \end{pmatrix}} S(-s) \oplus S(-t) \xrightarrow{\begin{pmatrix} F & G \end{pmatrix}} S \longrightarrow R_X \longrightarrow 0$$

and Theorem 6.3 we see that

$$\omega_X = \mathcal{E}xt_C^2(\mathcal{O}_X, \omega_{\mathbb{P}^3}) = \mathcal{E}xt^2(\mathcal{O}_X, \mathcal{O}_{\mathbb{P}^3}(-4)) = \mathcal{O}_X(s+t-4).$$

Note that for any ideals  $J \subset I$  in a ring  $A$  we have  $\text{Hom}_A(A/I, A/J) \cong (J : I)/J$ , where the isomorphism sends a homomorphism  $\phi$  to the element  $\phi(1)$ . From Theorem 6.3 we have

$$\omega_C = \mathcal{H}\text{om}_X(\mathcal{O}_C, \omega_X) = \mathcal{H}\text{om}_X(\mathcal{O}_C, \mathcal{O}_X)(s+t-4) = \frac{\mathcal{I}_X : \mathcal{I}_C}{\mathcal{I}_X}(s+t-4),$$

where we have identified  $\mathcal{O}_C$  with its pushforward under the inclusion map  $C \rightarrow X$ .

By Proposition 5.3  $\chi(\omega_C(m)) = -\chi(\mathcal{O}_C(-m))$ . It follows that the leading coefficient of the Hilbert polynomial of  $\omega_C$ , is equal to  $\deg C$ , and thus

$$st = \deg \mathcal{O}_X = \deg \mathcal{O}_{C'} + \deg \mathcal{O}_C,$$

as required by the formula for the sum of the degrees.

From Theorem 6.1 (or Theorem 6.3) we see that  $\chi(\mathcal{O}_X) = st(4 - s - t)/2$ . Since  $\mathcal{O}_{C'} = \mathcal{O}_{\mathbb{P}^3}/(\mathcal{I}_X : \mathcal{I}_C)$  and  $(\mathcal{I}_X : \mathcal{I}_C)/(\mathcal{I}_X) = \omega_C(4 - s - t)$  we have

$$\begin{aligned} & \frac{4-s-t}{2}(\deg C + \deg C') \\ &= \frac{4-s-t}{2}st \\ &= \chi(\mathcal{O}_X) \\ &= \chi(\mathcal{O}_{C'}) + \chi(\omega_C(4 - s - t)) \\ &= \chi(\mathcal{O}_{C'}) - \chi(\mathcal{O}_C(s + t - 4)) \\ &= \chi(\mathcal{O}_{C'}) - (s + t - 4)\deg C - \chi(\mathcal{O}_C) \\ &= (1 - p_a(\mathcal{O}_{C'})) - (1 - p_a(\mathcal{O}_C)) - (s + t - 4)\deg C, \end{aligned}$$

whence

$$p_a(\mathcal{O}_C) - p_a(\mathcal{O}_{C'}) = \frac{(s+t-4)}{2}(\deg C - \deg C').$$

□

Linkage also behaves in a simple way with respect to deficiency modules. The following result was attributed in [Peskine and Szpiro 1974] to Daniel Ferrand:

**Theorem 6.4.** *If  $C, C'$  are purely 1-dimensional subschemes of  $\mathbb{P}^3$  that are directly linked by a complete intersection of degrees  $s, t$  then*

$$D(C') = \text{Hom}_{\mathbb{C}}(D(C), \mathbb{C})(-s - t + 4).$$

as graded modules over the homogeneous coordinate ring of  $\mathbb{P}^3$ .

**Proof.** Suppose that the homogeneous ideal of  $C$  is generated by forms of degree  $a_i, i = 1, \dots, s$ . Since  $C$  is locally Cohen-Macaulay, the local rings  $\mathcal{O}_{C,p}$  have projective dimension 2 as modules over  $\mathcal{O}_{\mathbb{P}^3,p}$ , and  $\mathcal{I}_{C,p}$  has projective dimension 1. Thus we have an exact sequence

$$0 \rightarrow \mathcal{E} \rightarrow \bigoplus_i \mathcal{O}_{\mathbb{P}^3}(-a_i) \rightarrow \mathcal{I}_C \rightarrow 0.$$

Since the first and second cohomology groups of the twists of  $\mathcal{O}_{\mathbb{P}^3}$  vanish, we deduce an isomorphism

$$D(C) := \bigoplus_{m \in \mathbb{Z}} H^1(\mathcal{I}_C(m)) \cong \bigoplus_{m \in \mathbb{Z}} H^2(\mathcal{E}(m)).$$

Let  $X$  be the complete intersection of two hypersurfaces, of degrees  $s, t$ , containing  $C$ . From the inclusion we deduce a map of resolutions

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{E} & \longrightarrow & \bigoplus_i \mathcal{O}_{\mathbb{P}^3}(-a_i) & \longrightarrow & \mathcal{O}_{\mathbb{P}^3} \longrightarrow \mathcal{O}_C \longrightarrow 0 \\ & & \uparrow & & \uparrow & & \uparrow = \\ 0 & \longrightarrow & \mathcal{O}_{\mathbb{P}^3}(-s-t) & \longrightarrow & \mathcal{O}_{\mathbb{P}^3}(-s) \oplus \mathcal{O}_{\mathbb{P}^3}(-t) & \longrightarrow & \mathcal{O}_{\mathbb{P}^3} \longrightarrow \mathcal{O}_X \longrightarrow 0 \end{array}$$

We dualize this diagram, form the mapping cone, and twist by  $-s - t$ . Note that  $\text{Hom}_{\mathbb{P}^3}(\mathcal{O}_C, \mathcal{O}_{\mathbb{P}^3}) = 0$ . Also, since the vertical map  $\mathcal{O}_{\mathbb{P}^3} \rightarrow \mathcal{O}_{\mathbb{P}^3}$  on the right is the identity we may cancel these terms in the mapping cone. Noting that  $\omega_C = \text{Ext}^2(\mathcal{O}_C, \mathcal{O}_{\mathbb{P}^3}(-4))$  the result is a diagram with exact rows:

$$\begin{array}{ccccccc}
 0 & \longleftarrow & \omega_C(-s-t+4) & \longleftarrow & \mathcal{E}^*(-s-t) & \longleftarrow & \bigoplus_i \mathcal{O}_{\mathbb{P}^3}(a_i - s - t) & \longleftarrow & 0 \\
 & & \downarrow \phi & & \downarrow & & \downarrow & & \\
 0 & \longleftarrow & \mathcal{O}_X & \longleftarrow & \mathcal{O}_{\mathbb{P}^3} & \longleftarrow & \mathcal{O}_{\mathbb{P}^3}(-t) \oplus \mathcal{O}_{\mathbb{P}^3}(-s) & \longleftarrow & 0 \\
 & & \downarrow & & & & & & \\
 & & \mathcal{O}_{C'} & & & & & & \\
 & & \downarrow & & & & & & \\
 & & 0 & & & & & &
 \end{array}$$

(( Xianglong attributed this to Ferrand. Reference?? Jan 2: DE writes to Xianglong to ask ))

The map  $\phi$  is a monomorphism because  $(\mathcal{I}_X : \mathcal{I}_C)/\mathcal{I}_X \cong \omega_C(-s-t+4)$ , as explained above, so the column on the left is a short exact sequence. We can now write a resolution of  $\mathcal{I}_{C'}$  as the mapping cone:

$$0 \leftarrow \mathcal{I}_{C'} \leftarrow \mathcal{O}_{\mathbb{P}^3}(-t) \oplus \mathcal{O}_{\mathbb{P}^3}(-s) \oplus \mathcal{E}^*(-s-t) \leftarrow \bigoplus_i \mathcal{O}_{\mathbb{P}^3}(a_i - s - t) \leftarrow 0.$$

From this we see that

$$H^1(\mathcal{I}_{C'}(m)) \cong H^1(\mathcal{E}^*(-s-t+m)) \cong \text{Hom}_{\mathbb{C}}(H^2(\mathcal{E}(s+t-m-4)), \mathbb{C})$$

where the last equality is from Serre duality on  $\mathbb{P}^3$ . Summing over  $m$  we see that  $D(C') \cong \text{Hom}_{\mathbb{C}}(D(C)(s+t-4), \mathbb{C})$ , and since Serre duality is functorial, the isomorphism holds not only as graded vector spaces, but as graded  $S$ -modules.  $\square$

Sometimes the following consequence is a useful way to compute the deficiency module:

**Proposition 6.5.** *If  $C$  is a purely 1-dimensional subscheme of  $\mathbb{P}^3$  with homogeneous ideal  $I = I_C$  then*

$$D(C) \cong \text{Hom}_{\mathbb{C}}(\text{Ext}^3(S/I, S), \mathbb{C})(-4), \mathbb{C})$$

as graded modules over the homogeneous coordinate ring  $S$  of  $\mathbb{P}^3$ .

**Proof.** We may choose a surjection  $\psi : \bigoplus_i S(-a_i) \rightarrow I$ , and choose the map  $\phi : \bigoplus_i \mathcal{O}_{\mathbb{P}^3}(-a_i) \rightarrow \mathcal{I}_C$  in the proof of Theorem 6.4 to be the corresponding map of sheaves, so that  $\mathcal{E}$  is the sheafification of the graded module  $E = \ker \psi$ .

Since  $I$  is a saturated ideal, the depth of  $S/I$  is at least 1, so  $\text{pd } S/I \leq 3$ , and  $I$  has a free resolution of the form

$$0 \longrightarrow G \longrightarrow F \longrightarrow \bigoplus_i S(-a_i) \longrightarrow S \longrightarrow S/I \longrightarrow 0.$$

where  $G \rightarrow F$  is a free presentation of  $E$ . and there is an exact sequence

$$0 \rightarrow E^* \rightarrow F^* \rightarrow G^* \rightarrow \text{Ext}_S^3(S/I, S) \rightarrow 0.$$

By Theorem 3.1, the fact that  $C$  is Cohen-Macaulay implies that the projective dimension of each of its local rings is  $\leq 2$ , and it follows that  $\text{Ext}_S^3(S/I, S)$  has finite length. Writing  $(\widetilde{\phantom{x}})$  for the sheafification functor, we have a short exact sequence of sheaves

$$0 \rightarrow \mathcal{E}^* \rightarrow \widetilde{F^*} \rightarrow \widetilde{G^*} \rightarrow 0.$$

From this we see that

$$\text{Ext}_S^3(S/I, S) = H_*^1(\mathcal{E}^*) = \text{Hom}_{\mathbb{C}}(H_*^2(\mathcal{E}(-4)), \mathbb{C}) = H_*^1(\mathcal{I})(-4),$$

proving the assertion.  $\square$

Proposition 6.5 is actually a special case of the local duality isomorphism between local cohomology and the dual of Ext; see for example [Eisenbud 2005, Theorem A.1.9].

## 7. The linkage equivalence relation

As an immediate consequence of Theorem 6.4 we have:

**Corollary 7.1.** (Hartshorne) *If two curves  $C, C'$  are linked by an even length chain of direct linkages, then  $D(C)$  and  $D(C')$  are isomorphic up to a shift in grading.*  $\square$

As we mentioned at the beginning of this Chapter, the converse is also true: the Hartshorne-Rao modules, up to shift in grading, provide a complete invariant of linkage. Even more precise results are known (and the characteristic 0 hypothesis is largely unnecessary); here is a sample:

### Cheerful Fact 7.2.

**Theorem 7.3.** *Let  $S = \mathbb{C}[x_0, \dots, x_3]$  be the homogeneous coordinate ring of  $\mathbb{P}^3$ , and let  $M$  be a graded  $S$ -module of finite length.*

- (1) *There is a smooth curve  $C$  with  $D(C) = M(m)$  for some integer  $m$ .*
- (2) *There is a minimum value of  $m$  such that  $M(m) = D(C_0)$  for some purely one-dimensional scheme  $C_0$ .*

Moreover, each Liaison class has a relatively simple structure, known as the Lazarsfeld-Rao property. We say that  $C'$  is obtained from  $C$  by an ascending double link if  $I_{C'} = fI_C + (g)$  for some regular sequence contained in  $I_C$ —see Exercise 10.6.

**Theorem 7.4.** [Ballico et al. 1991] Let  $M = D(C_0)$  the the Hartshorne Rao invariant of a purely 1-dimensional subscheme of  $\mathbb{P}^3$ , and suppose that  $M$  is minimal in the sense that no  $M(m)$  with  $m > 0$  is the invariant of a purely 1-dimensional scheme.

- (1) Any curve in  $\mathbb{P}^3$  with  $D(C) = M$  is a deformation of  $C_0$  through curves with invariant  $M$ .
- (2) Every curve in the even linkage class of  $C_0$  is the result of a series of ascending double links followed by a deformation.

In [Lazarsfeld and Rao 1983] it is shown that general curves in  $\mathbb{P}^3$  that have reasonably large degree compared to their genus are minimal in the sense of Theorem 7.4.

## 8. Comparing the canonical sheaf with that of the normalization

In Chapter 15 we boasted in that we could effectively compute linear series on a smooth curve  $C$  given any plane curve  $C_0$  with normalization  $C$ , and we showed how to do this when the plane curve has only nodes. To complete the discussion we need to compare the canonical sheaf of  $C$  with that of  $C_0$ ; that is, we need a formula for the adjoint ideal of any curve singularity.

**Theorem 8.1.** If  $\nu : C \rightarrow C_0$  is the normalization of a reduced connected projective curve then the adjoint ideal

$$\mathfrak{A}_{C/C_0} := \text{ann}_{\mathcal{O}_{C_0}} \frac{\omega_{C_0}}{\nu_* \omega_C}$$

is equal to the conductor ideal

$$\mathfrak{f}_{C/C_0} := \text{ann}_{\mathcal{O}_{C_0}} \frac{\nu_* \mathcal{O}_C}{\mathcal{O}_{C_0}}$$

Moreover, if  $C_0$  is a plane curve, then

$$\delta(C_0) = \text{length} \frac{\nu_* \mathcal{O}_C}{\mathcal{O}_{C_0}} = \text{length} \frac{\mathcal{O}_{C_0}}{\mathfrak{f}_{C/C_0}}.$$

Here  $\delta(C_0)$  is the “number of nodes equivalent to the singularities of  $C_0$ ” as explained in Chapter 2. The property in the last statement of the theorem was first noted (for plane curves) in Daniel Gorenstein’s thesis under Oscar

Zariski<sup>1</sup>. This is the reason why Grothendieck gave the name “Gorenstein” to Cohen-Macaulay rings that have cyclic canonical modules—see [Bass 1963]. This condition is always satisfied for singularities of curves in the plane, and we exploited this fact in Chapter 15. See Example 8.3 for a singularity that is not locally planar, and behaves differently.

**Proof of Theorem 8.1.** Let  $\rho : C_0 \rightarrow \mathbb{P}^1$  be a finite morphism. Both  $\rho_*\nu_*\mathcal{O}_C$  and  $\rho_*\mathcal{O}_{C_0}$  are torsion free coherent sheaves over  $\mathcal{O}_{\mathbb{P}^1}$ , and are thus locally free. Since  $\rho_*$  is left-exact, the inclusion  $\mathcal{O}_{C_0} \subset \nu_*\mathcal{O}_C$  pushes forward to an inclusion

$$\alpha : \rho_*\mathcal{O}_{C_0} \hookrightarrow \rho_*\nu_*\mathcal{O}_C$$

and since  $\mathcal{O}_C$  is equal to  $\mathcal{O}_{C_0}$  generically on  $\mathbb{P}^1$ , the cokernel  $\text{coker } \alpha$  has finite length; indeed, it is supported on the image in  $\mathbb{P}^1$  of the singular locus of  $C_0$ . Since the maps  $\nu$  and  $\rho$  are finite, we may harmlessly think of both  $\mathcal{O}_{C_0}$  and  $\mathcal{O}_C$  as coherent sheaves on  $\mathbb{P}^1$ , and we will simplify the notation by dropping  $\nu_*$  and  $\rho_*$ . Taking duals into  $\omega_{\mathbb{P}^1} = \mathcal{O}_{\mathbb{P}^1}(-2)$  and defining  $\alpha^\vee := \text{Hom}_{\mathbb{P}^1}(\alpha, \omega_{\mathbb{P}^1})$  we get a map that fits into the long exact sequence of  $\text{Ext}_{\mathbb{P}^1}(-, \omega_{\mathbb{P}^1})$ :

$$\begin{aligned} 0 \longrightarrow \text{Hom}_{\mathbb{P}^1}(\text{coker } \alpha, \omega_{\mathbb{P}^1}) &\longrightarrow \omega_C \xrightarrow{\alpha^\vee} \omega_{C_0} \\ &\longrightarrow \text{Ext}_{\mathbb{P}^1}^1(\text{coker } \alpha, \omega_{\mathbb{P}^1}) \longrightarrow \text{Ext}_{\mathbb{P}^1}^1(\mathcal{O}_C, \omega_{\mathbb{P}^1}) \longrightarrow \dots \end{aligned}$$

Since  $\text{coker } \alpha$  has finite support,  $\text{Hom}_{\mathbb{P}^1}(\text{coker } \alpha, \omega_{\mathbb{P}^1}) = 0$  and  $\text{Ext}_{\mathbb{P}^1}^1(\text{coker } \alpha, \omega_{\mathbb{P}^1})$  has the same length and the same annihilator as  $\text{coker } \alpha$ . Furthermore, because  $\mathcal{O}_C$  is locally free as an  $\mathcal{O}_{\mathbb{P}^1}$ -module, the term  $\text{Ext}_{\mathbb{P}^1}^1(\mathcal{O}_C, \omega_{\mathbb{P}^1})$  vanishes, and we get the more manageable exact sequence

$$0 \longrightarrow \omega_C \xrightarrow{\alpha^\vee} \omega_{C_0} \longrightarrow \text{Ext}_{\mathbb{P}^1}^1(\text{coker } \alpha, \mathcal{O}_{\mathbb{P}^1}) \longrightarrow 0$$

It follows that the sheaves  $\nu_*\mathcal{O}_C/\mathcal{O}_{C_0}$  and  $\omega_{C_0}/\nu_*\omega_C$  have the same length  $\delta(C_0)$ . Note that the *conductor*  $\mathfrak{f}_{C/C_0}$  of  $C/C_0$ , which is defined to be the annihilator of  $\mathcal{O}_C/\mathcal{O}_{C_0}$  in  $\mathcal{O}_{C_0}$ , is at the same time an ideal sheaf of  $\mathcal{O}_{C_0}$  and an ideal sheaf of  $\mathcal{O}_C$  via the inclusion  $\mathcal{O}_{C_0} \subset \mathcal{O}_C$ . The argument above shows that  $\mathfrak{f}_{C/C_0}$  is also the annihilator ideal of  $\omega_{C_0}/\omega_C$ . By definition, this is the adjoint ideal of  $C_0$ , proving the first statement of the theorem.

A further simplification occurs when  $C_0 \subset \mathbb{P}^2$  is a plane curve, or more generally any Gorenstein curve. If the defining equation of  $C_0$  is the form  $F$  of degree  $d$ , then there is a locally free resolution of  $\mathcal{O}_C$  of the form

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^2}(-d) \xrightarrow{F} \mathcal{O}_{\mathbb{P}^2} \longrightarrow \mathcal{O}_C \longrightarrow 0$$

and thus  $\omega_C \cong \text{Ext}_{\mathbb{P}^2}^1(\mathcal{O}_{C_0}, \omega_{\mathbb{P}^2})$  is the cokernel of the map  $\mathcal{O}_{\mathbb{P}^2}(-3) \rightarrow \mathcal{O}_{\mathbb{P}^2}(d-3)$  given by multiplication by  $F$ . Thus  $\omega_{C_0}$  is locally cyclic, and since the support of  $\omega_{C_0}/\omega_C$  is finite, this module is globally cyclic, so  $\omega_{C_0}/\omega_C \cong \mathcal{O}_{C_0}/\mathfrak{f}_{C/C_0}$ . Since

<sup>1</sup>Gorenstein is better remembered for his work on the classification of finite simple groups.

the length of  $\omega_{C_0}/\omega_C$  is equal to the length of  $\mathcal{O}_C/\mathcal{O}_{C_0}$ , the last statement of the theorem follows.  $\square$

**Example 8.2.** Working locally, consider the germ of a node, represented by the ring  $R_0 := k[[x, y]]/(xy)$ , and the projection to a line represented by the inclusion

$$P := k[[t]] \subset R_0 : t \mapsto x + y.$$

The normalization of  $R_0$  is the map  $R_0 \rightarrow R := k[[x]]e_1 \times k[[y]]e_2$ , where  $e_1 = x/t, e_2 = y/t$  are orthogonal idempotents. Writing  $Q_1 = k((t)) \subset Q := k((x)) \times k((y))$  for the map of total quotient rings, we know that, because the extension is separable, the trace map  $Tr := Tr_{Q/Q_1} : Q \rightarrow Q_1$  generates  $\text{Hom}_{Q_1}(Q, Q_1)$  as a  $Q$ -vector space. Thus we may write the elements of  $\omega_{R_0} = \text{Hom}_P(R_0, P)$  and  $\omega_R = \text{Hom}_P(R, P)$  as multiples of  $Tr$  by elements of  $Q$ .

Since  $R \cong Pe_1 \oplus Pe_2$  as a  $P$ -module, the module  $\text{Hom}_P(R, P)$  is generated by the two projections, and it is easy to check that these are the maps  $(x/t)Tr$  and  $(y/t)Tr$ . One can also check easily that

$$g := \frac{x-y}{t^2} Tr \in \text{Hom}_P(R_0, R).$$

Since  $xg = x/t$  and  $yg = y/t$  in  $Q$  we have  $xg = x/tTr$  and  $yg = y/tTr$ , the generators of  $\text{Hom}_P(R, P)$ . The ring  $R_0$ , regarded as a  $P$ -module, is freely generated by 1 and  $x$ . Immediate computation shows that  $gTr(1) = 0$  while  $gTr(x) = 1$ . Furthermore  $(x-y)g = 1$  in  $Q$ , and thus  $\frac{1}{2}(x-y)gTr$  takes 1 to 1 and  $x$  to  $t$ , proving that  $gTr$  generates  $\text{Hom}_P(R_0, P)$ . We also see directly from this that the adjoint ideal of  $\omega_{R_0}/\omega_R$  is the conductor ideal  $(x, y)R_0 = (x, y)R = \mathfrak{f}_{R/R_0}$ , as shown by Theorem 8.1.

**Example 8.3.** We showed earlier that the adjoint ideal of a planar triple point is the square of the ideal of the point. Consider now the germ of a spatial triple point singularity  $R_0 := k[[x, y, z]]/(xy, xz, yz)$ , the vertex of the cone over 3 non-colinear points in the plane, with normalization  $R = k[[x]] \times k[[y]] \times k[[z]]$ . If we set  $t = x + y + z \in R_0 \subset R$ , then the three idempotents that generate  $R$  as an  $S$ -module can be written as  $x/t, y/t, z/t$ . We have  $x^2/t = xt/t = x$  and similarly for  $y$  and  $z$ , so the conductor is given by  $\mathfrak{f}_{R/R_0} = (x, y, z)R = (x, y, z)R_0$ .

The ring  $R$  is also finite over  $S := k[[x, y, z]]$ , and is isomorphic as such to  $S/(x, y) \oplus S/(x, z) \oplus S/(y, z)$ , so we can compute its canonical module from the resolution that is the direct sum of the three Koszul complexes resolving these summands, and as such the canonical module  $\omega_R = \text{Ext}^2(R, S)$  is isomorphic to  $R$  as an  $S$ -module.

The  $S$ -module  $R_0$ , on the other hand, has free resolution

$$0 \longrightarrow S^2 \xrightarrow{\begin{pmatrix} 0 & z \\ y & -y \\ -x & 0 \end{pmatrix}} S^3 \xrightarrow{(xy \ xz \ yz)} S \longrightarrow R_0 \longrightarrow 0.$$

The canonical module of  $R_0$  (which would be the germ of the canonical module of a global curve with such a singularity) is thus

$$\omega_{R_0} = \text{Ext}^2(R_0, S) = \text{coker} \left( \begin{pmatrix} 0 & y & -x \\ z & -y & 0 \end{pmatrix} \right)$$

a module requiring 2 generators.

The inclusion  $\omega_R \subset \omega_{R_0}$  is induced by the dual of the last map in a comparison map between the two resolutions. Computation confirms the conclusion of Theorem 8.1 that  $\mathfrak{A}_{R/R_0} = \mathfrak{f}_{R/R_0} = (x, y, z)$ . However, the length of  $R/R_0$  is 2, and the length of  $R_0/\mathfrak{f}_{R/R_0} = R_0/(x, y, z)$  is 1, so the last conclusion of Theorem 8.1 fails, along with its hypothesis.

## 9. A general Riemann-Roch theorem

Using dualizing sheaves for singular curves, we can state a general version of the Riemann-Roch theorem. We will not make use of the generality further, so we only sketch the argument.

We first need a general definition of the degree of a sheaf:

**Definition 9.1.** If  $\mathcal{F}$  is a sheaf of generic rank  $r$  on a projective reduced and irreducible curve  $C$  over  $\mathbb{C}$  we define the degree of  $\mathcal{F}$  to be  $\chi(\mathcal{F}) - r\chi(\mathcal{O}_C)$ .

**Cheerful Fact 9.2.** *The degree of  $\mathcal{F}$  is actually the degree of a divisor class called the first Chern class of  $\mathcal{F}$ . See for example [Eisenbud and Harris 2016a, Chapter 5] for more information.*

Thus:

**Proposition 9.3.**

$$\chi(\mathcal{F}) = \deg \mathcal{F} + r\chi(\mathcal{O}_C) = \deg \mathcal{F} + r(1 - p_a(C)).$$

This statement would be a tautology if there were no other way to compute  $\deg(\mathcal{F})$ , but there is:

**Cheerful Fact 9.4.** *Let  $C$  be a reduced and irreducible projective curve, let  $\mathcal{F}$  be a coherent sheaf on  $C$  of generic rank  $r$ , and let  $\mathcal{L}$  be an invertible sheaf on  $C$ .*

- (1) If  $\mathcal{F}$  is generated by its global sections and  $\sigma_1, \dots, \sigma_r$  is a maximal generically independent collection of global sections of  $\mathcal{F}$ , then  $M = \text{coker}(\mathcal{O}_C^r \xrightarrow{(\sigma_1, \dots, \sigma_r)} \mathcal{F})$  has finite support, and

$$\deg(\mathcal{F}) = r\chi(\mathcal{O}_C) + \dim_{\mathbb{C}} H^0(M);$$

- (2)  $\deg(\mathcal{L} \otimes \mathcal{F}) = \deg(\mathcal{F}) + r \deg(\mathcal{L})$ .

Thus if  $\mathcal{O}_C(1)$  is a very ample invertible sheaf on  $C$  and  $m$  is a sufficiently large integer so that  $\mathcal{F}(m)$  is generated by global sections, then the degree of  $\mathcal{F}(m)$  and the degree of  $\mathcal{O}_C(1)$  are computed by the formula in item (1) and  $\deg \mathcal{F} = \deg(\mathcal{F}(m)) - m \text{rank}(\mathcal{F}) \deg(\mathcal{O}_C(1))$ . ■

The proof of these assertions uses nothing more than the additivity of the Euler characteristic.

Using the dualizing property of  $\omega_C$  we can reformulate Theorem 9.3 as

**Theorem 9.5.** If  $C$  is a reduced and irreducible curve and  $\mathcal{F}$  is a coherent sheaf on  $C$ , then

$$h^0(\mathcal{F}) = \deg(\mathcal{F}) + \text{rank}(\mathcal{F})(1 - p_a(C)) + h^0(\text{Hom}_C(\mathcal{F}, \omega_C)).$$

## 10. Exercises

**Exercise 10.1.** Verify that the genus formula in Theorem 5.1 agrees with the usual calculation of degrees and genera for divisors on a quadric of classes  $(a, b)$  and  $(d - a, d - b)$ .

**Exercise 10.2.** Let  $C$  be a reduced and irreducible projective curve, and let  $\mathcal{E}$  be a locally free sheaf of rank  $r$  on  $C$ . Show that  $\deg(\mathcal{E}) = \deg(\wedge^r(\mathcal{E}))$ .

Hint: First show that any locally free sheaf on  $C$  is an iterated extension of invertible sheaves.

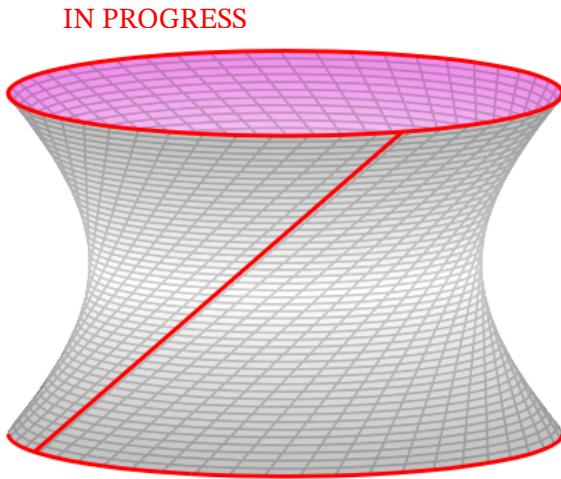
**Exercise 10.3.** Let  $C$  be the disjoint union of 3 skew lines.

- (1) prove that  $C$  lies on a unique quadric, and that  $H^2(\mathcal{I}_C) = 0$
- (2) compute the Hartshorne-Rao module  $D(C)$ .
- (3) show that if  $\Gamma$  is the union of 3 points in  $\mathbb{P}^3$  then  $H^1(\mathcal{I}(\Gamma)) = 0$  if and only if the three points are collinear.
- (4) Using the exact sequence in cohomology coming from the short exact sequence

$$0 \rightarrow \mathcal{I}_C \xrightarrow{\ell} \mathcal{I}_C(1) \rightarrow \mathcal{I}_{\Gamma}(1) \rightarrow 0$$

where  $\ell$  is a linear form, show that the map of vector spaces

$$H^1(\mathcal{I}_C) \xrightarrow{\ell} H^1(\mathcal{I}_C(1))$$



**Figure 2.** A family of disjoint lines on a smooth quadric. [Being redrawn]

has rank  $< 2$  if and only if  $\ell$  vanishes on 3 collinear points on the three lines (including the case when  $\ell$  vanishes identically on one of the lines). Conclude that if a different union  $C'$  of 3 skew lines is linked to  $C$ , then  $C'$  lies on the same quadric as  $C$ .

See [[Migliore 1986](#)] for more examples of this type.

**Exercise 10.4.** Compute the Hilbert function of the Hartshorne-Rao module of a curve of type  $(a, b)$  on a smooth quadric surface.

Hint: The ideal sheaf of the curve on the quadric  $Q$  is an extension of the ideal sheaf of the quadric in  $\mathbb{P}^3$  with the ideal sheaf of the curve on the quadric, which is

$$\mathcal{O}_Q(-a, -b) = \pi_1^*(\mathcal{O}_{\mathbb{P}^1}(-a)) \otimes \pi_2^*(\mathcal{O}_{\mathbb{P}^1}(-b)),$$

where  $\pi_1, \pi_2$  are the projections to  $\mathbb{P}^1$ . Use the Künneth formula

$$H^1(\mathcal{O}_Q(p, q)) = H^1(\mathcal{O}_{\mathbb{P}^1}(p)) \otimes H^0(\mathcal{O}_{\mathbb{P}^1}(q)) \oplus H^0(\mathcal{O}_{\mathbb{P}^1}(p)) \otimes H^1(\mathcal{O}_{\mathbb{P}^1}(q))$$

to compute the necessary cohomology.

**Exercise 10.5.** (Liaison addition) (From the unpublished Brandeis thesis of Phillip Schwartau): Suppose that  $I, J$  are saturated ideals defining purely 1-dimensional subschemes of  $\mathbb{P}^3$  and that  $f, g$  is a regular sequence with  $f \in I$  and  $g \in J$ . Prove that  $gI \cap fJ = (fg)$ , and conclude that if  $I, J$  are saturated codimension 2 ideals defining purely 1-dimensional schemes  $C, C'$  in  $\mathbb{P}^3$  then  $(gI + fJ)$  is a saturated ideal defining a purely 1-dimensional scheme  $C''$  with  $D(C'') = D(C)(-\deg g) \oplus D(C')(-\deg f)$ .

Hint: Use the exact sequence

$$0 \rightarrow (fg) \rightarrow gI \oplus fJ \rightarrow gI + fJ \rightarrow 0$$

and the corresponding exact sequence of quotients by these ideals.

**Exercise 10.6.** (Basic double links) The special case of the construction in Exercise 10.5 in which  $C'$  is trivial is already interesting.

- (1) Show that if  $I$  is a saturated ideal of codimension 2 defining a purely 1-dimensional scheme  $C$  in  $\mathbb{P}^3$  and  $(f, g)$  is a regular sequence with  $g \in I$ , then  $fI + (g)$  defines a scheme  $C'$  with  $D(C') = D(C)(-\deg f)$ .
- (2) Show directly that, with notation as above,  $C'$  is directly linked to  $C$  in two steps. Since the degrees of the generators of  $D(C')$  are more positive, this is sometimes called an *ascending double link*. Geometrically it amounts to taking the union of  $C$  with some components that are complete intersections.

**Exercise 10.7.** Here is a more general form of the last step in the proof of part 1 of Propositions 5.3. Suppose that  $B \rightarrow A$  is a homomorphism of rings,  $X$  is an  $A$ -module and  $Y$  is a  $B$ -module. Show that there is a natural transformation

$$\phi : \text{Hom}_A(X, \text{Hom}_B(A, Y)) \cong \text{Hom}_B(X, Y)$$

and that if  $X = \text{Hom}_B(A, Y)$ , then the map

$$A \rightarrow \text{Hom}_A(\text{Hom}_B(A, Y), \text{Hom}_B(A, Y))$$

taking an element  $a \in A$  to multiplication by  $a$  on the  $A$ -module  $\text{Hom}_B(A, Y)$  is sent by  $\phi$  to the evaluation map  $\alpha \mapsto \alpha(a)$  for  $\alpha \in \text{Hom}_B(A, Y)$ .

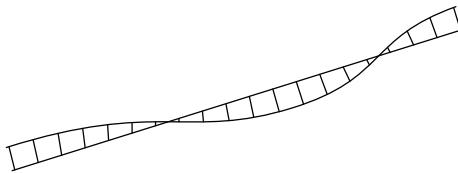
**10.1. Ropes and Ribbons.** The simplest way to construct well-behaved nonreduced curves is to take neighborhoods of smooth ones. Ropes and ribbons are examples of this sort:

**Definition 10.8.** The *rope defined from a curve*  $C \subset \mathbb{P}^n$  is the scheme  $V(I_C^2)$  defined by the square of the ideal  $C$ .

**Exercise 10.9.** If  $C$  is the rope defined from a line  $L \subset \mathbb{P}^3$  then the Hilbert function  $h_C(m)$  and Hilbert polynomial  $p_C(m)$  are both equal to  $3m+1$ . Thus  $C$  has degree 3 and arithmetic genus 0. Note that the degree can also be computed as the degree of a general hyperplane section, since this is defined by the square of the ideal of a point in  $\mathbb{P}^2$ .

Hint: Count the monomials of each degree in square of the ideal of a line.

**Exercise 10.10.** To see why the rope in Exercise 10.9 should look like a twisted cubic, show that it is the flat limit of a twisted cubic as follows: Let  $X \subset \mathbb{P}^3$  be



**Figure 3.** A ribbon can be thought of as the first infinitesimal neighborhood of a curve on a smooth surface. Silvio: the two curves don't actually intersect. possible to indicate?

the twisted cubic with parametrization  $x_i = s^i t^{3-i}$ . Consider the one-parameter subgroup of  $PGL_4$  given in homogeneous coordinates  $x_0, \dots, x_3$  on  $\mathbb{P}^3$  by

$$A_t : (x_0, \dots, x_3) \mapsto (tX_0, X_1, X_2, tX_3).$$

Show that the flat limit, as  $t \rightarrow 0$ , of the twisted cubics  $A_t(C)$  is the rope  $V(X_0^2, X_0X_1, X_1^2)$ .

Hint: Use the description of  $I(X)$  as the ideal of  $2 \times 2$  minors of

$$\begin{pmatrix} x_0 & x_1 & x_2 \\ x_1 & x_2 & x_3 \end{pmatrix}$$

**Exercise 10.11.** We saw in Section 1 that a twisted cubic curve is linked to a line by the complete intersection of two quadrics. Show that the same is true for the rope of Exercise 10.9

If  $C$  is the rope defined from a line in  $\mathbb{P}^2$ , then the Zariski tangent space to  $C$  at any point is 2-dimensional; that is, it looks like a ribbon. More generally:

**Definition 10.12.** By a *ribbon*  $X \subset \mathbb{P}^n$  we will mean a scheme of pure dimension 1 and multiplicity 2 whose support is a smooth, irreducible curve  $C \subset \mathbb{P}^n$  and whose Zariski tangent space at every point is 2-dimensional.

**Exercise 10.13.** Suppose that  $C \subset \mathbb{P}^n$  is a ribbon. Show that  $C$  is contained in the rope defined from  $C_{red}$ , and show that the degree of  $C$  is twice the degree of  $C_{red}$ .

Hint: look at hyperplane sections of  $C$ .

Unlike ropes, there are many different ribbons  $C$  with the same smooth curve  $C_{red}$ , and they can have different arithmetic genera. Suppose that  $C \subset \mathbb{P}^3$  is a ribbon such that  $X = C_{red}$  is the line  $V(x_0, x_1)$ . Since  $C$  is contained in the rope defined from  $X$  we must have  $(x_0^2, x_0x_1, x_1^2) \subset I(C)$ . The tangent space to  $C$  at a point  $(0, 0, s, t)$  meets the line  $X' = V(x_2, x_3)$  at some point  $(F(s, t), G(s, t), 0, 0)$ , so  $F$  and  $G$  define a morphism  $\mathbb{P}^1 \rightarrow \mathbb{P}^1$ ; thus they are homogeneous polynomials of the same degree  $d$ . It follows that  $I(C)$  also contains

the element  $x_0G(x_2, x_3) - x_1F(x_2, x_3)$ . Show that the ideal of  $C$  is obtained by adding this form to the ideal of the rope, that is,

$$I_C = (X_0^2, X_0X_1, X_1^2, F(X_3, X_4)X_0 + G(X_3, X_4)X_1)$$

In case  $d = 1$ , show that  $C$  lies on a smooth quadric

**10.2. General adjunction.** The next two exercises illustrate Theorem 6.3:

**Exercise 10.14.** Show that if  $C \rightarrow D$  is a map of smooth curves with ramification index  $e$  at  $p \in C$ , and  $t$  is a local analytic parameter at  $p$ , then locally analytically at  $p$  the sheaf  $\mathcal{H}\text{om}_C(\mathcal{O}_C, \omega_D)$  is  $\mathcal{O}_C(e)$ .

**Exercise 10.15.** Show that if  $C \subset S$  is a Cartier divisor on a surface  $S$  with canonical sheaf  $\omega_S$ , then  $\mathcal{E}\text{xt}^1(\mathcal{O}_C, \omega_S) \cong \mathcal{O}_C \otimes \mathcal{O}_S(C)$ , and thus  $K_C = (K_S + C) \cap C$ .

# Scrolls and the Curves They Contain

The naming of cats is a difficult matter,  
It isn't just one of your everyday games.  
You may think that I am as mad as a hatter,  
When I tell you each cat must have three different names.  
The first is the name that the family use daily ...  
But I tell you, a cat needs a name that's particular ...  
But above and beyond there's still one name left over, ...  
[his] deep and inscrutable, singular name.

—T. S. Eliot, Old Possum's Book of Practical Cats<sup>1</sup>

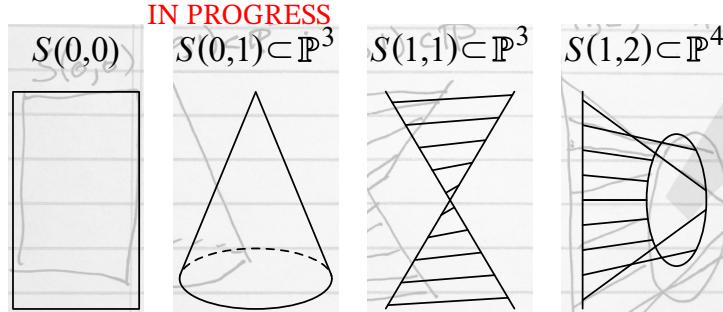
## Introduction

Some of the simplest subvarieties in projective space are the *rational normal scrolls*. They appear in many contexts in algebraic geometry, and are useful for describing the embeddings of curves of low degree and genus.

Like the cats in T. S. Eliot's poem, scrolls can be seen from three rather different points of view, each useful in a different context, and we will explore them in turn: First a classical geometric construction, then an algebraic description that allows one to “find” the scrolls containing a given variety, and then a more modern geometric definition that makes it easy to understand the

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<sup>1</sup>[Eliot 2009]



**Figure 1.** The smallest scrolls: from the left,  $S(0, 1) \cong \mathbb{P}^2$   $S(0, 2)$ , the cone over a conic,  $S(1, 1) \subset \mathbb{P}^3$  the union of lines joining corresponding points of two skew lines, isomorphic to a smooth quadric, and  $S(1, 2) \subset \mathbb{P}^4$ .

divisors on a scroll. Finally, we turn to some of the applications to the embeddings of curves. We will focus on the case of 2-dimensional scrolls because this is the case that occurs in our applications.

In this chapter we will refer to rational normal scrolls simply as scrolls. The third characterization we will give lends itself to a natural generalization to varieties swept out by a 1-parameter family of linear spaces; in the literature the word “scroll” is often used for this wider class.

## 1. Some classical geometry

To construct a scroll of dimension 2 in  $\mathbb{P}^r$ , we start by choosing integers  $0 \leq a_1$  and  $1 \leq a_2$  with  $a_1 + a_2 = r - 1$ , and consider a pair of complementary linear subspaces  $\mathbb{P}^{a_1}$  and  $\mathbb{P}^{a_2} \subset \mathbb{P}^r = \mathbb{P}(V)$ —that is, we express an  $(r + 1)$ -dimensional vector space  $V$  as a direct sum  $V = V_1 \oplus V_2$  of subspaces  $V_1, V_2 \subset V$  of dimensions  $a_1 + 1$  and  $a_2 + 1$ . For simplicity we will always assume, without loss of generality, that  $a_1 \leq a_2$ .

Next, for  $i = 1, 2$ , we take  $\phi_i : \mathbb{P}^1 \rightarrow \mathbb{P}^{a_i}$  to be the parametrization of the rational normal curve of degree  $a_i$  given by a basis of homogeneous polynomials of degree  $a_i$  (if  $a_i = 0$  this is just the constant map from  $\mathbb{P}^1$  to a point.) Finally, we define the scroll  $S(a_1, a_2)$  to be the union of the lines

$$S(a_1, a_2) := \bigcup_{p \in \mathbb{P}^1} \overline{\phi_1(p), \phi_2(p)}.$$

We call the curve  $C_{a_1}$  the *directrix* of the scroll, and we call the lines  $\overline{\phi_1(p), \phi_2(p)}$  the *rulings* of the scroll. For example a smooth quadric surface in  $\mathbb{P}^3$  is the union  $S(1, 1)$  of the lines joining corresponding points on two skew lines.

In the degenerate case  $a_1 = 0$ , the surface  $S(0, a_2)$  is the cone in  $\mathbb{P}^{a_2+1}$  over a rational normal curve of degree  $a_2$ . Note that  $S(0, 1) = \mathbb{P}^2$ . Since  $S(0, a_2)$  is

singular when  $a_2 \geq 2$ , it is useful to consider the surface

$$\tilde{S}(0, a_2) := \left\{ (t, q) \in \mathbb{P}^1 \times \mathbb{P}^r \mid q \in \overline{\phi_1(t), \phi_2(t)} \right\}.$$

This is the blow-up of the cone  $S(0, a_2)$  at its vertex; like the surfaces  $S(a_1, a_2)$  with  $a_1 > 0$  it is a  $\mathbb{P}^1$  bundle over  $\mathbb{P}^1$  and thus is smooth. As we shall see,  $\tilde{S}(0, a_2)$  is isomorphic to the scroll  $S(1, a_2 + 1)$ . It is not hard to prove directly that  $S(a_1, a_2)$  is an algebraic variety, and we shall soon write down its defining equations.

From the description above we can immediately deduce the dimension and degree of a scroll:

**Proposition 1.1.** (1)  $S(a_1, a_2)$  is a nondegenerate surface.

- (2)  $S(a_1, a_2)$  has degree  $a_1 + a_2$ , and codimension  $a_1 + a_2 - 1$ .
- (3)  $S(a_1, a_2)$  is non-singular if  $a_1 > 0$ .

**Proof.** The rational normal curves separately span the spaces  $\mathbb{P}^{a_i}$ , so a hyperplane containing both of them would contain  $\overline{\mathbb{P}^{a_1}, \mathbb{P}^{a_2}} = \mathbb{P}^r$ , proving nondegeneracy.

It is clear from our description that  $S$  is 2-dimensional, and thus of codimension  $a_1 + a_2 + 1 - 2 = a_1 + a_2 - 1$ .

To compute the degree, we choose a general hyperplane  $H$  containing  $\mathbb{P}^{a_1}$ . The intersection  $H \cap C_2$  consists of  $a_2$  reduced points. Thus the intersection  $H \cap S$  consists of  $C_1$  and the  $a_2$  reduced lines connecting the points of  $H \cap C_2$  with their corresponding points on  $C_1$ ; this union has degree  $a_1 + a_2$ .

If  $0 < a_1$  we also see from this argument that, given any point  $p \in S(a_1, a_2)$ , there is a hyperplane section that is non-singular at  $p$ , and thus  $S(a_1, a_2)$  is nonsingular at  $p$ .  $\square$

A completely parallel construction creates rational normal scrolls of any dimension  $s$ . Start with a series of integers  $0 \leq a_1 \leq \dots \leq a_s$ ; set  $r + 1 = \sum_{i=1}^s (a_i + 1)$ , and decompose  $\mathbb{C}^{r+1}$  as

$$\mathbb{C}^{r+1} = \bigoplus_{i=1}^s \mathbb{C}^{a_i+1}.$$

Let  $\mathbb{P}^{a_i} \subset \mathbb{P}^r$  be the subspaces corresponding to the summands, choose for each  $i$  a map  $\phi_i : \mathbb{P}^1 \rightarrow \mathbb{P}^{a_i}$  given by a basis of homogeneous polynomials of degree  $a_i$ , and define the scroll  $S \subset \mathbb{P}^r$  by

$$S = S(a_1, \dots, a_s) := \bigcup_{p \in \mathbb{P}^1} \overline{\phi_1(p), \phi_2(p), \dots, \phi_s(p)}.$$

For example,  $S(a_1)$  is the rational normal curve of degree  $a_1$ . In general, the variety  $S$  is nondegenerate of codimension  $r - s$  and degree  $\sum a_i = r - s + 1$ . The proof is similar to the one we gave for  $s = 2$ .

To put this construction in context, we recall that by Corollary 1.6 Any irreducible, nondegenerate variety  $X$  of codimension  $c$  in  $\mathbb{P}^r$  has degree  $\geq c + 1$ .

Thus scrolls are *varieties of minimal degree*. The reader already knows that the rational normal curves of degree  $a$  in  $\mathbb{P}^a$  are the only irreducible, nondegenerate curves of degree  $a$  and codimension  $a - 1$ . A celebrated theorem of del Pezzo (for surfaces) and Bertini (in general) generalizes this statement:

**Cheerful Fact 1.2.** *Any irreducible, nondegenerate variety  $X \subset \mathbb{P}^r$  with  $\deg X = \text{codim } X + 1$  is either a rational normal scroll, a quadric hypersurface, the Veronese surface in  $\mathbb{P}^5$  or a cone over the Veronese surface.*

A proof may be found in [Eisenbud and Harris 1987c].

One interesting way to view the construction of a scroll is that we chose subvarieties  $C_i \subset \mathbb{P}^{a_i}$  and a one-to-one correspondence between them, that is, a subscheme  $\Gamma \subset \prod C_i$  that projects isomorphically onto each  $C_i$ ; the scroll is then the union of the planes spanned by sets of points  $p_i \in C_i$  that are “in correspondence”. There are other interesting varieties constructed starting with other choices of subvarieties  $C_i$  and subschemes—not necessarily reduced—of  $\prod C_i$ . See [Eisenbud and Sammartano 2019] for an exploration of this idea.

We tend to speak of “the” rational normal scroll rather than “a” rational normal scroll”, despite the choices made in the definition, for the following reason:

**Proposition 1.3.** *The scroll  $S(a_1, a_2)$  is, up to a linear automorphism of  $\mathbb{P}^{a_1+a_2+1}$ , independent of the choices made in its definition.*

**Proof.** To simplify the notation, set  $S := S(a_1, a_2)$  and  $\mathbb{P} := \mathbb{P}^{a_1+a_2+1}$ . To construct  $S$  we chose

- (1) disjoint subspaces  $\mathbb{P}^{a_i} \subset \mathbb{P}$ ;
- (2) a rational normal curve in each subspace; and
- (3) an isomorphism between these curves.

Elementary linear algebra shows that there are automorphisms of  $\mathbb{P}$  carrying any choice of disjoint subspaces to any other choice. Further, since the rational normal curve of degree  $a$  is unique up to an automorphism of  $\mathbb{P}^a$ , the two choices in (2) differ by a linear automorphism. Finally, any automorphism of  $C_{a_2} \cong \mathbb{P}^1$  extends to an automorphism of  $\mathbb{P}^{a_2} = |\mathcal{O}_{\mathbb{P}^1}(a_2)|$ , and this extends

to an automorphism of  $\mathbb{P}$  fixing  $\mathbb{P}^{a_1}$  pointwise, showing that  $S(a_1, a_2)$  is independent, up to an automorphism of the ambient space, of the choice in (3) as well.  $\square$

## 2. 1-generic matrices and the equations of scrolls

Suppose that a scheme  $X$  is embedded in  $\mathbb{P}^r$  by a complete linear series, and that  $\mathcal{O}_X(1)$  can be “factored” as a tensor product  $\mathcal{L} \otimes \mathcal{M}$  of invertible sheaves on  $X$ . If we pick bases of  $p$  elements  $\{\ell_i\} \subset H^0(\mathcal{L})$  and  $q$  elements  $\{m_i\} \subset H^0(\mathcal{M})$  then the multiplication map

$$\mu : H^0(\mathcal{L}) \otimes H^0(\mathcal{M}) \rightarrow H^0(\mathcal{O}_X(1)) = H^0(\mathcal{O}_{\mathbb{P}^r}(1))$$

gives rise to a  $p \times q$  matrix  $M_\mu$  of linear forms on  $\mathbb{P}^r$  whose  $i, j$  entry is  $\mu(\ell_i m_j)$ . More abstractly, this is a linear space of matrices obtained from the “adjunction” isomorphism  $\text{Hom}(A \otimes B, C) \cong \text{Hom}(A, \text{Hom}(B, C))$ .

Regarding the sections of invertible sheaves as rational functions on  $X$ , we see from the commutativity of multiplication that the  $2 \times 2$  minors of

$$\det \begin{pmatrix} \ell_{i_1} m_{j_1} & \ell_{i_1} m_{j_2} \\ \ell_{i_2} m_{j_1} & \ell_{i_2} m_{j_2} \end{pmatrix},$$

vanish on  $X$ —that is, the ideal of  $2 \times 2$  minors  $I_2(M_\mu)$  is contained in the homogeneous ideal of  $X$ . We will show that the ideal of a rational normal scroll is generated by such minors.

**Example 2.1.** We have already seen this phenomenon in the case of the rational normal curve  $C_a \subset \mathbb{P}^a$  if  $X = \mathbb{P}^1$ . Here  $C_a$  is embedded by the complete linear series  $|\mathcal{O}_{\mathbb{P}^1}(a)|$ , and we can write  $\mathcal{O}_{\mathbb{P}^1}(a) = \mathcal{O}_{\mathbb{P}^1}(1) \otimes \mathcal{O}_{\mathbb{P}^1}(a-1)$ . If we take bases  $s^i t^j$  in each of  $H^0(\mathcal{O}_{\mathbb{P}^1}(1))$ ,  $H^0(\mathcal{O}_{\mathbb{P}^1}(a-1))$  and  $H^0(\mathcal{O}_{\mathbb{P}^1}(a))$  we get the  $2 \times a$  matrix

$$M_\mu := \begin{pmatrix} x_0 & x_1 & \dots & x_{a-1} \\ x_1 & \dots & x_{a-1} & x_a \end{pmatrix}.$$

When restricted to  $\mathbb{P}^1$ , this becomes

$$M_a = \frac{s}{t} \begin{pmatrix} s^{a-1} & s^{a-2}t & \dots & t^{a-1} \\ s^a & s^{a-1}t & \dots & st^{a-1} \\ s^{a-1}t & s^{a-2}t^2 & \dots & t^a \end{pmatrix}$$

where we have written  $s, t$  for the basis of  $H^0(\mathcal{O}_{\mathbb{P}^1}(1))$ , and bordered the matrix with the corresponding bases of  $H^0(\mathcal{O}_{\mathbb{P}^1}(1))$  and  $H^0(\mathcal{O}_{\mathbb{P}^1}(a-1))$ , and it is obvious that the minors of  $M_a$  are 0.

By a *generalized row* of  $M_a$ , we mean a  $\mathbb{C}$ -linear combination of the given rows of  $M_a$ . Note that the points at which the  $2 \times 2$  minors of  $M_a$  vanish are the points at which the evaluations of the two rows are linearly dependent; that is, the points at which some generalized row of  $M_a$  vanishes identically.

**Definition 2.2.** A matrix of linear forms  $M$  is *1-generic* if every generalized row of  $M$  consists of  $\mathbb{C}$ -linearly independent forms.

**Proposition 2.3.** Let  $X$  be an irreducible, reduced variety, and suppose that  $\mathcal{L}, \mathcal{M}$  are invertible sheaves on  $X$ . The matrix  $M_\mu$  coming from the map  $\mu : H^0(\mathcal{L}) \otimes H^0(\mathcal{M}) \rightarrow H^0(\mathcal{L} \otimes \mathcal{M})$  is 1-generic.

**Proof.** The entries of the generalized row of  $M_\mu$  corresponding to  $s \in H^0(\mathcal{L})$  are a basis of  $s \cdot H^0(\mathcal{M}) \cong H^0(\mathcal{M})$ , and are thus linearly independent.  $\square$

**Example 2.4.** For example, the matrix

$$M = \begin{pmatrix} x & y \\ z & x \end{pmatrix}$$

over  $\mathbb{C}[x, y, z]$  is 1-generic, since if a row and column transformation produced a 0 the determinant would be a product of linear forms, whereas  $\det M = x^2 - yz$  is irreducible. (This matrix corresponds to the multiplication  $H^0(\mathcal{O}_{\mathbb{P}^1}(1)) \otimes H^0(\mathcal{O}_{\mathbb{P}^1}(1)) \rightarrow H^0(\mathcal{O}_{\mathbb{P}^1}(2))$ ).

On the other hand, the matrix

$$M' = \begin{pmatrix} x & y \\ -y & x \end{pmatrix}$$

over  $\mathbb{C}[x, y]$  is not 1-generic, since

$$\begin{pmatrix} 1 & 0 \\ -i & 1 \end{pmatrix} M' \begin{pmatrix} 1 & 0 \\ i & 1 \end{pmatrix} = \begin{pmatrix} x+iy & 0 \\ 0 & x-iy \end{pmatrix}$$

(but note that it would be 1-generic if we restricted scalars to  $\mathbb{R}$ —thus the definition depends on the field).

**Lemma 2.5.** There exist 1-generic  $p \times q$  matrices of linear forms in  $r+1$  variables over  $\mathbb{C}$  if and only if  $r \geq p+q$ .

If  $M$  is 1-generic and involves more than  $p+q-1$  variables, then the restriction of  $M$  to a general hyperplane is still 1-generic.

**Proof.** Consider a map of vector spaces  $\mu : A \otimes B \rightarrow C$ , where we regard  $C$  as a space of linear forms. With notation as above, if  $\ell_i \otimes m_j \in \ker \mu$ , then the  $i, j$  entry of  $M_\mu$  is 0 and similarly for any *pure* tensor  $\ell \otimes m \in A \otimes B$ . Thus  $M_\mu$  is 1-generic if and only if the linear subspace  $\ker \mu \subset A \otimes B$  is disjoint from the set of pure tensors. Under the isomorphism  $A \otimes B \cong \text{Hom}(A^*, B)$  the set of pure tensors corresponds to matrices of rank 1, and this set has codimension  $(p-1)(q-1) = pq - p - q + 1$  in  $\text{Hom}(A^*, B)$  (proof: a rank 1 matrix corresponds to the choice of a 1-quotient of  $A^*$  and a 1-dimensional subspace of  $B$ , thus a point in  $\mathbb{P}^q \times \mathbb{P}^p$ ). Thus for  $\mu$  to correspond to a 1-generic matrix, we must have

$\text{codim } \ker \mu \geq q + p - 1$ . Since  $\text{codim } \ker \mu = \dim \text{im } \mu \subset C$  we see that any 1-generic matrix must involve at least  $q + p + 1$  variables.

Furthermore, the restriction of  $M_\mu$  to a hyperplane corresponds to the composite homomorphism  $A \otimes B \rightarrow C \rightarrow C/\langle x \rangle$ , or equivalently to the addition of one element to  $\ker \mu$ , and thus if  $M_\mu$  is 1-generic and involves  $> q + p - 1$  variables, then the restriction to a general hyperplane is again 1-generic.  $\square$

We will see that 1-generic  $2 \times r$  matrices correspond to scrolls. The beginning of the story is the calculation of the codimension of the ideal  $I_2(M)$  generated by the  $2 \times 2$  minors of  $M$ :

**Lemma 2.6.** *If  $M$  is a 1-generic  $2 \times b$  matrix of linear forms in  $\mathbb{C}[x_0, \dots, x_r]$  then  $V(I_2(M))$  is irreducible of codimension  $b - 1$ .*

**Proof.** The algebraic set  $V := V(I_2(M))$  is the set of points on which a generalized row  $\rho_\lambda$ ,  $\lambda \in \mathbb{P}^1$  of  $M$  vanishes. Thus the map

$$W := \{(p, \lambda) \in \mathbb{P}^r \times \mathbb{P}^1 \mid p \in V(\rho_\lambda)\} \rightarrow V$$

is surjective. All the fibers of the second projection  $\pi_2 : W \rightarrow \mathbb{P}^1$  are isomorphic to  $\mathbb{P}^{r-b}$ , so  $W$  is irreducible of dimension  $r-b+1$ , and thus  $V$  is irreducible. Each fiber  $\pi_2^{-1}(\lambda)$  injects into  $V$ , and the images of these fibers are distinct since otherwise the entries of  $M$  would all be contained in a single generalized row, contradicting Lemma 2.5. It follows that  $V$  also has dimension  $r-b+1$ , as required.  $\square$

We have seen in Proposition 1.1 that the ideal of minors of

$$M_a := \begin{pmatrix} x_0 & x_1 & \dots & x_{a-1} \\ x_1 & x_2 & \dots & x_a \end{pmatrix}$$

generates the ideal of the rational normal curve of degree  $a$  in  $\mathbb{P}^a$ , and is thus prime. More generally:

**Theorem 2.7.** *Let  $I = I_2(M)$  be the ideal generated by the  $2 \times 2$  minors of a 1-generic,  $2 \times a$  matrix  $M$  of linear forms in  $S = \mathbb{C}[x_0, \dots, x_n]$ .*

- (1) *The ideal  $I$  is prime, and  $V(I)$  either is smooth, or is a cone over a smooth variety.*
- (2) *If  $a = r$  then  $V(I)$  is a rational normal curve. More generally, the variety  $V = V(I) \subset \mathbb{P}^r$  has degree  $a$  and codimension  $a-1$  and is thus a variety of minimal degree.*

**Proof.** By Lemma 2.6,  $V(I)$  has codimension  $a-1$ . If the span of the linear forms in  $M$  is not the whole space of linear forms on  $\mathbb{P}^r$ , then  $V(I)$  is a cone, so we may assume that the entries of  $M$  generate the maximal ideal.

Let  $p \in V(I)$  be a point, so that there is a generalized row of  $M$ —without loss of generality the second row—whose entries all vanish at  $p$ . Since not all the linear forms of  $S$  can vanish at a point of  $\mathbb{P}^r$ , we may make column transformations to reduce to the case where  $\ell_{1,j}$  also vanishes at  $p$  for all  $j \neq 1$ . Since the entries of each generalized row are linearly independent, the ideal generated by the entries of the second row define a plane of  $\Lambda$  codimension  $a$  containing  $p$ .

Over the local ring  $\mathcal{O}_{\mathbb{P}^r, p}$  the element  $\ell_{1,1}$  becomes a unit. Thus, locally at  $p$ , the elements  $m_j := \ell_{2,j} + \ell_{1,1}^{-1}\ell_{2,1}\ell_{1,j}$  for  $j = 2, \dots, a$  are in the ideal generated by the minors. Since the  $\ell_{2,j}$  are independent regular parameters in  $\mathcal{O}_{\mathbb{P}^r, p}$  and  $\ell_{1,1}^{-1}\ell_{2,1}\ell_{1,j}$  is in the square of the maximal ideal of  $\mathcal{O}_{\mathbb{P}^r, p}$  the  $a - 1$  elements  $m_j$  are also regular parameters. Since  $\text{codim } I = a - 1$ , it follows that  $I$  is prime and defines a smooth variety, as claimed.

If  $a = r$  then  $V(I)$  is a smooth, nondegenerate curve. In Chapter 18 we will construct the Eagon-Northcott complex  $EN(M)$ . By Theorem 4.6, the complex  $EN(M)$  is a free resolution, and if  $M'$  is the ideal of the rational normal curve, then  $EN(M')$  is again a resolution. Since the Hilbert function of  $V(I)$  can be computed from the free resolution, it follows that  $V(I)$  has the same Hilbert function as  $V(I_2(M))$ , and thus  $V(I)$  is a rational normal curve.

If  $a < r$  then by Lemma 2.3 there is linear form  $\ell$  such that  $M$  remains 1-generic modulo  $\ell$ . Since  $I_2(M)$  is prime it has the same degree and codimension as  $I_2(M) + (x)/(x) \subset S/(x)$ , so by induction its degree is  $a$ .  $\square$

We can use Theorem 2.7 to prove a classic result of Castelnuovo, characterizing large sets of points on a rational normal curve. It is a key element in Castelnuovo's classification of Castelnuovo curves of high degree, Theorem 4.5 below.

**Corollary 2.8.** (*Castelnuovo's 2r+3 lemma*) *If  $\Gamma \subset \mathbb{P}^r$  is a set of  $d \geq 2r+3$  distinct points in linearly general position, then  $\Gamma$  is contained in a rational normal curve if and only if  $\Gamma$  imposes only  $2r + 1$  conditions on quadrics.*

For a classical proof of Corollary 2.8 see for example [Griffiths and Harris 1978, p. 531].

To understand the approach taken in the proof below, recall that in the case of a twisted cubic, whose equations are the minors of the matrix

$$\begin{pmatrix} x_0 & x_1 & x_2 \\ x_1 & x_2 & x_3 \end{pmatrix}$$

the first column defines a secant line to the twisted cubic, and the two minors involving the first column are a regular sequence defining the union of the twisted cubic and that secant line. In general a rational normal curve may

be defined by the 1-generic matrix associated to the product  $H^0(\mathcal{O}_{\mathbb{P}^1}(1)) \otimes H^0(\mathcal{O}_{\mathbb{P}^1}(r-1)) \rightarrow H^0(\mathcal{O}_{\mathbb{P}^1}(r))$ , and it follows from that description that the columns of the matrix define the  $(r-1)$ -secant  $(r-2)$ -planes to the curve. In the proof below we reconstruct the matrix starting from such a secant plane.

**Proof.** Suppose first that  $\Gamma$  is contained in a rational normal curve  $C$  of degree  $r$  in  $\mathbb{P}^r$ . If  $Q$  is a quadric vanishing on  $\Gamma$  then by Bézout's theorem  $C \subset Q$ . Since  $C$  is arithmetically Cohen-Macaulay, the dimension of the space of quadrics vanishing on  $C$  is

$$h^0(\mathcal{O}_{\mathbb{P}^r}(2)) - h^0(\mathcal{O}_C(2r)) = h^0(\mathcal{O}_{\mathbb{P}^r}(2)) - (2r+1)$$

so  $\Gamma$  imposes  $2r+1$  conditions on quadrics.

Conversely, suppose that  $\Gamma$  imposes only  $2r+1$  conditions on quadrics. By Theorem 2.7 it suffices to construct a 1-generic  $2 \times r$  matrix  $M$  whose minors vanish on  $\Gamma$ . Note that the dimension of the space of quadrics on  $\mathbb{P}^r$ , which is  $\binom{r+2}{2}$ , is equal to the sum  $\binom{r}{2} + 2r + 1$ , so the vector space of quadrics containing  $\Gamma$  has dimension  $\binom{r}{2}$ .

Enumerate the points  $p_i \in \Gamma$ , and let  $\Lambda = V(a_1, b_1)$  be the  $(r-2)$ -dimensional linear space spanned by  $p_1, \dots, p_{r-1}$ .

The number of conditions  $\Lambda$  imposes on quadrics is  $\binom{r}{2}$ , but the  $r-1$  points of  $\Gamma \cap \Lambda$  already impose  $r-1$  conditions, so the dimension of the space of quadrics containing  $\Lambda \cup \Gamma$  is  $\binom{r}{2} - (\binom{r}{2} - (r-1)) = r-1$ . These quadrics are contained in the ideal  $(a_1, b_1)$ , so a basis for them may be written as the  $2 \times 2$  minors that involve the first column of a matrix

$$M' := \begin{pmatrix} a_1 & a_2 & \dots & a_r \\ b_1 & b_2 & \dots & b_r \end{pmatrix}.$$

We will show first that  $M'$  is 1-generic.

If  $M'$  were not 1-generic then we could perform row and column operations that do not change the span of the first column or the span of the minors involving the first column to arrive at a matrix with an entry equal to 0. The minor involving the first column and that column is nonzero, because the quadrics defined by these minors are linearly independent. But a reducible quadric can contain only  $2r$  linearly independent points, a contradiction proving that  $M'$  is 1-generic.

It now suffices to show that all the minors of  $M$  vanish at all the points of  $\Gamma$ . At a point in  $p \in \Gamma$  that is not in  $\Lambda$ , at least one of  $a_1, b_1$  is nonzero. Since each minor of  $M$  involving the first column vanishes at  $p$ , each pair of scalars  $(a_i(p), b_i(p))$  is a multiple of  $(a_1(p), b_1(p))$ . Thus all the minors of  $M$  vanish at  $p$ , and thus vanish on at least  $\geq 2r+3-(r-1)=r+4$  points of  $\Gamma$ .

Because  $a_2, \dots, a_r$  are linearly independent, we may perform column operations to ensure that for  $i = 2, \dots, r$  the linear form  $a_i$  vanishes at all of  $\{p_2, \dots, p_r\}$  except possibly at  $p_i$ . Thus the minor of  $M$  involving columns  $i, j$  vanishes at each of  $p_2, \dots, p_r$  except possibly  $p_i, p_j$ , thus at  $r-3$  additional points, for a total of  $2r + 1$  points of  $\Gamma$ . Since  $\Gamma$  imposes only  $2r + 1$  conditions on quadrics, the minors of  $M$  vanish on all of  $\Gamma$ , as required.  $\square$

The number  $2r + 3$  in Corollary 2.8 is sharp: if  $C$  is a canonical curve of genus  $r + 2$  in  $\mathbb{P}^{r+1}$ , then, since  $C$  is arithmetically Cohen-Macaulay (Corollary ??), Corollary 3.4 implies that the points of a hyperplane section lie on the same number of independent quadrics as does  $C$ , and so, by Corollary 2.6 such points lie on the same number of quadrics as do  $2r + 2$  points on a rational normal curve.

**Cheerful Fact 2.9.** *In [Harris 1982b] a similar argument is used to show that if  $\Gamma$  is a set of  $\geq 2r + 1 + 2d$  points in uniform position imposing only  $2r + d$  conditions on quadrics, then  $\Gamma$  lies on a rational normal scroll of dimension  $d$ . We do not know whether the requirement of uniform position can be weakened to linearly general position, as in the case  $d = 1$ .*

**Corollary 2.10.** *Let  $a_1, \dots, a_d$  be non-negative integers, and let  $r = d-1+\sum_{i=1}^d a_i$ . The ideal of  $S(a_1, \dots, a_d) \subset \mathbb{P}^r$  is generated by the  $2 \times 2$  minors of the matrix*

$$M = \begin{pmatrix} x_{1,0} & x_{1,1} & \dots & x_{1,a_1-1} & | & x_{2,0} & \dots & x_{2,a_2-1} & | & \dots & | & x_{r,0} & \dots & x_{r,a_r-1} \\ x_{1,1} & x_{1,2} & \dots & x_{1,a_1} & | & x_{2,1} & \dots & x_{2,a_2} & | & \dots & | & x_{r,1} & \dots & x_{r,a_r} \end{pmatrix}$$

Moreover, the scroll admits a linear projection to each  $C_{a_i}$ .

**Proof.** We may think of the matrix  $M$  as consisting of  $d$  blocks,  $M_{a_i}$ . These blocks are 1-generic by Proposition 2.3. Since they involve distinct variables, it follows that  $M$  is 1-generic. Thus by Theorem 2.7, the ideal  $I_2(M)$  is prime and of codimension  $d = \sum a_i - 1$ , as is the ideal of the scroll. Thus it suffices to show that the minors of  $M$  vanish on the scroll.

Let  $C_i$  be the rational normal curve in the subspace  $\mathbb{P}^{a_i} \subset \mathbb{P}^r$ . As always, the set  $V(M)$  is the union of the linear spaces on which generalized rows of  $M$  vanish; and each such space is the space spanned by the points in the curves  $C_{a_i}$  corresponding to the part of that row in the block  $M_{a_i}$ —that is,  $V(I_2(M))$  is the union of the spans of sets of corresponding points on the  $C_{a_i}$ , as required.  $\square$

In Theorem 2.1 we will show that every 1-generic  $2 \times (r-d)$  matrix of linear forms in  $r$  variables can be transformed by row and column transformations and a linear change of variables to one of the type shown in Corollary 2.10, and thus the minors of any 1-generic matrix define a scroll.

### 3. Scrolls as Images of Projective Bundles

Recall that if  $X$  is a scheme and  $\mathcal{E}$  is a locally free sheaf on  $X$  then  $\mathbb{P}(\mathcal{E}) = \text{Proj}(\text{Sym}(\mathcal{E}))$  is a *projective space bundle*, and comes equipped with a projection  $\pi : \mathbb{P}(\mathcal{E}) \rightarrow X$  and a *tautological invertible sheaf*  $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$  that is a quotient of  $\pi^*(\mathcal{E})$  such that  $\pi_*(\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)) = \mathcal{E}$ , and thus with

$$H^0(\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)) = H^0(\mathcal{E}).$$

Our third description of 2-dimensional scrolls is that they are the images of projective space bundles

$$\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(a_1) \oplus \mathcal{O}_{\mathbb{P}^1}(a_2))$$

under the map given by the complete series associated to the tautological line bundle. When both  $a_1$  and  $a_2$  are strictly positive, we will see that this is an embedding, and in any case we will focus on the projective bundle itself, which is always smooth. The case of higher dimensional scrolls is similar. We follow [Hartshorne 1977, Chapter V]), but restrict to the 2-dimensional rational case.

The following description is given in [Hartshorne 1977, Proposition V.2.2 and V.2.3]).

**Theorem 3.1.** *Suppose that  $0 \leq a_1 \leq a_2$  are integers and let*

$$\mathcal{E} = \mathcal{O}_{\mathbb{P}^1}(a_1) \oplus \mathcal{O}_{\mathbb{P}^1}(a_2).$$

*Let  $X = \mathbb{P}(\mathcal{E})$  be the corresponding  $\mathbb{P}^1$  bundle over  $\mathbb{P}^1$  and let  $\pi : X \rightarrow \mathbb{P}^1$  be the natural projection. Set  $\mathcal{L} = \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ , the tautological 1-quotient of  $\pi^*(\mathcal{E})$ .*

*The complete linear series  $|\mathcal{L}|$  is base-point free, and is very ample if  $0 < a_1$ . Let  $\phi : X \rightarrow \mathbb{P}^{a_1+a_2+1} = \mathbb{P}^r$  be the corresponding morphism. The image of  $\phi$  is the rational normal scroll  $S(a_1, a_2)$ . More explicitly:*

- (1) *If  $C_1, C_2 \subset X$  are the curves defined by the projections  $\mathcal{E} \rightarrow \mathcal{O}_{\mathbb{P}^1}(a_i)$ , then  $\phi(C_1)$  and  $\phi(C_2)$  are defined by the vanishing of the sections of  $\mathcal{O}_{\mathbb{P}^1}(a_2)$  and  $\mathcal{O}_{\mathbb{P}^1}(a_1)$  respectively. Thus  $C_i \cong \mathbb{P}^1$ , and the restriction of  $\phi$  to  $C_i$  embeds it in  $\mathbb{P}^{a_i}$  as the rational normal curve of degree  $a_i$ .*
- (2) *The restriction of  $\mathcal{L}$  to the fiber  $\mathbb{P}^1$  of  $\pi$  is  $\mathcal{O}_{\mathbb{P}^1}(1)$ .*
- (3) *The fibers of  $\pi$  meet each  $C_i$  in a point. The images of  $C_1, C_2$  are contained in disjoint subspaces of  $\mathbb{P}^r$ , and the fibers of  $\pi$  are mapped to lines joining the corresponding points of the  $C_i$ .*

**Corollary 3.2.** [Hartshorne 1977, Section V.2] *Let  $0 \leq a_1 \leq a_2$  be integers. The divisor class group of the rational normal scroll*

$$X := S(a_1, a_2) \subset \mathbb{P}^r = \mathbb{P}^{a_1+a_2+1}$$

is generated by the class of the hyperplane section and the class of a ruling. If  $a_1 = 0$ , then the blowup of  $X$  at its singular point is  $S(1, a_2 + 1) \subset \mathbb{P}^{r+2}$ , and  $C_1 \subset S(1, a_2 + 1)$  is the exceptional divisor. The blowup map  $S(1, a_2 + 1) \rightarrow S(0, a_2)$  corresponds to the isomorphism

$$\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(a_2 + 1)) \rightarrow \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(a_2))$$

induced by tensoring with  $\pi^*(\mathcal{O}_{\mathbb{P}^1}(-1))$ .

**Proposition 3.3.** Suppose that  $0 < a_1 \leq a_2$  and  $\mathcal{E} = \mathcal{O}_{\mathbb{P}^1}(a_1) \oplus \mathcal{O}_{\mathbb{P}^1}(a_2)$ . Let  $\pi : X := \mathbb{P}(\mathcal{E}) \rightarrow \mathbb{P}^1$  be the projection. The sections  $\sigma : \mathbb{P}^1 \rightarrow C \subset X$  of  $\pi$ —that is, maps  $\sigma$  such that  $\pi\sigma$  is the identity—correspond to surjections  $\mathcal{E} \rightarrow \mathcal{L}$  for some line bundle  $\mathcal{L}$  on  $\mathbb{P}^1$ . Considering  $X$  as embedded in  $\mathbb{P}^{a_1+a_2+1}$ , we have  $\mathcal{L} = \sigma^*\mathcal{O}_C(1)$ , so the degree of  $C$  is equal to the degree of  $\mathcal{L}$ .

Thus  $\pi$  admits a section of degree  $e$  as a curve in  $\mathbb{P}^{a_1+a_2+1}$  if and only if  $0 < e = a_1$  or  $e \geq a_2$ .

**Proof.** Apply [Hartshorne 1977, II.7.12]). □

## 4. Curves on a 2-dimensional scroll

The reason we are interested in scrolls is for the study of the curves contained in them. We may begin with the curve, and try to “find” a scroll containing it; or we may begin with the scroll and ask what curves it contains. We start with the first approach:

### 4.1. Finding a scroll containing a given curve.

**Proposition 4.1.** Suppose that  $C \subset \mathbb{P}^r$  is a linearly normal reduced and irreducible curve, and  $D$  is a Cartier divisor on  $C$  such that  $|D|$  is base-point free. If the linear span of  $D$ , regarded as a subscheme of  $\mathbb{P}^r$ , is  $t$ -dimensional with  $t \leq r - 2$ , then  $C$  lies on a scroll of dimension  $t + 1$ .

**Proof.** Choose a base-point-free pencil  $\mathbb{C}^2 \cong V \subset H^0(\mathcal{O}_C(D))$ , and let  $H$  be a hyperplane section of  $C$ . Since the span of  $D$  is  $t$ -dimensional,  $W := H^0(\mathcal{O}_C(H - D))$  has dimension  $r - t$ , and the natural 1-generic mapping  $V \otimes W \rightarrow H^0(\mathcal{O}_C(H))$  corresponds, as in Theorem 2.7 and Theorem 2.1, to the desired scroll. □

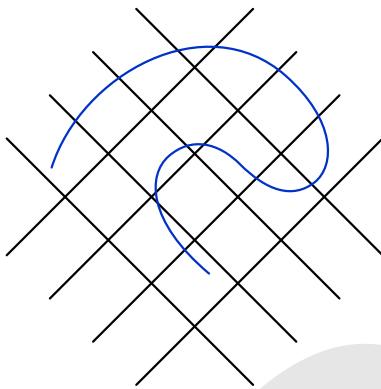
**Corollary 4.2.** Suppose that one of the following holds:

- (1)  $C \subset \mathbb{P}^r$  is a linearly normal hyperelliptic curve, and

$$\{D_\lambda \mid \lambda \in \mathbb{P}^1\}$$

are the divisors of the  $g_2^1$  on  $C$ ; or

- (2)  $C \subset \mathbb{P}^{g-1}$  is a trigonal canonical curve, and  $\{D_\lambda \mid \lambda \in \mathbb{P}^1\}$  are the divisors of a  $g_3^1$ .



**Figure 2.** A curve of type  $(2,3)$  on a smooth quadric meets one ruling twice and the other 3 times.

The union of the lines spanned by the  $D_\lambda$  is a scroll  $S(a_1, a_2)$  and  $a := \max\{a_1, a_2\}$  is the maximal integer such that  $aD_\lambda$  is a special divisor.

**Remark 4.3.** In the case of a trigonal curve the difference  $|a_2 - a_1|$  is called the *Maroni invariant*.

**Proof.** Let  $\mathcal{L}$  be the invertible sheaf  $\mathcal{O}_C(D_\lambda)$  corresponding to the  $g_2^1$  in case 1 or the  $g_3^1$  in case 2 and let  $s_\lambda$  be the section vanishing on  $D_\lambda$ . Setting  $\mathcal{M} = \mathcal{L}^{-1} \otimes \mathcal{O}_C(1)$ , we see that  $s_\lambda \cdot H^0(\mathcal{M}) \subset H^0(\mathcal{O}_C(1))$  is the space of linear forms vanishing on  $D_\lambda$ . In both cases, this space is a line: this is obvious in case 1, and follows from the geometric Riemann-Roch theorem in case 2.

These forms make up the generalized rows of the matrix  $M_\mu$  corresponding to the multiplication  $\mu : H^0(\mathcal{L}) \otimes H^0(\mathcal{M}) \rightarrow H^0(\mathcal{O}_C(1))$ , we see that the union of the lines is a scroll  $S(a_1, a_2)$  cut out by  $I_2(M_\mu)$ .

The proof is completed by the more general Proposition 4.4.  $\square$

**Proposition 4.4.** Let  $S(a_1, a_2) \subset \mathbb{P}^{a_1+a_2+1}$  be a scroll with  $a_1 \leq a_2$ . The maximal number of rulings contained in a proper subspace of  $\mathbb{P}^{a_1+a_2+1}$  is  $a_2$ .

Equivalently, if the scroll is defined from a multiplication map  $V \otimes H^0(\mathcal{L}_2) \rightarrow H^0(\mathcal{O}_X(1))$ , where  $V$  is a basepoint-free pencil in  $H^0(\mathcal{L}_1)$ , then  $a_2$  is the maximal integer such that  $\mathcal{L}_1^{-a_2} \mathcal{O}_X(1)$  is effective.

**Proof.** A hyperplane containing  $C_{a_1}$  meets  $C_{a_2}$  in  $a_2$  points, and thus contains  $a_2$  rulings of the scroll. If  $H$  were a hyperplane containing more than  $a_2$  rulings, then  $H$  would meet each curve  $C_{a_i}$  in more than  $a_i$  points, and thus  $H$  would contain both these curves, so that  $S(a_1, a_2) \subset H$ . Since  $S(a_1, a_2)$  is nondegenerate, this is impossible.

The second characterization follows because the rulings of the scroll are the divisors of elements of  $V$ .  $\square$

**Theorem 4.5.** *If  $C \subset \mathbb{P}^r$  is a reduced and irreducible curve of degree  $d \geq 2r + 1$  with arithmetic genus equal to the Castelnuovo bound  $\pi(r, d)$ , then  $C$  lies on a 2-dimensional scroll or  $r = 5$  and  $C$  lies on the Veronese surface.*

For the classes in which these Castelnuovo curves lie, see [Harris 1982b, Theorem 3.11] or [Griffiths and Harris 1978, p. 533].

**Proof.** From the proof of Castelnuovo's theorem (Theorem 2.1) we see that a general hyperplane section  $H \cap C$  is a set of  $d \geq 2(r - 1) + 3$  points in linearly general position in  $H$  imposing  $2(r - 1) + 1$  conditions on quadrics. Moreover the curve is arithmetically Cohen-Macaulay, so the dimension of the space of quadrics containing the hyperplane section is the same as the dimension of the space of quadrics containing  $C$ . By Theorem 2.8 the hyperplane section lies on a rational normal curve whose ideal is generated by the quadrics containing the points; thus the quadrics containing  $C$  intersect in a surface of minimal degree containing  $C$ .  $\square$

Both scrolls and the Veronese surface occur (Exercises 5.7 and 5.10).

**4.2. Finding curves on a given scroll.** We now turn to the reverse approach: given a 2-dimensional scroll, what are the curves that lie on it? The key is to understand the divisor class group and the canonical class.

**Notation:** Throughout this section we consider the vector bundle

$$\mathcal{E} = \mathcal{O}_{\mathbb{P}^1}(a_1) \oplus \mathcal{O}_{\mathbb{P}^1}(a_2)$$

with  $0 \leq a_1, 1 \leq a_2$  and  $a_1 \leq a_2$ . The scroll  $S(a_1, a_2) \subset \mathbb{P}^r$ , where  $r = a_1 + a_2 + 1$ . This scroll is the image of  $X = \mathbb{P}(\mathcal{E})$  by a map that is an isomorphism if  $0 < a_1$  and is the blow-down of  $C_1$  if  $a_1 = 0$ . We write  $\pi : X \rightarrow \mathbb{P}^1$  for the projection, and  $C_{a_i} \subset X$  for the directrix of degree  $a_i$ . The degree of  $S(a_1, a_2)$  is  $d := a_1 + a_2 = r - 1$ .

**4.2.1. The case of a smooth scroll.** Now assume in addition that  $1 \leq a_1$ , so that  $S(a_1, a_2)$  is smooth.

**Theorem 4.6.** (1) *The Picard group of  $X = S(a_1, a_2)$  is  $\text{Pic } X \cong \mathbb{Z}^2$ , freely generated by the class  $F$  of a ruling and the class  $H$  of a hyperplane section.*

(2) *The intersection form on  $\text{Pic}(X)$  is given by*

$$\begin{array}{c|cc} \cdot & F & H \\ \hline F & 0 & 1 \\ H & 1 & d \end{array}$$

- (3) The canonical class of the scroll is  $K := -2H + (d - 2)F$ , so  $K^2 = 8$ .
- (4) The degree of a curve in the class  $pH + qF$  is  $pd + q$ , while its arithmetic genus is  $\binom{p}{2}d + pq - p - q + 1$ .
- (5) The class of  $C_{a_1}$  is  $H - a_2F$ , and the class of  $C_{a_2}$  is  $H - a_1F$ .
- (6) If  $C \subset \mathbb{P}^{g-1}$  is a trigonal canonical curve and  $X$  is the scroll swept out by the trisecants of  $C$ , then the class of  $C$  is  $3H + (4 - g)F = H - K$ .

**Proof.** We have

$$H^2 = \deg X = a_1 + a_2 = d; \quad H \cdot F = \deg F = 1; \quad F^2 = 0$$

where the last equality follows because any two fibers are linearly equivalent and disjoint. It follows in particular that  $H$  and  $F$  are linearly independent.

We next show that  $\text{Pic } X$  is generated by  $H$  and  $F$ . If  $D$  is any divisor, then  $D' = D - (F \cdot D)H$  meets  $F$  in degree 0, and it now suffices to show that  $D' \sim aF$  for some integer  $a$ . Since  $\mathcal{O}_X(D')|_F = \mathcal{O}_F$  for any fiber  $F$ , we see that  $\pi_*(\mathcal{O}_X(D'))$  is a torsion-free sheaf on  $\mathbb{P}^1$  whose fiber at each point is one-dimensional, so  $\mathcal{L}$  is a line bundle on  $\mathbb{P}^1$ . Possibly replacing  $D'$  by  $-D'$ , and using the fact that  $\pi_*(\mathcal{O}_X(-D')) = (\pi_*(\mathcal{O}_X(D')))^{-1}$  we may assume that  $\mathcal{L}$  is globally generated, and it follows that the natural map of line bundles  $\pi^*\mathcal{L} = \pi^*\pi_*\mathcal{O}_X(D') \rightarrow \mathcal{O}_X(D')$  is a surjection, whence  $\mathcal{O}_X(D') \cong \pi^*\mathcal{L}$ . Thus if  $q = \deg \mathcal{L}$  then  $D' \sim qF$ . Note that we could also recover  $q$  as  $H \cdot D'$ . This completes the proof of parts 1 and 2.

To compute the canonical class  $K_X = pH + qF$  we use the adjunction formula on the rational curves  $H$  and  $F$ . Thus  $-2 = (F + K) \cdot F = p$  and

$$-2 = (H + K) \cdot H = d + pd + q = d + (-2)d + q$$

whence  $q = d - 2$  as required for part 3.

Part 4 is a direct computation from the adjunction formula.

For part 5 we observe that a hyperplane containing  $C_{a_1}$  meets  $X$  in  $C_{a_1}$  plus  $a_2$  rulings; thus  $C_{a_1} \sim H - a_2F$ . Similar reasoning holds for  $C_{a_2}$ .

If  $C \subset S(a_1, a_2) \subset \mathbb{P}^{g-1}$  is a trigonal canonical curve of genus  $g$ , then the degree  $d$  of the scroll must be  $g - 2$ . Moreover  $C \cdot F = 3$  and  $C \cdot (C + K) = 2g - 2$ . These equations have the unique solution  $C \sim H - K = 3H + (4 - g)F$ .  $\square$

Now we can say exactly which classes on the scroll contain curves:

**Theorem 4.7.** *Again, suppose that  $0 < a_1$ . There are reduced curves in the class  $D = pH + qF$  if and only if one of the following holds:*

- (1)  $D \sim qF$ ; that is,  $p = 0, q > 0$ ; or

- (2)  $D \sim C_{a_1}$ ; that is,  $p = 1, q = -a_2$ ; or
- (3)  $p \geq 0$  and  $D \cdot C_{a_1} > 0$ ; that is,  $q \geq -pa_1$ .

In case (3) the linear series  $|D|$  is basepoint free. When, in addition,  $a_2 > a_1$  or  $q > -pa_1$  the class  $|D|$  contains irreducible smooth curves.

Note that in case (1) we have  $D^2 = 0$ , because any two fibers of  $\pi$  are disjoint; in case (2) we have  $D^2 = a_1 - a_2 \leq 0$  and in case (3) we have  $D^2 \geq 0$ , and  $D^2 = 0$  only if  $d = 2, q = -pa_1$ . In particular, no irreducible curve on  $X$  other than  $C_{a_1}$  can have negative self-intersection.

**Proof.** The existence of smooth curves of types 1 and 2 is obvious; the following result will show that the ones of type 3 move in a base-point free linear series. By Bertini's theorem 5.4 such a series must contain smooth curves unless the associated map factors through a curve, in which case  $D^2 = dp^2 - 2pq = p(pa_1 + pa_2 - 2q) = 0$ , which implies that  $a_2 = a_1$  and  $q = -pa_1$ .

Theorem 4.8, below, thus completes the proof.  $\square$

**Theorem 4.8.** Again, suppose that  $0 < a_1$ . Suppose that  $D$  is a divisor on the scroll  $X$  as above. If  $D \sim pH + qF$  is effective, then  $p \geq 0$  and

$$\begin{aligned} H^0(\mathcal{O}_X(D)) &= H^0(\mathcal{O}_{\mathbb{P}^1}(q) \otimes \text{Sym}^p \mathcal{E}) \\ &= \bigoplus_{0 \leq i \leq p} H^0(\mathcal{O}_{\mathbb{P}^1}(q + (p-i)a_1 + ia_2)). \end{aligned}$$

The linear series  $|D|$  is base-point free if and only if every summand in the last expression is nonzero. Thus, numerically,

$$h^0(\mathcal{O}_X(D)) = \sum_{i|q+(p-i)a_1+ia_2 \geq 0} 1 + (q + (p-i)a_1 + ia_2),$$

and  $|D|$  is base-point free iff  $p \geq 0$  and  $q \geq -pa_1$ .

**Proof.** First, If  $q < -pa_1$ , then

$$D \cdot C_{a_1} = (pH + qF) \cdot (H - a_2F) = p(a_1 + a_2) - pa_1 + q = pa_1 + q < 0,$$

so any effective divisor in the class of  $D$  must have a component in common with  $C_{a_1}$ .

Let  $\pi : X \rightarrow \mathbb{P}^1$  be the structure map of the projective bundle  $X = \mathbb{P}_{\mathbb{P}^1}(\mathcal{E})$ . We have  $H^0(\mathcal{O}_X(pH + qF)) = H^0(\pi_*(\mathcal{O}_X(pH + qF)))$ . Also, we may write  $\mathcal{O}_X(pH + qF)$  as  $\mathcal{O}_X(p) \otimes \pi^*\mathcal{O}_{\mathbb{P}^1}(q)$ , and since  $\mathcal{O}_{\mathbb{P}^1}(q)$  is a line bundle we see that

$$\pi_*(\mathcal{O}_X(p) \otimes \pi^*\mathcal{O}_{\mathbb{P}^1}(q)) = \pi_*(\mathcal{O}_X(p)) \otimes \mathcal{O}_{\mathbb{P}^1}(q).$$

The projective bundle  $\mathbb{P}(\mathcal{E})$  is by definition  $\text{Proj}$  of the symmetric algebra  $\mathbb{P}(\mathcal{E}) := \text{Proj}(\text{Sym}(\mathcal{E}))$ . Over any open subset  $U$  of the base  $\mathbb{P}^1$  over which  $\mathcal{E}$  is free, this is  $\pi^{-1}(U) = U \times \mathbb{P}^1$ , and it follows that  $\pi_*(\mathcal{O}_{\mathbb{P}(\mathcal{E})}(p)) = \text{Sym}^p \mathcal{E}$ . Thus

$$\begin{aligned}\pi_*(\mathcal{O}_X(pH + qF)) &= \pi_*(\mathcal{O}_X(p) \otimes \pi^*\mathcal{O}_{\mathbb{P}^1}(q)) \\ &= \pi_*(\mathcal{O}_X(p)) \otimes \mathcal{O}_{\mathbb{P}^1}(q) \\ &= \text{Sym}^p(\mathcal{E}) \otimes \mathcal{O}_{\mathbb{P}^1}(q) \\ &= \left( \bigoplus_{0 \leq i \leq p} \mathcal{O}_{\mathbb{P}^1}((p-i)a_1 + ia_2) \right) \otimes \mathcal{O}_{\mathbb{P}^1}(q),\end{aligned}$$

and the first formula follows.

The term  $H^0(\mathcal{O}_{\mathbb{P}^1}(q + (p-i)a_1 + ia_2))$  is nonzero for all  $i$  if and only if  $H^0(\mathcal{O}_{\mathbb{P}^1}(q + pa_1))$  is nonzero, which holds if and only if  $q \geq -pa_1$ . If  $\sigma = \sum \sigma_i$  is a section of  $\mathcal{O}_X(D)$  written according to the decomposition above, then the restriction of  $\sigma$  to the rational normal curve  $C_{a_1} = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(a_1))$  is the component  $\sigma_0$ , and similarly for  $C_{a_2}$  and  $\sigma_p$ . Thus when all the summands are nonzero there are sections vanishing on  $C_{a_1}$  but not  $C_{a_2}$ , and vice versa, so the system is base-point free.  $\square$

**4.2.2. The case of a singular scroll  $S(0, a_2)$ .** We now assume that  $a_1 = 0$ . We will use a general fact about projective bundles:

**Proposition 4.9.** *If  $X$  is a scheme,  $\mathcal{E}$  a locally free sheaf on  $X$ ,  $\mathcal{L}$  an invertible sheaf on  $X$  and  $\pi : \mathbb{P}(\mathcal{E}) \rightarrow X$  the natural projection. Under the isomorphism*

$$\mathbb{P}(\mathcal{L} \otimes \mathcal{E}) \cong \mathbb{P}(\mathcal{E})$$

*the invertible sheaf  $\mathcal{O}_{\mathbb{P}(\mathcal{E} \otimes \mathcal{L})}(1)$  corresponds to the invertible sheaf  $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1) \otimes \pi^*\mathcal{L}$ .*

*Moreover, the singular scroll  $S(0, a_2)$ , which is the cone over a rational normal curve of degree  $a_2$ , is the image of  $S(1, a_2 + 1)$  under the map corresponding to the complete linear series*

$$|\mathcal{O}_{S(1, a_2 + 1)}(1) \otimes \pi^*(\mathcal{O}_{\mathbb{P}^1}(-1))|,$$

*which blows down the line  $C_1$ .*

Note that  $S(1, a_2 + 1)$  is isomorphic to the surface  $\tilde{S}(0, a_2)$  of Section 1.

**Proof.** The isomorphism  $\mathbb{P}(\mathcal{E}) \rightarrow \mathbb{P}(\mathcal{E} \otimes \mathcal{L})$  corresponds to the surjection  $\pi^*(\mathcal{E} \otimes \mathcal{L}) \cong \pi^*(\mathcal{E}) \otimes \pi^*(\mathcal{L}) \rightarrow \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1) \otimes \pi^*(\mathcal{L})$ , and the inverse map is formed similarly.

When  $\mathcal{E} = \mathcal{O}_{\mathbb{P}^1}(a_1) \oplus \mathcal{O}_{\mathbb{P}^1}(a_2 + 1)$  and  $a_1 = 1$  the invertible sheaf

$$\mathcal{O}_{S(a_1, a_2 + 1)}(1) \otimes \pi^*(\mathcal{O}_{\mathbb{P}^1}(-1))$$

corresponds to the divisor class  $H - F$ , which meets  $C_{a_1}$  in degree 0, so the image of  $C_{a_1}$  under the corresponding linear series is a point. However the description of  $H^0(\mathcal{E})$  in Theorem 4.8 shows that the restriction to  $F$  is still the complete linear series  $|\mathcal{O}_{\mathbb{P}^1}(1)|$ , and the restriction to  $C_{a_2+1} \cong \mathbb{P}^1$  is  $|\mathcal{O}_{\mathbb{P}^1}(a_2+1)|$ . Thus the displayed linear series in the proposition is base-point free and does define a map as claimed.  $\square$

**Theorem 4.10.** *Suppose that  $C$  is a smooth curve lying on the scroll  $S(0, d)$ , with  $1 \leq d$ . Let  $H, F$  be the (Weil) divisor classes of the hyperplane section and ruling, respectively.*

- (1)  $C \sim mH$  or  $C \sim mH + F$  for some  $m \geq 0$ .
- (2) If  $C \sim mH$  then  $\deg C = md$  and the genus of  $C$  is  $g = \binom{m}{2}d - m + 1$ .
- (3) If  $C \sim mH + F$  then  $\deg C = md + 1$  and the genus of  $C$  is  $g = \binom{m}{2}d$ .

In particular,  $\deg C$  is congruent to 0 or 1 modulo  $d$ .

**Proof.**  $d = 1$ : In this case,  $S(0, 1) \cong \mathbb{P}^2$ , with a distinguished point  $\phi_1(\mathbb{P}^1)$  and  $mH + F = (m+1)H$ . The formulas for degree and genus are familiar from the theory of curves in the plane.

We now assume  $2 \leq d$ , so  $S(0, d)$  is a cone over the rational normal curve of degree  $d$ . Let  $\pi : S(1, d+1) = \tilde{S}(0, d) \rightarrow S(0, d)$  be the blowup of the vertex of the cone, and let  $E, L$  be the exceptional divisor and ruling on  $S(1, d+1)$ , so that  $E = C_1$  and  $E^2 = -d$ . The proper transform of the hyperplane class  $H$  on  $S(0, d)$  has intersection number 1 with  $L$  and 0 with  $E$ , and is thus of class  $E + dL$ . Let  $\tilde{C} \cong C$  be the proper transform of  $C$  on  $S(1, d)$ . Since  $C$  is smooth,  $\tilde{C}$  meets  $E$  at most once, whence the first assertion.

The degree assertions of items 2,3 are immediate, and the genus assertions follow by direct computation from the adjunction formula on  $S(1, d+1)$ .  $\square$

In these terms we can also see which classes on a singular scroll correspond to hyperelliptic or canonical trigonal curves in the way described by Corollary 4.2:

- Corollary 4.11.**
- (1) *For a smooth canonical curve  $C \subset \mathbb{P}^{g-1}$  to lie on a singular scroll it is necessary that  $g \leq 4$ . If  $g = 4$  then  $C$  is a complete intersection of the cone over a conic with a cubic surface not passing through the vertex of the cone; thus  $C$  is trigonal, as is also the case for  $g = 3$ .*
  - (2) *If  $C$  is a smooth hyperelliptic curve of genus  $g \geq 2$ , and the union of the secant lines to  $C$  corresponding to the hyperelliptic involution sweep out a singular scroll  $S(0, d)$  then  $C \sim 2H$  or  $2H+F$  on  $\tilde{S}(0, d) = S(1, d+1)$*

in which cases the degree of  $C$  is  $2d$  or  $2d + 1$  and the genus of  $C$  is  $d - 1$  or  $d$  respectively. Conversely, any curve in these classes is hyperelliptic and the lines on the singular scroll  $S(0, d) \subset \mathbb{P}^d$  meet the image of  $C$  in the pairs of points that are conjugate under the hyperelliptic involution.

**Proof.** 1: If  $C \subset \mathbb{P}^{g-1}$  is a smooth non-hyperelliptic canonical curve lying on a singular 2-dimensional scroll in  $\mathbb{P}^{g-1}$  then the scroll is  $S(0, g - 2)$ . By Theorem 4.10,  $\deg C = 2g - 2$  must be congruent to 0 or 1 modulo  $g - 2$ , and this is the case only for  $g = 3, 4$ , the cases of a plane quartic or the complete intersection of a quadric cone with a cubic surface not passing through the vertex of the cone.

2: By hypothesis the strict transform  $\tilde{C} \subset \tilde{S} = S(1, d + 1)$  has class  $2H + \epsilon F$ , and by Theorem 4.10,  $\epsilon$  is 0 or 1. The formulas for degree and genus follow are given in that Theorem.  $\square$

## 5. Exercises

**Exercise 5.1.** Give a proof of Proposition ?? without using Bertini's theorem, by projecting  $X$  from a general point and using induction on  $\text{codim } X$ .

Hint: Let  $p \in L \cap X$  be a point. If every secant to  $X$  through  $p$  lies entirely in  $X$ , then  $X$  is a cone over  $p$ ; but since  $p$  was a general point, this would imply that  $X$  is a linear space, contradicting non-degeneracy.

It follows that the projection  $\pi_p : X \rightarrow \mathbb{P}^{N-1}$  is a generically finite (rational) map from  $X$  to  $X' := \pi_p(X)$ , and thus  $\dim X' = \dim X$  and  $\text{codim } X' = \text{codim } X - 1$ . The plane  $\pi_p(L)$  meets  $X'$  in the images of the points of  $L \cap X$  other than  $p$ , so  $\deg X \geq \deg X' + 1$ . By induction,  $\deg X' \geq \text{codim } X' + 1 = \text{codim } X$ , completing the argument.

**Exercise 5.2.** Suppose  $1 \leq a_1 < a_2$ . Show that the projection of  $S(a_1, a_2)$  from any point on  $C_{a_1}$  is  $S(a_1 - 1, a_2)$ , and that the projection from any point on  $C_{a_2}$  is  $S(a_1, a_2 - 1)$ . (If  $a_1 = a_2$ , projection of  $S(a_1, a_2)$  from any point is  $S(a_1 - 1, a_2)$ .)

This gives another proof that the blowup of the cone  $S(0, a_2)$  over the rational normal curve of degree  $a_2$  is  $S(1, a_2 + 1)$ .

**Exercise 5.3.** Show that a matrix  $M$  of linear forms is 1-generic iff, even after arbitrary row and column transformations, its entries are all non-zero.

**Exercise 5.4.** Let  $M$  be a 1-generic  $p \times q$  matrix of linear forms, with  $p \leq q$ . Show that the codimension of  $I_p(M)$  is  $q - p + 1$ . Hint: imitate the proof of Lemma 2.6.

**Exercise 5.5.** Show that if  $X \subset \mathbb{P}^r$  is a projective variety whose homogeneous ideal  $I$  contains  $m$  independent quadratics, then the ideal of the general hyperplane section of  $X$  in  $\mathbb{P}^{n-1}$  contains at least  $m$  independent quadratics. Use this

to prove that if  $X$  has codimension  $c$  then  $m \leq \binom{c+1}{2}$ , and that equality holds if and only if  $X$  is a variety of minimal degree (that is,  $\deg X = c + 1$ .)

**Exercise 5.6.** If  $X \subset \mathbb{P}^r$  is an irreducible, nondegenerate projective variety of codimension  $c$  whose homogeneous ideal contains  $\binom{c}{2}$  independent quadratic forms, then  $X$  is a variety of degree  $c$ .

Hint: Let  $Y = X \cap H$  be a general hyperplane section of  $X$  and consider the restriction map  $H^0(\mathcal{I}_{X/\mathbb{P}^r}(2)) \rightarrow H^0(\mathcal{I}_{Y/\mathbb{P}^{n-1}}(2))$ ; repeat  $n - c$  times.

**Exercise 5.7.** Show that every reduced, irreducible curve on the Veronese surface is a Castelnuovo curve.

**Exercise 5.8.** Use the Leray spectral sequence to compute the cohomology  $H^i(\mathcal{O}_X(D))$  on a scroll  $X \subset \mathbb{P}^r$ . Show that  $X$  is arithmetically Cohen-Macaulay (this also follows from the length of the Eagon-Northcott resolution described in Section 4). Hint: if you get stuck see [Hartshorne 1977, Section V.1].

**Exercise 5.9.** Show that if  $X \subset \mathbb{P}^r$  is an arithmetically Cohen-Macaulay surface and  $D \subset X$  is a curve, then  $D$  is arithmetically Cohen-Macaulay if and only if  $H_*^1(\mathcal{J}_{D/X}) = 0$ . Use this and Exercise 5.8 to determine the possible divisor classes of arithmetically Cohen-Macaulay curves on a 2-dimensional scroll.

**Exercise 5.10.** By Theorem 2.1 all Castelnuovo curves are arithmetically Cohen-Macaulay. Show that all the arithmetically Cohen-Macaulay curves on a 2-dimensional scroll are Castelnuovo curves ( Hint: See [Harris 1982b, Section 3c].)

**Exercise 5.11.** State and prove an analogue of Proposition 4.4 for curves on scrolls of dimension  $> 2$ . Hint:  $a_1$  is again the maximal number of rulings. What's the cleanest statement after that? Does one have to project the scroll to  $S(a_2, \dots)$  to see the rest?

**Exercise 5.12.** Improve the result of Exercise 5.2 by showing that, if  $0 \leq a_1 \leq a_2$ , then the projection of  $S(a_1, a_2)$  from any point not on  $C_{a_1}$  (or not the vertex, in case  $a_1 = 0$ ) is  $S(a_1, a_2 - 1)$ .

**Exercise 5.13.**

If  $C \subset S(0, a_2) \subset \mathbb{P}^{a_2+1}$  is a smooth curve with class  $m$  times the hyperplane section (that is, corresponding to a curve on  $S(1, a_2 + 1)$  with class  $mH - mF$ ), show that

$$\deg(C) = ma_2 \quad \text{and} \quad g(C) = \binom{m}{2}a_2 - m + 1.$$

If  $C \subset S(0, a_2) \subset \mathbb{P}^{a_2+1}$  is a smooth curve with class  $m$  times the hyperplane section plus a ruling (that is, corresponding to a curve on  $S(1, a_2 + 1)$  with class

$mH - (m - 1)F$ , show that

$$\deg(C) = ma_2 + 1 \quad \text{and} \quad g(C) = \binom{m}{2}a_2.$$

**Exercise 5.14.** Show that 3 general quadrics meet in 8 general points in  $\mathbb{P}^3$ , and deduce that the bound  $d \geq 2r + 3$  in Theorem 2.8 is sharp.

**Exercise 5.15.** Trigonal curves of genus 5. In Section 5.15 we analyzed trigonal curves  $C \subset \mathbb{P}^4$  of genus 5, showing that there are three independent quadrics in  $I_C$  that are either a complete intersection (and generate  $I_C$ ) or meet in a non-degenerate surface). From Corollary 4.2 we know that in the second case the surface is a scroll. Complete the analysis by deciding between the two types of scrolls in  $\mathbb{P}^4$ :  $S(1, 2)$  and  $S(0, 3)$ . What is the divisor class of the curve on the scroll?

Hint:  $S(0, 3)$  does not occur here, because by Theorem 4.10 the smooth curves on  $S(0, 3)$  are either hypersurface sections, and thus of degree 3 for some integer  $a$ , or in the rational equivalence class of a hypersurface section plus one line, of degree  $3a + 1$ , and thus not of degree 8. One can see this directly too: if the scroll were the cone over a twisted cubic, then the lines on the cone would cut out the  $g_3^1$  on  $C$ . Since a pair of lines on the cone span only a 2-plane, the sum  $D$  of the two divisors would be a divisor of degree 6 and thus  $K - D$  would be a  $g_2^1$ , so the curve would be hyperelliptic.

In the case when  $C$  lies on  $S := S(1, 2)$ , we may write the class of  $C$  in terms of the hyperplane class  $H$  and the class  $F$  of a ruling,  $C \sim pH + qF$ , and we see from the projection of  $S$  to  $\mathbb{P}^1$  that  $C$  admits a degree  $p$  covering of  $\mathbb{P}^1$ . Since we have assumed that  $C$  is not hyperelliptic, we must have  $p \geq 3$ . By Theorem 4.7 we must have  $q \geq -p$ . Since  $\deg C = C \cdot H = 3p + q = 8$ , we must have either  $C \sim 3H - F$  or  $C \sim 4H - 4F$ . By the adjunction formula, in the second case,

$$2g(C) - 2 = 8 = (C + K_S) \cdot C = (4H - 4F) + (-2H + F) \cdot (4H - 4F) = 4$$

whereas a similar computation in the first case yields 8; thus  $C \sim 3H - F$ .

The key thing to note here is that the curve  $C$  has intersection number 3 with the lines of the ruling of  $S$ , meaning that  $C$  is a trigonal curve. (We also see that the  $g_3^1$  on  $C$  is unique: if  $D = p + q + r$  is a divisor moving in a pencil, the points  $p, q$  and  $r$  must lie on a line; since the surface  $S$  is the intersection of quadrics, those lines must lie on  $S$  and so must be lines of the ruling.) We see also that the locus  $W_4^1(C)$  has two components: there are pencils with a base point, that is, consisting of the  $g_3^1$  plus a base point; and there are the residual series  $K_C - g_3^1 - p$ . Each of these components of  $W_4^1(C)$  is a copy of the curve  $C$  itself, and they meet in two points, corresponding to the points of intersection of  $C$  with the directrix of the scroll  $S$ .

**Exercise 5.16.** If  $C$  is a smooth trigonal curve, canonically embedded in  $\mathbb{P}^{g-1}$ , then  $C$  lies on a 2-dimensional scroll  $S(a_1, a_2)$ , as in Theorem 4.6, then

$$a_2 - a_1 \leq \frac{g+2}{3}.$$

Hint: use intersection theory on the scroll.

# Free resolutions and canonical curves

In Chapter 16 we related the resolutions of curves in  $\mathbb{P}^3$  to their Hartshorne-Rao modules. The simplest case was that in which the Hartshorne-Rao module vanishes, that is, the case of arithmetically Cohen-Macaulay curves. In this chapter we first quote a result that is a step toward understanding the structure of free resolutions, and use it to explain how the condition that a curve in  $\mathbb{P}^r$  is arithmetically Cohen-Macaulay manifests itself in the free resolution of the homogeneous ideal of the curve. We will then introduce the Eagon-Northcott complexes, and explain their relation to the free resolutions of canonical curves. We close the chapter with an explanation of Green's conjecture, which proposes a way in which the intrinsic geometry of a curve may be connected to the shape of its minimal free resolution of its ideal in the canonical embedding.

**Remark 0.1.** There are two related contexts for the results of this section: modules over a local ring, and graded modules over a polynomial ring whose variables have positive degree. These are parallel, essentially because Nakayama's lemma works in both cases. We will simply work with local rings or polynomial rings over a field with variables in degree 1 and leave the translations to the reader.

## 1. Free resolutions

A more complete presentation of the material in this section may be found in [[Eisenbud 1995a](#), Chapter 19].

Let  $M$  be a finitely generated graded module over  $S := \mathbb{C}[x_0, \dots, x_r]$ . A *free resolution* of  $M$  is an exact complex of graded free modules, with maps of degree 0:

$$(\mathbb{F}, \phi) : \quad 0 \longleftarrow N \xleftarrow{\epsilon} \bigoplus_j S(-j)^{\beta_{0,j}} \longleftarrow \dots \xleftarrow{\phi_t} \bigoplus_j S(-j)^{\beta_{t,j}} \longleftarrow 0.$$

Here  $S(-j)$  denotes the graded free module of rank 1 with generator in degree  $j$ . (The map to  $N$  is not considered part of the free resolution). The resolution is *minimal* if a minimal set of generators of  $F_i$  maps to a minimal set of generators of the kernel of the following map, or equivalently (by Nakayama's lemma) the maps in  $\mathbb{F} \otimes_S \mathbb{C}$  are all 0.

The first examples of a minimal free resolution are the Koszul complexes (first defined, despite the name, in [Cayley 1848], and repeated, as examples, in [Hilbert 1890]), which resolve  $S/I$  when  $I = (f_1, \dots, f_t)$  is a complete intersection, that is,  $f_1, \dots, f_t$  is a regular sequence, with  $\deg f_i = d_i$ . For  $t = 2, 3$ , these look like

$$S \xleftarrow{(f_1 \ f_2)} S(-d_1) \oplus S(-d_2) \xleftarrow{\begin{pmatrix} -f_2 \\ f_1 \end{pmatrix}} S(-d_1 - d_2) \longleftarrow 0,$$

and

$$S \xleftarrow{(f_1 \ f_2 \ f_3)} F_1 \xleftarrow{\begin{pmatrix} 0 & f_3 & -f_2 \\ -f_3 & 0 & f_1 \\ f_2 & -f_1 & 0 \end{pmatrix}} F_2 \xleftarrow{\begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix}} F_3 \longleftarrow 0,$$

where

$$F_1 = \bigoplus_{j=1}^3 S(-d_j), \quad F_2 = \bigoplus_{1 \leq i < j \leq 3} S(-d_i - d_j) \text{ and } F_3 = S(-d_1 - d_2 - d_3).$$

Hilbert's Basis and Syzygy theorems together imply that every finitely generated  $S$ -module has a finite free resolution of length no greater than the number of variables,  $r + 1$ . (Fundamental results of Auslander, Buchsbaum and Serre say that a local ring  $R$  is *regular*—that is, the Krull dimension of  $R$  is equal to the minimal number of generators of  $\mathfrak{m}$ —if and only if the minimal free resolution of the residue field is finite, in which case the minimal free resolution of every module is finite.)

To construct this, suppose that a minimal homogeneous set of generators of  $N$  contains  $\beta_{0,j}$  generators of degree  $j$  for each  $j$ ; this defines a degree 0 map  $\epsilon$  from  $\bigoplus_j S(-j)^{\beta_{0,j}}$  onto  $N$ . We proceed similarly with the kernel of  $\epsilon$ , and continue to construct the whole resolution. Hilbert's syzygy theorem [Eisenbud

[[1995a](#), Corollary 19.7] implies that the process ends after  $t \leq r$  steps with a map  $\phi_t$  whose kernel is 0.

The number  $t$  is then the projective dimension of  $N$ . Equivalently,

$$\text{pd } M = \max\{t \mid \text{Ext}_S^t(M, k) \neq 0\}.$$

It follows from the Auslander-Buchsbaum theorem [[Eisenbud 1995a](#), Theorem 19.9] that  $t \geq \text{codim ann}_S(N)$ , the codimension of the support of  $N$ . Minimal free resolutions of a given module  $M$  are all isomorphic.

## 2. Classification of 1-generic $2 \times f$ matrices

We can use the uniqueness of minimal free resolutions to give a simple proof of Kronecker's classification of 1-generic  $2 \times f$  matrices [[Gantmacher 1959](#)], which was announced in Chapter 17.

**Theorem 2.1.** *Every 1-generic  $2 \times f$  matrix of linear forms can be transformed by row and column operations and a linear change of variables to one of the type shown in Corollary 2.10, and thus the minors of any 1-generic matrix define a scroll.*

To prove this result we first reinterpret the 1-generic condition: We have observed that an  $a \times b$  matrix of linear forms in  $c$  variables is the same as a  $\mathbb{C}$ -linear map of vector spaces  $A \otimes B \rightarrow C$ , where  $A, B$  and  $C$  have dimensions  $a, b$  and  $c$  respectively. Such a map can be viewed in several ways, for example as a map  $C^* \otimes B \rightarrow A^*$ —in other words, a  $c \times b$  matrix in  $a$  variables—or equivalently an  $a$ -dimensional family of  $c \times b$  matrices (and similarly for other permutations of  $A, B, C$ ). This bit of trivial formalism pays off in the following observation:

**Proposition 2.2.** *An  $a \times b$  matrix of linear forms in  $c$  variables  $M$  corresponding to  $A \otimes B \rightarrow C$  is 1-generic if and only if the  $c \times b$  matrix  $N$  of linear forms in  $a$  variables corresponding to  $C^* \otimes B \rightarrow A$  has constant rank  $b$ ; that is, for point  $x$  in  $\mathbb{P}^{a-1}$ , the rank of  $N$  evaluated at  $x$  is  $b$ .*

**Proof.** The  $b$  columns of  $N$  correspond to the  $b$  columns of  $M$ , while the rows of  $N$  are indexed by the  $c$  variables in  $M$  and the variables in  $N$  are indexed by the  $a$  rows of  $M$ . Thus the evaluation of  $N$  at a point  $x$  corresponds to a generalized row of  $M$ ; and the image of  $N(x)$  is the span of the variables in that generalized row. The matrix  $M$  is 1-generic if the dimension of that span is  $b$  for every generalized row.  $\square$

Thus the classification of 1-generic  $2 \times f$  matrices of linear forms in  $r + 1$  variables of Theorem 2.1 is equivalent to the classification of *matrix pencils* of constant maximal rank—that is  $f \times (r + 1)$  matrices of linear forms over  $\mathbb{P}^1$

with constant rank  $r$ . Such ‘‘matrix pencils’’ were first classified by Kronecker; see [Gantmacher 1959, Vol. II, Chapter 12] for an exposition, and [Eisenbud and Harris 1987c] for a geometric approach. We will give a still simpler proof:

**Proof of Theorem 2.1.** Let  $M$  be a 1-generic  $2 \times b$  matrix in  $r+1$  variables. We may assume that the span of the entries is equal to the vector space of linear forms in  $\mathbb{P}^r$ . The associated  $(r+1) \times b$  matrix

$$N : \mathcal{O}_{\mathbb{P}^1}(-1)^b \rightarrow \mathcal{O}_{\mathbb{P}^1}^{r+1}$$

of linear forms over  $\mathbb{P}^1$  has constant rank  $b$ . For every point  $x \in \mathbb{P}^1$  the scalar matrix  $N(x)$  is a split inclusion, and thus  $\text{coker } N$  is a vector bundle, necessarily isomorphic to  $\sum_{i=1}^d \mathcal{O}_{\mathbb{P}^1}(a_i)$  for some integers  $a_i \geq 0$ .

Regarded as a matrix over  $\mathbb{C}[s, t]$  it follows that the exact sequence

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^1}(-1)^b \xrightarrow{N} \mathcal{O}_{\mathbb{P}^1}^{r+1} \longrightarrow \bigoplus_{i=1}^d (\bigoplus_{j=0}^{\infty} H^0(\mathcal{O}_{\mathbb{P}^1}(j - a_i))) \longrightarrow 0$$

is a minimal free resolution. It follows that the map  $N$  is the direct sum of the minimal free resolutions over  $\mathbb{C}[s, t]$  of the modules

$$\bigoplus_{j=0}^{\infty} H^0(\mathcal{O}_{\mathbb{P}^1}(j - a_i)) = (s, t)^{a_i}.$$

As we will explain in Example 4.4, the minimal presentation of  $(s, t)^{a_i}$  is the  $(a_i + 1) \times a_i$  matrix

$$\begin{pmatrix} s & 0 & 0 & \dots & 0 \\ -t & s & 0 & \dots & 0 \\ 0 & -t & s & \dots & 0 \\ 0 & 0 & -t & \dots & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & & \dots & \ddots & s \\ 0 & & \dots & \dots & -t \end{pmatrix}$$

The direct sum decomposition of  $N$  corresponds to a block decomposition of  $M$ , and thus after a change of bases and variables the matrix  $N$  is transformed into the direct sum of these matrices. Translating this form back to a  $2 \times (r+d-1)$  matrix  $M$ , we have transformed  $M$  into a matrix of the desired form.  $\square$

**2.1. How to look at a resolution.** As is apparent even in the example  $t = 3$  above, free resolutions can be bulky to describe; the *Betti table* is a compact representation of the numerical information in the resolution. Suppose that  $F$  is a minimal free resolution of a module  $M$  as illustrated in the beginning of the previous section. Since we choose a minimal set of generators at each stage, the matrices of the  $\phi_i$  have entries in the maximal ideal  $(x_0, \dots, x_r)$ , and thus each

$\beta_{i+1,j}$  must be strictly greater than some  $\beta_{i,j}$ . For this reason it is convenient to tabulate the numbers so that  $\beta_{i,j}$  is in the  $i$ -th column and  $(j-i)$ -th row:

$j \setminus i$	0	1	$\cdots$	$n$
:	:	:	$\cdots$	:
0	$\beta_{0,0}$	$\beta_{1,1}$	$\cdots$	$\beta_{n,n}$ .
1	$\beta_{0,1}$	$\beta_{1,2}$	$\cdots$	$\beta_{n,n+1}$
:	:	:	$\cdots$	:

**Example 2.3.** The Koszul complex that resolves the homogeneous coordinate ring  $S/(Q, F_1, F_2)$  of the complete intersection of 2 quadrics and a cubic in  $\mathbb{P}^3$  has the form

$$S \longleftarrow S(-2) \oplus S(-3)^2 \longleftarrow S(-5)^2 \oplus S(-6) \longleftarrow S(-8) \longleftarrow 0,$$

which has Betti table:

$j \setminus i$	0	1	2	3
0	1	—	—	—
1	—	1	—	—
2	—	2	—	—
3	—	—	2	—
4	—	—	1	—
5	—	—	—	1

To simplify our language, we will speak of the *Betti table of a scheme  $X$*  rather than the “Betti table of the minimal free resolution of the ideal of  $X$ ”.

**2.2. When is a finite free complex a resolution?** How does a free resolution over  $S := \mathbb{C}[x_0, \dots, x_r]$  “know” to end no later than the  $(r+1)$ -st step? The following result describes a sense in which the maps in the resolution change as the resolution continues. The result is proven in [Eisenbud 1995a, Theorem 20.9].

A central role in the theorem is played by the ideals of minors of the differentials in the complex: If  $\phi : F \rightarrow G$  is a map of finitely generated free  $R$ -modules with cokernel  $M$ , then  $I_t(\phi)$  denotes the ideal generated by all the  $t \times t$  minors (=subdeterminants) of a matrix representing  $\phi$ ; this is independent of the choice of bases used to represent  $\phi$  as a matrix. The ideal  $I_{\text{rank } G-j}(\phi)$  depends only on  $M$ ; it called the  $j$ -th Fitting ideal of  $\text{coker } \phi$  and usually written  $\text{Fitt}_j(M)$ . Some basic properties of these ideals are given in Exercise 6.12, where the reader may show that the annihilator of  $M$  has the same radical as  $\text{Fitt}_0(M)$ . Actually more is true:

**Cheerful Fact 2.4.** *If  $F \rightarrow G \rightarrow M \rightarrow 0$  is an exact sequence of finitely generated  $R$ -modules, then*

$$\text{ann}_R(\text{coker } \phi)^{\text{rank } G} \subset \text{Fitt}_0(\phi) \subset \text{ann}_R(\text{coker } \phi).$$

*For this and other such inequalities, see [Buchsbaum and Eisenbud 1977b].*

The *rank* of  $\phi : F \rightarrow G$  is the largest size of a nonvanishing minor of a matrix for  $\phi$ , or equivalently the largest  $k$  such that the exterior power  $\wedge^k \phi : \wedge^k F \rightarrow \wedge^k G$  is nonzero. If  $r := \text{rank } \phi = \text{rank } G$ , then the ideal  $I(\phi) := I_{\text{rank } \phi}(\phi)$  plays a special role: the cokernel of  $\phi$  is projective (= locally free) if and only if  $I(\phi) = R$ . If  $I(\phi)$  contains a non-zerodivisor, then the construction localizes, and we see that  $I(\phi)$  defines the locus  $P \in \text{Spec } R$  where  $(\text{coker } \phi)_P$  is not free.

Recall that the grade of an ideal is the length of a maximal regular sequence contained in it, or  $\infty$  if  $I = R$ . If  $R$  is a Cohen-Macaulay ring—for example  $\mathbb{C}[x_0, \dots, x_r]$ —then the grade of any proper ideal is equal to its codimension, so grade becomes a geometric notion. Readers less familiar with commutative algebra will lose little if they stick with the case when  $R$  is regular, or even the case when  $R$  is  $\mathbb{C}[x_0, \dots, x_r]$ . This suffices, for example for the applications of the Eagon-Northcott complex described below.

**Theorem 2.5.** *Let  $R$  be a Noetherian ring and let*

$$\mathbb{F} : F_0 \xleftarrow{\phi_1} F_1 \longleftarrow \cdots \longleftarrow F_{n-1} \xleftarrow{\phi_n} F_n \longleftarrow 0$$

*be a finite complex of free  $S$ -modules. Set  $r_i := \text{rank } \phi_i$ . The complex  $\mathbb{F}$  is acyclic (that is,  $H_i(\mathbb{F}) = 0$  for all  $i > 0$ ) if and only if*

- (1)  $\text{rank } F_i = r_i + r_{i+1}$ ; and
- (2)  $\text{grade } I_{r_i}(\phi_i) \geq i$ .

*for all  $i$ .*

□

In particular, if  $R$  is Noetherian, then a map of finitely generated free modules  $G \xleftarrow{\phi} F$  is injective if and only if  $I_{\text{rank } F}(\phi)$  contains a non-zerodivisor.

A familiar case occurs when  $r = 1$  and  $R$  is a domain. In this case the theorem says that a map  $F_1 \rightarrow F_0$  is a monomorphism iff it becomes a monomorphism after tensoring with the field of rational functions  $K$ , which follows from the flatness of localization and the fact that  $F_1$  is torsion-free, so that  $F_1 \subset F_1 \otimes K$ .

To see the relevance of the second hypothesis to the conclusion, suppose for a moment that  $R$  is a regular local ring of dimension  $r$ , and suppose that the complex  $\mathbb{F}$  is acyclic. The hypothesis  $\text{grade } I(\phi_{d+1}) \geq d+1$  can only be satisfied if  $I(\phi_{d+1}) = R$  (so that its grade is  $\infty$  by convention). This is equivalent to the cokernel of  $\phi_{d+1}$  being free. Thus the theorem “explains” why a minimal free resolution has length  $\leq r+1$ .

### 3. Depth and the Cohen-Macaulay property

If  $M$  is a graded  $\mathbb{C}[x_0, \dots, x_r]$ -module then an  $M$ -regular sequence is a sequence of homogeneous polynomials  $f_1, \dots, f_m \in (x_0, \dots, x_r)$  such that  $f_1$  is a non-zerodivisor on  $M$ ,  $f_2$  is a non-zerodivisor on  $M/(f_1M)$ , and so on. The maximal length of such a sequence is called the *depth* of  $M$ , or more properly the depth of  $(x_0, \dots, x_r)$  on  $M$ . The length of any all maximal  $M$ -regular sequences are the same, as one shows by proving

**Theorem 3.1.** (Auslander-Buchsbaum) *If  $M$  is a finitely generated  $S := \mathbb{C}[x_0, \dots, x_r]$ -module, then the length of every  $M$ -regular sequence is the smallest integer  $m$  such that  $\text{Ext}_S^m(S/(x_0, \dots, x_r), M) \neq 0$  and is also  $r + 1 - \text{pd } M$ .*

The depth of a module  $M$  is bounded above by  $\dim M$ , the Krull dimension. The reason is that if the dimension of  $M$  is  $d$ , and  $f_1 \in (x_0, \dots, x_r)$  is a non-zerodivisor on  $M$ , then  $\dim M/(f_1M) = \dim M - 1$ . Thus by induction, if  $f_1, \dots, f_d$  is  $M$ -regular then  $M/(f_1, \dots, f_d)M$  has dimension 0, which is equivalent to its being Artinian. Thus any  $f_{d+1} \in (x_0, \dots, x_r)$  acts as a nilpotent endomorphism of  $M/(f_1, \dots, f_d)M$ .

It follows from these facts that the depth of an  $S$ -module  $M$  is equal to the dimension of  $M$  if and only if the projective dimension of  $M$  is equal to the codimension of  $M$ ; in this case we say that  $M$  is a *Cohen-Macaulay module*.

As we showed in Chapter 16, a curve  $C \subset \mathbb{P}^3$  is linked to a complete intersection if and only if  $H_*^1(\mathcal{I}_C) := \bigoplus_{m \in \mathbb{Z}} H^1(\mathcal{I}_C(m)) = 0$ , in which case  $C$  is said to be arithmetically Cohen-Macaulay. From the Auslander-Buchsbaum theorem and Theorem 3.4 we see that  $C$  is arithmetically Cohen-Macaulay if and only if the homogeneous coordinate ring  $R_C$  is Cohen-Macaulay, and this is true if and only if  $\text{pd } R_C = \text{codim } C$ .

**3.1. The Gorenstein property.** Another important homological condition is the condition that  $\omega_X$  is an invertible sheaf; when this holds, we say that  $X$  is *quasi-Gorenstein*. When, in addition,  $X$  is Cohen-Macaulay we say that  $X$  is *Gorenstein*. Any scheme that is locally a complete intersection, such as any smooth scheme, is Gorenstein. Since the restriction of  $\mathcal{O}_{\mathbb{P}^r}(1)$  to a subvariety is always invertible, saying that a scheme  $X$  is canonically embedded implies that  $X$  is at least quasi-Gorenstein. As with the Cohen-Macaulay property, the Gorenstein property is interpreted locally on a scheme. We say that a projective scheme  $X$  is *arithmetically Gorenstein* if its homogeneous coordinate ring is Gorenstein, and it follows that  $\omega_{S/I} \cong S/I(a)$  for some integer  $a = a(X)$ .

In Chapter 16, we expressed  $\omega_X$  for a subscheme  $X \subset Y$  of a scheme  $Y$  as  $\text{Ext}_{\mathcal{O}_Y}^{\text{codim } X}(\mathcal{O}_X, \omega_Y)$ . Slightly extending this idea, if  $C \subset \mathbb{P}^r$  is a curve with homogeneous coordinate ring  $R_C$ , we define  $\omega_{R_C}$  to be  $\text{Ext}_S^{r-1}(R_C, S(-r-1))$ ,

where  $S$  is the homogeneous coordinate ring of  $\mathbb{P}^r$ . Since  $\omega_{\mathbb{P}^r} = \mathcal{O}_{\mathbb{P}^r}(-r - 1)$ , the sheafification of this module is  $\omega_C$ .

If  $C$  is arithmetically Cohen-Macaulay, so that  $\text{pd}(C) = \text{codim}(C) = r - 1$ , then computing  $\text{Ext}_S^{r-1}(R_C, S(-r - 1))$  from the minimal free resolution

$$(\mathbb{F}, \phi) : 0 \longleftarrow R_C \longleftarrow S \xleftarrow{\phi_1} F_1 \longleftarrow \dots \longleftarrow F_{r-2} \xleftarrow{\phi_{r-1}} F_r \longleftarrow 0$$

we see that  $\omega_{R_C} = \text{coker } \phi_{r-1}^*$ . Theorem 2.5 implies that the complex  $(\mathbb{F}^*, \phi^*)$  which is the dual of the resolution  $(\mathbb{F}, \phi)$  is again acyclic, so it is the minimal free resolution of  $\omega_{R_C}$ . Thus  $\omega_{R_C}$  is a Cohen-Macaulay module. Just as the Cohen-Macaulay property of  $R_C$  implies that  $R_C = H_*^0(\mathcal{O}_C)$ , it follows that  $\omega_{R_C} = H_*^0(\omega_C)$ .

If, in addition,  $C$  is a canonical curve, so that  $\omega_C = \mathcal{O}_C(1)$ , then we derive:

$$\omega_{R_C} = \text{coker } (\phi_{r-2}^*)(-r - 1) = R_C(1)$$

so  $\text{coker } \phi_{r-2}^* = R_C(r)$ . Thus  $(\mathbb{F}^*(-r), \phi^*)$  is a minimal free resolution of  $R_C$ , and is therefore isomorphic to  $(\mathbb{F}^*, \phi^*)$ ; that is,  $(\mathbb{F}^*, \phi^*)$  is self-dual. We have seen an example already in the Koszul complex (a complete intersection is arithmetically Gorenstein).

Taking into account that in a resolution  $(\mathbb{F}, \phi)$  each summand  $S(-j)$  of  $F_{i+1}$  can only map to summands  $S(-l)$  of  $F_i$  with  $\ell < j$ , and similarly for the dual, we see that the Betti table of the minimal free resolution of a canonical curve of genus  $g$  must have the form:

$j \setminus i$	0	1	2	$\dots$	$g-3$	$g-2$
0	1	–	–	$\dots$	–	–
1	–	$b_1$	$b_2$	$\dots$	$b_{g-3}$	–
2	–	$b_{g-3}$	$b_{g-4}$	$\dots$	$b_1$	–
3	–	–	–	$\dots$	–	1

where  $–$  represents 0. It turns out that the  $b_i$  depend on the particular canonical curve, but since the Hilbert function of the curve is the alternating sum of the Hilbert functions in the resolution, the differences  $b_i - b_{g-i-1}$  are independent of the curve.

To go further, we can make use of the invertible sheaves  $\mathcal{L}$  on  $C$  with  $h^0(\mathcal{L}) = 2$  and  $h^1(\mathcal{L}) \geq 2$ . The existence of such a series guarantees that the ideal of  $C$  contains the ideal of  $2 \times 2$  minors of the  $2 \times h^1(\mathcal{L})$  matrix corresponding to the multiplication map  $H^0(\mathcal{L}) \otimes H^0(\mathcal{L}^{-1} \otimes \omega_C) \rightarrow H^0(\omega_C) = H^0(\mathcal{O}_C(1))$  as in Chapter 17. The mechanism is a resolution of the ideal generated by the minors of this matrix. ■

#### 4. The Eagon-Northcott complex

The Eagon-Northcott complex  $EN(\phi)$  [Eagon and Northcott 1962] associated with a matrix, or a map of free modules  $\phi : F \rightarrow G$ , is a generalization of

the Koszul complex, which is the case  $\text{rank } G = 1$ . Like the Koszul complex, it is tautological: its existence depends only on the properties of commutative rings; and like the Koszul complex it is exact or not depending on a property of the matrix  $\phi$  related to regular sequences. It is part of a family of complexes described in [Eisenbud 1995a, Appendix A2], and, from a more conceptual and general point of view, in [Weyman 2003].

We are interested in  $EN(\phi)$  because its shape influences the shape of the free resolutions of canonical curves in an interesting way, leading to Green's conjecture. This conjecture, one of the central open problems in the theory of algebraic curves, is described in the last section of this chapter. We will also use the the Eagon-Northcott complex, in a special case, to give a proof of the classification of matrix pencils and an analysis of the ideals of ACM curves in  $\mathbb{P}^3$ .

To prepare for the description of the Eagon-Northcott complex we revisit the Koszul complex, which is equal to the Eagon-Northcott complex in the case  $\text{rank } G = 1$ , and then consider the case  $\text{rank } F = \text{rank } G + 1$ .

**4.0.1. rank  $G = 1$ .** In this subsection we let  $\phi : F = R^f \rightarrow R$  be a homomorphism from a free module to a ring  $R$ , and we will analyze the Koszul complex  $K(\phi)$ .

We may write  $K(\phi) = EN(\phi)$  in the form

$$S \xleftarrow{\delta_1} F \xleftarrow{\delta_2} \wedge^2 F \xleftarrow{\delta_3} \dots \xleftarrow{\delta_f} \wedge^f F \xleftarrow{} 0$$

where  $\delta_1 = \phi$ .

To define the complex, we must construct the differentials  $\delta_i$  and prove that  $\delta_i \delta_{i+1} = 0$ . Since the modules are free, it suffices to do this for the dual maps

$$\partial_i : \wedge^i F^* \rightarrow \wedge^{i+1} F^*,$$

and it turns out that this is in a sense even more natural.

It is convenient to think of  $R$  as an  $S := \mathbb{Z}[x_1, \dots, x_f]$ -algebra by the map sending  $x_i$  to  $\phi_i$ ; we define the Koszul complex of  $\phi$  over  $R$  by tensoring the Koszul complex of  $(x_1, \dots, x_f)$  with  $R$ .

Thus for the definition we take the map  $\phi$  to be

$$\phi : S^f \xrightarrow{(x_0 \ \dots \ x_f)} S.$$

First of all, the map  $\partial_i$  (like the map  $\delta_i$ ) is *linear*: the image of a basis vector of  $\wedge^i F^*$  is a sum of variables times basis vectors of  $\wedge^{i+1} F^*$ . We may write  $S$  as  $\text{Sym}(V)$ , where  $V$  is the free  $\mathbb{Z}$ -module generated by  $x_1, \dots, x_f$ , and we may think of  $F$  as the module  $V \otimes S$  with the map  $\phi$  sending  $V \otimes 1 \subset F$  by the

identity to  $V = S_1 \subset S$ —the “tautological map”. Let  $t \in V \otimes V^* \subset S \otimes \wedge V^*$  be the “trace element” represented in terms of any basis  $\{x_i\}$  of  $V$  and dual basis  $\{\hat{e}_i\}$  of  $V^*$  as  $t = \sum x_i \otimes \hat{e}_i$ . Because  $\wedge V^*$  is an anti-commutative algebra, we have  $t^2 = 0$ .

We define the map

$$\partial_i : S \otimes_{\mathbb{C}} \wedge^i V^* = \wedge^i F^* \rightarrow \wedge^{i+1} F^* = S \otimes_{\mathbb{C}} \wedge^{i+1} V^*$$

to be multiplication by  $t$ , and thus  $\partial_{i+1} \partial_i$  is multiplication by  $t^2 = 0$ .

Having defined the complex  $K(\phi) = EN(\phi)$  in the case  $\text{rank } G = 1$ , we next ask what conditions on  $\phi$  make it acyclic (that is, a free resolution of  $\text{coker}(\delta_1)$ ).

**Theorem 4.1.** *Suppose that  $R$  is a ring and  $\phi : F \rightarrow R$  is a map from a free  $R$ -module of rank  $f$ . The complex  $K(\phi)$  is acyclic if and only if the ideal  $I := I_1(\phi)$  has grade  $\geq f$ .*

**Proof.** Theorem 2.5 immediately implies that if  $K(\phi)$  is acyclic then  $\text{rank } \delta_f = 1$  and  $\text{grade } I(\delta_f) \geq f$ . Since  $I(\delta_f) = I$ , this proves one implication.

Now assume that  $\text{grade } I \geq f$ . We first prove that  $K(\phi)$  is split exact when  $I = R$ . The condition  $I = R$  implies that  $\delta_1$  is a split injection. In this case we may write  $F^* = R \oplus F'^*$  in such a way that  $\delta_1$  is the injection into the first summand, and we may choose a basis  $\{\hat{e}_i\}$  of  $F^*$  so that the last  $f - 1$  basis elements are a basis for  $F'^*$ . Specializing the sequence  $x_1, x_2, \dots, x_f$  to the sequence  $1, 0, \dots, 0$ , the differential of  $K(\phi)^*$  becomes the multiplication by  $1 \otimes e_1$ .

The module  $\wedge^i F$  decomposes as

$$\wedge^i F = (Re_1 \otimes_R \wedge^{i-1} F^*) \oplus \wedge^i F'^*.$$

Because  $e_1 \wedge e_1 = 0$  the differential  $\partial_i$  has the form

$$\begin{array}{ccccccc} & & Re_1 \otimes \wedge^{i-2} F'^* & \xrightarrow{0} & Re_1 \otimes \wedge^{i-1} F'^* & & \\ & & \oplus & \nearrow \cong & \oplus & = & \wedge^i F^*. \\ \wedge^{i-1} F^* & = & & & & & \\ & & \wedge^{i-1} F'^* & \xrightarrow{0} & \wedge^i F'^* & & \end{array}$$

Thus we see that  $K(\phi)$  is split exact when  $\phi$  is a split surjection.

We now assume only that  $\text{grade } I \geq f$ . From what we just proved we see that if we localize  $R$  by inverting any element of  $I$  the complex  $K(\phi)$  becomes split exact. Since  $\text{grade } f \geq 1$ , we can find such a non-zerodivisor in  $I$ , and inverting it does not change the ranks of the maps  $\phi_i$ . Because ranks of free modules are additive in direct sums, it is obvious that in the split exact case the

condition on the ranks of the  $\phi_i$  is satisfied; more precisely,  $\text{rank}(\delta_i) = \binom{i-1}{f-1}$ . We also see that after inverting a non-zerodivisor in  $I(\delta_1)$  we have  $I(\delta_i) = R$ ; equivalently,

$$I \subset \sqrt{I(\delta_i)}.$$

(In fact  $I(\delta_i) = I^{(i-1)}_{(f-1)}$ , though this requires a separate argument.) Thus if  $\text{grade } I = f$  then  $\text{grade } I(\delta_i) \geq i$  for all  $i$ , so  $K(\phi)$  is acyclic.  $\square$

**4.0.2. rank F = rank G + 1.** We set  $g = \text{rank } G$  and  $f = \text{rank } F = g + 1$ . In this case the Eagon-Northcott complex has the form:

$$EN(\phi) : 0 \longrightarrow G^* \otimes \wedge^f F \xrightarrow{\delta_2} \wedge^{f-1} F \xrightarrow{\delta_1} \wedge^g G.$$

Here  $\delta_1 = \wedge^g \phi$ , so that the entries of a matrix for  $\delta_1$  are the  $g \times g$  minors of  $\phi$ .

We choose an identification  $\wedge^f F = S$ , called an *orientation* of  $F$ , and get a perfect pairing

$$\wedge^g F \times F \rightarrow \wedge^f F = S$$

so that we may identify  $\wedge^g F$  with  $F^*$ . With this identification, we define  $\delta_2$  as

$$\delta_2 : G^* \xrightarrow{\phi^*} F^* = \wedge^g F.$$

We also choose an orientation  $\wedge^g G = S$ , in terms of which the image of  $\delta_1$  is the ideal generated by the  $(f-1) \times (f-1)$  minors of  $\phi$ .

We first claim that  $EN(\phi)$  is a complex; that is,  $\delta_1 \delta_2 = 0$ . As with the Koszul complex, it is convenient to dualize and consider the maps

$$EN(\phi)^* : 0 \longrightarrow \wedge^g G \rightarrow \wedge^g F^* = F \xrightarrow{\phi} G$$

The fact that this composition is 0 is often taught as Cramer's rule for solving a system of homogeneous equations represented by a  $g \times (g+1)$  matrix of rank  $g$ . The solutions—that is, the elements of  $\ker \phi$ —are multiples of the column  $\Delta_1, \dots, \Delta_g$  where the  $\Delta_j$  is  $-1^j$  times the determinant of the matrix obtained from  $\phi$  by leaving out the  $j$ -th column. This works because the composition of the two maps is a column matrix whose  $i$ -th entry is the expansion of the  $(g+1) \times (g+1)$  determinant of the matrix obtained from  $\phi$  by repeating the  $i$ -th row.

**Theorem 4.2.** Suppose that  $R$  is a ring and  $\phi : F \rightarrow G$  is a map of free  $R$ -modules, where  $G$  has rank  $g$  and  $F$  has rank  $f = g + 1$ . The complex  $EN(\phi)$  is acyclic if and only if the ideal  $I := I_g(\phi)$  has grade  $\geq 2$ .

**Example 4.3.** We have seen that the ideal of the twisted cubic is generated by the  $2 \times 2$  minors of the matrix

$$\phi := \begin{pmatrix} x_0 & x_1 & x_2 \\ x_1 & x_2 & x_3 \end{pmatrix}$$

and it follows that the free resolution of its homogeneous coordinate ring is the Eagon-Northcott complex

$$0 \longrightarrow S^2(-3) \xrightarrow{\begin{pmatrix} x_0 & x_1 \\ x_1 & x_2 \\ x_2 & x_3 \end{pmatrix}} S^3(-2) \xrightarrow{\wedge^2 \phi} S$$

**Example 4.4.** In Section 2 we asserted that the  $(a+1) \times a$  matrix

$$\phi_a := \begin{pmatrix} s & 0 & 0 & \dots & 0 \\ -t & s & 0 & \dots & 0 \\ 0 & -t & s & \dots & 0 \\ 0 & 0 & -t & \dots & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & \dots & \ddots & s & \\ 0 & \dots & \dots & -t & \end{pmatrix}$$

is the minimal presentation of the ideal  $(s^a, s^{a-1}t, \dots, t^a) \subset R := \mathbb{C}[s, t]$ . It is not hard to check this directly, but in any case it's easy to see that its  $a \times a$  minors of  $\phi_a$  generate this ideal, which has grade 2, so the Eagon-Northcott resolution  $EN(\phi)$  has the form

$$R \xleftarrow{\wedge^a \phi_a} R^{a+1} \xleftarrow{\phi_a} R^a \longleftarrow 0$$

verifying the assertion.

**Proof.** Once having shown that  $EN(\phi)$  is a complex, as we did above, the proof of the equivalence in the theorem follows the same pattern as the proof given above for the Koszul complex.

If  $EN(\phi)$  is acyclic, then by Theorem 2.5 the  $g \times g$  minors of  $\phi = \delta_2$  must have grade  $\geq 2$ . For the converse, suppose first that  $I_g(\phi)$  is the unit ideal. We may split  $F$  as  $S \oplus G$  with  $\Delta_1 = 1$  and  $\Delta_j = 0$  for  $j > 1$ , and then  $EN(\phi)^*$  has the form:

$$\begin{array}{ccccccc} & & & & 0 & & \\ & & & & \nearrow & & \\ 0 & \longrightarrow & G & \longrightarrow & S & \longrightarrow & 0 \\ & & \oplus & \cong & \oplus & \cong & \oplus \\ & & \downarrow & & \downarrow & & \downarrow \\ & & G & \xrightarrow{0} & S & \longrightarrow & 0 \end{array}$$

Thus we see that  $EN(\phi)$  is split exact in this case.

We now apply Theorem 2.5: From what we just proved we see that if we localize  $S$  by inverting any element of  $I$  then the complex  $EN(\phi)$  becomes split exact, and therefore, before localizing,  $\text{rank } \delta_1 = 1$  and  $\text{rank } \delta_2 = g$ . In this case it follows from the definition that  $I_1(\delta_1) = I_g(\phi) = I_g(\delta_2)$  so if  $I_g(\phi)$  has grade 2 then both conditions of Theorem 2.5 are satisfied.  $\square$

**4.1. The Hilbert-Burch theorem.** In a regular local ring any ideal of codimension 1 is principal (divisors are all Cartier). What about ideals of codimension 2? The answer is the content of the *Hilbert-Burch theorem*, proven in 1890 by David Hilbert in the case of homogeneous ideals in  $\mathbb{C}[x_0, x_1]$  and in general by Lindsay Burch [Burch 1967]. We can deduce it as an application of the Eagon-Northcott complex in the case  $f = g + 1$ :

**Corollary 4.5** (Hilbert-Burch theorem). *Suppose that  $R$  is a local ring. Any ideal  $I \subset R$  of projective dimension 1 and grade 2 has the form  $aI'$  where  $I'$  is an ideal of grade 2 generated by the  $g \times g$  minors of a  $g \times (g+1)$  matrix and  $a$  is a non-zerodivisor of  $R$ ; and conversely any ideal of this form has projective dimension 1.*

*In particular, if  $C \subset \mathbb{P}^3$  is an ACM curve whose homogeneous ideal  $I$  is generated by  $f$  elements, then  $I$  is minimally generated by the  $(f-1) \times (f-1)$  minors of the syzygy matrix of  $I$ .*

**Proof.** If  $C \subset \mathbb{P}^3$  is ACM, then the projective dimension of the homogeneous coordinate ring  $R_C$  is 2 by the Auslander-Buchsbaum theorem, and thus the ideal of  $C$  has projective dimension 1.

Now suppose that  $I \subset R$  is an ideal of projective dimension 1 in any local ring, and suppose that  $I$  is generated by  $f$  elements, so that we have a surjection  $F := R^f \rightarrow I$ . The module  $R/I$  has free resolution of the form

$$\mathbb{F} : R \xleftarrow{\alpha} R^f \xleftarrow{\phi} G \xleftarrow{} 0$$

where  $I = I_{1(\alpha)}$ , so by Theorem 2.5 the free module  $G$  has rank  $g = f - 1$ , the  $g \times g$  minors of  $\phi$  generate an ideal  $I'$  of grade  $\geq 2$ , and the ideal  $I$  has grade  $\geq 1$ . Theorem 2.5 implies that both the Eagon-Northcott complex  $EN(\phi)$  and its dual are acyclic.

The dual of the complex  $\mathbb{F}$  will not be acyclic unless grade  $I = 2$ , but there is at least a comparison map

$$\begin{array}{ccccccc} \mathbb{F}^* : & G^* & \xleftarrow{\phi^*} & G^* & \xleftarrow{\alpha^*} & R & \xleftarrow{} 0 \\ & \downarrow = & & \downarrow = & & \downarrow a & \\ EN(\phi)^* : & G^* & \xleftarrow{\phi^*} & F^* & \xleftarrow{\wedge^g \phi^*} & R & \xleftarrow{} 0 \end{array}$$

It follows that  $I = aI'$ , and since  $I$  has grade 1,  $a$  must be a non-zerodivisor.

Conversely, if  $\phi : R^f \rightarrow R^g$  is a map with  $f = g + 1$  and grade  $I_g(\phi) \geq 2$ , then the acyclicity of  $EN(\phi)$  shows that  $I_g(\phi)$  has projective dimension 1; and if  $a$  is a non-zerodivisor, then  $I := aI_g(\phi) \cong I_g(\phi)$  as  $R$ -modules, so  $I$  has projective dimension 1 as well.  $\square$

This argument applies, for example to the case of a non-hyperelliptic curve of genus 3 and degree 6 in  $\mathbb{P}^3$ , discussed in Section 4.1.

**4.2. The general case of the Eagon-Northcott complex.** With these two special cases in mind, we are ready for the general case. First the definition: If  $\phi : F \rightarrow G$  is a map of free  $S$ -modules with  $f := \text{rank } F \geq g := \text{rank } G$  then there is a complex of free  $S$ -modules

$$\begin{aligned} EN(M) := S &\xleftarrow{\delta_1 := \wedge^g \phi} \bigwedge^g F \xleftarrow{\delta_2} G^* \otimes \bigwedge^{g+1} F \xleftarrow{\delta_3} (\text{Sym}^2 G)^* \otimes \bigwedge^{g+2} F \\ &\quad \xleftarrow{\delta_4} \dots \xleftarrow{\delta_{f-g+1}} (\text{Sym}^{f-g} G)^* \otimes \bigwedge^f F \xleftarrow{f} 0 \end{aligned}$$

where:

- (1) After identifying  $\wedge^g G$  with  $S$ , the map  $\delta_1$  is identified with  $\wedge^g \phi$ .
- (2) It is convenient to give a formula for  $\delta_i = \delta_i^*$  by taking advantage of the algebra structures of  $\text{Sym}(G)$  and  $\wedge F^*$ . To do this, choose dual bases  $\{e_i\}$  and  $\{\hat{e}_i\}$  for  $F$  and  $F^*$ . In these terms

$$\delta_{i^*} = \delta_i : \text{Sym}^{i-2} G \otimes \wedge^{g+i-2} F^* \rightarrow \text{Sym}^{i-1} G \otimes \wedge^{g+i-1} F^*$$

is multiplication by the element  $\sum_{i=1}^f \phi(e_i) \otimes \hat{e}_i$ .

To show that  $\delta_1 \delta_2 = 0$  is almost the same as in the case  $f = g + 1$  because a basis element

$$b := e_{i_1} \wedge \dots \wedge e_{i_{g+1}} \in \wedge^{g+1} F$$

can be thought of as coming from a rank  $g + 1$  summand of  $F$ , and the value of  $\delta_2 \delta_1 b$  is the same as it would be if  $F$  were replaced by this summand. Thus from the case  $f = g + 1$  we see that  $\delta_1 \delta_2 b = 0$ , and thus  $\delta_2 \delta_1 = 0$ .

On the other hand, for  $i \geq 1$  the map  $\delta_i \delta_{i+1}$  is multiplication by

$$\left( \sum_{i=1}^f \phi(e_i) \otimes \hat{e}_i \right)^2,$$

which we may think of as the square of an element of degree 1 in the exterior algebra of the free  $\text{Sym}(G)$ -module  $\wedge(\text{Sym}(G) \otimes F)$ , and hence this square is 0. Thus the given maps do define a complex.

**Theorem 4.6.** Suppose that  $S$  is a ring and  $\phi : F \rightarrow G$  is a map of free  $R$ -modules, where  $G$  has rank  $g$  and  $F$  has rank  $f \geq g$ . The complex  $EN(\phi)$  is acyclic if and only if the ideal  $I := I_g(\phi)$  has grade  $\geq f - g + 1$ .

**Example 4.7.** If  $\phi$  is a matrix of linear forms, then the first map of  $EN(\phi)$  is represented by the row of  $g \times g$  minors of  $\phi$ , which are forms of degree  $g$ , but all the rest of the maps are represented by matrices of linear forms. Thus, for

example, the Betti table of the Eagon-Northcott complex of a  $2 \times f$  matrix of linear forms is:

$j \setminus i$	0	1	2	3	$f-1$
0	1	—	—	...	—
1	—	$\binom{f}{2}$	$2\binom{f}{3}$	...	$(f-1)\binom{f}{f}$

**Proof.** The dual of the last differential of  $EN(\phi)$  is

$$\partial_{f-g+1} : \text{Sym}^{f-g-1} G \otimes \wedge^{f-1} F^* \rightarrow \text{Sym}^{f-g}(G) \otimes \wedge^f F^*.$$

With our usual identifications  $\wedge^f F^* = S$  and  $\wedge^{f-1} F^* = F$  this becomes the map

$$\text{Sym}^{f-g-1} G \otimes F \xrightarrow{1 \cdot \phi} \text{Sym}^{f-g} G$$

whose cokernel is  $\text{Sym}^{f-g}(\text{coker } \phi)$  by the right exactness of the symmetric algebra functor [Eisenbud 1995a, Proposition A2.2]. The support of  $\text{Sym}^{f-g}(\text{coker } \phi)$  is obviously contained in the support of  $\text{coker } \phi$ , but in fact they are equal: if a localization of  $\text{coker } \phi$ , over a local ring  $S_P$ , is nonzero, then by Nakayama's lemma it surjects onto  $S_P/P_P = \kappa(P)$ , and again by the right exactness of the symmetric algebra functor  $\text{Sym}^{f-g}(\text{coker } \phi)_P$  surjects onto  $\text{Sym}^{f-g}(\kappa(P)) = \kappa(P)$ .

By Theorem 2.5 we see from this that if  $EN(\phi)$  is acyclic, then the support of  $\text{coker } \phi$  has grade  $\geq f - g + 1$ . This support is defined by the radical of  $I_g(\phi)$ , so  $\text{codim } I_g(\phi) \geq f - g + 1$  as required.

Conversely, to show that  $EN(\phi)$  is acyclic under the given hypothesis we first treat the case  $I_g(\phi) = S$ , and prove that  $EN(\phi)^*$  is split exact. This is the most complicated part of the proof, but it is purely formal:

As before we may split  $F$  and assume that  $F = G \oplus F'$ , the map  $\phi$  being the projection onto the first summand. Assuming that the first summand corresponds to the basis elements  $e_1, \dots, e_g \in G \subset F$  the dual differential  $\delta_i$  takes the form  $\sum_{i=1}^g e_i \otimes \hat{e}_i$ .

The map

$$\wedge^g G \xleftarrow{\delta_1} \wedge^g F = \bigoplus_{j=0}^g \wedge^j G \otimes \wedge^{g-j} F'$$

is the projection onto the  $j = g$  summand, so we must show that the rest of  $EN(\phi)$  is a split surjection ending with the terms of the source of  $\delta_1$  other than  $\wedge^g G$ .

Once again, it will be convenient to treat the dual complex. Using the splitting we may write the terms of the dual as

$$EN_i(\phi)^* = \text{Sym}^{i-2} G \otimes \wedge^{g+i-2} F^* = \bigoplus_j \text{Sym}^{i-2} G \otimes \wedge^j G^* \otimes \wedge^{g+i-2-j} F'^*$$

for  $i \geq 1$ . The map  $\delta_i = \delta_i^*$  is a direct sum from  $j = 0$  to  $g$  of the maps

$$\text{Sym}^{i-2} G \otimes \wedge^j G^* \otimes \wedge^{g+i-2-j} F'^* \longrightarrow \text{Sym}^{i-1} G \otimes \wedge^{j+1} G^* \otimes \wedge^{g+i-2-j} F'^*$$

that are all equal to multiplication by  $\sum_{k=1}^g e_k \otimes \hat{e}_k \otimes 1$  where the last tensor factor is the identity map of  $\wedge^{g+i-2-j} F'^*$ .

Thus it suffices to show that the complexes

$$(*_j) \quad \text{Sym}^0 G \otimes \wedge^j G^* \rightarrow \cdots \rightarrow \text{Sym}^i G \otimes \wedge^{i+j} G^* \rightarrow \cdots$$

are split exact for  $0 \leq j < g$ .

Let  $R = \text{Sym } G = S[e_1, \dots, e_g]$ . The Koszul complex over  $R$  of the sequence  $\phi_i = e_i$  may be written as

$$R \otimes_S \wedge G^* = \bigoplus_{p,q} \text{Sym}^p G \otimes \wedge^q G^*$$

and we have proven in Theorem 4.1 that it is a free resolution of  $R/(e_1, \dots, e_g) = S$ , which appears as  $\text{Sym}^0 G \otimes \wedge^0 G^*$ . The  $R$ -dual of this complex has terms  $(\text{Sym } G)^* \otimes \wedge^j G$  and the complexes  $(*_j)$  above are summands as  $S$ -modules. It follows that these finite complexes have no homology at all, and since the modules are free over  $S$ , they are split exact.

Since the complex  $EN(\phi)$  is split exact after inverting any element of  $I = (\phi_1, \dots, \phi_f)$ , it follows that the rank condition of Theorem 2.5 is satisfied, and

$$I \subset \sqrt{I_{\text{rank } \delta_i}(\delta_i)}.$$

Since the length of  $EN(\phi)$  is  $f - g + 1$ , Theorem 2.5 implies that it is acyclic when  $\text{grade } I \geq f - g + 1$ , completing the proof.  $\square$

**Corollary 4.8.** *With notation as in Theorem 4.6, if the ideal  $I_g(\phi)$  has codimension  $\geq f - g + 1$  then it has codimension exactly  $f - g + 1$ , the ring  $S/I_g(M)$  is Cohen-Macaulay, and the  $\binom{f}{g}$  forms that are the  $g \times g$  minors of a matrix for  $\phi$  are linearly independent over  $\mathbb{C}$ .*

**Proof.** From the resolution  $EN(M)$  we see that the projective dimension of  $S/I_2(M)$  is  $n - 1$ . Since the projective dimension of a module is at least the codimension of its annihilator, the equality follows, and the Auslander-Buchsbaum formula implies that  $S/I_2(M)$  is Cohen-Macaulay. The linear independence of the minors of  $M$  follows because  $EN(M)$  is a resolution and there  $EN(M)_2$  is

generated in degree 3, so all the relations on the minors have coefficients of degree 1.  $\square$

In general when  $X \subset Y \subset \mathbb{P}^r$ , so that  $I_X \supset I_Y$ , it may be hard to see which syzygies of  $X$  come from syzygies of  $Y$ . But when the degrees of the syzygies of  $Y$  are smaller than those from  $X$ , the situation is simpler. Here is the special case we will use:

**Proposition 4.9.** *Suppose that  $C \subset \mathbb{P}^r$  is a nondegenerate curve. If  $C \subset X \subset \mathbb{P}^r$ , where  $X$  is a rational normal scroll, then the Eagon-Northcott complex that is the minimal free resolution of  $I_X$  is termwise a direct summand of the minimal free resolution of  $I_C$ . Thus the Betti table of the resolution of  $I_C$  is termwise  $\geq$  that of  $I_X$ .*

**Proof.** Let  $EN$  be the minimal resolution of  $I_X$ , and let  $\mathbb{F}$  be the minimal resolution of  $I_C$ . The inclusion  $I_X \subset I_C$  induces a map  $\phi : EN \rightarrow \mathbb{F}$ , unique up to homotopy. Since the minimal generators of  $I_X$  are quadratic, and  $I_C$  contains no linear forms,  $\phi_0 : EN_0 \rightarrow \mathbb{F}_0$  is a split monomorphism.

By induction, we may assume that  $\phi_{i-1}$  is a split monomorphism. The free module  $EN_i$  is generated in degree  $i+1$ , while the free module  $\mathbb{F}_i$  is generated in degrees  $\geq i+1$ . It follows that the relations represented by  $EN_i$ , extended by 0, are among the minimal generators of the relations represented by  $\mathbb{F}_i$ , completing the proof.  $\square$

## 5. Green's Conjecture

Corollary 2.6 implies that the dimension of the vector space of forms of degree  $d$  vanishing on a canonical curve is independent of the curve; for example, for  $d = 2$  we get  $\dim(I_C)_2 = \binom{g-2}{2}$ . The Hilbert function of  $I_C$  is determined by the Betti table of its resolution, so that the table generally has more information.

For example when  $C$  is trigonal then by the geometric Riemann-Roch theorem,  $C$  has a 1-dimensional family of trisecant lines, and any quadric containing  $C$  must contain all these. As we have seen in Chapter 17, these lines sweep out the 2-dimensional rational normal scroll defined by the 1-generic  $2 \times (g-2)$  matrix  $M$  corresponding to the decomposition of  $\mathcal{O}_C(1)$  into a tensor product of the line bundle  $\mathcal{L}$  associated to the  $g^1_3$  and the residual line bundle  $\omega_C \otimes \mathcal{L}^{-1}$ . The latter has  $g-2$  sections, and we see from Section 16.2 that the scroll itself lies on the  $\binom{g-2}{2}$  quadrics defined by the minors of  $M$ . The exactness of the Eagon-Northcott complex associated to this matrix shows that there are no relations of degree 0 on these minors—that is, they are linearly independent over the ground field. It follows that they generate the vector space of all quadrics containing  $C$ .

Furthermore, if  $g = 6$  and  $C$  is isomorphic to a plane quintic curve, then the canonical series of the plane quintic is  $5 - 3 = 2$  times the hyperplane series, and it follows that the canonical image of  $C$  lies on the Veronese surface in  $\mathbb{P}^5$ . Thus the Veronese surface is contained in (in fact, equal to) the intersection of the quadrics defined by the  $2 \times 2$  minors of a generic symmetric matrix, coming from the multiplication map

$$H^0(\mathcal{O}_{\mathbb{P}^2}(1)) \otimes H^0(\mathcal{O}_{\mathbb{P}^2}(1)) \rightarrow H^0(\mathcal{O}_{\mathbb{P}^2}(2)) = H^0(\mathcal{O}_{\mathbb{P}^5}(1))$$

and there are  $6 = \binom{g-2}{2}$  independent quadrics in this ideal. Again in this case, they cannot generate the ideal of the curve.

One might fear that this is the beginning of some long series of examples, but in fact it is not:

**Theorem 5.1** (Petri). *The ideal of a canonical curve of genus  $\geq 5$  is generated by the  $\binom{g-2}{2}$ -dimensional space of quadrics it contains unless the curve is either trigonal or isomorphic to a plane quintic; in the latter cases, the ideal of the curve is generated by quadrics and cubics.*

For a modern treatment of Petri's theorem in this level of generality see [Schreyer 1991]; for a different treatment see [Arbarello and Sernesi 1978].

The two exceptions can be described simultaneously by using the Clifford index:

**Definition 5.2.** The *Clifford index*  $\text{Cliff } \mathcal{L}$  of a line bundle  $\mathcal{L}$  on a curve  $C$  is  $d - 2r$ , where  $d := \deg \mathcal{L}$  and  $r := h^0(\mathcal{L}) - 1$ . The Clifford index  $\text{Cliff } C$  of a curve  $C$  of genus  $\geq 2$  is the minimum of the Clifford indices of special line bundles with at least 2 sections.

Clifford's theorem (Corollaries 5.1 and 3.4) says that  $\text{Cliff } C \geq 0$ , and that  $\text{Cliff } C = 0$  if and only if  $C$  is hyperelliptic. If  $C$  is not hyperelliptic, then it turns out that  $\text{Cliff } C = 1$  if and only if  $C$  is either trigonal or isomorphic to a plane quintic. The Clifford index of any smooth curve of genus  $g \geq 2$  is  $\leq \lceil g/2 \rceil + 1$ , with equality for a general curve, as one sees from the Brill-Noether Theorem 2.1, and for “most” curves the line bundle  $\mathcal{L}$  of maximal Clifford index has only 2 sections, though there is an infinite sequence of examples where this “Clifford dimension” is greater.

Moving to cubic forms, we see that  $\dim(I_C)_3 = \binom{g+2}{3} - (5g - 5)$ . Comparing this number with the number of (possibly linearly dependent) cubics obtained by multiplying  $g$  linear forms and  $\binom{g-2}{2}$  quadrics, we see that the ideal of the curve has at least

$$\binom{g-2}{2} - \binom{g+2}{3} - (5g - 5)$$

independent syzygies of total degree 3 (that is, linear syzygies on the quadrics). For example when  $g = 4$  so that  $C \subset \mathbb{P}^3$  there is one quadric and 5 independent cubics, at most 4 of which are multiples of the quadric. Since the curve has degree  $6 = 2 \times 3$ , the ideal of the curve must be generated by the quadric and one cubic. When  $g = 5$  there are genuinely two possibilities: the three quadrics in the ideal might be a complete intersection (then they generate the ideal), so the Betti table would be

$j \setminus i$	0	1	2
0	1	—	—
1	—	2	—
2	—	—	1

or the curve could be trigonal, in which case the 3 quadrics generate the ideal of a surface scroll  $F$ . In the latter case, the Eagon-Northcott complex resolves the homogeneous coordinate ring  $S_F$  of the scroll,

$$0 \rightarrow S^2(-3) \rightarrow S^3(-2) \rightarrow S \rightarrow S_F \rightarrow 0$$

which has Betti table

$j \setminus i$	0	1	2
0	1	—	—
1	—	3	2

and we see that there are 2 linear relations among the quadrics. Thus the minimal generators of  $I_C$  must include exactly 2 cubics as well as the 3 quadrics. Since the homogeneous ring of a canonical curve is Gorenstein, its minimal free resolution is symmetric, and this is enough for us to fill in its Betti table:

$j \setminus i$	0	1	2	3
0	1	—	—	—
1	—	3	2	—
2	—	2	3	—
3	—	—	—	1

Note that we can “see” the scroll reflected in the top two lines of the table.

From the analogue of the Hilbert-Burch theorem for Gorenstein rings of codimension 3 one can show that the 5 generators can be written as the Pfaffians of a skew symmetric  $5 \times 5$  matrix whose entries are of degrees 1 and 2, in the following pattern (we give just the degrees, and put — in the places that are 0):

$$\begin{pmatrix} - & - & 1 & 1 & 1 \\ - & - & 1 & 1 & 1 \\ 1 & 1 & - & 2 & 2 \\ 1 & 1 & 2 & - & 2 \\ 1 & 1 & 2 & 2 & - \end{pmatrix}$$

Here the  $2 \times 2$  minors of the upper  $2 \times 3$  block of linear forms generate the ideal of the scroll.

Applying this logic more generally we get the following result about the canonical embedding of curves with low degree maps to  $\mathbb{P}^1$ :

**Theorem 5.3.** *Let  $C \subset \mathbb{P}^{g-1}$  be a reduced, irreducible canonical curve. If  $C$  has a line bundle  $\mathcal{L}$  of degree  $d \leq g - 1$  with  $h^0(\mathcal{L}) = 2$  then there is a 1-generic  $2 \times (g+1-d)$  matrix of linear forms whose minors define a scroll of codimension  $g-d$  containing  $C$ ; and thus an Eagon-Northcott complex of length  $g-d$  is a subcomplex of the minimal free resolution of  $R_C$ . In particular, the Betti table of  $R_C$  is termwise  $\geq$  that of the homogeneous coordinate ring of the scroll.*

(We have stated this theorem for canonical curves, but in fact the construction applies much more generally to a linearly normal variety  $X \subset \mathbb{P}^n$  of any dimension: if  $X$  has a divisor  $D$  that moves in a pencil and is contained in a subspace  $\mathbb{P}^k$  with  $k \leq n-2$ , the planes spanned by the divisors of the pencil  $|D|$  sweep out a rational normal scroll.)

Thus the existence of the  $g_d^1$  on  $C$ , together with the symmetry of the resolution of the Gorenstein ring  $R_C$ , implies that the Betti table of  $R_C$  has the form

$j \backslash i$	0	1	2	...	$d-3$	$d-2$	...	$g-d-1$	$g-d$	...	$g-3$	$g-2$
0	1	—	—	...	—	—	—	—	—	—	—	—
1	—	*	*	...	*	*	...	*	?	...	?	?
2	—	?	?	...	?	*	...	*	*	...	*	*
3	—	—	—	...	—	—	—	—	—	—	—	1

where we have assumed for illustration that  $d-2 < g-d-1$ . The places marked — are definitely 0 and those marked \* are definitely nonzero. The entries of the rows marked 0 and 1 are  $\geq$  the corresponding entries of the Betti table of the scroll.

We can summarize this by saying that if the curve  $C$  has a line bundle  $\mathcal{L}$  of degree  $d$  with exactly 2 sections (which is thus of Clifford index  $c = d-2$ ) the row labeled 2 in the Betti diagram definitely has  $\beta_{c,c-2} \neq 0$ . As with the case of the plane quintics above, one can make a similar argument for any line bundle of Clifford index  $c$ . Thus:

**Corollary 5.4.** *If  $\text{Cliff } C \leq c$  then  $\beta_{c,c+2}(S/I_C) \neq 0$ .*

Starting from examples such as the case of genus 6, Mark Green made a bold conjecture that is still open as of this writing:

**Conjecture 5.5 (Green's Conjecture).** *If  $C$  is a smooth canonical curve of genus  $g$  and  $S/I_C$  is the homogeneous coordinate ring of  $C$  in its canonical embedding, then the Clifford index of  $C$  is  $\leq c$  if and only if  $\beta_{c,c+2}(S/I_C) \neq 0$ .*

The conjecture was made for curves over a field of characteristic 0, and is known in many cases, though it is also known to fail in small finite characteristics (see [Bopp and Schreyer 2021] for an amended conjecture that may hold in all characteristics.) In [Voisin 2002] and [Voisin 2005] the conjecture was proven for generic curves of each Clifford index. See [Aprodu et al. 2019], [Kemeny 2021] and [Rathmann 2022] for various simpler proofs. As of this writing the full conjecture is known up to genus 9, for plane curves, and in a number of other special cases. See for example [Farkas 2017] for a survey on this and related topics.

**5.1. Low genus canonical embeddings.** In his 1983 Brandeis thesis [Schreyer 1986], Frank Schreyer analyzed all the possibilities for resolutions of smooth canonical curves up to genus 8. The project was carried further to genus 9 in a Saarbrücken thesis [Sagraloff 2018] of a student of Schreyer.

## 6. Exercises

**Exercise 6.1.** Let  $S = k[x_0, \dots, x_r]$ , and let

$$\mathbb{F} : F_0 \xleftarrow{\phi_1} F_1 \xleftarrow{\quad} \cdots \xleftarrow{\quad} F_{n-1} \xleftarrow{\phi_n} F_n \xleftarrow{\quad} 0$$

be a finite complex of free  $S$ -modules. Set

$$X_i = \{p \in \mathbb{A}^{n+1} \mid H_i(\mathbb{F} \otimes \kappa(p)) \neq 0\}$$

Use Theorem 2.5 to prove that the complex  $\mathbb{F}$  is *acyclic* (that is,  $H_i(\mathbb{F}) = 0$  for all  $i > 0$ ) if and only if

$$\text{codim } X_i \geq i$$

for all  $i > 0$ . Moreover,  $X_0 \supseteq X_1 \supseteq \cdots \supseteq X_n$

Hint: Elementary linear algebra shows that, if  $k$  is a field, then a complex  $k^p \xleftarrow{\phi} k^q \xleftarrow{\psi} k^r$  is exact at  $k^q$  if and only if  $\text{rank } \phi + \text{rank } \psi = q$ .

**Exercise 6.2.** Prove that if  $X \subset \mathbb{P}^r$  is arithmetically Cohen-Macaulay then the dual of the minimal free resolution of  $S/I_X$  is the minimal free resolution of  $\omega_X$ . Hint: Use Theorem 2.5 and the characterization of  $\omega_{S/I_X}$  as an Ext module.

**Exercise 6.3.** The *depth lemma* states that if

$$0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$$

is an exact sequence of nonzero finitely generated modules over a local ring  $R$  then

$$\text{depth } C \geq \min\{\text{depth } B, \text{depth } A - 1\}$$

$$\text{depth } A \geq \min\{\text{depth } B, \text{depth } C + 1\}$$

Prove this in the special case when  $R$  is regular using the characterization of depth via projective dimension.

**Exercise 6.4.** (1) Prove that if  $\phi : F \rightarrow G$  is a free presentation of a finitely generated module  $M$  then

$$\text{ann}_R(\text{coker } \phi)^{\text{rank } G} \subset \text{Fitt}_0(\phi) \subset \text{ann}_R(\text{coker } \phi).$$

(2) Prove that over a local ring every projective module is free; and show that  $I(\phi)$  defines the non-free locus of  $\text{coker } \phi$ .

**Exercise 6.5.** In dealing with arithmetically Gorenstein schemes, we used the fact that if  $\omega_{S/I}$  is an invertible sheaf (over  $\text{Spec}(S/I)$ ) then it is isomorphic to  $S/I(a)$  for some  $a$ . Why is this true?

Hint: Nakayama's Lemma can be used to prove that projectives are free in some cases.

**Exercise 6.6.** Find a degree 6 embedding of a curve of genus 3 that is not arithmetically Cohen-Macaulay, and another that is.

Hint: Show that the  $3 \times 3$  minors of a general  $4 \times 3$  matrix of linear forms defines a Cohen-Macaulay curve of genus 3. Show that a curve of type  $(2, 4)$  on a smooth quadric in  $\mathbb{P}^3$  is not arithmetically Cohen-Macaulay.

**Exercise 6.7.** Referring to the Betti table of a canonical curve just before Section 4, give a formula for the differences  $b_i - b_{g-i-1}$  that depends only on  $i$  and  $g$ .

**Exercise 6.8.** Show that if  $I \subset S := \mathbb{C}[x_0, \dots, x_r]$  is a codimension 2 ideal, then  $S/I$  is Cohen-Macaulay if and only if the minimal  $S$ -free resolution of  $S/I$  has the form

$$0 \longrightarrow S^{n-1} \rightarrow S^n \rightarrow S$$

for some  $n$ . Show that  $S/I$  is Gorenstein if and only if  $I$  is a complete intersection. Hint: for the first part, tensor with the field of rational functions.

**Exercise 6.9.** Which sets of 4 distinct points in  $\mathbb{P}^2$  are arithmetically Gorenstein? Which sets of 5 points? Which sets of 6 points?

**Exercise 6.10.** Give an example of a set of points in  $\mathbb{P}^3$  that is arithmetically Gorenstein but not a complete intersection.

Hint: take the hyperplane section of a trigonal canonical curve of genus 5.

**Exercise 6.11.** Let  $Q \subset \mathbb{P}^3$  be the quadric defined by the determinant of the  $2 \times 2$  matrix

$$F \xrightarrow{q=\begin{pmatrix} x_0 & x_1 \\ x_2 & x_3 \end{pmatrix}} G$$

where  $F = S^2(-1)$  and  $G = S^2$ .

- (1) Show that the sheafification of the graded module  $M := \text{coker } q$  is  $\mathcal{O}_Q(1, 0)$  and the sheafification of  $\text{coker } q^*$  is  $\mathcal{O}_Q(0, 1)$  by computing the vanishing locus of the two sections corresponding to the generators of the module.
- (2) Show that the sheafification of  $\text{Sym}^a(M)$  is  $\mathcal{O}_Q(a, 0)$ . Conclude that if  $a \leq b$  then the relative ideal sheaf  $\mathcal{I}_{C/Q} = \mathcal{O}(-a, -b)$  of a curve  $C$  of type  $(a, b)$  is the sheafification of the module  $\text{Sym}^{b-a}(M)(-b)$ .
- (3) Let  $S = \mathbb{C}[x_0, \dots, x_3]$ . Show that the minimal  $S$ -free resolution of  $\text{Sym}^a(M)$  has the form

$$0 \longrightarrow \wedge^2 F \otimes \text{Sym}^{a-2} G(-2) \longrightarrow F \otimes \text{Sym}^{a-1} G(-1) \longrightarrow \text{Sym}^a G.$$

Hint: use multilinear algebra (as in [Eisenbud 1995a]) to define the maps, and use Theorem 2.5 to prove that this is a resolution.

- (4) Show that if  $C \subset \mathbb{P}^3$  is a curve of type  $(a, b)$  with  $a \leq b$  then there is a free resolution of a module whose sheafification is  $\mathcal{I}_{C/\mathbb{P}^3}$  that is a mapping cone of the map of complexes:

$$\begin{array}{ccccccc} 0 & \longrightarrow & 0 & \longrightarrow & \wedge^2 F & \xrightarrow{\wedge^2 q} & \wedge^2 G \\ & & \uparrow & & \uparrow & & \uparrow \\ 0 & \longrightarrow & \wedge^2 F \otimes \text{Sym}^{b-a-2} G(-b-2) & \rightarrow & F \otimes \text{Sym}^{b-a-1} G(-b-1) & \longrightarrow & \text{Sym}^{b-a} G(-b) \end{array}$$

- (5) Conclude that the deficiency module of a curve of type  $(a, b)$  with  $a \leq b$  is the cokernel of a map

$$\wedge^2 F^* \otimes \text{Sym}^{b-a-2}(G^*)(b+2) \longleftarrow F^* \otimes \text{Sym}^{b-a-1} G^*(b+1)$$

and thus, after choosing a basis of  $F = S^2$ , may be identified as the sheafification of the cokernel of

$$\text{Sym}^{b-a-2}(G^*)(b+2) \longleftarrow F \otimes \text{Sym}^{b-a-1} G^*(b+1)$$

where the map is the action of  $F$  on  $\wedge G^*$  via the map  $q : F \rightarrow G$ .

Hint: Show that the sheafification of the graded module  $\text{coker } q$  is  $\mathcal{O}_Q(1, 0)$  by computing the vanishing locus of the two sections corresponding to the generators of the module. Imitate the proof of Theorem 4.6 to prove the exactness of the given resolution of  $\text{Sym}^a(M)$ . See also [Eisenbud 1995a, Appendix A2.6].

### Exercise 6.12.

Let  $\phi : F \rightarrow G \rightarrow M \rightarrow 0$  be an exact sequence of finitely generated  $R$ -modules, with  $F$  and  $G$  free.

- (1) Show that the annihilator of  $\text{coker } \phi$  has the same radical as the ideal  $I_{\text{rank } G}(\phi)$ .
- (2) Set  $r := \text{rank } \phi$  and  $I(\phi) := I_{\text{rank } \phi}(\phi)$ . Show that the cokernel of  $\phi$  is locally free if and only if  $I(\phi) = R$ .



# Hilbert Schemes

In earlier chapters, we described some degree  $d$  embeddings of curves of genus  $g$  in projective spaces  $\mathbb{P}^r$  for small  $g, r, d$ . In this chapter, we will try to describe the *restricted Hilbert scheme*  $H_{g,3,d}^\circ$ , defined to be the open subscheme of the Hilbert scheme  $H_{g,3,d} := \text{Hilb}_{dm-g+1}(\mathbb{P}^3)$  parametrizing smooth, irreducible and nondegenerate curves of degree  $d$  and genus  $g$  in  $\mathbb{P}^3$ .

Three basic questions about the schemes  $H_{g,r,d}^\circ$  are:

- Is  $H_{g,r,d}^\circ$  irreducible?
- What is its dimension (or the dimensions of its components)?
- Where is it smooth, and where is it singular?

Of course, there are many more questions one could ask about the geometry of  $H_{g,r,d}^\circ$ , many of which are open. For example, what is the closure  $\overline{H_{g,r,d}^\circ} \subset H_{g,r,d}$  in the whole Hilbert scheme? (In other words, when is a subscheme  $X \subset \mathbb{P}^r$  with Hilbert polynomial  $dm - g + 1$  *smoothable*, in the sense that it is the flat limit of a family of smooth curves?) In general, no one knows!

## 1. Degree 3

The first case to consider is that of the Hilbert scheme  $H_{0,3,3}$ . The corresponding restricted Hilbert scheme  $H_{0,3,3}^\circ$ , parameterizing twisted cubics, is one we have encountered and described already: in Proposition 3.5 we showed that  $H_{0,3,3}^\circ$  is irreducible of dimension 12, and we gave another proof, based on linkage, in Chapter 16. By Exercise 12.1, the normal bundle of a twisted cubic  $C$  is

$\mathcal{N}_{C/\mathbb{P}^3} = \mathcal{O}_{\mathbb{P}^1}(5) \oplus \mathcal{O}_{\mathbb{P}^1}(5)$  so by Theorem 3.2 the tangent space to the Hilbert scheme at  $C$  is  $H^0(\mathcal{N}_{C/\mathbb{P}^3}) = \mathbb{C}^{12}$ . Thus  $H_{0,3,3}$  is smooth at this point.

**1.1. The other component of  $H_{0,3,3}$ .** We might naively expect that the closure  $\overline{H_{0,3,3}}$  would be all of  $H_{0,3,3}$ , but this is not the case:  $H_{0,3,3}$  has two irreducible components.

One component is the closure of  $H_{0,3,3}$ . To describe the second (“extraneous”) component, suppose we start with a plane cubic curve  $C_0 \subset \mathbb{P}^2 \subset \mathbb{P}^3$ . This is of course a curve of degree 3, but its Hilbert polynomial is  $p(m) = 3m$  rather than  $3m + 1$ , reflecting the fact that the genus of  $C_0$  is 1, not 0.

But that can be fixed: adding a point  $p \in \mathbb{P}^3 \setminus C_0$  to  $C_0$  has the effect of increasing the Hilbert polynomial by 1, so that the Hilbert polynomial of  $C := C_0 \cup \{p\}$  is  $3m + 1$ . Thus  $C$  corresponds to a point of  $H_{0,3,3}$ . In fact, the locus in  $H_{0,3,3}$  of curves  $C$  of this form is open, and its closure, which includes plane cubics with an embedded point, is a second irreducible component of  $H_{0,3,3}$ .

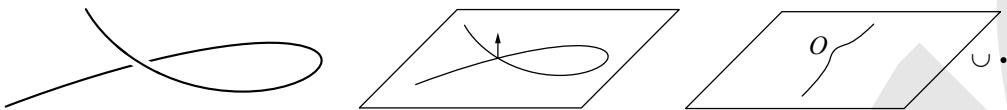
**Theorem 1.1.**  $H_{0,3,3}$  has two irreducible components. One has generic point corresponding to a twisted cubic, and the other has generic point corresponding to the union of a smooth plane cubic and a point outside the plane. They have dimension 12 and 15 respectively.

**Proof.** Let  $C'$  be the purely 1-dimensional scheme defined by the intersection of the 1-dimensional primary ideals in the decomposition of  $I_C$ . If the Hilbert polynomials of  $C$  and  $C'$  are  $p(m)$  and  $p'(m)$  then  $p(m) \geq p'(m)$  for all large  $m$ ; equality for large  $m$  would imply that  $C' = C$ .

Whatever 0-dimensional components  $C'$  may have do not contribute to the degree (= leading coefficient of the Hilbert polynomial) so  $\deg C' = \deg C = 3$ . Thus the curve  $C'$  is either irreducible or the union of two or three irreducible components. In the first case  $C'$  is either nondegenerate, in which case it is a twisted cubic by Theorem 4.1 and  $C' = C$ ; or a plane cubic. If  $C'$  is a plane cubic, then it has Hilbert function  $3m$ , so  $\mathcal{I}_{C'}/\mathcal{I}_C$  corresponds to a point in  $\mathbb{P}^3$ , either embedded in  $C'$  or not. Such a union is specified by the choice of the plane, the cubic in it, and the point,<sup>1</sup> and thus has dimension  $3 + 9 + 3 = 15$ .

On the other hand, if  $C$  is not planar and not irreducible, then  $C'$  consists of 3 lines or the union of a (planar) conic and a line not in the plane. In this case each connected component has arithmetic genus 0 and thus Hilbert polynomial with constant term 1; so the curve must be connected. All such curves

<sup>1</sup>if the point is an embedded point, specifying  $C'$  and  $p$  does not determine  $C$ —if  $p$  is a smooth point of  $C'$ , for example, there is a one-parameter family of curves  $C$  with support  $C'$  and an embedded point at  $p$ ; the tangent space to  $C$  at  $p$  can be any plane containing the tangent line to  $C'$  at  $p$ —but these are all flat limits of disjoint unions  $C' \sqcup \{p\}$ , so they don’t contribute a separate component of  $H_{0,3,3}$ .



**Figure 1.**  $\mathcal{H}_{0,3,3}$  has two components, whose generic members are shown on the left and right, with the generic member of the intersection shown in the middle. On the left is the “principal component” whose generic point is a smooth twisted cubic. This degenerates to a singular plane cubic with an embedded point at the node. In the other component, the singular plane cubic becomes smooth, and the extra point is free to move anywhere in space. Silvio: The left-hand picture is (mathematically) a twisted cubic; the middle curve is a plane cubic with a node and the vertical arrow represents an infinitesimal (non-planar) thickening; the right picture is supposed to be a smooth plane elliptic curve, and a point, floating in 3-space away from the plane. I suggest dropping the  $\cup$  and moving the floating point above the plane, to indicate that all three pictures are in the same 3-space. The three curves should all be red. The elliptic curve can be the usual beautiful one (two ovals, one extending to the edge of the plane). The middle picture actually represents a degeneration of each of the ones on the left and right. Maybe place it a little lower, and put in arrows?? An additional possibility to consider: add a “plane” that would be seen as “below” the curve on the left.

can be realized as divisors of type  $(1, 2)$  on a quadric, and thus have Hilbert function equal to that of  $C$ , whence again  $C' = C$ .  $\square$

**Cheerful Fact 1.2.** *In [Piene and Schlessinger 1985] it is also shown that the two components of  $H_{0,3,3}$  are smooth and rational and that their intersection is smooth and rational of dimension 11.*

The presence of components of the Hilbert scheme whose general member is not smooth, irreducible and nondegenerate is not exceptional, and in the following section we’ll describe some of the “extraneous components” more generally. But one aspect of the geometry of  $H_{0,3,3}$  is special: the action of the group  $PGL_4$  on  $H_{0,3,3}$  has only finitely many orbits (if you want to see the orbits in the principle component  $\overline{H}_{0,3,3}^\circ$ , see [Harris 1982a]). It’s not known if there are other examples of this, beyond Grassmannians, Hilbert schemes of quadric hypersurfaces and Hilbert schemes of triples of points. In Section 8.3 we will discuss the opposite situation: the possibility of “rigid curves,” corresponding to points in the Hilbert scheme  $H_{dm-g+1}^\circ$  with  $g > 0$  whose  $PGL_{r+1}$  orbit is open.

## 2. Extraneous components

Let  $H_d(\mathbb{P}^r)$  be the Hilbert scheme of subschemes of  $\mathbb{P}^r$  with Hilbert polynomial of degree 0, say equal to the constant  $d$ . There is an open subset  $H_d^\circ(\mathbb{P}^r)$  of dimension  $dr$  whose points correspond to reduced  $d$ -tuples of points in  $\mathbb{P}^r$ . This

open subset is isomorphic to the complement of the diagonals in the  $d$ th symmetric power of  $\mathbb{P}^r$ . We call the closure of this open set the *principal component* of  $H_d(\mathbb{P}^r)$ .

The Hilbert scheme  $H_d(\mathbb{P}^2)$  is smooth and irreducible of dimension  $2d$ , and a similar result holds for 0-dimensional schemes on any smooth surface, but Iarrobino showed that, for any  $r \geq 3$  and any sufficiently large  $d$ , there are components of  $H_d(\mathbb{P}^r)$  having dimension strictly larger than  $dr$  [Iarrobino 1985]; see Exercise 12.17. There are also examples of extraneous components of dimension  $< dr$ ; see [Cartwright et al. 2009]. No one knows how many irreducible components the Hilbert scheme  $H = H_d(\mathbb{P}^r)$  has, or what their dimensions might be.

This in turn infects the Hilbert schemes of curves. For example, the Hilbert scheme  $H_{g,r,d}$  has a component whose general point corresponds to a union of a plane curve of degree  $d$  and  $\binom{d-1}{2} - g$  points; moreover, if  $\Gamma$  is any irreducible component of the Hilbert scheme of zero-dimensional subschemes of degree  $\binom{d-1}{2} - g$  in  $\mathbb{P}^3$ , there is a component of  $H_{g,3,d}$  whose general point corresponds to a union of a plane curve of degree  $d$  and the subscheme corresponding to a general point of  $\Gamma$ . Further, we can replace the plane curves in this construction with any component of the Hilbert scheme of curves of degree  $d$  and genus  $g' > g$ . There can also be components of  $H_{d,r,3}$  whose general point corresponds to a subscheme of  $\mathbb{P}^3$  with a spatial embedded point—see [Chen and Nollet 2012].

Bottom line: it's a mess. For many  $g, d$  the Hilbert scheme  $H_{g,3,d}$  has many components. In most cases no one knows how many, or what their dimensions are, which is why we most often focus on the restricted Hilbert scheme.

### 3. Degree 4

By Clifford's theorem, an irreducible nondegenerate curve of degree 4 in  $\mathbb{P}^3$  must have genus 0 or 1; we consider these cases in turn.

**3.1. Genus 0.** We can deal with rational quartics by a slight variant of the first method we used to deal with twisted cubics. A rational curve of degree 4 is the image of a map  $\phi_F : \mathbb{P}^1 \rightarrow \mathbb{P}^3$  given by a four-tuple  $F = (F_0, F_1, F_2, F_3)$  with  $F_i \in H^0(\mathcal{O}_{\mathbb{P}^1}(4))$ . The space of all such four-tuples up to scalars is a projective space of dimension  $4 \times 5 - 1 = 19$ ; let  $U \subset \mathbb{P}^{19}$  be the open subset of four-tuples such that the map  $\phi$  is a nondegenerate embedding. By the universal property of the Hilbert scheme there is a surjective map  $\pi : U \rightarrow H^0$ , whose fiber over a point  $C$  is the space of maps with image  $C$ . Since any two such maps differ by an automorphism of  $\mathbb{P}^1$ —that is, an element of  $PGL_2$ —the fibers of  $\pi$  are three-dimensional, proving that  $H_{0,3,4}^0$  is irreducible of dimension 16.

The same analysis can be used on rational curves of any degree  $d$  in any projective space  $\mathbb{P}^r$ :

**Proposition 3.1.** *The open set  $H_{0,r,d}^\circ$  parametrizing smooth, irreducible nondegenerate rational curves  $C \subset \mathbb{P}^3$  is irreducible of dimension  $(r+1)(d+1) - 4$ ; in case  $r = 3$  in particular it has dimension 4d.*

**Proof.** The space  $U$  of nondegenerate embeddings  $\mathbb{P}^1 \rightarrow \mathbb{P}^r$  of degree  $d$  is an open subset of the projective space  $\mathbb{P}^{(r+1)(d+1)-1}$  of  $(r+1)$ -tuples of homogeneous polynomials of degree  $d$  on  $\mathbb{P}^1$  modulo scalars; and the fibers of the corresponding map  $U \rightarrow H_{dm+1}^\circ$  are copies of  $PGL_2$ .  $\square$

In contrast to the case of twisted cubics, smooth rational curves in  $\mathbb{P}^r$  of the same degree may have different normal bundles. This gives an interesting stratification of the restricted Hilbert scheme of rational curves; see [[Coskun and Riedl 2018](#)] for a discussion.

**3.2. Genus 1.** As we saw in Section 4.4, a quartic curve  $C \subset \mathbb{P}^3$  of genus 1 is the intersection of two quadric surfaces, and by Lasker's theorem, every quadric containing  $C$  is a linear combination of those two. Conversely, the intersection of two general quadrics in  $\mathbb{P}^3$  is a quartic curve of genus 1, as follows from Bézout's theorem and the adjunction formula. We can thus construct a family of quartics of genus 1: let  $V = H^0(\mathcal{O}_{\mathbb{P}^3}(2))$  be the 10-dimensional vector space of homogeneous quadric polynomials in  $\mathbb{P}^3$  and  $G(2, V)$  the Grassmannian of 2-planes in  $V$ , and consider the incidence correspondence

$$\Gamma = \{(\Lambda, p) \in G(2, V) \times \mathbb{P}^3 \mid F(p) = 0 \ \forall F \in \Lambda\}.$$

The fiber of  $\Gamma$  over a point  $\Lambda \in G(2, V)$  is thus the base locus of the pencil of quadrics represented by  $\Lambda$ ; let  $B \subset G(2, V)$  be the Zariski open subset over which the fiber is smooth, irreducible and nondegenerate of dimension 1. By the universal property of Hilbert schemes, the family  $\pi_1 : \Gamma_B \rightarrow U$  induces a map  $\phi : B \rightarrow H_{1,3,4}^\circ$  that is one-to-one on points; it follows that the reduced subscheme of  $H_{1,3,4}^\circ$  is birational to an open subset of the Grassmannian  $G(2, 10)$ . We conclude that  $H_{1,3,4}^\circ$  is irreducible of dimension 16. Exercise 12.6 shows that this map is actually an isomorphism.

The argument here—where we constructed a family  $C \rightarrow B$  of curves of given type, and then invoked the universal property of the Hilbert scheme to get a map  $B \rightarrow H$ —is typical in analyses of Hilbert schemes.

#### 4. Degree 5

Let  $C \subset \mathbb{P}^3$  be a smooth, irreducible, nondegenerate quintic curve of genus  $g$ . By Clifford's theorem the bundle  $\mathcal{O}_C(1)$  must be nonspecial, so by the Riemann-Roch theorem we must have  $0 \leq g \leq 2$ . We have already seen that the space  $H_{0,3,5}^\circ$  of rational quintic curves is irreducible of dimension 20. We will treat the case  $g = 2$  in detail, and leave the case  $g = 1$  as Exercise 12.10. (Degree 5 will be covered in a different way in Section 11.)

**4.1. Genus 2.** We have considered curves of genus 2 in Section 3. Recall that a curve of genus 2 and degree 5 in  $\mathbb{P}^3$  is contained in the intersection of a unique quadric surface  $Q$  and a cubic surface  $S$  not containing  $Q$ . The intersection  $Q \cap S$  has degree 6, and is thus the union of  $C$  and a line. If  $Q$  is smooth then, in terms of the isomorphism  $Q \cong \mathbb{P}^1 \times \mathbb{P}^1$ , the curve  $C$  is a divisor of type  $(2, 3)$  on the quadric  $Q$ . Note that conversely if  $L \subset \mathbb{P}^3$  is a line and  $Q$  and  $S \subset \mathbb{P}^3$  are general quadric and cubic surfaces containing  $L$ , and if we write

$$Q \cap S = L \cup C$$

then the curve  $C$  is a curve of type  $(2, 3)$  on the quadric  $Q$  and hence, by the adjunction formula, a quintic of genus 2.

This suggests two ways of describing the family  $H_{2,3,5}^\circ$  of such curves. First, we can use the fact that  $C$  is linked to a line to make an incidence correspondence

$$\Psi = \{(C, L, Q, S) \in H^\circ \times \mathbb{G}(1, 3) \times \mathbb{P}^9 \times \mathbb{P}^{19} \mid Q \cap S = C \cup L\},$$

where the  $\mathbb{P}^9$  (respectively,  $\mathbb{P}^{19}$ ) is the space of quadric (respectively, cubic) surfaces in  $\mathbb{P}^3$ . Given a line  $L \in \mathbb{G}(1, 3)$ , the space of quadrics containing  $L$  is a  $\mathbb{P}^6$ , and the space of cubics containing  $L$  is a  $\mathbb{P}^{15}$ ; thus the fiber of the projection  $\pi_2 : \Psi \rightarrow \mathbb{G}(1, 3)$  over  $L$  is an open subset of  $\mathbb{P}^6 \times \mathbb{P}^{15}$ , and we see that  $\Psi$  is irreducible of dimension  $4 + 6 + 15 = 25$ .

On the other hand, the fiber of  $\Psi$  over a point  $C \in H_{2,3,5}^\circ$  is an open subset of the  $\mathbb{P}^5$  of cubics containing  $C$ ; and we conclude that  $H_{2,3,5}^\circ$  is irreducible of dimension 20.

Another approach to describing the restricted Hilbert scheme  $H_{2,3,5}^\circ$  would be to use the fact that the quadric surface  $Q$  containing a quintic curve  $C \subset \mathbb{P}^3$  of genus 2 is unique. We thus have a map

$$H^\circ \rightarrow \mathbb{P}^9,$$

whose fiber over a point  $Q \in \mathbb{P}^9$  is the space of quintic curves of genus 2 on  $Q$ .

The general fiber of this map, the space of quintic curves of genus 2 on a smooth quadric  $Q$  is reducible: it consists of the disjoint union of the open subsets of smooth elements in the two linear series of curves of type  $(2, 3)$  and  $(3, 2)$

on  $Q$ , each of which is a  $\mathbb{P}^{11}$ , while over a quadric cone the fiber is irreducible, since the cone has a unique family of lines. We can conclude immediately that  $H_{2,3,5}^o$  is of pure dimension 20; to conclude that it is irreducible we use the fact that in the family of all smooth quadric surfaces, the monodromy exchanges the two rulings (Example 1.2).

## 5. Degree 6

Again the Clifford and Riemann-Roch theorems suffice to compute the possible genera of a curve of degree 6. To start with, if the line bundle  $\mathcal{O}_C(1)$  is nonspecial, then by the Riemann-Roch theorem we have  $g \leq 3$ . Suppose on the other hand that  $\mathcal{O}_C(1)$  is special. Since  $h^0(\mathcal{O}_C(1)) \geq 4$ , we have equality in Clifford's theorem, and either  $C$  is hyperelliptic and  $\mathcal{O}_C(1)$  is a multiple of the  $g_2^1$  or  $C$  is a canonically embedded curve of genus 4. The first case cannot occur, since no special series on a hyperelliptic curve is very ample; thus  $C$  must be a canonical curve of genus 4. Thus a smooth irreducible, nondegenerate curve of degree 6 in  $\mathbb{P}^3$  has genus at most 4.

The cases of genera 0, 1 and 2 are covered under Proposition 7.2 below, leaving us the cases  $g = 3$  and 4. In both these cases we can describe the ideal of the curve.

**5.1. Genus 4.** As we've seen in Section 1.1 a canonical curve of genus 4 is the complete intersection of a (unique) quadric  $Q$  and a cubic surface  $S$ . We thus have a map

$$\alpha : H_{4,3,6}^o \longrightarrow \mathbb{P}^9$$

sending a curve  $C$  to the quadric  $Q$  containing it. Moreover, the fibers of this map are open subsets of the projective space  $\mathbb{P}V$ , where  $V$  is the quotient of the space of all cubic polynomials modulo cubics containing  $Q$ ,

$$V = \frac{H^0(\mathcal{O}_{\mathbb{P}^3}(3))}{H^0(\mathcal{I}_{Q/\mathbb{P}^3}(3))}.$$

Since this vector space has dimension 16, the fibers of  $\alpha$  are irreducible of dimension 15, and we deduce that the space  $H_{4,3,6}^o$  is irreducible of dimension 24.

See Exercise 12.12 for the generalization to arbitrary smooth complete intersections in  $\mathbb{P}^3$ .

**5.2. Genus 3.** We leave this to the reader in Exercises 12.13, 12.14

## 6. Degree 7

Using the tools above, we invite the reader to show that each component of the restricted Hilbert schemes of curves of degree 7 in  $\mathbb{P}^3$  has expected dimension 28. See Exercise 12.16 for an outline.

## 7. The expected dimension of $H_{g,r,d}^\circ$

The sharp-eyed reader will have noticed that in, every case analyzed so far, the Hilbert scheme  $H_{g,3,d}^\circ$  parametrizing smooth curves of degree  $d$  and genus  $g$  in  $\mathbb{P}^3$  has dimension  $4d$ . This is the “expected dimension,” in a sense we will now make precise:

Let  $C \subset \mathbb{P}^r$  be a smooth curve of genus  $g$  and degree  $d$ . In Section 3.1 we computed the tangent space to  $H_{g,r,d}^\circ$  at the point  $[C]$  as  $H^0(\mathcal{N}_{C/\mathbb{P}^r})$ , so the dimension  $h^0(\mathcal{N}_{C/\mathbb{P}^r})$  is an upper bound for  $\dim H_{g,r,d}^\circ$ . We can compute a lower bound as well:

**Cheerful Fact 7.1.** *The completion of the local ring*

$$(R, \mathfrak{m}) = \mathbb{C}[[x_1, \dots, x_t]]/J$$

of  $H_{g,r,d}^\circ$  at the point  $[C]$  representing a smooth curve can, in principle, be computed by deformation theory. Though this can actually be carried out in small cases, it is hard to get much qualitative information from the process except for two numbers: the dimension of the Zariski tangent space  $(\mathfrak{m}/\mathfrak{m}^2)^*$ , which we have computed as  $t := h^0(\mathcal{N}_{C/\mathbb{P}^r})$ ; and an upper bound for the number of generators of the ideal  $J$ , which is  $h^1(\mathcal{N}_{C/\mathbb{P}^r})$ . See [Fantechi and Göttsche 2005, Corollaries 6.2.5, 6.4.11 and Proposition 6.5.2], where a similar result is given for any locally complete intersection subscheme of  $C \subset \mathbb{P}^r$ .

Thus from the principal ideal theorem [Eisenbud 1995a, Theorem 10.2] it follows that if  $C$  is a smooth curve, then

$$\chi(\mathcal{N}_{C/\mathbb{P}^r}) \leq \dim(H_{g,r,d}^\circ) \leq h^0(\mathcal{N}_{C/\mathbb{P}^r}) \text{ locally at } [C].$$

If the upper bound is achieved, then  $[C]$  is a smooth point of  $H_{g,r,d}^\circ$ , and if the lower bound is achieved then the local ring of  $H_{g,r,d}^\circ$  at  $[C]$  is a complete intersection.

Using the deformation bound 7.1 we can at least control the Hilbert scheme at a point representing a nonspecial embedding.

**Theorem 7.2.** *For any smooth curve  $C \subset \mathbb{P}^r$  of genus  $g$  and degree  $d$*

$$\chi(\mathcal{N}_{C/\mathbb{P}^r}) = (r+1)d - (r-3)(g-1),$$

which is a lower bound for the dimension of the Hilbert scheme locally at  $[C]$ . If  $\mathcal{O}_C(1)$  is nonspecial, then  $H^1(\mathcal{N}_{C/\mathbb{P}^r}) = 0$  so  $H_{g,r,d}^\circ$  is smooth and of dimension  $(r+1)d - (r-3)(g-1)$  at  $[C]$ . If  $d > 2g-2$ , then  $H_{g,r,d}^\circ$  is irreducible as well.

Note that in the case of  $\mathbb{P}^3$  we have  $\chi(\mathcal{N}_{C/\mathbb{P}^r}) = 4d$ , and we have seen that when  $d \leq 7$  this is always equal to the dimension of the restricted Hilbert scheme, whether or not the embedding is special. In view of this, we define the *expected dimension* of  $H_{g,r,d}^o$  to be

$$h(g, r, d) := (r+1)d - (r-3)(g-1).$$

We will give yet another argument for calling this the expected dimension in Section 11.

**Proof.** From the exact sequence

$$0 \rightarrow \mathcal{T}_C \rightarrow \mathcal{T}_{\mathbb{P}^r}|_C \rightarrow \mathcal{N}_{C/\mathbb{P}^r} \rightarrow 0$$

we see that  $\chi(\mathcal{N}_{C/\mathbb{P}^r}) = \chi(\mathcal{T}_{\mathbb{P}^r}|_C) - \chi(\mathcal{T}_C)$  and that if  $H^1(\mathcal{T}_{\mathbb{P}^r}|_C) = 0$  then  $\chi(\mathcal{N}_{C/\mathbb{P}^r}) = h^0(\mathcal{N}_{C/\mathbb{P}^r})$ . Since  $\mathcal{T}_C = \omega_C^{-1}$ , the Riemann-Roch theorem gives  $\chi(\mathcal{T}_C) = -3g + 3$ .

To compute  $\chi(\mathcal{T}_{\mathbb{P}^r}|_C)$  we restrict the Euler sequence

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^r} \rightarrow \mathcal{O}_{\mathbb{P}^r}(1)^{r+1} \rightarrow \mathcal{T}_{\mathbb{P}^r} \rightarrow 0$$

to  $C$ . Using the Riemann-Roch theorem again we deduce that

$$\chi(\mathcal{T}_{\mathbb{P}^r}) = (r+1)\chi(\mathcal{O}_C(1)) - \chi(\mathcal{O}_C) = (r+1)(d-g+1) - (1-g) = (r+1)d + r(1-g)$$

From the restriction of the sequence above we also see that if  $\mathcal{O}_C(1)$  is nonspecial then  $H^1(\mathcal{T}_{\mathbb{P}^r}|_C) = 0$ .

Putting these values together, we get  $\chi(\mathcal{N}_{C/\mathbb{P}^r}) = (r+1)d - (r-3)(g-1)$  as required.

Whenever  $\dim H_{g,r,d}^o = h^0(\mathcal{N}_{C/\mathbb{P}^r})$  the Hilbert scheme is smooth at  $C$  by Theorem 3.2. From the deformation theory argument in 7.1 we know that this dimension equality is true for any nonspecial embedding since in this case  $H^1(\mathcal{N}_{C/\mathbb{P}^r}) = 0$ , so the upper and lower estimates for the dimension coincide. If  $d > 2g - 2$ , then every curve in  $H_{g,r,d}^o$  is nonspecial.

We can also prove the dimension statement for nonspecial embeddings invoking only the existence of the relative Picard scheme from Chapter 5: If  $\mathcal{O}_C(1)$  is nonspecial, then the nonspecial invertible sheaves of degree  $d$  on curves of genus  $g$  form an open subset of the  $(3g - 3 + g)$ -dimensional relative Picard scheme. The dimension of the space of sections is  $d - g + 1$ , so the family of  $r$ -dimensional linear series associated to each invertible sheaf is the dimension of the Grassmannian,  $\dim G(r+1, d-g+1) = (r+1)(d-g-r)$ , and the choice of a basis of the linear series, up to scalars, adds  $(r+1)^2 - 1 = \dim PGL_{r+1}$ . Thus the dimension of  $H_{g,r,d}^o$  near  $[C]$  is

$$3g - 3 + g + (r+1)(d-g-r) + (r+1)^2 - 1 = (r+1)d - (r-3)(g-1)$$

as required.

The irreducibility of the Hilbert scheme in the case  $d > 2g - 2$  also follows from this argument together with the existence of a connected family containing all curves of genus  $g$ , for example over the Hilbert scheme of tricanonical curves.  $\square$

## 8. Some open problems

**8.1. Brill-Noether in low codimension.** If  $C \subset \mathbb{P}^r$  is a nonspecial embedding, then the set of isomorphism classes of the curves represented in the component of  $H_{d,g,r}^\circ$  containing  $[C]$  is open in  $M_g$ . We shall see in Section 11 that every component of  $H_{d,g,r}^\circ$  dominating  $M_g$  in this sense has dimension  $h(g, r, d)$ . What about “smaller” components? Observations suggest that components of  $H^\circ$  whose images in  $M_g$  have low codimension still have the expected dimension  $h(g, r, d)$ : among the examples we know of components of the Hilbert scheme whose dimension is strictly greater than the expected  $h(g, r, d)$ , there are none whose image in  $M_g$  has codimension less than  $g - 4$ .

**Conjecture 8.1.** *If  $K \subset H_{d,g,r}^\circ$  is any component of a restricted Hilbert scheme, and the image of  $K$  in  $M_g$  has codimension  $\leq g - 4$ , then  $\dim K = h(g, r, d)$ .*

A few cases of this conjecture are known; see 11.2

**8.2. Maximally special curves.** How special can a linear series on a special curve be?

To make such a question precise, let  $\tilde{M}_{g,d}^r \subset M_g$  be the closure of the image of the map  $\phi : H_{g,r,d}^\circ \rightarrow M_g$  sending a curve to its isomorphism class. We ask,

- (1) What is the smallest possible dimension of  $H_{g,r,d}^\circ$ ?
- (2) What is the smallest possible dimension of  $\tilde{M}_{g,d}^r$ ? and
- (3) Modifying the last question slightly, let  $M_{g,d}^r \subset M_g$  be the closure of the locus of curves  $C$  that possess a  $g_d^r$  (in other words, we are dropping the condition that the  $g_d^r$  be very ample). What is the smallest possible dimension of  $M_{g,d}^r$ ?

One might guess that the most special curves, from the point of view of questions 2 and 3, are hyperelliptic curves. A hyperelliptic curve is determined by a set of  $2g + 2$  points in  $\mathbb{P}^1$ , modulo the action of  $PGL_2$ , so the locus in  $M_g$  of hyperelliptic curves has dimension  $2g - 1$ . Smooth plane curves are a better guess – the locus in  $M_g$  of smooth plane curves has dimension asymptotic to  $g$  (Exercise 12.22) – but there are still a lot of them.

Can we do better? Consider the locus of smooth complete intersections of two surfaces of degree  $m$  in  $\mathbb{P}^3$ . (Exercise ?? suggests why we are choosing complete intersections of surfaces of the same degree.) As we saw in Exercise 12.7, these comprise an open subset  $H_{ci}^\circ$  of the Hilbert scheme of curves of degree  $d = m^2$ , and genus  $g$  given by the relation

$$2g - 2 = \deg K_C = m^2(2m - 4),$$

or, asymptotically,

$$g \sim m^3.$$

Moreover, the dimension of this component of the Hilbert scheme is easy to compute: as we saw in Exercise 12.8, it is isomorphic to an open subset of the Grassmannian  $G(2, \binom{m+3}{3})$ , and so has dimension

$$2 \left( \binom{m+3}{3} - 2 \right) \sim \frac{m^3}{3}.$$

To compute the dimension of the fibers of the map from this Hilbert scheme to  $M_g$  we use the facts that if  $C \subset \mathbb{P}^r$  is a complete intersection curve of genus  $g > 1$  then the canonical bundle  $K_C$  is a positive power of  $\mathcal{O}_C(1)$ , and by Exercise 4.16  $C$  is arithmetically Cohen-Macaulay. For a given abstract curve  $C$  there are only finitely many invertible sheaves having the canonical sheaf as a power; and since an arithmetically Cohen-Macaulay curve is necessarily embedded by a complete linear series, there are only finitely many embeddings of a given curve as a complete intersection, up to  $PGL_{r+1}$ . Thus the fibers of  $H_{ci}^\circ$  over  $M_g$  have dimension  $\dim(PGL_{r+1}) = r^2 + 2r$ .

This construction gives us a sequence of components of the restricted Hilbert scheme  $H_{g,3,d}^\circ$  whose images in  $M_g$  have dimension tending asymptotically to  $g/3$ .

More generally, we can consider complete intersections of  $r - 1$  hypersurfaces of degree  $m$  in  $\mathbb{P}^r$ . Such curves have genus  $g = m^{r-1}((r-1)m - r - 1)/2 + 1$  in a similar fashion we can calculate that their images in  $M_g$  have dimension asymptotically approaching  $2g/r! = (m-1)(r-1)/r!$  as  $m \rightarrow \infty$ , as we ask you to verify in Exercise 12.24.

These components have the smallest images in  $M_g$  of any we know. To pose a precise question: if we fix  $r$ , can we find a sequence of components  $H_n$  of restricted Hilbert schemes  $H_{g_n,r,d_n}^\circ$  of curves in  $\mathbb{P}^r$  such that

$$\lim \frac{\dim H_n}{g_n} = 0?$$

**8.3. Rigid curves?** In the last section, we considered components of the restricted Hilbert scheme whose image in  $M_g$  was “as small as possible.” Let’s go now all the way to the extreme, and ask: is there a component of the restricted Hilbert scheme  $H_{g,r,d}^\circ$  whose image in  $M_g$  is a single point? (Of course  $M_0$  itself is a single point, so we exclude genus 0.) We can give three flavors of this question, in order of ascending preposterousness. Let  $C \subset \mathbb{P}^r$  be a smooth irreducible nondegenerate curve.

- (1) We call  $C$  *moduli rigid* if it lies in a component of the restricted Hilbert scheme whose image in  $M_g$  is just the point  $[C] \in M_g$ —in other words, if the linear series  $|\mathcal{O}_C(1)|$  does not deform to any nearby curves.
- (2) We call  $C$  *rigid* if it lies in a component  $H_{g,r,d}^\circ$  of the restricted Hilbert scheme such that  $PGL_{r+1}$  acts transitively on  $H_{g,r,d}^\circ$ . This is saying that  $C$  is moduli rigid, plus the line bundle  $\mathcal{O}_C(1)$  does not deform to any other  $g_d^r$  on  $C$ .
- (3) We call  $C$  *deformation rigid* if the curve  $C \subset \mathbb{P}^r$  has no nontrivial infinitesimal deformations other than those induced by  $PGL_{r+1}$ —in other words, every global section of the normal bundle  $\mathcal{N}_{C/\mathbb{P}^r}$  is the image of the restriction of a vector field on  $\mathbb{P}^r$ .

Do any such curves exist? This is not so much a question as a howl of frustration. The existence of irrational rigid curves seems outlandish; we don’t know anyone who thinks there are such things. But then, why can’t we prove that they don’t exist?

## 9. Degree 8, genus 9

So far, with the reader’s presumed assistance, we have examined all the  $H_{g,3,d}^\circ$  with  $d \leq 7$  and found only irreducible varieties whose components have the expected dimension  $4d = h(g, 3, d)$ . But it turns out that this is not typical.

We start with an example of a component of  $H_{9,3,8}^\circ$  whose dimension is strictly greater than  $4d$ . Let  $C$  be a curve in this Hilbert scheme, and consider the restriction map

$$\rho_2 : H^0(\mathcal{O}_{\mathbb{P}^3}(2)) \longrightarrow H^0(\mathcal{O}_C(2)).$$

The source of  $\rho_2$  has dimension 10. The Riemann-Roch theorem admits two possibilities for the dimension of the target.

$$h^0(\mathcal{O}_C(2)) = \begin{cases} 9, & \text{if } \mathcal{O}_C(2) \cong K_C; \\ 8, & \text{if } \mathcal{O}_C(2) \not\cong K_C. \end{cases}$$

However, if  $h^0(\mathcal{O}_C(2))$  were 8 then  $C$  would lie on two distinct quadrics  $Q$  and  $Q'$ . Since  $C$  is nondegenerate, it cannot lie on a reducible quadric; thus  $Q$  and  $Q'$

would be irreducible, violating Bézout's theorem. We deduce that  $\mathcal{O}_C(2) \cong K_C$ , and thus that  $C$  lies on a unique quadric surface  $Q$ .

Similarly, since  $\deg C > 2 \cdot 3$ , the curve  $C$  cannot lie on any cubic not containing  $Q$ . Moving on to quartics, we look again at the restriction map

$$\rho_4 : H^0(\mathcal{O}_{\mathbb{P}^3}(4)) \longrightarrow H^0(\mathcal{O}_C(4)).$$

The dimensions here are, respectively, 35 and  $4 \cdot 8 - 9 + 1 = 24$ ; and we deduce that  $C$  lies on at least an 11-dimensional vector space of quartic surfaces. On the other hand, only a 10-dimensional vector subspace of these vanish on  $Q$ . Thus  $C$  lies on a quartic surface not containing  $Q$ . It follows from Bézout's theorem and Lasker's theorem that  $C = Q \cap X$  and moreover, by Lasker's theorem, the homogeneous ideal of  $C$  is generated by the forms defining  $Q$  and  $X$ . Thus  $\ker(\rho_4)$  has dimension exactly 11, and  $X$  is unique modulo quartics vanishing on  $Q$ .

From these facts it is easy to compute the dimension of  $H_{9,3,8}^\circ$ : Associating to  $C$  the unique quadric on which it lies gives a map  $H_{9,3,8}^\circ \rightarrow \mathbb{P}^9$  with dense image, and each fiber is an open subset of the projective space  $\mathbb{P}V$ , where  $V$  is the 25-dimensional vector space

$$V = \frac{H^0(\mathcal{O}_{\mathbb{P}^3}(4))}{H^0(\mathcal{I}_{Q/\mathbb{P}^3}(4))}.$$

By Exercise 5.5,  $H_{9,3,8}^\circ$  is generically smooth, as well.

In sum, we have proven:

**Proposition 9.1.** *The scheme  $H_{9,3,8}^\circ$  of curves of genus is generically smooth and irreducible of dimension 33—one larger than the “expected”  $4d$ .*

This is a special case of Exercise 12.12. See Exercise 12.21 for a rich set of examples of components whose dimension is  $> h(g, r, d)$ .

## 10. Degree 9, genus 10

For an example of a restricted Hilbert scheme that is reducible, consider  $H_{9,3,10}^\circ$ , the scheme of smooth irreducible curves of degree 9 and genus 10 in  $\mathbb{P}^3$ .

**Proposition 10.1.** *The scheme  $H_{10,3,9}^\circ$  of smooth, irreducible, nondegenerate curves  $C \subset \mathbb{P}^3$  of degree 9 and genus 10 has two irreducible components, each generically smooth of dimension 36, the expected dimension. One consists of the complete intersections of two cubics. The other consists of curves of type  $(3, 6)$  or  $(6, 3)$  on a smooth quadric surface.*

**Proof.** To describe a smooth irreducible nondegenerate curve  $C$  of degree 9 and genus 10 in  $\mathbb{P}^3$  we look at the restriction maps  $\rho_m : H^0(\mathcal{O}_{\mathbb{P}^3}(m)) \longrightarrow H^0(\mathcal{O}_C(m))$ .

The Riemann-Roch theorem admits the possibilities:

$$h^0(\mathcal{O}_C(2)) = \begin{cases} 10, & \text{if } \mathcal{O}_C(2) \cong K_C \text{ ("type 1,") and} \\ 9, & \text{if } \mathcal{O}_C(2) \not\cong K_C \text{ ("type 2.")} \end{cases}$$

Unlike the situation in degree 8, both occur.

1. Suppose first that  $C$  does not lie on any quadric surface (so that  $C$  is of type 1), and consider the map  $\rho_3 : H^0(\mathcal{O}_{\mathbb{P}^3}(3)) \rightarrow H^0(\mathcal{O}_C(3))$ . By the Riemann-Roch theorem, the dimension of the target is  $3 \cdot 9 - 10 + 1 = 18$ , from which we conclude that  $C$  lies on at least a pencil of cubic surfaces. Since  $C$  lies on no quadrics, all of these cubic surfaces must be irreducible, and it follows by Bézout's theorem that the intersection of two such surfaces is exactly  $C$ . At this point, Lasker's theorem assures us that  $C$  lies on exactly two cubics.

By Exercise 12.8 the space  $H_1^\circ$  of curves of this type is an open subset of the Grassmannian  $G(2, 20)$  of pencils of cubic surfaces, which is irreducible of dimension  $36 = 4d$ . By Exercise 5.5,  $H_1^\circ$  is generically smooth.

2. Next, suppose that  $C$  has type 2—that is,  $C$  does lie on a quadric surface  $Q \subset \mathbb{P}^3$ ; let  $H_2^\circ \subset H_{10,3,9}^\circ$  be the locus of such curves. First, suppose that the quadric  $Q$  is singular. Since  $C$  is irreducible and nondegenerate,  $Q$  must be the cone over a conic, that is, the scroll  $S(0, 2)$ . Since  $\deg C = 9$  we must have  $C \sim 4H + F$ , where  $H$  is the hyperplane and  $F$  the ruling on  $Q$ . By Theorem 4.10 the genus of such a curve is 12, not 10, a contradiction proving that the quadric  $Q$  is smooth.

If  $C$  has class  $(a, b)$  on  $Q$  then  $a+b=9, (a-1)(b-1)=10$  has the solutions  $(a, b) = (3, 6)$  or  $(6, 3)$ .

We can show that  $H_2^\circ$  is generically smooth by computing its tangent space  $H^0(\mathcal{N}_{C/\mathbb{P}^3})$ . We start with the normal bundle sequence

$$0 \rightarrow \mathcal{N}_{Q/\mathbb{P}^3}|_C \rightarrow \mathcal{N}_{C/\mathbb{P}^3} \rightarrow \mathcal{N}_{C/Q} \rightarrow 0$$

and note that, by the adjunction theorem  $\omega_C = \mathcal{O}_Q(1, 4)|_C$ . Thus both  $\mathcal{N}_{Q/\mathbb{P}^3}|_C = \mathcal{O}_Q(2, 2)|_C$  and  $\mathcal{N}_{C/Q} = \mathcal{O}_Q(3, 6)|_C$  are nonspecial, so

$$h^0(\mathcal{N}_{C/\mathbb{P}^3}) = h^0(\mathcal{N}_{\mathcal{O}_Q(2,2)}|_C) + h^0(\mathcal{O}_Q(3,6)|_C) = 10 - 1 + (4 \cdot 7 - 1) = 36$$

We outline two proofs that  $H_2^\circ$  is irreducible in Exercise 12.18.  $\square$

These examples are far from exhausting the possibilities for the schemes  $H_{g,3,d}^\circ$ . For example the scheme  $H_{14,3,24}^\circ$  has 3 components, one of which is everywhere non-reduced [Mumford 1962] and [Nasu 2008].

## 11. Estimating the dimension of the restricted Hilbert schemes using the Brill-Noether theorem

The Brill-Noether theorems lead to an understanding of at least one component of  $H_{g,r,d}^\circ$  when the Brill-Noether number is nonnegative:

**Theorem 11.1.** *Let  $g \geq 2, d$  and  $r$  be nonnegative integers such that the Brill-Noether number  $\rho(g, r, d) = g - (r + 1)(g - d + r) \geq 0$  and  $r \geq 3$ . There is a unique component  $H_0$  of the restricted Hilbert scheme  $H_{g,r,d}^\circ$  dominating the moduli space  $M_g$ ; and this component has the “expected dimension”*

$$\dim H_0 = h(g, r, d).$$

The component  $H_0$  identified in Theorem 11.1 is called the *principal component* of the Hilbert scheme; there may be others as well, of possibly different dimension, and we do not know precisely for which  $d, g$  and  $r$  these occur. In case  $\rho < 0$ , the Brill-Noether theorem tells us that there is no component of  $H_{g,r,d}^\circ$  dominating  $M_g$ .

**Proof.** Consider the spaces

$$H_{g,r,d}^\circ \longrightarrow P_{d,g} = \{(C, \mathcal{L}) \mid \mathcal{L} \in W_d^r(C)\} \longrightarrow M_g.$$

Starting from the right, we compute dimensions:

- $M_g$  is irreducible of dimension  $3g - 3$ . Let  $C$  be a general curve of genus  $g$ .
- The Brill-Noether theorem tells us that the variety  $W_d^r(C)$  has dimension  $\rho$ , and the general point of  $W_d^r(C)$  corresponds to a very ample line bundle with exactly  $r + 1$  sections. If  $\rho > 0$  then  $W_d^r(C)$  is irreducible, while if  $\rho = 0$  then the monodromy action on  $W_d^r(C)$  is transitive.
- Therefore, over a general point of  $W_d^r(C)$  the fiber of  $H_{g,r,d}^\circ$  is isomorphic to  $PGL_{r+1}$ . Thus there is a unique component of  $H_{g,r,d}^\circ$  dominating  $W_d^r(C)$  and therefore a unique component of  $H_{g,r,d}^\circ$  dominating  $M_g$ .
- Adding up the dimensions of base and fiber, we see that the dimension of this unique component is  $(3g - 3) + \rho(g, r, d) + ((r + 1)^2 - 1)$ , and arithmetic shows that this is  $h(g, r, d)$ .

□

**Cheerful Fact 11.2.** *Although when  $\rho(g, r, d) < 0$  there is no component of  $H_{g,r,d}^\circ$  dominating moduli, one could hope that when  $\rho$  is not very negative one could*

understand a component dominating a subset of  $M_g$  of codimension  $-\rho$ , leading to a proof of Conjecture 8.1 in such cases. Indeed, this can be done when  $\rho(g, r, d) \geq -2$ : In [Eisenbud and Harris 1989], it is shown that if  $\Sigma \subset M_g$  is any subvariety of codimension 1, then the curve  $C$  corresponding to a general point of  $\Sigma$  has no linear series with Brill-Noether number  $\rho < -1$ ; and Edidin in [Edidin 1993] proves the analogous (and much harder) result for subvarieties of codimension 2.

## 12. Exercises

**Exercise 12.1.** Let  $C \cong \mathbb{P}^1 \subset \mathbb{P}^3$  be a twisted cubic. Show that the normal bundle  $\mathcal{N}_{C/\mathbb{P}^3} \cong \mathcal{O}_{\mathbb{P}^1}(5)^{\oplus 2}$ , the direct sum of two line bundles of degree 5. Use this to prove that the restricted Hilbert scheme  $H^\circ$  of twisted cubics is everywhere smooth.

Hint: Restricting the presentation matrix of  $I_C$  to  $C$ —that is, substituting the forms of degree 3 in 2 variables for the variables in the presentation matrix—we get a presentation

$$R(-9)^2 \xrightarrow{A} R(-6)^3 \rightarrow I_C/I_C^2 \rightarrow 0$$

The kernel of the dual of  $A$  is the normal bundle; it has 2 linear generators,

$$\begin{pmatrix} t & 0 \\ -s & t \\ 0 & -s \end{pmatrix}$$

as a module over  $\mathbb{C}[s, t]$ .

**Exercise 12.2.** Show that the locus  $\Sigma$  of schemes  $X$  consisting of a nodal plane cubic curve  $C$  with a spatial embedded point of multiplicity 1 at the node is dense in the intersection  $\overline{H^\circ} \cap \overline{H'}$ .

**Exercise 12.3.** Compute the dimension of each of the following subsets of  $Hilb_{3m+1}$ :

- (1) unions of a conic and a line meeting it in 1 point.
- (2) the connected union of 3 lines not all contained in the same plane
- (3) nodal plane cubics together with an embedded point at the node that is not contained in the plane of the cubic.

Hint: the dimensions are 11, 10 and 11.

**Exercise 12.4.** Give an argument for Proposition 3.1 in case  $d = 4$  using linkage.

Hint: a rational quartic curve is residual to the union of two skew lines in the complete intersection of a quadric and a cubic.

**Exercise 12.5.** Let  $C \cong \mathbb{P}^1 \subset \mathbb{P}^3$  be a smooth rational curve of any degree  $d$ .

- (1) Show that  $h^1(\mathcal{N}_{C/\mathbb{P}^3}) = 0$ ; that is, the normal bundle of  $C$  is nonspecial.
- (2) Using this, the Riemann-Roch formula for vector bundles on a curve and Proposition 3.1, show that the Hilbert scheme  $H$  is smooth at the point  $[C]$ .

Hint: on a smooth rational curve any locally free sheaf that is generated by global sections is nonspecial.

**Exercise 12.6.** Let  $C = Q \cap Q' \subset \mathbb{P}^3$  be a smooth curve of degree 4 and genus 1. Identify the normal bundle  $\mathcal{N}_{C/\mathbb{P}^3}$  of  $C$ , and use this to conclude that  $H_{1,3,4}^\circ$  is itself reduced, and even smooth, and thus isomorphic to an open subset of the Grassmannian  $G(2, 10)$ .

Hint:  $\mathcal{N}_{C/\mathbb{P}^3} \cong \mathcal{O}_C(2)^{\oplus 2}$ .

**Exercise 12.7.** Let  $m \geq n > 0$  be two positive integers. Show that the locus  $U_{m,n} \subset H^\circ$  of curves  $C \subset \mathbb{P}^3$  that are smooth complete intersections of surfaces of degrees  $m$  and  $n$  is an open subset of the Hilbert scheme.

Hint: if  $C = S \cap T$ , show that every deformation of  $C$  lies on a deformation of  $S$  and a deformation of  $T$ .

**Exercise 12.8.** Consider the locus  $U_{n,n} \subset H^\circ$  of curves  $C \subset \mathbb{P}^3$  that are smooth complete intersections of two surfaces of degrees  $n$ . Show that  $U_{n,n}$  is isomorphic to an open subset of the Grassmannian  $G(2, H^0(\mathcal{O}_{\mathbb{P}^3}(n)))$ .

**Exercise 12.9.** Let  $C \subset \mathbb{P}^3$  be a smooth curve of degree 5 and genus 2, and assume that the quadric surface  $Q$  containing  $C$  is smooth. From the exact sequence

$$0 \rightarrow \mathcal{N}_{C/Q} \rightarrow \mathcal{N}_{C/\mathbb{P}^3} \rightarrow \mathcal{N}_{Q/\mathbb{P}^3}|_C \rightarrow 0,$$

calculate  $h^0(\mathcal{N}_{C/\mathbb{P}^3})$  and deduce that  $H_{2,3,5}^\circ$  is smooth at the point  $[C]$ . Does this conclusion still hold if  $Q$  is singular?

**Exercise 12.10.** Show that a smooth, irreducible, nondegenerate curve  $C \subset \mathbb{P}^3$  of degree 5 and genus 1 is residual to a rational quartic in the complete intersection of two cubics, and use the result of subsection 3.1 to deduce that the space of genus 1 quintics is irreducible of dimension 20.

**Exercise 12.11.** (1) Show that all genera  $g \leq 4$  occur among curves of degree 6 in  $\mathbb{P}^3$ ; that is, there exists a smooth irreducible, nondegenerate curve of degree 6 and genus  $g$  in  $\mathbb{P}^3$  for all  $g \leq 4$ .

- (2) What is the largest possible genus of a smooth irreducible, nondegenerate curve  $C \subset \mathbb{P}^3$  of degree  $d = 7$ ? Can you do this with Clifford and Riemann-Roch, or do you need to invoke Castelnuovo?

**Exercise 12.12.** As before, let  $U_{m,n} \subset H^\circ$  be the locus of curves  $C \subset \mathbb{P}^3$  that are smooth complete intersections of surfaces of degrees  $m$  and  $n$ . In case  $m > n$ , show that  $U_{m,n}$  is isomorphic to an open subset of a projective bundle over the projective space  $\mathbb{P}(H^0(\mathcal{O}_{\mathbb{P}^3}(n))) \cong \mathbb{P}^{\binom{n+3}{3}-1}$  of surfaces of degree  $n$ , with fiber over the point  $[S] \in \mathbb{P}(H^0(\mathcal{O}_{\mathbb{P}^3}(n)))$  the projective space  $\mathbb{P}(H^0(\mathcal{O}_{\mathbb{P}^3}(m)))/H^0(\mathcal{I}_{S/\mathbb{P}^3}(m)) \cong \mathbb{P}^{\binom{m+3}{3}-\binom{m-n+3}{3}-1}$ .

Note that by Exercise 5.5 the scheme  $H^\circ$  is smooth.

**Exercise 12.13.** Let  $C$  be a curve of degree 6 and genus 3, and assume that  $C$  does not lie on any quadric surface. Show that  $C$  is residual to a twisted cubic in the complete intersection of two cubic surfaces, and use this to deduce that the space of such curves is irreducible of dimension 24.

**Exercise 12.14.** Now let  $C$  again be a curve of degree 6 and genus 3, but now assume that  $C$  does lie on a quadric surface  $Q$ . Show that such a curve is a flat limit of curves of the type described in the last exercise, and conclude that  $H_{3,3,6}^\circ$  is irreducible of dimension 24. (Hint: Let  $L$ ,  $Q$  and  $F$  denote a general linear form, a general quadratic form and a general cubic form, and consider the pencil of surfaces  $S_t = V(tF + LQ) \subset \mathbb{P}^3$  specializing from the cubic surface  $V(F)$  to the reducible cubic  $V(LQ)$ .)

**Exercise 12.15.** By analyzing the geometry of linear series of degrees  $2g - 1$  and  $2g$  on a curve of genus  $g$ , extend Proposition 7.2 to the cases  $d = 2g - 1$  and  $2g$ . What goes wrong if  $d \leq 2g - 2$ ?

**Exercise 12.16.** Show that each component of the restricted Hilbert schemes of curves of degree 7 in  $\mathbb{P}^3$  has expected dimension 28:

- (1)  $g \geq 6$ : By Castelnuovo's theorem, the largest possible genus of a curve of degree 7 in  $\mathbb{P}^3$  is 6, and by Theorem 4.5 or a comparison of the Hilbert polynomial with the Hilbert polynomial of  $\mathbb{P}^3$ , these lie on a quadric.
- (2)  $g = 5$ : Show that there is a unique component, and it dominates the moduli space of curves of genus 5. Show that such a curve is residual to a conic in the complete intersection of two cubics.
- (3)  $g \leq 4$ : Since these curves are nonspecial, there is a unique irreducible component.

**Exercise 12.17.** (Iarrobino) Show that ideals generated by  $a$  independent forms of degree  $d$  together with all the forms of degree  $d + 1$  in 3 variables form a family of 0-dimensional schemes of degree  $d := \binom{3+d}{3} - a$  in  $\mathbb{P}^3$ , which can be interpreted as a subvariety of the Hilbert scheme  $Hilb_d(\mathbb{P}^3)$ . Show that the dimension of this subvariety is  $h := a(\binom{2+d}{2} - a)$ . Find values of  $d$  and  $a$  such

that  $h > 3d$ , and conclude that the Hilbert scheme  $Hilb_d(\mathbb{P}^3)$  has more than one component.

Hint: the ideals in question are sandwiched between two successive powers of the maximal ideal of a point in  $\mathbb{P}^3$

The next three exercises refer to the discussion of the two types of curves of degree 9 and genus 10 in Section 10:

**Exercise 12.18.** In Section 10 we identified a locus  $H_2^\circ$  of degree 9, genus 10 curves lying on a quadric. One can show that  $H_2^\circ$  is irreducible by a monodromy argument using Example 1.2, but one can also prove it via a liaison: a curve in  $H_2^\circ$  is residual to a union of three skew lines in the intersection of a quadric and a sextic surface. Use this to establish that  $H_2^\circ$  is irreducible.

**Exercise 12.19.** In this and the next exercise we refer to the classification of curves of degree 9 and genus 10 in  $\mathbb{P}^3$  given in Proposition 10.1. We used a dimension count to conclude that a general curve of type 1 could not be a specialization of a curve of type 2, and vice versa. Prove these assertions directly: specifically, argue that

- (1) by upper-semicontinuity of  $h^0(\mathcal{I}_{C/\mathbb{P}^3}(2))$ , argue that a curve  $C$  not lying on a quadric cannot be the specialization of curves  $C_t$  lying on quadrics; and
- (2) show that for a general curve of type (3, 6) on a quadric,  $K_C \not\cong \mathcal{O}_C(2)$ , and deduce that a general curve of type 2 is not a specialization of curves of type 1.

**Exercise 12.20.** Let  $\Sigma_1$  and  $\Sigma_2 \subset H_{10,3,9}^\circ$  be the loci of curves of types 1 and 2 respectively.

- (1) What is the intersection of the closures of  $\Sigma_1$  and  $\Sigma_2$  in  $H_{10,3,9}^\circ$ ?
- (2) What is the intersection of the closures of  $\Sigma_1$  and  $\Sigma_2$  in the whole Hilbert scheme  $H_{9m-9}(\mathbb{P}^3)$ ?

**Exercise 12.21.** Let  $H^\circ$  be a component of the Hilbert scheme parametrizing curves of degree  $d$  and genus  $g$  in  $\mathbb{P}^3$  that dominates the moduli space  $M_g$ . For  $s, t \gg d$ , let  $K^\circ$  be the family of smooth curves residual to a curve  $C \in H^\circ$  in a complete intersection of surfaces of degrees  $s$  and  $t$ .

- (1) Show that  $K^\circ$  is open and dense in a component of the Hilbert scheme of curves of degree  $st - d$  and the appropriate genus.
- (2) Calculate the dimension of  $K^\circ$ , and in particular show that it is strictly greater than  $h(g, r, d)$ .

**Exercise 12.22.** (1) Let  $C \subset \mathbb{P}^2$  be a smooth plane curve of degree  $d \geq 4$ . Show that the  $g_d^2$  cut by lines on  $C$  is unique; that is,  $W_d^2(C)$  consists of one point.

- (2) Using this, find the dimension of the locus of smooth plane curves in  $M_g$ .

Hint: the first part is Corollary 2.8.

**Exercise 12.23.** Consider the locus of curves  $C \subset \mathbb{P}^3$  that are complete intersections of a quadric surface and a surface of degree  $m$ . Show that these comprise components of the restricted Hilbert scheme, and that their images in moduli have dimension asymptotically approaching  $g$  as  $m \rightarrow \infty$ .

**Exercise 12.24.** Consider the locus  $H_{ci}^\circ$ , in the Hilbert scheme  $H^\circ$ , of smooth, irreducible, nondegenerate curves  $C \subset \mathbb{P}^r$  that are complete intersections of  $r - 1$  hypersurfaces of degree  $m$ .

- (1) Show that  $H_{ci}^\circ$  is open in  $H^\circ$ ;
- (2) Calculate the dimension of  $H_{ci}^\circ$  (and observe that it is irreducible); and
- (3) Show that the dimension of the image of  $H_{ci}^\circ$  in  $M_g$  is asymptotically  $2g/r!$  as  $m \rightarrow \infty$

# A historical essay on some topics in algebraic geometry

by  
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This essay offers a brief overview of aspects of the long history of plane curves and algebraic geometry.<sup>1</sup> It starts with some historical remarks about conic sections, moves on to look at some aspects of the early study of cubic and quartic curves and the emergence of real and complex projective space, then nods briefly at Bernhard Riemann's ideas about algebraic curves and Luigi Cremona's ideas about plane transformations before concluding with an introduction to the work of Alexander Brill and Max Noether in the late 19th century.

## 1. Greek mathematicians and conic sections

Quadratic problems were part of high-level teaching in ancient Mesopotamia around 1800 BCE. One such tablet (BM 13901) begins<sup>2</sup>

I have added up the area and the side of my square: 0;45.

---

<sup>1</sup>I am grateful to Andrea del Centina for letting me read the manuscript of his forthcoming book *The History of Projective Geometry*, which has been very helpful and will surely become the definitive work on the subject.

<sup>2</sup>This is tablet number 13901 in the British Museum collection.

The unstated task is to find the side of the square. The number 0;45 is  $\frac{3}{4}$  in sexagesimal notation, so if we rewrite this as an equation for  $s$ , the unknown side, we have

$$s^2 + s = \frac{3}{4}.$$

The tablet then goes through the steps of an algorithm much as we would for finding the positive root:  $s = \frac{1}{2}$ .

Other problems involve knowledge of the Pythagorean theorem, known also to Chinese and Indian mathematicians. But the first mathematicians to have studied curves more complicated than the straight line and the circle seem to have been those of ancient Greece. Surviving sources suggest that the study of conic sections, literally sections of a cone, may well have arisen with Menaechmus's work on doubling the cube around 350 BCE. This was a long-standing problem that was given a theological spin in the form of the Delian problem, which, according to one story, asked for an altar double the size of a given one, apparently to get rid of a plague. Finding this difficult, workers consulted Plato, who said that the real reason for the task was to reproach the Greeks for their neglect of geometry.<sup>3</sup> It may well also have stood out because Plato in the *Meno* dialogue made such a fuss of doubling the square. Hippocrates of Chios, who flourished around 470 BCE, had already reduced the problem to that of finding two mean proportionals between 1 and 2; i.e. quantities  $x$  and  $y$  such that

$$(3) \quad 1 : x = x : y = y : 2.$$

For Greek mathematicians, these quantities would have been line segments of those lengths.

Many solutions to the problem of finding two mean proportionals were proposed, a younger associate of Plato and a pupil of Eudoxus, may have been the first to connect the problem of doubling the cube to the idea of conic sections, or perhaps we should say quadratic curves, because it has been argued that he considered equation (3) as defining or generating curves that would have been drawn and understood pointwise. At all events, he expressed the solution to the Delian problem in terms of the intersection of a hyperbola and a parabola.<sup>4</sup>

About a century after Menaechmus, Diocles, and soon after him Apollonius, made significant progress with the conic sections.<sup>5</sup> Diocles identified the

<sup>3</sup>Theon of Smyrna, 2nd century CE, in (Thomas 1939, Vol. 1, 257).

<sup>4</sup>See Eutocius *Commentary on Archimedes' Sphere and Cylinder*, in (Cohen and Drabkin 1948, 62–66). Among Eudoxus's many achievements is to have provided the first theory of magnitudes that went beyond the rational numbers; it is now found in Book V of Euclid's *Elements*.

<sup>5</sup>The first places to go for an account of Greek mathematics are the analytical account in (Netz 2022) and the detailed if sometimes dated century-old account (Heath 1921).

curve that would focus the sun's rays to a point as the parabola, and knew precisely how to cut a cone so as to obtain it.<sup>6</sup>

Apollonius produced the first *theory* of conic sections as sections of a cone. The books are very dry. Van der Waerden called him a virtuoso "in dealing with geometric algebra, and also a virtuoso in hiding his original line of thought", while going on to say that "his reasoning was crystal clear and elegant."<sup>7</sup> Apollonius began with the basic definitions of a cone (on a circle) and its three types of section: the hyperbola, parabola, and ellipse; the names are due to him. He showed that all the known sections that had been studied as sections of a cone with vertical angle a right angle could be obtained as sections of a suitable but otherwise arbitrary cone.

It's arguable that a system of coordinates was present in his work, inasmuch as he generally referred everything to a pair of distinguished lines in the plane of section of the cone, but it is buried in the heavy use of proportion theory between different line segments. Apollonius produced a theory of the principal diameters of conics, studied such problems as finding the tangents to a conic from an exterior point, and came close to observing the properties of cross-ratio that modern writers were to detect. He also had a theory about normals to conics from which it's a short step to obtaining their evolutes. The biggest weakness in his theory was that very often he had to treat the three kinds of conic section separately, although you can see him trying for greater generality, as, for example, when he introduced the concept of conjugate hyperbolas in Book I to make the proofs of theorems about conjugate diameters run equally elegantly and simply for hyperbolas as ellipses.

Mathematicians of the Islamic era (active from the 9th to the 15th century C.E.) also studied conic sections and the various problems they inherited from the Greeks. One of the most outstanding is 'Umar ibn Ibrāhīm al-Khāyyamī (usually known in the West as Omar Khayyām), who was born in Nishapur in northern Iran in the 11th century CE and spent much of his life in Isfahan where he led a group of Islamic astronomers at the observatory there. In his book the *Risāla* he gave a complete account of how to solve cubic equations using pairs of conic sections where necessary.<sup>8</sup> Other Islamic mathematicians advanced the study of trigonometry, investigated the foundations of geometry (Euclid's parallel postulate, which al-Khāyyamī also investigated), invented methods for the numerical solution of polynomial equations, and studied various properties of conic sections, but there was little study of other kinds of curves.

<sup>6</sup>Apollonius's four books of his conics survive in Greek and three more in Arabic; the eighth and final volume is lost.

<sup>7</sup>See (Van der Waerden 1961, 248).

<sup>8</sup>See (al-Khāyyamī 1950).

## 2. The first appearance of complex numbers

In the 16th century, various Italian mathematicians, notably Gerolamo Cardano, Ludovico Ferrari, and Rafael Bombelli, succeeded in finding algebraic methods for solving cubic and quartic equations. This inevitably led them to have views about complex numbers. Cardano recognised in his (1545, Ch. 37) that they will appear in the formulae for the solution of these equations, even though he said “So progresses arithmetic subtlety, the end of which, as is said, is as refined as it is useless.” Bombelli was more hospitable to the new numbers in his (1572), and gave rules for manipulating them, but they remained mysterious and suspect for many years.

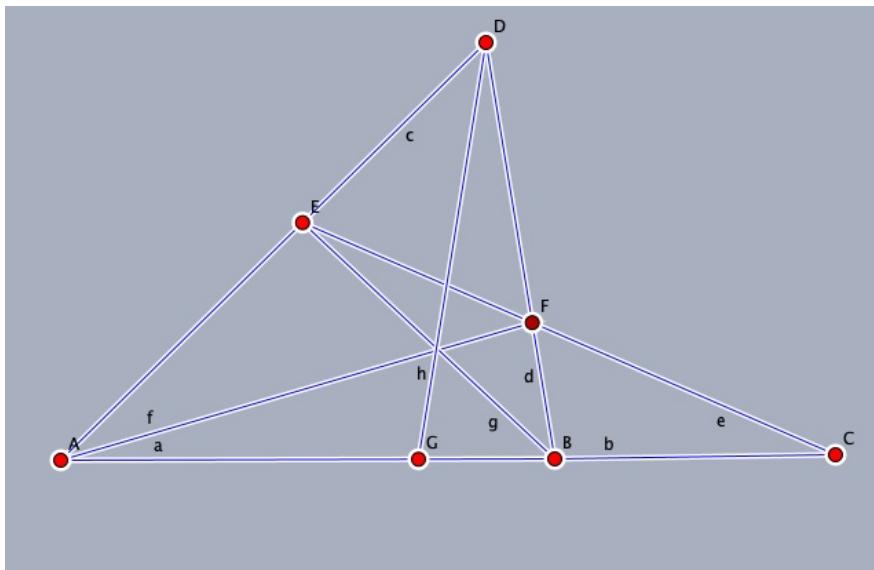
At this point, the history of algebraic geometry broadly divides into two. One part concerns the theory of conic sections – algebraically, curves of degree two – and the other the theory of curves of higher degree. The first can regard Girard Desargues as a major innovator, the second his contemporary, René Descartes. Together, they lead to the concept of a projective space. Let us take the study of conic sections first.

## 3. Conic sections from the 17th to the 19th centuries

In 1639 Desargues published fifty copies of a short pamphlet known as the *Brouillon Project* (1639), on what we could call the projective theory of conics. He was perhaps hoping for useful critical feedback, but he never returned to rewrite his account. In the essay, all non-degenerate conics are treated on a par, and treated as perspective images of the circle. The idea that all (non-degenerate) conics are the projective image of a circle is not original with Desargues. It was stated by Francesco Maurolico, for example, in his book (1611), which it is likely Desargues knew, and in any case it is visually apparent to anyone who thinks of the conics as sections of a cone on a circular base. The trick is to make this insight work.

Desargues's key idea is that of properties of figures invariant under projection, such as the cross-ratio of four points on a line: if  $A, B, C, D$  are four points on a line, their cross-ratio can be defined to be  $\frac{AC \cdot DB}{AD \cdot CB}$ . The special case, when the cross-ratio is  $-1$  and  $AC/CB = -AD/DB$  and people said  $B$  and  $D$  separate  $A$  and  $C$  in the same ratio, arises frequently in the theory of the poles and polars of conic section.<sup>9</sup> Desargues also had a more complicated property of six points, which, like the four-point property, was invariant under a projection. He connected all this to the study of the “complete quadrilateral” (see Figure 1) – take four points, no three collinear, and join them in all possible ways – and thence to the study of conics through those four points.

<sup>9</sup>Desargues did not use the concepts of pole and polar but of four or six points in involution.



**Figure 1.** A complete quadrilateral, consisting of the four points  $A, B, C, D$  and the three lines joining them in pairs

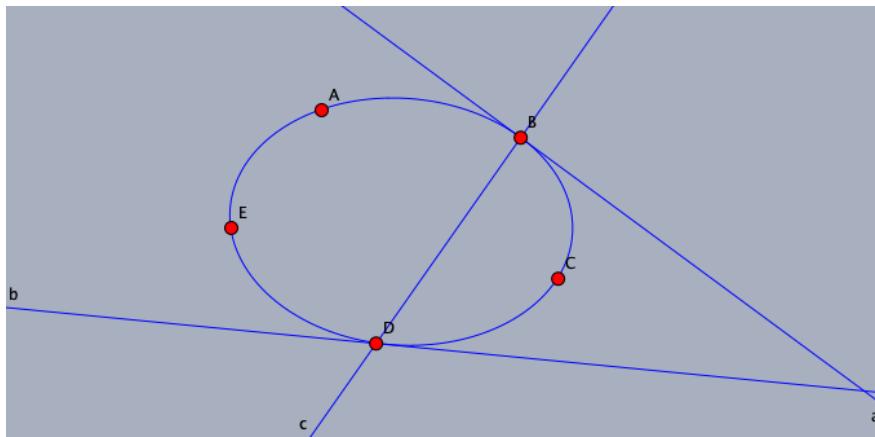
Jeremy says [\* Note to Editor: In Figure 1 please relabel the 7 points  $A, B, C, D, E, F, G$  as  $A, B, E, F, D, C, H$ ; remove the label  $G$ ; label the point where the lines  $AC$  and  $BD$  cross (on the new labels) as  $G$ ; delete the lower case letters that label the edges. \*]

The *Brouillon Project* is fiercely unreadable.<sup>10</sup> It was also lost for a long time and known only through commentaries by later authors until in 1845 Michel Chasles found a handwritten copy made by Philippe de la Hire.<sup>11</sup> In particular, de la Hire, a generation after Desargues, wrote some much more readable, and longer, works in Desargues's spirit, illuminating the role of cross-ratio in the theory of tangents that came close to a theory of duality. Desargues's famous theorem on two triangles in perspective was published separately in (Bosse 1648).

The best attention Desargues's little book got was from his younger contemporary Blaise Pascal, who evidently produced a virtually complete theory of conics around what he called the “mystical hexagram”. Unhappily, much of it is lost, and known to us only from some notes on it made by Leibniz, but the idea is that while there is always a conic through five points something happens if you want a conic through six points: we call it Pascal's theorem. Then, if you let the sixth point collapse onto one of the other five you get a tangent to

<sup>10</sup>Happily, it has been the subject of sequence of detailed analyses in recent papers by Marie Anglade and Jean-Yves Briand; see (Anglade and Briand 2022) and the references to their earlier papers cited there.

<sup>11</sup>The only known copy of the original *Brouillon project* turned up in 1950.



**Figure 2.** The tangents from a point  $P$  outside the conic touch it at  $A$  and  $B$ ; the line  $AB$  is the *polar* of  $P$  and  $P$  is the *pole* of the line  $AB$ . The construction can be extended to points inside the conic.

the conic through those five points. One way or another all the key properties of conics are wrapped up in this idea, or so Pascal seems to have shown, and much of the early 19th century work in France can be seen as attempts to recover such a theory, which would include such topics as duality, in the form of the pole and polar relationship with respect to a conic (see Figure 2).<sup>12</sup>

**Jeremy says** [\* Note to Editor: In Figure 2 please delete the points  $A, C, E$ ; relabel the points  $B, D$  as  $A, B$ ; make sure the lines through  $A$  and  $B$  that are claimed to be tangents really are, and label their point of intersection  $P$ . \*]

Thereafter, truly projective geometry languished for much of the 18th century, despite some insightful contributions by Isaac Newton that we shall mention below (see p. 384) and a few others. Its revival is conventionally dated from the early work of Gaspard Monge, who as a young man seeking a job in the French army was interested in how to depict three dimensions on two. He devised a method of plan and elevation (projections onto a horizontal and a vertical plane) that he could couple to some simple algebra in a way that was easy to use; as a result he was offered a job at the military Academy in Mézières and his discovery made a military secret. During the French Revolution, Monge was influential in setting up the École Polytechnique, where he was an inspiring teacher of geometry, and this did much to revive the subject.

<sup>12</sup>For a thorough analysis of Pascal's work and its context, see (Del Centina 2020).

Among those so inspired there was Jean Victor Poncelet, who promoted a much more general theory of transformations with a view to unifying the theory of conics that he began to develop while a prisoner-of-war in Saratov during Napoleon's disastrous invasion of Russia. His *Traité des Propriétés Projectives des Figures* (1822) is a visionary book that relies on some rather mysterious arguments about ideal points of intersection (Augustin-Louis Cauchy, who reviewed the book, urged that they be regarded as points with complex coordinates; Poncelet never agreed, and reproduced the review between the Preface and the Introduction to his *Traité* presumably to show his disdain). As a result, some of the transformations it invokes necessarily require complex coordinates. The most famous result in the book is Poncelet's closure theorem, which has continued to attract attention to this day, but it is also notable for many other theorems involving pairs of conics.

In the 1820s, Poncelet had a dispute with Joseph Diaz Gergonne, the editor of the only journal at the time entirely devoted to mathematics, about what duality in the plane actually is.<sup>13</sup> Poncelet always saw it as pole and polar with respect to a conic, Gergonne saw it as a new and fundamental feature of projective geometry.<sup>14</sup> Gergonne's view led to confusion when it was applied to curves of degree three or more, a matter that began to be sorted out only with Julius Plücker's work, as we shall see below.

Another mathematician who was inspired by Monge was Michel Chasles, who used the projective invariance of the cross-ratio of four points to eliminate much of the weirdness of Poncelet's ideas. Independently, Jakob Steiner did very similar work in Germany. He can be credited with the first truly projective definition of a conic section. Even so, there remained an irritating feature of cross-ratio: it was given as a function of four lengths, but length is a property of Euclidean geometry, not projective geometry. If projective geometry is to be regarded as more fundamental than Euclidean geometry, because it rests on fewer assumptions or axioms, then the intrusion of Euclidean length in the definition of cross-ratio is at the very least unfortunate. Nor can one easily speak of there being a concept of projective space in the 1820s; rather, much of geometry at this time was about figures in the plane subject to a variety of projective transformations. The first truly foundational work on real and complex projective geometry that avoided deriving it from Euclidean geometry was the achievement of Karl von Staudt in his *Geometrie der Lage* (1847), a long and difficult work that influenced Felix Klein when he succeeded von Staudt as a Professor at Erlangen twenty-five years later.<sup>15</sup>

<sup>13</sup>Gergonne's journal was called the *Annales de Mathématiques Pures et Appliquées*. It ran from 1810 to 1832, and it was succeeded in by Liouville's *Journal de Mathématiques Pures et Appliquées* in 1836. Crelle's *Journal für die reine und angewandte Mathematik* was founded in 1826.

<sup>14</sup>See (Gergonne 1826).

<sup>15</sup>See, for example, (Gray 2015).

All in all, a surprising amount of work was done on the theory of conic sections at the start of the 19th century, and before it is dismissed as arcane it should be stressed that the subject was a proving ground for the development of projective geometry. Two approaches stand out: the search for entirely general methods that would treat all non-degenerate conics on a par; and the emergence of the property of duality. Matters are complicated by the existence of two separate traditions, usually called the synthetic and the analytic, supposedly divided into classical geometric methods and more algebraic ones that introduce coordinates. Recent historical work suggests that it was all a bit murky, and algebraic methods were also often used in synthetic geometry.<sup>16</sup> Several things promoted the use of synthetic methods. They can be elegant when algebraic methods are blunt; they correspond to the visual form of the conics; they provide a language for describing what is apparent or to be found in a problem. Against them is the obstinate fact that algebra is more general: it does not care if some quantities become negative, but what is a negative length? Once ways round this obstacle were found (by Poncelet and then Chasles) the way was open to a truly systematic synthetic theory of conics.

How then did this change, and purely synthetic projective geometry begin to wane? Very few of the original protagonists disdained algebra outright, and as the (projective) theory of conics reached completion in the 1820s and established its fundamental character, being more general than metrical Euclidean geometry, it also had its baroque aspects. But worse, it did not generalise at all readily to the study of curves of higher degree. For that, as even Newton had recognised, a hefty dose of algebra was required.

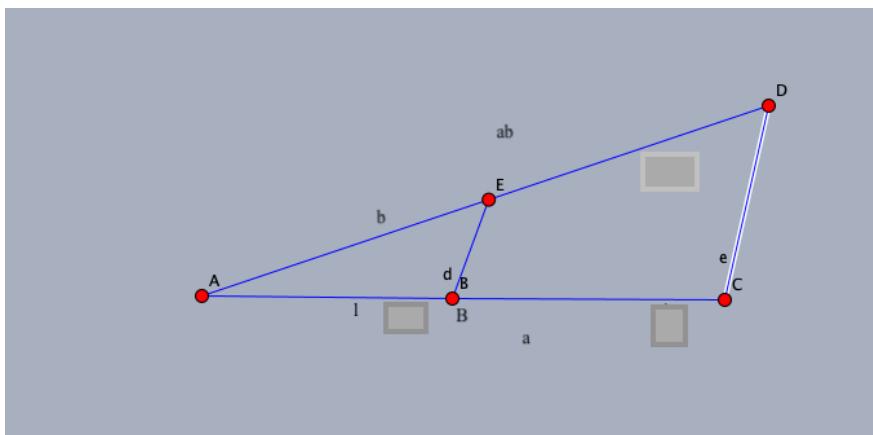
#### 4. Curves of higher degree from the 17th to the early 19th century

A common way to think of curves in antiquity was pointwise: some length depends in a given way on some other length. Accordingly, at least in principle, if you know the independent length (the ordinate, or  $x$  coordinate) you know the dependent length (the abscissa, or  $y$  coordinate).

By Descartes' time there was already some sophisticated algebra expressed in a formalism that hadn't quite shaken off the Greek insistence on seeing everything as geometrical magnitudes: lengths, areas, volumes, and, well, what exactly? The French mathematician François Viète in his *Isagoge* boldly spoke of magnitudes having the dimensions of side, square, cube, square-square, square-cube and so on.<sup>17</sup> It was possible to write polynomial equations in this language, as Viète did in the early 1600s. First Fermat in 1636, and then much

<sup>16</sup>See (Lorenat 2016).

<sup>17</sup>See his collected works, (Viète 1646).



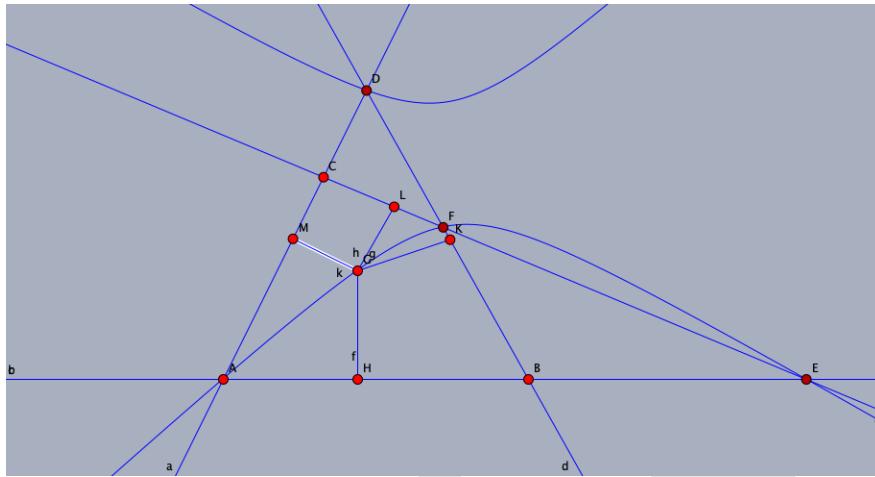
**Figure 3.** If  $AB$  is of length 1,  $AC$  of length  $a$ ,  $AE$  of length  $b$ , and  $BE$  and  $CE$  are parallel then  $AD$  is of length  $ab$ .

more boldly Descartes in 1637, realised that you could extend the language to two variables and so describe curves in the plane.

Jeremy says [\* Note to Editor: In Figure 3 please move the lower case letters in that denote lengths to better positions, and eliminate the shaded rectangles. \*]

Descartes's first achievement in his *La Géométrie* (1637) was to eliminate the dimensional aspect. A simple use of similar triangles allowed him to show that the product of two lengths could be seen as another length (not an area) so all geometrical quantities could be regarded as one-dimensional and the idea of dimension quietly dropped (see Figure 3). He also replaced Viète's cumbersome algebra, which was written in capital letters with verbal abbreviations for the algebraic operations, with something much more like what we use today. He wrote  $x$  and  $y$  for the key variables (not  $A$  and  $E$  in the manner of Viète). Descartes was clear that he was using coordinates, although his  $x$  and  $y$  coordinates could have oblique axes.

Then came the real work. Almost all mathematical problems in his day were expressed in the language of geometry, except for some problems we would call diophantine and were implicitly about integers and rational numbers. Accordingly, the answer had to be expressed geometrically. Descartes's idea was to give letters to all the lengths involved in a problem, use the statement of the problem to express relationships between the letters, and reduce the equations to a single equation. Then solve the equation and express the answer again in geometrical terms.



**Figure 4.** The Pappus problem: the product of the distances of  $P$  from the lines  $AB$  and  $BC$  is proportional to the product of the distances of  $P$  from the lines  $CD$  and  $DA$

In this way he solved the famous Pappus problem (see Figure 4): given four lines and four angles (nothing is lost if we take these angles to be right angles), find the locus of points  $P$  such that the product of the distances of  $P$  from the first two lines is proportional to the product of the distances of  $P$  from the last two lines. As he showed, and Pappus had known, the answer is a conic section, but Descartes went further and claimed that in this way he could solve the Pappus problem for any number of lines.

**Jeremy says [**\* Note to Editor: in Figure 4 please relabel  $F$  as  $C$ ,  $C$  as  $D$ , and  $G$  as  $P$ ; remove the lower case letters; label the point in the middle  $P$ , and make sure that the four lines out of  $P$  meet the four lines at right angles. **]\*]**

This was to infuriate Newton, who showed that Greek methods were indeed adequate to the original Pappus problem. The background here is that Newton was also engaged in demolishing Descartes's theory of planetary motion in favour of his own, which led him into the theory of conic sections and to pose and solve the problem of finding the conic (or conics) through  $n$  points and tangent to  $5 - n$  lines, which he did in Book I, Section 5 of his *Principia Mathematica*.

It can be argued that the theory of specifically algebraic curves has its origin in Descartes's method for finding normals to a curve at a point, which involved considering all the circles through the given point and imposing the condition that the equation for the circle be such that it passes twice through the given

point; this circle has its centre on the normal to the curve. On the basis of three examples, he claimed that the method always worked if the curve had an algebraic equation, and he described a system of linkages (sliding rulers) that he said could be adapted to draw any such curve. Curves like the cycloid that were patently not algebraic he sought to exclude from geometry — this exclusion was something else that annoyed Newton.

In the 1660s and 1670s, but only published the 1690s and again in 1704 as an Appendix to his *Opticks* Newton worked on a classification of cubic curves. He claimed that there were 72 distinct types which could be derived by simplifying the general equation using what we would call affine or linear transformations, although at one point he used a simple birational transformation. For some reason, in writing up his work for publication he omitted four of the possible cubics, and they were speedily found by various other mathematicians. Newton also made the striking remark that this collapses to five types if projective transformations are allowed.<sup>18</sup>

A plane cubic curve is defined by an equation involving ten coefficients, or more precisely by the nine ratios between the coefficients, which suggests that it should be determined by nine points in the plane because their coordinates provide nine equations that should determine these nine ratios. Moreover, it seemed to mathematicians in the early 18th century that any set of nine points and therefore any set of nine equations in the nine unknowns should have a unique solution — there was no theory of linear equations at the time — and as a result even the best mathematicians got into difficulties. For example, the Scottish mathematician Colin MacLaurin knew in 1720 that there were problems with the idea that nine points in the plane should determine a cubic: any two cubics will meet in nine points, and so these nine points do not determine a unique cubic, and indeed there will be infinitely many cubics through these nine points.<sup>19</sup> MacLaurin did not, however, know how to solve this puzzle. This apparent contradiction with the claim that nine points in the plane always determine a unique cubic became known as Cramer's paradox when the Swiss mathematician Gabriel Cramer addressed the problem in his book (1750, Chapter 3). Part of the confusion at the time was a lack of insight into systems of linear equations, and part was the lack of understanding about what are the implications of configurations that do not determine a unique curve.

More interestingly, Cramer also suggested that if the  $\frac{1}{2}n(n+3)$  points needed to determine a curve of degree  $n$  contain  $tn$  points common to a curve of degree  $t < n$  then the curve through the  $\frac{1}{2}(n+1)(n+2)$  points breaks up into two or

<sup>18</sup>"The five divergent parabolas, by their shadows, generate all other curves of the second genus [i.e. cubic curves]." See (Newton 1704) and Talbot (1860, p. 25).

<sup>19</sup>MacLaurin also knew that a general cubic curve has nine flexes, and the line joining any two passes through a third.

more curves, one of which passes through the  $tn$  points. We note that Cramer considered only irreducible curves: a circle and a line, for example, would be considered as two curves, not one.<sup>20</sup>

Cramer conveyed the (incorrect) claim about cubic curves, and more generally the analogous claim for curves of degree  $n$  — that they are determined by  $\frac{1}{2}n(n+3)$  general points — to his friend Leonhard Euler, who repeated them in the second volume of his *Introductio* (1748, § 81). Euler then wrote a paper (Euler 1750) in which he first spelled out the problem. It is a general proposition, he said, that  $k$  linear equations in  $k$  unknowns have a unique solution. Accordingly, 9 points in the plane will determine a unique cubic curve. But nine points common to two cubics plainly do not determine a unique cubic curve. To resolve this apparent contradiction, as he called it, he began with three equations in three unknowns and showed by example that there will not be a unique solution if one of these equations is contained in (that is, is a consequence of) the others. He drew the same conclusion about four linear equation in four unknowns, and stated more generally that there will not be a unique solution to a system of  $n$  equations in  $n$  unknowns if one or more are contained in all the others.

He then turned to the geometrical implications. In the case of conics, he showed that the containment condition corresponds to the case when four, or all five, of five given points lie on a line. For cubics, he remarked that one easily understands that not just one, but two or more of the nine equations specified by nine points may be contained in the others, and to obtain a unique cubic it will be necessary to specify that the curve passes through one or more additional points. However, he said, it was very difficult to see what the consequences were when this is the case, because there were so many points and coefficients that things become too complicated, although it was possible to draw conclusions in simple cases. For example, if the nine equations arise in part from four points lying on a line then the remaining five equations must determine a conic, which might itself consist of two lines. Euler concluded his paper with a few remarks about curves of higher degree.

The idea that algebraic curves of degrees  $k$  and  $m$  in the plane should meet in  $km$  points was something of a folklore result in the early 18th century, but it travelled without a proof for many years. Euler discussed it in the second volume of his *Introductio* (1748, Ch. 19) and noted that even in simple cases for the counting to work one would have to take care of multiple points (such as tangents), points ‘at infinity’ (consider a parabola and a line parallel to its axis), and allow the coordinates of intersection points to be complex (consider the intersection of two circles). The first person to find a persuasive way of tackling

<sup>20</sup>Curves were taken to be irreducible even in Salmon’s book *Higher Plane Curves* (1852).

the problem was Étienne Bézout, who lived from 1739 to 1783, and made his living teaching mathematics at the French military and naval academies. He published the theorem that bears his name in a book of 1779; it is based on his theory of the resultant of two polynomial equations that he developed in a paper of 1764. His proof of the theorem is gappy and intuitive by any standards, but so much better than what had been done before that his immediate successors were willing to give him real credit for doing as much as he did. His results inspired later work by Cauchy and James Joseph Sylvester.

To state Bézout's theorem in general requires a notion of the multiplicity of an intersection of two plane curves without common components. Such a notion was developed by Max Noether, and refined by Francis Sowerby Macaulay. A modern proof would invoke Noether's Fundamental Theorem — see p. 396 — and a computation of the Hilbert polynomial. This is a story that has been pursued, with successive new generalisations, right up to the present day.

As this activity suggests, throughout the 18th century mathematicians had become more and more comfortable with the idea of complex numbers in algebra, and with the idea of proving that a polynomial of degree  $n$  has  $n$  roots, possibly with repetitions (the so-called fundamental theorem of algebra).<sup>21</sup> Euler's attitude to complex numbers was that there was nothing to explain. Although they cannot be ordered, expressions of the form  $a + bi$  behave arithmetically like numbers, so they can reasonably be considered numbers and “are usually called *imaginary quantities*, because they exist merely in the imagination.”<sup>22</sup> Cauchy's view later was more explicit and very close to regarding the field of complex numbers as  $\mathbb{R}[x]/(x^2 + 1)$ . This works for finding a proof of the fundamental theorem of algebra, which he gave a proof of in his (1817a, b), but was not productive in contexts involving contour integration.

Credit for the first rigorous proof of the fundamental theorem is often given to Carl Friedrich Gauss for his paper (1799), although he did base his argument on the claim that an algebraic curve that enters a bounded region of the plane also leaves it, which is surely no easier to prove. Gauss went on to give three more proofs of the theorem, and soon any doubts about the algebraic nature of complex numbers were resolved.<sup>23</sup> Niels Abel and Carl Gustav Jacob Jacobi, in their work on elliptic functions, took a formal, non-geometrical attitude to complex numbers. Both men were algebraists at heart – formidable algebraists – and the exact nature the plane of complex numbers did not really interest them.

<sup>21</sup>For a history of attempts on this theorem, see (Gilain 1991).

<sup>22</sup>See Euler, *Algebra* § 143, p. 43. Euler here rejected the idea that ultimately mathematical quantities must be exhibited in nature: three sheep, a length of  $\sqrt{2}$ , and so on.

<sup>23</sup>William Rowan Hamilton published his rigorous theory of ordered pairs of real numbers in his (1837).

All things considered, however, progress in the study of cubic, quartic, and higher degree curves in the 18th century was piecemeal and slight, and the sheer enormity of the equations involved seems to have baffled even Euler. New methods for handling such curves would have to be found, such as came in with Plücker in the 1820s. His key idea was to study families of curves, using the symbolic notation devised by Gergonne.<sup>24</sup> If  $S_1$  and  $S_2$  stand for the equations of two curves of the same degree, then  $S_1 + \lambda S_2$  is generally the equation of another curve of that degree. This may be the first occurrence of the idea of a linear series, with which this book is much concerned. Using this idea allowed Plücker to pull out geometrical properties of cubic and quartic plane curves while avoiding the algebraic complexities that had defeated Euler.

Plücker also resolved the paradox about dual curves in the theory of plane curves in his (1834) that had stumped Gergonne and Poncelet. The paradox arises because the dual of a smooth curve of degree  $d$  has degree  $d(d - 1)$  so the dual of the dual “ought” to have degree  $d(d - 1)(d(d - 1) - 1)$ . However, the dual of the dual is the original curve. Plücker’s response, which would strike us today as at best heuristic, was that any line through a double point, say, is counted in this way as a tangent, and this throws the count of genuine tangents off. More precisely, he showed that each node on a curve reduces the degree of the dual curve by 2, and each cusp reduces it by 3. Consequently, he claimed that a curve of degree  $d$  with  $\delta$  double points and  $\kappa$  cusps has a dual curve of degree  $d(d - 1) - 2\delta - 3\kappa$ .

Plücker also showed in his (1839, 207–227) that the nodes of  $C^*$ , the dual curve of  $C$ , corresponded to the bitangents of  $C$ , while the cusps of  $C^*$  corresponded to the flexes of  $C$ , and vice versa. To see how he resolved the duality paradox, consider the case of a non-singular cubic curve. It has no nodes, cusps, or bitangents, and as yet an indeterminate number,  $j$  of flexes. Its dual has degree 6 and  $j$  cusps. The dual of the dual curve therefore has degree  $6.5 - 3j$ , which will equal 3 if  $j = 9$ . As we saw, the result that a non-degenerate cubic curve has nine flexes had been known since MacLaurin. For curves of higher degree, it is necessary to consider a curve of degree  $n$  that has  $\delta$  double points,  $\kappa$  cusps,  $\tau$  bitangents, and  $\iota$  inflections. Suppose that its dual has degree  $n'$ . Plücker argued that

$$n' = n(n - 1) - 2\delta - 3\kappa,$$

$$\text{and } \iota = 3n(n - 2) - 6\delta - 8\kappa,$$

with another formula for  $\tau$ , along with the corresponding formula for the dual curve and its inflection points and bitangents.

The number  $\iota$  is most easily found using an argument due to Ludwig Otto Hesse, who showed in his (1844) that the set of flexes  $\Gamma \subset G$  on a curve  $G$  of

<sup>24</sup>See (Gergonne 1826–1827), (Plücker 1835) and (Plücker 1839).

degree  $d$  with equation  $F(x, y, z) = 0$  could be characterized as the intersection of  $G$  with the curve  $D$  defined by the vanishing of the *Hessian determinant*:

$$H = \det \begin{pmatrix} \partial^2 F / \partial x^2 & \partial^2 F / \partial x \partial y & \partial^2 F / \partial x \partial z \\ \partial^2 F / \partial x \partial y & \partial^2 F / \partial y^2 & \partial^2 F / \partial y \partial z \\ \partial^2 F / \partial x \partial z & \partial^2 F / \partial y \partial z & \partial^2 F / \partial z^2 \end{pmatrix}.$$

Since the entries of this matrix have degree  $n - 2$ , the degree of  $H$  is  $3(n - 2)$ . For general  $F$  the curves  $G$  and  $D$  meet transversely, so Bézout's Theorem shows that a general curve of degree  $m$  has exactly  $\iota = 3n(n - 2)$  flexes.<sup>25</sup>

The most famous case Plücker established in his book (1839, 247), concerns a nonsingular plane curve  $C$  of degree  $n = 4$  (no nodes, cusps, or other singular points) having  $\tau$  bitangents and  $\iota$  flexes. Its dual curve has degree 12 and will have  $\tau$  double points and  $\iota$  cusps, and the dual of the dual will have degree  $12 \cdot 11 - 2\tau - 3h = 4$ . Hesse's theorem shows that  $h = 24$  (Plücker had an ad hoc argument to the same effect) and so  $\tau = 28$ , and we see that  $C$  has exactly 28 bitangents. Plücker showed that they can all be real (see Figure 5).

In 1849, Plücker gave up mathematics for physics, particularly the study of cathode rays; he was awarded the Royal Society of London's Copley medal (its highest honour) for this work in 1866. It is sometimes said that he gave up geometry because he tired of the criticisms of Steiner, who had a secure position in Berlin and influenced the decisions of the *Journal für Mathematik*, and only returned to geometry after Steiner died in 1863. Now he took up the field of line geometry, which he transformed into a new branch of the subject.

Plücker's formulae work well for curves of degrees 3 and 4, but not so well thereafter because other types of singularity can appear. He listed all the solutions he could find to the equations that bear his name up to curves of degree 10, but he did not discuss the kinds of singularities a curve may have that are not of the type he had considered.<sup>26</sup> Progress was only made with a paper by Arthur Cayley (1866), who drew on Victor Puiseux's paper (1850) that analysed how curves are ramified at a singular point, showed how the branches are permuted in cycles, and how this is captured by the local power series expansions, which begin with fractional indices.<sup>27</sup>

Puiseux was one of a number of mathematicians in the circle around Cauchy. Cauchy had spent the 1830s and early 1840s following the Bourbon Court around Europe from a strange belief that the oath of allegiance he had sworn to the crown on becoming a professor compelled him to do so. As a result, by 1850

<sup>25</sup>This result was first proved in a different way in (Plücker 1835, 264), as Hesse acknowledged.

<sup>26</sup>See (Plücker 1839, 214).

<sup>27</sup>Cayley's paper was corrected by Otto Stolz in his (1875), who found that the conclusions were correct but the proof "as he [Cayley] himself remarks", was not quite complete.

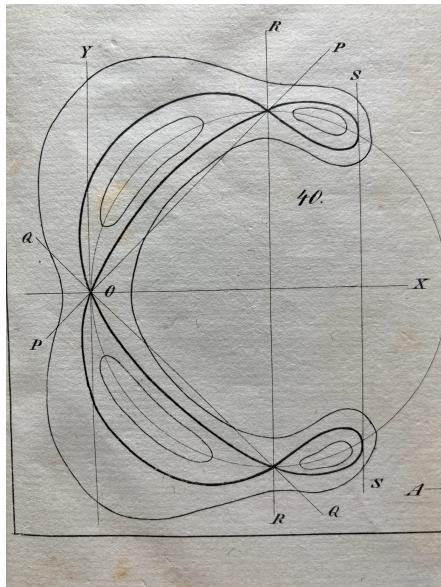


Figure 5. Plücker's quartic curve has 28 bitangents, from (Plücker 1839)

few people knew the work he had done in those years, and even he seems to have forgotten what he had done in complex variable theory back in the 1820s, which included a version of what we call the Cauchy integral theorem (Cauchy 1825) restricted to rectangular contours. Only on his return to Paris did he begin to think much more geometrically; previously his attitude to a many-valued ‘function’ was to cut the plane and study just a branch of it on what remains. His understanding of branch points was quite limited, and this left space for Puiseux to study integrals on arbitrary contours and what happens to integrals taken around branch points.

## 5. The birth of projective space

When did projective space come to be regarded as something more rigorous than Euclidean space with a “line at infinity”?<sup>28</sup> This might simply be a suitable set of coordinates. Plücker used coordinates for the plane with a line at infinity in his (1830) with a view to enabling homogeneous equations to correspond to curves; in this system the coordinates  $(p, q, r)$  denote the signed distances of a point from the three sides of a triangle of reference. However, in his study of curves (1835) he would first discuss them in the plane, and then as they went off to infinity; he didn’t say that the line at infinity could be mapped by a projective

<sup>28</sup>For an interesting set of essays analysing what projective space might be and how it came about, including an extensive analysis by J.-P. Friedelmeyer of the work of Poncelet, see (Biosemat-Martagon 2010), and the discussion in del Centina (to appear).

transformation into the finite part of the plane. The way forward was indicated by August Möbius, who introduced barycentric coordinates in his (1827). Pick three points forming a triangle, say  $ABC$ , and attach weights, positive, zero, or negative (not all zero) to these points. The barycentre or centre of gravity of these three weighted points is a point  $P$  which can be said to have those three weights (or, better, their ratios) as its barycentric coordinates. If you put the points  $A, B, C$  at, say,  $(0, 0), (1, 0), (0, 1)$  you get an easy way to relate points in what could be called the Cartesian and barycentric coordinate planes. The big plus is that the line at infinity, which is invisible in Cartesian coordinates, is a perfectly sensible line in barycentric coordinates. In this system, a line has an equation of the form  $ax + by + cz = 0$ , and by treating  $a, b, c$  as the coordinates of the line Möbius obtained a simple theory of conics and duality in the plane. (He also showed that there are dualities in  $P^3$  that are not pole-polar dualities.)

If you drop the talk about weights, and keep the idea that barycentric coordinates are best thought of as ratios of three numbers, you have projective coordinates; this was one of the contributions of Hesse in the 1840s.<sup>29</sup> Mathematicians were still reluctant, however, to decide if this space was  $\mathbb{P}_{\mathbb{R}}^2$  or, less likely,  $\mathbb{P}_{\mathbb{C}}^2$ .

Mathematicians found it hard to accept complex coordinates, even though Plücker used the term ‘imaginary’ (as in ‘imaginary’ points, ‘imaginary’ straight lines, ‘imaginary’ tangents) over two hundred times in his (1828–1831).<sup>30</sup> He employed the term confidently in his (1839) when, for example, he discussed how many of the 28 bitangents to a quartic curve can be real. In fact, the whole question of a complex space was obscure for a long time. As late as 1878 Cayley could write about complex curves as sets of points in  $\mathbb{C} \times \mathbb{C}$  and remark “I was under the impression that the theory was a known one; but I have not found it set out anywhere in detail.”<sup>31</sup> For quite some time attention was fixed on real curves in the real plane, which could conveniently sprout points with complex coordinates when they intersected other curves. This unstable situation could not last, but how it was to be swept away was not clear to mathematicians initially.

## 6. Riemann's theory of algebraic curves and its reception

It was, however, entirely clear to Riemann. The crucial issue in defining complex valued functions of a complex variable, where a complex variable can be taken as an expression of the form  $x + iy$ , is defining what it is for such a function to be something more particular than a mapping from  $\mathbb{R}^2$  to  $\mathbb{R}^2$ , and this comes

<sup>29</sup>See, for example, his (1844).

<sup>30</sup>Del Centina, forthcoming.

<sup>31</sup>See (Cayley 1878, 32).

down to defining what it is for such a function to be differentiable as a function of a complex variable. As is well-known, Riemann solved this problem in the opening pages of his inaugural dissertation (1851) by identifying – much more clearly than Cauchy – the role of what we call the Cauchy–Riemann equations. He proceeded to give a thoroughly geometric theory of complex functions, which he extended in his great paper on Abelian functions (1857). There he developed a strikingly topological account, which classified orientable, boundaryless surfaces by the number,  $2p + 1$ , of cuts needs to disconnect them. He called this number the order of connectivity of the surface, when  $p = 0$  he called the surface simply connected. The theory is too complicated to be described fully here, but briefly, he showed that the dimension of the space of “everywhere finite” differential forms on such a surface is  $p$ , and the integral of each such differential gives rise to a many-valued function, which can be expressed in the form  $f(z, w) = 0$ , where  $z$  and  $w$  are complex variables. He also deduced the Riemann inequality in this form (1857, § 5): the number of arbitrary constants in a function  $w$  that has  $m$  first-order poles on a surface of order of connectivity  $2p + 1$  is  $m - p + 1$  when  $m \geq p + 1$ . He gave a more detailed account that involves special cases that were interpreted by his student Roch to give us the Riemann–Roch theorem.

Riemann’s health started to decline in the early 1860s, and he died in Italy in July 1866. By then, Rudolf Clebsch had decided to develop Riemann’s ideas and to try to persuade people around him to take up the cause. In his (1864) he applied the theory of Abelian functions to the study of plane algebraic curves. Here he used coordinates that belonged either to  $\mathbb{C}^2$  or  $\mathbb{P}_{\mathbb{C}}^2$  and passed easily between them as required, so it would seem that he and the people he influenced had become comfortable with complex projective space in all but name.

In his papers (1865a, 43) and (1865b, 98) Clebsch sorted curves into different genera according to the value of the number  $p$  associated to them, but he did not speak of the genus of a curve.<sup>32</sup> If we allow ourselves to do so, we may say that Clebsch defined the genus of a plane algebraic curve with  $\delta$  double points as<sup>33</sup>

$$p = \frac{1}{2}(n - 1)(n - 2) - \delta.$$

In his (1865b) and his book with Paul Gordan (1866) he extended this to encompass curves with  $\kappa$  cusps, so  $p = \frac{1}{2}(n - 1)(n - 2) - \delta - \kappa$ . This tells us incidentally that no singular points more complicated than those considered by Plücker in the late 1830s had been looked at. Clebsch’s formula relied on being able to count the number of constants in an integral of an everywhere finite differential correctly, a result that requires the completeness of the adjoint

<sup>32</sup>See (Lê 2020), who suggests that Felix Klein was the first to do so.

<sup>33</sup>See (Clebsch 1864, 192).

series. Clebsch and Gordan, in their book (1866, Chapter 3) offered a proof that the genus of a plane algebraic curve is invariant under a birational transformation, but it was valid only for the case of a curve with simple cusps and double points.

Clebsch became Riemann's successor in Göttingen in 1868, but he died of diphtheria in 1872 at the age of 39. His plans for the study of algebraic geometry now devolved upon Alexander Brill and Max Noether, with whose theory, modernised and made rigorous, this book is in part concerned.<sup>34</sup>

Alexander Wilhelm Brill was born in Darmstadt in 1842.<sup>35</sup> His uncle was the mathematician Christian Wiener, an expert in descriptive geometry. He entered the University of Giessen intending to study architecture but his mathematical ability brought him to the attention of Clebsch, who was then at Giessen and who encouraged him to go to Berlin, where Ernst Kummer, Leopold Kronecker, and Karl Weierstrass taught. This broadened Brill's horizons considerably, but he returned to Giessen in 1867 to take his Habilitation under Clebsch.

In Giessen, Brill met Max Noether. Noether had been born in Mannheim in 1844, but polio at the age of 14 left him paralysed in one leg and delayed his education. He had to be privately schooled, which gave him broad literary and cultural interests. At university he initially intended to study astronomy, but then he switched to mathematics, first at Heidelberg and then at Giessen and Göttingen in 1868 and 1869. He habilitated in Heidelberg in 1870 with a thesis on surfaces possessing a family of rational curves. In 1875 he became a Professor at Erlangen, where he remained until his death in 1921.<sup>36</sup>

Noether worked not only on plane algebraic curves, including the analysis of their singular points, but the geometry of algebraic surfaces, algebraic curves in space, and Cremona transformations of the plane; van der Waerden justly remarked that "Algebraic geometry was created by Max Noether."<sup>37</sup> He drew largely on the work of Plücker on algebraic curves and their singular points, and was less interested in the computational side of the theory that Clebsch had emphasised. As a result, his obituarists — Guido Castelnuovo, Federigo Enriques, and Francesco Severi — noted that Noether's work was inclined to be qualitatively valuable even though he may not have paid sufficient attention to "contingent aspects of the formulae". However, they added that:<sup>38</sup>

<sup>34</sup>A modern account of much of the material described above can be found in (Brieskorn and Knörrer 1986). (Coolidge 1940) remains a historically informative if not entirely rigorous account of many of these developments, more additional mathematical details are in (Coolidge 1931).

<sup>35</sup>See (Severi 1922), and Brill's obituary (Finsterwalder 1936).

<sup>36</sup>See his obituary (Brill 1923).

<sup>37</sup>See (Waerden 1971, 171) who immediately commented that the logical foundations are shaky.

<sup>38</sup>See (Castelnuovo, Enriques, and Severi 1925, 162)

geometric intuition, which was always his guide, saved him from error, and the work, even if it was not perfect, and perhaps for that reason, was richly suggestive.

Brill chafed under the direction of Richard Baltzer, who was Clebsch's successor at Giessen, and moved to the new Polytechnic in Darmstadt in 1869, which was academically a step downwards. There, he did his work on the Cayley–Brill correspondence principle, and in 1875 he accepted Klein's offer of becoming a Professor at the Polytechnic in Munich, with responsibility for reforming high school education. Brill found Klein's inventiveness hard to match, but he enjoyed getting to know some of the other mathematicians there, including Philipp Seidel and the young Alfred Pringsheim. Among his pupils were Walther Dyck, Isaak Bacharach, and Ferdinand Lindemann, who went on to produce the two-volume geometry textbook *Vorlesungen über Geometrie* (known as Clebsch–Lindemann) before achieving fame as the mathematician who proved that  $\pi$  is transcendental.

## 7. First ideas about the resolution of singular points

In their work on algebraic curves, Brill and Noether found Riemann's use of the Dirichlet principle at the heart of his theory of complex analytic functions to be unsound, so they studied curves firmly in the complex plane  $\mathbb{C}^2$  or the complex projective plane  $\mathbb{P}_{\mathbb{C}}^2$ . They were concerned to treat such curves in full generality, and this led them to confront the issue of arbitrary singularities of plane curves. Their preferred method was to argue that any curve with whatever singularities could be reduced to one of the same genus but having only double points by a sequence of Cremona transformations, and then to deal in detail with these simpler curves. It was clear from the formula for the genus of a plane curve that if the genus of the curve is not a triangular number then the curve will always have singular points, so the process of desingularising cannot always lead to a nonsingular curve.

Luigi Cremona had outlined a general theory of transformations of the plane in his papers (1863) and (1865). He looked for geometrical transformations of the plane mapping a given figure one-to-one onto its image, and reciprocally; notably transformations that map straight lines to curves of order  $n$ , for some  $n$ . When  $n = 2$  (the case of quadratic transformations) as he showed, almost all straight lines must have images that are conics passing through 3 fixed points.<sup>39</sup>

<sup>39</sup>Such examples had been studied before, Cremona cited papers by Steiner, Magnus, and Schiaparelli, but for higher values of  $n$ , which Cremona analysed in his (1865), everything was new.

Noether in his (1871a) was the first to appreciate that Cremona transformations could be used to simplify singular points. He wrote a Cremona transformation of the plane algebraically as

$$[x, y, z] \mapsto [x', y', z'] = [\phi(x, y, z), \psi(x, y, z), \chi(x, y, z)],$$

where the functions  $\phi, \psi$  and  $\chi$  are homogeneous polynomials in  $x, y$  and  $z$  of the same degree. To simplify a singular point he used transformations defined as above where  $\phi, \psi$ , and  $\chi$  all vanish at the singular point. In the affine plane this becomes

$$(x, y) \mapsto (x_1, y_1) = \left( \frac{\varphi(x, y)}{\chi(x, y)}, \frac{\psi(x, y)}{\chi(x, y)} \right),$$

where  $\varphi, \psi$ , and  $\chi$  are polynomial functions that vanish at the singular point. To see what this does, it is convenient to take the familiar example of the simplest Cremona transformation, other than a projective transformation. It is given algebraically by the quadratic transformation

$$(4) \quad [x, y, z] \mapsto \left[ \frac{1}{x}, \frac{1}{y}, \frac{1}{z} \right] = [yz, zx, xy]$$

The union of the lines  $x = 0, y = 0, z = 0$  is called the exceptional triangle, and the images of these lines are the points  $[1, 0, 0], [0, 1, 0]$  and  $[0, 0, 1]$  respectively. The transformation is many-valued at these points, each of which is “blown up” into a line but it is well-defined away from those three points and it is one-to-one away from the lines  $x = 0, y = 0$ , and  $z = 0$ .

Noether’s argument that any plane algebraic curve can be reduced to one with only double points is sketchy at best. His use of Puiseux’s work suggests that he appreciated that quadratic transformation might have to be used repeatedly to reduce a complicated singularity to a simpler one, but he did not fully appreciate what happens to the rest of the curve at the points where it crosses the exceptional lines. In his more rigorous account (1876), he gave a better definition of a singular point, and tried to redefine the genus of a plane algebraic curve and prove its invariance under birational transformations.<sup>40</sup>

## 8. The work of Brill and Noether

Noether began, in his (1873), with the question of when a plane curve  $E$  with equation  $H = 0$  can be expressed in the form  $H = AF + BG$ , where  $F = 0$  and  $G = 0$  are the equations of two curves  $C$  and  $D$ . He rightly pointed out that people hitherto had assumed that it was necessary and sufficient that the curve  $E$  passes simply through the intersection points of the curves  $C$  and  $D$ , but in fact this is insufficient when the curves  $C$  and  $D$  have singular points or

<sup>40</sup>Noether also claimed the remarkable theorem that every such Cremona transformation is a composition of quadratic transformations and projective transformations. See his (1871b), with a small correction in his (1873).

intersections with multiple tangents. His own account, which was based on counting the number of independent equations imposed on  $H = 0$  by the equations  $F = 0$  and  $G = 0$ , was a significant advance, but it too had flaws and he and other authors, notably Eugenio Bertini, offered improvements. Eventually, Noether offered this formulation of his Fundamental Theorem:<sup>41</sup>

**Theorem 8.1.**  *$H(x, y)$  is representable in the form  $AF + BG$  if and only if in some neighbourhood of each common point  $(x_0, y_0)$  of  $F = 0$  and  $G = 0$  there are power series  $a(x - x_0)$  and  $b(x - x_0)$  in  $\mathbb{C}[y][[x]]$  whose coefficients are polynomials in  $y$  of orders  $n - 1$  and  $m - 1$  respectively, such that  $H = aF + bG$ .*

Only in the monumental joint work with Brill (Brill and Noether 1894, 352) was it shown that in the plane case there is nothing more to be said, the local conditions are indeed sufficient even in the projective case.

Today we would say that Noether's Fundamental Theorem is true for two polynomials  $F$  and  $G$  because  $F, G$  is a regular sequence and so the ideal they generate is, in later terminology due to Macaulay (1916), unmixed.<sup>42</sup>

Brill and Noether recognised that the Riemann-Roch Theorem was of central importance to the theory of plane algebraic curves; indeed, they were the first to name it.<sup>43</sup> Clebsch had studied what we would call holomorphic differential forms on a curve  $C$  of degree  $d$  in his (1864, 193) by supposing that they are given by homogeneous polynomials of degree  $d - 3$  that vanish at the double points of the curve. By Bézout's Theorem, such an intersection also has degree  $d(d - 3) = 2p - 2$ , as it should. The dimension of the family of divisors cut out on  $C$  in this way is the dimension of the projective space of forms of degree  $d - 3$  in  $\mathbb{C}[x_0, x_1, x_2]$ , which is conveniently equal to  $(d - 1)(d - 2)/2$ . For this to be correct and account for all the canonical divisors, it has to be shown that every divisor  $E$  on the curve  $C$  that differs from an intersection  $D \cap C$ , where  $D$  is the curve defined by the equation  $G = 0$ , by the divisor of the restriction to  $C$  of a rational function  $P/Q$  (where  $P, Q$  are forms of the same degree), is again of the form  $D' \cap C$  for some curve  $D'$ . This is the content of Noether's Fundamental Theorem. His Fundamental Theorem is essential when it comes to handling sets of points cut out on a fixed curve  $C$  by families of curves, counting the degree of an intersection of two curves at multiple points, dividing an arbitrary intersection into two subsets, and delivering a satisfactory version of the Riemann-Roch Theorem.

<sup>41</sup>See (Noether 1887, 413).

<sup>42</sup>See (Eisenbud and Gray 2023) and (Eisenbud and Gray 2024) for a historical account of Macaulay's work.

<sup>43</sup>See (Brill and Noether 1874, § 5).

For reasons of space, it is impossible to pursue the history of curves after the Riemann–Roch Theorem here, except to note that some of the consequences are the subject of the present book to which this essay is appended.

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