Personalities of Curves

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Chapter 0

Basic Questions

((The following is more material for a preface than a preface...))

I'm very well acquainted, too, with matters mathematical,
I understand equations, both the simple and quadratical,
About binomial theorem I am teeming with a lot o' news,
With many cheerful facts about the square of the hypotenuse.

—Gilbert and Sullivan, Pirates of Penzance, Major General's Song

Be simple by being concrete. Listeners are prepared to accept unstated (but hinted) generalizations much more than they are able, on the spur of the moment, to decode a precisely stated abstraction and to re-invent the special cases that motivated it in the first place.

–Paul Halmos, How to Talk Mathematics

Another damned thick book! Always scribble, scribble, scribble! Eh, Mr. Gibbon? — Prince William Henry, upon receiving the second volume of The History of the Decline and Fall of the Roman Empire from the author.

The most primitive objects of algebraic geometry are affine algebraic sets—subsets of $\mathbb{R}^n$ or $\mathbb{C}^n$ defined by the vanishing of polynomial functions—and the maps between them. But already in the first half of the 19th century geometers realized that there was a great advantage in working with varieties in complex projective space, treating affine varieties as projective varieties minus the intersection with the plane at infinity and real varieties as the fixed points of the complex involution. One sees this in the simplest examples: the ellipses, hyperbolas and parabolas in the real affine plane are all the same in the complex projective plane; the difference is only in how they intersect the line at infinity. A difficulty with the projective point of view is that on a connected projective variety there are no non-constant functions at all (reason: a function on a projective variety is a map to the affine line; since the image of a projective variety is again projective, the image would be a single point.)

Starting with Riemann in the 1860s and culminating in the scheme theory of Grothendieck in the 1950s, algebraic varieties were treated in a way independent of any embedding: An algebraic variety is is a topological space with a sheaf of locally defined polynomial functions. Many interesting aspects of geometry have to do not with single abstract varieties, but with maps between them, and in particular with embeddings in projective spaces. In general, maps between varieties can be described by their graphs, which are again varieties. But for the special case of maps to projective spaces, the theory of *linear series* is usually a more convenient description. The collection of all linear series on a variety reflects some of its best understood invariants.

The basic objects of study in this book are smooth, connected projective algebraic curves over an algebraically closed field of characteristic 0, which we take to be the complex numbers $\mathbb{C}$. Though we assume that the reader has been exposed to this theory in some form before, perhaps from Chapter IV of Hartshorne's *Algebraic Geometry*, we will review the elements in the form we will use.

0.1 Algebraic Curves and Riemann Surfaces

These objects can be viewed in two distinct but equivalent ways: as *compact Riemann surfaces*, or compact complex manifolds of dimension 1; and as *smooth projective algebraic curves over* $\mathbb{C}$. (Here, when we use the term projective variety, we mean a variety isomorphic to a closed subset of projective space, not a variety with a specified embedding in $\mathbb{P}^n$.) There are advantages to each point of view—the complex analytic point of view is more concrete, and requires relatively minimal amount of preliminaries; the algebraic point of view is substantially broader.

First, if $C \subset \mathbb{P}^n$ is a smooth, projective curve over $\mathbb{C}$, then it is a submanifold of complex projective space, and so a Riemann surface.
((discuss geometric genus vs. arithmetic genus here?))

The other direction—going from a compact Riemann surface C to a smooth projective curve over $\mathbb{C}$, or equivalently embedding C as a complex submanifold of $\mathbb{P}^n$, after which Chow's theorem says that it is in fact a projective variety—is much deeper. The first, and hardest step is to show that a compact Riemann surface admits a nonconstant meromorphic function $f : C \to \mathbb{C}$, and the corresponding statement is not true in higher dimensions. The function f can be viewed as a rational map $f' : C \to \mathbb{P}^1$. The next step is to see that the field $K(C)$ of all meromorphic functions on $\mathbb{C}$ is a finite extension of the field of rational functions on $\mathbb{P}^1$; the sheaf of regular functions on C is then the integral closure of the sheaf of regular functions on $\mathbb{P}^1$ in $K(C)$.

Though equivalent for curves defined over $\mathbb{C}$, these approaches have a very different flavors. For example, given a map $f : C \to C'$ from a smooth curve C to a possibly singular curve C' that is generically one-to-one, we can reconstruct C. From the algebraic point of view this can be done by *normalization*, or more concretely by blowing up the singular points of C'. From the analytic point of view, we can use the Weierstrass preparation theorem, which implies that there is the neighborhood U of any point $p \in C'$ such that the punctured neighborhood $U \setminus p$ is isomorphic to a disjoint union of punctured discs; and C is obtained by completing this to the corresponding disjoint union of discs.

0.2 Families of varieties

0.2.1 Hilbert schemes

Definition, universal property; construction

examples of hypersurfaces and linear spaces

tangent space

Fundamental problem: irreducible components of Hilb parametrizing smooth curves and their dimensions

Example 0.2.1. conics in $\mathbb{P}^3$ (refer to 3264)

0.2.2 Moduli spaces of curves

basic properties of M_g (coarse rather than fine; fine over automorphism-free curves)

dimension $3g-3$, irreducible

(just statements, w/ref to Harris-Morrison)

0.3 Moduli problems

It is a fundamental aspect of algebraic geometry that the objects we deal with often vary in families, and can often be parametrized by a "universal" such family. For example, the family of plane curves of degree d may be thought of as the projective space $\mathbb{P}(H^0\mathcal{O}_{\mathbb{P}^2}(d))$, and similarly with hypersurfaces in any projective space. This notion of objects varying with parameters underlies many of the constructions and theorems we will discuss.

0.3.1 What is a moduli problem?

Briefly, a *moduli problem* consists of two things: a class of objects, or isomorphism classes of objects; and a notion of what it means to have a *family* of these objects parametrized by a given scheme B. To make this relatively explicit, the four main examples of moduli problems we'll be discussing here are:

1. smooth curves: objects are isomorphism classes of smooth, projective curves C of a given genus g. A family over B is a subscheme $\mathcal{X} \subset B \times \mathbb{P}^r$, smooth, over B, whose fibers are curves of genus g.

2. the Hilbert scheme: objects are subchemes of $\mathbb{P}^r$ with a given Hilbert polynomials. A family is a subscheme $\mathcal{X} \subset B \times \mathbb{P}^r$, with cX flat over B, whose fibers have the given Hilbert polynomial. We will be interested in the case of Hilbert polynomial $p(m) = dm - g + 1$ and the open subscheme corresponding to smooth projective curves $C \subset \mathbb{P}^r$ of degree d and genus g.

3. effective divisors on a given curve: objects are effective divisors of a given degree d on a given smooth, projective curve C. A family over B will be a subscheme $\mathcal{D} \subset B \times C$ flat over B, with fibers of degree d

4. invertible sheaves on a given curve C: objects are invertible sheaves of a given degree d on C. A family over B is an invertible sheaf on the product $B \times C$ whose restriction to each fiber over B has degree d. We identify two such sheaves if they differ by tensor product with an invertible sheaf pulled back from B.

Given a moduli problem, our goal will be to describe a corresponding *moduli space*. By this we mean a scheme M whose points are in *natural* one-to-one correspondence with the objects in our moduli problem. This will realize the objects of the moduli problem as the points of the underlying set of the scheme M.

If the moduli space in question and the base of the family are varieties, then the crucial condition that the correspondence be *natural* is simple to express: that given a family of the objects in our moduli problem over a variety B, the map from underlying set of B to the underlying set of M

taking each fiber to the corresponding point of M should be a morphism of varieties. But in the world of schemes the set-theoretic mapping does not determine the morphism of schemes (think, for example, of the morphisms from $\mathrm{Spec}(\mathbb{C}[x]/x^2)$ into the plane with the closed point mapping to the origin. The situation is even worse when the moduli space itself is not a variety.)

To deal with the general case, we recast the naturality condition in functorial terms. We observe first that a moduli problem defines a functor $\mathcal{M}$ from the category of schemes to the category of sets: the value of the functor at a scheme B is the set of families of objects parametrized by B; a morphism $B' \to B$ of schemes gives rise, via pullback, to a map of sets $\mathcal{M}(B) \to \mathcal{M}(B')$. We define a *fine moduli space* for the moduli problem to be a scheme M that represents this functor, in the sense that there is an isomorphism of functors

$$\mathcal{M} \to \mathrm{Mor}(\bullet, M)$$

In other words, for every scheme B we have a bijection between families of our objects over B and morphisms from B to M. In particular, applying this to $B = \mathrm{Spec}\,\mathbb{C}$, we have a bijection between the set of objects and the closed points of M; and for any family over an arbitrary scheme B, the map from $B(\mathbb{C})$ to $M(\mathbb{C})$ sending each closed point $b \in B$ to the point in $M(\mathbb{C})$ corresponding to the fiber over b is the underlying map of a morphism $B \to M$ of schemes.

If a fine moduli space for a given problem exists at all, then Yoneda's Lemma shows that it is unique up to a unique isomorphsm. This is a real problem: there is no fine moduli space for the first and most important of the examples above—the isomorphism classes of smooth curves—though there is for the others. We'll defer the discussion of why this is, and what we can do about it, until Chapter Moduli chapter ??.

Looking ahead, we'll discuss the third and fourth example in Chapter ??, where we'll describe the moduli spaces for effective divisors of given degree d on a given curve C (the symmetric powers of the curve) and for invertible sheaves of a given degree on C (the *Jacobian* and *Picard variety* of C). These, as we'll see, are smooth, irreducible projective varieties of dimensions d and g respectively.

We'll take up the moduli space M_g of smooth curves in Chapter ??, Moduli chapter
where we'll see that this space (or rather the closest approximation to it we

can cook up) is irreducible of dimension $3g-3$ for $g \geq 2$, though not smooth or projective.

Finally, the Hilbert scheme will be described (to the extent that we can!) in Chapter HilbertSchemesChapter ??; this will turn out to be much wilder and more varied in its behavior than any of the above.

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