ON THE TANGENTIALLY DEGENERATE CURVES

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Introduction

It seems to have been widely believed that if a space curve C is not contained in any plane, then it has only a finite number of points whose tangent lines meet C again. However, to the best of my knowledge, there is no proof in the literature. One of the purposes of this article is to give a proof of this statement under the condition that the characteristic of the ground field is zero. If, on the contrary, the characteristic is positive, then this is not true. Our second purpose is to give examples in this case. Then, we propose to introduce the concept of 'tangential degeneration'. Strictly speaking, we say that a space curve C is *tangentially degenerate* if there are infinitely many points of C whose tangent lines meet C again.

On the other hand, it seems also to have been thought that a general plane curve of degree not less than 3 is tangentially degenerate, but that there might exist such curves which are not tangentially degenerate. The third purpose of this article is to prove that the only (non-singular) plane curves which are not tangentially degenerate are conics. In characteristic zero, of course, this result is well known; in fact, it holds in this case for singular curves as well. This follows from the fact that a plane curve other than a line has only finitely many inflexions (see, for example, [**2**, IV, Exercise 2.3] or, for full details, almost any book on algebraic curves). In positive characteristic, there exist (non-singular) curves with infinitely many inflexions (see, for example, [**2**, IV, Exercise 2.4]).

The contents of this article are as follows. In §1 we study the natural morphisms from the projective tangent bundles to the tangential surfaces, and make precise the definition of tangential degeneration. In §2 we prove that the only tangentially non-degenerate plane curves are conics. In §3 we prove that if the characteristic of the ground field is zero, then a space curve not contained in any plane is never tangentially degenerate. Finally, in §4 we give examples in positive characteristic of tangentially degenerate space curves which are not contained in any plane.

Throughout this article we work over an algebraically closed field k with arbitrary characteristic p. We use the word *curve* to mean an irreducible, non-singular, projective curve defined over k, unless otherwise specified. To avoid trivial exceptions we always assume that a curve in $\mathbb{P}^n$ has degree not less than 2.

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1. *Projective tangent bundles and tangential surfaces*

Let C be a curve in $\mathbb{P}^n$, let ϕ be the closed immersion of C in $\mathbb{P}^n$, let V be the k-vector space $H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$, and let $\mathcal{P}^1 = \mathcal{P}^1(\mathcal{O}_C(1))$ be the bundle of principal

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parts of $\mathcal{O}_C(1)$ of first order, where $\mathcal{O}_C(1) = \phi^*\mathcal{O}_{\mathbb{P}^n}(1)$ (see, for example, [4]). Then we have the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \phi^*\Omega^1_{\mathbb{P}^n} \otimes \mathcal{O}_C(1) \arrow[r] \arrow[d] & V_C \arrow[r] \arrow[d] & \mathcal{O}_C(1) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \Omega^1_C \otimes \mathcal{O}_C(1) \arrow[r] & \mathcal{P}^1 \arrow[r] & \mathcal{O}_C(1) \arrow[r] & 0
\end{tikzcd}$$

with exact rows, where $V_C = V \otimes \mathcal{O}_C$. Note that these vertical maps are surjective since ϕ is unramified.

Lemma 1.1. *Let C be a curve of degree d and genus g in $\mathbb{P}^n$. Then, with the same notation as above, the bundle $\mathcal{P}^1 = \mathcal{P}^1(\mathcal{O}_C(1))$ has degree $2d+2g-2$.*

Proof. Since the bundle $\mathcal{P}^1$ is an extension of $\mathcal{O}_C(1)$ by $\Omega^1_C \otimes \mathcal{O}_C(1)$, we have $\deg \mathcal{P}^1 = 2d+2g-2$.

Let TC be the projective bundle over C associated to the bundle $\mathcal{P}^1$, with projection $p: TC \to C$. Let C_0 be the section of $p: TC \to C$ associated with the surjection $\mathcal{P}^1 \to \mathcal{O}_C(1)$. Let q be the composition of the projection $\mathbb{P}^n \times C \to \mathbb{P}^n$ with the closed immersion $TC \to \mathbb{P}^n \times C$, and let X be the image of q. Intuitively, C_0 is the set of tangent points of C on TC, and X is the union of all tangent lines of C.

Definition 1.2. The projective bundle TC over C is called the *projective tangent bundle* of C, and the image X in $\mathbb{P}^n$ is called the *tangential surface* of C.

From the above diagram, we get the following diagram.

$$\begin{tikzcd}
C_0 \arrow[r, hook] \arrow[d, "\iota"'] & TC \arrow[r, hook] \arrow[dl, "p"'] \arrow[d, "q"] & \mathbb{P}^n \times C \arrow[d] \\
C \arrow[r, hook, "\phi"] & X \arrow[r, hook] & \mathbb{P}^n
\end{tikzcd}$$

Remark 1.3. The triangle in the above diagram, formed by TC, C and X, is *not* commutative.

Next, let G be the Grassmann manifold parametrizing every line in $\mathbb{P}^n$. For a curve C in $\mathbb{P}^n$, we define a morphism $g: C \to G$ by the surjection $V_C \to \mathcal{P}^1$. Let C^* be the image of g in G, let C' be the normalization of C^*, let TC' be the projective bundle over C' associated with the pull-back of the universal quotient bundle over G, and let h be the natural morphism from C to C'. Then we have $TC = TC' \times_{C'} C$, and the morphism $q: TC \to X$ factors through TC'. Therefore, we get the following commutative diagram.

$$\begin{tikzcd}
 & & C \arrow[dll, "g"'] \arrow[d, "h"] & TC \arrow[l, "p"'] \arrow[d, "Th"] \arrow[dr, "q"] & & \\
G & C^* \arrow[l, hook'] & & & X \arrow[r, hook] & \mathbb{P}^n \\
 & & C' \arrow[ul] & TC' \arrow[l, "p'"] \arrow[ur, "q'"'] & &
\end{tikzcd}$$

Let $C'_0 = Th(C_0)$ be the image of C_0 in TC'. Then, since the composition of the morphism $q: TC \to X$ with the closed immersion $C_0 \to TC$ is a closed immersion, the morphism $Th: C_0 \to C'_0$ is an isomorphism. Thus we have proved the following.

LEMMA 1.4. *The intersection number of the divisor C'_0 with a fibre of the ruled surface $p'\colon TC' \to C'$ is equal to the degree of the morphism $h\colon C \to C'$.*

LEMMA 1.5. $\deg g = \deg h = \deg Th$.

Proof. This follows from the definitions of these morphisms.

LEMMA 1.6. (a) $\deg C^* \cdot \deg g = \deg \mathscr{P}^1$;

(b) $\deg X \cdot \deg q = \deg \mathscr{P}^1$.

Proof. (a) Since the morphism $g\colon C \to G$ is defined by the surjection $V_C \to \mathscr{P}^1$, we have $g^*\mathcal{O}_G(1) = \det \mathscr{P}^1$, where $\mathcal{O}_G(1)$ is the determinant of the universal quotient bundle over G. Thus, it follows that

$$\deg C^* \cdot \deg g = \deg g^*\mathcal{O}_G(1) = \deg \mathscr{P}^1.$$

(b) Since the morphism $q\colon TC \to \mathbb{P}^n$ is also defined by the surjection $V_C \to \mathscr{P}^1$, we have $q^*\mathcal{O}_{\mathbb{P}^n}(1) = \mathcal{O}_{TC}(1)$, where $\mathcal{O}_{TC}(1)$ is the tautological line bundle on $TC = \mathbb{P}(\mathscr{P}^1)$ determined by the bundle $\mathscr{P}^1$. Therefore, it follows that

$$\deg X \cdot \deg q = \deg q^*\mathcal{O}_{\mathbb{P}^n}(1) = \deg \mathcal{O}_{TC}(1).$$

On the other hand, we have $\deg \mathcal{O}_{TC}(1) = \deg \mathscr{P}^1$. This completes the proof.

COROLLARY 1.7. $\deg C^* = \deg q' \cdot \deg X$.

Proof. Combine Lemmas 1.5 and 1.6.

REMARK 1.8. It is well known that if the characteristic p is zero, then $\deg g = 1$ and $C = C'$ (see, for example, [**2**, IV, Exercise 2.3]).

REMARK 1.9. We can prove that if $\deg X \neq 1$, then $\deg q' = 1$. Since we do not use this, we omit the proof.

REMARK 1.10. It is clear that the morphism $Th\colon TC \to TC'$ is finite. On the other hand, the morphism $q'\colon TC' \to X$ is not finite if and only if the curve C in $\mathbb{P}^n$ is strange, where a curve C in $\mathbb{P}^n$ is *strange* if there is a point in $\mathbb{P}^n$ which lies on all the tangent lines of C (see, for example, [**2**, IV, 3]).

Now we give a precise definition of tangential degeneration.

DEFINITION 1.11. A curve C in $\mathbb{P}^n$ is *tangentially degenerate* if, for a general point P on C, there exists another point Q on C which lies on the tangent line to C at P.

REMARK 1.12. Similarly, we can define the tangential degeneration for a general variety as follows. A projective variety M in a projective space is *tangentially degenerate* if, for a general point P on M and a general line tangent to M at P, there exists another point Q on M which lies on the line.

Next, we rephrase the definition of tangential degeneration in terms of projective tangent bundles.

Let q'^*C be the effective divisor on TC' which is the maximal divisor contained in the scheme-theoretic inverse image $q'^{-1}(C)$ of C, and let $q^*C = Th^*(q'^*C)$ be the pull-back of q'^*C.

LEMMA 1.13. *We can write*

$$q^*C = m \cdot C_0 + D$$

for some integer $m \geqslant 2$, and some effective divisor D on TC which does not contain the component C_0.

Proof. Since the morphism $q: TC \to X$ is ramified along the section C_0, the divisor $q^*C - 2 \cdot C_0$ is effective.

REMARK 1.14. We can write

$$q'^*C = m' \cdot C_0' + D'$$

for some integer $m' \geqslant 1$, and some effective divisor D' on TC' which does not contain the component C_0'.

By the definition of tangential degeneration, we can prove the following proposition.

PROPOSITION 1.15. *Let C be a curve in $\mathbb{P}^n$. Then the following conditions are equivalent:*

(1) *C is tangentially degenerate;*

(2) *the divisor D on TC is not empty;*

(3) *the divisor D' on TC' is not empty, or the morphism $p': C_0' \to C'$ has separable degree not less than 2.*

To conclude this section, we state some lemmas, which enable us to investigate the ruled surface $p: TC \to C$.

LEMMA 1.16. *Let C be a curve, and let $\mathscr{L}$ be a line bundle on C. Let $c_1: H^1(C, \mathcal{O}_C^\times) \to H^1(C, \Omega_C^1)$ be the homomorphism defined by $d\log: \mathcal{O}_C^\times \to \Omega_C^1$. Then we have that $c_1(\mathscr{L}) = 0$ if and only if $\deg \mathscr{L} \equiv 0 \pmod{p}$.*

Proof. See, for example, [**3**, Proposition 2.3].

LEMMA 1.17. *Let C, $\mathscr{L}$, c_1 be as above. Let $\mathscr{P}^1(\mathscr{L})$ be the bundle of principal parts of $\mathscr{L}$ of first order, and let $e \in H^1(C, \Omega_C^1)$ be the extension class defined by the natural exact sequence*

$$0 \longrightarrow \Omega_C^1 \otimes \mathscr{L} \longrightarrow \mathscr{P}^1(\mathscr{L}) \longrightarrow \mathscr{L} \longrightarrow 0.$$

Then we have $c_1(\mathscr{L}) = -e$.

Proof. See, for example, [**1**, Proposition 12].

Combining Lemmas 1.16 and 1.17, we get the following.

COROLLARY 1.18. *Let C, $\mathscr{L}$, c_1 and $\mathscr{P}^1(\mathscr{L})$ be as above. Let $\mathscr{E}$ be the unique non-trivial extension of $\mathscr{L}$ by $\Omega_C^1 \otimes \mathscr{L}$. Then we have that*

$$\mathscr{P}^1(\mathscr{L}) = \begin{cases} \mathscr{L} \oplus \Omega_C^1 \otimes \mathscr{L} & \text{if } p \text{ divides } \deg \mathscr{L}, \\ \mathscr{E} & \text{otherwise.} \end{cases}$$

2. *Tangentially non-degenerate plane curves*

Let C be a curve of degree d in $\mathbb{P}^2$. With the same notation as in §1, we see that the tangential surface X of C is equal to the projective plane $\mathbb{P}^2$, and that the scheme-theoretic inverse image $q'^{-1}(C)$ has no embedded component. Therefore, we have $q'^{-1}(C) = q'^{*}(C)$.

Now, assume that C is tangentially non-degenerate. Then, for any point P of C, the tangent line $T_P C$ to C at P does not meet C again, and it follows from Bézout's theorem that the intersection multiplicity of $T_P C$ with C at P is equal to d. Conversely, if, for any point P of C, the intersection multiplicity of $T_P C$ with C at P is equal to d, then C must be tangentially non-degenerate.

Hence we have proved the following.

LEMMA 2.1. *Let C be a curve of degree d in $\mathbb{P}^2$. Then the following conditions are equivalent:*

(1) *C is tangentially non-degenerate;*

(2) *for any point P of C, the intersection multiplicity of $T_P C$ with C at P is equal to d, where $T_P C$ is the tangent line to C at P.*

THEOREM 2.2. *Let C be a curve in $\mathbb{P}^2$. Then the following conditions are equivalent:*

(1) *C is a conic;*

(2) *C is tangentially non-degenerate.*

Proof. (1) $\Rightarrow$ (2) This is obvious.

(2) $\Rightarrow$ (1) Let g be the morphism from C to the Grassmann manifold G parametrizing every line in $\mathbb{P}^2$, defined by the surjection $V_C \to \mathscr{P}^1$, and let C^* be the image of g in G as in §1. Let d be the degree of C in $\mathbb{P}^2$, and let d^* be the degree of C^* in $G \cong \mathbb{P}^2$. Note that d is not less than 2.

Let $f: C \to \mathbb{P}^1$ be a projection from a general point of $\mathbb{P}^2$. According to Bertini's theorem, we have that f is a separable finite morphism of degree d. Furthermore, we see that the number of ramification points of f is equal to the degree d^* of C^*. On the other hand, by a local computation, we find that the length of the stalk $(\Omega^1_{C/\mathbb{P}^1})_P$ at each ramification point P on C is at least $d-1$, where $\Omega^1_{C/\mathbb{P}^1}$ is the relative differential sheaf of C over $\mathbb{P}^1$. Therefore, it follows that the ramification divisor R of f has degree at least $d^*(d-1)$. According to Hurwitz's theorem (see, for example, [**2**, IV, 2]), we have

$$2g(C) - 2 = d(-2) + \deg R,$$

where $g(C)$ is the genus of C. Thus, combining these facts, we get the following inequality:

$$d^* \leqslant d. \tag{$*$}$$

Note that $g(C) = \frac{1}{2}(d-1)(d-2)$ since C is non-singular.

By our hypothesis, the morphism $g: C \to C^*$ is injective. Thus, we see that the morphism g is birational or purely inseparable, and have

$$g(C) = g(C^*).$$

On the other hand, we have

$$g(C^*) \leqslant p_a(C^*) = \tfrac{1}{2}(d^*-1)(d^*-2),$$

where $p_a(C^*)$ is the arithmetic genus of C^*. Combining these facts, we find the inequality

$$\tfrac{1}{2}(d-1)(d-2) \leqslant \tfrac{1}{2}(d^*-1)(d^*-2). \qquad (**)$$

If $d^* < d$, then it follows from the inequality (**) that

$$d^* = 1 \quad \text{and} \quad d = 2.$$

This completes our proof. Thus, by the inequality (*), we may assume that

$$d^* = d.$$

Therefore, we have

$$\deg g = d-1,$$

$$g(C^*) = p_a(C^*).$$

and

$$C^* = C'$$

with the same notation as in §1.

On the other hand, by the hypothesis and Proposition 1.15, we can write

$$q'^*C = m' \cdot C'_0$$

for some integer $m' \geqslant 1$. Since the intersection number of C with a tangent line of C is equal to d, we have that the intersection number of q'^*C with a fibre of p' is also equal to d. It follows from Lemma 1.4 that the intersection number of C'_0 and a fibre of p' is equal to the degree of the morphism h. Thus we find that

$$d = m'(d-1),$$

since $\deg h = \deg g = d-1$. Hence, we have

$$d = 2.$$

REMARK 2.3. The plane curve with $d^* = 1$ and $d = 2$, which appears in the proof above, is a conic in characteristic 2. Hence, this curve is strange (see Example 2.7 below).

REMARK 2.4. Let C be a curve of degree d in $\mathbb{P}^2$, and let $\{g_0, g_1, g_2\}$ be the gap sequence of the linear system which defines the closed immersion ϕ of C in $\mathbb{P}^2$. Note that $g_0 = 1$ and $g_1 = 2$ since ϕ is base-point-free and unramified. According to Lemma 2.1, we have that C is tangentially non-degenerate if and only if $g_2 = d+1$. Therefore, we have proved that a curve of degree d in $\mathbb{P}^2$ has the gap sequence $\{1, 2, d+1\}$ if and only if $d = 2$.

REMARK 2.5. It is clear that if the characteristic p is zero, then there are not even any singular, tangentially non-degenerate curves in $\mathbb{P}^2$ besides conics.

REMARK 2.6. Using Bertini's theorem, we get the following result. Let M be a non-singular hypersurface in a projective space. Then M is tangentially non-degenerate if and only if M is a quadric.

For singular curves in positive characteristic, we have an example of tangentially non-degenerate plane curves as follows.

EXAMPLE 2.7. Assume that the characteristic p is positive. Let ϕ be a morphism from $\mathbb{P}^1$ to $\mathbb{P}^2$, defined by triples $(s^d : s^{d-1}t : t^d)$, where $d = p^l$, $l \geqslant 1$, and $\mathbb{P}^1 = \operatorname{Proj} k[s, t]$. Let C be the image of ϕ. Then we see that if $d = 2$ then ϕ is a closed immersion, and that if $d > 2$ then ϕ is injective, birational onto its image C and ramified at $\phi(0:1) = (0:0:1)$, which is the only singular point of C. Note that C has degree d.

Let $\mathscr{P}^1$ be the image of the natural homomorphism $V_{\mathbb{P}^1} \to \mathscr{P}^1(\phi^*\mathcal{O}_{\mathbb{P}^2}(1))$. Then $\mathscr{P}^1$ is locally free of rank 2 on $\mathbb{P}^1$ since ϕ is unramified at a general point of $\mathbb{P}^1$. Therefore, we can define the morphism $g: \mathbb{P}^1 \to G$ by the surjection $V_{\mathbb{P}^1} \to \mathscr{P}^1$ as in § 1, although $V_{\mathbb{P}^1} \to \mathscr{P}^1(\phi^*\mathcal{O}_{\mathbb{P}^2}(1))$ is not surjective.

Now, let A be the point $(0:1:0)$ of $\mathbb{P}^2$, and let L be any line in $\mathbb{P}^2$ passing through the point A. Then we see that L and C meet at a single point with multiplicity d. Therefore, we have that C is a strange curve whose tangent lines all pass through the point A, and that C is tangentially non-degenerate by Lemma 2.1.

In this case, we have $\mathscr{P}^1 = \mathcal{O}_{\mathbb{P}^1}(d) \oplus \mathcal{O}_{\mathbb{P}^1}$, and the morphism $q': TC' \to X = \mathbb{P}^2$ is the blow-up with centre A. Furthermore, the dual curve C^* is a line in $G \cong \mathbb{P}^2$, and the morphism $g: \mathbb{P}^1 \to C^*$ is a composition of Frobenius morphisms of degree $d = p^l$.

3. *Tangentially degenerate space curves in characteristic zero*

In this section, we prove the following theorem.

THEOREM 3.1. *Assume that the characteristic p of the ground field k is zero. Let C be a curve in $\mathbb{P}^n$, with $n \geqslant 3$. If C is tangentially degenerate, then C is contained in some 2-plane.*

First, we prove the following.

LEMMA 3.2. *Assume that the characteristic p is zero. Let C be a curve, let $\mathscr{E}$ be a vector bundle of rank 2 over C, and let $\mathscr{L}$ be a line bundle over C. Let C_0 be a section of the ruled surface $\mathbb{P}(\mathscr{E}) \to C$ corresponding to a surjection $\mathscr{E} \to \mathscr{L}$. If there exists an effective divisor D on $\mathbb{P}(\mathscr{E})$ such that C_0 and D are disjoint, then the surjection $\mathscr{E} \to \mathscr{L}$ splits.*

Proof. We may assume that D is irreducible. Let D_1 be the normalization of D. Making a base extension $D_1 \to D \to C$, we obtain two sections from C_0 and D. Moreover, we see that these sections are disjoint since C_0 and D are disjoint. Therefore, the pull-back of the surjection $\mathscr{E} \to \mathscr{L}$ by $D_1 \to C$ splits. Thus, it follows that the surjection $\mathscr{E} \to \mathscr{L}$ splits since the characteristic p is zero.

COROLLARY 3.3. *Assume that the characteristic p is zero. Let C be a curve in $\mathbb{P}^n$. With the same notation as in § 1, if C is tangentially degenerate, then the section C_0 meets every irreducible component of the divisor D on TC.*

Proof. Combine Proposition 1.15, Corollary 1.18 and Lemma 3.2.

Next, we recall an algebraic result which is known as the theorem of Eneström and Kakeya and will be used in the proof of Theorem 3.1.

LEMMA 3.4. *Let $f(X) \in \mathbb{R}[X]$ be a polynomial in one variable, written*

$$f(X) = a_0 + \ldots + a_{n-1} X^{n-1} + a_n X^n.$$

Let x be a root of $f(X)$. If $0 < a_0 \leqslant \ldots \leqslant a_{n-1} \leqslant a_n$, then we have

$$|x| \leqslant 1.$$

Here we give a geometric proof of the fact above. On a two-dimensional real vector space $\mathbb{R}^2$, take the vector $\boldsymbol{a}_i$ corresponding to the complex number $a_i x^i$ on the Gaussian plane. Then the required result follows directly from the following.

LEMMA 3.5. *Let $\{\boldsymbol{a}_0, \ldots, \boldsymbol{a}_{n-1}, \boldsymbol{a}_n\}$ be a sequence of vectors in a two-dimensional real vector space $\mathbb{R}^2$ satisfying the equalities*

$$\angle(\boldsymbol{a}_0, \boldsymbol{a}_1) = \angle(\boldsymbol{a}_1, \boldsymbol{a}_2) = \ldots = \angle(\boldsymbol{a}_{n-1}, \boldsymbol{a}_n)$$

and the inequalities

$$0 < |\boldsymbol{a}_0| < \ldots < |\boldsymbol{a}_{n-1}| < |\boldsymbol{a}_n|,$$

where $\angle(\boldsymbol{a}_{i-1}, \boldsymbol{a}_i)$ is the argument from $\boldsymbol{a}_{i-1}$ to $\boldsymbol{a}_i$, and $|\boldsymbol{a}_i|$ is the length of $\boldsymbol{a}_i$. Then we have

$$\sum_{i=0}^{n} \boldsymbol{a}_i \neq 0.$$

Proof. We use the induction on the number n. For the sequence $\{\boldsymbol{a}_i\}_{i=0}^{n}$ of the vectors, we consider the following sequence of the vectors

$$\{(|\boldsymbol{a}_{n-i}|/|\boldsymbol{a}_i|)\, \boldsymbol{a}_i\}_{i=0}^{n}.$$

Assume that $\sum_{i=0}^{n} \boldsymbol{a}_i = 0$. Then we should see that the summation of the new sequence is also zero because it is just the reflection of the given sequence $\{\boldsymbol{a}_i\}_{i=0}^{n}$ in the line bisecting the angle between $\boldsymbol{a}_n$ and $\boldsymbol{a}_0$. Now, set

$$\boldsymbol{b}_i = (1 - (|\boldsymbol{a}_0|\,|\boldsymbol{a}_{n-i}|)/(|\boldsymbol{a}_n|\,|\boldsymbol{a}_i|))\, \boldsymbol{a}_i.$$

Since the sequence $\{\boldsymbol{b}_i\}_{i=0}^{n}$ is a linear combination of the two sequences above, we see that $\sum_{i=0}^{n} \boldsymbol{b}_i = 0$. On the other hand, we can apply our induction hypothesis to the sequence $\{\boldsymbol{b}_i\}_{i=1}^{n}$. Note that $\boldsymbol{b}_0 = 0$. Therefore it follows that $\sum_{i=0}^{n} \boldsymbol{b}_i \neq 0$. This is a contradiction. Thus we have $\sum_{i=0}^{n} \boldsymbol{a}_i \neq 0$.

COROLLARY 3.6. *With the same notation as in Lemma 3.4, if $a_i > 0$ for any i, then we have*

$$r \leqslant |x| \leqslant R,$$

where

$$r = \min\{a_0/a_1, a_1/a_2, \ldots, a_{n-1}/a_n\},$$

$$R = \max\{a_0/a_1, a_1/a_2, \ldots, a_{n-1}/a_n\}.$$

Proof. Put $X = rX'$ (or $X = RX'$), and apply Lemma 3.4.

EXAMPLE 3.7. Let $f_n(X) \in \mathbb{Z}[X]$ be a polynomial in one variable such that

$$f_n(X) = (d/dX)((X^n - 1)/(X-1))$$

for any integer $n \geqslant 3$. Then, for integers a, b, with $3 \leqslant a < b$, the polynomials $f_a(X)$ and $f_b(X)$ have no common root. Because we see that $f_a(X)$ and $f_b(X)$ have a common root if and only if $f_a(X)$ and $f_b(X) - f_a(X)$ have a common root. This contradicts Corollary 3.6.

Now, we prove Theorem 3.1.

Proof of Theorem 3.1. By Lemma 1.15, we can write

$$q^*C = m \cdot C_0 + D$$

with non-empty D. Let D_0 be any component of D, and let $n: D_1 \to D_0$ be the normalization of D_0. Let p_1 be the composition of the morphism $p: D_0 \to C$ with $n: D_1 \to D_0$, and let q_1 be the composition of the morphism $q: D_0 \to X$ with $n: D_1 \to D_0$. Then we see that q_1 is a morphism from D_1 to C. According to Corollary 3.3, there exists a point y of D_1 such that $n(y)$ is contained in both C_0 and D_0. Since $n(y)$ lies on C_0, we have that $p \circ n(y) = q \circ n(y)$ on C. Let x be the point $p_1(y) = q_1(y)$, let $t \in \mathcal{O}_{C,x}$ be a local parameter at x, and let $u \in \mathcal{O}_{D_1,y}$ be a local parameter at y. We may assume that, in an affine neighbourhood of x in C, the closed immersion $\phi: C \to \mathbb{P}^n$ is determined by n-tuples $(x_1, \ldots, x_n)$ of rational functions $x_1, \ldots, x_n$ in $\mathcal{O}_{C,x}$. Therefore, the morphism $p_1: D_1 \to \mathbb{P}^n$ is determined by the n-tuples $(p_1^* x_1, \ldots, p_1^* x_n)$ of rational functions in $\mathcal{O}_{D_1,y}$, and the morphism $q_1: D_1 \to \mathbb{P}^n$ is determined by the n-tuples $(q_1^* x_1, \ldots, q_1^* x_n)$ of rational functions in $\mathcal{O}_{D_1,y}$. Here, for any point y' in D_1, the point $q_1(y')$ lies on the tangent line to C at $p_1(y')$. Thus, we find that the rational map from D_1 to $\mathbb{P}^{n-1}$, determined by the n-tuples $(q_1^* x_1 - p_1^* x_1 : \ldots : q_1^* x_n - p_1^* x_n)$ of rational functions in $\mathcal{O}_{D_1,y}$, and the rational map from D_1 to $\mathbb{P}^{n-1}$, determined by the n-tuples $(p_1^* \dot{x}_1 : \ldots : p_1^* \dot{x}_n)$ of rational functions in $\mathcal{O}_{D_1,y}$, are identical, where $\dot{x}_i = dx_i/dt$ in $\mathcal{O}_{C,x}$. In particular, we get the following relations in $\mathcal{O}_{D_1,y}$

$$p_1^* \dot{x}_i (q_1^* x_j - p_1^* x_j) = p_1^* \dot{x}_j (q_1^* x_i - p_1^* x_i) \quad i, j = 1, \ldots, n. \qquad (*_{i,j})$$

Now, we assume that C is not contained in any 2-plane. Therefore, we may assume that the rational functions $x_1, \ldots, x_n$ in $\mathcal{O}_{C,x}$ have the following expressions

$$x_i = t^{a_i} + (\text{terms of higher order}) \quad i = 1, \ldots, n,$$

in the completion $\hat{\mathcal{O}}_{C,x}$, with $1 \leqslant a_1 < a_2 < a_3 \leqslant \ldots \leqslant a_n \leqslant \infty$, where $t^\infty = 0$. Then we see that $a_1 = 1$ since the closed immersion $\phi: C \to \mathbb{P}^n$ is unramified. Furthermore, we may assume that

$$p_1^* t = u^d + (\text{terms of higher order}),$$

with $d \geqslant 1$, so that we can write

$$q_1^* t = c' u^e + (\text{terms of higher order}),$$

with $e \geqslant 1$, and $c' \in k^\times$. Then, by the relation $(*_{1,2})$ (or $(*_{1,3})$), we find that $d \leqslant e$. Therefore we can write

$$q_1^* t = c u^d + (\text{terms of higher order}),$$

with $c \in k$. Substituting for $p_1^* t$ and $q_1^* t$ in the relations $(*_{1,2})$ and $(*_{1,3})$, taking the coefficients of the lowest degree in u, we get the following relations:

$$a_1(c^{a_2}-1) = a_2(c^{a_1}-1) \qquad (**_{1,2})$$

and

$$a_1(c^{a_3}-1) = a_3(c^{a_1}-1). \qquad (**_{1,3})$$

If we consider the relation $(*_{1,2})$ as an equation of $q_1^* t$, then it has an irrelevant root $q_1^* t = p_1^* t$, of multiplicity at least 2. Therefore, the equation $(**_{1,2})$ of c also has an irrelevant root $c = 1$, of multiplicity at least 2. Thus, we find that the coefficient c of $q_1^* t$ in u^d satisfies the following:

$$c^{a_2-2}+2c^{a_2-3}+\ldots+(a_2-1) = 0 \qquad (***_{1,2})$$

since $a_1 = 1$. Similarly, the relation $(**_{1,3})$ implies the following:

$$c^{a_3-2}+2c^{a_3-3}+\ldots+(a_3-1) = 0. \qquad (***_{1,3})$$

In other words, the equations $(***_{1,2})$ and $(***_{1,3})$ have a common root c. But this contradicts Example 3.7 since the characteristic p of the ground field k is zero. Thus, the curve C is contained in some 2-plane.

REMARK 3.8. This proof is valid for an unramified morphism ϕ from a curve C to a projective space $\mathbb{P}^n$, namely, for a curve in $\mathbb{P}^n$ which has only ordinary singularities. Indeed, we have used only the assumption that ϕ is unramified.

REMARK 3.9. Using Bertini's theorem, we can prove the following. Assume that the characteristic p is zero. Let M be an irreducible non-singular projective variety of dimension m in $\mathbb{P}^n$. If M is tangentially degenerate, then M is contained in some $(m+1)$-plane.

4. *Tangentially degenerate space curves in positive characteristic*

Let us give examples of tangentially degenerate curves in $\mathbb{P}^3$ which are not contained in any planes.

EXAMPLE 4.1. Let ϕ be a morphism from $\mathbb{P}^1$ to $\mathbb{P}^3$, determined by quadruples $(s^d : s^{d-1}t : st^{d-1} : t^d)$, where $d \geqslant 3$, and $\mathbb{P}^1 = \operatorname{Proj} k[s, t]$, and let C be the image of ϕ. For any characteristic p of the ground field k, the morphism ϕ is a closed immersion. Thus C is a curve in $\mathbb{P}^3$. It is clear that C is not contained in any plane but contained in a smooth quadric Q in $\mathbb{P}^3$, defined by the equation $wz = xy$, where $\mathbb{P}^3 = \operatorname{Proj} k[w, x, y, z]$. We see that, by an isomorphism $Q \cong \mathbb{P}^1 \times \mathbb{P}^1$, there exist two pencils of lines on Q.

Now, assume that the characteristic p is positive, and that p divides $d-1$. We can write

$$d-1 = p^l n,$$

where $l \geqslant 1$, $n \geqslant 1$, and $(p, n) = 1$. Then we find that C and one of the two pencils on Q have the intersection number $d-1$, and that, if $n \geqslant 2$, then C and a general member of the pencil meet at n distinct points with multiplicity p^l respectively. In other words, every tangent line of C is a member of the pencil of lines on Q. Therefore, we have $Q = X$ in $\mathbb{P}^3$ with the same notation as in §1, and we see that, if $n \geqslant 2$, then

C is tangentially degenerate. Assume, for example, that $d = 7$. Then, C is tangentially degenerate if $p = 2$ or $p = 3$.

In this case, we have that the curve C^* is a conic in $G \subset \mathbb{P}^5$, and $C^* = C'$, and that the morphism $g: C \to C^*$ has degree $d-1 = p^l n$ and separable degree n. Furthermore, $Q = X = TC'$ and $C = C'_0$, and the pencil on Q corresponds to the morphism $p': TC' \to C'$.

EXAMPLE 4.2 (Singular strange curves). Let ϕ be a morphism from $\mathbb{P}^1$ to $\mathbb{P}^3$, determined by quadruples $(s^d : s^{d-1}t : s^{d-c}t^c : t^d)$, where $3 \leqslant c < d-1$, and $\mathbb{P}^1 = \operatorname{Proj} k[s, t]$, and let C be the image of ϕ. Then the morphism ϕ is ramified at the point $\phi(0:1) = (0:0:0:1)$, so C is a singular curve. It is clear that C is not contained in any plane. Put $P = \phi(1:0) = (1:0:0:0)$. Let H be a plane in $\mathbb{P}^3$, defined by the equation $w = 0$, where $\mathbb{P}^3 = \operatorname{Proj} k[w, x, y, z]$. Moreover, we consider the projection from the point P to the plane H, let $\bar{C}$ be the image of C, and let Q be the cone with the vertex P over $\bar{C}$.

Now, assume that the characteristic p is positive, and that p divides both $c-1$ and $d-1$. Then we find that C and a general generating line of the cone Q meet at the points besides P with multiplicity not less than p. In other words, every tangent line of C is a generating line of the cone Q. Thus, we see that $Q = X$ in $\mathbb{P}^3$ with the same notation as in §1, and that C is a strange curve whose tangent lines all pass through the point P on C. In particular, C is tangentially degenerate.

In this case, we have that the curve C^* is contained in some 2-plane and isomorphic to the curve $\bar{C}$, and that the morphism $g: C \to C^*$ is compatible with the projection from the point P via the isomorphism $C^* \cong \bar{C}$. Furthermore, g is not separable, and the morphism $q': TC' \to X$ is a composition of the blow-up at the point P with the normalization.

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