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Syzygies of Canonical Curves and Special Linear Series

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0. Introduction

(0.1) In this paper we study the syzygies of canonical curves C of genus $g \leqq 8$. The main result (0.5) relates the ranks of the syzygy-modules to the existence of special linear series on C. These findings confirm and were partially motivated by a conjecture of Green, see (0.7). Before we formulate the main result we recall some algebraic and geometric notation and some classical results.

(0.2) Let C be a smooth algebraic curve of genus $g \geqq 3$ defined over an algebraically closed field $\mathfrak{k}$. ω_C denoted the canonical sheaf on C and

$$j: C \to \mathbb{P}(H^0(C, \omega_C)) = \mathbb{P}^{g-1}$$

the canonical map.

j is an embedding unless C is hyperelliptic. Furthermore

Theorem (Noether [N]). *If C is not hyperelliptic then*

$$\Omega = \sum_{n \geqq 0} H^0(C, \omega_C^{\otimes n})$$

is the homogeneous coordinate ring of $C \subset \mathbb{P}^{g-1}$.

If C is hyperelliptic then $j: C \to \mathbb{P}^{g-1}$ is a $2:1$ map onto a rational normal curve

$$\begin{tikzcd} C \arrow[r, "j"] \arrow[d, "2:1"'] & \mathbb{P}^{g-1} \\ \mathbb{P}^1 \arrow[ur, hook] & \end{tikzcd}$$

and $\Omega = \sum_{n \geq 0} H^0(C, \omega_C^{\otimes n})$ regarded as a module over the homogenous coordinate ring $S = \operatorname{Sym} H^0(C, \omega_C)$ of $\mathbb{P}^{g-1}$ is the module of global sections of the rank 2 vectorbundle $j_* \mathcal{O}_C \cong \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-g-1)$ on the rational normal curve $\mathbb{P}^1 \subset \mathbb{P}^{g-1}$.

(0.3) Our basic object of studies are the syzygies of

$$\Omega = \sum_{n \geq 0} H^0(C, \omega_C^{\otimes n})$$

regarded as a graded module over the homogeneous coordinate ring

$$S = \operatorname{Sym} H^0(C, \omega_C) \cong \mathfrak{k}[x_0, \dots, x_{g-1}]$$

of $\mathbb{P}^{g-1}$. Let

$$F_* : 0 \to F_{g-2} \to \dots \to F_1 \to F_0 \to \Omega \to 0$$

be the *minimal free resolution* of Ω, i.e. an exact complex of graded free S-modules

$$F_i = \bigoplus_{j \in \mathbb{Z}} S(-j)^{\beta_{ij}}$$

such that $\operatorname{Im}(F_i) \subset \mathfrak{m} F_{i-1}$, where $\mathfrak{m}$ denotes the homogenous maximal ideal of S and $S(-j)$ the free S-module with one generator in degree $+j$, i.e. $S(-j)_n = S_{n-j}$.

The minimality gives

$$\beta_{ij} = \dim \operatorname{Tor}_i^S(\Omega, \mathfrak{k})_j,$$

we call these integers the *graded betti-numbers* of Ω (resp. C). $\operatorname{Im}(F_i) \subset F_{i-1}$ is the i^{th} *syzygy-module* of Ω. By the formula of Auslander-Buchsbaum-Serre the length of the resolution is $g-2$, since Ω is a Cohen-Macaulay module. Frequently we will replace F_* by the corresponding sequence of sheaves of $\mathcal{O} = \mathcal{O}_{\mathbb{P}^{g-1}}$-modules:

$$0 \to \bigoplus_j \mathcal{O}(-j)^{\beta_{g-2j}} \to \dots \to \bigoplus_j \mathcal{O}(-j)^{\beta_{0j}} \to j_* \mathcal{O}_C \to 0.$$

Since F_* is just the sequence of global sections, we loose no information.

(0.4) With this notation Noether's result says: $\beta_{0j} = 0$ for $j > 0$ iff C is not hyperelliptic. A classical result of Petri is a statement about the β_{1j}'s:

Theorem (Petri [P]). *The homogenous ideal I_C of a non-hyperelliptic canonical curve C is generated by quadrics unless*

(a) *C is trigonal (i.e. $C \xrightarrow{3:1} \mathbb{P}^1$) or*

(b) *C is isomorphic to a smooth plane quintic, $g = 6$.*

In the exceptional cases the quadrics contained in I_C generated the homogenous ideal of a surface of minimal degree, which is

(a) *a rational normal scroll or*

(b) *the Veronese surface $\mathbb{P}^2 \hookrightarrow \mathbb{P}^5$.*

For a proof see also [S-D].

Thus $\beta_{1j} = 0$ for $j > 2$ unless C is one of the exceptional curves.

(0.5) These results suggest that the values of the β_{ij} are related to the existence of special linear series of divisors on C. The symbol g_d^r will denote an *r-dimensional linear series of divisors of degree d* on C.

For $g \leq 6$ all possible values of the β_{ij} can be deduced from the results of Noether and Petri. Our main result is

Theorem. *For C a curve of genus $g=7$ or $g=8$ the values of the graded betti-numbers β_{ij} depend on and determine the existence of a*

$$g_2^1,\ g_3^1,\ g_6^2 \text{ or } g_4^1$$

on C.

The precise values of the β_{ij} in either case are given in the following table:

Table 1. The graded betti-numbers of Ω for curves up to genus 8 for char($\mathfrak{k}$)$\neq 2$

genus	β_{01}	β_{12}	β_{23}			special linear series
	β_{00}	β_{11}	β_{22}			
g = 3		1				general case
	1					
		1				g_2^1
	1					
		1				
	1					
g = 4			1			general case
		1				
		1				
	1					
			1			g_2^1
	2	3				
		3	2			
	1					
g = 5				1		general case
			3			
		3				
	1					
				1		g_3^1
		2	3			
		3	2			
	1					
				1		g_2^1
	3	8	6			
		6	8	3		
	1					
g = 6					1	general case
			5	6		
		6	5			
	1					
					1	g_3^1 or g_5^2
		3	8	6		
		6	8	3		
	1					
					1	g_2^1
	4	15	20	10		
		10	20	15	4	
	1					

genus	β_{01}	β_{12}	β_{23}					special linear series
	β_{00}	β_{11}	β_{22}					
g = 7						1		general case *
				16	10			
		10	16					
	1							
						1		g_4^1
			3	16	10			
		10	16	3				
	1							
						1		g_6^2
			9	16	10			
		10	16	9				
	1							
						1		g_3^1
		4	15	20	10			
		10	20	15	4			
	1							
						1		g_2^1
	5	24	45	40	15			
		15	40	45	24	5		
	1							
g = 8							1	general case
				21	35	15		
		15	35	21				
	1							
							1	g_4^1
			4	25	35	15		
		15	35	25	4			
	1							
							1	g_6^2
			14	35	35	15		
		15	35	35	14			
	1							
							1	g_3^1
		5	24	45	40	15		
		15	40	45	24	5		
	1							
							1	g_2^1
	6	35	84	105	70	21		
		21	70	105	84	35	6	
	1							

* For char($\mathfrak{k}$)$=2$: As above, but the general case for $g=7$:

g =7						1		general case
			1	10	16			
		16	10	1				
	1							

(0.6) *Remarks.* 1) The symmetry of the values

$$\beta_{ij} = \beta_{g-2-i,g+1-j}$$

reflects the self duality of F_*:

$$\operatorname{Hom}(F_*, S(-g-1))[g-2] \cong F_*,$$

which follows from the adjunction formula

$$j_*\omega_C = \mathscr{E}xt_{\mathcal{O}}^{g-2}(j_*\mathcal{O}_C, \omega_{\mathbb{P}^{g-1}})$$

and

$$\omega_{\mathbb{P}^{g-1}} = \mathcal{O}(-g), \qquad \omega_C = j^*\mathcal{O}_C(1).$$

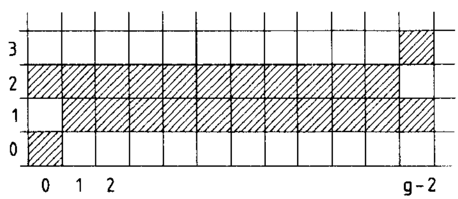
2) The graded betti-number satisfy the identities

$$\sum_i (-1)\beta_{ij} = \sum_i (-1)^i \binom{g-1}{i} h^0(\omega_C^{\otimes j-i})$$

by a result of Hilbert [Hi].

3) In general for a graded module little is known what kind of information is carried by its graded betti-numbers other than the Hilbert function and the depth.

(0.7) For curves of higher genus it is easy to see that the tuples $(i, j-i)$ such that $\beta_{ij} \neq 0$ lie in the range indicated below:



Green gives a conjecture on the precise range, which generalizes the result of Noether and Petri:

Conjecture (Green [G]). *For* $\operatorname{char}(\mathfrak{k}) = 0$ *the following is equivalent:*

(1) $\beta_{i,i+1} \neq 0$;

(2) *there exists a linear series* g_d^r *on* C *with* $r \geq 1$, $d \leq g-1$ *and Clifford-index* $d - 2r = g - 2 - i$.

A proof of (2) $\Rightarrow$ (1) is contained in [G]. Our result verifies this conjecture for $g \leq 8$ over fields of $\operatorname{char}(\mathfrak{k}) \neq 2$. If $\operatorname{char}(\mathfrak{k}) = 2$ the corresponding statement is false for curves of genus 7.

(0.8) To prove the result we take the following approach to the syzygies of Ω. Start with a base point free complete pencil g_d^1 of divisors of degree d on C. The variety swept out by the linear spans of these divisors

$$X = \bigcup_{D \in g_d^1} \bar{D} \subset \mathbb{P}^{g-1}$$

is a rational normal scroll of dimension $d-1$ by a classical result of Bertini, cf. Sects. 1, 2, 4.

We resolve $\mathcal{O}_C$ as an $\mathcal{O}_{\mathbb{P}^{g-1}}$-module in two steps:

1) resolve $\mathcal{O}_C$ as an $\mathcal{O}_X$-module by direct sums of line bundles on X;

2) take the resolution of each of these line bundles as $\mathcal{O}_{\mathbb{P}^{g-1}}$-module and make a mapping cone construction to obtain a resolution of $\mathcal{O}_C$ as an $\mathcal{O}_{\mathbb{P}^{g-1}}$-module. (cf. Sects. 1, 3, 4). This approach to the syzygies of was suggested to me by E. Sernesi. For small d at least in as much as equations of C are concerned it can already be found in Petri's paper.

(0.9) For $d=3$, $d=4$ we obtain the minimal resolution of $\mathcal{O}_C$ in this way. We study these cases in Sect. 6 in some details. For example we obtain:

Theorem. *A morphism $C \to \mathbb{P}^1$ of degree 4 on a curve of genus g factors through a curve E of genus g' with $6g' < g+3$, i.e.*

$$\begin{tikzcd} C \arrow[rr, "4:1"] \arrow[dr, "2:1"'] & & \mathbb{P}^1 \\ & E \arrow[ur, "2:1"'] & \end{tikzcd}$$

iff

$$\beta_{i,i+1} = i\binom{g-3}{i+1} \quad \textit{for} \quad i > g-2-2g'$$

and

$$\beta_{i,i+1} > i\binom{g-3}{i+1} \quad \textit{for} \quad i = g-2-2g'.$$

(0.10) Unfortunately for $d \geqq 5$ the same approach gives only a non-minimal resolution of Ω. However in Sect. 7 a detailed study of these complexes for $d=5$ and $g=7$ or $g=8$ allows us to deduce the existence of a rational determinantal surface Y on X with

$$C \subset Y \subset X \subset \mathbb{P}^{g-1}$$

whose degree depends on the values of the β_{ij}. The classification of these surfaces (Sect. 5) allows us to finish the proof.

1. Line Bundles on Scrolls

(1.1) Let $\mathscr{E} = \mathcal{O}(e_1) \oplus \ldots \oplus \mathcal{O}(e_d)$ be a locally free sheaf of rank d on $\mathbb{P}^1$ and let

$$\pi : \mathbb{P}(\mathscr{E}) \to \mathbb{P}^1$$

denote the corresponding $\mathbb{P}^{d-1}$-bundle. A *rational normal scroll* X of type $S(e_1, \ldots, e_d)$ with $e_1 \geqq \ldots \geqq e_d \geqq 0$ and

$$f = e_1 + \ldots + e_d \geqq 2$$

is the image of $\mathbb{P}(\mathscr{E})$ in $\mathbb{P}^r = \mathbb{P}H^0(\mathbb{P}(\mathscr{E}), \mathscr{O}_{\mathbb{P}(\mathscr{E})}(1))$:

$$j: \mathbb{P}(\mathscr{E}) \to X \subset \mathbb{P}^r, \quad r = f + d - 1.$$

X is a non-degenerate irreducible variety of minimal degree

$$\deg X = f = r - d + 1 = \operatorname{codim} X + 1$$

in $\mathbb{P}^r$ [Ha].

If all $e_i > 0$ then X is smooth and $j: \mathbb{P}(\mathscr{E}) \to X$ an isomorphism. If some of the $e_i = 0$, then X is singular and $j: \mathbb{P}(\mathscr{E}) \to X$ is a resolution of singularities. The singularities of X are rational, i.e.

$$j_* \mathscr{O}_{\mathbb{P}(\mathscr{E})} = \mathscr{O}_X, \quad R^i j_* \mathscr{O}_{\mathbb{P}(\mathscr{E})} = 0 \quad \text{for} \quad i > 0$$

[Ke]. Consequently we may replace X by $\mathbb{P}(\mathscr{E})$ for most cohomological considerations, even if X is singular.

(1.2) The *Picard group* of $\mathbb{P}(\mathscr{E})$ is generated by the *hyperplane class* $H = [j^* \mathscr{O}_{\mathbb{P}^r}(1)]$ and the *ruling* $R = [\pi^* \mathscr{O}_{\mathbb{P}^1}(1)]$:

$$\operatorname{Pic} \mathbb{P}(\mathscr{E}) = \mathbb{Z}H \oplus \mathbb{Z}R,$$

the intersection product is given by

$$H^d = f, \quad H^{d-1} \cdot R = 1, \quad R^2 = 0$$

[Ha].

In this section we recall from [E2] the description of the *syzygies* of the sheaves

$$\mathscr{O}_X(aH + bR) := j_* \mathscr{O}_{\mathbb{P}(\mathscr{E})}(aH + bR), \quad a, b \in \mathbb{Z}$$

regarded as $\mathscr{O}_{\mathbb{P}^r}$-modules, at least in case $b \geqq -1$.

(1.3) The *cohomology* of a line bundle $\mathscr{O}_{\mathbb{P}(\mathscr{E})}(aH + bR)$ can explicitly be calculated with the Leray spectral sequence:

$$H^i(\mathbb{P}^1, R^j \pi_* \mathscr{O}_{\mathbb{P}(\mathscr{E})}(aH + bR)) \Rightarrow H^{i+j}(\mathbb{P}(\mathscr{E}), \mathscr{O}_{\mathbb{P}(\mathscr{E})}(aH + bR)).$$

For example, if $a \geqq 0$ and $S_a \mathscr{E}$ denotes the a^{th}-symmetric power of $\mathscr{E}$ we have

$$H^0(\mathbb{P}(\mathscr{E}), \mathscr{O}_{\mathbb{P}(\mathscr{E})}(aH + bR)) \cong H^0(\mathbb{P}^1, (S_a \mathscr{E})(b)).$$

More precisely: Let $\mathfrak{k}[s, t]$ denote the homogenous coordinate ring of $\mathbb{P}^1$ and let

$$\varphi_i \in H^0(\mathbb{P}(\mathscr{E}), \mathscr{O}_{\mathbb{P}(\mathscr{E})}(H - e_i R)), \quad i = 1, \ldots, d$$

denote the *basic section* obtained from the inclusion of the i^{th}-summand

$$\mathscr{O}_{\mathbb{P}^1} \to \mathscr{E}(-e_i) \cong \pi_* \mathscr{O}_{\mathbb{P}(\mathscr{E})}(H - e_i R).$$

Then we can identify sections

$$\psi \in H^0(\mathbb{P}(\mathscr{E}), \mathscr{O}_{\mathbb{P}(\mathscr{E})}(aH + bR))$$

with homogenous polynomials

$$\psi = \sum_\alpha P_\alpha(s, t) \varphi_1^{\alpha_1} \cdot \ldots \cdot \varphi_d^{\alpha_d}$$

of degree $a=\alpha_1+\ldots+\alpha_d$ in the φ_i's and coefficients homogenous polynomials $P_\alpha \in \mathfrak{k}[s,t]$ of degree

$$\deg P_\alpha = \alpha_1 e_1 + \ldots + \alpha_d e_d + b.$$

In particular we notice that for $b \geqq -1$ the dimension

$$h^0(\mathcal{O}_{\mathbb{P}(\mathscr{E})}(aH+bR)) = f\binom{a+d-1}{d} + (b+1)\binom{a+d-1}{d-1}$$

does not depend on the type $S(e_1, \ldots, e_d)$ of the scroll (resp. $\mathscr{E}$), but only on its degree $f = e_1 + \ldots + e_d$.

(1.4) The defining equations of X are *determinantal*: Choose the basis

$$x_{ij} = t^j s^{e_i - j} \varphi_i$$

for $i=1, \ldots, d$, $j=0, \ldots, e_i$ of $H^0\mathcal{O}_{\mathbb{P}(\mathscr{E})}(H) \cong H^0\mathcal{O}_{\mathbb{P}^r}(1)$ and consider the matrix

$$\Phi = \begin{pmatrix} x_{10} \ldots x_{1e_1-1} & x_{20} \ldots & \ldots x_{de_d-1} \\ x_{11} \ldots x_{1e_1} & x_{21} \ldots & \ldots x_{de_d} \end{pmatrix}.$$

By definition the 2×2 minors of Φ vanish identically on X. We will see in (1.6) that they actually generate the homogenous ideal of X.

In more intrinsic terms we can obtain Φ from the multiplication map

$$H^0(\mathcal{O}_{\mathbb{P}(\mathscr{E})}(R)) \otimes H^0(\mathcal{O}_{\mathbb{P}(\mathscr{E})}(H-R)) \to H^0(\mathcal{O}_{\mathbb{P}(\mathscr{E})}(H)).$$

(1.5) Let

$$\Phi: F \to G$$

be a map of locally free sheaves of rank f and g, $f \geqq g$, respectively on a smooth variety V. We recall from [B-E 2] the family of complexes $\mathscr{C}^b$, $b \geqq -1$, of locally free sheaves on V, which resolve the b^{th}-symmetric power of coker Φ under suitable hypothesis on Φ.

Define the j^{th}-term in the complex $\mathscr{C}^b$ by

$$\mathscr{C}_j^b = \begin{cases} \bigwedge^j F \otimes S_{b-j} G, & \text{for } 0 \leqq j \leqq b \\ \bigwedge^{j+g-1} F \otimes D_{j-b-1} G^* \otimes \bigwedge^g G^*, & \text{for } j \geqq b+1 \end{cases}$$

and differential

$$\mathscr{C}_j^b \to \mathscr{C}_{j-1}^b$$

by the multiplication with $\Phi \in H^0(V, F^* \otimes G)$ for $j \neq b+1$ and $\bigwedge^g \Phi \in H^0\left(V, \bigwedge^g F^* \otimes \bigwedge^g G\right)$ for $j=b+1$ in the appropriate term of the exterior $(\wedge F)$, symmetric (S.G) or divided power (D.G) algebra.

E.g.: For an open set $U \subset V$ the differential of a term

$$f_1 \wedge \ldots \wedge f_j \otimes g \in H^0(U, \mathscr{C}_j^b); \quad j \leqq b,$$

with $f_i \in H^0(U, F)$, $g \in H^0(U, S_{b-j} G)$ is given by

$$\sum_{i=1}^{j} (-1)^i f_1 \wedge \ldots \hat{f}_i \ldots \wedge_j \otimes \Phi(f_i) \cdot g \in H^0(U, \mathscr{C}_{j-1}^b).$$

$\mathscr{C}^0$ is the well-known *Eagon-Northcott* complex [E-N] associated to Φ. The image of $\mathscr{C}_1^0 \to \mathscr{C}_0^0 \cong \mathcal{O}_V$ is the idealsheaf $I_g(\Phi)$ generated by the $g \times g$ minors of Φ. $\mathscr{C}^1$ is the *Buchsbaum-Rim* complex. The cokern M of $\mathscr{C}_1^1 \to \mathscr{C}_0^1$ is supported on $V(I_g(\Phi))$. For $b > 1$ the cokern $\mathscr{C}_1^b \to \mathscr{C}_0^b$ is (by definition) the b^{th}-symmetric power of M.

Theorem (Buchsbaum-Eisenbud [B-E2]). *The complexes $\mathscr{C}^b$ for $b \geqq -1$ is acyclic if the ideals $I_{g-k}(\Phi)$ of $(g-k) \times (g-k)$ minors of Φ have*

$$\operatorname{depth} I_{g-k}(\Phi) \geqq f - g + k + 1$$

for $k = 0$ and $1 \leqq k \leqq b - f + g - 1$. □

(1.6) We regard the matrix Φ from (1.4) as a map of bundles

$$\Phi: F(-1) \to G$$

on $\mathbb{P}^r$ with $F = \mathcal{O}_{\mathbb{P}^r}^f$, $G = \mathcal{O}_{\mathbb{P}^r}^2$. Since the condition (1.5) on the depth of the ideal of the minors on Φ is satisfied, all of the complexes $\mathscr{C}^b$ and their twist $\mathscr{C}^b(a) = \mathscr{C}^b \otimes \mathcal{O}_{\mathbb{P}^r}(a)$, are acyclic. Hence

Corollary (Eisenbud [E2]). *$\mathscr{C}^b(a)$ for $b \geqq -1$ is the minimal resolution of $\mathcal{O}_X(aH + bR)$ as an $\mathcal{O}_{\mathbb{P}^r}$-module.*

Proof. It remains to identify

$$\operatorname{coker}(\mathscr{C}_1^b \to \mathscr{C}_0^b) \cong \mathcal{O}_X(bR).$$

For $b = 0$, we obtain from the exactness of $\mathscr{C}^0$ that the variety X' defined by the 2×2-minors of Φ has dimension d and degree f. Furthermore, since the length of the complex $\mathscr{C}^0$ is $f - 1 = \operatorname{codim} X'$, X' is arithmetically Cohen-Macaulay. Since $X' \supset X$ and both varieties have the same degree and dimension, they coincide. In particular we obtain that the 2×2-minors of Φ generate the homogenous ideal of X.

For $b = 1$ we identify

$$\operatorname{coker}(\mathscr{C}_1^1 \to \mathscr{C}_0^1) = \operatorname{coker} \Phi \cong \mathcal{O}_X(R)$$

via global sections: Choose an orientation

$$\bigwedge^2 (H^0 \mathcal{O}_{\mathbb{P}(\mathscr{E})}(R)) \cong \mathfrak{k}$$

with the induced isomorphism

$$H^0 \mathcal{O}_{\mathbb{P}(\mathscr{E})}(R) \cong (H^0 \mathcal{O}_{\mathbb{P}(\mathscr{E})}(R))^*.$$

With this identification the diagram

$$\begin{tikzcd}
H^0(\mathbb{P}^r, F(a-1)) \arrow[r] \arrow[d, "\wr"] \arrow[dr, phantom, "\square"] & H^0(\mathbb{P}^r, G(a)) \arrow[r] \arrow[d, "\wr"] & H^0(\mathbb{P}^r, (\operatorname{coker} \Phi)(a)) \arrow[r] \arrow[d, "\exists\,\wr"] & 0 \\
H^0\mathcal{O}_{\mathbb{P}^r}(a-1) \otimes H^0\mathcal{O}_{\mathbb{P}(\mathscr{E})}(H-R) \arrow[r] & H^0\mathcal{O}_{\mathbb{P}^r}(a) \otimes (H^0\mathcal{O}_{\mathbb{P}(\mathscr{E})}(R))^* \arrow[r] & H^0\mathcal{O}_{\mathbb{P}(\mathscr{E})}(aH+R) \arrow[r] & 0
\end{tikzcd}$$

commutes, i.e. gives the desired isomorphism.

For $b>1$ the result follows from

$$\operatorname{coker}(\mathscr{C}_1^b \to \mathscr{C}_0^b) = S_b(\operatorname{coker}\Phi) \cong \mathcal{O}_X(bR).$$

Finally for $b=-1$ we use the duality

$$\mathscr{H}om(\mathscr{C}^{f-2-i}, \mathcal{O}_{\mathbb{P}^r}) \cong \mathscr{C}^i(f)$$

for $i=-1,\dots,f-1$ of the complexes. From the case $i=f-2$ we obtain the dualizing sheaf of X:

$$\begin{aligned}\omega_X = H_0(\mathscr{H}om(\mathscr{C}^0, \omega_{\mathbb{P}^r})) &\cong H_0(\mathscr{C}^{f-2}(-d))\\ &\cong \mathcal{O}_X(-dH+(f-2)R).\end{aligned}$$

Consequently

$$\begin{aligned}H_0(\mathscr{C}^{-1}) &\cong H_0(\mathscr{H}om(\mathscr{C}^{f-1}, \omega_{\mathbb{P}^r}(d))\\ &\cong \mathscr{H}om(\mathcal{O}_X(f-1)R, \omega_X(dH))\end{aligned}$$

by duality,

$$\begin{aligned}&\cong \mathscr{H}om(\mathcal{O}_X(f-1)R, \mathcal{O}_X(f-2)R)\\ &\cong \mathcal{O}_X(-R). \quad \square\end{aligned}$$

(1.7) *Remarks.* a) We calculated

$$\omega_X \cong \mathcal{O}_X(-dH)+(f-2)R)$$

during the proof of (1.6).

b) Notice that the complexes $\mathscr{C}^b$, $b \geqq -1$, for our matrix are 1-*regular*, i.e. all but one differential is given in terms of a basis by a matrix with *linear* entries, and the exceptional differential $\mathscr{C}_{b+1}^b \to \mathscr{C}_b^b$ is given by a matrix with *quadratic* entries.

c) For the line bundles $\mathcal{O}_X(aH+bR)$ with $b<-1$ a similarly simple description of the syzygies is not possible. One reason is that the dimensions $h^0\mathcal{O}_{\mathbb{P}(\mathscr{E})}(aH+bR)$ for $b<-1$ depend on the type $S(e_1,\dots,e_d)$ of the scroll.

2. Scrolls and Pencils

(2.1) In this section we survey for a smooth variety V and linearly normal map

$$j: V \to \mathbb{P}^r = \mathbb{P}(H^0(V, \mathcal{O}_V(H))$$

the rational normal scrolls $X \subset \mathbb{P}^r$ which contain the image $j(V)$, cf. [Ha].

Let $X \subset \mathbb{P}^r$ be a scroll of degree f containing $j(V)$. The ruling R on X cuts out on V a pencil of divisors (possibly with basepoints)

$$\{D_\lambda\}_{\lambda \in \mathbb{P}^1} \subset |D|$$

with $h^0(V, \mathcal{O}_V(H-D)) = f$.

(2.2) Conversely we can construct from any pencil of divisors $\{D_\lambda\}_{\lambda\in\mathbb{P}^1}$ on V which satisfies $h^0(V, \mathcal{O}_V(H-D)) = f \geqq 2$ a scroll of degree f as follows: Let $G \subset H^0(V, \mathcal{O}_V(D))$ be the 2-dimensional subspace which defines the pencil. The multiplication map

$$G \otimes H^0(V, \mathcal{O}_V(H-D)) \to H^0(V, \mathcal{O}_V(H))$$

yields a $2 \times f$ matrix Φ with linear entries whose 2×2 minors vanish on $j(V)$. The variety X defined by these minors contains $j(V)$ and is a scroll of degree f [cf. (1.4)].

(2.3) Geometrically we may construct X as follows: Let

$$\overline{D_\lambda} := \bigcap_{\substack{H \text{ with} \\ j^*H \geq D_\lambda}} H$$

be the linear span of $j(D_\lambda)$ in $\mathbb{P}^r$, i.e. the linear space defined by the linear forms of

$$H^0(V, \mathcal{O}_V(H-D_\lambda)) \to H^0(V, \mathcal{O}_V(H)) = H^0(\mathbb{P}^r, \mathcal{O}(1)).$$

Then X is the variety swept out by these linear spaces:

$$X = \bigcup_{\lambda \in \mathbb{P}^1} \overline{D_\lambda} \subset \mathbb{P}^r.$$

[Proof: The 2×2-minors of the matrix Φ vanish in a point $p \in \mathbb{P}^r$ iff the rank $\Phi(p) \leq 1$, i.e. iff the two rows of $\Phi(p)$ are linearly dependent. Say the first row vanishes identically. Then by definition $p \in \overline{D_{(1,0)}}$.]

(2.4) The type $S(e_1, \ldots, e_d)$ of the scroll can be determined as follows. Decompose the pencil

$$D_\lambda = F + E_\lambda, \quad \lambda \in \mathbb{P}^1$$

into its fixed and moving part and consider the following partition of $r+1$:

$$\begin{aligned}
d_0 &:= h^0\mathcal{O}_V(H) - h^0\mathcal{O}_V(H-D), \\
d_1 &:= h^0\mathcal{O}_V(H-D) - h^0\mathcal{O}_V(H-F-2E), \\
&\vdots \\
d_i &:= h^0\mathcal{O}_V(H-F-iE) - h^0\mathcal{O}_V(H-F-(i+1)E), \\
&\vdots
\end{aligned}$$

We use the dual partition to define the number e_i:

$$e_i + 1 = \# \{j \mid d_j \geq i\}.$$

(2.5) **Theorem** (Harris, Bertini). *With the notation as above* (2.2–2.4) *X is a d_0-dimensional rational normal scroll of type $S(e_1, \ldots, e_{d_0})$.*

Proof. We are looking for a basis

$$x_{ij}, \quad i = 1, \ldots, d_0, \quad j = 0, \ldots, e_i,$$

of $H^0\mathcal{O}_{\mathbb{P}^r}(1) \cong H^0\mathcal{O}_V(H)$ as in (1.4). Let G now denote the 2-dimensional subspace of $H^0(V, \mathcal{O}_V(E))$ corresponding to the pencil $\{E_\lambda\}_{\lambda \in \mathbb{P}^1}$ without fixed components. We consider the exact sequences (cf. pencil trick [S-D]):

$$\begin{aligned}
0 \to \bigwedge^2 G \otimes \mathcal{O}_V(H-F+(i-2)E) &\to G \otimes \mathcal{O}_V(H-F+(i-1)E) \\
&\to \mathcal{O}_V(H-f+iE)
\end{aligned}$$

and take global sections. The resulting sequence

$$0 \to \bigwedge^2 G \otimes M \to G \otimes M \to M$$

with

$$M = \sum_{i \in \mathbb{Z}} H^0 \mathcal{O}_V(H - F + iE)$$

is exact, where we regard M as a graded module over the polynomial ring $S_* G$. So

$$\operatorname{Tor}_j^{S_* G}(M, \mathfrak{k}) = 0 \quad \text{for} \quad j > 0$$

and

$$M \cong S_* G \bigotimes_{\mathfrak{k}} M/GM$$

as a graded $S_* G$-module. In particular

$$M_{-1} = \bigoplus_{i: e_i \geqq 1} S_{e_i - 1} G$$

and the composition

$$G \otimes M_{-1} = G \otimes H^0 \mathcal{O}_V(H - D) \to M_0 = H^0 \mathcal{O}_V(H - F) \subset H^0 \mathcal{O}_V(H)$$

gives a $2 \times h^0 \mathcal{O}_V(H - D)$ matrix of the desired type. □

3. Syzygies of a Subvariety of a Scroll

(3.1) Let $V \subset \mathbb{P}(\mathscr{E})$ be a subvariety of a $\mathbb{P}^{d-1}$-bundle $\mathbb{P}(\mathscr{E})$ over $\mathbb{P}^1$. In the first part of this section we want to construct a resolution F_* of $\mathcal{O}_V$ by locally free $\mathcal{O}_{\mathbb{P}(\mathscr{E})}$-modules F_i, which restricts to the *minimal* resolution of $\mathcal{O}_{V_\lambda}$ on each fibre $\mathbb{P}(\mathscr{E})_\lambda \cong \mathbb{P}^{d-1}$, $\lambda \in \mathbb{P}^1$.

Necessary and sufficient for the existence of such a complex F_* is that V has *constant betti-numbers* $\beta_{ij} = \beta_{ij}(\lambda)$ *over* $\mathbb{P}^1$, i.e. the betti-numbers $\beta_{ij}(\lambda)$ of V_λ do not depend on $\lambda \in \mathbb{P}^1$. This is a somewhat stronger condition than flatness of V over $\mathbb{P}^1$.

In the second part of this section we consider the image $V' \subset X \subset \mathbb{P}^r$ of V in a rational normal scroll X corresponding to $\mathbb{P}(\mathscr{E})$ and construct a resolution of $\mathcal{O}_{V'}$ as an $\mathcal{O}_{\mathbb{P}^r}$-module using the complex F_*, the complexes $\mathscr{C}^b(a)$ (1.6) and a mapping cone construction provide F_* satisfies a suitable hypothesis.

(3.2) **Theorem.** *Let $V \subset \mathbb{P}(\mathscr{E})$ be a subvariety with constant betti-numbers $\beta_{ij} = \beta_{ij}(\lambda)$ over $\mathbb{P}^1$. $\mathcal{O}_V$ has a resolution*

$$0 \to F_C \to F_{C-1} \to \dots \to F_1 \to \mathcal{O}_{\mathbb{P}(\mathscr{E})} \to \mathcal{O}_V \to 0$$

by vectorbundles F_i which admit a filtration

$$\dots \subset F_{ij-1} \subset F_{ij} \subset F_{ij+1} \subset \dots \subset F_i$$

such that the quotients

$$F_{ij}/F_{ij-1}=\sum_{k=1}^{\beta_{ij}}\mathcal{O}_{\mathbb{P}(\mathscr{E})}(-jH+b_{ij}^{(k)}R)$$

split as above. If the fibres V_λ have pure resolutions (i.e. for each i $\beta_{ij}(\lambda)\neq 0$ for at most one j) then F_* is unique up to isomorphism.

(3.3) *Remark.* The integers $b_{ij}^{(k)}$ satisfy a system of linear equations, which can be obtained from the identity

$$\chi\mathcal{O}_V(vH)=\sum_i(-1)^i\chi(F_i(vH))$$

of the Hilbert polynomials.

(3.4) *Proof.* Let $F_*^{k(t)}$ be the minimal resolution of the homogeneous coordinate ring of the generic fibre V_ξ over $\mathfrak{k}(t)[\varphi_1,\dots,\varphi_d]$. Clearing denominators we obtain a complex $F_*^{\mathfrak{k}[t]}$ defined over $R=\mathfrak{k}[t][\varphi_1,\dots,\varphi_d]$ whose homology is finite dimensional in each degree:

$$\dim_{\mathfrak{k}}H_*(F_*^{\mathfrak{k}[t]})_j<\infty\,.$$

We will modify the differential of $F_*^{\mathfrak{k}[t]}$ to obtain exactness. Suppose $H_*(F_*^{\mathfrak{k}[t]})_j\neq 0$, while the homology vanishes in all degrees $<j$, say $H_i(F_*^{\mathfrak{k}[t]})_j$ has support at a point $\lambda\in\operatorname{Spec}\mathfrak{k}[t]$. Then there exists a free summand

$$F_{i+1}^{\mathfrak{k}[t]}=\tilde{F}_{i+1}^{\mathfrak{k}[t]}\oplus R(-j)$$

whose image under the differential

$$d_{i+1}(R(-j))\subset\lambda\cdot F_i^{\mathfrak{k}[t]}\,.$$

(Indeed, if no summand of this type exists, than since V_ξ and V_λ have the same betti-numbers and $F_*^{\mathfrak{k}[t]}$ is exact in all degrees $<j$ the $\mathfrak{k}$-vector spaces

$$\operatorname{Im}(d_{i+1})_j\otimes\mathfrak{k}[t]/\lambda\subset\ker(d_i)_j\otimes\mathfrak{k}[t]/\lambda$$

have the same dimension, hence are equal. But this implies the exactness of $F_*^{\mathfrak{k}[t]}$ in degree ij at λ.)

Let $(t-p)\in\lambda$ be a local parameter. The differential d_{i+1} factors as follows

$$\begin{tikzcd}
F_{i+2}^{\mathfrak{k}[t]} \arrow[r, "d_{i+2}"] \arrow[dr, "d'_{i+2}"'] & \tilde{F}_{i+1}^{\mathfrak{k}[t]}\oplus R(-j) \arrow[r, "d_{i+1}"] \arrow[d, "\mathrm{id}\oplus(t-p)\mathrm{id}"] & F_i^{\mathfrak{k}[t]} \arrow[r, "d_i"] & F_{i-1}^{\mathfrak{k}[t]} \\
& \tilde{F}_{i+1}^{\mathfrak{k}[t]}\oplus R(-j) \arrow[ur, "d'_{i+1}"'] & &
\end{tikzcd}$$

By the resulting complex $(F_*^{\mathfrak{k}[t]},d')$ satisfies:

$$\dim_k H_*(F_*^{\mathfrak{k}[t]},d')_j=\dim H_*(F_*^{\mathfrak{k}[t]},d)_j-1\,.$$

Using this construction repeatedly we obtain exactness in degree j and by induction on j exactness in all degrees after finitely many steps.

Similarly we can construct an exact complex $F_*^{\mathfrak{k}[t^{-1}]}$ defined over $\mathfrak{k}[t^{-1}][\psi_1,\dots,\psi_d]$. After identifying

$$\mathfrak{k}[t,t^{-1}][\varphi_1,\dots,\varphi_d]\cong\mathfrak{k}[t,t^{-1}][\psi_1,\dots,\psi_d]$$

via the transition functions

$$\psi_i = t^{e_i}\varphi_i$$

of the bundle $\mathbb{P}(\mathscr{E})$ we obtain comparison maps

$$F^{\mathfrak{k}[t]}\otimes\mathfrak{k}[t,t^{-1}]\to F^{\mathfrak{k}[t^{-1}]}\otimes\mathfrak{k}[t,t^{-1}]$$

over $\operatorname{Spec}\mathfrak{k}[t,t^{-1}]$, that is the sequence

$$M_i = (M_i^{jl})$$

of invertible block-matrices with M_i^{jl} a $\beta_{ij}\times\beta_{il}$-matrix with entries in $\mathfrak{k}[t,t^{-1}][\varphi_1,\dots,\varphi_d]$ of degree $l-j$ in the φ_i's. In particular

$$(M_i^{jl})=0 \quad \text{for} \quad l<j$$

and

$$M_{il}^{ij}\in GL(\beta_{ij},\mathfrak{k}[t^{-1},t]).$$

Since bundles on $\mathbb{P}^1$ split, we may assume that M_i^{jj} are diagonal matrices. Thus the sheafications $F_*^{\mathfrak{k}[t]}$ and $F_*^{\mathfrak{k}[t^{-1}]}$ glue together to give a resolution of $\mathcal{O}_V$ of the desired type.

For the uniqueness we notice that given two complexes F_* and $\tilde{F}_*$ as in (3.2), we always have comparison maps locally over $\mathbb{P}^1$, i.e. for any affine $U\subset\mathbb{P}^1$ there exist comparison maps $F_*^U\to\tilde{F}_*^U$ of graded $\mathcal{O}_{\mathbb{P}^1}(U)[\varphi_1,\dots,\varphi_d]$-modules. These are unique up to a homotopy. By our additional assumption every homotopy is zero for degree reasons. Hence these comparison maps glue to give a uniquely determined isomorphism $F_*\to\tilde{F}_*$. □

(3.5) Suppose $V\subset\mathbb{P}(\mathscr{E})$ is a subvariety as in (3.2) and assume furthermore that all invariants $b_{ij}^{(k)}\geqq -1$. We consider now the image V' of V in a rational normal scroll X:

$$j:\mathbb{P}(\mathscr{E})\to X\subset\mathbb{P}^r.$$

Starting from the resolution F_* of V in $\mathbb{P}(\mathscr{E})$ we can construct a resolution of $\mathcal{O}_{V'}$ as an $\mathcal{O}_{\mathbb{P}^r}$-module using the complexes $\mathscr{C}^b(a)$. First notice that the induced complex

$$0\to j_*F_C\to\dots\to j_*F_1\to\mathcal{O}_X\to j_*\mathcal{O}_V\to 0$$

is still exact, in particular $j_*\mathcal{O}_V=\mathcal{O}_{V'}$. This is clear, if $j:\mathbb{P}(\mathscr{E})\to X\subset\mathbb{P}^r$ is an embedding. Otherwise it follows from the fact that $j:\mathbb{P}(\mathscr{E})\to X$ is a rational resolution of singularities, i.e.

$$R^ij_*\mathcal{O}_{\mathbb{P}(\mathscr{E})}=0 \quad \text{for} \quad i>0,$$

which implies the vanishing

$$R^ij_*\mathcal{O}_{\mathbb{P}(\mathscr{E})}(aH+bR)=0 \quad \text{for} \quad i>0$$

for all $a\in\mathbb{Z}$ and $b\geqq -1$.

Since each of the terms j_*F_l is an extension of sheaves $\mathcal{O}_X(aH+bR)$, $b\geqq -1$, a suitable mapping cone, between the Eagon-Northcott type complexes $\mathscr{C}^b(a)$'s (1.6) gives a resolution

$$\mathscr{A}_*^{(l)}:\dots\to\mathscr{A}_1^{(l)}\to\mathscr{A}_0^{(l)}\to j_*F_l\to 0.$$

Then finally an iterated mapping cone

$$[[\dots[\mathscr{A}^c_* \to \mathscr{A}^{c-1}_*] \to \dots] \to \mathscr{A}^{(1)}_*] \to \mathscr{C}^0 \to \mathcal{O}_{V'} \to 0$$

is a resolution of $\mathcal{O}_{V'}$ as an $\mathcal{O}_{\mathbb{P}^r}$-module.

Caution. In general the resulting complex is *not* minimal.

(3.6) *Example.* Let $C \subset X \subset \mathbb{P}^r$ be a "complete intersection" of divisors

$$Y_i \sim a_i H - b_i R, \qquad i = 1, \dots, d-1$$

on a d-dimensional rational normal scroll X of degree f with $b_i \geqq 0$. The resolution of $\mathcal{O}_C$ as an $\mathcal{O}_X$-module is a Koszul-complex:

$$\begin{aligned}&0 \to \mathcal{O}_X(-(a_1 + \dots + a_{d-1})H + (b_1 + \dots + b_{d-1})R) \to \dots \\ &\dots \to \sum_{i_1 < i_2} \mathcal{O}_X(-(a_{i_1} + a_{i_2})H + (b_{i_1} + b_{i_2})R) \to \sum_i \mathcal{O}_X(-a_i H + b_i R) \to \mathcal{O}_X .\end{aligned}$$

If all $a_i \geqq 2$, i.e. $C \subset \mathbb{P}^r$ is non-degenerate, then, because the complexes $\mathscr{C}^b$ are 1-regular, the resulting mapping cone is the *minimal* resolution of $\mathcal{O}_C$. C has

$$\deg C = a_i \cdot \ldots \cdot a_{d-1} f - \sum_i a_i \cdot \ldots \cdot a_{i-1} b_i \cdot a_{i+1} \cdot \ldots \cdot a_{d-1}$$

and arithmetic genus $p_a C$

$$2p_a C - 2 = \deg C \cdot (a_1 + \dots + a_{d-1} - d) + (a_1 \cdot \ldots \cdot a_{d-1})(b_1 + \dots + b_{d-2} - f + 2).$$

C is arithmetically Cohen-Macaulay iff

$$b_1 + \dots + b_{d-1} \leqq f - 1 .$$

4. Canonical Curves

In this section we construct for a canonical curve $C \subset \mathbb{P}^{g-1}$ with a basepoint free complete pencil of divisors of degree d an approximation of the minimal resolution with methods of Sects. 2, 3.

(4.1) Let

$$C \subset \mathbb{P}^{g-1}$$

be a canonical curve,

$$g^1_d = \{D_\lambda\}_{\lambda \in \mathbb{P}^1}$$

a basepoint free complete pencil of divisors of degree d, $d \leqq g-1$, on C. By the *geometric version* of the theorem of *Riemann-Roch*:

$$\dim \bar{D} = \deg D - 1 - \dim |D|$$

for D an effective divisor, $\bar{D}$ its linear span in $\mathbb{P}^{g-1}$ and $|D|$ the complete linear series (cf. [G-H, p. 248]), we have

$$\dim \overline{D_\lambda} = d - 2 .$$

So by Sect. 2

$$X = \bigcup_{\lambda \in \mathbb{P}^1} \overline{D_\lambda} \subset \mathbb{P}^{g-1}$$

is a $(d-1)$-dimensional rational normal scroll of degree $f=g-d+1$, whose type depends on and determines the dimensions $h^0(C,\mathcal{O}_C(iD))$ for $i\geqq 0$.

Let $\mathbb{P}(\mathscr{E})\to\mathbb{P}^1$ denote the corresponding $\mathbb{P}^{d-1}$-bundle. Since the g_d^1 has no base points, C does not intersect the (possibly empty) singular set of X and we may regard C as a subvariety of $\mathbb{P}(\mathscr{E})$. We will construct an $\mathcal{O}_{\mathbb{P}(\mathscr{E})}$-module resolution of $\mathcal{O}_C$.

(4.2) Consider

$$D_\lambda\subset\overline{D_\lambda}\cong\mathbb{P}^{d-1}$$

as a zero-dimensional subscheme.

Lemma. *Let $D\subset\mathbb{P}^{d-1}$ be a zero-dimensional non-degenerate subscheme of degree $d-2$. The following assertions are equivalent:*

1) *The homogenous coordinate ring S_D is Gorenstein.*
2) *$\mathcal{O}_D$ has an $\mathcal{O}_{\mathbb{P}^{d-2}}$-module resolution of type:*

$$0\to\mathcal{O}(-d)\to\mathcal{O}(-d+2)^{\beta_{d-3}}\to\ldots\to\mathcal{O}(-3)^{\beta_2}\to\mathcal{O}(-2)^{\beta_1}\to\check{\mathcal{O}}\to\mathcal{O}_D\to 0$$

with

$$\beta_i=\frac{i(d-2-i)}{d-1}\binom{d}{i+1}.$$

3) *No subscheme $E\subset D$ of degree $d-1$ is contained in a hyperplane of $\mathbb{P}^{d-2}$.*

Proof. The equivalence of 1) and 2) is well known see e.g. [Be]. The equivalence of 1) and 3) is proved in [G-O] for a collection of d distinct points. To prove the equivalence of 1) and 3) in general, we choose coordinates $x_0,\ldots,x_{d-2}$ of $\mathbb{P}^{d-2}$ such that x_0 is a non-zero divisor of the homogenous coordinate ring S_D, i.e. the hyperplane $\{x_0=0\}$ does not intersect D. $R=S_D/x_0S_D$ is a graded artinian ring with Hilbert function $(1,d-1,1,0,0,\ldots)$. Recall that S_D is Gorenstein iff the multiplication

$$R_1\times R_1\to R_2\cong k$$

gives a non-degenerate pairing [Be].

If a subscheme $E\subset D$ of degree $d-1$ is contained in a hyperplane $\{y_0=0\}$, then $y_0\notin(x_0)$ but $y_0\cdot R_1=0$. So the pairing is degenerate.

Conversely suppose the pairing is degenerate, say $y_0R_1=0$ for a linear form $y_0\notin(x_0)$. Then there exist linear forms $y_1,\ldots,y_{d-2}$ such that the quadrics

$$x_iy_0-x_0y_i$$

are contained in the homogenous ideal I_D of D. Since x_0 in a non-zero divisor, the relations

$$x_0(x_iy_j-x_jy_i)=x_i(x_0y_j-x_jy_0)-x_j(x_0y_i-x_iy_0)$$

imply that all 2×2 minors of the matrix

$$\begin{pmatrix} x_0x_1\ldots x_{d-2}\\ y_0y_1\ldots y_{d-2}\end{pmatrix}$$

are contained in I_D. Since $x_0,\ldots,x_{d-2}$ is a basis of $H^0\mathcal{O}_{\mathbb{P}^{d-2}}(1)$ the entries of the second row become linearly dependent, if we add a suitable multiple of the first row to the second. Thus without loss of generality we may assume that $y_{k+1}, \ldots, y_{d-2}$ are identically zero while $y_0, \ldots, y_k$ are linearly independent for some k with $0 \leqq k < d-2$. Consequently D is contained in the union of two linear spaces

$$L_1 = \{y_0 = \ldots = y_k = 0\} \quad \text{and} \quad L_2 = \{x_{k+1} = \ldots = x_{d-2} = 0\}$$

of dimension $d-3-k$ and k respectively. Since

$$\deg(L_1 \cap D) + \deg(L_2 \cap D) \geqq \deg D = d$$

we conclude

$$\deg(L_i \cap D) \geqq \dim L_i + 2$$

for $i=1$ or $i=2$. Thus a suitable subscheme E with $L_i \cap D \subset E \subset D$ of degree $d-1$ is contained in a linear space of dimension $d-3$, i.e. a hyperplane. □

(4.3) **Proposition.** *$C \subset \mathbb{P}(\mathscr{E})$ has constant betti-numbers over $\mathbb{P}^1$.*

Proof. For each $\lambda \in \mathbb{P}^1$ it suffices to prove that the divisor $D_\lambda \subset \overline{D_\lambda}$ regarded as a zero-dimensional subscheme satisfies the equivalent conditions of Lemma 4.2. We prove (3): If a subscheme $E \subset D_\lambda$ is contained in a hyperplane of $\overline{D_\lambda}$ then the corresponding divisor $E \subset C$ moves in a pencil by the geometric version of Riemann-Roch (cf. [G-H, p. 248]).

Since the g_d^1 is complete this forces the remaining point $D_\lambda - E$ to be a fixed point of the g_d^1, a contradiction to our assumption. □

(4.4) **Corollary.** i) *$C \subset \mathbb{P}(\mathscr{E})$ has a resolution F_* of type*

$$0 \to \mathcal{O}_{\mathbb{P}(\mathscr{E})}(-dH + (f-2)R) \to \sum_{k=1}^{\beta_{d-2}} \mathcal{O}_{\mathbb{P}(\mathscr{E})}(-(d+2)H + b_{d-2}^{(k)}R) \to$$
$$\to \sum_{k=1}^{\beta_1} \mathcal{O}_{\mathbb{P}(\mathscr{E})}(-2H + b_1^{(k)}R) \to \mathcal{O}_{\mathbb{P}(\mathscr{E})} \to \mathcal{O}_C \to 0.$$

ii) *F_* is self dual:*

$$\mathscr{H}om(F_*, \mathcal{O}_{\mathbb{P}(\mathscr{E})}(-dH + (f-2)R) \cong F_*.$$

iii) *If all $b_i^{(k)} \geqq -1$ then an iterated mapping cone*

$$\left[\ldots \left[\mathscr{C}^{(f-2)}(-d) \to \sum_{k=1}^{\beta_{d-3}} \mathscr{C}^{b_{d-3}^{(k)}}(-d+2) \right] \to \ldots \right] \to \mathscr{C}^0$$

is a (not necessarily minimal) resolution of $\mathcal{O}_C$ as an $\mathcal{O}_{\mathbb{P}^{g-1}}$-module.

Proof. i) follows from (3.2), iii) from (3.5). The duality follows from adjunction:

$$\mathscr{E}xt^i_{\mathbb{P}(\mathscr{E})}(\mathcal{O}_C, \omega_{\mathbb{P}(\mathscr{E})}) = \begin{cases} \omega_C & i = d-2 \\ 0 & \text{otherwise} \end{cases}$$

(cf. [Gr, Corollary 2]), $\omega_C \cong \mathcal{O}_C(H)$ and

$$\omega_{\mathbb{P}(\mathscr{E})} \cong \mathcal{O}_{\mathbb{P}(\mathscr{E})}(-(d-1)H + (f-2)R)$$

and the uniqueness of F_* (3.2). □

Remark. In Sect. 6 we will discuss this result for small d in further details.

5. On the Equation of Rational Surfaces

Summary

For H a sufficiently positive divisor on a rational surface S the image S' of $j: S \to \mathbb{P}H^0(S, \mathcal{O}_S(H))$ can be described as a subvariety of a scroll X: It will be a "*rational normal curve bundle*." We give an outline of this approach to the equation of S' in the first part of this section (5.1)–(5.5).

What will be important to us is partial converse of this approach: Given a certain determinantal subvariety of a scroll, we may conclude that it is a rational surface and how it can be obtained from minimal model via a rational map with assigned basepoints. In particular we can calculate the "number" of exceptional curves.

(5.1) We consider a rational ruled surface

$$\pi: P_k = \mathbb{P}(\mathcal{O}(k) \oplus \mathcal{O}) \to \mathbb{P}^1, \qquad k \geqq 0,$$

and a surface S which is obtained from P_k via a sequence of blow ups:

$$\sigma: S \to P_k.$$

Every rational smooth surface S except $\mathbb{P}^2$ can be obtained in this way (cf. [Beau]).

With A and B we denote the hyperplane class and the ruling of P_k, and by abuse of notation also their pullbacks to S. $E = \cup E_i$ denotes the exceptional divisor of σ with its components. Let

$$H \sim dA + dB - \sum e_i E_i$$

be a divisor such that the complete linear series H has no base points. We consider the image S' of

$$j: S \to \mathbb{P}H^0(S, \mathcal{O}_S(H)) = \mathbb{P}^r.$$

Suppose that

i) $h^0(\mathcal{O}_S(H - B)) \geqq 2$,

ii) $H^1(\mathcal{O}_S(kH - B)) = 0$, $k \geqq 1$, and

iii) the map $S_k H^0 \mathcal{O}_S(H) \to H^0 \mathcal{O}_S(kH)$ is surjective.

By Sect. 2, i), ii) the variety

$$X = \bigcup_{B_\lambda \in |B|} \overline{B_\lambda} \subset \mathbb{P}^r$$

is a $(d+1)$-dimensional rational normal scroll. Let $\pi: \mathbb{P}(\mathscr{E}) \to \mathbb{P}^1$ denote the corresponding $\mathbb{P}^d$-bundle and S'' the strict transform of S' in $\mathbb{P}(\mathscr{E})$. Blowing-up S further we may assume that $S \to S'$ factors through S'':

$$\begin{tikzcd}
 & S \arrow[r] \arrow[dl] \arrow[dr] \arrow[ddr, "\pi_S"'] & S' \arrow[r, hook] & X \arrow[r, hook] & \mathbb{P}^r \\
P_k \arrow[drr, "\pi_{P_k}"'] & & S'' \arrow[u] \arrow[r] \arrow[d] & \mathbb{P}(\mathscr{E}) \arrow[u] \arrow[dl, "\pi_{\mathbb{P}(\mathscr{E})}"] & \\
 & & \mathbb{P}^1 & &
\end{tikzcd}$$

We want to describe the syzygies of $\mathcal{O}_{S''}$ as an $\mathcal{O}_{\mathbb{P}(\mathscr{E})}$-modules with the methods of Sect. 3.

(5.2) **Lemma.** *Let B be an e-dimensional non-degenerate subvariety of degree d in $\mathbb{P}^{d+e-1}$. The following assertions are equivalent:*

1) *The homogenous coordinate ring S_B is Cohen-Macaulay.*
2) *$\mathcal{O}_B$ has an $\mathcal{O}=\mathcal{O}_{\mathbb{P}^{d+e-1}}$-module resolution of type*

$$0\to\mathcal{O}(-d)^{\beta_{d-1}}\to\dots\to\mathcal{O}(-2)^{\beta_1}\to\mathcal{O}\to\mathcal{O}_B\to 0$$

with

$$\beta_i=i\binom{d}{i+1}.$$

For a proof see e.g. [E-R-S].

(5.3) A general fibre $B_\lambda\subset\overline{B_\lambda}=\mathbb{P}^d$ is a rational normal curve, so 1) of Lemma 5.2 is satisfied. To prove this condition for all fibres we consider the commutative diagram

$$\begin{array}{ccc} H^0\mathcal{O}_{\mathbb{P}(\mathscr{E})}(kH) & \longrightarrow & H^0\mathcal{O}_{\overline{B_\lambda}}(kH) \\ \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} \\ H^0\mathcal{O}_S(kH) & \xrightarrow[\alpha]{} & H^0\mathcal{O}_{B_\lambda}(kH) \end{array}$$

α is surjective by ii), β as a consequence of iii). Thus γ is surjective, which proves that $B_\lambda\subset\overline{B_\lambda}$ is arithmetically Cohen-Macaulay.

Consequently:

$$S''\subset\mathbb{P}(\mathscr{E})\quad\textit{has constant betti-numbers over } \mathbb{P}^1.$$

(5.4) **Corollary.** *$\mathcal{O}_{S''}$ has an $\mathcal{O}_{\mathbb{P}(\mathscr{E})}$-module resolution of type F_**

$$0\to\sum_{j=1}^{\beta_{d-1}}\mathcal{O}_{\mathbb{P}(\mathscr{E})}(-dH+b_{d-1}^{(j)}R)\to\dots\to\sum_{j=1}^{\beta_1}\mathcal{O}_{\mathbb{P}(\mathscr{E})}(-2H+b_1^jR)$$
$$\to\mathcal{O}_{\mathbb{P}(\mathscr{E})}\to\mathcal{O}_{S''}\to 0$$

with

$$\beta_i=i\cdot\binom{d}{i+1}.$$

(5.5) In some cases the resolution of S'' on $\mathbb{P}(\mathscr{E})$ is actually given by an Eagon-Northcott complex: Let

$$\mathscr{F}'=\pi_{S*}\mathcal{O}_S(H-A),\qquad \mathscr{G}'=\pi_{S*}\mathcal{O}_S(A)$$

$\mathscr{F}'$ is locally free of rank d, say

$$\mathscr{F}'=\mathcal{O}(a_1)\oplus\dots\oplus\mathcal{O}(a_d)$$

and

$$\mathscr{G}'\cong\mathcal{O}(k)\oplus\mathcal{O}.$$

Consider

$$\mathscr{F}=\pi^*_{\mathbb{P}(\mathscr{E})}\mathscr{F}',\qquad \mathscr{G}=\pi^*_{\mathbb{P}(\mathscr{E})}\mathscr{G}'$$

and the map

$$\psi : \mathscr{F}(-H) \to \mathscr{G}^*$$

obtained from the composition

$$\mathscr{F} \otimes \mathscr{G} \to \pi^*_{\mathbb{P}(\mathscr{E})}(\pi_{S*})\mathcal{O}_S(H) \cong \pi^*_{\mathbb{P}(\mathscr{E})}\mathscr{E} \to \mathcal{O}_{\mathbb{P}(\mathscr{E})}(H).$$

ψ is given by a matrix with entries as indicated below

$$(\psi) \sim \begin{pmatrix} H-a_1R, & \ldots, H-a_dR \\ H-(a_1+k)R, & \ldots, H-(a_d+k)R \end{pmatrix}.$$

The Eagon-Northcott complex associated to ψ resolves $\mathcal{O}_{S''}$ iff this is true for every fibre. A necessary condition is that every fibre of $S'' \subset \mathbb{P}(\mathscr{E})$ over $\mathbb{P}^1$ is determinantal. We refer to [X] for a description of determinantal curves of minimal degree.

(5.6) We now give a partial converse of the result above: Let $X \subset \mathbb{P}^r$ denote a rational normal scroll of dimension $d+1$ and $\mathbb{P}(\mathscr{E})$ the corresponding $\mathbb{P}^d$-bundle over $\mathbb{P}^1$. Let $S'' \subset \mathbb{P}(\mathscr{E})$ be an irreducible surface defined by the 2×2 minors of a matrix ψ with entries section in line bundles as indicated below:

$$(\psi) \sim \begin{pmatrix} H-a_1R, & \ldots, H-a_dR \\ H-(a_1+k)R, & \ldots, H-(a_d+k)R \end{pmatrix}.$$

We assume that the general fibre of $S'' \subset \mathbb{P}(\mathscr{E})$ over $\mathbb{P}^1$ is a rational normal curve of degree d. We want to describe the image S' of S'' in $\mathbb{P}^r$.

Set

$$a = a_1 + \ldots + a_d, \qquad f = \deg X$$

and let P_k denote the rational ruled surface $\mathbb{P}(\mathcal{O}(k) \oplus \mathcal{O})$ with hyperplane class A and ruling B as above. Then we have:

(5.7) **Theorem.** *$S' \subset \mathbb{P}^r$ is the image of P_k under a rational map defined by a subseries of*

$$H^0(P_k, \mathcal{O}_{P_k}(dA + (f - dk - a)B)$$

which has

$$\delta = d \cdot f - \frac{d(d+1)}{2} \cdot k - (d+1)a$$

assigned base points. Furthermore, if $S' \subset X \subset \mathbb{P}^r$ contains a canonical curve C of genus $r+1$, then the ruling of X cuts on C a g^1_{d+2} and the strict transform C' of C in P_k is a divisor of class

$$C' \sim (d+2)A + (f - (d+1)k - a + 2)B$$

and arithmetic genus

$$P_a C' = r + 1 + \delta.$$

Proof. Let

$$\varphi_i \in H^0(\mathcal{O}_{\mathbb{P}(\mathscr{E})}(H - e_i R) \quad \text{for} \quad i = 1, \ldots, d+1$$

denote a collection of basic sections of $\mathbb{P}(\mathscr{E})$ [cf. (1.3)]. The entries of the matrix ψ are linear forms

$$\psi_{1j} = \sum_{i=1}^{d+1} p_{ij}(s,t)\varphi_i, \qquad \psi_{2j} = \sum_{i=1}^{d+1} q_{ij}(s,t)\varphi_i$$

whose coefficients $p_{ij}, q_{ij} \in \mathfrak{k}[s,t]$ are homogenous polynomials of degree

$$\deg p_{ij} = e_i - a_j, \qquad \deg q_{ij} = e_i - a_j - k.$$

A fibre $S''_{(s,t)}$ of S'' is given by the union

$$S''_{(s,t)} = \bigcup_{(\lambda:\mu)\in\mathbb{P}^1} \{\lambda\psi_{1i}(s,t) + \mu\psi_{2i}(s,t) = 0;\ i = 1, \dots, d+1\}.$$

Consider the $(d+1)\times d$ matrix of coefficients:

$$M = (\lambda p_{ij} + \mu q_{ij})_{\substack{i=1,\dots,d+1\\ j=1,\dots,d}}.$$

The fibre $S''_{(s,t)}$ is singular iff rank $M(s,t,\lambda,\mu) < d$ for a suitable choice of $(\lambda:\mu)\in\mathbb{P}^1$.

The main point of the proof is simply to identify

$$\lambda \in H^0\mathcal{O}_{P_k}(A - kB), \qquad \mu \in H^0\mathcal{O}_{P_k}(A)$$

with a collection of basic section of P_k. We then may interpret M as a map of bundles on P_k and can consider the associated Eagon-Northcott complex:

$$0 \longrightarrow \sum_{j=1}^{d} \mathcal{O}_{P_k}(-A + (a_j + k)B) \xrightarrow{{}^tM} \sum_{i=1}^{d+1} \mathcal{O}_{P_k}(e_iB)$$
$$\xrightarrow{\overset{d}{\wedge} M} \mathcal{O}_{P_k}(dA + (f - dk - a)B). \qquad (*)$$

Claim. S' is the image of P_k under the rational map defined by the subseries V given by

$$H^0\mathcal{O}_{P_k}(e_iB) \twoheadrightarrow V \subset H^0\mathcal{O}_{P_k}(dA + (f - dk - a)B). \qquad (**)$$

We first treat the case that $(*)$ is exact, i.e. the subseries defined by V has no fixed components cf. (1.5). In this case $(*)$ resolves the structure sheaf of a zero-dimensional subscheme Δ, which is the base locus of V. Its degree is

$$\delta = h^0\mathcal{O}_\Delta = h^0(\mathcal{O}_{P_k}(dA + (f - dk - a)B) - \sum_{i=1}^{d+1}(e_i + 1)$$
$$= df - \frac{d(d+1)}{2}k - da.$$

Clearly the image of V in $\mathbb{P}^r$ is contained in a scroll of type $S(e_1, \dots, e_{d+1})$. To exhibit the defining equation of the strict transform in $\mathbb{P}(\mathscr{E})$ we proceed in the same manner as above (5.5):

Let $\sigma: S \to P_k$ be the blow-up of Δ. We have to prove that the multiplication map

$$\pi_{S*}\mathcal{O}_S(H - A) \otimes \pi_{S*}\mathcal{O}_S(A) \to \pi_{S*}\mathcal{O}_S(H)$$

induces the matrix ψ.

$$\pi_{S*}\mathcal{O}_S(A) \cong \pi_{P_k*}\mathcal{O}_{P_k}(A) \cong \mathcal{O}_{\mathbb{P}^1}(k) \oplus \mathcal{O}_{\mathbb{P}^1}.$$

For $\pi_{S*}\mathcal{O}_S(H)$ we notice

$$\sigma_*\mathcal{O}_S(H) \cong \mathscr{J} = \operatorname{Im}\left(\overset{d}{\wedge} M\right) \subset \mathcal{O}_{P_k}(dA + (f - dk - a)B)$$

thus
$$\pi_{S*}\mathcal{O}_S(H)\cong\pi_{P_k*}\mathcal{J}\,.$$
For the first row of ψ consider the diagram

$$\begin{tikzcd}
0 \arrow[r] & \sum_{j=1}^{d}\mathcal{O}_{P_k}(-A+a_jB) \arrow[r, "{}^tM"] & \sum_{i=1}^{d+1}\mathcal{O}_{P_k}(e_iB) \arrow[r] & \mathcal{J} \arrow[r] & 0 \\
0 \arrow[r] & \sum_{j=1}^{d}\mathcal{O}_{P_k}(-2A+a_jB) \arrow[r, "{}^tM"] \arrow[u, "\cdot\mu"] & \sum_{i=1}^{d+1}\mathcal{O}_{P_k}(-A+e_iB) \arrow[r] \arrow[u, "\cdot\mu"] & \mathcal{J}(-A) \arrow[r] \arrow[u, "\cdot\mu"] & 0
\end{tikzcd}$$

and apply $(\pi_{P_k})_*$. We obtain a map

$$\mathcal{O}_{\mathbb{P}^1}(a_1)\oplus\ldots\oplus\mathcal{O}_{\mathbb{P}^1}(a_d)\cong(\pi_{P_k})_*\mathcal{J}(-A)\to(\pi_{P_k})_*\mathcal{J}\cong\mathcal{O}_{\mathbb{P}^1}(e_1)\oplus\ldots\oplus\mathcal{O}_{\mathbb{P}^1}(e_{d+1})$$

which we will identify with the map given by the matrix ${}^t(p_{ij})$.

Since both maps are well defined it suffices to prove this generically. Over the generic point of $\mathbb{P}^1$ this follows easily, chasing e.g. the corresponding diagram of Čech-cocycles. Similarly we can identify the second row.

Consequently the minors defining S'' vanish on the image of P_k in $\mathbb{P}(\mathscr{E})$. Since S'' is irreducible and has the same dimension as the image of P_k they coincide. This proves the first part of the theorem in case V has no fixed component.

If V has fixed components, then the complex $(*)$ is no longer exact, it has homology in the middle. But if we replace $\mathcal{J}$ by coker $({}^tM)$ then the calculation above goes through, and all what remains is to prove that the map

$$\sum_{i=1}^{d+1}H^0\mathcal{O}_{P_k}(e_iB)\to H^0\mathcal{O}_{P_k}(dA+(f-dk-a)B)$$

is still injective, hence has codimension δ.

Consider the commutative diagram

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \sum_j\mathcal{O}_{P_k}(-A+(a_j+k)B) \arrow[r] \arrow[d, equal] & \operatorname{Ker}\bigwedge^d M \arrow[r] \arrow[d] & H_1 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \sum_j\mathcal{O}_{P_k}(-A+(a_j+k)B) \arrow[r] & \sum_i\mathcal{O}_{P_k}(e_iB) \arrow[r] \arrow[d] & \operatorname{Coker}({}^tM) \arrow[r] \arrow[d] & 0 \\
 & & \mathcal{J} \arrow[r, equal] \arrow[d] & \mathcal{J} \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd}$$

of exact sequences.

H_1 has support at the fixed components of V. These are finitely many fibres of $P_k \to \mathbb{P}^1$ since the general fibre of S'' is a rational normal curve. Thus $(\pi_{P_k})_* H_1$ is concentrated in at most finitely many points of $\mathbb{P}^1$.

Since the sheaf

$$\pi_{P_k *} \operatorname{coker}({}^t M) \cong (\pi_{P_k})_* \sum_i \mathcal{O}_{P_k}(e_i B) \cong \sum_i \mathcal{O}(e_i)$$

has no torsion, we conclude $(\pi_{P_k})_* H_1 = 0$ and hence

$$H^0\left(\ker\left(\bigwedge^d M\right)\right) \cong H^0(P_k, H_1) \cong H^0(\mathbb{P}^1, \pi_{P_k *} H_1) = 0 .$$

This finishes the proof of the first part of the theorem.

For the second part of the theorem we notice that if $C \subset S' \subset X \subset \mathbb{P}^r$ is a canonical curve, then the rational map

$$P_k \dashrightarrow S' \subset \mathbb{P}^r$$

is defined by the adjoint series

$$V \subset H^0(P_k, \omega_{P_k}(C'))$$

which has assigned base points at the singularities of the strict transform C' of C. Hence

$$C' \sim H - K_{P_k} \sim (d+2)A + (f-(d+1)k - a + 2)B$$

and

$$P_a C' = \tfrac{1}{2} C'(C' + K_{P_k}) + 1 = r + 1 + \delta . \quad \square$$

6. d-Gonal Curves for $d \leqq 5$

In this section we discuss the result of Sect. 4 for $d \leqq 5$ in more details.

(6.1) *Trigonal Curves* (cf. [Ma, P]). A trigonal canonical curve C is contained in a two-dimensional rational normal scroll

$$X = \bigcup_{D \in g_3^1} \bar{D} \subset \mathbb{P}^{g-1}$$

of type $S(e_1, e_2)$ and degree $f = e_1 + e_2 = g-2$ (cf. Sect. 2). From $H^0(C, \omega_C(-nD)) = 0$ for $n > \frac{2g-2}{3}$ we obtain the bounds

$$\frac{2g-2}{3} \geqq e_1 \geqq e_2 \geqq \frac{g-4}{3}$$

by (2.5). C is a divisor of class

$$C \sim 3H - (f-2)R$$

on X (cf. Sect. 4). The mapping cone

$$\mathscr{C}^{f-2}(-3) \to \mathscr{C}^0$$

is a *minimal* resolution of $\mathcal{O}_C$ as an $\mathcal{O}_{\mathbb{P}^{g-1}}$-module. The contribution of $\mathscr{C}^0$ is distinguished by its degrees from the part of $\mathscr{C}^{f-2}(-3)$. This proves the well-known fact that X and hence g_3^1 is uniquely determined by C for a trigonal curve of genus $g \geqq 5$.

(6.2) *Tetragonal Curves* (cf. [P]). A tetragonal canonical curve C of genus $g \geqq 5$ is contained in a 3-dimensional rational normal scroll

$$X = \bigcup_{D \in g_4^1} \bar{D} \subset \mathbb{P}^{g-1}$$

of type $S(e_1, e_2, e_3)$ with

$$\frac{2g-2}{4} \geqq e_1 \geqq e_2 \geqq e_3 \geqq 0$$

and degree $f = e_1 + e_2 + e_3 = g-3$ (cf. Sect. 2).

By Sect. 4 C is a complete intersection of two divisors

$$Y \sim 2H - b_1 R, \qquad Z \sim 2H - b_2 R$$

on X with

$$b_1 + b_2 = f - 2$$

(4.4), say $b_1 \geqq b_2$. We will verify

$$f - 1 \geqq b_1 \geqq b_2 \geqq -1$$

in (6.3) below. Hence we can obtain a *minimal* resolution of $\mathcal{O}_C$ as $\mathcal{O}_{\mathbb{P}^{g-1}}$-module via an iterated mapping cone:

$$[\mathscr{C}^{f-2}(-4) \to \mathscr{C}^{b_1}(-2) \oplus \mathscr{C}^{b_2}(-2)] \to \mathscr{C}^0 .$$

In particular we note that the invariants b_1, b_2 are determined by the graded-betti-numbers of C. X and hence g_4^1 is uniquely determined by C unless $b_1 \geqq f-2$. It is the support of the cokernel of the dual map to the term

$$\mathcal{O}^{\beta_{g-4,g-3}}(-g+3) \to \mathcal{O}^{\beta_{g-5,g-4}}(-g+4)$$

in the resolution of C. For a similar reason the surface Y is uniquely determined by C if $b_1 > b_2$ (e.g. if g is even).

(6.3) We study the geometry of Y next. In terms of a collection of basic sections (1.3)

$$\varphi_i \in H^0(\mathbb{P}(\mathscr{E}, \mathcal{O}_{\mathbb{P}(\mathscr{E})}(H - e_i R)), \qquad i = 1, 2, 3$$

for the corresponding $\mathbb{P}^2$-bundle $\mathbb{P}(\mathscr{E})$, the surface Y is defined by a quadratic form

$$\psi = \sum_{i,j} P_{ij}(s,t) \varphi_i \varphi_j$$

with coefficients homogenous polynomials $P_{ij} \in \mathfrak{k}[s,t]$ of degree

$$\deg P_{ij} = e_i + e_j - b_1$$

with C also Y is irreducible. This gives a bound on b_1: Certainly we must have

$$2e_2 - b_1 \geqq 0.$$

In particular $f \geqq e_1 + e_2 \geqq 2e_2 \geqq b_1$. This bound can be slightly improved: Suppose $b_1 = f$ then $e_1 = e_2$ and $e_3 = 0$, hence Y is defined by a quadratic form in φ_1, φ_2 only, which has constant coefficients. Since the ground field $\mathfrak{k}$ is algebraically closed Y would be reducible. So

$$f - 1 \geqq b_1 \geqq b_2 \geqq -1.$$

Actually $f - 1 = b_1$ occurs only for $g \leqq 0$ and in that case there exists a g_3^1 or a g_5^2 on C, which can easily be checked with the methods below.

(6.4) The fibres of $Y' \subset \mathbb{P}(\mathscr{E})$ over $\mathbb{P}^1$ are conics. If the general fibre is a non-singular conic, then the number of singular fibres is given by

$$\delta = 2f - 3b_1$$

i.e. the degree of the determinant of the associated symmetric matrix of coefficients of the quadratic form (char $\mathfrak{k} \neq 2$).

One can derive the same result with methods of Sect. 5. Suppose ψ is given by determinant of a matrix with entries as indicated below:

$$\begin{pmatrix} H - a_1 R & H - a_2 R \\ H - (a_1 + k)R & H - (a_2 + k)R \end{pmatrix}$$

then (5.7) gives us the same number of singular fibres. We may identify Y with image of $P_k = \mathbb{P}(\mathcal{O}(k) \oplus \mathcal{O})$ under a rational map defined by a linear series with δ basepoints.

Moreover the composition

$$C \subset Y \to P_k \to \mathbb{P}(H^0(P_k, \mathcal{O}_{P_k}(A))) \cong \mathbb{P}^{k+1}$$

defines a linear series of degree

$$b_2 + 2 + 2(k+1) \quad (= A \cdot C', \text{ cf. } (5.7))$$

hence of *Clifford-index* $b_2 + 2$.

However, a determinantal presentation of ψ is not unique. For example taking the transposed matrix corresponds to blow-down the "other" line in each singular fibre of Y. In general the different determinantal presentations correspond to the various different ways Y can be blown down to a minimal smooth surface, such that the image of C is a curve with admost double point singularities. We leave it to reader to establish these facts with the methods of Sect. 5 and in particular to prove the existence of a determinantal presentation for a suitable choice of a_1, a_2, k with $a_1 + a_2 + k = b_1$.

(6.5) If all fibres of Y are degenerate conics the picture is quite different. In that case the g_4^1 is composed by an elliptic or hyperelliptic involution

$$C \xrightarrow{2:1} E \xrightarrow{2:1} \mathbb{P}^1$$

and Y is a birational ruled surface over E with a rational curve $\tilde{E}$ of double points. Since C is smooth $\tilde{E} \cap Z = \emptyset$, in particular

$$0 = \tilde{E} \cdot (2H - b_2 R) = 2 \deg \tilde{E} - b_2$$

i.e. b_2 is *even*. A general divisor of class $H - \frac{b_2}{2} R$ intersects Y in a smooth curve isomorphic to E, so the geometric genus is given by

$$2p_a E - 2 = \left(H - \frac{b_2}{2} R\right)(2H - b_1 R)\left(f - 2 - b_1 - \frac{b_2}{2}\right) R$$

[cf. (3.6)] i.e.

$$p_a E = \frac{b_2}{2} + 1$$

(and we may identify $\tilde{E}$ with the canonical image of E). Similarly to the first case (6.4) we may use Y to discover a linear series with *Clifford-index* $b_2 + 2$ on C: The composition of $C \xrightarrow{2:1} E$ with the map induced by a general divisor of degree $2p_a E$ on E induces a g^r_d with $r = p_a E$, $d = 4p_a E$, hence $d - 2r = 2p_a E = b_2 + 2$.

(6.6) Finally we note the following numerical characterization of decomposable g^1_4's:

Theoerm. *A g^1_4 on a curve of genus g is composed by an involution of genus g' with $6g' < g - 3$ iff*

$$\beta_{i,i+1} = i\binom{g+3}{i+1} \quad \textit{for} \quad i > g - 2 - 2g'$$

and

$$\beta_{i,i+1} > i\binom{g-3}{i+1} \quad \textit{for} \quad i = g - 2 - 2g'.$$

Proof. The condition on the graded betti-numbers gives $b_2 = 2g' - 2$. So our assumption $6g' < g - 3$ means that

$$\delta = 2f - 3b_1 < 0.$$

So by (6.4) a general fibre of Y over $\mathbb{P}^1$ cannot be non-singular. The factorization

$$\begin{tikzcd} C \arrow[rr, "4:1"] \arrow[dr, "2:1"', "\pi"] & & \mathbb{P}^1 \arrow[dl, "2:1"] \\ & E & \end{tikzcd}$$

over a curve of genus $g' = \frac{b_2}{2} + 1$ follows then from (6.5). Conversely given a factorization we obtain a conic bundle

$$Y = \overline{\bigcup_{P \in E} \pi^{-1}(P)} \subset X \subset \mathbb{P}^{g-1}$$

with a rational curve $\tilde{E}$ of singular points of degree $g'-1$. As above $\tilde{E}\cap Z=\emptyset$ implies

$$b_2=2g'-2, \quad b_1=g-5-b_2$$

and these invariants determine the graded betti-numbers of C. $\square$

(6.7) *Pentagonal Curves* (cf. [P]). From a complete base point free g_5^1 on a canonical curve of genus $g\geqq 7$, we obtain a 4-dimensional rational normal scroll

$$X=\bigcup_{D\in g_5^1}\bar{D}\subset\mathbb{P}^{g-1}$$

of type $S(e_1,e_2,e_3,e_4)$ with

$$\frac{2g-2}{5}\geqq e_1\geqq e_2\geqq e_3\geqq e_4\geqq 0$$

and degree $f=e_1+e_2+e_3+e_4=g-4$ (cf. Sect. 2).

The resolution of $\mathcal{O}_C$ on the corresponding $\mathbb{P}^3$-bundle $\mathbb{P}(\mathscr{E})$ is of type

$$0\longrightarrow\mathcal{O}_{\mathbb{P}(\mathscr{E})}(-5H+(f-2)R)\longrightarrow\sum_{i=1}^{5}\mathcal{O}_{\mathbb{P}(\mathscr{E})}(-3H+b_iR)$$

$$\xrightarrow{\psi}\sum_{i=1}^{5}\mathcal{O}_{\mathbb{P}(\mathscr{E})}(-2H+a_iR)\longrightarrow\mathcal{O}_{\mathbb{P}(\mathscr{E})}\longrightarrow\mathcal{O}_C\longrightarrow 0$$

with

$$a_i+b_i=f-2$$

since the complex is "selfdual," and

$$a_1+\ldots+a_5=2g-12$$

by (3.3). The corresponding iterated mapping cone (in case $-1\leqq a_i,\ b_i\leqq f-1$)

$$\left[\left[\mathscr{C}^{f-2}(-5)\to\sum_i\mathscr{C}^{b_i}(-3)\right]\to\sum_i\mathscr{C}^{a_i}(-2)\right]\to\mathscr{C}^0$$

is a *not necessarily minimal* resolution of C.

From the structure theorem for Gorenstein ideals in codimension 3 [B-E3] we obtain further information:

The matrix ψ is shew-symmetric and its 5 Pfaffians generate the ideal of C in $\mathbb{P}(\mathscr{E})$, i.e. form the entries of

$$\sum_{i=1}^{5}\mathcal{O}_{\mathbb{P}(\mathscr{E})}(-2H+a_iR)\to\mathcal{O}_{\mathbb{P}(\mathscr{E})}.$$

Thus C is determined by the entries of ψ. If one of the off-diagonal entries is zero, say $\psi_{12}=\psi_{21}=0$, then C is contained in the determinantal surface Y defined by the matrix

$$\begin{pmatrix}\psi_{13} & \psi_{14} & \psi_{15}\\ \psi_{23} & \psi_{24} & \psi_{25}\end{pmatrix}$$

since in this case the 2×2 minors of that matrix are among the Pfaffians of ψ. With C also Y is irreducible, so a general fibre of $Y\subset\mathbb{P}(\mathscr{E})$ over $\mathbb{P}^1$ is a twisted cubic. Consequently Y is one of the rational surfaces classified in (5.7). We will take this approach to the syzygies of canonical curves of genus $g=7,8$ in Sect. 7.

(6.8) Finally we note:

Corollary (Petri). *The Hurwitz-scheme*

$$H_{d,g}=\{C\to\mathbb{P}^1\colon d\text{-sheeted covers of curves } C \text{ of genus } g \text{ over } \mathbb{P}^1\}$$

is unirational for $d=3$, $g\geqq 4$, $d=4$, $g\geqq 5$ *and* $d=5$, $g\geqq 6$.

Proof. One can describe C by sections

$$\psi\in H^0\mathcal{O}_{\mathbb{P}(\mathscr{E})}(3H-(f-2)R),$$

$$\psi_i\in H^0\mathcal{O}_{\mathbb{P}(\mathscr{E})}(2H-b_iR),\qquad i=1,2$$

or

$$\psi_{ij}\in H^0\mathcal{O}_{\mathbb{P}(\mathscr{E})}(H-(b_j-a_i)R),\qquad 1\leqq i<j\leqq 5$$

respectively. The corresponding open subsets of these spaces, which describe smooth curves give for a suitable choice of $e_1,\ldots,e_{d-1}$ and the b_i's a rational variety which dominates $H_{d,g}$. □

7. Proof of the Main Result

(7.1) **Theorem.** *For a curve C of genus* $g=7$ *or* $g=8$ *the distribution of the graded betti-numbers depends on and determines the existence of a* g_2^1, g_3^1, g_6^2 *or* g_4^1 *on C.*

Proof. Every curve C of genus $g=7$ or $g=8$ admits a basepoint free complete pencil g_d^1 for some d with $2\leqq d\leqq\left[\frac{g}{2}+1\right]=5$ by [Ke, K-L]. For a curve with a g_2^1, g_3^1 or g_4^1 we have a description of the minimal resolution (cf. Sect. 6). In case of a g_4^1 there are two cases: Either the g_4^1 is uniquely determined by the resolution of C or C is contained in a surface Y of degree $g-3$. Either Y is a Del-Pezzo surface, i.e. $\mathbb{P}^2$ blown-up in $10-g$ points, and the composition

$$C\to Y\to\mathbb{P}^2$$

gives a g_6^2 on C or Y is a cone over an elliptic curve E and the composition

$$C\xrightarrow{2:1}E\hookrightarrow\mathbb{P}^2$$

with any embedding of E as a plane cubic gives a g_6^2 on C [cf. (6.4–5)].

It remains to handle the case that C admits a g_5^1.

(7.2) Given a g_5^1 on C let
$$X=\bigcup_{D\in g_5^1}\bar{D}\subset\mathbb{P}^{g-1}$$

be the associated 4-dimensional rational normal scroll [cf. (6.7)].

We first prove the theorem for curves of genus $g=7$, whose invariants

$$(a_1,\ldots,a_5)=(0,0,0,1,1),$$

i.e. the resolution of C on the corresponding $\mathbb{P}^3$-bundle $\mathbb{P}(\mathscr{E})$ is of type

$$0 \to \mathcal{O}_{\mathbb{P}(\mathscr{E})}(-5H+R) \to \begin{matrix} \mathcal{O}_{\mathbb{P}(\mathscr{E})}(-3H+R)^{\oplus 3} \\ \oplus \\ \mathcal{O}_{\mathbb{P}(\mathscr{E})}(-3H)^{\oplus 2} \end{matrix} \xrightarrow{\psi} \begin{matrix} \mathcal{O}_{\mathbb{P}(\mathscr{E})}(-2H)^{\oplus 3} \\ \oplus \\ \mathcal{O}_{\mathbb{P}(\mathscr{E})}(-2H+R)^{\oplus 2} \end{matrix} \to \mathcal{O}_{\mathbb{P}(\mathscr{E})} \to \mathcal{O}_C \to 0$$

with ψ a skew-symmetric matrix with appropriated entries. We shall see that $(a_1, \ldots, a_5)$ take these values for a "general" curve.

The only non-minimal map in the mapping cone

$$\left[\left[\mathscr{C}^1(-5) \to \begin{matrix} \mathscr{C}^1(-3)^{\oplus 3} \\ \oplus \\ \mathscr{C}^0(-3)^{\oplus 2} \end{matrix}\right] \to \begin{matrix} \mathscr{C}^0(-2)^{\oplus 3} \\ \oplus \\ \mathscr{C}^1(-2)^{\oplus 2} \end{matrix}\right] \to \mathscr{C}^0$$

arises from the 3×3 block

$$\mathcal{O}_{\mathbb{P}(\mathscr{E})}(-3H+R)^{\oplus 3} \xrightarrow{\begin{pmatrix} 0 & \psi_{12} & \psi_{13} \\ -\psi_{12} & 0 & \psi_{23} \\ -\psi_{13} & -\psi_{23} & 0 \end{pmatrix}} \mathcal{O}_{\mathbb{P}(\mathscr{E})}(-2H)^{\oplus 3}$$

of ψ.

$$\begin{tikzcd} G(-3)^{\oplus 3} \arrow[r] & \mathcal{O}(-2)^{\oplus 3} \\ F(-4)^{\oplus 3} \arrow[r, "\alpha"] \arrow[u] & \bigwedge^2 F(-4)^{\oplus 3} \arrow[u] \\ \bigwedge^3 F(-6)^{\oplus 3} \arrow[r] \arrow[u] & \bigwedge^3 F \otimes G^*(-5)^{\oplus 3} \arrow[u] \end{tikzcd}.$$

The graded betti-numbers if C depend on the rank of α. Recall (1.4), (1.6)

$$\psi_{12}, \psi_{13}, \psi_{23} \in H^0 \mathcal{O}_{\mathbb{P}(\mathscr{E})}(H-R) \cong H^0(\mathbb{P}^{g-1}, F).$$

α is given by the wedge-product with the corresponding section of F:

$$F^{\oplus 3} \xrightarrow{\begin{pmatrix} 0 & \psi_{12} & \psi_{13} \\ -\psi_{12} & 0 & \psi_{23} \\ -\psi_{13} & -\psi_{23} & 0 \end{pmatrix}} \bigwedge^2 F^{\oplus 3}.$$

Thus α has maximal possible rank, i.e. is invertible for $\operatorname{char}(\mathfrak{k}) \neq 2$ and has rank 8 for $\operatorname{char}(\mathfrak{k}) = 2$ iff $\psi_{12}, \psi_{13}, \psi_{23}$ are linearly independent. In that case the betti-numbers take the minimal value as predicted in Table 1.

On the other hand, if $\psi_{12}, \psi_{13}, \psi_{23}$ are dependent, then there exist a g^1_4: After suitable row and column operations on ψ we may assume $\psi_{12} = 0$. Thus C is contained in the determinantal surface Y defined by the matrix

$$\begin{pmatrix} \psi_{13} & \psi_{14} & \psi_{15} \\ \psi_{23} & \psi_{24} & \psi_{25} \end{pmatrix} \sim \begin{pmatrix} H-R & H & H \\ H-R & H & H \end{pmatrix}$$

Table 2. Canonical curves of genus 7 with a g_5^1

$a_1, \ldots, a_5$	Special linear series	Determinantal surface Y $C \subset Y \subset X \subset \mathbb{P}^{g-1}$	$\deg Y$
$0,0,0,1,1$	General case	$\mathbb{P}^2$ blown-up in the 8 double points of a g_7^2	8
$0,0,0,1,1$	g_4^1	$\mathbb{P}^1 \times \mathbb{P}^1$ blown-up in 5 double points of $g_5^1 \times g_4^1$	7
$-1,0,1,1,1$	g_6^2	$\mathbb{P}^2$ blown-up in 3 double points of a g_6^2, the g_5^1 is a projection	6
$-1,0,0,1,2$	g_6^2	As above, but the double points lie on a line	6
$-1,-1,1,1,2$	g_3^1	$\mathbb{P}^1 \times \mathbb{P}^1$ blown-up in the double point of a $g_5^1 \times g_3^1$	5

on $\mathbb{P}(\mathscr{E})$. By Theorem (5.6) Y is a blow-up of $\mathbb{P}^1 \times \mathbb{P}^1$ in 5 points and the image of C in $\mathbb{P}^1 \times \mathbb{P}^1$ is a divisor of class

$$C' \sim 5A + 4B.$$

The projection onto the second factor of $\mathbb{P}^1 \times \mathbb{P}^1$ is a g_4^1.

(7.3) Other values $(a_1, \ldots, a_5)$ for curves of genus $g=7$ are possible. But in all other cases we can deduce already from $h^0 \mathcal{O}_{\mathbb{P}(\mathscr{E})}(H-R)=3$ the vanishing of an off-diagonal entry of ψ (possibly after row and column operations on ψ), hence we deduce the existence of a special linear series g_3^1, g_4^1 or g_6^2 by Theorem (5.3). The result is summarized in Table 2.
All other distributions of $(a_1, \ldots, a_5)$ are impossible: In those case it is easy to see that at least 2 off-diagonal entries of ψ contained in one row are zero for degree reasons. So the Pfaffians of ψ would contain an reducible element. But with C also the generators of the ideal $\mathscr{J}_C$ of C in $\mathbb{P}(\mathscr{E})$ have to be irreducible.

(7.4) For $g=8$ we proceed similarly. For a general curve with a g_5^1 it will turn out that

$$(a_1, \ldots, a_5) = (0,1,1,1,1).$$

The resolution of C on the associated $\mathbb{P}^3$-bundle $\mathbb{P}(\mathscr{E})$ is of type

$$0 \to \mathcal{O}_{\mathbb{P}(\mathscr{E})}(-5H+2R) \to \begin{matrix} \mathcal{O}_{\mathbb{P}(\mathscr{E})}(3H+2R) \\ \oplus \\ \mathcal{O}_{\mathbb{P}(\mathscr{E})}(-3H+R)^{\oplus 4} \end{matrix} \xrightarrow{\psi} \begin{matrix} \mathcal{O}_{\mathbb{P}(\mathscr{E})}(-2H) \\ \oplus \\ \mathcal{O}_{\mathbb{P}(\mathscr{E})}(-2H+R) \end{matrix} \to \mathcal{O}_{\mathbb{P}(\mathscr{E})} \to \mathcal{O}_C \to 0$$

with ψ a skew-symmetric matrix with entries as indicated below:

$$(\psi)\sim\begin{pmatrix} \emptyset & H-R & H-R & H-R & H-R \\ H-R & \emptyset & H & H & H \\ H-R & H & \emptyset & H & H \\ H-R & H & H & \emptyset & H \\ H-R & H & H & H & \emptyset \end{pmatrix}.$$

The only non-minimal maps in the corresponding mapping cone arise from the first row of ψ and its dual in the resolution, that is the first column:

$$\begin{array}{ccc}
\mathcal{O}_{\mathbb{P}(\mathscr{E})}(-3H+R)^{\oplus 4} & \xrightarrow{(\psi_{12},\psi_{13},\psi_{14},\psi_{15})} & \mathcal{O}_{\mathbb{P}(\mathscr{E})}(-2H) \\
\uparrow & & \uparrow \\
\hline
G(-3)^{\oplus 4} & \longrightarrow & \mathcal{O}(-2) \\
\uparrow & & \uparrow \\
F(-4)^{\oplus 4} & \xrightarrow{\alpha} & \bigwedge^2 F(-4) \\
\uparrow & & \uparrow \\
\bigwedge^3 F(-6)^{\oplus 4} & \longrightarrow & \bigwedge^3 F\otimes G^*(-5) \\
\uparrow & & \uparrow \\
\bigwedge^4 F\otimes G(-7)^{\oplus 4} & \longrightarrow & \bigwedge^4 F\otimes D_2G^*(-6).
\end{array}$$

Identifying

$$\psi_{12},\ldots,\psi_{15}\in H^0\mathcal{O}_{\mathbb{P}(\mathscr{E})}(H-R)\cong H^0(\mathbb{P}^{g-1},F),$$

α is given by the wedge-product with the corresponding section of F. Thus α has maximal rank, consequently the betti-numbers of C the minimal value, if $\psi_{12},\ldots,\psi_{15}$ span an at least 3-dimensional subspace of $H^0(\mathbb{P}^{g-1},F)$. But this is satisfied, since if two off-diagonal entries in one row or column of ψ vanish, we obtain a reducible generator of the ideal of C in $\mathbb{P}(\mathscr{E})$, which contradicts the irreducibility of C.

Notice that in case $\psi_{12},\psi_{13},\psi_{14},\psi_{15}$ only span a 3-dimensional subspace, C is contained in a determinantal surface Y of type

$$\begin{pmatrix} H & H & H \\ H-R & H-R & H-R \end{pmatrix}.$$

By Theorem (5.3) Y is isomorphic to $\mathbb{P}^2$ blown-up in the 7 double points of a g^2_- on C.

(7.5) Other values $(a_1,\ldots,a_5)$ are possible. The result is summarized in Table 3:

Table 3. Canonical curves of genus 8 with a basepoint free g^1_5

$a_1, \ldots, a_5$	Special linear series	Determinantal surface Y $C \subset Y \subset X \subset \mathbb{P}^{g-1}$	$\deg Y$
$0,1,1,1,1$	General case	$\mathbb{P}^1 \times \mathbb{P}^1$ blown-up in 8 double points of a $g^1_5 \times g^1_5$	10
$0,1,1,1,1$	g^2_7	$\mathbb{P}^2$ blown-up in 7 double points of a g^2_7, the g^1_5 is a projection	9
$0,0,1,1,2$	g^2_7	As above, but all double points but the projection point lie on a conic	9
$0,0,1,1,2$	g^1_4	$\mathbb{P}^1 \times \mathbb{P}^1$ blown-up in 4 double points of a $g^1_5 \times g^1_4$	8
$0,0,0,2,2$	g^1_4	As above, but the double points lie on a divisor of type $(0,2)$	8
$-1,0,1,2,2$	g^2_6	$\mathbb{P}^2$ blown-up in 2 double points of a g^2_6	7
$-1,-1,2,2,2$	g^1_3	$\mathbb{P}^1 \times \mathbb{P}^1$, C embedded by a $g^1_5 \times g^1_3$	6

Most interesting is the distinction in case

$$(a_1, \ldots, a_5) = (0,0,1,1,2).$$

In this case the skew-symmetric matrix ψ is of type

$$(\psi) \sim \begin{pmatrix} \emptyset & H-2R & H-R & H-R & H \\ H-2R & \emptyset & H-R & H-R & H \\ H-R & H-R & \emptyset & H & H+R \\ H-R & H-R & H & \emptyset & H+R \\ H & H & H+R & H+R & \emptyset \end{pmatrix}.$$

The only non-minimal map in the corresponding mapping cone arises from the first 2×4-block of ψ and its dual.

$$\begin{tikzcd}
\mathcal{O}_{\mathbb{P}(\mathscr{E})}(-3H+2R)^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}(\mathscr{E})}(-3H+R)^{\oplus 2} \arrow[r] & \mathcal{O}_{\mathbb{P}(\mathscr{E})}(-2H)^{\oplus 2} \\
S_2G(-3)^{\oplus 2} \oplus G(-3)^{\oplus 2} \arrow[r] \arrow[u, dashed] & \mathcal{O}(-2)^{\oplus 2} \arrow[u, dashed] \\
F \otimes G(-4)^{\oplus 2} \oplus F(-4)^{\oplus 2} \arrow[r, "\alpha_1 \oplus \alpha_2"] \arrow[u] & \bigwedge^2 F(-4)^{\oplus 2} \arrow[u] \\
\bigwedge^2 F(-5)^{\oplus 2} \oplus \bigwedge^3 F(-6)^{\oplus 2} \arrow[r, "\beta_1 \oplus \beta_2"] \arrow[u] & \bigwedge^3 F \otimes G^*(-5)^{\oplus 2} \arrow[u] \\
\bigwedge^4 F(-7)^{\oplus 2} \oplus \bigwedge^4 F \otimes G^*(-7)^{\oplus 2} \arrow[r] \arrow[u] & \bigwedge^4 F \otimes D_2G^*(-6)^{\oplus 2}. \arrow[u]
\end{tikzcd}$$

Notice that α_1 and β_1 are dual to each other.

It suffices to determine the rank of $\alpha_1 \oplus \alpha_2$. Clearly, if $\psi_{12}=0$, then $\alpha_1 \oplus \alpha_2$ is not surjective and C is contained in a determinantal surface of type

$$\begin{pmatrix} H-R & H-R & H \\ H-R & H-R & H \end{pmatrix}.$$

By Theorem (5.3) there exists a g^1_4 on C. If $\psi_{12} \neq 0$ we have to prove that $\alpha_1 \oplus \alpha_2$ is surjective. Identifying

$$H^0\mathcal{O}_{\mathbb{P}(\mathscr{E})}(H-R) = H^0(\mathbb{P}^{g-1}, F)$$

we obtain from $\psi_{12} \in H^0\mathcal{O}_{\mathbb{P}(\mathscr{E})}(H-2R)$ a 2-dimensional subspace

$$G \cdot \psi_{12} \subset F.$$

Thus one component of α_1 is defined by

$$F \otimes G \to \bigwedge^2 F$$

$$f \otimes g \to f \wedge g \cdot \psi_{12}.$$

α_2 is given by

$$F^{\oplus 2} \xrightarrow{\begin{pmatrix} \psi_{13} & \psi_{14} \\ \psi_{23} & \psi_{24} \end{pmatrix} \wedge -} \bigwedge^2 F^{\oplus 2}.$$

Since we can achieve $\psi_{14}=0$ after row and column operations on ψ and $\psi_{13}, \psi_{24} \notin G \cdot \psi_{12}$, because no two entries on one row or column of ψ can be zero $\alpha_1 \oplus \alpha_2$ is surjective, i.e. the betti-numbers of C are minimal.

Finally we note that C is contained in a determinantal surface Y of type

$$\begin{pmatrix} H-R & H & H+R \\ H-2R & H-R & H \end{pmatrix}$$

that is $\mathbb{P}^2$ blown-up in the 7 double points of a g^2_7 on C. The special position of the double points follows from Theorem (2.5) applied to Y.

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