We are dealing here with a fundamental and almost paradoxical difficulty. Stated briefly, it is that learning is sequential but knowledge is not. A branch of mathematics... consists of an intricate network of interrelated facts, each of which contributes to the understanding of those around it. When confronted with this network for the first time, we are forced to follow a particular path, which involves a somewhat arbitrary ordering of the facts.
–Robert Osserman[]

$\geq 2Cg \geq 2d$

$\mathrm{Pic}_d(C) W_d^r(C)$

$g > 1 \mathbb{P}^3 g + 3$

$g 2g + 2 \mathbb{P}^1$

≥ 2

g

$g + 2gg$

g

$\mathbb{P}^3 \mathbb{P}^3 f^!$

$\mathbb{P}^3$

$\mathrm{H}_{g,3,d} g d \mathbb{P}^3$

$\mathbb{C}$

$f_1, \dots, f_c \subset \mathbb{C}[x_0, \dots, x_n] c (f_1, \dots, f_c) c$

R=RRR

$\mathbb{P}(V) V V \mathrm{Sym}(V) \mathbb{P}(V) V \mathbb{P}(V^*)$

$G(k, V) k V G(1, V) = \mathbb{P}(V*) \mathbb{G}(k, r) k \mathbb{P}^r$

$X, Y \subset \mathbb{P}^r \operatorname{codim}(X \cap Y) = \operatorname{codim} X + \operatorname{codim} Y \deg(X \cap Y) = \deg(X) \deg(Y)$

$X \subset \mathbb{P}^r \{H_\lambda \mid \lambda \in \Lambda\} \mathbb{P}^r \lambda \in \Lambda H_\lambda \cap X \cap_{\lambda \in \Lambda} H_\lambda X$

$\phi : X \to Y X' \subset X \phi(X') \subset Y$

$\phi : C \to D \phi$

DD

$\phi : C \to D \phi$

$X \subset \mathbb{P}^r X \subset \mathbb{P}^r R_X = R_{X/\mathbb{P}^r} := S/I(X) S = \mathbb{C}[x_0, \dots, x_r] \mathbb{P}^r X \mathbb{P}^r$

$X \kappa(X) X p \kappa(p) \cong \mathbb{C} p$

H^i

$\mathcal{F}XX \subset Y\mathcal{F}\iota_*\mathcal{F}\iota : X \to Y\mathcal{F}YH^i(X,\mathcal{F}) = H^i(Y,\iota_*\mathcal{F})H^i(\mathcal{F})$

$h^i(\mathcal{F})Dh^i(D)\dim_{\mathbb{C}} H^i(\mathcal{F})\dim_{\mathbb{C}} H^i(\mathcal{O}_X(D))$

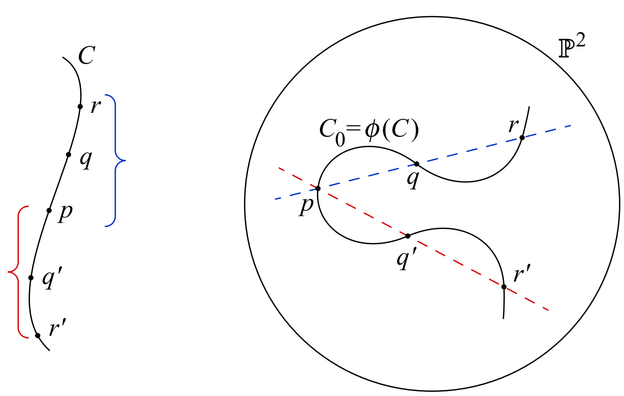
$\mathcal{F}H^i_*(\mathcal{F})\bigoplus_{m\in\mathbb{Z}}H^i(\mathcal{F}(m))$

C^{∞}

$\phi : C \to \mathbb{P}^r p \in C p$

$\mathbb{P}^r \phi(C) \phi(p) p$

$X C_i X X \setminus \cup C_i$



$\phi : C \to \mathbb{P}^2 p$

$CC\sum_{p\in C} m_p \cdot p m_p = 0 m_p p D m_p D Div(C) C$

$D = \sum_{p\in C} m_p \cdot p \sum m_p \mathcal{O}_{C,p} p C$

$CU \subset C f \in \mathcal{O}_C(U) p \in U f p \mathrm{ord}_p(f) m_p \subset \mathcal{O}_p f (f) f$

$$(f) := \sum_{p\in U} \mathrm{ord}_p(f) \cdot p.$$

$hC f/g f g h$

$$(h) := (f) - (g);$$

$f g h Div_0(C) Div(C) D E C D - E Div(C)/Div_0(C) C \mathrm{Pic}(C)$

$C \to \mathbb{P}^r \mathbb{P}^r \ell_1 = 0 \ell_2 = 0 \ell_1/\ell_2$

XXX

$XXXU_i \mathcal{O}_X(U_i)$

$C \subset \mathbb{A}^2 p = (0,0) \in C p(x,y) \Gamma = V(x^2, xy, y^2) \subset C p$

$$\Gamma_{\alpha,\beta} := V(\alpha x + \beta y, x^2, xy, y^2) \subset C$$

$\alpha x + \beta y = 0 L \Gamma_L \Gamma_L \subset L \Gamma_L L C \Gamma_{\alpha,\beta} C L \cap C p \Gamma_{\alpha,\beta}$

$XU \subset X K_X(U) \mathcal{O}_X(U) K_X \mathcal{O}_X^* K_X^* \mathcal{O}_X(U) K_X(U) X K_X^*/\mathcal{O}_X^* X$

$C\tilde{C}\tilde{C}$

$C f \in K(C)(f) C$

$\mathbb{P}^1 \phi C \pi : C \to \mathbb{P}^1$

$$(\phi) = \pi^{-1}(0) - \pi^{-1}(\infty).$$

$C \to D D' D C' \subset C \mathcal{O}_D(D') \to \mathcal{O}_C(C') \mathcal{O}_C(C') \mathcal{O}_D(D') C D \mathcal{O}_D(D') \mathcal{O}_C(C') \kappa(D) \subset \kappa(C)$

$C \to \mathbb{P}^1 \pi^{-1}(0) \pi^{-1}(\infty)(\phi)$ $\square$

$C \to \mathbb{P}^r$

$\mathrm{Pic}(C)$

$$\mathrm{Pic}(C) = \bigsqcup_{d\in\mathbb{Z}} \mathrm{Pic}_d(C),$$

$\mathrm{Pic}_d(C) d$

$\mathcal{L}X$

- $\{U_i\}X$
- $i\mathcal{O}_X(U_i)L_i$
- $i, j\sigma_{i,j} : L_i|_{U_i\cap U_j} \to L_j|_{U_i\cap U_j}\sigma_{j,k}\sigma_{i,j} = \sigma_{i,k}$

$\mathcal{L}t_i \in L_i\sigma_{i,j}t_i = t_j\mathcal{O}_X \to \mathcal{L}$

$\mathcal{L}L_iXL_i\mathcal{L}L|_U \cong \mathcal{O}_UX$

$\mathcal{L}_1 \otimes_{\mathcal{O}_X} \mathcal{L}_2\mathcal{L}^{-1} := \mathcal{H}\mathrm{om}_{\mathcal{O}_X}(\mathcal{L}, \mathcal{O}_X)\mathcal{F} \otimes \mathcal{H}\mathrm{om}_{\mathcal{O}_X}(\mathrm{F}, \mathcal{O}_X)\mathcal{F} = \mathcal{L}$

$CD = \sum m_p \cdot p\mathcal{O}_C(D)U$

$$\mathcal{O}_C(D)(U) := \{f \mid \mathrm{ord}_p(f) + m_p \geq 0p \in U\}.$$

$pm_p > 0fm_pm_p \leq 0fp{-}m_p$

$XD \subset XDX\mathcal{I}_{D/X}\mathcal{O}_X(-D)\mathcal{I}_{D/X}\mathcal{O}_X(D)\mathcal{O}_X(-D) \subset \mathcal{O}_X\sigma := \mathcal{O}_X \to \mathcal{O}_X(D)D$

$D = E - F$

$$\mathcal{O}_X(D) := \mathcal{O}_X(E) \otimes \mathcal{O}_X(F)^{-1}.$$

$DECfC(f) = D - Ef\mathcal{O}_C(E) \xrightarrow{\cong} \mathcal{O}_C(D)\mathcal{O}_C(D)D$

$\sigma \in H^0(\mathcal{L})Xp \in X\sigma(p)\sigma p\sigma H^0(\mathcal{L})\kappa(p) \otimes \mathcal{L}_p \cong \mathbb{C}\sigma Xp\mathcal{O}_{X,p}\sigma(\sigma)_0XX\sigma/\tau$ $\sigma(p)/\tau(p)\tau(p)(\sigma)_0\sigma \in H^0(\mathcal{L})$

$\sigma\mathcal{L}\sigma'\mathcal{L}'\mathcal{L} \otimes \mathcal{L}'\sigma\sigma'\mathrm{Pic}(X)\otimes$

$\mathcal{L}Xp \in X\mathcal{L}_p\mathcal{L}\mathcal{O}_{X,p}\mathfrak{m}_{X,p}\mathcal{O}_{X,p}\mathcal{L}$

$$\kappa(p) \otimes \mathcal{L} = \mathcal{L}_p/\mathfrak{m}_{X,p}\mathcal{L}_p \cong \mathbb{C}$$

$\kappa(p) := \mathcal{O}_{X,p}/\mathfrak{m}_{X,p}XpXL \to XL|_{U_i} \cong U_i \times \mathbb{C}^1\{U_i\}X$

$LX\mathcal{L}L\mathcal{L}(U)LU$

$pf : Y \to Xf^*(\mathcal{L})\mathcal{L}X\mathcal{O}_Y$

$\mathbb{P}^r\ \mathbb{C}[x_0, \dots, x_r]\mathbb{P}^rd\mathbb{P}^r\mathbb{Z}D = V(F) \subset \mathbb{P}^rFd\mathcal{O}_{\mathbb{P}^r}(d)\mathcal{O}_{\mathbb{P}^r}(D)$

$DH^0(\mathcal{O}_{\mathbb{P}^r}(-D)) = 0D$

$H^0(\mathcal{O}_{\mathbb{P}^1}(d))D = z_1 + z_2 + \cdots + z_dz_i\mathcal{O}_{\mathbb{P}^1}(D)\mathbb{P}^1z_i\mathbb{P}^1 \setminus \{\infty\} = \mathbb{A}^1\mathbb{C}$

$$\frac{\mathrm{g}(z)}{(z - z_1)(z - z_2)\cdots(z - z_d)}$$

$\mathrm{g}\deg(g) \leq dd + 1$

$\mathbb{P}^r\mathcal{O}_{\mathbb{P}^r}(d)dH^0(\mathcal{O}_{\mathbb{P}^r}(d))\binom{r+d}{r}d\mathbb{P}^r$

$\sigma p\sigma(p) = 0\sigma\mathcal{L}_p = \mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{O}_{X,p}\mathfrak{m}_{X,p}\mathcal{O}_{X,p}\mathcal{L}_p\sigma\mathcal{L}_p\sigma mp\sigma(p)\mathfrak{m}_{X,p}^m\mathcal{L}_p$

$\mathcal{L}\mathbb{P}^r\mathcal{L} \cong \mathcal{O}_{\mathbb{P}^r}(m)m = \deg \mathcal{L} \in \mathbb{Z}$

$$H^0(\mathcal{O}_{\mathbb{P}^r}(m)) = \mathbb{C}[x_0, \dots, x_r]_m$$

$mr + 1$

X

$X\mathcal{V} = (\mathcal{L}, V)\mathcal{L}XV\mathcal{L}$

$$\dim \mathcal{V} := \dim_{\mathbb{C}} V - 1.$$

$\sigma\mathcal{L}X(\sigma) = (\sigma)_0\sigma\tau\sigma H^0(\mathcal{O}_X) = \mathbb{C}X\mathcal{L}$

$|\mathcal{L}|(\mathcal{L}, H^0(\mathcal{L}))\mathcal{L} = \mathcal{O}(D)|D|$

$\mathcal{V} = (\mathcal{L}, V)X\mathbb{P}V^*\sigma \in VV\mathcal{V}V\mathcal{V}C\mathcal{V}V = H^0(\mathcal{L})$

$C\mathcal{V}dr\mathcal{V}g^r_d\mathcal{V}dr$

$X\phi : X \to \mathbb{P}^r PGL_{r+1}rX$

$\phi PGL_{r+1}\mathbb{P}^r PGL_{r+1}\mathbb{P}VV$

$(\mathcal{L}, V)CD_0D_0\mathcal{L}\mathcal{L}(-D_0)V\sigma\mathcal{O}_C(D_0)D_0$

$f : X \to \mathbb{P}^r\mathcal{V} = (\mathcal{L}, V)X\mathcal{L} = f^*\mathcal{O}_{\mathbb{P}^r}(1)\mathcal{O}_{\mathbb{P}^r}(1)$

$$V = f^*H^0(\mathcal{O}_{\mathbb{P}^r}(1)) \subset H^0(\mathcal{L}).$$

$X\mathbb{P}^r\mathbb{P}^rX$

$X\mathcal{V} = (\mathcal{L}, V)rXf : X \to \mathbb{P}^r\sigma_0, \dots, \sigma_rVD_i = (\sigma_i) \subset X\sigma_iU_i := X \setminus D_i\sigma_j/\sigma_iU_i$
$f_i : U_i \to \mathbb{P}^r$

$$f_i : p \mapsto \left(\frac{\sigma_0}{\sigma_i}(p), \dots, \frac{\sigma_r}{\sigma_i}(p)\right),$$

$\sigma_i/\sigma_i = 1f_if_jU_i \cap U_j\mathcal{V}U_iXfX\mathbb{P}^r$

$\mathcal{V}p \in XH_p := \{\sigma \in V \mid \sigma(p) = 0\}Vf : X \to \mathbb{P}V$

$\mathbb{P}^r(\mathcal{O}_{\mathbb{P}^r}(d), H^0(\mathcal{O}_{\mathbb{P}^r}(d))\mathbb{P}^{\binom{r+d}{r}-1}(a_0, \dots a_r)dx_0, \dots x_rd\mathbb{P}^r\mathbb{P}^1$

$$(x_0, x_1) \mapsto (x_0^d, x_0^{d-1}x_1, \dots, x_1^d).$$

$\mathbb{P}^1dd = 2d = 3\mathbb{P}^2$

---

gg^r_dC

$h^0(\mathcal{L})$

$C\mathcal{L}d \geq 0Ch^0(\mathcal{L}) \leq d+1C \cong \mathbb{P}^1$

$p_1,\dots,p_{d+1}Ch^0(\mathcal{L}) \geq d+2\sigma \in H^0(\mathcal{L})p_1,\dots,p_{d+1}(\sigma)\geq d+1\deg(\mathcal{L}) = d$

$h^0(\mathcal{L}) = d+1p_1,\dots,p_dC\sigma\in H^0(\mathcal{L})p_1,\dots,p_ddCp,q\in CfCC\cong\mathbb{P}^1$ □

$C\subset\mathbb{P}^dC\geq dC$ □

$d\mathbb{P}^1d\mathrm{PGL}_{d+1}$

$C\subset \mathbb{P}^d dECe\leq d+1EEe-1$

C

$LEEeEe-1$

$ECE+E'\deg E' = d-eCEd-e\mathbb{P}^{d-(\dim span(E))-1}d-e\geq d-(\dim span(E)-1)$ □

$\lambda_1,\dots,\lambda_{d+1}\in C\cong\mathbb{P}^1\ell=d+1$

$$\begin{vmatrix} 1 & \lambda_1 & \lambda_1^2 & \dots & \lambda_1^d \\ 1 & \lambda_2 & \lambda_2^2 & \dots & \lambda_2^d \\ \vdots & & & & \vdots \\ 1 & \lambda_{d+1} & \lambda_{d+1}^2 & \dots & \lambda_{d+1}^d \end{vmatrix} = \prod_{1\leq i<j\leq d+1}(\lambda_j-\lambda_i)$$

$V\subset H^0(\mathcal{L})|\mathcal{L}|V\subset W\subset H^0(\mathcal{L})WVV\pi:\mathbb{P}W\to\mathbb{P}VV\mathbb{P}(W/V)\subset\mathbb{P}W\pi$ $W^*V^*(W/V)^*=Ann(V)\subset W^*$

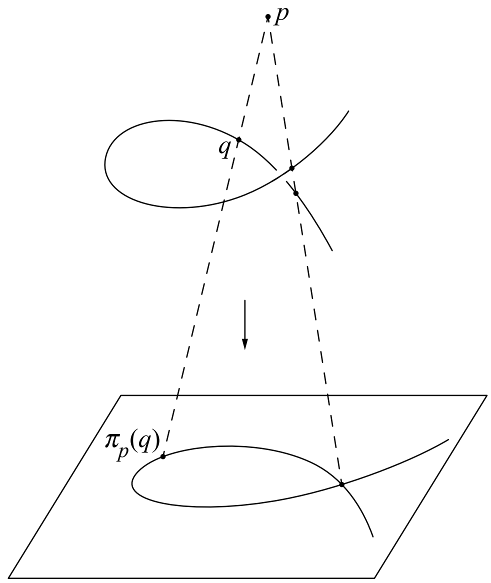
$$\begin{tikzcd} & \mathbb{P}W^* \arrow[d, "\pi"] \\ C \arrow[ur, "\phi_W"] \arrow[r, "\phi_V"'] & \mathbb{P}V^*. \end{tikzcd}$$

$WV\pi\phi_W(C)\pi C$

$C\subset\mathbb{P}^r|\mathcal{L}|$

$$H^0(\mathcal{O}_{\mathbb{P}^r}(1))\to H^0(\mathcal{L})$$

$C\tilde{C}\subset\mathbb{P}^{r+1}$



$p\mathbb{P}^2$

$\mathrm{D} = (\mathcal{L}, V)\mathrm{E} = (\mathcal{M}, W)C\mathrm{D} + \mathrm{E}\mathrm{D}\mathrm{E}$

$$\mathrm{D} + \mathrm{E} = (\mathcal{L} \otimes \mathcal{M}, U)$$

$U \subset H^0(\mathcal{L} \otimes \mathcal{M})V \otimes WH^0(\mathcal{L}) \otimes H^0(\mathcal{M}) \to H^0(\mathcal{L} \otimes \mathcal{M})|\mathcal{L} \otimes \mathcal{M}|D + ED \in \mathrm{D}$
$E \in \mathrm{E}$

$C \xrightarrow{\phi} \mathbb{P}^r C \xrightarrow{\psi} \mathbb{P}^s(\phi, \psi) : C \to \mathbb{P}^r \times \mathbb{P}^s \sigma : \mathbb{P}^r \times \mathbb{P}^s \to \mathbb{P}^{(r+1)(s+1)-1}$

$$((x_0, \dots, x_r), (y_0, \dots, y_s)) \mapsto (x_0 y_0, \dots, x_i y_j, \dots, x_r y_s).$$

$\mathrm{D} + \mathrm{E}\sigma \circ (\phi, \psi)$

$\mathrm{D}\mathrm{E}C$

$$\dim(\mathrm{D} + \mathrm{E}) \geq \dim \mathrm{D} + \dim \mathrm{E}.$$

$\dim \mathrm{D} \geq mD \in \mathrm{D}mC\mathrm{D} + \mathrm{E}D + ED \in \mathrm{D}E \in \mathrm{E}F \in \mathrm{D} + \mathrm{E}\dim \mathrm{D} + \dim \mathrm{E}C$ $\square$

$\mathcal{V} = (\mathcal{L}, V)DXD|D|DmDm > 0$

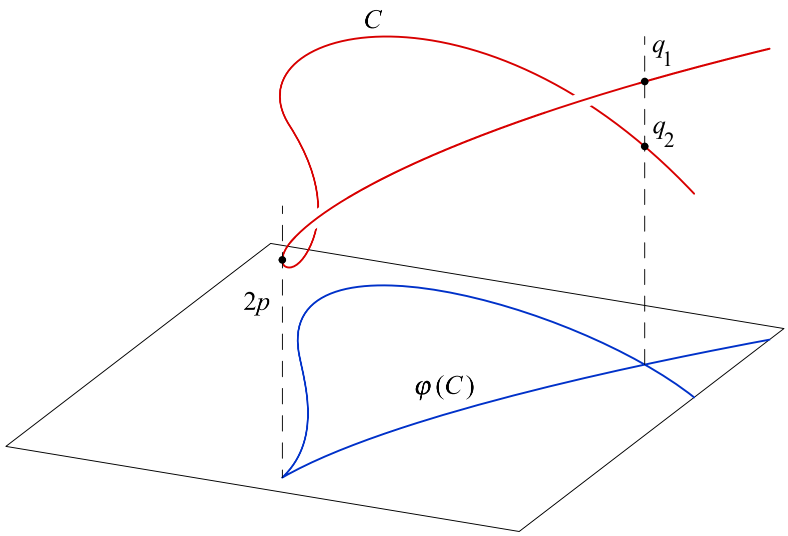
$\mathcal{V}D\mathcal{V}DCC_0$

$\mathcal{V} = (\mathcal{L}, V)DC\mathcal{V}(-D) = (\mathcal{L}(-D), V(-D))$

$$\mathcal{L}(-D) := \mathcal{L} \otimes \mathcal{O}(-D)V(-D) := \{\sigma \in V \mid \sigma(D) = 0\}.$$

$\dim \mathcal{V} - \dim \mathcal{V}(-D)D\mathcal{V}D\mathcal{V}\dim \mathcal{V} - \dim \mathcal{V}(-D) = \deg(D)$

$\phi : C \to \mathbb{P}^r H^0(\mathcal{L})\mathcal{L}h^0(\mathcal{L})H^0(\mathcal{L})$



a

$\mathcal{L}C|\mathcal{L}|$

$$h^0(\mathcal{L}(-p)) = h^0(\mathcal{L}) - 1 \quad \forall p \in C;$$

$\mathcal{L}$

$$h^0(\mathcal{L}(-p-q)) = h^0(\mathcal{L}) - 2 \quad \forall p, q \in C.$$

$\mathcal{L}\mathcal{L}h^0(\mathcal{L}(-D)) \geq h^0(\mathcal{L}) - \deg D D$

$|\mathcal{L}| p \in C \mathcal{L} p p H^0(\mathcal{L}) h^0(\mathcal{L}(-p)) = h^0(\mathcal{L}) - 1 p \mathcal{L} p$

$d d h^0(\mathcal{L}(-p-q)) = h^0(\mathcal{L}) - 2 p, q \phi_{\mathcal{L}}(p) \neq \phi_{\mathcal{L}}(q)$

$\phi : C \to \mathbb{P}^r \mathcal{L} \phi p$

$$\phi^* : \mathcal{O}_{\mathbb{P}^r, \phi(p)} \to \mathcal{O}_{C,p}$$

$C C \to \phi(C) \phi^* \mathcal{O}_{C,p} \mathcal{O}_{\mathbb{P}^r,\phi(p)} \mathcal{O}_{C,p} / \phi^*(\mathfrak{m}_{\mathbb{P}^r,\phi(p)}) \mathcal{O}_{\mathbb{P}^r,\phi(p)} / \mathfrak{m}_{\mathbb{P}^r,\phi(p)} = \mathbb{C}.$

$\mathbb{C} = \mathcal{O}_{\mathbb{P}^r,\phi(p)} / \mathfrak{m}_{\mathbb{P}^r,\phi(p)} \mathcal{O}_{C,p} / \mathfrak{m}_{C,p}$

$$\frac{\mathcal{O}_{C,p}}{\phi^*(\mathfrak{m}_{\mathbb{P}^r,\phi(p)}) \mathcal{O}_{C,p}}$$

$p h^0(\mathcal{L}(-2p) \neq h^0(\mathcal{L}(-p))$ □

$\phi C p$

$\phi : X \to \mathbb{P}^r \phi \phi(X) D := \sum_{p \in C} n_p p |D| |D| \deg D := \sum_{p \in C} n_p$

$\mathbb{P}^r \to \mathbb{P}^s s < r < r\mathbb{P}^r$

$$V = \langle s^{a_0} t^{d-a_0}, \dots, s^{a_r} t^{d-a_r} \rangle \subset \mathbb{C}[s,t]_d,$$

$\mathcal{V} = (\mathcal{O}_{\mathbb{P}^1}(d), V)$

$\mathcal{V}$

$\mathcal{V}$

$\mathcal{V}$

$\mathcal{V}$

$(\mathcal{L}, V)$

$$\dim\big(V \cap H^0(\mathcal{L}(-p-q))\big) = \dim V - 2 \quad \forall p, q \in C.$$

$\mathbb{P}^r \mathcal{V} = (\mathcal{O}_{\mathbb{P}^r}(1), H^0(\mathcal{O}_{\mathbb{P}^r}(1))) \mathrm{Aut}\,\mathbb{P}^r = PGL(r+1)$

$C \subset \mathbb{P}^r \phi : C \to C\phi\mathbb{P}^r\phi\mathcal{O}_C(1)\phi^*(\mathcal{O}_C(1)) \cong \mathcal{O}_C(1) C \subset \mathbb{P}^d \mathbb{P}^d PGL_2 \mathbb{P}^1 \mathbb{P}^1 C$

$\mathbb{C}[s^d, s^{d-1}t, \dots, t^d] {=} \mathbb{C}[s,t] f f$

$C = V(y^2 - x^2 - x^3) \subset \mathbb{A}^2 C p = (0,0) \in C$

$p(x,y)$

$\Gamma = V(x^2, xy, y^2) C C$

$\Gamma_{\alpha,\beta} := V(\alpha x + \beta y, x^2, xy, y^2) \subset C \beta \neq \pm\alpha \Gamma_{\alpha,\beta} C$

$\Gamma_{\alpha,\beta} \beta = \pm\alpha \Gamma_{\alpha,\beta} C$

$\beta \neq \pm\alpha \Gamma_{\alpha,\beta} C L \subset \mathbb{A}^2 \alpha x + \beta y = 0 \beta = \pm\alpha L \cap C p \Gamma_{\alpha,\beta}$

$\mathcal{L}$

$h^0(\mathcal{L})$

$$\chi(\mathcal{L}) := \sum_{i \geq 0} (-1)^i h^i(\mathcal{L}).$$

$h^0(\mathcal{L})$

$\mathcal{F} X n i H^i(\mathcal{F}) i > n X \subset \mathbb{P}^m d \gg 0 \mathcal{F}(d) H^i(\mathcal{F}(d)) = 0 i > 0$ □

d

□

$X \mathcal{L} X \mathcal{L} = \mathcal{O}_C(D) D X X \div$

$H \subset \mathbb{P}^r E C H n \gg 0 \mathcal{L}(n) F$

$$\mathcal{L} = \mathcal{O}_C(F - nE).$$

$X \div \div$ □

$X \subset \mathbb{P}^r \chi(\mathcal{L}(d)) = h^0(\mathcal{L}(d)) d C \subset \mathbb{P}^r \chi(\mathcal{L}) = h^0(\mathcal{L}) - h^1(\mathcal{L})$

$C \mathcal{L} C \chi(\mathcal{L}) = \deg \mathcal{L} + \chi(\mathcal{O}_C)$

$\mathcal{L} = \mathcal{O}_C C \mathcal{L} = \mathcal{O}_C(D) D p \in C \kappa(p) p$

$$0 \to \mathcal{L}(-p) \to \mathcal{L} \to \mathcal{L} \otimes \kappa(p) \to 0$$

$\mathcal{L} \otimes \kappa(p) \cong \kappa(p)$

$$\chi(\mathcal{L}) = \chi(\mathcal{L}(-p)) + \chi(\kappa(p)) = \chi(\mathcal{L}(-p)) + 1.$$

C □

$C \deg \mathcal{L} := \chi(\mathcal{L}) - \chi(\mathcal{O}_C)$

$h^1(\mathcal{L})$

$C n n \omega_C K_C$

$\mathbb{P}^r$

$\mathbb{P}^r \mathcal{O}_{\mathbb{P}^r}(-r-1)$

$x_0, \ldots, x_r \mathbb{P}^r U = \mathbb{P}^r \setminus H x_0 \neq 0 U \cong \mathbb{A}^r z_1 := x_1/x_0, \ldots, z_r := x_r/x_0 r U$ $d(x_1/x_0) \wedge \cdots \wedge d(x_r/x_0) U$

$$d \frac{x_i}{x_0} = \frac{x_0 dx_i - x_i dx_0}{x_0^2}$$

$$d(x_1/x_0) \wedge \cdots \wedge d(x_r/x_0) = \frac{dx_1 \wedge \cdots \wedge dx_r}{x_0^r} - \sum_{i=1}^{r} x_i \frac{dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_r}{x_0^{r+1}}$$

$r + 1 H x_0 - (r+1) H$ □

$$0 \to \Omega_{\mathbb{P}^r} \to \mathcal{O}_{\mathbb{P}^r}^{r+1}(-1) \to \mathcal{O}_{\mathbb{P}^r} \to 0.$$

$H^0_* H^0_*(\Omega_{\mathbb{P}^r})$

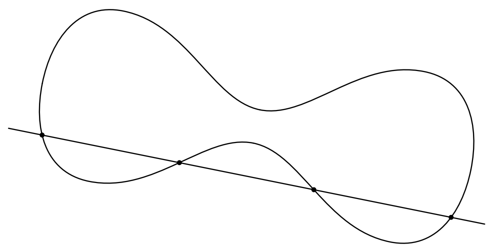
$$0 \to H^0_*(\Omega_{\mathbb{P}^r}) \longrightarrow S^{r+1}(-1) \xrightarrow{\delta_1} S \longrightarrow \mathbb{C} \to 0,$$

$S \mathbb{P}^r \delta_1 i S^{r+1}(-1) i S H^0_*(\Omega_{\mathbb{P}^r}) \mathbb{C} S \mathbb{C}$

$$0 \to S(-r-1) \xrightarrow{\delta_{r+1}} \bigwedge^r S^{r+1}(-r) \xrightarrow{\delta_r} \cdots \longrightarrow S^{r+1}(-1) \longrightarrow S \to \mathbb{C} \to 0.$$

$i i H^0_*(\Omega_{\mathbb{P}^r}) \longrightarrow S^{r+1}(-1) \bigwedge^i (\Omega_{\mathbb{P}^r})(i+1) \mathbb{C} \omega_{\mathbb{P}^r} = \bigwedge^r (\Omega_{\mathbb{P}^r})(r+1) S(-r-1)$

$C g(C) H^0(\omega_C)$



$X \subset Y X Y \mathcal{I}_{X/Y}/\mathcal{I}_{X/Y}^2 X Y$

$$\mathcal{N}_{X/Y} = \mathcal{H}\mathrm{om}(\mathcal{I}_{X/Y}/\mathcal{I}_{X/Y}^2, \mathcal{O}_Y).$$

$X Y X Y \mathcal{I}_{X/Y}/\mathcal{I}_{X/Y}^2 X \dim Y - \dim X X \mathcal{I}_X = \mathcal{O}_Y(-X)$

$$\mathcal{N}_{X/Y} = \mathcal{O}_X(X).$$

$X \subset Y c Y K_Y Y K_X X$

$$\omega_X = \bigwedge^c \mathcal{N}_{X/Y} \otimes \omega_Y.$$

$X K_X X K_Y + X Y$

$X Y$

$$0 \to \mathcal{I}_{X/Y}/\mathcal{I}_{X/Y}^2 \to \Omega_Y|_X \to \Omega_X \to 0$$

$\Omega_X X \mathcal{I}_{X/Y}|_X = \mathcal{O}_Y(-X)|_X = \mathcal{O}_X(-X)$ $\square$

$$0 \to \mathcal{E} \to \mathcal{F} \to \mathcal{G} \to 0$$

$e, f, g X$

$$\wedge^e \mathcal{E} \otimes \wedge^g \mathcal{G} \to \wedge^f \mathcal{F}.$$

$\wedge^e \mathcal{E} \otimes \wedge^g \mathcal{G} \to \wedge^f \mathcal{F}$

$$(\epsilon_1 \wedge \cdots \wedge \epsilon_e) \otimes (\gamma_1 \wedge \cdots \wedge \gamma_g) \mapsto \epsilon_1 \wedge \cdots \wedge \epsilon_e \wedge \gamma_1 \wedge \cdots \wedge \gamma_g.$$

$\gamma_i \mathcal{E}$

$\mathcal{G} X U$

$$0 \to \mathcal{E}|_U \to \mathcal{F}|_U \to \mathcal{G}|_U \to 0$$

$\mathcal{F}|_U = \mathcal{E}|_U \oplus \mathcal{G}|_U$

$$\bigwedge^f \mathcal{F}|_U = \bigoplus_{i+j=f} \bigwedge^i \mathcal{E}|_U \otimes \bigwedge^j \mathcal{G}|_U.$$

$\mathcal{E} e \mathcal{G} g$

$$\bigwedge^{f} \mathcal{F}|_U = \wedge^e \mathcal{E}|_U \otimes \bigwedge^{g} \mathcal{G}|_U,$$

$\square$

$C \subset \mathbb{P}^2 d \omega_C = \mathcal{O}_C(d-3) X \subset \mathbb{P}^r d_1, \dots, d_c \mathbb{P}^r \omega_X = \mathcal{O}_X(\sum d_i - r - 1).$

$\mathcal{N}_{X/Y} = \bigoplus_{i=1}^{c} \mathcal{O}_X(d_i)$ $\square$

$f : C \to X C X f f p \in C \mathrm{ram}(f, p)$

$$f^{-1}(q) = \sum_{p \in C | f(p) = q} (\mathrm{ram}(f, p) + 1) \cdot p$$

$q \in X$

$f : C \to X p \in C \mathrm{ram}(f, p) > 0$

f

$$R = \sum_{p \in C} \mathrm{ram}(f, p) \cdot p \ \in \ \mathrm{Div}(C).$$

$$B = \sum_{q \in X} \Big(\sum_{p \in f^{-1}(q)} \mathrm{ram}(f, p) \Big) \cdot q \ \in \ \mathrm{Div}(X).$$

$R B \sum_{p \in C} \mathrm{ram}(f, p)$

$K(X) \to K(C) p$

$z C p w X f(p) z \mapsto w = z^m m > 0 w X z f^*$

$$\mathbb{C}\{\{w\}\} \cong \widehat{\mathcal{O}}_{X, f(p)} \xrightarrow{\widehat{f^*}} \widehat{\mathcal{O}}_{C, p} \cong \mathbb{C}\{\{z\}\}$$

$w u z^m u \mathrm{ram}(f, p) = m - 1 p f C$ $\square$

$C X$

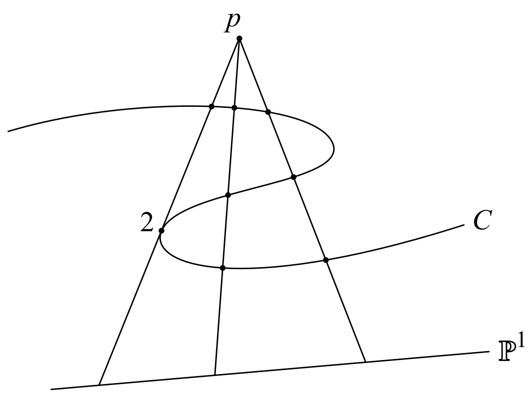
$f : C \to X R$

$$K_C = f^*(K_X) + R,$$

$\omega_C = (f^* \omega_X)(R).$

$\omega X \eta = f^*(\omega) C \omega B \omega X$

$\omega B \omega m d \eta m \omega B \omega p f z \mapsto w = z^e \omega = dw, \eta = z^{e-1} dz \eta \mathrm{ram}(f, p) p K_C \eta K_C = f^*(K_X) + R$ $\square$



$p\mathbb{P}^1\mathbb{P}'\mathbb{P}^1$

$C \subset \mathbb{P}^2 p\mathbb{P}^2 C\mathbb{P}^2 pW = \langle x_0, x_1\rangle(\mathcal{O}_C(1), W)Cp\mathbb{P}^1 d = \deg C$

$\mathbb{P}^1 -2K_C -2d + \deg RRx_0 = 0x_0 \neq 0z = x_1/x_0 Cf(x,y) = 0q \in CCqpdx$

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$$

$\partial f/\partial yqC\partial f/\partial y = 0d(d-1)Rd(d-1)C\deg K_C = -2d + d(d-1) = d(d-3)$

$VdV = H^0(\mathcal{O}_{\mathbb{P}^1}(d))\mathbb{P}(V^*) \cong \mathbb{P}^d\Delta\Delta$

$\Delta\Delta W \subset Vd\mathbb{P}^1\mathcal{W} = (\mathcal{O}_{\mathbb{P}^1}, W)\phi_{\mathcal{W}} : \mathbb{P}^1 \to \mathbb{P}(W) \cong \mathbb{P}^1\mathbb{P}(W)fd\{f = 0\} \subset \mathbb{P}^1$ $\Delta W\phi_{\mathcal{W}} mm - 1Bd\mathbb{P}^1\mathbb{P}^1$

$$\deg B = \deg \omega_{\mathbb{P}^1} - d\deg \omega_{\mathbb{P}^1} = 2d - 2.$$

$\deg \Delta = 2d - 2$

$h^0(\mathcal{L})\mathcal{L}\chi(\mathcal{L}) = h^0(\mathcal{L}) - h^1(\mathcal{L})h^1(\mathcal{L})$

CDC

$$H^1(D) = H^0(K_C - D)^* := Hom_{\mathbb{C}}(H^0(K_C - D), \mathbb{C}),$$

$h^1(D) = h^0(K_C - D)$

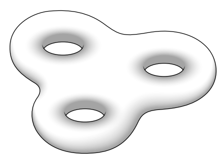
$Ch^1(\mathcal{O}_C) = h^0(K_C) = \mathrm{g}(C)\chi(\mathcal{O}_C) = 1 - \mathrm{g}(C).$

DC

$$h^0(D) - h^0(K_C - D) = \deg D - \mathrm{g}(C) + 1.$$

$\square$

$D = K_C \deg K_C = 2g(C) - 2$



$CH^1(C;\mathbb{C})X$

$$H^i(X,\mathbb{C}) = H^i_{deRham}(X) = \bigoplus_{j=0}^{i} H^j(\bigwedge^{i-j} \Omega_X).$$

C

$$H^1(C;\mathbb{C}) = H^0(\omega_C) \oplus H^1(\mathcal{O}_C) = H^0(\omega_C) \oplus (H^0(\omega_C))^\vee,$$

$h^0(\omega_C)$

$EH^0(E) = 0$

Dd

$$h^0(D) \geq d - g + 1,$$

$d > 2g - 2$

$h^0(K_C - D)h^0(K_C - D) = h^1(D)Dd - g + 1h^1(\mathcal{L})$

Ddg

$d > 2g - 2H^1(\mathcal{O}_C(D)) = 0$

$d \geq 2g\mathcal{O}_C(D)|D|\mathcal{O}_C(D)D = K_C + EE$

$d \geq 2g + 1\mathcal{O}_C(D)\phi_D : C \to \mathbb{P}^{d-g}DC\mathbb{P}^{d-g}$

$d > 2g - 2K - Dh^1(D) = h^0(K - D) = 0.$ $\square$

$C\Gamma \subset CC \setminus \Gamma$

$D\Gamma D\phi : C \to \mathbb{P}^n CHDC \setminus \Gamma\mathbb{A}^n = \mathbb{P}^n \setminus H$ $\square$

$C \subset \mathbb{P}^r dg$

$$0 \longrightarrow \mathcal{I}_{C/\mathbb{P}^r}(m) \longrightarrow \mathcal{O}_{\mathbb{P}^r}(m) \longrightarrow \mathcal{O}_C(m) \longrightarrow 0$$

$$H^0(\mathcal{O}_{\mathbb{P}^r}(m)) \xrightarrow{\rho_m} H^0(\mathcal{O}_C(m)) \longrightarrow H^1(\mathcal{I}_{C/\mathbb{P}^r}(m)) \longrightarrow 0.$$

$h_C C R_C C$

$$h_C(m) = \dim_{\mathbb{C}}(R_C)_m = \operatorname{rank}(\rho_m),$$

$(R_C)_m m C \mathbb{P}^n$

$H^1(\mathcal{I}_{C/\mathbb{P}^r}(m)) = 0 m h_C(m) = h^0(\mathcal{O}_C(m)) m dm - g + 1 m C \subset \mathbb{P}^r p_C(m) = dm - g + 1$

$C \subset \mathbb{P}^n p_C(m) = \chi(\mathcal{O}_C(m)) p_a(C) 1 - \chi(\mathcal{O}_C) = 1 - p_C(0) C g(C) C$

$C p_a(C) = g(C) = h^0(\omega_C) C p_a(C) \geq g(C) C$

$D h^0(K_C - D) > 0 D h^0(D) h^1(D)$

$\mathbb{C} D = \sum a_i p_i X L(D) X \leq a_i p_i$

$X g D d X K - D K L(D) d - g + 1 + \dim_{\mathbb{C}} L(K - D)$

$L(D) = H^0(\mathcal{O}_C(D))$

$\phi p X \Delta \subset X p \phi p Res_p(\phi) \frac{1}{2\pi i} \phi \Delta z \Delta p \phi$

$$\phi = \sum_{i=-n}^{\infty} a_i z^i dz$$

$\phi p a_{-1}$

$\phi X \phi \phi$

ϕ $\square$

$D = \sum a_i p_i X d = \sum a_i = \deg D n p z p \sum_{i=-n}^{\infty} a_i z^i \sum_{i=-n}^{-1} a_i z^i L(D) p_i \dim L(D) \leq d + 1$

$c_1, \ldots, c_d \in \mathbb{C}[z^{-1}] f X \phi \in L(K) X$

$$\sum Res_{p_i}(f \cdot \phi) = 0.$$

$g c_i \phi p_i f g - \dim L(K - D)$

$$\dim L(D) \leq d + 1 - g + \dim L(K - D).$$

$K - D$

$$\begin{aligned}\dim L(K-D) &\leq \deg(K-D) + 1 - g + L(K-(K-D)) \\ &= 2g - 2 - d + 1 - g + L(D) \\ &= g - d - 1 + L(D)\end{aligned}$$

$$L(D) + L(K-D) \leq L(D) + L(K-D).$$

□

CC

$C_0 \phi : C \to C_0 C_0 CC(C, \phi)$

$=CC$

□

$C_0 C_0$

$$\mathcal{O}_{C_0} \to \nu_* \mathcal{O}_C.$$

$\mathcal{F} C_0 \mathbb{C}$

$$0 \to \mathcal{O}_{C_0} \to \nu_* \mathcal{O}_C \to \mathrm{F} \to 0.$$

$C_0 \nu : C \to C_0 \mathcal{F} = \nu_* \mathcal{O}_C / \mathcal{O}_{C_0} \delta(C_0) := h^0(\mathcal{F})$

$$p_a(C_0) - g(C) = \delta(C_0)$$

$\nu : C \to C_0 R^i \nu_* \mathcal{O}_C i > 0 \chi(\nu_* \mathcal{O}_C) = \chi(\mathcal{O}_C)$

$$p_a(C_0) - g(C) = \chi(\mathcal{O}_C) - \chi(\mathcal{O}_{C_0}) = \chi(\mathrm{F}) = h^0(\mathrm{F})$$

F □

$\mathcal{F} R U \subset C_0 \mathcal{O}_C|_U \overline{R} R \mathcal{F}|_U \overline{R}/R \mathcal{F}|_U \mathfrak{f}_{\overline{R}/R} \overline{R} R \nu^{-1}(\mathfrak{f}_{C/C_0}) \mathcal{O}_C$

$\delta(C_0) f C C_0 \mathrm{F} p \in C_0 \delta \delta_p$

$$p_a(C_0) - g(C) = \sum_{p \in (C_0)_{sing}} \delta_p$$

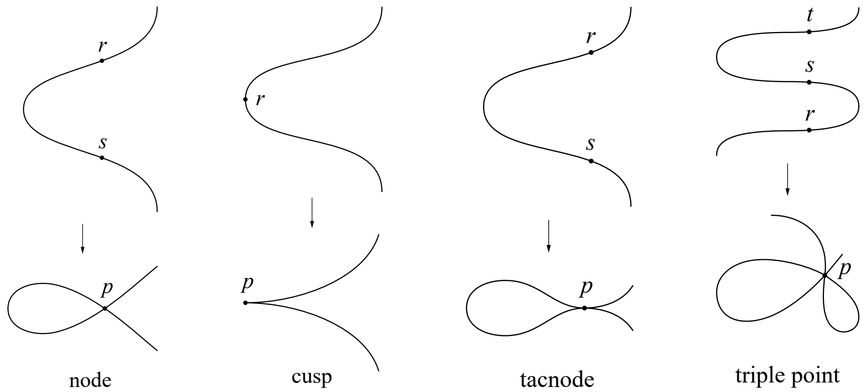
$f : X \to Y \mathcal{G} X$

$$H^p(R^q f_*(\mathcal{G})) \Longrightarrow H^{p+q}(\mathcal{G}).$$

$H^0 f_*$

δ

$C_0 p p C_0 p \mathcal{O}_{C_0,p} k[[x,y]]/(xy) p \in C_0 C C_0 r, s \in C f C C_0 C_0 f(r) = f(s) \delta_p = 1$



$C_0 p p C_0 y^2 = x^3 p \in C_0 C C_0 r \in C f C f'(r) = 0 \delta_p = 1$

$\mathbb{P}^1 |K_{\mathbb{P}^1}| = \emptyset C 1 2g - 2 = 0 \omega_C = \mathcal{O}_C K_C = 0 C g \geq 2$

$C \geq 2$

$|K_C|$

$|K_C| C \mathbb{P}^1$

$\geq 2 f : C \to \mathbb{P}^1$

$C g \geq 2 C \mathcal{L} \leq 2 \mathcal{L} |\mathcal{L}| \mathbb{P}^1 C g(C) = 2 |K_C| \mathbb{P}^1 C$

$C \ncong \mathbb{P}^1 C < 2 \mathcal{L} \mathbb{P}^1$ □

$p |K_C| h^0(K_C - p) = h^0(K_C) = g h^0(p) = 2 C \cong \mathbb{P}^1 K_C$

$p, q \in C$

$$h^0(K_C(-p-q)) = h^0(K_C) - 2 = g - 2.$$

$h^0(\mathcal{O}_C(p+q)) \geq 2 p, q \in C C C D = p + q h^0(D) = 2 h^0(K - p - q) = h^0(K) - 1$ □

$g > 2$

$C \subset \mathbb{P}^r DCC\overline{D}DH \subset \mathbb{P}^r D\phi : C \to \mathbb{P}^r DC\overline{\phi(D)} \subset \mathbb{P}^r$

$$\overline{\phi(D)} := \bigcap_{\{H \subset \mathbb{P}^r | D \subset \phi^{-1}(H)\}} H.$$

$Cg \geq 2\phi_K : C \to \mathbb{P}^{g-1} DCd\mathbb{P}^{g-1}\phi_K(D)K_C D$

$$h^0(K_C - D) = \operatorname{codim} \overline{\phi_K(D)} \subset \mathbb{P}^{g-1} = g - 1 - \dim \overline{\phi_K(D)}.$$

$DC \geq 2$

$$r(D) = d - 1 - \dim \overline{\phi_K(D)}.$$

$D = \sum_{i=1}^d p_i |D| p_i CD = p + q + r$

$\mathrm{p_a}) \in \mathbb{P}^r dpp\Gamma\gamma\gamma\Gamma\gamma$

$$0 \longrightarrow H^0(\mathcal{I}_\Gamma(d) \longrightarrow H^0(\mathcal{O}_{\mathbb{P}^r}(d)) \xrightarrow{ev} H^0(\mathcal{O}_\Gamma(d)) \longrightarrow 0,$$

$$H^0(\mathcal{O}_\Gamma(d)) \cong H^0(\mathcal{O}_\Gamma) \cong \mathbb{C}^\gamma.$$

$ev < \gamma\Gamma d\gamma - \operatorname{rank}(ev)$

$|D|CDC$

$C_0 \omega_{C_0} C_0$

$Cg \geq 2C\pi : C \to \mathbb{P}^1 \phi_K : C \to \mathbb{P}^{g-1} \pi g - 1\mathbb{P}^1 g - 1g - 1\pi$

$D\pi : C \to \mathbb{P}^1 D = p + qr(D) = 1DK_C\phi_K(p) = \phi_K(q)\phi_K\mathbb{P}^{g-1} \leq (2g-2)/2 = g - 1\phi_K g - 1\phi_K D\pi K g - 1\pi$ $\square$

$C\pi : C \to \mathbb{P}^1 Dds\pi d - 2sp_1, \dots, p_{d-2s} r(D) = sp_i|D|$

$DEs\pi D = E + D' \deg D' = d - 2s$

$\phi_K : C \to \mathbb{P}^{g-1}\phi_K(D')d - 2s\phi_K(E)s\phi_K(D)\min\{g-1, d-s-1\}r(D) = d - 1 - \min\{g-1, d-s-1\} = \max\{s, d-g\}Dr(D) > d - gr(D) = sr(E) = sE D'D$ $\square$

$r(\mathcal{L}) := h^0(\mathcal{L}) - 1 \geq \deg \mathcal{L} - g \deg \mathcal{L} > 2g - 2 \deg \mathcal{L} \leq 2g - 2$

$C g \mathcal{L} d \leq 2g - 2$

$$r(\mathcal{L}) \leq \frac{d}{2}.$$

$\mathcal{L} g \geq d/2 + 1 r(\mathcal{L}) = d - g + 1 \leq d/2$

$$r(K_C \otimes \mathcal{L}^{-1}) = r(\mathcal{L}) + g - d - 1$$

$$g = r(K_C) + 1 \geq r(\mathcal{L}) + r(K_C \otimes \mathcal{L}^{-1}) + 1 = 2r(\mathcal{L}) + g - d;$$

$r(\mathcal{L}) \leq d/2$ □

$C g \mathcal{L} d \leq 2g - 2$

$$r(\mathcal{L}) = \frac{d}{2},$$

$d = 0 \mathcal{L} = \mathcal{O}_C$

$d = 2g - 2 \mathcal{L} = K_C$

$C |\mathcal{L}| g^1_2 C$

$S \mathrm{Pic}(S) S$

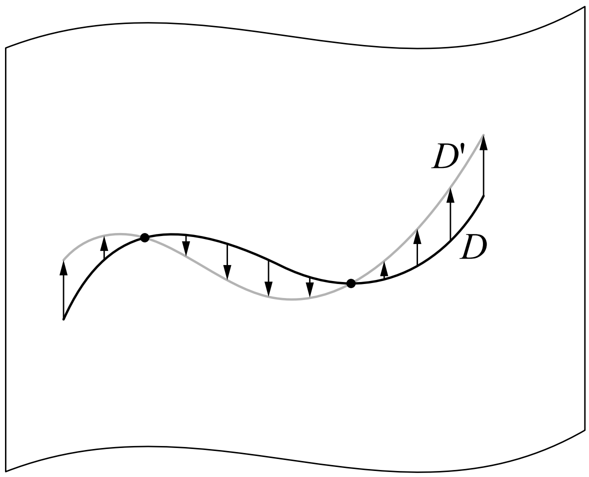
$D, E S D \cdot E D \cdot E D = E S H^2(X, \mathbb{Z})$

$$H^2(S, \mathbb{Z}) \times H^2(S, \mathbb{Z}) \to H^4(S, \mathbb{Z}) \cong \mathbb{Z},$$

$D \cdot D \sigma D D' D D \sigma D D \cdot D \sigma \mathbb{C}$

$D E m_S(D, E, p) p \in S \mathcal{O}_{S,p} / (\mathcal{I}_{D,p} + \mathcal{I}_{E,p})$

$$D \cdot E = \sum_p m_S(D, E, p).$$



$D'DD$

$\mathcal{L} := \mathcal{O}_S(D)\mathcal{M} := \mathcal{O}_S(E)$

$$D \cdot E = \chi(\mathcal{O}_S) - \chi(\mathcal{L}^{-1}) - \chi(\mathcal{M}^{-1}) + \chi(\mathcal{L}^{-1} \otimes \mathcal{M}^{-1})$$

$(D, E) \mapsto (D \cdot E)\mathrm{Pic}(S) \times \mathrm{Pic}(S) \to \mathbb{Z}S$

$D \cdot D = D^2 C^\infty$

$DSD \cdot D\mathcal{N}_{D/S} := \mathcal{O}_D(D).$

$\mathcal{O}_S(-D)\mathcal{L}^{-1}\mathcal{M}^{-1}$

$$0 \to \mathcal{O}_S(-D) \to \mathcal{O}_S \to \mathcal{O}_D \to 0,$$
$$0 \to \mathcal{O}_S(-2D) \to \mathcal{O}_S(-D) \to \mathcal{O}_D(-D) \to 0,$$

$$\chi(\mathcal{O}_S) - \chi(\mathcal{O}_S(-D)) = \chi(\mathcal{O}_D) = -p_a(D) + 1,$$
$$\chi(\mathcal{O}_S(-D)) - \chi(\mathcal{O}_S(-2D)) = \chi(\mathcal{O}_D(-D)) = -\deg \mathcal{O}_D(D) - p_a(D) + 1.$$

$D \cdot D = \deg \mathcal{O}_D(D)$ □

CS

$$p_a(C) = \frac{(K_X + C) \cdot C}{2} + 1.$$

$\mathbb{P}^3$ $\mathbb{P}^3$

$Q \subset \mathbb{P}^3 x_i q = \sum_{i=0}^3 x_i^2 r \leq 4Q\mathbb{P}^r m \leq r+1M(q(x+y)-q(x)-q(y))/2$

-
-
- $QQ\mathbb{Z}$
- $Q\mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^3\mathbb{P}^1 \times \{p\}\{p\} \times \mathbb{P}^1 Q\mathbb{Z} \times \mathbb{Z}$
- $\mathbb{P}^r\mathbb{P}(\mathcal{O}_{\mathbb{P}^r}(2)) = \mathbb{P}^{\binom{r+2}{2}-1} r+1(r+1)\times(r+1)Mk(k+1)\times(k+1)M$

-
- $CQ\mathbb{P}^3 C2mCmC(m-1)^2$

  $C2m+1CL_1, L_2$

  $$C \cup L_1 = Q \cap F_1, C \cup L_2 = Q \cap F_2$$

  $Qm+1CQ, F_1F_2Cm(m-1)CQ$
- $CQQ \cong \mathbb{P}^1 \times \mathbb{P}^1 C(a,b) \in \operatorname{Pic}(Q) = \mathbb{Z} \oplus \mathbb{Z}Q(1,0)^2 = (0,1)^2 = 0(1,0)\cdot(0,1) = 1(a,b)\cdot(d,e) = ae+bd(1,1)(K_{\mathbb{P}^3}+Q)\mid_Q = (-2,-2)$

  $C(a,b)Ca+bC(a-1)(b-1)a = bb = a+1b$

  $\mathcal{O}_Q(a,b)$

  $$H^i(\mathcal{O}_Q(a,b)) = \bigoplus_{i=s+t} H^s(\mathcal{O}_{\mathbb{P}^1}(a)) \otimes H^t(\mathcal{O}_{\mathbb{P}^1}(b)).$$

  C

  $$0 \to \mathcal{I}_{Q/\mathbb{P}^3} \to \mathcal{I}_{C/\mathbb{P}^3} \to \mathcal{I}_{C/Q} \to 0.$$

  $\mathcal{I}_{C/Q}(m) = (m-a, m-b)0 < a \leq b1 < bqQ < bb-a+1bq$

$\mathcal{L}S\chi(D) := h^0(\mathcal{L}) - h^1(\mathcal{L}) + h^2(\mathcal{L})\mathcal{L} = \mathcal{O}_S(D)$

$$\chi(D) = \chi(\mathcal{O}_S) + \frac{(D-K_S)\cdot D}{2} + 1$$

$\pi : X \to YXYpE = \pi^{-1}(p)$

$\operatorname{Pic} X = \pi^*(\operatorname{Pic} Y) \oplus \mathbb{Z}E$

$XK_X = \pi^*(K_Y) + E$

$\operatorname{Pic} X$

- $\pi^*(D)\cdot\pi^*(D') = D\cdot D'D, D' \in \operatorname{Pic} Y$
- $\pi^*(D)\cdot E = 0D \in \operatorname{Pic} Y$
- $E \cdot E = -1$

- $K_X \cdot E = -1$
- $Cmp\pi^{-1}(C)Em\tilde{C}XC \setminus \{p\}$

$$\tilde{C} \sim \pi^* C - mE.$$

$CS\tilde{C} \subset \tilde{S}$

$$\tilde{C} \cdot \tilde{C} = C \cdot C - m^2 \qquad \tilde{C} \cdot K_{\tilde{S}} = C \cdot K_S + m.$$

$$p_a(\tilde{C}) = p_a(C) - \binom{m}{2},$$

$\delta p \in C\binom{m}{2}\delta\tilde{C}p\delta$

$\tilde{C}p \in Cmm\delta p\binom{m}{2}$

$E \subset XXE^2 = E \cdot K_X = -1EXYp \in YE$

$X \subset \mathbb{P}^r XX$

$X \subset \mathbb{P}^r n0Xr - nn!XX$

$C \subset \mathbb{P}^r gdCH_C(m) = dm - g + 1$

$C \subset \mathbb{P}^r \phi : \widetilde{C} \to CX \subset \mathbb{P}^r C mult(C, X; p)p \in X \cap C\phi^{-1}(X)\phi^{-1}(p)$

$C p mult(C, X; p) \geq 2XpH \subset \mathbb{P}^r \deg C = \sum_{p \in C} mult(C, H; p)$

$C p \deg C - mult(C, H; p)H$
$\sum_{p \in C} mult(C, H; p)p\widetilde{C} \to \mathbb{P}^r C$

$C^\circ := V(f(x, y)) \subset \mathbb{A}^2 C^\circ$

$CC^\circ := V(y^3 + x^3 - 1)\pi : C \to \mathbb{P}^1 x$

$\overline{C^\circ} C^\circ \mathbb{P}^2 C = \overline{C^\circ}$

πC

$p, q \in C|p + q|r\overline{pq}C^\circ r$

$\eta : C \to \mathbb{P}^1 \eta((1, 0)) = \eta((0, 1))\eta$

$Cy^2 + x^3 = 1$

$C^\circ y^2 - x^6 + 1CC^\circ \to \mathbb{A}^1(x, y) \mapsto x\pi : C \to \mathbb{P}^1 C\mathbb{P}^1 1, \zeta, \ldots, \zeta^5 \zeta pq \in C$
$\infty \in \mathbb{P}^1$

$C^\circ C C^\circ \to \mathbb{A}^1 (x,y) \mapsto x \pi : C \to \mathbb{P}^1 C \mathbb{P}^1 1, \zeta, \dots, \zeta^5 \zeta p q \in C \infty \in \mathbb{P}^1 C \infty$

C

C

$r_\alpha \zeta^\alpha$

$$p + q \sim 2r_\alpha \qquad \sum_{\alpha=0}^{5} r_\alpha \sim 3p + 3q.$$

$H^0(\mathcal{O}_C(D)) D = r_0 + r_2 + r_4 E C E + r_1 \sim r_0 + r_2 + r_4$

$C D D = p + q + r_0 + r_3$

$H^0(\mathcal{O}_C(D))$

$\phi_{|D|} : C \to \mathbb{P}^2$

$\phi_{|D|}(C) \subset \mathbb{P}^2$

$x y C D = p + q + r_0 + r_3 \mathrm{Sym}^v H^0(\mathcal{O}_C(D)) \to H^0(\mathcal{O}_C(4D))$

$C y^3 = x^5 - 1 \pi : C \to \mathbb{P}^1 x C \mathbb{P}^1 \infty \eta = e^{2\pi i/5} r_\alpha (\eta^\alpha, 0) \in C \eta^\alpha p \infty \in \mathbb{P}^1$

$p \in C \infty \in \mathbb{P}^1 p$

C

$$3p \sim 3r_\alpha \qquad r_1 + \cdots + r_5 \sim 5p.$$

$H^0(K_C) C$

C

$\phi_K : C \to \mathbb{P}^3$

$D D = r_1 + \cdots + r_5 h^0(K_C(-D)) = 1 r(D) = 2 H^0(\mathcal{O}_C(D))$

$E = 3p r(E) = 1 |E| g_3^1 C 2E \sim K$

-1

$L \subset Q \subset \mathbb{P}^3 \mathbb{P}^3 3L p_a(3L) = -2$

$g \geq 3 \mathbb{P}^1$

$C = V(xy(x-y) \subset \mathbb{A}^2 \mathrm{Spec}(\mathbb{C}[x] \times \mathbb{C}[y] \times \mathbb{C}[z]) f, g, h$

$p \in C C \delta_p = 1 p$

$C p$

$C \subset Q \subset \mathbb{P}^3 Q C(a,b) 0 \leq a \leq b$

$Ca = 0 (a,b) = (1,1)$

b

$C_0 \subset \mathbb{P}^2 dmm\mathbb{P}^2 \hookrightarrow \mathbb{P}^{\binom{m+2}{2}-1} C_0 Cmd\mathbb{P}^N$

$$N = md - \binom{d-1}{2}.$$

$\mathbb{P}^{\binom{m+2}{2}-1} C_0 H^0(\mathcal{O}_{\mathbb{P}^2}(m)) \to H^0(\mathcal{O}_C(m))$

$m_p(C) p \in CC \subset \mathbb{P}^r H \cap CH \subset \mathbb{P}^r p\pi_p : C \to \mathbb{P}^{r-1} pC_0 = \pi_p(C) \subset \mathbb{P}^{r-1}$ $\deg(C_0) = \deg(C) - m_p(C)\pi_p : C \to \mathbb{P}^{r-1} k$

$$\deg(C_0) = \frac{\deg(C) - m_p(C)}{k}.$$

$C \subset \mathbb{P}^N C_1 \subset \mathbb{P}^{N-1} C_n \subset \mathbb{P}^{N-n}$

nC_n

$\mathbb{C}\mathbb{Q}\mathbb{P}^r$

$\mathbb{P}^1$

$C\mathbb{P}^1$

$C\mathcal{L}dCh^0(\mathcal{L}) \geq d+1C \cong \mathbb{P}^1$ $\square$

$\mathbb{C}$

$C \to DCDCD$

$K\mathbb{C} \subsetneq K \subset \mathbb{C}(x)K = \mathbb{C}(y)y \in \mathbb{C}(x)$

$\tilde{C} \to \tilde{D}$

$\mathbb{C}K\mathbb{C}(x)z \in K \setminus \mathbb{C}x\mathbb{C}(z)\mathbb{C}(x)\mathbb{C}(z)K\mathbb{C}(z)K\mathbb{C}KDK \subset \mathbb{C}(x)\mathbb{P}^1 \to DD \cong \mathbb{P}^1K \cong \mathbb{C}(y)y$ $\square$

$\mathbb{C}\mathbb{P}^2X\mathbb{P}^3 \to X$

CK_C-2

$k\mathbb{P}^1\mathbb{P}^1kk = \mathbb{R}\mathbb{R}\mathbb{P}^1\mathbb{R}x^2+y^2+z^2=0$

$\mathbb{P}^1\mathbb{P}^n\mathbb{P}^1_k k2\times 2k\mathbb{P}^1k{=}kx^2+y^2+z^2=0\mathbb{R}$

$\mathcal{V} = (V, \mathcal{L}) X V \subset H^0(\mathcal{L})$

$$\rho_{\mathcal{V}} : \mathrm{Sym}(V) \to \bigoplus_{n \in \mathbb{Z}} H^0(\mathcal{L}^n).$$

$\mathcal{V} X \mathbb{P}^r = \mathbb{P} V R_X X$

$C \subset \mathbb{P}^d d \mathcal{V} = (\mathcal{O}_{\mathbb{P}^1}(d), \mathbb{C}[s,t]_d) n d \mathbb{C}[s,t] n d$

$$\mathbb{C}[s,t]_{(d)} := \bigoplus_n (\mathbb{C}[s,t]_{nd}),$$

$\rho_{\mathcal{V}} C C \subset \mathbb{P}^r C n$

$$\rho_m : H^0(\mathcal{O}_{\mathbb{P}^r}(m)) \to H^0(\mathcal{O}_C(m))$$

$m = 1 m = 2, \dots, m = n C \rho_m m R_C$

$s, t \mathbb{P}^1 d \mathbb{P}^1 \to \mathbb{P}^d$

$$\phi_d : (s,t) \mapsto (s^d, s^{d-1}t, \dots, t^d)$$

$C \phi_d x_i x_j - x_{i+1} x_{j-1} i, j 2 \times 2$

$$M = \begin{pmatrix} x_0 & x_1 & \dots & x_{d-1} \\ x_1 & x_2 & \dots & x_d \end{pmatrix}.$$

$s^i t^{(d-i)} x_i H^0(\mathcal{O}_{\mathbb{P}^1}(i)) \mathbb{C}[s,t]_i$

$$H^0(\mathcal{O}_{\mathbb{P}^1}(1)) \times H^0(\mathcal{O}_{\mathbb{P}^1}(d-1)) \to H^0(\mathcal{O}_{\mathbb{P}^1}(d)).$$

$C 2 \times 2 M s = 1 \mathbb{P}^1 x_0 = 1 \mathbb{P}^d t \mapsto (t, t^2, \dots, t^d) x_1 = t x_0 x_i = x_1 x_{i-1} x_i = t^i M p p$
C

$$\mathbb{P}^1 \longrightarrow\!\!\!\!\!\!\rightarrow C \subset \mathbb{P}^d \quad (s,t) \mapsto (s^d, s^{d-1}t, \dots t^d)$$

$2 \times 2 M$

$I \subset \mathbb{C}[x_0, \dots, x_d] 2 \times 2 M$

$$\phi : \mathbb{C}[x_0, \dots, x_d]/I \longrightarrow\!\!\!\!\!\!\rightarrow \mathbb{C}[s,t]_{(d)} : \quad x_i \mapsto s^{d-i} t^i$$

$\phi n n d + 1 \phi n d$

$0 < i \leq j < d$

$$x_i x_j \equiv x_{i-1} x_{j+1} (mod I).$$

$x_i t I$

$(*)$ $$x_0^a x_1^{\epsilon_1} \cdots x_{d-1}^{\epsilon_{d-1}} x_d^b$$

$\epsilon_i n + 1 n \epsilon_i = 0 n (d-1) \epsilon_i = 1 n d + 1 \phi$ $\square$

$\mathbb{P}^2 2 \times 2$

$$\begin{pmatrix} x_0 & x_1 & x_2 \\ x_1 & x_3 & x_4 \\ x_2 & x_4 & x_5 \end{pmatrix}$$

$$H^0(\mathcal{O}_{\mathbb{P}^2}(1)) \otimes H^0(\mathcal{O}_{\mathbb{P}^2}(1)) \to H^0(\mathcal{O}_{\mathbb{P}^2}(2)).$$

nd

$$H^0(\mathcal{I}_{C/\mathbb{P}^d}(n)) = \binom{n+d}{n} - (nd+1).$$

$\binom{d}{2} M$

$$\mathbb{C}[s,t]_{(d)} \subset \mathbb{C}[s,t].$$

$\mathbb{C}[x_0, \ldots, x_d]_n \mathbb{C}[s,t]_{nd}$ $\square$

$C \subset \mathbb{P}^d$

$$h^0(\mathcal{I}_{C/\mathbb{P}^d}(2)) \leq \binom{d}{2};$$

C

$X \subset \mathbb{P}^r H^0(\mathcal{I}_X(1)) = H^1(\mathcal{I}_X) = 0 H \mathbb{P}^r h$

$$0 \to \mathcal{I}_{X/\mathbb{P}^r} \xrightarrow{h} \mathcal{I}_{X/\mathbb{P}^r}(1) \to \mathcal{I}_{(H \cap X)/H}(1) \to 0$$

$\mathcal{I}_{X/\mathbb{P}^r} H H \cap X H H^0(\mathcal{I}_{(H\cap X)/H}(1)) = 0$

$X \subset \mathbb{P}^r c \deg X \geq c+1$

$X \Lambda c \Lambda$ $\square$

$X h \mathcal{I}_{X/\mathbb{P}^r} h \mathcal{I}_{X/\mathbb{P}^r} = (h) \cap \mathcal{I}_{X/\mathbb{P}^r}(h) = h \mathcal{O}_X \Gamma = H \cap X$

$$\begin{aligned} \mathcal{I}_{X/\mathbb{P}^r}(1)/h\mathcal{I}_{X/\mathbb{P}^r} &= \mathcal{I}_{X/\mathbb{P}^r}(1)/((h) \cap \mathcal{I}_{X/\mathbb{P}^r}) \\ &= (\mathcal{I}_{X/\mathbb{P}^r}(1) + (h))/(h) \\ &= \mathcal{I}_{\Gamma/H}(1). \end{aligned}$$

$X H^0(\mathcal{O}_X) H^0(\mathcal{O}_{\mathbb{P}^r}) \to H^0(\mathcal{O}_X) H^1(\mathcal{I}_{X/\mathbb{P}^r}) = 0. H^0(\mathcal{I}_{X/\mathbb{P}^r}(1)) \to H^0(\mathcal{I}_{\Gamma/H}(1))$
$X H^0(\mathcal{I}_{X/\mathbb{P}^r}(1)) = 0$ $\square$

$CH \cong \mathbb{P}^{d-1} \subset \mathbb{P}^d \Gamma = H \cap C$

$$0 \to \mathcal{I}_{C/\mathbb{P}^d}(1) \to \mathcal{I}_{C/\mathbb{P}^d}(2) \to \mathcal{I}_{\Gamma/\mathbb{P}^{d-1}}(2) \to 0.$$

$h^0(\mathcal{I}_{C/\mathbb{P}^d}(1)) = h^1(\mathcal{I}_{C/\mathbb{P}^d}(1)) = 0 \Gamma C$

$H \Gamma \Gamma C d \geq d$

$$h^0(\mathcal{I}_{\Gamma/\mathbb{P}^{d-1}}(2)) \leq h^0(\mathcal{O}_{\mathbb{P}^{d-1}}(2)) - d = \binom{d+1}{2} - d = \binom{d}{2}$$

$C \subset \mathbb{P}^d \deg C > d$

$$\Gamma' \subset H \cong \mathbb{P}^{d-1}$$

$d H \cap C \Gamma = \Gamma' \cup \{p\} \subset H$

$p \Gamma' C \Gamma' d - 1 (d-2) H \Lambda p H \Lambda \Gamma' \setminus \Gamma' \cap \Lambda \Gamma' p$ ☐

$m m \mathbb{P}^d$

$C \subset \mathbb{P}^d G \subset PGL_{d+1} \mathbb{P}^d C C$

$X \subset \mathbb{P}^n \mathbb{P}^r |\mathcal{O}_{\mathbb{P}^r}(d)| X$

$\mathbb{P}^r \operatorname{Aut} \mathbb{P}^r \sigma \mathcal{O}_{\mathbb{P}^r}(d) d \mathbb{P}^r \sigma^* \mathcal{O}_{\mathbb{P}^r}(d) = \mathcal{O}_{\mathbb{P}^r}(d) \sigma \phi H^0(\mathcal{O}_{\mathbb{P}^r}(d)) \overline{\phi} \mathbb{P}^N := \mathbb{P} H^0(\mathcal{O}_{\mathbb{P}^r}(d))$
$d \ell \in H^0(\mathcal{O}_{\mathbb{P}^r}(d)) D \subset X \phi(\ell) = \ell \circ \sigma \sigma^{-1}(D) \overline{\phi}^{-1} \sigma \mathbb{P}^r$ ☐

$C \subset \mathbb{P}^d \mathbb{P}^d$

$\mathbb{P}^d k + 1 (k-1) k \leq n$

$p_1, \dots, p_{d+3} \in \mathbb{P}^d d + 3 \mathbb{P}^d C \subset \mathbb{P}^d$

$\Phi : \mathbb{P}^d \to \mathbb{P}^d p_1, \dots, p_{n+1} (0, \dots, 0, 1, 0, \dots, 0) \in \mathbb{P}^d p_{d+2} (1, 1 \dots, 1) p_{d+3} [\alpha_0, \dots, \alpha_n]$
$\mathbb{P}^1 \to \mathbb{P}^d z \mathbb{P}^1$

$$z \mapsto \left(\frac{1}{z - \nu_0}, \frac{1}{z - \nu_1}, \dots, \frac{1}{z - \nu_d} \right)$$

$\nu_0, \dots, \nu_d d + 1 \mathbb{P}^d z = \nu_0, \dots, \nu_d \in \mathbb{P}^1 z = \infty (1, 1, \dots, 1) \alpha_i \nu_i = -1/\alpha_i z = 0$
$(\alpha_0, \dots, \alpha_d)$ ☐

$d + 3 d + 3$

$\mathbb{P}^r m m \mathbb{P}^r$

$\mathrm{D}d\mathbb{P}^1|\mathcal{O}_{\mathbb{P}^1}(d)|\phi : \mathbb{P}^1 \to \mathbb{P}^r d\phi_d : \mathbb{P}^1 \to \mathbb{P}^d\mathbb{P}^1\pi : \mathbb{P}^d \to \mathbb{P}^r d\mathbb{P}^1 = \mathbb{P}(V)$ $\mathbb{P}V \to \mathbb{P}(\mathrm{Sym}^d(V))\mathbb{P}WW \subset \mathrm{Sym}^d(V)$

$s, tVF_i d\mathbb{P}^1 W$

$$(s,t) \mapsto (F_0(s,t), \dots, F_r(s,t)).$$

$F_i G(s,t) G = 0$

$$(s,t) \mapsto \left(\frac{F_0(s,t)}{G(s,t)}, \dots, \frac{F_r(s,t)}{G(s,t)}\right).$$

$d - \deg GW$

$\mathbb{A}^1 t = 1\mathbb{P}^1 z = s/tF_i f_i = F(s,1) \leq d$

$$z \mapsto (f_0(z), \dots, f_r(z)).$$

$F_i t f_i dz \mapsto (1, z, z^2, z^3)$

$f(z) = f(s/t) \leq d\mathbb{P}^1 d(1,0) \in \mathbb{P}^1 F_i(s,t)/G_i(s,t) \deg F_i - \deg G_i = d\phi_i(z) = f_i(z)/g_i(z)$

$C4\mathbb{P}^3\mathbb{P}^1\mathrm{Sym}^4(V)V \cong \mathbb{C}^2$

$PGL_4\mathbb{P}^3$

$$\mathbb{P}^1 \ni (s,t) \mapsto (s^4, s^3t, st^3, t^4) \in \mathbb{P}^3,$$

$t \mapsto (t, t^3, t^4)$

$C \subset \mathbb{P}^3 CS(1,3)CS$

$$\rho_e : H^0(\mathcal{O}_{\mathbb{P}^3}(e)) \to H^0(\mathcal{O}_C(e)).$$

$C \cong \mathbb{P}^1 H^0(\mathcal{O}_C(e)) H^0(\mathcal{O}_{\mathbb{P}^1}(4e)) C\rho_1$

$H^0(\mathcal{O}_{\mathbb{P}^3}(2))\rho_2 H^0(\mathcal{O}_{\mathbb{P}^1}(8))\rho_2 CCC$

$CQ(a,b)\mathrm{Pic}(Q) = \mathbb{Z} \oplus \mathbb{Z}(a,b) a+b(a-1)(b-1) a+b = 4, (a-1)(b-1) = 0$ $C \sim (1,3) C \sim (3,1) C \sim (1,3)$

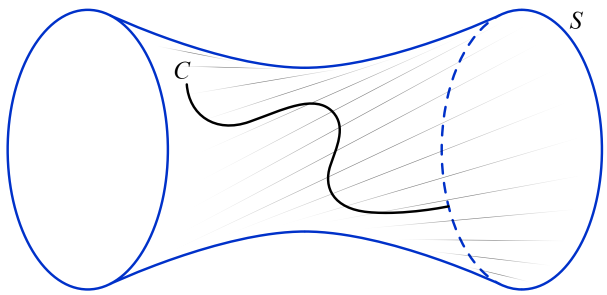
$\rho_3 C \subset \mathbb{P}^3\mathbb{P}^3 I_{C/\mathbb{P}^3} C$

$$0 \to I_{Q/\mathbb{P}^3} \to I_{C/\mathbb{P}^3} \to I_{C/Q} \to 0.$$

$I_{C/\mathbb{P}^3} I_{C/Q} I_{C/Q} \subset H^0_*(I_{C/Q})$

$$0 \to \mathcal{J}_{Q/\mathbb{P}^3} \to \mathcal{J}_{C/\mathbb{P}^3} \to \mathcal{J}_{C/Q} \to 0$$

$H^1_*(\mathcal{J}_{Q/\mathbb{P}^3}) = H^1_*(\mathcal{O}_{\mathbb{P}^3}(-2)) = 0 H^0_*(I_{C/Q})$



$(1, 3)$

$C(1,3)$

$$\mathcal{I}_{C/Q}(d) = \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(d-1, d-3)$$

$$h^0(\mathcal{I}_{C/Q}(3)) = h^0(\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(2,0)) = h^0(\mathcal{O}_{\mathbb{P}^1}(2)) \cdot h^0(\mathcal{O}_{\mathbb{P}^1}(0)) = 3$$

$I(C)Q$

$h^0(\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(2,0))$

$$H^0_*(\mathcal{I}_{C/Q}(3)) = H^0_*(\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(2,0)) = \oplus_{m\geq 0} H^0(\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(m+2, m)).$$

$$\rho_m : H^0(\mathcal{O}_{\mathbb{P}^3}(1)) \otimes H^0(\mathcal{I}_{C/Q}(m+3)) \to H^0(\mathcal{I}_{C/Q}(m+1+3)$$

$m \geq 0$

$H^0(\mathcal{O}_{\mathbb{P}^3}(1))$

$$H^0(\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(1,1)) = H^0(\mathcal{O}_{\mathbb{P}^1}(1)) \otimes H^0(\mathcal{O}_{\mathbb{P}^1}(1)),$$

ρ_m

$$H^0(\mathcal{O}_{\mathbb{P}^1}(1)) \otimes H^0(\mathcal{O}_{\mathbb{P}^1}(m+2)) \to H^0(\mathcal{O}_{\mathbb{P}^1}(m+1+2)$$

$$H^0(\mathcal{O}_{\mathbb{P}^1}(1)) \otimes H^0(\mathcal{O}_{\mathbb{P}^1}(m)) \to H^0(\mathcal{O}_{\mathbb{P}^1}(m+1)$$

$m \geq 0$ □

$\mathbb{P}^3 d d \mathbb{P}^3 \mathbb{P}^r r > 3 C \subset \mathbb{P}^r d$

Z

$d\mathbb{P}^r (r+1) H^0(\mathcal{O}_{\mathbb{P}^1}(d)) (r+1) H^0(\mathcal{O}_{\mathbb{P}^1}(d)) g$

$C \subset \mathbb{P}^r$

$$\rho_m : H^0(\mathcal{O}_{\mathbb{P}^r}(m)) \to H^0(\mathcal{O}_C(m)) = H^0(\mathcal{O}_{\mathbb{P}^1}(md)).$$

$V(r+1)dmV$

$\rho_m \binom{m+r}{r} md + 1 md + 1 \rho_m m$

gd

$d\mathbb{P}^3 d$

$C \subset \mathbb{P}^r esCs\Lambda \cong \mathbb{P}^s \subset \mathbb{P}^r \Lambda \cap C \geq e\Lambda \cap C\Lambda esCD \subset Ces$

$C \subset \mathbb{P}^r ess\mathbb{P}^r \mathbb{G} = \mathbb{G}(s,r)(s+1)(r-s)\mathbb{G}Cr-s-1pp\mathbb{G}(s-1,r-1)r-sC \subset \mathbb{P}^r$
es

$$e \ \leq \ (s+1)\frac{r-s}{r-s-1},$$

ers

$C \subset \mathbb{P}^r$

$$\rho_m : H^0(\mathcal{O}_{\mathbb{P}^r}(m)) \to H^0(\mathcal{O}_C(m))$$

m

$f_1, \dots, f_c R f_{i+1}(f_1, \dots, f_i) i = 0, \dots, c-1 (f_1, \dots, f_c)$

$IRIRI\infty I = R$

$(R, \mathfrak{m})\text{grade}\,\mathfrak{m} = \dim RRR$

- $x_0, \dots, x_r \subset S := \mathbb{C}[x_0 \dots, x_r]S$
- $S := \mathbb{C}[x_0 \dots, x_r]$
-
- R
- $IkExt^k(R/I, R) \neq 0$
- $I \subsetneq R\text{grade}\, I \leq \text{codim}\, IR\text{grade}\, I = \text{codim}\, IIR$
- $X \subset \mathbb{P}^r R_X X \subset \mathbb{P}^r R_X XXR_X C \subset \mathbb{P}^r R_C \to H^0_*(\mathcal{O}_C)R_C$
- $R_C CCC\rho_2$

$C \subset \mathbb{P}^r S = H^0_*(\mathcal{O}_{\mathbb{P}^r})\mathbb{P}^r R_C = S/I_C C$

$R_C \to H^0_*(\mathcal{O}_C(1))$

$H^1_*(\mathcal{J}_{C/\mathbb{P}^r}) = 0.$

$R_C Ch, h'\mathbb{P}^r R_C R_C C$

$H \subset \mathbb{P}^r CH \cap CI_C + (h)hH$

$\Leftrightarrow r \geq 2H^1_*(\mathcal{O}_{\mathbb{P}^r}(n)) = 0n$

$$0 \longrightarrow \mathcal{J}_{C/\mathbb{P}^r} \longrightarrow \mathcal{O}_{\mathbb{P}^r} \longrightarrow \mathcal{O}_C \longrightarrow 0$$

$H^0(\mathcal{O}_{\mathbb{P}^r}(n)) \to H^0(\mathcal{O}_C(n)) nH^1(\mathcal{J}_{C/\mathbb{P}^r}(n)) = 0n$

$\Leftrightarrow C \subset \mathbb{P}^r R = H^0_*(\mathcal{O}_C) := \bigoplus_{n\in\mathbb{Z}} H^0(\mathcal{O}_C(n)) H h = 0 h C$

$$0 \longrightarrow \mathcal{O}_C(-1) \xrightarrow{h} \mathcal{O}_C \longrightarrow \mathcal{O}_{C\cap H} \longrightarrow 0$$

$H^0_* h R R/hR H^0_*(\mathcal{O}_{C\cap H}) h' C \cap H h' H^0_*(\mathcal{O}_{C\cap H}) R/hR$

$R_C S \to R C R_C = R.C R_C C$

$\Leftrightarrow h C h R_C I_{C\cap H} I_C + (h) I_{C\cap H} = I_C + (h) h' C \cap H I_C + (h) C I_C + (h)$ □

$X f : X \to P P f_*(\mathcal{O}_X)$

$X X X \geq 2$

$\mathcal{V} = (\mathcal{O}_{\mathbb{P}^1}(d), V) d \mathbb{P}^1 V = \langle s^d, s^{d-1}t, st^{d-1}, t^d \rangle s, t \mathbb{P}^1 \mathcal{V} \mathbb{P}^1(1, d-1)$

$coker\, M \mathcal{O}^d_{\mathbb{P}^d}(-1) \to \mathcal{O}^2_{\mathbb{P}^d} M C C \to \mathbb{P}^1$

$\mathbb{P}^n \operatorname{Proj} \mathbb{C}[x_0, \dots, x_n] z_p \mathbb{P}^{\binom{n+d}{d}-1} p d x_i M_{n,d}(n+1) \times \binom{n+d-1}{n} \mathbb{P}^{\binom{n+d}{d}-1} x_i m d - 1 x_i$ $(i, m) z_{x_i m} M M_{1,d} 2 \times 2 M_{n,d} V_{n,d} \mathbb{P}^n \to \mathbb{P}^{\binom{n+d}{d}-1} M_{n,d} V_{n,d}$

$\nu_d : \mathbb{P}^r \to \mathbb{P}^{\binom{r+d}{r}-1} d C \subset \mathbb{P}^r r \nu_d(C)$

$H^0(\mathcal{O}_{\mathbb{P}^N}(1)) \to H^0(\mathcal{O}_{\nu_d(C)}(1))$

$C_1 \mathbb{P}^3 C_2 \mathbb{P}^3 C_1 C_2$

$X \subset \mathbb{P}^r Z \mathbb{P}^r F \in H^0(\mathcal{O}_{\mathbb{P}^r}(m)) F \mathcal{O}_X(m)$

$$\mathcal{I}_X(d-m) \xrightarrow{F} \mathcal{I}_X(m) \longrightarrow \mathcal{I}_{X\cap Z}(m) \to 0.$$

$C \subset \mathbb{P}^3 h H I_C + (h) I_C(1)/h I_c C \cap H$

$\mathrm{g}^3_4 \mathbb{P}^1 \mathrm{g}^1_1 \mathrm{g}^1_3 \mathbb{P}^1$

C

$\mathbb{P}^1 \mathbb{P}^3$

$\mathrm{g}^1_3 PGL_2 \mathrm{g}^1_3 \mathbb{P}^1$

$C, C' \subset \mathbb{P}^n n + 3 C = C'$

$n n = 2$

$V = \mathbb{C} \cdot e_1 \oplus \mathbb{C} \cdot e_2$

$SL_2 = SL(V) d \mathbb{P}^d \mathbb{P}(\mathrm{Sym}^d(V))$

$SL(V) \mathrm{Sym}^e(V) e \geq 0$

$$\alpha := \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \in SL(V).$$

$\mathrm{Sym}^e(V) \equiv \alpha w_s := e_1^{e-s} e_2^s \alpha w_s = t^{e-2s} s = 0, \dots e W W \mathrm{Sym}^{e_i} V \alpha w \in W \alpha \alpha \alpha w = t^m w m w \mathrm{Sym}^m(V) SL_2 \mathrm{Sym}^m(V)$

$$\mathrm{Sym}^d(V) \otimes \mathrm{Sym}^d(V) = \mathrm{Sym}^{2d}(V) \oplus \mathrm{Sym}^{2d-2}(V) \oplus \mathrm{Sym}^{2d-4}(V) \cdots$$
$$\mathrm{Sym}^2(\mathrm{Sym}^d(V)) = \mathrm{Sym}^{2d}(V) \oplus \mathrm{Sym}^{2d-4}(V) \oplus \mathrm{Sym}^{2d-8}(V) \cdots$$
$$\bigwedge^2 (\mathrm{Sym}^d(V)) = \mathrm{Sym}^{2d-2}(V) \oplus \mathrm{Sym}^{2d-6}(V) \oplus \mathrm{Sym}^{2d-10}(V) \cdots$$

$\mathrm{Sym}^m(V) = 0 m < 0$

SL_2

$$\mathrm{Sym}^{2d-4}(V) \oplus \mathrm{Sym}^{2d-8}(V) \cdots$$

$\mathbb{P}^3 \mathbb{P}^3$

d

$\mathbb{P}^1 \hookrightarrow C \subset \mathbb{P}^3 \mathcal{N}_{C/\mathbb{P}^3} T_{\mathbb{P}^3}|_C C \mathbb{P}^3 T_C$

$$\mathcal{N}_{C/\mathbb{P}^3} \cong \mathcal{O}_{\mathbb{P}^1}(5) \oplus \mathcal{O}_{\mathbb{P}^1}(5).$$

$p \in C L_p \subset \mathcal{N}_{C/\mathbb{P}^3} \mathcal{N}_{C/\mathbb{P}^3} q \neq p \in C (\mathcal{N}_{C/\mathbb{P}^3})_q \overline{p, q} \mathcal{N}_{C/\mathbb{P}^3} C \setminus \{p\} \mathcal{N}_{C/\mathbb{P}^3} C p \neq p'$

$$\mathcal{N}_{C/\mathbb{P}^3} = L_p \oplus L_{p'}.$$

$\mathbb{P}^1 \hookrightarrow C \subset \mathbb{P}^d \mathcal{N}_{C/\mathbb{P}^d}$

$$\mathcal{N}_{C/\mathbb{P}^d} \cong \bigoplus_{i=1}^{d-1} \mathcal{O}_{\mathbb{P}^1}(d+2).$$

$p \in C L_p \subset \mathcal{N}_{C/\mathbb{P}^d} p_1, \dots, p_{d-1} L_{p_i}$

$\mathcal{N}_{C/\mathbb{P}^d} \mathbb{P}^{d-2} \mathbb{P}^d C \mathbb{P}^{d-2}$

$SL(V) \mathrm{Sym}^{d-2} V$

$C = \bigcap_{i=1}^{r-1} X_i \subset \mathbb{P}^r C$

$r - 1 r - 1 \mathbb{C}[x_0, \dots, x_r] r + 1$

$CD3 = 2g + 1CdCd + 1DCDC$

$\mathbb{P}^2 = \mathbb{P}^2_{\mathbb{C}}$

$\mathbb{P}^1_{\mathbb{C}}$

$\mathbb{P}^r r \geq 3CC \subset \mathbb{P}^r \mathbb{P}^r \to \mathbb{P}^2 CC_0 CC_0 CC_0 CC_0 CC_0$

$D \subset C_0 D' C_0 C'_0 E := D' - DD' - E = DD'$

$I \subset S := k[x_0, \dots, x_n] S/IS/I$

$DCCD$

$F(X, Y, Z) dCdH^0(K_C)$

$D = D_+ - D\_C|D|C$

DC

$ECE \sim D$

$H^0(\mathcal{O}_C(D))$

$|D|$

$C \subset \mathbb{P}^2 F(X,Y,Z) dCCd-3Cd-3$

$x = X/Zy = Y/ZU \cong \mathbb{A}^2 Z \neq 0 f(x,y) = F(x,y,1)FC^\circ = C \cap UV(f) \subset \mathbb{A}^2$

$\mathbb{P}^2\mathbb{P}^2[0,1,0]C\pi : C \to \mathbb{P}^1(0,1,0)[X,Y,Z] \mapsto [X,Z](x,y) \mapsto xdDCZ = 0$

$dx\mathbb{A}^2 dx\mathbb{A}^1 dx\mathbb{P}^1 dx|_C 2D$

$dx\mathbb{P}^2 h(x,y)m\mathbb{A}^2 mLhdx/hh(x,y)C^\circ dx/hhC^\circ dxhdx$

$$h(x,y) = f_y := \frac{\partial f}{\partial y}(x,y).$$

$$\varphi_0 = \frac{dx}{f_y}$$

C°

dfC°

$$0 \equiv df|_{C^\circ} = f_x dx|_{C^\circ} + f_y dy|_{C^\circ}.$$

$\varphi_0 p f_y(p) \neq 0 dx|_{C^\circ} C^\circ dy|_{C^\circ} \neq 0 \varphi_0 f_y(p) = 0 C^\circ f_x(p) \neq 0 dx|_{C^\circ} f_y \varphi_0 = (dx/f_y)|_{C^\circ}$ $\blacksquare$

$LU = \mathbb{P}^2 \setminus L \cong \mathbb{A}^2 U \Omega_U = \mathcal{O}_U dx \oplus \mathcal{O}_U dy, \Omega_{C^\circ} C^\circ \omega_{C^\circ} \mathcal{O}_C(-d)|_U \Omega_U C^\circ f$
$f_x dx + f_y dy \in \Omega_U f_x f_y d_y \in \Omega_{C^\circ} dx/f_y \mathcal{O}_{C^\circ} \omega_C|_{C^0} dx/f_y$

$f_y d - 11/f_y d - 1D\varphi_0 d - 3D$

$$(\varphi_0) = (d-3)D.$$

$d \geq 3\varphi_0 Cd(d-3)gC2g-2 = d(d-3)$

$$g = \frac{d(d-3)}{2} + 1 = \binom{d-1}{2}.$$

$\binom{d-1}{2}\varphi_0 e(x,y)d-3$

$Cf = 0$

$$\left\{\frac{e(x,y)dx}{f_y} \mid \leq d-3\right\}. \quad \square$$

$CD = D_+ - D\_DD + (H/G)G, HmmGmD_+ AC$

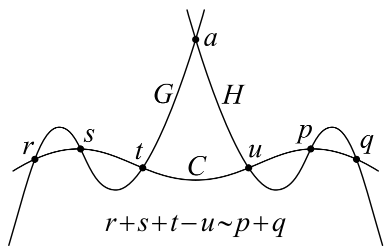
$D = D_+ - D\_CGmD_+(G) = D_+ + AD(H) - A - D\_HmD\_ + AC$

$HmA + D\_CDHmG$

C

$$H^0(\mathcal{O}_{\mathbb{P}^n}(m)) \to H^0(\mathcal{O}_C(m))$$

m



$GHr + s + t - u \sim p + q$

$CDmm$

$CF = 0DCLmCGHtCD + (G/H)(LG)C(H)C\mathbb{P}^2(LG, F)(H, F)HFLG = AH + BFA, B\deg A = \deg LG - \deg H = mD + (G/H) = (A)$ $\square$

$CCopq \in Cp + q - om = 1Lp + qGr \in CLCHo + rCsCC$

$HmD' = (H) - (D_- + A)$

$$D' = D + (H/G) = D_+ - D_- - (D_+ + A) + (D_- + A + D')$$

$D'D$

$D' \sim DD'D' \sim D$

$$\mathcal{O}_C(A + D_- + D') = \mathcal{O}_C(A + D_- + D) = \mathcal{O}_C(m),$$

$A + D_- + D' \equiv (G)GmA + D_+ = (H)m$ $\square$

$F(X, Y, Z) = 0Q(X, Y, Z) = 0\Gamma\mathbb{P}^2E(X, Y, Z) = 0E = QH + LFHL$

$CdE', E''CE := E' + E'' = C \cap C'CC'd'0 \leq k \leq d + d' - 3kE''EE'E'd'' := d + d' - 3 - kE'd''$

$HCCs := d + d' - 3 - k$

$$h^0(kH - E) - h^0(kH - E') = \deg E'' - \big(h^0(sH) - h^0(sH - E'')\big)$$

$e' := \deg E', e'' := \deg E''e = e' + e'' = \deg E.CK := (d - 3)H$

$$kd - e - h^0(K - (kH - E)) - \big(kd - e' - h^0(K - (kH - E'))\big).$$

$K - (kH - E) = K - (kH - d'H) = sHK - (kH - E') = sH + E''e'' - h^0(sH) + h^0(sH + E'')$ $\square$

C

$E'C \subset \mathbb{P}^2 d r E' r d - 3$

$d' C' d' E C E = E' + E'' E'' E' C \cap C' E + E' C C' d' E C r |E'| d' E + E' E' d'' = d + d' - d' - 3 = d - 3$ $\square$

$1 \leq n \Gamma k \leq n \mathbb{P}^2 n - 1 \Gamma \Gamma n - 2$

$\Gamma d p \in \Gamma d$

$$\Gamma_p := \Gamma \setminus \{p\}$$

$p \Gamma d \Gamma d + 1 k = n$

$\Gamma_p p$

$\Gamma p \in \Gamma p \in \Gamma n - 2 \Gamma_p p \Gamma \mathbb{P}^2 p, q, r \Gamma q, r n - 3 \Gamma_p \setminus \{q_1, q_2\} n - 2 \Gamma_p p$ $\square$

$C \subset \mathbb{P}^2 d \geq 3$

$\mathcal{V} g^1_e C e \leq d - 1 e = d - 1 \mathcal{V} C$

$\mathcal{V} g^2_e C e \leq d \geq 4 e = d C \mathbb{P}^2 \mathbb{P}^2$

$E \mathcal{V} E d - 3 E d - 1 L C p |E| p p$

$\mathcal{V} d - 3 d - 2 E d L \mathcal{V} C$ $\square$

$d = 2, 3, 45$

$E \mathcal{O}_C h^0(\mathcal{L}) = \deg \mathcal{L} \mathcal{L} o \in E E o o$

$E o o \in E p + q = r r p + q - o E D n > 0 p D \sim n p$

$E o \in E p, q E \mathcal{O}_E(p + q - o) r p + q - o p + q = r r 2 o - p p + r - o \sim o r = -p D \Sigma D \Sigma D - (\deg D - 1) o \Sigma D = \Sigma D'$

$r \sim p + q - o E \mathbb{P}^2 L p + q L C s s p q s o C r r \sim p + q - o r s o$

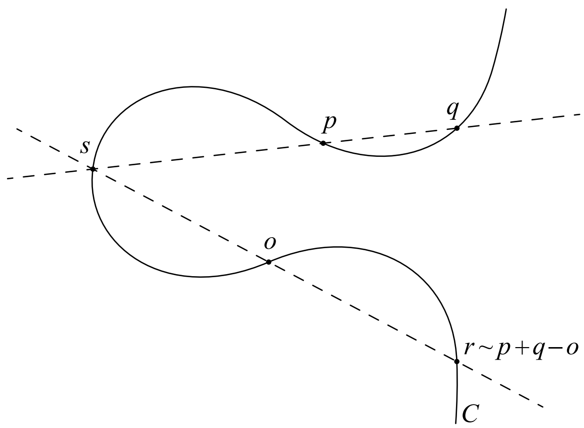
$E n : E \to E n p n p = q n \neq 0$ $\square$

$C \mathbb{P}^2 |3o| p, q, r \in C p + q + r = o$

$E \to \mathrm{Pic}_0(E) p \mathcal{O}_E(p - o) o \mathrm{Pic}_d(E) C \mathrm{Pic}_0(C) C g g \mathrm{Pic}_g(C)$

$\mathcal{L}, \mathcal{L}' E \sigma : E \to E \sigma^* \mathcal{L} = \mathcal{L}'$

$\mathcal{L} \cong \mathcal{O}_E(n p) \mathcal{L}' \cong \mathcal{O}_E(n p') p, p' p - p' E$ $\square$



p, qo

$gM_g gM_g$

M_1

$\mathbb{P}^1$ $E\mathcal{L}Eh^0(\mathcal{L}) = 2|\mathcal{L}|\phi : E \to \mathbb{P}^1\phi\mathbb{P}^1E\mathcal{L}$

$E\mathbb{P}^3\mathbb{P}^1EM_1$

$\mathbb{P}^1\mathrm{H}\mathbb{P}^1\mathrm{H} \to M_1PGL_2\mathbb{P}^1M_1 4 - 3 = 1$

$\mathcal{L}E|\mathcal{L}|E$

$\mathbb{P}^9|\mathcal{L}|, |\mathcal{L}'|EE\mathcal{L} = \mathcal{L}'PGL_3\mathbb{P}^2PGL_3$

$E\mathbb{P}^2E \subset \mathbb{P}^2DE \subset \mathbb{P}^2DEDD'DD'DD'EDDnEn - 1\dim|D| = \deg D - 1D$

$\mathbb{P}^3$ $EE\mathbb{P}^3EDEED + D'D'|D|D'D'\mathbb{P}^2D'\mathbb{P}^2\mathbb{P}^3\mathbb{P}^3D'D'\mathbb{P}^3D' = p + qpqLpqrLL\mathbb{P}^3$

E

$E\mathbb{P}^3\mathbb{P}^3$

$$\rho_2 \ : \ H^0(\mathcal{O}_{\mathbb{P}^3}(2)) \ \to \ H^0(\mathcal{O}_E(2)) = H^0(\mathcal{L}^2).$$

$H^0(\mathcal{L}^2)EQQ'EQQ'Q \cap Q'$

$$E = Q \cap Q'.$$

$Q, Q'(Q, Q')I(E)$

$E := Q \cap Q'QQ'EQ_0EQ_0E(2,2)Q_0$ $\square$

$\mathbb{P}^3 G(2, H^0(\mathcal{O}_{\mathbb{P}^3}(2))) = G(2,10) G(2,10) PGL_4 \mathbb{P}^3$

$E \mathbb{P}^3 E \mathbb{P}^1 E A, B E s A + t B (s,t) det(sA + tB)$

$$\Phi := \{(\lambda, \mathcal{L}) \mid \mathcal{L} \in \mathrm{Pic}(Q_\lambda) Q_\lambda\}$$

$\mathbb{P}^1 \lambda$

$\Phi E \Phi \mathbb{P}^1 \mathbb{P}^1 \mathbb{P}^1 E$

$o \in E E \to \Phi \Phi \to E$

$q \in E o M \subset \mathbb{P}^3 \overline{o, q} o, q Q_\lambda o, q \in M Q_\lambda r \in M \lambda r M Q_\lambda q Q_\lambda E \to \Phi$

$Q_\lambda Q_\lambda M \subset Q_\lambda o M E q \Phi \to E$

$\square$

$Q \subset \mathbb{P}^{2g+1} g 2g+1 \mathbb{P}^{2g} \{Q_\lambda\}_{\lambda \in \mathbb{P}^1} X = \cap_{\lambda \in \mathbb{P}^1} Q_\lambda 2g+2 2g+1 \Phi Q_\lambda \mathbb{P}^1 2g+2 g \det(sA + tB) 2g+2$

$g F_{g-1}(X)(g-1) X \Phi g = 2g$

$\mathbb{P}^4$ $D E \subset \mathbb{P}^2 \phi_D : E \hookrightarrow \mathbb{P}^4 |D| D D D C D C \cap E D + p p \in E |D| p E p$

$\phi_D(E) X \mathbb{P}^2 p \mathbb{P}^2 p X p \deg X = 3 X \mathbb{P}^2 p p \phi_D(E) E \cap L p g^1_2 E p$

$\phi_D(D) D \mathbb{P}^2 E' D E E \cap E' = D + D' D' \subset E 9 - 5 = 4 |D| E D'$

$\mathbb{P}^2 D' E 6 \mathbb{P}^2 Y \subset \mathbb{P}^5 E D' \phi_D(E) Y Y \mathbb{P}^5 \mathbb{P}^2 D' \mathbb{P}^5 Y \binom{6+2}{2} - 3 * 4 = 16 Y Y Y 5 = \binom{5+2}{2} - 16$

$\phi_D(E)$

$$H^0(\mathcal{O}_{\mathbb{P}^4}(2)) \to H^0(\mathcal{O}_{\phi_D(E)}(2)) = H^0(\mathcal{O}_E(5)).$$

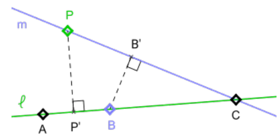
$\phi_D(E) Y$

$Y \phi_D(E)$

$\mathbb{P}^4$ $AAAA$

$$M = \begin{pmatrix} 0 & x_{1,1} & x_{1,2} & x_{1,3} \\ -x_{1,1} & 0 & x_{2,2} & x_{2,3} \\ -x_{1,2} & -x_{2,2} & 0 & x_{3,3} \\ -x_{1,3} & -x_{2,3} & -x_{3,3} & 0 \end{pmatrix}$$

$M x_{1,1} x_{3,3} - x_{1,2} x_{2,3} + x_{1,3} x_{2,2} \tilde{A} 5 \times 5 I 4 \times 4 A$



$I\tilde{Y}\mathbb{P}^9Y\phi_D(E)\tilde{Y}I(\phi_D(E))5\times 5$

$$B=\begin{pmatrix} 0 & 0 & x_0 & x_1 & x_2\\ 0 & 0 & x_1 & x_2 & x_3\\ -x_0 & -x_1 & 0 & \ell_1 & \ell_2\\ -x_1 & -x_2 & -\ell_1 & 0 & \ell_3\\ -x_2 & -x_3 & -\ell_2 & -\ell_3 & 0\end{pmatrix}$$

$\phi_D(E)4\times 4X2\times 22\times 3AB$

$CdCC\to\mathbb{P}^1d-1CC\to\mathbb{P}^1d-2$

$d-2d-3$

$\mathbb{P}^2$

$p\in CF(x_0,x_1,x_2)=0d>1CpCm+2Cm$

$p=(0,0)\in\mathbb{A}^2\subset\mathbb{P}^2\mathbb{A}^2x_0\neq 0f(x,y)=0Cx=x_1/x_0, y=x_2/x_0$

$$\det Hess(C)=$$

$$x_0^2\det\begin{pmatrix} d(d-1)F & (d-1)\partial F/\partial x_1 & (d-1)\partial F/\partial x_2\\ (d-1)\partial F/\partial x_1 & \partial^2F/\partial x_1\partial x_1 & \partial^2F/\partial x_1\partial x_2\\ (d-1)\partial F/\partial x_2 & \partial^2F/\partial x_2\partial x_1 & \partial^2F/\partial x_2\partial x_2\end{pmatrix}.$$

$$\det\begin{pmatrix} d(d-1)f & (d-1)f_x & (d-1)f_y\\ (d-1)f_x & f_{xx} & f_{xy}\\ (d-1)f_y & f_{xy} & f_{yy}\end{pmatrix}$$

$\mathbb{A}^2$

$Cpy=0f$

$$f=x^{m+2}\phi(x)+y\psi(x,y)$$

$\phi(0)\neq 0\psi(0,0)\neq 0f$

$$f_x^2f_{yy}+f_y^2f_{xx}-2f_xf_yf_{xy}.$$

$xpfmp$

$\mathbb{P}^2_{\mathbb{C}}\mathbb{P}^2_{\mathbb{R}}$

$\Gamma \subset \mathbb{P}^2_{\mathbb{R}} \Gamma$

$\Gamma \subset \mathbb{R}^2 P, LP \in Gamma \ell \Gamma PLp \Gamma L \Gamma L'$

C

$$\{d\} \xrightarrow{\mu} \{d\}.$$

$D\mathcal{O}_C(D)$

μ

$CdCD \subset CdC_d dCdC_d C$

$Cd\mathrm{Pic}_d(C)CJac(C)\mathrm{Pic}_0(C)$

$\pi : \mathrm{C} \to B\mathrm{Pic}_d(\mathrm{C}/B) \to Bb \in B\mathrm{Pic}_d(C_b)C_b gg$

$C_d \mathrm{Pic}_d g + 3\mathrm{g}\mathbb{P}^3 \mathrm{g} + 3\mathrm{g} + 2g + 1$

CC

$BdCBX \subset B \times C\{b\} \times C \cong CBdC$

$$F : (schemes) \to (sets),$$

$BdCB\pi : B' \to BF(B) \to F(B')\mathrm{D} \subset B \times C\mathrm{D}' := B' \times_B \mathrm{D} \subset B' \times CC_d dC$

$$F \cong \mathrm{Hom}_{Schemes}(-, C_d).$$

$\mathrm{D} \subset C_d \times CX \subset B \times CCB\phi : B \to C_d X = (\phi \times id_C)^{-1}(\mathrm{D})$

dC

$GX := \operatorname{Spec} AX/G\operatorname{Spec}(A^G)A^GAA^G$

$\pi : X \to X/GX\pi G$

$X \to X/G\dim X/G = \dim X$

G

G

$XdX^{(d)}XX^ddX\pi_d : X^d \to X^{(d)}$

$$\mathrm{D} = \{(x,y) \in X^{(d)} \times X \mid y\pi_d^{-1}(x)\};$$

$x \in X^{(d)}dX\mathrm{D}xx$

$X = \mathbb{A}^1X^d = \mathbb{A}^d = \operatorname{Spec}\mathbb{C}[x_1,\dots,x_d]\mathbb{C}[x_1,\dots,x_d]dX^{(d)}\mathbb{A}^dX^{(d)}d\mathbb{C}[z]$

$X = \mathbb{P}^1(\mathbb{P}^1)^di\mathbb{P}^1(s_i,t_i)d+1d$

$$t_1t_2\cdots t_d, \sum_i s_it_1\cdots\hat{t}_i\cdots t_d, \dots, s_1\cdots s_d$$

$$(s_1\lambda + t_1\mu)(s_2\lambda + t_2\mu)\cdots(s_d\lambda + t_d\mu)$$

$\mathbb{P}^1d(\mathbb{P}^1)^{(d)} \to \mathbb{P}^ddd(\mathbb{P}^1)^{(d)}Fd\mathrm{D} \subset \mathbb{P}^1 \times \mathbb{P}^d \to \mathbb{P}^dFF$

$CC^{(d)}$

$CC^{(d)}\mathrm{D} \subset C \times C^{(d)}$

$\{p_1,\dots,p_s\}\overline{p} \in C^{(d)}C^dGXp_iG\{p_1,\dots,p_s\}\mathbb{A}^1(\mathbb{A}^1)^{(d)}$ $\square$

$\dim X \geq 2X^{(d)}d \geq 2(\mathbb{A}^2)^{(2)}$

$C^{(d)}dC$

$CdC^{(d)}CC_ddC\mathrm{D} \subset C \times C_d \to C_d$

B

$C_dC^{(d)}$

$C_d\operatorname{Pic}_d(C)C$

$BCB\mathcal{L}B \times C\mathcal{L}\mathcal{L}'\pi_1 : B \times C \to B$

$$\mathcal{L}' \cong \mathcal{L} \otimes \pi_1^*\mathrm{F}$$

$\mathrm{F}B\mathcal{L}d\mathcal{L}\{b\} \times Cdb \in B$

$p \in C \mathrm{F} B \pi_1^*(\mathrm{F}) \mid_{B \times p} = \mathrm{F} \mathcal{L} B \times \{p\}$

$$Pic_d : (schemes) \to (sets)$$

$BdB \times CBPic_0$

$\mathrm{Pic}_d(C) dC \mathrm{Pic}_d(C) \mathrm{P} d \mathrm{Pic}_d(C) \times CBdBBB \to \mathrm{Pic}_d(C) \mathrm{P} \mathrm{Pic}_0(C) \mathrm{Pic}_d(C) \mathrm{P} C$

$\mathrm{P} \mathrm{Pic}_0(C) \mathrm{Pic}_0(C) \mathrm{Spec}\, \mathbb{C} C \mathrm{Spec}\, \mathbb{C} d \mathrm{Pic}_d(C) \mathrm{Pic}_0(C)$

$\mathrm{Pic}_d(C) \times CC\mathcal{C} \to S\sigma : S \to \mathcal{C} \mathrm{Pic}_d(\mathcal{C}/S) \to SdSB \to S$

$\mathcal{L} eCdBd + eBB\pi_2^* \mathcal{L} \mathrm{Pic}_d(C) \cong \mathrm{Pic}_{d+e}(C) \mathcal{L}$

$C_d \mathrm{Pic}_d$

$$C_d \to \mathrm{Pic}_d \quad D \mapsto \mathcal{O}_c(D)$$

$\mathrm{D} \subset B \times C \mathcal{O}_{B \times C}(\mathrm{D})$

$\mathrm{Pic}_d(C) \mathrm{Pic}_d(C) Jac(C)$

$$\int_{t_0}^{t} \frac{dx}{\sqrt{x^2+1}}.$$

$y^2 = x^2 + 1 \mathbb{P}^1 xyz$

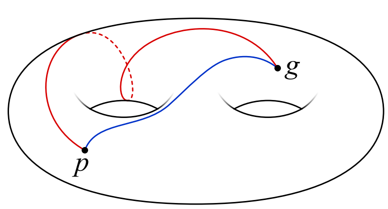
$$\int_{s_0}^{s} R(z) dz,$$

R

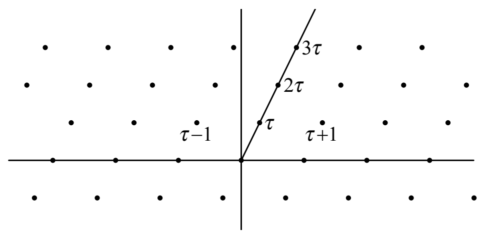
$$\int_{t_0}^{t} \frac{dx}{\sqrt{x^3+1}},$$

$y^2 = x^3 + 1 \mathbb{P}^1 H_1(C, \mathbb{Z}) \cong \mathbb{Z}^2 t \mathbb{Z}^2 \subset \mathbb{C} \mathbb{C} \mathcal{P}$

$p_0, p \in CH_1(C, \mathbb{Z}) \cong \mathbb{Z}^{2g} \mathbb{Z}^2 \int_p^q \omega \omega$



pq



$\mathbb{C}\mathbb{Z} \oplus \mathbb{Z}\tau$

$C g \mathbb{C} \omega_C C C \omega_C$

C

$$\iota : \mathbb{Z}^{2g} = H_1(C, \mathbb{Z}) \;\to\; H^0(\omega_C)^* \cong H^1(\mathcal{O}_C) = \mathbb{C}^g.$$

$$H^1(C, \mathbb{C}) \cong H^1(C, \mathcal{O}_C) \oplus \overline{H^1(C, \mathcal{O}_C)}$$

$H^1(C, \mathbb{C}) \iota H_1(C, \mathbb{C}) = \mathbb{C} \otimes_{\mathbb{Z}} H_1(C, \mathbb{Z}) H_1(C, \mathbb{Z}) H^1(C, \mathbb{C}) H^1(C, \mathbb{C})$

$\iota H^0(\omega_C)^* 2g H^0(\omega_C)^* \mathbb{C}^g / \iota(\mathbb{Z}^{2g}) g$

$J(C) := \mathbb{C}^g / \iota(\mathbb{Z}^{2g}) C$

$p, q \in C \int_q^p H^0(\omega_C) J(C) p - q p - q \mapsto \int_q^p \mu_0 J(C) p', q'$

$$\int_q^p + \int_{q'}^{p'} - (\int_q^{p'} + \int_{q'}^p)$$

$q \to p \to q' \to p' \to q$

$p \in C$

$$\mu \;:\; C \;\to\; J(C); \quad q \mapsto \int_p^q$$

$$\mu_d \;:\; C_d \;\to\; J(C) : \qquad (q_1, \ldots, q_d) \mapsto \sum_i \int_p^{q_i}.$$

$d \mu \mu(-D) - \mu(D) \mu D, E \mu(D + E) = \mu(D) + \mu(E)$

$D, D' C d \mu(D) = \mu(D') \mu_d d C \mu_0 \mathrm{Pic}_0(C) \to J(C) p \mathrm{Pic}_d(C) \to Jac(C)$

$$\mathrm{Pic}_d(C) \xrightarrow{-\otimes \mathcal{O}_C(-dp)} \mathrm{Pic}_0(C) \to J(C).$$

$DD' \mathcal{O}_C(D) \cong \mathcal{O}_C(D') \mathcal{L} DD' \sigma, \sigma' \in H^0(\mathcal{L}) \sigma \sigma' \{D_\lambda\}_{\lambda \in \mathbb{P}^1} C$

$$D_\lambda = V(\lambda_0 \sigma + \lambda_1 \sigma'),$$

$\alpha : \mathbb{P}^1 \to C_d$

$$\phi = \mu \circ \alpha : \mathbb{P}^1 \to J(C).$$

$zV = H^0(\omega_C)^* dz V J(C) V J(C) J(C) \omega J(C) \phi^* \omega \mathbb{P}^1 d\phi \phi \mu(D) = \mu(D')$ $\square$

$C g n \mathcal{O}_C n^{2g}$

$n \mathbb{C}^g / \mathbb{Z}^{2g} \cong (S^1)^{2g}$ $\square$

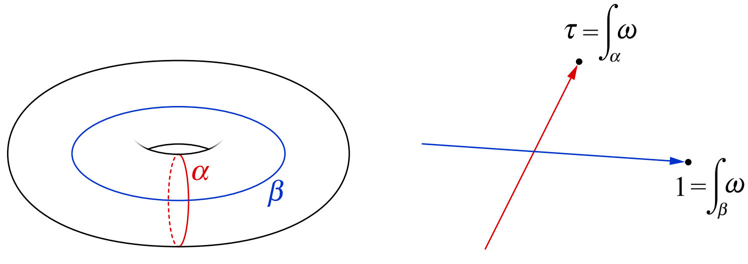
$C g \mu_g : C_d \to J(C) d \geq g d \leq g \mu_g : C_g \to J(C)$

$d \leq g = \dim H^0(\omega_C) D d p_1, \ldots, p_d \in C \omega_C$

$$h^0(\omega_C(-D)) = g - d.$$

$h^0(\mathcal{O}_C(D)) = 1 \mu_d$

$d \geq g h^0(\omega_C(-D)) = 0 r(D) = d - g = \dim C_d - \dim J(C) C_d g C_d J(C) \mu_d$ $\square$



$\mu \Lambda = \mathbb{Z}\langle w_1, w_2 \rangle$

$\mu_1 : C \to \mathrm{Pic}_1(C) p \in C g > 0 C p \kappa(p) \otimes \omega_C Pic_1 \mu_1(p) H^0(\omega_C) \kappa(p) \cong \mathbb{C} p$

$$T^*(\mu_1) : H^0(\omega_C) \to \kappa(p) \otimes \omega_C$$

$p H^0(\omega_C(-p))$

$g(C) > 0 p, q \in C \mu_1$

$\mu_1 p \in C z p p f(z) dz C p \kappa(p) \otimes \omega_C \cong \mathbb{C} dz J(C) = H^0(\omega_C)^* / H_1(C, \mathbb{Z}) H^0(\omega_C)^*$ $H^0(\omega_C)$

$\mu_1 q$

$$\mu_1 : C \to J(C) := H^0(\omega_C)^* / H_1(C, \mathbb{Z})$$

$$q \mapsto \int_p^q .$$

$q \in C p$

$$\frac{\partial}{\partial q} \int_p^q f(z) dz = f(q).$$

$T^*(\mu_1) \mu_1 H^0(\omega_C) \to \kappa(p) \otimes \omega_C p T^*(\mu_1) p p H^0(\omega_C(-p)) \omega_C p T^*(\mu_1) \mu_1$ $\square$

$\mu_1 : C \to Pic_1(C) \mu_1(p) \mu_1(C) p$

$T^*(\mu_1)(p) H^0 \omega_C(-p)) \subset H^0(\omega_C) \mu_1 C p H^0(\omega_C)^* H^0(\omega_C(-p)) \subset H^0(\omega_C) p \mathbb{P}(H^0(\omega_C)) =$ $\mathbb{P}^{g-1}$ $\square$

$g + 3$

$d C g$

$\geq 2g + 1 C d \leq 2g D d C$

$g + 3$ $C g D \in C_{g+3} g + 3 C D g \mathbb{P}^3 g + 3$

$g \geq 2 C \leq g + 2 C$

$C C \mathbb{P}^3 d = \lceil 3g/4 \rceil + 3 \mathbb{P}^2 d = \lceil 2g/3 \rceil + 2$

$D g + 3 D h^0(\mathcal{O}_C(D)) = 4 D h^0(\mathcal{O}_C(D - F)) = 2 F = p + q C$

$h^0(\mathcal{O}_C(D - F)) \geq 3 h^0(\omega_C(-D + F)) \geq 1 K_C - D + F E g - 3$

$$\nu : C_{g-3} \times C_2 \ \to \ J(C)$$

$$\nu : (E, F) \ \mapsto \ \mu_{g-3}(K_C - E + F) = \mu_{2g-2}(K_C) - \mu_{g-3}(E) + \mu_2(F),$$

$+ - J(C)$

$D\mu(D) \notin Im(\nu)\nu g - 3 + 2 = g - 1J(C)\mu_{g+3}D \in C_{g-3}J(C)Im(\nu)$

$g + 3C\geq 2d \geq 4\omega_C = \mathcal{O}_C(d-3)$

$Dh^0(D) \geq 3\deg D \leq g + 2DDrD_0D_0 g_2^1 r \leq (g+2)/2D_0(g+1)D_0 g \geq 2$ $(g+2)/2 < g + 1D$ □

$g + 2\mathbb{P}^2$

$g + 2$ $CgDg + 2C$

$C\phi_D : C \to \mathbb{P}^2 C_0 C_0 g + 2\binom{g}{2}$

$C\phi_D : C \to \mathbb{P}^2 C_0 C_0 g + 2g$

$W_d^r(C)$

$\mathrm{Pic}_d(C)d\geq r$

$$W_d^r(C) := \{\mathcal{L} \in \mathrm{Pic}_d(C) \mid h^0(\mathcal{L}) \geq r+1\}.$$

$W_d^0(C)\mu : C_d \to \mathrm{Pic}_d(C)W\,_d(C)$

$W_d^r(C)$

$$W_d^r(C) = \left\{\mathcal{L} \in \mathrm{Pic}_d(C) \mid \dim(\mu^{-1}(\mathcal{L})) \geq r\right\}.$$

$\mu : C_d \to \mathrm{Pic}_d(C)W_d^r(C)W_d^r(C)\mu^{-1}(W_d^r) \subset C_d Dr(D) \geq rC_d^r$

$W_d^r(C)r + 1B\mathcal{L}B \times C(\pi_2)_*\mathcal{L}r + 1BBCW_3^1(C)CW_4^1(C)\mathrm{Spec}\,\mathbb{C}[\epsilon]/(\epsilon^5)$

$C\mu : C \to \mathrm{Pic}_1(C)\mathrm{Pic}_d(C) \cong Cdp \in CC \to \mathrm{Pic}_0(C)q \in C\mathcal{O}_C(q-p)$

$\mathbb{C}\Lambda \subset \mathbb{C}$

$Ep \in E\phi : E \to E\phi(p) = p$

$pC\mathbb{C}/\Lambda\Lambda.\phi\tilde{\phi} : \mathbb{C} \to \mathbb{C}\Lambda$ □

$\mu_2 : C_2 \to J(C)\Gamma \subset C_2 g_2^1 CC_2 CJ(C)g_2^1$

$C\mu_2 C\mu_2 C_2 g_2^1 W_2(C)C\mu_2 W_2(C)$

$\mu_3 = \mu_g J(C) \cong \mathrm{Pic}_3(C)W_3^1(C)$

$$W_3^1(C) = K - W_1(C),$$

$\mathcal{L}g_3^1\omega_C \otimes \mathcal{L}'\mathcal{L}'h^0(\mathcal{L}') = 1W_3^1 CC_3 J(C)C$

$Cgr \leq d - gW_d^r = Pic_d Cgd \leq 2g - 2W_d^r(C)d - 2r < 0\dim W_d^r \leq d - 2rr > d - g$

$Cgr > d - g$

$$\dim(W_d^r(C)) \leq d - 2r;$$

$rd = 2g - 2r = g - 1W_d^r = \{\omega_C\}d = r = 0W_d^r = \{\mathcal{O}_C\}Cdrd \geq 2r > 2(d - g)$

$C \subset \mathbb{P}^r \Sigma_d^r \subset C_d DdC\dim \overline{D} \leq d - r - 1\Sigma_d^r$

$C \subset \mathbb{P}^r d \leq rr \geq 0$

$$\dim \Sigma_d^r \leq d - r.$$

$\Sigma_d^0 = C^d r = 0$

$D \in \Sigma_d^r r > 0\dim \overline{D} \leq d - r - 1 \leq d - 2\Lambda DD' := D - pd - 1Dq \notin \Lambda \cap C$
$q + D' \leq d - rq + D' \in \Sigma_d^{r-1}.D$

$$\{q + D' \mid q \notin \Lambda \cap C\}.$$

$\Sigma_d^r \setminus \Sigma_d^{r-1}\Sigma_d^{r-1}$

$$\dim \Sigma_d^0 > \dim \Sigma_d^1 > \cdots > \dim \Sigma_d^r,$$

$\dim \Sigma_d^r \leq d - r$ $\square$

$r(\mathcal{L}) = s\mathcal{L}_0^{\otimes s}(p_1 + \cdots + p_{d-2s})\mathcal{L}_0 g_2^1 C p_i d - 2s$

$r > d - gW_d^r \mathcal{L} h^1(\mathcal{L}) \neq 0\mathcal{L} \to \omega_C \otimes \mathcal{L}^{-1}$

$$\{\mathcal{L} \in W_d^r(C) \mid h^1(\mathcal{L}) \neq 0\} = \{\mathcal{L} \in W_{2g-2-d}^{g-1-d+r} \mid h^1(\mathcal{L}) \neq 0\}$$

$$2g - 2 - d - 2(g - 1 - d + r) = d - 2r,$$

$d \leq g - 1$

$C_d^r \subset C_d W_d^r(C) C_d^r W_d^r r$

$$\dim C_d^r \ \leq \ d - r.$$

$\phi_K : C \to \mathbb{P}^g - 1r(D) = \deg D - 1 - \dim \overline{\phi_K(D)} D \in C_d^r \dim \overline{\phi_K(D)} \leq d - r - 1$
$d \leq g - 1D \leq d - r$ $\square$

$Cg_2^1 = |D| r \leq g - 1\mu(rD) \in W_{2r}^r$

$$W_d^r(C) \supset W_{d-2r}(C) + \mu(rD).$$

$W_{d-2r}(C)d - 2r$

$\dim(W_d^r(C)) < d - 2r$

GXX

$GXgx = gygx = yGXX/G$

$X = (\mathbb{A}^2)^2 G := \mathbb{Z}/2X\mathbb{A}^2(\mathbb{A}^2)^2 = \operatorname{Spec} SS = k[x_1, x_2, y_1, y_2]\sigma \in G\sigma(x_i) = y_i$

G

$\neq 2S^G A^2 z_i := x_i + y_i, w_i := x_i - y_i$

$(\mathbb{A}^2)^2$

$MXF, GX\eta : F \to Ga \in X\eta_a : F(a) \to G(a)f : a \to bXG(f) \circ \eta_a \eta_b \circ F(f)$

XFX

$$\operatorname{Hom}_X(\operatorname{Hom}_X(-, Z), F) = F(Z)$$

$\operatorname{Hom}_X(-, Z)\operatorname{Hom}_X(-, Z')Z \cong Z'X\operatorname{Hom}_X(-, Z)Z$

$r \geq d - gW_d^r(C) \setminus W_d^{r+1}(C)W_d^r(C)W_d^{r+1}(C)W_d^r(C)$

$\mathcal{L}dh^0(\mathcal{L}) > d - g + 1p, q \in Ch^0(\mathcal{L}(p - q)) = h^0(\mathcal{L}) - 1$

$CC \subset J(C)\mu_1 C$

$C \subset J(C)$

$C + p \subset C_2\mu_2$

$\mu_2 : C_2 \to J(C)J(C)$

$C\nu : C \times C \to \operatorname{Pic}_0(C)(p, q) \in C \times C\mathcal{O}_C(p - q)$

$p, q \in C(p', q') \neq (p, q)p - q \sim p' - q'$

$C_d^r W_d^r r C_d \to J(C)C_d \setminus C_d^1 \mathbb{P}^{g-1} D$

$g + 1 \;\; CgD \in C_{g+1} g + 1CD\mathbb{P}^1 2g + 2$

$\geq g + 1h^0(\mathcal{O}_C(D)) = (g + 1) - g + 1 = 2|D|DD' + ph^0(D')gh^0(D') = 1 + h^0(K - D')D'K - D'K - D'g - 2C_{g-2}g - 2D' + ph^0(D') \geq 2g - 2 + 1Dg + 1$ $C \to \mathbb{P}^1$

$2g + 2D' + 3pD'' + 2p + 2qg - 1C_{g-2} \times CC_{g-3} \times C_2 D$

$C\phi_K : C \to \mathbb{P}^1 C\mathbb{P}^1 \tau : C \to C\phi_K p \in Ch^0(K_C(-p)) = 1\tau p\sigma \in H^0(K_C(-p))$ $\Gamma \subset C \times C\tau$

$C_2^1 \Gamma C_2 - 1$

$\Gamma C \times C$

$g \geq 2g - 1\mathbb{P}^{g-1}g^1_2$

$C \geq 2\pi : C \to \mathbb{P}^1\mathbb{P}^1C\mathbb{P}^1z \mapsto z^2 2g + 2\mathbb{P}^1$

$C\mathbb{P}^1 2g + 2\{q_1, \dots, q_{2g+2}\}$

$q_i \in \mathbb{P}^1\lambda_i C$

$$C^\circ = \Big\{(x, y) \in \mathbb{A}^2 \;\mid\; y^2 = \prod_{i=1}^{2g+2}(x - \lambda_i)\Big\}.$$

$x\mathbb{P}^1 x = \infty q_i \infty \in \mathbb{P}^1 r, s \in C x \to \infty y^2/x^{2g+2} \to 1$

$$\lim_{x\to\infty} \frac{y}{x^{g+1}} = \pm 1.$$

$r, s \in C$

$CC^\circ \subset \mathbb{A}^2\mathbb{P}^2\mathbb{P}^1 \times \mathbb{P}^1$

$\square$

$C2g + 2g + 1q_1, \dots, q_{g+1}q_{g+2}, \dots, q_{2g+2}C\mathbb{P}^1 \times \mathbb{P}^1$

$$\Big\{(x, y) \in \mathbb{A}^2 \;\mid\; y^2 \prod_{i=1}^{g+1}(x - \lambda_i) = \prod_{i=g+2}^{2g+2}(x - \lambda_i)\Big\};$$

$$C \;=\; \{((X_0,X_1),(Y_0,Y_1)) \in \mathbb{P}^1\times\mathbb{P}^1 \;\mid\; Y_1^2 \prod_{i=1}^{g+1}(X_1-\lambda_i X_0) = Y_0^2 \prod_{i=g+2}^{2g+2}(X_1-\lambda_i X_0)\}.$$

$C \subset \mathbb{P}^1 \times \mathbb{P}^1 (2, g+1) \mathbb{P}^1 \times \mathbb{P}^1 \mathbb{P}^1 \times \mathbb{P}^1 g$

$C p_1, \dots, p_{2g+2} \in C C \to \mathbb{P}^1 \{1, \dots, 2g+2\} A, B g+1$

$$\sum_{i\in A} p_i \;\sim\; \sum_{i\in B} p_i.$$

$C \subset \mathbb{P}^1 \times \mathbb{P}^1 p_i g+1 p_1, \dots, p_{2g+2} \{p_1, \dots, p_{g+1}\} \{p_{g+2}, \dots, p_{2g+2}\} (0,1)(1,0)$ □

$\iota : C \to C C \to \mathbb{P}^1 C ((X_0,X_1),(Y_0,Y_1)) \mapsto ((X_0,X_1),(Y_0,-Y_1)) \iota C$

$C C^\circ = V(f) \subset \mathbb{A}^2$

$$f(x,y) = y^2 - \prod_{i=1}^{2g+2}(x-\lambda_i).$$

$C \setminus C^\circ r s r + s D \pi : C \to \mathbb{P}^1 C^\circ (x,y) \in C x$

$C dx \mathbb{P}^1 \pi^* dx C x C^\circ \lambda_i \pi \pi^* dx C^\circ q_i = (\lambda_i, 0) dx \infty \in \mathbb{P}^1 \pi r s \pi^* dx r s C$

$$(*) \qquad K_C \sim (dx) \sim R - 2D,$$

R

$C dx x^2 x C^\circ C$

$dx r s dx \lambda_i f_y := \partial f/\partial y = 2y q_i g + 1 r s y / x^{g+1} \pm 1 x x \infty \in \mathbb{P}^1 r, s g \geq 1$

$$\omega = \pi^*\left(\frac{dx}{f_y}\right)$$

$$(\omega) = (g-1)r + (g-1)s = (g-1)D.$$

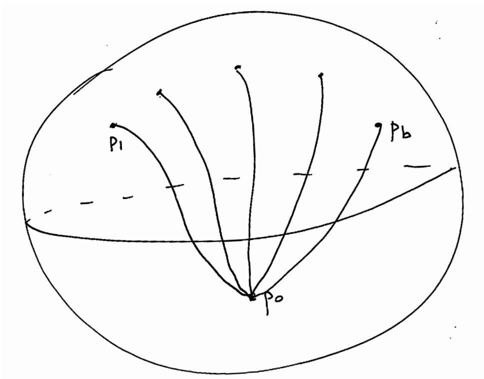
$C x \omega x^k k = 0, 1, \dots, g-1 g$

$$\omega, x\omega, \dots, x^{g-1}\omega$$

$H^0(K_C)$

$$\omega, x\omega, \dots, x^{g-1}\omega$$

$x^i g - 1$



$B\Delta = \{p_1, \dots, p_b\} \subset BB\pi : C \to Bdp_iB$

$U = B \setminus \Delta U$

$B\Delta \subset BU := B \setminus \Delta\pi^\circ : V \to UV\pi^\circ VC\pi^\circ\pi : C \to B$

$VUD \subset U(\pi^\circ)^{-1}(D)dDV$

$VD^* = \{z \in \mathbb{C} \mid 0 < |z| < 1\}D \to D : z \mapsto z^n nD^* \to D^*\pi_1(D^*) = \mathbb{Z}En\pi E$
D

$D_ip_i \in BVD_i^* := D_i \cap UE_{i,j}^*E_{i,j}^* \to D_i^*z \mapsto z^{n_{i,j}}n_{i,j}Vz \mapsto z^{n_{i,j}}E_{i,j} \to D_i$
$V \cup \bigcup E_{i,j}$ □

$C\pi : C \to BdU$

$\mathbb{P}^1$ $U = B \setminus \Delta B = \mathbb{P}^1_{\mathbb{C}}\pi : V \to U$

$p_0 \in U\gamma_i p_0 p_i U\Sigma\Sigma V d\Sigma\Sigma_1, \dots, \Sigma_d$

$UV \to UUdVU$

$$M : \pi_1(U, p_0) \to S_d,$$

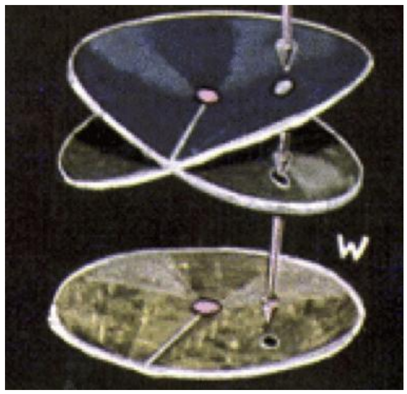
$dS_dMV \to UdVpM\beta Up_0\pi^{-1}(p_0)q \in \pi^{-1}(p_0)\beta q\sigma M'M\sigma$

$\pi_1(U, p_0)\beta_i p_0\gamma_i p_i p_i p_0\gamma_i U\beta_1, \dots \beta_b \prod_{i=1}^b \beta_i = 1\beta_i$

$dVdp_0\tau_i\{1, 2, \dots, d\}\beta_i V\tau_i S_d$

$C \to \mathbb{P}^1V\tau_i$

$p_1, \dots, p_b \in \mathbb{P}^1b$



$z \mapsto z^2$

$\pi : C \to \mathbb{P}^1 d p_i \mathbb{P}^1$

$b\tau_1, \dots, \tau_b \in S_d \prod \tau_i = id\tau_1, \dots, \tau_b S_d S_d$

$\square$

$d = 2S_2\mathbb{P}^1 p_1, \dots, p_b$ $\square$

$C \to \mathbb{P}^1 \tau \in S_3 b\tau_1, \dots, \tau_{b-1} \in S_3 \tau_1 \cdot \dots \tau_{b-1} b\tau_1, \dots, \tau_b \in S_3 \prod \tau_i 3^{b-1} \tau_i \tau_i S_3 bb$
$\tau_1, \dots, \tau_b \mathbb{P}^1 q_1, \dots, q_b \in \mathbb{P}^1$

$$\frac{3^{b-1} - 3}{6} = \frac{3^{b-2} - 1}{2}.$$

$\phi_K : C \to \mathbb{P}^1 C\mathbb{P}^1\mathbb{P}^1\mathbb{P}^1$

$CC \to \mathbb{P}^1$

$C\mathbb{P}^1 \;\; C\mathbb{P}^1 |K_C| \mathbb{P}^1$

$LC3 > 2g - 2h^0(L) = 2$

$|L| p \in Ch^0(L(-p)) = 2LL = K_C(p)L = K_C(p)h^0(L(-p)) = h^0(L)p|L|L$

$LL = K_C(p)|L|\phi_L : C \to \mathbb{P}^1$

$\mathrm{Pic}_3(C)g = 2$

$d \geq 4 = 2g\mathbb{P}^{d-2}\mathbb{P}^1$

$C\mathbb{P}^2$ $CLCh^0(L) = 3|L|\phi_L : C \to \mathbb{P}^2\mathcal{L} \otimes \omega_C^{-1}\omega_C\mathcal{L} \otimes \omega_C^{-1} = \mathcal{O}_C(p+q)p, q$

$p + q = K_C\mathcal{L} = \omega_C^2 H^0(\omega_C)\omega, x\omega$

$$\mathrm{Sym}^2 H^0(\omega_C) \to H^0(\mathcal{L}).$$

$D \sim 2K_C D_1, D_2 \in |K_C|\phi_L\phi_K : C \to \mathbb{P}^1\nu_2 : \mathbb{P}^1 \to \mathbb{P}^2\mathbb{P}^1\phi_L$

$p+q \neq K_C h^0(p+q) = 1p, qh^0(\mathcal{L}-p) = 2 = h^0(\mathcal{L}(-p-q))H^0(\mathcal{L}(-p)) = H^0(\mathcal{L}(-q))\phi_{\mathcal{L}}(p) = \phi_{\mathcal{L}}(q)\delta p \neq qp = q$

$\mathcal{L}\mathrm{Pic}_4(C)\mathrm{Pic}_4(C)\mathrm{Pic}_4(C)$

$\mathbb{P}^3$ $L\phi_{\mathcal{L}} : C \to \mathbb{P}^3|L|\phi_L\phi_L(C) \subset \mathbb{P}^3C\mathcal{O}_C(1)L$

$\mathbb{P}^3C$

$$H^0(\mathcal{O}_{\mathbb{P}^3}(2)) \to H^0(\mathcal{O}_C(2)) = H^0(L^2).$$

$h^0(L^2) = 2 \cdot 5 - 2 + 1 = 9CQCC\deg(C) \leq 4Q$

QL

$C \subset \mathbb{P}^3Q \subset \mathbb{P}^3CL = \mathcal{O}_C(1) \in \mathrm{Pic}_5(C)Q$

$$L \cong K^2(p)$$

$p \in CpQ$

$\mathrm{Pic}_5(C)K^2(p)QL$

$L \cong K^2(p)p \in CL(-p) \cong K^2\pi : C \to \mathbb{P}^2p\phi_{K^2} : C \to \mathbb{P}^2\phi_{K^2}E \subset \mathbb{P}^2CQEpC$

$LK^2(p)$

$$L = K \otimes M,$$

$MK(p)|M|C \to \mathbb{P}^1$

$\phi_L : C \to \mathbb{P}^3\phi_K : C \to \mathbb{P}^1\phi_M : C \to \mathbb{P}^1\sigma : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^3$

$$C \xrightarrow{\phi_K \times \phi_M} \mathbb{P}^1 \times \mathbb{P}^1 \xrightarrow{\sigma} \mathbb{P}^3$$

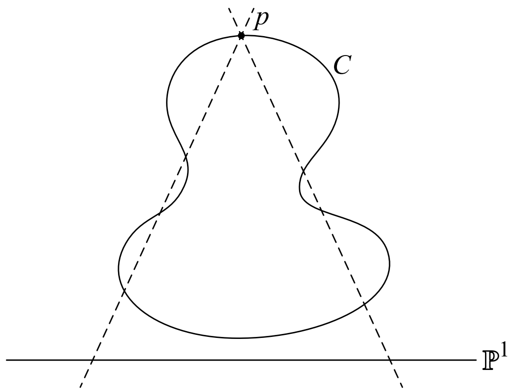
$\phi_L C(2,3)Q \subset \mathbb{P}^3$ $\square$

$I(C) \subset \mathbb{C}[x_0, x_1, x_2, x_3]$

$$H^0(\mathcal{O}_{\mathbb{P}^3}(3)) \to H^0(\mathcal{O}_C(3)).$$

$15 - 2 + 1 = 14CQCQ$

$QC(2,3)QL \subset QC + LQS_LQSCQS \cap QCLI(Q)S = S_LQLQ$



$C\mathbb{P}^1$

C

$\mathbb{P}^1 PGL_2 \mathbb{P}^1 \dim(PGL_2) = 36 - 3 = 3$

$\mathbb{P}^1 PGL_2 5 - 2 = 3$

$PGL_3 \mathbb{P}^2 PGL_3 5 - 2 = 3$

$\mathbb{P}^3 QQC(2,3)QQ(2,3)Q \cong \mathbb{P}^1 \times \mathbb{P}^1 3 \cdot 4 - 1 = 11 PGL_4 \mathbb{P}^3 CPGL_4 5 - 2 = 3$

$CC\phi_K : C \to \mathbb{P}^2 C$

$\dim PGL(3) = 8$

$C\mathbb{P}^1 L \in \mathrm{Pic}_3(C)$

$$h^0(L) = \begin{cases} 2, & L \cong K - pp \in C \\ 1 \end{cases}$$

$C\mathbb{P}^1 \phi_{K-p} : C \to \mathbb{P}^1 \phi_K : C \to \mathbb{P}^2 p$

$CC g^2_3 g^2_4 L|L| g^2_5 LK + p\phi_L CC_0 \subset \mathbb{P}^2 L2K - p - q - rp, q, r \in CC_0 Cp, qrp, q, r$

$DK + p + qp, q \in CC$

$C \subset \mathbb{P}^3 CC(a,b)abC(2,4)$

$CCh^0(\mathcal{O}_{\mathbb{P}^3}(3)) = 20h^0(\mathcal{O}_C(3)) = 18 - 3 + 1 = 16CCDCDDD2 \times 2$

$$\begin{pmatrix} \ell_0 & \ell_1 & \ell_2 \\ \ell_1 & \ell_2 & \ell_3 \end{pmatrix}$$

$\ell_i 3\times 3$

$$\begin{pmatrix}\ell_0 & \ell_1 & \ell_2\\ \ell_1 & \ell_2 & \ell_3\\ \ell_4 & \ell_5 & \ell_6\\ \ell_7 & \ell_8 & \ell_9\end{pmatrix}$$

$\ell_4,\dots,\ell_9 C 3\times 3$

$C\subset\mathbb{P}^2 C L\subset\mathbb{P}^2 C\geq 4CC2DCL\subset\mathbb{P}^2$

$L\in{\mathbb{P}^2}^* Cd\geq 4$

$$12\binom{d+1}{4}-4d(d-2),$$

C

$C4=2\times 2D=p+q2DC$

$$2D\sim K_C$$

$\mathcal{O}_C(D)C$

$\mathcal{L}\mathcal{M}\mathcal{L}^2=\mathcal{M}^2=K\mathcal{L}\mathcal{M}$

$$\mathcal{M}=\mathcal{L}\otimes \mathrm{F},\qquad \mathrm{F}\otimes\mathrm{F}\sim\mathcal{O}_C.$$

$\mathrm{F}\mathcal{L}\mathcal{M}\mathrm{Pic}_0(C)\mathrm{Pic}_0(C)=Jac(C)\mathbb{C}^3\Lambda\cong\mathbb{Z}^6 2^6=64\mathcal{L}\mathcal{L}^2\cong K_C 64=2^{2g}$

$D2D\sim K\mathcal{L}h^0(\mathcal{L})=0h^0(\mathcal{L})C$

$h^0(\mathcal{L})$

$\mathrm{C}\to B\mathcal{L}_b\mathcal{L}\mathrm{C}(\mathcal{L}|_{C_b})^2\cong K_{C_b} b\in Bf:B\to\mathbb{Z}/2$

$$f(b)=h^0(\mathcal{L}|_{C_b})\quad (2)$$

f

$\mathcal{L}h^0(\mathcal{L})gg$

$Cg2^{2g}C2^{g-1}(2^g+1)2^{g-1}(2^g-1)$

C

$Ch^0(\mathcal{L})\mathcal{L}\geq 22^{g-1}(2^g-1)=28$

$QV\Lambda\subset VQ(\Lambda,\Lambda)=0$

$V2nQ$

Qn

$QG(n,V)\binom{n}{2}Q$

$\Lambda, \Lambda' \subset V$

$$\dim(\Lambda \cap \Lambda') \equiv n \qquad \Longleftrightarrow \qquad \Lambda, \Lambda'$$

$\widetilde{Q} : V \xrightarrow{\cong} V^* Q > n\Lambda \subset V < n\Lambda v \overline{Q}(v,v) = 0 \overline{Q} ann(\Lambda)/\Lambda$

- $n = 2Q\mathbb{P}^3\mathbb{C}^4$
- $n = 3\mathbb{G}(1,3)\mathbb{P}^5 p \in \mathbb{P}^3 H \subset \mathbb{P}^3$

$Cg\mathcal{L}C\mathcal{L}^2 \cong K_C D = p_1 + \cdots + p_n n > g - 1V2n$

$$V := H^0(\mathcal{L}(D)/\mathcal{L}(-D)).$$

$$0 \to \mathcal{L}(-D) \to \mathcal{L}(D) \to \mathcal{L}(D)/\mathcal{L}(-D) \to 0$$

$\mathcal{L}(D)/\mathcal{L}(-D)D\mathcal{O}_p/\mathfrak{m}^2_{C,p} p \in DV$

$$Q(\sigma,\tau) := \sum_i {}_{p_i}(\sigma \cdot \tau)$$

$\mathcal{L}^2 \cong K_C \sigma\tau$

Q

$$\Lambda := H^0(\mathcal{L}/\mathcal{L}(-D)).$$

$\sigma\tau\Lambda$

$$\Lambda' := \mathrm{im}\big(H^0(\mathcal{L}(D)) \to H^0(\mathcal{L}(D)/\mathcal{L}(-D))\big)$$

$H^0(\mathcal{L}(-D)) = 0h^0(\mathcal{L}(D)) = nnVC$

$$H^0(\mathcal{L}) \cong \Lambda \cap \Lambda',$$

$SJac(C)_2 \cong (\mathbb{Z}/2\mathbb{Z})^{2g}$

$CS^- Jac(C)_2 CJac(C)\mathcal{L}_1, \ldots, \mathcal{L}_4 \in S^-$

$$\mathcal{L}_1 + \cdots + \mathcal{L}_4 = 2K_C.$$

$g = 3CCS^-$

gg

n

$$\sum_{k=0}^{n}\binom{2n}{2k} = \sum_{k=0}^{n-1}\binom{2n}{2k+1} = 2^{2n-1}$$

$$\sum_{k=0}^{n}\binom{4n}{4k} = 2^{4n-2}+(-1)^n2^{2n-1} \qquad \sum_{k=0}^{n-1}\binom{4n}{4k+2} = 2^{4n-2}-(-1)^n2^{2n-1}$$

$$\sum_{k=0}^{n}\binom{4n+2}{4k+1} = 2^{4n}+(-1)^n2^{2n} \qquad \sum_{k=0}^{n-1}\binom{4n}{4k+3} = 2^{4n}-(-1)^n2^{2n}$$

$$2^{2n} = (1+1)^{2n} = \sum_{l=0}^{2n}\binom{2n}{l} \qquad 0 = (1-1)^{2n} = \sum_{l=0}^{2n}(-1)^l\binom{2n}{l}$$

$(1+i)^{4n} = (-1)^n2^{2n}$

$$\sum_{k=0}^{n}\binom{4n}{4k} - \sum_{k=0}^{n-1}\binom{4n}{4k+2} = (-1)^n2^{2n}$$

$$\sum_{k=0}^{n}\binom{4n}{4k} + \sum_{k=0}^{n-1}\binom{4n}{4k+2} = 2^{4n-1};$$

$(1+i)^{4n+2} = (-1)^n2^{2n+1}i$

$$\sum_{k=0}^{n}\binom{4n+2}{4k+1} - \sum_{k=0}^{n-1}\binom{4n+2}{4k+3} = (-1)^n2^{2n+1}$$

$$\sum_{k=0}^{n}\binom{4n+2}{4k+1} + \sum_{k=0}^{n-1}\binom{4n+2}{4k+3} = (-1)^n2^{4n+1}$$

□

$C g \mathbb{P}^1 p_1, \dots, p_{2g+2} I_1, I_2 \{1, \dots 2g+2\}$

$$D_i = \sum_{j\in I_i} p_j.$$

$D_1, D_2 I_1 = I_2 g + 1 I_1 \cup I_2 = \{1, \dots, 2g+2\}$

$D_1 D_2 I_1 \cap I_2 = \emptyset D_1 \sim D_2 d \leq g+1 d = 0 d = g+1$

$D_1 \sim D_2 D_1 + D_2 \equiv 2D_1 d \leq g r(2D_1) = dd < gd = gd \leq g |2D_1| d C \mathbb{P}^1 p_i d \quad \square$

$Cg\mathbb{P}^1 p_1, \ldots, p_{2g+2}$

$\mathrm{g}_2^1 CEDD + E$

$$D \sim mE + F$$

$-1 \leq m \leq \frac{g-1}{2} Fg - 1 - 2mp_i m = -1g + 1Cg + 1g$

$$\binom{2g+2}{0} + \binom{2g+2}{2} + \binom{2g+2}{4} + \cdots + \binom{2g+2}{g-1} + \frac{1}{2}\binom{2g+2}{g+1}$$

g

$$\binom{2g+2}{1} + \binom{2g+2}{3} + \binom{2g+2}{5} + \cdots + \binom{2g+2}{g-1} + \frac{1}{2}\binom{2g+2}{g+1}.$$

$(2g+2)\frac{1}{4} \cdot 2^{2g+2} = 2^{2g}$

$gC2^{g-1}(2^g - 1)2^{g-1}(2^g + 1)gg$

$Cg = 1\mathbb{P}^1 C^\circ \subset \mathbb{A}^2 y^2 - \prod_{i=1}^4 (x - \lambda_i) \overline{C^\circ} C^\circ \subset \mathbb{A}^2 \mathbb{P}^2 \mathbb{P}^1 \times \mathbb{P}^1 C^\circ \overline{C^\circ}$

$\overline{C^\circ} \setminus C^\circ C \mathbb{P}^2 \mathbb{P}^1 \times \mathbb{P}^1$

$C \to \mathbb{P}^1 g$

$2g + 2$

BhB

$\pi_1(C) H_1(C, \mathbb{Z}) \cong \mathbb{Z}^{2g}$

$C\mathcal{L}C\mathcal{L}^2 \cong \mathcal{O}_C Jac(C)$

$f : X \to Cf_*(\mathcal{O}_X) C\mathbb{Z}/2{+}1\mathcal{O}_C{-}1\mathcal{L}C\mathcal{L}^2 \cong \mathcal{O}_C$

$Eq_1, \ldots, q_b \in EC \to Eq_i$

$H_1(E, \mathbb{Z}) \cong \mathbb{Z}^2$

$L, L' QQS_L S_{L'} I(C) I(C)$

$M \subset QH, H' \mathbb{P}^3 L \cup ML' \cup M$

$C\mathbb{P}^1 p_1, \ldots, p_6$

$C\mathcal{L} = \mathcal{O}_C(p_i) \mathcal{L} = \mathcal{O}_C(p_i + p_j - p_k) i, j, k$

$h^0(\mathcal{L}) = 1 h^0(\mathcal{L}) = 0$

$2^4 = 16$

$h^0(\mathcal{L}) = 0 p_k h^0(\mathcal{L}(p_k)) = 1 |\mathcal{L}(p_k)|$

$C \mathcal{L} \in \mathrm{Pic}_4(C) L = K_C(p+q) p \neq q p + q \nsim K_C$

$h^0(L(-2r)) = 1 r \in C$

$h^0(L(-2p-2q)) = 0$

$\phi_L \phi_L(C) \phi_L(p) = \phi_L(q) \phi_L(p) = \phi_L(q) \phi_L(C)$

$\phi_L h^0(L K_C^{-1}) = 1 p [q d \phi_L p q$

$\mathbb{P}^3 2 \times 4 2 \times 2$

$$\{p_{i,j}\}_{0 \leq i < j \leq 3}$$

$\mathbb{G}(1,3) \mathbb{P}^5$

$p_{i,j} 2 \times 2 4 \times 4$

$$p_{0,1} p_{2,3} - p_{0,2} p_{1,3} + p_{0,3} p_{1,2} = 0.$$

$Q = p_{0,1} p_{2,3} - p_{0,2} p_{1,3} + p_{0,3} p_{1,2} \mathbb{G}(1,3) \mathbb{P}^5$

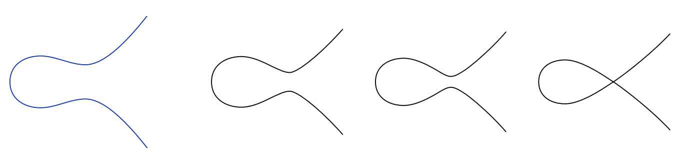
$Q \mathbb{G}(1,3)$

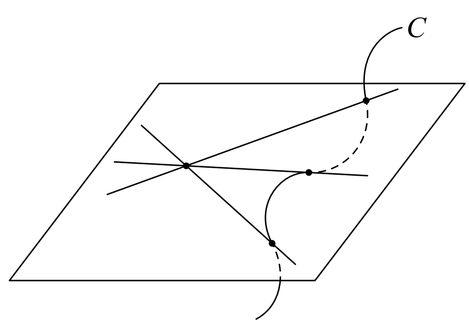
Q

$d d \mathbb{P}(H^0(\mathcal{O}_{\mathbb{P}^2}(d))) = \mathbb{P}^{\binom{d+2}{2}-1} \mathbb{P}^9 \mathrm{C} \to B \phi : B \to \mathbb{P}^9$

$d C g g + 3$

$\mathbb{P}^3 \mathbb{G}(1,3) d d \mathbb{P}^3$





$\mathbb{P}^3 C \deg C$

BBB

$C\mathrm{D} \subset B \times CdBdddC_dC$

$CB\mathcal{L}B \times CB \times CBd\mathcal{L}\mathcal{L}'B \times C\mathcal{L} \cong \mathcal{L}' \otimes \mathcal{M}\mathcal{M}Bb \times Cd$

$gBf : \mathrm{C} \to Bgf, f'ff'$

$$\begin{tikzcd}
\mathrm{C} \arrow[rr, "\cong"] \arrow[dr, "f"'] & & \mathrm{C}' \arrow[dl, "f'"] \\
& B &
\end{tikzcd}$$

M_g

$Cf : C \to \mathbb{P}^1$

$$\begin{tikzcd}
C \arrow[rr, "\cong"] \arrow[dr, "f"'] & & C' \arrow[dl, "f'"] \\
& \mathbb{P}^1 &
\end{tikzcd}.$$

$\mathrm{C} \to B\mathrm{C} \to B \times \mathbb{P}^1 d$

$dgg \neq \binom{d-1}{2}$

$B\mathrm{C} \subset B \times \mathbb{P}^r BB$

$C \subset \mathbb{P}^r$

$2^{\aleph_0} M\mathbb{C}M$

$\phi : \mathcal{X} \to MMBMB \to M.M\phi$

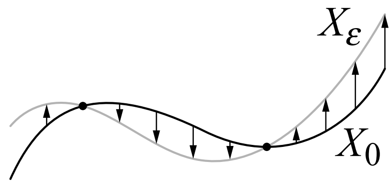
$$\{B \mapsto XB\} \cong \{B \mapsto Mor_{Schemes}(B, M)\}.$$

$MM \to M\phi : \mathcal{X} \to MDiv_d(C)\mathrm{Pic}_d(C)$

$MM'MM \to M'M' \to MM$

$(R, \mathfrak{m})m\mathcal{M}MmHom_{R/\mathfrak{m}}(\mathfrak{m}/\mathfrak{m}^2, R/\mathfrak{m})Rk := R/\mathfrak{m}\mathfrak{m}/\mathfrak{m}^2R \to k[\epsilon]/(\epsilon^2)k\mathcal{M}F$ $F(E)E := \mathrm{Spec}(k[\epsilon]/(\epsilon^2))$

$X_i \to EE \subset E \times EX_1 \times X_2 \to E \times E$



$\mathbb{P}^r B \mathbb{C} \mathcal{X} \subset B \times \mathbb{P}^r B f : B' \to B f$

$p(m) := dm + 1 - \binom{d-1}{2} d Hilb_{p(m)}(\mathbb{P}^2) X \subset \mathbb{P}^2 p \dim X = \deg p = 1 X d p(m)$
$p(m) X d Hilb_{p(m)}(\mathbb{P}^2) \mathbb{P}^{\binom{d+2}{2}-1} d$

$$\{(x, F) \in \mathbb{P}^2 \times \mathbb{P}^{\binom{d+2}{2}-1} \mid F(x) = 0\} \to \mathbb{P}^{\binom{d+2}{2}-1}.$$

$C \subset \mathbb{P}^r d g \mathbb{P}^r p(m) = dm - g + 1$

$\mathbb{P}^3$

$E := \operatorname{Spec} \mathbb{C}[\epsilon]/(\epsilon^2) X \subset \mathbb{P}^r \mathcal{N}_{X/\mathbb{P}^r} X \mathbb{P}^r X \mathbb{P}^r$

$$0 \longrightarrow \mathcal{T}_X \xrightarrow{\phi} \mathcal{T}_{\mathbb{P}^r}\mid_X \longrightarrow \mathcal{N}_{X/\mathbb{P}^r} \longrightarrow 0.$$

$\mathcal{N}_{X/\mathbb{P}^r} X$

$X \phi \Omega_{\mathbb{P}^r}|_X \to \Omega_X X$

$$0 \to \mathcal{I}_{X/\mathbb{P}^r}/\mathcal{I}^2_{X/\mathbb{P}^r} \to \Omega_{\mathbb{P}^r}|_X \to \Omega_X \to 0.$$

$\mathcal{N}_{X/\mathbb{P}^r} := \mathcal{H}\mathrm{om}(\mathcal{I}_{X/\mathbb{P}^r}/\mathcal{I}^2_{X/\mathbb{P}^r}, \mathcal{O}_X)$

$X \subset \mathbb{P}^r$

$$\mathcal{N}_{X/\mathbb{P}^r} := \mathcal{H}\mathrm{om}(\mathcal{I}_{X/\mathbb{P}^r}/\mathcal{I}^2_{X/\mathbb{P}^r}, \mathcal{O}_X).$$

X

$X \subset \mathbb{P}^r p(m) E = \operatorname{Spec} \mathbb{C}[\epsilon]/(\epsilon)^2 \mathcal{X} \subset \mathbb{P}^r \times E X(\epsilon) Hom_{\mathbb{P}^r}(\mathcal{I}_{X/\mathbb{P}^r}, \mathcal{O}_X) = H^0(\mathcal{N}_{X/\mathbb{P}^r})$
$Hilb_p(\mathbb{P}^r)[X]$

$X \subset \mathbb{A}^r = \operatorname{Spec} S S = \mathbb{C}[x_1, \ldots, x_r]$

$E M E \operatorname{Tor}_1^E(\mathbb{C}, M) = 0 \mathbb{C} E$

$$\cdots \xrightarrow{\epsilon} E \xrightarrow{\epsilon} E \xrightarrow{\epsilon} E \longrightarrow \mathbb{C} \longrightarrow 0$$

$\operatorname{Tor}_1^E(\mathbb{C}, M) = 0 M \epsilon \epsilon M$

$I = (g_1, \dots, g_t) \subset S = \mathbb{C}[x_1, \dots, x_r] X Hom_S(I/I^2, S/I) = Hom_S(I, S/I) \phi : I \to S/I h_i \in S \phi(g_i) I$

$$I' := (g_1 + \epsilon h_1, \dots, g_t + \epsilon h_t) \subset S[\epsilon]/(\epsilon^2) =: S'$$

$\operatorname{Spec} S'/I' E X(\epsilon) I' + (\epsilon) = I$

$I' k_1, \dots, k_t \in I g_i + \epsilon(h_i + k_i) I'' k_i = \sum r_{i,j} g_j$

$$g_i + \epsilon(h_i + k_i) = g_i + \epsilon h_i + \sum \epsilon r_{i,j}(g_j + \epsilon h_j)$$

$I'' \subset I I' \subset I''$

$a \in S' \epsilon a \in I' \epsilon a = \sum (r_i + \epsilon s_i)(g_i + \epsilon h_i) r_i, s_i \in S \sum r_i g_i = 0 \phi \sum r_i h_i \in I.$ $\epsilon a \in \epsilon I \subset S' a \in I + (\epsilon) a = \sum p_i g_i + \epsilon b' g_i \equiv -\epsilon h_i I' a \equiv \epsilon(-\sum p_i h_i + b') I'$

$S'/I' E I' + (\epsilon) = I + \epsilon J I' S'/(\epsilon I) = S \oplus (\epsilon S)/(\epsilon I) J \phi : I \to (\epsilon S)/(\epsilon I) \cong S/I$

$I' + \epsilon S = I + \epsilon S S \subset S' J I J J (\epsilon S)/(\epsilon I) r \in S \epsilon r \in I' \epsilon r \in \epsilon I$

$\epsilon r \in I' r S'/I' \epsilon(S'/I') r \in I' + \epsilon S = I + \epsilon S r \in I + \epsilon S, \epsilon r \in \epsilon I$

$J \phi I' \{g + \phi g \mid g \in I\}$ □

$C d F C \mathcal{O}_{\mathbb{P}^2}(-d)$

$$\mathcal{H}\mathrm{om}(\mathcal{I}_{C/\mathbb{P}^2}/\mathcal{I}^2_{C/\mathbb{P}^2}, \mathcal{O}_C) = \mathcal{O}_{\mathbb{P}^2}(d)|_C = \mathcal{O}_C(d).$$

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^2} \xrightarrow{F} \mathcal{O}_{\mathbb{P}^2}(d) \longrightarrow \mathcal{O}_C(d) \longrightarrow 0$$

$Hilb_{p(m)}(\mathbb{P}^2) = \mathbb{P}^{\binom{d+2}{2}-1} C p(m) = dm + 1 - \binom{d-1}{2} h^0(\mathcal{O}_C(d)) = h^0(\mathcal{O}_{\mathbb{P}^2}(d)) - 1 = \dim \mathbb{P}^{\binom{d+2}{2}-1}$

$\mathbb{P}^3 Hilb_{3m+1}(\mathbb{P}^3) C \subset \mathbb{P}^3 Q_1, Q_2 Q_3 \mathbb{P}^3 3 \times 10 = 30 Q_i C V = \langle Q_1, Q_2, Q_3 \rangle \subset H^0(\mathcal{O}_{\mathbb{P}^3}(2))$

$$h : \{C \subset \mathbb{P}^3\} \to G = G(3, H^0(\mathcal{O}_{\mathbb{P}^3}(2)))$$

C

$\mathbb{P}^{\binom{d+2}{2}-1} Q_i h G$

$X \to B X \subset B \times \mathbb{P}^n b \in B X_b$

$B = \operatorname{Spec} A x_0 \dots, x_r \mathbb{P}^r X \subset \mathbb{P}^r_{\mathbb{C}} I_X = (F_1, \dots, F_n) \in A[x_0, \dots x_r] X B d X_b X_b J (r - d) \times (r - d)(\partial F_i / \partial x_j)$

$Y \subset B \times \mathbb{P}^r J Y \cap X \subset X X X \to B X \cap Y X_b h^0(\mathcal{O}_{X_b}) < 2$ □

H°$Hilb_{3m+1}(\mathbb{P}^3)$

$C_0 \subset \mathbb{P}^3 C_0 A \in \mathrm{PGL}_4 \mathbb{P}^3$

$$\mathrm{C} = \{(A, p) \in \mathrm{PGL}_4 \times \mathbb{P}^3 \ \mid \ p \in A(C_0)\}.$$

$\pi : \mathrm{C} \to \mathrm{PGL}_4$

$$\phi : \mathrm{PGL}_4 \to \mathrm{H}^\circ.$$

$C_0 C_0 \mathrm{PGL}_4 \mathbb{P}^3 C_0 C_0 \mathbb{P}^3 \mathrm{PGL}_2 \mathrm{PGL}_4 \mathrm{H}^\circ$ □

$3m + 1 \mathbb{P}^3 3m + 1$

$p(m) \in \mathbb{Q}[m] m_0$

$X \subset \mathbb{P}^r p_X = p$

$$h^0(\mathcal{I}_{X/\mathbb{P}^r}(m)) = \binom{m+r}{r} - p(m) \quad m \geq m_0$$

$X p_X = p m \geq m_0$

$X \subset \mathbb{P}^r p_X = p X \leq m$

$X m_0 m_0 X p m_0$

$X \subset \mathbb{P}^r$

$$\chi(\mathcal{O}_X(m) = \binom{m + a_1}{a_1} + \binom{m + a_2 - 1}{a_2} + \cdots + \binom{m + a_s - (s-1)}{a_s},$$

$$a_1 \geq \cdots \geq a_s \geq 0$$

$m X \leq s m_0 = s$

$a_j H^i(\mathcal{I}_X(m))$

$3m + 1$

$$3m + 1 = \binom{m+1}{1} + \binom{m+1-1}{1} + \binom{m+1-2}{1} + \binom{m+0-(3)}{0},$$

$s = 4$

$0 \leq k \leq r \mathbb{G}(k, r) k \mathbb{P}^r G(k+1, r+1)(k+1)(r+1)(r+1) V G(k+1, V)$

$G(k+1, V) \mathbb{P}(\wedge^{r-k} V) = \mathbb{P}^{\binom{r+1}{r-k}-1} W \subset V \wedge^{r-k} V \to \wedge^{r-k}(V/W)$

$V/W V \to V/W (r-k) \times (r+1) A W (r-k) \times (r-k) A W (r-k) \times (r-k)$
$k \times (r-k) G(k, V) k(r-k) G(k+1, r+1) = \mathbb{G}(k, r) k(r-k)$

$\mathbb{G}(1,3)\mathbb{P}^3 Lq, r \in \mathbb{P}^3 2\times 22\times 4$

$$\begin{pmatrix} q_0 & q_1 & q_2 & q_3 \\ r_0 & r_1 & r_2 & r_3 \end{pmatrix}.$$

$p_{i,j}$

$$p_{0,1}p_{2,3} - p_{0,2}p_{1,3} + p_{0,3}p_{1,2} = 0,$$

$\mathbb{G}(1,3) \subset \mathbb{P}^5$

$\mathcal{W} \subset V \times G(k+1, r+1) W \subset V G(k+1, V) W X (k+1) \mathcal{W}' V \times X X \to G(k+1, V)$ $\mathcal{W}\mathcal{W}'$

$$h : \{X \subset \mathbb{P}^r p_X = p\} \to G\left(\binom{m_0+r}{r} - p(m_0), \binom{m_0+r}{r}\right)$$

$XH^0(\mathcal{I}_{X/\mathbb{P}^r}(m_0))$

$V\mathbb{P}^3$

$$V \otimes H^0(\mathcal{O}_{\mathbb{P}^3}(1)) \to H^0(\mathcal{O}_{\mathbb{P}^3}(3)).$$

$H^0(\mathcal{O}_{\mathbb{P}^1}(9))V$

$\mathrm{E} G = G(3, H^0(\mathcal{O}_{\mathbb{P}^3}(2)) H^0(\mathcal{O}_{\mathbb{P}^3}(d)) \otimes \mathcal{O}_G$

$$\mu : \mathrm{E} \otimes H^0(\mathcal{O}_{\mathbb{P}^3}(1)) \to H^0(\mathcal{O}_{\mathbb{P}^3}(3)).$$

$11 \times 11 G$

$p(m) m_0$

$$G = G\left(\binom{m_0+r}{r} - p(m_0), \binom{m_0+r}{r}\right)$$

$h\mathbb{P}^r pGXH^0(\mathcal{I}_{X/\mathbb{P}^r}(m_0))G$

$$\mathcal{E} \otimes H^0(\mathcal{O}_{\mathbb{P}^r}(1)) \to H^0(\mathcal{O}_{\mathbb{P}^r}(m_0+1)).$$

$Gh\binom{r+m_0+1}{r} - p(m_0+1)h$

$\mathbb{P}^1 CDg, h \geq 2C \to Dgh$

$g, h \geq 2N(g,h)CgDh$

$g \geq 2$

$df : C \to Dd$

$SHilb_{p(m)}(\mathbb{P}^r_S)SX \to SX\mathbb{P}^r_X Xp$

$ghS_g \to Sg2g+1\mathbb{P}^{g+1}_S S_h h([C],[D])S_g \times_S S_h S_g S_h \mathcal{C} \to S = S_g \times_S S_h sgh$ $\mathbb{P}^{g+1}_s \times_S \mathbb{P}^{h+1}_s \mathbb{P}^N_S N = (g+2)(h+2)-1$

$\Gamma \subset C \times D \subset \mathbb{P}^N f : C \to Dd\Gamma g2g+1+d(2h+1)pCdDSdgh$

$dC, D\phi\Gamma_\phi$

$$\mathcal{N}_{\Gamma_\phi/(C\times D} = \mathcal{I}_{C\times D}/\mathcal{I}_{\Gamma_\phi}$$

$\mathcal{I}_{\Sigma_\phi}\mathcal{I}_C\phi^*\mathcal{I}_D 2-2h<0$

$SN(g,h)$ $\square$

$g = hd = 1\mathcal{X} \subset B \times \mathbb{P}^r_S \mathcal{Y} \subset B \times \mathbb{P}^s_S Isom(\mathcal{X},\mathcal{Y}) \to SSX_b \to Y_b bsM_g$

$X \subset \mathbb{P}^n Y, ZXYZ$

$p_X = p_Y + p_Z h^1(\mathcal{I}_{Y\cup Z}(m))mh_X \le h_Y + h_Z h_X(m) = h_Y(m) + h_Z(m)YZ$ $|\mathcal{O}_{\mathbb{P}^n}(m)|$

$X \subset \mathbb{P}^n Y, ZXYZY \cap Z$

$$0 \to \mathcal{I}_{Y\cup Z} \to \mathcal{I}_Y \oplus \mathcal{I}_Z \to \mathcal{I}_{Y\cap Z} \to 0$$

$H \subset \mathbb{P}^3 C \subset Hp \in H \setminus CHCX = C \cup \{p\}$

$Xp_X(m) = 3m+1$

m_0

XH

$\mathrm{H}^\circ C \subset \mathbb{P}^r r^2 + 2r - 3$

$PGL_{r+1}\mathrm{H}^\circ PGL_2$

$C = X \cap Y \subset \mathbb{P}^3 d, eHilb[C]2\binom{3+d}{3} - 4d = e\binom{3+d}{3} + \binom{3+e}{3} - \binom{3+e-d}{3} - 2d < e$

$\mathcal{N} = \mathcal{O}_C(d) + \mathcal{O}_C(e)H^0(\mathcal{N})$

$M_g g M_g$

$C C \mathcal{O}_C(2p) \mathbb{P}^1 C \mathbb{P}^1 p \mathbb{P}^1 C$

$$y^2 = f(x)$$

f

$$f(x) = \prod_{i=1}^{4} (x - \lambda_i).$$

$M_1 \{\lambda_1, \ldots, \lambda_4\} \mathbb{P}^1 \mathrm{Aut}(\mathbb{P}^1) = PGL_2$

$\mathbb{P}^1 \lambda_1, \lambda_2, \lambda_3 0, 1 \infty \in \mathbb{P}^1 C$

$$y^2 = x(x-1)(x-\lambda)$$

$\lambda \in \mathbb{P}^1 \setminus \{0, 1, \infty\} C_\lambda \lambda_i \lambda \infty \lambda 1/\lambda S_4 \mathbb{A}^1 \setminus \{0, 1, \infty\} \lambda$

$$\left\{\lambda, 1 - \lambda, \frac{1}{\lambda}, \frac{1}{1-\lambda}, \frac{\lambda - 1}{\lambda}, \frac{\lambda}{\lambda - 1}\right\}.$$

$K = \mathbb{Z}/2 \times \mathbb{Z}/2 \subset S_4 S_4 / K \cong S_3$

$S_3 \mathbb{P}^1 \setminus \{0, 1, \infty\} \mathbb{C}(\lambda) S_3 \mathbb{C}(j) j(\lambda)$

$$j(\lambda) := 256 \frac{(\lambda^2 - \lambda + 1)^3}{\lambda^2 (\lambda - 1)^2},$$

$j \lambda \mathbb{P}^1 \setminus \{0, 1, \infty\} j(\lambda) \mathbb{A}^1$

$M_1 \cong \mathbb{A}^1 y^2 = x(x-1)(x-\lambda) j(\lambda) \in \mathbb{A}^1$

M_1 M_1

$\pi : \mathrm{C} \to B B \phi : B \to M_1 b \in B j C_b$

M_1

$B b_0 \in B U b_0 \in B \mathrm{C}_U = \pi^{-1}(U) C_b \mathbb{P}^1 \mathrm{C}_U C_b B \rho : U \to \mathrm{C}_U \pi : \mathrm{C} \to B D = \rho(U) \subset \mathrm{C}_U \mathcal{L} := \mathcal{O}_{\mathrm{C}_U}(2D)$

$f : X \to Y \mathcal{F} X H^1(\mathcal{F}|_{f^{-1}(y)}) = 0 y \in Y h^0(\mathcal{F}|_{f^{-1}(y)}) y f_*(\mathcal{F})$ □

$\mathrm{E} := \pi_*(\mathcal{O}_{\mathrm{C}_U}(2D)) \mathrm{C}_U \to \mathbb{P}(\mathrm{E}) C_b \mathbb{P}(\mathrm{E}_b) B \mathrm{E}$

$$\begin{tikzcd}
\mathrm{C}_U \arrow[rr] \arrow[dr] & & U \times \mathbb{P}^1 \arrow[dl] \\
& U &
\end{tikzcd}$$

$U \mathrm{C}_U \to U$

$$y^2 = \prod_1^4 (x - \lambda_i),$$

$\lambda_i U j \lambda_i U \to M_1 j B \to M_1$ □

M_1

B

$j : B \to \mathbb{A}^1 B \alpha : B' \to B j \circ \alpha j B'$

$\pi : \mathrm{C} \to B \eta : \mathrm{D} \to B j \alpha : B' \to B \mathrm{C} \times_B B' \cong \mathrm{D} \times_B B'$

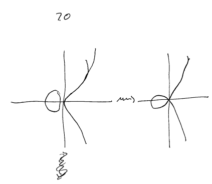
$$\begin{tikzcd}
\mathrm{C} \times_B B' \arrow[rr] \arrow[dr] & & \mathrm{D} \times_B B' \arrow[dl] \\
& B' &
\end{tikzcd}$$

$$B' := \{(b, \lambda) \in B \times (\mathbb{A}^1 \setminus \{0,1\}) \mid j(b) = j(\lambda)\}$$

$j(\lambda) \lambda \mathbb{A}^1 \setminus \{0,1\} B'$

$\sigma : B \to \mathrm{C} \tau : B \to \mathrm{D}$

$$B' := \{(b, \phi) \mid b \in B,\ \phi : C_b \xrightarrow{\cong} D_b \phi(\sigma(b)) = \tau(b)\};$$



$$y^2 = x(x-1)(x-\lambda)$$

$B'BB'B'$ $\square$

$M_1 Bj : B \to M_1$

M_1 $j \in \mathbb{A}^1 C_j C_j j \infty C_\lambda$

$$y^2 = x(x-1)(x-\lambda)$$

$\lambda \infty y^2 = x^2(x-1)$

$M_1 \overline{M}_1 \infty M_1 \cong \mathbb{A}^1 \overline{M}_1 \cong \mathbb{P}^1$

$M\overline{M}$

C_t

$$C_t = V(y^2 - x^3 - t^2 x - t^3)$$

$jt \neq 0 C_0 M_1$

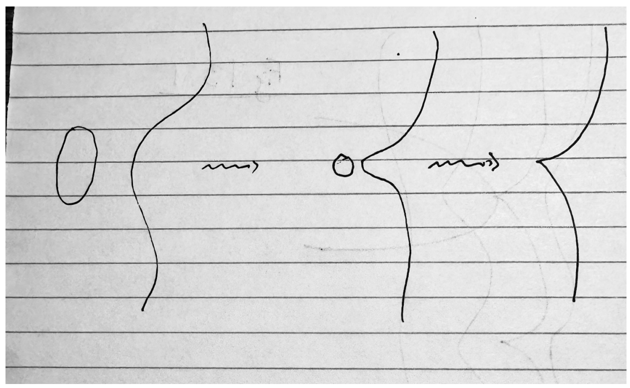
$MM\overline{M}M_g$

$\overline{M}_1 j(\lambda)$

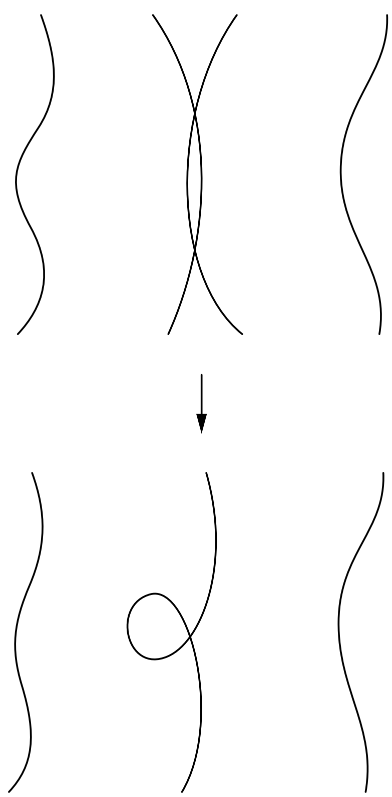
$$y^2 = x(x-t)(tx-1)$$

$t \to 0 \infty$

$Cg \geq 2 |3K_C| C 6g - 6 \mathbb{P}^{5g-6} C \mathbb{P}^{5g-6}$



$C_t = V(y^2 - x^3 - t^2x - t^3)$



$XHilb_{(6g-6)m+1-g}(\mathbb{P}^{5g-6})$g6g $- 6\mathbb{P}^{5g-6}$

$$Hilb^\circ = Hilb^\circ_{(6g-6)m+1-g}(\mathbb{P}^{5g-6}) \subset Hilb_{(6g-6)m+1-g}(\mathbb{P}^{5g-6})$$

C $\subset Hilb^\circ \times \mathbb{P}^{5g-6}\mathcal{O}_{\mathbb{P}^{5g-6}}(1)K^3Hilb^\circ X \subset Hilb^\circ$

$PGL_{5g-5}\mathbb{P}^{5g-6}XgM_gXPGL_{5g-5}\mathbb{C}^*\mathbb{C}M_g$

$PGL_{5g-5}M_g$

$Y \to BgB \to M_g p \in BM_gY_b$

$G(-)Mor_{schemes}(-, M_g)G \to Mor_{schemes}(-, M')$

$M_1 = \mathbb{A}^1\overline{M}_1 = \mathbb{P}^1M_1M_1$

$M_g g \geq 2 \geq 2$

$\overline{M}_g g$

$X \subset \mathbb{P}^N G \subset PGL_{N+1}XAXX$

$AXGAGASL_{N+1}SL_{N+1} \to PGL_{N+1}N + 1N + 1PGL_{N+1}(N + 1)$

$G = SL_{N+1}A^G \subset A$

$\mathrm{Proj}(A^G)X/\!\!/GGX\mathrm{Proj}(A^G)X/\!\!/G$

$X/\!\!/GX$

$X^{stable} \subset XX/\!\!/GX^{stable}/GPGL_3\mathbb{P}^9j$

$ppX^{semistable} \subset Xp, qX/\!\!/G\overline{Gp} \cap \overline{Gq} \cap X^{semistable} \neq \emptyset PGL_3\mathbb{P}^9$

$p\mathrm{Proj}\, A \to \mathrm{Proj}(A^G)pX/\!\!/GXM_gX'/\!\!/G'$

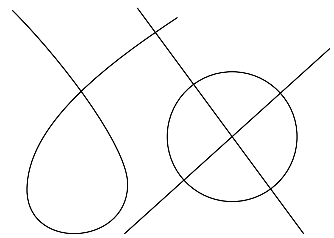
$\overline{M}_g M_g$

C

M_g

$\overline{M}_g$

$\overline{M}_g$



$\mathrm{C} \to BB \to M_g b \in B\mathrm{C}bG(-)Mor_{schemes}(-, \overline{M}_g)G \to Mor_{schemes}(-, M')$

$M_g \overline{M}_g \overline{M}_g \setminus M_g$

$g \geq 1\overline{M}_g \setminus M_g 1 + \lfloor g - 1/2 \rfloor$

$g - 1$

$i = 1, \dots \lfloor (g-1)/2 \rfloor C_i \cup C_{g-i-1} ig - 1 - i$

$\overline{M}_g \ \overline{M}_g$

$Cg\mathcal{L}\mathcal{L}^m \cong \mathcal{O}_C(\mathbb{Z}/m)^{2g} m2g\mathcal{L}_1, \dots, \mathcal{L}_{2g} C \geq 2mM_g[m]mM_g M_g M_g$

$M_g A \subset M_g g - 2M_g^\circ = M_g \setminus Ag \geq 4M_g M_g^\circ$

g

g

$M_g \overline{M}_g g M_g y^2 = x(x-1)(x-\lambda)\mathbb{A}^n M_g$

45

$S \subset \mathbb{P}^5 Q \mathbb{P}^5 SQ$

$7p_1, \dots, p_8 \in \mathbb{P}^2 p_1, \dots, p_8 \in \mathbb{P}^2 p_i 3 \times 8 = 24\mathbb{P}^{35} p_i \mathbb{P}^{11}$

$$\Sigma := \{(p_1, \dots, p_8, C) \in (\mathbb{P}^2)^8 \times \mathbb{P}^{35} \mid C p_1, \dots, p_8\}$$

$\Sigma^\circ \Sigma (\mathbb{P}^2)^8 \mathbb{P}^{11} (\mathbb{P}^2)^8 M_7 M_7$

$g + 2\binom{9}{2} - 11 = 253 \times 25 > 65$

$$\Sigma := \{(p_1, \dots, p_{25}, C) \in (\mathbb{P}^2)^{25} \times \mathbb{P}^{65} \mid C p_1, \dots, p_{25}\}$$

$\Sigma \to (\mathbb{P}^2)^{25}\Sigma\overline{M}_g\overline{M}_g$

$\geq 22\overline{M}_g \setminus M_g\overline{M}_g nX\omega_X^{\otimes p} p > 0\mathbb{P}^n \to X\mathbb{P}^n\mathbb{P}^n\overline{M}_g gM_g\overline{M}_g \setminus M_g$

$g \geq 22M_g$

$\overline{M}_g$

$\mathbb{P}^1$

$Hur^\circ_{g,d}(C, f)Cgf : C \to \mathbb{P}^1 d$

$$Hur^\circ_{g,d} = \{(C, f) \mid C \in M_g f : C \to \mathbb{P}^1 d\}.$$

$f\pi : Hur^\circ_{g,d} \to M_g(C, f) \in Hur^\circ_{g,d} B \subset \mathbb{P}^1 b\mathbb{P}^1 b(\mathbb{P}^1)_b \cong \mathbb{P}^b$

$$\begin{tikzcd}
 & Hur^\circ_{g,d} \arrow[dl, "\pi"'] \arrow[dr, "\beta"] & \\
M_g & & U \subset \mathbb{P}^b
\end{tikzcd}$$

$U \subset \mathbb{P}^b\mathbb{P}^b bUM_g$

$\beta B \subset \mathbb{P}^1\mathbb{P}^1 BBHur^\circ_{g,d}$

$$\dim(Hur^\circ_{g,d}) = b = 2d + 2g - 2$$

$M_g \;\; \pi : Hur^\circ_{g,d} \to M_g dgd \geq g + 1dd > 2g$

$d \geq g + 1\pi : Hur^\circ_{g,d} \to M_g 2d - g + 1$

$Cf : C \to \mathbb{P}^1 dg + 1\pi$

$f : C \to \mathbb{P}^1 D \in C_d f^{-1}(\infty)dCEf^{-1}(0)|D|d - gf^{-1}(\infty)f^{-1}(0)f\mathbb{P}^1\pi$

$$d + (d - g) + 1 = 2d - g + 1.$$

□

$g \geq 2$

$$\dim(M_g) = (2d + 2g - 2) - (2d - g + 1) = 3g - 3.$$

$dPGL_2\mathbb{P}^1\varphi \in PGL_2(C, f)(C, \varphi \circ f)\pi : Hur^\circ_{g,d} \to M_g\pi$

$d < \lceil \frac{g}{2} \rceil + 1Cgd\mathbb{P}^1$

$r = 1$

M_g $M_g \beta : Hur^{\circ}_{g,d} \to U \subset \mathbb{P}^b U Hur^{\circ}_{g,d} Hur^{\circ}_{g,d} M_g d M_g$

$CgCC$

$g \leq 5 d > g - 1$

$\mathbb{P}^1$

$d\binom{d-1}{2} d \binom{d-1}{2} d$

$\mathbb{P}^N := \mathbb{P}^{\binom{d+2}{2}-1} d \mathbb{P}^N$

$$\mathbb{P}^{\binom{d_1+2}{2}-1} \times \mathbb{P}^{\binom{d_2+2}{2}-1} \to \mathbb{P}^N$$

$d_1 + d_2 = d$

$V_{g,d} \subset \overline{V}_{g,d} d \delta = \binom{d-1}{2} - g \mathbb{P}^N$

$V_{g,d} \overline{V}_{g,d} \overline{V}_{g,d} V_{g,d}$

$V_{g,d} V_{g,d}$

$$\Phi := \{(C, p) \in \mathbb{P}^N \times \mathbb{P}^2 \mid p \in C_{sing}\}$$

$\Delta \subset \mathbb{P}^N$

$\Phi N - 1 \Delta \mathbb{P}^N$

$\Phi \mathbb{P}^{N-3} \mathbb{P}^2 [X, Y, Z] \mathbb{P}^2 \{a_{i,j,k} \mid i + j + k = d\} \mathbb{P}^N$

$$\mathbb{C} := \{(C, p) \in \mathbb{P}^N \times \mathbb{P}^2 \mid p \in C\}$$

$(1, d)$

$$F([a_{i,j,k}], [X, Y, Z]) = \sum a_{i,j,k} X^i Y^j Z^k;$$

$\partial F / \partial X \partial F / \partial Y \partial F / \partial Z$

$F \Delta N - 1$ $\square$

$\pi : \Phi \to \mathbb{P}^N$

$(C, p) \in \Phi p C$

$$d\pi : T_{(C,p)} \Phi \to T_C \mathbb{P}^N$$

$H_p \subset \mathbb{P}^N p$

$pCC\Delta C(C,p)\in\Phi pH_p$

$\mathbb{P}^2\mathbb{P}^N[1,0,0]\notin Cp[0,0,1]x=X/Zy=Y/ZZ\neq 0F(x,y,1)$

$$f(x,y)=\sum_{i+j\leq d}a_{i,j}x^iy^j$$

$a_{d,0}$

g,hf

$$g(x,y):=\frac{\partial f}{\partial x}=\sum_{i+j\leq d}ia_{i,j}x^{i-1}y^j$$
$$h(x,y):=\frac{\partial f}{\partial y}=\sum_{i+j\leq d}ja_{i,j}ix^iy^{j-1}.$$

$f,gh\Phi x,ya_{0,0}(C,p)$

	f	g	h
$\frac{\partial}{\partial x}$		$a_{2,0}$	$a_{1,1}$
$\frac{\partial}{\partial y}$		$a_{1,1}$	$a_{0,2}$
$\frac{\partial}{\partial a_{0,0}}$			

$pC2\times 2d\pi a_{0,0}=0\mathbb{P}^Np$ □

$q_iCdd-3m\geq d-3$

$g\widetilde{C}C\mathcal{O}_C(d-3)h^0(\mathcal{O}_C(d-3))=\binom{d-1}{2}$ □

$Cdg=\binom{d-1}{2}-\delta C\in\mathbb{P}^N\delta V_{g,d}$

$C\in\mathbb{P}^N\overline{V}_{g,d'}g'=\binom{d-1}{2}-\delta'>g\binom{\delta}{\delta'}N-\delta'\{p_1,\ldots,p_{\delta'}\}\delta'$

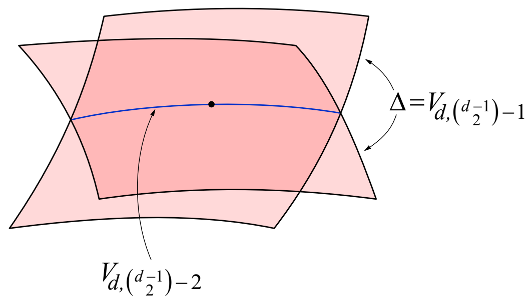
$C\in\mathbb{P}^N\Delta\delta$ □

$V_{g,d}N-\delta\delta=\binom{d-1}{2}-g$

$h(d,g,r)dg\mathbb{P}^r$

$$h(g,r,d):=4g-3+(r+1)(d-g+1)-1.$$

$r=2r=1$



$V_{d,\binom{d-1}{2}-2}$

$\alpha\Delta\Delta = V_{d,\binom{d-1}{2}-1}$

$G_m\mathbb{P}^3$

$$t : (x_0, x_1, x_2, x_3) \mapsto (tx_0, tx_1, t^{-1}x_2, t^{-1}x_3)$$

$t \in G_m = \mathbb{C}^*$

$\mathbb{C}[x_0, \ldots, x_3]$

$$x_0x_3, x_0x_2, x_1x_3, x_1x_2$$

$\mathbb{P}^3 /\!\!/ G_m \cong \mathbb{P}^1 \times \mathbb{P}^1$

$x_0 = x_1 = 0 x_2 = x_3 = 0$

$G_m p \mathbb{P}^1 \setminus \{0, \infty\} \cong G_m p$

$G_m\mathbb{P}^3$

$$t : (x_0, x_1, x_2, x_3) \mapsto (tx_0, tx_1, tx_2, t^{-1}x_3)$$

$t \in G_m = \mathbb{C}^*$

$\mathbb{C}[x_0, \ldots, x_3]$

$$F(x_0, x_1, x_2)x_3$$

$F\mathbb{P}^2\mathbb{P}^3 /\!\!/ G_m \cong \mathbb{P}^2$

$x_0 = x_1 = x_2 = 0 x_3 = 0$

$G_m p \mathbb{P}^1 \setminus \{0, \infty\} \cong G_m p$

$jj : B \to M_1 = \mathbb{A}^1\mathrm{C} \to BjB \to M_1 M_1 M_1 j(\lambda) = 1728$

$M_1 jL := \mathbb{A}^1 \setminus \{0, 1728\}\mathcal{X} \to LtjtB\tau : B \to BEE\mathcal{X} \to LE \times B(e, b) \sim (-e, \tau(b))$
$B/\tau E/(\pm) \cong EE \times B/\tau \to B/\tau$

$K_{\mathcal{X}}$

$\mathbb{P}^r$

$\mathbb{P}^1$

$C\phi_K : C \hookrightarrow \mathbb{P}^3 C \mathbb{P}^3 CIC$

$$\rho_m : H^0(\mathcal{O}_{\mathbb{P}^3}(m)) \to H^0(\mathcal{O}_C(m)) = H^0(mK_C).$$

$m = 2h^0(\mathcal{O}_{\mathbb{P}^3}(2)) = \binom{5}{3} = 10$

$$h^0(\mathcal{O}_C(2)) = 12 - 4 + 1 = 9.$$

$C \subset \mathbb{P}^3 QQQ \neq Q' \subset \mathbb{P}^3 Q \cap Q' CQ\rho_2$

$$\rho_3 : H^0(\mathcal{O}_{\mathbb{P}^3}(3)) \to H^0(\mathcal{O}_C(3)) = H^0(3K_C).$$

$H^0(\mathcal{O}_{\mathbb{P}^3}(3))\binom{6}{3} = 20$

$$h^0(\mathcal{O}_C(3)) = 18 - 4 + 1 = 15.$$

$CFCQ5 > 4CSQQ \cap SC$

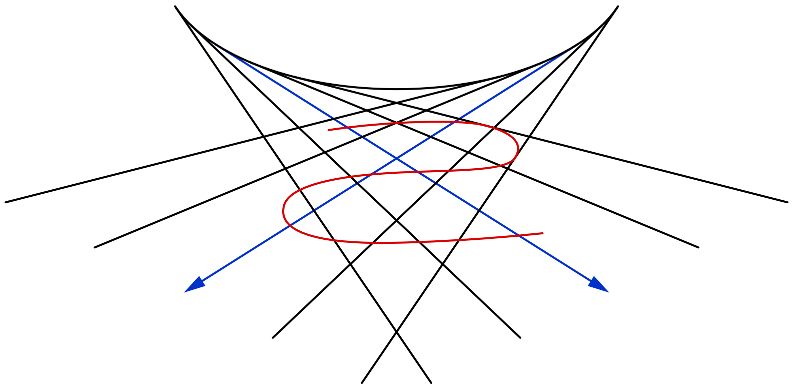
$G = 0S(F, G)C$

$C = Q \cap SQSC$

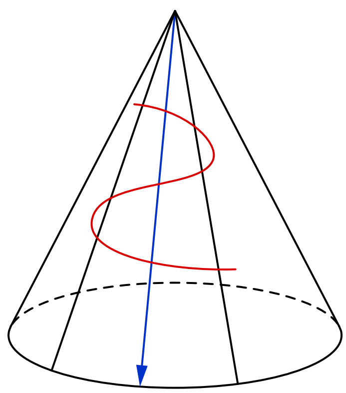
$$\omega_C = ((\omega_Q \otimes \mathcal{O}_Q(3))|_C = \mathcal{O}_C(-2 + 3) = \mathcal{O}_C(1)$$

CC

$Q = V(F) S = V(G) \mathbb{P}^3$ $\square$



g^1_3



g^1_3
g^1_3

$QCCCSS \cap Q = CSCCSQSC$

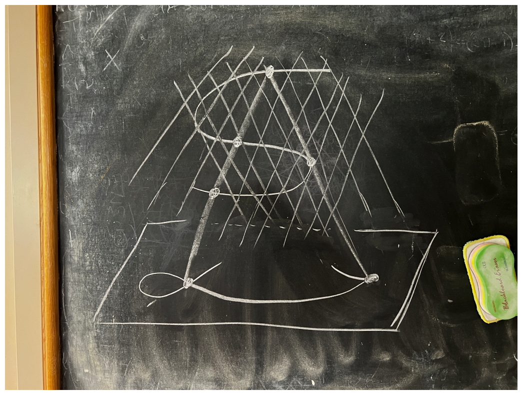
$\mathbb{P}^1$ $\mathbb{P}^1 DCr(D) \geq 1 D\pi : C \to \mathbb{P}^1 D = p + q + r$

$D = p + q + rC \subset \mathbb{P}^{g-1} r(D) \geq 1 p, q, r \in C p, q, r \in C \subset QL \subset \mathbb{P}^3 QLLLQ$ $D = C \cap L = S \cap LCQ$

$\mathbb{P}^1 \times \mathbb{P}^1 C g^1_3 C$

$\mathbb{P}^1$

$\mathbb{P}^1 g \leq 4$



$CCp \in C$

$\mathbb{P}^2$ $CCpCDr(D) = 2h^0(K - D) = 1DK - pp \in C\mathbb{P}^2D\pi_p\pi_p : C \to \mathbb{P}^2 g^2_4$

$\pi_p(C)C = Q \cap SQLpCp \geq 2LQpQC\pi(C)C > 3CCC\pi_p(C)$

Q

$L \subset S \subset \mathbb{P}^3 S \subset \mathbb{P}^3 d \geq 2S_{sing}S\mathrm{G} : S \to \mathbb{P}^{3^*} p \in S \setminus S_{sing}\mathbb{T}_p(S)L\mathbb{P}^3 LSL\mathrm{G}d - 1$
SL

$SLH \cong \mathbb{P}^2 LH \cap S = L \cup DDd - 1HSpH \cap Spd - 1DL(d-1)$

$SLLd - 1$

$[X, Y, Z, W]\mathbb{P}^3 LX = Y = 0FS$

$$F(X, Y, Z, W) = X \cdot G(Z, W) + Y \cdot H(Z, W) + J(X, Y, Z, W)$$

$JLJ \in (X, Y)^2 \mathrm{G}|_L L$

$$[0, 0, Z, W] \mapsto [G(Z, W), H(Z, W), 0, 0].$$

$GHd - 1SL$ □

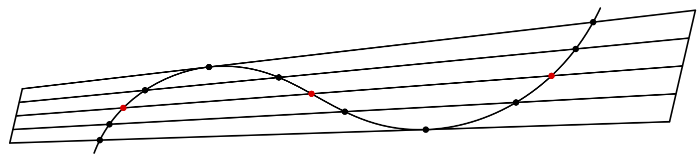
$Q \subset \mathbb{P}^3 L \subset QX = Y = 0QXZ + YW = 0QLLQ$

$H \subset \mathbb{P}^3 L \subset QHQLMQ \cap H = L \cup Mp \in LLL$

$\pi_p(C)C = Q \cap S$

$QL_1, L_2QpCpE_1E_2E_iL_i\pi(C)E_i$
$E_i 2qL_iCq \neq pCq = p\pi(C)$
$\pi(C)g^1_3 C$

$QQL \subset Qpp + ECCQLE = q_1 + q_2\pi_p(C)\pi_p(q_1) = \pi_p(q_2)\pi_p(C)E = 2qLC$
$q \neq pCpg^1_3 C\pi(C)$



Cg^1_3

C

$C\mathbb{P}^1 C g^1_3$

$C\mathbb{P}^1 C g^1_4$

$Cg^1_3 C g^1_3 C \alpha, \beta : C \to \mathbb{P}^1$

$$\alpha \times \beta : C \to \mathbb{P}^1 \times \mathbb{P}^1$$

$C(3,3)\mathbb{P}^1 \times \mathbb{P}^1 \mathbb{P}^1 \times \mathbb{P}^1 C$

$deCgg^1_d g^1_e$

$$g \leq (d-1)(e-1).$$

$Cg^1_d d$

$\alpha\beta : C \to \mathbb{P}^1 g^1_d g^1_e \alpha \times \beta : C \to \mathbb{P}^1 \times \mathbb{P}^1 C(d,e)\mathbb{P}^1 \times \mathbb{P}^1$ $\square$

$d = mae = mb\alpha \times \beta : C \to \mathbb{P}^1 \times \mathbb{P}^1 CmC_0 \subset \mathbb{P}^1 \times \mathbb{P}^1 (a,b)C$

$CC \subset \mathbb{P}^4 C \mathbb{P}^4 C$

$$\rho_2 : H^0(\mathcal{O}_{\mathbb{P}^4}(2)) \to H^0(\mathcal{O}_C(2)).$$

$\mathbb{P}^4 \binom{6}{4} = 15$

$$2 \cdot 8 - 5 + 1 = 12.$$

$C\rho_2 CCQ_1 \cap Q_2 \cap Q_3$

$Q_1 \cap Q_2 \cap Q_3 CC = Q_1 \cap Q_2 \cap Q_3 Q_i CC$

$Cg^1_3 CLCLCCg^1_3$

$g^1_4 C \subset \mathbb{P}^4 D = p_1 + \cdots + p_4 \subset C\Lambda$

$$H^0(\mathcal{J}_{C/\mathbb{P}^4}(2)) \to H^0(\mathcal{J}_{D/\Lambda}(2)).$$

Λ

$\Gamma \subset \mathbb{P}^2 \Gamma L \subset \mathbb{P}^2 \Gamma h^0(\mathcal{I}_{\Gamma/\mathbb{P}^2}(2)) = 2$

$\Gamma\Gamma q \in \mathbb{P}^2 h^0(\mathcal{I}_{\Gamma/\mathbb{P}^2}(2)) \geq 3C', C'' \subset \mathbb{P}^2\Gamma\cup\{q\}LC' = L\cup L'C'' = L\cup L''L', L'' \subset \mathbb{P}^2$
$C' \cap C''LL' \cap L''\Gamma \cup \{q\}q\Gamma L' \cap L'' = \{q\}\Gamma \subset L$ □

$n \leq 2d + 1dd + 2$

$\Lambda DQ \subset \mathbb{P}^4 CV = \mathbb{C}^3 \subset \mathbb{C}^5 V$

$Q\mathbb{P}^3\mathbb{P}^3 g^1_4 CC$

$Q \subset \mathbb{P}^4 C = Q_1 \cap Q_2 \cap Q_3 \Lambda \subset QQ'Q''QC$

$$\Lambda \cap C = \Lambda \cap Q' \cap Q'',$$

$D = \Lambda \cap CCr(D) = 1CCC$

$C\mathbb{P}^2 C\mathbb{P}^{14}\mathbb{P}^4\mathbb{P}^{14} CCBCg^1_4\mathbb{P}^1 g^1_4$

$C \subset \mathbb{P}^4 C\mathbb{P}^4 C\mathbb{P}^1 W^1_4(C) g^1_4 CB$

BC

$Z_1, \ldots, Z_k \subset \mathbb{P}^n d_1, \ldots, d_k \Gamma_1, \ldots, \Gamma_m \bigcap_1^k Z_j$

$$\sum_{\alpha=1}^{m} \deg(\Gamma_\alpha) \ \leq \ \prod_{i=1}^{k} d_i.$$

$kk = 1k - 1V_i \bigcap_1^{k-1} Z_j Z_k V_i V_i \bigcap_1^k Z_j V_i \cap Z_k d_k \deg V_i \bigcap_1^k Z_j d_i \bigcap_1^{k-1} Z_j$ □

$C \subset \mathbb{P}^4 X = Q_1 \cap Q_2 \cap Q_3 CCXXCSXS$

$C \subset \mathbb{P}^4 C$

$CCW^1_4(C)C\mathbb{P}^1$

$CCg^1_3 W^1_4(C)C$

$I_C = (Q_1, Q_2, Q_3, F_1, F_2) Q_i F_i Q_i$

$Q_i I_C 4 \times 45 \times 5$

$$\begin{pmatrix} 0 & g_1 & g_2 & \ell_0 & \ell_1 \\ -g_1 & 0 & g_3 & \ell_2 & \ell_3 \\ -g_2 & -g_3 & 0 & \ell_3 & \ell_4 \\ -\ell_0 & -\ell_2 & -\ell_3 & 0 & 0 \\ -\ell_1 & -\ell_3 & -\ell_4 & 0 & 0 \end{pmatrix}$$

$\ell_0, \ldots, \ell_3 g_1, g_2, g_3 2 \times 2 Q_i C Q_i 5 \times 5 I_C 4 \times 4$

$C|D|g^1_3C$

$D \sim K - D$

$\mu : H^0(D) \otimes H^0(K-D) \to H^0(K)$

QC

$|D|g^1_3C$

g^1_3CQ

$CCC_0p, q \in C_0CpqDD \cap C$

$CDCg + 3h^0(D) = 4\phi_D : C \to \mathbb{P}^3C \subset \mathbb{P}^3STS \cap T$

$C \subset \mathbb{P}^3S \cap T = C \cup DD$

$CC\phi_D : C \to \mathbb{P}^3Q(2,5)QDD \sim 2g^1_2 + p + q + rQ\phi_D(C)Q$

$Eg^1_2\mathbb{P}^3$

$M^r_{d,g}g^r_dg$

$$M^r_{d,g} = \{(C,L) \mid Cg, L \in \mathrm{Pic}^d(C)h^0(L) \geq r+1\}.$$

$M^1_{3,4}M_4$

$$\Gamma := \{(Q,L) \in \mathbb{P}^9 \times \mathbb{G}(1,3) \mid L \subset Q\}$$

$\mathbb{P}^9Q \subset \mathbb{P}^3$

$S \subset \mathbb{P}^4$

C

$Cg + 3C\mathbb{P}^3$

$|K_C - p|C$

Γ

$\mathbb{P}^r C \subset \mathbb{P}^r \Gamma = C \cap H C$

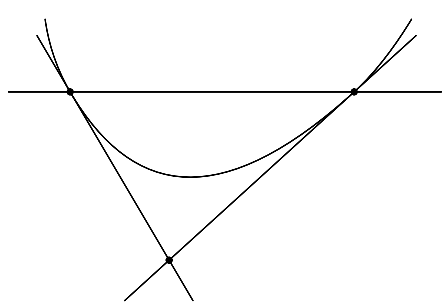
$r C r \Gamma$

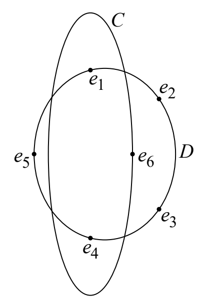
$C \subset \mathbb{P}^r k C p \in \mathbb{P}^r C C p > 0$

$C \subset \mathbb{P}^n C$

$n \geq 3 C \subset \mathbb{P}^r H r H \cap C$

$C \subset \mathbb{P}^3 C p \neq q C \mathbb{T}_p(C), \mathbb{T}_q(C) C p q$





$\mathbb{P}^3 C \mathbb{P}^2 \mathbb{P}^3 e_1, \dots e_6 CDC$

$C_2 C$

$$mult(C) := \{(p,q) \in C_2 \mid \overline{p,q} C \geq 3\};$$
$$stat(C) := \{(p,q) \in C_2 \setminus \Delta \mid C p, q\},$$

$\overline{p,q} p, q p = q \Delta \subset C_2 C stat(C) = C_2 C \deg C > 2 mult(C) = C_2$

$C \subset \mathbb{P}^r mult(C) \subset C_2 stat(C) \subset C_2 \setminus \Delta \leq 1$

$mult(C)$

$$\{((p,q),r) \in C_2 \times \mathbb{P}^r \mid r \in \overline{p,q}\} \to C_2$$

$((p,q),r)(p,q) \geq 3 mult(C) mult(C)$

$$\{((p,q),r) \in (C_2 \setminus \Delta) \times \mathbb{P}^r \mid r \in T_p(C) \cup T_q(C)\}$$

$(C_2 \setminus \Delta) \times \mathbb{P}^r stat(C) C_2 \setminus \Delta$

$\dim stat(C) > 1 stat(C) C_2 \setminus \Delta stat(C) = C_2 \setminus \Delta CCC \dim stat(C) \leq 1$

$\dim mult(C) > 1 \dim stat(C) = 2$

$\pi_p : C \to \mathbb{P}^{r-1}$

$$I = \{(p,r,r') \in C^3 \mid p, r, r' \pi_p(r) = \pi_p(r'), \pi_p(C).$$

$T_r(C) T_{r'}(C) p T_{\pi_p(r)}(\pi_C) p (r,r') \in stat(C) I \to C_2 : (p,r,r') \mapsto (r,r') \overline{r,r'} C$ $\dim I \leq \dim stat(C)$

$pCpCCp \in Cq \in \pi_p(C) \pi_p^{-1}(q) CI \to C : (p,r,r') \mapsto pIC2 = \dim I \leq \dim stat(C)$ $\square$

$n = 3$

$$I := \{((p,q),H) \in mult(C) \times \mathbb{P}^{3*} \mid \overline{p,q} \in H\}.$$

$\dim \overline{p,q} = 1 I mult(C) \dim I = 1 + \dim mult(C) \leq 2 I \mathbb{P}^{3*} \leq 2$

$\mathbb{P}^n n + 1 \mathbb{P}^n n$

$n \geq 4 C' C \pi_p : C \to C' C C \subset \mathbb{P}^r r, r' \in C \subset \mathbb{P}^r T_r(C) \cap T_{r'}(C) = \emptyset T_r(C)$ $T_{r'}(C) L C p \in C L \pi_p L T_{\pi_p(r)}(C') \cap T_{\pi_p(r')}(C') = \emptyset C' n \geq 4C'$

Cr

$$I_1 \subset I \subset \{(p, H) \in C \times \mathbb{P}^{n*}\}$$

$p \in H I_1 p_2, \dots, p_r \in H p, p_2, \dots, p_r I_1 I$

$I_1 I \mathbb{P}^{n*} I C \mathbb{P}^{r-1} I_1 = I p \{p, p_2, \dots, p_r\} p$

$H' \mathbb{P}^{n-1} H = \pi_p^{-1}(H') p H \cap C r \{p, p_2, \dots, p_r\} \pi_p(p_2), \dots \pi_p(p_n) H' \cap C' C$ $\square$

$g C C \mathbb{P}^r \mathrm{D} = (\mathcal{L}, V) \phi_{\mathrm{D}}$

$d \mathbb{P}^r d r$

$$M := M(d, r) := \lfloor (d-1)/(r-1) \rfloor,$$

$$d - 1 = M(r-1) + \epsilon \quad \epsilon = \epsilon(d, r) 0 \leq \epsilon \leq r - 2.$$

$C \subset \mathbb{P}^r d M \epsilon$

$$p_a(C) \leq \pi(d, r) := \frac{M(M-1)}{2}(r-1) + M\epsilon.$$

$p_a(C) = \pi(d, r) C H \cap C \dim(R_{H \cap C})_m = \min\{m(r-1), d\}.$

$C \mathbb{P}^r h^0(\mathcal{O}_C(1)) \geq r + 1$

$$g = d + 1 - h^0(\mathcal{O}_C(1)) + h^1(\mathcal{O}_C(1)),$$

$h^1(\mathcal{O}_C(1))$

$h^0(\mathcal{O}_C(1)) h^0(\mathcal{O}_C(m)) m h^1(\mathcal{O}_C(m)) = 0 m h^0(\mathcal{O}_C(m)) m g$

$h^0(\mathcal{O}_C(m)) h^0(\mathcal{O}_C(m)) - h^0(\mathcal{O}_C(m-1)) \Gamma = C \cap H C H^0(\mathcal{O}_C(m-1)) H^0(\mathcal{O}_C(m))$ $\Gamma \Gamma C$

$\mathcal{V} = (V, \mathcal{L}) X \Gamma \Gamma \mathcal{V} V \Gamma V H^0(\mathcal{L}|_\Gamma) = H^0(\mathcal{L} \otimes \mathcal{O}_\Gamma)$

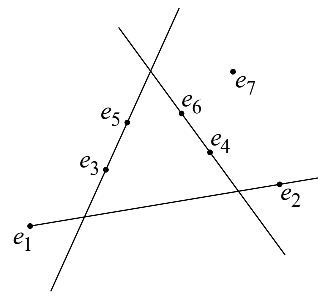
$$\dim(V) - \dim\big(V \cap H^0(\mathcal{L} \otimes \mathcal{I}_{\Gamma/X})\big).$$

$\Gamma \subset \mathbb{P}^r \Gamma H^0(\mathcal{O}_{\mathbb{P}^r}(m)) h_\Gamma(m) \Gamma m \Gamma \Gamma V d \Gamma \Gamma V$

$C \subset \mathbb{P}^r m h^0(\mathcal{O}_C(m)) = md - p_a(C) + 1 h^0(\mathcal{O}_C(m))$

$\Gamma = C \cap H C V_m \subset H^0(\mathcal{O}_C(m)) C m \mathbb{P}^r$

$$H^0(\mathcal{O}_{\mathbb{P}^r}(m)) \to H^0(\mathcal{O}_C(m)).$$



$d = 3 * 2 + 1 e_7 e_1, \dots, e_6$

$\Gamma V_m \rho_m : V_m \to \mathcal{O}_\Gamma(m) \Gamma H^0(\mathcal{O}_C(m))$

$$\begin{aligned} h^0(\mathcal{O}_C(m)) - h^0(\mathcal{O}_C(m-1)) &= \Gamma H^0(\mathcal{O}_C(m)) \\ &\geq \Gamma V_m \\ &= \Gamma H^0(\mathcal{O}_{\mathbb{P}^r}(m)) \\ &= h_\Gamma(m). \end{aligned}$$

$h^0(\mathcal{O}_C(m))$

$$h^0(\mathcal{O}_C(m)) \geq \sum_{k=0}^{m} h_\Gamma(k).$$

$C \Gamma \Gamma \Gamma$

$\Gamma \subset \mathbb{P}^r d \mathbb{P}^r$

$$h_\Gamma(m) \geq \begin{cases} mr + 1 m \leq M(d, r+1) \\ d \end{cases}$$

$mr + 1 \Gamma C \subset \mathbb{P}^r r h_\Gamma(m) = \min\{\deg \Gamma, mr + 1\} m$

$d \geq mr + 1 p_1, \dots, p_{mr+1} \in \Gamma mr + 1 \Gamma' = \{p_1, \dots, p_{mr+1}\} H^0(\mathcal{O}_{\mathbb{P}^r}(m)) p_i \in \Gamma'$
$X \subset \mathbb{P}^r m p_1, \dots, \hat{p}_i, \dots, p_{mr+1} p_i$

$X m r \Gamma' \setminus \{p_i\} m \Gamma_k r \Gamma_k H_k \subset \mathbb{P}^r X = H_1 \cup \cdots \cup H_m$

$d < mr + 1 mr + 1 - dm$ $\square$

$C \subset \mathbb{P}^r d M = M(d,r) = \lfloor \frac{d-1}{r-1} \rfloor$

$$\begin{aligned} h^0(\mathcal{O}_C(M)) &= \sum_{k=0}^{M} h^0(\mathcal{O}_C(k)) - h^0(\mathcal{O}_C(k-1)) \\ &\geq \sum_{k=0}^{M} (k(r-1)+1) \\ &= \frac{M(M+1)}{2}(r-1) + M + 1 \end{aligned}$$

$$h^0(\mathcal{O}_C(M+m)) \geq \frac{M(M+1)}{2}(r-1) + M + md + 1$$

$m \mathcal{O}_C(M+m)$

$$\begin{aligned} g &= (M+m)d - h^0(\mathcal{O}_C(M+m)) + 1 \\ &\leq (M+m)d - \big(\frac{M(M+1)}{2}(r-1) + M + 1 + md\big) + 1 \\ &= M\big(M(r-1) + 1 + \epsilon\big) - \left(\frac{M(M+1)}{2}(r-1) + M\right) \\ &= \frac{M(M-1)}{2}(r-1) + M\epsilon. \end{aligned}$$

$C V_m = H^0(\mathcal{O}_C(m)) m$

$$\begin{tikzcd} 0 \arrow[r] & H^0(\mathcal{O}_{\mathbb{P}^r}(m-1)) \arrow[r] \arrow[d, "surjection"'] & H^0(\mathcal{O}_{\mathbb{P}^r}(m)) \arrow[r] \arrow[d, "surjection"'] & H^0(\mathcal{O}_\Gamma(m)) \arrow[d, "="'] \\ 0 \arrow[r] & V_{m-1} \arrow[r] \arrow[d, "injection"'] & V_m \arrow[r] \arrow[d, "injection"'] & H^0(\mathcal{O}_\Gamma(m)). \arrow[d, "="'] \\ 0 \arrow[r] & H^0(\mathcal{O}_C(m-1)) \arrow[r] & H^0(\mathcal{O}_C(m)) \arrow[r] & H^0(\mathcal{O}_\Gamma(m)) \end{tikzcd}$$

$V_{m-1} \to V_m V_m \to H^0(\mathcal{O}_\Gamma(m)) \Gamma V_m \dim V_m - \dim V_{m-1}$

$g = \pi(d,r)$

$$\begin{aligned} &\Gamma H^0(\mathcal{O}_C(m)) \\ &\geq \Gamma V_m, \end{aligned}$$

$$h^0(\mathcal{O}_C(m)) - h^0(\mathcal{O}_C(m-1)) \geq \dim V_m - \dim V_{m-1}$$

$mmH^0(\mathcal{O}_{\mathbb{P}^r}(m)) \to H^0(\mathcal{O}_C(m))$

$$\begin{aligned}&\sum_{k=0}^{m}(h^0(\mathcal{O}_C(k)) - h^0(\mathcal{O}_C(k-1))\\ &= h^0(\mathcal{O}_C(m)) = \dim V_m\\ &= \sum_{k=0}^{m}(\dim V_k - \dim V_{k-1}).\end{aligned}$$

k

$$\dim V_k - \dim V_{k-1} \geq h^0(\mathcal{O}_C(k)) - h^0(\mathcal{O}_C(k-1)),$$

$k \dim V_k = h^0(\mathcal{O}_C(k)) kC$

$p_a(C) = \pi(d,r)\Gamma = H \cap C h_\Gamma(m) = \min\{m(r-1)+1, d\} m \geq 0 C h_\Gamma(m) = h^0(\mathcal{O}_C(m)) - h^0(\mathcal{O}_C(m-1))\Gamma$ □

$r = 2\pi(d,r) = \binom{d-1}{2}$

$r = 3$

$$\pi(d,r) = \begin{cases} \left(\frac{d-2}{2}\right)^2, & d\\ \left(\frac{d-1}{2}\right)\left(\frac{d-3}{2}\right), & d\end{cases}$$

$(\frac{d}{2}, \frac{d}{2}) d (\frac{d+1}{2}, \frac{d-1}{2}) dr = 3$

dr

$C \subset \mathbb{P}^r p_a dC$

$d < 2rd \geq 2p_a + 1$

$d = 2rC$

$r = 3, d > 6CmHmH + LHL$

$d > 2rC$

$d \geq 2p_a + 1 d < 2r p_a = \pi(d,r) p_a = \pi(d,r) d < 2rd \geq 2p_a + 1$

$d = 2r\pi(d,r) = r + 1C$

$r = 3, d > 6\Gamma C h_\Gamma(2) = \min(2 \cdot 2 + 1, d) = 5\Gamma CCCr > 3$ □

$C \subset \mathbb{P}^{g-1} S_C C g$

$$\dim(S_C)_n = h^0(\mathcal{O}_C(n)) = \begin{cases} 0 & d < 0 \\ 1 & d = 0 \\ g & d = 1 \\ (2g-2)n + 1 - g & d > 1 \end{cases}$$

$C \bigoplus_{n \in \mathbb{Z}} H^0(\mathcal{O}_C(n))$ $\square$

$> 0 d g \mathbb{P}^3$

$$g \leq \pi_1(d,3) := \frac{d^2 - 3d}{6} + 1$$

$\pi_1(9,3) = 10 < \pi(9,3) = 12 \mathbb{P}^r r$

$C C_0 \subset \mathbb{P}^2$

$C \subset \mathbb{P}^r \Lambda \cong \mathbb{P}^k \subset \mathbb{P}^r k \pi_\Lambda : C \to \mathbb{P}^{n-k-1} \Lambda C n - k - 1 \geq 3k \leq n - 4$ $\pi_\Lambda : C \to \mathbb{P}^{n-k-1} C n - k - 1 = 2k = n - 3 \pi_\Lambda$

$C \overline{q,r} q, r \in C \mathbb{T}_q(C) C_2 C \lambda \mathbb{P}^r \overline{\lambda}$

$$I := \{(\lambda, p) \mid \lambda \in C_2 C, p \in \overline{\lambda} \subset \mathbb{P}^r\}$$

$C_2 \mathbb{P}^1 \mathbb{P}^r C \leq 3(n-4)$

$n > 3(n-4) \Lambda n = 3 \Lambda \mathbb{P}^3 C \Lambda \pi_\Lambda$

$\pi_\Lambda \Lambda$

$C \pi_\Lambda \leq 2 I \mathbb{P}^3 \Lambda \pi_\Lambda$

$\pi_\Lambda C_0 \subset \mathbb{P}^2 q, r \in C \Lambda \mathbb{T}_q(C) \mathbb{T}_r(C) \mathbb{P}^2 \pi_p(\mathbb{T}_q(C)) = \pi_p(\mathbb{T}_r(C)) \mathbb{T}_q(C) \mathbb{T}_r(C)$

$C C C$ $\square$

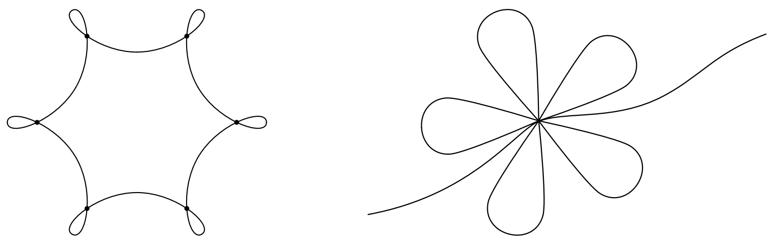
$W_d^r(C) d C r(D) \geq r$

$C g r > d - g$

$$\dim W_d^r(C) \leq d - 2r.$$

$C d = r = 0 d = 2g - 2, r = g - 1 W_d^r$

$\dim W_d^r(C) \leq d - 2r \dim C_d^r \ \leq \ d - r$



$g+2g\binom{g}{2}g$

$C \subset \mathbb{P}^n \Sigma_d^r \subset C_d D d C \dim \overline{D} \leq d-r-1 d \leq r r > 0$

$$\dim \Sigma_d^r \leq d-r-1.$$

$$\Gamma := \left\{(D,H) \in \Sigma_d^r \times \mathbb{P}^{r*} \mid \overline{D} \subset H\right\}.$$

$\Gamma \to \mathbb{P}^{n*} \Gamma \Sigma_d^r n-d+r \dim \Sigma_d^r \geq d-r \dim \Gamma \geq n \Gamma \to \mathbb{P}^{r*}$ □

$C d = r = 0 d = 2g-2, r = g-1 \dim W_d^r(C) \leq d-2r-1 \dim W_d^r(C) = d-2r$
C □

$C g D C \leq 2g-2 \deg D = 2r(D) D = 0 D = K_C C$

$C D \deg D = 2r(D) \deg D = 2(\deg D - g + h^1(D)) \deg D = 2g-2h^1(D) \deg D \leq 2g-2h^1(D) \geq 1 r(D) > \deg D - g W_{\deg D}^{r(D)} D \geq 0$ □

$g+2 \quad g+2$

$C g D g+2 C |D| \phi_D : C \to \mathbb{P}^2 C_0 \phi_D C_0$

$C C_0 \binom{g}{2}$

$C C_0 g$

- $D \deg D > h^0(K) D h^0(D) = (g+2) - g + 1 = 3.$
- $|D| p \in C 3 = h^0(D) = h^0(D-p) = (g+1) - g + 1 + h^0(K_C - D + p) K_C - D$ $2g-2-(g+1) = g-3 D g-3+1 \mu(D) \mathrm{Pic}_{g+2}(C).$
- $\phi_D(p) = \phi_D(q) p, q \in C h^0(D-p-q) = 2 D-p-q = K-E E g-2 \mathcal{O}_C(D)$

$$\nu : C_2 \times C_{g-2} \to \mathrm{Pic}_{g+2}(C)$$

$(p+q, E) K_C - E + p + q$
$\nu C_2 \times C_{g-2} \to C_g C_g \to Pic_g \to Pic_{g+2} \nu \nu D p, q \in C \phi_D(p) = \phi_D(q) \phi_D$

- $p,q \in CC_0 h^0(D-p-q) \geq 2h^0(D-2p-2q) \geq 1E = D-2p-2q$ $\overline{h^0(D-2p-2q) \geq 1E\phi_K : C \to \mathbb{P}^{g-1} r(E) = \deg E - 1 - \dim \phi_K(E)}$ $\overline{\phi_K(E)} g - 3h^0(D-p-q) \geq 2\dim \overline{\phi_K(E+p+q)} = g - 2\overline{p,q}(g-3)\overline{\phi_K(E)}$ $C\Lambda \subset \mathbb{P}^{g-1} g - 3\pi_\Lambda : C \to \mathbb{P}^1 \mu(D) C_{g-2} \times C_2 (E, p+q) \mapsto \mathcal{O}(E+2p+2q)$ $C_{g-2} \times C_2 \mathrm{Pic}_{g+2}(C) < gC' \mathcal{O}_C(D) \mathrm{Pic}_{g+2}(C) C_{g+2} \to Pic_{g+2}(C) D$
  $CDg + 2CC_0 = \phi_D(C)$
- $p \in CC_0 d\phi_D p h^0(D-2p) \geq 2D - 2pg^1_g W^1_g = K_C - W_{g-2}\phi_D$

$$\mu(D) \in 2W_1 + K_C - W_{g-2},$$

  $g - 1J(C)|D|$
- $C_0 E = p + q + rh^0(D-E) \geq 1$

$$\mu(D) \in W_3 + W^1_{g-1}$$

  $\dim W^1_{g-1} \leq g - 4 \dim W^1_{g-1} \leq g - 4CC_0$
  $C_0 = \phi_D(C)\binom{g}{2}$

$C|E|g^1_2 CDg + 2D - Eg$

$$D \sim E + p_1 + \cdots + p_g$$

$gp_i D p_i$

$$h^0(D - p_1 - \cdots - p_g) = h^0(E) = 2 = h^0(D) - 1$$

$\phi_D p_i C_0 g p_a(C) = \binom{g+1}{2} C g \delta \binom{g+1}{2} - g = \binom{g}{2} > gg\delta\binom{g}{2}$ □

$C \subset \mathbb{P}^r C \nu_m : \mathbb{P}^r \to \mathbb{P}^N m\widetilde{C} = \nu_m(C) C$

$p \ \widetilde{C}\mathbb{P}^{N-3}$

$C_0 q_1, q_2 \nu : C \to C_0 o \in CC_0 CCo$

$st \in CDs, t, q_1 q_2 uvC_0 \cap DD'u, v, q_1, q_2 os + tD' \cap C_0$

$kp > 0q = p^e e \geq 1C \subset \mathbb{P}^r C_0$

$$\mathbb{A}^1 \ni t \mapsto (t, t^q, t^{q^2}, \ldots, t^{q^r}) \in \mathbb{A}^r$$

$\mathbb{A}^r \subset \mathbb{P}^r x_0 = 1$

C

$$x_0^{q-1} x_2 - x_1^q, x_0^{q-1} x_3 - x_2^q, \ldots, x_0^{q-1} x_r - x_{r-1}^q.$$

$Cq = r = 2$

$C_0 q C_0 a_1, \ldots, a_r C_0 a_1, \ldots, a_r q^{(r-1)} C_0 rq$

$g+2CgDg+1C|E|g_2^1C\phi : C \to \mathbb{P}^1 \times \mathbb{P}^1 \phi_D : C \to \mathbb{P}^1 \phi_E : C \to \mathbb{P}^1 |D||E|$

$\phi C(g+1,2)\mathbb{P}^1 \times \mathbb{P}^1$

$\mathbb{P}^1 \times \mathbb{P}^1 \mathbb{P}^3 Q p \in C \subset QCpC_0 g+2g$

$h^0(D-K_C)=0\phi_D \times \phi_E LpCLQ$

$m\mathbb{P}^d mC \subset \mathbb{P}^d \Gamma C$

$$0 \to \mathcal{I}_{C/\mathbb{P}^d}(l-1) \to \mathcal{I}_{C/\mathbb{P}^d}(l) \to \mathcal{I}_{\Gamma/\mathbb{P}^{d-1}}(l) \to 0.$$

$2 \le l \le m$

$$h^0(\mathcal{I}_{C/\mathbb{P}^d}(m)) \le \binom{d+m}{m} - (md+1)$$

C

$D \subset \mathbb{P}^r \Gamma \subset DdDDd\Gamma$

$$h_\Gamma(m) = \min\{d, mr+1\}$$

$d > mrm\Gamma mDd \le mrm\Gamma D|\mathcal{O}_D(m)(-\Gamma))| = |\mathcal{O}_{\mathbb{P}^1}(mr-d)|$

$C\Gamma d-1 = M(r-1)+\epsilon\epsilon < r-1$

$\epsilon > 0\mathcal{O}_C(M)\mathcal{O}_C(M-1)$

$\epsilon = 0\mathcal{O}_C(M-1)\mathcal{O}_C(M-2)$

$h^0(\mathcal{O}_C(m+1)) - h^0(\mathcal{O}_C(m)) \le \deg Cm \ge m_0 \mathcal{O}_C(m) m \ge m_0$

$C \subset \mathbb{P}^r d \ge 2r|D| = |\mathcal{O}_C(1)|g_d^r C$

$|D'|g_d^r C h^0(mD+D') > h^0((m+1)D)$

$C = Q \cap S \subset \mathbb{P}^3 QSkg = \pi(2k,3)2k\mathbb{P}^3 C \subset \mathbb{P}^3 2kg = \pi(2k,3) = (k-1)^2$

$\Gamma \subset M_g$

$H \subset M_g$

$\mathrm{H}^\circ(Q,C)Q\mathbb{P}^3 C \subset Q(k,k)\mathrm{H}^\circ \to M_g PGL_4 2g-12g+2\mathbb{P}^1 PGL_2$

k

$C \subset \mathbb{P}^r$

$C \subset \mathbb{P}^r d \mathbb{C} H_0 \subset \mathbb{P}^r C C \cap H_0 = \{p_1, \dots, p_d\} H_0 \{H_t\} U \subset \mathbb{P}^{r*} C p_i(t) C H_t$

$H_t H_0 t = 1 \{H_t\}_{0 \leq t \leq 1} \subset U H_1 = H_0 p_i \{p_i(t) \in C \cap H_t\}_{0 \leq t \leq 1} H_1 = H_0 p_i(1)$ $p_j \in C \cap H_0 C \cap H_0 C \cap H_0$

$f : Y \to X \mathbb{C} X U \subset X U V = f^{-1}(U) f V d K(Y)/K(X)$

$f : V \to U p_0 \in U \Gamma := f^{-1}(p_0) = \{q_1, \dots, q_d\} \gamma U p_0 i = 1, \dots, d \gamma \tilde{\gamma}_i V$ $\tilde{\gamma}_i(0) = q_i \tilde{\gamma}_i(1) = q_j j \in \{1, 2, \dots, d\} j i i \mapsto j \{1, 2, \dots, d\} \Gamma \gamma \pi_1(U, p_0)$

$$\pi_1(U, p_0) \to Perm(\Gamma) \cong S_d.$$

$M f \Gamma S_d$

$U U' \subset U U \setminus U' \geq 2 \pi_1(U', p_0) \to \pi_1(U, p_0) \pi_1(U', p_0) S_d$

$Y f^* K(Y) K(X) f K(Y) K(X)$

$X U V V V \mathbb{P}^3$

$$X := \{(p, Q) \mid p \in Q \mathbb{P}^3\}$$

$\mathbb{P}^3$

$$V := \{((p, Q), L) \mid (p, Q) \in X, L p \in L \subset Q\}.$$

$V V \mathbb{P}^3 \mathbb{P}^1$

$C \subset \mathbb{P}^r d X = \mathbb{P}^{r^*} \mathbb{P}^r C f : Y \to \mathbb{P}^{r^*}$

$$Y = \{(H, p) \in \mathbb{P}^{r^*} \times C \mid p \in H\}.$$

$f C f U \subset \mathbb{P}^{r^*} C f V U d Y p \in C p \mathbb{P}^{r-1} Y f$

$C \subset \mathbb{P}^r S_d$

C

$\phi : Y \to X \tilde{Y}^n / X$

$$\tilde{Y}^n / X := \{(x, y_1, \dots, y_n) \in X \times Y^n \mid \phi(y_i) = x y_i \neq y_j \ \forall i \neq j\}$$

$\tilde{Y}^n / X n \phi$

$f : Y \to X d M \subset S_d M n \tilde{Y}^n / X$

$U \subset X V = f^{-1}(U) \subset Y U V f|_V : V \to U Y^n / X n \tilde{Y}^n / X \tilde{Y}^n / X$ □

$p C \subset \mathbb{P}^r C p C p$

$r > 1 C \subset \mathbb{P}^r C$

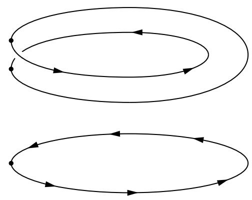
$C \hookrightarrow \mathbb{P}^r v(t) \mathbb{C}^{r+1} t = 0 v(0) v'(0) \in \mathbb{C}^{r+1} v(t) v(t), v'(t) v''(t) t$

$$v(t) \wedge v'(t) \wedge v''(t) \ \equiv \ 0.$$

$v(t) \wedge v'(t) \wedge v''(t)$

$$v(t) \wedge v'(t) \wedge v'''(t) \equiv 0,$$

$v'''(t) v(t) v'(t) v^{(l)}(p) v(t) v'(t) l C$ □



p

$C \subset \mathbb{P}^r r > 1 p \in C \mathbb{T}_p(C) \subset \mathbb{P}^r C$

$D \xrightarrow{v} \mathbb{P}^r D \xrightarrow{w} \mathbb{P}^r$

$C_1, C_2 v, w \tilde{v}, \tilde{w} \mathbb{C}^{r+1}$

$$\Sigma := \{(p,q) \in D \times D \mid \mathbb{T}_{v(p)}(C_1) = \mathbb{T}_{w(q)}(C_2)\}$$

$C v(t) w(t) \tilde{v}(t), \tilde{v}'(t), \tilde{w}(t) \tilde{w}'(t) \Lambda \subset \mathbb{C}^{r+1}$

$$\tilde{v}(t) \wedge \tilde{v}'(t) \wedge \tilde{w}(t) \equiv 0 \qquad \tilde{v}(t) \wedge \tilde{w}(t) \wedge \tilde{w}'(t) \equiv 0$$

$\tilde{v}(t) \tilde{w} t = 0 \Lambda C_1 C_2 \mathbb{P}^r \Lambda$ □

$C C p r = 2 p > 0$

$C \subset \mathbb{P}^r d f : Y \subset \mathbb{P}^{r*} \times C \to X = \mathbb{P}^{r*} U \subset \mathbb{P}^{r*} C V = f^{-1}(U) M \subset S_d V U M M M$

$M \tilde{V}^2 / U$

$$\Sigma := \left\{(H,p,q) \in \mathbb{P}^{r*} \times C \times C \mid p, q \in H p \neq q\right\}.$$

$\Sigma \mathbb{P}^{r-2} C \times C \setminus \Delta C \times C \Sigma \tilde{V}^2 / U \Sigma$

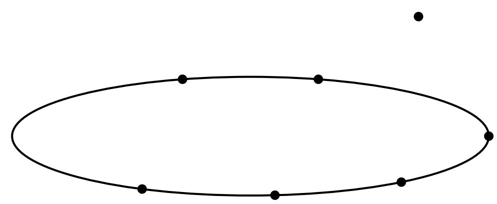
$f : Y \to X d X M \subset S_d p \in X f^{-1}(p) \subset V d - 2 p_1, \dots, p_{d-2} q q Y M$

$Y p U \subset X X V := f^{-1}(U) f|_V : V \to U V d U U, V p \in X$

$B_i \subset Y f^{-1}(p) f|_V A := \cap f(B_i) X A p$

$p' \in A \cap U d f^{-1}(p') B q q' q'' B \cap V B \gamma : [0,1] \to B \cap V q' q'' f^{-1}(p') f \circ \gamma q' q''$
$d - 2 f^{-1}(p')$ □

$C d d - 2 L C C H L C p L \cap C C \cap H H p L H p L H \cap C$ □



$X \subset \mathbb{P}^r k \mathbb{G}(r-k,r)(r-k)\Lambda \subset \mathbb{P}^r(r-k)X$

$$Y = \{(\Lambda, p) \in \mathbb{G}(r-k,r) \times X \mid p \in \Lambda\}.$$

$Y\mathbb{G}(r-k,r)$

$(r-k)kX \subset \mathbb{P}^r S_d$

$(r-k+1)\Gamma \subset \mathbb{P}^r X \subset \mathbb{P}^r k \geq 2C := \Gamma \cap X(r-k)X\mathbb{G}(r-k,\Gamma) \cong \mathbb{P}^{r*}(r-k)\Gamma C$
S_d □

$C \subset \mathbb{P}^r$

$C \subset \mathbb{P}^r dk \leq d\tilde{Y}^k/XC$

$C \subset \mathbb{P}^r \Gamma = H \cap C\Gamma$

$U = \mathbb{P}^{r*} \setminus C^* CY \to U\tilde{V}^n/U$

$m\Gamma m\tilde{V}^n/U\tilde{V}^n/UZ \dim \tilde{V}^n/U = \dim UH \in \mathbb{P}^{r*} Zk\Gamma \subset C \cap H$ □

$C\mathbb{P}^n nCn$

$DCr(D) = \dim |D| = h^0(\mathcal{O}_C(D)) - 1$

D, EC

$$r(D+E) \geq r(D) + r(E).$$

$C> 0|D+E|$

$\mathbb{P}^1 Dr(D) = \deg DC = \mathbb{P}^1$

$|D||E|r(D)+r(E)D+E$

$D+Er(D+E)=r(D)+r(E)C|D+E|H\cap CD$

$Y\deg D+\deg Ey\in Y\phi : Y\to \mathrm{Pic}_d(C)ydY'\mathrm{Pic}_d(C)DDY'\mathbb{P}^{n*}YY'=YddD$

$p\in Dq\notin DD-p+q\equiv Dq\equiv pr(p)\geq 1C\cong \mathbb{P}^1.$ □

$V_{d,g}\mathbb{P}^N dd\delta := \binom{d-1}{2}-gV_{d,g}dg$

$$\Phi := \{(C,p)\in V_{d,g}\times \mathbb{P}^2 \mid p\in C_{sing}\}.$$

$\delta V_{d,g}g=0$

$\Phi V_{d,0}S_\delta\delta=\binom{d-1}{2}$

$C\subset \mathbb{P}^2\tilde{C}\subset \mathbb{P}^d(d-3)\Lambda\subset \mathbb{P}^d\Lambda\Lambda X\subset \mathbb{P}^d\tilde{C}$ □

$V_{d,g}V_{d,0}V_{d,g}V_{d,0}V_{d,g}V_{d,g}V_{d,0}$

g

$\Phi V_{d,g}S_\delta\delta=\binom{d-1}{2}-g$

$X\subset \mathbb{P}^r k\geq 2X$

$k=2XX(r-2)\Lambda\Lambda H\cap XH\Lambda X\cap\Lambda$

$\mathbb{P}^3$

$$Q_t := V(X^2+Y^2+Z^2+tW^2)$$

$t=0$

$p=[1,i,0,0]Q_t pY-iX=Z-\pm i\sqrt{t}W=0t$

$C(1,d-1)\mathbb{P}^2_k kkp>0d-1=p^\ell n\ell>\geq 2npCCd-1Cnp^\ell$

$C\subset \mathbb{P}^r C_i d_i C\prod S_{d_i}$

$C_i^*\subset \mathbb{P}^{r*}H\cap C_iH\cap C_jj\neq iC_i^*\subset \mathbb{P}^{r*}(C_i^*)^*=C_i$

$de\mathbb{P}^M d\mathbb{P}^N e$

$$\Phi := \{(D,E,p)\in \mathbb{P}^M\times\mathbb{P}^N\times\mathbb{P}^2 \mid p\in D\cap E\}.$$

$\pi : \Phi\to \mathbb{P}^M\times\mathbb{P}^N S_{de}de$

$ED \cap ED\nu_d(E)(D,E)D \cap Ede - 2$

$E \subset \mathbb{P}^2 p \in E\mathcal{O}_E(3p) \cong \mathcal{O}_E(1)g = 1\mathbb{P}^9$

$$\Phi := \{(E,p) \in \mathbb{P}^9 \times \mathbb{P}^2 \mid pE\}.$$

$\Phi \to \mathbb{P}^9 S_9$

E

$C\pi : C \to \mathbb{P}^1 n\pi S_n$

$S_n S_n$

$Cg^r_d C \mathcal{L} d C h^0(\mathcal{L}) \geq r+1$

$Cg \mathcal{L} d C h^0(\mathcal{L}) \geq r+1$

$$r \leq \begin{cases} d-g, & d \geq 2g-1; \\ d/2, & 0 \leq d \leq 2g-2. \end{cases}$$

$gg^r_d d \geq \sqrt{g(2r-2)}$

$r, d g g^r_d$

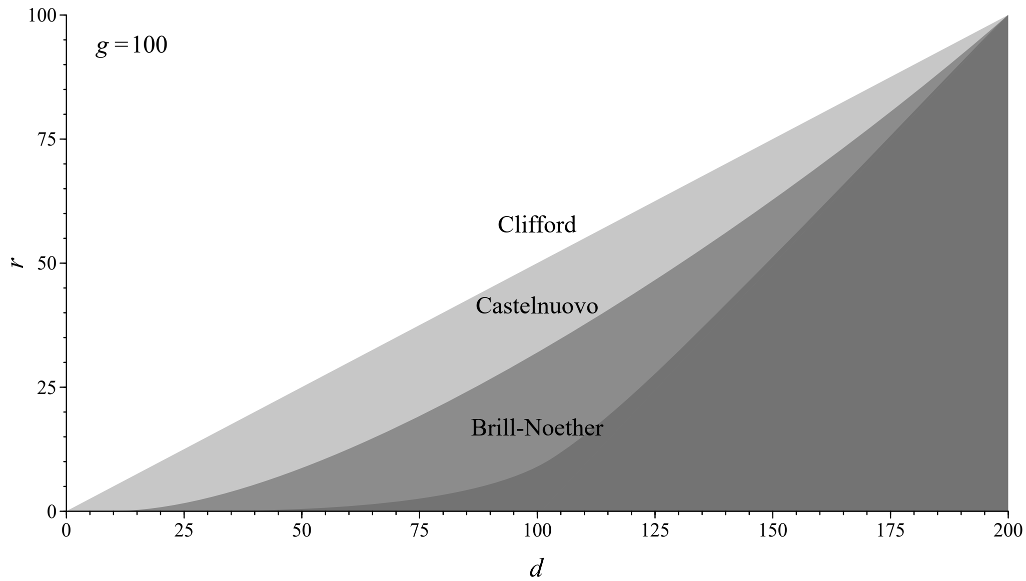
$r \geq 0$

$$\rho(g,r,d) := g-(r+1)(g-d+r) \geq 0,$$

$gg^r_d \rho < 0 C g g^r_d$

d, r

dg, r



(d, r)

$$d \geq \min\{r + g, 2r\}$$
$$d \geq \sqrt{(2r-2)g}$$
$$d \geq r + g - \frac{g}{r+1}$$

$r = 1$

$CgC\mathbb{P}^1 dd \leq \lceil \frac{g+2}{2} \rceil$

$\mathbb{P}^1 g \leq 5$

W_d^r

$CgD = p_1 + \cdots + p_d C p_i$

$Drh^0(D) \geq r + 1 H^0(K - D) Dg - d + r$

$$H^0(K) \to H^0(K|_D) = \bigoplus k_{p_i}$$

$d - r$

$g \times d\omega_1, \dots, \omega_g H^0(K) C U_j p_j \in D z_j U_j p_j$

$$\omega_i = f_{i,j}(z_j) dz_j$$

$U_j r(D) \geq r$

$$A(z_1, \dots, z_d) = \begin{pmatrix} f_{1,1}(z_1) & f_{2,1}(z_1) & \dots & f_{g,1}(z_1) \\ f_{1,2}(z_2) & f_{2,2}(z_2) & \dots & f_{g,2}(z_2) \\ \vdots & \vdots & & \vdots \\ f_{1,d}(z_d) & f_{2,d}(z_d) & \dots & f_{g,d}(z_d) \end{pmatrix}$$

$d - r(z_1, \dots, z_d) = (0, \dots, 0)$

$M_{d,g} d \times gd - rr(g - d + r) D dh^0(D) \geq r + 1 D \in C_d C_d^r d - r(g - d + r) A M_{d,g}$
$d - r\mu : C_d^r \to W_d^r(C) r$

$$\dim W_d^r(C) \geq d - r(g - d + r) - r = g - (r+1)(g - d + r)$$

$W_d^r(C) C W_d^r(C)$

$Cg\rho = g - (r+1)(g - d + r) d \leq g + r$

$\dim(W_d^r(C)) = \rho$

$W_d^r(C) W_d^{r+1}(C)$

$\rho > 0 W_d^r(C)$

$\rho = 0 W_d^r(C)$

$$\# W_d^r(C) = g! \prod_{\alpha=0}^{r} \frac{\alpha!}{(g - d + r + \alpha)!}$$

$M_g \mathcal{W}_d^r W_d^r$

$\mathcal{L} C$

$$m : H^0(L) \otimes H^0(\omega_C \otimes L^{-1}) \longrightarrow H^0(\omega_C)$$

$W_d^r(C) L T_L \mathrm{Pic}_d(C) = H^0(\omega_C)^* mm$

$C \mathcal{L} W_d^r(C) r \geq 2 \mathcal{L}^m m \geq 2$

$$\mathcal{L}^m \omega_C \otimes \mathcal{L}^{-m} = E H^0(\mathcal{L}) = H^0(\omega_C \otimes \mathcal{L}^{-m+1}(-E)) \hookrightarrow H^0(\omega_C \otimes \mathcal{L}^{-m+1})$$

$$m : H^0(\mathcal{L}^{m-1}) \otimes H^0(\omega_C \otimes \mathcal{L}^{-m+1}) \longrightarrow H^0(\omega_C)$$

$$H^0(\mathcal{L}^{m-1}) \otimes H^0(\mathcal{L}) \subset H^0(\mathcal{L}^{m-1}) \otimes H^0(\omega_C \otimes \mathcal{L}^{-m+1})$$

$\sigma, \tau \in H^0(\mathcal{L}) \sigma^{m-1} \otimes \tau - \sigma^{m-2} \tau \otimes \sigma \mathcal{L}^m$ $\square$

$g_d^1 \rho = 0 g = 2d - 2 C_{d-1} := \frac{1}{d}\binom{2d}{d}$

$|K| g_2^1 g_3^1 g_4^1$

$\mathcal{L} \in W^r_d(C) \setminus W^{r+1}_d(C) W^r_d \mathcal{L} (H^0(\omega_C))^* \mu\mu W^r_d \mathcal{L}$

$C G^r_d(C) d r$

$$G^r_d(C) = \left\{ (\mathcal{L}, V) \mid \mathcal{L} \in Pic_d(C), V \subset H^0(\mathcal{L}) \dim V = r+1 \right\}.$$

$G^r_d(C) W^r_d(C) W^r_d(C) \setminus W^{r+1}_d(C) W^{r+1}_d(C) G^r_d(C) d r$

$g+1 g+2 g+3 \mathbb{P}^1, \mathbb{P}^2 \mathbb{P}^3 . \geq g g+1, g+2, g+3$

$C g |D| g^r_d C$

$r \geq 3 D \phi_D : C \to \mathbb{P}^r C \mathbb{P}^r$

$r = 2 \phi_D : C \to \mathbb{P}^2 C$

$r = 1 \phi_D : C \to \mathbb{P}^1 C \mathbb{P}^1$

$\rho = 0 g^r_d C g^r_d C$

$C \subset \mathbb{P}^n H^0(\mathcal{O}_{\mathbb{P}^n}(d)) \to H^0(\mathcal{O}_C(d))$

$C g \mathcal{L} \in W^r_d(C) m > 0$

$$\rho_m : \operatorname{Sym}^m H^0(\mathcal{L}) \to H^0(\mathcal{L}^m)$$

$\binom{m+r}{r} \leq h^0(\mathcal{L}^m) \binom{m+r}{r} \geq h^0(\mathcal{L}^m)$

$\mathcal{L} \in W^r_d(C) h^0(\mathcal{L}^m) = md - g + 1 m \geq 2d$

$$h_C(m) = \min\left(\binom{m+r}{r}, md - g + 1 \right).$$

$d, g r \rho(d, g, r) \geq 0 d g n \mathbb{P}^r$

$$(r-1)n \leq (r+1)d - (r-3)(g-1)$$

$(d, g, r) = (5, 2, 3), (6, 4, 3), (7, 2, 5) (10, 6, 5)$

$C \subset \mathbb{P}^r m C m \rho_m C m_0 m I(C)_m \neq 0 m \binom{m+r}{r} > md - g + 1 I(C) I(C)_{m_0} I(C) I(C)$
$m m + 1$

$I(C)_m m$

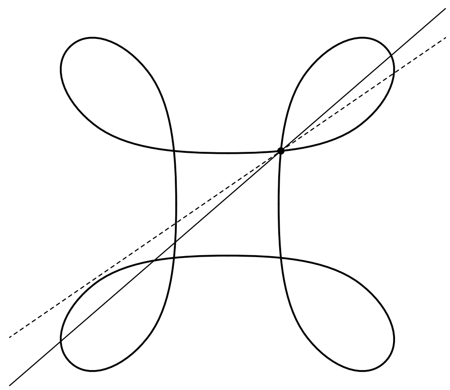
$$\sigma_m : I(C)_m \otimes H^0(\mathcal{O}_{\mathbb{P}^r}(1)) \to I(C)_{m+1}$$

$m \sigma I(C)$

$\rho \geq 0$

$C \subset \mathbb{P}^r s t s (r - t - 1) \leq (t+1)(r-t) C \subset \mathbb{P}^3 \mathbb{P}^r$

$C g D g + 3 C \phi_D : C \hookrightarrow \mathbb{P}^3 C \mathbb{P}^3$



g^1_4

$g^r_d W^r_d(C)$

$W^r_d(C) W^r_d(C)$

$C \subset \mathbb{P}^5$

$C g^2_6 C W^2_6(C) W^1_4(C) W^2_5 = W^1_3 = \emptyset g^2_6 g^1_4 g^2_5 g^1_3 g^2_6$

C

$C C_0 \binom{6-1}{2} = 10 C_0$

$C g^2_6 g^1_4 g^1_6 f : C \to \mathbb{P}^1 g^1_4 g^1_4$

$f : X \to S X S \mathcal{L} S V \subset H^0(\mathcal{L}) X f^* V \subset H^0(f^*\mathcal{L}) X X V$

$g^1_4 C C f : C \to \mathbb{P}^2 C_0 p \in C_0 (\mathcal{O}_{\mathbb{P}^2}(1), V) \mathbb{P}^2 p f^* \mathcal{O}_{\mathbb{P}^2}(1) f^* V q, r \in C p C V g^1_4$

$g^1_4 C C_0 f^* \mathcal{O}_{\mathbb{P}^2}(2) g^1_4 C C_0 W^1_4(C)$

$C 3 \cdot 6 - 8 = 10 C$

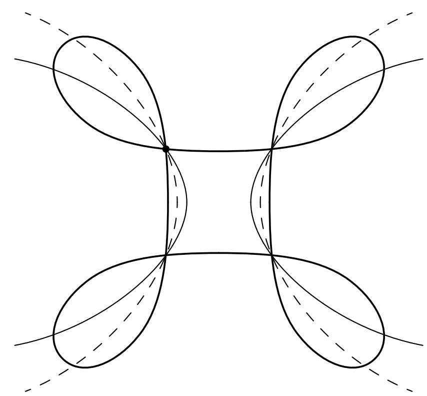
$f : C \to C_0 \subset \mathbb{P}^2 g^2_6 g^2_6 C g^1_4 g^2_6 C C_0 C_0 \mathbb{P}^2$

$\leq 5 C \mathbb{P}^5 h^0(\mathcal{O}_C(2)) = 2(2g-2) - g + 1 = 15 C 6 > \operatorname{codim} C$

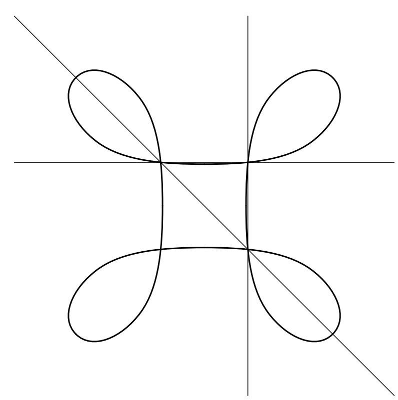
$\mathbb{P}^5$

$\mathbb{P}^n - K_S 3 \leq n \leq 9 \mathbb{P}^3$

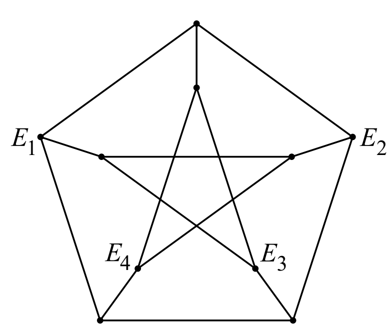
$\mathbb{P}^n n \mathbb{P}^2 9 - n \mathbb{P}^2 9 - n n = 8 (2,2) \mathbb{P}^1 \times \mathbb{P}^1$



g^1_4



g^2_6



$p_1, \dots, p_4 p_1, \dots, p_4 S \subset \mathbb{P}^5 5$

$S \subset \mathbb{P}^5 S \mathbb{P}^2 p_1, \dots, p_4 \in \mathbb{P}^2 p_i \mathbb{P}^5 G(2,5) \subset \mathbb{P}^9 S 5 \times 5 \mathbb{P}^5$

$\mathbb{P}^2 S \phi_{-K}(S)$

$C C_0 C S \subset \mathbb{P}^5$

C

$C \subset \mathbb{P}^5 S \subset \mathbb{P}^5 C C S C$ □

$g = 7, 8, 9$

$C g^2_6 \phi_D C g^1_4 \mathbb{P}^2$

$C g^3_6 g^2_6 |D| D$

C

$|D| C$

$\phi_D C E \subset \mathbb{P}^2 E \subset \mathbb{P}^4$

ϕ_D

$C S(a, 4-a) |D| \phi_D C$

$a W^1_4$

$|D|$ $g^2_3 |D| g^2_5$

$C \to C_0 \subset \mathbb{P}^r g^r_d d C_0 C \to C_0 \phi_D : C \to \mathbb{P}^2 \phi_D(C) C$

$C |K_C| C$

$\mathbb{P}^2 \mathbb{P}^5 C C 2 \times 2 3 \times 3$

$$H^0(\mathcal{O}_{\mathbb{P}^2}(1)) \otimes H^0(\mathcal{O}_{\mathbb{P}^2}(1)) \to H^0(\mathcal{O}_{\mathbb{P}^2}(2))$$

$C \mathbb{P}^2$

$g^1_4 C C W^1_4(C) \cong C g^1_3$

$C \phi_D$ $E C C C$

$K_C \mathcal{O}_E(F) E |K_C| |\mathcal{O}_E(F)| \phi_K : C \to \mathbb{P}^5 \mathbb{P} H^0(K_C) X \in \mathbb{P}^5 \pi^* H^0(F) \hookrightarrow H^0(K_C)$
$C \phi_F(E) \subset \mathbb{P}^4 C S = \overline{X, E} E \subset \mathbb{P}^4$

$\phi_F(E) \subset \mathbb{P}^4 S \subset \mathbb{P}^5 Q \subset \mathbb{P}^5 C S$

CE

$CC\mathbb{P}^5$

$FE\phi_F(E) \subset \mathbb{P}^4\phi_F(E)S \subset \mathbb{P}^5$

$Q \subset \mathbb{P}^5CS$

$\phi_F(E)$

$CEg^1_4Cg^1_2Eg^2_6Cg^2_3EW^1_4(C)W^2_6(C)E$

C

$Cg^1_3|E|p \in C|K_C - E - p|g^2_6C \to \mathbb{P}^2C$

$CC_0g^2_6C$

Cg^2_6

$CC_0W^1_4(C) \cong W^2_6(C)$

$CW^1_4(C)W^1_4(C) \cong \operatorname{Spec}\mathbb{C}[\epsilon]/(\epsilon^5)$

$g^1_4g^1_4C$

$Cg^2_6C|K_C| + pp \in C$

$CSCSCSC > 2$

$C_0\mu : H^0(D) \otimes H^0(K - D) \to H^0(K)g^1_4C$

$C\mathbb{P}^1$

$\mathcal{D} = (\mathcal{L}, V)C$

$V\mathcal{L}C p \in C\sigma_0, \dots, \sigma_r V p$

$$\{\mathrm{ord}_p(\sigma) \mid \sigma \neq 0 \in V\}$$

$\dim V$

$\tau_0, \dots, \tau_r V \tau_i \tau_j p \tau_i' := a\tau_i + b\tau_j ab\tau_i\tau_i'\sigma_i \deg \mathcal{L}(r+1)\deg \mathcal{L}$ □

$\mathrm{D} = (\mathcal{L}, V)\mathrm{g}_d^r C$

$$\{\mathrm{ord}_p(\sigma) \mid \sigma \neq 0 \in V\} = \{a_0, \dots, a_r\} \;\; 0 \leq a_0 < a_1 < \cdots < a_r.$$

$a_i = a_i(\mathrm{D}, p)\mathrm{D}pa_i \geq i\alpha_i = \alpha_i(\mathrm{D}, p) := a_i - i0 \leq \alpha_0 \leq \alpha_1 \leq \cdots \leq \alpha_r \mathrm{D}p$

$p\mathrm{D}(\alpha_0, \dots, \alpha_r) \neq (0, \dots, 0)\alpha_r > 0p$

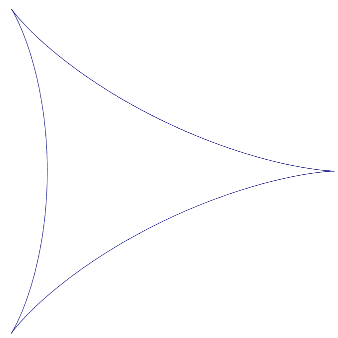
$$w(\mathrm{D}, p) = \sum_{i=0}^{r} \alpha_i(\mathrm{D}, p).$$

$\mathrm{D}CC \subset \mathbb{P}^r pa_r > rH \subset \mathbb{P}^r r + 1Cp$

$\alpha_0(\mathrm{D}, p)p\mathrm{D}\alpha_0(\mathrm{D}, p) = 0\alpha_1(\mathrm{D}, p) = 0\phi_{\mathrm{D}}p$

$Cg\mathrm{D}dr$

$$\sum_{p \in C} w(\mathrm{D}, p) \;=\; (r+1)d + r(r+1)(g-1).$$



$C \subset \mathbb{P}^r C r$

$C \subset \mathbb{P}^r d g C$

$$(r+1)d + r(r+1)(g-1) = 0.$$

$g = 0 (r+1)(d-r) = 0 d = r C$ □

$C \subset \mathbb{P}^r r$

C

C

$r+1 C \mathbb{P}^r$

C

$p \in C (r+1) p C \subset \mathbb{P}^r \Gamma \subset C C r + 1 \Gamma \mathbb{P}^r$ □

$g d$

$C g \mathcal{L} \in W^r_d(C) \subset \mathrm{Pic}_d(C) d h^0(\mathcal{L}) = r + 1 V = H^0(\mathcal{L}) \mathrm{D} = (\mathcal{L}, V)(0, \dots, 0, 1)$

$d g = \binom{d-1}{2} 3(d-2) d$

$\sum w(C, p) p C 3(d-2) C F(x_0, x_1, x_2) d C$

$$Hess(C) := \begin{pmatrix} \partial^2 F/\partial x_0 \partial x_0 & \partial^2 F/\partial x_0 \partial x_1 & \partial^2 F/\partial x_0 \partial x_2 \\ \partial^2 F/\partial x_1 \partial x_0 & \partial^2 F/\partial x_1 \partial x_1 & \partial^2 F/\partial x_1 \partial x_2 \\ \partial^2 F/\partial x_2 \partial x_0 & \partial^2 F/\partial x_2 \partial x_1 & \partial^2 F/\partial x_2 \partial x_2 \end{pmatrix}$$

$CCC\det Hess(C)$

$C|K_C|$

$pCCgpw_p p \in Cw(|K_C|, p)p$

C

Cg

$$\sum_{p\in C} w_p = g^3 - g.$$

□

$Cgg^3 - gC$

$CC\phi_K : C \to \mathbb{P}^1 C$

$CC \to \mathbb{P}^1 C\alpha C\alpha 24 - 2\alpha\alpha$

$$h^0(\mathcal{O}_C(gp)) = g - g + 1 + h^0(K_C(-gp))$$

$h^0(K_C(-gp)) \neq 0ph^0(\mathcal{O}_C(gp)) > 1$

$p \in CCC \setminus \{p\}gp$

$pC \setminus \{p\}$

$$W(C,p) := \left\{-\operatorname{ord}_p(f) \mid f \in K(C)fC \setminus \{p\}\right\}.$$

$\mathbb{N}p$

$CC \setminus \{p\}kp$

$$h^0(\mathcal{O}_C(kp)) = h^0(\mathcal{O}_C((k-1)p)) + 1.$$

$$h^0(K_C(-kp)) = h^0(K_C((-k+1)p)).$$

$CC \setminus \{p\}kpCk - 1p\mathbb{N} \setminus W(C,p)pgp$

$\mathbb{N}w \leq g/2$

CC

$C \geq 2\mathrm{Aut}(C)$

$Aut(C)\mathbb{Z}/2$

$C \times CN(S)SN(S)H, H'DSH \cdot D = H' \cdot DN(S)N(S)S$

$H \subset SD \neq 0 \in N(S)D \cdot H = 0D^2 < 0$

□

$DD^2 > 0$

$Cg \geq 2f : C \to CC$

$f2g + 3f$

$f2g + 2fCf$

$S = C \times C\Delta\Gamma \subset Sf\Phi_1\Phi_2 \subset S\delta, \gamma, \varphi_1\varphi_2 N(S)fb = \delta \cdot \gamma$

$\varphi_1\varphi_2$

$$0 \to \mathcal{T}_\Delta \to \mathcal{T}_{C\times C}|_\Delta \to \mathcal{N}_{\Delta/C\times C} \to 0$$

C

$$\delta^2 = 2 - 2g.$$

$id_C \times f : C \times C \to C \times C\Delta\Gamma\gamma^2 = 2 - 2g$

$$\delta' = \delta - \varphi_1 - \varphi_2 \qquad \gamma' = \gamma - \varphi_1 - \varphi_2,$$

$\delta'\gamma'\varphi_1 + \varphi_2\varphi_1 + \varphi_2\langle\delta', \gamma'\rangle \subset N(S)$

	δ'	γ'
δ'	$-2g$	$b-2$
γ'	$b-2$	$-2g$

$b = \gamma \cdot \delta b \leq 2g + 2$ □

fC

$Cg \geq 2C2g + 22g + 2$

$p \in C w_1 = w_1(p), \ldots, w_g = w_g(p) |K_C| p$

$$h^0(K_C(-(w_i + i)p)) = g - i.$$

$$g - i \leq \frac{2g - 2 - w_i - i}{2} + 1;$$

$$w_i \leq i$$

$$w_p \leq \binom{g}{2}$$

$w_p p C g^3 - g$

$$\frac{g^3 - g}{\binom{g}{2}} = 2g + 2.$$

$\square$

$g > 2$

$M_g g \geq 3g - 2$

$C g \phi : C \to C \phi p \langle \phi \rangle = \{1, \phi, \phi^2, \ldots, \phi^{p-1}\} C \phi B = C / \langle \phi \rangle C \pi : C \to B C p B b$ $q_1, \ldots, q_b p - 1 B B C$

$[C] \in M_g p 3h - 3 + b$

$$3g - 3 - (3h - 3 + b) \geq g - 2$$

$$2g - 1 - (3h - 3) - b \geq 0.$$

$C \to B$

$$2g - 2 = p(2h - 2) + b(p - 1).$$

$$p(2h - 2) + b(p - 1) + 1 - (3h - 3) - b \geq 0$$

$$(2p - 3)h - 2p + b(p - 2) + 4 \geq 0.$$

$h = 0$

$$(b - 2)(p - 2) \geq 0;$$

$h = 0 b$ $\square$

$\mathbb{P}^1$

$\mathbb{P}^1$

$g^r_d \mathbb{P}^1 (r+1) W H^0(\mathcal{O}_{\mathbb{P}^1}(d)) g^r_d G(r+1, H^0(\mathcal{O}_{\mathbb{P}^1}(d))) = G(r+1, d+1)$

$g^r_d \mathbb{P}^1 \alpha = (\alpha_0, \dots, \alpha_r) p \in \mathbb{P}^1 G(r+1, d+1) |\alpha| := \sum \alpha_i$

$\ell := r+1, e := d+1$

$V = H^0(\mathcal{O}_{\mathbb{P}^1}(d)) p \in \mathbb{P}^1 V_i(p) d \geq e - i p \mathcal{V}(p) p$

$$0 \subset V_1(p) \subsetneq V_2(p) \subsetneq \cdots \subsetneq V_e(p) = V.$$

$\dim V_i(p) = i W \ell = r+1 \geq a = a_0, \dots a_r$

$$\geq \alpha = (\alpha_0 = a_0 - 0, a_1 - 1 \dots, \alpha_r = a_r - r)$$

$p \dim W \cap V_{e-a_{\ell-i}}(p) \geq i i = 0, \dots, \ell W G(\ell, e) \beta_i = \alpha_{\ell-i}$

$$d - r = e - \ell \geq \beta_1 \geq \cdots \geq \beta_\ell \geq 0,$$

$\beta \beta_i$

$\ell W \subset H^0(\mathcal{O}_{\mathbb{P}^1}(d)) g^r_d \alpha = (\alpha_0, \dots, \alpha_r) p$

$$\dim W \cap V_{e-\ell+i-\beta_i}(p) \geq i 0 \leq i \leq \ell.$$

$e := d + 1 \beta_i = \ell - \alpha_i$

$\mathrm{V} e V$

$$0 \subset V_1 \subset V_2 \subset \cdots \subset V_e = V.$$

$\dim V_i = i$

$V \ell W \subset V e W$

$$0 \subset W \cap V_1 \subset W \cap V_2 \subset \dots \subset W \cap V_d \subset W \cap V_e = W$$

$e - \ell \ell \ell W W \cap V_{e-\ell} = 0 \ell i$

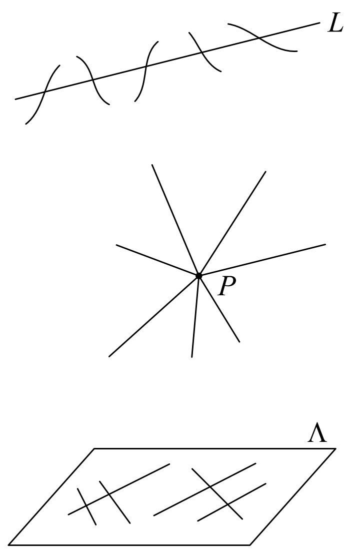
$G(\ell, e) \beta = (\beta_1, \dots, \beta_\ell) e - \ell \geq \beta_1 \geq \beta_2 \geq \cdots \geq \beta_\ell \geq 0 \Sigma_\beta(\mathrm{V}) \subset G \mathcal{V} \mathbb{C}^e \beta$

$$\Sigma_\beta(\mathrm{V}) := \left\{ W \in G \mid \dim(W \cap V_{e-\ell+i-\beta_i}) \geq i \ \forall i \right\}$$

$\mathcal{V}(p) d p \in \mathbb{P}^1 g^r_d \geq \alpha p W \Sigma_\beta(\mathcal{V}(p)) \beta_i = \alpha_{\ell-i}$

$\Sigma_{0\dots,0} \Sigma_{1,0\dots,0}(\mathrm{V}) \ell V_{e-\ell} \Sigma_\gamma(\mathcal{V}) := \Sigma_{\gamma,0,\dots,0}(\mathcal{V}) \ell V_{\gamma-\ell-\beta+1} U = V_{e-\ell-\gamma+1} \Sigma_\gamma(U)$

$\beta \Sigma_\beta(\mathrm{V}) G(\ell, e) |\beta| := \sum \beta_i$



$\Sigma_1(L)$

$\Sigma_2(P)$

$\Sigma_{11}(\Lambda)$

$\Sigma_1(L) = L \subset \mathbb{P}^3 \Sigma_2(p) p \in \mathbb{P}^3 \Sigma_{1,1}(\Lambda) \Lambda \subset \mathbb{P}^3 \Lambda$

$\mathbb{C}^e \Sigma_\beta(\mathrm{V}) \subset G A^*(G(\ell, e)) \mathcal{V} \sigma_\beta \sigma_\beta$

$\sigma_\beta \cdot \sigma_{\beta'} A^*(G(\ell, e))$

$\sigma_\gamma \sigma_\beta$

$$\sigma_\gamma \cdot \sigma_\beta \ = \ \sum \sigma_\delta$$

$\delta = (\delta_1, \dots \delta_\ell)$

$$\sum \delta_i = \gamma + \sum \beta_i \qquad \beta_i \leq \delta_i \leq \beta_{i-1} i$$

$\Sigma_1 \cdot \Sigma_1 \Sigma_2 \cup \Sigma_{1,1}$

$\sigma_\beta \beta_1 \beta_2 \gamma$

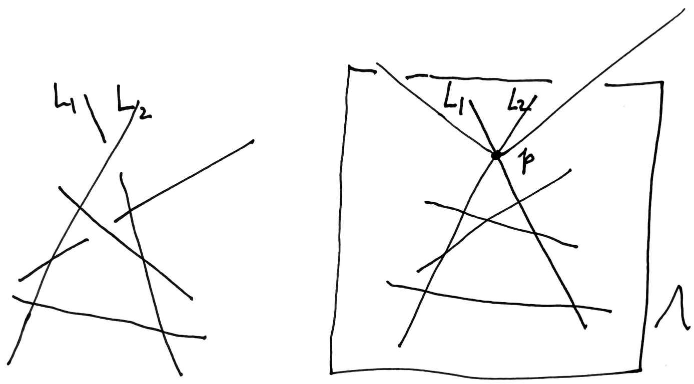
$\sigma_\gamma \sigma_\beta m \geq 0 m \gamma + \sum \beta_i \leq \dim G(\ell, e) = \ell(e - \ell)$

$$(\sigma_\gamma)^l \cdot \sigma_\beta \neq 0 \in A^*(G(\ell, e), \mathbb{Z}).$$

$C \subset \mathbb{P}^d (V_W, \mathcal{O}_C(1)) g^r_d \mathbb{P}^1 \mathbb{P}^d W \subset \mathbb{P}^d d - r - 1 \mathcal{V} = 0 \subset V_1 \subset \cdots \subset V_d = \mathbb{P}^d C p$ $\alpha(V_W, p)$

$$W \in \Sigma_a(\mathrm{V}) \iff \alpha_i(V_W, p) \geq a^*_{r+1-i} = \#\{j \mid a_j \geq r + 1 - i\}.$$

$\alpha(V_W, p) V_W p \widetilde{W} W G(d - r, d + 1)$



$\Sigma_1(L_1) \cap \Sigma_1(L_2) \rightsquigarrow \Sigma_2(p) \cup \Sigma_{1,1}(\Lambda)$

$L_1 L_2 P \Lambda \Sigma_1(L_1) \cap \Sigma(L_2) \Sigma_2(P) \cup \Sigma_{1,1}(\Lambda) L_1 L_2 p p \Lambda$

$\widetilde{W}, \widetilde{\mathcal{V}} = (\widetilde{V}_1, \dots, \widetilde{V}_{d+1}) p \dim \widetilde{W} = d - r \widetilde{V}_W(V_W, \mathcal{O}_{\mathbb{P}^1}(d)) \widetilde{V}_{\widetilde{W}}$

$\widetilde{W} \in \Sigma_a(\widetilde{\mathcal{V}})$

$$\dim \widetilde{W} \cap \widetilde{V}_{r+1+i-a_i} \geq i$$

$i\widetilde{W} \leq r - a_i + 1 \widetilde{V}_{r+i-a_i}(r + i - a_i + 1) p \subset C \leq r - a_i + 1 \widetilde{V}_{\widetilde{W}} \geq r - a_i + 1 + ip$

$\widetilde{V}_{\widetilde{W}} b_0 < b_1 < \cdots < b_r \widetilde{W} \in \Sigma_a a_i b_j \geq r + i + a_i + 1 b_{r-a_i+1} \geq r - a_i + 1 + i$ $\alpha_{r-a_i+1} \geq i i \alpha_i \leq \alpha_{i+1} \leq \alpha_r \{j \mid \alpha_j \geq i\} a_i \alpha' = \alpha_r \geq \alpha_{r-1} \geq \cdots \geq 0 a_i \geq i$ $a'_{a_i} = \#\{j \mid a_j \geq a_i\} = i a \alpha' \geq a'$ □

$g^r_d \mathbb{P}^1$

$p_1, \dots, p_\delta \in CC \subset \mathbb{P}^d \mathrm{V}^1, \dots, \mathrm{V}^\delta \beta^1, \dots, \beta^\delta \delta G(d-r, d+1) \Sigma_{\beta^1}(\mathrm{V}^1), \dots, \Sigma_{\beta^\delta}(\mathrm{V}^\delta) \subset G(d-r, d+1) \sum_{j=1}^{\delta} |\beta^j| \Sigma_{\beta^j}[\Sigma_{\beta^j}] A^*(G(d-r, d+1))$

$\Sigma_1 G(d-r, r+1) \Phi \subset Gmm \Sigma_1$

$$X := \bigcap_{i=1}^{\delta} \Sigma_{\beta^i}(\mathrm{V}^i)$$

$$\rho := (r+1)(d-r) - \sum_{i=1}^{\delta} |\beta^i|,$$

$\rho + 1 q_1, \dots, q_{\rho+1} C \mathrm{V}^1, \dots, \mathrm{V}^{\rho+1} X \Sigma_1(\mathrm{V}^i)$

$$\sum_{i=1}^{\delta} |\beta^i| \ > \ (r+1)(d-r) = \dim G(d-r, d+1).$$

$$W \ \in \ \bigcap_{i=1}^{\delta} \Sigma_{\beta^i}(\mathrm{V}^i),$$

$(W, \mathcal{O}_{\mathbb{P}^1}(d))|\beta_i| p_i (r+1)(d-r)$ $\square$

$\mathbb{P}^1 . p_1, \dots, p_\delta \in \mathbb{P}^1 p_i$

$$\alpha^i = (\alpha_0^i, \alpha_1^i, \dots, \alpha_r^i) \qquad 0 \le \alpha_0^i \le \alpha_1^i \le \cdots \le \alpha_r^i \le d-r.$$

β^i

$$\beta^i \ = \ (\alpha_r^i, \alpha_{r-1}^i, \dots, \alpha_1^i, \alpha_0^i)$$

$b^i \beta^i$

$\alpha^i b^i g_d^r \mathbb{P}^1 p_i (\alpha_0^i, \alpha_1^i, \dots, \alpha_r^i)$

$$\prod_{i=0}^{\delta} \sigma_{b^i} \ \neq \ 0 \qquad A^*(G(d-r, r+1)).$$

$g_d^r \mathbb{P}^1 \alpha^i p_i \alpha^i g_d^r \alpha^i p_i \alpha^i p_i$ $\square$

$C \subset \mathbb{P}^d \mathbb{P}^d$

$$0 \subset V_1 \subset V_2 \subset \cdots \subset V_{d-1} \subset V_d = \mathbb{P}^d$$

$V_i C p_1, p_2, \dots, p_{d+1} \in C$

$$V_i \ = \ \overline{p_1 + p_2 + \cdots + p_{i+1}}$$

p_i

$k p y = x^{p+1} + 1 k$

$\Sigma_\beta(\mathrm{V}) G \sum \beta_i$

$\Lambda \in \Sigma_\beta(\mathrm{V}) \Lambda \Lambda \cap V_i$

$\beta^* \beta$

$|\beta^*| = |\beta|$

$(\beta^*)^* = \beta$

$$\phi : G(k+1, V) \xrightarrow{\cong} G(n-k, V^*)\Sigma_\beta(\mathrm{V})\Sigma_{\beta^*}(\mathrm{V}^*)$$

$p \in C \subset \mathbb{P}^d d\Lambda \subset \mathbb{P}^d d - r - 1\mathcal{U} := (\mathcal{O}_{\mathbb{P}^1}(d), V)g^r_d\Lambda$

$$U(p) := \{U_1, \dots, U_d\}$$

$pU_t tptC\mathcal{U}p$

$$W \in \Sigma_a(U(p)) \iff \alpha_i(\mathcal{U}, p) \geq a^*_{r+1-i} = \#\{j \mid a_j \geq r+1-i\}.$$

$\alpha(\mathcal{U}, p)\mathcal{U}pL\Lambda G(d-r, d+1)C$

$C\mathbb{P}^r(p_i, q_i) \in C^2 tiS_i k\mathbb{P}^r\overline{p_i q_i} p_i, q_i S_i$

$d = 2r = 1$

$C \subset \mathbb{P}^d pp\overline{mp}(m-1)$

$$\{p\} \subset \overline{2p} \subset \overline{3p} \subset \dots$$

$$\{p\} \subset \overline{b_2 p} \subset \overline{b_3 p} \subset \cdots \subset \overline{b_{d+1} p} = \mathbb{P}^d$$

$\sum b_i - ip$

$E \subset \mathbb{P}^{n-1} n^2 EnE \cong Jac(E)$

$p \in E\mathcal{O}_E(np) \cong \mathcal{O}_E(1)$

$H \subset \mathbb{N}g = \#(\mathbb{N} \setminus H)gp(0^{12}, 6, 7, 9, 9)p$

$\mathbb{P}^1$

$$\rho(g,r,d) := g-(r+1)(g-d+r) \geq 0,$$

$g g^r_d C \dim W^r_d(C) = \rho \rho < 0 C g g^r_d$

$2d-2 g^1_d g = 2, 4 g C_0 g \mathbb{P}^1 g^1_d$

$r_1, \ldots, r_g C_0 p_i, q_i \in \mathbb{P}^1 r_i g^1_d C_0 C_0 \mathbb{P}^1 \mathbb{P}^1 \mathbb{P}^d d(d-2) \Lambda \subset \mathbb{P}^d C_0 p_i q_i \Lambda \cap \overline{p_i, q_i} \neq \emptyset$

$G(d-1, d+1)(d-2)\mathbb{P}^d L_i = \overline{p_i, q_i} \Sigma_1(L_i) g^1_d C_0$

$$W^1_d(C_0) = \bigcap_{i=1}^{2d-2} \Sigma_1(L_i).$$

$p_i, q_i \in \mathbb{P}^1 L_i \sigma_1^{2d-2} A(G(d-1,d+1)) C g = 2d-2$

$$\# W^1_{d+1}(C) = \frac{(2d-2)!}{(d-1)!d!}$$

$d p_i, q_i \Sigma_1(L_i) g^1_d$

$W^r_d(C) \rho \geq 0 r = 1$

g^1_d

$g, d, r g C_0 g$

$W^r_d(C)$ $C g W^r_d(C) C^r_d \subset C_d r \mathrm{Pic}(C) \Delta \mathbb{C} \mathrm{Spec}$

$\mathcal{C}/\Delta g C_0 \dim W^r_d(C_0) = \rho(g, r, d) \geq 0 \mathcal{L}_0 \dim W^r_d(C_b) = \rho b 0 \in \Delta \mathcal{L}_0 \mathrm{Pic}_d(\mathcal{C}/\Delta)$ $W^r_d(C_b) b 0 \in \Delta$

$\mathcal{C}/\Delta \Delta \to \Delta m p_i(b) m D_b = \sum_i p_i(b) m h^1(\mathcal{O}_{C_b}(D_b)) = 0 \mathcal{L}_b d C_b h^1(\mathcal{L}_b(D_b)) = 0$

$$h^0(\mathcal{L}_b(D_b)) = d + m - g + 1.$$

$\mathrm{P} \mathrm{Pic}_{d+m}(\mathcal{C}/\Delta) \times \mathcal{C} \mathrm{Pic}_{d+m}(\mathcal{C}/\Delta) \mathcal{E} d + m - g + 1 \Delta \mathcal{L}_b H^0(\mathcal{L}(D_b)) \pi_{1*}(\mathrm{P}|_{D_b}) \mathcal{F}$ $m \bigoplus_{i=1}^m \mathcal{L}(E)|_{p_i} D \mathcal{C}$

$$\mathrm{P} \longrightarrow \mathrm{P}_D$$

$\phi : \mathrm{E} \to \mathrm{F} \mathcal{C} b \sigma \in H^0(\mathcal{L}_b(D_b)) p_i$

$\mathcal{L}_b d C_b H^0(\mathcal{L}_b) \phi \mathcal{L}_b(D_b) \in \mathrm{Pic}_{d+m}(\mathcal{C}/\Delta) \mathcal{C} \phi (d+m) \times (d+m-g+1) W^r_d(C_b)$ $\otimes \mathcal{O}_{\mathcal{C}}(D) \phi d + m - g - r \leq (r+1)(g - d + r) \mathrm{Pic}_{d+m}(C) g W^r_d(C) \rho$ □

$C_0 g \rho(g, r, d) \geq 0 \dim W^r_d(C_0) = \rho C_0$

g

$\{C_t\} g C_t t \neq 0 C_0 g C_0 g$

$C p \in C C_0 f : C \to C_0 f C \setminus \{p\} C_0 \setminus \{r\} r = f(p) \in C_0 C_0$

$C p C_0$

$C_0 C \mathcal{O}_{C_0} \mathcal{O}_C C p$ □

$\mathbb{P}^1 g p_1, \ldots, p_g \in \mathbb{P}^1 p_i g C_0$

$C_0 \mathrm{C} \to \Delta C_0 C_0$

$p \in C_0 p C_0 y^2 = x^3$

$$y^2 = x(x-t)(x-2t)$$

$t \in \Delta$

$\mathrm{C} \to \Delta$

$p_1, \ldots, p_g \in \mathbb{P}^1 C_0 \mathbb{P}^1 p_i \pi : \mathrm{C} \to \Delta$

$\Delta\mathbb{C}$

$b \neq 0 \in \Delta C_b = \pi^{-1}(b) g$

$0 C_0$

$C_0 \mathbb{P}^n$ $\square$

$\mathrm{C} \to \Delta g g C_0$

C_0

$p \in C_0 C q \in C \nu : C \to C_0$

$$0 \longrightarrow (\mathbb{C},+) \longrightarrow \mathrm{Pic}_0(C_0) \xrightarrow{\nu^*} \mathrm{Pic}_0(C) \longrightarrow 0.$$

$\nu^{-1}(p) 2q \subset C \mathcal{L} C_0 \nu^*(\mathcal{L}) C q 2q \mathcal{L}' C a : \mathcal{L}'|_U \cong \mathcal{O}_U U \subset C q \mathcal{L} C_0 a : \mathcal{L}|_{\nu(U)} \cong \mathcal{O}_{\nu(U)} \nu(U) \mathcal{L}' \mathcal{L}' a \mathcal{L} C_0 \mathcal{L}' C \widetilde{\mathcal{L}} \nu^{-1}(p) = 2q$

$2q \mathcal{O}_{2q} \cong \mathbb{C}[\epsilon]/(\epsilon^2) \mathbb{C}[\epsilon]/(\epsilon^2) a + b\epsilon a \neq 0 b \in \mathbb{C} \mathcal{L}' a^{-1} a = 1 \mathbb{C} b \in \mathbb{C} 1 \mapsto 1 + b\epsilon$ $1 \mapsto 1 + b'\epsilon 1 \mapsto 1 + (b + b')\epsilon \nu^* : \mathrm{Pic}_d(C_0) \to \mathrm{Pic}_d(C)(\mathbb{C},+)$ $\square$

$C_0 g \mathrm{Pic}_d(C_0) \cong \mathbb{C}^g \mathrm{Pic}_d(C_0) \mathbb{C}^g$

$C_0 d \mathrm{Pic}_0(C_0) g$ $\square$

$\mathrm{Pic}_d(C_0) g g \mathrm{Pic}_d(C_0)$

$\pi : \mathrm{C} \to \Delta \mathrm{Pic}_d(C_t) \mathrm{Pic}_d(\mathrm{C}/\Delta) W^r_d(C_t) \mathcal{W}^r_d(\mathrm{C}/\Delta) W^r_d p_i \mathcal{W}^r_d(\mathrm{C}/\Delta) \subset \mathrm{Pic}_d(\mathrm{C}/\Delta)$ $\leq (r+1)(g-d+r)$

$W^r_d(C_0) \mathcal{W}^r_d(\mathrm{C}/\Delta) \rho(g,r,d) \rho(g,r,d) \geq 0 \Delta W^r_d(C_0) = \emptyset W^r_d(C_t) \neq \emptyset t \neq 0 \mathcal{L}_t C_t$ $\mathrm{Pic}_d(C_0)$

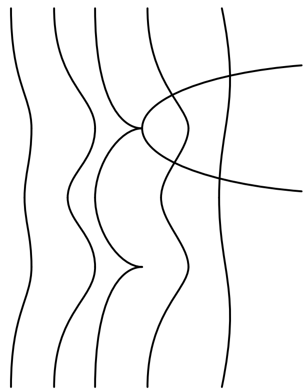
$\pi : \mathrm{C} \to \Delta g C_0 g$

$$\pi^\circ : \mathrm{C}^\circ := \mathrm{C} \setminus C_0 \to \Delta^\circ := \Delta \setminus 0,$$

$\mathcal{L}^\circ \mathrm{C}^\circ h^0(\mathcal{L}^\circ|_{C_b}) \geq r + 1 b \neq 0 \in \Delta \mathcal{L}^\circ|_{C_b} \in W^r_d(C_b) \mathcal{L} b \to 0$

$\mathcal{L} \mathrm{C} \Delta \mathrm{C} \mathcal{L}|_{\mathrm{C}^\circ} \cong \mathcal{L}^\circ$

$C_0 C_0$



$\mathcal{M}\mathrm{C}e > d + 2g\mathcal{M}^\circ\mathcal{M}C^\circ$

$$\mathcal{N}^\circ = (\mathcal{L}^\circ)^* \otimes \mathcal{M}^\circ.$$

$\sigma\mathcal{N}^\circ D^\circ \subset \mathrm{C}^\circ D \subset \mathrm{C}D^\circ\mathrm{C}D\mathcal{O}_\mathrm{C}/\mathcal{I}_{D/\mathrm{C}}\mathcal{O}_\Delta$

$\Delta\mathcal{O}_\mathrm{C}/\mathcal{I}_{D/\mathrm{C}}\Delta\mathcal{I}_{D/\mathrm{C}}|_{C_0}\mathcal{O}_\mathrm{C} \to \mathcal{O}_\mathrm{C}/\mathcal{I}_{D/\mathrm{C}}$

$$\mathcal{L} := \mathcal{I}_{D/\mathrm{C}} \otimes \mathcal{M}$$

□

R_0

R_0R_0R/R_0RR_0

R_0RR_0R

$R_0R/R_0 \cong kRR_0\mathfrak{m}_0R_0$ □

$R_0R_0R_0 \subset Rk[[x^2, x^3]] \subset k[[x]]kx \cong k$ □

$pC_0\pi : C \to C_0R_0 = \mathcal{O}_{C_0,p}\mathfrak{m}_0 = \mathcal{I}_pR_0R_0 \xrightarrow{\pi^*} RI_0 \neq 0R_0I_0 \cong R_0I_0 \cong \mathfrak{m}_0$

$I_0 \cong RaR_0/I_0vR/RI_0 = R/I_0v + 1$

$$0 \to R/\mathfrak{m}_0R \to R \otimes I_0 \to RI_0 \to 0$$

$RI_0 \cong R$

$p \in C_0\pi : C \to C_0C_0pp_a(C) = p_a(C_0) - 1q \in Cp \in C_0$

$\mathcal{F}_0C_0\mathcal{F}_0p\mathcal{F}C$

$$0 \to \mathcal{O}_{\pi^{-1}(p)} \to \pi^*\mathcal{F}_0 \to \mathcal{F} \to 0.$$

$\chi(\mathcal{F}) = \chi(\mathcal{F}_0) - 2$

$$\deg \mathcal{F} = \chi(\mathcal{F}) - \chi(\widetilde{\mathcal{O}}_C) = \deg(\mathcal{F}_0) - 1.$$

$H^0(\mathcal{F}_0) \xrightarrow{\pi^*} H^0(\mathcal{F}(-q))$ □

$I_0 R_0 R_0$

$$R_0 \subset End(I_0) \subset R.$$

$R/R_0 \cong k End(I_0) R_0 R$

$End(I_0) = RRI_0 = Ra\mathfrak{m}_0 RRRR_0 \subset RI \cong \mathfrak{m}_0 R_0 \widehat{R}_0 \cong \mathbb{C}[[t^2, t^3, \dots]] adR$ $R_0/(aR)d-1$

$End(I) = R_0$

$$\mathfrak{m}_0 I \subset I \subset RI.$$

$RR/\mathfrak{m}_0 I/\mathfrak{m}_0 II \cong R_0$ □

$\mathrm{C} \to \Delta t W^r_d(C_t) \neq \emptyset \Delta \mathcal{W}^r_d(\mathcal{C}/\Delta) \to \Delta \mathcal{L} \mathcal{C} \mathcal{L} C_t t \neq 0 r + 1 g C_0 \mathcal{L}_0 d \mathcal{L}_0 r + 1$

$r_i C_0 \mathcal{L}_0 p_1, \dots, p_k C_0 \mathcal{L}_0 C'_0 C_0 C'_0 g - kg - k \mathcal{L}_0 C'_0 r p_1, \dots, p_k g^r_{d-k} (g-k) D_0$

$\rho(g, r, d) < 0 W^r_d(C) = \emptyset M_g g g^r_d g C_0 g^r_d \rho(g, r, d) < 0 d \geq r + 1 (g-k) C'_0 k \geq 0$

$$\rho(g-k, r, d-k) = \rho(g, r, d) - k < 0$$

$\mathrm{C} \to \Delta g C_0 W^r_d(C_0) \rho(g, r, d) \mathcal{W}^r_d(\mathrm{C}/\Delta) \subset \mathrm{Pic}_d(\mathrm{C}/\Delta)(r+1)(g-d+r) t W^r_d(C_t)$ $\rho(g, r, d)$

$g^r_d C$

$C g p_1, \dots, p_n \in C C \mathrm{D} = (L, V) C d r \mathrm{D} p_k$

$$\rho(\mathrm{D}; p_1, \dots, p_k) := g - (r+1)(g-d+r) - \sum_{k=1}^{n} w(\mathrm{D}, p_k).$$

$(C; p_1, \dots, p_n) n g C p_1, \dots, p_n \in C \mathrm{D} C$

$$\rho(\mathrm{D}; p_1, \dots, p_n) \geq 0.$$

$\mathrm{C} \to \Delta\Delta \to \Delta\sigma_1, \dots, \sigma_n : \Delta \to \mathrm{C}\mathrm{C} \to \Delta\sigma_k(0) C_0 C_b \mathrm{g}^r_d \mathrm{D}$

$$\rho(\mathrm{D}; \sigma_1(b), \dots, \sigma_n(b)) < 0$$

$\Delta \to \Delta\{\mathrm{D}_b\} C_b b \neq 0 \mathrm{g}^r_d \mathrm{D}_0 \mathbb{P}^1$

$$w(\mathrm{D}_0, q_i) \geq r$$

$g q_i \in \mathbb{P}^1 C_0$

$$w(\mathrm{D}_0, r_k) \geq w(\mathrm{D}_b, \sigma_k(b))$$

$r_k \in \mathbb{P}^1 \mathbb{P}^1 \sigma_k(0) \in C_0$

$$\begin{aligned} \sum_{i=1}^{g} w(\mathrm{D}_0, q_i) + \sum_{k=1}^{n} w(\mathrm{D}_0, r_i) \geq rg + \sum_{k=1}^{n} w(\mathrm{D}_b, \sigma_k(b)) \\ > rg + g - (r+1)(g-d+r) = (r+1)(d-r) \end{aligned}$$

$$\rho(\mathrm{D}_b; \sigma_1(b), \dots, \sigma_n(b)) = g - (r+1)(g-d+r) - \sum_{k=1}^{n} w(\mathrm{D}_b, \sigma_k(b)) < 0.$$

$\mathbb{P}^1$

$$\sum_{p \in \mathbb{P}^1} w(\mathrm{D}_0, p) = (r+1)(d-r),$$

□

$(C; p_1, \dots, p_n) n g \alpha^1, \dots, \alpha^n \mathrm{g}^r_d \mathrm{D} C \alpha_i(\mathrm{D}, p_k) \geq \alpha^k_i k = 1, \dots, n i = 0, \dots, r G(d - r, d+1)$

$C g p_1, \dots, p_n \in C k = 1, \dots, n \alpha^k = (\alpha^k_0, \dots \alpha^k_r)$

$$G^r_d(p_1, \dots, p_n; \alpha^1, \dots, \alpha^n) = \{\mathrm{D} \in G^r_d(C) \mid \alpha_i(\mathrm{D}, p_k) \geq \alpha^k_i\}.$$

$(C, p_1, \dots, p_n) n G^r_d(p_1, \dots, p_n; \alpha^1, \dots, \alpha^n)$

$$\dim G^r_d(p_1, \dots, p_n; \alpha^1, \dots, \alpha^n) = \rho(g, r, d) - \sum_{k+1}^{n} \sum_{i=0}^{r} \alpha^k_i.$$

$\mathrm{D} \mathrm{g}^r_d \mathrm{D}$

$$w(\mathrm{D}, p) \leq 1 \quad p \in C.$$

$d = 2g - 2r = g - 1 C g$

$g^1_4 C p, q \in C$

$C pq$

$C\{(p_i, q_i) \mid i = 1 \dots h\} 2hC$

$C \mathbb{P}^r C (r-4) \Lambda \overline{p_i, q_i} C h \mathbb{P}^3$

ΛC

$\Lambda C \overline{p_i, q_i}$

$\Lambda \overline{\mathbb{T}_{p_i} C, \mathbb{T}_{q_i} C}$

$C \subset \mathbb{P}^d d\{L_i = \overline{p_i, q_i}\} C p_i, q_i \in C \Sigma_{a_i}(L_i)$

p_i, q_i

$R_0 \widehat{R} \mathbb{C}[x, y]/(xy) RRRC \to C_0 0 \to \mathbb{C}^* \to \mathrm{Pic}_0(C_0) \to \mathrm{Pic}_0(C) \to 0$

$C_0 pC \xrightarrow{\nu} C_0 pq, r \in \widetilde{C} p \mathcal{L} C \mathcal{M} := \nu^*(\mathcal{L}) \mathcal{L} \widetilde{C} \mathcal{M} \widetilde{C} qr \mathcal{L}_p \mathcal{L} p \mathcal{M} \widetilde{C} \mathcal{M}_q \mathcal{M}_r \mathcal{L} C$
$\nu_* \mathcal{M} qr$

$C\mathbb{P}^2 C\mathbb{P}^2 C C_0 \subset \mathbb{P}^2 C_0 C C$

$C_0 \subset \mathbb{P}^2$

$\nu : C \to C_0 \subset \mathbb{P}^2 B \subset C_0 C m B B \nu^{-1}(B) C C m B \mathcal{V} C \nu^{-1}(B) C_0 m B' := \nu^{-1}(B) C$
$L \mathbb{P}^2 L' C \mathcal{V} \mathcal{O}_C(mL' - B') \mathcal{O}_C(mL' - B')$

$$H^0(\mathcal{I}_{B/\mathbb{P}^2}(m)) \to H^0(\mathcal{O}_C(mL' - B')).$$

$C_0 \subset \mathbb{P}^2 d \delta g C C_0 p_a(C_0) = \binom{d-1}{2} C_0 \delta$

$$g = \binom{d-1}{2} - \delta.$$

gC

$[X, Y, Z] \mathbb{P}^2 C_0 L = V(Z) D C_0 [0, 1, 0] C_0 U = \mathbb{P}^2 \setminus L C_0 C_0 q_1, \dots, q_\delta r_i, s_i \in C q_i$
$\Delta \sum r_i + \sum s_i C$

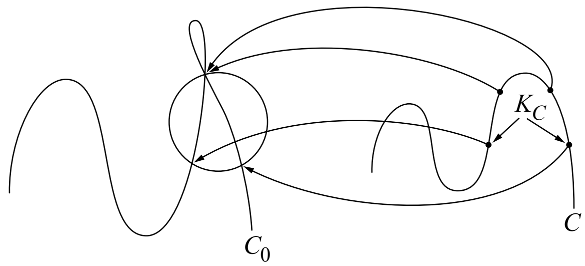
$F(X, Y, Z) d C_0 f(x, y) = F(x, y, 1) C_0^\circ := C_0 \cap U C_0 \nu : C \to C_0 \nu^*(dx) C^\circ :=$
$\nu^{-1}(C_0^\circ)$

$C_0 = C C^\circ f_x f_y C_0 f_x f_y q_i \nu^*(f_x) \nu^*(f_y) r_i s_i$

$\nu^* dx D C_0 \mathbb{P}^1 e(x, y) \leq d - 3$

$$\nu^* \left(\frac{e(x, y) dx}{f_y} \right)$$

$r_i s_i$



CC_0C_0C, C_0

eq_iee

$C_0 d\nu : C \to C_0 C$

$$\nu^*\left(\frac{e(x,y)dx}{f_y}\right)$$

$e(x,y) \leq d-3C_0.$

$\mathfrak{F}(C_0) \subset C_0 C |\omega_C| C d - 3\mathfrak{F}(C_0)$

$e(x,y)d-3\binom{d-1}{2}\delta\delta ee \mapsto \nu^*(edx/f_y)\binom{d-1}{2}-\delta$ □

$Cd-3\mathfrak{F}(C_0)Cm\mathfrak{F}(C_0)m$

CdC_0d-3

C_0 □

$g\mathbb{P}^{g-1}$

$dd-3$ □

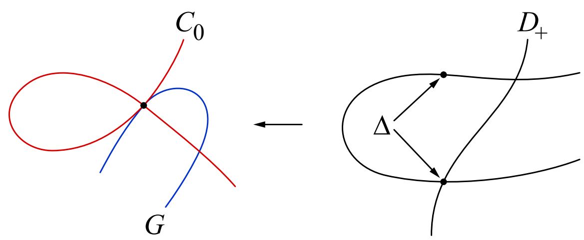
C_0CC_0

CC_0

$\nu : C \to C_0DCDC_0C\Delta C_0$

$C_0 p \in C_0 q, rC_0 paq + bra, bC_0C' p(a-1)(b-1)$ □

$C_0\nu : C \to C_0DCD_+ - D_-C_0|D|\mathfrak{F}(C_0)C\Delta\mathfrak{F}(C_0)C$



$G C_0 D_+ \Delta C D_+$

$D = D_+ - D_- C G \mathbb{P}^2 D_+ + \mathfrak{F}(C_0) C_0$

$G m A = (\nu^* G) - D_+ - \Delta C D (\nu^* H) - D_- - A - \Delta H m \mathfrak{F}(C_0) + A C_0$

$\mathfrak{F}(C_0)$

$C |D| m \mathfrak{f}_{C/C_0} D_- + A$ □

$m G D_+ \mathfrak{F}(C_0) \nu^*(G) D_+ + \Delta C D_+ p \Delta G C_0$

GC

$$(\nu^* G) = D_+ + \Delta + A,$$

$H m A + D_- \mathfrak{F}(C_0) C_0 H D H D'$

$$D' = (\nu^* H) - (D_+ + \Delta),$$

$D'(D_+ + \Delta)(\nu^* H)$

$\nu^*(G/H) C$

$$D_- + \Delta + A + D' = (\nu^* H) \sim (\nu^* G) = D_+ + \Delta + A,$$

$D' D = D_+ - D_- C$

$D' C S C_0 S$

$C_0 m C C_0 m$

$\pi : S \to \mathbb{P}^2 \mathbb{P}^2 q_i C_0 C_0 \subset \mathbb{P}^2 S C_0 C$

$L S \mathbb{P}^2 E \mathfrak{F}(C_0) h = L \cap C e = E \cap C = \sum (p_i + q_i) C C q_i C \sim dL - 2E K_S \sim -3L + E$

$m A \subset \mathbb{P}^2 q_i \pi^* A - E$

$$H^0(\mathcal{I}_{\{q_1, \dots, q_\delta\}/\mathbb{P}^2}(m)) \cong H^0(\mathcal{O}_S(mL - E)).$$

$C m C_0$

$$H^0(\mathcal{O}_S(mL - E)) \to H^0(\mathcal{O}_C(mL - E)),$$

$$0 \to \mathcal{O}_S((m-d)L+E) \to \mathcal{O}_S(mL-E) \to \mathcal{O}_C(mL-E) \to 0,$$

$H^1(\mathcal{O}_S((m-d)L+E)) = 0$

S

$$H^1(\mathcal{O}_S((m-d)L+E)) \cong H^1(\mathcal{O}_S((d-m-3)L))^*.$$

$\mathcal{O}_S((d-m-3)L) S \mathcal{O}_{\mathbb{P}^2}(d-m-3) H^1$ $\square$

$X \pi : S \to X \mathcal{L} X$

$$H^1(S, \pi^*\mathcal{L}) = H^1(X, \mathcal{L}).$$

$\mathbb{P}^2 \pi_*(\mathcal{O}_S) \mathcal{O}_{\mathbb{P}^2} \pi_*(\mathcal{O}_S) = \mathcal{O}_{\mathbb{P}^2} \mathcal{L} \mathbb{P}^2 \mathcal{O}_{\mathbb{P}^2}$

$$0 \to H^1(\pi_*(\mathcal{L})) \to H^1(\mathcal{L}) \to H^0(R^1(\pi_*(\mathcal{L})) \to 0$$

$\pi^*(\mathcal{L}) \pi H^1 H^0(R^1(\pi_*(\mathcal{L}))) = 0 \pi_* \pi^* \mathcal{O}_{\mathbb{P}^2}(1) \pi_* \pi^*(\mathcal{L}) \to \mathcal{L}$

$$H^1(S, \pi^*\mathcal{L}) = H^1(\pi_*\pi^*\mathcal{L}) = H^1(\mathcal{L}),$$

$\square$

$\square$

$C S \nu : S' \to S S p C' C$

$$p_a(C') = p_a(C) - \binom{m}{2},$$

$m p \in C$

$S S'$

$$p_a(C) = \frac{C^2 + K_S \cdot C}{2} + 1.$$

$S' E$

$$K_{S'} = \nu^* K_S + E,$$

C'

$$C' \sim \nu^* C - mE.$$

$$(C')^2 = C^2 + m^2 E^2 = C^2 - m^2 \qquad K_{S'} \cdot C' = K_S \cdot C + m.$$

S'

$$p_a(C') = \frac{{C'}^2 + K_{S'} \cdot C'}{2} + 1 = \frac{C^2 + K_S \cdot C - m(m-1)}{2} + 1 = p_a(C) - \binom{m}{2}$$

$\square$

C

$\delta C_0 p \binom{m_q}{2} q m_q C_0 q$ $\square$

$C_0 \subset \mathbb{P}^2 \nu : C \to C_0 L' C L \mathbb{P}^2$

$\mathfrak{F}(C_0) \subset C_0 \mathfrak{f}_{C/C_0} \mathcal{O}_{C_0} \nu_*(\mathcal{O}_C)/\mathcal{O}_{C_0} \Delta \subset C \mathfrak{f}_{C/C_0} C$

$C \Delta C_0$

$D C \Delta D C_0$

$C_0 \mathcal{O}' \mathcal{O} C \Delta \mathcal{O} \mathfrak{f}_{\mathcal{O}/\mathcal{O}'} \mathcal{O}' \mathcal{O} \mathcal{O} \mathcal{I}_D \mathcal{O}' \mathbb{C} \mathcal{I}_D \subset \mathcal{O}'(x) \mathcal{O}' n x \mathcal{I}_D^n = \mathcal{I}_D^{n+1}$

$D_0 C_0 x C$

$$\nu^*(D_0) + nD = (n+1)D$$

$\nu^*(D_0) = D$ $\square$

$C \mathrm{E} - \Delta \mathrm{E} \Delta C_0 \mathfrak{F}(C_0) \subset C_0$

$D = D_+ - D_- C G \mathbb{P}^2 \mathfrak{F}(C_0) C D_+ C$

$G m (\nu^* G) = D_+ + A + \Delta C D (\nu^* H) - D_- - A - \Delta H m \mathfrak{F}(C_0) + A H^0(\mathcal{O}_C(D)) m$ $\mathfrak{f}_{C/C_0} D_- + A + \Delta$

$C|D|$

$C|D| m \mathfrak{f}_{C/C_0} D_- + A$ $\square$

$H \in \mathfrak{f}(C_0) A C_0$

$$D_- + \Delta + A + D' = (\nu^* H) \sim (\nu^* G) = D_+ + \Delta + A,$$

$D' = (\nu^* H) - D_- - A - \Delta D D' D$ $\square$

$m \geq 0 C m \mathbb{P}^2 \mathfrak{F}(C_0)$

$R_0 \subset R \mathfrak{f}_{R/R_0} := ann_{R_0}(R/R_0) \subset R_0 R_0 R f \in \mathfrak{f}_{R/R_0} r \in R f R \subset R_0 (rf) R = frR \subset fR \subset R_0$

$R_0 R Q(R) R \mathfrak{f}_{R/R_0} \cong \mathrm{Hom}_{R_0}(R, R_0) R_0 R Q(R_0) Hom_{R_0}(R, R_0) \subset Hom_Q(Q, Q) =$ $Q\{\alpha \in Q \mid \alpha R \subset R_0\} \alpha \alpha \cdot 1 = \alpha \in R_0$

$C_0 (\nu^*(\mathcal{O}_{C_0}(m))(-\Delta) C U \mathcal{O}_{C_0}(U) U m \mathfrak{f}_{C/C_0} \widetilde{\mathfrak{f}_{C/C_0}}(m) C$

$S = \mathbb{C}[x_0, x_1, x_2] \mathbb{P}^2 \mathfrak{f}_{C/C_0} H^0_*(\widetilde{\mathfrak{f}_{C/C_0}}) \mathfrak{f}_{C/C_0} S R_0 = H^0_*(\mathcal{O}_{C_0}) R = H^0_*(\nu_*(\mathcal{O}_C))$

$$\mathfrak{f}_{R/R_0} = Hom_{R_0}(R, R_0).$$

$M Hom(P, M) P (a\phi)(p) = a(\phi(p)) R_0 = S/(F) a, b R_0$

$$0 \longrightarrow R_0 \xrightarrow{a} R_0 \longrightarrow R_0/(a) \longrightarrow 0$$

$$0 \longrightarrow Hom_{R_0}(R, R_0) \xrightarrow{a} Hom_{R_0}(R, R_0) \longrightarrow Hom_{R_0}(R, R_0/(a)).$$

$$Hom_{R_0}(R, R_0)/aHom_{R_0}(R, R_0) \subset Hom_{R_0}(R, R_0/(a))$$

$bHom_{R_0}(R, R_0/(a))Hom_{R_0}(R, R_0)/aHom_{R_0}(R, R_0)$ $\square$

$Dm\Delta$

$C_0^\circ C_0 \mathbb{A}^2 \cong U \subset \mathbb{P}^2 Z \neq 0 C^\circ \subset C C_0^\circ$

$C_0 L C_0 (0, 1, 0) \nu : C \to C_0$

$$\frac{e(x, y)dx}{f_y}$$

$e(x, y) \leq d - 3\mathfrak{f}_{C^\circ/C_0^\circ}$

$g\mathbb{P}^{g-1}$

$C C_0 d d - 3\mathfrak{f}_{C/C_0}$ $\square$

$(0, 1, 0) C x d C \mathbb{P}^1 C_0 L d x \mathbb{P}^1 d x \nu^{-1}(C_0 \cap L)\varphi_0 := dx/f_y d - 3 C C_0 \cap L$

$C_0^\circ x \mathbb{A}^1 \kappa(C) = \kappa(C_0^\circ)\mathbb{C}(x)\omega_{\kappa(C)/\mathbb{C}}\kappa(C_0^\circ) dx C dx \varphi_0 := dx/f_y d - 3 C_0 \cap L C$ $e(x, y)\varphi_0 e(x, y) \leq d - 3$

$\omega_C C$

$$\mathcal{H}\mathrm{om}_{\mathbb{P}^1}(\nu_*(\mathcal{O}_C), \omega_{\mathbb{P}^1}),$$

$\nu^! \pi^! \mathcal{H}\mathrm{om}_{\mathbb{P}^1}(\pi_* \nu_*(\mathcal{O}_C), \omega_{\mathbb{P}^1}) \pi C_0 \to \mathbb{P}^1 x \mathcal{O}_{C_0} \nu_*(\mathcal{O}_C)\mathbb{P}^1$

$\mathcal{O}_{C_0} \subset \nu_*(\mathcal{O}_C)$

$$\mathcal{H}\mathrm{om}_{\mathbb{P}^1}(\mathcal{O}_{C_0}, \omega_{\mathbb{P}^1}).$$

$$\mathfrak{f}_{C/C_0} = \frac{\mathcal{H}\mathrm{om}_{\mathbb{P}^1}(\mathcal{O}_{C_0}, \omega_{\mathbb{P}^1})}{\mathcal{H}\mathrm{om}_{\mathbb{P}^1}(\nu_*(\mathcal{O}_C), \omega_{\mathbb{P}^1})}.$$

$$\omega_{C_0} := \mathcal{H}\mathrm{om}_{\mathbb{P}^1}(\mathcal{O}_{C_0}, \omega_{\mathbb{P}^1}) = \mathcal{H}\mathrm{om}_{\mathbb{P}^1}(\mathcal{O}_{C_0}, \mathcal{O}_{\mathbb{P}^1})(-2)$$

$C_0 C \omega_{C_0} \varphi_0 \omega_{C_0} \mathcal{O}_{C_0}$

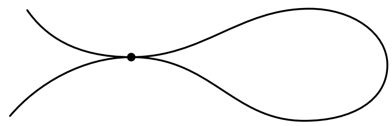
$\kappa := \kappa(C)\kappa(C) = \kappa(C_0)$

$$Hom_{\kappa(\mathbb{P}^1)}(\kappa, \kappa(\mathbb{P}^1))$$

$\kappa T \mathcal{O}_{C_0} \mathcal{O}_{\mathbb{P}^1}$

$$T(\mathcal{O}_{C_0}) \subset \mathcal{O}_{\mathbb{P}^1}.$$

$C_0^\circ f(x, y) = 0 \mathbb{C}[x, y]/(f)\mathbb{C}[x] Hom_{\mathbb{C}[x]}(\mathbb{C}[x, y]/(f), \mathbb{C}[x])(1/f_y) T$



$\mathbb{C}(x) \cong \mathbb{C}(x)dxdx(1 \mapsto dx) \in Hom_{\mathbb{C}(x)}(\kappa, \mathbb{C}(x)dx)C_0^\circ \mathcal{O}_{C_0^\circ}\phi_0$ $\square$

$qC_0C\varphi_0 m_1 = m_2 = 1\mathcal{J}_q qy^2 - x^3 rCqx = t^2, y = t^3)$

$$\varphi_0 = \frac{dx}{f_y} = \frac{2tdt}{2t^3} = \frac{dt}{t^2}$$

$Cgqrq\mathbb{C}[[t]]/\mathbb{C}[[t^2, t^3]]$

$y^2 - x^4 x = t, y = t^2 x = t, y = -t^2$

Cq

$$\varphi_0 = \frac{dt}{2t}\frac{-dt}{2t}$$

q

n $nnnr_i Cqf_y n - 1qdx/f_y n - 1r_i e(x, y)dx/f_y en - 1r_i$

$(n-1)(x, y)$

$$\nu^* : \frac{\mathbb{C}[x, y]}{\prod_{i=1}^n (x - \alpha_i y)} \to \prod_{i=1}^n \frac{\mathbb{C}[x, y]e_i}{(x - \alpha_i y)}$$

$\alpha_i \mathbb{C} e_i 1 = \sum_i e_i e_i x - \alpha_i y n g_j := \prod_{i \neq j}(x - \alpha_i y)\mathbb{P}^1 g_i g_j(\alpha_j, 1) g_i (x, y)^{n-1} n(x, y)^{n-1} =$ $(g_1, \dots, g_n)$

$\delta\binom{n}{2}k[x, y]/(x, y)^{n-1}$

$p\mathbb{A}^3$

$$\nu^* : R := \frac{\mathbb{C}[x, y, z]}{(xy, xz, yz)} \to \frac{\mathbb{C}[x, y, z]}{(x, y)} \times \frac{\mathbb{C}[x, y, z]}{(x, z)} \times \frac{\mathbb{C}[x, y, z]}{(y, z)} =: \overline{R}.$$

$x\overline{R} = x\mathbb{C}[x, y, z]/(y, z)\mathbb{C}[x, y, z]/(xy, xz, yz)yz(x, y, z)\delta(\overline{R})/Rf\overline{R}Rf$

$C_0 d \geq 4pCCC \to \mathbb{P}^1 d - 2C \to \mathbb{P}^1 d - 3$

$D = q_1 + \cdots + q_{d-2} r(D) \geq 1 q_1 + \cdots + q_{d-2} pd - 3$

$C_0 d \geq 5 p p' C C C \to \mathbb{P}^1 d - 2 C \to \mathbb{P}^1 d - 3$

$D = q_1 + \cdots + q_{d-2} r(D) \geq 1 q_1 + \cdots + q_{d-2} p, p' d - 3 d - 1$

$d \delta \leq d + 3 d - 2 g^1_d$

$n \leq 2d + 2 d d + 2 n = 2d + 2$

$g^1_{d-2} d - 3$

$C \nu : C \to C_0 D = D_+ - D_- C |D| D$

$y^3 - x^4$

$C_0 p$

$\nu : C \to C_0 C_0 p \in C_0 C_0 p B_1, \ldots, B_k r_i B_i p B_i C_0 p$

$$m_i = \sum_{j \neq i} mult_p (B_i \cdot B_j)$$

$C_0 p g \mathrm{ord}_{r_i}(\nu^* g) \geq m_i$

$R = \widehat{\mathcal{O}_{C_0,p}} C_0 r_i R_i = \widehat{\mathcal{O}_{B_i}} \cong k[[t_i]] R_i = R / P_i P_i R m_i \sum_{j \neq i} P_j \subset R$

$\mathbb{P}^3$

$\mathbb{P}^3 d, ed, e C X = C \cup D D C X D$

$C \subset \mathbb{P}^3$

$$D(C) := H^1_*(\mathcal{I}_C) := \bigoplus_{m \in \mathbb{Z}} H^1(\mathcal{I}_C(m)),$$

$C S = H^0_*(\mathcal{O}_{\mathbb{P}^3}) S$

$\mathbb{P}^3$

$Hilb_{3m+1}(\mathbb{P}^3)$H°

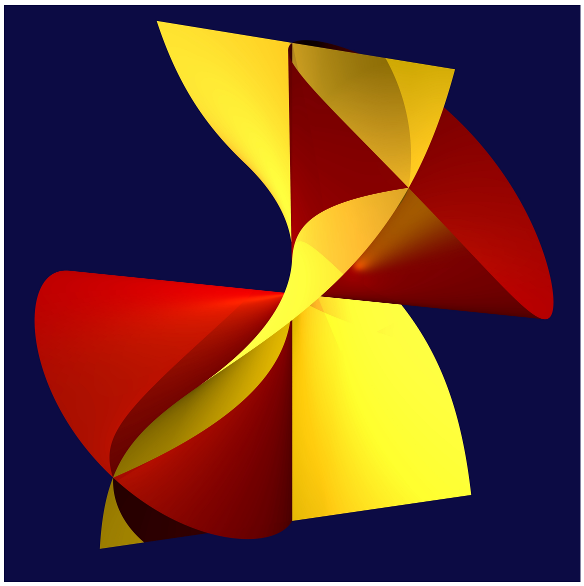
$C \subset \mathbb{P}^3 (1,2) L (1,0)(2,2) L C$

$L C C 2 \times 2$

$$\begin{pmatrix} x_0 & x_1 & x_2 \\ x_1 & x_2 & x_3 \end{pmatrix}.$$

$Q_{1,2} Q_{2,3} L : x_1 = x_2 = 0 x_0 = x_1 = x_2 = 0 x_1 = x_2 = x_3 = 0 L C Q Q_{1,2} Q_{2,3}$ $\mathbb{P}^1 \times \mathbb{P}^1 C (1,2)(2,1)(1,0)(0,1)(2,2) Q_{1,2}$

$Q_1, Q_2 C Q_1 \cap Q_2 C$



$\mathbb{P}^9\mathbb{P}^3$

$$\Phi = \{(C, L, Q, Q') \in \mathrm{H}^\circ \times \mathbb{G}(1,3) \times \mathbb{P}^9 \times \mathbb{P}^9 \;\mid\; Q \cap Q' = C \cup L\}.$$

$\Phi\mathrm{H}^\circ\mathbb{G}(1,3)$

$$\begin{tikzcd} & \Phi \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ \mathrm{H}^\circ & & \mathbb{G}(1,3) \end{tikzcd}$$

$\pi_2 : \Phi \to \mathbb{G}(1,3) L \in \mathbb{G}(1,3) \mathbb{P}^6 \times \mathbb{P}^6 \mathbb{P}^6 L \dim \mathbb{G}(1,3) = 4 \Phi 4 + 2 \times 6 = 16$ $\pi_1 : \Phi \to \mathrm{H}^\circ C \mathbb{P}^2 \times \mathbb{P}^2 \mathbb{P}^2 C \mathrm{H}^\circ PGL_4/PGL_2$

$\mathbb{P}^3$

$\mathbb{P}^3$

$C, C_2 \subset \mathbb{P}^3 c, dS, TSS, Ts, t$

$$\deg C_1 + \deg C_2 = st$$
$$g(C_1) - g(C_2) = \frac{s+t-4}{2}(\deg C_1 - \deg C_2).$$

$C_1 C_2 (s+t-4)/2 (s+t-4)/2 = 0$

$S \cap T = C_1 \cup C_2 st$

$\mathbb{P}^3 S K_S = (s-4)HS$

$$g(C_i) = \frac{C_i^2 + C_i \cdot K_S}{2} + 1 = \frac{C_i^2 + (s-4)\deg C_i}{2} + 1.$$

$$g(C_1) - g(C_2) = \frac{C_1^2 - C_2^2 + (s-4)(\deg C_1 - \deg C_2)}{2}.$$

$C_1 + C_2 tHS$

$$C_1^2 - C_2^2 = (C_1 - C_2)(C_1 + C_2) = t(\deg C_1 - \deg C_2)$$

□

$S\mathbb{P}^3 C \subset STtCC'CS, TtH - CT't'C'' = C + (t-t')Ht = t'CSS\mathbb{P}^3 SCT, T'|C|S$

$\mathbb{P}^3$

$C \cup C'XI_X = I_C \cap I_{C'} I_C \cap I_{C'} I_C I_{C'} (I_X : I_C) := \{F \mid FI_C \subset I_X\} I_{C'}$

$F \notin I_{C'} G \in I_C \setminus I_{C'} FG \notin I_{C'} F \notin (I_X : I_C)(I_X : I_C) = I_{C'} C, C'\mathbb{P}^3$

$C, C'\mathbb{P}^3 C'CXC, C'(I_X : I_C) = I_{C'} C'CC'C$

$CC'C \subset \mathbb{P}^3 \mathcal{I}_{L/\mathbb{P}^3} L \subset \mathbb{P}^3 L$

$C_1 \subset \mathbb{P}^3 I_1 C_1 X := S \cap TI_2 = (I_X : I_1) I_1 = (I_X : I_2)$

$XI_X I_2 = (I_X : I_1) I_1 = (I_X : I_2) PI_X$

$RPXI_X RR$

$$I_{C_2} R = ann_R(I_{C_1} R) \cong Hom_R(R/I_{C_1} R, R)$$

□

$\mathbb{P}^3$

$C_1, C_2 \subset \mathbb{P}^3 S, Ts, t$

$$\deg C_1 + \deg C_2 = st$$

$$p_a(C_1) - p_a(C_2) = \frac{s+t-4}{2}(\deg C_1 - \deg C_2).$$

$Xd\mathbb{C}X\omega_X\eta : H^d(\omega_X) \to \mathbb{C}\mathcal{F}$

$$H^d(\mathcal{F}) \times \mathrm{Hom}(\mathcal{F}, \omega_X) \to H^d(\omega_X) \xrightarrow{\eta} \mathbb{C}$$

$\mathrm{F} = \mathcal{L}C\mathrm{Hom}(\mathcal{L}, \omega_X) = \mathcal{L}^{-1} \otimes \omega H^1(\mathcal{L})H^0(\mathcal{L}^{-1} \otimes \omega_C)$

$(\omega_X, \eta)H^0_*(\omega_X) = \oplus_{n\in\mathbb{Z}}(Hom_X(\mathcal{O}_X(n), \omega_X))H^d_*(\mathcal{O}_X)\eta$

$C\mathbb{C}$

$\mathcal{H}\mathrm{om}_C(\omega_C, \omega_C) = \mathcal{O}_C C\omega_C$

$\mathcal{L}C$

$$H^1(\mathcal{L}^{-1}) = H^0(\mathcal{L} \otimes \omega_C); \quad H^0(\mathcal{L}^{-1}) = H^1(\mathcal{L} \otimes \omega_C).$$

$$\chi(\mathcal{L}^{-1}) = -\chi(\mathcal{L} \otimes \omega_C).$$

$\mathcal{O}_C \to \mathcal{H}\mathrm{om}_C(\omega_C, \omega_C)$

$Cf : C \to \mathbb{P}^1\omega_C \cong \mathcal{H}\mathrm{om}_{\mathbb{P}^1}(\mathcal{O}_C, \omega_{\mathbb{P}^1})Cf^!\omega_{\mathbb{P}^1}C\mathcal{O}_C\mathcal{O}_{\mathbb{P}^1}\mathbb{P}^1\omega_{\mathbb{P}^1}\mathcal{O}_{\mathbb{P}^1}BABB$ $A \to Hom_A(Hom_B(A, B), Hom_B(A, B)) \cong Hom_B(Hom_B(A, B), B)aaf \mapsto f(a)$ AB

$\omega_C\mathcal{L}$

$$\begin{aligned} H^1(\mathcal{L}^{-1}) &= \mathrm{Hom}_C(\mathcal{L}^{-1}, \omega_C)) \\ &= H^0(\mathcal{H}\mathrm{om}_C(\mathcal{L}^{-1}, \omega_C)) \\ &= H^0(\mathcal{L} \otimes_C \omega_C). \end{aligned}$$

$$\begin{aligned} H^0(\mathcal{L}^{-1}) &= H^0(\mathcal{L}^{-1} \otimes_C \mathcal{O}_C) \\ &= H^0\big(\mathcal{L}^{-1} \otimes_C \mathcal{H}\mathrm{om}_C(\omega_C, \omega_C)\big) \\ &= \mathrm{Hom}_C(\mathcal{L} \otimes_C \omega_C, \omega_C)) \\ &= H^1(\mathcal{L} \otimes_C \omega_C) \end{aligned}$$

□

- $Xd\mathbb{C}\omega_X = \wedge^d\Omega_{X/\mathbb{C}}$
- $f : X \to Y\omega_X = f^*(\omega_Y)(\mathrm{ram}_{X/Y})\mathrm{ram}$

- $X \subset Y \omega_X = \omega_Y(X)\mid_X$

$X \to Y \mathcal{F} X$

$U := \operatorname{Spec} A \subset XV := \operatorname{Spec} B \subset Yf^* : B \to AF := \mathcal{F}_U A f_* F := f_*(\mathcal{F})(V) F B f^*$

$BM \operatorname{Hom}_B(A, M) A(a\phi)(m)\phi(am) f^!(-) := \operatorname{Hom}_B(A, -) : mod_B :\to mod_A$ $f_*(-) : mod_A \to mod_B$

$$\operatorname{Hom}_B(f_*F, -) \cong \operatorname{Hom}_A(F, \operatorname{Hom}_B(A, -)) = Hom_A(F, f^!(-)).$$

$Y \omega_Y o_Y := \omega_Y(V) \operatorname{Hom}_B(f_*F, o_Y) \cong \operatorname{Hom}_A(F, f^! o_Y), \eta f_* f^!$

$\eta : f_* f^!(M) = f_*(\operatorname{Hom}_B, (A, M)) = \operatorname{Hom}_B, (A, M) \to M : \quad \eta(\phi) = \phi(1)$

$AM\eta(f_*, f^!)$

$f^!(-) \operatorname{Hom}_B(A, -) f^! f_* \eta : f_* f^!(\mathcal{F}) \to \mathcal{F}$

$f : X \to Y d Y \omega_Y \eta_Y : H^d(\omega_Y) \to \mathbb{C}$

$$\omega_X := f^! \omega_Y,$$

$$\rho_X : H^d(f^! \omega_X) = H^d(f_* f^! \omega_X) \xrightarrow{H^d(\eta)} H^d(\omega_Y) \xrightarrow{\rho_Y} \mathbb{C},$$

$\eta(f_*, f^!) X$

$\mathcal{F} X f H^d(\mathcal{F}) = H^d(f_*(\mathcal{F})) f^!(-) f_*$

$$H^d(\mathcal{F}) = H^d(f_*\mathcal{F}) \cong^{\rho_Y} \operatorname{Hom}_Y(f_*\mathcal{F}, \omega_Y)^\vee \cong \operatorname{Hom}_X(\mathcal{F}, f^! \omega_Y)^\vee,$$

$\cong^{\rho_Y} \rho_Y \rho_X \omega_X = f^!(\omega_Y)$ $\square$

$f^! f f f^! R f_*$

$f : X \to Y Y \omega_Y$

$$f^!(\omega_Y) \cong \mathcal{E}xt_Y^{\dim Y - \dim X}(\mathcal{O}_X, \omega_Y)$$

X

$Y f^! \omega_Y \mathcal{E}xt_Y^{\dim Y - \dim X}(\mathcal{O}_X, \omega_Y)$

$Xs, tCR_X = S/(F,G)S = \mathbb{C}[x_0, \dots, x_3]\mathbb{P}^3$

$$0 \longrightarrow S(-s-t) \xrightarrow{\begin{pmatrix} G \\ -F \end{pmatrix}} S(-s) \oplus S(-t) \xrightarrow{\begin{pmatrix} F & G \end{pmatrix}} S \longrightarrow R_X \longrightarrow 0$$

$$\omega_X = \mathcal{E}xt^2_C(\mathcal{O}_X, \omega_{\mathbb{P}^3}) = \mathcal{E}xt^2(\mathcal{O}_X, \mathcal{O}_{\mathbb{P}^3}(-4)) = \mathcal{O}_X(s+t-4).$$

$J \subset IAHom_A(A/I, A/J) \cong (J : I)/J\phi\phi(1)$

$$\omega_C = \mathcal{H}\mathrm{om}_X(\mathcal{O}_C, \omega_X) = \mathcal{H}\mathrm{om}_X(\mathcal{O}_C, \mathcal{O}_X)(s+t-4) = \frac{\mathcal{I}_X : \mathcal{I}_C}{\mathcal{I}_X}(s+t-4),$$

$\mathcal{O}_C C \to X$

$\chi(\omega_C(m)) = -\chi(\mathcal{O}_C(-m))\omega_C \deg C$

$$st = \deg \mathcal{O}_X = \deg \mathcal{O}_{C'} + \deg \mathcal{O}_C,$$

$\chi(\mathcal{O}_X) = st(4-s-t)/2\mathcal{O}_{C'} = \mathcal{O}_{\mathbb{P}^3}/(\mathcal{I}_X : \mathcal{I}_C)(\mathcal{I}_X : \mathcal{I}_C)/(\mathcal{I}_X) = \omega_C(4-s-t)$

$$\begin{aligned}
&\frac{4-s-t}{2}(\deg C + \deg C') \\
&= \frac{4-s-t}{2} st \\
&= \chi(\mathcal{O}_X) \\
&= \chi(\mathcal{O}_{C'}) + \chi(\omega_C(4-s-t)) \\
&= \chi(\mathcal{O}_{C'}) - \chi(\mathcal{O}_C(s+t-4)) \\
&= \chi(\mathcal{O}_{C'}) - (s+t-4)\deg C - \chi(\mathcal{O}_C) \\
&= (1 - p_a(\mathcal{O}_{C'})) - (1 - p_a(\mathcal{O}_C)) - (s+t-4)\deg C,
\end{aligned}$$

$$p_a(\mathcal{O}_C) - p_a(\mathcal{O}_{C'}) = \frac{(s+t-4)}{2}(\deg C - \deg C').$$

□

$C, C'\mathbb{P}^3 s, t$

$$D(C') = Hom_{\mathbb{C}}(D(C), \mathbb{C})(-s-t+4).$$

$\mathbb{P}^3$

$Ca_i, i = 1, \dots, sC\mathcal{O}_{C,p}\mathcal{O}_{\mathbb{P}^3,p}\mathcal{I}_{C,p}$

$$0 \to \mathcal{E} \to \oplus_i \mathcal{O}_{\mathbb{P}^3}(-a_i) \to \mathcal{I}_C \to 0.$$

$\mathcal{O}_{\mathbb{P}^3}$

$$D(C) := \oplus_{m\in\mathbb{Z}} H^1(\mathcal{I}_C(m)) \cong \oplus_{m\in\mathbb{Z}} H^2(\mathcal{E}(m)).$$

Xs, tC

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{E} \arrow[r] & \oplus_i \mathcal{O}_{\mathbb{P}^3}(-a_i) \arrow[r] & \mathcal{O}_{\mathbb{P}^3} \arrow[r] & \mathcal{O}_C \arrow[r] & 0 \\
0 \arrow[r] & \mathcal{O}_{\mathbb{P}^3}(-s-t) \arrow[r] \arrow[u] & \mathcal{O}_{\mathbb{P}^3}(-s) \oplus \mathcal{O}_{\mathbb{P}^3}(-t) \arrow[r] \arrow[u] & \mathcal{O}_{\mathbb{P}^3} \arrow[r] \arrow[u, "="'] & \mathcal{O}_X \arrow[r] \arrow[u] & 0
\end{tikzcd}$$

$-s-t\mathrm{Hom}_{\mathbb{P}^3}(\mathcal{O}_C, \mathcal{O}_{\mathbb{P}^3}) = 0\mathcal{O}_{\mathbb{P}^3} \to \mathcal{O}_{\mathbb{P}^3}\omega_C = \mathcal{E}xt^2(\mathcal{O}_C, \mathcal{O}_{\mathbb{P}^3}(-4))$

$$\begin{tikzcd}
0 & \omega_C(-s-t+4) \arrow[l] \arrow[d, "\phi"'] & \mathcal{E}^*(-s-t) \arrow[l] \arrow[d] & \oplus_i \mathcal{O}_{\mathbb{P}^3}(a_i - s - t) \arrow[l] \arrow[d] & 0 \arrow[l] \\
0 & \mathcal{O}_X \arrow[l] \arrow[d] & \mathcal{O}_{\mathbb{P}^3} \arrow[l] & \mathcal{O}_{\mathbb{P}^3}(-t) \oplus \mathcal{O}_{\mathbb{P}^3}(-s) \arrow[l] & \\
& \mathcal{O}_{C'} \arrow[d] & & & \\
& 0 & & &
\end{tikzcd}$$
.

$\phi(\mathcal{I}_X : \mathcal{I}_C)/\mathcal{I}_X \cong \omega_C(-s-t+4)\mathcal{I}_{C'}$

$$0 \leftarrow \mathcal{I}_{C'} \leftarrow \mathcal{O}_{\mathbb{P}^3}(-t) \oplus \mathcal{O}_{\mathbb{P}^3}(-s) \oplus \mathcal{E}^*(-s-t) \leftarrow \oplus_i \mathcal{O}_{\mathbb{P}^3}(a_i - s - t) \leftarrow 0.$$

$$H^1(\mathcal{I}_{C'}(m)) \cong H^1(\mathcal{E}^*(-s-t+m)) \cong Hom_{\mathbb{C}}(H^2(\mathcal{E}(s+t-m-4)), \mathbb{C})$$

$\mathbb{P}^3 mD(C') \cong \mathrm{Hom}_{\mathbb{C}}(D(C)(s+t-4), \mathbb{C})S$ $\square$

$C\mathbb{P}^3 I = I_C$

$$D(C) \cong Hom_{\mathbb{C}}(Ext^3(S/I, S), \mathbb{C})(-4), \mathbb{C})$$

$S\mathbb{P}^3$

$\psi : \oplus_i S(-a_i) \longrightarrow I\phi : \oplus_i \mathcal{O}_{\mathbb{P}^3}(-a_i) \longrightarrow \mathcal{I}_C \mathcal{E} E = \ker \psi$

$IS/I\mathrm{pd}S/I \leq 3I$

$$0 \longrightarrow G \longrightarrow F \longrightarrow \oplus_i S(-a_i) \longrightarrow S \longrightarrow S/I \longrightarrow 0.$$

$G \to FE$

$$0 \to E^* \to F^* \to G^* \to Ext^3_S(S/I, S) \to 0.$$

$C \leq 2Ext^3_S(S/I, S)(\widetilde{\ })$

$$0 \to \mathcal{E}^* \to \widetilde{F^*} \to \widetilde{G^*} \to 0.$$

$$Ext^3_S(S/I, S) = H^1_*(\mathcal{E}^*) = Hom_{\mathbb{C}}(H^2_*(\mathcal{E}(-4)), \mathbb{C}) = H^1_*(\mathcal{I})(-4),$$

$\square$

$C, C' D(C) D(C')$ $\square$

$S = \mathbb{C}[x_0, \ldots, x_3] \mathbb{P}^3 MS$

$CD(C) = M(m)m$

$mM(m) = D(C_0)C_0$

$C'CI_{C'} = fI_C + (g)I_C$

$M = D(C_0)\mathbb{P}^3 MM(m)m > 0$

$\mathbb{P}^3 D(C) = MC_0 M$

C_0

$\mathbb{P}^3$

CC_0CCC_0

$\nu : C \to C_0$

$$\mathfrak{A}_{C/C_0} := ann_{\mathcal{O}_{C_0}} \frac{\omega_{C_0}}{\nu_* \omega_C}$$

$$\mathfrak{f}_{C/C_0} := ann_{\mathcal{O}_{C_0}} \frac{\nu_* \mathcal{O}_C}{\mathcal{O}_{C_0}}$$

C_0

$$\delta(C_0) = \text{length} \frac{\nu_* \mathcal{O}_C}{\mathcal{O}_{C_0}} = \text{length} \frac{\mathcal{O}_{C_0}}{\mathfrak{f}_{C/C_0}}.$$

$\delta(C_0)C_0$

$\rho : C_0 \to \mathbb{P}^1 \rho_* \nu_* \mathcal{O}_C \rho_* \mathcal{O}_{C_0} \mathcal{O}_{\mathbb{P}^1} \rho_* \mathcal{O}_{C_0} \subset \nu_* \mathcal{O}_C$

$$\alpha : \rho_* \mathcal{O}_{C_0} \hookrightarrow \rho_* \nu_* \mathcal{O}_C$$

$\mathcal{O}_C \mathcal{O}_{C_0} \mathbb{P}^1 coker\, \alpha \mathbb{P}^1 C_0 \nu \rho \mathcal{O}_{C_0} \mathcal{O}_C \mathbb{P}^1 \nu_* \rho_* \omega_{\mathbb{P}^1} = \mathcal{O}_{\mathbb{P}^1}(-2) \alpha^\vee := \mathrm{Hom}_{\mathbb{P}^1}(\alpha, \mathcal{O}_{\mathbb{P}^1})$ $\mathcal{E}xt_{\mathbb{P}^1}(-, \omega_{\mathbb{P}^1})$

$$0 \longrightarrow \mathrm{Hom}_{\mathbb{P}^1}(coker\, \alpha, \omega_{\mathbb{P}^1}) \longrightarrow \omega_C \xrightarrow{\alpha^\vee} \omega_{C_0}$$
$$\longrightarrow \mathcal{E}xt^1_{\mathbb{P}^1}(coker\, \alpha, \omega_{\mathbb{P}^1}) \longrightarrow \mathcal{E}xt^1_{\mathbb{P}^1}(\mathcal{O}_C, \omega_{\mathbb{P}^1}) \longrightarrow \cdots$$

$coker\, \alpha \mathrm{Hom}_{\mathbb{P}^1}(coker\, \alpha, \omega_{\mathbb{P}^1}) = 0 \mathcal{E}xt^1_{\mathbb{P}^1}(coker\, \alpha, \omega_{\mathbb{P}^1}) coker\, \alpha \mathcal{O}_C \mathcal{O}_{\mathbb{P}^1} \mathcal{E}xt^1_{\mathbb{P}^1}(\mathcal{O}_C, \omega_{\mathbb{P}^1})$

$$0 \longrightarrow \omega_C \xrightarrow{\alpha^\vee} \omega_{C_0} \longrightarrow \mathcal{E}xt^1_{\mathbb{P}^1}(coker\, \alpha, \mathcal{O}_{\mathbb{P}^1}) \longrightarrow 0$$

$\nu_* \mathcal{O}_C / \mathcal{O}_{C_0} \omega_{C_0} / \nu_* \omega_C \delta(C_0) \mathfrak{f}_{C/C_0} C/C_0 \mathcal{O}_C / \mathcal{O}_{C_0} \mathcal{O}_{C_0} \mathcal{O}_{C_0} \mathcal{O}_C \mathcal{O}_{C_0} \subset \mathcal{O}_C \mathfrak{f}_{C/C_0} \omega_{C_0} / \omega_C$
C_0

$C_0 \subset \mathbb{P}^2 C_0 F d \mathcal{O}_C$

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^2}(-d) \xrightarrow{F} \mathcal{O}_{\mathbb{P}^2} \longrightarrow \mathcal{O}_C \longrightarrow 0$$

$\omega_C \cong \mathcal{E}xt^1_{\mathbb{P}^2}(\mathcal{O}_{C_0}, \omega_{\mathbb{P}^2}) \mathcal{O}_{\mathbb{P}^2}(-3) \to \mathcal{O}_{\mathbb{P}^2}(d-3) F \omega_{C_0} \omega_{C_0} / \omega_C \omega_{C_0} / \omega_C \cong \mathcal{O}_{C_0} / \mathfrak{f}_{C/C_0}$
$\omega_{C_0} / \omega_C \mathcal{O}_C / \mathcal{O}_{C_0}$ □

$R_0 := k[[x,y]]/(xy)$

$$P := k[[t]] \subset R_0 : t \mapsto x + y.$$

$R_0 R_0 \to R := k[[x]]e_1 \times k[[y]]e_2 e_1 = x/t, e_2 = y/t Q_1 = k((t)) \subset Q := k((x)) \times k((y)) Tr := Tr_{Q/Q_1} : Q \to Q_1 \mathrm{Hom}_{Q_1}(Q, Q_1) Q \omega_{R_0} = \mathrm{Hom}_P(R_0, P)$ $\omega_R = \mathrm{Hom}_P(R, P) Tr Q$

$R \cong Pe_1 \oplus Pe_2 P \mathrm{Hom}_P(R, P)(x/t)Tr(y/t)Tr$

$$g := \frac{x-y}{t^2} Tr \in \mathrm{Hom}_P(R_0, R).$$

$xg = x/tyg = y/tQxg = x/tTryg = y/tTr\mathrm{Hom}_P(R,P)R_0PxgTr(1) = 0$ $gTr(x) = 1(x-y)g = 1Q\frac{1}{2}(x-y)gTrxtgTr\mathrm{Hom}_P(R_0,P)\omega_{R_0}/\omega_R(x,y)R_0 = (x,y)R = \mathfrak{f}_{R/R_0}$

$R_0 := k[[x,y,z]]/(xy,xz,yz)R = k[[x]] \times k[[y]] \times k[[z]] t = x+y+z \in R_0 \subset R$ $RSx/t, y/t, z/tx^2/t = xt/t = xyz\mathfrak{f}_{R/R_0} = (x,y,z)R = (x,y,z)R_0$

$RS := k[[x,y,z]]S/(x,y) \oplus S/(x,z) \oplus S/(y,z) \omega_R = Ext^2(R,S)RS$

SR_0

$$0 \longrightarrow S^2 \xrightarrow{\begin{pmatrix} 0 & z \\ y & -y \\ -x & 0 \end{pmatrix}} S^3 \xrightarrow{\begin{pmatrix} xy & xz & yz \end{pmatrix}} S \longrightarrow R_0 \longrightarrow 0.$$

R_0

$$\omega_{R_0} = Ext^2(R_0, S) = coker \begin{pmatrix} 0 & y & -x \\ z & -y & 0 \end{pmatrix}$$

$\omega_R \subset \omega_{R_0} \mathfrak{A}_{R/R_0} = \mathfrak{f}_{R/R_0} = (x, y, z)R/R_0 R_0/\mathfrak{f}_{R/R_0} = R_0/(x, y, z)$

$\mathcal{F} r C \mathbb{C} \mathcal{F} \chi(\mathcal{F}) - r\chi(\mathcal{O}_C)$

$\mathcal{F}\mathcal{F}$

$$\chi(\mathcal{F}) = \deg \mathcal{F} + r\chi(\mathcal{O}_C) = \deg \mathcal{F} + r(1 - p_a(C)).$$

$\deg(\mathcal{F})$

$C\mathcal{F}Cr\mathcal{L}C$

$\mathcal{F}\sigma_1, \dots, \sigma_r \mathcal{F} M = coker\,(\mathcal{O}_C^r \xrightarrow{(\sigma_1,\dots,\sigma_r)} \mathcal{F})$

$$\deg(\mathcal{F}) = r\chi(\mathcal{O}_C) + \dim_{\mathbb{C}} H^0(M)\,;$$

$\deg(\mathcal{L} \otimes \mathcal{F}) = \deg(\mathcal{F}) + r\deg(\mathcal{L}).$

$\mathcal{O}_C(1) C m \mathcal{F}(m) \mathcal{F}(m) \mathcal{O}_C(1) \deg \mathcal{F} = \deg(\mathcal{F}(m)) - m\,\mathrm{rank}(\mathcal{F})\deg(\mathcal{O}_C(1))$

ω_C

$C\mathcal{F}C$

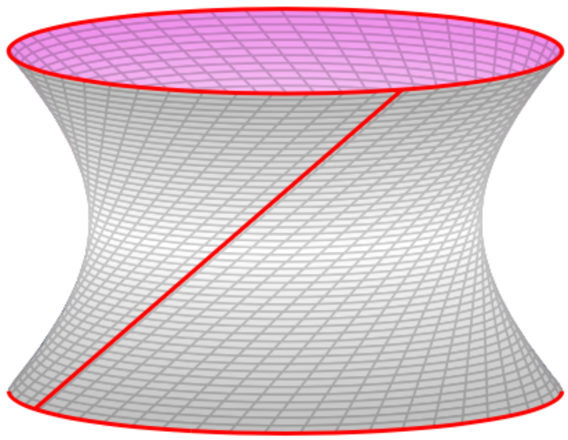
$$h^0(\mathcal{F}) = \deg(\mathcal{F}) + \mathrm{rank}(\mathcal{F})(1 - p_a(C)) + h^0(\mathrm{Hom}_C(\mathcal{F}, \omega_C)).$$

$(a, b)(d - a, d - b)$

$C\mathcal{E}rC\deg(\mathcal{E}) = \deg(\wedge^r(\mathcal{E}))$

C

C



$CH^2(\mathcal{I}_C) = 0$

$D(C)$

$\Gamma\mathbb{P}^3 H^1\mathcal{I}(\Gamma) = 0$

$$0 \to \mathcal{I}_C \xrightarrow{\ell} \mathcal{I}_C(1) \to \mathcal{I}_\Gamma(1) \to 0$$

ℓ

$$H^1(\mathcal{I}_C) \xrightarrow{\ell} H^1(\mathcal{I}_C(1))$$

$< 2\ell\ell C'CC'C$

(a, b)

$Q\mathbb{P}^3$

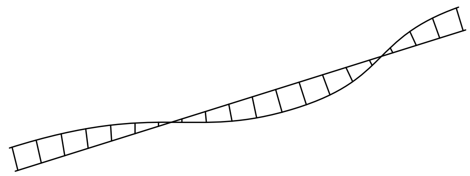
$$\mathcal{O}_Q(-a, -b) = \pi_1^*(\mathcal{O}_{\mathbb{P}^1}(-a)) \otimes \pi_2^*(\mathcal{O}_{\mathbb{P}^1}(-b)),$$

$\pi_1, \pi_2\mathbb{P}^1$

$$H^1(\mathcal{O}_Q(p,q)) = H^1(\mathcal{O}_{\mathbb{P}^1}(p)) \otimes H^0(\mathcal{O}_{\mathbb{P}^1}(q)) \oplus H^0(\mathcal{O}_{\mathbb{P}^1}(p)) \otimes H^1(\mathcal{O}_{\mathbb{P}^1}(q))$$

$I, J\mathbb{P}^3 f, gf \in Ig \in JgI \cap fJ = (fg)I, JC, C'\mathbb{P}^3(gI + fJ)C'' D(C'') = D(C)(-\deg g) \oplus D(C')(-\deg f)$

$$0 \to (fg) \to gI \oplus fJ \to gI + fJ \to 0$$



C'

$IC\mathbb{P}^3(f,g)g \in IfI + (g)C'D(C') = D(C)(-\deg f)$

$C'CD(C')C$

$B \to AXAYB$

$$\phi : \mathrm{Hom}_A(X, \mathrm{Hom}_B(A,Y)) \cong \mathrm{Hom}_B(X,Y)$$

$X = \mathrm{Hom}_B(A,Y)$

$$A \to \mathrm{Hom}_A(\mathrm{Hom}_B(A,Y), \mathrm{Hom}_B(A,Y))$$

$a \in AaA\mathrm{Hom}_B(A,Y)\phi\alpha \mapsto \alpha(a)\alpha \in \mathrm{Hom}_B(A,Y)$

$C \subset \mathbb{P}^n V(I_C^2)C$

$CL \subset \mathbb{P}^3 h_C(m)p_C(m)3m+1C\mathbb{P}^2$

$X \subset \mathbb{P}^3 x_i = s^i t^{3-i} PGL_4 x_0, \ldots, x_3 \mathbb{P}^3$

$$A_t : (x_0, \ldots, x_3) \mapsto (tX_0, X_1, X_2, tX_3).$$

$t \to 0 A_t(C) V(X_0^2, X_0X_1, X_1^2)$

$I(X) 2 \times 2$

$$\begin{pmatrix} x_0 & x_1 & x_2 \\ x_1 & x_2 & x_3 \end{pmatrix}$$

$C\mathbb{P}^2 C$

$X \subset \mathbb{P}^n C \subset \mathbb{P}^n$

$C \subset \mathbb{P}^n CC_{red}CC_{red}$

C

$CC_{red}C \subset \mathbb{P}^3 X = C_{red} V(x_0, x_1) C X(x_0^2, x_0 x_1, x_1^2) \subset I(C) C(0,0,s,t) X' = V(x_2, x_3)(F(s,t), G(s,t), 0, 0) F G \mathbb{P}^1 \to \mathbb{P}^1 d I(C) x_0 G(x_2, x_3) - x_1 F(x_2, x_3) C$

$$I_C = \left(X_0^2, X_0 X_1, X_1^2, F(X_3, X_4) X_0 + G(X_3, X_4) X_1\right)$$

$d = 1C$

$C \to Dep \in Ctpp\mathcal{H}\mathrm{om}_C(\mathcal{O}_C, \omega_D)\mathcal{O}_C(e)$

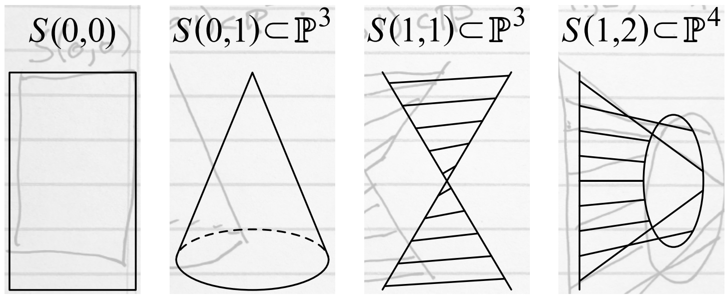
$C \subset SS\omega_S \mathcal{E}xt^1(\mathcal{O}_C, \omega_S) \cong \mathcal{O}_C \otimes \mathcal{O}_S(C) K_C = (K_S + C) \cap C$

```
The naming of cats is a difficult matter,
It isn't just one of your everyday games.
You may think that I am as mad as a hatter,
When I tell you each cat must have three different names.
The first is the name that the family use daily ...
But I tell you, a cat needs a name that's particular ...
But above and beyond there's still one name left over,  ...
[his] deep and inscrutable, singular name.
```

$\mathbb{P}^r 0 \leq a_1 1 \leq a_2 a_1 + a_2 = r - 1 \mathbb{P}^{a_1} \mathbb{P}^{a_2} \subset \mathbb{P}^r = \mathbb{P}(V)(r+1)VV = V_1 \oplus V_2$ $V_1, V_2 \subset V a_1 + 1 a_2 + 1 a_1 \leq a_2$

$i = 1, 2 \phi_i : \mathbb{P}^1 \to \mathbb{P}^{a_i} a_i a_i a_i = 0 \mathbb{P}^1 S(a_1, a_2)$

$$S(a_1, a_2) := \bigcup_{p \in \mathbb{P}^1} \overline{\phi_1(p), \phi_2(p)}.$$



$S(0,1)\cong\mathbb{P}^2 S(0,2) S(1,1)\subset\mathbb{P}^3 S(1,2)\subset\mathbb{P}^4$

$C_{a_1}\overline{\phi_1(p),\phi_2(p)}\mathbb{P}^3 S(1,1)$

$a_1 = 0 S(0,a_2)\mathbb{P}^{a_2+1} a_2 S(0,1) = \mathbb{P}^2 S(0,a_2) a_2 \geq 2$

$$\tilde{S}(0,a_2) := \left\{(t,q)\in\mathbb{P}^1\times\mathbb{P}^r \mid q\in\overline{\phi_1(t),\phi_2(t)}\right\}.$$

$S(0,a_2)S(a_1,a_2)a_1 > 0\mathbb{P}^1\mathbb{P}^1\tilde{S}(0,a_2)S(1,a_2+1)S(a_1,a_2)$

$S(a_1,a_2)$

$S(a_1,a_2)a_1+a_2 a_1+a_2-1.$

$S(a_1,a_2)a_1 > 0$

$\mathbb{P}^{a_i}\overline{\mathbb{P}^{a_1},\mathbb{P}^{a_2}} = \mathbb{P}^r$

$S a_1+a_2+1-2 = a_1+a_2-1$

$H\mathbb{P}^{a_1}H\cap C_2 a_2 H\cap S C_1 a_2 H\cap C_2 C_1 a_1+a_2$

$0 < a_1 p\in S(a_1,a_2)pS(a_1,a_2)p$ □

$s 0\leq a_1\leq\cdots\leq a_s r+1 = \sum_{i=1}^{s}(a_i+1)\mathbb{C}^{r+1}$

$$\mathbb{C}^{r+1} = \bigoplus_{i=1}^{s}\mathbb{C}^{a_i+1}.$$

$\mathbb{P}^{a_i}\subset\mathbb{P}^r i\phi_i : \mathbb{P}^1\to\mathbb{P}^{a_i} a_i S\subset\mathbb{P}^r$

$$S = S(a_1,\dots,a_s) := \bigcup_{p\in\mathbb{P}^1}\overline{\phi_1(p),\phi_2(p),\dots,\phi_s(p)}.$$

$S(a_1)a_1 S r-s\sum a_i = r-s+1 s = 2$

$X c\mathbb{P}^r \geq c+1$

$a\mathbb{P}^a a a-1$

$X \subset \mathbb{P}^r \deg X = \operatorname{codim} X + 1 \mathbb{P}^5$

$C_i \subset \mathbb{P}^{a_i} \Gamma \subset \prod C_i C_i p_i \in C_i C_i \prod C_i$

$S(a_1, a_2) \mathbb{P}^{a_1+a_2+1}$

$S := S(a_1, a_2) \mathbb{P} := \mathbb{P}^{a_1+a_2+1} S$

$\mathbb{P}^{a_i} \subset \mathbb{P}$

$\mathbb{P} a \mathbb{P}^a C_{a_2} \cong \mathbb{P}^1 \mathbb{P}^{a_2} = |\mathcal{O}_{\mathbb{P}^1}(a_2)| \mathbb{P} \mathbb{P}^{a_1} S(a_1, a_2)$ □

$X \mathbb{P}^r \mathcal{O}_X(1) \mathcal{L} \otimes \mathcal{M} X p \{\ell_i\} \subset H^0(\mathcal{L}) q \{m_i\} \subset H^0(\mathcal{M})$

$$\mu : H^0(\mathcal{L}) \otimes H^0(\mathcal{M}) \to H^0(\mathcal{O}_X(1)) = H^0(\mathcal{O}_{\mathbb{P}^r}(1))$$

$p \times q M_\mu \mathbb{P}^r i, j \mu(\ell_i m_j) \operatorname{Hom}(A \otimes B, C) \cong \operatorname{Hom}(A, \operatorname{Hom}(B, C))$

$X 2 \times 2$

$$\det \begin{pmatrix} \ell_{i_1} m_{j_1} & \ell_{i_1} m_{j_2} \\ \ell_{i_2} m_{j_1} & \ell_{i_2} m_{j_2} \end{pmatrix},$$

$X 2 \times 2 I_2(M_\mu) X$

$C_a \subset \mathbb{P}^a X = \mathbb{P}^1 C_a |\mathcal{O}_{\mathbb{P}^1}(a)| \mathcal{O}_{\mathbb{P}^1}(a) = \mathcal{O}_{\mathbb{P}^1}(1) \otimes \mathcal{O}_{\mathbb{P}^1}(a-1) s^i t^j H^0(\mathcal{O}_{\mathbb{P}^1}(1))$ $H^0(\mathcal{O}_{\mathbb{P}^1}(a-1)) H^0(\mathcal{O}_{\mathbb{P}^1}(a)) 2 \times a$

$$M_\mu := \begin{pmatrix} x_0 & x_1 & \dots & x_{a-1} \\ x_1 & \dots & x_{a-1} & x_a \end{pmatrix}.$$

$\mathbb{P}^1$

$$M_a = \begin{array}{c c} & \begin{array}{cccc} s^{a-1} & s^{a-2}t & \dots & t^{a-1} \end{array} \\ \begin{array}{c} s \\ t \end{array} & \begin{pmatrix} s^a & s^{a-1}t & \dots & st^{a-1} \\ s^{a-1}t & s^{a-2}t^2 & \dots & t^a \end{pmatrix} \end{array}$$

$s, t H^0(\mathcal{O}_{\mathbb{P}^1}(1)) H^0(\mathcal{O}_{\mathbb{P}^1}(1)) H^0(\mathcal{O}_{\mathbb{P}^1}(a-1)) M_a$

$M_a \mathbb{C} M_a 2 \times 2 M_a M_a$

$M M \mathbb{C}$

$X \mathcal{L}, \mathcal{M} X M_\mu \mu : H^0(\mathcal{L}) \otimes H^0(\mathcal{M}) \to H^0(\mathcal{L} \otimes \mathcal{M})$

$M_\mu s \in H^0(\mathcal{L}) s \cdot H^0(\mathcal{M}) \cong H^0(\mathcal{M})$ □

$$M = \begin{pmatrix} x & y \\ z & x \end{pmatrix}$$

$\mathbb{C}[x, y, z] \det M = x^2 - yz H^0(\mathcal{O}_{\mathbb{P}^1}(1)) \otimes H^0(\mathcal{O}_{\mathbb{P}^1}(1)) \to H^0(\mathcal{O}_{\mathbb{P}^1}(2)).$

$$M' = \begin{pmatrix} x & y \\ -y & x \end{pmatrix}$$

$\mathbb{C}[x, y]$

$$\begin{pmatrix} 1 & 0 \\ -i & 1 \end{pmatrix} M' \begin{pmatrix} 1 & 0 \\ i & 1 \end{pmatrix} = \begin{pmatrix} x + iy & 0 \\ 0 & x - iy \end{pmatrix}$$

$\mathbb{R}$

$p \times q r + 1 \mathbb{C} r \geq p + q$

$M p + q - 1 M$

$\mu : A \otimes B \to C C \ell_i \otimes m_j \in \ker \mu i, j M_\mu \ell \otimes m \in A \otimes B M_\mu \ker \mu \subset A \otimes B$ $A \otimes B \cong \mathrm{Hom}(A^*, B)(p-1)(q-1) = pq - p - q + 1 \mathrm{Hom}(A^*, B) A^* B \mathbb{P}^q \times \mathbb{P}^p \mu$ $\mathrm{codim} \ker \mu \geq q + p - 1 \mathrm{codim} \ker \mu = \dim \mathrm{im} \mu \subset C q + p + 1$

$M_\mu A \otimes B \to C \to C/\langle x \rangle \ker \mu M_\mu > q + p - 1$ □

$2 \times r I_2(M) 2 \times 2 M$

$M 2 \times b \mathbb{C}[x_0, \dots, x_r] V(I_2(M)) b - 1$

$V := V(I_2(M)) \rho_\lambda, \lambda \in \mathbb{P}^1 M$

$$W := \{(p, \lambda) \in \mathbb{P}^r \times \mathbb{P}^1 \mid p \in V(\rho_\lambda)\} \to V$$

$\pi_2 : W \to \mathbb{P}^1 \mathbb{P}^{r-b} W r - b + 1 V \pi_2^{-1}(\lambda) V M V r - b + 1$ □

$$M_a := \begin{pmatrix} x_0 & x_1 & \dots & x_{a-1} \\ x_1 & x_2 & \dots & x_a \end{pmatrix}$$

$a \mathbb{P}^a$

$I = I_2(M) 2 \times 2 2 \times a M S = \mathbb{C}[x_0, \dots, x_n]$

$I V(I)$

$a = r V(I) V = V(I) \subset \mathbb{P}^r a a - 1$

$V(I) a - 1 M \mathbb{P}^r V(I) M$

$p \in V(I) M p S \mathbb{P}^r \ell_{1,j} p j \neq 1 \Lambda a p$

$\mathcal{O}_{\mathbb{P}^r, p} \ell_{1,1} p m_j := \ell_{2,j} + \ell_{1,1}^{-1} \ell_{2,1} \ell_{1,j} j = 2, \dots, a \ell_{2,j} \mathcal{O}_{\mathbb{P}^r, p} \ell_{1,1}^{-1} \ell_{2,1} \ell_{1,j} \mathcal{O}_{\mathbb{P}^r, p} a - 1$ $m_j \mathrm{codim} I = a - 1 I$

$a = rV(I)EN(M)EN(M)M'EN(M')V(I)V(I)V(I_2(M))V(I)$

$a < r\ell M\ell I_2(M)I_2(M) + (x)/(x) \subset S/(x)a$ $\square$

$2r+3\Gamma \subset \mathbb{P}^r d \geq 2r+3\Gamma\Gamma 2r+1$

$$\begin{pmatrix} x_0 & x_1 & x_2 \\ x_1 & x_2 & x_3 \end{pmatrix}$$

$H^0(\mathcal{O}_{\mathbb{P}^1}(1)) \otimes H^0(\mathcal{O}_{\mathbb{P}^1}(r-1)) \to H^0(\mathcal{O}_{\mathbb{P}^1}(r))(r-1)(r-2)$

$\Gamma C r \mathbb{P}^r Q \Gamma C \subset QCC$

$$h^0(\mathcal{O}_{\mathbb{P}^r}(2)) - h^0(\mathcal{O}_C(2r)) = h^0(\mathcal{O}_{\mathbb{P}^r}(2)) - (2r+1)$$

$\Gamma 2r+1$

$\Gamma 2r + 12 \times rM\Gamma\mathbb{P}^r\binom{r+2}{2}\binom{r}{2} + 2r+1\Gamma\binom{r}{2}$

$p_i \in \Gamma\Lambda = V(a_1, b_1)(r-2)p_1, \ldots, p_{r-1}$

$\Lambda\binom{r}{2}r-1\Gamma \cap \Lambda r - 1\Lambda \cup \Gamma\binom{r}{2} - \left(\binom{r}{2} - (r-1)\right) = r-1.(a_1,b_1)2\times 2$

$$M' := \begin{pmatrix} a_1 & a_2 & \ldots & a_r \\ b_1 & b_2 & \ldots & b_r \end{pmatrix}.$$

M'

$M'2rM'$

$M\Gamma p \in \Gamma\Lambda a_1, b_1 M p(a_i(p), b_i(p))(a_1(p), b_1(p))Mp \geq 2r+3-(r-1) = r+4$
Γ

$a_2, \ldots, a_r i = 2, \ldots, r a_i\{p_2, \ldots, p_r\} p_i M i, j p_2, \ldots, p_r p_i, p_j 2r+1\Gamma\Gamma 2r+1M\Gamma$ $\square$

$2r+3Cr+2\mathbb{P}^{r+1}CC2r+2$

$\Gamma \geq 2r+1+2d2r+d\Gamma dd = 1$

$a_1, \ldots, a_d r = d-1+\sum_{i=1}^d a_i S(a_1, \ldots, a_d) \subset \mathbb{P}^r 2\times 2$

$$M = \left(\begin{array}{cccc|ccc|c|ccc} x_{1,0} & x_{1,1} & \ldots & x_{1,a_1-1} & x_{2,0} & \ldots & x_{2,a_2-1} & \ldots & x_{r,0} & \ldots & x_{r,a_r-1} \\ x_{1,1} & x_{1,2} & \ldots & x_{1,a_1}. & x_{2,1} & \ldots & x_{2,a_2} & \ldots & x_{r,1} & \ldots & x_{r,a_r} \end{array}\right)$$

C_{a_i}

$MdM_{a_i}MI_2(M)d = \sum a_i - 1M$

$C_i\mathbb{P}^{a_i} \subset \mathbb{P}^r V(M)MC_{a_i}M_{a_i}V(I_2(M))C_{a_i}$ $\square$

$2\times(r-d)r$

$X\mathcal{E}X\mathbb{P}(\mathcal{E}) = \mathrm{Proj}(\mathrm{Sym}(\mathcal{E}))\pi : \mathbb{P}(\mathcal{E}) \to X\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)\pi^*(\mathcal{E})\pi_*(\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)) = \mathcal{E}$

$$H^0(\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)) = H^0(\mathcal{E}).$$

$$\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(a_1) \oplus \mathcal{O}_{\mathbb{P}^1}(a_2))$$

$a_1 a_2$

$0 \le a_1 \le a_2$

$$\mathcal{E} = \mathcal{O}_{\mathbb{P}^1}(a_1) \oplus \mathcal{O}_{\mathbb{P}^1}(a_2).$$

$X = \mathbb{P}(\mathcal{E})\mathbb{P}^1\mathbb{P}^1\pi : X \to \mathbb{P}^1\mathcal{L} = \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)\pi^*(\mathcal{E})$

$|\mathcal{L}| 0 < a_1\phi : X \to \mathbb{P}^{a_1+a_2+1} = \mathbb{P}^r\phi S(a_1, a_2).$

$C_1, C_2 \subset X\mathcal{E} \to \mathcal{O}_{\mathbb{P}^1}(a_i)\phi(C_1)\phi(C_1)\mathcal{O}_{\mathbb{P}^1}(a_2)\mathcal{O}_{\mathbb{P}^1}(a_1)C_i \cong \mathbb{P}^1\phi C_i\mathbb{P}^{a_i}a_i$

$\mathcal{L}\mathbb{P}^1\pi\mathcal{O}_{\mathbb{P}^1}(1)$

$\pi C_i C_1, C_2\mathbb{P}^r\pi C_i$

$0 \le a_1 \le a_2$

$$X := S(a_1, a_2) \subset \mathbb{P}^r = \mathbb{P}^{a_1+a_2+1}$$

$a_1 = 0XS(1, a_2+1) \subset \mathbb{P}^{r+2}C_1 \subset S(1, a_2+1)S(1, a_2+1) \to S(0, a_2)$

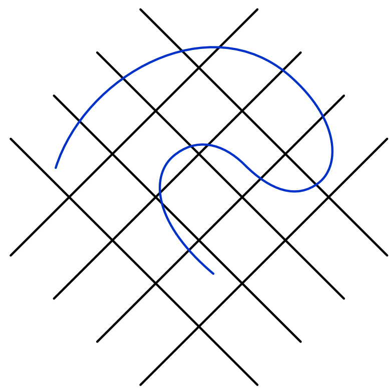
$$\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(a_2+1)) \to \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(a_2))$$

$\pi^*(\mathcal{O}_{\mathbb{P}^1}(-1))$

$0 < a_1 \le a_2\mathcal{E} = \mathcal{O}_{\mathbb{P}^1}(a_1) \oplus \mathcal{O}_{\mathbb{P}^1}(a_2)\pi : X := \mathbb{P}(\mathcal{E}) \to \mathbb{P}^1\sigma : \mathbb{P}^1 \to C \subset X\pi\sigma$
$\pi\sigma\mathcal{E} \to \mathcal{L}\mathcal{L}\mathbb{P}^1 X\mathbb{P}^{a_1+a_2+1}\mathcal{L} = \sigma^*\mathcal{O}_C(1)C\mathcal{L}$

$\pi e\mathbb{P}^{a_1+a_2+1} 0 < e = a_1 e \ge a_2$

$\square$



$C \subset \mathbb{P}^r D C |D| D \mathbb{P}^r t t \leq r-2 C t+1$

$\mathbb{C}^2 \cong V \subset H^0(\mathcal{O}_C(D)) H C D t W := H^0(\mathcal{O}_C(H-D)) r-t V \otimes W \rightarrow H^0(\mathcal{O}_C(H))$

$\square$

$C \subset \mathbb{P}^r$

$$\{D_\lambda \mid \lambda \in \mathbb{P}^1\}$$

$\mathrm{g}_2^1 C$

$C \subset \mathbb{P}^{g-1}\{D_\lambda \mid \lambda \in \mathbb{P}^1\} \mathrm{g}_3^1$

$D_\lambda S(a_1, a_2) a := \max\{a_1, a_2\} a D_\lambda$

$|a_2-a_1|$

$\mathcal{L} \mathcal{O}_C(D_\lambda) \mathrm{g}_2^1 \mathrm{g}_3^1 s_\lambda D_\lambda \mathcal{M}=\mathcal{L}^{-1} \otimes \mathcal{O}_C(1) s_\lambda \cdot H^0(\mathcal{M}) \subset H^0(\mathcal{O}_C(1)) D_\lambda$

$M_\mu \mu : H^0(\mathcal{L}) \otimes H^0(\mathcal{M}) \rightarrow H^0(\mathcal{O}_C(1)) S(a_1, a_2) I_2(M_\mu)$

$\square$

$S(a_1, a_2) \subset \mathbb{P}^{a_1+a_2+1} a_1 \leq a_2 \mathbb{P}^{a_1+a_2+1} a_2$

$V \otimes H^0(\mathcal{L}_2) \rightarrow H^0(\mathcal{O}_X(1)) V H^0(\mathcal{L}_1) a_2 \mathcal{L}_1^{-a_2} \mathcal{O}_X(1)$

$C_{a_1} C_{a_2} a_2 a_2 H a_2 H C_{a_i} a_i H S(a_1, a_2) \subset H S(a_1, a_2)$

$V.$ $\square$

$C \subset \mathbb{P}^r d \geq 2r+1 \pi(r, d) C r=5 C$

$H \cap C d \geq 2(r-1)+3 H 2(r-1)+1 C C C$ $\square$

$$\mathcal{E}=\mathcal{O}_{\mathbb{P}^1}(a_1) \oplus \mathcal{O}_{\mathbb{P}^1}(a_2)$$

$0 \leq a_1 1 \leq a_2 a_1 \leq a_2 S(a_1, a_2) \subset \mathbb{P}^r r=a_1+a_2+1 X=\mathbb{P}(\mathcal{E}) 0<a_1 C_1 a_1=0$ $\pi: X \rightarrow \mathbb{P}^1 C_{a_i} \subset X a_i S(a_1, a_2) d:=a_1+a_2=r-1$

$1 \leq a_1 S(a_1, a_2)$

$X=S(a_1, a_2) \operatorname{Pic} X \cong \mathbb{Z}^2 F H$

$\operatorname{Pic}(X)$

$$\begin{array}{c|c} \cdot & \begin{array}{cc} F & H \end{array} \\ \begin{array}{c} F \\ H \end{array} & \begin{pmatrix} 0 & 1 \\ 1 & d \end{pmatrix} \end{array}$$

$K:=-2 H+(d-2) F K^2=8$

$p H+q F p d+q\binom{p}{2} d+p q-p-q+1$

$C_{a_1} H-a_2 F C_{a_2} H-a_1 F$

$C \subset \mathbb{P}^{g-1} X C C 3 H+(4-g) F=H-K$

$$H^2=\operatorname{deg} X=a_1+a_2=d ; \quad H . F=\operatorname{deg} F=1 ; \quad F^2=0$$

$H F$

$\operatorname{Pic} X H F D D^{\prime}=D-(F \cdot D) H F D^{\prime} \sim a F a \mathcal{O}_X(D^{\prime})|_F=\mathcal{O}_F F \pi_*(\mathcal{O}_X(D^{\prime})) \mathbb{P}^1$ $\mathcal{L} \mathbb{P}^1 D^{\prime}-D^{\prime} \pi_*(\mathcal{O}_X(-D^{\prime}))=(\pi_*(\mathcal{O}_X(D^{\prime}))^{-1} \mathcal{L} \pi^* \mathcal{L}=\pi^* \pi_* \mathcal{O}_X(D^{\prime}) \rightarrow \mathcal{O}_X(D^{\prime})$ $\mathcal{O}_X(D^{\prime}) \cong \pi^* \mathcal{L} q=\operatorname{deg} \mathcal{L} D^{\prime} \sim q F q H \cdot D^{\prime}$

$K_X=p H+q F H F-2=(F+K) \cdot F=p$

$$-2=(H+K) \cdot H=d+p d+q=d+(-2) d+q$$

$q=d-2$

$C_{a_1} X C_{a_1} a_2 C_{a_1} \sim H-a_2 F C_{a_2}$

$C \subset S(a_1, a_2) \subset \mathbb{P}^{g-1} g d g-2 C . F=3 C .(C+K)=2 g-2 C \sim H-K=$ $3 H+(4-g) F .$ $\square$

$0<a_1 D=p H+q F$

$D \sim qF p = 0, q > 0$

$D \sim C_{a_1} p = 1, q = -a_2$

$p \geq 0 D \cdot C_{a_1} > 0 q \geq -pa_1.$

$|D| a_2 > a_1 q > -pa_1 |D|$

$D^2 = 0 \pi D^2 = a_1 - a_2 \leq 0 D^2 \geq 0 D^2 = 0 d = 2, q = -pa_1 X C_{a_1}$

$$D^2 = dp^2 - 2pq = p(pa_1 + pa_2 - 2q) = 0 a_2 = a_1 q = -pa_1$$

□

$0 < a_1 D X D \sim pH + qF p \geq 0$

$$\begin{aligned} H^0(\mathcal{O}_X(D)) &= H^0(\mathcal{O}_{\mathbb{P}^1}(q) \otimes \operatorname{Sym}^p \mathcal{E}) \\ &= \bigoplus_{0 \leq i \leq p} H^0\big(\mathcal{O}_{\mathbb{P}^1}(q + (p-i)a_1 + ia_2)\big). \end{aligned}$$

$|D|$

$$h^0(\mathcal{O}_X(D)) = \sum_{i | q + (p-i)a_1 + ia_2 \geq 0} 1 + (q + (p-i)a_1 + ia_2),$$

$|D| p \geq 0 q \geq -pa_1$

$q < -pa_1$

$$D \cdot C_{a_1} = (pH + qF) \cdot (H - a_2 F) = p(a_1 + a_2) - pa_1 + q = pa_1 + q < 0,$$

$D C_{a_1}$

$\pi : X \to \mathbb{P}^1 X = \mathbb{P}_{\mathbb{P}^1}(\mathcal{E}) H^0(\mathcal{O}_X(pH + qF)) = H^0(\pi_*(\mathcal{O}_X(pH + qF)))$ $\mathcal{O}_X(pH + qF) \mathcal{O}_X(p) \otimes \pi^* \mathcal{O}_{\mathbb{P}^1}(q) \mathcal{O}_{\mathbb{P}^1}(q)$

$$\pi_*\big(\mathcal{O}_X(p) \otimes \pi^* \mathcal{O}_{\mathbb{P}^1}(q)\big) = \pi_*\big(\mathcal{O}_X(p)\big) \otimes \mathcal{O}_{\mathbb{P}^1}(q).$$

$\mathbb{P}(\mathcal{E}) \mathbb{P}(\mathcal{E}) := \operatorname{Proj}(\operatorname{Sym}(\mathcal{E})) U \mathbb{P}^1 \mathcal{E} \pi^{-1}(U) = U \times \mathbb{P}^1 \pi_*(\mathcal{O}_{\mathbb{P}(\mathcal{E})}(p)) = \operatorname{Sym}^p \mathcal{E}$

$$\begin{aligned} \pi_*(\mathcal{O}_X(pH + qF)) &= \pi_*\big(\mathcal{O}_X(p) \otimes \pi^* \mathcal{O}_{\mathbb{P}^1}(q)\big) \\ &= \pi_*\big(\mathcal{O}_X(p)\big) \otimes \mathcal{O}_{\mathbb{P}^1}(q) \\ &= \operatorname{Sym}^p(\mathcal{E}) \otimes \mathcal{O}_{\mathbb{P}^1}(q) \\ &= \Big(\bigoplus_{0 \leq i \leq p} \mathcal{O}_{\mathbb{P}^1}((p-i)a_1 + ia_2)\Big) \otimes \mathcal{O}_{\mathbb{P}^1}(q), \end{aligned}$$

$H^0(\mathcal{O}_{\mathbb{P}^1}(q + (p-i)a_1 + ia_2)) i H^0(\mathcal{O}_{\mathbb{P}^1}(q + pa_1)) q \geq -pa_1 \sigma = \sum \sigma_i \mathcal{O}_X(D) \sigma$ $C_{a_1} = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(a_1)) \sigma_0 C_{a_2} \sigma_p C_{a_1} C_{a_2}$ □

$S(0,a_2)$ $a_1=0$

$X\mathcal{E}X\mathcal{L}X\pi : \mathbb{P}(\mathcal{E})\to X$

$$\mathbb{P}(\mathcal{L}\otimes\mathcal{E})\cong\mathbb{P}(\mathcal{E})$$

$\mathcal{O}_{\mathbb{P}(\mathcal{E}\otimes\mathcal{L})}(1)\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)\otimes\pi^*\mathcal{L}$

$S(0,a_2)a_2S(1,a_2+1)$

$$|\mathcal{O}_{S(1,a_2+1)}(1)\otimes\pi^*(\mathcal{O}_{\mathbb{P}^1}(-1))|,$$

C_1

$S(1,a_2+1)\tilde{S}(0,a_2)$

$\mathbb{P}(\mathcal{E})\to\mathbb{P}(\mathcal{E}\otimes\mathcal{L})\pi^*(\mathcal{E}\otimes\mathcal{L})\cong\pi^*(\mathcal{E})\otimes\pi^*(\mathcal{L})\to\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)\otimes\pi^*(\mathcal{L})$

$\mathcal{E}=\mathcal{O}_{\mathbb{P}^1}(a_1)\oplus\mathcal{O}_{\mathbb{P}^1}(a_2+1)a_1=1$

$$\mathcal{O}_{S(a_1,a_2+1)}(1)\otimes\pi^*(\mathcal{O}_{\mathbb{P}^1}(-1))$$

$H-FC_{a_1}C_{a_1}H^0(\mathcal{E})F|\mathcal{O}_{\mathbb{P}^1}(1)|C_{a_2+1}\cong\mathbb{P}^1|\mathcal{O}_{\mathbb{P}^1}(a_2+1)|$ □

$CS(0,d)1\le dH,F$

$C\sim mHC\sim mH+Fm\ge 0$

$C\sim mH\deg C=mdCg=\binom{m}{2}d-m+1$

$C\sim mH+F\deg C=md+1Cg=\binom{m}{2}d$

$\deg Cd$

$d=1S(0,1)\cong\mathbb{P}^2\phi_1(\mathbb{P}^1)mH+F=(m+1)H$

$2\le dS(0,d)d\pi : S(1,d+1)=\tilde{S}(0,d)\to S(0,d)E,LS(1,d+1)E=C_1$ $E^2=-dHS(0,d)L0EE+dL\widetilde{C}\cong CCS(1,d)C\widetilde{C}E$

$S(1,d+1)$ □

$C\subset\mathbb{P}^{g-1}g\le 4g=4CCg=3$

$Cg\ge 2CS(0,d)C\sim 2H2H+F\tilde{S}(0,d)=S(1,d+1)C2d2d+1Cd-1d$ $S(0,d)\subset\mathbb{P}^dC$

$C\subset\mathbb{P}^{g-1}\mathbb{P}^{g-1}S(0,g-2)\deg C=2g-2g-2g=3,4$

$\widetilde{C}\subset\tilde{S}=S(1,d+1)2H+\epsilon F\epsilon$ □

$X \operatorname{codim} X$

$p \in L \cap X X p X X p p X$

$\pi_p : X \to \mathbb{P}^{N-1} X X' := \pi_p(X) \dim X' = \dim X \operatorname{codim} X' = \operatorname{codim} X - 1$ $\pi_p(L) X' L \cap X p \deg X \geq \deg X' + 1 \deg X' \geq \operatorname{codim} X' + 1 = \operatorname{codim} X$

$1 \leq a_1 < a_2 S(a_1, a_2) C_{a_1} S(a_1 - 1, a_2) C_{a_2} S(a_1, a_2 - 1) a_1 = a_2 S(a_1, a_2) S(a_1 - 1, a_2)$

$S(0, a_2) a_2 S(1, a_2 + 1)$

M

$M p \times q p \leq q I_p(M) q - p + 1$

$X \subset \mathbb{P}^r I m X \mathbb{P}^{n-1} m X c m \leq \binom{c+1}{2} X \deg X = c + 1$

$X \subset \mathbb{P}^r c \binom{c}{2} X c$

$Y = X \cap H X H^0(\mathcal{I}_{X/\mathbb{P}^r}(2)) \to H^0(\mathcal{I}_{Y/\mathbb{P}^{n-1}}(2)) n - c$

$H^i(\mathcal{O}_X(D)) X \subset \mathbb{P}^r X$

$X \subset \mathbb{P}^r D \subset X D H^1_*(\mathcal{I}_{D/X}) = 0$

$> 2 a_1 S(a_2, \dots)$

$0 \leq a_1 \leq a_2 S(a_1, a_2) C_{a_1} a_1 = 0 S(a_1, a_2 - 1)$

$C \subset S(0, a_2) \subset \mathbb{P}^{a_2+1} m S(1, a_2 + 1) m H - m F$

$$\deg(C) = m a_2 \qquad g(C) = \binom{m}{2} a_2 - m + 1.$$

$C \subset S(0, a_2) \subset \mathbb{P}^{a_2+1} m S(1, a_2 + 1) m H - (m - 1) F$

$$\deg(C) = m a_2 + 1 \qquad g(C) = \binom{m}{2} a_2.$$

$\mathbb{P}^3 d \geq 2r + 3$

$C \subset \mathbb{P}^4 I_C I_C \mathbb{P}^4 S(1, 2) S(0, 3)$

$S(0, 3) S(0, 3) 3 a 3 a + 1 g^1_3 C D K - D g^1_2$

$CS := S(1,2) C H F C \sim pH + qF S \mathbb{P}^1 C p \mathbb{P}^1 C p \geq 3q \geq -p \deg C = C \cdot H = 3p + q = 8 C \sim 3H - F C \sim 4H - 4F$

$2g(C) - 2 = 8 = (C + K_S) \cdot C = (4H - 4F) + (-2H + F)) \cdot (4H - 4F) = 4$

$C \sim 3H - F$

$C S C g^1_3 C D = p + q + r p, q r S S W^1_4(C) g^1_3 K_C - g^1_3 - p W^1_4(C) C C S$

$C \mathbb{P}^{g-1} C S(a_1, a_2)$

$$a_2 - a_1 \leq \frac{g+2}{3}.$$

$\mathbb{P}^3 \mathbb{P}^r$

$MS := \mathbb{C}[x_0, \dots x_r]M$

$$(\mathbb{F}, \phi): \quad 0 \longleftarrow N \xleftarrow{\epsilon} \bigoplus_j S(-j)^{\beta_{0,j}} \longleftarrow \cdots \xleftarrow{\phi_t} \bigoplus_j S(-j)^{\beta_{t,j}} \longleftarrow 0.$$

$S(-j) j N F_i \mathbb{F} \otimes_S \mathbb{C}$

$S/II = (f_1, \dots, f_t) f_1, \dots, f_t \deg f_i = d_i t = 2, 3$

$$S \xleftarrow{\begin{pmatrix} f_1 & f_2 \end{pmatrix}} S(-d_1) \oplus S(-d_2) \xleftarrow{\begin{pmatrix} -f_2 \\ f_1 \end{pmatrix}} S(-d_1 - d_2) \longleftarrow 0,$$

$$S \xleftarrow{\begin{pmatrix} f_1 & f_2 & f_3 \end{pmatrix}} F_1 \xleftarrow{\begin{pmatrix} 0 & f_3 & -f_2 \\ -f_3 & 0 & f_1 \\ f_2 & -f_1 & 0 \end{pmatrix}} F_2 \xleftarrow{\begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix}} F_3 \longleftarrow 0,$$

$$F_1 = \bigoplus_{j=1}^{3} S(-d_j), F_2 = \bigoplus_{1 \le i < j \le 3} S(-d_i - d_j) F_3 = S(-d_1 - d_2 - d_3).$$