Geometry of Algebraic Curves

Fall 2011, taught by Joe Harris.

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1 Notation

Let X be a compact Riemann surface, or equivalently a projective algebraic curve over $\mathbb{C}$. A divisor D is a finite linear combination of points in X:

$$D = \sum_p n_p p. \tag{1.1}$$

We say that D is effective, or $D \geq 0$, if every n_p is nonnegative. We write $\deg(D) = \sum_p n_p$.

Given a meromorphic function f on X, we define the principal divisor

$$(f) = \sum_p \operatorname{ord}_p(f) p. \tag{1.2}$$

Given $D = \sum_p n_p p$, we write

$$\begin{aligned} \mathcal{L}(D) &= \{f : \operatorname{ord}_p(f) \geq -n_p\} && (1.3)\\ &= \{f : (f) + D \geq 0\}. && (1.4) \end{aligned}$$

Let $\ell(D) = \dim \mathcal{L}(D)$ and $r(D) = \ell(D) - 1$.

We say that D is linearly equivalent to E ($D \sim E$) if for some meromorphic g on X, we have $D - E = (g)$. Observe that in this situation, $\mathcal{L}(D) \cong \mathcal{L}(E)$ via multiplication by g. As $\deg(g) = 0$, $D \sim E$ requires $\deg D = \deg E$. Let

$$\operatorname{Pic}^d(X) = \{\text{divisors of degree } d\}/\sim. \tag{1.5}$$

For ω a meromorphic 1-form, if $\omega = f(z)\,dz$ near p, define $\operatorname{ord}_p(\omega) = \operatorname{ord}_p(f)$. If ω and η are any two meromorphic 1-forms, then $\frac{\omega}{\eta}$ is a global rational function, so $(\omega) \sim (\eta)$. Define the canonical class K_X of X to be the equivalence class of (ω).

If $D = \sum_p n_p p$ is a divisor on X, let $\mathcal{O}_X$ be the sheaf of regular functions on X, and $\mathcal{O}_X(D)$ the sheaf of regular functions on X with zeros and poles as described by D. Specifically,

$$\mathcal{O}_X(D)(U) = \{f : U \to \widehat{\mathbb{C}} : \operatorname{ord}_p(f) \geq -n_p \,\forall\, p \in U\}. \tag{1.6}$$

$L = \mathcal{O}_X(D)$ is locally free of rank 1; in other words, a line bundle on X. And if $D \sim E$, then $\mathcal{O}_X(D) \cong \mathcal{O}_X(E)$, so {line bundles on X} $\leftrightarrow$ {divisors}$/\sim$; this is typically how $\operatorname{Pic} X$ is defined. The canonical class arises from T_X^*, the cotangent bundle of X.

In the above situation, $\mathcal{L}(D) \cong H^0(L)$, the global sections of L.

Say L is a line bundle on X and $\sigma_0, \ldots, \sigma_r \in H^0(L)$ with no common zeros. We get a map $f : X \to \mathbb{P}^r$ by $p \mapsto [\sigma_0(p), \ldots, \sigma_r(p)]$. Modulo $PGL_{r+1} = \operatorname{Aut}(\mathbb{P}^r)$, f depends only on $V = \langle \sigma_0, \ldots, \sigma_r \rangle \subseteq H^0(L)$. We get a correspondence between

$$\{(L, V) : L \text{ a line bundle of degree } d \text{ on } X, V^{r+1} \subseteq H^0(L) \text{ with no common zeros}\} \tag{1.7}$$

and

$$\{\text{nondegenerate maps } f : X \to \mathbb{P}^r \text{ of degree } d\}/\operatorname{Aut}(\mathbb{P}^r). \tag{1.8}$$

A linear series of degree d and dimension r on X is a pair (L, V) for L a line bundle of degree d and $V^{r+1} \subseteq H^0(L)$ (V may have common zeros here). This is called a g^r_d.

In the case of the canonical divisor, $\ell(K)$ is the dimension of the vector space of global holomorphic 1-forms, equal to g. We get a canonical map $\phi = \phi_K : X \to \mathbb{P}^{g-1}$ associated to $(K, H^0(K))$.

We conclude by stating the following result:

Theorem 1.1 (Riemann-Roch)**.** *For any divisor D on X, we have*

$$\ell(D) = d + 1 - g + \ell(K - D). \tag{1.9}$$

2 Riemann Surfaces Associated to a Polynomial

Let $f(x, y) \subseteq \mathbb{C}[x, y] = (\mathbb{C}[x])[y]$. Then $f(x, y) = 0$ implicitly gives y as a function of x. Let

$$X = \{(x, y) : f(x, y) = 0\} \subseteq \mathbb{C}^2 \tag{2.1}$$

and X^0 the smooth locus of X. For $p \in X$ and $q = \pi(p) \in \mathbb{C}_x$, if $\frac{\partial f}{\partial y}(p) \neq 0$, the X is smooth at p and π is locally an isomorphism. If $\frac{\partial f}{\partial y}(p) = 0$ but $\frac{\partial f}{\partial x}(p) \neq 0$, then X is still smooth at p, but π locally has the form $z \mapsto z^m$ for some positive integer m.

Finally, it may be that X has a singularity at p. There exists a disc Δ of $q = \pi(p)$ such that for $\Delta^* = \Delta \setminus \{q\}$, $\pi^{-1}(\Delta^*) \to \Delta^*$ is a covering space. So $\pi^{-1}(\Delta^*)$ will be a disjoint union of punctured discs. We can then complete each disc by adding a point to each.

Here are some examples:

- $y^2 - x^2 = (y - x)(y + x)$ is two discs.
- $y^2 - x^3$ is given by $\{(t^2, t^3) : t \in \mathbb{C}^\times\}$ outside the origin. We get a single punctured disc after removing the origin; this completes to a disc.
- $y(x - y^2)$ is again two discs.

To do Possibly draw pictures of curves? (1)

Now complete the map to a map to $\mathbb{P}^1_x$. For $R \gg 0$, $\pi^{-1}(\mathbb{C}_x \setminus \Delta_R) \to \mathbb{C}_x \setminus \Delta_R$ is again a covering space. X is completed to a compact space accordingly.

As an example, for $y^2 = x^3 - 1$ near ∞, winding once around a large complex number increases $\arg(x)$ by 2π, so $\arg(y)$ increases by 3π. We end up on the other branch, so $y^2 = x^3 - 1$ is connected near ∞. On the other hand, $y^2 = x^4 - 1$ is disconnected near ∞.

It is also possible to carry out this process analytically, by taking the closure $\mathbb{CP}^2$ of $\mathbb{C}^2$ and then normalizing by blowing up. It is usually easier to work analytically, though.

3 Riemann-Hurwitz

Let X, Y be compact Riemann surfaces with $g(X) = g$ and $g(Y) = h$, and suppose $f : X \to Y$ is nonconstant of degree d. At each point $p \in X \mapsto q \in Y$, f is locally z^m. We say that p is a ramification point of order $m - 1 = v_p(f)$. Let $R = \sum_p v_p(f)p$ be the ramification divisor, and

$$B = \sum_q \left(\sum_{f(p)=q} v_p(f) \right) q = \sum_q \eta_q q \tag{3.1}$$

be the branch divisor. Then $\#f^{-1}(q) = d - \eta_q$. After removing all branch points and their preimages, we get a d-sheeted covering map. So if $y_1, \ldots, y_\delta$ the branch points (so $\delta = \deg B$), we get

$$\chi(X \setminus f^{-1}(q_1) \setminus \cdots \setminus f^{-1}(q_d)) = d\chi(Y \setminus \{q_1, \ldots, q_d\}). \tag{3.2}$$

This means $2 - 2g = d(2 - 2h) = \sum_q \eta_q q$, or:

Theorem 3.1 (Riemann-Hurwitz)**.** *$g - 1 = d(h - 1) + \frac{b}{2}$, where $b = \deg B$.*

Again consider $f : X \to Y$, and let ω be a meromorphic 1-form on Y such that $\operatorname{supp}((\omega)) \cap \operatorname{supp}(B) = \varnothing$. Pulling back ω, we get

$$(f^*\omega) = f^*(\omega) + R \tag{3.3}$$

since at the points of ramification,

$$z \mapsto z^m = w \implies dw = mz^{m-1}\, dz. \tag{3.4}$$

Hence if $\deg(K_Y) = 2h - 2$, then $\deg(K_X) = 2g - 2$.

Since every compact Riemann surface admits a map to $\mathbb{P}^1$, and $\deg(K_{\mathbb{P}^1}) = -2$, we conclude that $\deg(K_X) = 2g - 2$ for every X.

4 Linear Series

Fix Me We may rename this. (2)

Consider a linear series on X given by (L, V) for L a line bundle of degree d and $V^{r+1} \subseteq H^0(L)$, called a g^r_d. A base point is a common zero of every $\sigma \in V$. (L, V) is base point free if no common zero exists.

A global section $\sigma \in H^0(L)$ is determined up to multiplication by its divisors of zeros, so we get a family of effective divisors D of degree d parameterized by $\mathbb{P}V$. So a g^r_d corresponds to a family of effective divisors parameterized by $\mathbb{P}^r$.

If (L, V) is base point free, we get a map $X \to \mathbb{P}^r$ by $p \mapsto V(-p) \subseteq V$. (For E an effective divisor on X, $V(-E) = \{\sigma \in V : (\sigma) \geq E\}$.) If $f_0, \ldots, f_r \in \mathcal{L}(D)$ form a basis, we have $\phi(z) = [f_0(z) : \cdots : f_r(z)]$. Observe that ϕ is characterized (up to $\mathrm{Aut}(\mathbb{P}^r)$) by the property that

$$\{\phi^{-1}(H) : H \subseteq \mathbb{P}^r \text{ a hyperplane}\} = D. \tag{4.1}$$

Proposition 4.1. *For (L, V) a g^r_d on X, then $\phi : X \to \mathbb{P}V^*$ is an embedding if and only if $\forall\, p, q \in X$, $\dim V(-p - q) = \dim(V) - 2$.*

Proof. ϕ separates points if and only if $V(-p) \neq V(-q)$ for all $p \neq q$, and ϕ is an immersion if and only if $V(-2p) \subsetneq V(-p)$ for all p. $\square$

As a consequence. if $\deg L \geq 2g + 1$ and $V = H^0(L)$, then ϕ_V is an embedding.

Consider the canonical map $\phi_K : X \to \mathbb{P}^{g-1}$. This fails to be an embedding if and only if $\ell(p + q) \geq 2$ for some p, q; that is, there exists a g^1_2 on X (a nonconstant meromorphic function of degree 2). In this case, we say that X is hyperelliptic. If X is not hyperelliptic, we call $\phi_K(X) \subseteq \mathbb{P}^{g-1}$ the canonical model of X.

Theorem 4.2 (Geometric Riemann-Roch)**.** *Assume that X is not hyperelliptic, giving $X \hookrightarrow \mathbb{P}^{g-1}$ of degree $2g - 2$, and $D = p_1 + \cdots + p_d$ a divisor consisting of d distinct points. Then $r(D)$ equals the number of linear relations of points $p_1, \ldots, p_d$.*

Proof. $\mathcal{L}(K - D)$ is the set of holomorphic differentials vanishing on D, therefore the set of linear forms on $\mathbb{P}^{g-1}$ vanishing at $p_1, \ldots, p_d$. Now apply Riemann-Roch. $\square$

5 Facts About Higher Dimensional Varieties

Let X be a smooth projective variety (or complex manifold) of dimension n. A divisor D on X is a formal linear combination of irreducible subvarieties of dimension $n - 1$. Then we can talk about effective divisors, linear equivalence, etc. Also

$$\mathrm{Pic}(X) = \{\text{line bundles on } X\} = \{\text{divisors on } X\}/\sim \tag{5.1}$$

if X is quasi-projective (so every line bundle has meromorphic sections).

Consider the canonical bundle $K_X = \Lambda^n T_X^*$, the line bundle whose sections are holomorphic forms of top degree, expressible locally as $f(z_1, \ldots, z_n)\, dz_1 \wedge \cdots \wedge dz_n$. For ω an n-form on X, write (ω) for the corresponding divisor.

For $X = \mathbb{P}^n$, every irreducible subvariety is the zero locus of a homogeneous polynomial F. And $V(F) \sim V(G)$ requires $\deg(F) = \deg(G)$, but the converse is also true, since $[V(F)] - [V(G)] = \left(\frac{F}{G}\right)$. We find that $\mathrm{Pic}(\mathbb{P}^n) \cong \mathbb{Z}$, generated by the hyperplane class $\mathcal{O}_{\mathbb{P}^n}(1)$.

The meromorphic differential $\omega = dz_1 \wedge \cdots \wedge dz_n$ is holomorphic and nonvanishing on $\mathbb{A}^n$, and on the hyperplane at ∞, it has a pole of order $n+1$. So $K_{\mathbb{P}^n} = -(n+1)H = \mathcal{O}_{\mathbb{P}^n}(-n-1)$.

Theorem 5.1 (Adjunction)**.** *For X a smooth variety (or complex manifold) and $Y \subseteq X$ of codimension 1, then*

$$K_Y = (K_X \otimes \mathcal{O}_X(Y))|_Y. \tag{5.2}$$

Here are some examples:

- Suppose $X \subseteq \mathbb{P}^2$ is a smooth curve of degree d, so that $\mathcal{O}_{\mathbb{P}^2}(X) = \mathcal{O}_{\mathbb{P}^2}(d)$. Since $K_{\mathbb{P}^2} = \mathcal{O}_{\mathbb{P}^2}(-3)$,

  $$K_X = \mathcal{O}_{\mathbb{P}^2}(d-3)|_X = \mathcal{O}_X(d-3). \tag{5.3}$$

  In particular, since $\deg \mathcal{O}_X(1) = d$, $d(d-3) = \deg K_X = 2g-2$, implying $g = \binom{d-1}{2}$.

- Suppose $X = \mathbb{P}^1 \times \mathbb{P}^1 = Q \subseteq \mathbb{P}^3$. Then letting e and f be two lines in different rulings of Q, we have $\mathrm{Pic}(X) = \langle e, f\rangle$ and $K_X \sim -2e - 2f$. The formula for K_X may be found either by considering Q as $\mathbb{P}^1 \times \mathbb{P}^1$, or by adjunction:

  $$K_Q = (K_{\mathbb{P}^3} \otimes \mathcal{O}_{\mathbb{P}^3}(2))|_Q = \mathcal{O}_Q(-2). \tag{5.4}$$

  Observe that the hyperplane class $\mathcal{O}_Q(1)$ is $e + f$.

  A smooth curve $X \subseteq \mathbb{P}^1 \times \mathbb{P}^1$ has type (a, b) if $X \sim ae + bf$. Equivalently, X meets a line of the first ruling b times and a line of the second ruling a times. Equivalently, X is given by the zero locus of a bihomogenous polynomial of bidegree (a, b). In this case

  $$K_X = (K_{\mathbb{P}^1\times\mathbb{P}^1} \otimes \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(X)|_X = (a-2)e + (b-2)f. \tag{5.5}$$

  Now $2g - 2 = \deg K_X = (a-2)b + (b-2)a$, implying $g = (a-1)(b-1)$.

- Suppose X has genus 0. First, $X \cong \mathbb{P}^1$, since for $p \in X$, $L = \mathcal{O}_X(p)$ gives a degree 1 map $X \to \mathbb{P}^1$. Then $\mathrm{Pic}(X) = \mathbb{Z}$, generated by p. If $D = d\infty$, then $\mathcal{L}(D) = \langle 1, z, \ldots, z^d\rangle$, and $\phi_D : \mathbb{P}^1 \to \mathbb{P}^d$ is $z \mapsto [1 : z : z^2 : \cdots : z^d]$, called the rational normal curve of degree d.

6 Overview of Brill-Noether Theory

For each g, we have a space $\mathfrak{M}_g$ parameterizing curves of genus g. $\mathfrak{M}_g$ can be "stratified" according to how curves can be embedded in projective space.

The Brill-Noether problem: For *general* curves X of genus g, describe the space of maps $X \to \mathbb{P}^1$ which are nondegenerate and have degree d. For example, what is the smallest degree of a nonconstant map $X \to \mathbb{P}^1$, or the smallest degree of a plane curve birational to X, or the smallest degree of an embedding in $\mathbb{P}^3$?

The classical approach: instead of $\mathfrak{M}_g$, consider the Hurwitz scheme

$$\mathfrak{H}_{d,g} = \{(X, f) : f : X \to \mathbb{P}^1 \text{ of degree } d, \text{ simply branched over } b \text{ points}\}. \tag{6.1}$$

Letting Δ be the diagonal in $\mathbb{P}^b$, there exist maps

$$\begin{tikzcd} & \mathfrak{H}_{d,g} \arrow[dl] \arrow[dr] & \\ \mathfrak{M}_{d,g} & & \mathbb{P}^b \setminus \Delta \end{tikzcd} \tag{6.2}$$

The map $\mathfrak{H}_{d,g} \to \mathbb{P}^b \setminus \Delta$ is a covering map, so $\mathfrak{H}_{d,g}$ has degree b.

To describe fibers of $\mathfrak{H}_{d,g} \to \mathfrak{M}_g$, take $d \gg 0$ ($d \geq 2g+1$ suffices), and specify a divisor D on X of degree d, so that $\ell(D) = d - g + 1$. An open subset of $\mathcal{L}(D)$ then consists entirely of simply branched functions. As there are d parameters for D, we conclude that

$$\dim \mathfrak{M}_g = b - (d + d - g + 1) = 3g - 3 \tag{6.3}$$

since $b = 2d + 2g - 2$. On the other hand, $\mathfrak{H}_{d,g} \to \mathfrak{M}_g$ must *always* (even for small g) have fibers of dimension at least 3, since PGL_2 acts on these fibers. So $\mathfrak{H}_{d,g}$ can dominate $\mathfrak{M}_g$ only when $b \geq 3g - 3 + 3$, or $d \geq \frac{g}{2} + 1$. It turns out that the converse is true!

Next, consider

$$V_{d,g} = \{(X, f : X \to \mathbb{P}^2) : f \text{ of degree } d, \text{ birational onto a nodal curve}\}. \tag{6.4}$$

Such a nodal curve must have $\delta = \binom{d-1}{2} - g$ nodes. We have maps

$$\begin{tikzcd} & V_{d,g} \arrow[dl] \arrow[dr] & \\ \mathfrak{M}_g & & (\mathbb{P}^2)^\delta \setminus \Delta \end{tikzcd} \tag{6.5}$$

The fibers of the map $V_{d,g} \to (\mathbb{P}^2)^\delta \setminus \Delta$ have dimension

$$\underbrace{\frac{d(d+3)}{2}}_{\text{all plane curves}} - \underbrace{3\delta}_{\text{number of conditions for functions to be doubled at nodes}} \tag{6.6}$$

so that $V_{d,g}$ has dimension $\frac{d(d+3)}{2} - \delta = 3d + g - 1$. (Actually, this argument only works for small δ, but can be shown in general using deformation theory.)

Now PGL_3 acts on the fibers of $V_{d,g} \to \mathfrak{M}_g$, so the fibers have dimension at least 8. So $V_{d,g}$ can dominate $\mathfrak{M}_g$ only if $3d + g - 9 \geq 3g - 3$. or $d \geq \frac{2}{3}g + 2$.

In general, there exists a nondegenerate map $X \to \mathbb{P}^r$ of degree d in general if and only if $d \geq \frac{r}{r+1}g + r$. However, for $\mathbb{P}^3$, the above method fails! The analog of $\mathfrak{H}_{d,g}$ and $V_{d,g}$ are very poorly behaved.

Theorem 6.1. *Let $\rho = g - (r+1)(g - d + r)$. Then a general Riemann surface of genus g admits a nondegenerate map to $\mathbb{P}^r$ of degree d if and only if $\rho \geq 0$, and the dimension of the space of such maps is ρ. Moreover, for a general such f, f is an embedding if $r \geq 3$, and is a birational embedding if $r = 2$.*

Once X is embedded as a curve $C \subseteq \mathbb{P}^r$, it acquires a great deal of structure:

- Geometric structure: secant planes, tangent lines, etc.
- Algebraic structure: an ideal $I_C \trianglelefteq K[z_0, \ldots, z_r]$.

Here is a problem: for general X and a general $f : X \to \mathbb{P}^r$ of degree d, for $r \geq 3$, describe (for example) a minimal set of generators for the ideal I_C of the image.

As an example, a general X of genus 2 can be embedded in $\mathbb{P}^3$ of degree 5, but no lower. Pick a general divisor D of degree 5, so $\ell(D) = 4$. We get an embedding $X \to C \subseteq \mathbb{P}^3$ given by $[1 : f_1, f_2, f_3]$ for $1, f_1, f_2, f_3$ a basis of $\mathcal{L}(D)$.

C can't lie on a plane, since it's nondegenerate. Does it lie on a quadric? We have a map

$$\underbrace{\{\text{ homogeneous quadratic polynomials on } \mathbb{P}^3\}}_{\dim 10} \to \underbrace{\mathcal{L}(2D)}_{\dim 9} \tag{6.7}$$

This map must have a nontrivial kernel, forcing such a quadric to exist. By Bezout, it's unique. Call it Q. Similarly, the space of cubics has dimension 20, while $\ell(3D) = 14$, so C lies on (at least) 6 dimensions of cubics. Four of them come from the union of Q and a plane, so there are 2 cubics modulo Q.

In Q, C will have type $(2, 3)$. Taking a line formed by the ruling of Q, we find that the union of C and this line is the intersection of Q and some cubic.

As an exercise, we have an exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-4)^{\oplus 2} \to \mathcal{O}_{\mathbb{P}^3}(-3)^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^3}(-2) \to \mathcal{I}_C \to 0. \tag{6.8}$$

Conjecture 6.2 (Maximal rank)**.** *Let X a general curve of genus g, and $f : X \hookrightarrow \mathbb{P}^r$ a general map of degree d, for $r \geq 3$. We have a map*

$$r_m : \{\textit{homogeneous polynomials of degree } m \textit{ on } \mathbb{P}^r\} \to \mathcal{L}(mD). \tag{6.9}$$

The conjecture is that r_m has maximal rank.

7 Abel's Theorem

For C a Riemann surface of genus g, consider $\int_{p_0}^{p}$ as a linear function on the space $H^0(K_C)$ of holomorphic 1-forms, modulo those functionals obtained by integrating over closed loops. We have a map $H^1(C, \mathbb{Z}) \to H^0(K_C)^*$; define the Jacobian to be

$$J(C) = H^0(K_C)^*/H_1(C, \mathbb{Z}). \tag{7.1}$$

We have a map $C \to J(C)$ by $p \mapsto \int_{p_0}^{p}$ (fixing p_0 in advance). This map can be extended additively to all divisors, and in particular, by fixing a degree d, we get a map $u_d : C_d \to J$, where C_d is the dth symmetric product.

Remark (Abel). If $D \sim E$, so that $D - E = (f)$, we get a family of divisors $D_t = (f - t) + E$, with $D_0 = D$ and $D_\infty = E$. Then we get a map $\mathbb{P}^1(J)$ by $t \mapsto u_d(D_t) = \sum_i \int_{p_0}^{p_i(t)}$.

But $T_p^* J$ is generated by global holomorphic 1-forms on J, and no nonzero holomorphic 1-forms exist on $\mathbb{P}^1$, so this map must be constant! Alternatively, we can lift to the universal cover $\mathbb{P}^1 \to \mathbb{C}^g$ since $\mathbb{P}^1$ is simply connected, and any map $\mathbb{P}^1 \to \mathbb{C}^g$ must be constant.

Therefore $u_d(D)$ just depends on the equivalence class of D'.

Theorem 7.1 (Clebsch)**.** *$u(D) = u(E)$ if and only if $D \sim E$.*

We conclude that the fibers of the map $u : C_d \to J$ are exactly the complete linear systems. Here are some consequences:

- For $d \leq g$, the generic fiber of $u : C_d \to J$ is a single point, so u is birational onto its image. In particular, C_g is birational to J, called the Jacobi inversion.
- For $d \geq 2g - 1$, $C_d \to J$ is a $\mathbb{P}^{d-g}$-bundle.
- In genus 1, $C \cong J$.
- In genus 2, $C_2 \to J$ is one-to-one except at linear series of degree 2 and dimension at least 1, which must be $|K|$. So C_2 is the blow-up of J at a point.

8 Plane Curves

Given a smooth projective curve C and a divisor D, we would like to find $|D|$, or equivalently $\mathcal{L}(D)$. In particular, finding $|K|$ is equivalent to finding all holomorphic 1-forms.

8.1 Smooth Plane Curves

For starters, we'll assume that $C \subseteq \mathbb{P}^2$ is a smooth plane curve of degree d. Fix a line L_∞, then $\mathbb{A}^2 = \mathbb{P}^2 \setminus L_\infty$ is an affine open. Choose coordinates (x, y) on $\mathbb{A}^2$, equivalently $[X : Y : Z]$ on $\mathbb{P}^2$, so that $L_\infty = \{Z = 0\}$. By taking L_∞ sufficiently general, we can assume that L_∞ is transverse to C (so the intersection is d distinct points) and $[0 : 1 : 0] \notin C$. In $\mathbb{A}^2$, let C be the zero locus of f.

Consider $\pi_{[0:1:0]}C \to \mathbb{P}^1_x$ expressing C as a d-sheeted cover which is unramified over ∞. Start with the meromorphic differential $\omega_0 = dx$. On C, $\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = 0$, and smoothness implies dx and dy do not have common zeros, and neither do $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$. Therefore $(dx) = \left(\frac{\partial f}{\partial y}\right)$ as a divisor on $C \cap \mathbb{A}^2$. At ∞, we have d poles of order 2. So for $D_\infty = C \cap L_\infty$, we conclude that

$$(dx) = \left(\frac{\partial f}{\partial y}\right)|_{\mathbb{A}^2} - 2D_\infty. \tag{8.1}$$

We can kill the poles at ∞ by dividing by an appropriate meromorphic function, namely $\frac{\partial f}{\partial y} = f_y$. Then $\frac{dx}{f_y}$ is holomorphic and nonzero on $\mathbb{A}^2 \cap C$, and has zeros of order $d - 3$ along D_∞, since f_y has poles of order $d - 1$ there. We get

$$\left(\frac{dx}{f_y}\right) = (d-3)D_\infty. \tag{8.2}$$

We can produce more holomorphic functions by multiplying by any polynomial $g(x, y)$ of degree at most $d - 3$. We get

$$H^0(C, K) = \left\{\frac{g\,dx}{f_y} : g \text{ of degree at most } d-3\right\} \tag{8.3}$$

having dimension $\binom{d-1}{2}$.

Now suppose D is an effective divisor on C; we want to find $|D|$, or $\mathcal{L}(D)$. Start by choosing a polynomial $g(x, y)$ of degree m vanishing at D, giving a curve $G = V(g) \subseteq \mathbb{P}^2$ of degree m, containing D.

Write $G \cap C = D + E$. Now choose a polynomial h, also of degree d, vanishing on E. Let $H \cap C = E + F$; then

$$E + F = H \cap C \sim mD_\infty \sim G \cap C = D + E \implies F \sim D. \tag{8.4}$$

Specifically, $F - D = \left(\frac{h}{g}\right)$. (If $D = D' - D''$ with D' and D'' effective, choose G containing D' and H containing $E + D''$.)

Claim. Choose any g. Then

$$\left\{\frac{h}{g} : \deg h = m, h(E) = 0\right\} = \mathcal{L}(D). \tag{8.5}$$

Say $\deg D = n$. Then $\deg E = md - n$, so

$$\ell(D) \geq \underbrace{\binom{m+2}{2}}_{\deg m} - \underbrace{\binom{m-d+2}{2}}_{\text{vanishing on } C} - \underbrace{(md-n)}_{\substack{\text{vanishing on } E \\ \text{(this forces inequality)}}} \tag{8.6}$$

$$= n - \binom{d-2}{2} + 1. \tag{8.7}$$

8.2 Nodal Plane Curves

Now let C be a smooth projective curve of genus g, with $\nu : C \to C_0 \subseteq \mathbb{P}^2$ of degree d, having nodal singularities $r_1, \ldots, r_\delta$. For simplification, we assume that there are no vertical tangents at nodes. Then at each node r_i, let $\nu^{-1}(r_i) = p_1 + q_i$. Then at the nodes, f_y has a simple zero while dx does not vanish, so $\left(\frac{dx}{f_y}\right)$ (really $\nu^*\left(\frac{dx}{f_y}\right)$) equals $(d-3)D_\infty - \Delta$, for $\Delta = \sum_i (p_i + q_i)$. So the space of holomorphic differentials is

$$\left\{ \frac{g\,dx}{f_y} : \deg(g) \leq d-3, g(r_i) = 0 \,\forall\, i \right\}. \tag{8.8}$$

This has dimension at least $\binom{d-1}{2} - \delta = g$ (which can be deduced by several methods, such as Riemann-Hurwitz).

First, this is the complete set of holomorphic differentials. Second, since we have equality, the conditions $g(r_i) = 0$ are linearly independent, so $r_1, \ldots, r_\delta$ impose independent conditions on polynomials of degree $d-3$. Also, recall that $K_C = (d-3)D_\infty - \Delta$.

Now consider an effective divisor D of degree n, where, for notational convenience, D and Δ have disjoint support. Let $g(x, y)$ be of degree m vanishing on D and on Δ. So $(g) = D + \Delta + E - mD_\infty$. Let h of degree m vanish on E and Δ. Then $(h) = E + \Delta + F - mD_\infty$, so $\left(\frac{h}{g}\right) = F - D$.

Claim.

$$\mathcal{L}(D) = \left\{ \frac{h}{g} : \deg h = m, h(E) = h(r_i) = 0 \right\}. \tag{8.9}$$

Counting degrees, $\deg E = md - n - 2\delta$, and so

$$\ell(D) \geq \binom{m+2}{2} - \binom{m-d+2}{2} - \underbrace{(md-n-2\delta)}_{\text{reason for inequality}} - \delta \tag{8.10}$$

$$= n - \binom{d-1}{2} + \delta + 1 \tag{8.11}$$

$$= n - g + 1. \tag{8.12}$$

8.3 More General Singularities

Although the case of nodal plane curves contains all curves up to birational equivalence, we would like to directly deal with plane curves having arbitrary singularities. We can find the complete linear series by replacing $g(r_i) = 0$ with "g satisfies the adjoint conditions at the r_i" (g is in the conductor ideal, equal to $\mathrm{Ann}\left(\frac{\nu_* \mathcal{O}_C}{\mathcal{O}_{C_0}}\right)_{r_i}$).

9 Riemann-Roch

We'll now show that the space of holomorphic differentials has dimension at most g (showing the dimension equals g, and that we get all of the differentials by the methods of §§8.2).

Nonzero holomorphic differentials can't be of the form df, so $H^0(K)$ embeds into $H^1_{\mathrm{dR}}(C) \cong \mathbb{C}^{2g}$, and also $\overline{H^0(K)}$ embeds into $H^1_{\mathrm{dR}}(C)$.

Claim. $H^0(K) \cap \overline{H^0(K)} = 0$.

Proof. We have a positive definite Hermitian form on $H^0(K)$ given by

$$H(\omega) = i \int_C \omega \wedge \overline{\omega}. \tag{9.1}$$

For if $\omega = f(z)\, dz$, then $\overline{\omega} = \overline{f}(z)\, d\overline{z}$, so that

$$\omega \wedge \overline{\omega} = -i|f(z)|^2\, dx \wedge dy. \tag{9.2}$$

On the other hand, if $\overline{\omega}$ is homologous to η, then

$$\int_C \omega \wedge \overline{\omega} = \int_C \omega \wedge \eta = 0. \tag{9.3}$$

□

The claim implies $H^0(K)$ and $\overline{H^0(K)}$ have dimension at most g.

Now we will prove Riemann-Roch. First suppose D is effective, with $D = p_1 + \cdots + p_d$, and $f \in \mathcal{L}(D)$. For any holomorphic differential on C, $\sum \mathrm{Res}_{p_i}(f\omega) = 0$. This imposes $g - \ell(K - D)$ linear conditions on the principal parts of f (which determine f up to a constant), so

$$\ell(D) \leq d + 1 - (g - \ell(K - D)). \tag{9.4}$$

If $K - D$ is also effective, then $\ell(K - D) \leq (2g - 2 - d) + 1 - (g - \ell(D))$ as well, forcing equality. If $\ell(K - D) > 0$, then the argument still applies, since $K - D \sim E$ for some effective E.

On the other hand, if D is effective but $\ell(K - D) = 0$, then $\ell(D) \leq d + 1 - g$. The reverse inequality was proven in §§8.2!

Finally, if $\ell(D) = \ell(K - D) = 0$, then the inequality $\ell(D) \geq d - g + 1$ forces $d \leq g - 1$, and similarly $\deg(K - D) \leq g - 1$. As $\deg(K) = 2g - 2$, this is only possible if D and $K - D$ both have degree $g - 1$, and in this case, we're done.

We also asserted that

$$\left\{ \frac{h}{g} : \deg h = m, h(E) = h(r_i) = 0 \right\} \subseteq \mathcal{L}(D) \tag{9.5}$$

was the complete linear series, for g of degree m satisfying $g(D) = g(r_i) = 0$, $(g) = D + \Delta + E - mD_\infty$. Completeness of the adjoint series asserts that we have equality for all m (this is left as an exercise, and is easier to show if $m \geq d - 3$).

As an example, a quartic with three nodes is rational. For if p is a point on the quartic, and the nodes are r_1, r_2, r_3, then pick a conic through p, r_1, r_2, r_3. This intersects the quartic with multiplicity 8, so its intersection divisor has the form $p + 2r_1 + 2r_2 + 2r_3 + q$ for some q. We get a point $q(p)$. Taking another conic gives $q'(p)$, and so $q \sim q'$. Now $\frac{q}{q'}$ maps to $\mathbb{P}^1$.

10 Independent Conditions on Polynomials

Suppose $\Gamma = \{p_1, \dots, p_\delta\} \subseteq \mathbb{P}^2$. Then the set of polynomials of degree m vanishing at $p_1, \dots, p_\delta$ embeds into the set of all polynomials of degree m. In other words,

$$H^0(\mathcal{I}_\Gamma(m)) \hookrightarrow H^0(\mathcal{O}_{\mathbb{P}^2}(m)). \tag{10.1}$$

We would like to know the codimension of this embedding.

We say that Γ imposes independent conditions on polynomials or curves of degree m if this codimension is δ. (A priori, it's at most δ.) Equivalently, $\mathcal{O}_{\mathbb{P}^2}(m) \to \mathcal{O}_\Gamma(m)$ is surjective on global sections.

Now we have an exact sequence of sheaves

$$0 \to \mathcal{I}_\Gamma(m) \to \mathcal{O}_{\mathbb{P}^2}(m) \to \mathcal{O}_\Gamma(m) \to 0 \tag{10.2}$$

Then the long exact sequence of cohomology gives us

$$0 \to H^0(\mathcal{I}_\Gamma(m)) \to H^0(\mathcal{O}_{\mathbb{P}^2}(m)) \to H^0(\mathcal{O}_\Gamma(m)) \to H^1(\mathcal{I}_\Gamma(m)) \to H^1(\mathcal{O}_{\mathbb{P}^2}(m)) = 0 \tag{10.3}$$

so independent conditions is equivalent to having $H^1(\mathcal{I}_\Gamma(m)) = 0$.

Suppose $C \subseteq \mathbb{P}^3$ is a quartic curve, and let $\Gamma = C \cap H$ for H a hyperplane. Γ consists of four points, so we have two quadratics on $H \cong \mathbb{P}^2$ vanishing on Γ. Are they restrictions to H of quadratic polynomials in $\mathbb{P}^3$ vanishing on C?

We have an exact sequence

$$0 \to \mathcal{I}_C(1) \to \mathcal{I}_C(2) \to \mathcal{I}_\Gamma(2) \to 0. \tag{10.4}$$

Theorem 10.1. *If $\delta \leq m+1$, then any Γ of degree δ imposes independent conditions on polynomials of degree m. If $\delta = m+2$, then Γ fails to impose independent conditions if and only if Γ is contained in a line.*

We can find a plane curve of degree m passing through all but one of these points (by using m lines).

Suppose $C \subseteq \mathbb{P}^2$ is a smooth plane curve of degree d. What is the smallest degree of a nonconstant meromorphic function on C? Equivalently, what is the smallest m such that there exists $C \to \mathbb{P}^1$ of degree m?

This is certainly possible for $m = d-1$; choose two lines through a point on C and then take their quotient. On the other hand, consider $D = p_1 + \cdots + p_m$. Then $r(D) \geq 1$ if and only if $p_1, \dots, p_m$

fail to impose independent conditions on $|K|$. But $|K|$ consists of the intersections of C with plane curves of degree $d-3$. By Theorem 10.1, this can only happen if $m \geq d-1$, with equality only if $p_1, \ldots, p_m$ are all collinear.

11 Curves of Low Genus

Let C be a curve of genus g, and L (or D) a line bundle (or divisor class) of degree d. We get a map $\varphi : C \to \mathbb{P}^r$ of degree d. For $g = 0, 1$, the behavior of φ depends only on d. For $g = 2$, the behavior can depend on the specifics of L (for example, if $d = 4$). For $g \geq 3$, in general, the behavior can depend on both L and C.

- $g = 3, d = 4$: We get $\varphi_K : C \to \mathbb{P}^2$. In the nonhyperelliptic case, this embeds C as a smooth plane quartic.

- Hyperelliptic curves (of any genus g): these can be expressed in the form

$$y^2 = \prod_{i=1}^{2g+2} (x - \lambda_i). \tag{11.1}$$

  Let p and q be the points lying over ∞ and $r_i = (\lambda_i, 0)$ be the ramification points. The holomorphic differentials are $\frac{P(x)\,dx}{y}$ for P of degree at most $g-1$.

  For $g \geq 2$, $\varphi_K = [1 : x : \cdots : x^{g-1}]$, the composition of the two-to-one map $\pi : C \to \mathbb{P}^1$ and the rational normal curve in $\mathbb{P}^{g-1}$. For $\pi^*(\mathrm{pt}) = g^1_2$, we get $K = (g-1)g^1_2$. For any two points $u, v \in C$, by Riemann-Roch, $r(u+v) \geq 1$ if and only if $|K(-u)| = |K(-v)|$ if and only if $\varphi_K(p) = \varphi_K(q)$, so the g^1_2 is unique.

  Recall geometric Riemann-Roch: for $\varphi_K : C \to \mathbb{P}^{g-1}$ (even if C is hyperelliptic), and D any effective divisor of degree d on C, let

$$\overline{D} = \bigcap_{\varphi_K^{-1}(D) \supseteq H} H. \tag{11.2}$$

  Then $r(D) = d - 1 - \dim \overline{D}$.

  In the hyperelliptic case, for C_0 the image of φ_K, C_0 is a rational normal curve, so for $d \leq g$, any divisor of degree $d \leq g$ on C_0 is linearly independent (spans a $\mathbb{P}^{d-1}$). Geometric Riemann-Roch implies that for any special D, $|D| = mg^1_2 + D_0$ for D_0 a fixed divisor of degree $d - 2m$. Then φ_D factors through π.

  So, to get a birational embedding of a hyperelliptic C, we have to take $d \geq g+2$ and $r = d-g$. To get an embedding, we need $d \geq g+3$ and $r = d-g$. (This is the most resistant to embedding in $\mathbb{P}^r$.)

- $g = 3$: If C is hyperelliptic, $d = 6$, and $D \neq K + p + q$ for any $p, q \in C$, we get an embedding $C \to \mathbb{P}^3$ as a curve of type $(2, 4)$ on a quadric. For $D - g^1_2 \not\supseteq g^1_2$, implying $D - g^1_2$ is nonspecial of degree 4. We get $r = 1$, therefore a four-to-one map φ to $\mathbb{P}^1$. Then $\pi \times \varphi : C \to \mathbb{P}^1 \times \mathbb{P}^1 \subseteq \mathbb{P}^3$.

  For C nonhyperelliptic, $C \hookrightarrow \mathbb{P}^2$ is a smooth quadric. In particular, it can be expressed as a three-sheeted cover of $\mathbb{P}^1$ in ∞^1 ways.

- $g = 4$ and C nonhyperelliptic: Can C be expressed as a three-sheeted cover of $\mathbb{P}^1$? Look at the canonical model $C_0 \hookrightarrow \mathbb{P}^3$ of degree 6. Then φ_K induces $H^0(\mathcal{O}_{\mathbb{P}^3}(2)) \to H^0(K^2)$, of dimensions 10 and 9, and $H^0(\mathcal{O}_{\mathbb{P}^3}(3)) \to H^0(K^3)$ of dimensions 20 and 15. So C_0 lies on a quadric, and five dimensions of cubics, four of which contain the quadric. Then $C \subseteq Q$ irreducible, and $C_0 \subseteq S$ not containing Q, and so $C_0 = Q \cap S$. By the Noether theorem, the homogeneous ideal of C_0 is generated by Q and S. In particular, φ_K^* on $H^0(\mathcal{O}_{\mathbb{P}^3}(2,3))$ have maximal rank. This implies every quadratic and cubic differential form is generated by the linear forms.

  Theorem 11.1 (Noether, $AF + BG$)**.** *If $V(F)$ and $V(G)$ are transverse, and $H = 0$ on $V(F) \cap V(G)$, then $H = AF + BG$ for some A, B.*

  Conversely, if C is the smooth complete intersection of any quadric and cubic in $\mathbb{P}^3$, then C has degree 6, and $K_C = \mathcal{O}_C(1)$ (in particular, C has genus 4). (Q could be smooth, or Q could be a cone with vertex not in S.)

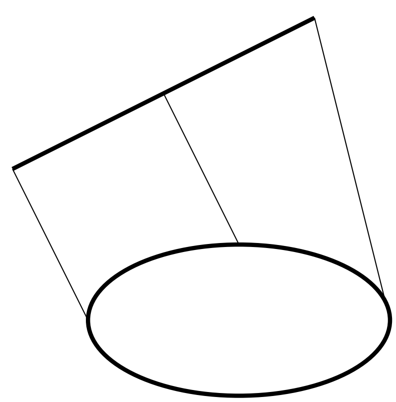
  Now for D of degree 3 on C, $r(D) \geq 1$ if and only if D is contained in a line. Such a line is contained in Q. We get that C is a 3-sheeted cover of $\mathbb{P}^1$ in 1 or 2 ways, depending on whether Q is a cone or smooth.

- $g = 5$ and C nonhyperelliptic: then C is canonically embedded in $\mathbb{P}^4$ as a curve of degree 8. C lies on at least three linearly independent quadrics Q_1, Q_2, Q_3. C can be the complete intersection of $Q_1 \cap Q_2 \cap Q_3$. Conversely, if $C = Q_1 \cap Q_2 \cap Q_3$, then $K_C = \mathcal{O}_C(1)$; this happens in general.

  But this can fail if all of the quadrics contain a surface. By the $AF + BG$ theorem, $Q_1 \cap Q_2$ must be reducible, and then Q_3 contains a component. We can only have that $Q_1 \cap Q_2$ is the union of a cubic and a plane, with Q_3 containing the cubic (any quadric surface must be contained in some hyperplane).

  So there are two possibilities: either $C = Q_1 \cap Q_2 \cap Q_3$, or $C \subseteq S = Q_1 \cap Q_2 \cap Q_3$.

  If D is a divisor of degree 3 on C, $r(D) \geq 1$ if and only if D is contained in a line. This line must be contained in Q_i. In the first case, this is impossible. In the second case, it turns out that S is a cubic (rational normal) scroll.



(11.3)

This is $\mathbb{P}^2$ blown up at a point p, embedded in $\mathbb{P}^4$ by a linear system $2H - E$ (the conics in $\mathbb{P}^2$ through p). When C lies on a cubic scroll, C meets the ruling of the scroll three times, giving a unique g_3^1.

12 Automorphism Groups of Curves

Fix a smooth projective curve X of genus g; what can we say about $\mathrm{Aut}(X)$?

For $g = 0$, $\mathrm{Aut}(\mathbb{P}^1) = PGL_2(\mathbb{C})$, which is simply 3-transitive.

For $g = 1$, fix $p \in X$; then we get a group law with origin p. For every $q \in X$, $T_q : x \mapsto x \oplus q$ is an automorphism. Also, $\varphi : x \mapsto -x$ is an automorphism.

Are there others? After translating by T_q, we reduce to finding $\widetilde{\varphi}$ such that $\widetilde{\varphi}(p) = p$. Consider $|2p|$, giving a branched cover $X \to \mathbb{P}^1$. This induces an automorphism of $\mathbb{P}^1$ permuting the four branch points (therefore the three ones other than ∞ since $\varphi(p) = p$). Conversely, given such an automorphism $\overline{\varphi}$ of $\mathbb{P}^1$, we can lift to an automorphism $\widetilde{\varphi}$ of the cover. $\widetilde{\varphi}$ is unique up to the hyperelliptic involution (which is negation).

The only triples of points for which there is a nontrival automorphism are $(1, 0, -1)$ and $(1, \omega, \omega^2)$. In these cases, there are two and three classes of $\overline{\varphi}$, respectively.

(Alternatively, view the curve as $\mathbb{C}/\Lambda$ and lift $\widetilde{\varphi}$ to $\mathbb{C}$.)

Theorem 12.1. *For X a smooth projective curve of genus $g \geq 2$, $\mathrm{Aut}(X)$ is finite, with at most $84(g-1)$ elements.*

Step 1 is to show that $\mathrm{Aut}(X)$ is finite. Step 2 is to bound the size by $84(g-1)$, given that it is finite.

12.1 Finiteness of the Automorphism Group

Let $G = \mathrm{Aut}(X)$ and $V = H^0(K_X)$, so G acts on V by pulling back holomorphic differentials, so we get a map $\rho : G \to GL(V)$.

We first claim that $\operatorname{im}\rho$ is finite. Under the positive definite hermitian form

$$\langle \omega, \eta \rangle = i \int_X \omega \wedge \overline{\eta}, \tag{12.1}$$

ρ actually has image in the unitary group U_g, which is a compact group. But also ρ preserves the lattice $H^1(X; \mathbb{Z}) \hookrightarrow H^0(K)$, so its image lies in the space of integer matrices, which is discrete. Therefore the image is finite.

Now we claim that ρ is injective. Suppose $g \in G$ acts trivially on V. Then

$$H^*_{\mathrm{dR}}(X, \mathbb{C}) = H^0(X, \mathbb{C}) \oplus (H^0(K) \oplus \overline{H^0(K)}) \oplus H^2(X, \mathbb{C}) \tag{12.2}$$

implies g acts trivially on H^*_{dR}. Therefore the Lefschetz number of g is $\chi(X) = 2 - 2g < 0$. The Lefschetz fixed point theorem implies there are infinitely many fixed points (if the number of fixed points is finite, it equals the Lefschetz number), and so g is the identity.

An alternative proof: First observe that if $\varphi : X \to X$ fixes more than $2g + 2$ points, then it is the identity (for the Lefschetz number is at most $2g + 2$). The associate to a Riemann surface X a finite set $W \subseteq X$, such that any automorphism permutes W. If W is not hyperelliptic, then W has more than $2g + 2$ elements, so $\mathrm{Aut}(X)$ is finite.

Yet another proof sketch (purely algebraic): $\mathrm{Aut}(X)$ can be made into a quasi-projective scheme. We can show it's zero-dimensional. In fact, the tangent space is always zero. For $T_C|\mathrm{Aut}(C)| = H^0(T_X)$, which is zero for $g \geq 2$.

12.2 Bounding the Number of Automorphisms

Now we know that G is finite. Form the quotient $Y = X/G$, the set of G-orbits, with a map $\pi : X \to Y$.

Claim. Y can be given the structure of a Riemann surface such that $\pi : X \to Y$ is holomorphic.

To show this, give Y the quotient topology. If G is fixed point free, π' is a covering map. In general, the points of X with nontrivial stabilizers is finite. And for all $p \in X$, $G_p = \{g \in G : g(p) = p\}$ is cyclic. Fo we have a map $G_p \to \mathbb{C}^\times$ by $g(z) = a_1 z + a_2 z^2 + \cdots$. We claim it's injective. If $g(z) = z + az^m + \cdots$, then $g^k(z) = z + kaz^m + \cdots$, and g^k is the identity for some k, so $a = 0$.

Now given p, if it has a stabilizer, choose a neighborhood U of p disjoint from its translates modulo G. We can choose a coordinate z on U such that G_p acts by $z \mapsto \zeta_n z$, so z^n acts on U/G_p.

Now if X has genus g and Y has genus h, Riemann-Hurwitz gives $2g - 2 = |G|(2h - 2) + R$. Suppose Y has branch points $p_1, \ldots, p_b$, and over p_i there are f_i ramification points. The action on ramification points is uniform, so all z are mapped to z^{e_i}, with $e_i f_i = |G|$. We have

$$R = \sum_i f_i(e_i - 1) \tag{12.3}$$

$$= |G| \sum_{i=1}^{b} \left(1 - \frac{1}{e_i}\right) \tag{12.4}$$

so that

$$2g - 2 = |G| \left(\underbrace{(2h-2) + \sum_{i=1}^{b} \left(1 - \frac{1}{e_i}\right)}_{Q} \right). \tag{12.5}$$

We want to determine the minimum positive value of Q. We have $h \geq 0$, $b \geq 0$, and $e_i \geq 2$ integers. If $h \geq 2$, then $Q \geq 1$. If $h = 1$, then $Q = \sum_{i=1}^{b} \left(1 - \frac{1}{e_i}\right)$. In order to have $Q > 0$, we must have $b \geq 1$, so $Q \geq \frac{1}{2}$.

It remains to deal with $h = 0$. Then $Q = -2 + \sum_{i=1}^{b} \left(1 - \frac{1}{e_i}\right)$. We need $b \geq 3$. If $b \geq 4$, then

$$Q \geq -2 + \left(1 - \frac{1}{2}\right) + \left(1 - \frac{1}{2}\right) + \left(1 - \frac{1}{2}\right) + \left(1 - \frac{1}{3}\right) = \frac{1}{6}. \tag{12.6}$$

If $b = 3$, it turns out the minimum possible Q is $\frac{1}{42}$. So $Q \geq \frac{1}{42}$, implying $|G| \leq 84(g-1)$.

12.3 Strength of the Bound

It turns out this bound is sharp for infinitely many genera. For $g = 3$, we have the Klein quartic

$$X^3Y + Y^3Z + Z^3X = 0. \tag{12.7}$$

$\mathrm{Aut}(C)$ contains $[X : Y : Z] \mapsto [Y : Z : X]$ of order 3 and $[X : Y : Z] \mapsto [\zeta^4 X : \zeta^2 Y : \zeta Z]$ of order 7, where ζ is a primitive 7th root of unity. Other automorphisms are harder to write down, but there are 168 of them.

Let $N(g)$ be the size of the largest automorphism group that a curve of genus g can have. Then $N(g) \leq 84(g-1)$, with equality for infinitely many genera. On the other hand, the curve $y^2 = x^{2g+2} - 1$ has $8(g+1)$ automorphisms, implying $N(g) \geq 8(g+1)$. This lower bound is sharp for infinitely many genera.

13 Special Linear Series

For C a smooth projective curve of genus g and D a divisor (class) of degree d, what can we say about $h^0(D) = \ell(D)$ or $r(D) = \ell(D) - 1$?

We know that $h^0 = 0$ for $d < 0$, $h^0 = d - g + 1$ for $d \geq 2g - 2$, and $h^0 \leq d + 1$ in general.

Theorem 13.1 (Clifford)**.** *If $0 \leq d \leq 2g-2$, then $h^0(D) \leq \frac{d}{2} + 1$. (Equivalently, $r(D) \leq \frac{d}{2}$.) Equality holds if and only if either $D = 0$, $D = K$, or if C is hyperelliptic and D is a multiple of the g^1_2.*

Let $\mathcal{D} \subseteq |D|$ be a g^r_d and $\mathcal{E} \subseteq |E|$ be a g^s_e; then define $\mathcal{D} + \mathcal{E}$ to be the subspace of $|D + E|$ spanned by the divisors of the form $D' + E'$ for $D' \in \mathcal{D}$ and $E' \in \mathcal{E}$. Letting $\mathcal{D} = (L, V \subseteq H^0(L))$ and $\mathcal{E} = (M, W \subseteq H^0(M))$, amp $V \otimes W \to H^0(L \otimes M)$.

Fact. If $\varphi : V \otimes W \to U$ is such that φ is injective on $\langle v \rangle \otimes W$ for every v and $V \otimes \langle w \rangle$ for every w, then $\dim(\operatorname{im} \varphi) \geq \dim V + \dim W - 1$.

(This fact is actually false for real spaces!)

Lemma 13.2. *$r(\mathcal{E} + \mathcal{D}) \geq r(\mathcal{E}) + r(\mathcal{D})$.*

This is true since $r(D) \geq r$ if and only if for every $p_1, \ldots, p_r \in C$, there exists $D \in \mathcal{D}$ such that $p_1, \ldots, p_r \in D$.

Proof of Clifford's Theorem. Apply Lemma 13.2 to $\mathcal{D} = |E|$ and $\mathcal{E} = |K - D|$. So $\mathcal{D} + \mathcal{E} \subseteq |K|$, implying $h^0(D) + h^0(K-D) \leq g+1$. But also $h^0(D) - h^0(K-D) = d - g + 1$. So $h^0(D) \leq \frac{d}{2} + 1$. $\square$

However, what linear series exist on a general curve? That is, for which r, d does every curve of genus g possess a g^r_d? This is much smaller than Clifford's result, and is the subject of Brill-Noether theory.

Also, what very ample line bundles exist? (That is, linear series giving an embedding or birational embedding.)

So now suppose that $C \hookrightarrow \mathbb{P}^r$ os a smooth, irreducible, nondegenerate curve of degree d. How large can g be? In the case $r = 3$, Clifford's bound is sufficient for $d \leq 6$, but not for $d \geq 7$.

14 Castelnuovo's Approach

Start with $C \hookrightarrow \mathbb{P}^r$ a smooth, irreducible, nondegenerate curve, and let D be the divisor class given by a hyperplane ($\mathbb{P}^{r-1}$) section. Then we will bound $h^0(mD)$ from below. For $m \gg 0$, m will be nonspecial, and so we can apply Riemann-Roch to get $g = md - h^0(mD) + 1$, so that we obtain an upper bound for g.

To do Draw curve and hyperplane picture. (3)

Let $D = p_1 + \cdots + p_d = C \cap H \hookrightarrow \mathbb{P}^r$ for H a general hyperplane. We have an evaluation map

$$\rho_m : H^0(\mathcal{O}_C(mD)) \to H^0(\mathcal{O}_D(mD)) \cong \mathbb{C}^d. \tag{14.1}$$

Here $\mathcal{O}_C(mD)$ consists of sections of the line bundle $L_m = \mathcal{O}_C(mD)$.

The kernel turns out to be $H^0(\mathcal{O}_C((m-1)D))$. So $h^0(mD) = h^0((m-1)D) + \dim(\operatorname{im} \rho_m)$. But $\operatorname{rank}(\rho_m)$ equals the number of linear conditions on a section of $\mathcal{O}_C(mD)$ to vanish at $p_1, \ldots, p_d$. This is at least the number of linear conditions on a polynomial of degree m on $\mathbb{P}^r$ to vanish at $p_1, \ldots, p_d$, since the polynomials form a subseries of $\mathcal{O}_C(mD)$. This is always at least $\min\{d, m+1\}$, but we want to do better.

Lemma 14.1 (General Position)**.** *For C an irreducible, nondegenerate curve in $\mathbb{P}^r$, H a general hyperplane, and $D = p_1 + \cdots + p_d = H \cap C$, then the points of D are in linearly general position in $\mathbb{P}^{r-1}$ (so no r are linearly dependent).*

We will prove this later.

Now suppose that $p_1, \ldots, p_d \in \mathbb{P}^n$ span $\mathbb{P}^n$ and are in linearly general position; how many conditions do they impose on hypersurfaces of degree m? (This is the Hilbert function $h_D(m)$.)

Claim. $h_D(m) \geq \min\{d, mn+1\}$.

Proof. Say $d \geq mn + 1$. Given $mn + 1$ points of D, we claim we can find a hypersurface Z of degree m containing all but one of them. (We can just take m hyperplanes.) So the points impose independent conditions. □

It turns out that $h_D(m) = \min\{d, mn + 1\}$ for any configuration D on a rational normal curve in $\mathbb{P}^n$, so this bound is actually sharp!

Conclusion: for $C \hookrightarrow \mathbb{P}^r$ of degree d, $n = r-1$ gives $h^0(\mathcal{O}_C(mD)) \geq h^0((m-1)D) + \min\{d, m(r-1)+1\}$. Also $h^0(D) \geq r + 1$, so $h^0(2D) \geq r + 1 + (2r - 1) = 3r$, $h^0(3D) \geq 3r + (3r - 2) = 6r - 2$, etc. if d is large.

Write $d = m_0(r-1) + \epsilon$ for $0 \leq \epsilon \leq r - 2$. Then

$$h^0(m_0 D) \geq \binom{m_0}{2}(r-1) + m_0 + 1. \tag{14.2}$$

After this, we keep adding d, so $h^0((m_0 + k)D) \geq \binom{m_0}{2}(r-1) + m_0 + 1 + kd$. For $k \gg 0$, Riemann-Roch implies

$$
\begin{aligned}
g &= \deg((m_0+k)D) - h^0((m_0+k)D) + 1 && (14.3)\\
&\le (m_0+k)d - \binom{m_0}{2}(r-1) - m_0 - 1 - kd + 1 && (14.4)\\
&= \binom{m_0+1}{2}(r-1) + m_0\epsilon. && (14.5)
\end{aligned}
$$

We denote this last expression by $\pi(d,r)$. What this tells us:

d	$m = \lfloor \frac{d-1}{r-1} \rfloor$	ϵ	π	
r	1	0	0	only rational normal curve
$r+1$	1	1	1	Clifford implies Γ is nonspecial
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
$2r-2$	1	$r-2$	$r-2$	$\cdots$
$2r-1$	2	0	$r-1$	can use Clifford here, too
$2r$	2	1	$r+1$	the special curves are canonical curves
$2r+1$	2	2	$r+3$	
$\cdots$	$\cdots$	$\cdots$	$\cdots$	
$3r-3$	2	$r-2$	$3r-5$	
$3r-2$	3	0	$3r-3$	
$3r-1$	3	1	$3r$	
$\cdots$	$\cdots$	$\cdots$	$\cdots$	

(14.6)

We have $\pi(d,r) \sim \frac{d^2}{2(r-1)}$ for fixed r as $d \to \infty$.

Lemma 14.2. *For $C \subseteq \mathbb{P}^r$ irreducible and nondegenerate, $H \subseteq \mathbb{P}^r$ a general hyperplane, and $\Gamma = H \cap C$, Γ is in linearly general position in $\mathbb{P}^{r-1}$.*

15 Monodromy in General

Suppose $X \to Y$ is a dominant, generically finite morphism of degree d. Then for some open U, we have $|\pi^{-1}(p)| = d$ for $p \in U$, and $f^{-1}(U) \to U$ is a topological covering space.

Now given a covering space, $\pi_1(U,p)$ acts on $\pi^{-1}(p)$ by lifting loops to paths. The induced subgroup G of permutations of the set $\pi^{-1}(p)$ is called the monodromy group.

Algebraically, $K(Y) \hookrightarrow K(X)$ is an extension of finite degree d. Let L be the Galois closure of $K(X)$ over $K(Y)$, Then $G = \mathrm{Gal}(L/K(Y))$.

We have that G is transitive if and only if X is irreducible, and G is 2-transitive if and only if $X \times_U X \setminus \Delta$ as a covering space over U acts transitively on fibers if and only if $X \times_U X \setminus \Delta$ is irreducible. More generally, taking

$$
X_U^{(r)} = X \times_U \cdots \times_U X \setminus \Delta = \{(q, p_1, \ldots, p_r) : q \in U, p_i \in \pi^{-1}(q) \text{ distinct}\}, \tag{15.1}
$$

G is r-transitive if and only if $X_U^{(r)}$ is irreducible.

Theorem 15.1 (Uniform Position)**.** *For $C \subseteq \mathbb{P}^r$ an irreducible nondegenerate curve and $C^* \subseteq (\mathbb{P}^r)^*$ the dual hypersurface (the set of non-transverse hyperplanes), let $U = (\mathbb{P}^r)^* \setminus C^*$ and*

$$X = \{(H,p) : p \in H \cap C\} \subseteq U \times C \tag{15.2}$$

so that X is a covering space over U. Then $G(X/U) = S_d$.

This is only true in characteristic 0.

Proof. 1. We first show that G is 2-transitive. We have that

$$X_U^{(2)} = \{(H,p,q) : H \in U, p \neq q \in H \cap C\} \tag{15.3}$$

projects dominantly onto $C \times C \setminus \Delta$ by $(H,p,q) \mapsto (p,q)$. The fibers are all open in $\mathbb{P}^{r-2}$ and $C \times C \setminus \Delta$ is irreducible, so $X_U^{(2)}$ is irreducible.

2. We claim that G contains a transposition. To see this, choose $H_0 \in (\mathbb{P}^r)^*$ simply tangent to C, so $H_0 \cap C = 2p + p_1 + \cdots + p_{d-2}$. (This requires characteristic zero.) Now choose a small analytic neighborhood V of H_0 in $(\mathbb{P}^r)^*$. Look at $X_{V\cap U}$. As a covering space, we have

   To do Draw discs with multiplicities. (4)

   We claim that $\overline{X} = \{(H,p) : p \in H \cap C\}$ is locally irreducible, so we can move from one component to another. This induces a transposition.

□

Proof that Uniform Position Theorem implies General Position Lemma. Let

$$U = (\mathbb{P}^r)^* \setminus C^* = \{\text{hypeplanes } H \subseteq \mathbb{P}^r \text{ transverse to } C\} \tag{15.4}$$

and

$$X = \{(H,p) : p \in H \cap C\} \subseteq U \times C. \tag{15.5}$$

$X \to U$ is a d-sheeted cover, with monodromy all of S_d. So

$$X_U^{(r)} = \{(H,p_1,\dots,p_r) : p_i \in H \cap C \text{ distinct}\} \subseteq U \times C^r. \tag{15.6}$$

Let $Z = \{(H,p_1,\dots,p_r) : p_i \text{ linearly dependent}\}$. Then Z is a closed subvariety (given by vanishing of the determinant), and Z is a proper subset of $X_U^{(r)}$, since there exist r linearly independent points in C. As $X_U^{(r)}$ is irreducible, it must be that $\dim Z < \dim X = r$, implying $\operatorname{im} Z \subset U$ is contained in a proper closed subvariety of U. □

16 Extremal Curves

For $C \subseteq \mathbb{P}^r$ smooth, we say that C is projectively normal if the hypersurfaces of degree m in $\mathbb{P}^r$ cut out a complete linear series on C. That is, if D is a divisor on C and $D \sim mH$, then there exists Z on $\mathbb{P}^r$ such that $D = Z \cap C$. Equivalently, $H^0(\mathcal{O}_{\mathbb{P}^r}(m)) \to H^0(\mathcal{O}_C(m))$ is surjective, or $H^1(\mathcal{I}_C(m)) = 0$. (This needs to hold for all m.)

Another equivalent definition, which is not entirely obvious: If $S = \overline{pC}$ is the cone over C in $\mathbb{P}^{r+1}$, then C is projectively normal if and only if $\mathcal{O}_{S,p}$ is Cohen-Macaulay.

Now assume C is irreducible and nondegenerate. Let

$$E_k = H^0(\mathcal{O}_{\mathbb{P}^r}(k))|_C \subseteq H^0(\mathcal{O}_C(k)). \tag{16.1}$$

We proved that $\dim E_k - \dim E_{k-1} \geq \min\{d, k(r-1)+1\}$, then concluded an inequality on E_k, and then $g \leq \pi(d, r)$.

If we had $g = \pi(d, r)$, then not only is $\dim E_k - \dim E_{k-1} = \min\{d, k(r-1)+1\}$, but also $E_k = H^0(\mathcal{O}_C(k))$. So C must be projectively normal!

As an example, if $C \subseteq \mathbb{P}^{g-1}$ is a canonical curve, then C is an extremal curve, so must be projectively normal. As a result, every quadratic differential on C is a quadratic polynomial in the holomorphic differentials.

Now our goal is to find curves having genus $\pi(d, r)$. If $\Gamma = H \cap C$ is a general hyperplane section, we need $h_\Gamma(k) = \min\{d, k(r-1)+1\}$. An example of such a Γ is any set of points on a rational normal curve $B \subseteq \mathbb{P}^{r-1}$.

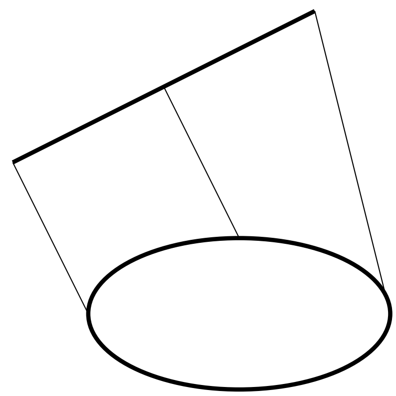
As an example, if $C \subseteq \mathbb{P}^3$ lies on a quadric, then $\Gamma = C \cap H$ lies on a conic. If C has type (a, b), then $d = a + b$ and $g = (a-1)(b-1)$. For fixed d, this is maximal if $|a - b| \leq 1$.

For curves of unbalanced type, we still have $h_\Gamma(k) = \min\{d, k(r-1)+1\}$, but C is not extremal. So C cannot be projectively normal.

In general, we're looking for $C \subseteq \mathbb{P}^r$ such that $H \cap C$ lies on a rational normal curve. Can we find a surface $S \subseteq \mathbb{P}^r$ such that $H \cap S$ is a rational normal curve? We could take the cone of a rational normal curve, but this doesn't always work.

If $X \subseteq \mathbb{P}^r$ is irreducible and nondegenerate of dimension k, then X must have degree at least $r + 1 - k$. We will construct smooth surfaces of degree $r - 1$.

Choose two complementary linear subspaces $\mathbb{P}^a$ and $\mathbb{P}^b$ (they span $\mathbb{P}^r$ and are disjoint), so $a + b = r - 1$. Next, choose a rational normal curve in each subspace; call them C_a and C_b. Then choose an isomorphism $\varphi : C_a \to C_b$, and define $S = \bigcup_{p \in C_a} \overline{p, \varphi(p)}$, the union of lines joining C_a to C_b.



(16.2)

To calculate the degree of S, choose a general hyperplane H containing $\mathbb{P}^a$, and consider $H \cap S$. The intersection contains C_a. Also, $H \cap \mathbb{P}^b$ is a hyperplane in $\mathbb{P}^b$, intersecting C_b in points $p_1, \ldots, p_b$. So $H \cap S$ contains the lines of S through the points $p_1, \ldots, p_b$.

We claim this is it, so $\deg S = a + b = r - 1$. Since if H contained a point on some line, not on $\mathbb{P}^a$, then H contains the line, so intersects at a pointof C_b.

We can choose parameterizations $\varphi : \mathbb{P}^1 \to C_a$ and $\varphi' : \mathbb{P}^1 \to C_b$, and $S = \bigcup_{p \in \mathbb{P}^1} \overline{\varphi(p), \varphi'(p)}$. If a or b is 0, this degenerates to a cone over a rational normal curve.

S is called $S_{a,b}$, a rational normal surface scroll. It turns out this depends only on a and b; any two choices differ by a linear change of coordinates.

Here are two other descriptions of these surfaces:

- $S \cong \mathbb{P}(\mathcal{O}(a) \oplus \mathcal{O}(b)) \xhookrightarrow{\phi_{|\mathcal{O}_S(1)|}} \mathbb{P}^r$ (using post-Grothendieckian projectivization). We have $S_{a,b} \cong S_{a',b'}$ if and only if $|a - b| = |a' - b'|$.
- Given a $2 \times r$ matrix of linear forms

$$A = \begin{pmatrix} L_1 & \cdots & L_r \\ M_1 & \cdots & M_r \end{pmatrix} \tag{16.3}$$

(a general matrix for now), $\Phi = \{p \in \mathbb{P}^r : \operatorname{rank}(A(p)) = 1\}$ is a rational normal curve.

Now if

$$A = \begin{pmatrix} L_1 & \cdots & L_{r-1} \\ M_1 & \cdots & M_{r-1} \end{pmatrix} \tag{16.4}$$

is a $2 \times (r-1)$ matrix of linear forms, $\Phi = \{p \in \mathbb{P}^r : \operatorname{rank}(A(p)) = 1\}$ turns out to be a scroll.

It turns out that:

Theorem 16.1. *If $S \subseteq \mathbb{P}^r$ is an irreducible nondegenerate surface of degree $r - 1$, then S is a scroll (or the cone over a rational normal curve), or $r = 5$ and S is the Veronese surface.*

Given $X^k \subseteq \mathbb{P}^r$ irreducible and nondegenerate of dimension k, how many quadrics can contain X? It turns out that the maximum possible is $\binom{r+1-k}{2}$.

For $Y = H \cap C$, the sequence

$$0 \to \mathcal{I}_{X,\mathbb{P}^r}(1) \to \mathcal{I}_{X,\mathbb{P}^r}(2) \to \mathcal{I}_{Y,\mathbb{P}^{r-1}}(2) \to 0 \tag{16.5}$$

is exact, so $H^0(\mathcal{I}_{X,\mathbb{P}^r}(2)) \hookrightarrow H^0(\mathcal{I}_{Y,\mathbb{P}^{r-1}}(2))$.

We claim that the maximum is achieved by rational normal scrolls, cones over the Veronese surface in $\mathbb{P}^5$, and quadric hypersurfaces. For a surface scroll, 2×2 minors of

$$\begin{pmatrix} L_1 & \cdots & L_{r-1} \\ M_1 & \cdots & M_{r-1} \end{pmatrix} \tag{16.6}$$

give quadrics.

To find curves of maximal genus $\pi(d, r)$, look for curves $C \subseteq \mathbb{P}^r$ whose hyperplane section lies on a rational normal curve. If $d > 2r - 2$, then $Q \subseteq \mathbb{P}^{r-1}$ containing a hyperplane section Γ cut out a rational normal curve in $\mathbb{P}^{r-1}$, so by projective normality, the quadrics in $\mathbb{P}^r$ containing C cut out a surface $S \subseteq \mathbb{P}^r$ of degree $r - 1$.

A few facts about scrolls: let $X_{a,b}$ for $a \leq b$ be the scroll connecting C_a to C_b. Then C_a is called the directrix.

- Projecting from a point on the line of the ruling gives $X_{a,b} \cong X_{a-1,b-1}$ if $a > 1$.
- We have $X_{a,a} \cong X_{1,1} \cong \mathbb{P}^1 \times \mathbb{P}^1$, and $X_{a,a+1} \cong X_{1,2}$, the cubic scroll in $\mathbb{P}^4$. We claim $X_{1,2} \cong Bl_p\mathbb{P}^2$. This is given by the conics through p (carrying L to a conic, and blowing up takes p to a line).

Divisors on $X_{a,b}$: Let F be a line connecting C_a to C_b, and let H be the hyperplane divisor. Then:

- $\operatorname{Pic} X$ is freely generated by $f = [F]$ and $h = [H]$. (First, $U = X \setminus (C_a \cup F) = \mathbb{A}^2$, so the rank is 2. Now given C in X not equal to F or C_a, take $C \cap U$ and $\varphi_t : U \to U$ to be multiplication by t, and $C_t = \overline{\varphi_t(C \cap U)}$. We get a family of curves C_t with $\lim_{t \to 0} C_t$ supported on $C_a \cup F$, so of the form $\alpha F + \beta C_a$.)

  The intersection pairing is given by $H.H = r - 1$, $H.F = 1$, and $F.F = 0$ (so h, f are independent).
- $C_a.F = 1$ and $C_a.H = a$, so $C_a = h - (r - 1 - a)f = h - bf$, Similarly, $C_b = h - af$. (We've seen this before, by considering a hyperplane section through C_a.)
- $C_a^2 = a - b \leq 0$ and $C_b^2 = b - a \geq 0$.
- For the canonical class K_X, we have $K_X = \alpha h + \beta f$ for some integers α, β. Applying adjunction:
  - To F, a line, its genus is 0, so $-2 = (K_X + F).F = \alpha$, implying $\alpha = -2$.
  - To H, a rational normal curve, its genus is also 0. Then $-2 = (K_X + H).H = \beta - (r - 1)$, so $\beta = r - 3$.

Therefore $K_X = -2h + (r-3)f$.

Now suppose $C \subseteq X \subseteq \mathbb{P}^r$ is any curve; let its class be $\alpha h + \beta f$. We have $\deg(C) = C.H = \alpha(r-1) + \beta$. To get the genus,

$$\begin{aligned} 2g - 2 &= C.(K_X + C) && (16.7)\\ &= (\alpha h + \beta f)((\alpha-2)h + (\beta + r - 3)f) && (16.8)\\ &= \alpha(\alpha-2)(r-1) + \beta(\alpha-2) + \alpha(\beta + r - 3) && (16.9)\\ &= \alpha(\alpha-1)(r-1) + 2\alpha\beta - 2\alpha - 2\beta && (16.10) \end{aligned}$$

so that

$$g = \binom{\alpha}{2}(r-1) + (\alpha-1)(\beta-1). \qquad (16.11)$$

Now given a fixed degree, we try to maximize the genus of curves on the scroll. Take $\alpha = m+1$ and $\beta = \epsilon - r + 2$. We get $\deg(C) = m(r-1) + \epsilon + 1$, and

$$\begin{aligned} g(C) &= \binom{m+1}{2}(r-1) + m(\epsilon - r + 1) && (16.12)\\ &= \binom{m}{2}(r-1) + m\epsilon. && (16.13) \end{aligned}$$

(If $\epsilon = 1$, we could also take $\alpha = m$ and $\beta = 1$.)

So the curve C should be residual to $r - 2 - \epsilon$ lines in a hypersurface of degree $m+1$. But does such C exist?

Lemma 16.2. *$\alpha h + \beta f$ is represented by a smooth curve if (but not quite only if) $\alpha > 0$ and $\beta \geq -\alpha a$.*

Fix Me *Can we make this precise? (5)*

$\beta \geq -\alpha a$ means $(\alpha h + \beta f).C_a \geq 0$. If not, C must contain C_a.

Proof. Use Bertini. Write $\alpha h + \beta f = \alpha C_b + (\beta + \alpha a)F$. Check that $|C_b|$ has no basepoints. Then taking α copies of C_b and $\beta + \alpha a$ copies of F, we get a linear series with no basepoints, so the general member is smooth.

To show irreducibility, check that if the curve C is a sum of divisors, then there is a positive intersection between the two, so C would have to be singular. $\square$

Also, in the case $r = 5$ and $d = 2k$ even, we can take C to be the image of a plane curve of degree k under the Veronese map.

17 Castelnuovo's Lemma

Our goal is to show that if $\Gamma \subseteq \mathbb{P}^n$ is a configuration of d points in linearly general position with $d \geq 2n+3$, imposing only $2n+1$ conditions on quadrics, then Γ lies on a rational normal curve.

The Steiner construction: say $\Lambda_1, \ldots, \Lambda_n \cong \mathbb{P}^{n-2} \subseteq \mathbb{P}^n$. Let $\{H_t^i\}_{t\in\mathbb{P}^1}$ be the pencil of hyperplanes containing Λ_i. Consider $H_t^1 \cap \cdots \cap H_t^n$. If this is just a point for each value of t, then the union of these points will be a curve; it turns out to be a rational normal curve.

More generally, say $\Lambda_1, \ldots, \Lambda_\ell \cong \mathbb{P}^{n-2} \subseteq \mathbb{P}^n$, and consider $H_t^1 \cap \cdots \cap H_t^\ell$, expected to be a dimension $n-\ell$ space. If this is the case, we claim that $\bigcup_t (H_t^1 \cap \cdots \cap H_t^\ell)$ is a rational normal scroll of dimension $n-\ell+1$.

Proof. The mp $\mathbb{P}^1 \to \mathbb{G}(n-\ell, n)$ by $t \mapsto H_t^1 \cap \cdots \cap H_t^\ell$ is a wedge product of ℓ linear forms, so has degree ℓ. Hence $\bigcup_t (H_t^1 \cap \cdots \cap H_t^\ell) = X$ has degree ℓ. X is also irreducible and nondegenerate, and can't be a Veronese surface since it's swept out by linear subspaces of codimension 1, so is a scroll. $\square$

Alternatively, writing $H_t^i = V(t_0 L^i + t_1 M^i)$, X is the locus

$$\left\{ \begin{pmatrix} L^1 & \cdots & L^\ell \\ M^1 & \cdots & M^\ell \end{pmatrix} \text{ has rank } 1 \right\}, \tag{17.1}$$

which is a scroll.

Lemma 17.1. *In $\mathbb{P}^n$, if $p_1, \ldots, p_{n+3}$ are points in linearly general position, then $p_1, \ldots, p_{n+3}$ lie on a rational normal curve.*

Proof. Let $\Lambda_i = \operatorname{span}(p_1, \ldots, \widehat{p_i}, \ldots, p_n)$, and choose a parameterization of each pencil $\{H_t^i\}$ so that $p_{n+1} \in H_0^i$, $p_{n+2} \in H_\infty^i$, and $p_{n+3} \in H_1^t$. Then taking $X = \bigcup_{t\in\mathbb{P}^1}(H_t^1 \cap \cdots \cap H_r^n)$, X contains $p_1, \ldots, p_n$, and $p_{n+1}, p_{n+2}, p_{n+3}$. (It contains p_1 since $H_t^2, \ldots, H_t^n$ contain p_1 for every t, and H_t^1 does for some t.) $\square$

Lemma 17.2 (Castelnuovo)**.** *For $\Gamma = \{p_1, \ldots, p_d\} \subseteq \mathbb{P}^n$ in linearly general position, if $d \geq 2n+3$ and Γ imposes only $2n+1$ conditions on quadrics (that is, $h_\Gamma(2) = 2n+1$), then Γ is contained in a rational normal curve.*

Proof. Again, take $\Lambda_i = \operatorname{span}(p_1, \ldots, \widehat{p_i}, \ldots, p_n)$, and choose a parameterization of each pencil $\{H_t^i\}$ so that $p_{n+1} \in H_0^i$, $p_{n+2} \in H_\infty^i$, and $p_{n+3} \in H_1^i$. We get a rational normal curve containing $p_1, \ldots, p_{n+3}$.

Next, take $\Lambda = p_{n+4}, \ldots, p_{2n+2}$, and H_t the pencil of hyperplanes containing Λ, so that $p_{n+1} \in H_0, p_{n+2} \in H_\infty, p_{n+3} \in H_1$. Consider $Q_i = \bigcup_{t\in\mathbb{P}^1}(H_t^i \cap H_t)$; then Q_i is a quadric hypersurface. (In fact, it is a cone over a quadric in $\mathbb{P}^3$.)

Q_i contains the points in Λ_i, so $p_1, \ldots, p_{i-1}, p_{i+1}, \ldots, p_n$, and also $p_{n+1}, p_{n+2}, p_{n+3}$, as well as $p_{n+4}, \ldots, p_{2n+2}$; these are $2n+1$ points of Γ. By hypothesis, we need Q_i to contain al of Γ.

Now for the points $p \in \{p_{2n+3}, \ldots, p_d\}$, we have $p \in Q_i$, so the values of t for which $p \in H_r^i$ and $p \in H_t$ are equal. As a result, the values of t for which $p \in H_t^i$ and $p \in H_t^j$ are equal.

We get $p_{2n+3}, \ldots, p_d \in X$. We also know that $p_1, \ldots, p_{n+3} \in X$. X is the unique rational normal curve through $n+3$ points, so by swapping out points, we get that $p_{n+4}, \ldots, p_{2n+2} \in X$. $\square$

18 Generalizations of Castelnuovo's Bound

The uniform position theorem implies that for $\Gamma', \Gamma'' \subseteq \Gamma$ of the same cardinality, then $h_{\Gamma'}(m) = h_{\Gamma''}(m)$ for every m. This statement is usually referred to as uniform position.

Castelnuovo showed that for $\Gamma \subseteq \mathbb{P}^n$ in linearly general position, $h_\Gamma(m) \geq h_0(m) = \min\{d, mn+1\}$ with equality attained if Γ is contained in a rational normal curve B.

So $h_0(m)$ is the smallest possible Hilbert function among Γ in uniform position. What is the second smallest?

This turns out to occur for an elliptic normal curve (take E an elliptic curve, and map E into $\mathbb{P}^n$ by a divisor L of degree $n+1$). For B the image of the curve, $h_B(m) = h^0(\mathcal{O}_B(L^m))$ which is $m(n+1)$ by Riemann-Roch.

If $\Gamma \subseteq B$, then

$$h_\Gamma(m) = \begin{cases} m(n+1) & d > m(n+1) \\ d-1 & d = m(n+1) \\ d & d < m(n+1). \end{cases} \tag{18.1}$$

Call this function $h_1(m)$. Then if Γ doesn't lie on a rational normal curve, then $h_\Gamma(m) \geq h_1(m)$. We get a bound $\pi_1(d,r)$ on the genus. Contrapositively, if $g > \pi_1(d,r)$, then C lies on a rational normal scroll (and we can determine such curves).

An analog of Castelnuovo's lemma: if $\Gamma \subseteq \mathbb{P}^n$ is in uniform position and $d \geq 2n+5$, and $h_\Gamma(2) = 2n+2$, then Γ lies on an elliptic normal curve.

Conjecture 18.1. *For $\alpha = 0, \ldots, n-3$, Γ in uniform position and $d \geq 2n+3+2\alpha$, $h_\Gamma(2) \leq 2n+1+\alpha$ implies Γ is contained in a curve $B \subseteq \mathbb{P}^n$ of degree at most $n+\alpha$.*

Let $h_\alpha(m)$ be the minimal Hilbert function of Γ contained in a curve of degree $m+\alpha$ in $\mathbb{P}^n$. Derive a corresponding bound on the genus: $g(C) \leq \pi_\alpha(d,r)$ for $C \subseteq \mathbb{P}^r$ such that Γ does not lie on a curve of degree at most $r-1+\alpha$.

So for $g(C) > \pi_\alpha(d,r)$, C lies on a surface of degree at most $r-2+\alpha$. For $g \leq \pi_{r-2}(d,r)$, it turns out every genus occurs.

Now suppose $S \subseteq \mathbb{P}^r$ is a surface of degree $r-1+\alpha < 2r-2$. Let $B \cap H \subseteq \mathbb{P}^{r-1}$. Clifford implies B is nonspecial, so $g(B) \leq d-(r-1)-\alpha$. But also, $B.B = \deg(S) = r-1+\alpha$, and

$$\begin{aligned} K_S.B &= 2g(B) - 2 - (r-1+\alpha) && (18.2)\\ &\leq 2\alpha - 2 - (r-1+\alpha) && (18.3)\\ &\leq \alpha - (r-1) && (18.4)\\ &< 0. && (18.5) \end{aligned}$$

It turns out this implies S is ruled, birational to a $\mathbb{P}^1$-bundle over a curve. The curves on such surfaces can then be classified.

As an example, for $\alpha = 1$, let S be a surface of degree r in $\mathbb{P}^r$. For H a hyperplane section, $H \cap S$ is rational or elliptic. If rational, then S is the projection of a scroll in $\mathbb{P}^{r+1}$. If elliptic, then S could be the cone over an elliptic normal curve, or a del Pezzo surface.

A del Pezzo surface is given by $\mathbb{P}^2$ blown up at δ points $p_1, \ldots, p_\delta$ embedded in $\mathbb{P}^{9-\delta}$ by cubics through the p_i. This can only occur if $r \leq 9$. So in the case of linearly normal curves, if $r \geq 10$, an appropriate Γ must lie on a cone over an elliptic normal curve.

Let p be the vertex of the cone S. If $p \notin C$ and C meets a line of the ruling in n points, then $d = rn$. If $p \in C$, and C meets a line of the ruling in n other points, then $d = rn + 1$. So $d \equiv 0, 1 \pmod{r}$.

Here is a problem: what can you say about the degrees and dimensions of linear series on a curve? Suppose we also require $\phi_{\mathcal{D}}$ to be a (birational) embbeding. We obtained Clifford's theorem at first, then Castelnuovo's bound.

We might also ask what happens for C a general curve. This will be the subject of Brill-Noether theory later.

19 Inflectionary Points

Suppose $C \subseteq \mathbb{P}^2$ is a smooth nondegenerate plane curve. Then (in characteristic 0) a tangent line to a general point has intersection multiplicity $(m_p(C.T_p))$ exactly 2. p is a flex point if this is not the case.

We can generalize this notion for $C \subseteq \mathbb{P}^r$ smooth and nondegenerate. For Λ a k-plane in r, define $\mathrm{ord}_p(\Lambda.C)$ to be $\min_{H \supseteq \Lambda} m_p(C.H)$. The osculating plane in $\mathbb{P}^3$ is the plane H such that $m_p(C.H) \geq 3$. For some $p \in C$, we have $m_p(C.H) \geq 4$.

In general, say C is a smooth curve, and $\mathcal{D} = (L, V)$ for L a line bundle of degree d and $V^{r+1} \subseteq H^0(L)$. So $\mathcal{D}$ is a g^r_d.

Proposition 19.1. $\#\{\mathrm{ord}_p \sigma\}_{\sigma \in V \setminus \{0\}} = r + 1$.

Proof. $\leq$ is immediate. For $\geq$, choose a basis $\{\sigma_0, \ldots, \sigma_r\}$, and if $\mathrm{ord}_p \sigma_i = \mathrm{ord}_p \sigma_j$, replace σ_j by some linear combination of σ_i and σ_j vanishing to higher order. □

Write $\{\mathrm{ord}_p \sigma\}_{\sigma \in V \setminus \{0\}} = \{a_0, \ldots, a_r\}$ with $a_i < a_{i+1}$. (This is written $a_i(V, p)$ if necessary to avoid confusion.) This is called the vanishing sequence.

Let $\alpha_i = a_i - i$ so $0 \leq \alpha_0 \leq \cdots \leq \alpha_r$, called the ramification sequence. The total ramification $\alpha = \alpha(V, p)$ is $\sum_i \alpha_i$.

We say that p is an inflectionary point of V if $\alpha > 0$, or equivalently $\alpha_r > 0$, or equivalently $a_r > r$, or equivalently there exists a nonzero $\sigma \in V$ vanishing to order at least $r + 1$. Observe:

- $a_0 = \alpha_0 > 0$ if and only if p is a base point. We could remove this point, but don't need to.
- If $\alpha_0 = 0$, then $\alpha_1 > 0$ (so $a_1 \geq 2$) if and only if $d\phi_V = 0$ at p (so ϕ_V fails to be an immersion at p).

Lemma 19.2 (characteristic 0)**.** *For any (L, V) on C, and general $p \in C$, $\alpha(V, p) = 0$.*

Proof. Say the map $\phi_V : C \to \mathbb{P}^r$ is given locally by the vector-valued function $v(z) = [\sigma_0(z) : \cdots : \sigma_r(z)]$. Then p is inflectionary if and only if $v(p) \wedge v'(p) \wedge \cdots \wedge v^{(r)}(p) = 0$. So now suppose that $v \wedge v' \wedge \cdots \wedge v^{(r)}$ were identically zero. Let k be the smallest integer such that $v \wedge \cdots \wedge v^{(k)}$ is identically zero. Then $v^{(k)} \in \operatorname{span}(v, v', \ldots, v^{(k-1)})$ at a general point p. And taking the derivative of $v \wedge v' \wedge \cdots \wedge v^{(k)} \equiv 0$, we get $v \wedge v' \wedge \cdots \wedge v^{(k-1)} \wedge v^{(k+1)} = 0$. So $v^{(k+1)} \in \operatorname{span}(v, v', \ldots, v^{(k-1)})$ as well.

Taking another derivative,

$$0 = v \wedge v' \wedge \cdots \wedge v^{(k-1)} \wedge v^{(k+2)} \pm v \wedge v' \wedge \cdots \wedge v^{(k-2)} \wedge v^{(k)} \wedge v^{(k+1)}. \tag{19.1}$$

But the second term is zero, since

$$v, v', \ldots, v^{(k-2)}, v^{(k)}, v^{(k+1)} \in \operatorname{span}(v, v', \ldots, v^{(k-1)}), \tag{19.2}$$

a space of dimension $k-1$. So

$$v^{(k+2)} \in \operatorname{span}(v, v', \ldots, v^{(k-1)}). \tag{19.3}$$

Repeating, all $v^{(\ell)}$ are in

$$\operatorname{span}(v, v', \ldots, v^{(k-1)}) \subsetneq \mathbb{C}^{r+1}. \tag{19.4}$$

In characteristic 0, this implies the image $\phi_V(C)$ is contained in a proper subspace of $\mathbb{P}^r$, a contradiction. $\square$

Observation: $v \wedge v' \wedge \cdots \wedge v^{(r)} \in \Lambda^{r+1}\mathbb{C}^{r+1} \cong \mathbb{C}$, the Wronskian determinant. It does not vanish identically, and $\operatorname{ord}_p(v \wedge v' \wedge \cdots \wedge v^{(r)}) = \alpha(V,p) = \sum_i \alpha_i(V,p)$. (Choose a basis vanishing to distinct orders of increasing size.)

Say L has transition functions $f_{\alpha\beta}$. Choose local coordinates z_α in U_α; then the derivative is multiplied by $\frac{dz_\beta}{dz_\alpha} = g_{\alpha\beta}$. Then $\frac{\det \text{ in } U_\beta}{\det \text{ in } U_\alpha}$ equals

$$f_{\alpha\beta} \cdot (f_{\alpha\beta} g_{\alpha\beta}) \cdot \cdots \cdot (f_{\alpha\beta} g_{\alpha\beta}^r) = f_{\alpha\beta}^{r+1} g_{\alpha\beta}^{\binom{r+1}{2}}. \tag{19.5}$$

So the determinant is a section of the line bundle $L^{r+1} \otimes K^{\binom{r+1}{2}}$. Hence the total number of zeros is

$$\begin{aligned}\deg(L^{r+1} \otimes K^{\binom{r+1}{2}}) &= d(r+1) + \binom{r+1}{2}(2g-2) && (19.6)\\ &= (r+1)(d + r(g-1)). && (19.7)\end{aligned}$$

We conclude the Plücker formula:

$$\sum_p \alpha(V,p) = (r+1)(d + r(g-1)). \tag{19.8}$$

Here are some examples:

- $g = 0$, $V = H^0(\mathcal{O}_{\mathbb{P}^1}(d))$ complete; we have $d = r$, so we get no inflectionary points.

- A rational quartic in $\mathbb{P}^3$: there are four flex points. The quartic is the projection of a rational normal curve in $\mathbb{P}^4$; the flex points correspond to the points in the rational normal curve whose osculating hyperplane passes through the point of projection. So the osculating hyperplanes fill up $\mathbb{P}^4$ four times.

- $g = 1$, V complete: then $d = r + 1$ and $C \hookrightarrow \mathbb{P}^r$. The formula gives $(r+1)^2$ points. They turn out to be the torsion points of order $r + 1$ (after changing one such point to be the origin of the elliptic curve). Equivalently, they consist of p such that $(r + 1)p$ is the hyperplane class.

Given $C \subseteq \mathbb{P}^r$ (or more generally $\phi : C \to C_0 \subseteq \mathbb{P}^r$), we have the Gauss map $\phi^{(1)} : C \to \mathbb{G}(1, r)$ by $p \mapsto T_pC$. If ϕ is given by $[v(z)]$, then $\phi^{(1)} : z \mapsto [v(z) \wedge v'(z)]$. More generally we can define $\phi^{(k)}C \to \mathbb{G}(k, r)$ for $k \leq r - 1$, with

$$\phi^{(k)}(z) = [v(z) \wedge \cdots \wedge v^{(k)}(z)]. \tag{19.9}$$

This is called the kth associated map. Define the osculating k-plane at p to be $\phi^{(k)}(p)$ to be $\phi(k)(p)$.

For $k = 2$ and $r - 1$, vanishing of $\phi^{(1)}$ corresponds to cusps of C_0.

For $r = k - 1$, $\phi^{(k-1)} : C \to (\mathbb{P}^r)^*$ maps p to its osculating hyperplane. $\operatorname{im} \phi^{(k-1)}$ is called the dual curve.

As an example, if $C \subseteq \mathbb{P}^3$ is a twisted cubic, then $\operatorname{Aut}(\mathbb{P}^3)$ acts transitively on C, so $\operatorname{Aut}((\mathbb{P}^3)^*)$ acts transitively on the dual curve. So the dual curve can't have any inflectionary points, therefore is a twisted cubic.

20 Plücker Formulas for Plane Curves

Consider $C \subseteq \mathbb{P}^2$ not contained in a line, and let C^* be the dual curve in $(\mathbb{P}^2)^*$.

Fact. In characteristic 0, $\phi^{(1)}$ is birational. Equivalently, not every tangent is a bitangent.

We then have $g^* = g$. We would like to know all of the properties of C^*.

Fact. Again in characteristic 0, $(C^*)^* = C$.

To do Draw an intersection of tangent lines. (6)

Idea: let r be the point corresponding to the secant line p^*q^* in C^*. We claim that $r \to p$ as $q \to p$.

Singularities of C^*: If $\overline{pq}$ is a bitangent (at p and q), then $p^* = q^*$, giving us a node. Conversely, a node of C^* os a bitangent of $C/$

Also, if p is a flex point, then C^* has a cusp at p^*. (Say $v(t) = [1 : t : t^3]$; then $v'(t) = [0 : 1 : 3t^2]$, so $v(t) \wedge v'(t) = [1 : 3t^2 : 2t^3]$.)

We say that $C \subseteq \mathbb{P}^2$ has traditional singularities if C and C^* have only nodes and cusps. Equivalently, C has only nodes and cusps as singularities, and only simple flexes and bitangents.

Let C have traditional singularities, and let $d = \deg(C)$, $d^* = \deg(C^*)$, $g = g(C) = g(C^*) = g^*$ the (geometric) genus, δ the number of nodes, κ the number of cusps, b the number of bitangents, and f the number of flexes. We have $\delta^* = b$, $\kappa^* = f$, $b^* = \delta$, and $f^* = \kappa$.

The first formula: $g = \binom{d-1}{2} - \delta - \kappa$. As $g = g^*$, this equals $\binom{d^*-1}{2} - b - f$.

Finally, compute d^*, equal to $\deg(C^*) = \#(C^* \cap L)$, for L a line in $(\mathbb{P}^2)^*$, equal to the number of tangent lines to C passing through L^* in $\mathbb{P}^2$.

If C were smooth, with $C = V(F)$, we get $\deg(C^*) = V(F, \frac{\partial F}{\partial x}) = d(d-1)$. If F had nodes, then the secant line intersects with multiplicity 2; with cusps, it intersects with multiplicity 3.

We get $d^* = d(d-1) - 2\delta - 3\kappa$. Dualizing, $d = d^*(d^*-1) - 2b - 3f$.

As an example, suppose C is smooth of degree d, so $\delta = \kappa = 0$. First, $g = \binom{d-1}{2}$ and $d^* = d(d-1)$. We can then compute the number of flexes and bitangents. We know that $d = d^*(d^*-1) - 2b - 3f$, so $2b + 3f = (d^2-d)(d^2-d-1) - d = d^4 - 2d^3$. Also $2b + 2f = (d^*-1)(d^*-2) - 2g = (d^*-1)(d^*-2) - (d-1)(d-2) = d^4 - 2d^3 = 3d^2 + 6d$.

We obtain $f = 3d^2 - 6d = 3d(d-2)$, and $b = \frac{1}{2}(d^4 - 2d^3 + 9d^2 + 18d) = \frac{1}{2}d(d-2)(d-3)(d+3)$.

21 Weierstrass Points

The inflectionary points of a canonical curve are intrinsic to the curve itself. (The same goes for pluricanonical curves, maps given by K^n for some n. We have

$$\begin{array}{ccc} \mathcal{P}_{d,g} & = & \{(C,L) : C \in \mathfrak{M}_g, L \in \mathrm{Pic}^d(C)\} \\ \downarrow & & \\ \mathfrak{M}_g & = & \{\text{smooth projective curves of genus } g\}/\text{isomorphism} \end{array} \tag{21.1}$$

It turns outs that the only rational sections $\mathfrak{M}_g \dashrightarrow \mathcal{P}_{d,g}$ are given by powers of K.

Now consider Weierstrass points. The basic idea is to look at meromorphic functions on C which are holomorphic except at p; that is, $\mathcal{L}(mp)$ for each m.

If D is an effective divisor on C of degree d, then $r(D) = d - g + h^0(K - D)$ and $r(D+p) = d + 1 - g + h^0(D - K - p)$. We have $r(D) \le r(D+p) \le r(D) + 1$. It is $r(D) + 1$ if p is a base point of $|K - D|$.

Applying this to $0, p, 2p, \ldots, mp, \ldots$, there are exactly g values $m_1, \ldots, m_g$ such that $h^0(K - (m_i p)) < h^0(K - (m_i - 1)p)$. For other m, $h^0(K - mp) = h^0(K - (m-1)p)$, so there exists a meromorphic f on C with $(f)_\infty = mp$.

We conclude that there are exactly g natural numbers m for which there does not exist f with $(f)_\infty = mp$. So $H = \{m : \exists, f \text{ with } (f)_\infty = mp\}$ is a semigroup of $\mathbb{N}$ with finite complement, and in fact $\#(\mathbb{N} \setminus H) = g$. The numbers in $\mathbb{N} \setminus H$ are called the Weierstrass gap sequence, and H is called the Weierstrass semigroup.

Theorem 21.1. *Fix C an arbitrary curve. For general $p \in C$, $H = \{g+1, g+2, \ldots\}$.*

We say that p is a Weierstrass point of $H \neq \{g+1, g+2, \ldots\}$, so $G \neq \{1, \ldots, g\}$. Equivalently, $r(gp) > 0$. The weight $w(p)$ of a Weierstrass point is $\sum_{m \in G} m - \binom{g+1}{2}$.

Theorem 21.2. $\sum_{p \in C} w(p) = g^3 - g$.

Proof. $m \in G$ if and only if $h^0(K - mp) < h^0(K - (m-1)p)$ if and only if there exists a holomorphic differential ω on C with $\operatorname{ord}_p \omega = m - 1$. So Weierstrass gaps correspond to ramification of the canonical curve. $\square$

Fact. On a general curve, all Weierstrass points have weight 1 (these are said to be normal Weierstrass points).

22 Real Algebraic Geometry

Consider now a real algebraic plane curve $C = V(F)$, with F a homogeneous polynomial in $\mathbb{R}[X, Y, Z]$. We obtain sets of real and complex points $C_{\mathbb{R}} \subseteq \mathbb{RP}^2$ and $C_{\mathbb{C}} \subseteq \mathbb{CP}^2$.

If C is smooth, $C_{\mathbb{R}}$ is a disjoint union of S^1's. The connected components of $C_{\mathbb{R}}$ are called "ovals" or "circuits". If $\gamma \approx S^1$ in $\mathbb{RP}^2$, there are two possible situations:

- $\gamma \sim 0$ in $H^1(\mathbb{RP}^2; \mathbb{Z})$. Then $\mathbb{RP}^2 \setminus \gamma$ is disconnected. One component, called the interior, is a disc, while the other is a Möbius band and called the exterior. γ is called an even oval.
- $\gamma \not\sim 0$ in $H^1(\mathbb{RP}^2; \mathbb{Z})$, so $\mathbb{RP}^2 \setminus \gamma$ is connected. γ is called an odd oval.

Any two odd ovals must meet. In particular, a smooth C can have at most one odd oval. (There will be an odd oval if and only if $\deg(C)$ is odd.)

Also, there are two possible types of nodes: a crunode (real tangent vectors) and an acnode (nonreal tangent vectors, so an isolated point).

Theorem 22.1 (Harnack)**.** *For $C \subseteq \mathbb{P}^2$ a smooth plane curve of degree d, C has at most $\binom{d-1}{2} + 1$ ovals.*

We say that γ and γ' are nested if one lies in the interior of the other. A further question is to describe the possible nestings.

Proof of Harnack. Suppose C has degree d and $m > \binom{d-1}{2}$ ovals $\gamma_1, \dots, \gamma_m$. Without loss of generality, assume that $\gamma_1, \dots, \gamma_{m-1}$ are even. Choose $p_i \in \gamma_i$, and take $Y \subseteq \mathbb{P}^2$ a plane curve of degree $d - 2$ passing through $p_1, \dots, p_{\binom{d-1}{2}+1}$, and also $q_1, \dots, q_{d-3} \in \gamma_m$. Now Y has to exit each oval $\gamma_1, \dots, \gamma_{\binom{d-1}{2}+1}$, so intersects each at least twice.

The number of points of $Y \cap C$ is then at least $(d-1)(d-2) + 2 + (d-3) = d(d-2) + 1$, a contradiction. $\square$

If C is a smooth projective curve of genus g over $\mathbb{R}$, then $C_{\mathbb{R}} \subseteq C_{\mathbb{C}}$, a genus g surface. We can consider $(C_{\mathbb{C}} \setminus C_{\mathbb{R}})/\sigma$, where σ is complex conjugation. This quotient is a manifold (and $C_{\mathbb{C}}/\sigma$ is a manifold with boundary). We can complete this to a compact surface by adding discs.

If $C_{\mathbb{R}}$ has δ ovals, then δ discs will be added, Considering Euler characteristics, $\chi(C_{\mathbb{C}}) = 2 - 2g$ and $\chi(C_{\mathbb{R}}) = 0$. This implies $\chi((C_{\mathbb{C}} \setminus C_{\mathbb{R}})/\sigma) = 1 - g$, and so the compact surface has $\chi = 1 - g + \delta$. But $\chi \leq 2$, implying Harnack. (This also works for arbitrary real curves, not just $C \subseteq \mathbb{P}^2$.)

23 Brill-Noether Theory

Consider the variety $M_{a,b} = \{a \times b \text{ matrices}\}/\text{scalars} \cong \mathbb{P}^{ab-1}$, and the subvariety $M_{a,b}^k$ given by matrices of rank at most k, for $k < a, b$. Then:

1. $\operatorname{codim}(M^k \subseteq M) = (a-k)(b-k)$.
2. $(M^k)_{\text{sing}} = M^{k-1}$.
3. If $A \in M^k \setminus M^{k-1}$, then $T_A(M^k) = \{\varphi : \varphi(\ker A) \subseteq \operatorname{im} A\}$. For

$$(N_{M^k/M})_A = \operatorname{Hom}(\ker A, \operatorname{coker} A). \tag{23.1}$$

Let C be a smooth curve of genus g, and C_d the set of effective divisors of C, equal to C^d/S_d. $\operatorname{Pic}^d(C)$ is the set of line bundles of degree d on C up to isomorphism, isomorphic to $J(C) = H^0(K)^*/H_1(C, \mathbb{Z})$. We have a map $u : C_d \to J(C)$ given by $\sum p_i \mapsto \sum \int_{p_0}^{p_i}$. Abel's theorem implies u factors through $C_d \to \operatorname{Pic}^d(C)$.

Now introduce $C_d^r = \{D : r(D) \geq r\} \subseteq C_d$ and $W_d^r = \{L : h^0(L) \geq r+1\} \subseteq \operatorname{Pic}^d(C)$. We have $C_d^r = u^{-1}(W_d^r)$. Equivalently, W_d^r is the set of L such that $u^{-1}(L)$ has dimension at least r. By upper semicontinuity, this is closed in $\operatorname{Pic}^d(C)$.

Say $D = p_1 + \cdots + p_d$ with p_i distinct. Then $T_D C_d \cong \bigoplus T_{p_i} C$, and du maps this onto $T_{u(D)} J = H^0(K)^*$, by $du : v_i \mapsto [\ell : H^0(K) \to \mathbb{C} : \ell(w) = \langle \omega(p_i), v_i \rangle]$. The transpose is the evaluation map

$$H^0(K) \to \bigoplus T_{p_i}^* C = \bigoplus K_{p_i} = H^0(K/K(-D)). \tag{23.2}$$

For D arbitrary, $T_D^* C_d = H^0(K/K(-D))$ and $T_D C_d = H^0(\mathcal{O}_C(D)/\mathcal{O}_C)$, with the pairing given by $\langle \omega, f \rangle = \sum \operatorname{Res}(f\omega)$.

We have $\operatorname{im}(du_D) = \operatorname{Ann}(H^0(K-D)) \subseteq H^0(K)^*$, and $\dim(\ker(d_D)) = d - (h^0(K) - h^0(K-D))$. So we obtain

$$C_d^r = \{D : \operatorname{rank}(du_D) \leq d - r\}. \tag{23.3}$$

Our objectives are to find dimension estimates on W_d^r, C_d^r, and their tangent spaces.

Theorem 23.1 (Marten)**.** *$\dim(W_d^r) \leq d - 2r$, with equality only if $r = 0$ or C is hyperelliptic.*

Proof. If C is nonhyperelliptic, C_d^r is the locus of divisors $D \subseteq C \subseteq \mathbb{P}^{g-1}$ lying in a $\mathbb{P}^{d-r-1}$. Set

$$\begin{tikzcd}
& \Sigma \arrow[r, equal] \arrow[dl] \arrow[drr, "\text{general fiber} \cong \mathbb{P}^{g-d+r-1} \times C_d^r"] & \{(H, D) : D \subseteq H \cap C\} \arrow[r, hook] & (\mathbb{P}^{g-1})^* \times C_d^r \\
(\mathbb{P}^{g-1})^* & & & C_d^r \arrow[r, "\text{general fiber} \cong \mathbb{P}^r"] & W_d^r
\end{tikzcd} \tag{23.4}$$

(We have that $W_d^r \setminus W_d^{r+1}$ is dense in W_d^r.) We get $\dim \Sigma = \dim W_d^r + r + (g - d + r - 1)$. On the other hand, $\Sigma \to (\mathbb{P}^{g-1})^*$ is finite and (by the general position theorem if $r > 0$) not dominant, so $\dim \Sigma \leq g - 2$. We get $\dim(W_d^r) \leq d - 2r - 1$. $\square$

Inside W^r_d, we have

$$G^r_d = \{(L,V) : L \in \operatorname{Pic}^d(C), V^{r+1} \subseteq H^0(L)\}. \tag{23.5}$$

Recall that $T_D C_d = H^0(\mathcal{O}_C(D)/\mathcal{O}_C)$, $T^* C_d = H^0(K/K(-D))$, and $T_L J = H^0(K)^*$, $T_L^* J = H^0(K)$. $(du)^T : T_L^* J \to T_D^* C_d$ is $H^0(K) \to H^0(K/K(-D))$.

For $D = p_1 + \cdots + p_d \in C_d$ with p_i distinct, we have

$$C^r_d = \left\{ \operatorname{rank} \begin{pmatrix} \omega_1(p_1) & \cdots & \omega_g(p_1) \\ \vdots & \ddots & \vdots \\ \omega_1(p_d) & \cdots & \omega_g(p_d) \end{pmatrix} \leq d - r \right\} \tag{23.6}$$

where ω_i is a basis for $H^0(K)$. We locally get a map

$$\begin{tikzcd} C_d & U \arrow[l, hook'] \arrow[r] & M_{d,g} \\ & U \cap C^r_d \arrow[u, hook] \arrow[r] & M^{d-r}_{d,g} \arrow[u, hook] \end{tikzcd} \tag{23.7}$$

We conclude that $\dim C^r_d \geq d - r(g - d + r)$ everywhere.

Also $\dim W^r_d = \dim C^r_d - r \geq g - (r+1)(d - g + r)$. This number is called the Brill-Noether number and denoted by ρ.

Consider the maps $\alpha : H^0(K) \to H^0(K/K(-D)) = H^0(K|_D)$, and $\mu : H^0(D) \otimes H^0(K - D) \to H^0(K)$. If $D \in C^r_d \setminus C^{r+1}_d$, then $T_D C^r_d = \operatorname{Ann}(\operatorname{im}(\alpha \circ \mu))$; if $L = \mathcal{O}(D)$, then $T_L W^r_d = \operatorname{Ann}(\operatorname{im} \mu^*)$.

Note that if μ is injective at L, then $\dim(\operatorname{im} \mu) = (r+1)(g - d + r)$. We conclude that W^r_d is smooth of dimension ρ at L.

Theorem 23.2 (Brill-Noether existence). *If $\rho \geq 0$, then $W^r_d \neq \varnothing$.*

Theorem 23.3. *If C is a general curve of genus g, then $\rho < 0$ implies $W^r_d = \varnothing$, and if $\rho \geq 0$, then $\dim W^r_d = \rho$.*

Corollary 23.4 (of proof). *For C general and $L \in W^r_d$ general, then ϕ_L is an embedding for $r \geq 3$. If $r \geq 2$, then ϕ_L is birational onto a plane curve C_0 with only nodes. If $r = 1$, then ϕ_L is simply branched.*

As an example, if C is a general curve of genus g, the smallest degree of a nonconstant meromorphic function on C is $\lceil \frac{g}{2} \rceil + 1$, the smallest degree of a plane curve birational to C is $\lceil \frac{2}{3} g \rceil + 2$, and for $g \geq 4$, the smallest degree of an embedding is $\lceil \frac{3}{4} g \rceil + 3$.

Theorem 23.5 (Gieseker-Petri). *For C general and L arbitrary, $\mu : H^0(L) \otimes H^0(K - L) \to H^0(K)$ is injective. In particular, $(W^r_d)_{\text{sing}} = W^{r+1}_d$, $(C^r_d)_{\text{sing}} = C^{r+1}_d$, and G^r_d is smooth.*

In fact, if $\rho \geq 0$, the number of g^r_d's on C is

$$g! \prod_{\alpha=0}^{r} \frac{\alpha!}{(g - d + r + \alpha)!}. \tag{23.8}$$

Also, $W^r_d = K - W^{g-d+r-1}_{2g-2-d}$, so this number is also

$$g! \prod_{\alpha=0}^{g-d+r-1} \frac{\alpha!}{(r+1-\alpha)!}. \tag{23.9}$$

Theorem 23.6. *For C general, if $\rho > 0$, then W^r_d is irreducible.*

Theorem 23.7 (Brill-Noether with inflection)**.** *Given C, $p_1, \dots, p_\delta \in C$, and nondecreasing sequences*

$$\alpha^1, \dots, \alpha^\delta, \tag{23.10}$$

let

$$G^r_d(p, \alpha) = \{(L, V) : \alpha_k(p_i, V) \geq \alpha^i_k\}. \tag{23.11}$$

For $C, p_1, \dots, p_d$ general, $\dim G^r_d(p, \alpha) = \rho - \sum \alpha^i_k$.

We will first prove that if C is general and V is any linear series on C, then $\rho(V) \geq 0$. (It's enough to find a single curve C with this property.)

Given a curve C, and $p_1, \dots, p_m \in C$, for V any g^r_d on C, define $\rho(V, p_1, \dots, p_m)$ to be $\rho(V) = \sum_{k=1}^m \alpha(V, p_k)$.

Theorem 23.8 (Adjusted Brill-Noether)**.** *If $C, p_1, \dots, p_m$ are general, then for V any linear series on C, $\rho(V, p_1, \dots, p_m) \geq 0$.*

(This proves Theorem 23.3, including the dimension.)

Let Δ be a disc, or Spec(dvr). Consider a family of curves $\pi : \mathcal{C} \to \Delta$, whose fibers are smooth projective curves of genus g, $C_t = \pi^{-1}(t)$. A family of line bundles is $\{L_t \in \mathrm{Pic}^d(C_t)\}$ expressible as $L_t = \mathcal{L}|_{C_t}$ for some line bundle $\mathcal{L}$ on $\mathcal{C}$. A family of linear series on $\{C_t\}$ is a set $\{V_t \subseteq H^0(L_t)\}$ given by a locally free subsheaf (vector subbundle) $\mathcal{V} \subseteq \pi_*\mathcal{L}$ of rank $r+1$. (So $V_t = \mathcal{V}_t$.)

Also consider families of marked points, given by sections $\sigma_1, \dots, \sigma_m : \Delta \to \mathcal{C}$.

Claim. $\rho(V_t, \sigma_1(t), \dots, \sigma_m(t))$ is lower semicontinuous.

Proof. $\alpha(V_t, \sigma_k(t))$ is upper semicontinuous. □

Consider a family $\mathcal{C} \to \Delta$ where $\mathcal{C}$ is a smooth surface, and C_t are smooth projective curves of genus g for $t \neq 0$, but $C_0 = X \cup Y$, with X, Y smooth with intersection a node p of C_0. Let $\sigma_1, \dots, \sigma_m$ be sections such that $\sigma_k(0) \in X$ for $k = 1, \dots, \delta$ and $\sigma_k(0) \in Y$ for $k = \delta + 1, \dots, m$. Then we have a reduction statement:

Theorem 23.9. *If the Adjusted Brill-Noether theorem holds for*

$$(X, \sigma_1(0), \dots, \sigma_\delta(0), p) \tag{23.12}$$

and

$$(Y, \sigma_{\delta+1}(0), \dots, \sigma_m(0)), \tag{23.13}$$

then it holds for

$$(C_t, \sigma_1(0), \dots, \sigma_m(0)) \tag{23.14}$$

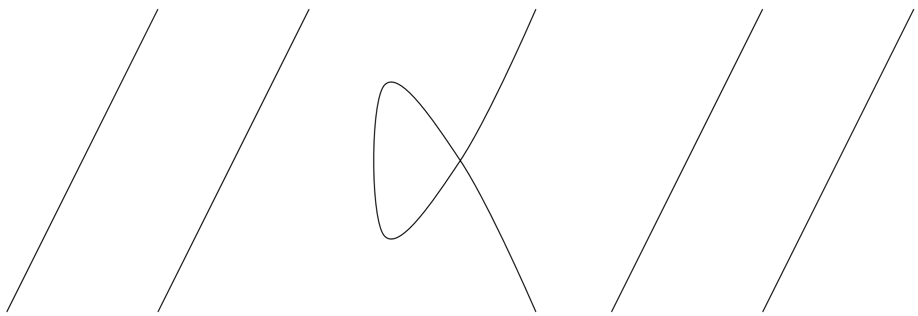
for $0 < |t| < \epsilon$.

Fix Me *Should it be $\sigma(t)$ at the end? (7)*

To construct such families: start with X, Y general and $p \in X$, $p \in Y$ general. Let U be the union of two annuli, one around each p. Then look at $(X \cup Y \setminus U) \times \Delta$:

To do Draw attaching of surfaces. (8)

Also consider $\eta = \{(z, w, t) : zw = t, |z|, |w|, |t| < 1\}$.



(23.15)

Use η to fill in the missing band.

Now we need to find sections $\sigma_i(t)$ with $\sigma_i(0)$ as specified.

Suppose we have a family of linear series on $\mathcal{C}^* \to \Delta^* = \Delta \setminus \{0\}$ (here $\mathcal{C}^* = \pi^{-1}(\Delta^*)$). This is a line bundle $\mathcal{L}^*$ on $\mathcal{C}^*$ and $\mathcal{V}^* \subseteq \pi_* \mathcal{L}_*$. Can $\mathcal{L}_*$ be extended to a line bundle on $\mathcal{C}$, and is this unique?

Extension is always possible, for $\mathcal{L}_* = \mathcal{O}_{\mathcal{C}^*}(D^*)$ for some D^*; we can take $D = \overline{D^*}$ and $\mathcal{L} = \mathcal{O}_{\mathcal{C}}(D)$. But the extension need not be unique. For any α, β, $\mathcal{O}_{\mathcal{C}}(\alpha X + \beta Y)$ is trivial on $\mathcal{C}^*$. Also $\mathcal{O}_{\mathcal{C}}(X + Y)$ is trivial on $\mathcal{C}$.

Claim. $\mathcal{O}_C(mX)$ is nontrivial for $m \neq 0$.

We conclude that any two $\mathcal{L}$'s extending $\mathcal{L}_*$ differ by tensoring with $\mathcal{O}_{\mathcal{C}}(mX)$ for some m.

$(\mathcal{O}_C(mX)|_Y = \mathcal{O}_Y(mp)$ so is nontrivial. Also $\mathcal{O}_{\mathcal{C}}(mX)|_X = \mathcal{O}_X(-mp)$.)

As a result, given $\mathcal{L}^*$ and $\alpha \in \mathbb{Z}$, there exists a unique $\mathcal{L}$ extending $\mathcal{L}^*$ with $\deg(\mathcal{L}|_X) = \alpha$ and $\deg(\mathcal{L}|_Y) = d - \alpha$. (We may write $\mathcal{L}_\alpha$ for this $\mathcal{L}$.) We will focus on $\mathcal{L}_d$ and $\mathcal{L}_0$.

For $\mathcal{V}^*$ a family of linear series on C_t for $t \neq 0$, $\mathcal{V}^*$ extends to $\mathcal{V}_\alpha \subseteq \pi_* \mathcal{L}_\alpha$. We have

$$(\mathcal{V}_d)_0 \subseteq H^0(\mathcal{L}_d|_{\mathcal{C}_0}) \hookrightarrow H^0(\mathcal{L}_d|_X). \tag{23.16}$$

We get a g^r_d on X. Similarly,

$$(\mathcal{V}_0)_0 \subseteq H^0(\mathcal{L}_0|_{\mathcal{C}_0}) \hookrightarrow H^0(\mathcal{L}_0|_Y), \tag{23.17}$$

giving a g^r_d on Y.

Initial cases of the Adjusted Brill-Noether theorem:

- $g = 0$, $C = \mathbb{P}^1$. The adjusted Brill-Noether number is $(r+1)(d-r) - \sum_{k=1}^{m} \alpha(V, p_k)$. Plücker implies $(r+1)(d-r) = \sum_{p \in C} \alpha(V, p)$, so the adjusted Brill-Noether number is *always* non-negative.
- $g = 1$, $C = E$, and the case $m = 1$: Let $V \subseteq H^0(L)$ of degree d. We have $a_r(V) \leq d$, but also $a_{r-1}(V) \leq d-2$. So $\alpha_r(V) \leq d-r$, and $\alpha_{r-1}(V) \leq d-r-1$, implying $\alpha(V,p) \leq (r+1)(d-r+1)+1$. This satisfies the theorem.

To do...

- ☐ 1 (p. 4): Possibly draw pictures of curves?
- ☐ 2 (p. 5): **Fix Me** We may rename this.
- ☐ 3 (p. 19): Draw curve and hyperplane picture.
- ☐ 4 (p. 21): Draw discs with multiplicities.
- ☐ 5 (p. 25): **Fix Me** Can we make this precise?
- ☐ 6 (p. 30): Draw an intersection of tangent lines.
- ☐ 7 (p. 36): **Fix Me** Should it be $\sigma(t)$ at the end?
- ☐ 8 (p. 36): Draw attaching of surfaces.