Odds and Ends

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0.1 Principles

1. Model book: Beauville Complex Surfaces
2. 10-12 Chapters, average 20pp, each of which can be covered in 1 week (3 hours lecture)
3. Work over $\mathcal{C}$, singular curves are images of smooth ones (no multiple components or embedded points.)
4. Background: able to write H^0, H^1, and use the terms scheme and Hilbert Function without blushing. Assume Ch 2 of Geometry of Schemes.
5. Unproven assertions segregated in subsections labeled "Many Cheerful Facts".

0.2 Abbreviated version organization

1. Consider moving moduli problems to beginning of Jacobians chapter. However, maybe M_g in "curves of genus 1" .
2. Clifford – Do the easy half – sum of linear series – in ch 1. Do the other half as application of general position. add not every secant is tri in char 0 via projection from tan line (exercise)
3. If we include the appendix on scrolls, and assuming that we talk about the curves on them, we could easily add a few lines about castelnuovo curves etc.
4. Torelli: factify it, starting with the general case of the differential of Abel-Jacobi
5. Temptations:
   - add theta char in Jacobians ch.
   - do tan vectors to Div, use it to give an algebraic formula for the derivative of Abel-Jacobi.

0.3 Notation and terminology

1. ring means commutative, noetherian, with unit.

2. Convention: "open set" without modifier should mean Zariski open set; if we want to work with open sets in the analytic/classical/complex topology we should say "analytic open set" or something.

3. Use "morphism" for what it is, and use "map" for rational maps.

4. form or homogeneous polynomial?

5. nonzerodivisor or non-zerodivisor?

6. Ideal sheaf, normal bundle, tangent bundle, differentials, principal parts: we write things like $\mathcal{T}_p X$ and NOT $\mathcal{T}_{X,p}$. (DE votes for the former. Joe points out that this will leave many things to change.) However we write $\mathcal{I}_X$ and NOT $\mathcal{I}(X)$ and, for the homogeneous ideal, I_X and not $I(X)$.

7. Check notation for the direct sum of a number of copies of a bundle. Replace $\mathcal{O}_X^{\oplus k}$ with $\mathcal{O}_X^k$ but use $\mathcal{L}^{\oplus k}$.

8. Secant varieties: the locus in the Grassmannian in denoted $\Psi_m(X)$; the locus in projective space is $\mathrm{Sec}_m(X)$. (need to search for Sec (with and without curly braces) outside of Ch 10).

9. The notation for ideal sheaves varies. If $Y \subset X$, the ideal sheaf should be $\mathcal{I}_{Y/X}$, and $\mathcal{I}_Y$ (or $\mathcal{I}_p$ in unambiguous cases.

10. Symbols: Silvio: Would it be possible to get a symbol for a wedge power of a vector space that's a little larger than the one for the wedge product of two vectors? The symbol in "$\wedge^k V$" just strikes me as too small. Also, is there a good alternative to using $\cup$ for both unions and cup products?

11. Silvio: "base point free" is three words, no hyphens. (I'm happy with pretty much any convention, but we might as well be consistent.)

12. We should respect the convention that the sheaf Hom of sheaves is $\mathcal{H}om(\mathcal{F}, \mathcal{G})$, whereas $\mathrm{Hom}(\mathcal{F}, \mathcal{G})$ is the vector space.

13. Silvio: When a lemma/proposition/theorem gets promoted or demoted, references to it are not always updated. Please check to make sure references to numbered statements are accurate.

14. IN ch 10 we use $\dashrightarrow$ for rational maps. grep for "ational map" and fix them (there aren't too many) (Also, it might be nice to come up with a better arrow than $\dashrightarrow$.)

15. Is it "local complete intersection" or "locally complete intersection"? The latter is epsilonically more descriptive, the former more standard. Use "locally complete intersection"

16. David or Silvio: the hooked arrow in diagrams.tex ($\hookrightarrow$) looks really crappy (and the arrowheads don't match the ones in $\longrightarrow$). Can we do better?

17. Sym^m rather than Sym_m for symmetric products (same for Sym).

18. Ideal sheaf, normal bundle, tangent bundle, differentials, principal parts: do we write things like $\mathcal{T}_{X,p}$ or T_pX? Do we write $\mathcal{I}_X$ or $\mathcal{I}(X)$? (DE votes for the former. Joe points out that this will leave many things to change.)

19. the residue field at p is $\kappa(p)$.

20. non-zerodivisor, non-negative, etc have hyphens.

0.4 Odds and Ends

Capitalize words in section names?

Basic results used in this section [does this refer to the "personalities chapter?" or chapter 1?]: Bézout, Riemann-Roch, Lasker (aka AF+BG), Clifford, Adjunction.

Let's explicitly allow things like $\mathrm{H}^0(D)$ where D is a divisor, as well as $\mathrm{H}^0(\mathcal{O}(D))$, but be careful not to mix the two too much.

Would it be more confusing or less to use the same letter for a polynomial vanishing on C and the surface it defines?

When we first introduce/set notation for linear series, define *sum* of two linear series (with special case $\mathcal{E} + E$).

We never put the classification of double covers of a variety with given branch divisor in 3264; this is a second chance. It would be very useful in several places in Personalities to have it; I propose we put it in.

Check references to the two rulings of a smooth quadric $Q \subset \mathbb{P}^3$. I keep getting confused about which is which—the convention should be that a line of the first ruling means a curve of type $(1, 0)$, that is, a fiber of the first projection map.

Expand the discussion of deg $\geq 2g + 1$ implies very ample: for example, line bundles of degree $2g$ are very ample unless L is of the form $K(p + q)$ which means

a) in genus $g \geq 3$, a general such line bundle is very ample;

b) if C is non-hyperelliptic, then any such line bundle is birationally very ample;

c) and in genus 2, it's birationally very ample unless $L = K^2$.

Also, when $\deg L = 2g - 1$, $|L|$ is base-point-free iff L is not of the form $K(p)$

Convention: "open set" without modifier should mean Zariski open set; if we want to work with open sets in the analytic/classical/complex topology we should say "analytic open set" or something.

basepoint = one word, so that e.g. "basepoint-free" has one hyphen.

rational maps are denoted DashTo (with initial letter r, l, d or u to indicate direction)

Notation: where do we introduce the notation g^r_d? How about geometric vs. arithmetic genus?

We should have a discussion of minimal degree of irreducible, nondegenerate varieties in $\mathbb{P}^n$, possibly in Chapter 1

Chapter 2: as consequence of RR, there is only one curve of genus 0

We go on about how Abel proved only the easy part of what is now called Abel's theorem; shouldn't we be calling it the Abel-Clebsch theorem instead?

where do we define "rational normal curve"? What about "twisted cubic"?

refer to "Bézout's theorem" rather than "Bézout"

Attributions: "Kodaira-Serre duality" should be Serre duality; "Riemann-Hurwitz" should be Hurwitz; "Hartshorne-Rao" should be Rao

Chapter 6 has a reference to "the push-pull formula"; put this terminology into the pre-requisite section?

Nodes: when we first use the term, we should define it.

We should adopt the convention that "hyperelliptic" applies only to curves of genus $g \geq 2$.

Chapter 8: (and everywhere) Cheerful facts (environment fact) should be numbered along with the theorems etc.

Look for occurrences of "specialization"; replace with "flat limit" where appropriate

conflations: polynomial vs hypersurface it defined; divisor/divisor class/line bundle, etc. In each case, make statement of equivalence clearly (and put references in index?)

Clifford index/Voisin's theorem – where should this be introduced/discussed first?

In a number of places we use the same symbol to denote a polynomial and the hypersurface it defines. Is this OK? If so, we should say somewhere that we're doing it; and if not, we should stop doing it.

"Noether-Lasker" should be "Lasker's theorem"

"parametrize" should be "parameterize"

0.5 Notes from DE visit 6/19/21

Chapter 2: start with brief ($\tilde{2}$ p) discussion of algebraic curves vs. Riemann surfaces (one direction via normalization; the other via Riemann existence)

Ch. 2: exercise on proof of resolution by projection (maybe in later chapter, after we establish $d \geq r$ for an irreducible, nondegenerate curve of degree d in $\mathbb{P}^r$, if indded that's necessary)

Ch. 4, p. 5: avoid invoking Hodge theory; just use the fact that there exist g independent holomorphic differentials

intro to Chapter 4: say we have two geometric objects – effective divisors and line bundles – and we want to parametrize each and describe the map between parameter spaces; do symmetric products first, since they're much more elementary than Jacobians.

Ch. 4: Poincare bundles, issue of when $\mathrm{Pic}_{d,g} \cong \mathrm{Pic}_{d',g}$

Ch. 5 p. 5: do the case of Q a cone by blowing up and describing the Picard group of the blow-up; say this will be a special case of a rational normal scroll as described in Chapter 10.

Ch. 5: in genus 2, do plane models: if $D = K + p + q$, image is a double conic if $p + q = K$; a cuspidal quartic if $p = q$ and a nodal quartic otherwise. Redo via projection from the quintic model.

Ch. 5 genus 3: do bitangents and flexes. Describe W^1_3?

Ch. 4: introduce W^r_d (or in Ch. 6, maybe?)

Ch. 6: title = "families of curves." Start with general discussion of families: family = map; need condition for fibers to vary nicely (remark local triviality too restrictive; ur-example = plane curves) reproduce discussion of flatness from GoS.

Notion of fine moduli space. do examples of:

1. Hilbert schemes (do basic construction; mention truncated Hilbert schemes in Cheerful Facts)
2. Chow varieties
3. symmetric products and Jacobians
4. Hurwitz spaces
5. Severi varieties
6. M_g

Chapter on curves on scrolls (currently 11):

for hyperelliptic curves, add description of transcanonical embeddings from DE

for trigonal curves: introduce Maroni earlier; answer question: what scrolls occur? Finally, finish description of special linear series on trigonal curves.

(In Cheerful Facts: curves of higher gonality and the scrolls that contain them (ref. Patel))

Castelnuovo: use description of curves on scrolls to show Castelnuovo bound is sharp; put characterization of Castelnuovo curves in Cheerful Facts

Chapter 7: We're not going to do trigonal case here; refer to chapter on curves of scrolls.

In the genus 6, forward ref. to Brill-Noether thm. to assert existence of g^2_6s; find Del Pezzo surface.

Chapter 9: do automorphism groups of curves in Cheerful Facts