Practical Curves Notes/Typos

September 28, 2022

Dear Prof. Eisenbud,

Here are some typos/points of confusion/stylistic concerns I noted when reading the second chapter of your book. I may eventually go back and look at Chapter 1 again more closely for new errors, but probably not until I get to finish the rest of the book.

Best, CJ Dowd

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- I would find it helpful if you clarified at the beginning that treating a sheaf $\mathcal{F}$ on X as a sheaf on $Y \supseteq X$ formally means considering $\iota_*\mathcal{F}$, where $\iota : X \hookrightarrow Y$ is the inclusion map.
- p30, first sentence: missing end parentheses on $\chi(\mathcal{L}(d)$ and $h^0(\mathcal{L}(d)$
- Proof of Theorem 2.0.2: It might be helpful to justify explicitly $\chi(\kappa(p)) = 1$ using $\kappa(p)$ has 0-dimensional support + Vanishing Theorem $\implies h^1(\kappa(p)) = 0$.

2.1

- p31, first line: $z_{:=}x_1/x_0$ has the := in subscript.
- Similarly p31, second formatted equation: $\widehat{d_{x_i}}$ should be $\widehat{dx_i}$.

2.1.1

- I'm glad you added in Lemma 2.1.5; I don't think it was in previous drafts. It helped me a lot because I wan't sure what exactly to do with the top exterior powers. You should probably separate the statement of the lemma from its proof more clearly. Judging by the italicization, probably only the word "If" and the exact sequence $0 \to \mathcal{E} \to \mathcal{F} \to \mathcal{G} \to 0$ are included in your LaTeX theorem environment. There's no proof environment for the proof of the lemma, but you might not want nested proofs so it might be fine as is.
- In proof of Lemma 2.1.6: extraneous period in "We may define a map $\wedge^e\mathcal{E} \otimes \wedge^g\mathcal{G} \to \wedge^f\mathcal{F}$. in terms of local sections as..."

2.1.2

- Title of subsubsection and many places elsewhere: Hurwitz' should be Hurwitz's. I don't think that the rules for a possessive noun ending in "s" also apply to a possessive noun ending in "z," and I can't find any other sources that use Hurwitz' instead of Hurwitz's.
- p34, near end of proof of Theorem 2.1.8: "At a point p where (locally) f has the form $z \mapsto w = z^e$ and $\omega = \omega = dw, \eta dz$ we have..." I don't think you meant to include ηdz here, or you meant to write something else.

- p33, Example 2.1.9: Grammatically, the comma in the first sentence should be removed. But also semantically, including the comma made me confused about which variety the ideal sheaf was associated to. Writing "Let $C \subseteq \mathbb{P}^2$ be a smooth plane curve and let $p \in \mathbb{P}^2$ be a general point with ideal sheaf generated by the vector space of linear forms $W = \langle x_0, x_1 \rangle$" makes it obvious that the ideal sheaf is associated to the point p, but with the comma it took me a few re-reads to clarify whether the ideal sheaf is associated to C or p.

- Example 2.1.10: It might be helpful to clarify that the branch locus of $\varphi_{\mathcal{W}}$ is the same as the zeros with multiplicity of the polynomial defining the discriminant restricted to the subspace W. My understanding is that by determining the degree of the discriminant restricted to a general pencil W of degree d forms, we can determine its degree as a function on $\mathbb{P}^d$. This isn't necessarily immediately obvious from the text, but it might be reasonable for a reader to make this connection, so how much you want to elaborate is up to you.

2.2

- p36, Corollary 2.2.6: Γ must consist of Cartier divisors of C, i.e. closed points, since otherwise the set $C \backslash \Gamma$ probably won't be a scheme and we certainly won't be able to define a divisor associated to Γ. This only matters if we're thinking scheme-theoretically rather than classically, so it might still be fine to ignore this issue since the book has largely ignored scheme-theoretic subtleties thus far.

- p36, in the definition of the Hilbert function: The only place you have previously used the notation S_C is in Cheerful Fact 2.1.3, so it might be helpful to tell us what it is here.

2.2.1

- p37: Missing period at end of first paragraph.

- p37, 4th paragraph: extraneous "t" in "... we can relate the arithmetic and geometric genera of t C_0 using the map..."

- p37: In $p_a(C_0) - g(C) = \sum_{p \in (C_0)_{sing}} \delta_p$, the "sing" should be romanized, i.e. write $(C_0)_{\text{sing}}$.

- Theorem 2.3.1 and elsewhere: the hyphenation of "base-points" vs. "basepoint," and "basepoint-free" vs "base-point free" etc. is still inconsistent. It seems that most of this was fixed in Chapter 1, where you seemed to have settled on "basepoint-free," but I found at least one inconsistency: p25 last paragraph you write "basepoint freeness" instead of "basepoint-freeness." Using "base-point" seems to have proliferated throughout Chapter 2.

- p39, Lemma 2.3.2: Replace the second comma of the second sentence with "and" so that it reads "If C has an invertible sheaf $\mathcal{L}$ of degree ≤ 2 with two independent sections, then $\mathcal{L}$ has degree 2 and $|\mathcal{L}|$ defines a morphism of degree 2 to $\mathbb{P}^1$, so if $g(C) \geq 2$ then C is hyperelliptic."

- p39, proof of Theorem 2.3.1: Remove "If K_C had a base-point then" from the first sentence. It seems like you started writing one sentence but decided to redo it, but then forgot to remove the start of the original sentence.

- p40, after Corollary 2.3.3: $\phi(C) \subset \mathbb{P}^{g-1}$ should be $\phi_K(C) \subset \mathbb{P}^{g-1}$.

- p40, right before the start of section 2.3.2: You've used the terminology of allowing a divisor to "move" in lecture before, and now here: "the linear series $|D|$ in which D moves". What does this actually mean?

- p41, proof of Corllary 2.3.5, second paragraph: $\phi_K(E$ missing parenthesis. The span of $\phi_K(D)$ should have dimension $\min\{g-1, d-s-1\}$ rather than $\min\{g-1, d-s\}$, since $d-s$ linearly independent points define a plane of dimension $d-s-1$. Replacing $d-s$ with $d-s-1$ makes the arithmetic $r(D) = d-1-\min\{g-1, d-s\} = max\{s, d-g\}$ correct. Finally, $max\{s, d-g\}$ should have max in Roman font.

2.3

- p41, first paragraph of 2.4: Missing period at end of paragraph.
- p41, Proof of Corollary 2.4.1: the first inequality

$$r(\mathcal{L}) + r(K_C \otimes \mathcal{L}^{-1}) \geq r(K_C) = g-1$$

should be flipped, since the correct statement is $r(\mathcal{D}) + r(\mathcal{E}) \leq r(\mathcal{D}+\mathcal{E})$. You use the correct version of the inequality two lines down, but why not write

$$g = r(K_C) + 1 \geq r(\mathcal{L}) + r(K_C \otimes \mathcal{L}^{-1}) + 1 = 2r(\mathcal{L}) + g - d$$

instead of

$$g = r(K_C) + 1 \geq r(\mathcal{L}) + r(K_C \otimes \mathcal{L}^{-1}) + 1 \geq 2r(\mathcal{L}) + g - d?$$

I think the second inequality is an equality.

2.4

- General: Sometimes you use $D \cdot E$ for the intersection number and sometimes you use $D.E$
- p42, right before Theorem 2.5.1: Do you want to write $\chi(L^{-1} \otimes M^{-1})$ instead of $\chi(\mathcal{L}^{-1}\mathcal{M}^{-1})$? Or is it an acceptable convention to omit the tensor? There's also a sign error: it should be

$$D \cdot E = \chi(\mathcal{O}_X) - \chi(\mathcal{L}^{-1}) - \chi(\mathcal{M}^{-1}) + \chi(\mathcal{L}^{-1} \otimes \mathcal{M}^{-1})$$

with the last term positive. (Based on Hartshorne Exercise V.1.1.)

- p43, Theorem 2.5.2: Missing period at the end of the theorem after the second formula.
- p43, Example 2.5.3: The first item in the list is missing a period.
- p43, Example 2.5.3: The second subitem of the second item should start with a capital letter.
- p43, Example 2.5.3, 4th item: need a space between $2H$ and "in" in "Treating the quadric Q as a divisor $2H$in $\mathbb{P}^3$..." Later, it should be "a divisor C of type (a,b) on Q" instead of "a divisor C (a,b) on Q".
- p44, Theorem 2.5.4: Second sentence needs to be capitalized. For consistency, do you want to write $\chi(\mathcal{O}_S)$ in the formula instead of just $\chi(\mathcal{O})$?
- p44, Theorem 2.5.5: Remove the comma in "If X is the blowup of Y at a point p, with exceptional divisor $E = \pi^{-1}(p)$ then:"