
Contents

Chapter 0. Introduction	1
Why you want to read this book	1
Why we wrote this book	2
What's with practice?	3
What's in this book	4
▸ Exercises and hints	7
▸ Relation of this book to other texts	7
Prerequisites, notation and conventions	8
▸ Commutative algebra	8
▸ Projective geometry	8
▸ Sheaves and cohomology	9
Chapter 1. Linear series and morphisms to projective space	11
1.1 Divisors	12
1.2 Divisors and rational functions	13
▸ Generalizations	13
▸ Divisors of functions	14
▸ Invertible sheaves	15
▸ Invertible sheaves and line bundles	17
1.3 Linear series and maps to projective space	18
1.4 The geometry of linear series	20
▸ An upper bound on $h^0(\mathcal{L})$	20
▸ Incomplete linear series	21
▸ Sums of linear series	23
▸ Which linear series define embeddings?	23

Exercises	26
Chapter 2. The Riemann–Roch theorem	29
2.1 How many sections?	29
▸ Riemann–Roch without duality	30
2.2 The most interesting linear series	31
▸ The adjunction formula	32
▸ Hurwitz’s theorem	34
2.3 Riemann–Roch with duality	37
▸ Residues	40
▸ Arithmetic genus and geometric genus	41
2.4 The canonical morphism	43
▸ Geometric Riemann–Roch	45
▸ Linear series on a hyperelliptic curve	46
2.5 Clifford’s theorem	47
2.6 Curves on surfaces	48
▸ The intersection pairing	48
▸ The Riemann–Roch theorem for smooth surfaces	49
▸ Blowups of smooth surfaces	50
2.7 Quadrics in \mathbb{P}^3 and the curves they contain	51
▸ The classification of quadrics	51
▸ Some classes of curves on quadrics	51
2.8 Exercises	52
Chapter 3. Curves of genus 0	57
3.1 Rational normal curves	58
3.2 Other rational curves	64
▸ Smooth rational quartics	64
▸ Some open problems about rational curves	66
3.3 The Cohen–Macaulay property	68
3.4 Exercises	71
Chapter 4. Smooth plane curves and curves of genus 1	75
4.1 Riemann, Clebsch, Brill and Noether	75
4.2 Smooth plane curves	77
▸ 4.2.1 Differentials on a smooth plane curve	77
▸ 4.2.2 Linear series on a smooth plane curve	79
▸ 4.2.3 The Cayley–Bacharach–Macaulay theorem	80
4.3 Curves of genus 1 and the group law of an elliptic curve	82
4.4 Low degree divisors on curves of genus 1	84

• The dimension of families	84
• Double covers of \mathbb{P}^1	85
• Plane cubics	85
4.5 Genus 1 quartics in \mathbb{P}^3	86
4.6 Genus 1 quintics in \mathbb{P}^4	88
4.7 Exercises	90
Chapter 5. Jacobians	93
5.1 Symmetric products and the universal divisor	94
• Finite group quotients	95
5.2 The Picard varieties	96
5.3 Jacobians	98
5.4 Abel's theorem	101
5.5 The $g + 3$ theorem	103
5.6 The schemes $W_d^r(C)$	105
5.7 Examples in low genus	105
• Genus 1	105
• Genus 2	106
• Genus 3	106
5.8 Martens' theorem	106
5.9 Exercises	108
Chapter 6. Hyperelliptic curves and curves of genus 2 and 3	111
6.1 Hyperelliptic curves	111
• The equation of a hyperelliptic curve	111
• Differentials on a hyperelliptic curve	113
6.2 Branched covers with specified branching	114
• Branched covers of \mathbb{P}^1	115
6.3 Curves of genus 2	117
• Maps of C to \mathbb{P}^1	118
• Maps of C to \mathbb{P}^2	118
• Embeddings in \mathbb{P}^3	119
• The dimension of the family of genus 2 curves	120
6.4 Curves of genus 3	121
• Other representations of a curve of genus 3	121
6.5 Theta characteristics	123
• Counting theta characteristics (proof of Theorem 6.8)	127
6.6 Exercises	129
Chapter 7. Fine moduli spaces	133

7.1	What is a moduli problem?	133
7.2	What is a solution to a moduli problem?	136
7.3	Hilbert schemes	137
▸ 7.3.1	The tangent space to the Hilbert scheme	138
▸ 7.3.2	Parametrizing twisted cubics	140
▸ 7.3.3	Construction of the Hilbert scheme in general	141
▸ 7.3.4	Grassmannians	142
▸ 7.3.5	Equations defining the Hilbert scheme	143
7.4	Bounding the number of maps between curves	144
7.5	Exercises	146
Chapter 8. Moduli of curves		149
8.1	Curves of genus 1	149
▸ M_1	is a coarse moduli space	150
▸	The good news	151
▸	Compactifying M_1	152
8.2	Higher genus	154
▸	Stable, semistable, unstable	156
8.3	Stable curves	157
▸	How we deal with the fact that \overline{M}_g is not fine	159
8.4	Can one write down a general curve of genus g ?	159
8.5	Hurwitz spaces	161
▸	The dimension of M_g	162
▸	Irreducibility of M_g	163
8.6	The Severi variety	163
▸	Local geometry of the Severi variety	164
8.7	Exercises	167
Chapter 9. Curves of genus 4 and 5		169
9.1	Curves of genus 4	169
▸	The canonical model	169
▸	Maps to projective space	170
9.2	Curves of genus 5	174
9.3	Canonical curves of genus 5	175
▸	First case: the intersection of the quadrics is one-dimensional	175
▸	Second case: the intersection of the quadrics is a surface	178
9.4	Exercises	179
Chapter 10. Hyperplane sections of a curve		181

10.1 Linearly general position	181
10.2 Castelnuovo's theorem	185
• Proof of Castelnuovo's bound	186
• Consequences and special cases	190
10.3 Other applications of linearly general position	191
• Existence of good projections	191
• The case of equality in Martens' theorem	192
• The $g + 2$ theorem	194
10.4 Exercises	196
Chapter 11. Monodromy of hyperplane sections	199
11.1 Uniform position and monodromy	199
• The monodromy group of a generically finite morphism	200
• Uniform position	201
11.2 Flexes and bitangents are isolated	202
• Not every tangent line is tangent at a flex	202
• Not every tangent is bitangent	203
11.3 Proof of the uniform position lemma	203
• Uniform position for higher-dimensional varieties	205
11.4 Applications of uniform position	206
• Irreducibility of fiber powers	206
• Numerical uniform position	206
• Sums of linear series	207
• Nodes of plane curves	207
11.5 Exercises	208
Chapter 12. Brill–Noether theory and applications to genus 6	211
12.1 What linear series exist?	211
12.2 Brill–Noether theory	211
• 12.2.1 A Brill–Noether inequality	213
• 12.2.2 Refinements of the Brill–Noether theorem	214
12.3 Linear series on curves of genus 6	217
• 12.3.1 General curves of genus 6	218
• 12.3.2 Del Pezzo surfaces	219
• 12.3.3 The canonical image of a general curve of genus 6	221
12.4 Other curves of genus 6	221
• $ D $ has a basepoint	222
• C is not trigonal and the image of ϕ_D is two-to-one onto a plane curve of degree 3.	222
12.5 Exercises	223

Chapter 13. Inflection points	225
13.1 Inflection points, Plücker formulas and Weierstrass points	225
• Definitions	225
• The Plücker formula	226
• Flexes of plane curves	228
• Weierstrass points	228
• Another characterization of Weierstrass points	229
13.2 Finiteness of the automorphism group	230
13.3 Curves with automorphisms are special	232
13.4 Inflections of linear series on \mathbb{P}^1	233
• Schubert cycles	234
• Special Schubert cycles and Pieri's formula	235
• Conclusion	237
13.5 Exercises	239
Chapter 14. Proof of the Brill–Noether Theorem	243
14.1 Castelnuovo's approach	243
• Upper bound on the codimension of $W_d^r(C)$	245
14.2 Specializing to a g -cuspidal curve	246
• Constructing curves with cusps	246
• Smoothing a cuspidal curve	246
14.3 The family of Picard varieties	247
• The Picard variety of a cuspidal curve	247
• The relative Picard variety	248
• Limits of invertible sheaves	249
14.4 Putting it all together	252
• Nonexistence	252
• Existence	252
14.5 Brill–Noether with inflection	252
14.6 Exercises	254
Chapter 15. Using a singular plane model	257
15.1 Nodal plane curves	257
• 15.1.1 Differentials on a nodal plane curve	258
• 15.1.2 Linear series on a nodal plane curve	260
15.2 Arbitrary plane curves	263
• The conductor ideal and linear series on the normalization	264
• Differentials	266
15.3 Exercises	269
Chapter 16. Linkage and the canonical sheaves of singular curves	273

16.1 Introduction	273
16.2 Linkage of twisted cubics	274
16.3 Linkage of smooth curves in \mathbb{P}^3	276
16.4 Linkage of purely 1-dimensional schemes in \mathbb{P}^3	277
16.5 Degree and genus of linked curves	278
• Dualizing sheaves for singular curves	278
16.6 The construction of dualizing sheaves	279
16.7 The linkage equivalence relation	284
16.8 Comparing the canonical sheaf with that of the normalization	285
16.9 A general Riemann–Roch theorem	288
16.10 Exercises	289
• Ropes and ribbons	291
• General adjunction	292
Chapter 17. Scrolls and the curves they contain	293
Introduction	293
17.1 Some classical geometry	293
17.2 1-generic matrices and the equations of scrolls	296
17.3 Scrolls as Images of Projective Bundles	302
17.4 Curves on a 2-dimensional scroll	303
• 17.4.1 Finding a scroll containing a given curve	303
• 17.4.2 Finding curves on a given scroll	305
17.5 Exercises	310
Chapter 18. Free resolutions and canonical curves	315
18.1 Free resolutions	315
18.2 Classification of 1-generic $2 \times f$ matrices	317
• 18.2.1 How to look at a resolution	318
• 18.2.2 When is a finite free complex a resolution?	319
18.3 Depth and the Cohen-Macaulay property	320
• 18.3.1 The Gorenstein property	321
18.4 The Eagon-Northcott complex	322
• 18.4.1 The Hilbert-Burch theorem	326
• 18.4.2 The general case of the Eagon-Northcott complex	327
18.5 Green’s Conjecture	331
• 18.5.1 Low genus canonical embeddings	334
18.6 Exercises	335
Chapter 19. Hilbert Schemes	339

19.1 Degree 3	339
• 19.1.1 The other component of $\mathcal{H}_{0,3,3}$	340
19.2 Extraneous components	341
19.3 Degree 4	342
• 19.3.1 Genus 0	342
• 19.3.2 Genus 1	343
19.4 Degree 5	343
• 19.4.1 Genus 2	344
19.5 Degree 6	345
• 19.5.1 Genus 4	345
• 19.5.2 Genus 3	345
19.6 Degree 7	345
19.7 The expected dimension of $\mathcal{H}_{g,r,d}^\circ$	345
19.8 Some open problems	348
• 19.8.1 Brill-Noether in low codimension	348
• 19.8.2 Maximally special curves	348
• 19.8.3 Rigid curves?	349
19.9 Degree 8, genus 9	350
19.10 Degree 9, genus 10	351
19.11 Estimating the dimension of the restricted Hilbert schemes using the Brill-Noether theorem	352
19.12 Exercises	353
Chapter 20. A historical essay on some topics in algebraic geometry	359
20.1 Greek mathematicians and conic sections	359
20.2 The first appearance of complex numbers	361
20.3 Conic sections from the 17th to the 19th centuries	362
20.4 Curves of higher degree from the 17th to the early 19th century	365
20.5 The birth of projective space	373
20.6 Riemann's theory of algebraic curves and its reception	374
20.7 First ideas about the resolution of singular points	376
20.8 The work of Brill and Noether	378
20.9 Bibliography	379
Chapter 21. Hints to selected exercises	385
Bibliography	387

Linkage and the canonical sheaves of singular curves

16.1. Introduction

In this chapter, curves are purely 1-dimensional projective schemes, not always reduced or irreducible.

Linkage is an equivalence relation on varieties and schemes of a given dimension embedded in a common space. It was a key element in the classification of curves in \mathbb{P}^3 for which [Max Noether](#) and [Georges-Henri Halphen](#) received the [Steiner prize](#) of the Prussian Academy of Sciences in 1880, and it was a necessary ingredient in the work of [Clebsch](#), [Brill](#), [Noether](#) and [Macaulay](#) toward a version of the [Riemann–Roch theorem](#) couched in terms of the algebra of plane curves near the end of the nineteenth century. It was put on a firm modern footing in [[Peskin and Szpiro 1974](#)], and this foundation was used for further progress in projective geometry by [Hartshorne](#), [Rao](#) and others. In this chapter we will explain some of these developments, starting with a simple example, and including the algebra necessary for a formulation in the natural generality of purely 1-dimensional schemes.

As we have seen, any plane curve is [arithmetically Cohen–Macaulay](#), and its arithmetic genus is determined by its degree. Similarly, a curve in \mathbb{P}^3 that is a [complete intersection](#) of surfaces of degrees d, e is arithmetically Cohen–Macaulay (Theorem 0.1) and has arithmetic genus determined by d, e . Next simplest, perhaps, is a curve C that is [directly linked](#) to a complete intersection, which means roughly that its union $X = C \cup D$ with a complete intersection

curve D is again a complete intersection (see Definition 16.3 for the general definition). We will see that, once again, such a curve C is arithmetically Cohen–Macaulay, and its genus is determined by the degrees of the equations of X and the degree and genus of D . Allowing sequences of direct links we define an equivalence relation called *linkage* or *liaison*, and curves in the linkage class of a complete intersection are said to be *licci*.

A famous theorem of Hartshorne and Rao [Prabhakar Rao 1978/79] shows that the linkage class of a curve $C \subset \mathbb{P}^3$ is classified by the finite-dimensional graded module

$$D(C) := H_*^1(\mathcal{I}_C) := \bigoplus_{m \in \mathbb{Z}} H^1(\mathcal{I}_C(m)),$$

called the *deficiency module* or *Hartshorne–Rao module* of C . (This is a module over the homogeneous coordinate ring $S = H_*^0(\mathcal{O}_{\mathbb{P}^3})$.) The correspondence is explicit: from a finite-dimensional graded module over S one can actually construct curves.

In the first sections of this chapter we will examine the equivalence relation on curves in \mathbb{P}^3 that is defined by linkage. Much of the story extends to the case of singular curves. This extension requires an understanding of the dualizing sheaves of singular curves, to which we turn in Section 16.5. We conclude the chapter with an analysis of the adjoint ideal, completing a result from Chapter 15, and allowing us to formulate the Riemann–Roch theorem for general curves and coherent sheaves.

Aside from the classification result above, linkage is useful in analyzing [Hilbert schemes](#). We will exploit this systematically in cases of low degree and genera in Chapter 19, and we begin this chapter with what is perhaps the simplest example, computing the dimension of the component of the Hilbert scheme $\text{Hilb}_{3m+1}(\mathbb{P}^3)$ that is the closure of the open subset \mathcal{H}° parametrizing twisted cubics (see Proposition 7.11 for another proof).

16.2. Linkage of twisted cubics

The simplest example of linkage is that of the union of a twisted cubic and one of its secant lines, pictured in Figure 16.1, and we will start with that.

Any twisted cubic curve $C \subset \mathbb{P}^3$ lies on a nonsingular quadric in class $(1, 2)$. Adding a line L of class $(1, 0)$ we get a divisor of class $(2, 2)$, the class of the complete intersection of two quadrics. Since L is also a complete intersection, C is *licci*.

We can make the relation of L and C explicit as follows: The ideal of C is minimally generated by the three 2×2 minors of the matrix

$$\begin{pmatrix} x_0 & x_1 & x_2 \\ x_1 & x_2 & x_3 \end{pmatrix}.$$

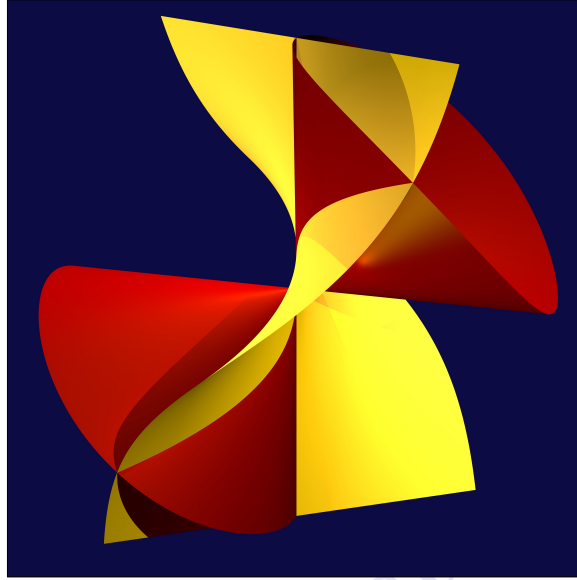


Figure 16.1. A quadratic cone (red) intersecting a smooth quadric (yellow) in the union of a vertical line and a twisted cubic (credit: Herwig Hauser).

The minor $Q_{1,2}$ involving the first two columns and the minor $Q_{2,3}$ involving the last two columns both vanish on the line $L : x_1 = x_2 = 0$, which meets the twisted cubic in the two points $x_0 = x_1 = x_2 = 0$ and $x_1 = x_2 = x_3 = 0$. Thus L is a secant line to C . A general linear combination Q of $Q_{1,2}$ and $Q_{2,3}$ defines a smooth quadric, which is thus isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. The curve C necessarily lies in the divisor class $(1, 2)$ (or, symmetrically, $(2, 1)$), and the line in class $(1, 0)$ (respectively, $(0, 1)$), summing to the complete intersection $(2, 2)$ of Q with (say) $Q_{1,2}$. See Figure 16.1.

Conversely, if two irreducible quadrics Q_1, Q_2 both contain a twisted cubic C then, by [Bézout's theorem](#), $Q_1 \cap Q_2$ is the union of C with a line. If at least one of the quadrics is smooth, we are in the situation above.

This suggests that we set up an incidence correspondence between twisted cubics and their secant lines. Let \mathbb{P}^9 denote the projective space of quadrics in \mathbb{P}^3 , and consider

$$\Phi = \{(C, L, Q, Q') \in \mathcal{H}^\circ \times \mathbb{G}(1, 3) \times \mathbb{P}^9 \times \mathbb{P}^9 \mid Q \cap Q' = C \cup L\}.$$

We'll analyze Φ by considering the projection maps to \mathcal{H}° and $\mathbb{G}(1, 3)$; that is, by looking at the diagram

$$\begin{array}{ccc} & \Phi & \\ \pi_1 \swarrow & & \searrow \pi_2 \\ \mathcal{H}^\circ & & \mathbb{G}(1, 3) \end{array}$$

Consider the projection map $\pi_2 : \Phi \rightarrow \mathbb{G}(1, 3)$ on the second factor. By what we just said, the fiber over any point $L \in \mathbb{G}(1, 3)$ is an open subset of $\mathbb{P}^6 \times \mathbb{P}^6$, where \mathbb{P}^6 is the space of quadrics containing L . Since $\dim \mathbb{G}(1, 3) = 4$ we see that Φ is irreducible of dimension $4 + 2 \times 6 = 16$. On the other hand, the map $\pi_1 : \Phi \rightarrow \mathcal{H}^\circ$ is surjective, with fiber over a curve C an open subset of $\mathbb{P}^2 \times \mathbb{P}^2$, where \mathbb{P}^2 is the projective space of quadrics containing C ; we conclude that \mathcal{H}° is irreducible of dimension 12, in accord with our computation in Proposition 7.11 of the space of twisted cubics as $\text{PGL}_4 / \text{PGL}_2$.

16.3. Linkage of smooth curves in \mathbb{P}^3

If the union of two smooth curves in \mathbb{P}^3 is a [complete intersection](#) then the degrees and genera of the curves are related:

Theorem 16.1. *Let $C_1, C_2 \subset \mathbb{P}^3$ be distinct smooth irreducible curves whose union is the complete intersection of two surfaces S, T of degrees s, t , with S smooth. Then $\deg C_1 + \deg C_2 = st$ and*

$$g(C_1) - g(C_2) = \frac{s + t - 4}{2}(\deg C_1 - \deg C_2).$$

In words, the difference between the genera of C_1 and C_2 is proportional to the difference in their degrees, with constant of proportionality $(s + t - 4)/2$. In the example of a complete intersection of two quadrics described above, the multiplier $(s + t - 4)/2$ is zero, and indeed the line and the twisted cubic have the same genus. The relation of degrees and genera is true more generally, as we shall see in the next section, but the special case is already useful.

Proof. The sum of degrees of C_1 and C_2 is the degree of $C_1 \cup C_2 = S \cap T$ which, by Bézout's theorem, is st .

By the [adjunction formula](#) in \mathbb{P}^3 the [canonical divisor](#) of S has class $K_S = (s - 4)H$. Thus, from the adjunction formula on the surface S we get

$$g(C_i) = \frac{C_i^2 + C_i \cdot K_S}{2} + 1 = \frac{C_i^2 + (s - 4)\deg C_i}{2} + 1.$$

Subtracting,

$$g(C_1) - g(C_2) = \frac{C_1^2 - C_2^2 + (s - 4)(\deg C_1 - \deg C_2)}{2}.$$

Because $C_1 + C_2$ is in the class tH on S we have

$$C_1^2 - C_2^2 = (C_1 + C_2)(C_1 - C_2) = t(\deg C_1 - \deg C_2).$$

Combining the last two displays yields the second formula of the theorem. \square

Remark 16.2. Linkage is closely related to linear equivalence. Here is a special case: suppose that S is a smooth surface in \mathbb{P}^3 , and $C \subset S$ is a curve. If T is a sufficiently general surface of degree t containing C then the curve C' that is

the link of C with respect to S, T lies in the class $tH - C$. If we link again with respect to another surface T' of degree t' we thus arrive at $C'' = C + (t - t')H$. Thus if $t = t'$ we get a curve in the same linear equivalence class as C . Moreover, since every rational function on S is the restriction to S of the ratio of two forms of the same degree on \mathbb{P}^3 , the set of curves on S that can be obtained from C by two linkages with surfaces T, T' of the same degree is exactly the linear series $|C|$ on S . This idea is generalized in the notion of a *basic double link*; see Exercise 16.8.

Exercise 16.1. Let C_1, C_2 be irreducible curves on a sm

16.4. Linkage of purely 1-dimensional schemes in \mathbb{P}^3

To say that the union X of distinct reduced irreducible curves $C \cup C'$ is a complete intersection means that the ideal I_X equals $I_C \cap I_{C'}$. Since the latter contains $I_C I_{C'}$, the ideal quotient $(I_X : I_C) := \{F \mid FI_C \subset I_X\}$ contains $I_{C'}$.

On the other hand, if $F \notin I_{C'}$ and we choose $G \in I_C \setminus I_{C'}$, then $FG \notin I_{C'}$, so $F \notin (I_X : I_C)$, and thus $(I_X : I_C) = I_{C'}$. This relationship underlies the formulas connecting the degrees and genera of C, C' , which hold for arbitrary purely 1-dimensional subschemes of \mathbb{P}^3 , as we shall see in Theorem 16.5.

Definition 16.3. Let C, C' be purely 1-dimensional subschemes of \mathbb{P}^3 . We say that C' is *directly linked* to C if there is a complete intersection X containing C, C' and satisfying $(I_X : I_C) = I_{C'}$. We say that C' is *linked* to C if they are connected by a chain of such direct linkages, and we say that C' is *evenly linked* to C if the chain involves an even number of direct linkages.

Note that in this setting, C and C' can have components in common. For example the subscheme $C \subset \mathbb{P}^3$ defined by the square of the ideal $\mathcal{J}_{L/\mathbb{P}^3}$ of a line $L \subset \mathbb{P}^3$ is linked to the reduced line L in the complete intersection of two quadrics. This is an example of a *rope* as discussed in Exercise 16.12. (This makes sense, since this scheme is a flat limit of twisted cubics.)

As in the smooth case treated above, direct linkage is a symmetric relation:

Proposition 16.4. Let $C_1 \subset \mathbb{P}^3$ be a purely 1-dimensional subscheme with saturated homogeneous ideal I_1 and suppose that C_1 is contained in a complete intersection of hypersurfaces $X := S \cap T$. The ideal $I_2 = (I_X : I_1)$ is a saturated ideal, defining a purely 1-dimensional subscheme and $I_1 = (I_X : I_2)$ as well.

Proof. The ideal I_X is unmixed of codimension 2, since X is a complete intersection [Eisenbud 1995a, Proposition 18.13]. It follows that $I_2 = (I_X : I_1)$ is also unmixed of codimension 2, and therefore saturated. Thus it suffices to prove that $I_1 = (I_X : I_2)$ after localizing at a codimension 2 prime P that contains I_X .

Write R for the localization at P of the homogeneous coordinate ring of X . Because I_X is a complete intersection, the ring R is zero-dimensional and [Gorenstein](#). By [[Eisenbud 1995a](#), Propositions 21.1 and 21.5], every finitely generated R -module is [reflexive](#). Since $I_{C_2}R = \text{ann}_R(I_{C_1}R) \cong \text{Hom}_R(R/I_{C_1}R, R)$, the proposition follows. \square

16.5. Degree and genus of linked curves

The degrees and [arithmetic genera](#) of directly linked schemes are related exactly as in the pilot case of Section 16.3.

Theorem 16.5. *If $C_1, C_2 \subset \mathbb{P}^3$ are purely 1-dimensional schemes that are directly linked by surfaces S, T of degrees s, t , then $\deg C_1 + \deg C_2 = st$ and*

$$p_a(C_1) - p_a(C_2) = \frac{s+t-4}{2}(\deg C_1 - \deg C_2).$$

Since we have left the realm of smooth curves and surfaces, we will need a more sophisticated [duality theory](#), and we postpone the proof to explain the necessary ideas.

Dualizing sheaves for singular curves. In Chapter 2 we said that the canonical sheaf of a smooth curve — the sheaf of differential forms — was the most important invertible sheaf after the structure sheaf. In the general setting of Cohen–Macaulay schemes, the analogue of the canonical sheaf is known as the dualizing sheaf. The general definition of the dualizing sheaf of a pure dimensional projective scheme this notion is not very illuminating; what is useful is how it is constructed and its cohomological properties relating to duality. However, having a definition may be comforting.

Definition 16.6. Let X be a projective scheme of pure dimension d over \mathbb{C} . The *dualizing sheaf* for X is a coherent sheaf ω_X together with a *residue map* $\eta : H^d(\omega_X) \rightarrow \mathbb{C}$ such that for every coherent sheaf \mathcal{F} the composite map

$$H^d(\mathcal{F}) \times \text{Hom}(\mathcal{F}, \omega_X) \rightarrow H^d(\omega_X) \xrightarrow{\eta} \mathbb{C}$$

is a [perfect pairing](#).

If $\mathcal{F} = \mathcal{L}$ is an invertible sheaf on a projective curve C then $\text{Hom}(\mathcal{L}, \omega_X) = \mathcal{L}^{-1} \otimes \omega$, so we recover [Serre duality](#): $H^1(\mathcal{L})$ is the dual of $H^0(\mathcal{L}^{-1} \otimes \omega_C)$.

It follows from the definition that the pair (ω_X, η) is unique up to canonical isomorphism if it exists: The module $H_*^0(\omega_X) = \bigoplus_{n \in \mathbb{Z}} (\text{Hom}_X(\mathcal{O}_X(n), \omega_X))$ is determined as the graded vector space dual of $H_*^d(\mathcal{O}_X)$, and the choice of η simply fixes the isomorphism. It may not be apparent that such a sheaf exists, but we will give a construction in Section 16.6.

Several properties of the dualizing sheaf on a purely 1-dimensional scheme are the same as in the smooth case, and follow easily from the definition:

Proposition 16.7. *Let C be a purely 1-dimensional projective scheme.*

- (1) $\mathcal{H}om_C(\omega_C, \omega_C) = \mathcal{O}_C$; thus if C is integral then the generic rank of ω_C is 1.
- (2) For any invertible sheaf \mathcal{L} on C we have

$$H^1(\mathcal{L}^{-1}) = H^0(\mathcal{L} \otimes \omega_C) \quad \text{and} \quad H^0(\mathcal{L}^{-1}) = H^1(\mathcal{L} \otimes \omega_C).$$

In particular, $\chi(\mathcal{L}^{-1}) = -\chi(\mathcal{L} \otimes \omega_C)$.

Proof. (1) We will show that the natural map $\mathcal{O}_C \rightarrow \mathcal{H}om_C(\omega_C, \omega_C)$ is an isomorphism. Since the map is globally defined, it suffices to prove that it is an isomorphism locally.

Choose a [Noether normalization](#) of C , that is, a finite map $f : C \rightarrow \mathbb{P}^1$. We shall see in Theorem 16.8 below that $\omega_C \cong \mathcal{H}om_{\mathbb{P}^1}(\mathcal{O}_C, \omega_{\mathbb{P}^1})$, regarded as a sheaf on C (in Theorem 16.8 this is the sheaf $f^!\omega_{\mathbb{P}^1}$). Since C is purely 1-dimensional, \mathcal{O}_C is torsion-free as an $\mathcal{O}_{\mathbb{P}^1}$ -module, and is thus locally free. Also, since \mathbb{P}^1 is smooth, $\omega_{\mathbb{P}^1}$ is locally isomorphic to $\mathcal{O}_{\mathbb{P}^1}$. But if B is any commutative ring and A is a B -algebra that is finitely generated and free as a B -module, then the composite map

$$A \rightarrow \text{Hom}_A(\text{Hom}_B(A, B), \text{Hom}_B(A, B)) \cong \text{Hom}_B(\text{Hom}_B(A, B), B)$$

sending an element a to the multiplication by a and thence to the map $f \mapsto f(a)$ may be identified with the isomorphism of A to its double dual as a B -module this completes the argument. (See Exercise 16.9 for a generalization of this last step.)

- (2) The definition of ω_C shows that, if \mathcal{L} is an invertible sheaf, then

$$H^1(\mathcal{L}^{-1}) = \text{Hom}_C(\mathcal{L}^{-1}, \omega_C) = H^0(\mathcal{H}om_C(\mathcal{L}^{-1}, \omega_C)) = H^0(\mathcal{L} \otimes \omega_C).$$

Using part (1), we have

$$H^0(\mathcal{L}^{-1}) = H^0(\mathcal{L}^{-1} \otimes_C \mathcal{O}_C)$$

$$= H^0(\mathcal{L}^{-1} \otimes_C \mathcal{H}om_C(\omega_C, \omega_C))$$

$$= \text{Hom}_C(\mathcal{L} \otimes_C \omega_C, \omega_C)$$

$$= H^1(\mathcal{L} \otimes_C \omega_C) \quad \square$$

16.6. The construction of dualizing sheaves

Dualizing sheaves do exist on any purely 1-dimensional projective scheme, and more generally on any projective Cohen–Macaulay scheme. We have already seen constructions in three cases:

- If X is a smooth scheme of dimension d over \mathbb{C} then $\omega_X = \bigwedge^d \Omega_{X/\mathbb{C}}$ is a dualizing sheaf [[Hartshorne 1977](#), Section III.7; [[1978](#), p. 648, 708]].
- If $f : X \rightarrow Y$ is a map of smooth curves, then $\omega_X = f^*(\omega_Y)(\text{ram}_{X/Y})$, where ram denotes the [ramification divisor](#).

- If $X \subset Y$ is a **Cartier divisor** on a surface, then $\omega_X = \omega_Y(X)|_X$.

How can such different looking formulas all be correct? **Grothendieck** provided a general scheme that unifies them and gives many more. To understand what is needed for the general case, we first consider a setting generalizing **Hurwitz's theorem**. Suppose that $X \rightarrow Y$ is a **finite map** of projective schemes, and that \mathcal{F} is a coherent sheaf on X .

If we restrict ourselves to open affine subsets $U := \text{Spec } A \subset X$ mapping to $V := \text{Spec } B \subset Y$ via the map of rings $f^* : B \rightarrow A$, then $F := \mathcal{F}_U$ is an A -module. Moreover, $f_* F := f_*(\mathcal{F})(V)$ is just F regarded as a B -module via the map f^* .

For any B -module M the module $\text{Hom}_B(A, M)$ has a natural A -module structure, where $(a\phi)(m)$ is defined to be $\phi(am)$. The functor

$$f^!(-) := \text{Hom}_B(A, -) : \text{mod}_B \rightarrow \text{mod}_A$$

defined in this way is the right adjoint of the functor $f_*(-) : \text{mod}_A \rightarrow \text{mod}_B$, which means that there is a natural isomorphism of functors

$$\text{Hom}_B(f_* F, -) \cong \text{Hom}_A(F, \text{Hom}_B(A, -)) = \text{Hom}_A(F, f^!(-)).$$

Thus if Y has dualizing sheaf ω_Y and we set $o_Y := \omega_Y(V)$, then

$$\text{Hom}_B(f_* F, o_Y) \cong \text{Hom}_A(F, f^! o_Y).$$

Also, there is a natural transformation η from $f_* f^!$ to the identity functor given by the formula

$$\eta : f_* f^!(M) = f_*(\text{Hom}_B(A, M)) = \text{Hom}_B(A, M) \rightarrow M, \quad \eta(\phi) = \phi(1),$$

for any A -module M . The transformation η called the *counit* of the adjoint pair $(f_*, f^!)$.

We also write $f^!(-)$ for the sheafification of the functor $\text{Hom}_B(A, -)$. Again $f^!$ is right adjoint to f_* on coherent sheaves, and again there are natural maps $\eta : f_* f^!(\mathcal{F}) \rightarrow \mathcal{F}$.

Theorem 16.8. *Let $f : X \rightarrow Y$ be a finite map of d -dimensional projective schemes. If Y has a dualizing sheaf ω_Y , with residue map $\eta_Y : H^d(\omega_Y) \rightarrow \mathbb{C}$, then*

$$\omega_X := f^! \omega_Y,$$

with residue map

$$\rho_X : H^d(f^! \omega_X) = H^d(f_* f^! \omega_X) \xrightarrow{H^d(\eta)} H^d(\omega_Y) \xrightarrow{\rho_Y} \mathbb{C},$$

where η is the counit of the adjoint pair $(f_*, f^!)$, is a dualizing sheaf on X .

Proof. Let \mathcal{F} be a coherent sheaf on X . Since f is finite, $H^d(\mathcal{F}) = H^d(f_*(\mathcal{F}))$. Thus, since $f^!(-)$ is a right adjoint of f_* there are natural isomorphisms

$$H^d(\mathcal{F}) = H^d(f_* \mathcal{F}) \cong \text{Hom}_Y(f_* \mathcal{F}, \omega_Y)^\vee \cong \text{Hom}_X(\mathcal{F}, f^! \omega_Y)^\vee,$$

the first being induced by ρ_Y . One can check that the composite isomorphism is the one induced by ρ_X , so $\omega_X = f^!(\omega_Y)$, completing the proof. \square

Cheerful Fact 16.9. Given this theorem, it seems natural to look for an adjoint functor $f^!$ for a wider class of morphisms f , but... in most cases, for example when f is the inclusion of a divisor on a smooth surface, no such functor exists on the category of coherent sheaves! An adjoint functor $f^!$ does exist on the derived category, where it is the right adjoint to Rf_* , leading to a theory of dualizing complexes.

Fortunately for the reader who is mostly interested in curves, this level of complication is unnecessary, and there is an intermediate level of generality that suffices for all the purposes of this book and more:

Theorem 16.10. *Suppose that $f : X \rightarrow Y$ is a finite map of projective schemes. If Y is Gorenstein with dualizing module ω_Y , then*

$$f^!(\omega_Y) \cong \mathcal{E}xt_Y^{\dim Y - \dim X}(\mathcal{O}_X, \omega_Y)$$

is a dualizing module for X .

The hypothesis is satisfied for any Y that is smooth, or even **locally a complete intersection**. The reason this works is that the complex $f^!\omega_Y$ can be identified with its one nonvanishing cohomology module, $\mathcal{E}xt_Y^{\dim Y - \dim X}(\mathcal{O}_X, \omega_Y)$. See for example [Altman and Kleiman 1970] for a thorough and accessible exposition and the Wikipedia page [?].

Proof of Theorem 16.5. Let X be the complete intersection of surfaces of degrees s, t containing C , and let $R_X = S/(F, G)$ be its homogeneous coordinate ring, where $S = \mathbb{C}[x_0, \dots, x_3]$ is the homogeneous coordinate ring of \mathbb{P}^3 . From the free resolution

$$0 \rightarrow S(-s-t) \xrightarrow{\begin{pmatrix} G \\ -F \end{pmatrix}} S(-s) \oplus S(-t) \xrightarrow{\begin{pmatrix} F & G \end{pmatrix}} S \rightarrow R_X \rightarrow 0$$

and Theorem 16.10 we see that

$$\omega_X = \mathcal{E}xt_C^2(\mathcal{O}_X, \omega_{\mathbb{P}^3}) = \mathcal{E}xt^2(\mathcal{O}_X, \mathcal{O}_{\mathbb{P}^3}(-4)) = \mathcal{O}_X(s+t-4).$$

Note that for any ideals $J \subset I$ in a ring A we have $\text{Hom}_A(A/I, A/J) \cong (J : I)/J$, where the isomorphism sends a homomorphism ϕ to the element $\phi(1)$. From Theorem 16.10 we have

$$\omega_C = \mathcal{H}om_X(\mathcal{O}_C, \omega_X) = \mathcal{H}om_X(\mathcal{O}_C, \mathcal{O}_X)(s+t-4) = \frac{\mathcal{I}_X : \mathcal{I}_C}{\mathcal{I}_X}(s+t-4),$$

where we have identified \mathcal{O}_C with its pushforward under the inclusion map $C \rightarrow X$.

By Proposition 16.7 we have $\chi(\omega_C(m)) = -\chi(\mathcal{O}_C(-m))$. It follows that the leading coefficient of the Hilbert polynomial of ω_C is equal to $\deg C$, and thus

$$st = \deg \mathcal{O}_X = \deg \mathcal{O}_{C'} + \deg \mathcal{O}_C,$$

as required by the formula for the sum of the degrees.

From Theorem 16.8 (or Theorem 16.10) we see that $\chi(\mathcal{O}_X) = st(4-s-t)/2$. Since $\mathcal{O}_{C'} = \mathcal{O}_{\mathbb{P}^3}/(\mathcal{I}_X : \mathcal{I}_C)$ and $(\mathcal{I}_X : \mathcal{I}_C)/(\mathcal{I}_X) = \omega_C(4-s-t)$ we have

$$\begin{aligned} \frac{4-s-t}{2}(\deg C + \deg C') &= \frac{4-s-t}{2}st \\ &= \chi(\mathcal{O}_X) \\ &= \chi(\mathcal{O}_{C'}) + \chi(\omega_C(4-s-t)) \\ &= \chi(\mathcal{O}_{C'}) - \chi(\mathcal{O}_C(s+t-4)) \\ &= \chi(\mathcal{O}_{C'}) - (s+t-4)\deg C - \chi(\mathcal{O}_C) \\ &= (1-p_a(\mathcal{O}_{C'})) - (1-p_a(\mathcal{O}_C)) - (s+t-4)\deg C, \end{aligned}$$

whence

$$p_a(\mathcal{O}_C) - p_a(\mathcal{O}_{C'}) = \frac{s+t-4}{2}(\deg C - \deg C'). \quad \square$$

Linkage also behaves in a simple way with respect to deficiency modules:

Theorem 16.11. *If C, C' are purely 1-dimensional subschemes of \mathbb{P}^3 that are directly linked by a complete intersection of degrees s, t then*

$$D(C') = \text{Hom}_C(D(C), \mathbb{C})(-s-t+4)$$

as graded modules over the homogeneous coordinate ring of \mathbb{P}^3 .

A more general form of this result appears as Proposition 2.5 in [Peskin and Szpiro 1974], with an attribution to Daniel Ferrand.

Proof. Suppose that the homogeneous ideal of C is generated by forms of degree a_i , $i = 1, \dots, s$. Since C is **locally Cohen–Macaulay**, the local rings $\mathcal{O}_{C,p}$ have projective dimension 2 as modules over $\mathcal{O}_{\mathbb{P}^3,p}$, and $\mathcal{I}_{C,p}$ has projective dimension 1. Thus we have an exact sequence

$$0 \rightarrow \mathcal{E} \rightarrow \bigoplus_i \mathcal{O}_{\mathbb{P}^3}(-a_i) \rightarrow \mathcal{I}_C \rightarrow 0.$$

Since the first and second cohomology groups of the twists of $\mathcal{O}_{\mathbb{P}^3}$ vanish, we deduce an isomorphism

$$D(C) := \bigoplus_{m \in \mathbb{Z}} H^1(\mathcal{I}_C(m)) \cong \bigoplus_{m \in \mathbb{Z}} H^2(\mathcal{E}(m)).$$

Let X be the [complete intersection of two hypersurfaces](#), of degrees s, t , containing C . From the inclusion we deduce a map of resolutions

$$\begin{array}{ccccccccc}
 0 & \longrightarrow & \mathcal{E} & \longrightarrow & \bigoplus_i \mathcal{O}_{\mathbb{P}^3}(-a_i) & \longrightarrow & \mathcal{O}_{\mathbb{P}^3} & \longrightarrow & \mathcal{O}_C \longrightarrow 0 \\
 \uparrow & & \uparrow & & & & \uparrow = & & \uparrow \\
 0 & \longrightarrow & \mathcal{O}_{\mathbb{P}^3}(-s-t) & \longrightarrow & \mathcal{O}_{\mathbb{P}^3}(-s) \oplus \mathcal{O}_{\mathbb{P}^3}(-t) & \longrightarrow & \mathcal{O}_{\mathbb{P}^3} & \longrightarrow & \mathcal{O}_X \longrightarrow 0
 \end{array}$$

We dualize this diagram, form the mapping cone, and twist by $-s-t$. Note that $\mathrm{Hom}_{\mathbb{P}^3}(\mathcal{O}_C, \mathcal{O}_{\mathbb{P}^3}) = 0$. Also, since the vertical map $\mathcal{O}_{\mathbb{P}^3} \rightarrow \mathcal{O}_{\mathbb{P}^3}$ on the right is the identity we may cancel these terms in the mapping cone. Noting that $\omega_C = \mathcal{E}xt^2(\mathcal{O}_C, \mathcal{O}_{\mathbb{P}^3}(-4))$ the result is a diagram with exact rows:

$$\begin{array}{ccccccc}
 0 & \longleftarrow & \omega_C(-s-t+4) & \longleftarrow & \mathcal{E}^*(-s-t) & \longleftarrow & \bigoplus_i \mathcal{O}_{\mathbb{P}^3}(a_i-s-t) \longleftarrow 0 \\
 \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longleftarrow & \mathcal{O}_X & \longleftarrow & \mathcal{O}_{\mathbb{P}^3} & \longleftarrow & \mathcal{O}_{\mathbb{P}^3}(-t) \oplus \mathcal{O}_{\mathbb{P}^3}(-s) \\
 & & \downarrow & & & & \\
 & & \mathcal{O}_{C'} & & & & \\
 & & \downarrow & & & & \\
 & & 0 & & & &
 \end{array}$$

The map ϕ is a monomorphism because $(\mathcal{I}_X : \mathcal{I}_C)/\mathcal{I}_X \cong \omega_C(-s-t+4)$, as explained above, so the column on the left is a short exact sequence. We can now write a resolution of $\mathcal{I}_{C'}$ as the mapping cone:

$$0 \leftarrow \mathcal{I}_{C'} \leftarrow \mathcal{O}_{\mathbb{P}^3}(-t) \oplus \mathcal{O}_{\mathbb{P}^3}(-s) \oplus \mathcal{E}^*(-s-t) \leftarrow \bigoplus_i \mathcal{O}_{\mathbb{P}^3}(a_i-s-t) \leftarrow 0.$$

From this we see that

$$H^1(\mathcal{I}_{C'}(m)) \cong H^1(\mathcal{E}^*(-s-t+m)) \cong \mathrm{Hom}_{\mathbb{C}}(H^2(\mathcal{E}(s+t-m-4)), \mathbb{C}),$$

where the last equality is from [Serre duality](#) on \mathbb{P}^3 . Summing over m we see that $D(C') \cong \mathrm{Hom}_{\mathbb{C}}(D(C)(s+t-4), \mathbb{C})$, and since Serre duality is functorial, the isomorphism holds not only as graded vector spaces, but as graded S -modules. \square

Sometimes the following consequence is a useful way to compute the deficiency module:

Proposition 16.12. *If C is a purely 1-dimensional subscheme of \mathbb{P}^3 with homogeneous ideal $I = I_C$ then*

$$D(C) \cong \mathrm{Hom}_{\mathbb{C}}(\mathrm{Ext}^3(S/I, S), \mathbb{C})(-4), \mathbb{C})$$

as graded modules over the homogeneous coordinate ring S of \mathbb{P}^3 .

Proof. We may choose a surjection $\psi : \oplus_i S(-a_i) \rightarrow I$, and choose the map $\phi : \oplus_i \mathcal{O}_{\mathbb{P}^3}(-a_i) \rightarrow \mathcal{I}_C$ in the proof of Theorem 16.11 to be the corresponding map of sheaves, so that \mathcal{E} is the sheafification of the graded module $E = \ker \psi$.

Since I is a saturated ideal, the depth of S/I is at least 1, so $\text{pd } S/I \leq 3$, and I has a free resolution of the form

$$0 \rightarrow G \rightarrow F \rightarrow \oplus_i S(-a_i) \rightarrow S \rightarrow S/I \rightarrow 0.$$

where $G \rightarrow F$ is a free presentation of E . and there is an exact sequence

$$0 \rightarrow E^* \rightarrow F^* \rightarrow G^* \rightarrow \text{Ext}_S^3(S/I, S) \rightarrow 0.$$

By Theorem 18.7, the fact that C is Cohen–Macaulay implies that the projective dimension of each of its local rings is ≤ 2 , and it follows that $\text{Ext}_S^3(S/I, S)$ has finite length. Writing $(\widetilde{})$ for the sheafification functor, we have a short exact sequence of sheaves

$$0 \rightarrow \mathcal{E}^* \rightarrow \widetilde{F^*} \rightarrow \widetilde{G^*} \rightarrow 0.$$

From this we see that

$$\text{Ext}_S^3(S/I, S) = H_*^1(\mathcal{E}^*) = \text{Hom}_{\mathbb{C}}(H_*^2(\mathcal{E}(-4)), \mathbb{C}) = H_*^1(\mathcal{I})(-4),$$

proving the assertion. \square

Proposition 16.12 is actually a special case of the local duality isomorphism between local cohomology and the dual of Ext ; see for example [Eisenbud 2005, Theorem A.1.9].

16.7. The linkage equivalence relation

As an immediate consequence of Theorem 16.11 we have:

Corollary 16.13 (Hartshorne). *If two curves C, C' are linked by an even length chain of direct linkages, then $D(C)$ and $D(C')$ are isomorphic up to a shift in grading.* \square

As we mentioned at the beginning of this chapter, the converse is also true: the **Hartshorne–Rao modules**, up to shift in grading, provide a complete invariant of linkage.

Cheerful Fact 16.14. Even more precise results are known (and the characteristic 0 hypothesis is largely unnecessary); here is a sample:

Theorem 16.15. *Let $S = \mathbb{C}[x_0, \dots, x_3]$ be the homogeneous coordinate ring of \mathbb{P}^3 , and let M be a graded S -module of finite length.*

- (1) *There is a smooth curve C with $D(C) = M(m)$ for some integer m .*
- (2) *There is a minimum value of m such that $M(m) = D(C_0)$ for some purely one-dimensional scheme C_0 .*

Moreover, each linkage class has a relatively simple structure, known as the *Lazarsfeld-Rao property*. We say that C' is obtained from C by an *ascending double link* if $I_{C'} = fI_C + (g)$ for some regular sequence contained in I_C — see Exercise 16.8.

Theorem 16.16 [Ballico et al. 1991]. *Let $M = D(C_0)$ the Hartshorne–Rao module of a purely 1-dimensional subscheme of \mathbb{P}^3 , and suppose that M is minimal in the sense that no $M(m)$ with $m > 0$ is the invariant of a purely 1-dimensional scheme.*

- (1) *Any curve in \mathbb{P}^3 with $D(C) = M$ is a deformation of C_0 through curves with invariant M .*
- (2) *Every curve in the even linkage class of C_0 is the result of a series of ascending double links followed by a deformation.*

In [Lazarsfeld and Rao 1983] it is shown that general curves in \mathbb{P}^3 that have reasonably large degree compared to their genus are minimal in the sense of Theorem 16.16.

16.8. Comparing the canonical sheaf with that of the normalization

In Chapter 15 we boasted in that we could effectively compute linear series on a smooth curve C given any plane curve C_0 with normalization C , and we showed how to do this when the plane curve has only nodes. To complete the discussion we need to compare the canonical sheaf ω_C of C with the dualizing sheaf ω_{C_0} of C_0 ; that is, we need a formula for the adjoint ideal of any curve singularity. A simplification occurs when the dualizing sheaf is invertible, that is, when the curve is Gorenstein (as is every plane curve).

Theorem 16.17. *If $\nu : C \rightarrow C_0$ is the normalization of a reduced connected projective curve then the adjoint ideal*

$$\mathfrak{A}_{C/C_0} := \operatorname{ann}_{\mathcal{O}_{C_0}} \frac{\omega_{C_0}}{\nu_* \omega_C}$$

is equal to the conductor ideal

$$\mathfrak{f}_{C/C_0} := \operatorname{ann}_{\mathcal{O}_{C_0}} \frac{\nu_* \mathcal{O}_C}{\mathcal{O}_{C_0}}.$$

Moreover, if C_0 is Gorenstein, then

$$\delta(C_0) = \operatorname{length} \frac{\nu_* \mathcal{O}_C}{\mathcal{O}_{C_0}} = \operatorname{length} \frac{\mathcal{O}_{C_0}}{\mathfrak{f}_{C/C_0}}.$$

As explained in Chapter 2, we can think of $\delta(C_0)$ as the number of nodes equivalent to the singularities of C_0 . The formula for $\delta(C_0)$ in the theorem was

first noted in Daniel Gorenstein's thesis ¹ under [Oscar Zariski](#). [Hyman Bass \[1963\]](#) explains that this is why [Grothendieck](#) named Cohen–Macaulay rings that have [cyclic canonical modules](#) after Gorenstein.

In Examples 15.19–15.21 we used the second equality in the formula for $\delta(C_0)$ in Theorem 16.17 to compute the δ invariant for several plane curve singularities. It fails for many space curve singularities; see Example 16.19 for a singularity that is not Gorenstein and behaves differently.

Proof of Theorem 16.17. Let $\rho : C_0 \rightarrow \mathbb{P}^1$ be a finite morphism. Both $\rho_* \nu_* \mathcal{O}_C$ and $\rho_* \mathcal{O}_{C_0}$ are torsion free coherent sheaves over $\mathcal{O}_{\mathbb{P}^1}$, and are thus locally free. Since ρ_* is left-exact, the inclusion $\mathcal{O}_{C_0} \subset \nu_* \mathcal{O}_C$ pushes forward to an inclusion

$$\alpha : \rho_* \mathcal{O}_{C_0} \hookrightarrow \rho_* \nu_* \mathcal{O}_C$$

and since \mathcal{O}_C is equal to \mathcal{O}_{C_0} generically on \mathbb{P}^1 , the cokernel $\text{coker } \alpha$ has finite length; indeed, it is supported on the image in \mathbb{P}^1 of the singular locus of C_0 . Since the maps ν and ρ are finite, we may harmlessly think of both \mathcal{O}_{C_0} and \mathcal{O}_C as coherent sheaves on \mathbb{P}^1 , and we will simplify the notation by dropping ν_* and ρ_* . Taking duals into $\omega_{\mathbb{P}^1} = \mathcal{O}_{\mathbb{P}^1}(-2)$ and defining $\alpha^\vee := \text{Hom}_{\mathbb{P}^1}(\alpha, \omega_{\mathbb{P}^1})$ we get a map that fits into the long exact sequence of $\mathcal{E}xt_{\mathbb{P}^1}(-, \omega_{\mathbb{P}^1})$:

$$\begin{aligned} 0 \rightarrow \text{Hom}_{\mathbb{P}^1}(\text{coker } \alpha, \omega_{\mathbb{P}^1}) &\rightarrow \omega_C \xrightarrow{\alpha^\vee} \omega_{C_0} \\ &\rightarrow \mathcal{E}xt_{\mathbb{P}^1}^1(\text{coker } \alpha, \omega_{\mathbb{P}^1}) \rightarrow \mathcal{E}xt_{\mathbb{P}^1}^1(\mathcal{O}_C, \omega_{\mathbb{P}^1}) \rightarrow \dots \end{aligned}$$

We know that $\text{coker } \alpha$ has finite support, so $\text{Hom}_{\mathbb{P}^1}(\text{coker } \alpha, \omega_{\mathbb{P}^1})$ is trivial and $\mathcal{E}xt_{\mathbb{P}^1}^1(\text{coker } \alpha, \omega_{\mathbb{P}^1})$ has the same length and the same annihilator as $\text{coker } \alpha$. Because \mathcal{O}_C is locally free as an $\mathcal{O}_{\mathbb{P}^1}$ -module, the term $\mathcal{E}xt_{\mathbb{P}^1}^1(\mathcal{O}_C, \omega_{\mathbb{P}^1})$ vanishes, and we get the more manageable exact sequence

$$0 \rightarrow \omega_C \xrightarrow{\alpha^\vee} \omega_{C_0} \rightarrow \mathcal{E}xt_{\mathbb{P}^1}^1(\text{coker } \alpha, \omega_{\mathbb{P}^1}) \rightarrow 0.$$

It follows that the sheaves $\nu_* \mathcal{O}_C / \mathcal{O}_{C_0}$ and $\omega_{C_0} / \nu_* \omega_C$ have the same length $\delta(C_0)$. Note that the [conductor](#) \mathfrak{f}_{C/C_0} (the annihilator of $\mathcal{O}_C / \mathcal{O}_{C_0}$ in \mathcal{O}_{C_0}) is at the same time an ideal sheaf of \mathcal{O}_{C_0} and an ideal sheaf of \mathcal{O}_C via the inclusion $\mathcal{O}_{C_0} \subset \mathcal{O}_C$. The argument above shows that \mathfrak{f}_{C/C_0} is also the annihilator ideal of ω_{C_0} / ω_C . By definition, this is the adjoint ideal of C_0 , proving the first statement of the theorem.

A further simplification occurs when $C_0 \subset \mathbb{P}^2$ is a plane curve, or more generally any Gorenstein curve. If the defining equation of C_0 is the form F of degree d , then there is a locally free resolution of \mathcal{O}_{C_0} of the form

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^2}(-d) \xrightarrow{F} \mathcal{O}_{\mathbb{P}^2} \rightarrow \mathcal{O}_{C_0} \rightarrow 0.$$

¹Gorenstein is better remembered for his work on the classification of finite simple groups.

Thus $\omega_{C_0} \cong \text{Ext}_{\mathbb{P}^2}^1(\mathcal{O}_{C_0}, \omega_{\mathbb{P}^2})$ is the cokernel of the map $\mathcal{O}_{\mathbb{P}^2}(-3) \rightarrow \mathcal{O}_{\mathbb{P}^2}(d-3)$ given by multiplication by F . It follows that ω_{C_0} is locally cyclic, and since ω_{C_0}/ω_C has finite support, ω_{C_0}/ω_C is globally cyclic, so $\omega_{C_0}/\omega_C \cong \mathcal{O}_{C_0}/\mathfrak{f}_{C/C_0}$. Since the length of ω_{C_0}/ω_C is equal to the length of $\mathcal{O}_C/\mathcal{O}_{C_0}$, the last statement of the theorem follows. \square

Example 16.18. Working locally, consider the [germ](#) of a [node](#), represented by the ring $R_0 := k[[x, y]]/(xy)$, and the projection to a line represented by the inclusion

$$P := k[[t]] \subset R_0 : t \mapsto x + y.$$

The normalization of R_0 is the map $R_0 \rightarrow R := k[[x]]e_1 \times k[[y]]e_2$, where $e_1 = x/t$ and $e_2 = y/t$ are orthogonal idempotents. Writing

$$Q_1 = k((t)) \subset Q := k((x)) \times k((y))$$

for the map of total quotient rings, we know that, because the extension is separable, the trace map $\text{Tr} := \text{Tr}_{Q/Q_1} : Q \rightarrow Q_1$ generates $\text{Hom}_{Q_1}(Q, Q_1)$ as a Q -vector space. Thus we may write the elements of $\omega_{R_0} = \text{Hom}_P(R_0, P)$ and $\omega_R = \text{Hom}_P(R, P)$ as multiples of Tr by elements of Q .

Since $R \cong Pe_1 \oplus Pe_2$ as a P -module, the module $\text{Hom}_P(R, P)$ is generated by the two projections, and it is easy to check that these are the maps $(x/t)\text{Tr}$ and $(y/t)\text{Tr}$. One can also check easily that

$$g := \frac{x-y}{t^2} \text{Tr} \in \text{Hom}_P(R_0, R).$$

Since $xg = x/t$ and $yg = y/t$ in Q we have $xg = x/t \text{Tr}$ and $yg = y/t \text{Tr}$, the generators of $\text{Hom}_P(R, P)$. The ring R_0 , regarded as a P -module, is freely generated by 1 and x . Immediate computation shows that $g \text{Tr}(1) = 0$ while $g \text{Tr}(x) = 1$. Furthermore $(x-y)g = 1$ in Q , and thus $\frac{1}{2}(x-y)g \text{Tr}$ takes 1 to 1 and x to t , proving that $g \text{Tr}$ generates $\text{Hom}_P(R_0, P)$. We also see directly from this that the adjoint ideal of ω_{R_0}/ω_R is the conductor ideal $(x, y)R_0 = (x, y)R = \mathfrak{f}_{R/R_0}$, as shown by Theorem 16.17.

Example 16.19. In Example ?? we showed that if C has a spatial triple point at 0, so that the completion of its local ring has the form $R_0 := k[[x, y, z]]/(xy, xz, yz)$, with normalization R , then it has δ invariant 2 but the conductor is $\mathfrak{f}_{R/R_0} = (x, y, z)R = (x, y, z)R_0$, and thus

$$\delta(C_0) \neq \text{length} \frac{\mathcal{O}_{C_0}}{\mathfrak{f}_{C/C_0}}.$$

The S -module R_0 , on the other hand, has free resolution

$$0 \longrightarrow S^2 \xrightarrow{\begin{pmatrix} 0 & z \\ y & -y \\ -x & 0 \end{pmatrix}} S^3 \xrightarrow{(xy \ xz \ yz)} S \longrightarrow R_0 \longrightarrow 0.$$

The canonical module of R_0 (which would be the germ of the canonical module of a global curve with such a singularity) is thus

$$\omega_{R_0} = \text{Ext}^2(R_0, S) = \text{coker} \begin{pmatrix} 0 & y & -x \\ z & -y & 0 \end{pmatrix},$$

a module requiring 2 generators. This shows that R_0 is not Gorenstein.

16.9. A general Riemann–Roch theorem

Using [dualizing sheaves](#) we can state a more [version of the Riemann–Roch theorem](#), applicable to any reduced, irreducible projective curve and any coherent sheaf thereupon. We will not make use of this generality later, so we only sketch the argument. We first need to extend the notion of degree of a sheaf:

Definition 16.20. If \mathcal{F} is a sheaf of generic rank r on a projective reduced and irreducible curve C we define the *degree* of \mathcal{F} as $\deg \mathcal{F} := \chi(\mathcal{F}) - r\chi(\mathcal{O}_C)$. Keeping in mind the definition of the arithmetic genus, we can write

$$(*) \quad \chi(\mathcal{F}) = \deg \mathcal{F} + r\chi(\mathcal{O}_C) = \deg \mathcal{F} + r(1 - p_a(C)).$$

This formula would be pointless if there were no other way to compute $\deg \mathcal{F}$, but that is not the case:

Cheerful Fact 16.21. The degree of \mathcal{F} coincides with that of a certain divisor class, the *first Chern class* of \mathcal{F} . See [\[Eisenbud and Harris 2016a, Chapter 5\]](#) for more information.

More to the point, if \mathcal{F} is a coherent sheaf on the reduced, irreducible projective curve C of generic rank r , and if \mathcal{L} is an invertible sheaf on C , then the following assertions can be proved using nothing more than the additivity of the Euler characteristic:

- (1) If \mathcal{F} is generated by its global sections and $\sigma_1, \dots, \sigma_r$ is a maximal generically independent collection of global sections of \mathcal{F} , then

$$M = \text{coker}(\mathcal{O}_C^r \xrightarrow{(\sigma_1, \dots, \sigma_r)} \mathcal{F})$$

has finite support, and

$$\deg \mathcal{F} = r\chi(\mathcal{O}_C) + \dim_{\mathbb{C}} H^0(M).$$

- (2) $\deg(\mathcal{L} \otimes \mathcal{F}) = \deg \mathcal{F} + r \deg \mathcal{L}$.

Thus if $\mathcal{O}_C(1)$ is a [very ample](#) invertible sheaf on C and m is an integer that is large enough so that $\mathcal{F}(m)$ is generated by global sections, then the degree of $\mathcal{F}(m)$ and the degree of $\mathcal{O}_C(1)$ are computed by the formula in item (1) and $\deg \mathcal{F} = \deg \mathcal{F}(m) - m \text{rank}(\mathcal{F}) \deg \mathcal{O}_C(1)$.

Using these facts and the dualizing property of ω_C we can reexpress the equality (*) in the definition of $\deg \mathcal{F}$ as follows:

Theorem 16.22. *If C is a reduced and irreducible curve and \mathcal{F} is a coherent sheaf on C , then*

$$h^0(\mathcal{F}) = \deg \mathcal{F} + \operatorname{rank}(\mathcal{F})(1 - p_a(C)) + h^0(\operatorname{Hom}_C(\mathcal{F}, \omega_C)).$$

16.10. Exercises

Exercise 16.2. Verify that the genus formula in Theorem 16.5 agrees with the usual calculation of degrees and genera for divisors on a quadric of classes (a, b) and $(d - a, d - b)$.

Exercise 16.3. Let $C_1, C_2 \subset \mathbb{P}^3$ be distinct smooth irreducible curves whose union is the complete intersection of two surfaces S, T of degrees s, t , with S smooth. Compute the intersection number $(C_1 \cdot C_2)$ in terms of the degrees and genera of C_1 and C_2 .

Exercise 16.4. Let C be a reduced and irreducible projective curve, and let \mathcal{E} be a locally free sheaf of rank r on C . Show that $\deg \mathcal{E} = \deg \bigwedge^r(\mathcal{E})$.

Hint: First show that any **locally free sheaf** on C is an iterated extension of invertible sheaves.

Exercise 16.5. Let C be the disjoint union of 3 **skew lines** (see Figure 16.2).

- (1) Prove that C lies on a unique quadric, and that $H^2(\mathcal{I}_C) = 0$.
- (2) Compute the **Hartshorne–Rao module** $D(C)$.
- (3) Show that if Γ is the union of 3 points in \mathbb{P}^3 then $H^1\mathcal{I}(\Gamma) = 0$ if and only if the three points are collinear.

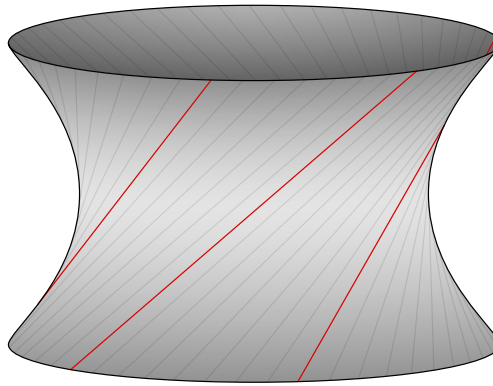


Figure 16.2. Any three skew lines in space lie on a unique (and necessarily smooth) quadric surface, and all belong to the same **ruling**.

- (4) Using the exact sequence in cohomology coming from the short exact sequence

$$0 \rightarrow \mathcal{I}_C \xrightarrow{\ell} \mathcal{I}_C(1) \rightarrow \mathcal{I}_\Gamma(1) \rightarrow 0,$$

where ℓ is a linear form, show that the map of vector spaces

$$H^1(\mathcal{I}_C) \xrightarrow{\ell} H^1(\mathcal{I}_C(1))$$

has rank < 2 if and only if ℓ vanishes on 3 collinear points on the three lines (including the case when ℓ vanishes identically on one of the lines). Conclude that if a different union C' of 3 skew lines is linked to C , then C' lies on the same quadric as C .

See [Migliore 1986] for more examples of this type.

Exercise 16.6. Compute the [Hilbert function](#) of the [Hartshorne–Rao module](#) of a curve of type (a, b) on a smooth quadric surface.

Hint: The ideal sheaf of the curve on the quadric Q is an extension of the ideal sheaf of the quadric in \mathbb{P}^3 with the ideal sheaf of the curve on the quadric, which is

$$\mathcal{O}_Q(-a, -b) = \pi_1^*(\mathcal{O}_{\mathbb{P}^1}(-a)) \otimes \pi_2^*(\mathcal{O}_{\mathbb{P}^1}(-b)),$$

where π_1, π_2 are the projections to \mathbb{P}^1 . Use the [Künneth formula](#)

$$H^1(\mathcal{O}_Q(p, q)) = H^1(\mathcal{O}_{\mathbb{P}^1}(p)) \otimes H^0(\mathcal{O}_{\mathbb{P}^1}(q)) \oplus H^0(\mathcal{O}_{\mathbb{P}^1}(p)) \otimes H^1(\mathcal{O}_{\mathbb{P}^1}(q))$$

to compute the necessary cohomology.

Exercise 16.7 (linkage addition [Schwartz 1982]). Suppose that I, J are saturated ideals defining purely 1-dimensional subschemes of \mathbb{P}^3 and that f, g is a regular sequence with $f \in I$ and $g \in J$. Prove that $gI \cap fJ = (fg)$, and conclude that if I, J are saturated codimension 2 ideals defining purely 1-dimensional schemes C, C' in \mathbb{P}^3 then $(gI + fJ)$ is a saturated ideal defining a purely 1-dimensional scheme C'' with $D(C'') = D(C)(-\deg g) \oplus D(C')(-\deg f)$.

Hint: Use the exact sequence

$$0 \rightarrow (fg) \rightarrow gI \oplus fJ \rightarrow gI + fJ \rightarrow 0$$

and the corresponding exact sequence of quotients by these ideals.

Exercise 16.8 (basic double links). The special case of the construction in Exercise 16.7 in which C' is trivial is already interesting.

- (1) Show that if I is a [saturated ideal of codimension 2](#) defining a purely 1-dimensional scheme C in \mathbb{P}^3 and (f, g) is a regular sequence with $g \in I$, then $fI + (g)$ defines a scheme C' with $D(C') = D(C)(-\deg f)$.
- (2) Show directly that, with notation as above, C' is directly linked to C in two steps. Since the degrees of the generators of $D(C')$ are more positive, this is sometimes called an *ascending double link*. Geometrically it amounts to

taking the union of C with some components that are [complete intersections](#).

Exercise 16.9. Here is a more general form of the last step in the proof of Proposition 16.7(1). Suppose that $B \rightarrow A$ is a homomorphism of rings, X is an A -module and Y is a B -module. Show that there is a natural transformation

$$\phi : \operatorname{Hom}_A(X, \operatorname{Hom}_B(A, Y)) \cong \operatorname{Hom}_B(X, Y)$$

and that if $X = \operatorname{Hom}_B(A, Y)$, then the map

$$A \rightarrow \operatorname{Hom}_A(\operatorname{Hom}_B(A, Y), \operatorname{Hom}_B(A, Y))$$

taking an element $a \in A$ to multiplication by a on the A -module $\operatorname{Hom}_B(A, Y)$ is sent by ϕ to the evaluation map $\alpha \mapsto \alpha(a)$ for $\alpha \in \operatorname{Hom}_B(A, Y)$.

Ropes and ribbons. The simplest way to construct well-behaved nonreduced curves is to take neighborhoods of smooth ones. Ropes and ribbons are examples of this sort:

Definition 16.23. The *rope defined from a curve* $C \subset \mathbb{P}^n$ is the scheme $V(I_C^2)$ defined by the square of the ideal C .

Exercise 16.10. If C is the rope defined from a line $L \subset \mathbb{P}^3$ then the Hilbert function $h_C(m)$ and Hilbert polynomial $p_C(m)$ are both equal to $3m+1$. Thus C has degree 3 and arithmetic genus 0. Note that the degree can also be computed as the degree of a general hyperplane section, since this is defined by the square of the ideal of a point in \mathbb{P}^2 .

Hint: Count the monomials of each degree in square of the ideal of a line.

Exercise 16.11. To see why the rope in Exercise 16.10 should look like a twisted cubic, show that it is the flat limit of a twisted cubic as follows: Let $X \subset \mathbb{P}^3$ be the twisted cubic with parametrization $x_i = s^i t^{3-i}$. Consider the one-parameter subgroup of PGL_4 given in homogeneous coordinates x_0, \dots, x_3 on \mathbb{P}^3 by

$$A_t : (x_0, \dots, x_3) \mapsto (tX_0, X_1, X_2, tX_3).$$

Show that the [flat limit](#), as $t \rightarrow 0$, of the [twisted cubics](#) $A_t(C)$ is the rope $V(X_0^2, X_0X_1, X_1^2)$.

Hint: Use the description of $I(X)$ as the ideal of 2×2 minors of

$$\begin{pmatrix} x_0 & x_1 & x_2 \\ x_1 & x_2 & x_3 \end{pmatrix}.$$

Exercise 16.12. We saw in Section 16.1 that a twisted cubic curve is linked to a line by the [complete intersection of two quadrics](#). Show that the same is true for the rope of Exercise 16.10.

If C is the rope defined from a line in \mathbb{P}^2 , then the Zariski tangent space to C at any point is 2-dimensional; that is, it looks like a ribbon. More generally:

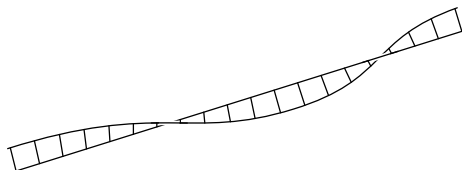


Figure 16.3. A ribbon supported on a line.

Definition 16.24. By a *ribbon* $X \subset \mathbb{P}^n$ we mean a scheme of pure dimension 1 and multiplicity 2 whose support is a smooth, irreducible curve $C \subset \mathbb{P}^n$ and whose [Zariski tangent space](#) at every point is 2-dimensional (Figure 16.3).

Exercise 16.13. Suppose that $C \subset \mathbb{P}^n$ is a ribbon. Show that C is contained in the rope defined from C_{red} , and show that the degree of C is twice that of C_{red} .

Hint: Look at hyperplane sections of C .

Unlike ropes, there are many different ribbons C with the same smooth curve C_{red} , and they can have different [arithmetic genera](#). Suppose that $C \subset \mathbb{P}^3$ is a ribbon such that $X = C_{\text{red}}$ is the line $V(x_0, x_1)$. Since C is contained in the rope defined from X we must have $(x_0^2, x_0x_1, x_1^2) \subset I(C)$. The tangent space to C at a point $(0, 0, s, t)$ meets the line $X' = V(x_2, x_3)$ at some point $(F(s, t), G(s, t), 0, 0)$, so F and G define a morphism $\mathbb{P}^1 \rightarrow \mathbb{P}^1$; thus they are homogeneous polynomials of the same degree d . It follows that $I(C)$ also contains the element $x_0G(x_2, x_3) - x_1F(x_2, x_3)$. Show that the ideal of C is obtained by adding this form to the ideal of the rope, that is,

$$I_C = (X_0^2, X_0X_1, X_1^2, F(X_3, X_4)X_0 + G(X_3, X_4)X_1).$$

In case $d = 1$, show that C lies on a smooth quadric.

General adjunction. The next two exercises illustrate Theorem 16.10:

Exercise 16.14. Show that if $C \rightarrow D$ is a map of smooth curves with [ramification index](#) e at $p \in C$, and t is a local analytic parameter at p , then locally analytically at p the sheaf $\mathcal{H}om_C(\mathcal{O}_C, \omega_D)$ is $\mathcal{O}_C(e)$.

Exercise 16.15. Show that if $C \subset S$ is a [Cartier divisor](#) on a surface S with canonical sheaf ω_S , then $\mathcal{E}xt^1(\mathcal{O}_C, \omega_S) \cong \mathcal{O}_C \otimes \mathcal{O}_S(C)$, and thus

$$K_C = (K_S + C) \cap C.$$