Curves

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1 pre-requisites and conventions

Basic results used in this section: Bézout, Riemann-Roch, Lasker (aka AF+BG), Clifford, Adjunction.

Let's explicitly allow things like $\mathrm{H}^0(D)$ where D is a divisor, as well as $\mathrm{H}^0(\mathcal{O}(D))$, but be careful not to mix the two too much.

Would it be more confusing or less to use the same letter for a polynomial vanishing on C and the surface it defines?

2 Personalities

The subject of algebraic curves abounds with examples amenable to explicit construction and analysis. In this chapter, we will survey the basic geometry and embeddings of the curves of genus 0 to 6. Our knowledge of the geometry of curves becomes increasingly less complete as the genus increases, and 6, as we shall see, is a natural turning point.

2.1 Curves of genus 0

rational curves as projections of rational normal curves. Rational quartic in $\mathbb{P}^3$ as curve of type 1,3 on quadric do dimension count. Branch points can be chosen. g^3_4 is sum of g^1_1 and a g^1_3. Cheerful fact: Set theor comp int problem.. Maximal rank for forms of degree d. Open questions: Hilbert functions? generators of the ideal? mention "secant conjecture"?

2.2 Curves of genus 1

Wonderful subject; refer to somewhere else. Double cover of $\mathbb{P}^1$, leading to $y^2 - f(x)$. Plane cubic, quartic in $\mathbb{P}^3$. Cheerful fact: elliptic quintic is Pfaffian. Cheerful fact: any g^5_6 is the product of two g^2_3s. Get a 3×3 matrix of linear forms. The image of the matrix and its transpose are g^2_3's. Prove this by going to the Segre embedding $\mathbb{P}^2 \times \mathbb{P}^2 \subset \mathbb{P}^8$.

2.3 Curves of genus 2

Canonical map to $\mathbb{P}^1$. Embedding in $\mathbb{P}^3$ as $(2,3)$ on a quadric, via any degree 5 line bundle. Ideal is 1 quadric, 2 cubics. Plane model of degree 4 with node or cusp.

2.4 Curves of genus 3

2.5 Curves of genus 4

As in the case of curves of genus 3, the study of curves of genus 4 bifurcates immediately into two cases: hyperelliptic and non-hyperelliptic; again, we will study the geometry of hyperelliptic curves in Chapter **??** and focus here on the nonhyperelliptic case.

In genus 4 we have a question that the elementary theory based on the Riemann-Roch formula cannot answer: are nonhyperelliptic curves of genus 4 expressible as three-sheeted covers of $\mathbb{P}^1$? The answer will emerge from our analysis in Proposition 2.2 below.

Let C be a non-hyperelliptic curve of genus 4. We start by considering the canonical map $\phi_K : C \hookrightarrow \mathbb{P}^3$, which embeds C as a curve of degree 6 in $\mathbb{P}^3$. We identify C with its image, and investigate the homogeneous ideal $I = I_C$ of equations it satisfies. As in previous cases we may try to answer this by considering the restriction maps
((replaced K_C^m with mK_C.))

$$r_m : \mathrm{H}^0(\mathcal{O}_{\mathbb{P}^3}(m)) \;\to\; \mathrm{H}^0(\mathcal{O}_C(m)) = \mathrm{H}^0(mK_C).$$

For $m = 1$, this is by construction an isomorphism; that is, the image of

C is non-degenerate (not contained in any plane).

For $m = 2$ we know that $\mathrm{h}^0(\mathcal{O}_{\mathbb{P}^3}(2)) = \binom{5}{3} = 10$, while by the Riemann-Roch Theorem we have

$$\mathrm{h}^0(\mathcal{O}_C(2)) = 12 - 4 + 1 = 9.$$

This shows that the curve $C \subset \mathbb{P}^3$ must lie on at least one quadric surface Q. The quadric Q must be irreducible, since any any reducible and/or non-reduced quadric must be a union of planes, and thus cannot contain an irreducible non-degenerate curve. If $Q' \neq Q$ is any other quadric then, by Bézout's Theorem, $Q \cap Q'$ is a curve of degree 4 and thus could not contain C. From this we see that Q is unique, and it follows that r_2 is surjective.

What about cubics? Again we consider the restriction map

$$r_3 : \mathrm{H}^0(\mathcal{O}_{\mathbb{P}^3}(3)) \ \rightarrow \ \mathrm{H}^0(\mathcal{O}_C(3)) = \mathrm{H}^0(3K_C).$$

The space $\mathrm{H}^0(\mathcal{O}_{\mathbb{P}^3}(3))$ has dimension $\binom{6}{3} = 20$, while the Riemann-Roch Theorem shows that

$$\mathrm{h}^0(\mathcal{O}_C(3)) = 18 - 4 + 1 = 15.$$

It follows that the ideal of C contains at least a 5-dimensional vector space of cubic polynomials. We can get a 4-dimensional subspace as products of the unique quadratic polynomial F vanishing on C with linear forms—these define the cubic surfaces containing Q. Since $5 > 4$ we conclude that the curve C lies on at least one cubic surface S not containing Q. Bézout's Theorem shows that the curve $Q \cap S$ has degree 6; thus it must be equal to C.

Let $G = 0$ be the cubic form defining the surface S. By Lasker's Theorem the ideal (F, G) is unmixed, and thus is equal to the homogeneous ideal of C. Putting this together, we have proven the first statement of the following result:

Theorem 2.1. *The canonical model of any nonhyperelliptic curve of genus 4 is a complete intersection of a quadric $Q = V(F)$ and a cubic surface $S = V(G)$ meeting along nonsingular points of each. Conversely, any smooth curve that is the intersection of a quadric and a cubic surface in $\mathbb{P}^3$ is the canonical model of a nonhyperelliptic curve of genus 4.*

Proof. Let $C = Q \cap S$ with Q a quadric and S a cubic. Because C is nonsingular and a complete intersection, both S and Q must be nonsingular at every point of their intersection Applying the Adjunction Formula to $Q \subset \mathbb{P}^3$ we get

$$\omega_Q = (\omega_{\mathbb{P}^3} \otimes \mathcal{O}_{\mathbb{P}^3}(2))|_Q = \mathcal{O}_Q(-4+2) = \mathcal{O}_Q(-2).$$

Applying it again to C on Q, and noting that $\emptyset_Q(C) = \emptyset_Q(3)$, we get

$$\omega_C = ((\omega_Q \otimes \mathcal{O}_3(3))|_C = \mathcal{O}_C(-2+3) = \mathcal{O}_C(1)$$

as required. $\square$

We can now answer the question we asked at the outset, whether a non-hyperelliptic curve of genus 4 can be expressed as a three-sheeted cover of $\mathbb{P}^1$. This amounts to asking if there are any divisors D on C of degree 3 with $r(D) \geq 1$; since we can take D to be a general fiber of a map $\pi : C \to \mathbb{P}^1$, we can for simplicity assume $D = p + q + r$ is the sum of three distinct points.

By the geometric Riemann-Roch theorem, a divisor $D = p + q + r$ on a canonical curve $C \subset \mathbb{P}^{g-1}$ has $r(D) \geq 1$ if and only if the three points $p, q, r \in C$ are colinear. If three points $p, q, r \in C$ lie on a line $L \subset \mathbb{P}^3$ then the quadric Q would meet L in at least three points, and hence would contain L. Conversely, if L is a line contained in Q, then the divisor $D = C \cap L = S \cap L$ on C has degree 3. Thus we can answer our question in terms of the family of lines contained in Q.

Any smooth quadric is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$, and contains two families of lines, or *rulings*. On the other hand, any singular quadric is a cone over a plane conic, and thus has just one ruling. By the argument above, the pencils of divisors on C cut out by the lines of these rulings are the g^1_3s on C. This proves:

Proposition 2.2. *A nonhyperelliptic curve of genus 4 may be expressed as a 3-sheeted cover of* $\mathbb{P}^1$ *in either one or two ways, depending on whether the unique quadric containing the canonical model of the curve is singular or smooth.*

((include this?))

(One might ask why the non-singularity of the cubic surface S plays no role. However, G is determined only up to a multiple of F, and it follows that the linear series of cubics in the ideal I_C has only base points along C. Bertini's Theorem says that a general element of this series will be nonsingular away from C; and since any every irreducible cubic in the family must be nonsingular along C, it follows that the general such cubic is nonsingular.)

A curve expressible as a 3-sheeted cover of $\mathbb{P}^1$ is called *trigonal*; by the analyses of the preceding sections, we have shown that *every curve of genus* $g \leq 4$ *is either hyperelliptic or trigonal.*

We can also describe the lowest degree plane models of nonhyperelliptic curves C of genus 4. We can always get a plane model of degree 5 by projecting C from a point p of the canonical model of C. Moreover, the Riemann-Roch Theorem shows that if D is a divisor of degree 5 with $r(D) = 2$ then, $\mathrm{h}^0(K - D) = 1$. Thus D is of the form $K - p$ for some point $p \in C$, and the map to $\mathbb{P}^2$ corresponding to D is π_p. These maps $\pi_p : C \to \mathbb{P}^2$ have the lowest possible degree (except for those whose image is contained in a line) because, by Clifford's Theorem a nonhyperelliptic curve of genus 4 cannot have a g^2_4.

We now consider the singularities of the plane quintic $\pi_p(C)$. Suppose as above that $C = Q \cap S$, with Q a quadric. If a line L through p meets C in p plus a divisor of degree ≥ 2 then, as we have seen, L must lie in Q. All other lines through p meet C in at most a single points, so π_p whose images are thus nonsingular points of $\pi(C)$, and π_C is one-to-one there. Moreover, a line that met C in > 3 points would have to lie in both the quadric and the cubic containing C, and therefore would be contained in C. Since C is irreducible there can be no such line.

We distinguish two cases:

1. Q is nonsingular: In this case there are two lines L_1, L_2 on Q that pass through p; they meet C in p plus divisors E_1 and E_2 of degree 2. If E_i consists of distinct points, then, since the tangent planes to the quadric along L_i are all distinct $\pi(C)$ will have a node at their common image.

   ((do we expect the reader to know this about quadrics, or should we prove it? or should we argue that since there are two distinct branches and a plane quintic of genus 2 can have

only the equivalent of two double points, these must be simple?? The first option is probably better.))

On the other hand, if E_i consists of a double point $2q$ (that is, L_i is tangent to C at $q \neq p$, or meets C 3 times at $q = p$), then $\pi(C)$ will have a cusp at the corresponding image point. In either case, $\pi(C)$ has two distinct singular points, each either a node or a cusp. The two g^1_3s on C correspond to the projections from these singular points.

2. Q is a cone: In this case, since the curve cannot pass through the singular point of Q there is a unique line $L \subset Q$ that passes through p. Let $p + E$ be the divisor on C in which this line meets C. The tangent planes to Q along L are all the same. Thus if $E = q_1 + q_2$ consists of two distinct points, the image $\pi_p(C)$ will have two smooth branches sharing a common tangent line at $\pi_p(q_1) = \pi_p(q_2)$. Such a point is called a *tacnode* of $\pi_p(C)$. On the other hand, if $E = 2q$, that is, if L meets C tangentially at one point $q \neq p$ (or meets C 3 times at p) then the image curve will have a higher order cusp, called a *ramphoid cusp*. In either case, the one g^1_3 on C is the projection from the unique singular point of $\pi(C)$.

((add pictures illustrating some of the possibilities above.))

2.6 Curves of genus 5

We consider now nonhyperelliptic curves of genus 5. There are now two questions that cannot be answered by simple application of the Riemann-Roch Theorem:

1. Is C expressible as a 3-sheeted cover of $\mathbb{P}^1$? In other words, does C have a g^1_3?
2. Is C expressible as a 4-sheeted cover of $\mathbb{P}^1$? In other words, does C have a g^1_4?

As we'll see, all other questions about the existence or nonexistence of linear series on C can be answered by the Riemann-Roch Theorem.

As in the preceding case, the answers can be found through an investigation of the geometry of the canonical model $C \subset \mathbb{P}^4$ of C. This is an octic curve in $\mathbb{P}^4$, and as before the first question to ask is what sort of polynomial equations define C. We start with quadrics, by considering the restriction map

$$r_2 : \mathrm{H}^0(\mathcal{O}_{\mathbb{P}^4}(2)) \ \to \ \mathrm{H}^0(\mathcal{O}_C(2)).$$

On the left, we have the space of homogeneous quadratic polynomials on $\mathbb{P}^4$, which has dimension $\binom{6}{4} = 15$, while by the Riemann-Roch Theorem the target is a vector space of dimension

$$2 \cdot 8 - 5 + 1 = 12.$$

We deduce that C lies on at least 3 independent quadrics. We will see in the course of the following analysis that it is exactly 3; that is, r_2 is surjective.) Since C is irreducible and, by construction, does not lie on a hyperplane, each of the quadrics containing C is irreducible, and thus the intersection of any two is a surface of degree 4. There are now two possibilities: The intersection of (some) three quadrics $Q_1 \cap Q_2 \cap Q_3$ containing the curve is 1-dimensional; or every such intersection is two dimensional.

We first consider the case where $Q_1 \cap Q_2 \cap Q_3$ is 1-dimensional. By the principal ideal theorem the intersection has no 0-dimensional components. By Bézout's Theorem the intersection is a curve of degree 8, and since C also has degree 8 we must have $C = Q_1 \cap Q_2 \cap Q_3$. Lasker's Theorem then shows that the three quadrics Q_i generate the whole homogeneous ideal of C.

We can now answer the first of our two questions for curves of this type. As in the genus 4 case the geometric Riemann-Roch Theorem implies that C has a g^1_3 if and only if the canonical model of C contains 3 colinear points or, more generally, meets a line L in a divisor of 3 points. When C is the intersection of quadrics, this cannot happen, since the line L would have to be contained in all the quadrics that contain C and $L \subset C$, which is absurd. Thus, in this case, C has no g^1_3.

What about g^1_4s? Again invoking the geometric Riemann-Roch Theorem, a divisor of degree 4 moving in a pencil lies in a 2-plane; so the question is, does $C \subset \mathbb{P}^4$ contain a divisor of degree 4, say $D = p_1 + \cdots + p_4 \subset C$, that lies in a plane Λ? Supposing this is so, we consider the restriction map

$$\mathrm{H}^0(\mathcal{I}_{C/\mathbb{P}^4}(2)) \;\to\; \mathrm{H}^0(\mathcal{I}_{D/\Lambda}(2)).$$

By hypothesis, the left hand space is 3-dimensional; but any four noncolinear points in the plane impose independent conditions on quadrics,
((this is a scheme of length 4; how is the reader supposed to cope with this if we don't assume the notion of a scheme, at least a finite one? And does the reader really know this fact about schemes of length 4 in the plane?))
so that the right hand space is 2-dimensional. It follows that Λ *must be contained in one of the quadrics Q containing C.*

The quadrics in ¶^{4} that contain 2-planes are exactly the singular quadrics: such a quadric is a cone over a quadric in ¶^{3}, and it is ruled by the (one or two) families of 2-planes it contains, which are the cones over the (one or two) rulings of the quadric in ¶^{3}. The argument above shows that the existence of a g^1_4s on C in this case implies the existence of a singular quadric containing C.

Conversely, suppose that $Q \subset \mathbb{P}^4$ is a singular quadric containing $C = Q_1 \cap Q_2 \cap Q_3$. Now say $\Lambda \subset Q$ is a 2-plane. If Q' and Q'' are "the other two quadrics" containing C, we can write

$$\Lambda \cap C = \Lambda \cap Q' \cap Q'',$$

from which we see that $D = \Lambda \cap C$ is a divisor of degree 4 on C, and so has $r(D) = 1$ by the geometric Riemann-Roch Theorem. Thus, the rulings of singular quadrics containing C cut out on C pencils of degree 4; and every pencil of degree 4 on C arises in this way.

Does C lie on singular quadrics? There is a $\mathbb{P}^2$ of quadrics containing C—a 2-plane in the space $\mathbb{P}^{14}$ of quadrics in $\mathbb{P}^4$—and the family of singular quadrics consists of a hypersurface of degree 5 in $\mathbb{P}^{14}$–called the *discriminant* hypersurface. By Bertini's Theorem, not every quadric containing C is singular. Thus the set of singular quadrics containing C is a plane curve B cut out by a quintic equation. So C does indeed have a g^1_4, and is expressible as a 4-sheeted cover of $\mathbb{P}^1$. In sum, we have proven:

Proposition 2.3. *Let $C \subset \mathbb{P}^4$ be a canonical curve, and assume C is the complete intersection of three quadrics in $\mathbb{P}^4$. Then C may be expressed as* *a 4-sheeted cover of $\mathbb{P}^1$ in a one-dimensional family of ways, and there is a map from the set of g^1_4s on C to a plane quintic curve B, whose fibers have cardinality 1 or 2.*

((could the "quintic curve" be reducible/multiple? Just a line?))
Of course, we can go further and ask about the geometry of the plane curve B and how it relates to the geometry of C; a fairly exhaustive list of possibilities is given in [**?**] [ACGH]. But that's enough for now.

In the second possibility above, that the canonical curve $C \subset \mathbb{P}^4$ is not a complete intersection; we will see in *** that the the intersection of the quadrics containing C is two-dimensional: a rational normalscroll; and C is trigonal, that is, a 3-sheeted cover of $\mathbb{P}^1$.

2.7 Curves of genus 6

Canonical model lies on at least 6 quadrics.

To prove projective quadratic normality, use general position: the general hyperplane section is 10 points in $\mathbb{P}^4$ 8 of them lie on the union of two hyperplanes – which won't contain the rest – so they impose exactly 9 conditions.

Prove monodromy of hyperplane sections is the symmetric group. Do this carefully. Explain the correspondence between monodromy and Galois theory.

Deduce projective normality from quadratic normality.

At this point, we're stuck: we still don't know what linear series exist on our curve, or much about the geometry of the canonical model. But if we invoke Brill-Noether, we have both: the curve has a g^2_6, which gives us a plane model as a sextic (with only double points, since no g^1_3s); the canonical series on the curve is cut out by cubics passing through the double points, which embeds the (blow-up of the) plane as a del Pezzo surface in ¶^{5}, of which the canonical curve is a quadric section. Also, use the count of g^2_6s on C to deduce the uniqueness of the del Pezzo.