Personalities of Curves

February 21, 2021

Contents

Chapter 1

Hilbert Schemes I: Examples

tSchemesChapter

In Chapter ??, we looked at curves of low genus and described the linear systems on them; that is, their maps to (and in particular their embeddings in) projective space. In this chapter we'll ask a more refined question: can we describe the family of all such curves in projective space?

((Add a section on basics of the Hilbert scheme explaining why Hilbert schemes; the universal property; and the tangent space **I think this should go in Chapter 6—we should have a "cast of characters" section there, where we introduce all the moduli spaces we'll be dealing with))**

Denote by $\mathcal{H} = \mathcal{H}_{g,r,d}$ the Hilbert scheme parametrizing subschemes of $\mathbb{P}^r$ with Hilbert polynomial $p(m) = dm - g + 1$ (which includes smooth curves of degree d and genus g in $\mathbb{P}^r$), and by $\mathcal{H}^\circ \subset \mathcal{H}$ the open subset parametrizing smooth, irreducible, nondegenerate curves $C \subset \mathbb{P}^r$ (called the *restricted Hilbert scheme*).

Three basic questions about the schemes $\mathcal{H}^\circ$ are:

- Is $\mathcal{H}^\circ$ irreducible? and
- What is its dimension or dimensions?

1. Where is it smooth, and where is it singular?

Of course, there are many more questions about the geometry of $\mathcal{H}^\circ$: for example, what is the closure $\overline{\mathcal{H}^\circ} \subset \mathcal{H}$ in the whole Hilbert scheme? (In other words, when is a subscheme $X \subset \mathbb{P}^r$ with Hilbert polynomial $dm - g + 1$ *smoothable*, in the sense that it is the flat limit of a family of smooth curves?) What is the Picard group of $\mathcal{H}^\circ$ or of its closure? We will for the most part not address these, though we will indicate the answers in special cases.

We'll limit ourselves in this chapter to looking at curves in $\mathbb{P}^3$. Most of the questions we raise in what follows could be asked, and many of them answered, in $\mathbb{P}^r$ for any $r \geq 3$, but for the most part the $r = 3$ case is enough to give us the flavor. We will start with curves of the lowest possible degree:

1.1 Degree 3

The smallest possible degree of an irreducible, nondegenerate curve $C \subset \mathbb{P}^3$ is 3. Any irreducible, nondegenerate curve $C \subset \mathbb{P}^3$ of degree 3 is a twisted cubic, so that in this case $\mathcal{H}^\circ$ is the parameter space for twisted cubics.

f twisted cubics **Proposition 1.1.1.** *The open subset $\mathcal{H}^\circ$ of the Hilbert scheme $\mathcal{H}_{0,3,3}$ parametrizing twisted cubics is irreducible of dimension 12.*

Proof. There are in fact several ways of establishing this statement. To start with the simplest, let $C_0 \subset \mathbb{P}^3$ be any given twisted cubic, and consider the family of translates of C_0 by automorphisms $A \in \mathrm{PGL}_4$ of $\mathbb{P}^3$: that is, the family

$$\mathcal{C} = \{(A, p) \in \mathrm{PGL}_4 \times \mathbb{P}^3 \ \mid\ p \in A(C_0)\}.$$

Via the projection $\pi : \mathcal{C} \to \mathrm{PGL}_4$, this is a family of twisted cubics, and so it induces a map

$$\phi : \mathrm{PGL}_4 \to \mathcal{H}^\circ.$$

Since every twisted cubic is a translate of C_0, this is surjective, with fibers isomorphic to the stabilizer of C_0, that is, the subgroup of PGL_4 of automorphisms of $\mathbb{P}^3$ carrying C_0 to itself. By the discussion in Section ??, every automorphism of C_0 is induced by an automorphism of $\mathbb{P}^3$, so the stabilizer is isomorphic to PGL_2 and thus has dimension 3. Since PGL_4 is irreducible of dimension 15, we conclude that $\mathcal{H}^\circ$ *is irreducible of dimension 12.* □

linear series 1

normal hilbert **Exercise 1.1.2.** Use an analogous argument to show that the restricted Hilbert scheme $\mathcal{H}^\circ \subset \mathcal{H}_{0,r,r}$ of rational normal curves $C \subset \mathbb{P}^r$ is irreducible of dimension $r^2 + 2r - 3$.

Second proof of Proposition hilb of twisted cubics 1.1.1

The argument above for Proposition hilb of twisted cubics 1.1.1 is based on a rather special fact, that all irreducible nondegenerate cubic curves $C \subset \mathbb{P}^3$ are translates of one another. There is another, less ad-hoc way of arriving at the conclusion above, called the method of *liaison*, or *linkage*, which we'll now describe. While it is more involved, it is more broadly applicable, at least in $\mathbb{P}^3$.

The idea behind this approach is the fact the intersection of any two distinct quadrics $Q, Q' \supset C$ containing a twisted cubic curve C has degree 4 and is unmixed; therefore it is the union of C and a line $L \subset \mathbb{P}^3$.

Conversely, suppose that $L \subset \mathbb{P}^3$ is any line and Q, Q' two general quadrics containing L; write the intersection $Q \cap Q'$ as a union $L \cup C$. Since smooth quadrics contain lines a general quadric containing L is smooth. The quadric Q' will intersect it in a curve of type $(2, 2)$, so the curve C will have class $(2, 1)$ or $(1, 2)$. The quadrics Q' containing L cut out on Q the complete linear system of curves of type $(2, 1)$, which has no base locus, so Bertini's theorem tells us that C will be smooth, so that the intersection $Q \cap Q' = L \cup C$ will be the union of L and a twisted cubic. This suggests that we set up an incidence correspondence: let $\mathbb{P}^9$ denote the projective space of quadrics in $\mathbb{P}^3$, and consider

$$\Phi = \{(C, L, Q, Q') \in \mathcal{H}^\circ \times \mathbb{G}(1, 3) \times \mathbb{P}^9 \times \mathbb{P}^9 \mid Q \cap Q' = C \cup L\}.$$

We'll analyze Φ by considering the projection maps to $\mathcal{H}^\circ$ and $\mathbb{G}(1, 3)$; that is, by looking at the diagram

$$\begin{tikzcd} & \Phi \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ \mathcal{H}^\circ & & \mathbb{G}(1,3) \end{tikzcd}$$

Consider first the projection map $\pi_2 : \Phi \to \mathbb{G}(1,3)$ on the second factor. By what we just said, the fiber over any point $L \in \mathbb{G}(1,3)$ is an open subset of $\mathbb{P}^6 \times \mathbb{P}^6$, where $\mathbb{P}^6$ is the space of quadrics containing L; it follows that Φ is irreducible of dimension $4 + 2 \times 6 = 16$. Going down the other side, we see that the map $\pi_1 : \Phi \to \mathcal{H}^\circ$ is surjective, with fiber over every curve C an open subsets of $\mathbb{P}^2 \times \mathbb{P}^2$, where $\mathbb{P}^2$ is the projective space of quadrics containing C; we conclude again that $\mathcal{H}^\circ$ *is irreducible of dimension 12.*

We'll see below several more instances of the application of liaison to the study of curves in $\mathbb{P}^3$. It should be said, though, that the method is largely limited to curves in $\mathbb{P}^3$ (and subvarieties $X \subset \mathbb{P}^r$ of codimension 2 in general); for example, you can't use it to do Exercise rational normal hilbert 1.1.2 for $r \geq 4$.

Third proof of Proposition hilb of twisted cubics 1.1.1

Yet another proof of Proposition hilb of twisted cubics 1.1.1 is based on a remarkable fact about twisted cubics, described in the next proposition; the application to $\mathcal{H}^\circ$ is carried out in the following exercise. In fact, the proposition here applies more generally to *rational normal curves*, and we'll state it in that generality.

points on rnc **Proposition 1.1.3.** *If* $p_1, \ldots, p_{n+3} \in \mathbb{P}^n$ *are any* $n+3$ *points in* $\mathbb{P}^n$ *in* linear general position*, that is, with no* $n+1$ *lying in a hyperplane, then there exists a unique rational normal curve* $C \subset \mathbb{P}^n$ *containing them.*

Proof. To start, we observe that there is an automorphism $\Phi : \mathbb{P}^n \to \mathbb{P}^n$ carrying the points $p_1, \ldots, p_{n+1}$ to the coordinate points $[0, \ldots, 0, 1, 0, \ldots, 0] \in \mathbb{P}^n$; denote the images of the remaining two points p_{n+2} and p_{n+3} by $[\alpha_0, \ldots, \alpha_n]$ and $[\beta_0, \ldots, \beta_n]$. We consider maps $\mathbb{P}^1 \to \mathbb{P}^n$ given in terms of an inhomogeneous coordinate z on $\mathbb{P}^1$ by

$$z \mapsto \left[\frac{\alpha_0}{z - \nu_0}, \frac{\alpha_1}{z - \nu_1}, \ldots, \frac{\alpha_n}{z - \nu_n}\right]$$

with $\nu_0, \ldots, \nu_n$ any distinct scalars, and $\mu_0, \ldots, \mu_n$ any nonzero scalars. Clearing denominators, we see that the image of such a map is a rational normal curve, and it passes through the $n+1$ coordinate points of $\mathbb{P}^n$, which are the images of the points $z = \nu_0, \ldots, \nu_n \in \mathbb{P}^1$. Moreover, the image of the point $z = \infty$ at infinity is the point $[\alpha_0, \ldots, \alpha_n]$; and we can adjust the values of $\nu_0, \ldots, \nu_n$ so that the image of the point $z = 0$ is $[\beta_0, \ldots, \beta_n]$. This proves existence; we'll leave uniqueness as the following exercise. $\square$

Exercise 1.1.4. Show that if $C, C' \subset \mathbb{P}^n$ are two rational normal curves and $\#(C \cap C') \geq n+3$, then $C = C'$. (Hint: use induction on n.)

There is another way to prove Proposition 1.1.3 that may provide more insight (it actually produces the equations defining the rational normal curve through the points $p_1, \dots, p_{n+3}$); this is described in [Harris 1982].

There are also a number of further statements and open problems involving generalizations of this construction. For example, in the statement of Proposition 1.1.3, we can generalize the points $p_1, \dots, p_{n+3} \in \mathbb{P}^n$ to an arbitrary *curvilinear scheme* $\Gamma \subset \mathbb{P}^n$, where by curvilinear scheme we mean a 0-dimensional scheme with Zariski tangent space of dimension at most 1 at every point (equivalently, such that every irreducible component of Γ is isomorphic to $\operatorname{Spec} K[\epsilon]/(\epsilon^k)$ for some k). In this setting the condition of "linear general position" is generalized to the condition that for any hyperplane $H \subset \mathbb{P}^n$ we have $\deg(\Gamma \cap H) \leq n+1$; and it's shown in [Eisenbud and Harris 2016] that the statement of Proposition 1.1.3 holds in this greater generality.

For an open problem related to Proposition 1.1.3, let's return to $\mathbb{P}^3$ and suppose $\mathcal{H}^\circ$ is any component of the restricted Hilbert scheme parametrizing curves of degree d and genus g in $\mathbb{P}^3$; say the dimension $\dim \mathcal{H}^\circ = 2m$. A straightforward dimension count then shows that if $p_1, \dots, p_m \in \mathbb{P}^3$ are general points, then there will be a finite number of curves in this component containing the points p_i; Proposition 1.1.3 asserts that in case $\mathcal{H}^\circ$ parametrizes twisted cubics, that number is 1. The question is, are there any other components of the restricted Hilbert scheme for which the number is similarly 1, other than components parametrizing complete intersections of two surfaces of the same degree?

In any case, returning to the case $n = 3$, we see that if $p_1, \dots, p_6 \in \mathbb{P}^3$ are any six points, with no four lying in a plane, then there is a unique twisted cubic containing all six; as promised, we can use this somewhat esoteric fact to deduce the dimension of the Hilbert scheme parametrizing twisted cubics.

Exercise 1.1.5. Consider the incidence correspondence

$$\Phi = \{(p_1, \dots, p_6, C) \in (\mathbb{P}^3)^6 \times \mathcal{H}^\circ \mid p_1, \dots, p_6 \in C\}.$$

Use the result above to show that $\mathcal{H}^\circ$ is irreducible of dimension 12. More generally, use Proposition 1.1.3 to give a second proof of Exercise 1.1.2.

1.1.1 Tangent spaces to Hilbert schemes

As we've said, our descriptions of Hilbert schemes of curves is primarily concerned with issues like the irreducibility and dimension of the restricted Hilbert scheme $\mathcal{H}^\circ$. Nonetheless, it is worth pointing out that we have at least one useful tool for answering questions about the smoothness or singularity of the restricted Hilbert scheme. In practice, it's very often the case that we can describe the Zariski tangent space $T_{[C]}\mathcal{H}^\circ$ to the Hilbert scheme at a point $[C] \in \mathcal{H}^\circ$, via the identification of $T_{[C]}\mathcal{H}^\circ$ with the space $H^0(\mathcal{N}_{C/\mathbb{P}^3})$ of global sections of the normal sheaf of C in $\mathbb{P}^3$, described in Section ??. In particular, we'll see in Section ?? below how to exhibit an everywhere nonreduced component of the restricted Hilbert scheme.

hilbert scheme section

mumford example

To illustrate how this may go, the following exercise gives a very simple and basic example.

ic normal bundle

Exercise 1.1.6. Let $C \cong \mathbb{P}^1 \subset \mathbb{P}^3$. Show that the normal bundle $\mathcal{N}_{C/\mathbb{P}^3} \cong \mathcal{O}_{\mathbb{P}^1}(5)^{\oplus 2}$; that is, the normal bundle of a twisted cubic is the direct sum of two line bundles of degree 5. Use this to prove that the restricted Hilbert scheme $\mathcal{H}^\circ$ of twisted cubics is everywhere smooth.

1.1.2 Extraneous components

Although $\mathcal{H}^\circ$ is open in the Hilbert scheme $\mathcal{H} = \mathcal{H}_{3m+1}(\mathbb{P}^3)$, its closure is not all of $\mathcal{H}$! There is a second irreducible component of $\mathcal{H}$, of dimension 15. This is an example of what is called an *extraneous component* of the Hilbert scheme; they are components of the Hilbert scheme whose general point does *not* correspond to a smooth, irreducible nondegenerate curve $C \subset \mathbb{P}^n$. They are the bane of anyone who works with Hilbert schemes; and while choosing to work just with the locus $\mathcal{H}^\circ \subset \mathcal{H}$ means that we won't be dealing with them directly, it's worth describing their behavior in at least the case of twisted cubics.

To start, observe that any plane cubic $C \subset \mathbb{P}^2 \subset \mathbb{P}^3$ has Hilbert polynomial $p(m) = 3m$. If $p \in \mathbb{P}^3 \setminus C$ is any point not on C, then, the union $C' = C \cup \{p\}$ is a subscheme of $\mathbb{P}^3$ with Hilbert polynomial $3m+1$, and so corresponds to a point of $\mathcal{H}$.

Now, let $\mathcal{H}' \subset \mathcal{H}$ be the open subset corresponding to unions $C' = C \cup \{p\}$ of a plane cubic and a point. By an argument analogous to the one given in

[Eisenbud and Harris 2016] for plane conics, the Hilbert scheme $\mathcal{H}_{3m}$ is a $\mathbb{P}^9$-bundle over the dual projective space $(\mathbb{P}^3)^*$, and so in particular is irreducible of dimension 12; the locus $\mathcal{H}'$ is then an open subset of the product $\mathcal{H}_{3m} \times \mathbb{P}^3$, and so is irreducible of dimension 15.

Exercise 1.1.7. Show that the Hilbert scheme $\mathcal{H}_{3m+1}$ is indeed the union of the closures of the loci $\mathcal{H}^\circ$ and $\mathcal{H}'$ above (in other words, any subscheme of $\mathbb{P}^3$ with Hilbert polynomial $3m+1$ is either a flat limit of twisted cubics, or a flat limit of subschemes of the form $C \cup \{p\}$ with C a plane cubic).

Given this, we conclude that the Hilbert scheme $\mathcal{H}_{3m+1}$ consists of two irreducible components: one, the closure of the locus $\mathcal{H}^\circ$ of twisted cubics, which has dimension 12; and a second, the closure of $\mathcal{H}'$, of dimension 15.

One further question: given that the Hilbert scheme $\mathcal{H}_{3m+1}$ consists of two irreducible components, it's natural to ask what their intersection is. The answer is suggested by an example in [Eisenbud and Harris 2000, II.3.4], where we take a general twisted cubic $C \subset \mathbb{P}^3$ and apply the family of linear maps $A_t : \mathbb{P}^3 \to \mathbb{P}^3$ given by

$$A_t = \begin{pmatrix} t & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix};$$

we see there that the flat limit $\lim_{t\to 0} A_t(C)$ is a nodal plane cubic, with a spatial embedded point of multiplicity 1 at the node. In fact, the intersection of the two components is exactly the closure of this locus, as the following exercise asks you to show.

Exercise 1.1.8. Show that the locus Σ of schemes X consisting of a nodal plane cubic curve C with a spatial embedded point of multiplicity 1 at the node is dense in the intersection $\overline{\mathcal{H}^\circ} \cap \overline{\mathcal{H}'}$.

Extraneous components in general

While we'll largely ignore the extraneous components of the Hilbert schemes that we'll be dealing with here, it's worth taking a moment out and seeing how they arise, and how numerous they are.

It starts already in dimension 0, actually. Let $\mathcal{H} = \mathcal{H}_d(\mathbb{P}^n)$ be the Hilbert scheme of subschemes of $\mathbb{P}^n$ with Hilbert polynomial the constant d. We have an open subset $\mathcal{H}^\circ \subset \mathcal{H}$ whose points correspond to reduced d-tuples of points in $\mathbb{P}^n$, and this open subset is easy to describe: it's just the complement of the diagonal in the dth symmetric power of $\mathbb{P}^n$. The closure of this open set will be called the *principal component* of $\mathcal{H}$.

You might think this would be all of the Hilbert scheme $\mathcal{H}$, but as the name suggests, it's not in general. Iarrobino in [Iarrobino 1985] first proved for any $n \geq 3$ and any sufficiently large d the existence of components of $\mathcal{H}_d(\mathbb{P}^n)$ having dimension strictly larger than dn—in particular, whose general point corresponded to a nonreduced subscheme of $\mathbb{P}^n$. Other such examples have been found (ref?); in general, no one knows how many irreducible components the Hilbert scheme $\mathcal{H} = \mathcal{H}_d(\mathbb{P}^n)$ has, or what their dimensions might be.

And that in turn infects the Hilbert schemes of curves. For example, if we're looking at the Hilbert scheme $\mathcal{H}_{dm-g+1}$ parametrizing curves of degree d and genus g in $\mathbb{P}^3$, we'll have a component whose general point corresponds to a union of a plane curve of degree d and $\binom{d-1}{2} - g$ points; moreover, if Γ is any irreducible component of the Hilbert scheme of zero-dimensional subschemes of degree $\binom{d-1}{2} - g$ in $\mathbb{P}^3$, there'll be a component of $\mathcal{H}_d(\mathbb{P}^n)$ whose general point corresponds to a union of a plane curve of degree d and the subscheme corresponding to a general point of Γ. And of course we can replace the plane curves in this construction with any component of the Hilbert scheme of curves of degree d and genus $g' > g$; in addition, there may also be components of $\mathcal{H}_{dm-g+1}$ whose general point corresponds to a subscheme of $\mathbb{P}^3$ with an embedded point—we don't know (see the paper by Dawei Chen and Scott Nollet, at https://arxiv.org/abs/0911.2221).

Bottom line, it's a mess. For many g, d the Hilbert scheme $\mathcal{H}_{dm-g+1}(\mathbb{P}^3)$ has many components. In most cases no one knows how many, or what their dimensions are. For that reason, we'll henceforth focus exclusively on the restricted Hilbert scheme, and ignore the extraneous components as much as possible.

1.2 Linkage

SLinkage

As the second proof of Proposition hilb of twisted cubics 1.1.1 suggests, when the union of two curves C and D forms a complete intersection we can use this fact to relate the geometry of their respective Hilbert schemes. This is a technique we'll use repeatedly. One thing we need in order to apply it is a formula relating the genera of the curves C and D. This is one aspect of the general theory of *liaison*, or *linkage*, of curves in $\mathbb{P}^3$.

n genus formula

Theorem 1.2.1. *Let $C \subset \mathbb{P}^3$ be a purely 1-dimensionsional subscheme of degrees c, and let $S = V(F)$ and $T = F(G)$ be surfaces of degrees s and t containing C and having no common component. If $D \subset \mathbb{P}^3$ is the subscheme defined by $\mathcal{I}_D = (F, G) : \mathcal{I}_C$ then D is purely one-dimensional and $\mathcal{I}_C = (F, G) : \mathcal{I}_D$. Furthermore $c + d = fg$ and*

d genus formula

$$p_a(C) - p_a(D) = \frac{s+t-4}{2}(c-d); \tag{1.1}$$

In words, the difference between the genera of C and D is proportional to the difference in their degrees, with constant of proportionality $(s+t-4)/2$.

We will prove Theorem v ?? in its full generality in Chapter DualityChapter ??, using a homological algebra argument. For now, we'll give a simple proof by intersection theory in a case sufficient for our needs in this chapter, and postpone the general proof to Chapter DualityChapter ??. For this, assume that C and $D \subset \mathbb{P}^3$ are smooth curves of degrees c and d with no common components. Let $S = V(F)$ and $T = V(G)$ be surfaces of degrees s and t respectively, such that that $C \cup D = S \cap T$ is a complete intersection, and assume in addition that S smooth. In this situation, Bézout's Theorem tells us that $c + d = st$; we want a formula relating the genera $g = p_a(C)$ and $h = p_a(D)$ of C and D.

To do this, we work in the Chow ring of S. By adjunction, the canonical divisor class of S is $K_S = (s-4)H)$, where H denotes the hyperplane class on S, so that by adjunction

$$2g - 2 = (C \cdot C) + (K_S \cdot C) = C \cdot C + (s-4)d,$$

or in other words,

$$(C \cdot C) = 2g - 2 - (s-4)d.$$

Next, since $C \cup D$ is a complete intersection of S with a surface of degree t, we have $C + D \sim tH$. Thus we have

$$(C \cdot D) = (C \cdot (tH - C)) = td - (C \cdot C) = td - 2g + 2 + (s-4)d$$

and similarly

$$(D \cdot D) = (D \cdot (tH - C)) = td - tc + 2g - 2 - (s-4)d.$$

Finally, we can apply the adjunction formula to D to arrive at

$$2h - 2 = (D \cdot D) + (K_S \cdot D) = (s-4)d + td - tc + 2g - 2 - (s-4)c.$$

Collecting terms, we can write this in the convenient form

ed genus formula

$$h - g = \frac{s+t-4}{2}(d-c); \tag{1.2}$$

We will see this formula used repeatedly in this chapter, and as we indicated it will be discussed as part of the larger theory of liaison for space curves in Chapter LinkageChapter ??. For now, you should just take a moment and reassure yourself that the right hand side of (linked genus formula 1.2) is indeed an integer!

1.3 Degree 4

By Clifford's Theorem an irreducible nondegenerate curve of degree 4 in $\mathbb{P}^3$ must have genus 0 or 1; we consider these cases in turn.

degree 4 genus 0

1.3.1 Genus 0

We can deal with rational quartics by a slight variant of the first method we used to deal with twisted cubics. A rational curve of degree 4 is the image of a map $\phi_F : \mathbb{P}^1 \to \mathbb{P}^3$ given by a four-tuple $F = (F_0, F_1, F_2, F_3)$ with $F_i \in H^0(\mathcal{O}_{\mathbb{P}^1}(4))$. The space of all such four-tuples up to scalars is a projective space of dimension $4 \times 5 - 1 = 19$; let $U \subset \mathbb{P}^{19}$ be the open subset of four-tuples such that the map ϕ is a nondegenerate embedding. We then have a surjective map $\pi : U \to \mathcal{H}^\circ$, whose fiber over a point C is the space of maps with image C. Since any two such maps differ by an automorphism of

$\mathbb{P}^1$—that is, an element of PGL_2—the fibers of π are three-dimensional; we conclude that $\mathcal{H}^\circ_{0,3,4}$ *is irreducible of dimension 16.*

The same analysis can be used on rational curves of any degree d: the space U of nondegenerate embeddings $\mathbb{P}^1 \to \mathbb{P}^3$ of degree d is an open subset of the projective space $\mathbb{P}^{4(d+1)-1}$ of four-tuples of homogeneous polynomials of degree d on $\mathbb{P}^1$ modulo scalars; and the fibers of the corresponding map $U \to \mathcal{H}^\circ_{dm+1}$ are copies of PGL_2. This yields the

rational curves

Proposition 1.3.1. *The open set $\mathcal{H}^\circ \subset \mathcal{H}_{0,3,d}$ parametrizing smooth, irreducible nondegenerate rational curves $C \subset \mathbb{P}^3$ is irreducible of dimension $4d$.*

dimension of rational curves

Exercise 1.3.2. Give an argument for Proposition 1.3.1 in case $d = 4$ using linkage.

twisted cubic normal bundle

One further remark. Following our discussion of twisted cubics, we were able to see in Exercise 1.1.6 that the restricted Hilbert scheme of twisted cubics is smooth by identifying the normal bundle of a twisted cubic and determining the dimension of its space of global sections. In fact, the same is true for rational curves of any degree, as the following exercise shows.

Exercise 1.3.3. Let $C \cong \mathbb{P}^1 \subset \mathbb{P}^3$ be a smooth rational curve of any degree d.

1. Show that $h^1(\mathcal{N}_{C/\mathbb{P}^3}) = 0$; that is, the normal bundle of C is nonspecial.

2. Using this, the Riemann-Roch formula for vector bundles on a curve and Proposition 1.3.1, show that the Hilbert scheme $\mathcal{H}$ is smooth at the point $[C]$. (dimension of rational curves)

We should point out that, in contrast to the case of twisted cubics, smooth rational curves in $\mathbb{P}^r$ of the same degree may have different normal bundles. This gives an interesting stratification of the restricted Hilbert scheme of rational curves; see [Coskun and Riedl 2017] for a discussion.

1.3.2 Genus 1

As we saw in Section ??, a quartic curve $C \subset \mathbb{P}^3$ of genus 1 is the intersection of two quadric surfaces, and by Lasker's theorem, every quadric containing

C is a linear combination of those two. Conversely, the intersection of two general quadrics in $\mathbb{P}^3$ is a quartic curve of genus 1. We can thus construct a family of quartics of genus 1: let $V = H^0(\mathcal{O}_{\mathbb{P}^3}(2))$ be the 10-dimensional vector space of homogeneous quadric polynomials in $\mathbb{P}^3$ and $G(2, V)$ the Grassmannian of 2-planes in V, and consider the incidence correspondence

$$\Gamma = \{(\Lambda, p) \in G(2, V) \times \mathbb{P}^3 \mid F(p) = 0 \; \forall \; F \in \Lambda\}.$$

The fiber of Γ over a point $\Lambda \in G(2, V)$ is thus the base locus of the pencil of quadrics represented by Λ; let $B \subset G(2, V)$ be the Zariski open subset over which the fiber is smooth, irreducible and nondegenerate of dimension 1. By the universal property of Hilbert schemes, the family $\pi_1 : \Gamma_B \to U$ induces a map $\phi : B \to \mathcal{H}^\circ$ that is one-to-one on points; it follows that the reduced subscheme of $\mathcal{H}^\circ$ is birational to an open subset of the Grassmannian $G(2, 10)$, and we conclude that *$\mathcal{H}^\circ_{1,3,4}$ is irreducible of dimension 16.* Exercise 1.3.4 shows that this map is actually an isomorphism.

hilb 1,3,4 **Exercise 1.3.4.** Let $C = Q \cap Q' \subset \mathbb{P}^3$ be a smooth curve of degree 4 and genus 1. Identify the normal bundle $\mathcal{N}_{C/\mathbb{P}^3}$ of C, and use this to conclude that $\mathcal{H}^\circ_{1,3,4}$ is itself reduced, and even smooth, and thus isomorphic to an open subset of the Grassmannian $G(2, 10)$.

The argument here—where we constructed a family $\mathcal{C} \to B$ of curves of given type, and then invoked the universal property of the Hilbert scheme to get a map $B \to \mathcal{H}$ is typical in analyses of Hilbert schemes. Here here are two slightly more general cases:

ntersection open **Exercise 1.3.5.** Let $m \geq n > 0$ be two positive integers. Show that the locus $U_{n,m} \subset \mathcal{H}^\circ$ of curves $C \subset \mathbb{P}^3$ that are smooth complete intersections of surfaces of degrees n and m is an open subset of the Hilbert scheme.

section exercise **Exercise 1.3.6.** Consider the locus $U_{n,n} \subset \mathcal{H}^\circ$ of curves $C \subset \mathbb{P}^3$ that are smooth complete intersections of two surfaces of degrees n. Show that $U_{n,n}$ is isomorphic to an open subset of the Grassmannian $G(2, H^0(\mathcal{O}_{\mathbb{P}^3}(n)))$.

1.4 Degree 5

Let $C \subset \mathbb{P}^3$ be a smooth, irreducible, nondegenerate quintic curve of genus g. By Clifford's theorem the bundle $\mathcal{O}_C(1)$ must be nonspecial, so by the

Riemann-Roch theorem we must have $0 \leq g \leq 2$. We have already seen that the space $\mathcal{H}^{\circ}_{5m+1}$ of rational quintic curves is irreducible of dimension 20. We will treat the case $g = 2$ in detail, and leave the case $g = 1$ as an exercise. estimating dim hilb
This case will be covered in a different way in Section 1.6.

1.4.1 Genus 2

We have considered curves of genus 2 in Section **??**. To recap the analysis, let $C \subset \mathbb{P}^3$ be a smooth, irreducible, nondegenerate curve of degree 5 and genus 2. By the Riemann-Roch theorem, $h^0(\mathcal{O}_C(2)) = 10 - 2 + 1 = 9 < h^0(\mathcal{O}_{\mathbb{P}^3}(2)) = 10$ so the restriction map

$$H^0(\mathcal{O}_{\mathbb{P}^3}(2)) \to H^0(\mathcal{O}_C(2))$$

has a kernel. Since $\deg C = 5 > 2 \times 2$, the curve C cannot lie on two independent quadrics; thus C lies on a unique quadric surface Q. Similarly, the restriction map

$$H^0(\mathcal{O}_{\mathbb{P}^3}(3)) \to H^0(\mathcal{O}_C(3))$$

has at least a 6-dimensional kernel; since cubics of the form LQ span only a 4-dimensional space, we see that C lies on a cubic surface S not containing Q. The intersection $Q \cap S$ has degree 6, and is thus the union of C and a line. If Q is smooth then, in terms of the isomorphism $Q \cong \mathbb{P}^1 \times \mathbb{P}^1$, we can say C is a curve of type $(2,3)$ on the quadric Q. Note that conversely if $L \subset \mathbb{P}^3$ is a line and Q and $S \subset \mathbb{P}^3$ are general quadric and cubic surfaces containing L, and if we write

$$Q \cap S = L \cup C$$

then the curve C is a curve of type $(2,3)$ on the quadric Q and hence, by the adjunction formula, a quintic of genus 2.

This suggests two ways of describing the family $\mathcal{H}^{\circ} \subset \mathcal{H}_{5m-1}$ of such curves. First, we can use the fact that C is linked to a line to make an incidence correspondence

$$\Psi = \{(C, L, Q, S) \in \mathcal{H}^{\circ} \times \mathbb{G}(1,3) \times \mathbb{P}^9 \times \mathbb{P}^{19} \mid Q \cap S = C \cup L\},$$

where the $\mathbb{P}^9$ (respectively, $\mathbb{P}^{19}$) is the space of quadric (respectively, cubic) surfaces in $\mathbb{P}^3$. Given a line $L \in \mathbb{G}(1,3)$, the space of quadrics containing L is a $\mathbb{P}^6$, and the space of cubics containing L is a $\mathbb{P}^{15}$; thus the fiber of the projection $\pi_2 : \Psi \to \mathbb{G}(1,3)$ over L is an open subset of $\mathbb{P}^6 \times \mathbb{P}^{15}$, and we see that Ψ *is irreducible of dimension* $4 + 6 + 15 = 25$.

On the other hand, the fiber of Ψ over a point $C \in \mathcal{H}^\circ$ is an open subset of the $\mathbb{P}^5$ of cubics containing C; and we conclude that $\mathcal{H}^\circ$ *is irreducible of dimension* 20.

Exercise 1.4.1. Let $C \subset \mathbb{P}^3$ be a smooth curve of degree 5 and genus 2, and assume that the quadric surface Q containing C is smooth. From the exact sequence

$$0 \to \mathcal{N}_{C/Q} \to \mathcal{N}_{C/\mathbb{P}^3} \to \mathcal{N}_{Q/\mathbb{P}^3}|_C \to 0,$$

calculate $h^0()$ and deduce that $\mathcal{H}^\circ_{2,3,5}$ *is smooth at the point* $[C]$. Does this conclusion still hold if Q is singular?

Another, in some ways more direct, approach to describing the restricted Hilbert scheme $\mathcal{H}^\circ_{2,3,5}$ would be to use the fact that the quadric surface Q containing a quintic curve $C \subset \mathbb{P}^3$ of genus 2 is unique. We thus have a map

$$\mathcal{H}^\circ \to \mathbb{P}^9,$$

whose fiber over a point $Q \in \mathbb{P}^9$ is the space of quintic curves of genus 2 on Q.

The problem is, the space of quintic curves of genus 2 on a given quadric Q is not in general irreducible: for a general, and thus smooth quadric Q it consists of the disjoint union of the open subsets of smooth elements in the two linear series of curves of type $(2,3)$ and $(3,2)$ on Q, each of which is a $\mathbb{P}^{11}$. We can conclude immediately that $\mathcal{H}^\circ$ is of pure dimension 20; but to conclude that it is irreducible we need to verify that, in the family of all smooth quadric surfaces, the monodromy exchanges the two rulings.
((refer to the place—earlier—where monodromy is discussed, and say this follows from the irreducibility of an appropriately modified incidence correspondence. Do this example where the monodromy if first discussed, too.))
This is not hard: it amounts to the assertion that the family

$$\Gamma = \{(Q, L) \in \mathbb{P}^9 \times \mathbb{G}(1,3) \mid L \subset Q\}$$

is irreducible, which can be seen via projection on the second factor.

Exercise 1.4.2. Show that a smooth, irreducible, nondegenerate curve $C \subset \mathbb{P}^3$ of degree 5 and genus 1 is residual to a rational quartic in the complete intersection of two cubics, and use the result of subsection degree 4 genus 0 1.3.1 to deduce that the space of genus 1 quintics is irreducible of dimension 20.

1.5 Degree 6

Again the Clifford and Riemann-Roch theorems suffice to compute the possible genera of a curve of degree 6. To start with, if the line bundle $\mathcal{O}_C(1)$ is nonspecial, then by the Riemann-Roch theorem we have $g \leq 3$. Suppose on the other hand that $\mathcal{O}_C(1)$ is special. Since $h^0(\mathcal{O}_C(1)) \geq 4$, we have equality in Clifford's theorem, and either C is hyperelliptic and $\mathcal{O}_C(1)$ is a multiple of the g^1_2 or C is a canonically embedded curve of genus 4. The first case cannot occur, since no special multiple of the hyperelliptic series of degree $\leq 2g-2$ can be very ample; thus C must be a canonical curve of genus 4. In sum, by applying Clifford's Theorem and the Riemann-Roch Theorem, we see that a smooth irreducible, nondegenerate curve of degree 6 in $\mathbb{P}^3$ has genus at most 4.

Exercise 1.5.1.

1. Show that all genera $g \leq 4$ do occur; that is, there exists a smooth irreducible, nondegenerate curve of degree 6 and genus g in $\mathbb{P}^3$ for all $g \leq 4$.

2. What is the largest possible genus of a smooth irreducible, nondegenerate curve $C \subset \mathbb{P}^3$ of degree $d = 7$? Can you do this with Clifford and Riemann-Roch, or do you need to invoke Castelnuovo?

The cases of genera 0, 1 and 2 are covered under Proposition nonspecial Hilbert 1.6.1, leaving us the cases $g = 3$ and 4. Both are well-handled by the Cartesian approach of describing their ideals.

1.5.1 Genus 4

As we've seen in Section ???? ?? a canonical curve of genus 4 is the complete intersection of a (unique) quadric Q and a cubic surface S. We thus have a map

$$\alpha : \mathcal{H}^\circ \longrightarrow \mathbb{P}^9$$

sending a curve C to the quadric Q containing it. Moreover, the fibers of this map are open subsets of the projective space $\mathbb{P}V$, where V is the quotient

$$V = \frac{H^0(\mathcal{O}_{\mathbb{P}^3}(3))}{H^0(\mathcal{I}_{Q/\mathbb{P}^3}(3))}$$

of the space of all cubic polynomials modulo cubics containing Q. Since this vector space has dimension 16, the fibers of α are irreducible of dimension 15, and we deduce that *the space $\mathcal{H}^\circ_{6m-3}$ is irreducible of dimension 24.*

first complete intersection exercise

In fact, Exercise 1.3.6 can be generalized in this way to smooth complete intersections of surfaces of any degree:

section exercise

Exercise 1.5.2. As before, let $U_{n,m} \subset \mathcal{H}^\circ$ be the locus of curves $C \subset \mathbb{P}^3$ that are smooth complete intersections of surfaces of degrees n and m. In case $m > n$, show that $U_{m,n}$ is isomorphic to an open subset of a projective bundle over the projective space $\mathbb{P}(H^0(\mathcal{O}_{\mathbb{P}^3}(n))) \cong \mathbb{P}^{\binom{n+3}{3}-1}$ of surfaces of degree n, with fiber over the point $[S] \in \mathbb{P}(H^0(\mathcal{O}_{\mathbb{P}^3}(n)))$ the projective space $\mathbb{P}(H^0(\mathcal{O}_{\mathbb{P}^3}(m))/H^0(\mathcal{I}_{S/\mathbb{P}^3}(m)) \cong \mathbb{P}^{\binom{m+3}{3}-\binom{m-n+3}{3}-1}$

1.5.2 Genus 3

We leave this to the reader to complete as follows:

Exercise 1.5.3. Let C be a curve of degree 6 and genus 3, and assume that C does not lie on any quadric surface. Show that C is residual to a twisted cubic in the complete intersection of two cubic surfaces, and use this to deduce that the space of such curves is irreducible of dimension 24.

Exercise 1.5.4. Now let C again be a curve of degree 6 and genus 3, but now assume that C *does* lie on a quadric surface Q. Show that such a curve is a flat limit of curves of the type described in the last exercise, and conclude that $\mathcal{H}^\circ_{3,3,6}(\mathbb{P}^3)$ is irreducible of dimension 24. (Hint: Let L, Q and F denote a general linear form, a general quadratic form and a general cubic form, and consider the pencil of surfaces $S_t = V(tF + LQ) \subset \mathbb{P}^3$ specializing from the cubic surface $V(F)$ the to reducible cubic $V(LQ)$.)

1.6 Why $4d$?

mating dim hilb

The sharp-eyed reader will have noticed that, in every case analyzed so far, the Hilbert scheme parametrizing smooth curves of degree d and genus g in $\mathbb{P}^3$ has dimension $4d$. While this is not the case in general (we will see shortly an example where it fails), $4d$ is indeed the "expected dimension" from certain points of view. In the following subsections we'll describe two such computations. For the remainder of this section, we will step outside $\mathbb{P}^3$ and consider, more generally, the restricted Hilbert scheme $\mathcal{H}^\circ$ of smooth, irreducible, nondegenerate curves in $\mathbb{P}^r$.

1.6.1 Estimating $\dim \mathcal{H}^\circ$ by Brill-Noether

One method of estimating the dimension of $\mathcal{H}^\circ$ is a generalization of the proof of Proposition dimension of rational curves 1.3.1, with two additional wrinkles: First, since not all line bundles of degree d on a curve C of genus $g > 0$ are linearly equivalent, we must invoke the Picard variety $\mathrm{Pic}_d(C)$ parametrizing line bundles of degree d on a given curve C, discussed in Chapter new Jacobians chapter ??. Second, since not all curves of genus $g > 0$ are isomorphic, we must involve the moduli space M_g parametrizing abstract curves of genus g, discussed in Chapter Moduli chapter ??.

To begin with a simple example, let $\mathcal{H}^\circ$ again be the space of smooth, irreducible, nondegenerate curves $C \subset \mathbb{P}^3$ of degree 5 and genus 2. By the property of M_2 as a coarse moduli space, we get a map

$$\mu : \mathcal{H}^\circ \longrightarrow M_2.$$

To analyze the fiber $\Sigma_C = \mu^{-1}(C)$ of the map μ over a point $C \in M_2$ we first use the map

$$\nu : \Sigma_C \longrightarrow \mathrm{Pic}_5(C),$$

obtained by sending a point in Σ_C to the line bundle $\mathcal{O}_C(1)$. Proposition ** ??, implies that any line bundle of degree 5 on a curve of genus 2 is very ample, so this map is surjective. Note that $h^0(\mathcal{L}) = 4$, so the linear series giving the embedding is complete. Thus, once we have specified the abstract curve C, and the line bundle $\mathcal{L} \in \mathrm{Pic}_5(C)$ the embedding is determined by giving a basis for $H^0(\mathcal{L})$, up to scalars. In other words, each fiber of ν is isomorphic to PGL_4. We can now work our way up from M_2:

- We know that M_2 is irreducible of dimension 3.
- It follows that the space of pairs $(C, \mathcal{L})$ with $C \in M_2$ a smooth curve of genus 2 and $\mathcal{L} \in \mathrm{Pic}_5(C)$ is irreducible of dimension $3 + 2 = 5$; and finally
- It follows that $\mathcal{H}^\circ$ is irreducible of dimension $5 + 15 = 20$.

In fact, this approach applies to a much wider range of examples: whenever $d \geq 2g + 1$ and $r \leq d - g$, we can look at the tower of spaces

$$\begin{array}{c} \mathcal{H}^\circ = \mathcal{H}^\circ_{dm-g+1}(\mathbb{P}^r) \\ \big\downarrow \\ \mathcal{P}_{d,g} = \{(C, \mathcal{L}) \mid \mathcal{L} \in \mathrm{Pic}_d(C)\} \\ \big\downarrow \\ M_g. \end{array}$$

Exactly as in the special case $(d, g, r) = (5, 2, 3)$ above, we can work our way up the tower:

- M_g is irreducible of dimension $3g - 3$;
- it follows from the fact that the Picard variety is irreducible of dimension g that $\mathcal{P}_{d,g}$ is irreducible of dimension $3g - 3 + g = 4g - 3$; and finally
- since the fibers of $\mathcal{H}^\circ \to \mathcal{P}_{d,g}$ consist of $(r + 1)$-tuples of linearly independent sections of $\mathcal{L}$ (mod scalars), it follows that $\mathcal{H}^\circ$ is irreducible of dimension $4g - 3 + (r + 1)(d - g + 1) - 1$.

In sum, we have the

special Hilbert **Proposition 1.6.1.** *Whenever $d \geq 2g+1$, the space $\mathcal{H}^\circ$ of smooth, irreducible, nondegenerate curves $C \subset \mathbb{P}^r$ is either empty (if $d - g < r$) or irreducible of dimension $4g-3+(r+1)(d-g+1)-1$; in particular, if $r = 3$, the dimension of $\mathcal{H}^\circ$ is $4d$.*

Exercise 1.6.2. By analyzing the geometry of linear series of degrees $2g-1$ and $2g$ on a curve of genus g, extend Proposition nonspecial Hilbert 1.6.1 to the cases $d = 2g-1$ and $2g$. What goes wrong if $d \leq 2g-2$?

Proposition nonspecial Hilbert 1.6.1 gives a simple and clean answer to our basic questions about the dimension and irreducibility of the restricted Hilbert scheme $\mathcal{H}^\circ$ in case $d \geq 2g-1$. But what happens outside of this range? In fact, we can use Brill-Noether theory to modify this analysis to extend this beyond the range $d \geq 2g+1$.

Basically, what's different in general is that the map $\mathcal{H}^\circ \to \mathcal{P}_{d,g}$ is no longer dominant; rather, over a point $[C] \in M_g$, its image is open in the subvariety $W^r_d(C) \subset \mathrm{Pic}_d(C)$ parametrizing line bundles $\mathcal{L}$ on C of degree d with at least $r+1$ sections. Now, as long as the Brill-Noether number $\rho(d,g,r)$ is non-negative, the Brill-Noether theorem tells us that for a general curve C, the variety $W^r_d(C)$ has dimension ρ, and (assuming $r \geq 3$) the general point of $W^r_d(C)$ corresponds to a very ample line bundle with exactly $r+1$ sections. In this situation, there is a unique component of $\mathcal{H}_0 \subset \mathcal{H}^\circ$ dominating M_g, and the map $\mathcal{H}^\circ \to \mathcal{P}_{d,g}$ carries this component to a subvariety $\mathcal{W}^r_d \subset \mathcal{P}_{d,g}$ of dimension $3g-3+\rho$. In sum, then, we have the basic theorem

cipal component **Theorem 1.6.3.** *Let g, d and r be any nonnegative integers, with Brill-Noether number $\rho(g,r,d) = g-(r+1)(g-d+r) \geq 0$. There is then a unique component $\mathcal{H}_0$ of the restricted Hilbert scheme $\mathcal{H}^\circ_{g,r,d}$ dominating the moduli space M_g; and this component has dimension*

$$\dim \mathcal{H}_0 = 3g-3+\rho+(r+1)^2-1 = 4g-3+(r+1)(d-g+1)-1.$$

The component $\mathcal{H}_0$ identified in Theorem principal component 1.6.3 is called the *principal component* of the Hilbert scheme; there may be others as well, of possibly different dimension, and we do not know precisely for which d, g and r these occur. Finally, in case $\rho < 0$, the Brill-Noether theorem tells us only that there is no component of $\mathcal{H}^\circ_{g,r,d}$ dominating M_g; we'll discuss some of the outstanding questions in this range in Section open problems ?? below.

1.6.2 Estimating $\dim \mathcal{H}^\circ$ by the Euler characteristic of the normal bundle

It is interesting to compare the estimate of $\dim \mathcal{H}^\circ$ above with what we get from deformation theory. Let $\mathcal{H}$ be a component of the scheme $\mathcal{H}^\circ$, with $C \subset \mathbb{P}^r$ a curve corresponding to a general point $[C]$ of $\mathcal{H}$.

We start with the idea that the dimension of the scheme $\mathcal{H}$ is approximated by the dimension of its Zariski tangent space $T_{[C]}\mathcal{H}$ at a general point $[C]$. In Section ?? we saw that the tangent space to $\mathcal{H}$ at $[C]$ is the space $H^0(\mathcal{N}_{C/\mathbb{P}^r})$ of global sections of the normal bundle $\mathcal{N} = \mathcal{N}_{C/\mathbb{P}^r}$. We can think of the dimension $h^0(\mathcal{N})$ as approximated by the Euler characteristic $\chi(\mathcal{N})$, with "error term" $h^1(\mathcal{N})$ coming from its first cohomology group.

Given these two approximations, we arrive at a number we can compute. From the exact sequence

$$0 \to T_C \to T_{\mathbb{P}^r}|_C \to \mathcal{N} \to 0$$

we deduce that

$$\begin{aligned} c_1(\mathcal{N}) &= c_1(T_{\mathbb{P}^r}|_C) - c_1(T_C) \\ &= (r+1)d - (2-2g). \end{aligned}$$

Now we can apply the Riemann-Roch Theorem for vector bundles on curves ([Eisenbud and Harris 2016, Theorem ???]) to conclude that

$$\begin{aligned} \chi(\mathcal{N}) &= c_1(\mathcal{N}) - \operatorname{rank}(\mathcal{N})(g-1) \\ &= (r+1)d - (r-3)(g-1). \end{aligned}$$

Note that our two "estimates" are actually inequalities. But, unfortunately, they go in opposite directions: we have

$$\dim \mathcal{H} \le \dim T_{[C]}\mathcal{H},$$

but

$$\dim T_{[C]}\mathcal{H} \ge \chi(\mathcal{N}).$$

Nonetheless, one can show that if $C \subset \mathbb{P}^r$ is a smooth curve then the versal deformation space of $C \subset \mathbb{P}^r$ has dimension at least $\chi(\mathcal{N})$. If we consider the family of Picard varieties over the family of smooth curves in a neighborhood of C and we can deduce that for any component of $\mathcal{H}^\circ$ containing C we have

$$\dim \mathcal{H}^\circ \ \ge \ (r+1)d - (r-3)(g-1)$$

1.6.3 They're the same!

nonspecial Hilbert

Proposition 1.6.1 suggests that the "expected dimension" of the restricted Hilbert scheme $\mathcal{H}^\circ$ of curves of degree d and genus g in $\mathbb{P}^r$ should be

$$h(g,r,d) := 4g - 3 + (r+1)(d-g+1) - 1.$$

But the calculation immediately above suggests it should be $(r+1)d - (r-3)(g-1)$. Which is it? The answer is both: they're the same number!

Bibliography

Riedl Coskun and Riedl 2017] Coskun and Riedl, "Normal bundles of rational curves in projective space", *arxiv:1607.06149v2* (2017).

GeomSchemes Eisenbud and Harris 2000] D. Eisenbud and J. Harris, *The geometry of schemes*, vol. 197, Graduate Texts in Mathematics, Springer-Verlag, New York, 2000.

3264 Eisenbud and Harris 2016] D. Eisenbud and J. Harris, *3264 and all that—a second course in algebraic geometry*, Cambridge University Press, Cambridge, 2016. Available at `https://doi-org.libproxy.berkeley.edu/10.1017/CBO9781139062046`.

Montreal Harris 1982] J. Harris, *Curves in projective space*, vol. 85, Séminaire de Mathématiques Supérieures [Seminar on Higher Mathematics], Presses de l'Université de Montréal, Montreal, Que., 1982. With the collaboration of David Eisenbud.

Iarrobino1985 Iarrobino 1985] A. Iarrobino, "Compressed algebras and components of the punctual Hilbert scheme", pp. 146–165 in *Algebraic geometry* (Sitges, Barcelona, 1983), edited by E. Casas-Alvero, Lecture Notes in Math. **1124**, Springer, Berlin, 1985.