

## MAXIMALLY INFLECTED REAL RATIONAL CURVES

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*Dedicated to V. I. Arnold on the occasion of his 65th birthday*

**ABSTRACT.** We begin the topological study of real rational plane curves all of whose inflection points are real. The existence of such curves is implied by the results of real Schubert calculus, and their study has consequences for the important Shapiro and Shapiro conjecture in real Schubert calculus. We establish restrictions on the number of real nodes of such curves and construct curves realizing the extreme numbers of real nodes. These constructions imply the existence of real solutions to some problems in Schubert calculus. We conclude with a discussion of maximally inflected curves of low degree.

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### INTRODUCTION

In 1876, Harnack [15] established the bound of  $g + 1$  for the number of ovals of a smooth curve of genus  $g$  in  $\mathbb{RP}^2$  and showed that this bound is sharp by constructing curves with  $g + 1$  ovals. Since then, such curves with maximally many ovals, or  $M$ -curves, have been primary objects of interest in the topological study of real algebraic plane curves (part of 16th problem of Hilbert) [44], [42]. We begin the study of maximally inflected curves, which can be considered analogues of  $M$ -curves among rational curves in the sense that, like the classical maximal condition, maximal inflection implies non-trivial restrictions on the topology. The case of a maximally inflected curve that we are most interested in is a rational plane curve of degree  $d$  all of whose  $3(d-2)$  inflection points are real. More generally, consider a parameterization of a rational curve of degree  $d$  in  $\mathbb{P}^r$  with  $d > r$ ; this is a map from  $\mathbb{P}^1$  to  $\mathbb{P}^r$  of degree  $d$ . Such a map is said to be *ramified* at a point  $s \in \mathbb{P}^1$  if its first  $r$  derivatives do not span  $\mathbb{P}^r$ . Over  $\mathbb{C}$ , there always are  $(r + 1)(d - r)$  such points of ramification, counting multiplicities. A *maximally inflected curve* is, by definition, a parameterized real rational curve all of whose ramification points are real.

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We first address the question of the existence of maximally inflected curves. The conjecture of B. Shapiro and M. Shapiro in real Schubert calculus [39] would imply the existence of maximally inflected curves for every possible arrangement of ramification points. Eremenko and Gabrielov [9] proved the conjecture of Shapiro and Shapiro in the special case of  $r = 1$ . In this case, a maximally inflected curve is a real rational function, all of whose critical points are real.

When  $r > 1$ , for special types of ramification (including flexes and cusps of plane curves), and when the ramification points are clustered very near to one another, there do exist maximally inflected curves by a result of real Schubert calculus [38]. When the degree  $d$  is even and the curve has only simple but arbitrarily arranged flexes, the existence of such curves follows from the results of Eremenko and Gabrielov on the degree of the Wronski map [8].

Suppose that  $r = 2$  (this is the case of plane curves). Consider a maximally inflected plane curve of degree  $d$  having the maximum number  $(3(d-2))$  of real flexes. By the genus formula, the curve has at most  $\binom{d-1}{2}$  ordinary double points. We deduce from the Klein [21] and Plücker [26] formulas that the number of real nodes of such a curve is in fact at most  $\binom{d-2}{2}$ . Then we show that this bound is sharp. We use Shustin's theorem on combinatorial patchworking for singular curves [33] to construct maximally inflected curves that realize the lower bound of 0 real nodes and another theorem of Shustin concerning deformations of singular curves [36] to construct maximally inflected curves for which the upper bound of  $\binom{d-2}{2}$  real nodes is attained. We also classify such curves in degree 4 and discuss some aspects of the classification of quintics.

The study of the relation between Schubert calculus and rational curves in projective space (linear series on  $\mathbb{P}^1$ ) was initiated by Castelnuovo [4] for  $g$ -nodal rational curves. This led to the application of Schubert calculus to Brill–Noether theory (see Chapter 5 of [16] for details). In a closely related vein, Schubert calculus plays a role in the local study of flattenings of curves in singularity theory [31], [19], [1]. In turn, the theory of limit linear series of Eisenbud and Harris [6], [7] provides essential tools for proving the reality of the special Schubert calculus [38], which implies the existence of many types of maximally inflected curves. The constructions suggested in this paper, which use patchworking and gluing, show the existence of real solutions to some problems in Schubert calculus. In particular, they show the existence of maximally inflected plane curves with a given type of ramification when the degree  $d$  is odd, extending the results of Eremenko and Gabrielov [8].

This paper is organized as follows. In Section 1, we describe the connection between the Schubert calculus and maps from  $\mathbb{P}^1$  with specified points of ramification. In Section 2, we show the existence of some special types of maximally inflected curves and discuss the conjecture of Shapiro and Shapiro in this context. Section 3 concentrates on maximally inflected plane curves and gives bounds for the number of solitary points and real nodes. In Section 4, we construct maximally inflected curves with extreme numbers of real nodes and in Section 5, discuss the classification of quartics. Section 6 considers some aspects of the classification of quintics.

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## 1. SINGULARITIES OF PARAMETERIZED CURVES

**1.1. Notation and conventions.** Given non-zero vectors  $v_1, v_2, \dots, v_n$  in  $\mathbb{A}^{r+1}$ , we denote their linear span in the projective space  $\mathbb{P}^r$  by  $\langle v_1, v_2, \dots, v_n \rangle$ . A subvariety  $X$  of  $\mathbb{P}^r$  is *non-degenerate* if its linear span is  $\mathbb{P}^r$ , or, equivalently, if it does not lie in a hyperplane. A rational map between varieties is denoted by a broken arrow  $X \dashrightarrow Y$ .

Let  $d > r$ . The center of a (surjective) linear projection  $\mathbb{P}^d \dashrightarrow \mathbb{P}^r$  is a  $(d - r - 1)$ -dimensional linear subspace of  $\mathbb{P}^d$ . This center determines the projection up to projective transformations of  $\mathbb{P}^r$ . Consequently, we identify the Grassmannian  $\text{Gr}_{d-r-1}\mathbb{P}^d$  of  $(d - r - 1)$ -dimensional linear subspaces of  $\mathbb{P}^d$  with the space of linear projections  $\mathbb{P}^d \dashrightarrow \mathbb{P}^r$  (modulo the projective transformations of  $\mathbb{P}^r$ ).

A flag  $F_\bullet$  in  $\mathbb{P}^d$  is a sequence

$$F_\bullet: F_0 \subset F_1 \subset \dots \subset F_d = \mathbb{P}^d$$

of linear subspaces with  $\dim F_i = i$ . Given a flag  $F_\bullet$  in  $\mathbb{P}^d$  and a sequence  $\alpha: 0 \leq \alpha_0 < \alpha_1 < \dots < \alpha_r \leq d$  of integers, the *Schubert cell*  $X_\alpha^\circ F_\bullet \subset \text{Gr}_{d-r-1}\mathbb{P}^d$  is the set of all linear projections  $\pi: \mathbb{P}^d \dashrightarrow \mathbb{P}^r$  such that

$$\dim \pi(F_{\alpha_i}) = i, \quad \text{and} \quad \dim \pi(F_{\alpha_{i-1}}) = i - 1 \quad \text{if } i \neq 0. \quad (1.1)$$

The Schubert variety  $X_\alpha F_\bullet$  is the closure of the Schubert cell; it is obtained by replacing the equalities in (1.1) by inequalities ( $\leq$ ).

Replacing  $\mathbb{P}^d$  by the dual projective space  $\hat{\mathbb{P}}^d$  gives an isomorphism  $\text{Gr}_{d-r-1}\mathbb{P}^d \simeq \text{Gr}_r\hat{\mathbb{P}}^d$ ; a  $(d - r - 1)$ -plane  $\Lambda$  in  $\mathbb{P}^d$  corresponds to the  $r$ -plane in  $\hat{\mathbb{P}}^d$  consisting of the hyperplanes in  $\mathbb{P}^d$  that contain  $\Lambda$ . Under this isomorphism, the Schubert variety  $X_\alpha F_\bullet$  is mapped isomorphically to the Schubert variety  $X_{\hat{\alpha}} \hat{F}_\bullet$ , where  $\hat{F}_\bullet$  is the flag dual to  $F_\bullet$  ( $\hat{F}_i$  consists of the hyperplanes containing  $F_{d-i}$ ) and  $\hat{\alpha}$  is the sequence  $0 \leq \beta_0 < \beta_1 < \dots < \beta_{d-r-1} \leq d$  such that

$$\{0, 1, \dots, d\} = \{\alpha_0, \alpha_1, \dots, \alpha_r\} \cup \{d - \beta_0, d - \beta_1, \dots, d - \beta_{d-r-1}\}. \quad (1.2)$$

**1.2. Parameterized rational curves.** Any  $r + 1$  coprime linearly independent binary homogeneous forms of the same degree  $d$  give a morphism  $\varphi: \mathbb{P}^1 \rightarrow \mathbb{P}^r$  whose image is non-degenerate and of degree  $d$  in the sense that  $\varphi_*([\mathbb{P}^1]) = d[\ell]$ , where  $[\ell]$  is the standard generator in  $H_2(\mathbb{P}^r)$ . Reciprocally, every morphism from  $\mathbb{P}^1$  to  $\mathbb{P}^r$  whose image is non-degenerate and of degree  $d$  is given by  $r + 1$  coprime linearly independent binary homogeneous forms of the same degree  $d$ . We consider two such maps to be equivalent when they differ by a projective transformation of  $\mathbb{P}^r$ . In what follows, we call an equivalence class of such maps a *rational curve of degree  $d$  in  $\mathbb{P}^r$* .

A rational curve  $\varphi: \mathbb{P}^1 \rightarrow \mathbb{P}^r$  of degree  $d$  is said to be *ramified* at  $s \in \mathbb{P}^1$  if the derivatives  $\varphi(s), \varphi'(s), \dots, \varphi^{(r)}(s) \in \mathbb{A}^{r+1}$  do not span  $\mathbb{P}^r$ . This occurs at the roots of the Wronskian

$$\det \begin{bmatrix} \varphi_0(s) & \dots & \varphi_r(s) \\ \varphi'_0(s) & \dots & \varphi'_r(s) \\ \dots & \dots & \dots \\ \varphi_0^{(r)}(s) & \dots & \varphi_r^{(r)}(s) \end{bmatrix},$$

of  $\varphi$ , which is a form of degree  $(r+1)(d-r)$ . (Here,  $\varphi_0, \dots, \varphi_r$  are the components of  $\varphi$ .) The Wronskian is defined by the curve  $\varphi$  up to multiplication by a scalar; in particular, its roots and their multiplicities are well-defined in the equivalence class of  $\varphi$ . An equivalence class of such maps containing a real map  $\varphi$  is a *maximally inflected curve* when the Wronskian has only real roots.

An equivalent definition can be given in terms of linear series on  $\mathbb{P}^1$ . A rational curve  $\varphi: \mathbb{P}^1 \rightarrow \mathbb{P}^r$  of degree  $d$  with non-degenerate image may be factored as

$$\mathbb{P}^1 \xrightarrow{\gamma} \mathbb{P}^d \xrightarrow{\pi} \mathbb{P}^r, \quad (1.3)$$

where  $\gamma$  is the *rational normal curve*, which is given by

$$[s, t] \mapsto [s^d, s^{d-1}t, s^{d-2}t^2, \dots, st^{d-1}, t^d],$$

and  $\pi$  is a projection whose center  $\Lambda$  is a  $(d-r-1)$ -plane disjoint from the curve  $\gamma$ . The hyperplanes of  $\mathbb{P}^d$  which contain  $\Lambda$  constitute a base point free linear series of dimension  $r$  in  $\mathbb{P}H^0(\mathbb{P}^1, \mathcal{O}(d)) = \mathbb{P}^d$ . (Base point freeness is equivalent to our requirement that the original forms  $\varphi_i$  have no common factor.) The center of projection  $\Lambda$  and hence the linear series depends only upon the original map. Thus, the space of rational curves  $\varphi: \mathbb{P}^1 \rightarrow \mathbb{P}^r$  of degree  $d$  is identified with the (open) subset of the Grassmannian  $\text{Gr}_{d-r-1} \mathbb{P}^d$  formed by the  $(d-r-1)$ -planes in  $\mathbb{P}^d$  which do not meet the image of the rational normal curve  $\gamma$ .

Let us define the *ramification sequence* of  $\varphi$  at a point  $s$  in  $\mathbb{P}^1$  to be the vector  $\alpha = \alpha(s)$  whose  $i$ th component is the smallest number  $\alpha_i$  such that the linear span of  $\varphi(s), \varphi'(s), \dots, \varphi^{(\alpha_i)}(s)$  in  $\mathbb{P}^r$  has dimension  $i$ . Equivalently,  $\mathbb{P}^r$  has coordinates such that  $\varphi = [\varphi_0, \varphi_1, \dots, \varphi_r]$  with  $\varphi_i$  vanishing to order  $\alpha_i$  at  $s$ . We will also call this sequence  $\alpha$  the *ramification* of  $\varphi$  at  $s$ . Note that  $\alpha_0 = 0$  and  $d \geq \alpha_i \geq i$  for  $i = 0, \dots, r$ , and the sequence  $\alpha$  is increasing. We call such an integer vector  $\alpha$  a *ramification sequence* for degree  $d$  curves in  $\mathbb{P}^r$  if  $\alpha_r > r$ . If  $s$  is a point of  $\mathbb{P}^1$  with ramification sequence  $\alpha$ , then the Wronskian vanishes to order  $|\alpha| := (\alpha_1 - 1) + (\alpha_2 - 2) + \dots + (\alpha_r - r)$  at  $s$ . When this is zero, i.e.,  $\alpha_r = r$ , we say that  $\varphi$  is *unramified* at  $s$ . Since the Wronskian has degree  $(r+1)(d-r)$ , the following assertion holds.

**Proposition 1.1.** *Let  $\varphi: \mathbb{P}^1 \rightarrow \mathbb{P}^r$  be a rational curve of degree  $d$ . Then*

$$\sum_{s \in \mathbb{P}^1} |\alpha(s)| = (r+1)(d-r). \quad \square$$

A collection  $\alpha^1, \dots, \alpha^n$  of ramification sequences for degree  $d$  curves in  $\mathbb{P}^r$  such that  $|\alpha^1| + \dots + |\alpha^n| = (r+1)(d-r)$  is called *ramification data* for degree  $d$  rational curves in  $\mathbb{P}^r$ .

A ramification sequence is expressed in terms of Schubert cycles in  $\text{Gr}_{d-r-1}\mathbb{P}^d$  as follows. For each  $s \in \mathbb{P}^1$ , let  $F_\bullet(s)$  be the flag of subspaces of  $\mathbb{P}^d$  which osculate the rational normal curve  $\gamma$  at the point  $\gamma(s)$ , and let  $\pi$  the linear projection defining  $\varphi$  as in (1.3). Then the image  $\pi(F_i(s))$  of the  $i$ -plane  $F_i(s)$  in  $F_\bullet(s)$  is the linear span of  $\varphi(s), \varphi'(s), \dots, \varphi^{(i)}(s)$  in  $\mathbb{P}^r$ .

Comparing the definitions of ramification sequences and of Schubert cells, we deduce that the curve  $\varphi$  has ramification  $\alpha$  at  $s \in \mathbb{P}^1$  when the center of the projection lies in the Schubert cell  $X_\alpha^\circ F_\bullet(s)$  of the Grassmannian. The closure of this cell is the Schubert cycle  $X_\alpha F_\bullet(s)$ , which has codimension  $|\alpha|$  in the Grassmannian. Thus, given ramification data  $\alpha^1, \dots, \alpha^n$  and distinct points  $s_1, \dots, s_n \in \mathbb{P}^1$ , a center  $\Lambda$  provides the ramification  $\alpha^i$  at the point  $s_i$  for each  $i$  if and only if it belongs to the intersection

$$\bigcap_{i=1}^n X_{\alpha^i}^\circ F_\bullet(s_i).$$

On the other hand,

$$\bigcap_{i=1}^n X_{\alpha^i}^\circ F_\bullet(s_i) = \bigcap_{i=1}^n X_{\alpha^i} F_\bullet(s_i).$$

To see this, suppose that a point in  $\bigcap_{i=1}^n X_{\alpha^i}^\circ F_\bullet(s_i)$  does not lie in a Schubert cell  $X_{\alpha^i}^\circ F_\bullet(s_i)$ . Then the corresponding rational curve has ramification exceeding  $\alpha^i$  at the point  $s_i$ , which contradicts Proposition 1.1.

The intersection of the above Schubert cycles has dimension at least  $(r+1)(d-r) - \sum_i |\alpha^i| = 0$ . If it had positive dimension, then it would have a non-empty intersection with any hypersurface Schubert variety  $X_\beta F_\bullet(s)$ , where  $|\beta| = 1$  and  $s$  is not among  $\{s_1, \dots, s_n\}$ . This gives a rational curve whose existence contradicts Proposition 1.1. Thus the intersection is zero-dimensional. Let  $N(\alpha^1, \dots, \alpha^n)$  be its degree, which can be computed by using the classical Schubert calculus of enumerative geometry [12]. Thus, we obtain the following corollary to Proposition 1.1.

**Corollary 1.2.** *Given a  $d > r$ , ramification data  $\alpha^1, \dots, \alpha^n$  for degree  $d$  rational curves in  $\mathbb{P}^r$ , and distinct points  $s_1, \dots, s_n \in \mathbb{P}^1$ , the number of non-degenerate rational curves in  $\mathbb{P}^r$  of degree  $d$  with ramification  $\alpha^i$  at the point  $s_i$  for each  $1 \leq i \leq n$  is  $N(\alpha^1, \dots, \alpha^n)$ , counting multiplicities.*  $\square$

Let us consider some special cases. A curve  $\varphi$  has a *cusp* at  $s$  if the center of the projection  $\Lambda$  meets the tangent line to the rational normal curve  $\gamma$  at  $\gamma(s)$ . The corresponding Schubert cycle has codimension  $r$ , and so a rational curve of degree  $d$  in  $\mathbb{P}^r$  has at most  $(d-r)(r+1)/r$  cusps. A curve  $\varphi$  is *simply ramified* (or has a *flex*) at  $s$  if  $\varphi(s), \varphi'(s), \dots, \varphi^{(r-1)}(s)$  span a hyperplane in  $\mathbb{P}^r$  which contains  $\varphi^{(r)}(s)$ . This occurs if the center of the projection  $\Lambda$  meets the osculating  $r$ -plane  $F_r(s)$  in a point. In this case, the codimension is 1, and so a rational curve with only flexes has  $(r+1)(d-r)$  flexes.

## 2. MAXIMALLY INFLECTED CURVES

**2.1. Existence.** Consider the following question.

**Question 2.1.** Given a  $d > r$ , ramification data  $\alpha^1, \dots, \alpha^n$  for degree  $d$  rational curves in  $\mathbb{P}^r$ , and distinct points  $s_1, \dots, s_n \in \mathbb{RP}^1$ , are there any *real* rational curves  $\varphi: \mathbb{P}^1 \rightarrow \mathbb{P}^r$  of degree  $d$  with ramification  $\alpha^i$  at the point  $s_i$  for each  $i = 1, \dots, n$ ?

Corollary 1.2 guarantees the existence of  $N(\alpha^1, \dots, \alpha^n)$  such *complex* rational curves, counting multiplicities. So the question is: if the ramification occurs at real points in the domain  $\mathbb{P}^1$ , are any of the resulting curves real? We call a real rational curve whose ramifications occur only at real points a *maximally inflected* curve.

In the general case, the answer to Question 2.1 is unknown. There are however many cases for which the answer is yes. Moreover, one, still open, conjecture of Shapiro and Shapiro in real Schubert calculus would imply a very strong result answering Question 2.1. We formulate this conjecture in terms of maximally inflected curves.

**Conjecture 2.2** (Shapiro–Shapiro). Let  $d > r$  be an integer, and let  $\alpha^1, \dots, \alpha^n$  be ramification data for degree  $d$  rational curves in  $\mathbb{P}^r$ . For any choice of distinct real points  $s_1, \dots, s_n \in \mathbb{RP}^1$ , every curve  $\varphi: \mathbb{P}^1 \rightarrow \mathbb{P}^r$  of degree  $d$  with ramification  $\alpha^i$  at  $s_i$  for each  $i = 1, \dots, n$  is real.

Eremenko and Gabrielov [9] proved this conjecture in the cases when  $r$  is 1 or  $d - 2$ . It is also known to be true for a few sporadic cases of ramification data, and some cases of the conjecture imply others. There is substantial computational evidence in support of Conjecture 2.2 and no known counterexamples. For an account of this conjecture, see [39] or the web page [37].

A ramification sequence is *special* if it is

$$(0, 1, \dots, r-1, r+a) \quad \text{or} \quad (0, 1, \dots, r-a, r-a+2, \dots, r+1)$$

for some  $a > 0$ . When  $a = 1$ , these sequences coincide and give the sequence of a flex. We can guarantee the existence of maximally inflected curves with these special singularities.

**Theorem 2.3.** Suppose that  $\alpha^1, \dots, \alpha^n$  are ramification data for degree  $d$  rational curves in  $\mathbb{P}^r$  with no more than two of the sequences  $\alpha^i$  not special. Then there do exist distinct points  $s_1, \dots, s_n \in \mathbb{RP}^1$  such that there are exactly  $N(\alpha^1, \dots, \alpha^n)$  real rational curves of degree  $d$  which have ramification  $\alpha^i$  at  $s_i$  for each  $i = 1, \dots, n$ .

The point of this theorem is that there are the expected number of such curves, and *all* of them are real.

*Proof.* Let  $\alpha^1, \dots, \alpha^n$  be ramification data for degree  $d$  rational curves in  $\mathbb{P}^r$  with at most two sequences  $\alpha^i$  not special. These are types of Schubert varieties in the Grassmannian of  $(d-r-1)$ -planes in  $\mathbb{P}^d$ , and all except possible two are special. By Theorem 1 of [38], there exist points  $s_1, \dots, s_n \in \mathbb{RP}^1$  so that the Schubert varieties  $X_{\alpha^i} F_{\bullet}(s_i)$  defined by flags osculating the rational normal curve at points  $s_i$  intersect transversally with all points of intersection real. Every  $(d-r-1)$ -plane  $\Lambda$  in such an intersection is a center of projection giving a maximally inflected real rational curve  $\varphi$  with ramification sequence  $\alpha^i$  at  $s_i$  for each  $i = 1, \dots, n$ .  $\square$

The proof given in [38] provides points of ramification that are clustered together. More precisely, suppose that the sequences  $\alpha^3, \dots, \alpha^n$  are special. Let

$$\forall s_3 \ll s_4 \ll \dots \ll s_n$$

mean

$$\forall s_3 > 0 \exists \mathcal{N}_4 > 0, \text{ such that } \forall s_4 > \mathcal{N}_4 \exists \mathcal{N}_5 > 0, \dots$$

$$\forall s_{n-1} > \mathcal{N}_{n-1} \exists \mathcal{N}_n > 0, \text{ such that } \forall s_n > \mathcal{N}_n.$$

The points  $s_i$  for which all curves are real in Theorem 2.3 are

$$s_1 = 0, \quad s_2 = \infty, \quad \text{and} \quad \forall s_3 \ll s_4 \ll \dots \ll s_n.$$

In short, Theorem 2.3 only guarantees the existence of many maximally inflected curves when almost all of the ramification data are special, and the points of ramification are clustered together as described above. A modification of the proof given in [38] (using the idea of the Pieri homotopy algorithm suggested in [18]) shows that there can be two ‘clusters’ of ramification points.

Eremenko and Gabrielov [8, Corollary 4] proved the following assertion.

**Proposition 2.4** (Eremenko–Gabrielov). *Suppose that  $0 < r < d$ ,  $d$  is even,  $m := \max\{r+1, d-r\}$ ,  $p := \min\{r+1, d-r\}$ , and  $M := (r+1)(d-r)$ . Then, for any distinct points  $s_1, \dots, s_M \in \mathbb{RP}^1$ , there exist at least*

$$\frac{1!2!\dots(p-1)!(m-1)!(m-2)!\dots(m-p+1)!\left(\frac{pm}{2}\right)!}{(m-p+2)!(m-p+4)!\dots(m+p-2)!\left(\frac{m-p+1}{2}\right)!\left(\frac{m-p+3}{2}\right)!\dots\left(\frac{m+p-1}{2}\right)!} \quad (2.1)$$

*maximally inflected curves of degree  $d$  in  $\mathbb{P}^r$  with flexes at the points  $s_1, \dots, s_M$ .*

The constructions of Section 4.1 show that there exist maximally inflected plane curves of any degree  $d$  with any number (up to  $d-2$ ) of cusps under some choices of ramification points not clustered together.

**Remark 2.5.** It is worthwhile to compare this number (2.1) of Eremenko–Gabrielov with the number of (complex) curves with the same flexes computed by Schubert [29]:

$$\frac{1!2!\dots(p-2)!(p-1)!(mp)!}{(m)!(m+1)!\dots(m+p-1)!}. \quad (2.2)$$

The ratio of (2.2) to (2.1) is

$$r(m, p) := \frac{\left(\frac{m-p+1}{2}\right)!\left(\frac{m-p+3}{2}\right)!\dots\left(\frac{m+p-1}{2}\right)!(mp)!}{(m-p+1)!(m-p+3)!\dots(m+p-1)!\left(\frac{pm}{2}\right)!}.$$

According to Stirling’s formula,  $\log r(m, p)$  grows as

$$mp \log \frac{mp}{2} - (m-p+1) \log \frac{m-p+1}{2} - \dots - (m+p-1) \log \frac{m+p-1}{2}.$$

Fixing  $p$  and letting  $m$  grow (that is, fixing the target  $\mathbb{P}^r$  and letting the degree  $d$  grow), we obtain the asymptotic value of  $\frac{1}{2}mp \log p$  for  $\log r(m, p)$ . Similar arguments show that, asymptotically, the logarithm of Schubert’s number grows as  $mp \log p$ . Thus, we see that the number of real solutions guaranteed by the result

of Eremenko and Gabrielov is approximately the square root of the total number of solutions, in this asymptotic limit.

This asymptotic result is reminiscent of (but different from) the results of Kostlan [23] and Shub and Smale [32] concerning the expected number of real solutions to a system of polynomial equations. The set of real polynomial systems on  $\mathbb{RP}^n$

$$f_1(x_0, x_1, \dots, x_n) = f_2(x_0, x_1, \dots, x_n) = \dots = f_n(x_0, x_1, \dots, x_n) = 0, \quad (2.3)$$

where  $f_i$  is homogeneous of degree  $d_i$ , is parameterized by  $\mathbb{RP}^{d_1} \times \mathbb{RP}^{d_2} \times \dots \times \mathbb{RP}^{d_n}$ . Integrating the number of real roots of system (2.3) against the Fubini–Study probability measure on this space of systems gives the expected number of real roots

$$(d_1 d_2 \cdots d_n)^{\frac{1}{2}},$$

which is the square root of the expected number  $d_1 d_2 \cdots d_n$  of complex roots.

**2.2. Deformations.** Deforming a maximally inflected curve  $\varphi: \mathbb{P}^1 \rightarrow \mathbb{P}^r$  means deforming the positions of its ramifications in  $\mathbb{RP}^1$ . A set  $S := \{s_1(t), \dots, s_n(t)\}$  of continuous functions  $s_i: [0, 1] \rightarrow \mathbb{RP}^1$  where, for each  $t$ , the points  $s_1(t), \dots, s_n(t)$  are distinct is an *isotopy* between  $\{s_1(0), \dots, s_n(0)\}$  and  $\{s_1(1), \dots, s_n(1)\}$ . Given such an isotopy  $S$ , suppose that  $\varphi$  has ramification  $\alpha^i$  at  $s_i(0)$  for  $i = 1, \dots, n$ . A continuous family  $\varphi_t$  ( $t \in [0, 1]$ ) of maximally inflected curves with  $\varphi_0 = \varphi$ , where  $\varphi_t$  has ramification  $\alpha^i$  at  $s_i(t)$  for  $i = 1, \dots, n$ , is a *deformation of  $\varphi$  along  $S$* , and  $\varphi_1$  is a deformation of  $\varphi$ . A maximally inflected curve  $\varphi$  is said to admit *arbitrary deformations* if  $\varphi$  has a deformation along any isotopy  $S$  deforming the ramification points of  $\varphi$ . Since reparameterization by a projective transformation of  $\mathbb{RP}^1$  does not change the image of  $\varphi$ , a basic question in the topology of maximally inflected curves is to classify them up to deformation and reparameterization.

Experimental evidence and geometric intuition suggest that maximally inflected curves admit arbitrary deformations in the strong sense that we make precise in Theorem 2.7(2) below. First we state a non-degeneracy conjecture.

**Conjecture 2.6** (Conjecture 5 of [38]). Suppose that  $d > r$  and  $\alpha^1, \dots, \alpha^n$  are ramification data for degree  $d$  rational curves in  $\mathbb{P}^r$ . If  $s_1, \dots, s_n \in \mathbb{RP}^1$  are distinct, then there are exactly  $N(\alpha^1, \dots, \alpha^n)$  complex rational curves of degree  $d$  in  $\mathbb{P}^r$  with ramification sequence  $\alpha^i$  at  $s_i$  for each  $i = 1, \dots, n$ .

That is, when the points  $s_i$  are real, there should be no multiplicities in Corollary 1.2.

**Theorem 2.7.** (1) Suppose that Conjecture 2.6 holds in all cases when the ramification consists only of flexes. Then Conjecture 2.2 holds for all ramification data.

(2) If Conjecture 2.6 holds for all ramification data, then, given  $s_1, \dots, s_n \in \mathbb{RP}^1$ , each of the  $N = N(\alpha^1, \dots, \alpha^n)$  maximally inflected curves with ramification sequence  $\alpha_i$  at  $s_i$  admit arbitrary deformations, and the  $N$  deformations along a given isotopy are distinct at each point  $t \in [0, 1]$ .

*Proof.* Statement 1 is precisely Theorem 6 of [38] adapted to maximally inflected curves.



To prove the second statement, let  $\{s_1(t), \dots, s_n(t)\}$  be continuous functions  $s_i: [0, 1] \rightarrow \mathbb{RP}^1$  with the points  $s_1(t), \dots, s_n(t)$  distinct for each  $t$ . For every  $t \in [0, 1]$ , consider the intersection of Schubert varieties

$$\bigcap_{i=1}^n X_{\alpha^i} F_{\bullet}(s_i(t)) \subset \mathrm{Gr}_{d-r-1} \mathbb{P}^d \times [0, 1].$$

By Conjecture 2.6, it consists of exactly  $N := N(\alpha^1, \dots, \alpha^n)$  points for each  $t$ , and by the first statement, they are all real. Since  $N$  is the degree of such an intersection, it must be transverse, and so the totality of these intersections define  $N$  continuous disjoint arcs at the real points of  $\mathrm{Gr}_{d-r-1} \mathbb{P}^d \times [0, 1]$ . Each arc is a deformation of the corresponding curve at  $t = 0$ , which proves the second statement.  $\square$

Note that Conjecture 2.6 is not true without the assumption that the points  $s_i$  are real. For example, there is a unique map  $\mathbb{P}^1 \rightarrow \mathbb{P}^1$  of degree 3 with simple critical points (simple ramification)  $0, 1, \omega$ , and  $\omega^2$ , where  $\omega$  is a primitive third root of unity. Given four critical points in general position, there are two such maps, which coincide for this particular choice. However, if all the critical points are real, there always are two such maps. (The details can be found in [6, Section 9].)

Conjecture 2.6 turned out to be true whenever it was tested. In particular, in the cases when  $r = 1$  or  $d = 2$  and the ramification consists only of flexes [9] and for some other ramification data. In the case  $r = 2$  and  $d = 4$  of plane quartics with arbitrary ramification, the conjecture has been proved earlier, by direct computation [39, Theorem 2.3].

**2.3. Constructions using duality.** We give two elementary constructions of new maximally inflected curves from given ones, each invoking a different notion of duality for these curves. The first relies on Grassmann duality, namely, on the isomorphism between the Grassmannian of  $(d - r - 1)$ -planes in  $\mathbb{P}^d$  and the Grassmannian of  $r$ -planes in  $\mathbb{P}^d$ . The second construction relies on projective duality and has new consequences concerning the Shapiro conjecture.

Let  $F_{\bullet}(s)$  be the flag of subspaces in  $\mathbb{P}^d$  osculating the rational normal curve  $\gamma(s) = F_0(s)$ . Then its dual flag  $\hat{F}_{\bullet}(s)$  is the flag of subspaces osculating the rational normal curve  $F_{d-1}(s)$  in the dual projective space. In particular,  $\hat{F}_i(s)$  is dual to  $F_{d-i}(s)$ . Consider a possible ramification sequence  $\alpha = (0, \alpha_1, \dots, \alpha_r)$  for degree  $d$  curves in  $\mathbb{P}^r$  with the additional constraint  $\alpha_r < d$ . It was mentioned in Section 1.1 that a  $(d - r - 1)$ -plane  $\Lambda$  lies in the Schubert variety  $X_{\alpha} F_{\bullet}(s)$  if and only if the dual  $r$ -plane lies in the Schubert variety  $X_{\hat{\alpha}} \hat{F}_{\bullet}(s)$ . Here, the sequence  $\hat{\alpha}$  is determined by  $\alpha$  according to (1.2). Identifying  $\mathbb{P}^d$  with its dual space, we obtain a bijection between the two algebraic sets

$$\left\{ \begin{array}{l} \text{Curves } \varphi \text{ of degree } d \text{ in } \mathbb{P}^r \\ \text{with ramification } \alpha^i \text{ at } s_i \text{ for} \\ i = 1, \dots, n \end{array} \right\} \iff \left\{ \begin{array}{l} \text{Curves } \varphi \text{ of degree } d \text{ in} \\ \mathbb{P}^{d-r-1} \text{ with ramification } \hat{\alpha}^i \\ \text{at } s_i \text{ for } i = 1, \dots, n \end{array} \right\}. \quad (2.4)$$

For a maximally inflected curve in  $\mathbb{P}^r$ , this *Grassmann duality* gives a maximally inflected curve of the same degree in a possibly different projective space ramified at the same points but with different ramification sequences.

Now, let us give another construction, which involves the usual projective duality. Given a curve  $C$  in  $\mathbb{P}^r$ , its dual curve  $\check{C}$  is the curve in the dual projective space  $\check{\mathbb{P}}^r$  which is the closure of the set of hyperplanes osculating  $C$  at general points. If  $C$  is the image of a maximally inflected curve  $\varphi$ , then  $\check{C}$  is also rational with the parameterization  $\check{\varphi}$  induced by  $\varphi$ . The ramification points of  $\check{\varphi}$  coincide with those of  $\varphi$ , but the ramification indices and degree of  $\check{\varphi}$  are different. Let us compute this degree and the transformation of ramification indices and thereby show that  $\check{C}$  is a maximally inflected curve.

The osculating hyperplane at a general point  $\varphi(s)$  of  $C$  is determined by the 1-form  $\psi(s)$  whose  $i$ th coordinate is the determinant

$$(\varphi_b^{(a)}(s))_{b=0,1,\dots,\widehat{r-i},\dots,r}^{a=0,1,\dots,r-1}, \quad (2.5)$$

where  $\varphi = (\varphi_0, \dots, \varphi_r)$ , the derivatives are taken with respect to some local coordinate of  $\mathbb{P}^1$  at  $s$ , and  $\widehat{r-i}$  indicates that the index  $r-i$  is omitted. We may assume for simplicity that, in this local coordinate,  $\varphi_i$  has degree  $d-i$ . Thus the 1-form  $\psi$  has degree  $r(d-r+1)$ , and its coordinates define a linear series of dimension  $r$  and degree  $r(d-r+1)$  on  $\mathbb{P}^1$ . In general, this linear series has base points, and so the degree of the resulting map is less than  $r(d-r+1)$ .

Let us find the base point divisor and the ramification of the map determined by  $\psi$ . Suppose that  $\alpha$  is the ramification sequence of  $\varphi$  at a point  $s$  of  $\mathbb{P}^1$ . We may assume that  $\varphi_i$  vanishes to order  $\alpha_i$  at  $s$ , and a calculation shows that the determinant (2.5) vanishes to order  $|\alpha| + r - \alpha_{r-i}$  at  $s$ . Thus,  $s$  has multiplicity  $|\alpha| + r - \alpha_r$  in the base point divisor. Removing this base point divisor from the map  $\psi$  gives a map  $\check{\varphi}$  whose  $i$ th coordinate vanishes to order  $\alpha_r - \alpha_{r-i}$  at  $s$ , and so the ramification sequence of the dual curve at a point  $s$  where  $\alpha = \alpha(s)$  is

$$\check{\alpha} := (0, \alpha_r - \alpha_{r-1}, \dots, \alpha_r - \alpha_1, \alpha_r). \quad (2.6)$$

Thus the map determined by  $\psi$ , and hence the dual curve, has degree

$$r(d-r+1) - \sum_{s \in \mathbb{P}^1} (|\alpha(s)| + r - \alpha(s)_r), \quad (2.7)$$

where  $\alpha(s)$  is the ramification sequence at  $s$ , which is typically  $(0, 1, \dots, r)$ . The following theorem is immediate.

**Theorem 2.8.** *Let  $d > r$  and suppose that  $\alpha^1, \dots, \alpha^n$  is ramification data for rational curves of degree  $d$  in  $\mathbb{P}^r$ . Then  $\check{\alpha}^1, \dots, \check{\alpha}^n$  is ramification data for rational curves in  $\mathbb{P}^r$  of degree*

$$m := r(d-r+1) - \sum_{s \in \mathbb{P}^1} (|\alpha(s)| + r - \alpha(s)_r).$$

(1) *For any distinct points  $s_1, \dots, s_n \in \mathbb{RP}^1$ , there is a one-to-one correspondence between the maximally inflected curves of degree  $d$  with ramification  $\alpha^i$  at  $s_i$  for  $i = 1, \dots, n$  and the maximally inflected curves of degree  $m$  with ramification  $\check{\alpha}^i$  at  $s_i$  for  $i = 1, \dots, n$ .*

(2) *Conjecture 2.2 holds for  $r, d$ , and sequences  $\alpha^1, \dots, \alpha^n$  if and only if it holds for the integers  $r, m$ , and the sequences  $\check{\alpha}^1, \dots, \check{\alpha}^n$ .  $\square$*

Let us compute the degree  $m$  of the dual curve for  $r = 2$ . Suppose that the original curve has ramification indices  $\alpha^1, \dots, \alpha^n$ . Formula (2.7) gives

$$m = 2(d - 1) - \sum_i (|\alpha^i| + 2 - \alpha_2^i) = 4 - d + \sum_i (\alpha_2^i - 2). \quad (2.8)$$

### 3. MAXIMALLY INFLECTED PLANE CURVES

For plane curves, we have  $r = 2$ . First, suppose that we have a rational plane curve whose only ramifications are cusps and flexes and the other singularities are ordinary double points. Such a curve of degree  $d$  with  $\kappa$  cusps has  $\iota = 3(d - 2) - 2\kappa$  flexes. This Plücker formula follows, for example, from Proposition 1.1, since a cusp has ramification sequence  $(0, 2, 3)$  and a flex  $(0, 1, 3)$ , and we have  $|(0, 2, 3)| = 2$  and  $|(0, 1, 3)| = 1$ . Since the curve is of genus zero, it must have  $g_\kappa := \binom{d-1}{2} - \kappa$  double points.

Now, suppose that the curve is real. Each visible (in the real part of  $\mathbb{P}^2$ ) node is either a *real node* (a real ordinary double point with real tangents) or a *solitary point* (a real ordinary double point with complex conjugate tangents). All other nodes are *complex nodes*; they occur in complex conjugate pairs. Let  $\eta$ ,  $\delta$ , and  $c$  be the numbers of real nodes, solitary points, and complex nodes, respectively.

Up to projective transformation and reparametrization, there are only three real rational plane cubic curves. They are specified by the equations  $y^2 = x^3 + x^2$ ,  $y^2 = x^3 - x^2$ , and  $y^2 = x^3$ , and they have the shapes shown in Figure 3.1. All

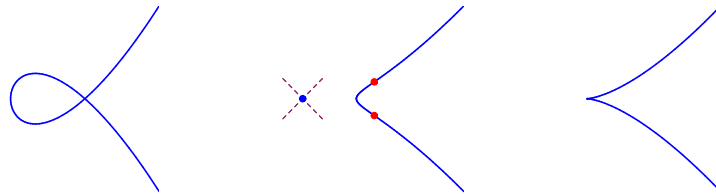


FIGURE 3.1. Real rational cubics

three have a real flex at infinity and are singular at the origin. The first has a real node and no other real flexes, the second has a solitary point and two real flexes at  $(\frac{4}{3}, \pm \frac{4}{3\sqrt{3}})$  (we show them by dots and the complex conjugate tangents at the solitary point by dashed lines), and the third has a real cusp. The last two curves are maximally inflected, while the first is not. In general, as is shown below, the number of real nodes is restricted for maximally inflected curves.

In what follows, we consider maximally inflected curves with arbitrary ramifications and define a *solitary point* of a real rational curve  $\varphi: \mathbb{P}^1 \rightarrow \mathbb{P}^2$  to be a pair of distinct complex conjugate points  $s, \bar{s} \in \mathbb{P}^1$  with  $\varphi(s) = \varphi(\bar{s})$ , which necessarily represents a point in  $\mathbb{RP}^2$ . Let  $\delta$  be the number of such solitary points. We similarly define solitary bitangents to be solitary points of the dual curve, and let  $\tau$  be their number.

**Theorem 3.1.** *Let  $\varphi$  be a maximally inflected plane curve of degree  $d$  with ramification  $\alpha^1, \dots, \alpha^n$ . Then the numbers  $\tau$  of solitary bitangents and  $\delta$  of solitary points satisfy*

$$\delta = \tau + d - 2 - \sum_i (\alpha_1^i - 1).$$

*Proof.* Let  $C$  be the real points of the image of the curve  $\varphi$  and  $\check{C}$  be its dual curve, the real points of the image of  $\check{\varphi}$ . The generalized Klein formula due to Schuh [30] (see also [42]) gives the relation

$$m + \sum_{z \in C} (\mu_z - r_z) = d + \sum_{z \in \check{C}} (\mu_z - r_z), \quad (3.1)$$

where  $\mu_z$  is the multiplicity of a point  $z$  on the curve and  $r_z$  is the number of real branches of the curve at  $z$ .

We first evaluate the sum

$$\sum_{z \in C} (\mu_z - r_z). \quad (3.2)$$

The multiplicity  $\mu_z$  at a point  $z \in C$  is the local intersection multiplicity (at  $z$ ) of the curve  $C$  with a general linear form  $f$  vanishing at  $z$ . This is the sum over all  $s \in \varphi^{-1}(z)$  of the order of vanishing at  $s$  of the pullback  $\varphi^*(f)$ . This order of vanishing is  $\alpha(s)_1$ , where  $\alpha(s)$  is the ramification sequence of  $\varphi$  at  $s$ . We have two cases to consider: either  $s$  is real ( $s \in \mathbb{RP}^1$ ) or it is not.

In the first case, the image of a neighborhood of  $s$  in  $\mathbb{RP}^1$  is a branch of  $C$  at  $z$ , so the contribution of  $s$  to the sum (3.2) is  $\alpha(s)_1 - 1$ . Since  $\alpha(s)_1 - 1$  vanishes except at some points of ramification, the contribution of the points from  $\mathbb{RP}^1$  to (3.2) is the sum

$$\sum_{i=1}^n (\alpha_1^i - 1).$$

In the second case, the complex conjugate  $\bar{s}$  of  $s$  is also in  $\varphi^{-1}(z)$ . Since  $\varphi$  is maximally inflected, it is unramified at these points, so  $\alpha(s)_1 = \alpha(\bar{s})_1 = 1$ . Thus, each solitary point contributes 2 to the sum (3.2). Combining these observations gives

$$\sum_{z \in C} (\mu_z - r_z) = 2\delta + \sum_{i=1}^n (\alpha_1^i - 1).$$

If  $s \in \mathbb{P}^1$  and  $\varphi$  has ramification  $(0, a, b)$  at  $s$ , then  $\check{\varphi}$  has ramification  $(0, b-a, b)$  at  $s$  (by (2.6)). Thus,

$$\sum_{z \in \check{C}} (\mu_z - r_z) = 2\tau + \sum_{i=1}^n (\alpha_2^i - \alpha_1^i - 1).$$

Substituting these expressions and the formula (2.8) for  $m$  into Schuh's formula (3.1), we obtain

$$4 - d + \sum (\alpha_2^i - 2) + 2\delta + \sum (\alpha_1^i - 1) = d + 2\tau + \sum (\alpha_2^i - \alpha_1^i - 1)$$

or  $2\delta = 2d - 4 + 2\tau - 2\sum (\alpha_1^i - 1)$ , which completes the proof.  $\square$

**Corollary 3.2.** *Let  $\varphi$  be a maximally inflected plane curve of degree  $d$ . Let  $\delta$ ,  $\eta$ , and  $c$  be the numbers of solitary points, real nodes, and complex nodes, respectively. Then*

$$d - 2 - \sum (\alpha_1^i - 1) \leq \delta \leq \binom{d-1}{2} - \frac{1}{2} \sum (\alpha_1^i - 1)(\alpha_2^i - 1)$$

and

$$0 \leq \eta + 2c \leq \binom{d-2}{2} - \frac{1}{2} \sum (\alpha_1^i - 1)(\alpha_2^i - 1) + \sum (\alpha_1^i - 1).$$

*Proof.* Since  $\tau \geq 0$ , the lower bound for  $\delta$  follows from the formula of Theorem 3.1. The upper bound is, in fact, an upper bound for the number of *virtual double points* which can appear, outside a given ramification, on a curve of degree  $d$ . Namely, the total number of virtual double points (including the virtual double points accounted for by the fixed ramification points) is equal to the genus of a nonsingular curve of degree  $d$ , and the number of virtual double points contained in a ramification point equals  $\frac{1}{2}(\mu + r - 1)$ , where  $\mu$  is the Milnor number and  $r$  is the number of local branches (this formula can be found in [2]; for a more modern treatment and generalizations, see [24]). Now, it remains to notice that  $\mu \geq (\alpha_1 - 1)(\alpha_2 - 1)$  at a ramification point of type  $(0, \alpha_1, \alpha_2)$ .

Since a maximally inflected curve has genus 0, the genus formula gives

$$\delta + \eta + 2c \leq g_\alpha, \quad (3.3)$$

where  $g_\alpha$  is the genus of a generic curve with the singularities prescribed by the ramification data. The upper bound for  $\eta + 2c$  now follows from the lower bound on  $\delta$ , and the upper bound on the genus  $g_\alpha \leq \binom{d-1}{2} - \frac{1}{2} \sum (\alpha_1^i - 1)(\alpha_2^i - 1)$  is given by the above bound on the number of virtual double points.  $\square$

**Corollary 3.3.** *Let  $\varphi$  be a maximally inflected plane curve of degree  $d$  such that it has  $\kappa$  (real) cusps and  $\iota = 3(d - 2) - 2\kappa$  (real) flexes and its all remaining singularities are nodes. Let  $\delta$ ,  $\eta$ , and  $c$  be the numbers of solitary points, real nodes, and complex nodes (respectively) of  $\varphi$ , which satisfy  $\delta + \eta + 2c = \binom{d-1}{2} - \kappa =: g_\kappa$ . If  $\kappa \leq d - 2$ , then they additionally satisfy*

$$d - 2 - \kappa \leq \delta \leq g_\kappa, \quad 0 \leq \eta + 2c \leq \binom{d-2}{2}. \quad (3.4)$$

If  $d - 2 \leq \kappa$  (which is at most  $3(d - 2)/2$ ), then  $\delta$ ,  $\eta$ , and  $c$  satisfy

$$0 \leq \delta \leq \binom{2d-4-\kappa}{2}, \quad g_\kappa - \binom{2d-4-\kappa}{2} \leq \eta + 2c \leq g_\kappa. \quad (3.5)$$

This can be deduced from the Klein [21] and Plücker [26] formulas alone.

*Proof.* Since the ramification sequence of a flex is  $(0, 1, 3)$  and that of a cusp is  $(0, 2, 3)$ , we have

$$\sum (\alpha_1^i - 1) = \frac{1}{2} \sum (\alpha_1^i - 1)(\alpha_2^i - 1) = \kappa,$$

and so (3.4) is just a special case of Corollary 3.2.

Now, suppose that  $d - 2 \leq \kappa$  and consider the dual curve to  $\varphi$ , which has degree  $m := 2(d - 1) - \kappa$  (by (2.8)),  $\kappa$  flexes, and  $\iota = 3(d - 2) - 2\kappa$  cusps. The total number of double points of this curve (bitangents of  $\varphi$ ) is given by the genus formula

$$\binom{m-1}{2} - \iota = \binom{2d-3-\kappa}{2} - (3(d-2) - \kappa).$$

This expression is an upper bound for the number  $\tau$  of solitary bitangents of  $\varphi$ . This gives the upper bound for  $\delta$ :

$$\begin{aligned} \delta &\leq \binom{2d-3-\kappa}{2} - (3(d-2) - 2\kappa) + d - 2 - \kappa \\ &= \binom{2d-3-\kappa}{2} - (2d-4-\kappa) = \binom{2d-4-\kappa}{2}. \end{aligned}$$

Combining the bounds for  $\delta$  with the genus formula  $\delta + \eta + 2c = g_\kappa$  gives the bounds for  $\eta + 2c$ .  $\square$

**Remark 3.4.** A fundamental question about the statement of Corollary 3.3 is whether its hypotheses are satisfied by any maximally inflected curve with  $\kappa$  cusps and  $3(d - 2) - 2\kappa$  flexes. That is, is there a curve with  $\kappa$  cusps whose other singularities are only ordinary double points not occurring at points of ramification? (If there is one such curve, then the general curve has this property.) As the construction in Section 4.1 shows, this is true when  $\kappa \leq d - 2$ .

We believe that a generic maximally inflected curve has only ordinary double points not occurring at ramification points, but we do not have a proof. A difficulty is that there are few constructions of maximally inflected curves.

Corollary 3.3 suggests our first question concerning the classification of maximally inflected curves by their topological invariants.

**Question 3.5.** Let  $d$ ,  $\iota$ , and  $\kappa$  be positive integers with  $\iota + 2\kappa = 3(d - 2)$ .

- (1) Which numbers  $\delta$  in the range allowed by Corollary 3.3 are realized as the number of solitary points in a maximally inflected plane curve of degree  $d$  with  $\iota$  flexes and  $\kappa$  cusps whose all other singularities are ordinary double points?
- (2) Given a number  $\delta$  of solitary points of a curve as above, we have  $\eta + 2c = g_\kappa - \delta$ . Which numbers  $\eta$  in the range  $[0, g_\kappa - \delta]$  with the same parity as  $g_\kappa - \delta$  are realized as the number of real nodes of such a maximally inflected curve?

**Remark 3.6.** For example, when  $d = 5$ ,  $\kappa = 4$ , and  $\iota = 1$ , we have  $\kappa \geq d - 2$ . The bounds of Corollary 3.3 give  $\delta = 0$  or  $1$ . Question 3.5(1) asks: Do both values of  $\delta$  occur? If  $\delta = 1$ , then  $\eta + 2c = 1$ , so a curve with  $\delta = 1$  has a single real node. If  $\delta = 0$ , then  $\eta + 2c = 2$ , so there are two possibilities for  $\eta$ , 0 or 2. Question 3.5(2) asks: Do both values of  $\eta$  occur? In fact, there are curves with both values of  $\delta$ , but when  $\delta = 0$ ,  $\eta$  can take only the value of 2.

The first part of the last statement follows from the results of Section 5, as explained in Remark 3.8. Two such curves are shown in Figure 6.1. The impossibility of the values  $\delta = \eta = 0$  for a rational quintic plane curve with four cusps and

one flex is explained by Rokhlin's theory of complex orientations [28] extended by Mishachev [25] and Zvonilov [46]. Let  $f: \mathbb{P}^1 \rightarrow \mathbb{P}^2$  be a real rational plane quintic with four cusps and no solitary points or nodes. We first perturb it to a new real rational quintic with four real nodes instead of the cusps, and then smooth each real node so as to obtain an oval; in this way, we get a dividing real quintic  $Y$  of genus 4. (This smoothing of a cusp is illustrated in Figure 3.2.) Since the Rokhlin

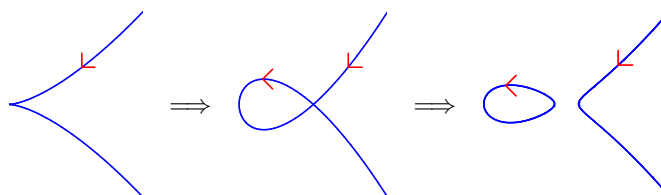


FIGURE 3.2. Smoothing a cusp and complex orientation

complex orientation formula extends to “flexible” curves, which are only almost holomorphic near the real plane, we do not need the existence of such an algebraic deformation; instead, we can simply attach a proper local model in place of the cusps.

A choice of one component of  $Y \setminus \operatorname{Re} Y$  determines a complex orientation on the ovals and the odd branch of the curve  $Y$ . This complex orientation must satisfy

$$25 = 1 + 4\mu + 4 + 4(R_+ - R_-),$$

where  $\mu = 0$  or  $2$  counts the intersection of the two components of  $Y \setminus \operatorname{Re} Y$  at the complex nodes and  $R_{\pm}$  counts the relative orientation of the ovals with respect to the odd branch. As we can see from Figure 3.2, the ovals are all oriented coherently with respect to the odd branch, so we have  $R_+ = 4$  and  $R_- = 0$ , which gives a contradiction.

We might ask a more general question, being the analog of Question 3.5 for maximally inflected curves with arbitrary ramification. We leave the formulation of this question to the reader.

**Remark 3.7.** There is another approach to showing the impossibility of the values  $\delta = \eta = 0$  for a rational quintic plane curve with four cusps and one flex. By the formula of Theorem 3.1,  $\tau \geq 1$  and thus the dual curve has at least one solitary point. Consider this dual curve. It is a rational quartic with four flexes and one cusp. Draw a straight line through such a solitary point and the cusp. By Bézout's theorem, this line contains no other points of the curve. Hence, choosing a nearby line as the line at infinity, we may assume that the real part of the quartic lies in an affine part of  $\mathbb{RP}^2$ .

According to the Fabricius–Bjerre formula [10], we have  $\iota + 2\eta + 2\kappa = 2(t_+ - t_-)$ , where  $\iota$  is the number of flexes,  $\eta$  is the number of nodes,  $\kappa$  is the number of cusps, and  $t_+$  ( $t_-$ ) is the number of one-sided (respectively, two-sided) double tangents. Here, a line passing through a cusp and tangent at another point is counted as a double tangent as well. Because this quartic curve is connected, any double

tangent has both germs of the curve at the tangencies on the same side, that is, it is a one-sided double tangent. The projection from a cusp is 2-sheeted and, thus, the number of such tangents through the cusp is at most 2. All this together implies that there is at least one ordinary (i. e., not passing through a cusp) double tangent. Hence, the quintic has a real node.

**Remark 3.8.** Yet another proof of the impossibility of  $\eta = 0$  for a maximally inflected quintic with four cusps and one flex uses the results of Section 5. As we have seen, the dual of such a curve is a maximally inflected rational quartic in  $\mathbb{RP}^2$  with 4 flexes and a single cusp. By Theorem 5.6, the possible topological types of such curves and of the arrangements of their bitangents with respect to the curve are those shown in the second column of Table 5.2 (the bitangents are not drawn there, but they can be inferred). One curve has one bitangent, while the other (nodal) curve has two bitangents. Thus, our original quintic must have one or two real nodes, so  $\eta = 0$  is impossible.

#### 4. TWO CONSTRUCTIONS OF MAXIMALLY INFLECTED CURVES

In the study of real plane curves, examples are typically constructed either by deforming a reducible curve (for example, in the constructions of Harnack [15] and Hilbert [17]) or by Viro's patchworking method of deforming a curve in a reducible toric variety [40], [41] (see also [27]), sometimes combined with Cremona transformations. These methods were used initially to construct smooth curves. Gudkov and, then, Shustin extended these methods to use and obtain singular curves [33], [36]. We use such extensions to construct maximally inflected plane curves of degree  $d$  with no complex nodes and the extreme numbers of 0 and  $\binom{d-2}{2}$  real nodes.

Gudkov and Shustin proved a number of theorems extending the classical result of Brusotti [3], which assert that, given a singular curve satisfying a numerical condition and local models for certain deformations of the singular points of the curve, there exists a deformation of the singular curve where each singularity is deformed according to the corresponding local model. It is this approach that we use in the proof of the following theorem. (We give the details of our proof, as the results in the literature are vastly more general than we need and use subtle hypotheses.)

**Theorem 4.1.** (1) *For any  $d$  and  $\kappa$  with  $0 \leq \kappa \leq d - 2$ , there exists a maximally inflected plane curve with  $\kappa$  cusps,  $3(d - 2) - 2\kappa$  flexes, and  $\binom{d-2}{2}$  real nodes (and, hence,  $d - 2 - \kappa$  solitary points).*

(2) *For any  $d$  there exists a maximally inflected plane curve of degree  $d$  with  $3(d - 2)$  flexes and  $\binom{d-1}{2}$  solitary points (and, hence, without real or imaginary nodes).*

We prove the first statement in Section 4.1 and the second in Section 4.2. They give the maximum and minimum possible numbers of real nodes  $\eta$  and solitary points  $\delta$  allowed by Corollary 3.3 for degree  $d$  curves with  $3(d - 2)$  flexes. In Section 6, we discuss some consequences of the constructions in Section 4.1.



**Remark 4.2.** Theorem 4.1 asserts the existence of maximally inflected curves with a given ramification for certain ramifications. The result is new for odd  $d$ . Eremenko and Gabrielov's result (Proposition 2.4) does not guarantee the existence of maximally inflected curves of odd degree with a given ramification. From the constructions described below, we can see that the ramification points are not 'clustered together', as in the discussion following Theorem 2.3, so Theorem 2.3 also does not apply.

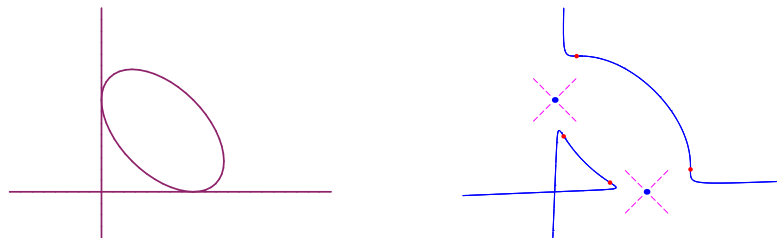
It follows from the relation between maximally inflected curves and real Schubert calculus that the classical constructions of the curves mentioned in Theorem 4.1 imply the existence of real solutions to some problems in Schubert calculus. When  $d$  is odd and the ramifications are such as specified in Theorem 4.1, this result is new and gives yet another evidence in support of the conjecture of Shapiro and Shapiro.

Note also that, whatever the value of  $d$  (even or odd), the proof of the theorem gives not merely the existence but also some explicit constructions of maximally inflected curves with well-controlled topology.

**4.1. Deformations of singular curves.** For a sufficiently small  $\epsilon > 0$ , the equation

$$12xy((x+y-1)^2 - xy) - \epsilon(x+y)^2(x+y-1)^2 = 0$$

defines a rational quartic curve  $C(\epsilon)$  with six flexes, one real node, and two solitary points. The curve  $C(0)$  is the union of the coordinate axes and the ellipse shown on the left below. The curve  $C(\frac{1}{20})$  is shown on the right below.



The quartic curve on the right has two solitary points at  $(1, 0)$  and  $(0, 1)$  and a node at the origin. By the genus formula, the curve is rational. It also has six flexes. Four of them are indicated by circles, and there are two more along the branches close to the coordinate axes, because the oriented geodesic curvature (which is given by the Wronskian  $\det(\phi, \phi', \phi'')$  with respect to the local orientation defined by  $\phi$ , where  $\phi$  is the parametrization of the curve) of the segment of the curve along such a branch takes values of opposite sign at the extremal points (close to the initial tangency points) of these branches. (The local orientation changes with respect to the affine one when the branch passes infinity.) Thus, any parameterization of this curve is a maximally inflected quartic with six flexes, one node, and two solitary points.

*Proof of Theorem 4.1(1).* Fix an integer  $d > 2$  throughout and let  $P_0$  be the union of a nonsingular conic and any  $d - 2$  distinct lines tangent to the conic. Then

$P_0$  is a reducible curve of degree  $d$ . The tangents intersect pairwise, but no three of them meet at one point, as the dual curve to the conic is another nonsingular conic. Thus,  $P_0$  has  $\binom{d-2}{2}$  real nodes and  $d-2$  other singularities at the points of tangency. We deform the tangency singularities while leaving the nodes intact.

In a neighborhood  $V$  of each point of tangency,  $P_0$  is isomorphic to the reducible curve  $I_0$  given by the equation

$$y(y - x^2) = 0$$

in some neighborhood  $U$  of the origin. For each  $t \in \mathbb{R}$ , let  $I_t$  be the deformation of  $I_0$  defined in  $U$  by

$$y(y - x^2) + tx^2 = 0. \quad (4.1)$$

When  $t$  is positive but sufficiently small,  $I_t$  has a solitary point at the origin and two flexes within  $U$ , with one flex near each branch of the parabola  $y = x^2$ . Moreover,  $I_t$  lies above the  $x$ -axis. Calculating the Whitney index by means of the Gauss map shows the existence of two such flexes.

To construct curves with cusps we replace the local deformation model  $I_t$  of the tangent points, which is defined by (4.1), by the local model  $K_t$  defined by

$$y(y - x^2) + tx^3 = 0. \quad (4.2)$$

For each  $t > 0$ ,  $K_t$  has a cusp at the origin and one flex near the origin.

Suppose that we can deform each tangency singularity (a tacnode) according to the local model  $I_t$  while preserving the nodes so that we obtain a deformation  $P_t$  of the curve  $P_0$  for  $t \in (0, \epsilon)$  such that (i)  $P_t$  has degree  $d$ , (ii)  $P_t$  has a node in a neighborhood of each node of  $P_0$ , and (iii) in a neighborhood  $V$  of each point of tangency of  $P_0$ ,  $P_t$  is isomorphic to  $I_t$  in the neighborhood  $U$  of the origin. For  $0 < t < \epsilon$ , the curve  $P_t$  has  $\binom{d-2}{2}$  nodes and  $d-2$  solitary points by the construction. Since it has degree  $d$ , it is rational. Each solitary point contributes two flexes, which gives  $2(d-2)$  flexes. Furthermore, there is an additional flex along each asymptote (the original tangent lines), as the concavity of  $P_t$  changes when passing through infinity. Thus any parameterization of the curve  $P_t$  gives a maximally inflected curve of degree  $d$  with  $3(d-2)$  flexes,  $\binom{d-2}{2}$  nodes, and  $d-2$  solitary points.

If we replace some local deformation models  $I_t$  by  $K_t$ , every tangency point where we use the local perturbation  $K_t$  gives us one cusp, no solitary points, and one flex (there is no flex along the tangent line, as the concavity of  $K_t$  does not change along this line). Applying this procedure to  $\kappa$  of the  $d-2$  tangency points of  $P_0$  gives a maximally inflected curve with  $\kappa$  cusps,  $3(d-2) - 2\kappa$  flexes,  $d-2-\kappa$  solitary points, and  $\binom{d-2}{2}$  real nodes, which proves statement (1) (the curve is rational because  $g = 0$ ).

The existence of such a simultaneous deformation of the tacnodes according to arbitrary independent local models while preserving the nodes follows from the transversality of the equisingularity strata of the singularities. The equisingularity stratum of a node is smooth, and its tangent space is given by polynomials vanishing at the node. The equisingularity stratum of a tacnode is also smooth, and its tangent space is given by polynomials vanishing at the tacnode and satisfying two additional conditions: their first derivatives at tangent directions of the branches

vanish, and the second derivatives along both branches (taken in the same direction) are equal.

So, to check the transversality, it is sufficient to merely calculate the dimension of the intersection of the tangent spaces. In the case under consideration, the intersection is contained in the linear span of the polynomials  $L'_0 Q_0 L_1 \cdots L_{d-3} + L'_1 Q_0 L_0 \cdots L_{d-3} + \cdots + L_0 L_1 \cdots L_{d-3} Q'_0$ , where  $Q_0$  is the initial nonsingular conic,  $L_0, \dots, L_{d-3}$  are the tangent lines, and  $L'_0, L'_1, \dots, Q'_0$  are equations of arbitrary lines and a conic. So, the condition on the second derivatives takes the form of a triangular linear system and, hence, the subspace we are looking for is of dimension at most  $2(d-2) + 6 - (d-2) = d+4$ , which implies transversality.  $\square$

Figure 4.1 shows these curves for  $d = 5$  and  $\kappa = 0, 1, 2, 3$ .

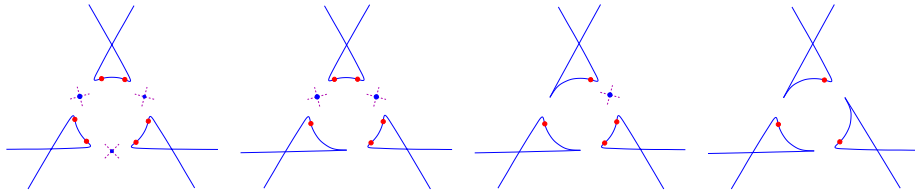


FIGURE 4.1. Maximally inflected degree 5 curves with three real nodes

**Remark 4.3.** For  $t > 0$ , the cusp of the curve  $K_t$  defined by (4.2) is on the branch to the left of the origin. If we used the perturbation  $K'_t$  given by

$$y(y - x^2) - tx^3 = 0$$

instead, then the cusp would be on the right branch for  $t > 0$ . In Section 6.1, we use this to study Question 5.7 concerning possible necklaces for a given set of ramifications and numbers of solitary points.

**4.2. Patchworking of Singular Curves.** Viro's method for constructing real plane curves with prescribed topology [40], [41] (see also [27]) begins with the subdivision of the simplex

$$\Delta_d := \{(i, j) \in (\mathbb{Z}_+)^2 : i + j \leq d\}$$

defined by a piecewise linear convex *lifting function*. Reflecting this subdivision in the coordinate axes and in the origin gives a subdivision of the region

$$\diamond_d := \{(x, y) \in \mathbb{R}^2 : |x| + |y| \leq d\}.$$

Gluing together the opposite edges of  $\diamond_d$  gives a PL-space homeomorphic to  $\mathbb{RP}^2$ .

The other ingredient is real polynomials  $f_F$ , where  $F$  are the facets of the subdivision, such that the Newton polytope of each  $f_F$  is  $F$  and  $f_F = 0$  define smooth curves in the torus  $(\mathbb{C}^\times)^2$ . These polynomials additionally satisfy the compatibility condition that, for an edge  $e$  common to two facets  $F$  and  $G$ , we have  $f_F|_e = f_G|_e$ ; the last expressions denote the truncations of the polynomials  $f_F$  and  $f_G$  to the edge  $e$  (i.e., the monomials whose exponent vectors are in  $e$ ). Furthermore, this common truncation has no multiple factors (except  $x$  and  $y$ ).

The real points of the real curve  $f_F = 0$  lie naturally in the four copies of the facet  $F$  in  $\diamond_d$  (with the boundary points representing the asymptotic behavior of solutions, or, equivalently, the zeros of the restriction of  $f_F$  to the corresponding edge). The pair consisting of these facets and curves is called the *Newton portrait* of  $f_F = 0$ . By the compatibility condition, the Newton portraits of the facet curves glue together in a natural way, determining a topological curve  $\Gamma$  in  $\mathbb{RP}^2$ . Viro's theorem asserts that there exists a curve  $C$  of degree  $d$  in  $\mathbb{RP}^2$  such that the pair  $(\mathbb{RP}^2, C)$  is homeomorphic to the pair  $(\mathbb{RP}^2, \Gamma)$ . The homeomorphism preserves each coordinate axis and each quadrant. The complex points of  $C$  are smooth, and  $\Gamma$  and  $C$  meet each coordinate axis in the same number of points.

Shustin [33] (see also [34], [35]) showed how to modify this construction when the facet curves ( $f_F = 0$ ) have singularities in  $(\mathbb{C}^\times)^2$ . If a certain numerical criterion is satisfied (it is given by Theorem 1.7 of [34]), there exists a curve of degree  $d$  in  $\mathbb{RP}^2$  whose singularities are the disjoint union of the singularities of the facet curves and whose topology is glued from that of the facet curves as before. (See Theorem 1.8 of [34], originally proven in [33].) In particular, Shustin proved the following assertion.

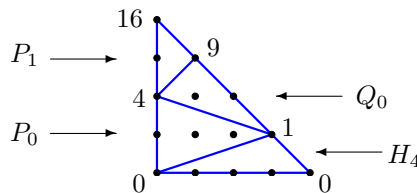
**Proposition 4.4** (See [34] and [33]). *If the singularities of the facet curves are only nodal, then there exists a curve of degree  $d$  in  $\mathbb{RP}^2$  whose only (complex) singularities are the disjoint union of the singularities of the facet curves and whose topology is obtained by gluing together the facet curves as in the Viro construction.*

We use this to prove statement (2) of Theorem 4.1. (Another approach is outlined in Remark 4.5 at the end of this section.)

*Proof of Theorem 4.1(2).* Consider the subdivision of  $\Delta_d$  given by the piecewise linear convex lifting function  $w$  defined for some of the vertices of  $\Delta_d$  as follows. We set  $w(0, 0) = w(d, 0) = 0$  and

$$\begin{aligned} w(0, 2i+2) &= (2i+2)^2, \\ w(d-1-2i, 2i+1) &= (2i+1)^2 \end{aligned} \quad \text{for } i = 0, \dots, \left\lfloor \frac{d-1}{2} \right\rfloor.$$

The resulting subdivision of  $\Delta_d$  and the values of the lifting function are



This regular subdivision of  $\Delta_d$  has three types of triangles:

- (i) the triangle  $H_d$  with vertices  $\{(0, 0), (d, 0), (d-1, 1)\}$ ;
- (ii) the triangles  $P_i$  with vertices  $\{(0, 2i), (0, 2i+2), (d-1-2i, 2i+1)\}$  for each  $i$  from 0 to  $\lfloor \frac{d-2}{2} \rfloor$ ;
- (iii) the triangles  $Q_i$  with vertices  $\{(0, 2i+2), (d-1-2i, 2i+1), (d-3-2i, 2i+3)\}$  for each  $i$  from 0 to  $\lfloor \frac{d-3}{2} \rfloor$ .

For each facet, we take polynomials  $f_F$  that define curves with only solitary points. They do not necessarily satisfy the compatibility condition but rather the weaker condition that a common edge  $e$  between two facets  $F$  and  $G$  contains only two lattice points, and, possibly after multiplying one of the facet polynomials by  $-1$ , the signs of the monomials from the two facet polynomials agree. This weak compatibility implies that there are monomial transformations with positive coefficients of the facet polynomials which do satisfy the compatibility criteria, possibly after changing the sign of one of the two polynomials. Since these monomial transformations do not change the geometry (the number of solitary points and the topology of the glued curve  $\Gamma$ ) and the dual graph to the triangulation is a chain, it suffices to construct polynomials satisfying the weaker criteria and giving the desired topology.

Let us describe the monomial transformations. First, consider a common edge  $e$  between adjacent facets  $F$  and  $G$  of the triangulation. Since  $e$  has no interior lattice points, the restrictions of the facet polynomials to  $e$  are binomials of the form

$$f_F|_e = Ax^ay^b + Bx^cy^{b+1}, \quad f_G|_e = Cx^ay^b + Dx^cy^{b+1}.$$

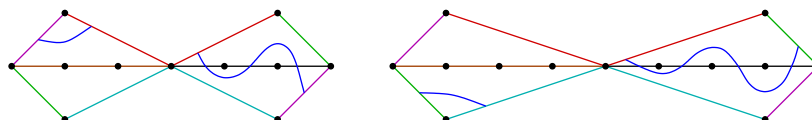
(Not only do the exponents of  $y$  differ by 1, but one of the exponents  $a$  and  $c$  is zero.) For  $z \neq 0$ , let  $\operatorname{sgn}(z) := z/|z|$ . We have

$$\begin{aligned} & \operatorname{sgn}\left(\frac{A}{C}\right) \left|\frac{C}{A}\right|^{\frac{-c}{a-c}} \left|\frac{D}{B}\right|^{-b} f_F\left(\left|\frac{C}{A}\right|^{\frac{1}{a-c}} x, \left|\frac{D}{B}\right| y\right) \\ &= \operatorname{sgn}\left(\frac{A}{C}\right) \left|\frac{C}{A}\right|^{\frac{-c}{a-c}} \left|\frac{D}{B}\right|^{-b} \left(A \left|\frac{C}{A}\right|^{\frac{a}{a-c}} \left|\frac{D}{B}\right|^b x^a y^b + B \left|\frac{C}{A}\right|^{\frac{c}{a-c}} \left|\frac{D}{B}\right|^{b+1} x^c y^{b+1}\right) \\ &= \operatorname{sgn}\left(\frac{A}{C}\right) \left(A \left|\frac{C}{A}\right| x^a y^b + B \left|\frac{D}{B}\right| x^c y^{b+1}\right) \\ &= Cx^ay^b + Dx^cy^{b+1}, \end{aligned}$$

because the weak compatibility criterion ensures that  $\operatorname{sgn}(\frac{A}{C}) = \operatorname{sgn}(\frac{D}{B})$ .

Since the dual graph of the triangulation is a chain, we encounter no obstructions when transforming the facet polynomials so that they satisfy the compatibility condition. More precisely, given a facet polynomial for  $H_d$ , we transform the facet polynomial for  $P_0$  as above, then we transform the facet polynomial for  $Q_0$ , then for  $P_1$ , etc.

Now, let us define the facet polynomials. The reader is invited to check that the weak compatibility conditions are satisfied. The facet  $H_d$  is the Newton polytope of the polynomial  $h_d := x^{d-1}y - (1-x)(2-x)\cdots(d-x)$ . The Newton portraits of  $h_3$  and  $h_4$  are shown below.



The construction of the remaining facet polynomials is based on an idea of Shustin [34, p. 849]. Recall that the Chebyshev polynomials  $\Upsilon_k(x)$  (defined recursively

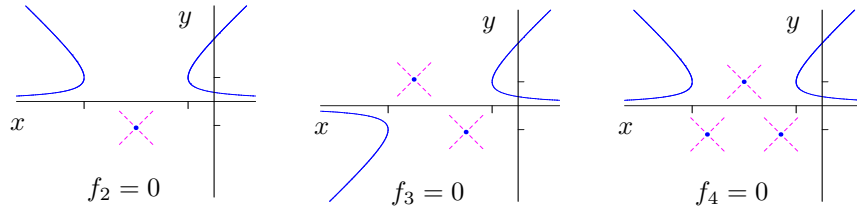
by

$$\mathfrak{U}_0 := 1, \quad \mathfrak{U}_1 := x, \quad \text{and} \quad \mathfrak{U}_k(x) := 2x\mathfrak{U}_{k-1}(x) - \mathfrak{U}_{k-2}(x) \text{ for } k > 1$$

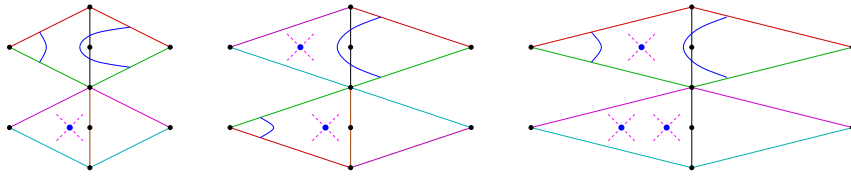
have the properties that  $\mathfrak{U}_k(x)$  has exactly  $k$  roots in the interval  $(-1, 1)$  and  $k-1$  local extrema in this interval with values  $\pm 1$ ,  $|\mathfrak{U}_k(x)| > 1$  for  $|x| > 1$ , and, finally, the leading term of  $\mathfrak{U}_k(x)$  is  $2^{k-1}x^k$ . Thus the Newton polytope of the polynomial  $f_k(x) := y^2 - 2y\mathfrak{U}_k(x+2) + 1$  is the triangle

$$\text{Conv}\{(0, 0), (1, k), (0, 2)\},$$

which is a translate of the polytope  $P_i$  by  $(0, -2i)$ , where  $d-1-2i = k$ . The curve  $f_k = 0$  in  $\mathbb{R}^2$  has two connected components and  $k-1$  solitary points. Below, these curves are shown for  $k = 2, 3$ , and  $4$ ; the  $y$ -axis is scaled by the transformation  $y \mapsto \text{sign}(y)|y|^{1/k}$ .



The Newton portraits of these curves are given below. The interior lattice points in the triangles are not shown.

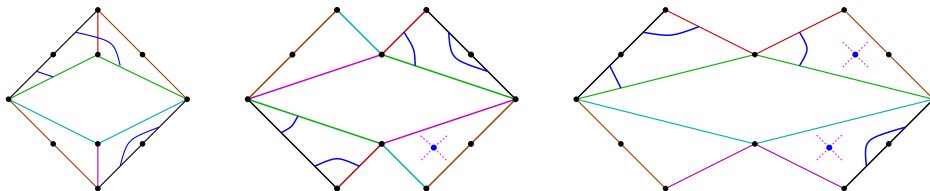


Let  $y^{2i}f_{d-1-2i}$  be the facet polynomial for the facet  $P_i$ .

Finally, we set

$$g_k(x, y) := f_k\left(-\frac{1}{x}, (-1)^k \frac{y}{x}\right) = \frac{y^2}{x^2} - (-1)^k 2 \frac{y}{x} \mathfrak{U}_k\left(2 - \frac{1}{x}\right) + 1.$$

The Newton polytope of  $g_k$  has vertices  $\{(0, 0), (-2, 2), (-k-1, 1)\}$ . We give the Newton portraits of  $g_1$ ,  $g_2$ , and  $g_3$ . (We first translate their Newton polytopes by  $(k+1, 0)$ , placing them into the positive quadrant.)



Let  $x^{d-1-2i}y^{1+2i}g_{d-2-2i}$  be the facet polynomial of the facet  $Q_i$ .

The curve  $C_d$  whose existence for these data is asserted by Proposition 4.4 has  $3(d-2)$  flexes and  $\binom{d-1}{2}$  solitary points. First, since the facet curves  $f_k$  and  $g_k$  each have  $k-1$  solitary points,  $C_d$  has  $(d-2) + (d-3) + \cdots + 1 + 0 = \binom{d-1}{2}$  solitary points, and so is rational (in fact, the solitary points correspond to internal integer points of the Newton polygon). From the Newton portrait of  $h_d$ , we see that  $C_d$  meets the  $x$ -axis in  $d$  points. Each facet  $P_k$  contributes two intersection points of  $C_d$  with the  $y$ -axis. This gives  $d$  intersection points when  $d$  is even and  $d-1$  intersection points when  $d$  is odd. When  $d$  is odd, the last facet  $Q_1$  (corresponding to  $g_1$ ) contributes an additional intersection point with the  $y$ -axis. Finally, each facet  $Q_k$  contributes two intersection points of  $C_d$  with the  $z$ -axis, which gives  $d-2$  intersection points when  $d$  is even and  $d-1$  intersection points when  $d$  is odd. The facet  $H_d$  contributes an additional point, and when  $d$  is even, the facet  $P_{\frac{d-2}{2}}$  (corresponding to  $f_2$ ) contributes one point.

As a result, the curve  $C_d$  has three separate segments, each intersecting a different coordinate axis in  $d$  points that appear on the curve in the same order as on the axis. Thus, calculating the Whitney indices by means of the Gauss map, we find at least  $d-2$  flexes on each segment. Hence, the curve constructed above has  $3(d-2)$  flexes.  $\square$

Figure 4.2 shows the curves  $C_4$  and  $C_5$ . The curve  $C_4$  has the topological type of the quartic in Table 5.2 with six flexes and no real nodes.

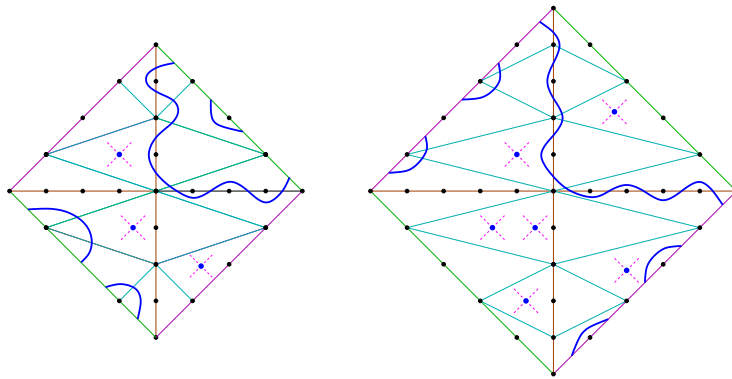


FIGURE 4.2. The curves  $C_4$  and  $C_5$

**Remark 4.5.** Another patchworking is via gluing together parameterizations of the facet curves  $f$  and  $g$  (which are rational). Since the dual graph of the triangulation is a chain, it is sufficient to have a gluing construction for a pair of rational plane curves intersecting transversally. For this purpose, we take parameterizations  $F$  and  $G$  such that  $F(0) = G(0)$  and consider the rational curves  $H_\epsilon$  given by  $\lambda F(u) + \mu G(v)$ , where  $uv = \epsilon$ , for generic  $\lambda$  and  $\mu$  and sufficiently small  $\epsilon > 0$ . The flexes and the nodes are preserved under this patchworking construction, since they are stable under small deformations.

## 5. MAXIMALLY INFLECTED PLANE CURVES OF DEGREES 3 AND 4

**5.1. Cubic plane curves.** In Figure 3.1, we saw that a plane cubic with three real flexes has a solitary point and no nodes and a plane cubic with one real cusp has no nodes. These are the only possible maximally inflected cubics, and they exhaust all the possibilities allowed by Corollary 3.3.

**5.2. Quartic plane curves.** Now, consider quartics whose only ramifications are flexes and cusps. The upper bound for  $\eta + 2c$  allowed by Corollary 3.3 for quartics is 1, so maximally inflected quartics have no complex nodes and either one or no real nodes. Table 5.1 summarizes the possible numbers  $\eta$  of real nodes and  $\delta$  of solitary points. Clearly, for quartics, these numbers determine the real part of the

TABLE 5.1. The topological invariants of quartics allowed by Corollary 3.3

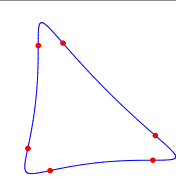
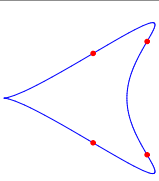
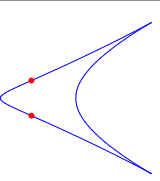
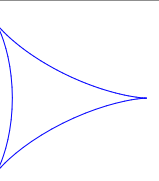
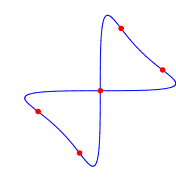
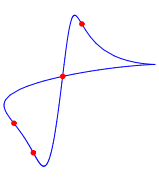
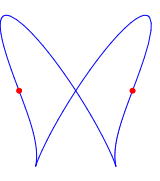
$\kappa$	0		1		2		3
$\iota$	6		4		2		0
$\eta$	0	1	0	1	0	1	0
$\delta$	3	2	2	1	1	0	0

image up to homeomorphism. In fact, for quartics, they determine it up to isotopy in  $\mathbb{RP}^2$  (see Theorem 5.2, Remark 5.4, and Theorem 5.6).

**Theorem 5.1.** *For every quadruple  $(\kappa, \iota, \eta, \delta)$  given in Table 5.1, there is a (real) plane quartic with  $\kappa$  cusps,  $\iota$  inflection points,  $\eta$  nodes, and  $\delta$  solitary points.*

*Proof.* The theorem can be deduced from the classification of real rational quartics given by Gudkov et. al. [14] (see also the papers [22] by Klein, [45] by Zeuthen, and [43] by Wall). But instead, we exhibit examples of curves with each possible ramification and singularity. In Table 5.2, we show a maximally inflected plane

TABLE 5.2. Quartics realizing all topological types allowed by Corollary 1

		$\kappa, \iota$			
		0,6	1,4	2,2	3,0
$\eta$	0				
	1				not allowed



quartic curve for each quadruple  $(\kappa, \iota, \eta, \delta)$  from Table 5.1. These were generated using a computer calculation, by first solving for the centers of projection (as described, for example in [39, Section 2] or in [37, Section 2]), and then drawing the resulting parameterized curve using MAPLE. (This method is used to draw most of the curves we display.) The positions of the flexes are marked with dots and the curve with 6 flexes and one node has 2 flexes at its node.  $\square$

Recall from Section 2.2 that Conjecture 2.6 holds for plane quartics, and so Theorem 2.7 provides strong information about deformations of plane quartics. We explore some consequences of this fact. There is a single isotopy class of sextuples of distinct points on  $\mathbb{RP}^1$ . Thus, given any sextuple  $S = \{s_1, \dots, s_6\}$  of distinct (non-ordered) points, there is an isotopy from the sextuple  $\{-3, -1, 0, 1, 3, \infty\}$  to  $S$ .

Schubert calculus gives five rational quartics with six given points of inflection; thus, each of the five maximally inflected quartics with flexes at  $S$  are deformations of one of the five maximally inflected curves with flexes at  $\{-3, -1, 0, 1, 3, \infty\}$ , which we show in Figure 5.1. (Each nodal curve has two flexes at its node.) We

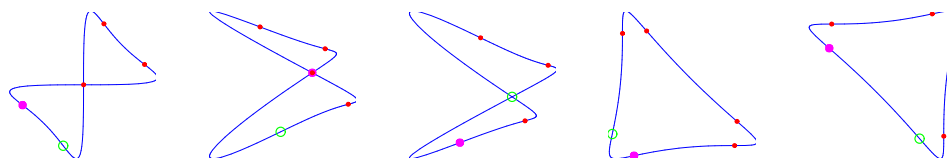


FIGURE 5.1. The 5 curves with flexes at  $\{-3, -1, 0, 1, 3, \infty\}$

indicate the differences in the parameterizations of these curves by labeling the flex at  $-3$  by the thick dot and the flex at  $-1$  by the circle.

**Theorem 5.2.** *For each sextuple  $S = \{s_1, \dots, s_6\}$  of distinct points in  $\mathbb{RP}^1$ , there are exactly five maximally inflected quartics with flexes at  $S$ . Two of these five quartics have three solitary points and no real nodes, and three have two solitary points and one real node. Moreover, the possible arrangements of solitary points and bitangents are as shown in Fig. 5.2 for each of these types of curves.*

*Proof.* Since the five quartics in Fig. 5.1 show that the statement is valid for the sextuple  $\{-3, -1, 0, 1, 3, \infty\}$ , it suffices to show that the number of solitary points does not change under a deformation of such a quartic curve.

Any deformation of a quartic with one real node must have one real node; in passing to a curve without a real node, the deformation would include a curve with some other singularity, which would necessarily be a ramification point that is not a flex. Since the curve already contains six flexes, this would contradict Proposition 1.1. Thus, every deformation of the first three curves has a single real node.

To see that every deformation of any of the five curves has only ordinary double points, consider the arrangements of bitangents and solitary points on an example. Fig. 5.2 shows the second and fourth curves of Fig. 5.1 together with their bitangents and solitary points. Here, each solitary point is separated from the other real,

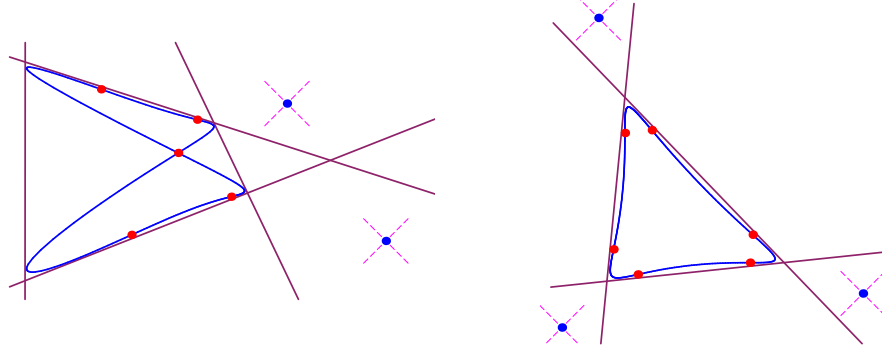


FIGURE 5.2. Bitangents and solitary points

solitary or not, points of the curve by real double tangents. The same phenomena takes place for each of the five maximally inflected curves.

Since the curves have degree 4, the bitangents cannot meet the curves in additional points, by Bézout's theorem. Similarly, a bitangent cannot be tangent to a flex. Thus in a deformation, the flexes are confined to the arcs of the curve between two points of tangency to bitangents, and the number of bitangents does not change. Similarly, the solitary points cannot meet another point of the curve, to do so, they would first have to meet a bitangent, which is not possible, by Bézout's theorem. This completes the proof of the theorem.  $\square$

**Remark 5.3.** For reference, in Fig. 5.3, we give the solitary points and bitangents, as well as parameterizations  $[\varphi_0(s, t), \varphi_1(s, t), \varphi_2(s, t)]$  for the two curves shown in Fig. 5.2. The decimals are numerical approximations.

Interestingly, the two solitary points of the nodal curve lie on the line  $y = 3 - \frac{17}{12}x$ , and this line also meets the two intersection points of the bitangents to the right of and above the quartic. Similarly, the node lies at the intersection of the pairs of lines passing through the other intersection points of the bitangents. The other two nodal quartics with flexes at  $\{-3, -1, 0, 1, 3, \infty\}$  shown in Fig. 5.1 also have this property. This is clear for the second asymmetric nodal quartic, whose image in  $\mathbb{RP}^2$  is isomorphic to this quartic. The symmetric nodal quartic has its solitary points at  $[1, 0, 0]$  and  $[0, 1, 0]$ , and the corresponding pairs of bitangents are parallel, so all four points lie on the line at infinity. The statement about its node easily follows from symmetry considerations.

By Theorem 3.1, for such quartics, we have  $\delta = \tau + 2$ , where  $\delta$  and  $\tau$  are the numbers of solitary points and solitary bitangents. Thus, the nodal curve has no solitary bitangents, while the other curve has a single solitary bitangent, which is the line at infinity. To see this, note that  $\varphi_2$  factors as  $(s^2 + 3t^2)^2$ , and so the line at infinity is a bitangent with tangencies at the points of the curve where  $[s, t] = [\pm\sqrt{-3}, 1]$ . Evaluating, we see that these are the points  $[-1 \pm 2\sqrt{-2}, 1, 0]$ .

**Remark 5.4.** Theorem 5.2 shows that the isotopy type of the embedding of a quartic with six flexes into  $\mathbb{RP}^2$  does not change under an isotopy of the positions

NODAL CURVE	ANODAL CURVE
Parameterization	Parameterization
$\varphi_0(s, t) = s^4 - 6t^2s^2 + 9t^4$	$\varphi_0(s, t) = ts^3 - \sqrt{6}t^2s^2 - \frac{3\sqrt{6}}{2}t^4$
$\varphi_1(s, t) = \frac{3}{4}s^4 + ts^3 - \frac{3}{2}t^2s^2 + \frac{9}{4}t^4$	$\varphi_1(s, t) = t^3s + \frac{\sqrt{6}}{2}t^4$
$\varphi_2(s, t) = s^4 + ts^3 - 3t^2s^2 - 2t^3s + \frac{15}{2}t^4$	$\varphi_2(s, t) = s^4 + 6t^2s^2 + 9t^4$
Solitary Points	Solitary Points
(1.514769, 0.854076)	$(\frac{7}{16} - \frac{\sqrt{6}}{32}, -\frac{7}{48} + \frac{13\sqrt{6}}{288})$
(2.088892, 0.040735)	$(-\frac{7}{16} - \frac{\sqrt{6}}{32}, \frac{7}{48} + \frac{13\sqrt{6}}{288})$
	$(-\frac{\sqrt{6}}{4}, -\frac{\sqrt{6}}{36})$
Bitangents	Bitangents
$x = 0$	$y = 0.03402069 - \frac{1}{3}x$
$y = 3 - \frac{25}{12}x$	$y = 1.4984552 + 3.2996598x$
$y = 1.07221014 - 0.31889744x$	$y = -.0015448 + .0336735x$
$y = -.17859312 + 0.39336553x$	

FIGURE 5.3. Parameterizations, solitary points, and bitangents for the curves shown in Fig. 5.2

of the flexes in  $\mathbb{RP}^1$ . In fact, something stronger is true. The space of all possible positions of flexes modulo orientation-preserving reparameterizations (i. e., the quotient of  $(\mathbb{RP}^1)^6$  minus all the diagonals by  $\mathrm{SL}(2, \mathbb{R})$ ) is contractible (to see this, fix the position of one flex), and because of the confinement property of flexes and the tangent points of the bitangents (see the proof of Theorem 5.2), none of the five quartics in Fig. 5.1 can be deformed into any other.

However, two flexes can converge, as, for example, the flexes shown in Fig. 5.1 by the thick dot and the open circle. When the positions of these two flexes coincide, the second, third, and fourth curve acquires a cusp, while the first and fifth acquire a *planar point* with ramification sequence  $(0, 1, 4)$ . Suppose that we fix five of the ramification points and let the sixth move along  $\mathbb{RP}^1 \simeq S^1$ . At every position of the sixth point, we get a maximally inflected curve which has six flexes, except when the sixth point collides with one of the fixed points, and then we get a curve with either a cusp or a planar point<sup>1</sup>.

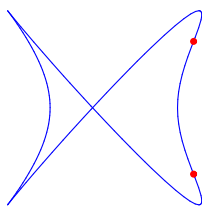
We have used symbolic methods to determine what happens when the sixth point moves. The number of nodes is always preserved, and when the sixth point returns to its original position, we get a curve different from the original one. In fact, this monodromy action cyclically permutes the three nodal quartics shown in

<sup>1</sup>Motion pictures of families of curves with such moving ramification points may be found at [www.math.tamu.edu/~sottile/pages/inflected](http://www.math.tamu.edu/~sottile/pages/inflected).

Fig. 5.1 and interchanges the two quartics without nodes. This can also be inferred by visualizing the effect of moving the flexes labelled by the open circles in Fig. 5.1.

**Remark 5.5.** The proof of Theorem 5.2 given below is based on the following property going back to Zeuthen [45]: each of the components of the complement in  $\mathbb{RP}^2$  of the double and solitary tangents contains at most one connected compact component of the real locus. It is valid for any quartic, and a similar argument shows that every maximally inflected rational quartic has only ordinary double points, apart from the singularities at the ramification points; thus, we obtain the analogue of Theorem 5.2 for all maximally inflected rational quartics, which we state below after introducing yet another necessary notion.

Underlying a deformation of a maximally inflected curve is the isotopy type of its ramification. Consider the curves with two flexes and two cusps in Table 5.2. In the curve with no real nodes,  $\eta = 0$ , the cusps ( $\kappa$ ) and flexes ( $\iota$ ) occur along  $\mathbb{RP}^1$  in the order  $\kappa\kappa\iota\iota$ , while for the curve with a single real node, the order is  $\kappa\iota\kappa\iota$ . There is another curve with two cusps, two flexes, and one node.

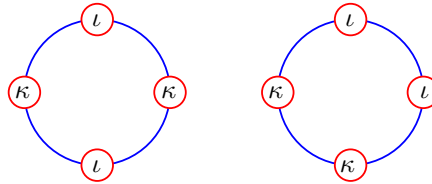


Here, the ramification occurs in the order  $\kappa\kappa\iota\iota$ . Thus quartics with the ramification  $\kappa\kappa\iota\iota$  may have no or one real node. Interestingly, all quartics with order  $\kappa\iota\kappa\iota$  have one real node; we explain this below.

Thus, we can refine the problem of classifying maximally inflected plane rational curves so as to take into account the isotopy type of the ramification in  $\mathbb{RP}^1$ . The different isotopy types of the placement of ramification data  $\alpha^1, \dots, \alpha^n$  on  $\mathbb{RP}^1$  are encoded by combinatorial objects called necklaces:  $n$  ‘beads’ with ‘colors’  $\alpha^1, \dots, \alpha^n$  are strung along  $S^1 = \mathbb{RP}^1$  to make a necklace. Given a maximally inflected curve  $\varphi: \mathbb{P}^1 \rightarrow \mathbb{P}^2$ , we may reverse the parameterization of  $\mathbb{RP}^1$  to obtain another maximally inflected curve, whose necklace is the mirror image of the necklace of the original curve. Thus, we identify two necklaces that differ only by the orientation of  $\mathbb{RP}^1$ . For example, Fig. 5.4 shows the two necklaces with two beads each of color  $\kappa$  and  $\iota$ . To a maximally inflected curve with ramification  $\alpha^1, \dots, \alpha^n$ , we associate a necklace with beads of colors  $\alpha^1, \dots, \alpha^n$ , where the bead with color  $\alpha^i$  is placed at the corresponding point of ramification on  $S^1 = \mathbb{RP}^1$ .

To state the promised generalization of Theorem 5.2, we denote the space of maximally inflected rational quartics with a given necklace  $\Omega$  by  $C(\Omega)$  and

the space of the placements of the necklace in  $\mathbb{RP}^1$  by  $P(\Omega)$ . This generalization can be proven in a manner similar to the proof of Theorem 5.2 (cf. Remark 5.5), and so we omit its proof.

FIGURE 5.4. Necklaces for  $\kappa = \iota = 2$ .

**Theorem 5.6.** *Maximally inflected rational quartic curves in  $\mathbb{RP}^2$  admit arbitrary deformations, and the isotopy type of the image (together with the bitangents and solitary points) in  $\mathbb{RP}^2$  is a deformation invariant. Moreover, for any necklace  $\Omega$ , the canonical projection  $C(\Omega) \rightarrow P(\Omega)$  is a trivial covering.*

*In particular, the only isotopy types of the image are those given in Table 5.2 and for any given necklace the number of maximally inflected quartics whose images in  $\mathbb{RP}^2$  have a given isotopy type does not depend on the placement of the necklace.*

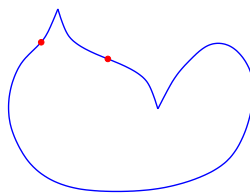
(Note that there are rational real quartics with one real node and nested loops, but such a quartic cannot have solitary points, and, as follows from Corollary 3.3, it cannot be maximally inflected.)

We refine Question 3.5 as follows.

**Question 5.7.** Given a necklace  $\Omega$  with beads of color  $\alpha^1, \dots, \alpha^n$ , which are ramification data for degree  $d$  plane curves, what are the possibilities for the numbers  $\delta$ ,  $\eta$ , and  $c$  of solitary points, real nodes, and complex nodes of a maximally inflected curve of degree  $d$  whose associated necklace is  $\Omega$ ? Is any of them (or their arrangements) a deformation invariant?

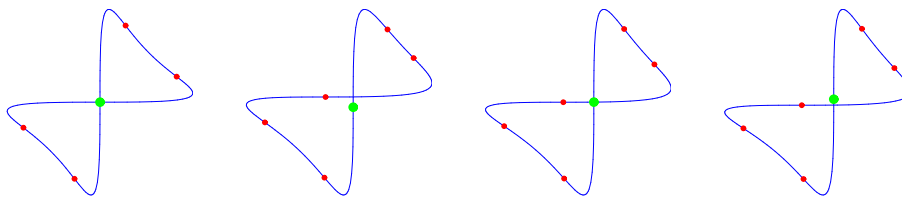
A weaker problem is to give bounds better than those of Corollary 3.3 for these numbers. For example,  $\eta = 0$  is not possible for the necklace on the left in Fig. 5.4. Indeed, the two curves for this necklace with ramification points at  $\{-1, 0, 1, \infty\}$  are obtained from each other by reversing the parameterization, and so their images in  $\mathbb{RP}^2$  are equal. This common image, which has a single node, is given in the second row of the third column of Table 5.2. An application of Theorem 5.6 completes the proof.

A more direct proof that  $\eta = 0$  is impossible is as follows. Suppose that such a curve has no real nodes. Then the real locus is a topologically embedded two-sided circle. The inflection points are the points where the concavity (which can be represented by an oriented curvature covector vanishing on the tangent direction) of the curves changes. Note that the concavity does not change at a cusp, and because of a flex between the cusps, one cusp is pointed outward and another inward, as shown below.

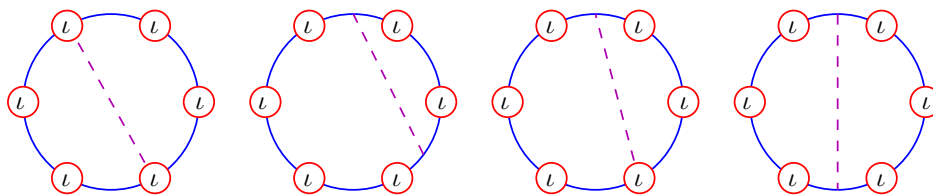


Now, consider the line through the cusps. It must meet the curve in at least one additional point, or the local intersection number of this line with a cusp is 3; thus, intersection number of the line with the quartic is at least 5. This contradicts Bézout's theorem and proves the impossibility of such a curve.

Now, consider the possible arrangements of the nodes on a maximally inflected curve. There are four possibilities for the arrangements of the node with respect to the flexes in a maximally inflected quartic with six flexes and one real node. These possibilities are



The position of the node can be represented in the associated necklace by drawing a chord joining the two points whose images coincide:



These are the only four possibilities. Indeed, since a bitangent to a quartic can neither be tangent at a flex nor meet another point of the curve, the flexes and nodes of such a quartic are constrained to lie on the arcs of the curve between the contacts of its bitangents. Since, by Theorem 5.2, every maximally inflected quartic with a single node can be deformed into one of those shown in Fig. 5.1, this constraint rules out the other possibilities for the chord in the necklace of such a quartic.

Let us refine Questions 3.5 and 5.7.

**Question 5.8.** Which necklaces  $\Omega$  with beads of color  $\alpha^1, \dots, \alpha^n$  and  $\eta$  chords occur as maximally inflected curves? Given such a chord diagram, what are the possible numbers of solitary points (and hence of complex nodes)?

Ignoring the beads, we obtain a circle with  $\eta$  chords, and the same questions may be asked about these diagrams. Such pure chord diagrams encode, together with the number of solitary points, the topology of the image as an abstract, not embedded in  $\mathbb{RP}^2$ , topological space, and therefore the classification of chord diagrams is a necessary part of any topological classification.

We make one final observation concerning the orientation of the cusp of a maximally inflected quartic with four flexes and one cusp. The examples in Table 5.2 both have cusps pointing into the unbounded region in the complement of the curve. This is necessarily the case. If a quartic has a cusp pointing into a bounded region, then it cannot have real nodes or solitary points, because the line joining the cusp

with such a real double point would meet the curve in at least two additional points, in contradiction to Bézout's theorem. But Corollary 3.3 requires such a maximally inflected curve to have at least one solitary point. While we are presently unable to formulate a general question concerning constraints on the arrangement of cusps (and other ramification), it is likely that there are further topological restrictions.

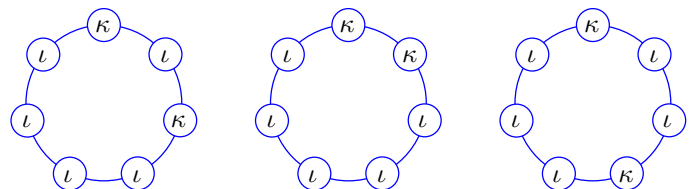
Further pictures of maximally inflected quartics with more general ramification and many quintics can be found on the web<sup>2</sup>.

## 6. MAXIMALLY INFLECTED PLANE QUINTICS

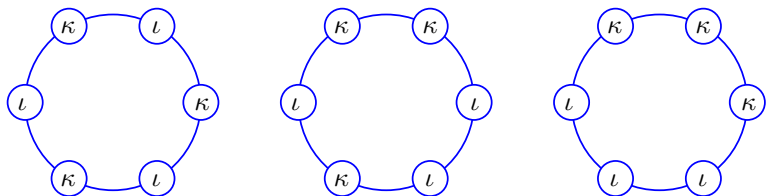
Unless explicitly stated otherwise, all quintics in this section are assumed to have cusps and flexes as their only ramification points and ordinary double points as their only other singularities. In Section 6.1, we additionally assume that all quintics have three real nodes.

**6.1. Quintics with cusps and flexes having three nodes.** The construction of Theorem 4.1 for quintics gives maximally inflected quintics whose number  $\kappa$  of cusps is 0, 1, 2, or 3. We obtain a quintic with the maximum number, four, cusps by taking the dual of a maximally inflected quartic with one cusp and four flexes, whose existence is asserted by Theorem 5.1. The existence of such curves is also a consequence of Theorem 2.3, as the ramification indices of cusps and flexes are special, in the sense of that theorem.

Since Theorem 2.3 is an existence result and gives no information about the geometry of the resulting curve, it is very instructive to look at specific examples coming from constructions. As an example, consider the possible necklaces of such curves, which is interesting only for two or three cusps. There are three possible necklaces of maximally inflected quintics with two cusps and five flexes,



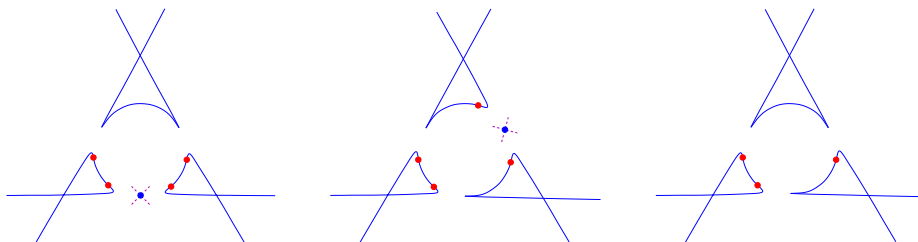
and three possible necklaces with three cusps and three flexes.



The leftmost necklace shown for both values of  $\kappa$  is that of the corresponding quintic in Fig. 4.1. We can realize three of the remaining four necklaces using the variant of the construction of Theorem 4.1 where we use a different local model for the

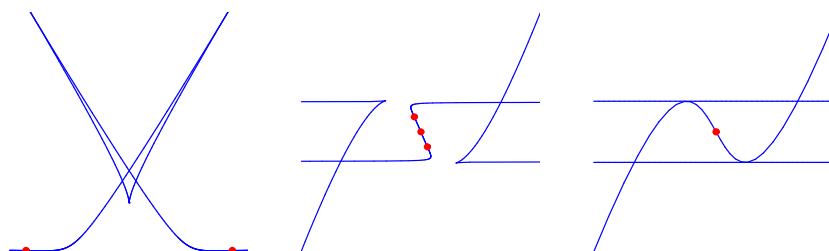
<sup>2</sup>See [www.math.tamu.edu/~sottile/pages/inflected](http://www.math.tamu.edu/~sottile/pages/inflected).

perturbation of a tacnode into a cusp, as explained in Remark 4.3. This gives the three quintics shown below.



These correspond to the remaining two necklaces of curves with two cusps and to the second necklace for curves with three cusps.

What is missing is a quintic with three consecutive cusps. There are six maximally inflected plane quintics with cusps at  $\infty$  and  $\pm 3$  and with flexes at 0 and  $\pm 1$ , which we have computed symbolically. One of these six curves is particularly interesting. The two pictures on the left below are two different views of this curve. In the first, we put one flex at infinity, and in the second, one cusp and one node at infinity.



The second view of this curve suggests that it is a perturbation of the singular reducible curve shown on the right.

**6.2. Solitary points of quintics with cusps and flexes.** In this section, we address the finer classification of Question 5.7 concerning the possible numbers of solitary points and nodes in a maximally inflected quintic having flexes and cusps with a given necklace. Here, the situation is quite intricate, and our knowledge at present is not definitive. It is, however, based upon extensive experimental numerical evidence. Briefly, for quintics with no or one cusp, all the possibilities occur, but for most necklaces with two or three cusps, we have not yet seen all the possibilities for the numbers of solitary points. We have also observed a fascinating global phenomenon related to (but weaker than) the very strong classification result (Theorem 5.6) we have obtained for quartics.

Let us recall Corollary 3.3 concerning quintics with flexes and cusps.

A rational quintic plane curve with  $\kappa$  cusps and  $9 - 2\kappa$  flexes has genus  $g_\kappa = 6 - \kappa$ . Maximally inflected quintics with four cusps ( $\kappa = 4$ ) are dual to maximally inflected quartics with four flexes and one cusp; they were considered from different points of



view in Remarks 3.6, 3.7, and 3.8. In particular, we have shown the impossibility of such a quintic with no solitary points and no nodes. In Fig. 6.1, we give examples of maximally inflected quintics with four cusps and one flex realizing the two remaining possibilities. The quintic on the left has its flex at infinity.

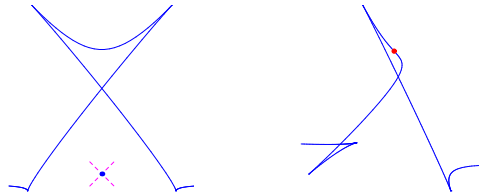


FIGURE 6.1. Quintics with four cusps

Consider quintics with three or fewer cusps. By Corollary 3.3, the number  $\delta$  of solitary points of such a quintic satisfies

$$3 - \kappa \leq \delta \leq 6 - \kappa =: g_\kappa,$$

or  $3 \leq \delta + \kappa \leq 6$ . The number  $c$  of complex nodes is an even number between 0 and  $g_\kappa - \delta$ , and the number  $\eta$  of nodes satisfies  $\eta + c = g_\kappa - \delta$ . Thus the possibilities for these numbers for quintics are in one-to-one correspondence with the cells of the diagram

		$\delta + \kappa$			
		3	4	5	6
$c$	0				
	2				

where the first and second rows correspond to the values of 0 and 2 for  $c$  and the columns correspond to the possible values for  $\delta + \kappa$ , from 3 on the left to 6 on the right. The first column of Table 6.1 lists the possible necklaces for quintics with at most three cusps. The filled cells of the diagrams in its second column indicate the observed cardinalities of solitary points, complex nodes, and nodes.

The third column of Table 6.1 records an interesting global phenomenon we have observed. Recall from Section 1.2 that, for a given collection of ramification data  $\alpha^1, \dots, \alpha^n$ , there is the same number  $N := N(\alpha^1, \dots, \alpha^n)$  of rational curves having that ramification at specified points. We observe that the number of these  $N$  curves having a given number of solitary points appears to depend only on the necklace, and not on the placement of the ramification. We record this in the third column, where we list the number of the curves for each of the possible values of  $\delta + \kappa$  between 3 and 6, from left to right. The last two columns of Table 6.1 give the number  $N(\alpha^1, \dots, \alpha^n)$  of rational quintics of this type having a given choice of ramification and the number of choices of ramification for which we computed all  $N(\alpha^1, \dots, \alpha^n)$  quintics and determined their numbers of solitary points, complex nodes, and real nodes.

The following conjecture is a more precise formulation of the observation made in the third column of Table 6.1.

TABLE 6.1. Observed numbers of nodes and solitary points of quintics

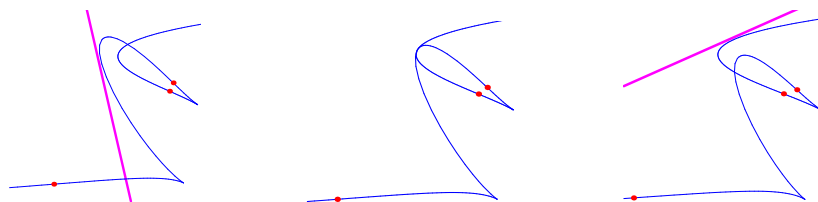
Necklace	Observed $\delta, \eta, c$	$\delta + \kappa$				$N$	Number Tested
		3	4	5	6		
$KKKUU$		1	3	2	0	6	985
$KKUKU$		2	3	1	0	6	986
$KUKUKU$		5	0	0	1	6	986
$KKUUUU$		3	5	3	0	11	731
$KUKUUUU$		5	4	1	1	11	731
$KUUUUUU$		4	5	2	0	11	731
$KUUUUUU$		7	9	4	1	21	1000
$UUUUUUU$		12	18	9	3	42	11,416

**Conjecture 6.1.** The number of solitary points in a maximally inflected plane quintic is a deformation invariant.

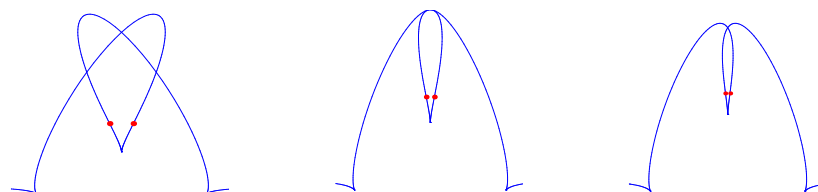
In particular, if Conjecture 2.6 (concerning non-degeneracy) holds for quintics, then the number of curves having a given ramification and given number of solitary points depends only on the necklace of the ramification, that is, only on the relative positions of the points of ramification. This assertion is similar to Theorem 5.6 for quartics, but it is weaker, because the topology of the embedding of a quintic can change under an isotopy. (We give an example below.) If Conjecture 2.6 holds for quintics, then an argument similar to that used in the proof of Theorem 5.6 and involving isolation of solitary points by bitangents and Bézout's theorem might be sufficient to prove Conjecture 6.1.

An instructive example is provided by quintics whose ramification consists of three cusps at  $\pm 1$  and  $\infty$  and three flexes at  $0$  and  $\pm t$ , where  $1 < t < \infty$ . For such quintics, the cusps and flexes alternate; for a given choice of ramification, there are six curves. Letting  $t$  vary, we obtain six families of curves, which are deformations of the curves at any value of  $t$ . All the curves in one family have three solitary points, while those in the other five families have no solitary points. Two of these five families consist of curves with three nodes. When  $t = 3$ , each curve in the remaining three families has two smooth branches with a point of contact of order three. Each of the other curves in two of these three families has a single node and two complex nodes. These two families differ only by a reparameterization of their curves. Below, we show curves from one of them at  $t = 2.9, 3$ , and  $3.1$ . All the

three curves have a cusp at infinity. The line joining the complex nodes is drawn in the first and third pictures. Despite appearances, it is not tangent to the curve. If it were tangent, then it would have intersection number with the curve at least 6, in contradiction to Bézout's theorem.



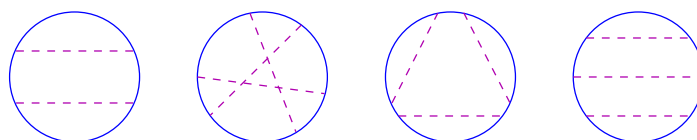
The curves in the third family have one node and two complex nodes for  $t > 3$  and three nodes for  $t < 3$ . We show curves from this family at  $t = \frac{5}{2}$ , 3, and  $\frac{7}{2}$ . The horizontal line in the third picture is the real line meeting the two complex nodes; all the three curves have a flex at infinity.



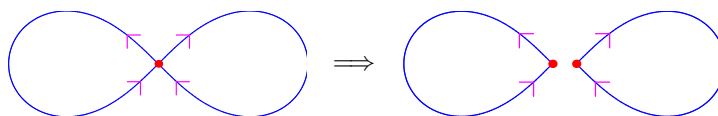
These curves, like those shown in Section 4 and in Fig. 6.2 below, were drawn using computer algebra systems MAPLE and SINGULAR [11]. We first computed the centers of projection by specifying all curves with a given ramification and formulating and solving the problem for the Grassmannian in local coordinates. Having the centers, we computed parameterizations and then set up and solved the equations for the double points. We plotted the curves whose embedding we wanted to study, and for those displayed in this paper, we created PostScript files which were then edited to show important features, such as flexes and solitary points.

**6.3. Chord diagrams of quintics and embeddings.** In this section, we classify the possible chord diagrams of maximally inflected quintics and discuss the possible topology of the image as a subset of  $\mathbb{RP}^2$  (ignoring possible solitary points). Since the proof we give does not use flexes, it also classifies the chord diagrams of quintics with at most three nodes.

**Theorem 6.2.** *There are six possibilities for the chord diagram of a maximally inflected quintic in  $\mathbb{RP}^2$ . The four possibilities for diagrams with two or more chords are shown below.*



*Proof.* By Corollary 3.3, a maximally inflected quintic can have at most three nodes. Recall that the degree of any (piecewise smooth) closed curve in  $\mathbb{RP}^2$  is well-defined modulo 2, and this degree is additive, again modulo 2. Consider an image of  $\mathbb{RP}^1$  in  $\mathbb{RP}^2$  with a node. Under the orientation inherited from  $\mathbb{RP}^1$ , the node has two incoming arcs and two outgoing arcs. We may split the curve into two pieces at the node by joining the incoming arc of one branch with the outgoing arc of the other branch. We illustrate this splitting below.



The first case to consider is that of a quintic with two nodes. In the chord diagram of such a quintic, the two chords cannot cross. If they did, then split the quintic at a node. The resulting two closed curves have exactly one intersection point (the other node), and so each piece necessarily has odd degree. It follows that their union, the original quintic, has even degree, which is a contradiction.

Suppose now that we have a quintic with three nodes. The same argument as above forbids any chord diagrams containing a chord that meets exactly one other chord, and so the only possibilities are as claimed. Thus, we get only one chord diagram with two chords (two non-intersecting chords) and three chord diagrams with three chords (three pairwise intersecting chords, three sides of a hexagon, and three parallel chords).  $\square$

Recall that a pseudoline is a closed curve in  $\mathbb{RP}^2$  which has odd degree and whose complement is connected (it is a one-sided curve). An oval is a two-sided closed curve which necessarily has even degree. We enumerate the possible topological embeddings of  $\mathbb{RP}^1$  given by a maximally inflected quintic, beginning with the classification of chord diagrams.

If there is a single node, then the curve must look like a pseudoline with a loop. Splitting the curve at the node and applying the Jordan curve theorem to the loop in the complement of the pseudoline shows that the loop is two-sided. Thus, there is a unique possible topological embedding. Such a maximally inflected quintic is shown in Fig. 6.2(a). We do not draw the solitary points.

Now, consider a quintic with two nodes whose chord diagram (necessarily) consists of two non-intersecting chords. We split the curve at both nodes to obtain three closed curves that meet only at the nodes. Deforming them slightly away from the nodes, we see that at most one of them is a pseudoline (because any two pseudolines meet); so exactly one resulting curve is a pseudoline, since the original curve is a quintic. As before, the other components are then 2-sided ovals. These ovals cannot be nested. Indeed, consider a line through a solitary point or a singular ramification point (a maximally inflected quintic must have one such point) that meets the nest. Its intersection number with the quintic is at least 6, which contradicts Bézout's theorem.

Thus, there are two possibilities. Either the pseudoline meets both ovals (see Fig. 6.2(b)) or the pseudoline meets only one oval, which then meets the other (see Fig. 6.2(c)). As above, the ovals cannot be nested. Each open circle in Fig. 6.2(c) represents two flexes that have nearly merged to create a planar point.

Now, suppose that there are three pairwise intersecting chords. If we split the curve at *one* node, we obtain a pseudoline and an oval, which have two additional points of intersection. There is only one possibility for this configuration, and we have already seen it in the last figures of Sections 6.1 and 6.2. We show yet another such curve below in Fig. 6.2(d).

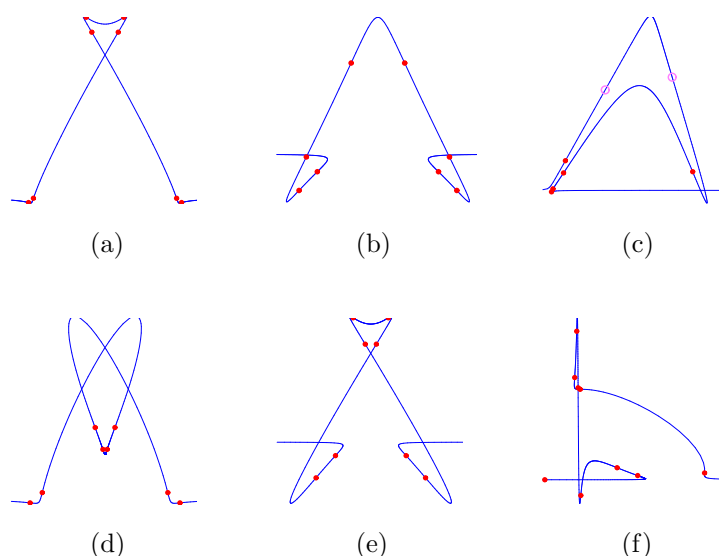


FIGURE 6.2. Maximally inflected quintics realizing different embeddings

If the three chords are sides of a hexagon, we can split the curve at each of its nodes to obtain a pseudoline and three loops. One piece meets the other three, and these three are disjoint from each other. If the pseudoline meets the three loops, there are two possibilities for the arrangement of the three loops along the pseudoline. They either alternate (as shown in Fig. 6.2(e) or in Fig. 4.1) or not. Non-alternating loops are prohibited by Proposition 6.3 below. We have not yet observed the remaining possibility of such a quintic where the pseudoline meets one loop, which then meets the other two loops. A schematic is given in Fig. 6.3(a).

Lastly, we have the possibility of three parallel chords. As before, we split the curve at each of its nodes to obtain one pseudoline and three ovals, and the ovals cannot be nested. There are two components each of which meets one component, and two remaining components each of which meets two components. Thus, the pseudoline either meets two ovals (see Fig. 6.2(f)) or it meets only one. This last case has not yet been observed, but we provide a schematic of it in Fig. 6.3(b).

**Proposition 6.3.** *In a maximally inflected quintic consisting of a pseudoline with three loops, the loops must alternate along the pseudoline.*



FIGURE 6.3. Possible embeddings for quintics that have not been observed

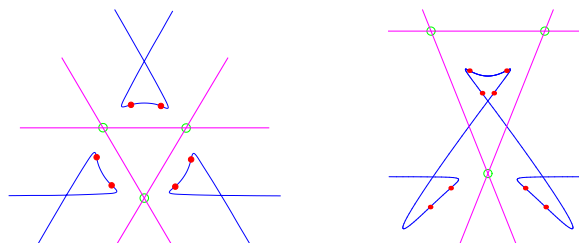
*Proof.* Suppose that we have a maximally inflected quintic whose embedding consists of a pseudoline with three loops attached to the pseudoline. Such a curve has three nodes. By Corollary 3.2, a maximally inflected quintic with three nodes can have only flexes, planar points, cusps, or points with ramification sequence  $(0, 1, 5)$ . Let  $\kappa$  be the number of its cusps, which is at most 3—otherwise Corollary 3.3 restricts the number of nodes to be at most 2. Since the curve must have at least  $3 - \kappa$  solitary points and  $6 - \kappa$  double points in all, it has exactly  $3 - \kappa$  solitary points.

Gudkov's extension [13] of Brusotti's Theorem [3] allow us to perform independent smoothings of the cusps and nodes to obtain a smooth quintic. We smooth each cusp as in the local model of Fig. 3.2, smooth each node to detach its loop from the pseudoline, and smooth each solitary point to obtain an oval. Thus, we obtain a smooth plane quintic with six ovals, which is an  $M$ -curve. Consider the complex orientations of the ovals of this  $M$ -curve. Each of the three ovals obtained from the loops has positive orientation with respect to the pseudoline (see Fig. 3.2), and the complex orientation formula [25] requires that the other three have the opposite orientation.

(This implies, in particular, yet another topological restriction on maximally inflected quintics: *all cusps must occur on loops, and none on the pseudoline*. If a cusp occurred on the pseudoline, then the oval obtained by smoothing it would have positive orientation with respect to the pseudoline, which is forbidden by the complex orientation formula. Furthermore, any cusp on a loop must point outward, for otherwise a line from the cusp to a different loop would have intersection number at least 6 with the quintic, which is impossible. It is clear from Fig. 3.2 that such outward-pointing cusps smooth to ovals negatively oriented with respect to the pseudoline.)

Let us pick a point inside each of the three negative ovals, connect every pair of points by a line, and study the configuration of these lines with respect to the quintic. By Bézout's theorem, each of these bisecants intersects the one-sided component at one point and its two ovals at two points each. Hence, the configuration does not depend on the choice of the three points. Since all  $M$ -quintics are deformation equivalent, i. e., belong to one connected family of nonsingular quintics (see [20]; a minor correction of the proof is given in [5]), the configuration does not depend on the choice of the  $M$ -quintic either. Examining any of the various known examples, we find a configuration similar to that in the pictures below, where we have drawn the lines joining the solitary points of a rational quintic (the two pic-

tures correspond to similar curves in different projections), leaving smoothing to the imagination of the reader. The solitary points are indicated by open circles.



The bisecants form four triangles in  $\mathbb{RP}^2$ , among which is a distinguished triangle not meeting the pseudoline and containing no ovals (i. e., no ovals arising from the loops). The pseudoline divides each of the other triangles into two components; the component adjacent to the edge of the distinguished triangle which is not intersected by the pseudoline contains an oval. This configuration is equivalent to the statement that the loops alternate along the pseudoline.  $\square$

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