Personalities of Curves

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Contents

Chapter 0

Basic Questions

((The following is more material for a preface than a preface...))

I'm very well acquainted, too, with matters mathematical,
I understand equations, both the simple and quadratical,
About binomial theorem I am teeming with a lot o' news,
With many cheerful facts about the square of the hypotenuse.

—Gilbert and Sullivan, Pirates of Penzance, Major General's Song

Be simple by being concrete. Listeners are prepared to accept unstated (but hinted) generalizations much more than they are able, on the spur of the moment, to decode a precisely stated abstraction and to re-invent the special cases that motivated it in the first place.

–Paul Halmos, How to Talk Mathematics

Another damned thick book! Always scribble, scribble, scribble! Eh, Mr. Gibbon? — Prince William Henry, upon receiving the second volume of The History of the Decline and Fall of the Roman Empire from the author.

The most primitive objects of algebraic geometry are affine algebraic sets—subsets of $\mathbb{R}^n$ or $\mathbb{C}^n$ defined by the vanishing of polynomial functions—and the maps between them. But already in the first half of the 19th century geometers realized that there was a great advantage in working with varieties in complex projective space, treating affine varieties as projective varieties minus the intersection with the plane at infinity and real varieties as the fixed points of the complex involution. One sees this in the simplest examples: the ellipses, hyperbolas and parabolas in the real affine plane are all the same in the complex projective plane; the difference is only in how they intersect the line at infinity. A difficulty with the projective point of view is that on a connected projective variety there are no non-constant functions at all (reason: a function on a projective variety is a map to the affine line; since the image of a projective variety is again projective, the image would be a single point.)

Starting with Riemann in the 1860s and culminating in the scheme theory of Grothendieck in the 1950s, algebraic varieties were treated in a way independent of any embedding: An algebraic variety is is a topological space with a sheaf of locally defined polynomial functions. Many interesting aspects of geometry have to do not with single abstract varieties, but with maps between them, and in particular with embeddings in projective spaces. In general, maps between varieties can be described by their graphs, which are again varieties. But for the special case of maps to projective spaces, the theory of *linear series* is usually a more convenient description. The collection of all linear series on a variety reflects some of its best understood invariants.

0.1 What is a family of varieties?

0.1.1 Hilbert schemes

Definition, universal property; construction

examples of hypersurfaces and linear spaces

tangent space

Fundamental problem: irreducible components of Hilb parametrizing smooth curves and their dimensions

Example 0.1.1. conics in $\mathbb{P}^3$ (refer to 3264)

0.1.2 Moduli spaces of curves

basic properties of M_g (coarse rather than fine; fine over automorphism-free curves)

dimension $3g-3$, irreducible

(just statements, w/ref to Harris-Morrison)

Bibliography

Walker [Walker] Walker.

Hartshorne [Hartshorne] Hartshorne Ch 4

Fulton [Fulton] Fulton Alg curves

Eisenbud-Harris [Schemes] Schemes

Eisenbud-Harris [3264] 3264

riffiths-Harris [Griffiths-Harris] (for the Abel-Jacobi stuff)

iffiths-Chinese [Griffiths] (for the Abel-Jacobi stuff)

Mumford [Mumford] -Curves and their Jacobians

Voisin [Voisin] Hodge Theory

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