Reviews: Eisenbud/Harris, “Practical Curves”

R1

This is a largely introductory book on the theory of algebraic curves and their moduli spaces. This is a central part of mathematics that has been studied without interruption for close to 200 years, as a result there are many very good books at various levels, as these authors mention in the Introduction. However, the subject being so prominent and having so many facets, there is a clear need for a different perspective and for presenting a path from the classic book of Hartshorne (on the foundations of algebraic geometry) to an active and modern research topic and this book does precisely this job.

All in all, I regard the book as being quite successful, it offers an informal introduction to the theory of algebraic curves and linear systems that is easier and more informal than the one found in the wonderful book of Arbarello-Cornalba-Griffiths-Harris written in the 1980s.

I warmly recommend that you go an try to get this book for the Cambridge University Press, I am pretty sure that other publishers will do the same.

The book could be used for a course in algebraic curves, and due to the immense reputation of these writers, I have no doubt that many people would do just that.

As often with these two authors, the strength of their writing lies in making readers fall in love with the subject, providing motivation to dig deeper into the subject. In this sense they are absolute masters, and here they show it again. The downside is that sometimes the proofs are somewhat sketchy and the book could use more references. But in general, I judge a book for what it contains and not for what is missing, and by this measure this book is quite successful.

Some comments:
-The title "Practical curves" is puzzling. When I first saw it, I thought of visualization and numerical analysis and not of algebraic geometry. I understand what the authors aim to achieve, but compared to "3264 and all that" (their previous book), this title is less inspired and I recommend they look for alternatives.

-The prerequisite for reading this book is knowing the foundations of modern algebraic geometry as presented in Hartshorne. This is a sound choice. Some facts are introduced as "Cheerful facts" throughout the book. This term is used somewhat liberally, for instance fundamental things like the Euler sequence on p. 36 or the Hodge decomposition, are in the same category as some much less consequential remarks like those in 3.1.4 involving Azumaya algebras. It would be good to achieve some uniformity, as not to confuse the reader.

-Is the book written in char 0, or arbitrary characteristic? First I thought char 0, but then I saw the discussion on Rathmann's mondromy results which are in positive characeristics. At other places, like resolution of canonical curves the authors revert to characteristic zero. It would be good to achieve some uniformity.

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-The choice and especially the order of the chapters is not always logical. Chapters 4 and 5 (Jacobians and theta characteristics) are quite nicely written and at their right place, but then Chapter 6 (moduli and Hurwitz spaces) comes too early, for a reader who is reading just this book. Then in Chapter 8 (curves of genus 4 and 5) the book reverts to a more leisurely pace. I would suggest placing the moduli section later in the book, especially since important facts like the dimension of the Hilbert scheme are only treated in Chapter 18.

-A key topic of the book, the Brill-Noether theorem appears first in Chapter 11 in a section on curves of genus 6! This si also not quite optimal.

-Chapter 13 is devoted to actually proving the Brill-Noether Theorem and here the authors follow their first proof of the theorem from 1983 using cuspidal curves. This is of course fine, though the proof via limit linear series (also given three years later by Eisenbud and Harris) is surely more elegant and instructive, not to mention that it has proved to be more influential. Perhaps the authors may reconsider.

-Chapters 16 (curves on scrolls) and respectively 17 (free resolutions) are quite nicely written.

-Notations: The authors should be more consistent with what they developed in their papers. For instance Example 8.4.5 they introduce the notation M_{g,d}^r to denote the parameter space of pairs (curve, linear systems). But the standard notation for that is \mathcal{G}_{g,d}^r and then M_{g,d}^r is the image inside the moduli space of curves. Eisenbud and Harris wrote several important papers using this notation, I see no reason to abandon it and create confusion.

-These authors, though supremely gifted writers, are not the most reliable when it comes to full references. Having accepted that, they should still make more of an effort to keep some things clear. For instance Thm 7.4.2 states their famous theorem on the Kodaira dimension of M_g for g>23, but them immediately after in 7.4.3 they mention g=22, which suggests that they have in mind more recent works, which however are not cited. They can choose to cite or not to cite, but like this it is confusing for the uninitiated reader. Also on p.143 they state that noone has written down a general curve of genus g>14, which is not quite correct, for Bruno and Verra gave a rational connected parametrization of M_15 that is extremely elegant and simple.

-Theorem 2.5.4 (Riemann-Roch for surfaces) is incorrect as stated.

R2

I did look a bit further into their book, and it confirms my impression from the front matter. I like the conversational, intuitive style, and the fact that exercises are at the end of chapters rather than interrupting the text, the fact that proofs seem to isolate the nontrivial bit rather than going into excessive formality. That said, it is for a more advanced reader (3rd course in algebraic geometry), and some editing is needed.

For instance, the proof of the g+3 theorem on pp. 88-89 illustrates all of this: super-efficient, though not the first proof I'd want a student to see, and with a typo in the definition of \nu: it should read \mu_{g-3}(K_C - E + F).

But I would strongly encourage you to pick up this book, because it seems to offer an excellent "practical" (and up to date) introduction to moduli of curves and Hilbert schemes, written by two experts. What I read of Chapter 6 was beautifully motivated and inviting, much easier to follow than some other (well-regarded) accounts out there.

<u>R3</u>

> Would you consider the subject of this book to be important? Is the subject area expanding, static, or contracting? Is there a sufficient body of established knowledge for the book to not date too quickly?

The subject matter is important, and as algebraic curves have been studied nearly continuously since the 19th century and yet remain relevant, it will not go out of date any time soon. There continue to be fundamental and important results obtained about algebraic curves, and a book like this is an entryway into appreciating that work.

> Is there a need for a book on this subject at the proposed level?

Yes. As the authors point out, there are many books on algebraic curves. As they also point out, this is between the level of Fulton's classical book (from 1969) and the four-author tome Arbarello, et al., which is necessary for advanced work. I found the list of topics very inviting, particularly the way that they build up the need for the theory through examples, and do this over many chapters. It really draws one in.

> Is the manuscript in line with the mainstream of the subject?

Yes.

> Who do you think will be interested in reading the book?

Me, students, and others wanting an inviting introduction to the main topics and players in the theory of algebraic curves.

> Is it written at the right level for this audience? Does it cover the right material for this audience?

Yes, like I said, the way this draws one in and enlivens the topics through examples is seductive and effective.

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> Does this fit a standard course? What course would benefit from this book?

Were I training students to work on curves, I could see teaching the first 1/2 to 1/3 to graduate students, or running a reading seminar on this, with the intent of seeing how fa we could go.

> What do you see as the main strengths of the book?

Excellent writing, the organization, the examples and how they draw the reader into the theory.

> What are the main weaknesses?

Possibly too many topics; it requires strong students to cover all the material. On the other hand, I have lots of books that I read to learn topics from, and if one does not get through everything in a course, then those who are interested may simply work through what is left.

> Are there enough exercises, examples, or illustrations to aid the reader in understanding the material?

These are promised. It would be nice to have more illustrations than the mere 40 that were mentioned.

> Are you able to comment on the standing of the author?

Yes. Both are excellent mathematicians and expositors, and they have fantastic taste.

> Is the title appropriate? Do you have alternative suggestions?

Yes, it is well-chosen for the material.

> Taking everything into consideration, would you recommend publication of this work (or a revised version)?

Yes.

R4
See attached PDF

R4

REVIEW OF PROPOSAL FOR "PRACTICAL CURVES," BY EISENBUD AND HARRIS

The proposed book is an addition to the already sizable literature on algebraic curves—and the even more sizable collection of graduate-level texts on algebraic geometry more generally—by two authors whose contributions to the field are enormous. The goal of the book is ambitious and exciting: to convey, to the fullest extent possible, the scope and limitations of our knowledge in the study of algebraic curves, so a reader is equipped to "read the current literature and work on open problems." In particular, it is example-driven and leaves some results unproven, a wise choice given the stated goal and one of the ways in which this book differs from the existing literature.

There is certainly a place for a book of this type, even given the number of similar books available; a graduate student hoping to work in this field might find a book like this particularly useful to complement an algebraic geometry course, since it quickly builds up enough techniques and motivation to discuss open problems. As a specific example, Chapter 3 includes a nice section on open problems about rational curves, which I suspect is not easily found in other graduate-level books.

All this being said, I find it difficult to recommend publication of the book in its current form. Alongside some smaller issues mentioned below, my primary concern is that the exposition is not clear enough to make the book readable by the intended audience. The authors specifically state that they do not intend for the book to be an encyclopedia; in fact, this seems to be one of the main ways in which they view their book as serving a different purpose from perhaps its closest cousin in the existing literature, the two-volume *Geometry of Algebraic Curves* by Arbarello et al. The effort not to write an encyclopedia is important and worthwhile, but there is more than one way to write an encyclopedia: one is by including too much, but another is by listing facts with no narrative flow. Unfortunately, it is my opinion that while the authors have succeeded impressively in avoiding the first pitfall (their book is quite short given the amount of content it covers), they have fallen prey to the second. An illustrative example is Chapter 3: there is no narrative segue from Section 3.1 to 3.2, and Section 3.2 itself reads like a laundry list of unmotivated and unrelated propositions.

To elaborate further on my concerns about the expositional clarity, the following are some specific examples from Chapter 1:

- The introduction of divisors seems in some sense to be written for a reader who has not yet met these objects, but it would likely be confusing to such a reader. On a smooth projective curve, the authors first define "divisor" (meaning Weil divisor), then define "Cartier divisor," but then immediately begin conflating the two concepts, for instance by referring in Definition 1.1.1 to "divisors modulo linear equivalence" when linear equivalence has been defined as a relation on Cartier divisors.
- Two other issues that might confuse a reader not yet familiar with divisors is that the generalization to higher-dimensional varieties is tucked in a paragraph after Definition 1.1.1, making it easy to miss, and the notion of pullback of a divisor is used on page 21 without being defined.

- In a similar vein, the definition of coherent sheaves seems to be written for an unclear audience. The definition begins quite precisely, but then the notation $L_i|_{U_i \cap U_j}$ is used without being defined. While an assumption of familiarity with the notion of restriction might make sense in some contexts, the definition as stated occupies an odd middle ground between precision and summary.
- Some of the text on page 23 seems to have been written out of order: the notation for a stalk $\mathcal{L}_p$ and maximal ideal $\mathfrak{m}_{X,p}$ is used in the first paragraph and then defined in the second.
- A similar issue appears on pages 24–25: the phrase "the 'family of effective divisors' point of view" appears in one paragraph but then the key observation "Thus a linear system $\mathcal{V} = (\mathcal{L}, V)$ gives rise to a family of effective divisors" appears in the next paragraph.
- On page 25, the term "base locus" is defined twice within a few paragraphs.

In addition to these concerns about the exposition, I would like to mention two other issues that I view as more minor. The first is the description of the contents given in the introduction. The organizational scheme of "alternat[ing] between chapters focused on special cases and chapters developing the general theory" sounds great but does not seem to be borne out by the book's current table of contents, so perhaps this description should be removed or changed. Furthermore, the description of the book's contents is presented as a beautifully clear narrative for the earlier chapters, but the summary of Chapters 14–17 seems to represent a sudden break from that story. The authors might consider whether the narrative could be written in a way that more closely ties in the material of those chapters. If this is not possible, it may be worth considering whether those chapters belong in the book at all, given the stated goal of writing a text that is not encyclopedic.

Lastly, given that a primary motivation for this book is to bring the reader up to speed on current research, a natural concern is that the book may become dated quickly. This may indeed be the case, but I do not view it as a reason not to publish the book. Even if it only connects with one generation of students, a book of this form could have a large impact on future researchers and bring many new minds into the field.