Personalities of Curves

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Chapter 0

Basic Questions

((The following is more material for a preface than a preface...))

I'm very well acquainted, too, with matters mathematical,
I understand equations, both the simple and quadratical,
About binomial theorem I am teeming with a lot o' news,
With many cheerful facts about the square of the hypotenuse.

—Gilbert and Sullivan, Pirates of Penzance, Major General's Song

Be simple by being concrete. Listeners are prepared to accept unstated (but hinted) generalizations much more than they are able, on the spur of the moment, to decode a precisely stated abstraction and to re-invent the special cases that motivated it in the first place.

–Paul Halmos, How to Talk Mathematics

Another damned thick book! Always scribble, scribble, scribble! Eh, Mr. Gibbon? — Prince William Henry, upon receiving the second volume of The History of the Decline and Fall of the Roman Empire from the author.

The most primitive objects of algebraic geometry are affine algebraic sets—subsets of $\mathbb{R}^n$ or $\mathbb{C}^n$ defined by the vanishing of polynomial functions—and the maps between them. But already in the first half of the 19th century geometers realized that there was a great advantage in working with varieties in complex projective space, treating affine varieties as projective varieties minus the intersection with the plane at infinity and real varieties as the fixed points of the complex involution. One sees this in the simplest examples: the ellipses, hyperbolas and parabolas in the real affine plane are all the same in the complex projective plane; the difference is only in how they intersect the line at infinity. A difficulty with the projective point of view is that on a connected projective variety there are no non-constant functions at all (reason: a function on a projective variety is a map to the affine line; since the image of a projective variety is again projective, the image would be a single point.)

Starting with Riemann in the 1860s and culminating in the scheme theory of Grothendieck in the 1950s, algebraic varieties were treated in a way independent of any embedding: An algebraic variety is is a topological space with a sheaf of locally defined polynomial functions. Many interesting aspects of geometry have to do not with single abstract varieties, but with maps between them, and in particular with embeddings in projective spaces. In general, maps between varieties can be described by their graphs, which are again varieties. But for the special case of maps to projective spaces, the theory of *linear series* is usually a more convenient description. The collection of all linear series on a variety reflects some of its best understood invariants.

The basic objects of study in this book are smooth, connected projective algebraic curves over an algebraically closed field of characteristic 0, which we take to be the complex numbers $\mathbb{C}$. Though we assume that the reader has been exposed to this theory in some form before, perhaps from Chapter IV of Hartshorne's *Algebraic Geometry*, we will review the elements in the form we will use.

0.1 Algebraic Curves and Riemann Surfaces

These objects can be viewed in two distinct but equivalent ways: as *compact Riemann surfaces*, or compact complex manifolds of dimension 1; and as *smooth projective algebraic curves over* $\mathbb{C}$. (Here, when we use the term projective variety, we mean a variety isomorphic to a closed subset of projective space, not a variety with a specified embedding in $\mathbb{P}^n$.) There are advantages to each point of view—the complex analytic point of view is more concrete, and requires relatively minimal amount of preliminaries; the algebraic point of view is substantially broader.

First, if $C \subset \mathbb{P}^n$ is a smooth, projective curve over $\mathbb{C}$, then it is a submanifold of complex projective space, and so a Riemann surface.
((discuss geometric genus vs. arithmetic genus here?))

The other direction—going from a compact Riemann surface C to a smooth projective curve over $\mathbb{C}$, or equivalently embedding C as a complex submanifold of $\mathbb{P}^n$, after which Chow's theorem says that it is in fact a projective variety—is much deeper. The first, and hardest step is to show that a compact Riemann surface admits a nonconstant meromorphic function $f : C \to \mathbb{C}$, and the corresponding statement is not true in higher dimensions. The function f can be viewed as a rational map $f' : C \to \mathbb{P}^1$. The next step is to see that the field $K(C)$ of all meromorphic functions on $\mathbb{C}$ is a finite extension of the field of rational functions on $\mathbb{P}^1$; the sheaf of regular functions on C is then the integral closure of the sheaf of regular functions on $\mathbb{P}^1$ in $K(C)$.

Though equivalent for curves defined over $\mathbb{C}$, these approaches have a very different flavors. For example, given a map $f : C \to C'$ from a smooth curve C to a possibly singular curve C' that is generically one-to-one, we can reconstruct C. From the algebraic point of view this can be done by *normalization*, or more concretely by blowing up the singular points of C'. From the analytic point of view, we can use the Weierstrass preparation theorem, which implies that there is the neighborhood U of any point $p \in C'$ such that the punctured neighborhood $U \setminus p$ is isomorphic to a disjoint union of punctured discs; and C is obtained by completing this to the corresponding disjoint union of discs.

0.2 Families of varieties

0.2.1 Hilbert schemes

Definition, universal property; construction

examples of hypersurfaces and linear spaces

tangent space

Fundamental problem: irreducible components of Hilb parametrizing smooth curves and their dimensions

Example 0.2.1. conics in $\mathbb{P}^3$ (refer to 3264)

0.2.2 Moduli spaces of curves

basic properties of M_g (coarse rather than fine; fine over automorphism-free curves)

dimension $3g - 3$, irreducible

(just statements, w/ref to Harris-Morrison)

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Chapter 1

Lasker's and Bézout's Theorems

1.1 Results with which the Reader should be comfortable

- definition of sheaves, schemes at the level of chapter I of [Eisenbud and Harris 2000]; subscheme of $\mathbb{P}^n$ is a homogeneous ideal up to saturation (Nullstellensatz).
- definition of cohomology, cohomology of line bundles on $\mathbb{P}^n$
- Serre's theorem: vanishing of cohomology above the dimension; vanishing of higher cohomology of high twists

1.2 The Noether-Lasker-Macaulay Theorems

A key element in the algebraic study of plane curves initiated by Brill and Noether following Riemann's discoveries was what Noether named the "Fundamental Lemma on Holomorphic Functions." In the original language it said that if the homogeneous forms $F(x_0, x_1, x_2) = 0$ and $G(x_0, x_1, x_2) = 0$ are the equations of two curves in $\mathbb{P}^2$ that have no component in common, then any form H that locally represents a function (the "holomorphic functions" of the name) vanishing on all the points of the intersection must have an expression $H = AF + BG$, where A, B are also homogeneous forms. It was extended by

Lasker to the case of many polynomials (homogeneous or not) in many variables. (This is sometimes called the "$AF + BG$ Theorem".)

Theorem 1.2.1. *Suppose that $I = (f_1, \dots, f_c) \subsetneq S := \mathbb{C}[x_0, \dots, x_n]$ is an ideal generated by c homogeneous forms in a polynomial ring. The scheme $X := V(I)$ is either empty or has codimension $\leq c$. If equality holds, then*

1. *Every primary component of I has codimension exactly c; in particular, if $c < n + 1$ then I is saturated.*

2. $\mathrm{H}^i(\mathcal{O}_X(d)) = 0$ *for all* $0 < i < \dim X$ *and all* $d \in \mathbb{Z}$.

3. *If* $\dim X \geq 1$ *then the map* $\mathrm{H}^0(\mathcal{O}_{\mathbb{P}^n}(d)) \to \mathrm{H}^0(\mathcal{O}_X(d))$ *is surjective for every d.*

By the Principal Ideal Theorem([Eisenbud 1995, Theorem ***] the codimension of $V(f_1, \dots, f_c)$ is at most c; the Theorem thus covers the case where the codimension is "as large as possible", that is, the $V(f_i)$ meet eachother in a dimensionally transverse way.

In modern language, the conclusion says that the ring S/I is Cohen-Macaulay. See for example [Eisenbud 1995, Chapter 18], where the result it proven in greater generality.

Theorem 1.2.1 immediately implies the original version of the theorem because (by the Nullstellensatz) the set of forms vanishing on $V(f_1, \dots, f_c)$ is the saturation of the ideal $(f_1, \dots, f_c)$.

We will make use of the following algebraic Lemma, proven in in [Eisenbud 1995, Theorem 18.***]:

Lemma 1.2.2. *Under the hypothesis of Theorem 1.2.1, the polynomial f_i is a nonzerodivisor in the ring $S/(f_1, \dots, f_{i-1})$ for $i = 1, \dots, c$.*

The conclusion is usually stated by saying that $f_1, \dots, f_c$ is a *regular sequence.*

Partial Proof. The result is obvious for $c = 1$. For $c = 2$ the hypothesis implies that f_1 and f_2 have no common factor. If $gf_2 \equiv 0 \bmod f_1$ then, since S has unique prime factorization, g must be divisible by f_1, that is, $g \equiv 0 \bmod f_1$, proving the result for $c = 2$. In general, the result is equivalent to the statement that S is a Cohen-Macaulay ring, [Eisenbud 1995, Proposition 18.9]. □

Proof of Theorem 1.2.1. We do induction on c. For $c = 0$ the saturation is trivial, and the vanishing in the case $c = 0$ is the usual computation of the cohomology of line bundles on $\mathbb{P}^n$.

Now suppose that the theorem is true for $X' := (f_1, \dots f_{c-1})$. Note that $\dim X' = n - c + 1 \geq 1$. The surjectivity of the maps $\mathrm{H}^0(\mathcal{O}_{\mathbb{P}^n}(d)) \to \mathrm{H}^0(\mathcal{O}_{X'}(d))$ shows that the homogeneous coordinate ring of X' is $R' := \oplus_d \mathrm{H}^0(\mathcal{O}_{X'}(d))$.

Write e for the degree of f_c . By Lemma 1.2.2 there is a short exact sequence

$$0 \to \mathcal{O}_{X'}(-e) \xrightarrow{f_c} \mathcal{O}_{X'} \to \mathcal{O}_X \to 0.$$

Tensoring with $\mathcal{O}_{\mathbb{P}^n}(d)$, and passing to cohomology, we obtain a long exact sequence that begins

$$\begin{aligned} 0 \to &\mathrm{H}^0(\mathcal{O}_{X'}(d-e_1)) \xrightarrow{f_1} \mathrm{H}^0(\mathcal{O}_{X'}(d)) \to \mathrm{H}^0(\mathcal{O}_X(d)) \to \\ &\mathrm{H}^1(\mathcal{O}_{X'}(d-e_1)) \to \cdots . \end{aligned}$$

From the exactness of the top row we see that every element of R that vanishes on X is a multiple of f_c, so f_c generates the homogeneous ideal of X in R. Lifting this back to the polynomial ring and using the inductive hypothesis, we see that $f_1, \dots, f_c$ generate the homogeneous ideal of X in S.

Moreover, if $\dim X \geq 1$ then $\dim X' \geq 2$, so $\mathrm{H}^1(\mathcal{O}_{X'}(d-e_1)) = 0$ for all d by the induction, whence the surjectivity statement holds for X.

Finally, the induction hypothesis and the exact sequences

$$\mathrm{H}^i(\mathcal{O}_{X'}(d)) \to \mathrm{H}^i(\mathcal{O}_X(d)) \to \mathrm{H}^{i+1}(\mathcal{O}_{X'}(d-e_1))$$

together show that $\mathrm{H}^i(\mathcal{O}_X(d)) = 0$ for $i < \dim X = \dim X' - 1$.

□

1.2.1 Determinantal ideals

There is a natural generalization of Theorem 1.2.2 that will be useful to us in constructing the ideals of embedded curves. Consider a homomorphism

$$\bigoplus_{j=1}^{q} \mathcal{O}_{\mathbb{P}^n}(d_j) \xrightarrow{M} \bigoplus_{i=1}^{p} \mathcal{O}_{\mathbb{P}^n}(e_i)$$

given by a $p \times q$ matrix of forms

$$M := \begin{pmatrix} f_{1,1} & f_{1,2} & \cdots & f_{1,q} \\ \vdots & & \ddots & \vdots \\ f_{p,1} & f_{p,2} & \cdots & f_{p,q} \end{pmatrix}$$

with $\deg f_{i,j} = \delta_{i,j} := e_i - d_j$.

When $p = 1$, this is just a sequence of forms of the type considered in Theorem 1.2.2.

Theorem 1.2.3. *(Macaulay) Suppose $p \leq q$, and let I be the ideal of $p \times p$ minors of the matrix M. The scheme $X := V(I)$ is either empty or has codimension $\leq c := q - p + 1$. If equality holds, then*

1. *Every primary component of I has codimension exactly c; in particular, if $c < n+1$ then I is saturated.*
2. $\mathrm{H}^i(\mathcal{O}_X(d)) = 0$ *for all* $0 < i < \dim X$ *and all* $d \in \mathbb{Z}$.
3. *if* $\dim X \geq 1$ *then the map* $\mathrm{H}^0(\mathcal{O}_{\mathbb{P}^n}(d)) \to \mathrm{H}^0(\mathcal{O}_X(d))$ *is surjective for every* d.

See *** for a proof.

Cheerful Fact 1.2.1. There are further theorems of this type, for lower order minors, for symmetric matrices, for skew symmetric matrices, etc. Perhaps the most general statement is that if R is a regular local ring and R/J is a Cohen-Macaulay ring then if $\phi : R \to S$ is a local homomomorphism to a Cohen-Macaulay ring, then $\operatorname{codim} JS \leq \operatorname{codim} J$ and, if equality holds then S/IS is Cohen-Macaulay. For example, Theorem 1.2.2 may be regarded as the case where R is a polynomial ring in c variables, and J is the ideal generated by the variables. See for example [Koh 1988].

1.3 Bézout's Theorem

The most basic invariants of a subvariety X of $\mathbb{P}^n$ are its dimension and degree; for example, they determine its cohomology class in the integral cohomology $\mathrm{H}^*(\mathbb{P}^n; \mathbb{Z}) \cong \mathbb{Z}[x]/(x^{n+1})$. It is convenient to compute these invariants in the case of schemes using the Hilbert polynomial. It is convenient that this definition extends at once to coherent sheaves:

Theorem 1.3.1. *Let $X \subset \mathbb{P}^n$ be a subscheme. The function $P_X(t) := \chi(\mathcal{O}_X(t)$ is a polynomial whose degree is equal to the dimension of X and whose leading coefficient is* $\deg X/(\dim X)!$.

Proof. We do induction on the dimension. If the dimension of X is zero, then $\mathcal{O}_X$ has a composition series whose successive factors have the form $\mathcal{O}_p$, for various points $p \in \mathbb{P}^n$, and the number of these points is the degree of X. By the additivity of the Euler characteristic, it suffices to prove the formula for a single point p. But $\mathcal{O}_p(t) \cong \mathcal{O}_p$, while $\mathrm{H}^0(\mathcal{O}_p) = 1$ and $\mathrm{H}^i\mathcal{O}_p = 0$. Thus $\chi(\mathcal{O}_p(t)) = 1$ for all t, as required.

Now suppose $\dim X \geq 1$. If If x is a general linear form on $\mathbb{P}^n$ then x is a nonzerodivisor on $\mathcal{O}_X$. Let H be the hyperplane defined by x. Then $X' = X \cap H$ has the same degree as X, and dimension 1 less. Twisting the exact sequence

$$0 \to \mathcal{O}_X(-1) \to \mathcal{O}_X \to \mathcal{O}_{X'} \to 0$$

by $\mathcal{O}_{\mathbb{P}^n}(t)$ we see that

$$\chi(\mathcal{O}_{X'}(t)) = \chi(\mathcal{O}_X(t)) - \chi(\mathcal{O}_X(t-1)).$$

so the degree of $\chi(\mathcal{O}_{X'}(t)$ is one less than the degree of $\chi(\mathcal{O}_{X'}(t)$, and the leading coefficient of $\chi(\mathcal{O}_{X'}(t)$ is $\dim X$ times the leading coefficient of $\chi(\mathcal{O}_X(t)$ $\square$

For example, $\mathbb{P}^n$ itself has degree 1 since

$$\chi(\mathcal{O}_{\mathbb{P}^n}(t)) = \binom{n+t}{n} = \frac{t^n}{n!} + \text{lower degree terms.}$$

Also, a hypersurface defined by a form of degree d does indeed have degree d—otherwise we would not have made this definition! More generally:

Lemma 1.3.2. *Let $X \subset \mathbb{P}^n$ be a subscheme, and suppose that $f \in \mathrm{H}^0(\mathcal{O}_{\mathbb{P}^n}(d))$ is form of degree d. Let H be the hypersurface $V(f)$. If the induced map $\mathcal{O}_X(-d) \to \mathcal{O}_X$ is a monomorphism, then $\deg(H \cap X) = d \cdot \deg X$. In particular $H = H \cap \mathbb{P}^n$ has degree d.*

Proof. Twisting the exact sequence

$$0 \to \mathcal{O}_X(-d) \to \mathcal{O}_X \to \mathcal{O}_{H\cap X} \to 0$$

by $\mathcal{O}_{\mathbb{P}^n}(t)$ and using the additivity of χ we see that $\chi(\mathcal{O}_{H\cap X}(t)) = \chi(\mathcal{O}_X(t)) - \chi(\mathcal{O}_X(t-d))$. An immediate computation shows that if $\chi(\mathcal{O}_X(t)) = at^m +$ lower degree terms then $\chi(\mathcal{O}_{H\cap X}(t)) = dat^{m-1} +$ lower degree terms. $\square$

The classic version of Bézout's Theorem says that plane curves of degrees e_1 and e_2 that have no common components meet in e_1e_2 points, counted with multiplicity; that is, the degree of the intersection is e_1e_2. More generally:

Corollary 1.3.3. *If $H_1, \dots, H_c$ are hypersurfaces in $\mathbb{P}^n$ whose intersection $X = \cap_{i=1}^c H_i$ has codimension c, then*

$$\deg X = \prod_{i=1}^{c} \deg H_i.$$

Proof. We do induction on c, the case $c = 1$ being covered by Lemma 1.3.2. The induction step follows at once from Lemma 1.2.2. $\square$

Using primary decomposition (see for example [Eisenbud and Harris 2000, Section II.3.3]) we can write any scheme $X \subset \mathbb{P}^n$ as a union of primary components X_i. The dimension of X is by definition the maximum of the dimensions of these components. Let X' be the union of those components whose dimension is equal to the dimension of X. Since $\mathcal{O}_X$ is a homomorphic image of $\oplus_i \mathcal{O}_{X_i}$, and the pairwise intersections $X_i \cap X_j$ all have dimension $< \dim X$, we see that

$$\deg X = \deg X' = \sum_{\{i \mid \dim X_i = \dim X\}} \deg X_i$$

If $f \in \mathrm{H}^0(\mathcal{O}_{\mathbb{P}^n}(d))$, and f does not contain the support of X_i, then f is a non-zerodivisor on $\mathcal{O}_{X_i}$ in the sense of Lemma 1.3.2 so $\deg H \cap X_i = \deg X_i$. In particular, if $\dim H \cap X = \dim X - 1$, then $\deg(H \cap X') = (\deg H) \deg X$.

On the other hand, if H does contain the support of X_i then from the exact squence

$$0 \to (\mathcal{I}_X : f)/\mathcal{I}_X(-d) \to \mathcal{O}_{X_i}(-d) \xrightarrow{f} \mathcal{O}_{X_i} \to \mathcal{O}_{H \cap X_i} \to 0$$

we deduce that $\deg(\mathcal{O}_{H \cap X_i}) \leq d \deg X_i$.

Putting these observations together we deduce a weak but surprisingly general form of Bézout's Theorem:

Proposition 1.3.4. *([Fulton 1984, Exericse 8.4.6]) Let $X \subset \mathbb{P}^n$ be a scheme, and let $H_1, \dots, H_c$ be hypersurfaces of degrees $d_1, \dots, d_c$. The sum of the degrees of the isolated components of*

$$H_1 \cap \cdots \cap H_c \cap X$$

is at most $(\prod_i d_i)$ times the sum of the degrees of the isolated components of X. □

Cheerful Fact 1.3.1. Bézout's theorem is the beginning of Intersection Theory, as described in [Fulton 1984] or [Eisenbud and Harris 2016]. One useful statement that can be extended well beyond the classical case we are considering is that subvariety of $\mathbb{P}^n$ has a fundamental class in $\mathrm{H}^*(\mathbb{P}^n; \mathbb{Z})$, and the cup product in cohomology can be realized algebraically in terms of linear equivalence classes in the Chow ring.

Chapter 2

Linear Systems

Morphisms of a smooth curve C (or indeed of any scheme) to a projective space are conveniently studied using the closely related notions of Divisors, linear systems and invertible sheaves.

2.1 Morphisms to projective space, and families of Cartier divisors

Let $\phi : C \to \mathbb{P}^r$ be a morphism from a smooth curve C. If $H \subset \mathbb{P}^r$ is a hyperplane that does not contain $\phi(C)$, then the preimage of $\phi(C) \cap H$ is a finite sets of points on C, with multiplicities when H is tangent to $\phi(C)$ or passes through a singular point of $\phi(C)$. Such a set of points with non-negative integer multiplicities is called an *effective divisor* on C; more generally, a *divisor* (sometimes called a *Weil divisor*)on a scheme X is an integral linear combination of codimension 1 subvarieties, and it is called *effective* if the coefficients are all non-negative. The divisors that arise as the pullbacks of general hyperplanes are special: since a hyperplane is defined by just one equation, which is locally given by the vanishing of a function, the pullback of a hyperplane will be locally defined by the vanishing of a single function that is a nonzerodivisor; that is, it is an *effective Cartier divisor*. See [**?**, pp. 140-146] for more information; on a smooth curve every divisor is Cartier, so the difference between Weil and Cartier divisors will not be an issue for us.)

The word "local" scattered through the previous paragraph is needed because, if X is a projective variety, then the only algebraic functions $X \to \mathbb{C}$ are constant functions. (Proof: the image of a projective variety is again projective, and the only projective subvarieties of an affine variety are points.)

If we are given the family of divisors on C that are the preimages of the intersections of hyperplanes with $\phi(C)$, we can recover the morphism ϕ set-theoretically: it takes a point $p \in C$ to the point of projective space that is the intersection of those hyperplanes whose preimages contain p.

The relationship of two divisors on C that are preimages of intersections of $\phi(C)$ with hyperplanes is simple to describe: If hyperplanes $H, H' \subset \mathbb{P}^r$ are defined by the linear form h, h' then $1/h$ has a simple pole along E—we may say that it "vanishes along E" to degree -1. In this sense the divisor $H - H'$ on $\mathbb{P}^n$ is defined by the rational function $\lambda = h'/h$. If neither H nor H' contain C then the pullback of λ is a well-defined, nonzero rational function on C, and the divisor $\phi^{-1}(\phi(C) \cap H') - \phi^{-1}(\phi(C) \cap H)$ is defined by the pullback $\phi^*(\lambda) := \lambda \circ f$. Thus the divisors arising from a given morphism to $\mathbb{P}^r$ differ by the divisors of zeros minus poles of rational functions on C.

If C is a smooth curve then the local ring $\mathcal{O}_{C,p}$ of C at a point p is a discrete valuation ring, and if π is a generator of the maximal ideal of $\mathcal{O}_{C,p}$, then any rational function λ on C can be expressed uniquely as $u\pi^k$ where $u \in \mathcal{O}_{C,p}$ is a unit and $k \in \mathbb{Z}$. We say that the *order* of λ at p, and write $k = \operatorname{ord}_p \lambda$. We associate λ to the divisor

$$(\lambda) := \sum_{p \in C} (\operatorname{ord}_p \lambda) p.$$

The *class group* of C is defined to be the the group of divisors on C modulo the divisors of rational functions. Thus the divisors on C that are preimages of intersections of $\phi(C)$ with different hyperplanes all belong to the same *divisor class*, and form a linear system in the sense of the following section.

2.2 Morphisms and linear systems

We want to understand morphisms to $\mathbb{P}^r$ more than set-theoretically, and we want to be able to produce them from data on C. For this we use the notion of linear system (sometimes called linear series).

Definition 2.2.1. A *linear system* on a scheme X is a pair $\mathcal{V} = (\mathcal{L}, V)$ where $\mathcal{L}$ is an invertible sheaf on X and V is a vector space of global sections of $\mathcal{L}$.

We will spend the next pages unpacking this notion. Our goal is to explain and prove:

Theorem 2.2.2. *There is a natural bijection between the set of nondegenerate morphisms $\phi : C \to \mathbb{P}^r$ modulo PGL_{r+1}, and basepoint-free linear systems of dimension r on C.*

Here "nondegenerate" means the image of the morphism ϕ is not contained in any hyperplane.

2.2.1 Invertible sheaves

Recall first that a *coherent sheaf* $\mathcal{L}$ on a scheme X may be defined by giving

- An open affine cover $\{U_i\}$ of X;
- For each i, a finitely generated $\mathcal{O}_X(U_i)$-module L_i;
- For each i, j, an isomorphism $\sigma_{i,j} : L_i \mid_{U_i \cap U_j} \to L_j \mid_{U_i \cap U_j}$ satisfying the compatibility conditions $\sigma_{j,k}\sigma_{i,j} = \sigma_{i,k}$.

A *global section* of $\mathcal{L}$ is a family of elements $t_i \in F_i$ such that $\sigma_{i,j} t_i = t_j$. Such a section may be realized as the image of the constant function 1 under a homomorphism of sheaves $\mathcal{O}_X \to \mathcal{L}$. By Theorem [**?**, Thm III.5.2] the space $H^0(\mathcal{L})$ of global sections is a finite-dimensional vector space. For example, $H^0(\mathcal{O}_X) = \mathbb{C}$ because the only globally defined functions on X are the constant functions.

The coherent sheaf $\mathcal{L}$ is said to be an *invertible sheaf* on X if there is an open cover as above with the additional property that $F_i \cong \mathcal{O}_X(U_i)$, the free module on one generator.

If $\sigma \in H^0\mathcal{L}$ is a global section of an invertible sheaf on X, and $p \in X$ is a point, then $\sigma(p)$ is in the stalk of $\mathcal{L}$ at p, a module isomorphic to $\mathcal{O}_{X,p}$. Since the isomorphism is not canonical, σ does not define a function on X at p; but since any two isomorphisms differ by a unit in $\mathcal{O}_{X,p}$, the vanishing locus, denoted $(\sigma)_0$ of σ *is* a well-defined subscheme of X. Moreover, if X is integral, then the ratio of two global sections is a well-defined rational function, so the divisor class of $(\sigma)_0$ is independent of the choice of σ.

Proposition 2.2.3. *The invertible sheaves on X form a group under $\otimes_X$, called the* Picard group of X*, denoted* $\mathrm{Pic}(X)$*.*

Proof. If $\mathcal{F}, \mathcal{G}$ are invertible sheaves then so are $\mathcal{F} \otimes_X \mathcal{G}$ and $\mathrm{Hom}_X(\mathcal{F}, \mathcal{G})$, as one sees immediately by restricting to the open sets where $\mathcal{F}$ and $\mathcal{G}$ are isomorphic to $\mathcal{O}_X$. Moreover the natural isomorphisms

$$\mathcal{F}(U) \otimes_X \mathrm{Hom}(\mathcal{F}(U), \mathcal{O}_X(U)) \to \mathcal{O}_X(U) \quad s \otimes f \mapsto f(s)$$

patch together to define a global isomorphism

$$\mathcal{F} \otimes_X \mathrm{Hom}(\mathcal{F}, \mathcal{O}_X) \to \mathcal{O}_X$$

justifying the definition $\mathcal{F}^{-1} := \mathrm{Hom}(\mathcal{F}, \mathcal{O}_X)$ and thus the name "invertible sheaf". □

If $D \subset X$ is an effective divisor, then we define $\mathcal{O}_X(-D)$ to be the ideal sheaf of D. If D is locally defined by the vanishing of a (locally defined) nonzerodivisor in $\mathcal{O}_X$, (that is, D is a Cartier divisor), then $\mathcal{O}_X(-D)$ is an invertible sheaf. We write $\mathcal{O}_X(D)$ for the inverse, $\mathcal{O}_X(-D)^{-1}$. The dual of the inclusion $\mathcal{O}_X(-D) \subset \mathcal{O}_X$ is a map $\mathcal{O}_X \to \mathcal{O}_X(D)$ sending the global section $1 \in \mathcal{O}_X$ to a section $\sigma \in \mathcal{O}_X(D)$ that vanishes precisely on D.

Example 2.2.4 (Invertible sheaves on $\mathbb{P}^r$)**.** If $H \subset \mathbb{P}^r$ is a hyperplane defined by the vanishing of a linear form $\ell = \ell(x_0, \dots x_r)$ then the ideal sheaf $\mathcal{O}_{\mathbb{P}^r}(-1) := \mathcal{I}_{H/\mathbb{P}^r} \subset \mathcal{O}_{\mathbb{P}^r}$ is generated on the open affine set $U_i := \{x_i \neq 0\} \cong \mathbb{A}^r$ by ℓ/x_i, and is thus an invertible sheaf. Moreover, if H' is the hyperplane defined by another linear form ℓ', then

$$\frac{\ell'}{\ell} \cdot \mathcal{I}_{H/\mathbb{P}^r} = \mathcal{I}_{H'/\mathbb{P}^r}$$

((check that this is out notation for ideal sheaf))
so the sheaves $\mathcal{I}_{H/\mathbb{P}^r}$ and $\mathcal{I}_{H'/\mathbb{P}^r}$ are isomorphic, justifying the name $\mathcal{O}_{\mathbb{P}^r}(-1)$.

The p-th tensor power of $\mathcal{O}_{\mathbb{P}^r}(-1)$ is called $\mathcal{O}_{\mathbb{P}^r}(-d)$; it is isomorphic to the ideal sheaf of any hypersurface of degree d. Because polynomials satisfy the unique factorization property, every effective divisor $D \subset \mathbb{P}^r$ is a hypersurface of some degree d, so $\mathcal{O}_{\mathbb{P}^r}(-D) \cong \mathcal{O}_{\mathbb{P}^r}(-d)$. Note that if $d > 0$ then $H^0(\mathcal{O}_{\mathbb{P}^r}(-D)) = 0$, since it may be realized as the sheaf of locally defined functions vanishing on D, and there are no such globally defined functions except 0.

We take $\mathcal{O}_{\mathbb{P}^r}(d)$ to be the inverse of $\mathcal{O}_{\mathbb{P}^r}(-d)$. If D is the hypersurface defined by a form F of degree d, then $\mathcal{O}_{\mathbb{P}^r}(-D)$ is generated on U_i by $F/(x_i^d)$, so $\mathcal{O}_{\mathbb{P}^r}(D)$ is generated on U_i by x_i^d/F. Starting from the inclusion $\mathcal{O}_{\mathbb{P}^r}(-D) \subset \mathcal{O}_{\mathbb{P}^r}$ and taking inverses, we see that $\mathcal{O}_{\mathbb{P}^r} \subset \mathcal{O}_{\mathbb{P}^r}(D)$ and the global section $1 \in H^0(\mathcal{O}_{\mathbb{P}^r}) \subset H^0(\mathcal{O}_{\mathbb{P}^r}(D))$, restricted to U_i, is $F/(x_0^d)$ times the local generator of $\mathcal{O}_{\mathbb{P}^r}(D)$ and thus vanishes on D. Because every rational function on $\mathbb{P}^r$ has degree 0, and any two global sections differ by a rational function, it follows that every global section of $\mathcal{O}_{\mathbb{P}^r}(d)$ vanishes on a divisor of degree d. Thus we may identify $H^0(\mathcal{O}_{\mathbb{P}^r}(d))$ with the $\binom{n+d}{n}$-dimensional vector space of forms of degree d on $\mathbb{P}^r$.

The proof of Theorem 2.2.2 is contained in the material of the next two subsections:

2.2.2 The morphism to projective space coming from a linear system

For any $\mathbb{C}$-vector space V of dimension $r + 1$ with basis $x_0, \dots, x_r$, we write $\mathrm{Sym}(V) \cong \mathbb{C}[x_0, \dots, x_r]$ for the symmetric algebra on V, and $\mathbb{P}(V) \cong \mathbb{P}^r_{\mathbb{C}}$ to be the projective space $\mathrm{Proj}(\mathrm{Sym}(V))$, which is naturally isomorphic to the space of lines in V^*. Note that the isomorphism $\mathbb{P}(V) \cong \mathbb{P}^r_{\mathbb{C}}$ is well-defined up to the action of $\mathrm{Aut}(\mathbb{P}^r) = PGL(r + 1)$.

Given a linear system $\mathcal{V} := (\mathcal{L}, V)$ of dimension r on a scheme X, where $\mathcal{L}$ is an invertible sheaf on X and $V = \langle \sigma_0, \dots \sigma_r \rangle$ is a vector space of global sections, we define the *base locus* of $\mathcal{V}$ to be the closed subscheme

$$B_{\mathcal{V}} := \bigcap_{i=0}^{r} \{\sigma_i = 0\}.$$

Let $W := X \setminus B_{\mathcal{V}}$ be the open subscheme where not all sections σ_i vanish.

For any point $q \in W$ we may choose an open neighborhood $W' \subset W$ of q, and an identification

$$t : \mathcal{L}\,|_{W'} \xrightarrow{\cong} \mathcal{O}_{W'}$$

and define $\phi_{\mathcal{V}} : W' \to \mathbb{P}(V)$ by

$$W' \ni p \mapsto \big(t(\sigma_0(p)), \dots, t(\sigma_r(p))\big) \in \mathbb{P}(V).$$

This is a morphism on W'. A change of neighborhoods W' or of identifications t would multiply each value $t(\sigma_i(p))$ by a unit, the same one for each i, and thus the construction would define the same morphism. It follows that the morphisms defined on different W' agree on overlaps, and thus define a morphism $W \to \mathbb{P}(V) \cong \mathbb{P}^r$. This is the reason that the dimension of $\mathcal{V}$ is defined to be $r = \dim V - 1$ instead of $\dim V$.

The most useful linear series are those that define morphisms defined on all of X. This happens when $B_{\mathcal{V}} = \emptyset$, that is, for every point $q \in X$, there is a section $\sigma \in V$ such that σ does not vanish at x. In this case we say that $(\mathcal{L}, \mathcal{V})$ is *basepoint free*.

Example 2.2.5. The morphism from $\mathbb{P}^r$ defined by the complete linear system $|\mathcal{O}_{\mathbb{P}^r}(d)|$ has target $\mathbb{P}^{\binom{r+d}{r}}$, and takes a point $x_0, \dots x_r$ to the point whose coordinates are all the monomials of degree d in $x_0, \dots x_r$. It is called the *d-th Veronese morphism* of $\mathbb{P}^r$. For example on $\mathbb{P}^1$, this has the form

$$(x_0, x_1) \mapsto (x_0^d,\ x_0^{d-1}x_1,\ \dots, x_1^d).$$

The image of $\mathbb{P}^1$ under this morphism is called the *rational normal curve* of degree d; in the case $d = 2$ is the *plane conic*, and if $d = 3$ it is called the *twisted cubic*. Veronese himself studied the image of $\mathbb{P}^2$ by the Veronese morphism of degree 2 now simply called *the Veronese surface*.

Exercise 2.2.6. Show that there is no non-constant morphism $\mathbb{P}^r \to \mathbb{P}^s$ when $s < r$ by showing that any nontrivial linear system of dimension $< r$ has a non-empty base locus.

2.2.3 The linear system coming from a morphism to projective space

Conversely, suppose that we are given a morphism $\phi : X \to \mathbb{P}^r$. With notation as in Example 2.2.4 we may choose an open affine cover $W_{i,j}$ of X such that

$\phi(W_{i,j}) \subset U_j$. Composing the regular functions $x_0/x_j, \ldots, x_r/x_j$ with ϕ we get functions $\sigma_0, \ldots, \sigma_r$ on $W_{i,j}$. The function σ_j, is the image under $\phi^* : \mathcal{O}_{U_j} \to \mathcal{O}_{W_{i,j}}$ of the function $x_j/x_j = 1$ on U_j, so it $\sigma_j = 1 \in \mathcal{O}_{W_{i,j}}$. In particular, the module $\mathcal{L}_{\phi^{-1}(U_j)}$ generated by the rational functions

$$\{(\sigma_i)_{\phi^{-1}(U_j)} = \phi^*(x_i/x_j)\}_{0 \leq i \leq n}$$

is a free $\mathcal{O}_{W_{i,j}}$-module on 1 generator. On the preimage of $U_j \cap U_k$ these sections differ by the common unit $\phi^*(x_k/x_j)$, and thus the collection of these modules defines an invertible sheaf $\mathcal{L}$ on X together with an $r+1$-dimensional space of global sections $\mathcal{V} := \langle \sigma_0, \ldots \sigma_r \rangle$ that forms a basepoint free linear system. Note that the subscheme $\{\sigma_k = 0\} \subset W_{i,j}$ is the scheme-theoretic preimage of the the hyperplane $\{x_k = 0\} \subset \mathbb{P}^r$. This completes the explanation and proof of Theorem 2.2.2

2.2.4 More about linear systems

Let $\mathcal{V} = (\mathcal{L}, V)$ be a linear sysytem on X. The linear system is said to be *complete* if $V = H^0(\mathcal{L})$; in this case it is sometimes denoted $|\mathcal{L}|$. If $\mathcal{L} \cong \mathcal{O}_C(D))$, we also write it as $|D|$. The *dimension of* $\mathcal{V}$ is $\dim V - 1$. If D is any divisor on C we write $r(D)$ for the dimension of the complete linear series $|D|$; that is, $r(D) = h^0(\mathcal{O}_C(D)) - 1$. Finally, a linear system of dimension 1 is called a *pencil*, a linear system of dimension 2 is called a *net* and, less commonly, a three-dimensional linear system is called a *web*.

A linear system $\mathcal{V} = (\mathcal{L}, V)$ is called *basepoint free* if it defines a morphism to $\mathbb{P}(V)$, or equivalently if the the sections in V generate $\mathcal{L}$ locally at each point of X. It is called *very ample* if it is basepoint-free and defines an embedding. If D is a Cartier divisor on X, then we say that D is *very ample* if the complete linear system $|D|$ is versy ample, and we say that D is *ample* if mD is very ample for some integer $m > 0$.

Via the correspondence of Theorem 2.2.2, the statements about the geometry of a morphism $\phi : C \to \mathbb{P}^r$ can be formulated as statements about the relevant linear systems. We will see this in many instances throughout this book. It will be most convenient to formulate this in terms of the vector space $H^0(\mathcal{L})$ of global sections of $\mathcal{L}$, and we write $h^0(\mathcal{L})$ for the dimension of this vector space. Here is a first example:

Proposition 2.2.7. *[**?**, Thm. IV.3.1] Let $\mathcal{L}$ be an invertible sheaf on a smooth curve C. The complete linear system $|\mathcal{L}|$ is base-point-free iff*

$$h^0(\mathcal{L}(-p)) = h^0(\mathcal{L}) - 1 \quad \forall p \in C;$$

and in this case the associated morphism $\phi_{\mathcal{L}}$ is an embedding, so $|\mathcal{L}|$ is very ample, iff

$$h^0(\mathcal{L}(-p-q)) = h^0(\mathcal{L}) - 2 \quad \forall p, q \in C.$$

Proof. The statement $h^0(\mathcal{L}(-p-q)) = h^0(\mathcal{L}) - 2$ for $p \neq q$ implies that $\phi_{\mathcal{L}}(p) \neq \phi_{\mathcal{L}}(q)$. The tangent space of C at p is $(\mathcal{I}_C(p)/\mathcal{I}_C(p)^2)^*$, so the condition that there is a section of $\mathcal{L}$ that vanishes at p, but does not vanish to order 2, implies that the differential $d\phi_{\mathcal{L}}$ is injective at p. ☐

((this uses a lot: even given the identification of the tangent space with m/m^2 we really only get an analytic isomorphism. To deduced the algebraic one we'd need a finiteness principal: projective maps with finite fibers are finite. Should we say some of this??))

We can also relate the geometry of the morphism associated to a incomplete linear system $V \subset H^0(\mathcal{L})$ to the geometry of the morphism associated to the complete linear system $|\mathcal{L}|$. In general, if $V \subset W \subset H^0(\mathcal{L})$ are a pair of nested linear systems, we have a linear map $W^* \to V^*$ dual to the inclusion $V \hookrightarrow W$, and a corresponding linear projection $\pi : \mathbb{P}W^* \dashrightarrow \mathbb{P}V^*$, with indeterminacy locus the subspace $\mathbb{P}(Ann(V)) \subset \mathbb{P}W^*$. In this case, we have

$$\phi_V = \pi \circ \phi_W;$$

that is, we have the diagram

$$\begin{tikzcd} & \mathbb{P}W^* \arrow[d, dashed, "\pi"] \\ C \arrow[ur, "\phi_W"] \arrow[r, "\phi_V"] & \mathbb{P}V^*. \end{tikzcd}$$

Note that in this case, given that W is base-point-free, the condition that V be base-point-free is equivalent to saying that the center $\mathbb{P}(Ann(V))$ of the projection π is disjoint from $\phi_W(C)$.

By way of language, we will say that a curve $C \subset \mathbb{P}^r$ embedded by a complete linear series is *linearly normal*; this is equivalent to saying that the restriction map

$$H^0(\mathcal{O}_{\mathbb{P}^r}(1)) \to H^0(\mathcal{O}_C(1))$$

is surjective, which is in turn equivalent to saying that C is not the regular projection of a nondegenerate curve $\tilde{C} \subset \mathbb{P}^{r+1}$.

Exercise 2.2.8. Extend the statement of Proposition 2.2.7 to incomplete linear systems; that is, prove that the morphism associated to a linear system $(\mathcal{L}, V)$ is an embedding iff

$$\dim \left(V \cap H^0(\mathcal{L}(-p-q))\right) = \dim V - 2 \quad \forall p, q \in C.$$

Exercise 2.2.9. An automorphism of $\mathbb{P}^r$ takes hyperplanes to hyperplanes. Deduce that it is given by the linear system $\mathcal{V} = \mathcal{O}_{\mathbb{P}^r}(1), H^0(\mathcal{O}_{\mathbb{P}^r}(1))$, and use this to show that $\operatorname{Aut}\mathbb{P}^r = PGL(r+1)$.

Exercise 2.2.10. Show that, if $s < r$, then the image of any morphism $\mathbb{P}^r \to \mathbb{P}^s$ is a single point.

For another example of the relationship between linear series on curves and morphisms of curves to projective space, consider a smooth curve $C \subset \mathbb{P}^r$ embedded in projective space, and assume that C is linearly normal. If $\phi : C \to C$ is any automorphism, we can ask whether ϕ is induced by an automorphism of $\mathbb{P}^r$; in other words, does there exist an automorphism $\Phi : \mathbb{P}^r \to \mathbb{P}^r$ such that $\Phi(C) = C$ and $\Phi|_C = \phi$? The answer is expressed in the following exercise.

Exercise 2.2.11. In the circumstances above, the automorphism ϕ is induced by an automorphism of $\mathbb{P}^r$ if and only if ϕ carries the invertible sheaf $\mathcal{O}_C(1)$ to itself; that is, $\phi^*(\mathcal{O}_C(1)) = \mathcal{O}_C(1)$.

Example 2.2.12. Consider the morphism of $\mathbb{P}^1 to \mathbb{P}^d$ given by the complete linear system $|\mathcal{O}_{\mathbb{P}^1}(d)|$; this is called the *rational normal curve*. Since there is a unique invertible sheaf of each degree n on C, and the curve is linearly normal, we see that *every automorphism of a rational normal curve $C \subset \mathbb{P}^d$ is projective*, so "the" rational normal curve of degree d is well-defined up to an automorphism of $\mathbb{P}^d$. A similar statement holds for the image of any Veronese morphism.

If $\mathcal{L}, \mathcal{L}'$ are linear systems on a smooth curve C and $D = (\sigma)_0, D' = (\sigma')_0$ are the divisors of zeros of sections of $\mathcal{L}$ and $\mathcal{L}'$ respectively, then $D + D'$ is the divisor of zeros of the section $\sigma \otimes \sigma'$ of $\mathcal{L} \otimes \mathcal{L}'$.

We often want to consider sections of a given invertible sheaf $\mathcal{L}$ with bounded singularities: if $D = \sum m_i p_i$ is a divisor, we define the invertible sheaf $\mathcal{L}(D)$ to be the sheaf of rational sections σ of $\mathcal{L}$ satisfying $\operatorname{ord}_{p_i}(\sigma) \geq -m_i$ for all i; as a line bundle, this is the same as $\mathcal{L} \otimes \mathcal{O}_C(D)$.

If $\phi : X \to \mathbb{P}^r$ is a generically finite morphism, then the *degree of* ϕ is the number of points in the preimage of a general point of $\phi(X)$. Thus, for example, if $D := \sum_{p \in C} n_p p$ is a divisor on a smooth curve, and the linear system $|D|$ is basepoint free, then the degree of the morphism associated to $|D|$ is $\deg D := \sum_{p \in C} n_p$.

2.2.5 The most interesting linear system

The most important invertible sheaf on a smooth variety X is the sheaf of global sections of the top exterior power of the the cotangent bundle of X, called the canonical sheaf ω_X of X (for canonical sheaves more generally, see Chapter **??**). A section of ω_X is thus a differential form of degree equal to the dimension of X, and the divisor class of such a form is usually denoted K_C.

Theorem 2.2.13. *The canonical sheaf of* $\mathbb{P}^r$ *is* $\mathcal{O}_{\mathbb{P}^n}(-r-1)$.

Proof. Let $x_0, \dots, x_r$ be the projective coordinates on $\mathbb{P}^r$ and let $U = \mathbb{P}^r \setminus H$ be the affine open set where $x_0 \neq 0$. Thus $U \cong \mathbb{A}^r$ with coordinates $z_{1:=}x_1/x_0, \dots, z_r := x_r/x_0$. The space of r-dimensional differential forms on U is spanned by $d(x_1/x_0) \wedge \cdots \wedge d(x_r/x_0)$, which is regular everywhere in U. In view of the formula

$$d\frac{x_i}{x_0} = \frac{x_0 dx_i - x_i dx_0}{x_0^2}$$

we get

$$d(x_1/x_0) \wedge \cdots \wedge d(x_r/x_0) = \frac{dx_1 \wedge \cdots \wedge dx_r}{x_0^r} - \sum_{i=1}^{r} x_i \frac{dx_1 \wedge \cdots \wedge \widehat{d_{x_i}} \wedge \cdots \wedge dx_r}{x_0^{r+1}}$$

which has a pole of order $r+1$ along the locus H defined by x_0. Thus the divisor of this differential form is $-(r+1)H$, and this is the canonical class. □

Cheerful Fact 2.2.1. A different derivation: there is a short exact sequence of sheaves of differentials, called the Euler sequence:

$$0 \to \Omega_{\mathbb{P}^r} \to \mathcal{O}_{\mathbb{P}^r}^{r+1}(-1) \to \mathcal{O}_{\mathbb{P}^r} \to 0.$$

. Taking exterior powers, we see that

$$\bigwedge^{r} \Omega_{\mathbb{P}^r} \otimes \bigwedge^{1} \mathcal{O}_{\mathbb{P}^r} = \bigwedge^{r+1} (\mathcal{O}_{\mathbb{P}^r}^{r+1}(-1)) = \mathcal{O}_{\mathbb{P}^r}(-r-1).$$

Computations of the canonical sheaf on a variety usually involve comparing the variety to another variety, such as projective space, where the canonical sheaf is already known. The most useful results of this type are the *adjunction formula* and the *Hurwitz' Theorem.*

Proposition 2.2.14. *(Adjunction Formula) Let* X *be a variety that is a Cartier divisor on a variety* Y. *If the canonical divisor of* S *is* K_Y, *then* K_X *is the restriction to* X *of the divisor* $K_Y + X$.

This is a special case of [**?**, ****].

Proof. There is an exact sequence of sheaves

$$0 \to \mathcal{I}_{X/Y} \mid_X \to \Omega_Y \mid_X \to \Omega_X \to 0$$

where Ω_X is the sheaf of differential forms on X (see [**?**, Theorem ***]), and $\mathcal{I}_{X/Y} \mid_X = \mathcal{O}_Y(-X) \mid_X = \mathcal{O}_X(-X)$. The proposition follows by taking top exterior powers. □

Corollary 2.2.15. *If* $C \subset \mathbb{P}^2$ *is a smooth plane curve of degree* d, *then* $\omega_C = \mathcal{O}_C(d-3)$; *more generally, if* $X \subset \mathbb{P}^r$ *is a complete intersection of hypersurfaces of degrees* $d_1, \dots, d_c$ *then* $\omega_X = \mathcal{O}_X(\sum_i d_i - r - 1)$.

Given a (nonconstant) morphism $f : C \to X$ of smooth projective curves, the Riemann-Hurwitz formula computes the canonical sheaf C in terms of that of X and the local geometry f. To do this we define the *ramification index* of f at p, denoted $\mathrm{ram}(f, p)$, by the formula of divisors

$$f^{-1}(f(p)) = \sum_{p \in C | f(p) = q} (\mathrm{ram}(f, p) + 1) \cdot p$$

In terms of a suitable choice of local coordinates z on C around p and w on X around $f(p)$, we can write the morphism as $z \mapsto w = z^m$ for some integer $m > 0$, and $\mathrm{ram}(f, p) = m - 1$.

It follows from complex analysis (or the separability of field extensions in characteristic 0) that there are only finitely many points on C where $\mathrm{ram}(f, p) \neq 0$ (this would be false in characteristic > 0 in the case where the induced extension of fraction fields was inseparable.) Thus we may define the *ramification divisor* of f to be the divisor

$$R = \sum_{p \in C} \mathrm{ram}(f, p) \cdot p \ \in \ \mathrm{Div}(C).$$

and the *branch divisor* to be

$$B = \sum_{q \in X} \Big(\sum_{p \in f^{-1}(q)} \mathrm{ram}(f, p) \Big) \cdot q \ \in \ \mathrm{Div}(X).$$

Note that R and B have the same degree $\sum_{p \in C} \mathrm{ram}(f, p)$.

Theorem 2.2.16. *(Hurwitz' Theorem) [**?**, ****] If $f : C \to X$ is a nonconstant morphism of smooth curves, with ramification divisor R, then*

$$\omega_C = f^* \omega_X(-R).$$

Proof. Choose a rational 1-form ω on X, and $\eta = f^*(\omega)$ be its pullback to C. For simplicity, we will assume that the zeroes and poles of ω lie outside the branch divisor B, so that ω will be regular and nonzero at each branch point. (Since we have the freedom to multiply by any rational function on X we can certainly find such a form, and in any event the calculation goes through without this assumption, albeit with more complicated notation.)

Since the zeroes of ω lie outside the branch divisor B, for every zero of ω of multiplicity m we have exactly d zeroes of η, each with multiplicity m; and likewise for the poles of ω. Meanwhile, at every point of B, the form ω is regular and nonzero. At a point p where (locally) f has the form $z \mapsto w = z^e$ and $\omega = dw$, ηdz we have $\eta = z^{e-1} dz$; that is η has a zero of multiplicity $\mathrm{ram}(f, p)$ at p. Thus the divisor K_C of η is $K_C = df^*(K_X) + R$. $\square$

Example 2.2.17. Let V be the vector space of homogeneous polynomials of degree d in two variables; that is, $V = H^0(\mathcal{O}_{\mathbb{P}^1}(d))$. In the projectivization

$\mathbb{P}(V^*) \cong \mathbb{P}^d$, let Δ be the locus of polynomials with a repeated factor. Since Δ is defined by the vanishing of the discriminant, it is a hypersurface. What is its degree?

To answer this, let $W^* \subset V^*$ be a general 2-dimensional linear subspace—that is, a general pencil of forms of degree d on $\mathbb{P}^1$. The linear system $\mathcal{W} = (\mathcal{O}_{\mathbb{P}^d}, W^*)$ defines a morphism $\phi_{\mathcal{W}} : \mathbb{P}^1 \to \mathbb{P}(W) \cong \mathbb{P}^1$ and the fiber over the point of $\mathbb{P}(W)$ corresponding to a form f of degree d is the divisor $f = 0 \subset \mathbb{P}^1$. Thus the locus of polynomials in W with a multiple root is the branch locus of $\phi_{\mathcal{W}}$, where we count an m-fold root $m-1$ times. By Hurwitz' formula, the degree of the branch locus B of a degree d morphism from $\mathbb{P}^1$ to $\mathbb{P}^1$ is

$$\deg B = \deg \omega_{\mathbb{P}^1} - d \deg \omega_{\mathbb{P}^1} = 2d - 2.$$

Cheerful Fact 2.2.2. A famous result asserted by Franchetta and proved by **** is that the canonical sheaf (and its powers) are the *only* sheaves that can be chosen uniformly among all, or even almost all, smooth curves. For a more precise statement, see ****.

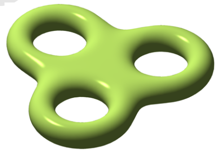
2.3 Genus, Riemann-Roch and Serre Duality

We will henceforward assume that the reader is acquainted with sheaf cohomology, at least sufficiently to write $H^i(X;(F)$ or $H^i(\mathcal{F})$ (our preferred form) without blushing. If D is a divisor on a scheme X we will often abbreviate $H^i(\mathcal{O}_X(D))$ to $H^i(D)$, and we write $h^i(\mathcal{F})$ or $h^i(D)$ for $\dim_{\mathbb{C}} H^i(\mathcal{F})$ or $\dim_{\mathbb{C}} H^i(D)$. Because $h^i(\mathcal{F})$, for $i > 0$, often appears as a kind of "error term" in formulas when one would like to compute $H^0(\mathcal{F})$, vanishing theorems have an important place in all of algebraic and analytic geometry. We will use the simplest of these often:

Theorem 2.3.1. *(Serre Vanishing Theorem) If $\mathcal{F}$ is a coherent sheaf on $\mathbb{P}^n$ then $H^i(\mathcal{F}(d)) = 0$ for all $i > 0$ and $d \gg 0$.*

2.3.1 The genus of a curve

The sole topological invariant of a smooth projective curve C is its genus. We can think of C as a submanifold of the complex projective space $\mathbb{P}^r(\mathbb{C})$ with the classical topology; as such, it is a compact, oriented surface, and its genus is the rank of its first integral homology, $H^1(C;\mathbb{Z})$—informally, the "number of holes":



**** Riemann Surface of genus 3, from Wikimedia ****

Of course this definition does not apply to curves over fields other than $\mathbb{C}$, and doesn't relate the genus to the algebra of the curve. However, we can relate the topological genus of a curve directly to its topological Euler characteristic $\chi_{top}(C) = 2 - 2g$. By the Hopf index theorem, the topological Euler characteristic is the degree of the tangent sheaf, or equivalently, minus the degree of the cotangent sheaf ω_C; that is, $\deg K_C = 2g - 2$, and thus

$$g(C) = \frac{\deg(K_C)}{2} + 1.$$

(This formula serves to define the genus of a smooth projective curve over any field).

Other characterizations of the genus require more machinery to establish. We will give some here, and use tools from the following section to prove equivalence.

1. $g(C)$ is the dimension of the vector space of regular 1-forms (that is, global sections of the cotangent sheaf) on C.

2. The (Zariski) Euler characteristic of the structure sheaf of C is $\chi(\mathcal{O}_C) = h^0(\mathcal{O}_C) - h^1(\mathcal{O}_C)$. Since $h^0(\mathcal{O}_C) = 1$,

$$g(C) = 1 - \chi(\mathcal{O}_C).$$

   Recall that if $X \subset \mathbb{P}^r = \mathbb{P}(V)$ is any projective scheme, the *homogeneous coordinate ring* of X is the ring $S/I(X)$ where $S = \operatorname{Sym} V \cong \mathbb{C}[x_0, \ldots, x_r]$ and $I(V) \subset S$ is the ideal of homogeneous forms that vanish on X.

3. Suppose that $C \subset \mathbb{P}^r = \mathbb{P}(V)$ is a smooth curve of degree d with homogeneous coordinate ring S_C, then the function $d \mapsto \dim_{\mathbb{C}}(S_C)_d$ is equal to a polynomial function $p_C(m)$ for large d. We have:

$$p_C(m) = dm - g + 1,$$

   so $g(C) = 1 - p_C(0)$.

2.3.2 The Riemann-Roch Theorem

To prove that these formulas for the genus are correct, we use the Riemann-Roch Theorem and Serre duality (sometimes called Kodaira-Serre duality, since Kodaira was responsible for the analytic version.)

Theorem 2.3.2 (Riemann-Roch Theorem)**.** *If C is a smooth, connected projective curve of genus g, and D a divisor of degree d on C then*

$$h^0(D) = d - g + 1 + h^0(K_C - D).$$

For example, if we take $D = 0$, this tells us that $h^0(K) = g$, proving the characterization (1) above. Also, since $h^0(D) = 0$ for any divisor D of negative degree, the formula gives the dimension of $h^0(D)$ when $\deg D$ is large:

Corollary 2.3.3. *For any divisor of degree $d \geq 2g - 1$, we have*

$$h^0(D) = d - g + 1.$$

Using this, we can apply Proposition 2.2.7 to show that all high degree divisors come from embeddings:

Corollary 2.3.4. *Let D be a divisor of degree d on a smooth, connected projective curve of genus g. If $d \geq 2g$, the complete linear series $|D|$ is base point free; and if $d \geq 2g + 1$ the associated morphism $\phi_D : C \to \mathbb{P}^{d-g}$ is an embedding, so that D is the preimage of the intersection of C with a hyperplane in $\mathbb{P}^{d-g}$.*

Since the complement of a hyperplane in projective space is an affine space, we get an affine embedding result too:

Corollary 2.3.5. *If C is any smooth, connected projective curve and $\emptyset \neq \Gamma \subset C$ a finite subset then $C \setminus \Gamma$ is affine.*

Proof. Let D be the divisor defined by Γ By Corollary 2.3.4 a high multiple of D is very ample, and gives an embedding $\phi : C \to \mathbb{P}^n$ such that the preimage of the intersection of C with some hyperplane H is a multiple of D. It follows that $C \setminus \Gamma$ is embedded in $\mathbb{P}^n \setminus H$. ☐

We can use Theorem 2.3.2in the simple case of Corollary 2.3.3 to determine the Hilbert polynomial of a projective curve. To do this, let $C \subset \mathbb{P}^r$ be a smooth curve of degree d and genus g, and consider the exact sequence of sheaves

$$0 \longrightarrow \mathcal{I}_{C/\mathbb{P}^r}(m) \longrightarrow \mathcal{O}_{\mathbb{P}^r}(m) \longrightarrow \mathcal{O}_C(m) \longrightarrow 0$$

and the corresponding exact sequence

$$H^0(\mathcal{O}_{\mathbb{P}^r}(m)) \xrightarrow{\rho_m} H^0(\mathcal{O}_C(m)) \longrightarrow H^1(\mathcal{I}_{C/\mathbb{P}^r}(m)) \longrightarrow 0.$$

The *Hilbert function* h_C of C is defined by

$$h_C(m) = \dim_{\mathbb{C}}(S_C)_m = \operatorname{rank}(\rho_m).$$

By Theorem 2.3.1 we have $H^1(\mathcal{I}_{C/\mathbb{P}^r}(m)) = 0$ for large m, so $h_C(m) = h^0(\mathcal{O}_C(m))$, for large m, which, by the Riemann-Roch Theorem , equals $md - g + 1$, again for large m. Thus, the Hilbert polynomial of $C \subset \mathbb{P}^r$ is $p_C(m) = dm - g + 1$, establishing the characterization (3).

The Riemann-Roch formula does *not* give us a formula for the dimension $h^0(D)$ when $h^0(K_C - D) > 0$; such divisors D are called *special divisors*, or *special divisor classes*. The existence or non-existence of divisors D with given $h^0(D)$ and $h^1(D)$ often serves to distinguish one curve from another, and will be an important part of our study.

Cheerful Fact 2.3.1. Classically, the dimension $h^0(K_C - D) = h^1(D)$ was called the *superabundance* of D: the idea was that a divisor of degree d had, at a minimum, $d - g + 1$ sections and $h^1(D)$ represented the number of "extra" sections. Even though the introduction of cohomology was still almost a century away, the ranks of cohomology groups h^1 had classical names, often involving the term superabundance—a premonition of the Riemann-Roch theorem in general.

Cheerful Fact 2.3.2. If k is a field that is not algebraically closed there may be genus 0 curves that are not isomorphic to $\mathbb{P}^1$. However, they must be"forms" of $\mathbb{P}^1$ in the sense that they become isomorphic to $\mathbb{P}^1$ after extension of scalars to the algebraic closure $\overline{k}$ of k. The unique example with $k = \mathbb{R}$ is the conic $x^2 + y^2 + z^2 = 0$. Indeed, any form of $\mathbb{P}^1$ over any field k can all be embedded in $\mathbb{P}^2_k$ (by using the anti-canonical linear system.

The curve $\mathbb{P}^1_k$ itself may be described as the scheme of left ideals of k-vector-space dimension 1 in the ring of 2×2 matrices over k (such an ideal can be embedded in the matrix ring as a linear combination of the 2 columns in an appropriate sense). More generally, any scheme that is a form of $\mathbb{P}^1$ over k may be described as the scheme of 1-dimensional left ideals in a central simple (= Azumaya) algebra over k—though as a set this scheme has no k-rational points unless the algebra is the algebra of 2×2 matrices!

2.3.3 Serre duality

In general, if $\mathcal{F}$ and $\mathcal{G}$ are coherent sheaves on a scheme X, we have for every i and j a cup product map

$$H^i(\mathcal{F}) \otimes H^j(\mathcal{G}) \to H^{i+j}(\mathcal{F} \otimes \mathcal{G}).$$

Theorem 2.3.6 (Serre Duality)**.** *Let C be a smooth connected projective curves with canonical divisor K. We have*

$$h^1(K) = 1$$

and the cup product map

$$H^1(D) \otimes H^0(K - D) \to H^1(K)$$

is a perfect pairing; that is, it induces a natural isomorphism

$$H^1(D) = H^0(K - D)^*.$$

2.3.4 A partial proof

Combining Theorem 2.3.2and Serre Duality we get:

Corollary 2.3.7. *If C is a smooth, connected projective curve and D is a divisor on C then*

$$\chi(\mathcal{O}_C(C)) := h^0(D) - h^1(D) = d - g + 1$$

or in other words, for any invertible sheaf $\mathcal{L}$ of degree d on C,

$$\chi(\mathcal{L}) = d - g + 1$$

which is pretty easy to prove. To see this, observe that for any invertible sheaf $\mathcal{L}$ on C and any point $p \in C$ we have an exact sequence of sheaves

$$0 \to \mathcal{L}(-p) \to \mathcal{L} \to \mathcal{L}_p \to 0.$$

It follows that $\chi(\mathcal{L}(-p)) = \chi(\mathcal{L}) - 1$, so that Riemann-Roch for $\mathcal{L}$ is equivalent to Riemann-Roch for $\mathcal{L}(-p)$. Since any divisor can be obtained from 0 by adding and subtracting points, the Riemann-Roch formula for an arbitrary $\mathcal{L}$ follows from the special case $\mathcal{L} = \mathcal{O}_C$.

2.4 The canonical morphism

Given the central role played by the canonical divisor class, it is natural to look at the geometry of the morphism $\phi_K : C \to \mathbb{P}^{g-1}$ associated to the complete canonical series $|K|$. By the Riemann-Roch theorem, $h^0(K) = g(C)$, so $|K|$ cannot define a non-constant morphism unless $g(C) \geq 2$, and cannot define an embedding unless $g(C) \geq 3$.

Definition 2.4.1. A curve C of genus $g \geq 2$ is said to be *hyperelliptic* if there exists a morphism $f : C \to \mathbb{P}^1$ of degree 2.

Proposition 2.4.2. *The canonical morphism $\phi_K : C \to \mathbb{P}^{g-1}$ is an embedding if and only if C is not hyperelliptic.*

Proof. By Corollary 2.3.4 we have to show that for any pair of points $p, q \in C$ we have

$$h^0(K_C(-p-q)) = h^0(K_C) - 2 = g - 2.$$

Applying the Riemann-Roch Theorem we see that this would fail if and only if $h^0(\mathcal{O}_C(p+q)) \geq 2$ for some $p, q \in C$, and by Lemma 2.4.3 $|p+q|$ would define a degree 2 morphism to $\mathbb{P}^1$. $\square$

Lemma 2.4.3. *Let C be a smooth, projective curve of genus $g \geq 2$. Any invertible sheaf of degree 2 on C defines a morphism to $\mathbb{P}^1$. In particular, if $g(C) = 2$ then the canonical series $|K_C|$ defines a 2 to 1 morphism to $\mathbb{P}^1$.*

Proof. If this happens, we claim that $\mathcal{O}_C(p+q)$ is basepoint free, so that C is hyperelliptic. To finish the proof, by Corollary 2.3.4 it suffices to show that an invertible sheaf $\mathcal{L}$ of degree 1 on C must have $h^0(\mathcal{L}) \leq 1$.

Suppose that σ_0, σ_1 were two linearly independent sections of $\mathcal{L}$. Each σ_i vanishes at a unique point p_i. If $p_0 = p_1$ then a linear combination of σ_0, σ_1 would be a section vanishing to order ≥ 2, which is impossible, so $\mathcal{L}$ is basepoint free, and defines a degree 1 morphism $C \to \mathbb{P}^1$. Such a morphism must be an isomorphism (because $\mathbb{P}^1$ is normal), contradicting $g(C) \geq 2$. $\square$

((the following argument is only set-theoretic. Admit this or make it precise))

Note that if C is hyperelliptic, the morphism ϕ_K factors through the degree 2 morphism $\pi : C \to \mathbb{P}^1$: if $\{p, q\} \subset C$ is a fiber of this morphism, we have $h^0(\mathcal{O}_C(p+q)) = 2$ and hence $\phi_K(p) = \phi_K(q)$. The image of the morphism ϕ_K is a nondegenerate curve of degree $g-1$ in $\mathbb{P}^{g-1}$, which we will see is a *rational normal curve*. This observation implies in particular that if C is hyperelliptic of genus $g \geq 2$, then the invertible sheaf $\mathcal{L}$ of degree 2 with $h^0(\mathcal{L}) = 2$ is in fact unique.

Among curves with $g \geq 3$ the hyperelliptic curves are very special: in the family of all curves, as we'll see, they comprise a closed subvariety. Also, the behavior of linear series and morphisms on a hyperelliptic curve is very different from that of series on a general curve; when we discuss the geometry of curves of low genus in the Chapter **??**, we will exclude the hyperelliptic case, and deal with this case in a separate chapter.

For non-hyperelliptic curves, however, the geometry of the canonical morphism, and its image, the canonical curve, are the keys to understanding the curve. We'll see this in detail in many cases in the following chapter; for now, we mention one highly useful result along these lines.

((add here: canonical series on plane curves cut by $|\mathcal{O}_{\mathbb{P}^2}(d-3)|$; consequence that no smooth plane curve can be hyperelliptic))

((maybe move initial discussion of hyperelliptic curves from Ch. 6 to a section here))

((maybe add to this chapter: differentials on plane curves C, possibly with nodes or more general singularities; adjoint conditions; algorithm for determining the complete linear system associated to a divisor D on C))

2.4.1 The geometric Riemann-Roch theorem

Let's state this first in a relatively simple case: let C be a nonhyperelliptic curve, embedded in $\mathbb{P}^{g-1}$ by its canonical series and let $D = p_1 + \cdots + p_d$ be a divisor consisting of d distinct points; let $\overline{D}$ be the span of the points $p_i \in C \subset \mathbb{P}^{g-1}$. Since the hyperplanes in $\mathbb{P}^{g-1}$ containing $\{p_1, \dots, p_d\}$ correspond (up to scalars) to sections of K_C vanishing at all the points p_i, we see that

$$h^0(K_C - D) = g - 1 - \dim \overline{D}.$$

Plugging this into the Riemann-Roch formula, we arrive at the statement

$$r(D) = d - 1 - \dim \overline{D};$$

or in other words, *the dimension of the linear series $|D|$ in which the divisor D moves is equal to the number of linear relations on the points p_i on the canonical curve.* Thus, for example, if $D = p_1 + p_2 + p_3$, we see that D moves in a pencil if and only if the points p_i are collinear.

We can extend this statement to the case of arbitrary effective divisors D (and even hyperelliptic curves) if we define our terms correctly. To do this, suppose $f : C \to \mathbb{P}^d$ is any morphism, and $D \subset C$ any divisor. We define the *span* of $f(D)$ to be the intersection

$$\overline{f(D)} = \bigcap_{H \mid f^{-1}(H) \supset D} H$$

of all hyperplanes in $\mathbb{P}^d$ whose preimage in C contains D.

Theorem 2.4.4 (Geometric Riemann-Roch Theorem)**.** *If C is any curve of genus $g \geq 2$, $\phi : C \to \mathbb{P}^{g-1}$ its canonical morphism and $D \subset C$ any effective divisor of degree d, then*

$$r(D) = d - 1 - \dim \overline{\phi(D)}.$$

((I moved the hyperelliptic section to be with the curves of genus 2))

2.5 Moduli problems

It is a fundamental aspect of algebraic geometry that the objects we deal with often vary in families, and can often be parametrized by a "universal" such family. For example, the family of plane curves of degree d may be thought of as the projective space $\mathbb{P}(H^0\mathcal{O}_{\mathbb{P}^2}(d))$, and similarly with hypersurfaces in any projective space. This notion of objects varying with parameters underlies many of the constructions and theorems we will discuss.

2.5.1 What is a moduli problem?

Briefly, a *moduli problem* consists of two things: a class of objects, or isomorphism classes of objects; and a notion of what it means to have a *family* of these objects parametrized by a given scheme B. To make this relatively explicit, the four main examples of moduli problems we'll be discussing here are:

1. smooth curves: objects are isomorphism classes of smooth, projective curves C of a given genus g. A family over B is a subscheme $\mathcal{X} \subset B \times \mathbb{P}^r$, smooth, over B, whose fibers are curves of genus g.

2. the Hilbert scheme: objects are subchemes of $\mathbb{P}^r$ with a given Hilbert polynomials. A family is a subscheme $\mathcal{X} \subset B \times \mathbb{P}^r$, with cX flat over B, whose fibers have the given Hilbert polynomial. We will be interested in the case of Hilbert polynomial $p(m) = dm - g + 1$ and the open subscheme corresponding to smooth projective curves $C \subset \mathbb{P}^r$ of degree d and genus g.

3. effective divisors on a given curve: objects are effective divisors of a given degree d on a given smooth, projective curve C. A family over B will be a subscheme $\mathcal{D} \subset B \times C$ flat over B, with fibers of degree d

4. invertible sheaves on a given curve C: objects are invertible sheaves of a given degree d on C. A family over B is an invertible sheaf on the product $B \times C$ whose restriction to each fiber over B has degree d. We identify two such sheaves if they differ by tensor product with an invertible sheaf pulled back from B.

Given a moduli problem, our goal will be to describe a corresponding *moduli space*. By this we mean a scheme M whose points are in *natural* one-to-one correspondence with the objects in our moduli problem. This will realize the objects of the moduli problem as the points of the underlying set of the scheme M.

If the moduli space in question and the base of the family are varieties, then the crucial condition that the correspondence be *natural* is simple to express: that given a family of the objects in our moduli problem over a variety B, the map from underlying set of B to the underlying set of M taking each fiber to the corresponding point of M should be a morphism of varieties. But in the world of schemes the set-theoretic mapping does not determine the morphism of schemes (think, for example, of the morphisms from $\mathrm{Spec}(\mathbb{C}[x]/x^2)$ into the plane with the closed point mapping to the origin. The situation is even worse when the moduli space itself is not a variety.)

To deal with the general case, we recast the naturality condition in functorial terms. We observe first that a moduli problem defines a functor $\mathcal{M}$ from the category of schemes to the category of sets: the value of the functor at a scheme B is the set of families of objects parametrized by B; a morphism $B' \to B$ of schemes gives rise, via pullback, to a map of sets $\mathcal{M}(B) \to \mathcal{M}(B')$. We define a *fine moduli space* for the moduli problem to be a scheme M that represents this functor, in the sense that there is an isomorphism of functors

$$\mathcal{M} \to \mathrm{Mor}(\bullet, M)$$

In other words, for every scheme B we have a bijection between families of our objects over B and morphisms from B to M. In particular, applying this to $B = \operatorname{Spec} \mathbb{C}$, we have a bijection between the set of objects and the closed points of M; and for any family over an arbitrary scheme B, the map from $B(\mathbb{C})$ to $M(\mathbb{C})$ sending each closed point $b \in B$ to the point in $M(\mathbb{C})$ corresponding to the fiber over b is the underlying map of a morphism $B \to M$ of schemes.

If a fine moduli space for a given problem exists at all, then Yoneda's Lemma shows that it is unique up to a unique isomorphsm. This is a real problem: there is no fine moduli space for the first and most important of the examples above—the isomorphism classes of smooth curves—though there is for the others. We'll defer the discussion of why this is, and what we can do about it, until Chapter 6.

Looking ahead, we'll discuss the third and fourth example in Chapter **??**, where we'll describe the moduli spaces for effective divisors of given degree d on a given curve C (the symmetric powers of the curve) and for invertible sheaves of a given degree on C (the *Jacobian* and *Picard variety* of C). These, as we'll see, are smooth, irreducible projective varieties of dimensions d and g respectively.

We'll take up the moduli space M_g of smooth curves in Chapter 6, where we'll see that this space (or rather the closest approximation to it we can cook up) is irreducible of dimension $3g-3$ for $g \geq 2$, though not smooth or projective.

Finally, the Hilbert scheme will be described (to the extent that we can!) in Chapter 12; this will turn out to be much wilder and more varied in its behavior than any of the above.

Chapter 3

Curves of genus 0 and 1

In this chapter, we will introduce the basic facts about the geometry and embeddings of curves of genus 0 and 1; in later chapters we will take up similarly results for curves of genus 2 through 6. Our knowledge of the geometry of curves becomes increasingly less complete as the genus increases, and 6, as we shall see, is a natural turning point.

3.1 pre-requisites and conventions

Basic results used in this section: Bézout, Riemann-Roch, Lasker (aka AF+BG), Clifford, Adjunction. To write: an appendix on cohomology covering RR, exact sequences. Section on Families: define family, define Hilbert Scheme and Chow variety; but say we're not going to treat them formally very much. Flatness referred to "Geom Schemes"; our families are smooth.

Would it be more confusing or less to use the same letter for a polynomial vanishing on C and the surface it defines?

3.2 Curves of genus 0

((To put in: Rational quartic in $\mathbb{P}^3$ as curve of type 1,3 on quadric do dimension count. Branch points can be chosen. g^3_4 is sum of g^1_1 and a g^1_3. Mention open questions: Set theor comp int problem, "secant conjecture"?))

As we saw in more generality in Example 2.2.4, there is for each $d \in \mathbb{Z}$

a unique invertible sheaf $\mathcal{O}_{\mathbb{P}^1}(d)$ of degree d on $\mathbb{P}^1$. To compute $H^0(\mathcal{O}_{\mathbb{P}^1}(d))$ directly, let $D = z_1 + z_2 + \cdots + z_d$ be a divisor of degree d and suppose that the coordinates are chosen so that none of the z_i are at infinity. The sections of $\mathcal{O}_{\mathbb{P}^1}(D)$ are the rational functions with poles only at the z_i. In affine coordinates, identifying the z_i with complex numbers, these can each be written

$$\frac{g(z)}{(z - z_1)(z - z_2)\cdots(z - z_d)}$$

with $\deg(g) \leq d$, the condition that infinity is not a pole. We see that these form a vector space of dimension $d + 1$.

((the following result is such a good exercise in the correspondence of linear systems and maps and divisors, maybe move it to Ch 2?))

By the Riemann-Roch Theorem , any invertible sheaf of degree d on a curve of genus 0, like $\mathbb{P}^1$, has at least $d+1$ sections. In fact (over $\mathbb{C}$, or any algebraically closed field, this characterizes $\mathbb{P}^1$:

Theorem 3.2.1. *Let C be a reduced, irreducible projective curve and let $\mathcal{L}$ be an invertible sheaf of degree d on C. If $\mathrm{h}^0(\mathcal{L}) \geq d+1$ then $C \cong \mathbb{P}^1$, so $\mathcal{L} \cong \mathcal{O}_{\mathbb{P}^1}(d)$, and $\mathrm{h}^0(\mathcal{L}) = d + 1$.*

Proof. Let $p_1, \dots p_{d-1}$ be general points of C, and set $\mathcal{L}' := \mathcal{L}(-p_1 - \cdots - p_{d-1})$. From the correspondence between divisors and invertible sheaves, we see that the degree of $\mathcal{L}'$ is 1. Since $\mathcal{L}$ is locally isomorphic to the sheaf of functions on C, the condition of vanishing at a point imposes at most 1 linear condition on the global sections of $\mathcal{L}$, and thus $H^0(\mathcal{L}') \geq 2$, so we may assume from the outset that $d = 1$.

The linear system $(\mathcal{L}, H^0(\mathcal{L}))$ cannot have any base points, since otherwise after subtracting one, we would get an invertible sheaf of degree ≤ 0 with two independent global sections. Again by the correspondence with divisors, neither of these sections could vanish at any point of C, so their ratio would be a non-constant function defined everywhere on C, a contradiction.

Thus we see that the linear system $(\mathcal{L}, H^0(\mathcal{L}))$ defines a morphism $\phi : C \to \mathbb{P}^1$ of degree 1 whose fibers—the divisors defined by sections of $\mathcal{L}$ are of degree 1. Thus if $p \in C$ is the preimage of $q \in \mathbb{P}^1$, the induced map of local rings $\phi^* : \mathcal{O}_{\mathbb{P}^1,q} \to \mathcal{O}_{C,p}$ is a finite, birational map. Since $\mathcal{O}_{\mathbb{P}^1,q}$ is integrally closed, this is an isomorphism. Thus ϕ is an isomorphism, as required. □

Note that we used the algebraic closure of the ground field in choosing points on C.

Corollary 3.2.2. *Every smooth curve C of genus 0 over an algebraically closed field is isomorphic to $\mathbb{P}^1$.*

Proof. By the Riemann-Roch Theorem , any linear system $\mathcal{L}$ of degree d on C has $h^0\mathcal{L} \geq d+1$. $\square$

The classification of curves of genus 0 over non-algebraically closed fields is a subject that goes back to Gauss.

Cheerful Fact 3.2.1. The left ideals of the ring of 2×2 matrices over any field can be indexed by the points of $\mathbb{P}^1$: to the point (λ, μ) we associate the set of matrices

$$\left\{ \begin{pmatrix} x_{1,1} & x_{1,2} \\ x_{2,1} & x_{2,2} \end{pmatrix} \mid \lambda x_{1,1} + \mu x_{1,2} = 0,\ \lambda x_{2,1} + \mu x_{2,2} = 0 \right\};$$

for example $(0,1)$ corresponds to the set of matrices with 0 in the second column. The quaternion algebra

$$\mathbb{H} := \mathbb{R} < i, j, k > /(i^2 = j^2 = -1, ij = k = -ji)$$

is a division ring, so it has no non-trivial left ideals, but we can still define the scheme of 2-dimensional left ideals to be the subscheme of the Grassmannian $G(2, \mathbb{H})$ consisting of 2-dimensional subspaces stable under multiplication by i, j, k. This subscheme, which can also be identified with the "pointless" conic $x^2 + y^2 + z^2 = 0$ in $\mathbb{P}^2_{\mathbb{R}}$ has no $\mathbb{R}$-rational points, but $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{H}$ is the ring of 2×2 matrices over $\mathbb{C}$, so the set of $\mathbb{C}$-points of the scheme of left ideals of $\mathbb{H}$ may be identified with $\mathbb{P}^1_{\mathbb{C}}$. It turns out that every scheme over a field k that becomes $\mathbb{P}^1$ over the algebraic closure can be constructed as the scheme of left ideals of a 4-dimensional Azumaya (that is, central simple) algebra, in this case a (generalized) quaterion algebra, and as a smooth conic in $\mathbb{P}^2_k$. There is also a cohomological description. See for example Serre ****
((I think its in Groupes Algebriques et Corps de Classes, but I'm not sure..))

3.3 Rational Normal Curves

Recall from Example 2.2.5 that the image of the d-th *Veronese map* $\phi_d : \mathbb{P}^1 \to \mathbb{P}(H^0((\mathcal{O}_{\mathbb{P}^1}(d)) \cong \mathbb{P}^d$ is called the *rational normal curve* of degree d.

Proposition 3.3.1. *If C is a nondegenerate curve in $\mathbb{P}^d$ then $\deg C \geq d$, with equality if and only if C is a rational normal curve.*

Proof. By the correspondence between morphisms and linear systems, the invertible sheaf $\mathcal{L}$ corresponding to such a morphism has degree d and $h^0(\mathcal{L}) \geq d+1$. The conclusion follows from Theorem 3.2.1. $\square$

We will see more generally that, if X is a non-degenerate variety in $\mathbb{P}^d$ of dimension k, then $\deg(X) \geq d-k+1$; and we will describe the varieties that achieve the minimum in Section **??**.

The points on a rational normal curve are "as independent as possible":

Proposition 3.3.2. *If $C \subset \mathbb{P}^d$ is a rational normal curve of degree d and $\Gamma \subset C$ is a subscheme of length $\ell \leq d+1$, then Γ lies on no plane of dimension $< \ell$. In particular, any $m \leq d+1$ distinct points on a rational normal curve $C \subset \mathbb{P}^d$ are linearly independent.*

The rational normal curve is the unique curve with this property, as we shall see in Chapter 9.

Proof. We can reduce to the case $\ell = d+1$ by adding points to Γ, so it suffices to do that case, which follows at once from Bezout's Theorem. $\square$

In the case of distinct points it is easy to make a direct argument: In affine coordinates chosen so that none of the points are at infinity we can identify the points $\lambda_1, \dots, \lambda_{d+1} \in C \cong \mathbb{P}^1$ with complex numbers, and the statement (for $\ell = d+1$) is tantamount to the nonvanishing of the Vandermonde determinant

$$\begin{vmatrix} 1 & \lambda_1 & \lambda_1^2 & \dots & \lambda_1^d \\ 1 & \lambda_2 & \lambda_2^2 & \dots & \lambda_2^d \\ \vdots & & & & \vdots \\ 1 & \lambda_{d+1} & \lambda_{d+1}^2 & \dots & \lambda_{d+1}^d \end{vmatrix}.$$

We say that a smooth curve $C \subset \mathbb{P}^d$ is *projectively normal* if the restriction map

$$H^0(\mathcal{O}_{\mathbb{P}^d}(m)) \ \to \ H^0(\mathcal{O}_C(m))$$

is surjective for every m. We'll this property it in many settings, in particular the discussion of *liaison* in Chapter **??**. Since every monomial of degree md on $\mathbb{P}^1$ is a product of m monomials of degree d, we see that the rational normal curve is projectively normal.

It is easy to write down equations that define a rational normal curve. Choosing a basis s, t for the linear forms on $\mathbb{P}^1$, we can write

$$\phi_d : (s,t) \mapsto (s^d, s^{d-1}t, \dots t^d)$$

from which we see that C lies in the zero locus of the homogeneous quadratic polynomial $z_i z_j - z_k z_l$ for every $i+j = k+l$. As a convenient way to package these, we can realize these forms the 2×2 minors of the matrix

$$M \ = \begin{pmatrix} z_0 & z_1 & \dots & z_{d-1} \\ z_1 & z_2 & \dots & z_d \end{pmatrix}.$$

Note that if we substitute $s^i t^{(d-i)}$ for z_i and identify $H^0(\mathcal{O}_{\mathbb{P}^1}(i)$ with $\mathbb{C}[s,t]_i$, this becomes the multiplication table

$$H^0(\mathcal{O}_{\mathbb{P}^1}(i) \times H^0(\mathcal{O}_{\mathbb{P}^1}(d-i-1) \to H^0(\mathcal{O}_{\mathbb{P}^1}(d);$$

we shall see a general version of this in Chapter 11.

In fact, the minors of this matrix generate the ideal of forms on $\mathbb{P}^d$ vanishing on C. For this result see for example [**?**, ****].
((We can prove this later, from the Eagon-Northcott resolution and homological considerations. Put in a ref when there is one.))
We can immediately prove two slightly weaker results:

First, C is set-theoretically defined by the 2×2 minors of M. Explicitly, suppose that $p = (z_0, \dots, z_d) \in \mathbb{P}^d$ is any point, and all the polynomials Q_{ijkl} above vanish at p. If $z_0 = 0$, then from the vanishing of $\det \begin{pmatrix} z_0 & z_1 \\ z_1 & z_2 \end{pmatrix}$ we see that $z_1 = 0$, and similarly we have $z_2 = \cdots = z_{d-1} = 0$; this the point $p = (0, \dots, 0, 1)$, which is a point on the rational normal curve. On the other hand, if $z_0 \neq 0$, set $\lambda = z_1/z_0$; we see in turn that $z_2/z_1 = \cdots = z_d/z_{d-1} = \lambda$; thus $p = (1, \lambda, \dots, \lambda^d)$, again a point of the rational normal curve.

Second, the $\binom{d}{2}$ distinct 2×2 minors of M are linearly independent, as one can see by first factoring out x_0 and x_d and noting that the resulting minors generate the square of the maximal ideal in $\mathbb{C}[x_1, \dots, x_{d-1}]$. Note that the restriction map

$$H^0(\mathcal{O}_{\mathbb{P}^d}(2)) \ \to \ H^0(\mathcal{O}_C(2)) = H^0(\mathcal{O}_{\mathbb{P}^1}(2d))$$

is surjective because every monomial of degree $2d$ on $\mathbb{P}^1$ is a product of two monomials of degree d. Comparing dimensions, we see that the dimension of the kernel—that is, the space of quadratic polynomials on $\mathbb{P}^d$ vanishing on C—has dimension

$$\binom{d+2}{2} - (2d+1) \ = \ \binom{d}{2}.$$

Another important property of rational normal curves $C \subset \mathbb{P}^d$ is that they are *projectively homogeneous*: the subgroup G of the automorphism group PGL_{d+1} of automorphisms of $\mathbb{P}^d$ that carries C to itself acts transitively on C. More generally, every $\mathbb{P}^r$ is a homogeneous variety in the sense that $\operatorname{Aut} \mathbb{P}^r$ acts transitively. If σ is an automorphism then, because $\mathcal{O}_{\mathbb{P}^r}(d)$ is the unique invertible sheaf of degree d on $\mathbb{P}^r$, we have $\sigma^*\mathcal{O}_{\mathbb{P}^r}(d) = \mathcal{O}_{\mathbb{P}^r}(d)$ so σ induces an automorphism ϕ on $H^0(\mathcal{O}_{\mathbb{P}^r}(d))$, and an automorphism $\overline{\phi}$ on the ambient space $\mathbb{P}H^0(\mathcal{O}_{\mathbb{P}^r}(d))$ of the target of the d-th Veronese map. If α is a rational function with divisor D, then $\phi(\alpha) = \alpha \circ \sigma$ has divisor $\sigma^{-1}(D)$, so $\overline{\phi}^{-1}$ induces σ on $\mathbb{P}^r$.

The rational normal curve $C \subset \mathbb{P}^r$ can also be characterized among irreducible, nondegenerate curves as the unique projectively homogeneous curve in $\mathbb{P}^r$, as we shall see in Chapter 9.

Finally, rational normal curves lie on more hypersurfaces than any other irreducible, nondegenerate curve in $\mathbb{P}^d$.

Proposition 3.3.3. *If $C \subset \mathbb{P}^d$ is any irreducible, nondegenerate curve, then*

$$h^0(\mathcal{I}_{C/\mathbb{P}^d}(2)) \leq h^0(\mathcal{I}_{C/\mathbb{P}^d}(2)) = \binom{d}{2};$$

and if equality holds then C is a rational normal curve

Proof. Consider the restriction of the quadrics containing C to a general hyperplane $H \cong \mathbb{P}^{d-1} \subset \mathbb{P}^d$, and let $\Gamma = H \cap C$. We have exact sequence:

$$0 \to \mathcal{I}_{C/\mathbb{P}^d}(1) \to \mathcal{I}_{C/\mathbb{P}^d}(2) \to \mathcal{I}_{\Gamma/\mathbb{P}^{d-1}}(2) \to 0.$$

Since C is nondegenerate, $h^0(\mathcal{I}_{C/\mathbb{P}^d}(1)) = 0$, and since $\deg C \geq d$, the hyperplane section Γ of C must contain at least d linearly independent points. Since linearly independent points impose independent conditions on quadrics, we have

$$h^0(\mathcal{I}_{\Gamma/\mathbb{P}^{d-1}}(2)) \leq h^0(\mathcal{O}_{\mathbb{P}^{d-1}}(2)) - d,$$

establishing the desired inequality. □

((will "linearly independent points impose independent conditions on quadrics" be obvious to our reader?))

Exercise 3.3.4. Establish the analogous statement for hypersurfaces of any degree d; that is, no irreducible, nondegenerate curve in $\mathbb{P}^r$ lies on more hypersurfaces of degree d than the rational normal curve.

Exercise 3.3.5. Prove directly
((clarify what "directly" means? Otherwise we're playing "read my mind".))
the special case $r = 3$: that the twisted cubic is the unique irreducible, nondegenerate space curve lying on three quadrics.

3.3.1 Other rational curves

What about other rational curves in projective space? Since any linear series $\mathcal{D}$ of degree d on $\mathbb{P}^1$ is a subseries of the complete series $|\mathcal{O}_{\mathbb{P}^1}(d)|$, we see that *any rational curve $C \subset \mathbb{P}^r$ of degree d is a projection of a rational normal curve in $\mathbb{P}^d$*. Slightly more generally, any map $\phi : \mathbb{P}^1 \to \mathbb{P}^r$ of degree d is given as

$$z \mapsto (f_0(z), \ldots, f_r(z))$$

for some $(r+1)$-tuple of polynomials f_α of degree d on $\mathbb{P}^1$, which is to say it is the composition of the embedding $\phi_d : \mathbb{P}^1 \to \mathbb{P}^d$ of $\mathbb{P}^1$ as a rational normal curve with a linear projection $\pi : \mathbb{P}^d \to \mathbb{P}^r$.

Given how easy it is to describe rational curves in projective space in this way, it is in some ways surprising how many open questions there are about such curves. We'll talk more about some of these questions in the following section; for now, we will try to give a sense of what we can say about such curves by considering one of the first and simplest cases: smooth rational curves of degree 4 in $\mathbb{P}^3$.

So: let $C \subset \mathbb{P}^3$ be a smooth, nondegenerate curve of degree 4 and genus 0 in $\mathbb{P}^3$. To describe the geometry of C, the first thing to determine is what surfaces it lies on—that is, what degree polynomials on $\mathbb{P}^3$ vanish on C.
((what's proven is that C lies on a smooth quadric and the ideal needs at least 3 additional 3-ic generators. State in advance, and maybe do better.))
To start with, we can ask: does C lie on a quadric surface? To answer this, we consider again the restriction map

$$H^0(\mathcal{O}_{\mathbb{P}^3}(2)) \;\to\; H^0(\mathcal{O}_C(2)) = H^0(\mathcal{O}_{\mathbb{P}^1}(8)).$$

Here the vector space on the left—homogeneous quadratic polynomials on $\mathbb{P}^3$—has dimension 10, while the one on the right, either by Riemann-Roch or by direct examination, has dimension 9. We conclude that *the curve C must lie on at least one quadric surface $Q \subset \mathbb{P}^3$.*

Since C is irreducible and nondegenerate, it can't lie on a union of planes, so the quadric Q must either be smooth or a cone over a conic curve. We'll see in a moment that the latter case can't occur, so let's assume for now that Q is smooth.

The natural follow-up question is, what is the class of C in the Picard group of Q? We know that $Q \cong \mathbb{P}^1 \times \mathbb{P}^1$, with the fibers of the two projections appearing as lines of the two rulings of Q. Lines L and M of the two rulings generate the Picard group [**?**, ***], so that we must have $C \sim aL + bM$ for some a, b (in other words, in terms of the isomorphism $Q \cong \mathbb{P}^1 \times \mathbb{P}^1$, C is the zero locus of a bihomogeneous polynomial of bidegree (a, b)), and we ask what a and b are. The choices are limited: since C is a quartic curve, we must have $a+b = 4$. Adjunction [**?**, ***] tells us which must be the case: the genus formula for curves on Q tells us that the genus of a smooth curve of class (a, b) on Q has genus $(a-1)(b-1)$, whence the class of our curve C must be $(1, 3)$ (for a suitable ordering of the two rulings).

It follows in particular that *Q is the unique quadric containing C*. One way to see this is that since C has class $(1, 3)$ it meets the lines of the first ruling three times; if Q' is any quadric containing C, then, it must contain all these lines and hence must equal Q. Equivalently, we may consider the exact sequence

$$0 \to \mathcal{I}_{C/Q}(2) \to \mathcal{O}_Q(2) \to \mathcal{O}_C(2) \to 0.$$

If C has class $L + 3M$, we have $\mathcal{I}_{C/Q}(2) = \mathcal{O}_Q(L - M)$. Since this bundle has negative degree on every line of the first ruling, it has no sections; hence the restriction map $H^0(\mathcal{O}_Q(2)) \to H^0(\mathcal{O}_C(2))$ is injective and so there are no quadrics in $\mathbb{P}^3$ containing C other than Q.

We can also describe the rest of the ideal of C similarly. For example, to find the cubic polynomials vanishing on C we consider the restriction map

$$H^0(\mathcal{O}_{\mathbb{P}^3}(3)) \ \to \ H^0(\mathcal{O}_C(3)) = H^0(\mathcal{O}_{\mathbb{P}^1}(12)).$$

The dimensions of these two vector spaces being 20 and 13 respectively, we see that C must lie on at least 7 cubics; four of these are simply products of Q with linear forms, and so we see that C must lie on at least three cubics modulo those containing Q. Indeed, these are easy to spot: if L and L' are any two lines of the first ruling, the divisor $C + L + L'$ has class $(3, 3)$ on Q and hence is the intersection of Q with a cubic surface. As $L + L'$ varies in a two-dimensional linear series, we get three cubics containing C modulo those containing Q. Conversely, any cubic containing C (but not containing Q) will intersect Q in the union of C with a curve of type $(2, 0)$ on Q, which is to say the sum of two lines of the first ruling, so these are all the cubics containing C.

((at least state that these are the generators. And state the set-theoretic intersection problem.))

Finally, we have to show that the quadric containing the curve C cannot be a cone over a conic plane curve. The key question here is whether or not C contains the vertex p of the cone: if not, the same adjunction-based calculation shows that C must have genus 1; while a parity argument (how many times does C meet a line of the ruling of Q?) shows that if a curve $C \subset Q$ of even degree contains p it must be singular there.
((this is pretty fast, compared to the level in the rest of the Ch. let's fill it in.))

Exercise 3.3.6. Find all possible Hilbert functions of smooth rational quintic curves $C \subset \mathbb{P}^3$. (There are only two, depending on whether or not C lies on a quadric, so this isn't so bad.)

Exercise 3.3.7. Every g^3_4 on $\mathbb{P}^1$ is uniquely expressible as a sum of the g^1_1 and a g^1_3

Exercise 3.3.8. There is a 1-parameter family of rational quartic curves in $\mathbb{P}^3$ up to projective equivalence. (Finding the invariants is a nice problem, which we should talk about. This is the cross-ratio of the roots of the quartic in 2 variables corresponding to the projection center.)

3.3.2 Further problems (open and otherwise) concerning rational curves in projective space

To begin with, we should remark that this one example of a non-linearly normal rational curve in projective space is misleading in that we can give such a complete description. For general d and r, we have no idea what may be the Hilbert function of a rational curve of degree d in $\mathbb{P}^r$. Indeed, even in the limited case of $r = 3$, our knowledge gives out around $d = 9$.

We can, however, say some things about a *general* rational curve $C \subset \mathbb{P}^r$ of given degree d. To make sense of this, let $C_0 \subset \mathbb{P}^d$ be a rational normal curve of degree d. As we've said, any rational curve of degree d in $\mathbb{P}^r$ is the projection $\pi_\Lambda(C_0)$ of C_0 from a $(d-r-1)$-plane $\Lambda \subset \mathbb{P}^d$. If we let $\mathbb{G} = \mathbb{G}(d-r-1, d)$ be the Grassmannian of $(d-r-1)$-planes in $\mathbb{P}^d$, and we let $U \subset \mathbb{G}$ be the open subset of planes disjoint from the secant variety of C_0, we have a family of rational curves in $\mathbb{P}^r$ parametrized by U and including every smooth rational curve $C \subset \mathbb{P}^r$ of degree d. Thus in particular we can talk about a *general rational curve* of degree d and genus g in $\mathbb{P}^r$, and ask about its geometry.

This is, in fact, still largely uncharted waters. Consider, for example, one of the most basic questions we might ask: what is the Hilbert function of a general rational curve $C \subset \mathbb{P}^r$ of degree d? As in the example, this is tantamount to looking at the restriction map

$$\rho_m : H^0(\mathcal{O}_{\mathbb{P}^r}(m) \to H^0(\mathcal{O}_C(m)) = H^0(\mathcal{O}_{\mathbb{P}^1}(md)).$$

Equivalently, we're asking: if V is a general $(r+1)$-dimensional vector space of homogeneous polynomials of degree d, what is the dimension of the space of polynomials spanned by m-fold products of polynomials in V? We might naively guess that the answer is, "as large as possible," meaning that the rank of ρ_m is $\binom{m+r}{r}$ when that number is less than $md + 1$, and equal to $md + 1$ when it is greater—in other words, the map ρ_m is either injective or surjective for each m.

This, it turns out, is true, but it is only relatively recently known: the case $g = 0$, as here, was done by Ballico in **** (??), and the analogous statement for curves of arbitrary genus, which we will describe in Chapter 8, was proved in 2019 by Eric Larson.

The secant plane conjecture

Another question we may ask about a curve in projective space is what secant planes it has. To frame the question, let's start with some language: given a smooth curve $C \subset \mathbb{P}^r$, we say that an e-secant s-plane to C is an s-plane $\Lambda \cong \mathbb{P}^s \subset \mathbb{P}^r$ such that the intersection $\Lambda \cap C$ has degree $\geq e$; if we exclude degenerate cases (for example, where $\Lambda \cap C$ fails to span Λ), this is the same as saying we have a divisor $D \subset C$ of degree e whose span is contained in an s-plane.

Do we expect a curve $C \subset \mathbb{P}^r$ to have any e-secant s-planes? The set of s-planes in $\mathbb{P}^r$ is parametrized by the Grassmannian $\mathbb{G} = \mathbb{G}(s, r)$, which had dimension $(s+1)(r-s)$. Inside $\mathbb{G}$, the locus of planes that meet C has codimension $r-s-1$ (the locus of planes containing a given point of C has codimension $r-s$); so our naive expectation might be that the locus of e-secant s-planes would have codimension $e(r-s-1)$ in $\mathbb{G}$. Thus we would expect a curve $C \subset \mathbb{P}^r$ to have e-secant s-planes when

$$e \;\leq\; (s+1)\frac{r-s}{r-s-1}.$$

Is this true of a general rational curve? For most e, r and s, we don't know!

3.4 Curves of genus 1

We cannot begin to describe everything that has been said or done with curves of genus 1, or *elliptic curves*[1]. They appeared, in the second half of the 19th century, as key objects in the developing subjects of geometry, number theory and complex analysis, and the literature is correspondingly rich. Though all curves of genus 0 are isomorphic to $\mathbb{P}^1$ and on a given curve of genus 0 all divisors of a given degree are linearly equivalent, neither of the analogous statements holds true for curves of genus 1. The ways in which 19th century geometers dealt with this fact has shaped much of algebraic geometry.

Specifically, classical geometers observed that there was a one-parameter family of curves of genus 1 up to isomorphism, and that on a given curve of genus 1 there was a one-dimensional family of divisors up to linear equivalence. These were perhaps the earliest examples of *moduli spaces*, and they were ultimately generalized to the moduli space M_g of curves of genus g, and the Picard variety $\mathrm{Pic}^d(C)$ parametrizing divisors of degree d on a given curve C up to linear equivalence.

Here we will focus on the geometric side, and try to describe maps of genus 1 curves to projective space. As a sort of through-line for our discussion, we will try to indicate in each case how the given projective model of a curve E of genus 1 gives rise to the expectation that there is a one-parameter family of curves of genus 1 up to isomorphism. For any d, the automorphism group of E acts transitively on the invertible sheaves of degree d on E. In other words, if $\phi, \phi' : E \to \mathbb{P}^r$ are two maps given by complete linear series $|L|$ and $|L'|$ of degree d on E, then there exists automorphisms $\alpha : \mathbb{P}^r \to \mathbb{P}^r$ and $\beta : E \to E$ such that $\phi' \circ \beta = \alpha \circ \phi$. In particular, if ϕ and ϕ' are embeddings—as will be the case when $d \geq 3$—then their images are projectively equivalent. As for the business of parametrizing invertible sheaves on a given curve C, we will take that up in the next chapter, and see it applied in the case of curves of genus $g \geq 2$ in Chapter 5.

[1]Technically, an elliptic curve is a smooth curve of genus 1 with a distinguished point, called the *origin*.

3.4.1 Double covers of $\mathbb{P}^1$

Let E be a smooth projective curve of genus 1. If L is any invertible sheaf of degree 1 on E, the Riemann-Roch Theorem says that $h^0(L) = 1$, so if we're looking for nonconstant maps to projective space we have to go to degree 2 and higher.

To start with, suppose L is an invertible sheaf of degree 2 on E. By the Riemann-Roch Theorem , $h^0(L) = 2$ and the linear series $|L|$ is base point free, so we get a map $\phi : E \to \mathbb{P}^1$ of degree 2. By the Riemann-Hurwitz Theorem , the map ϕ will have 4 branch points; by the remark above, these four points are determined, up to automorphisms of $\mathbb{P}^1$ by the curve E, and are independent of the choice of L. After composing with an automorphism of $\mathbb{P}^1$ we can take these four points to be $0, 1, \infty$ and λ for some $\lambda \neq 0, 1 \in \mathbb{C}$. Since there is a unique double cover of $\mathbb{P}^1$ with given branch divisor (see **??**) it follows that $E \cong E_\lambda$, where E_λ is the curve given by the affine equation

$$y^2 = x(x-1)(x-\lambda).$$

When are two curves E_λ and $E_{\lambda'}$ isomorphic? By what we've said, this will be the case if and only if there is an automorphism of $\mathbb{P}^1$ carrying the points $\{0, 1, \infty, \lambda\}$ to $\{0, 1, \infty, \lambda'\}$, in any order. This will be the case if and only if λ and λ' belong to the same orbit under the action of the group $G \cong S_3 \subset PGL(3)$ of automorphisms of $\mathbb{P}^1$ permuting the three points $0, 1$ and ∞. Direct computation shows that the orbit of λ is

$$\lambda' \in \{\lambda,\ 1-\lambda,\ \frac{1}{\lambda},\ \frac{1}{1-\lambda},\ \frac{\lambda-1}{\lambda},\ \frac{\lambda}{\lambda-1}\}.$$

Now, the quotient of $\mathbb{P}^1$ by the action of G is again isomorphic to $\mathbb{P}^1$ by Luroth's theorem, which means that the field of rational functions on $\mathbb{P}^1$ invariant under G is again a purely transcendental extension $K(j)$; explicitly, we can take

$$j \ = \ 256 \cdot \frac{\lambda^2 - \lambda + 1}{\lambda^2(\lambda-1)^2}.$$

(the factor of 256 is there for arithmetic reasons). In any case, we see explicitly that there is a unique smooth projective curve of genus 1 for each value of j; in particular, the family of all such curves is parametrized by a curve.

3.4.2 Plane cubics

Moving from degree 2 to degree 3, let L be an invertible sheaf of degree 3 on E. We see from Corollary 2.3.4 that the sections of L give an embedding of E as a smooth plane cubic curve; conversely, the genus formula tells us that a smooth plane cubic curve indeed has genus 1.

We won't delve into the geometry of plane cubics, except to point out that once more we can use this representation to argue that the isomorphism classes of elliptic curves form a 1-dimensional family. To see this, observe that the space of homogeneous polynomials of degree 3 in three variables is 10-dimensional, and the space of plane cubic curves is correspondingly parametrized by $\mathbb{P}^9$; the locus of smooth curves is a Zariski open subset of this $\mathbb{P}^9$. On the other hand, by what we've said, two plane cubics are isomorphic iff they are congruent under the group PGL_3 of automorphisms of $\mathbb{P}^2$. Since the group PGL_3 has dimension 8, we would expect that the family of such curves up to isomorphism has dimension 1.

3.4.3 Quartics in $\mathbb{P}^3$

Again, let E be a smooth projective curve of genus 1, and consider now the embedding of E into $\mathbb{P}^3$ given by the sections of an invertible sheaf L of degree 4. The first question we might ask is what polynomial equations in $\mathbb{P}^3$ cut out the image, and as before we will do this by looking at the restriction map

$$\rho_2 \;:\; H^0(\mathcal{O}_{\mathbb{P}^3}(2)) \;\to\; H^0(\mathcal{O}_E(2)) = H^0(L^2).$$

The space on the right—the space of homogeneous polynomials of degree 2 in four variables—has dimension 10, while by Riemann-Roch the space $H^0(L^2)$ has dimension 8. It follows that E lies on at least two linearly independent quadrics Q and Q'. Since E does not lie in any plane, neither Q nor Q' can be reducible; thus by Bezout's Theoremwe see that

$$E = Q \cap Q'$$

is the complete intersection of two quadrics in $\mathbb{P}^3$. Moreover, we also see from the Lasker-Noether "AF+BG" theorem that the kernel of ρ_2 is exactly the span of Q and Q'. Thus E determines a point in the Grassmannian $G(2, H^0(\mathcal{O}_{\mathbb{P}^3}(2))) = G(2, 10)$ of pencils of quadrics; and by Bertini's Theorem, a Zariski open subset of that Grassmannian correspond to smooth quartic curves of genus 1. We can use this to once more calculate the dimension of the family of curves of genus 1: the Grassmannian $G(2, 10)$ has dimension 16, while the group PGL_4 of automorphisms of $\mathbb{P}^3$ has dimension 15, so we may conclude that the family of curves of genus 1 up to isomorphism has dimension 1.

Projective normality II

((maybe this should be part of the homological algebra development much later))

Observe that last two cases (cubic and quartic genus 1 curves) are projectively normal; extend this to arbitrary smooth complete intersections.

Exercise: $C \subset Q \subset \mathbb{P}^3$ of class (a, b) is projectively normal iff $|a - b| \leq 1$.

Chapter 4

Jacobians

An essential construction in studying a curve C is the association to a given divisor of degree d of a invertible sheaf of that degree—in other words, the map

$$\mu : \big\{\text{effective divisors of degree } d\big\} \longrightarrow \big\{\text{invertible sheaves of degree } d\big\}.$$

A priori, this is a map of sets. But it is a fundamental fact that both sets may be given the structures of algebraic varieties in a natural way, so that the map between them is regular. The geometry of this map governs the geometry of the curve in many ways.

One part is relatively easy: the divisors on a smooth curve are parametrized by the *symmetric powers* $C^{(d)}$ of the curve C, described in Section 4.1. By contrast, the parametrization of the set of invertible sheaves on C of a given degree by the points of an algebraic variety $\mathrm{Pic}^d(C)$ is a major undertaking, one that historically brought complex analysis and algebraic geometry together. We'll describe the original construction of the varieties $\mathrm{Pic}^d(C)$ by complex analysis in Section 4.3 below, and touch briefly on the algebraic constructions.

The fact that invertible sheaves of a given degree of a curve C are parametrized by the points of a variety $\mathrm{Pic}^d(C)$ has many consequences. For example, applying dimension theory to $\mathrm{Pic}^d(C)$, we will show in Theorem 4.2.3 that every curve can be embedded in projective space as a curve of degree $g + 3$.

4.1 Symmetric products

If G is a finite group acting by automorphisms on an affine scheme $X := \operatorname{Spec} A$ then X/G is by definition $\operatorname{Spec}(A^G)$, the spectrum of the ring A^G of invariant elements of A. It is a basic theorem of commutative algebra that the map $X \to X/G$ induced by the inclusion of rings is finite, and the fibers of the map $X \to X/G$ are actually the orbits of G (see for example [**?**, Theorem ***]), something that often fails when G is infinite. Since the map $X \to X/G$ is finite, $\dim X/G = \dim X$. The construction commutes with the passage to G-invariant open affine sets, and thus passes to more general schemes—and in particular to projective schemes (see exercise **??**)—as well.

Exercise 4.1.1. Let G be a finite group acting on a quasi-projective scheme X. Show that there is a finite covering of X by invariant open affine sets. (Hint: consider the sum of the G-translates of a very ample divisor.)

For any variety X
((I guess we need to say in the intro that varieties are quasi-projective...
))
we define the d-th symmetric power of X to be the quotient of the Cartesian product X^d of d copies of X by the action of the group of all permutations of the factors. The resulting variety X^d/S_d is called the *d-th symmetric power*, or *d-th symmetric product*, of X, denoted $X^{(d)}$.

For example, if $X = \mathbb{A}^1$ then $X^d = \mathbb{A}^d$, and the ring of invariants of the symmetric group acting on $\mathcal{O}_{\mathbb{A}^d} = k[x_1, \dots, x_d]$ by permuting the variables is generated by the d elementary symmetric functions, which generate a polynomial subring. Since the symmetric functions of the roots of a polynomial are the coefficients of the polynomial, we may identify the scheme X^d with $\mathbb{A}^d$. ([**?**, Exercises 1.6, 13.2-13.4])

If $X = \mathbb{P}^1$ we can observe that on the product $(\mathbb{P}^1)^d$, taking the homogenesous coordinates of the i-th copy of $\mathbb{P}^1$ to be (s_i, t_i), the multilinear symmetric functions of degree d,

$$s_0 t_1 t_2 \cdots t_d, \dots, s_0 s_1 \cdots s_d$$

localize on each of the standard affine open sets $(\mathbb{A}^1)^d = \mathbb{A}^d$ to the usual ordinary symmetric functions, and define an isomorphism $\mathrm{Sym}^d(\mathbb{P}^1) \to \mathbb{P}^d$. Again, we may think of this map as taking a d-tuple of points to the homogeneous form of degree d vanishing on it, which is unique up to scalars.

Note that this argument does not say anything about the symmetric products of $\mathbb{A}^2$, which are in fact singular—see Exercise 4.1.2.

Since an effective divisor of degree d on a curve C is an unordered d-tuple of points on C, with repetitions allowed, it corresponds to a point in the *dth symmetric power* $C^{(d)}$.

There is one aspect of the symmetric powers that is special to the case of curves:

Proposition 4.1.2. *If X is a smooth curve then each symmetric power $X^{(d)}$ is smooth.*

Proof. The general case follows from the case of $\mathbb{A}^1$ because locally analytically the action of the symmetric group on C^d is the same as for $\mathbb{A}^1$: If $\overline{p} \in X^{(d)}$, then it suffices to show that the quotient of an invariant formal neighborhood of the preimage $p_1, \ldots, p_s$ of overline p is smooth. After completing the local rings, we get an action of the symmetric group G on the product of the completions of X at the p_i, and this depends only on the orbit structure of G acting on $\{p_1, \ldots, p_s\}$. Thus it would be the same for some orbit of points on $\mathbb{A}^1$. $\square$

By contrast, if $\dim X \geq 2$ then the symmetric powers $X^{(d)}$ are singular for all $d \geq 2$.

Exercise 4.1.3. 1. We say that a group G acts freely on X if $gx = gy$ only when $g = 1$ or $x = y$. Show that if G is a finite group acting freely on a smooth affine variety X then the quotient X/G is smooth.

2. Let $X = (\mathbb{A}^2)^2$ and let $G = \mathbb{Z}/2$ act on X by permuting the two copies of $\mathbb{A}^2$; algebraically, $(\mathbb{A}^2)^2 = \operatorname{Spec} S$, with $S = k[x_1, x_2, y_1, y_2]$ and the nontrivial element $\sigma \in G$ acts by $\sigma(x_i) = y_i$.

3. . Show that G acts freely on the complement of the diagonal, but fixes the diagonal pointwise.

4. Show that the algebra S^G has dimension 4 and is generated by the 5 elements

$$f_1 = x_1 + y_1, f_2 = x_2 + y_2, g_1 = x_1 y_1, g_2 = x_2 y_2, h = x_1 y_2 + x_2 y_1,$$

perhaps by appropriately modifying the steps given in [**?**, Exercise 1.6].

5. Show that h^2 lies in the subring generated by $f_1, \ldots, f_4$, and thus $S^{(2)}$ is a hypersurface, singular along the codimension 2 subset $f_1 = f_2 = 0$, which is the image of the diagonal subset of the cartesian product $(\mathbb{A}^2)^2$.

Exercise 4.1.4 (The universal divisor of degree d)**.** Let C be a smooth projective curve, and $C^{(d)}$ its dth symmetric power. Show that the locus

$$\mathcal{D} := \{(D, p) \in C^{(d)} \times C \mid p \in D\}$$

is a closed subvariety of the product $C^{(d)} \times C$, whose fiber over any point $D \in C^{(d)}$ is the divisor $D \subset C$.
((How do our readers do this? We need to have proven the universal property of the symmetric product – the fine moduli space for invariant divisors of degree d.))

The variety $\mathcal{D}$ is called the universal divisor on C by virtue of the fact that for any family of divisors of degree d on C—that is, a scheme B and a subscheme $\mathcal{E} \subset B \times C$ flat of degree d over B, there is a unique morphism $\phi : B \to C^{(d)}$ such that $\mathcal{E}$ is the pullback via ϕ of $\mathcal{D} \subset C^{(d)} \times C$. Indeed, this amounts to saying that $C^{(d)}$ is the *Hilbert scheme* parametrizing subschemes of C of degree d. These statements are not generally true for higher-dimensional varieties; see Chapter **??** and especially Exercise 6.6.1

4.2 Jacobians

To construct $\mathrm{Pic}^d(C)$ we start with $d = 0$, and identify $\mathrm{Pic}^0(C)$ with the *Jacobian* $J(C)$ of C using *abelian integrals* and the classical topology. This produces a complex manifold rather than an algebraic variety, but has the virtue of being relatively concrete.

((I think we should make the following into a formal theorem– in the characterization section — which maybe doesn't exist yet?))

The Jacobian $J(C)$ is in fact a projective variety, and may be constructed purely algebraically—so that, for example, if the curve C is defined over a given field K then $J(C)$ will be defined over K as well. The search for such a construction was one of the driving forces of algebraic geometry in the first half of the 20th century, giving rise to the notion of abstract algebraic varieties. See for example [?] [Kleiman must have something for this].

The goal of the 19th century mathematicians who first described abelian integrals was simply to make sense of integrals of algebraic functions. In the early development of calculus, mathematicians figured out how to evaluate explicitly integrals such as

$$\int_{t_0}^{t} \frac{dx}{\sqrt{x^2+1}}.$$

Such integrals can be thought of as path integrals of meromorphic differentials on the Riemann surface associated to the equation $y^2 = x^2 + 1$. This surface is isomorphic to $\mathbb{P}^1$, meaning that x and y can be expressed as rational functions of a single variable z; making the corresponding change of variables transformed the integral into one of the form

$$\int_{s_0}^{s} R(z)dz,$$

with R a rational function, and such integrals are readily evaluated by the technique of partial fractions.

When they tried to extend this to similar-looking integrals like

$$\int_{t_0}^{t} \frac{dx}{\sqrt{x^3+1}},$$

which arises when one studies the length of an arc of an elipse and was thus called an elliptic integral, they were stymied. The reason gradually emerged: the problem is that the Riemann surface associated to the equation $y^2 = x^3 + 1$ is not $\mathbb{P}^1$, but rather a curve of genus 1, and so has nontrivial homology group $H_1(C, \mathbb{Z}) \cong \mathbb{Z}^2$. In particular, if one expresses this "function" of t as a path integral, then the value depends on a choice of path; it is defined only modulo a lattice $\mathbb{Z}^2 \subset \mathbb{C}$. This implies that the inverse function is a doubly periodic meromorphic function on $\mathbb{C}$, and not an elementary function. Many new special functions, such as the Weierstrass $\mathcal{P}$-function were studied as a result. The name "elliptic curve" arose from these considerations too.

Once this case was understood, the next step was to extend the theory to path integrals of holomorphic differentials on curves of arbitrary genus. One problem is that the dependence of the integral on the choice of path is much worse; the set of homology classes of paths between two points $p_0, p \in C$ is identified with $H_1(C, \mathbb{Z}) \cong \mathbb{Z}^{2g}$ rather than $\mathbb{Z}^2$. The Jacobian arises when one considers the integrals of *all* holomorphic differentials on C simultaneously.

To express the resulting construction in relatively modern terms, let C be a smooth projective curve of genus g over $\mathbb{C}$, and let ω_C be the sheaf of differential forms on C. We will consider C as a complex manifold. Every meromorphic differential form is in fact algebraic [**?**], and we consider ω_C as a sheaf in the analytic topology.

We consider the space $V = H^0(\omega_C)^*$ of linear functions on the space of differentials $H^0(\omega_C)$. Integration over a closed loop in C defines a linear function on 1-forms, so that we have a map

$$\iota : \mathbb{Z}^{2g} = H_1(C, \mathbb{Z}) \;\to\; H^0(\omega_C)^* \cong H^1(\mathcal{O}_C) = \mathbb{C}^g.$$

Using Hodge theory[1] one can show that ι induces an injective map of vector spaces

$$\mathbb{R} \otimes H_1(C, \mathbb{Z}) = H_1(C, \mathbb{R}) \to H^0(\omega_C)^*$$

The complex structure on $H^0(\omega_C)^*$ yields a complex analytic structure on the quotient $\mathbb{C}^g/(\iota(\mathbb{Z}^{2g}))$, which is thus a torus of real dimension $2g$. We call this quotient, with its structure as a g-dimensional complex manifold, the Jacobian of C, denoted

$$J(C) \;=\; V/\Lambda.$$

The point of this construction is that for any pair of points $p, q \in C$, the expression $\int_q^p$ describes a linear functional on $H^0(\omega_C)$, defined up to functionals obtained by integration over closed loops, and thus a point of $J(C)$. Thus, for example, if we choose a "base point" $q \in C$, we get a holomorphic map

$$\mu \;:\; C \;\to\; J(C); \quad p \mapsto \int_q^p$$

Having chosen a base point $q \in C$ as above, we get for each $d \geq 0$ the *Abel-Jacobi* map

$$\mu_d \;:\; C^{(d)} \;\to\; J(C),$$

[1] By Hodge theory

$$H^1(C, \mathbb{C}) \cong H^1(C, \mathcal{O}_C) \oplus \overline{H^1(C, \mathcal{O}_C)}$$

where the bar denotes complex conjugation $H^1(C, \mathbb{C})$, and the map ι is the composition of the natural inclusion with the projection to the first summand. Now $H_1(C, \mathbb{C}) = \mathbb{C} \otimes_{\mathbb{Z}} H_1(C, \mathbb{Z})$, so any basis of $H_1(C, \mathbb{Z})$ maps to a basis of $H^1(C, \mathbb{C})$ invariant under conjugation in $H^1(C, \mathbb{C})$— See Voisin [] or Griffiths-Harris []. If there were a real dependence relation among elements of the image of this basis under ι, then it the same relation would hold after complex conjugation and thus hold on the image of the basis in $H_1(C, \mathbb{C})$, a contradiction.

defined by

$$\mu_d(p_1 + \cdots + p_d) \;=\; \sum \int_q^{p_i} .$$

When there is no ambiguity about d, we will denote them simply by μ. and we we define $\mu(-D)$ to be $-\mu(D)$. The map μ is a group homomorphism in the sense that if D, E are divisors, then $\mu(D+E) = \mu(D) + \mu(E)$; this is immediate when the divisors are effective, and follows in general because the group of divisors is a free group. The connection between the discussion above and the geometry of linear series is made by *Abel's theorem*:

Theorem 4.2.1. *Two divisors $D, D' \in C^{(d)}$ on C are linearly equivalent if and only if $\mu(D) = \mu(D')$; in other words, the fibers of μ_d are the complete linear systems of degree d on C.*

See [**?**, Section 2.2] for a complete proof; we will just prove the "only if" part. This was in fact the only part proved by Abel; the converse, which is substantially more subtle, was proved by Clebsch.

Proof of "only if". Suppose that D and D' are linearly equivalent; that is, $\mathcal{O}_C(D) \cong \mathcal{O}_C(D')$. Call this invertible sheaf $\mathcal{L}$, and suppose that D and D' are the zero divisors of sections $\sigma, \sigma' \in H^0(\mathcal{L})$. Taking linear combinations of σ and σ', we get a pencil $\{D_\lambda\}_{\lambda \in \mathbb{P}^1}$ of divisors on C, with

$$D_\lambda \;=\; V(\lambda_0\sigma + \lambda_1\sigma'),$$

and by Exercise 4.1.4 this corresponds to a regular map $\alpha : \mathbb{P}^1 \to C^{(d)}$.

Consider now the composition

$$\phi = \mu \circ \alpha \;:\; \mathbb{P}^1 \;\to\; J(C).$$

Now, $J(C)$ is the quotient of the complex vector space $V = H^0(\omega_C)^*$ by a discrete lattice. If z is any linear functional on V, then, the differential dz on V descends to a global holomorphic 1-form on the quotient $J(C)$, so that the regular one-forms on $J(C)$ generate the cotangent space to $J(C)$ at every point. But for any 1-form ω on $J(C)$, the pullback $\phi^*\omega$ is a global holomorphic 1-form on $\mathbb{P}^1$, and hence identically zero. It follows that the differential $d\phi$ vanishes identically, and hence (since we are in characteristic 0) that ϕ is constant; thus $\mu(D) = \mu(D')$. □

Abel's Theorem goes surprisingly far to describe the Jacobian. The first statement of the following Corollary suggests how to describe the structure of the Jacobian algebraically, and was used by Andre Weil in the first such construction.

Corollary 4.2.2. *If C is a smooth curve of genus g then the Abel-Jacobi map $\mu_g : C^{(g)} \to J(C)$ is a surjective birational map. More generally, μ_d is generically injective for $d \le g$ and surjective for $d \ge g$.*

Proof. For $d \leq g = \dim H^0(\omega_C)$, a divisor D that is the sum of d general points $p_1, \dots, p_d \in C$ will impose independent vanishing conditions on the sections of ω_C, and thus

$$h^1\mathcal{O}_C(D) = h^0(\omega_C(-D)) = g - d,$$

by Serre duality. Using this, the Riemann-Roch formula gives $h^0\mathcal{O}_C(D) = 1$, so the fiber of μ_d consists of a single point, proving generic injectivity. In particular when $d = g$, the image of μ_d has dimension g, and since $C^{(g)}$ is compact, the image is closed, so it must be equal to $J(C)$.

Similarly, if $d \geq g$, we will have $h^0(\omega_C(-D)) = 0$ and hence $h^0(\mathcal{O}_C(D)) = d - g + 1$. Since this is the affine dimension, the linear series $|D|$ has dimension $d - g = \dim C^{(d)} - \dim J(C)$, and again it follows that μ_d is surjective. □

4.2.1 Applications to linear series

To illustrate some of the power of Abel's theorem, we will use it to prove a basic result:

Theorem 4.2.3. *Let C be a smooth projective curve of genus g. If $D \in C_{g+3}$ is a general divisor of degree $g+3$ on C, then D is very ample. In particular, every curve of genus g may be embedded in $\mathbb{P}^3$ as a curve of degree $g+3$.*

We proved in Theorem 2.2.7 that *every divisor* of degree $\geq 2g+1$ is very ample; the difference here is that we are taking a *general* divisor. This result is sharp in the sense that hyperelliptic curves, for example, cannot be embedded in projective space as curves of any degree less than $g+3$, as we'll see in Chapter 11. However, if we consider only general divisors on general curves, we can do still better: "most" curves of genus g can in fact be embedded in $\mathbb{P}^3$ as curves of degree $d = \lceil 3g/4 \rceil + 3$ [].

Proof. If D is general of degree $g+3$ we have $h^0(\mathcal{O}_C(D)) = 4$. To show that it is very ample, we have to show that

1. for any point $p \in C$, we have $h^0(\mathcal{O}_C(D - p)) = 3$ (that is, $|D|$ has no base points, and so defines a regular map $\phi_D : C \to \mathbb{P}^3$); and
2. for any pair of points $p, q \in C$, we have $h^0(\mathcal{O}_C(D - p - q)) = 2$.

The second of these assertions immediately implies the first, and this is what we will prove.

Now let D be an arbitrary divisor of degree $g+3$. To say that $h^0(\mathcal{O}_C(D - p - q)) \geq 3$ is equivalent, by the Riemann-Roch theorem, to the condition $h^0(\omega_C(-D + p + q)) \geq 1$; fixing a divisor $K_C \in |\omega_C|$, this is the condition that there exists an effective divisor E of degree $g-3$ linearly equivalent to a divisor in $K_C - D + p + q|$.

Now consider the map

$$\nu : C^{(g-3)} \times C^{(2)} \;\to\; J(C)$$

given by

$$\nu : (E, F) \;\mapsto\; \mu_{2g-2}(K_C) - \mu_{g-3}(E) + \mu_2(p+q),$$

where the $+$ and $-$ on the right refer to the group law on $J(C)$.

By what we have just said, and Abel's theorem, the divisor D fails to be very ample only if $\mu(D) \in \mathrm{Im}(\nu)$. But the source $C^{(g-3)} \times C^{(2)}$ of ν has dimension $g-3+2 = g-1$, and so its image in $J(C)$ must be a proper subvariety; since μ_{g+3} is dominant, the image of a general divisor $D \in C^{(g-3)}$ is a general point of $J(C)$ and thus will not lie in $\mathrm{Im}(\nu)$. $\square$

Thus Abel's theorem, which was born out of an effort to evaluate calculus integrals, winds up proving a basic fact in the theory of algebraic curves!

((we said early on that we don't need to know that $J(C)$ is algebraic; for the present purposes, it's enough to know that $J(C)$ is a complex torus of dimension g. But in that case we do need to know that if $f : X \to Y$ is a holomorphic map of compact complex manifolds with $\dim X < \dim Y$, then $f(X)$ is a proper analytic subvariety of Y. we also need to know that the group law is algebraic. We need to have a formal statement of the existence as an algebraic group. Will be taken care of by the Characterization section, yet to be written.))

((DE rev to here 1/8/22))

4.3 Picard varieties

((the rest of this chapter is more like an outline than a chapter. Joe, we could go over the material you want to put in together, and I could try to write it, or you could write it.))

Possible contents for this section: definition and notation for $\mathrm{Pic}^d(C)$; the fact (cheerful or otherwise) that $\mathrm{Pic}^d(C)$ represents the functor of families of invertible sheaves of degree d on C (does it, by the way? I'm worried because invertible sheaves do have automorphisms). Explicit correspondence between first order deformations of a given invertible sheaf L on C and tangent vectors to $\mathrm{Pic}^d(C)$ at L (that is, elements of $H^0(\omega_C)^* = H^1(\mathcal{O}_C)$ via transition functions for L.

Existence of a universal family $\mathcal{P}_{d,g}$ of Picard varieties over M_g (that is, a coarse moduli space for pairs (C, L) with C of genus g and $L \in \mathrm{Pic}^d(C)$. Likewise, analogs $\mathcal{W}^r_d \subset \mathcal{P}_{d,g}$, or "universal Brill-Noether vareities." Cheerful fact: $\mathcal{P}_{d,g}$ is not in general isomorphic to $\mathcal{P}_{d',g}$, unless $d' \equiv \pm d \pmod{2g-2}$.

Chow and/or cohomology groups of $\mathrm{Pic}^d(C)$: the fact that the Neron-Severi group of a very general Jacobian is $\mathbb{Z}$; maybe even the fact that the algebraic cohomology of a very general Jacobian is $\mathbb{Z}$ in every dimension.

Introduce the varieties $W^r_d(C)$ here, or later? In any event, when we do we should give the full statement of the Brill-Noether theorem

Notation: we should say at some point that when there's no danger of confusion, we suppress the (C): that is, write J for $J(C)$, W^r_d for $W^r_d(C)$, etc.

4.4 Differential of the Abel-Jacobi map

In this section we will describe the differential $d\mu$ of the Abel-Jacobi map $\mu : C_d \to J(C)$; this yield a sharper form of Abel's theorem.

((clarify the structure of the next few pages: what is *the* theorem, what are special cases to get an intuitive feel.))
To start, suppose $D = p_1 + \cdots + p_d$ is a divisor consisting of d distt oints on our curve C. Since the quotient map $C^d \to C_d$ is unramified at D, the tangent space to C_d at the point D is naturally identified with the tangent space to C^d at $(p_1, \ldots, p_d)$; that is, the direct sum of the tangent spaces to C at the points p_i:

$$T_D(C_d) = \bigoplus T_{p_i}(C).$$

On the other hand, the tangent space to $J(C)$ at the image point $\mu(D)$ is simply the vector space $H^0(\omega_C)^*$ of which $J(C)$ is a quotient (as it is at every point!). The differential $d\mu_D$ is thus a linear map

$$\bigoplus T_{p_i}(C) \longrightarrow H^0(\omega_C)^*,$$

and the transpose of this a linear map

$$H^0(\omega_C) \longrightarrow \bigoplus T^*_{p_i}(C).$$

This last map is easy to describe: since the map μ is given by

$$\mu_d(p_1 + \cdots + p_d) \;=\; \sum \int_q^{p_i},$$

we can simply differentiate under the integral sign to conclude that *the codifferential d_μ^* is the map*

$$\begin{aligned} H^0(\omega_C) &\to \bigoplus T^*_{p_i}(C) \\ \omega &\mapsto (\omega(p_1), \ldots, \omega(p_d)). \end{aligned}$$

There is a natural extension of this to the case of non-reduced divisors D, that is, divisors with repeated points. We first need a description of the tangent space to C_d at the point D:

Proposition 4.4.1. *The tangent and cotangent spaces to C_d at the point corresponding to an arbitrary divisor $D = \sum a_i p_i$ are naturally identified with $H^0(\mathcal{O}_C(D)/\mathcal{O}_C)$ and $H^0(\omega_C/\omega_C(-D))$ respectively.*

Note that we have a natural pairing between the spaces $H^0(\mathcal{O}_C(D)/\mathcal{O}_C)$ and $H^0(\omega_C/\omega_C(-D))$, given by sending (f, ω) to $\sum_i Res_{p_i}(f\omega)$. Note also that the term "natural" has a precise meaning here: if we let

$$\mathcal{D} = \{(D, p) \in C_d \times C \mid p \in D\}$$

be the universal effective divisor of degree d on C, the proposition says that the cotangent sheaf $T^*_{C_d}$ is the direct image $\alpha_*(\beta^*\omega_C/\beta^*\omega_C(-\mathcal{D}))$, where α and β are the projections of $C_d \times C$ onto the two factors.

((give a real proof — better than ACGH's))

If you wanted to see an argument for Proposition 4.4.1 done out in coordinates (you really don't, but if you did) you could find it in [ACGH].

In any event, given Proposition 4.4.1, we can extend our earlier statement to the

Proposition 4.4.2. *The codifferential $d\mu^*$ of the Abel-Jacobi map is simply the natural restriction map*

$$H^0(\omega_C) \longrightarrow H^0(\omega_C/\omega_C(-D)).$$

Now, note that the codimension of the image of $d\mu^*$—equivalently, the dimension of the kernel of the differential $d\mu$—is by the geometric Riemann-Roch theorem exactly the dimension of the fiber of C_d over the point $\mu(D) \in J(C)$. In other words, the fibers of μ are smooth, and in particular reduced. Thus we can think of Proposition 4.4.2 as a strengthening of the Abel-Clebsch theorem: while Abel and Clebsch show that the fibers of μ are complete linear series set-theoretically, we see from the above that it is in fact true scheme-theoretically.

4.5 Further consequences

One consequence of the description of the Jacobian and the Abel-Jacobi map of a curve C is that *the set of linear series on C of given degree d and dimension r can be given the structure of a scheme in its own right.*

((all this needs a base point, and some care to state the universal property precisely. OK to state it intuitively, then translate))
To start with, we can define $W^r_d(C) \subset \mathrm{Pic}^d(C)$ to be simply the set of invertible sheaves $L \in \mathrm{Pic}^d(C)$ such that $h^0(L) \geq r+1$. We can see readily that this is a Zariski closed subset of $\mathrm{Pic}^d(C)$, for example by pointing out that it is exactly the locus where the fiber dimension of the Abel-Jacobi map $\mu : C_d \to \mathrm{Pic}^d(C)$ is at least r; this is closed by upper-semicontinuity of fiber dimension.

Note that among the subvarieties W^r_d are the images W^0_d of the Abel-Jacobi maps; in other words, the locus of *effective* divisor classes of degree d. The superscript is often omitted, meaning W^0_d is usually written simply W_d.

Two other constructions come up in this setting. For one, the subsets $W^r_d \subset \mathrm{Pic}^d(C)$ can be given the structure of a scheme, in a natural way. One way to characterize this scheme structure is to say that the scheme W^r_d represents the functor of families of invertible sheaves $L \in \mathrm{Pic}^d(C)$ on C with $h^0(L) \geq r+1$.

In the other construction, we can actually parametrize the set of linear series g^r_d on C: that is, there is a scheme $G^r_d(C)$ parametrizing pairs (L, V) with $L \in \mathrm{Pic}^d(C)$ and $V \subset H^0(L)$ a subspace of dimension $r+1$. Again, the scheme structure may be characterized by saying that $G^r_d(C)$ represents the functor of families of linear series on C. Note that the natural map $G^r_d \to W^r_d$ is an isomorphism over the dense open subset $W^r_d \setminus W^{r+1}_d$, and more generally its fiber over a point of $W^s_d \setminus W^{s+1}_d$ is a copy of the Grassmannian $\mathbb{G}(r, s)$.

4.5.1 Examples in low genus

Genus 2

There is not a lot going on here, but there are a couple observations to make. First of all, the map $\mu_1 : C \to J(C)$ embeds the curve C in $J(C)$. Secondly, the map $\mu_2 : C_2 \to J(C)$ is an isomorphism except along the locus $\Gamma \subset C_2$ of divisors of the unique g^1_2 on C; in other words, *the symmetric square C_2 of C is the blow-up of $J(C)$ at a point.*
((true that the fiber is $\mathbb{P}^1$, but is that enough? Maybe so for a birational map of smooth surfaces, but does the reader know this?))

Exercise 4.5.1. Let $C \subset J(C)$ be the image of the Abel-Jacobi map μ_1. Show that the self-intersection of the curve C is 2,

1. by applying the adjunction formula to $C \subset J(C)$; and
2. by calculating the self-intersection of its preimage $C + p \subset C_2$ and using the geometry of the map μ_2.

4.5.2 Genus 3

4.5.3 Genus 4

In genus 4 we encounter for the first time a scheme $W^r_d(C)$ that is neither of the form W_d or $K - W_e$. This is the subscheme $W^1_3(C)$ parametrizing g^1_3s on C.

4.5.4 Genus 5

Want: for general curve C of genus 5, the scheme $W^1_4(C)$ is smooth & irreducible; but when C becomes trigonal, $W^1_4(C)$ becomes reducible, with one component of the form $W^1_3 + C$ and the other $K - W^1_3 - C$.

4.6 Martens' theorem and variants

The general theorems we have described so far dealing with linear series on a curve C, like the Riemann-Roch and Clifford theorems, have to do with the existence or non-existence of linear series on C. Now that we've seen how to parametrize the set of linear series on C by the varieties $W^r_d(C)$, we can ask more quantitative questions: for example, what can the dimension of $W^r_d(C)$ be? One basic result, for example, is the following.

Theorem 4.6.1 (Martens' theorem)**.** *If C is any smooth projective curve of genus g, then for any d and g we have*

$$\dim(W^r_d(C)) \leq d - 2r;$$

moreover, if we have equality for any $r > 0$ and $d < 2g - 2$ the curve C must be hyperelliptic.

Note that if C is hyperelliptic with $g^1_2 = |D|$, we have

$$W^r_d(C) \supset W_{d-2r}(C) + \mu(rD).$$

(In fact, as we'll see in the following chapter, this is an equality.) Since this has dimension $d - 2r$, we see that Martens' theorem is sharp. Note also that Clifford's theorem is a special case of Martens' theorem!

Proof. □

There are extensions of Martens' theorem to the case $\dim(W^r_d(C)) = d - 2r - 1$ (Mumford) and $d - 2r - 2$ (Keem).

4.7 The full Brill-Noether theorem

The existence of the varieties $W_d^r(C)$ parametrizing linear series on an arbitrary curve C allows us to strengthen Clifford's theorem to Martens' theorem. If we ask about linear series on a *general* curve, similarly, it allows us to give a much more detailed version of the Brill-Noether theorem: instead of simply saying when there exists a g_d^r on a general curve C, we can give the dimension of the variety $W_d^r(C)$ parametrizing such linear series, and we can also talk about the geometry of a general linear series on C. We collect the basic facts into the

Theorem 4.7.1 (Brill-Noether theorem, omnibus version)**.** *Let C be a general curve of genus g. If we set $\rho = g-(r+1)(g-d+r)$, then*

1. $\dim(W_d^r(C)) = \rho$;
2. *the singular locus of $W_d^r(C)$ is exactly $W_d^{r+1}(C)$;*
3. *if $\rho > 0$ then $W_d^r(C)$ is irreducible;*
4. *if $\rho = 0$ then the variety $\mathcal{W}_d^r$ is irreducible;*
5. *if L is any invertible sheaf on C, the map*

$$H^0(L) \otimes H^0(\omega_C L^{-1}) \longrightarrow H^0(\omega_C)$$

   is injective;

6. *if $|D|$ is a general g_d^r on C, then*
   (a) *if $r \geq 3$ then D is very ample; that is, the map $\phi_D : C \to \mathbb{P}^2$ embeds C in $\mathbb{P}^r$;*
   (b) *if $r = 2$ the map $\phi_D : C \to \mathbb{P}^2$ gives a birational embedding of C as a nodal plane curve; and*
   (c) *if $r = 1$, the map $\phi_D : C \to \mathbb{P}^2$ expresses C as a simply branched cover of $\mathbb{P}^1$.*

4.8 The Torelli theorem

((consider making this a cheerful fact. or exercise?))
In the examples above, we see that a lot of information about a curve C is encoded in the geometry of its Jacobian. In fact, we can make this official: we have the celebrated

Theorem 4.8.1 (Torelli)**.** *A curve C is determined by the pair $(J(C), \Theta)$.*

Proof. In fact, there are many ways of reconstructing a curve from its Jacobian; this one is due to Andreotti, and makes essential use of our description of the differential of the Abel-Jacobi map.

A key fact is that the Jacobian $J(C) = H^0(\omega_C)^*/H_1(C,\mathbb{Z})$ is a torus, and so has trivial tangent bundle, with fiber $H^0(\omega_C)^*$ at every point. What this means is that if $X \subset J(C)$ is a smooth, k-dimensional subvariety, we have a *Gauss map*

$$\mathcal{G} : X \to G(k,g) = G(k, H^0(\omega_C)^*),$$

sending a point $x \in X$ to its tangent plane $T_xX \subset T_xJ(C) = H^0(\omega_C)^*$; more generally, if X is singular then $\mathcal{G}$ will be a rational map. In particular, if $X = \Theta = W_{g-1}$, we get a rational map

$$W_{g-1} \longrightarrow \mathbb{P}^{g-1} = \mathbb{P}(H^0(\omega_C))$$

between two $g-1$-dimensional varieties, and it is the geometry of this map from which we can recover the curve C.

To start with, let's identify an open subset of W_{g-1} where the Gauss map is defined. This is not hard: a point $L \in W_{g-1} \setminus W^1_{g-1}$ is the image of a unique point $D \in C_{g-1}$ under the map μ, and moreover we've seen that the differential $d\mu$ is injective at D; it follows that L is a smooth point of W_{g-1}.

Moreover, we've identified the tangent space to W_{g-1} at $L = \mu(D)$: as we saw, the differential $d\mu : T_D(C_{g-1}) \to T_L(J) = H^0(\omega_C)^*$ is just the transpose of the evaluation map $H^0(\omega_C) \to H^0(\omega_C(-D))$, and it follows that the tangent space to W_{g-1} at the point L is the hyperplane in $H^0(\omega_C)^*$ dual to the unique differential vanishing on D. To put it another way: if we think of C as canonically embedded in $\mathbb{P}(H^0(\omega_C)^*)$, then by geometric Riemann-Roch the divisor D will span a hyperplane in $\mathbb{P}(H^0(\omega_C)^*)$, and the Gauss map $\mathcal{G}$ sends L to the point in the dual projective space $\mathbb{P}(H^0(\omega_C))$ corresponding to that hyperplane.

Cheerful Fact 4.8.1. We have shown that the open subset $W_{g-1} \setminus W^1_{g-1}$ is contained in the smooth locus of W_{g-1}. In fact, they are equal; that is, W^1_{g-1} is exactly the singular locus of W_{g-1}. This is a special case of the beautiful *Riemann singularity theorem*, which says that for any point $L \in W_{g-1}$, the multiplicity $\mathrm{mult}_L(W_{g-1}) = h^0(L)$. For a proof of the Riemann singularity theorem, see for example [GH].

((David – can we find another reference for the RST? The proof in [GH] is clear but somewhat sketchy; I don't have a copy handy, but as I recall it implicitly assumes that the tangent cone is generically reduced.))

We are now in a position to describe the Gauss map

$$\mathcal{G} : W_{g-1} \longrightarrow \mathbb{P}(H^0(\omega_C))$$

explicitly in terms of the geometry of the canonical curve $C \subset \mathbb{P}(H^0(\omega_C)^*)$. To start, let $p \in \mathbb{P}(H^0(\omega_C))$ be a general point, dual to a general hyperplane $H \subset \mathbb{P}(H^0(\omega_C)^*)$. The hyperplane H will intersect the canonical curve C transversely in $2g-2$ points $p_1, \dots, p_{2g-2}$; these points will be in linear general position (in particular, any $g-1$ of them will be linearly independent and so span H). It follows that the fiber of $\mathcal{G}$ over the point H will consist of the invertible sheaves $L = \mathcal{O}_C(p_{\alpha_1} + \cdots + p_{\alpha_{g-1}})$, where $p_{\alpha_1}, \dots, p_{\alpha_{g-1}}$ is any subset of $g-1$ of the points p_i; in particular, we see that the degree of the map $\mathcal{G}$ is

$$\deg(\mathcal{G}) = \binom{2g-2}{g-1}.$$

The next question is, where does this analysis fail—in other words, for which hyperplanes $H \subset \mathbb{P}H^0(\omega_C)^*$ does the fiber of $\mathcal{G}$ not consist of $\binom{2g-2}{g-1}$ points, or equivalently, what is the branch divisor of the map $\mathcal{G}$? The answer is, the analysis above fails in two cases: when the points $p_1, \dots, p_{2g-2}$ are not in linear general position—specifically, when some $g-1$ of the points p_i fail to be linearly independent; and when the hyperplane H is not transverse to C, so that the hyperplane section $H \cap C$ consists of fewer than $2g-2$ distinct points.

The first of these occurs in codimension 2 in $\mathbb{P}H^0(\omega_C)$, and so does not contribute any components to the branch divisor of $\mathcal{G}$. It follows that the branch divisor of the map $\mathcal{G}$ is exactly the locus of hyperplanes $H \subset H^0(\omega_C)^*$ tangent to the canonical curve C; in other words, *the branch divisor of $\mathcal{G}$ is the hypersurface in $\mathbb{P}H^0(\omega_C)$ dual to the canonical curve $C \subset \mathbb{P}H^0(\omega_C)^*$.*

Now we can invoke the fact that the dual of the dual of a variety $X \subset \mathbb{P}^n$ is X itself (see for example [3264] or something by Kleiman). We thus have a way of recovering the curve C from the data of the pair (J, W_{g-1}): simply put, *the curve C is the dual of the branch divisor of the Gauss map on W_{g-1}*, and the Torelli theorem is proved.

□

The Torelli theorem for curves was the first instance of a class of theorems, called *Torelli theorems*, to the effect that certain classes of varieties are determined to some degree by their Hodge structure; there are, for example, Torelli theorems of varying strength for K3 surfaces, cubic threefolds and fourfolds and hypersurfaces in $\mathbb{P}^n$.

4.9 Additional topics

A couple of topics that would naturally go here, if we have the inclination and space.

4.9.1 Theta characteristics

Basically: introduce the notion of theta-characteristic (= square root of the canonical bundle), and prove the invariance of $h^0(\mathcal{L})$ mod 2. Describe the configuration of theta-characteristics on a given curve C as a principal homogeneous space for the group $J(C)[2] \cong (\mathbb{Z}/2)^{2g}$ of torsion of order 2 in the Jacobian.

Example: bitangents to a plane quartic; distinguished triples of bitangents

4.9.2 Intermediate Jacobians and the irrationality of cubic threefolds

First, describe the intermediate Jacobians $J(X)$ of higher-dimensional varieties X by analogy with the case of curves; introduce the Abel-Jacobi maps from parameter spaces of cycles on X to $J(X)$.

Application: show that the intermediate Jacobian of a cubic threefold is not the Jacobian of a curve by calculating the degree of the Gauss map on the theta-divisor and showing it's not 70 (which by the calculation above it would be if $J(X)$ were a Jacobian). Deduce irrationality of X.

I know this is a bit of a stretch for the current volume, but I'd really like to include it if at all possible: the proof in Clemens-Griffiths is a mess, and this is much simpler

Chapter 5

Hyperelliptic curves and curves of genus 2 and 3

5.1 Hyperelliptic Curves

In the world of curves, hyperelliptic curves are outliers: they behave differently from other curves, and the techniques used to analyze them are different from the techniques used for more general curves. Many theorems about curves contain the hypothesis "non-hyperelliptic," with the corresponding result for hyperelliptic curves arrived at directly by ad hoc methods. Because the methods of this section will not be used in other cases, it could be skipped in first reading

There will be a further discussion of hyperelliptic curves in Chapter 11, focussing on the algebra and geometry of their projective embeddings; the analysis here will cover most of the questions we'll be asking about curves in general in the next four chapters.

5.1.1 The equation of a hyperelliptic curve

By definition, a hyperelliptic curve C is one admitting a degree two map $\pi : C \to \mathbb{P}^1$. Because the degree is only 2, each point in $\mathbb{P}^1$ has either two distinct preimages, or one point of simple ramification. there can be no higher ramification, so at all but finitely many points $p \in C$ the map π is a local isomorphism ("local" here in the complex analytic/classical or étale topology, not the Zariski topology!); at any other point $p \in C$, the map is given in terms of local analytic coordinates on C and $\mathbb{P}^1$ simply by $z \mapsto z^2$. In particular, both the ramification divisor and the branch divisor

((where are these defined?))

are reduced. Thus by the Riemann-Hurwitz formula there are exactly $2g + 2$

branch points $q_1, \ldots, q_{2g+2} \in \mathbb{P}^1$. These points determine the curve:

Theorem 5.1.1. *There is a unique smooth projective hyperelliptic curve C expressible as a 2-sheeted cover of $\mathbb{P}^1$ branched over any set of $2g+2$ distinct points.*

We can easily construct such a curve, postponing for a moment the uniqueness: If the coordinate of the point $p_i \in \mathbb{P}^1$ is λ_i, it is the smooth projective model of the affine curve

$$C^\circ = \left\{(x,y) \in \mathbb{A}^2 \mid y^2 = \prod_{i=1}^{2g+2} (x - \lambda_i)\right\}.$$

Note that we're choosing a coordinate x on $\mathbb{P}^1$ with the point $x = \infty$ at infinity not among the q_i, so that the pre-image of $\infty \in \mathbb{P}^1$ is two points $r, s \in C$. Concretely, we see that as $x \to \infty$, the ratio $y^2/x^{2g+2} \to 1$, so that

$$\lim_{x \to \infty} \frac{y}{x^{g+1}} = \pm 1;$$

the two possible values of this limit correspond to the two points $r, s \in C$.

It's worth pointing out that C is *not* simply the closure of the affine curve $C^\circ \subset \mathbb{A}^2$ in either $\mathbb{P}^2$ or $\mathbb{P}^1 \times \mathbb{P}^1$: as you can see from a direct examination of the equation, each of these closures will be singular at the (unique) point at infinity.

Completion of the proof of Theorem 5.1.1. The proof (in characteristic 0) of uniqueness follows from elementary algebraic topology:

First, a punctured 2-disk has fundamental group $\mathbb{Z}$ and the unique n-sheeted covering is again a punctured disk; regarding these disks as neighborhoods of the origin in $\mathbb{C}^2$, the covering map can be taken to be $x \mapsto x^n$. This map can of course be extended (by the same formula) to a map analytic also at the origin, with ramification index (by definition) $n-1$.

Now suppose that $\Gamma = \{p_1, \ldots, p_d\}$ is the desired branch divisor. Globally, if γ_i is a small loop around p_i then the abelianization of the fundamental group π of the d-times punctured sphere

$$S' := \mathbb{P}^1 \setminus \Gamma$$

is its first homology group,

$$H := H_1(S', \mathbb{Z}) = \frac{\oplus \mathbb{Z} \cdot \gamma_i}{\mathbb{Z} \cdot \sum_i [\gamma_i]}$$

((Insert "lollipop picture".))
Since $\mathbb{Z}/2$ is abelian, a degree 2 unramified covering of S' corresponds to a map

$H \to \mathbb{Z}/2$, and this map must send $2\gamma_i$ to 0 for $i = 1 \ldots d$. There is such a map if and only if d is even, and in this case the map is unique.

Summarizing: there is, a unique degree 2 topological covering $C' \to \mathbb{P}^1 \setminus \Gamma$ by a surface C' that extends to a ramified covering of $\rho : C \to \mathbb{P}^1$, simply ramified over the points of Γ, as long as the number of ramification points is even.

A triangulation of $\mathbb{P}^1$ with V vertices including the points of Γ, E edges, and F triangles must have

$$V - E + F = \chi_{\text{top}}(S^2) = 2.$$

It lifts to a triangulation of C with $2V - d$ vertices, $2E$ edges, and $2F$ faces, so

$$\chi_{\text{top}}(C) = 2V - d - 2E + 2F = 4 - d,$$

so if $d = 2g + 2$ then $\chi_{\text{top}}(C) = 2 - 2g$, so C is a surface of genus g.

Though given as a topological surface, the map ρ is a local homeomorphism at every point not in the preimage of Γ, so C inherits a unique complex structure from the requirement that ρ be holomorphic; thus C is actually a smooth algebraic curve of genus g.
((we had better say topological and algebraic genus are the same in the intro.))

□

Exercise 5.1.2. In the case $g = 1$, show that the closure $\overline{C^\circ}$ of $C^\circ \subset \mathbb{A}^2$ in either $\mathbb{P}^2$ or $\mathbb{P}^1 \times \mathbb{P}^1$ consists of the union of C° with one additional point, with that point a tacnode of $\overline{C^\circ}$ in either case.

It is also possible to give a projective model of the hyperelliptic curve C with given branch divisor: if we divide the points $q_1, \ldots, q_{2g+2} \in \mathbb{P}^1$ into two sets of $g+1$—say, for example, $q_1, \ldots, q_{g+1}$ and $q_{g+2}, \ldots, q_{2g+2}$—then C is the closure in $\mathbb{P}^1 \times \mathbb{P}^1$ of the locus

$$\Big\{(x, y) \in \mathbb{A}^2 \;\mid\; y^2 \prod_{i=1}^{g+1}(x - \lambda_i) = \prod_{i=g+2}^{2g+2}(x - \lambda_i)\Big\};$$

in projective coordinates, this is

$$C \;=\; \Big\{(X, Y) \in \mathbb{P}^1 \times \mathbb{P}^1 \;\mid\; Y_1^2 \prod_{i=1}^{g+1}(X_1 - \lambda_i X_0) = Y_0^2 \prod_{i=g+2}^{2g+2}(X_1 - \lambda_i X_0)\Big\}.$$

(No local analysis is needed to see that $C \subset \mathbb{P}^1 \times \mathbb{P}^1$ is smooth: it is a curve of bidegree $(2, g+1)$ in $\mathbb{P}^1 \times \mathbb{P}^1$, and the genus formula tells us that such a curve has arithmetic genus g.)

5.1.2 Differentials on a hyperelliptic curve

We can give a very concrete description of the differentials, and thus the canonical linear series, on a hyperelliptic curve C by working with the affine model $C^\circ = V(f) \subset \mathbb{A}^2$, where

$$f(x,y) = y^2 - \prod_{i=1}^{2g-2}(x-\lambda_i).$$

We will again denote the two points at infinity—that is, the two points of $C \setminus C^\circ$ by r and s; for convenience, we'll denote the divisor $r+s$ by D.

To start, consider the simple differential dx on C. (Technically, we should write this as $\pi^* dx$, since we mean the pullback to C of the differential dx on $\mathbb{P}^1$, but for simplicity of notation we'll suppress the π^*.) The function x is regular on C°, and is a local parameter over points other than the λ_i; but since $2dy = \sum_i$ **((add formula))**
with zeros at the ramification points $q_i = (\lambda_i, 0)$. But it does not extend to a regular differential on all of C: it will have double poles at r and s. This can be seen directly: the differential dx extends to a rational differential on $\mathbb{P}^1$, and in terms of the local coordinate $w = 1/x$ around the point $x = \infty$ on $\mathbb{P}^1$, we have

$$dx = d\left(\frac{1}{w}\right) = \frac{-dw}{w^2}$$

so dx has a double pole at the point at ∞; since the map π is a local isomorphism near r and s the pullback of dx to C likewise has double poles at the points r and s.

We could also see that dx must have poles by degree considerations: as we said, dx has $2g+2$ zeros and no poles in C°, while the degree of K_C is $2g-2$, meaning that there must be a total of four poles at the points r and s. In any event, we have an expression for the canonical divisor class on C: denoting by $R = q_1 + \cdots + q_{2g+2}$ the sum of the ramifications points of π, we have

$$K_C \sim (dx) \sim R - 2D;$$

this is a case of the Riemann-Hurwitz formula above.

So, given that dx has poles at r and s, how do we find regular differentials on C? One thing to do would be simply to divide by x^2 (or any quadratic polynomial in x) to kill the poles. But that just introduces new poles in the finite part C° of C. Instead, we want to multiply dx by a rational function with zeros at p and q, but *whose poles occur only at the points where dx has zeroes*—that is, the points q_i. A natural choice is simply the reciprocal of the partial derivative $f_y = \partial f/\partial y = 2y$, which vanishes exactly at the points r_i, and has correspondingly a pole of order $g+1$ at each of the points r and s (reason: the involution $y \to -y$ fixes C° and x, and exchanges the points p, q). In other words, the differential

$$\omega = \frac{dx}{f_y}$$

is regular, with divisor

$$(\omega) = (g-1)r + (g-1)s = (g-1)D.$$

The remaining regular differentials on C are now easy to find: Since x has only a simple pole at the two points at infinity we can multiply ω by any x^k with $k = 0, 1, \ldots, g-1$. Since this gives us g independent differentials, these form a basis for $H^0(K_C)$.

5.1.3 The canonical map of a hyperelliptic curve

Given that a basis for $H^0(K_C)$ is given by

$$H^0(K_C) = \langle \omega, x\omega, \ldots, x^{g-1}\omega \rangle,$$

we see that the canonical map $\phi : C \to \mathbb{P}^{g-1}$ is given by $[1, x, \ldots, x^{g-1}]$. In other words, the canonical map ϕ is simply the composition of the map $\pi : C \to \mathbb{P}^1$ with the Veronese embedding $\mathbb{P}^1 \hookrightarrow \mathbb{P}^{g-1}$ of $\mathbb{P}^1$ into $\mathbb{P}^{g-1}$ as a rational normal curve of degree $g-1$.

Note that as a consequence of this fact, we see that *a hyperelliptic curve C has a* unique *linear series g^1_2 of degree 2 and dimension 1*, that is, a unique map of degree 2 to $\mathbb{P}^1$. Finally, we can give an explicit description of special linear series on a hyperelliptic curve: if $D = \sum p_i$ is any effective divisor on C, we can pair up points p_i that are conjugate under the involution ι exchanging sheets of the degree 2 map $C \to \mathbb{P}^1$; each conjugate pair is a divisor of the unique g^1_2 on C, and so we can write

$$D \sim r \cdot g^1_2 + q_1 + \cdots + q_{d-2r},$$

where no two of the points q_i are conjugate under ι. Now the geometric form of the Riemann-Roch formula tells us that the dimension $r(D)$ of the complete linear series $|D|$ is exactly r, so that in fact

$$|D| = |r \cdot g^1_2| + +q_1 + \cdots + q_{d-2r};$$

that is, the points q_i are base points of the linear series D.

One key observation is that, according to this analysis, *no special linear series on a hyperelliptic curve can be very ample*; the map associated to any special series factors through the degree 2 map $C \to \mathbb{P}^1$. This is in marked contrast to the case of non-hyperelliptic curves, for which the embeddings of minimal degree in projective space are given by special linear series.

5.2 Curves of genus 2

Canonical map to $\mathbb{P}^1$. Embedding in $\mathbb{P}^3$ as $(2,3)$ on a quadric, via any degree 5 line bundle. Ideal is 1 quadric, 2 cubics. Plane model of degree 4 with node or cusp.

Representations as double covers of $\mathbb{P}^1$

As with curves of genus 1, there are no nontrivial linear series of degree 0 or 1 on a curve of genus 2; the first positive-dimensional linear series occurs in degree 2. Unlike the case of genus 1, however, this series is unique: by Riemann-Roch, if D is any divisor of degree 2 on a curve C of genus 2, we have

$$h^0(D) = 1 + h^1(D) = 1 + h^0(K - D);$$

since $K - D$ has degree 0, this says that $h^0(D) > 1$ if and only if $D = K$, in which case $|D| = |K|$ is the canonical g^1_2 on C.

The canonical series gives a map $\phi_K : C \to \mathbb{P}^1$ expressing C as a double cover of $\mathbb{P}^1$; as in the case of genus 1, this means we can realize C as the smooth projective compactification of the affine curve given by

$$y^2 = x(x-1)(x-\alpha)(x-\beta)(x-\gamma)$$

for some triple $\alpha, \beta, \gamma \in \mathbb{C}$ distinct from each other and from 0 and 1. This representation shows us that the moduli space M_2 is the space of 6-tuples of distinct points in $\mathbb{P}^1$ modulo the action of PGL_2. This tells us immediately that M_2 is irreducible of dimension 3; with a fair amount of additional work, we can also use this to describe the coordinate ring of M_2 (**??**).

Embeddings in $\mathbb{P}^3$

For line bundles L of degree $d \geq 3$ on C, Riemann-Roch tells us simply that $h^0(D) = d - 1$; if we want to embed our curve C in projective space, accordingly, we had better take $d \geq 5$. Conversely, Corollary (2.3.4) tells us that any line bundle of degree 5 on C is very ample, so we'll consider first the embeddings of C given by those.

So: for the following, let L be any line bundle of degree 5 on our curve C, and $\phi_L : C \to \mathbb{P}^3$ the embedding given by the complete linear system $|L|$. By a mild abuse of language, we'll also denote the image $\phi_L(C) \subset \mathbb{P}^3$ by C.

The first question to ask is once more, what degree surfaces in $\mathbb{P}^3$ contain the curve C? We start with degree 2, where we consider the restriction map

$$H^0(\mathcal{O}_{\mathbb{P}^3}(2)) \to H^0(\mathcal{O}_C(2)) = H^0(L^2).$$

The space on the left has dimension 10 as always; on the right, Riemann-Roch tells us that $h^0(L^2) = 2 \cdot 5 - 2 + 1 = 9$. It follows that C must lie on a quadric surface Q; and by Bezout that Q is unique (since C can't lie on a union of planes, any quadric containing C must be irreducible; if there were more than one such, Bezout would imply that $\deg(C) \leq 4$).

We might ask at this point: is Q smooth or a quadric cone? The answer depends on the choice of line bundle L:

Proposition 5.2.1. *Let $C \subset \mathbb{P}^3$ be a smooth curve of degree 5 and genus 2 and $Q \subset \mathbb{P}^3$ the unique quadric containing C. If $L = \mathcal{O}_C(1) \in \operatorname{Pic}^5(C)$, then Q is singular if and only if we have*

$$L \cong K^2(p)$$

for some point $p \in C$; in this case, the point p is the vertex of Q.

Note that there is a 2-parameter family of line bundles of degree 5 on C, of which a one-dimensional subfamily are of the form $K^2(p)$, conforming to our naive expectation that "in general" Q should be smooth, and that it should become singular in codimension 1.

Proof. First, suppose that the line bundle $L \cong K^2(p)$ for some $p \in C$. Then $L(-p) \cong K^2$, meaning that the map $\pi : C \to \mathbb{P}^2$ given by projection from p is the map $\phi_{K^2} : C \to \mathbb{P}^2$ given by the square of the canonical bundle.

What does this map look like? □

Whether the quadric Q is smooth or not, we can describe a minimal set of generators of the homogeneous ideal $I(C) \subset \mathbb{C}[x_0, x_1, x_2, x_3]$ similarly. First, we look at the restriction map

$$H^0(\mathcal{O}_{\mathbb{P}^3}(3)) \to H^0(\mathcal{O}_C(3));$$

since the dimensions of these spaces are 20 and $15 - 2 + 1 = 14$ respectively, we see that vector space of cubics vanishing on C has dimension at least 6. Four of these are already accounted for: we can take the defining equation of Q and multiply it by any of the linear forms on $\mathbb{P}^3$; we conclude, accordingly, that *there are at least two cubics vanishing on C linearly independent modulo those vanishing on Q.*

In fact, we can prove the existence of these cubics geometrically, and show that there are no more than 2 linearly independent modulo the ideal of Q. Suppose first that Q is smooth, so that C is a curve of type $(2, 3)$ on Q. In that case, if $L \subset Q$ is any line of the first ruling, the sum $C + L$ is the complete intersection of Q with a cubic S_L, unique modulo the ideal of Q; conversely, if S is any cubic containing C but not containing S, the intersection $S \cap Q$ will be the union of C and a line L of the first ruling; thus, mod $I(Q)$, $S = S_L$. A similar argument applies in case Q is a cone, and L is any line of the (unique) ruling of Q.

Exercise 5.2.2. Show that for any pair of lines L, L' of the appropriate ruling of Q, the three polynomials Q, S_L and $S_{L'}$ generate the homogeneous ideal $I(C)$. Find relations among them. Write out the minimal resolution of $I(C)$.

Projective normality III

Theorem 5.2.3. *Let C be a smooth (is reduced, irreducible enough?) curve of arithmetic genus g, and let $\mathcal{L}$ be a line bundle on C of degree $\geq 2g+1$. The image of C under the complete linear series $|\mathcal{L}|$ is projectively normal (when C is singular, aritmetically Cohen-Macaulay).*

Proof. The line bundle $\mathcal{L}$ is very ample by **??**. Thus it suffices We must show that the multiplication map $H^0(\mathcal{L}) \otimes H^0(\mathcal{L}^m) \to H^0(\mathcal{L}^{m+1})$ is surjective for all $m \geq 1$. For $m = 1$ do it by number of quadrics, uniform position. For m¿1 the bpf pencil trick. □

5.3 Curves of genus 3

If C be a smooth projective curve of genus 3. The is an immediate bifurcation into two cases, hyperelliptic and non-hyperelliptic curves; we will discuss hyperelliptic curves of any genus in Section **??**, and so for the following we'll assume C is nonhyperellitic. By our general theorem **??**, this means that the canonical map $\phi_K : C \to \mathbb{P}^2$ embeds C as a smooth plane quartic curve; and conversely, by adjunction any smooth plane of degree 4 has genus 3 and is canonical (that is, $\mathcal{O}_C(1) \cong K_C$).
((maybe a reference to the plane curve chapter for differentials etc?))

Note that this gives us a way to determine the dimension of the moduli space M_3 of smooth curves of genus 3: if $\mathbb{P}^{14}$ is the space of all plane quartic curves, and $U \subset \mathbb{P}^{14}$ the open subset corresponding to smooth curves, we have a dominant map $U \to M_3$ whose fibers are isomorphic to the 8-dimensional affine group PGL_3. (Actually, the fiber over a point $[C] \in M_3$ is isomorphic to the quotient of PGL_3 by the automorphism group of C; but since $Aut(C)$ is finite this is still 8-dimensional.) We conclude, therefore, that

$$\dim M_3 = 14 - 8 = 6.$$

What about other linear series on C, and the corresponding models of C? To start with, by hypothesis C has no g^1_2s; that is, it is not expressible as a 2-sheeted cover of $\mathbb{P}^1$. On the other hand, it is expressible as a 3-sheeted cover:

if $L \in \mathrm{Pic}^3(C)$ is a line bundle of degree 3, by Riemann-Roch we have

$$h^0(L) = \begin{cases} 2, & \text{if } L \cong K - p \text{ for some point } p \in C\text{; and} \\ 1 & \text{otherwise.} \end{cases}$$

There is thus a 1-dimensional family of representations of C as a 3-sheeted cover of $\mathbb{P}^1$. In fact, these are plainly visible from the canonical model: the degree 3 map $\phi_{K-p} : C \to \mathbb{P}^1$ is just the composition of the canonical embedding $\phi_K : C \to \mathbb{P}^2$ with the projection from the point p.

There are of course other representations of C as the normalization of a plane curve. By Riemann-Roch, C will have no g^2_3s and the canonical series is the only g^2_4, but there are plenty of models as plane quintic curves: by Proposition **??**, if L is any line bundle of degree 5, the linear series $|L|$ will be a base-point-free g^2_5 as long as L is not of the form $K + p$, so that ϕ_L maps C birationally onto a plane quintic curve $C_0 \subset \mathbb{P}^2$. But these can also be described geometrically in terms of the canonical model: any such line bundle L is of the form $2K - p - q - r$ for some trio of points $p, q, r \in C$ that are not colinear in the canonical model, and we see correspondingly that C_0 is obtained from the canonical model of C by applying a Cremona transform with respect to the points p, q and r.

We can also embed C in $\mathbb{P}^3$ as a smooth sextic curve by Proposition **??**; in fact, a line bundle $L \in \mathrm{Pic}^6(C)$ of degree 6 will be very ample if and only if it is not of the form $K + p + q$ for any $p, q \in C$. One cheerful fact in this connection is that these curves are determinantal:

Exercise 5.3.1. Let $C \subset \mathbb{P}^3$ be a smooth non-hyperelliptic curve of degree 3 and genus 6. Show that there exists a 3×4 matrix M of linear forms on $\mathbb{P}^3$ such that

$$C = \{p \in \mathbb{P}^3 \mid \mathrm{rank}(M(p)) \leq 2\}.$$

Chapter 6

Moduli of Curves

Philosophy of this chapter: the book *Moduli of Curves* has already been written, and we don't want to write it again. But a summary of the basic information would be useful

6.1 What is a moduli problem?

We've already seen, in the preceding chapters, examples of *moduli*, or *parameter spaces*—spaces parametrizing algebro-geometric objects of a specified sort. In this chapter, we'll try to systematize this, giving a general framework for the notion of moduli space, introducing the main examples we'll be dealing with here, and mentioning some common variants.

As for the question of the title, in colloquial terms a *moduli problem* consist of two things: a class of objects in algebraic geometry—schemes, subschemes of a given scheme, sheaves on schemes, and the like—and a notion of what it means to have a *family* of these objects parametrized by a scheme B. As usual, some examples will make this somewhat vague notion more real. In each case, we'll discuss the notion as it applies to curves.

6.1.1 Examples

1. *Curves of genus g.* Here, the objects are simply (isomorphism classes of) smooth, projective curves of genus g. By a family of such curves, we'll mean a smooth, projective morphism $f : \mathcal{C} \to B$ whose fibers are curves of genus g.

2. *Curves in projective space* Here, the objects are simply subschemes $C \subset \mathbb{P}^r$ of degree d, isomorphic to smooth, projective curves of genus g. By a

family of such curves, we'll mean a subscheme $\mathcal{C} \subset B \times \mathbb{P}^r$, flat over B, whose fibers are smooth, projective curves of genus g.

3. *Effective divisors on a given curve.* The objects are divisors of degree d on a given smooth, projective curve C. To define a family of such divisors, we use the equivalence between effective divisors and subschemes to define a family to be subscheme $\mathcal{D} \subset B \times C$, flat of relative degree d over B.

4. *Line bundles on a given curve.* Now the objects are line bundles of degree d on a given smooth, projective curve C; by a family of line bundles we'll mean simply a line bundle $\mathcal{L}$ on $B \times C$, whose restriction to each fiber of $B \times C$ over B has degree d.

6.2 What is a solution to a moduli problem?

The basic idea here is straightforward: given a moduli problem, we want to construct a scheme M whose closed points are in natural 1-to-1 correspondence with the objects we're trying to parametrize. The problem, as it so often is, is the word "natural" appearing in that last sentence: given that most of the time, the set of objects has cardinality $\aleph_1$, as do all positive-dimensional varieties M over $\mathbb{C}$, the existence of a bijection between the points of M and the objects to be parametrized is not much of a condition!

In the pre-scheme days, the answer was not entirely satisfactory, but at least easy to understand: for any family of objects parametrized by a variety B, we have an induced map of sets from the closed points of B to the set of closed points in M, and the requirement is simply that this be the underlying set map of a regular morphism $B \to M$.

In the world of schemes, however, a map is not determined by its underlying set map, and this needs to be updated. In the end, rather than construct a morphism $B \to M$ associated to a family of objects over B and then require it to be a regular map, we simply require that there exist a natural bijection, for any scheme B, between the set of families $\mathcal{C}$ over B and the set of morphisms $\phi_{\mathcal{C}} : B \to M$. Moreover, here the word "natural" is easy to make precise: we require that if $\pi : B \to B_0$ is any morphism of schemes, $\mathcal{C}_0 \to B_0$ a family ovr B_0 and $\mathcal{C} := \mathcal{C}_0 \times_{B_0} B \to B$ the induced family over B, then the morphism

$$\phi_B = \phi_{B_0} \circ \pi.$$

In other words, we have two functors from the category of schemes to the category of sets: the functor h that associates to a scheme B the set of families over B; and the functor $\mathrm{Mor}(\bullet, \mathrm{M})$ that associates to a scheme B the set of morphisms $B \to M$. We'll say that M is a *fine moduli space* for the given moduli problem if there exists an isomorphism of functors $h \cong \mathrm{Mor}(\bullet, \mathrm{M})$

6.3 M_g

Assert existence as a coarse, rather than a fine moduli space (say what this means and give ref to Geometry of Schemes?)

6.4 Compactifying M_g

Describe the Deligne-Mumford compactification; mention alternatives?

6.5 Auxilliary constructions

It is hard to specify an abstract curve. It is much easier if the curve C comes to us with some additional structure, such as a map to projective space; if the map is a birational embedding, we can specify the curve just by specifying a set of polynomial equations cutting it out.

There are two special cases of this: we can look at pairs (C, f) where $f : C \to \mathbb{P}^1$ is a branched cover of degree d, which yields various flavors of the *Hurwitz spaces*; and we can look at pairs (C, f) where $f : C \to \mathbb{P}^2$ is a birational embedding of C as a plane curve of degree d, yielding the *Severi varieties*. Both of these spaces are more readily described and better-behaved than the moduli space M_g of abstract curves, or the Hilbert scheme $\mathcal{H}$ of curves in higher-dimensional space; for this reason they are useful in proving theorems about M_g and $\mathcal{H}$.

6.5.1 Hurwitz spaces

Fix integers $d \geq 2$ and $g \geq 0$. By the *small Hurwitz space* $\mathcal{H}^\circ_{d,g}$ we will mean a space parametrizing simply branched covers $f : C \to \mathbb{P}^1$ of degree d, with C a smooth projective curve of genus g. Here "simply branched" means that every fiber either is reduced—that is, consists of d reduced points—or consists of one double point and $d - 2$ reduced points. If a map $f : C \to \mathbb{P}^1$ is simply branched, in particular, the branch divisor $B \subset \mathbb{P}^1$ of the map will consist of $b = 2d + 2g - 2$ distinct points in $\mathbb{P}^1$

What does such a space look like? The answer is easiest to see if we work over $\mathbb{C}$ and use the classical or étale topology. To begin with, we've observed that the effective divisors of degree b on a curve C are parametrized by the bth symmetric product C_b of C; in the case of $C = \mathbb{P}^1$, since an effective divisor on $\mathbb{P}^1$ is given by a homogeneous polynomial $F \in H^0(\mathcal{O}_{\mathbb{P}^1}(b))$, the space of all effective divisors is $(\mathbb{P}^1)_b \cong \mathbb{P}^b$, and the locus of reduced divisors—divisors consisting of b distinct points—is an open subset $U \subset \mathbb{P}^b$.

In this setting, we consider the incidence correspondence

$$\begin{tikzcd} & \mathcal{H}_{d,g} \arrow[dl, "\alpha"'] \arrow[dr, "\beta"] & \\ U \subset \mathbb{P}^b & & M_g \end{tikzcd}$$

Here α is the map associating to a branched cover $f : C \to \mathbb{P}^1$ its branch divisor, and β the map sending $f : C \to \mathbb{P}^1$ to the point $[C] \in M_g$; the open set $U \subset \mathbb{P}^b$ is the open set of b-tuples of distinct points in the space $\mathbb{P}^b$ of all effective divisors of degree b on $\mathbb{P}^1$.

Over $\mathbb{C}$, we can describe a branched cover concretely: if we make a collection of cuts in $\mathbb{P}^1$ joining a base point p to each of the branch points $p_1, p_2, \dots, p_b$ of the map, the preimage in C of the complement of the cuts will consist of d disjoint copies of the complement of the cuts in $\mathbb{P}^1$ (the "sheets" of the cover), which we can label with the integers $1, 2, \dots, d$. In these terms, we can associate to each branch point $p_i \in \mathbb{P}^1$ the transposition $\tau_i \in S_d$ exchanging the two sheets that come together over p_i. We arrive at a sequence of transpositions $\tau_1, \tau_2, \dots, \tau_b \in S_d$, that satisfies two conditions:

1. the product $\tau_1 \cdot \tau_2 \cdots \tau_b$ is the identity; and
2. the τ_i together generate a transitive subgroup of S_d.

Note that the sequence $\tau_1, \tau_2, \dots, \tau_b \in S_d$ is determined by the cover $f : C \to \mathbb{P}^1$ up to simultaneous conjugation in S_d: we can revise our labelling of the sheets, which has the effect of conjugating all the τ_i by the relabelling permutation.

The conclusion is simply that *the map $\alpha : \mathcal{H}_{d,g} \to U$ is a covering space map*, which gives us a picture of the local geometry of $\mathcal{H}_{d,g}$.

We can also ask, and in many cases answer, questions about the global geometry of the map $\alpha : \mathcal{H}_{d,g} \to U$. For example, the degrees of the covering spaces $\alpha : \mathcal{H}_{d,g} \to U$ are what are called *Hurwitz numbers*; they arise in many contexts, and in many cases, they can be calculated **[]**. Another global aspect of the geometry of the maps α that can be described is their *monodromy*. Indeed, this was the basis of the first proof that M_g is irreducible: Clebsch, Hurwitz and others (**??**) analyzed the monodromy of the cover $\alpha : \mathcal{H}_{d,g} \to U$, and showed that it was indeed transitive; they concluded that $\mathcal{H}_{d,g}$ *is irreducible for all d and g* and hence, since $\mathcal{H}_{d,g}$ dominates M_g for $d \gg g$, that M_g is irreducible for all g. (Note that because of the reliance on the classical topology, this argument only works in characteristic 0; a proof of the irreducibility of M_g valid in arbitrary characteristic was not found until much later.
((reference!))

Exercise 6.5.1. Find the degree of the covering space $\alpha : \mathcal{H}_{3,g} \to U \subset \mathbb{P}^{2g+4}$

The Hurwitz spaces also give us a way to estimate the dimension of the moduli space M_g. The point is, while it may not be immediately obvious what the dimension of M_g is, the dimension of $\mathcal{H}_{d,g}$ is clear: it's a finite-sheeted cover of an open subset $U \subset \mathbb{P}^b$, and so has dimension $b = 2d + 2g - 2$. To find the dimension of M_g, accordingly, we simply have to choose $d \gg g$ (so that $\mathcal{H}_{d,g}$ dominates M_g), and estimate the dimension of the fibers of $\mathcal{H}_{d,g}$ over M_g.

This is straightforward, based on our previous constructions. Given a curve C, to specify a map $f : C \to \mathbb{P}^1$ we have to specify first a line bundle L of degree d on C (g parameters, as described in Chapter 4. We then have to specify a pair of sections of L (up to multiplying the pair by a scalar). By Riemann-Roch, we will have $h^0(L) = d - g + 1$, so to specify a pair of sections (mod scalars) is $2(d - g + 1) - 1$ parameters. Altogether, we have

$$2d + 2g - 2 = \dim \mathcal{H}_{d,g} = \dim M_g + g + 2(d - g + 1) - 1;$$

and solving, we arrive at

$$\dim M_g \;=\; 3g - 3.$$

6.5.2 Severi varieties

The Severi varieties behave in many ways like the Hurwitz spaces, even though they differ in virtually all particulars.

Just as the Hurwitz space $\mathcal{H}_{d,g}$ parametrizes pairs (C, f) consisting of a smooth curve C of genus g and a map $f : C \to \mathbb{P}^1$ of degree d, the Severi variety parametrizes pairs (C, f) consisting of a smooth curve C of genus g and a map $f : C \to \mathbb{P}^2$ of degree d. As, in the case of Hurwitz spaces we restrict our attention initially to the locus where the map f is simply branched, we initially make a similar restriction: we consider only those maps $f : C \to \mathbb{P}^2$ that are birational onto a plane curve C_0 having only nodes as singularities. In this case, the curve C is determined simply as the normalization of C_0, so we can define the *small Severi variety* $V_{d,g}$ to be the locally closed subset of the projective space $\mathbb{P}^N$ parametrizing plane curves of degree d corresponding to irreducible nodal curves of degree d and geometric genus g.

Since we have appended the adjective "small" to both the Hurwitz spaces and the Severi varieties, we should explain: in both cases, one goal is to find a good compactification or partial compactification of these spaces. "Good" here means that the larger space is still a moduli space, but for a larger class of pairs (C, f). For example, in the case of the Hurwitz space, we have the *space of admissible covers*, which parametrizes maps of degree d from nodal curves C of arithmetic genus g to nodal curves of arithmetic genus 0 satisfying certain local conditions. In the case of Severi varieties, we could simply take the closure $\overline{V_{d,g}}$ of $V_{d,g}$ in the space $\mathbb{P}^N = \mathbb{P}^{\binom{d+2}{2}-1}$ of all plane curves of degree d, but this is unsatisfactory in a number of ways: the singularities of $\overline{V_{d,g}}$ are arbitrarily awful, as are those of the curves $C_0 \subset \mathbb{P}^2$ corresponding to the added points.

In fact, the Severi varieties share two key attributes with Hurwitz spaces, the first of which is that their dimensions are readily calculable. In brief, the curve $C_0 \subset \mathbb{P}^2$ corresponding to a point on the small Severi variety $V_{d,g}$ will have $\delta = \binom{d-1}{2} - g$ nodes; to describe the locus of such points, we can introduce the incidence correspondence

$$\Sigma = \{(C_0, p_1, \dots, p_\delta) \in V_{d,g} \times (\mathbb{P}^2)^\delta \mid C_0 \text{ has a node at } p_i\}.$$

The fibers of the projection $\Sigma \to (\mathbb{P}^2)^\delta$ are linear subspaces of $\mathbb{P}^N$, and either a calculation in local coordinates or a little deformation theory shows that they have the expected dimension $N - 3\delta$. We see thus that

$$\dim V_{d,g} = \dim \Sigma = 2\delta + N - 3\delta = 3d + g - 1.$$

As in the case of Hurwitz spaces, this knowledge is enough for us to determine the dimension of M_g. Again, for any given g if we choose $d \gg g$ the map $V_{d,g} \to M_g$ sending the pair (C, f) to the point $C \in M_g$ will be dominant. And again, we can readily calculate the dimension of the fibers: if C is a given curve of genus g, to specify a map $f : C \to \mathbb{P}^2$ of degree d we have to choose a line bundle L of degree d on C (g parameters), and then choose 3 global sections of L up to simultaneous multiplication by a scalar. Since by Riemann-Roch we will have $h^0(L) = d - g + 1$, we arrive at

$$3d + g - 1 = \dim V_{d,g} = g + 3(d - g + 1) - 1,$$

and solving we see again that $\dim M_g = 3g - 3$.

The second key attribute that the Severi variety $V_{d,g}$ shares with the Hurwitz space $H_{d,g}$ and is that *it is irreducible for all d and $g \leq \binom{d-1}{2}$*. In particular, we can use the Severi varieties to prove that M_g is irreducible for all g. It should be said, however, that it's not so easy to prove irreducibility of the Severi variety[1]; the fastest way to prove irreducibility of M_g (in characteristic 0, at least) is still via the Hurwitz spaces.

6.6 Hilbert schemes

The Hilbert schemes—schemes $\mathcal{H}_{d,g,r}$ parametrizing curves of given degree and arithmetic genus in $\mathbb{P}^r$ for any r—are the subject of a later chapter in their own right. But it should be said at this point that the sort of regular behavior exhibited by the Hurwitz spaces and the Severi varieties is completely absent from the Hilbert scheme. Even if we restrict out attention to what we might call the "small Hilbert scheme"—the open subset corresponding to smooth,

[1]The second author of the current volume owes his current employment to this fact.

irreducible and nondegenerate curves—Hilbert schemes are truly wild. Locally, they can have arbitrarily bad singularities (ref to Vakil
((I thought he needed surfaces);))
globally, they can have many irreducible components—no one has any idea how many in general—of many different dimensions.

Exercise 6.6.1. 1. If X is a smooth curve, then the Hilbert scheme of finite subschemes of X of degree d is isomorphic to the symmetric product of d copies of X.

2. If X is a singular curve or any variety of dimension $r \geq 2$, the symmetric power $X^{(d)}$ is *not* the Hilbert scheme of subschemes of dimension 0 and degree d on X.
((maybe needs a hint, esp at this early stage in the book.))

6.7 Unirationality

The Brill-Noether theorem, in all its many forms, deals with a simple question: what linear series g^r_d does a curve C of genus g possess? Of course, the answer depends on C, but we know from general principles that it's constant on an open subset of M_g; so it's natural to ask what the answer is on that open set. That is the subject of the Brill-Noether theorem: what linear series exist on a general curve C of genus g? This raises, naturally, a question we'll take up next.

6.7.1 Can we write down a general curve of genus g?

Let's do this one genus at a time. For curves of genus 2, the family of curves given by
$$y^2 = x^6 + a_5x^5 + \cdots + a_1x + a_0$$
includes every curve of genus 2; in other words, the induced (rational) map $\mathbb{A}^6 \to M_2$ is dominant, so that a general choice of the coefficients a_i will yield a general curve.

For genus 3, we can consider the family
$$\sum_{i+j+k=4} a_{i,j,k}x^iy^jz^k = 0$$
of all plane quartic curves; again, the rational map $\mathbb{A}^{15} \to M_3$ is dominant, so a general choice of the $a_{i,j,k}$ yields a general curve. The same approach works in genera 4 and 5; in each case a general canonical curve is a complete intersection, so that if we take the coefficients of its defining polynomials to be general scalars we have a general curve.

This breaks down when we get to genus 6, where a canonical curve is not a complete intersection. But it's close enough: a general canonical curve of genus 6 is the intersection of a smooth del Pezzo surface $S \subset \mathbb{P}^5$ with a quadric hypersurface Q; since all smooth del Pezzo surfaces in $\mathbb{P}^5$ are isomorphic, we can just fix one such surface S and let Q be a general quadric.

What we are doing in each of these cases is exhibiting a family of curves of the relevant genus over a rational base that dominates M_g. Thus, to say that we can write down a general curve of genus g is tantamount to saying that *the moduli space M_g is unirational*

It gets harder as the genus increases. Let's do one more case, genus 7, which already calls for a different approach. Here we want to argue that, by Brill-Noether theory, a general curve of genus 7 can be realized as (the normalization of) a plane septic curve with 8 nodes $p_1, \dots, p_8 \in \mathbb{P}^2$. Equivalently, if we let $S = Bl_{p_1,\dots,p_8}(\mathbb{P}^2)$ be the blow-up, and let l and $e_1, \dots, e_8$ be the classes of the pullback of a line and of the eight exceptional divisors respectively, a divisor of class $7l - 2\sum e_i$ on S. Thus the curves on S form a linear series, parametrized by a projective space $\mathbb{P}^{11}$.

The problem is, there are many such surfaces S; we don't have a single linear system that includes the general curve of genus 7. The good news is, that's OK because the surfaces S themselves form a rationally parametrized family. Explicitly, if we look at the set Φ of pairs (S, C) with $S = Bl_{p_1,\dots,p_8}(\mathbb{P}^2)$ the blow-up of $\mathbb{P}^2$ at eight points and $C \subset S$ a curve of class $7l - 2\sum e_i$ on S, then Φ is a $\mathbb{P}^{11}$-bundle over $(\mathbb{P}^2)^8$, and so is again a rational variety; choosing a rational parametrization of Φ we get a family of curves of genus 7 parametrized by $\mathbb{P}^{27}$ and dominating M_7. As before, then, a general point in $\mathbb{P}^{27}$ yields a general curve of genus 7.

Things continued in this vein up through genus 10, but then this approach fails as well: if we represent a general curve of genus 11 as a plane curve with nodes, the nodes are no longer general points of $\mathbb{P}^2$, and the same argument doesn't work. Ad hoc (and much more difficult) arguments were given in general 11, 12 13 and 14, but that's where progress apparently stalled.

The reason why became apparent in
((ref1982,))
when it was shown that for g odd and > 23, *the moduli space M_g is of general type* (the restriction to odd g is unneccessary, as was shown in
((ref);))
in particular, it can't be unirational. Thus the answer to our naive question—can we write down a general curve of genus g—is "no" for large g!

Chapter 7

Curves of genus 4, 5 and 6

7.1 Curves of genus 4

As in the case of curves of genus 3, the study of curves of genus 4 bifurcates immediately into two cases: hyperelliptic and non-hyperelliptic; again, we will study the geometry of hyperelliptic curves in Chapter **??** and focus here on the nonhyperelliptic case.

In genus 4 we have a question that the elementary theory based on the Riemann-Roch formula cannot answer: are nonhyperelliptic curves of genus 4 expressible as three-sheeted covers of $\mathbb{P}^1$? The answer will emerge from our analysis in Proposition 7.1.2 below.

Let C be a non-hyperelliptic curve of genus 4. We start by considering the canonical map $\phi_K : C \hookrightarrow \mathbb{P}^3$, which embeds C as a curve of degree 6 in $\mathbb{P}^3$. We identify C with its image, and investigate the homogeneous ideal $I = I_C$ of equations it satisfies. As in previous cases we may try to answer this by considering the restriction maps
((replaced K_C^m with mK_C.))

$$r_m : \mathrm{H}^0(\mathcal{O}_{\mathbb{P}^3}(m)) \ \to \ \mathrm{H}^0(\mathcal{O}_C(m)) = \mathrm{H}^0(mK_C).$$

For $m = 1$, this is by construction an isomorphism; that is, the image of C is non-degenerate (not contained in any plane).

For $m = 2$ we know that $\mathrm{h}^0(\mathcal{O}_{\mathbb{P}^3}(2)) = \binom{5}{3} = 10$, while by the Riemann-Roch Theorem we have

$$\mathrm{h}^0(\mathcal{O}_C(2)) = 12 - 4 + 1 = 9.$$

This shows that the curve $C \subset \mathbb{P}^3$ must lie on at least one quadric surface Q. The quadric Q must be irreducible, since any any reducible and/or non-reduced quadric must be a union of planes, and thus cannot contain an irreducible non-degenerate curve. If $Q' \neq Q$ is any other quadric then, by Bézout's Theorem,

$Q \cap Q'$ is a curve of degree 4 and thus could not contain C. From this we see that Q is unique, and it follows that r_2 is surjective.

What about cubics? Again we consider the restriction map

$$r_3 : \mathrm{H}^0(\mathcal{O}_{\mathbb{P}^3}(3)) \ \to \ \mathrm{H}^0(\mathcal{O}_C(3)) = \mathrm{H}^0(3K_C).$$

The space $\mathrm{H}^0(\mathcal{O}_{\mathbb{P}^3}(3))$ has dimension $\binom{6}{3} = 20$, while the Riemann-Roch Theorem shows that

$$\mathrm{h}^0(\mathcal{O}_C(3)) = 18 - 4 + 1 = 15.$$

It follows that the ideal of C contains at least a 5-dimensional vector space of cubic polynomials. We can get a 4-dimensional subspace as products of the unique quadratic polynomial F vanishing on C with linear forms—these define the cubic surfaces containing Q. Since $5 > 4$ we conclude that the curve C lies on at least one cubic surface S not containing Q. Bézout's Theorem shows that the curve $Q \cap S$ has degree 6; thus it must be equal to C.

Let $G = 0$ be the cubic form defining the surface S. By Lasker's Theorem the ideal (F, G) is unmixed, and thus is equal to the homogeneous ideal of C. Putting this together, we have proven the first statement of the following result:

Theorem 7.1.1. *The canonical model of any nonhyperelliptic curve of genus 4 is a complete intersection of a quadric $Q = V(F)$ and a cubic surface $S = V(G)$ meeting along nonsingular points of each. Conversely, any smooth curve that is the intersection of a quadric and a cubic surface in $\mathbb{P}^3$ is the canonical model of a nonhyperelliptic curve of genus 4.*

Proof. Let $C = Q \cap S$ with Q a quadric and S a cubic. Because C is nonsingular and a complete intersection, both S and Q must be nonsingular at every point of their intersection Applying the Adjunction Formula to $Q \subset \mathbb{P}^3$ we get

$$\omega_Q = (\omega_{\mathbb{P}^3} \otimes \mathcal{O}_{\mathbb{P}^3}(2))|_Q = \mathcal{O}_Q(-4+2) = \mathcal{O}_Q(-2).$$

Applying it again to C on Q, and noting that $\emptyset_Q(C) = \emptyset_Q(3)$, we get

$$\omega_C = ((\omega_Q \otimes \mathcal{O}_3(3))|_C = \mathcal{O}_C(-2+3) = \mathcal{O}_C(1)$$

as required. □

We can now answer the question we asked at the outset, whether a nonhyperelliptic curve of genus 4 can be expressed as a three-sheeted cover of $\mathbb{P}^1$. This amounts to asking if there are any divisors D on C of degree 3 with $r(D) \geq 1$; since we can take D to be a general fiber of a map $\pi : C \to \mathbb{P}^1$, we can for simplicity assume $D = p + q + r$ is the sum of three distinct points.

By the geometric Riemann-Roch theorem, a divisor $D = p + q + r$ on a canonical curve $C \subset \mathbb{P}^{g-1}$ has $r(D) \geq 1$ if and only if the three points $p, q, r \in C$ are colinear. If three points $p, q, r \in C$ lie on a line $L \subset \mathbb{P}^3$ then the quadric Q

would meet L in at least three points, and hence would contain L. Conversely, if L is a line contained in Q, then the divisor $D = C \cap L = S \cap L$ on C has degree 3. Thus we can answer our question in terms of the family of lines contained in Q.

Any smooth quadric is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$, and contains two families of lines, or *rulings*. On the other hand, any singular quadric is a cone over a plane conic, and thus has just one ruling. By the argument above, the pencils of divisors on C cut out by the lines of these rulings are the g^1_3s on C. This proves:

Proposition 7.1.2. *A nonhyperelliptic curve of genus 4 may be expressed as a 3-sheeted cover of* $\mathbb{P}^1$ *in either one or two ways, depending on whether the unique quadric containing the canonical model of the curve is singular or smooth.*

((include this?))
(One might ask why the non-singularity of the cubic surface S plays no role. However, G is determined only up to a multiple of F, and it follows that the linear series of cubics in the ideal I_C has only base points along C. Bertini's Theorem says that a general element of this series will be nonsingular away from C; and since any every irreducible cubic in the family must be nonsingular along C, it follows that the general such cubic is nonsingular.)

A curve expressible as a 3-sheeted cover of $\mathbb{P}^1$ is called *trigonal*; by the analyses of the preceding sections, we have shown that *every curve of genus* $g \leq 4$ *is either hyperelliptic or trigonal.*

We can also describe the lowest degree plane models of nonhyperelliptic curves C of genus 4. We can always get a plane model of degree 5 by projecting C from a point p of the canonical model of C. Moreover, the Riemann-Roch Theorem shows that if D is a divisor of degree 5 with $r(D) = 2$ then, $\mathrm{h}^0(K - D) = 1$. Thus D is of the form $K - p$ for some point $p \in C$, and the map to $\mathbb{P}^2$ corresponding to D is π_p. These maps $\pi_p : C \to \mathbb{P}^2$ have the lowest possible degree (except for those whose image is contained in a line) because, by Clifford's Theorem a nonhyperelliptic curve of genus 4 cannot have a g^2_4.

We now consider the singularities of the plane quintic $\pi_p(C)$. Suppose as above that $C = Q \cap S$, with Q a quadric. If a line L through p meets C in p plus a divisor of degree ≥ 2 then, as we have seen, L must lie in Q. All other lines through p meet C in at most a single points, so π_p whose images are thus nonsingular points of $\pi(C)$, and π_C is one-to-one there. Moreover, a line that met C in > 3 points would have to lie in both the quadric and the cubic containing C, and therefore would be contained in C. Since C is irreducible there can be no such line.

We distinguish two cases:

1. Q is nonsingular: In this case there are two lines L_1, L_2 on Q that pass through p; they meet C in p plus divisors E_1 and E_2 of degree 2. If E_i

consists of distinct points, then, since the tangent planes to the quadric along L_i are all distinct $\pi(C)$ will have a node at their common image.

((do we expect the reader to know this about quadrics, or should we prove it? or should we argue that since there are two distinct branches and a plane quintic of genus 2 can have only the equivalent of two double points, these must be simple?? The first option is probably better.))

On the other hand, if E_i consists of a double point $2q$ (that is, L_i is tangent to C at $q \neq p$, or meets C 3 times at $q = p$), then $\pi(C)$ will have a cusp at the corresponding image point. In either case, $\pi(C)$ has two distinct singular points, each either a node or a cusp. The two g^1_3s on C correspond to the projections from these singular points.

2. Q is a cone: In this case, since the curve cannot pass through the singular point of Q there is a unique line $L \subset Q$ that passes through p. Let $p + E$ be the divisor on C in which this line meets C. The tangent planes to Q along L are all the same. Thus if $E = q_1 + q_2$ consists of two distinct points, the image $\pi_p(C)$ will have two smooth branches sharing a common tangent line at $\pi_p(q_1) = \pi_p(q_2)$. Such a point is called a *tacnode* of $\pi_p(C)$. On the other hand, if $E = 2q$, that is, if L meets C tangentially at one point $q \neq p$ (or meets C 3 times at p) then the image curve will have a higher order cusp, called a *ramphoid cusp*. In either case, the one g^1_3 on C is the projection from the unique singular point of $\pi(C)$.

((add pictures illustrating some of the possibilities above.))

7.2 Curves of genus 5

We consider now nonhyperelliptic curves of genus 5. There are now two questions that cannot be answered by simple application of the Riemann-Roch Theorem:

1. Is C expressible as a 3-sheeted cover of $\mathbb{P}^1$? In other words, does C have a g^1_3?
2. Is C expressible as a 4-sheeted cover of $\mathbb{P}^1$? In other words, does C have a g^1_4?

As we'll see, all other questions about the existence or nonexistence of linear series on C can be answered by the Riemann-Roch Theorem.

As in the preceding case, the answers can be found through an investigation of the geometry of the canonical model $C \subset \mathbb{P}^4$ of C. This is an octic curve in $\mathbb{P}^4$, and as before the first question to ask is what sort of polynomial equations define C. We start with quadrics, by considering the restriction map

$$r_2 : \mathrm{H}^0(\mathcal{O}_{\mathbb{P}^4}(2)) \ \to\ \mathrm{H}^0(\mathcal{O}_C(2)).$$

On the left, we have the space of homogeneous quadratic polynomials on $\mathbb{P}^4$, which has dimension $\binom{6}{4} = 15$, while by the Riemann-Roch Theorem the target is a vector space of dimension

$$2 \cdot 8 - 5 + 1 = 12.$$

We deduce that C lies on at least 3 independent quadrics. We will see in the course of the following analysis that it is exactly 3; that is, r_2 is surjective.) Since C is irreducible and, by construction, does not lie on a hyperplane, each of the quadrics containing C is irreducible, and thus the intersection of any two is a surface of degree 4. There are now two possibilities: The intersection of (some) three quadrics $Q_1 \cap Q_2 \cap Q_3$ containing the curve is 1-dimensional; or every such intersection is two dimensional.

We first consider the case where $Q_1 \cap Q_2 \cap Q_3$ is 1-dimensional. By the principal ideal theorem the intersection has no 0-dimensional components. By Bézout's Theorem the intersection is a curve of degree 8, and since C also has degree 8 we must have $C = Q_1 \cap Q_2 \cap Q_3$. Lasker's Theorem then shows that the three quadrics Q_i generate the whole homogeneous ideal of C.

We can now answer the first of our two questions for curves of this type. As in the genus 4 case the geometric Riemann-Roch Theorem implies that C has a g^1_3 if and only if the canonical model of C contains 3 colinear points or, more generally, meets a line L in a divisor of 3 points. When C is the intersection of quadrics, this cannot happen, since the line L would have to be contained in all the quadrics that contain C and $L \subset C$, which is absurd. Thus, in this case, C has no g^1_3.

What about g^1_4s? Again invoking the geometric Riemann-Roch Theorem, a divisor of degree 4 moving in a pencil lies in a 2-plane; so the question is, does $C \subset \mathbb{P}^4$ contain a divisor of degree 4, say $D = p_1 + \cdots + p_4 \subset C$, that lies in a plane Λ? Supposing this is so, we consider the restriction map

$$\mathrm{H}^0(\mathcal{I}_{C/\mathbb{P}^4}(2)) \ \to\ \mathrm{H}^0(\mathcal{I}_{D/\Lambda}(2)).$$

By hypothesis, the left hand space is 3-dimensional; but any four noncolinear points in the plane impose independent conditions on quadrics,
((this is a scheme of length 4; how is the reader supposed to cope with this if we don't assume the notion of a scheme, at least a finite one? And does the reader really know this fact about schemes of length 4 in the plane?))
so that the right hand space is 2-dimensional. It follows that Λ *must be contained in one of the quadrics Q containing C.*

The quadrics in $\P^4$ that contain 2-planes are exactly the singular quadrics: such a quadric is a cone over a quadric in $\P^3$, and it is ruled by the (one or two) families of 2-planes it contains, which are the cones over the (one or two) rulings of the quadric in $\P^3$. The argument above shows that the existence of a g^1_4s on C in this case implies the existence of a singular quadric containing C.

Conversely, suppose that $Q \subset \mathbb{P}^4$ is a singular quadric containing $C = Q_1 \cap Q_2 \cap Q_3$. Now say $\Lambda \subset Q$ is a 2-plane. If Q' and Q'' are "the other two quadrics" containing C, we can write

$$\Lambda \cap C = \Lambda \cap Q' \cap Q'',$$

from which we see that $D = \Lambda \cap C$ is a divisor of degree 4 on C, and so has $r(D) = 1$ by the geometric Riemann-Roch Theorem. Thus, the rulings of singular quadrics containing C cut out on C pencils of degree 4; and every pencil of degree 4 on C arises in this way.

Does C lie on singular quadrics? There is a $\mathbb{P}^2$ of quadrics containing C—a 2-plane in the space $\mathbb{P}^{14}$ of quadrics in $\mathbb{P}^4$—and the family of singular quadrics consists of a hypersurface of degree 5 in $\mathbb{P}^{14}$–called the *discriminant* hypersurface. By Bertini's Theorem, not every quadric containing C is singular. Thus the set of singular quadrics containing C is a plane curve B cut out by a quintic equation. So C does indeed have a g^1_4, and is expressible as a 4-sheeted cover of $\mathbb{P}^1$. In sum, we have proven:

Proposition 7.2.1. *Let $C \subset \mathbb{P}^4$ be a canonical curve, and assume C is the complete intersection of three quadrics in $\mathbb{P}^4$. Then C may be expressed as a 4-sheeted cover of $\mathbb{P}^1$ in a one-dimensional family of ways, and there is a map from the set of g^1_4s on C to a plane quintic curve B, whose fibers have cardinality 1 or 2.*

((could the "quintic curve" be reducible/multiple? Just a line?))
Of course, we can go further and ask about the geometry of the plane curve B and how it relates to the geometry of C; a fairly exhaustive list of possibilities is given in [**?**] [ACGH]. But that's enough for now.

In the second possibility above, that the canonical curve $C \subset \mathbb{P}^4$ is not a complete intersection; we will see in *** that the the intersection of the quadrics containing C is two-dimensional: a rational normalscroll; and C is trigonal, that is, a 3-sheeted cover of $\mathbb{P}^1$.

7.3 Curves of genus 6

Canonical model lies on at least 6 quadrics.

To prove projective quadratic normality, use general position: the general hyperplane section is 10 points in $\mathbb{P}^4$ 8 of them lie on the union of two hyperplanes – which won't contain the rest – so they impose exactly 9 conditions.

Prove monodromy of hyperplane sections is the symmetric group. Do this carefully. Explain the correspondence between monodromy and Galois theory.

Deduce projective normality from quadratic normality.

At this point, we're stuck: we still don't know what linear series exist on our curve, or much about the geometry of the canonical model. But if we invoke Brill-Noether, we have both: the curve has a g^2_6, which gives us a plane model as a sextic (with only double points, since no g^1_3s); the canonical series on the curve is cut out by cubics passing through the double points, which embeds the (blow-up of the) plane as a del Pezzo surface in ¶^{5}, of which the canonical curve is a quadric section. Also, use the count of g^2_6s on C to deduce the uniqueness of the del Pezzo.

Chapter 8

What linear series exist?

8.1 What linear series exist?

In the last few chapters, we have alternated between setting up a general description of linear series on curves, and showing how this plays out in examples. It's time to return to the general theory, and the next question to ask, naturally, is "What linear systems exist?"

There are various ways to interpret this question. Let's start by taking the question in its plain, unvarnished form—for which g, r and d does there exist a curve C of genus g and a linear system $(\mathcal{L}, V)$ on C of degree d and dimension r? In this form, the answer is given for line bundles of large degree $d \geq 2g-1$ by the Riemann-Roch theorem: on any curve, there exists a linear series of degree $d \geq 2g-1$ and dimension r iff $r \leq d-g$.

8.1.1 Clifford's theorem

Riemann-Roch still leaves open the question of what linear systems of degree $d \leq 2g-2$ may exist on a curve of genus g. The answer is given by the classical theorem of Clifford:

Theorem 8.1.1. *Let C be a curve of genus g and $\mathcal{L}$ a line bundle of degree $d \leq 2g-2$. Then*

$$r(\mathcal{L}) \leq \frac{d}{2}.$$

Moreover, if equality holds then we must have either

1. *$d = 0$ and $\mathcal{L} = \mathcal{O}_C$;*
2. *$d = 2g-2$ and $\mathcal{L} = K_C$; or*

3. C is hyperelliptic, and $|\mathcal{L}|$ is a multiple of the g^1_2 on C.

Proof. The proof of Clifford rests on a very basic construction and observation.

To start, let $\mathcal{D} = (\mathcal{L}, V)$ and $\mathcal{E} = (\mathcal{M}, W)$ be two linear series on a curve C. By the *sum* $\mathcal{D} + \mathcal{E}$ of $\mathcal{D}$ and $\mathcal{E}$, we will mean the pair

$$\mathcal{D} + \mathcal{E} = (\mathcal{L} \otimes \mathcal{M}, U)$$

where $U \subset H^0(\mathcal{L} \otimes \mathcal{M})$ is the subspace generated by the image of $V \otimes W$, under the multiplication/cup product map $H^0(\mathcal{L}) \otimes H^0(\mathcal{M}) \to H^0(\mathcal{L} \otimes \mathcal{M})$—in other words, it's the subspace of the complete linear series $|\mathcal{L} \otimes \mathcal{M}|$ spanned by divisors of the form $D + E$, with $D \in \mathcal{D}$ and $E \in \mathcal{E}$.

The observation is a simple one:

Lemma 8.1.2. *If $\mathcal{D}$ and $\mathcal{E}$ are two nonempty linear series on a curve C, then*

$$\dim(\mathcal{D} + \mathcal{E}) \geq \dim \mathcal{D} + \dim \mathcal{E}.$$

(To see this, we observe that to say $\dim \mathcal{D} \geq m$ means exactly that we can find a divisor $D \in \mathcal{D}$ containing any given m points of C; since $\mathcal{D} + \mathcal{E}$ contains all pairwise sums $D + E$ with $D \in \mathcal{D}$ and $E \in \mathcal{E}$, we can certainly find a divisor $F \in cD + \mathcal{E}$ containing any given $\dim \mathcal{D} + \dim \mathcal{E}$ points of C.)

Given this lemma, the proof of Clifford follows simply by applying it to the pair $|\mathcal{L}|$ and $|K_C \otimes \mathcal{L}^{-1}|$: by Riemann-Roch, we have

$$r(K_C \otimes \mathcal{L}^{-1}) = r(\mathcal{L}) + g - d - 1$$

and so we deduce that

$$g = r(K_C) + 1 \geq r(\mathcal{L}) + r(K_C \otimes \mathcal{L}^{-1}) + 1 \geq 2r(\mathcal{L}) + g - d;$$

hence $r(\mathcal{L}) \leq d/2$.

The proof of the second half of Clifford rests on a basic fact about the geometry of hyperplane sections of a curve in projective space; we'll defer it until we've established that fact. □

Combining Clifford with Riemann-Roch, we arrive at the answer to our initial question

Theorem 8.1.3. *There exists a curve C of genus g and line bundle $\mathcal{L}$ of degree d on C with $h^0(\mathcal{L}) \geq r + 1$ if and only if*

$$r \leq \begin{cases} d - g, & \text{if } d \geq 2g - 1; \text{ and} \\ d/2, & \text{if } 0 \leq d \leq 2g - 2. \end{cases}$$

Exercise 8.1.4. Prove a slightly stronger version of Theorem 8.1.3: that under the hypotheses of Theorem 8.1.3 there exists a *complete* linear series of degree d and dimension r.

8.1.2 Castelnuovo's theorem

Theorem 8.1.3 gives a complete and sharp answer to the question originally posed: for which d, r and g does there exists a triple $(C, \mathcal{L}, V)$ with C a curve of genus g, $\mathcal{L}$ a line bundle of degree d on C and $V \subset H^0(\mathcal{L})$ of dimension $r+1$.

But maybe that wasn't the question we meant to ask! After all, we're interested in describing curves in projective space as images of abstract curves C under maps given by linear systems on C. Observing that the linear series that achieve equality in Clifford's theorem give maps to $\mathbb{P}^r$ that are 2 to 1 onto a rational curve, we might hope that we would have a different—and more meaningful—answer if we restrict our attention to linear series $\mathcal{D} = (\mathcal{L}, V)$ for which the associated map $\phi_{\mathcal{D}}$ is at least a birational embedding. With this restriction, the question is tantamount to the

Question 8.1.5. *What is the largest possible genus of an irreducible, nondegenerate curve $C \subset \mathbb{P}^r$ of degree d?*

The answer to this question is indeed quite different from the inequality provided by Theorem 8.1.3. It is the content of *Castelnuovo's theorem*, which gives a sharp answer to this question. We'll sketch the derivation of the inequality here; we'll prove that it is in fact sharp and describe in detail the curves that achieve it in Chapter **??**.

To start, Castelnuovo's bound follows from a very straightforward approach: if C is a curve of degree d and genus g in $\mathbb{P}^r$, the idea is to prove successive lower bounds for the dimensions $h^0(\mathcal{O}_C(m))$ of multiples of the g^r_d cut on C by hyperplanes. For large values of m, of course, the line bundle $\mathcal{O}_C(m)$ is nonspecial, and so a lower bound on the dimension of its space of sections translates, via Riemann-Roch, into an upper bound on the genus g.

Definition 8.1.6. Let $\mathcal{L}$ be any line bundle on a smooth projective variety X, and $D = \{p_1, \ldots, p_d\}$ a collection of points of X. By the *number of conditions imposed by D on sections of $\mathcal{L}$* we will mean simply the difference

$$h^0(\mathcal{L}) - h^0(\mathcal{L} \otimes \mathcal{I}_{D/X});$$

that is, the codimension in $H^0(\mathcal{L})$ of the subspace of sections vanishing on D. More generally, if $V \subset H^0(\mathcal{L})$ is any linear system, by the number of conditions imposed by D on V we will mean the difference

$$\dim(V) - \dim\left(V \cap H^0(\mathcal{L} \otimes \mathcal{I}_{D/X})\right).$$

Thus, for example, if $X = \mathbb{P}^r$, the number of conditions imposed by D on $H^0(\mathcal{O}_{\mathbb{P}^r}(m))$ is the value $h_D(m)$ of the Hilbert function of D. Note that the number of conditions imposed by D on a linear system V is necessarily less than or equal to the degree d of D; if it is equal we say that D *imposes independent conditions on V*.

To apply this notion, suppose $C \subset \mathbb{P}^r$ is an irreducible, nondegenerate curve. Let $\Gamma = C \cap H$ be a general hyperplane section of C. Let $V_m \subset H^0(\mathcal{O}_C(m))$ be the linear series cut on C by hypersurfaces of degree m in $\mathbb{P}^r$, that is, the image of the restriction map

$$H^0(\mathcal{O}_{\mathbb{P}^r}(m)) \to H^0(\mathcal{O}_C(m)).$$

We have then a series of more or less trivial inequalities:

$$\begin{aligned} h^0(\mathcal{O}_C(m)) - h^0(\mathcal{O}_C(m-1)) &\geq \# \text{ of conditions imposed by } \Gamma \text{ on } H^0(\mathcal{O}_C(m)) \\ &\geq \# \text{ of conditions imposed by } \Gamma \text{ on } V_m \\ &\geq \# \text{ of conditions imposed by } \Gamma \text{ on } H^0(\mathcal{O}_{\mathbb{P}^r}(m)); \end{aligned}$$

in other words, the dimension $h^0(\mathcal{O}_C(m))$ is bounded below by the sum

$$h^0(\mathcal{O}_C(m)) \geq \sum_{k=0}^{m} h_\Gamma(k).$$

We need, in other words, a lower bound on the Hilbert function of a general hyperplane section Γ of our curve C. This is turn requires that we have some knowledge of the geometry of Γ, but in fact we don't need all that much: all we need is the basic

Lemma 8.1.7 (general position lemma)**.** *If $C \subset \mathbb{P}^r$ is an irreducible, nondegenerate curve and $\Gamma = C \cap H$ a general hyperplane section of C, then the points of Γ are in linear general position in $H \cong \mathbb{P}^{r-1}$, meaning no r points of Γ lie in a hyperplane $\mathbb{P}^{r-2} \subset H$.*

Thus, for example, if $C \subset \mathbb{P}^3$ is a space curve, no three points of $\Gamma = H \cap C$ will be collinear.

Exercise 8.1.8. Prove directly that a general plane section of an irreducible, nondegenerate space curve does not contain three collinear points. (Hint: estimate the dimension of the family of trisecant lines to the curve.)

The general position lemma was originally asserted by Castelnuovo. In more modern treatments, it is usually deduced as a special case of the more general *uniform position lemma*; we'll take a few pages out and describe this derivation.

Uniform position

We start by introducing the *monodromy group* of a generically finite cover. To set this up, let $f : Y \to X$ be a dominant rational map between irreducible varieties of the same dimension over $\mathbb{C}$. There is then an open subset $U \subset X$ such that the restriction of f to the preimage $V = f^{-1}(U)$ is a covering space in the classical topology; we'll denote by d the number of sheets.

Now choose a base point $p_0 \in U \subset X$, and label the preimage $\Gamma = f^{-1}(p_0)$ as $\{q_1, \dots, q_d\}$. If γ is any loop in U with base point p_0, for any $i = 1, \dots, d$ there is a unique lifting of γ to an arc $\tilde{\gamma}_i$ in V with initial point $\tilde{\gamma}_i(0) = q_i$ and end point $\tilde{\gamma}_i(1) = q_j$ for some $j \in \{1, 2, \dots, d\}$; in this way, we can associate to γ a permutation of $\{1, 2, \dots, d\}$. Since the permutation depends only on the class of γ in $\pi_1(U, p_0)$, we get a homomorphism to the symmetric group

$$\pi_1(U, p_0) \to \mathrm{Perm}(\Gamma) \cong S_d.$$

The image M of this map is called the *monodromy group* of the map f; it is well-defined as a subgroup of S_d up to conjugation (the choice of labelling of the points of Γ). Note that it is independent of the choice of open set U: if $U' \subset U$ is a Zariski open subset, the map $\pi_1(U', p_0) \to \pi_1(U, p_0)$ will be surjective, and so the image of $\pi_1(U', p_0)$ in S_d is again M.

Cheerful Fact 8.1.1. There is another characterization of the monodromy group M that will not be used here but that is worth knowing. In the situation described above, the pullback map f^* expresses the function field $K(Y)$ as a finite algebraic extension of $K(X)$; the degree d is the degree of this extension, and M is equal to the Galois group of the Galois normalization of $K(Y)$ over $K(X)$. (For a proof of this equality, see [**?**].)

One note: we have assumed here that both X and Y are irreducible. In fact, we need only have assumed that X is irreducible; we can apply the same construction in case Y is reducible (note that any irreducible components of Y that fail to dominate X simply won't appear in the construction). In this setting, we see that Y *is irreducible if and only if the monodromy group* $M \subset S_d$ *is transitive.*

There are two basic lemmas we can use to describe the monodromy group.

Lemma 8.1.9. *Let $f : Y \to X$ be a generically finite cover of degree d, with monodromy group $M \subset S_d$; let $U \subset X$ and $V = f^{-1}(U) \subset Y$ be open sets as above. For any $k = 1, 2, \dots, d$, let V_k^* be the complement of the large diagonal in the kth fiber power of $V \to U$; that is,*

$$V_k^* = \{(x; y_1, \dots, y_k) \in U \times V^k \mid f(y_i) = x \text{ and } y_i \neq y_j \ \forall i \neq j\}.$$

Then V_k^* is irreducible if and only if M is k times transitive.

Lemma 8.1.10. *Let $f : Y \to X$ be a generically finite cover of degree d, with monodromy group $M \subset S_d$; let $U \subset X$ and $V = f^{-1}(U) \subset Y$ be open sets as above. If for some point $p \in X$ the fiber $f^{-1}(p)$ consists of $d-2$ reduced points and one point of multiplicity 2, then M contains a transposition.*

Let's now introduce the specific cover to which we'll apply these lemmas. Let $C \subset \mathbb{P}^r$ be a smooth irreducible, nondegenerate curve of degree d, let $X = \mathbb{P}^{r*}$ be the space of hyperplanes in $\mathbb{P}^r$, and let

$$Y = \{(H, p) \in \mathbb{P}^{r*} \times C \mid p \in H\}$$

Here we can take U to be simply the complement of the dual $C^* \subset \mathbb{P}^{r*}$, that is, the locus of hyperplanes transverse to C. Our claim is

Proposition 8.1.11. *In the situation above, he monodromy group of the cover $Y \to X$ is the full symmetric group; that is, $M = S_d$*

Proof. There are two components in this proof: we show first that M is twice transitive, and then that it contains a transposition. Given this, M will contain all transpositions, and hence equal S_d.

For the double transitivity, we introduce a related cover: set

$$\Phi = \{(H, p, q) \in \mathbb{P}^{r*} \times C \times C \mid p + q \subset H\}$$

(Here $p+q$ is the divisor $p+q$ on C, viewed as a subscheme of C.) the projection $\pi_{2,3} : \Phi \to C \times C$ is a $\mathbb{P}^{r-2}$-bundle, and so irreducible; applying Lemma 8.1.9, we deduce that M is twice transitive.

Finally, for the existence of a transposition in M, we have to use the hypothesis of characteristic zero to say that *not every point of C is a flex.* Given this, let $p \in C$ be a non-flex point, and let $H \subset \mathbb{P}^r$ be a general hyperplane containing the tangent line T_pC. Under these hypotheses, the fiber of Φ over the point $H \in \mathbb{P}^{r*}$ consists of the point p with multiplicity 2, and $d-2$ reduced points; applying Lemma 8.1.10, we deduce that M contains a transposition. □

Finally, as a consequence of Proposition 8.1.11 (and Lemma 8.1.9) we can deduce the

Lemma 8.1.12 (uniform position lemma)**.** *With $C \subset \mathbb{P}^r$ and $\Gamma = C \cap H$ as above, any two subsets $\Gamma', \Gamma'' \subset \Gamma$ of the same cardinality k have the same Hilbert function, i.e., impose the same number of conditions on $\mathcal{O}_{\mathbb{P}^{r-1}}(m)$ for all m.*

Proof. In this situation, we restrict to the open set $U = \mathbb{P}^{r*} \setminus C^*$ of hyperplanes transverse to C, and introduce the fiber power

$$V_k^* = \{(x; y_1, \ldots, y_k) \in U \times V^k \mid f(y_i) = x \text{ and } y_i \neq y_j \ \forall i \neq j\}.$$

as above; V_k^* parametrizes subsets Γ of cardinality k in hyperplane sections $H \cap C$ of C. Applying Lemma 8.1.9, we see that V_k^* is irreducible of dimension r.

Now, the Hilbert function $h_\Gamma(m)$ is lower semicontinuous, so it achieves its maximum on a Zariski open subset of V_k^*. Since V_k^* is irreducible, the complement of this open will have dimension strictly less than r; a general hyperplane $H \in \mathbb{P}^{r*}$ will lie outside this image, meaning that $h_\Gamma(m)$ is the same for all $\Gamma \subset C \cap H$ of cardinality k. □

We return now to Castelnuovo's analysis.

The general position lemma is just the special case $m = 1$ of the uniform position lemma. This may not seem like much information about Γ, but in fact it's all we need to prove a sharp bound! The basic (and completely elementary) statement is

Proposition 8.1.13. *If $\Gamma \subset \mathbb{P}^n$ is a collection of d points in linear general position and spanning $\mathbb{P}^n$, then*

$$h_\Gamma(m) \geq \min\{d, mn+1\}$$

Proof. Suppose first that $d \geq mn+1$, and let $p_1, \ldots, p_{mn+1} \in \Gamma$ be any subset of $mn+1$ points. We want to show that $\Gamma' = \{p_1, \ldots, p_{mn+1}\}$ imposes independent conditions of $H^0(\mathcal{O}_{\mathbb{P}^n}(m))$, that is, for any $p_i \in \Gamma'$ we can find a hypersurface $X \subset \mathbb{P}^n$ of degree m containing all the points $p_1, \ldots, \hat{p_i}, \ldots, p_{mn+1}$ but not containing p_i.

This is easy: simply group the mn points of $\Gamma' \setminus \{p_i\}$ into m subsets Γ_k of cardinality n; each set Γ_k will span a hyperplane $H_k \subset \mathbb{P}^n$, and we can take $X = H_1 \cup \cdots \cup H_m$. □

This may seem like a crude argument, but the bound derived is sharp: any collection of point $\Gamma \subset \mathbb{P}^n$ lying on a rational normal curve $D \subset \mathbb{P}^n$ has exactly this Hilbert function.

At this point, all that remains is to add up the lower bounds in the proposition. To this end, let $C \subset \mathbb{P}^r$ be as above an irreducible, nondegenerate curve of degree d, and set $M = \lfloor \frac{d-1}{r-1} \rfloor$, so that we can write

$$d = M(r-1) + 1 + \epsilon \quad \text{with} \quad 0 \leq \epsilon \leq r-2.$$

We have then

$$\begin{aligned} h^0(\mathcal{O}_C(M)) &\geq \sum_{k=0}^{M} h^0(\mathcal{O}_C(k)) - h^0(\mathcal{O}_C(k-1)) \\ &\geq \sum_{k=0}^{M} k(r-1) + 1 \\ &= \frac{M(M+1)}{2}(r-1) + M + 1 \end{aligned}$$

and similarly

$$h^0(\mathcal{O}_C(M+m)) \geq \frac{M(M+1)}{2}(r-1) + M + 1 + md.$$

For sufficiently large m, the line bundle $\mathcal{O}_C(M+m)$ will be nonspecial, so we can plug this in to Riemann-Roch to arrive at

$$\begin{aligned} g &= (M+m)d - h^0(\mathcal{O}_C(M+m)) + 1 \\ &\le (M+m)d - \big(\frac{M(M+1)}{2}(r-1) + M + 1 + md\big) \\ &= M\big(M(r-1)+1+\epsilon\big) - \big(\frac{M(M+1)}{2}(r-1) + M + 1\big) \\ &= \frac{M(M-1)}{2}(r-1) + M\epsilon. \end{aligned}$$

To summarize our discussion: for positive integers d and r, we write

$$d = M(r-1) + 1 + \epsilon \quad \text{with} \quad 0 \le \epsilon \le r-2$$

and set

$$\pi(d,r) = \frac{M(M-1)}{2}(r-1) + M\epsilon.$$

In these terms, we have proved the

Theorem 8.1.14 (Castelnuovo's bound)**.** *If $C \subset \mathbb{P}^r$ is an irreducible, nondegenerate curve of degree d and genus g, then*

$$g \le \pi(d,r).$$

We will see in Chapter **??** that this is in fact sharp: for every r and $d \ge r$, there do exist such curves with genus exactly $\pi(d,r)$. For now, we make a few observations:

1. In case $r = 2$, all the inequalities used in the derivation of Castenuovo's bound are in fact equalities, and indeed we see that in this case $\pi(d,2) = \binom{d-1}{2}$ is the genus of a smooth plane curve of degree d.
2. In case $r = 3$, we have
$$\pi(d,3) = \begin{cases} (k-1)^2 & \text{if } d = 2k \text{ is even; and} \\ k(k-1) & \text{if } d = 2k+1 \text{ is odd.} \end{cases}$$
In this case again, it's not hard to see the bound is sharp: these are exactly the genera of curves of bidegree (k,k) and $(k+1,k)$ on a quadric surface $Q \cong \mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^3$.
3. In general, we see that for fixed r asymptotically
$$\pi(d,r) \sim \frac{d^2}{2(r-1)}.$$

Exercise 8.1.15. Show that with C as above, the line bundle $\mathcal{O}_C(M)$ is nonspecial. (We will see in Section **??** that this is sharp; that is, there exist such curves C with $\mathcal{O}_C(M-1)$ special).

8.2 Brill-Noether theory

((This needs to be amalgamate with the discussion in the Jacobians chapter–here or there))

8.2.1 Basic questions addressed by Brill-Noether theory

In the last section, we restricted our attention to the linear series most of interest to us: those corresponding to embeddings of our curve in projective space (or at any rate birational embeddings) and their limits. But there is one other respect in which Castelnuovo theory fails to address a basic concern: the curves with linear systems achieving Castelnuovo's bound are, like hyperelliptic curves, very special. (In fact, we'll see in Section **??** that in general they are even rarer than hyperelliptic curves.) That is, if we were to pick a curve C of genus g "at random" (we'll make this notion more precise when we describe the moduli space of curves in Chapter **??**), we would still have no idea what linear systems existed on C or how they behaved.

Brill-Noether theory addresses exactly this issue: it asks, "what linear series exist on *all* curves of a given genus?" To start with, we'll give the crudest form of the theorem:

Theorem 8.2.1. *Fix non-negative integers g, r and d. It is the case that every curve of genus g possesses a linear series of degree d and dimension r if and only if*

$$\rho(g, r, d) := g - (r + 1)(g - d + r) \geq 0.$$

In the following sections, we'll see why we might naively expect this to be the case, and we'll also describe some of the many refinements and strengthenings of the theorem (we will be able to give better versions in later chapters, after we have, for example, introduced the schemes parametrizing linear systems on a given curve). A proof of the existence half of the theorem (the "if" part of the statement) may be found in [Eisenbud and Harris 2016]; and we will give in the concluding chapter of this book a relatively simple proof of the nonexistence part (the "only if"). In the meantime, we'll mention here the special case $r = 1$:

Corollary 8.2.2. *If C is any curve of genus g, then C admits a rational function of degree d for some positive $d \leq \lceil \frac{g+2}{2} \rceil$.*

Thus, for example, any curve of genus 2 is hyperelliptic, any curve of genus 3 or 4 is either hyperelliptic or trigonal, and so on.

8.2.2 Heuristic argument leading to the statement of Brill-Noether

The Brill-Noether theorem, as we'll see, is a far-reaching description of the linear series to be found on a general curve. It starts, though, with a relatively simple dimension count—one that was first carried out almost a century and a half ago.

To set this up, let C be a smooth projective curve of genus g, and $D = p_1 + \cdots + p_d$ a divisor on C. We'll assume here the points p_i are distinct; the same argument (albeit with much more complicated notation) can be carried out in general.

When does the divisor D move in an r-dimensional linear series? Riemann-Roch gives an answer: it says that $h^0(D) \geq r + 1$ if and only if the vector space $H^0(K - D)$ of 1-forms vanishing on D has dimension at least $g - d + r$—that is, if and only if the evaluation map

$$H^0(K) \to H^0(K|_D) = \oplus K_{p_i}$$

has rank at most $d - r$.

We can represent this map by a $g \times d$ matrix. Choose a basis $\omega_1, \ldots, \omega_g$ for the space $H^0(K)$ of 1-forms on C; choose an analytic open neighborhood U_j of each point $p_j \in D$ and choose a local coordinate z_j in U_j around each point p_j, and write

$$\omega_i = f_{i,j}(z_j) dz_j$$

in U_j. We will have $r(D) \geq r$ if and only if the matrix-valued function

$$A(z_1, \ldots, z_d) = \begin{pmatrix} f_{1,1}(z_1) & f_{2,1}(z_1) & \ldots & f_{g,1}(z_1) \\ f_{1,2}(z_2) & f_{2,2}(z_2) & \ldots & f_{g,2}(z_2) \\ \vdots & \vdots & & \vdots \\ f_{1,d}(z_d) & f_{2,d}(z_d) & \ldots & f_{g,d}(z_d) \end{pmatrix}$$

has rank $d - r$ or less at $(z_1, \ldots, z_d) = (0, \ldots, 0)$.

The point is, we can think of A as a matrix valued function in the open set $U = U_1 \times U_2 \times \cdots \times U_d \subset C_d$; and for divisors $D \in U$, we have $r(D) \geq r$ if and only if $\operatorname{rank}(A(D)) \leq d - r$. Now, in the space $M_{d,g}$ of $d \times g$ matrices, the subset of matrices of rank $d - r$ or less has codimension $r(g - d + r)$, and so we might naively expect that the locus of divisors with $r(D) \geq r$ would have dimension $d - r(g - d + r)$. At the same time, if any divisor of degree d with $h^0(D) \geq r + 1$ exists, then there must be at least an r-dimensional family of them; so we'd suspect that such divisors exist only if

$$d - r(g - d + r) \;\geq\; r,$$

which is exactly the Brill-Noether statement.

As we indicated, Theorem 8.2.1 represents only the most bare-bones version of Brill-Noether. The full statement describes as well the space parametrizing linear series g^r_d on a general curve C—it says that it has dimension $\rho(g, r, d)$, is smooth and irreducible when $\rho > 0$—and also the geometry of C as mapped to projective space by a general such g^r_d. The problem is, all these versions involve the existence of a parameter space for linear series on a given curve (which we'll see how to construct in Chapter **??**), as well as the existence of a moduli space M_g parameterizing abstract curves of genus g (which we'll discuss further in Chapter **??**). For this reason, we will have to defer the full statement of Brill-Noether to that chapter. In the meantime, though, we'll see in the next chapter how the theory plays out in the case of curves of low genus.

Chapter 9

Inflections and Brill Noether

In this concluding chapter, we want to introduce one more aspect of the geometry of linear series on curves, the *inflectionary points* of a linear system, and use it to give a proof of at least half of the classical Brill-Noether theorem.

Inflectionary points in general are a direct generalization of the notion of flex point of a smooth plane curve to curves in higher-dimensional space. Just as a point $p \in C$ on a smooth plane curve $C \subset \mathbb{P}^2$ is called a *flex point* if there is a line $L \subset \mathbb{P}^2$ having contact of order 3 or more with C at p, a point on a smooth, nondegenerate curve $C \subset \mathbb{P}^r$ will be called an inflectionary point if there is a hyperplane $H \subset \mathbb{P}^r$ having contact of order $r+1$ or more with C at p. This notion can be extended to arbitrary linear series on smooth curves (as opposed to very ample ones); we'll see below that every linear series has finitely many inflectionary points, and how to count them.

9.1 Inflection points, Plücker formulas and Weierstrass points

9.1.1 Definitions

To start with the definition: let C be a smooth projective curve of genus g, and $\mathcal{D} = (\mathcal{L}, V)$ a g^r_d on C; that is, let $\mathcal{L} \in \mathrm{Pic}^d(C)$ be a line bundle of degree d on C and $V \subset H^0(\mathcal{L})$ an $(r+1)$-dimensional vector space of sections.

For any point $p \in C$, we can find a basis $\sigma_0, \ldots, \sigma_r$ of V consisting of sections vanishing to different orders at p (just start with any basis and if two elements vanish to the same order, replace one with a linear combination of the two vanishing to strictly higher order; since the order $\text{ord}_p(\sigma)$ of any section at p is bounded above by d, this process must terminate). The set

$$\{\text{ord}_p(\sigma) \mid \sigma \neq 0 \in V\}$$

thus has cardinality $r+1$, and we can write it as

$$\{\text{ord}_p(\sigma) \mid \sigma \neq 0 \in V\} = \{a_0, \dots, a_r\} \text{ with } a_0 < a_1 < \cdots < a_r;$$

the sequence $a_i = a_i(\mathcal{D}, p)$ is called the *vanishing sequence* of $\mathcal{D}$ at p. Finally, since $a_i \geq i$, we can set $\alpha_i = \alpha_i(\mathcal{D}, p) = a_i - i$; the sequence $0 \leq \alpha_0 \leq \alpha_1 \leq \cdots \leq \alpha_r$ is called the *ramification sequence* of $\mathcal{D}$ at p, and we say that p is an *inflectionary point* of the linear series $\mathcal{D}$ if $(\alpha_0, \dots, \alpha_r) \neq (0, \dots, 0)$. (Note that if $\mathcal{D}$ is very ample, so that it may be viewed at the linear series cut on C by hyperplanes for some embedding $C \subset \mathbb{P}^r$, then this coincides with the notion above: p is an inflectionary point if there is a hyperplane $H \subset \mathbb{P}^r$ having contact of order $r+1$ or more with C at p.)

Finally, we define the *weight* of an inflectionary point to be the sum

$$w(\mathcal{D}, p) = \sum_{i=0}^{r} \alpha_i(\mathcal{D}, p).$$

9.1.2 The Plücker formula

There are two essential facts about the inflectionary points of a linear series $\mathcal{D} = (\mathcal{L}, V)$ on a smooth curve C.

The first is simply that *not every point of C is an inflectionary point.* This may seem obvious—try to imagine a plane curve in which every point is a flex!—but in fact it's false in characteristic p: there exist what are called "strange curves," smooth curves in $\mathbb{P}^r$ such that every point is inflectionary. Luckily, we are in characteristic 0 here, so we don't have to worry about this phenomenon.

The second is that the total number of inflectionary points, properly counted, is determined by d, g and r. Here "properly counted" means that we count an inflectionary point $p \in C$ w times, where $w = w(\mathcal{D}, p)$ is the weight of p; the actual formula, called the *Plücker formula*, is

$$\sum_{p \in C} w(\mathcal{D}, p) \;=\; (r+1)d + r(r+1)(g-1). \tag{9.1}$$

Proofs of these two statements can be found in a variety of sources; see for example [] (3264?).

It should be emphasized that the Plücker formula, while extremely useful (as we'll see in the remainder of this chapter), leaves many questions unanswered: we don't know, for example, what combinations of inflectionary points are possible, or what the behavior of the inflectionary points on a suitably general curve may be.

9.1.3 Weierstrass points

As with any extrinsic invariant of a curve in projective space, we can derive an intrinsic invariant of an abstract curve by applying the notion of inflectionary point to the canonical linear series.

We define a *Weierstrass point* of a curve C to be an inflectionary point of the canonical linear series $|K_C|$. This amounts to saying a point p is a Weierstrass point if there exists a canonical differential on C vanishing to order g or more at p; by Riemann-Roch, this is tantamount to the condition that $h^0(\mathcal{O}_C(gp)) \geq 2$, or in other words to saying that there exists a rational function on C, regular away from p and having a pole of order g or less at p.

We can similarly characterize all the inflectionary indices of the canonical series at a point. We see from Riemann-Roch that for any $k \geq 0$, there exists a rational function on C, regular on $C \setminus \{p\}$ and having a pole of order exactly k at p—that is,

$$h^0(\mathcal{O}_C(kp)) > h^0(\mathcal{O}_C((k-1)p))—$$

if and only if

$$h^0(K_C(-kp)) = h^0(K_C((-k+1)p));$$

that is, if and only if there does *not* exist a regular differential on C with a zero of order exactly $k-1$ at p. To give the classical terminology, we see from the above that there will exist exactly g values of k such that there does *not* exist a rational function on C with a pole of order exactly k at p; these are called the *gap values* of the point $p \in C$, and by the above they comprise exactly the vanishing sequence of the canonical series $|K_C|$ at p, shifted by 1. Moreover, it is clear that the complement in $\mathbb{N}$ of the gap values—that is, the set of k such that there *does* exist a rational function on C with a pole of order exactly k at p—forms a semigroup, called the *Weierstrass semigroup* of $p \in C$. Finally, the *weight* w_p of a Weierstrass point $p \in C$ is defined to be the weight $w(|K_C|, p)$ of p as an inflectionary point of the canonical series.

From the general theory of ramification above, we see that a general point p on any curve C has gap sequence $(1, 2, \dots, g)$, and correspondingly the semigroup $W_p = (0, g+1, g+2, \dots)$. A Weierstrass point is called *normal* if it has weight 1; this is tantamount to saying that the gap sequence is $(1, 2, \dots, g-1, g+1)$, or that the semigroup is $(0, g, g+2, g+3, \dots)$. (The full Brill-Noether theorem tells us that a general curve C has only normal Weierstrass points; this will be a consequence of Theorem 9.4.5 below.) Finally, the Plücker formula tells us the total weight of the Weierstrass points on a given curve C: plugging in, we have

$$\sum_{p \in C} w(|K_C|, p) \ = \ g(2g-2) + (g-1)g(g-1) \ = \ g^3 - g.$$

and hence

Theorem 9.1.1. *The sum of the weights of the Weierstrass points on a curve C of genus g is*

$$\sum_{p\in C} w_p = g^3 - g.$$

There is still much we don't know about Weierstrass points in general. Most notably, we don't know what semigroups of finite index in $\mathbb{N}$ occur as Weierstrass semigroups; an example of Buchweitz shows that not all semigroups occur, but there are also positive results, such as the statement ([EH]) that every semigroup of weight $w \leq g/2$ occurs, and its refinement and strengthening by Pflueger ([**?**]).

9.2 Finiteness of the automorphism group

As an application of just a rudimentary knowledge of Weierstrass points, we will deduce a fundamental fact: that the automorphism group of a curve of genus $g \geq 2$ is finite. The idea behind the argument is simple: because the Weierstrass points of a curve C are intrinsically defined, *any automorphism of C must carry Weierstrass points to Weierstrass points.* Since there are only finitely many Weierstrass points, then, it will suffice to show that the subgroup of $Aut(C)$ of automorphisms of C that fix all the Weierstrass points individually is finite. In fact, the following two lemmas establish a strong version of this:

Lemma 9.2.1. *Let C be a smooth projective curve of genus $g \geq 2$, and $f : C \to C$ an automorphism of C.*

1. *If f has $2g + 3$ distinct fixed points, then f is the identity; and*
2. *If f has $2g + 2$ distinct fixed points, then either f is the identity or C is hyperelliptic and f is the hyperelliptic involution.*

Proof. There are two possible arguments here, one invoking the classical topology and applying the Lefschetz fixed point formula and the other more algebrao-geometric.

For the first, we recall the definition of the *Lefschetz number* of a map $f : M \to M$ of a compact oriented real n-manifold M. This is the alternating sum of the traces of the action of f on $H^i(X, \mathbb{C})$:

$$L(f) := \sum_{i=0}^{n} \operatorname{Trace}\left(f^* : H^i(X, \mathbb{C}) \to H^i(X, \mathbb{C})\right).$$

The Lefschetz fixed point formula then says that if f has isolated fixed points, the number of those points, properly counted, is equal to $L(f)$.

In the situation of a smooth projective curve over $\mathbb{C}$, any automorphism other than the identity has isolated fixed points, and since the map is orientation-preserving each fixed point contributes positively to the total; thus the number of distinct fixed points is at most $L(f)$.

Now suppose $f : C \to C$ is any automorphism. Of necessity, f acts as the identity on $H^0(C, \mathbb{C})$ and $H^2(C, \mathbb{C})$, so if we want to bound $L(f)$ we just have to say something about the action of f on $H^1(C, \mathbb{C})$. To do this, note that the action of f on $H^1(C, \mathbb{C})$ respects the *Hodge decomposition*

$$H^1(C, \mathbb{C}) = H^0(K_C) \oplus H^1(\mathcal{O}_C).$$

Moreover, the action of f on $H^0(K_C)$ preserves the definite Hermitian inner product

$$H(\eta, \phi) = \int_C \eta \wedge \overline{\phi},$$

and it follows that *the eigenvalues of the action of f on $H^0(K_C)$ are all complex numbers of absolute value 1*, and likewise for the action on $H^1(\mathcal{O}_C)$. The absolute value of the trace of $f^* : H^1(C, \mathbb{C}) \to H^1(C, \mathbb{C})$ is thus at most 2g, and hence

$$L(f) \leq 2 + 2g,$$

proving the stated inequality in general.

Finally, if we have equality then f must act as -1 on $H^1(C, \mathbb{C})$, and it follows (again from Lefschetz) that f^2 is the identity; applying Riemann-Hurwitz to the map from C to the quotient $B = C/\langle f \rangle$ we may deduce that $B = \mathbb{P}^1$, so C is hyperelliptic and f the hyperelliptic involution. $\square$

An alternative, more algebraic argument for the lemma may be given using the intersection pairing on the surface $S = C \times C$ and applying the index theorem for surfaces. To carry this out, let Δ and $\Gamma \subset S$ be the diagonal and the graph of f respectively, and let Φ_1 and $\Phi_2 \subset S$ be fibers of the two projection maps; let $\delta, \gamma, \varphi_1$ and $\varphi_2 \in N(S)$ be the classes of these curves in the Neron-Severi group of S. We are trying to estimate the intersection number $b = \delta \cdot \gamma$.

We know all the other pairwise intersection number of these classes: the ones involving φ_1 or φ_2 are obvious; we have

$$\delta^2 = 2 - 2g$$

and since the automorphism $id_C \times f : C \times C \to C \times C$ carries Δ to Γ, we see that $\gamma^2 = 2 - 2g$ as well.

We can now apply the index theorem for surfaces to deduce our inequality. To keep things relatively simple, let's introduce two new classes: set

$$\delta' = \delta - \varphi_1 - \varphi_2 \quad \text{and} \quad \gamma' = \gamma - \varphi_1 - \varphi_2,$$

so that δ' and γ' are orthogonal to the class $\varphi_1 + \varphi_2$. Since $\varphi_1 + \varphi_2$ has positive self-intersection, the index theorem tells us that the intersection pairing must be negative definite on the span $\langle \delta', \gamma' \rangle \subset N(S)$. In particular, the determinant of the intersection matrix

	δ'	γ'
δ'	$-2g$	$b-2$
γ'	$b-2$	$-2g$

(where again $b = \gamma \cdot \delta$) must be nonnegative, from which our inequality follows.

Having established an upper bound on the number of fixed points an automorphism f of C (other than the identity) may have, it remains to find a lower bound on the number of distinct Weierstrass points; this is the content of the next lemma.

Lemma 9.2.2. *If C is a smooth projective curve of genus $g \geq 2$, then C has at least $2g+2$ distinct Weierstrass points; and if it has exactly $2g+2$ Weierstrass points it is hyperelliptic.*

Proof. Let $p \in C$ be any point, and $w_1 = w_1(p), \ldots, w_g = w_g(p)$ the ramification sequence of the canonical series $|K_C|$ at p. By definition,

$$h^0(K_C(-(w_i+i)p)) = g - i.$$

Applying Clifford's theorem we have

$$g - i \leq \frac{2g-2-w_i-i}{2} + 1;$$

solving, we see that

$$w_i \leq i$$

and hence

$$w_p \leq \binom{g}{2}$$

where w_p is the total weight of p as a Weierstrass point. Since the total weight of the Weierstrass points on C is $g^3 - g$ by Plücker, we see that the number of distinct Weierstrass points must be at least

$$\frac{g^3-g}{\binom{g}{2}} = 2g+2.$$

Finally, by the strong form of Clifford, equality here implies that the curve is hyperelliptic. ☐

9.3 Proof of (half of) the Brill-Noether theorem

In its most basic form, the Brill-Noether theorem asserts for any d, g and r that

1. if $\rho(g,r,d) := g - (r+1)(g-d+r) \geq 0$, then every curve C of genus g possesses as g^r_d; and

2. if $\rho < 0$ then a general curve of genus g does not possess a g^r_d.

The first part, often called the "existence half" of Brill-Noether, was originally proved by Kempf ([]) and Kleiman-Laksov ([]); both proofs relied on an application of the Thom-Porteous formula to a particular map of vector bundles on the Jacobian of C. An account of this argument may also be found in Appendix A of [].

We will now apply the notion of inflectionary points, and the Plücker formula in particular, to deduce the second half of the statement above—the "nonexistence half." We will then go back and deduce some of the other parts of the full Brill-Noether statement from the same set-up.

The basic approach here is to consider a family of curves $\pi : \mathcal{C} \to B$, where

1. B is a smooth curve, with distinguished point $0 \in B$;
2. for all $b \neq 0 \in B$, the fiber $C_b = \pi^{-1}(b)$ is a smooth, projective curve of genus g; and
3. the fiber C_0 over 0 is a rational curve with g ordinary cusps.

We will establish the

Lemma 9.3.1. *If $\mathcal{C} \to B$ is a family of curves as above, then for general $b \in B$ the fiber C_b does not possess a g^r_d with $\rho < 0$.*

Once we establish the existence of such a family, we deduce the basic

Theorem 9.3.2. *A general curve C of genus g does not possess a g^r_d with $\rho(g, r, d) < 0$.*

The basic outline of the argument is by contradiction, but straightforward: we assume that the general curve C_b in the family does have a g^r_d, consider what the limit of those g^r_ds might look like and, using our knowledge of the relatively simple curve C_0, arrive at a contradiction. By way of notation, let $B^\circ = B \setminus \{0\}$ and let $\mathcal{C}^\circ = \pi^{-1}(B^\circ)$ be the complement in $\mathcal{C}$ of the special fiber. The proof proceeds essentially in four/five steps.

Step 0: Existence of such a family

Lemma 9.3.3. *For each g, there exists a family $\mathcal{C} \to B$ of curves with B smooth and one-dimensional; C_b a smooth curve of genus g for $b \neq 0 \in B$ and C_0 a rational curve with g cusps.*

Step 1: Finding a family of g^r_ds over B°

Suppose now that the general curve C_b in the family does have a line bundle of degree d with $r+1$ sections. The first thing to observe is that, possibly after a base change, we can pick out one such line bundle $\mathcal{L}_b$ for each $b \neq 0$, varying regularly with b; or, in other words *there exists a line bundle $\mathcal{L}^\circ$ on the complement $\mathcal{C}^\circ$ of the special fiber such that*

$$\deg(\mathcal{L}^\circ|_{C_b}) = d \quad \text{and} \quad h^0(\mathcal{L}^\circ|_{C_b}) \geq r+1$$

for all $b \neq 0 \in B$.
((need to give argument for this assertion))

Step 2: Extending the line bundle $\mathcal{L}^\circ$ to a sheaf on all of $\mathcal{C}$

Next, we want to extend $\mathcal{L}^\circ$ to a sheaf on all of $\mathcal{C}$. We claim that *there exists a torsion-free sheaf $\mathcal{L}$ on all of $\mathcal{C}$ such that $\mathcal{L}|_{\mathcal{C}^\circ} \cong \mathcal{L}^\circ$.*

To see this, we choose an auxiliary line bundle $\mathcal{M}$ on $\mathcal{C}$ with relative degree $e > d + 2g$ (for example, embed $\mathcal{C}$ is projective space and take $\mathcal{M} = \mathcal{O}_{\mathcal{C}}(m)$ for large m); in keeping with our notational conventions, let $\mathcal{M}^\circ$ be the restriction of $\mathcal{M}$ to $\mathcal{L}^\circ$. Consider the line bundle

$$\mathcal{N}^\circ = (\mathcal{L}^\circ)^* \otimes \mathcal{M}^\circ.$$

The bundle $\mathcal{N}^\circ$ has lots of sections: the direct image is locally free of rank $e - g + 1 > 0$, and after restricting to an open neighborhood of $0 \in B$ we can assume it's generated by them. Choose a section σ of $\mathcal{N}^\circ$; let $D^\circ \subset \mathcal{C}^\circ$ be its zero divisor, and let $D \subset \mathcal{C}$ be the closure of D° in $\mathcal{C}$. Now, away from C_0 we can write

$$\mathcal{L}^\circ = (\mathcal{N}^\circ)^* \otimes \mathcal{M}^\circ = \mathcal{I}_{D^\circ/\mathcal{C}^\circ} \otimes \mathcal{M}^\circ$$

and accordingly the sheaf

$$\mathcal{L} := \mathcal{I}_{D/\mathcal{C}} \otimes \mathcal{M}$$

is the desired sheaf. Note that this need not be locally free: the total space $\mathcal{C}$ of our family may not be smooth at the cusps of the special fiber C_0 (even if the family we originally started with had smooth total space, the base change called for in Step 1 would yield a family with total space singular at the cusps of C_0), and if D passes through any of these points it need not be Cartier.

In sum, if the general fiber C_b of our family has a g^r_d, we can conclude that the special fiber C_0 has a torsion-free sheaf $\mathcal{L}_0$ with

$$c_1(\mathcal{L}_0) = d;$$

and, by upper-semicontinuity of cohomolgy,

$$h^0(\mathcal{L}_0) \geq r+1.$$

Step 3: Local description of the sheaf $\mathcal{L}_0$

The next question is, what does $\mathcal{L}_0$ look like if it's not locally free? Here we have a basic lemma:

Lemma 9.3.4. *Let p be a cusp of a curve C. If $\mathcal{F}$ is a torsion-free sheaf on C, then in a neighborhood of p in C the sheaf $\mathcal{F}$ is either locally free or isomorphic to the ideal sheaf $\mathcal{I}_{p/C}$ of p in C.*

Proof. Consider the endomorphism ring of $\mathcal{F}$, and note that it is commutative and integral over $\mathcal{O}_{p/C}$; thus it is either $\mathcal{O}_{p/C}$ or it's integral closure $\mathcal{O}_{p/\tilde{C}}$. In the latter case it is free over $\mathcal{O}_{p/\tilde{C}} \cong \mathcal{I}_{p/C}$. In the former case....
((complete the argument))

□

Exercise 9.3.5.

1. Show that the conclusion of Lemma 9.3.4 holds in case p is a node of C
2. Show by example that the conclusion of Lemma 9.3.4 is false in case p is either a tacnode or a triple point of C.

Step 4: Applying the Plücker formula

Now, back to our family $\pi : \mathcal{C} \to B$ of curves. We have assumed that for some d and r with $\rho(g, r, d) < 0$ the general curve C_b has a g^r_d, and deduced that the special fiber C_0 has a rank 1 torsion-free sheaf $\mathcal{L}_0$ of degree d with at least $r + 1$ sections; we now have to derive from this a contradiction.

To see most clearly where this contradiction comes from, let's start with the simplest case: where $\mathcal{L}_0$ is indeed locally free. In this case, let $\nu : C^\nu \cong \mathbb{P}^1 \to C$ be the normalization of C and let $q_1, \dots, q_g \in \mathbb{P}^1$ be the points lying over the cusps of C_0. We have

$$\nu^*(\mathcal{L}) \cong \mathcal{O}_{\mathbb{P}^1}(d)$$

and

$$V = \nu^*(H^0(\mathcal{L}_0)) \subset H^0(\mathcal{O}_{\mathbb{P}^1}(d))$$

is an $(r + 1)$-dimensional space of sections. (If $H^0(\mathcal{L}_0) > r + 1$, just choose any $(r + 1)$-dimensional subspace.)

Now, given that any section $\sigma \in V \subset H^0(\mathcal{O}_{\mathbb{P}^1}(d))$ is pulled back from the cuspidal curve C, we see that σ *cannot vanish to order exactly 1 at the point $q_i \in \mathbb{P}^1$ lying over any of the cusps of C_0*. It follows that for each i the ramification index

$$\alpha_1(V, q_i) \geq 1$$

and hence in general $\alpha_1(q_i, V) \geq 1$ for all $i \geq 1$. In particular, the weight of the inflectionary point q_i for the linear series V satisfies

$$w(V, q_i) \geq r$$

and correspondingly

$$\sum_{i=1}^{g} w(V, q_i) \geq rg$$

But the Plücker formula **??** tells us that the total weight of all inflectionary points for the series V is

$$\sum_{p \in \mathbb{P}^1} w(V, p) = (r+1)(d-r)$$

and there's our contradiction: by the hypothesis that

$$\rho(g, r, d) := g - (r+1)(g-d+r) < 0$$

we have $rg > (r+1)(d-r)$.

Finally, the case where $\mathcal{L}_0$ is not locally free is if anything even easier. Suppose now that the sheaf $\mathcal{L}$ fails to be locally free at l of the cusps of C_0, say $\nu(p_1), \ldots, \nu(p_l)$. Again, we can pull $\mathcal{L}$ back to $\mathbb{P}^1$; again we have

$$\nu^*(\mathcal{L}_0) \cong \mathcal{O}_{\mathbb{P}^1}(d);$$

and again we pull back section of $\mathcal{L}$ to arrive at a linear system

$$V = \nu^*(H^0(\mathcal{L}_0)) \subset H^0(\mathcal{O}_{\mathbb{P}^1}(d))$$

of degree d and genus g on $\mathbb{P}^1$. The only difference here is that sections of V all vanish at $p_1, \ldots, p_l$, so that we have

$$w(V, p_k) \geq \begin{cases} r+1 & \text{if } k \leq l; \text{ and} \\ r & \text{if } k > l. \end{cases}$$

so that

$$\sum_{k=1}^{g} w(V, p_k) \geq rg + l$$

and our contradiction is even more of a contradiction!

9.4 Corollaries and extensions of our proof

We have proved a bare-bones version of the nonexistence half of Brill-Noether; in particular, the argument does not a priori tell us anything about the geometry of $W^r_d(C)$ for a general curve C, or the geometry of C as embedded by a general linear system. In the final section of this chapter, we'll see how we can deduce stronger forms of Brill-Noether from the basic set-up above.

9.4.1 Brill-Noether with inflection

Let's start with the low-hanging fruit. Since the basic ingredient of the proof of Theorem 9.3.2 is the Plücker formula, the argument tells us something about the inflectionary behavior of linear series on a general curve C as well.

We start with a definition.

Definition 9.4.1. Let C be a smooth curve of genus g and $p_1, \ldots, p_n \in C$ distinct points of C. If $\mathcal{D} = (L, V)$ is a linear system on C of degree d and dimension r, we define the *adjusted Brill-Noether number* of $\mathcal{D}$ relative to the points p_k to be

$$\rho(\mathcal{D}; p_1, \ldots, p_k) := g - (r+1)(g-d+r) - \sum_{k=1}^{n} w(\mathcal{D}, p_k).$$

In these terms, we have the basic (but powerful) extension of the Brill-Noether theorem proved above:

Theorem 9.4.2. *Let $(C; p_1, \ldots, p_n)$ be a general n-pointed curve of genus g (that is, let C be a general curve and $p_1, \ldots, p_n \in C$ general points; equivalently, let $(C; p_1, \ldots, p_n)$ correspond to a general point of $M_{g,n}$). If $\mathcal{D}$ is any linear system on C, then*

$$\rho(\mathcal{D}; p_1, \ldots, p_k) \geq 0.$$

Proof. The proof is just an extension of the argument for Lemma 9.3.1. To start, let $\mathcal{C} \to B$ be a family of curves as in the proof of Lemma 9.3.3. Let $\sigma_1, \ldots, \sigma_n : B \to \mathcal{C}$ be sections of $\mathcal{C} \to B$ with $\sigma_k(0)$ a smooth point of C_0 for all k (such sections can always be found after passing to an étale open neighborhood of $0 \in B$). Exactly as in the proof of Lemma 9.3.1, if the general curve C_b in our family admits a g^r_d $\mathcal{D}$ with

$$\rho(\mathcal{D}; \sigma_1(b), \ldots, \sigma_n(b)) < 0$$

we can choose a family $\{\mathcal{D}_b\}$ of such linear series on the fibers C_b for $b \neq 0$ and, taking limits, we arrive at a g^r_d $\mathcal{D}_0$ on $\mathbb{P}^1$ with

$$w(\mathcal{D}_0, q_i) \geq r$$

for each of the g points $q_i \in \mathbb{P}^1$ lying over the cusps of C_0, and in addition

$$w(\mathcal{D}_0, r_k) \geq w(\mathcal{D}_b, \sigma_k(b))$$

where $r_k \in \mathbb{P}^1$ is the point in $\mathbb{P}^1$ lying over $\sigma_k(0) \in C_0$. Adding up, we have

$$\begin{aligned}\sum_{i=1}^{g} w(\mathcal{D}_0, q_i) + \sum_{k=1}^{n} w(\mathcal{D}_0, r_i) &\geq rg + \sum_{k=1}^{n} w(\mathcal{D}_b, \sigma_k(b)) \\ &> rg + g - (r+1)(g-d+r) = (r+1)(d-r)\end{aligned}$$

since we assumed that

$$\rho(\mathcal{D}_b; \sigma_1(b), \dots, \sigma_n(b)) = g - (r+1)(g-d+r) - \sum_{k=1}^{n} w(\mathcal{D}_b, \sigma_k(b)) < 0.$$

But as before the Plücker formula for $\mathbb{P}^1$ tells us that

$$\sum_{p \in \mathbb{P}^1} w(\mathcal{D}_0, p) = (r+1)(d-r),$$

a contradiction. □

9.4.2 Brill-Noether with dimension

Theorem 9.4.2 might at first glance seem relevant only to problems involving inflection, but in fact in can be used to prove results that have nothing to do with inflection points. For example, one consequence is the stronger form of Brill-Noether:

Theorem 9.4.3. *If C is a general curve of genus g, then for any d and r with $\rho(g, r, d) \geq 0$,*

$$\dim W^r_d(C) = \rho(g, r, d).$$

Proof. The basic idea of the proof is simple: basically, we argue that if we had a $(\rho+1)$-dimensional family of $g^r_d s$ on C, then we could find one with nonzero ramification at $\rho + 1$ general points of C, violating Theorem 9.4.2.

This idea is easier to implement after specializing, so once more we go back to our family $\mathcal{C} \to B$ of smooth curves specializing to a g-cuspidal curve C_0, with normalization $\mathbb{P}^1$. The basic lemma is:

Lemma 9.4.4. *Let Σ be a complete curve and let $\{\mathcal{D}_\lambda\}_{\lambda \in \Sigma}$ be a (nonconstant) family of $g^r_d s$ on $\mathbb{P}^1$ parametrized by Σ. If $p \in \mathbb{P}^1$ is any fixed point, then for at least one $\lambda \in \Sigma$ we have $w(\mathcal{D}_\lambda, p) > 0$.*

Proof. Embed $\mathbb{P}^1$ in $\mathbb{P}^d$ as a rational normal curve of degree d. Given a $(d-r-1)$-plane $\Lambda \subset \mathbb{P}^d$, the hyperplanes in $\mathbb{P}^d$ containing Λ cut out a g^r_d $\mathcal{D}_\Lambda$ on $\mathbb{P}^1$, and indeed every g^r_d on $\mathbb{P}^1$ can be described in this way for a unique Λ. The g^r_ds on $\mathbb{P}^1$ are thus parametrized by the Grassmannian $\mathbb{G}(d-r-1, d)$, and we can think of Σ as a complete curve in $\mathbb{G}(d-r-1, d)$.

Consider now the hyperplanes $H \subset \mathbb{P}^d$ such that the divisor $H \cap \mathbb{P}^1$ has multiplicity $\geq r+1$ at p. These correspond to points in a linear space of codimension $r+1$ in $(\mathbb{P}^d)^*$; in particular, their intersection is an r-plane $\Omega \subset \mathbb{P}^d$, called the *osculating plane* to the rational normal curve at p. The condition that a g^r_d $\mathcal{D}_\Lambda$ have non-zero ramification at p—in other words, that $\mathcal{D}_\Lambda$ contains a divisor with multiplicity $\geq r+1$ at p—is simply that $\Lambda \cap \Omega \neq \emptyset$. But the set of such Λ is a hyperplane section of $\mathbb{G}(d-r-1, d)$ under trhe Plücker embedding; in particular, any complete curve $\Sigma \subset \mathbb{G}(d-r-1, d)$ must intersect it. □

Given this lemma, the proof of Theorem 9.4.3 proceeds as follows. We know from the basic dimension estimates of Chapter **??** that $\dim W^r_d(C) \geq \rho(g, r, d)$ for any C; we have to show that we cannot have $\dim W^r_d(C) > \rho(g, r, d)$ for a general curve C. We argue as follows:

First: if it were the case that $\dim W^r_d(C) > \rho(g, r, d)$ for a general curve C, we would have, after specializing and pulling back to $\mathbb{P}^1$, at least a $(\rho + 1)$-dimensional family of g^r_ds on $\mathbb{P}^1$, all of which had ramification weight at least r at the points q_i of $\mathbb{P}^1$ lying over the cusps of C_0.

Secondly, we pick any $\rho + 1$ points $r_k \in \mathbb{P}^1$ other than the q_i. Applying Lemma 9.4.4 repeatedly, we find that there is at least a ρ-dimensional subfamily of g^r_ds having nonzero ramification at p_1, a $(\rho - 1)$-dimensional subfamily of g^r_ds having nonzero ramification at p_1 and p_2, and so on; ultimately, we conclude that there is a g^r_d $\mathcal{D}$ on $\mathbb{P}^1$ with ramification index at least r at each q_i and nonzero ramification index at each r_k.

Finally, we observe that the linear series $\mathcal{D}$ has total ramification at least

$$rg + \rho + 1 = (r + 1)(d - r) + 1$$

at the points q_i and r_k, once more violating the Plücker formula. □

We can combine Theorem 9.4.3 and Theorem 9.4.2 into one theorem, more complicated but more inclusive:

Theorem 9.4.5. *Let C be a smooth curve of genus g and $p_1, \dots, p_n \in C$ distinct points; for $k = 1, \dots, n$ let $\alpha^k = (\alpha^k_0, \dots \alpha^k_r)$ be a nondecreasing sequence of nonnegative integers, and let*

$$G^r_d(p_1, \dots, p_n; \alpha^1, \dots, \alpha^n) = \{\mathcal{D} \in G^r_d(D) \mid \alpha_i(\mathcal{D}, p_k) \geq \alpha^k_i\}.$$

If $(C, p_1, \dots, p_n)$ is a general n-pointed curve, then either $G^r_d(p_1, \dots, p_n; \alpha^1, \dots, \alpha^n)$ is empty or

$$\dim G^r_d(p_1, \dots, p_n; \alpha^1, \dots, \alpha^n) = \rho(g, r, d) - \sum_{k+1}^{n} \sum_{i=0}^{r} \alpha^k_i.$$

Finally, we can combine this last theorem with a little dimension-counting to deduce a simple fact:

Theorem 9.4.6. *If $\mathcal{D}$ is a general g^r_d on a general curve, then $\mathcal{D}$ has only simple ramification; that is,*

$$w(\mathcal{D}, p) \leq 1 \quad \textit{for all } p \in C.$$

Note that applying this in case $d = 2g - 2$ and $r = g - 1$, we arrive at the statement made earlier: that a general curve C of genus g has only normal Weierstrass points!

Chapter 10

Rational Normal Scrolls

```
The naming of cats is a difficult matter,
It isn't just one of your everyday games.
You may think that I am as mad as a hatter,
When I tell you each cat must have three different names.
The first is the name that the family use daily ...
But I tell you, a cat needs a name that's particular ...
But above and beyond there's still one name left over,  ...
[his] deep and inscrutable, singular name.


--T.S.Eliot, Practical Cats
```

((Notation in this chapter the field often has to be algebraically closed. I am writing it as $\mathbb{C}$, but this will be easy to replace globally with k if this seems desirable – in which case we should add a blanket assumption. Also we write $\mathbb{P}$ instead of $\mathbb{P}_{\mathbb{C}}$.))

Some of the simplest subvarieties in projective space are the *rational normal scrolls*. They appear in many contexts in algebraic geometry, and are useful, in particular, for describing the embeddings of curves of low degree and genus.

We begin this chapter by giving three different characterizations of these varieties, each useful in a different context: First a classical geometric construction that gives a good picture, then an algebraic description that allows one to "find" the scrolls containing a given variety, and then a more modern geometric definition that makes it easy to understand the divisors on a scroll. Finally, we turn to some of the applications to the embeddings of curves.

In each section we will focus on the 2-dimensional case, both because this is the case that occurs in our applications, and to simplify the discussion. We will also indicate the surprisingly simple extensions to higher dimensions.

In this chapter we will refer to rational normal scrolls simply as scrolls. The third characterization we will give lends itself to a natural generalization to families of irrational ruled varieties, which we'll briefly mention, and the reader should be aware that in the literature the word "scroll" is used for this wider class.

[Notation: $\mathbb{P}^N$ and $\mathbb{P}^1$ mean $\mathbb{P}^N_{\mathbb{C}}$ and $\mathbb{P}^1_{\mathbb{C}}$.]

10.1 Some classical geometry (the name the family use daily)

Recall that the image of the map

$$\mathbb{P}^1 \to \mathbb{P}^a : (s,t) \mapsto (s^a, s^{a-1}t, \dots, t^a),$$

corresponding to the complete linear series $|\mathcal{O}_{\mathbb{P}^1}(a)|$, is called a *rational normal curve*; it is, up to linear transformations, the unique nondegenerate curve of degree a in $\mathbb{P}^a$.

Scrolls are one answer to the question: "What are the higher-dimensional analogues of rational normal curves?" To construct a scroll of dimension 2, we choose integers $0 < a_1, a_2$ and consider a projective space

$$\mathbb{P}^{a_1+a_2+1} = \mathbb{P}_B(\mathbb{C}^{a_1+1} \oplus \mathbb{C}^{a_2+1})$$

with its two subspaces $\mathbb{P}^{a_i} = \mathbb{P}_B(\mathbb{C}^{a_i+1})$. We choose a rational normal curve $C_i \subset \mathbb{P}^{a_i}$, and we choose an isomorphism $\phi : C_1 \to C_2$. We define the scroll $S(a_1, a_2)$ to be the union of the lines

$$S(a_1, a_2) := \bigcup_{p \in C_1} \overline{p, \phi(p)}.$$

We call the curves C_{a_1} and C_{a_2} the *directrices* (singular: directrix) of the scroll, and we call the lines $\overline{p, \phi(p)}$ the *rulings* of the scroll.

It is not hard to prove directly that $S(a_1, a_2)$ is an algebraic variety, but as we shall soon write down its defining equations we will not bother to do so.

From this description we can immediately deduce the dimension and degree of the scroll:

Proposition 10.1.1. *1. $S(a_1, a_2)$ is a nondegenerate surface.*

2. $S(a_1, a_2)$ has degree $a_1 + a_2$, and codimension $a_1 + a_2 - 1$.

Proof. The rational normal curves separately span the spaces $\mathbb{P}^{a_i}$, so a hyperplane containing both of them would contain $\overline{\mathbb{P}^{a_1}, \mathbb{P}^{a_2+1}} = \mathbb{P}$, proving nondegeneracy.

It is clear from our description that S is 2-dimensional, and thus of codimension $a_1 + a_2 + 1 - 2 = a_1 + a_2 - 1$.

To compute the degree, we choose a general hyperplane H containing $\mathbb{P}^{a_1}$. The intersection $H \cap C_2$ consists of a_2 reduced points. Thus the intersection $H \cap S$ consists of C_1 and the a_2 reduced lines connecting the points of $H \cap C_2$ with their corresponding points on C_1; this union has degree $a_1 + a_2$. □

A completely parallel construction creates rational normal scrolls of dimension r. Set $N = \sum_{i=1}^{r}(a_i + 1)$, where each $a_i > 0$ and decompose $\mathbb{C}^N$ as

$$\mathbb{C}^{\sum_{i=1}^{r}(a_i+1)} = \oplus_{i=1}^{r} \mathbb{C}^{a_i+1}.$$

Let $\mathbb{P}^{a_i} \subset \mathbb{P}^N$ be the subspaces corresponding to the summands, choose rational normal curves $C_i \subset \mathbb{P}^{a_i}$ and choose isomorphisms $C_1 \to C_i$. Set

$$S := S(a_1, \dots, a_r) = \bigcup_{p \in C_1} \overline{p, \phi_2(p), \dots, \phi_r(p)}.$$

The variety S is nondegenerate of codimension $N - r$ and degree $\sum_i a_i = N - r + 1$. The proof is similar to the one we gave for $r = 2$. Note the case $r = 1$, in which $S(a)$ is simply the rational normal curve of degree a.

To put this result in context, we recall an elementary fact of projective geometry:

Proposition 10.1.2. *Any nondegenerate variety of codimension c in $\mathbb{P}^N$ has degree $\geq c + 1$.*

Proof. We do induction on c.The case $c = 0$ being trivial, we may assume that $c \geq 1$. A general plane $L \subset \mathbb{P}^N$ meets X in $\deg X$ distinct general points, which must be nonsingular points of X.

Let $p \in L \cap X$ be a point. If every secant to X through p lies entirely in X, then X is a cone over p; but since p was a general point, this would imply that X is a plane, contradicting non-degeneracy.

It follows that the projection $\pi_p : X \to \mathbb{P}^{N-1}$ is a generically finite (rational) map from X to $X' := \pi_p(X)$, and thus $\dim X' = \dim X$ and $\operatorname{codim} X' = \operatorname{codim} X + 1$. The plane $\pi_p(L)$ meets X' in the images of the points of $L \cap X$ other than p, so $\deg X \geq \deg X' + 1$. By induction, $\deg X' \geq \operatorname{codim} X' + 1 = \operatorname{codim} X$, completing the argument. □

Thus scrolls are *varieties of minimal degree.* The reader already knows that the rational normal curves of degree a in $\mathbb{P}^a$ are the only curves of degree a and codimension $a - 1$. A celebrated Theorem of Del Pezzo (for surfaces) and Bertini (in general) generalizes this statement:

Theorem 10.1.3. *Any nondegenerate variety $X \subset \mathbb{P}^N$ with with* $\deg X = \operatorname{codim} X + 1$*, is either a scroll or the Veronese surface in $\mathbb{P}^5$ or a cone over one of these.*

A proof is given in the appendix to this chapter.

One interesting way to view the construction above is that we chose subvarieties $C_i \subset \mathbb{P}^{a_i}$ and a one-to-one correspondence between them, that is, a subscheme $\Gamma \subset \prod_i C_i$ that projects isomorphically onto each C_i; the scroll is then the union of the planes spanned by sets of points $p_i \in C_i$ that are "in correspondence". There are other interesting varieties constructed starting with other choices of subvarieties C_i and subschemes—not necessarily reduced—of $\prod_i C_i$. See [Eisenbud and Sammartano 2019] for an exploration of this idea.

We tend to speak of "the" rational normal scroll rather than "a" rational normal scroll, despite the choices made in the definition, for the following reason:

Proposition 10.1.4. *The scroll $S(a_1, a_2)$ is, up to a linear automorphism of $\mathbb{P}^{a_1+a_2+1}$, independent of the choices made in its definition.*

Proof. To simplify the notation, set $S := S(a_1, a_2)$ and $\mathbb{P} := \mathbb{P}^{a_1+a_2+1}$. To construct S we chose

1. disjoint subspaces $\mathbb{P}^{a_i} \subset \mathbb{P}$;
2. a rational normal curve in each subspace; and
3. an isomorphism between these curves.

Elementary linear algebra shows that there are automorphisms of $\mathbb{P}$ carrying any choice of disjoint subspaces to any other choice. Further, since the rational normal curve of degree a is unique up to an automorphism of $\mathbb{P}^a$, the choice in (2) can be undone by a linear automorphism. Finally, any automorphism of C_{a_2} extends to an automorphism of $\mathbb{P}^{a_2}$, and this extends to an automorphism of $\mathbb{P}$ fixing $\mathbb{P}^{a_1}$ pointwise, showing that $S(a_1, a_2)$ is independent, up to an automorphism of the ambient space, of the choice in (3) as well. □

10.2 1-generic matrices and the equations of scrolls (the name that's particular)

From the definition above it is easy to find the equations of a rational normal scroll. To warm up, we look at the case of the rational normal curve, $S(a) \subset \mathbb{P}^a$. For $a = 1$ there is of course no problem, so we assume $a > 1$. TConsider the expression of the hyperplane bundle $\mathcal{O}_{\mathbb{P}^1}(a)$ as the product

$$\mathcal{O}_{\mathbb{P}^1}(1) \otimes_{\mathbb{P}^1} \mathcal{O}_{\mathbb{P}^1}(a-1) = \mathcal{O}_{\mathbb{P}^1}(a).$$

This leads to the map of vector spaces

$$\mu : H^0\mathcal{O}_{\mathbb{P}^1}(1) \otimes_{\mathbb{C}} H^0\mathcal{O}_{\mathbb{P}^1}(a-1) \to H^0\mathcal{O}_{\mathbb{P}^1}(a),$$

which, in coordinates s, t on $\mathbb{P}^1$, is just the multiplication map

$$< s, t > \otimes_{\mathbb{C}} < s^{a-1}, s^{a-2}t, \ldots, t^{a-1}.$$

We may represent this map as a multiplication table, with matrix

$$\begin{array}{c c}
 & \begin{matrix} s^{a-1} & s^{a-2}t & \ldots & t^{a-1} \end{matrix} \\
\begin{matrix} s \\ t \end{matrix} & \begin{pmatrix} s^a & s^{a-1}t & \ldots & st^{a-1} \\ s^{a-1}t & s^{a-2}t^2 & \ldots & t^a \end{pmatrix}
\end{array}$$

More abstractly, what we have done is to use the equivalence between maps of vector spaces $U \otimes_{\mathbb{C}} V \to W$ and maps $W^* \to Hom_{\mathbb{C}}(V, U^*)$. Because this is a multiplication table, the 2×2 minors of any of the 2×2 submatrices

$$\begin{pmatrix} s^{a-i}t^i & s^{a-j}t^j \\ s^{a-i-1}t^{i+1} & s^{a-j-1}t^{j+1} \end{pmatrix}$$

vanish on $\mathbb{P}^1$. Thus, if we give the parametrization $\mathbb{P}^1 \to \mathbb{P}^a$ of the rational normal curve by

$$x_i = s^{a-i}t^i$$

we see that 2×2 minors of the matrix

$$(*) \qquad M_a = \begin{pmatrix} x_0 & x_1 & \ldots & x_{a-1} \\ x_1 & x_2 & \ldots & x_a \end{pmatrix}$$

vanish on the curve. In fact:

Proposition 10.2.1. *The ideal of forms vanishing on the rational normal curve in $\mathbb{P}^a$ given parametrically by $x_i = s^{a-i}t^i$ is generated by the 2×2 minors of M_a.*

Proof. The ideal of forms vanishing on the rational normal curve is the kernel of the map of graded rings

$$\mathbb{C}[x_0, \ldots, x_a] \twoheadrightarrow \mathbb{C}[s^a, s^{a-1}t, \ldots, t^a],$$

and the map is homogeneous of degree 0 if we take the monomials $s^{a-i}t^i$ in the target to have degree 1. Write $I_2(M_a)$ for the ideal generated by the 2×2 minors of M_a. By what we have seen above, this map factors through $\mathbb{C}[x_0, \ldots, x_a]/I_2(M_a)$.

We claim that, modulo the 2×2 minors, any monomial in the x_i of degree d can be reduced to the form

$$x_0^m x_i^\epsilon x_a^{m-d-\epsilon}$$

for some i, with $\epsilon \in \{0, 1\}$ and $0 \le m \le d + \epsilon$. To see this, Suppose that m is a monomial that cannot be so expressed, so that m contains $z_i z_j$ as a factor, with $1 \le i \le j \le a - 1$. We do induction on i. Since M_a contains the submatrix

$$\begin{pmatrix} z_{i-1} & z_j \\ z_i & z_{j+1} \end{pmatrix}$$

we see that $z_i z_j \equiv z_{i-1} z_{j+1} \mod I$, proving the claim.

Thus the dimension of the d-th graded component of $\mathbb{C}[x_0, \ldots, x_a]/I_2(M_a)$ is at most $ad + 1$. The d-graded component of $\mathbb{C}[s^a, s^{a-1}t, \ldots, t^a]$ is the ad-th graded component of $\mathbb{C}[s, t]$, which has dimension exactly $ad + 1$. Thus the map

$$\mathbb{C}[x_0, \ldots, x_a]/I_2(M_a) \cong \mathbb{C}[s^a, s^{a-1}t, \ldots, t^a]$$

is an isomorphism, as required. □

By a *generalized row* of M_a, we mean a $\mathbb{C}$-linear combination of the given rows of M_a. Note that the points at which the 2×2 minors of M_a vanish are the points at which the evaluations of the two rows are linearly dependent; that is, the points at which some generalized row of M_a vanishes identically. Given Proposition 10.2.1, we see that the points of the rational normal curve are exactly the points where all the linear forms in some generalized row of M_a vanish.

The matrix M_a has a special property: it is 1-generic in the sense below:

Definition 10.2.2. A matrix of linear forms M is said to be *1-generic* if every generalized row of M consists of $\mathbb{C}$-linearly independent forms..

Exercise 10.2.3. Show that a matrix M of linear form is 1-generic iff, even after arbitrary row and column transformations, it's entries are all non-zero.

For example, the matrix

$$M = \begin{pmatrix} x & y \\ z & x \end{pmatrix}$$

over $\mathbb{C}[x, y, z]$ is 1-generic, since if a row and column transformation produced a 0 the determinant would be a product of linear forms, whereas $\det M = x^2 - yz$ is irreducible.

On the other hand, the matrix

$$M' = \begin{pmatrix} x & y \\ -y & x \end{pmatrix}$$

over $\mathbb{C}[x, y]$ is not 1-generic, since

$$\begin{pmatrix} 1 & 0 \\ -i & 1 \end{pmatrix} M' \begin{pmatrix} 1 & 0 \\ i & 1 \end{pmatrix} = \begin{pmatrix} x + iy & 0 \\ 0 & x - iy \end{pmatrix}$$

(but note that it would be 1-generic if we restricted scalars to $\mathbb{R}$—thus the definition depends on the field).

The fact that M_a is 1-generic is most conveniently seen as a special case of the next proposition:

Let X be an irreducible, reduced variety, and suppose that $(\mathcal{L}, V)$ is a linear series. Suppose that there are two linear series $(\mathcal{L}_1, V_1), (\mathcal{L}_2, V_2)$ on X such that $\mathcal{L} = \mathcal{L}_1 \otimes \mathcal{L}_2$ and $V_1 \otimes V_2 \subset V$. If we choose bases $\{u_i\}$ of V_1 and $\{v_j\}$ of V_2 then we may regard $\ell_{i,j} := u_i v_j \in V$ as a linear form on $\mathbb{P}(V)$.

Proposition 10.2.4. *With notation above, the* $\dim V_1 \times \dim V_2$ *matrix* $M := (\ell_{i,j})$ *is 1-generic. Moreover, its* 2×2 *minors are contained in the image of* X *under the linear series* (cL, V).

Proof. The entries of M, after any row and column operations, have the form uv, where u and v are nonzero sections of $\mathcal{L}_1$ and $\mathcal{L}_2$, respectively. Since X is irreducible and reduced, u and v can vanish only on nowhere dense subsets of X, so uv is nonzero. This proves the first statement.

To see that the 2×2 minors of M vanish on X we interpret all the sections of the line bundles $\mathcal{L}_1, \mathcal{L}_2$ and $\mathcal{L}$ as elements of the field of rational functions on X, so a 2×2 minor

$$\det \begin{pmatrix} \ell_{i,j} & \ell_{i,j'} \\ \ell_{i',j} & \ell_{i',j'} \end{pmatrix} = (u_i v_j)(u_{i'} v_{j'}) - (u_i v_{j'})(u_{i'} v_j)$$

vanishes on X by the associativity of multiplication. □

The fact that the matrix M' above is *not* 1-generic can be seen from a more general point of view as well:

Lemma 10.2.5. *There exist 1-generic* $p \times q$ *matrices of linear forms in* $n+1$ *variables over* $\mathbb{C}$ *if and only if* $n \geq p+q$*; In particular, the dimension of the space of linear forms spanned by the* 1×1 *minors of a 1-generic matrix* M *of size* $p \times q$ *is at least* $p+q-1$*. Moreover, if this space of linear forms has dimension* $> p+q-1$*, then the restriction of* M *to a general hyperplane is still 1-generic.*

Proof. If we think of a polynomial ring $\mathbb{C}[z_0, \dots, z_n]$ as the symmetric algebra of a vector space V of rank $n+1$, then we may regard a $p \times q$ matrix of linear forms M as coming from a map $m : \mathbb{C}^p \otimes \mathbb{C}^q \to V$. The matrix is 1-generic if and only if no "pure" tensor $r \otimes s$ goes to zero, that is, iff the kernel K of m intersects the cone of pure tensors only in 0. The cone of pure tensors is the cone over the Segre embedding of $\mathbb{P}^{p-1} \times \mathbb{P}^{q-1}$, an thus has dimension $(p-1)+(q-1)+1$. Thus a general subspace K of codimension $\geq p+q-1$ will intersect the cone only in 0, but any larger subspace K will intersect the cone non-trivially, and the first two statements follow.

Moreover, if K is any space of codimension $> p+q-1$ that intersects the cone only in 0, then the general subspace $K' \supset K$ of dimension one larger still intersects the cone only in 0, proving the last statement. ☐

Note that the idea behind this argument is present in the usual proof of the bound in Clifford's Theorem: if $\mathcal{L}$ is a special line bundle on a curve C then the map

$$\mathrm{H}^0(\mathcal{L}) \otimes \mathrm{H}^0(\mathcal{L}^{-1} \otimes \omega_C) \to \mathrm{H}^0(\omega_C)$$

is 1-generic by Proposition 10.2.4, and thus

$$h^0(\mathcal{L}) + h^1(\mathcal{L}) - 1 \leq g.$$

By Riemann-Roch, $h^1(\mathcal{L}) = h^0(\mathcal{L}) - d + g - 1$, so this last relation becomes $h^0(\mathcal{L}) + (h^0(\mathcal{L}) - d + g - 1) - 1 \leq g$, or $2(h^0(\mathcal{L}) - 1) \leq d$.

We have already seen that the ideal of minors of the 1-generic $2 \times a$ matrix M_a associated to the rational normal curve is a prime ideal of codimension $a-1$ and degree a. This too is part of a more general pattern:

Theorem 10.2.6. *Let M be a 1-generic $2 \times a$ matrix of linear forms on $\mathbb{P}^n$. Let $I = I_2(M)$ be the ideal generated by the 2×2 minors of M.*

1. *The ideal I is prime, and the variety $V = V(I) \subset \mathbb{P}^n$ has degree a and codimension $a-1$.*

2. *We have $n \geq a$. If $n = a$ then the 1-generic matrix M is equivalent up to row and column transformations to the matrix M_a of Proposition 10.2.1; in particular, I is prime and $V(I)$ is a rational normal curve of degree a.*

Proof. 1) The inequality $a \leq n$ was established in Lemma 10.2.5. We will reduce the case $a < n$ to the case $a = n$. Thus we suppose $a < n$, and let ℓ be a general linear form. By induction on n, the image of $I_2(M)$ in $S/\ell \cong \mathbb{C}[x_0, \dots, x_{n-1}$ is prime of codimension $a-1$, and degree a so $V(I_2(M))$ is irreducible of degree a and codimension $a-1$. It remains to show that $I_2(M)$ is prime.

It follows from the induction that the image $\overline{\ell}$ of ℓ in $R := S/I_2(M)$ generates a prime ideal containing the unique minimal prime P of R. Every element $f \in P$ is divisible by ℓ. If $f \in P$ is a minimal generator, then $f = \ell f'$, and since $\ell \notin P$ we must have $f' \in P$, a contradiction. Thus $P = 0$; that is, $I_2(M)$ is prime, as required.

(2) The points of $X = V(I_2(M))$ are the points where some generalized row of M vanish. Since the family of generalized rows is $\mathbb{P}^1$, this variety is at most one-dimensional. If it were 0-dimensional then, since $\mathbb{P}^1$ is irreducible it would have to be a single point—that is, all the generalized rows would be contained in a single vector space of linear forms of dimension n, contradicting Lemma 10.2.5. Thus X is a curve in $\mathbb{P}^N$, and $I_2(M)$ has codimension $n-1 = a-1$.

We now use an important general result from commutative algebra, Theorem 10.2.3, from which we see that $I_2(M)$ is unmixed, and that, even after factoring out two (= dimension $I_2(M)$) general linear forms ℓ_1, ℓ_2, the 2×2 minors of M remain linearly indedpendent. Since the dimension of the space of quadratic forms in $T := S/(\ell_1, \ell_2)$ in $n-1$ variables is $\binom{n}{2}$, the same as the number of minors,we see that the vector space dimension of $\overline{S}/(I_2(M)+(\ell_1,\ell_2)$ is $1+(n-1)=n$; thus the curve defined (in the scheme-theoretic sense) by $I_2(M)$ has degree n. As it is nondegenerate in $\mathbb{P}^N$, X_{red} must be the rational normal curve, and since $I_2(M)$ is unmixed it follows that $I_2(M)$ is the whole homogeneous ideal defining the rational normal curve.

Let $\phi : \mathbb{P}^1 \to \mathbb{P}^N$ be the parametrization of this rational normal curve, and let

$$\begin{pmatrix} \ell_{0,0}, \dots, \ell_{0,a-1} \\ \ell_{1,0}, \dots, \ell_{1,a-1} \end{pmatrix}$$

Write $\overline{\ell_{i,j}}$ for the restriction of $\ell_{i,j}$ to $X \cong \mathbb{P}^1$; we thus consider $\overline{\ell_{i,j}}$ as a form of degree a in 2 variables. Rechoosing coordinates on $\mathbb{P}^1$, we may assume that the first row vanishes at the point $\phi(0,1)$, and the second row vanishes at $\phi(1,0)$ so that each $\overline{\ell_{0,j}}$ is divisible by s and each $\overline{\ell_{1,j}}$ is divisible by t. Since the vector space of forms of degree a divisible by s has dimension a, we may, rechoosing coordinates on $\mathbb{P}^n$, assume that $\overline{\ell_{0,i}} = s^{a-i}t^i$. It follows that each $\overline{\ell_{1,i}}$ is divisible by t, and that the restriction to $\mathbb{P}^1$ of the second row is proportional to $(t/s)(s^a, \dots, st^{a-1}$; thus, after multiplying by a scalar, it will become equal to $s^{a-1}t, \dots, t^a$; that is, the matrix M is equivalent under row and column operations to the matrix M_a defined above.

□

Corollary 10.2.7. *Let $a_1, \dots a_r$ be positive integers, and let $N = r - 1 + sum_{i=1}^r a_i$. The ideal of $S(a_1, \dots, a_r) \subset \mathbb{P}^N$ is generated by the 2×2 minors of the matrix*

$$M = \begin{pmatrix} x_{1,0} & x_{1,1} & \dots & x_{1,a_1-1} & | & x_{2,0} & \dots & x_{2,a_1-1} & | & \dots & | & x_{r,0} & \dots & x_{r,a_1-1} \\ x_{1,1} & x_{1,2} & \dots & x_{1,a_1}. & | & x_{2,1} & \dots & x_{2,a_1} & | & \dots & | & x_{r,1} & \dots & x_{r,a_1} \end{pmatrix}$$

Proof. We may think of the matrix M as consisting of r blocks, M_{a_i} of the form (*). These blocks are 1-generic by Proposition 10.2.4. Since they involve distinct variables, it follows that M is 1-generic. Thus by Theorem **??**, the ideal $I_2(M)$ is prime and of codimension $\sum a_i - 1$, as is the ideal of the scroll. Thus it suffices to show that the minors of M vanish on the scroll.

Let C_i be the rational normal curve in the subspace $\mathbb{P}^{a_i} \subset \mathbb{P}^N$. As always, the set $V(M)$ is the union of the linear spaces on which generalized rows of M vanish; and each such space is the space spanned by the points in the curves C_{a_i} corresponding to the part of that row in the block M_{a_i}—that is, $V(I_2(M))$ is the union of the spans of sets of corresponding points on the C_{a_i}, as required. □

More is true:

Cheerful Fact 10.2.1. Every 1-generic matrix of linear forms is equivalent to one of the type shown in Corollary 10.2.7, and thus the minors of any 1-generic matrix defines a scroll or the cone over a scroll.

References. A $2 \times a$ matrix of linear forms in $N + 1$ variables may be though of as a tensor in $\mathbb{C}^2 \otimes \mathbb{C}^a \otimes \mathbb{C}^{N+1}$, or, equivalently, as an $a \times N + 1$ matrix of linear forms in 2 variables. This, in turn is equivalent to a *pencil* (that is, a projective line) in the vector space of scalar $a \times N + 1$ matrices. These were first classified by Kronecker; see [Gantmacher 1959, Theorems *** and ***] for a modern exposition. □

((mention $2 \times n$ matrices in general; and the Kac classification of matrix formats with finite classification problems))

In fact it will be convenient to widen the definition of scrolls to allow cones over scrolls as well: indeed, if we think of a *rational normal curve of degree 0*, then the cone over $S(a_1, \dots, a_r)$with b-dimensional vertex is $S(0, \dots, 0, a_1, \dots a_r)$, where there are b zeros in the sequence. With this in mind, we can say that giving a scroll of codimension $a - 1$ is "the same" as giving a 1-generic $2 \times a$ matrix of linear forms.

There are not many types of varieties of minimal degree:

Cheerful Fact 10.2.2. Suppose that $X \subset \mathbb{P}^N$ is a nondegenerate variety of minimal degree; that is, $\deg X - 1 = \operatorname{codim} X$. Then X is one of the following:

- a quadric hypersurface;
- a (cone over the) Veronese surface $\mathbb{P}^2 \subset \mathbb{P}^5$; or
- a (cone over a) rational normal scroll.

We have used an important result from commutative algebra. We say that a matrix of forms is *homogeneous* if the entries $f_{i,j}$ satisfy

$$\deg f_{i,j} + \deg f_{k,l} = \deg f_{i,l} + \deg f_{k,j} \text{ for all } i, j, k, l \text{ where this makes sense;}$$

that is, if the determinant of each 2×2 submatrix is "naturally" homogeneous.

Cheerful Fact 10.2.3. Every minimal prime over the ideal I of $p \times p$ minors of a homogeneous $p \times q$ matrix forms has codimension $\leq q - p + 1$. If I has codimension $q - p + 1$, then it is unmixed—that is, there are no embedded primes—and the $p \times p$ minors are linearly independent over the ground field.

References.
((I think we're going to prove this in the free res chapter))
[?, Theorem ***] □

10.3 Scrolls as Images of Projective Bundles (the deep and inscrutable name)

Our third description of scrolls is that they are projective space bundles on $\mathbb{P}^1$, embedded by the complete series associated to the tautological line bundle. Before we review the general theory, we establish the facts about scrolls that characterize such bundles. For simplicity we focus on the 2-dimensional case; the case of a higher dimensional scroll is similar. We start from the description of $X := S(a_1, a_2)$ as the vanishing locus of the minors of the matrix

$$M := M_{a_1,a_2} = \begin{pmatrix} x_{1,0} & x_{1,1} & \dots & x_{1,a_1-1} & | & x_{2,0} & \dots & x_{2,a_1-1} \\ x_{1,1} & x_{1,2} & \dots & x_{1,a_1}. & | & x_{2,1} & \dots & x_{2,a_1} \end{pmatrix}$$

of Section 10.2. For $p = (s,t) \in \mathbb{P}^1$ we write R_p for the locus where the linear forms

$$sx_{1,0} + tx_{1,1}, \dots, sx_{2,a_1-1} + tx_{2,a_1}$$

all vanish, so that R_p is a ruling of X in $\mathbb{P}^N$

Proposition 10.3.1. *Let $X = S(a_1, a_2) \subset \mathbb{P}^N$, with $N = a_1 + a_2 + 1$, be a non-singular rational normal scroll. The rullings R_p of X are the preimages of points under a morphism $\pi : X \to \mathbb{P}^1$. Furthermore, the line bundle*

$$\mathcal{L} := \mathcal{O}_{\mathbb{P}^N}(1)|_X$$

restricts to $\mathcal{O}_{\mathbb{P}^1}(1)$ on each $R_p \cong \mathbb{P}^1$, and the pushforward $\mathcal{E} := \pi_*\mathcal{L}$ is isomorphic to $\mathcal{O}_{\mathbb{P}^1}(a_1) \oplus \mathcal{O}_{\mathbb{P}^1}(a_2)$.

Proof. We write M for the matrix M_{a_1,a_2}. Since there are $N+1$ independent entries of M the intersection of R_p with R_q is empty when $p \neq q$, so there is at least a set-theoretic map $X \to \mathbb{P}^1$ sending the points of R_p to p. To see that this is really a morphism, consider the sheaf

$$\mathcal{L} = \operatorname{coker} \phi : \mathcal{O}_{\mathbb{P}^N}(-1)^{a+b} \to \mathcal{O}_{\mathbb{P}^N}^2$$

given by the matrix M. Let p, q be distinct points of $\mathbb{P}^1$ and let $\tilde{p}$ be a point in the ruling L_p. Since L_q is disjoint from L_p, some linear form in the generalized row corresponding to q does not vanish at p. Thus the restriction of M to the point p is equivalent to the matrix

$$\begin{pmatrix} 0 & 0 & \dots & 0 \\ 1 & 0 & \dots & 0 \end{pmatrix}$$

and we see that the fiber of $\mathcal{L}$ at p is $\mathbb{C}$. It follows that $\mathcal{L}$ is a line bundle on X.

The images of the two basis vectors of $\mathcal{O}_{\mathbb{P}^N}^2$ map to two global sections σ_1, σ_2 of $\mathcal{L}$. By the argument above these two sections generated $\mathcal{L}$ locally everywhere on X, and indeed σ_1 fails to generate $\mathcal{L}$ locally precisely at the points where the second row of M vanishes. Thus the linear series defined by these sections corresponds to a morphism to $\mathbb{P}^1$ whose fibers are exactly the rulings of X.

Because L_p is a linear space, the general hyperplane in $\mathbb{P}^N$ meets L_p in a point; that is $\mathcal{O}_{\mathbb{P}^N}(1)$ restricts to $\mathcal{O}_{\mathbb{P}^1}(1)$ as claimed.

Since $\mathcal{L}$ is a line bundle and X is a variety, $\mathcal{L}$ is flat over $\mathbb{P}^1$ and $\mathcal{E}$ is a vector bundle on $\mathbb{P}^1$. Since the restriction of $\mathcal{L}$ to each fiber is $\mathcal{O}_{\mathbb{P}^1}(1)$, which has two global sections, we see that $\mathcal{E}$ has rank 2 Moreover, since $X \subset \mathbb{P}^N$ is non-degenerate, we see that $\mathcal{E}$ has at least $N+1 = a_1 + a_2 + 2$ independent global sections.

Now consider the directrices $C_i := C_{a_i} \subset X$. The restriction $\pi|_{C_i}$ is an isomorphism inverse to the parametrization $\mathbb{P}^1 \to C_i$, and $\mathcal{O}_{\mathbb{P}^N}|_{C_i}$ pulls back to $\mathcal{O}_{\mathbb{P}^1}(a_i)$, so $\pi_*(\mathcal{L}|_{C_i}) = \mathcal{O}_{\mathbb{P}^1}(a_i)$. Thus the maps $\mathcal{L} \to \mathcal{L}|_{C_{a_i}}$ induce maps $\mathcal{E} = \pi_*\mathcal{L} \to \pi_*(\mathcal{L}|_{C_i})$. Putting this together, we get a map

$$\mathcal{E} \to \mathcal{O}_{\mathbb{P}^1}(a_1) \oplus \mathcal{O}_{\mathbb{P}^1}(a_2) =: \mathcal{E}'.$$

Since the spaces spanned by C_1 and C_2 are complementary, this map of rank 2 vector bundles is an inclusion. Since $\mathcal{E}'$ has only $a_1 + a_2 + 2$ independent global sections, and is generated by them, the map is an isomorphism, completing the argument. $\square$

This is a special case of a very general situation, where, among other things, the Picard group is easy to compute, and which we now explain.

Recall that the projective space $\mathbb{P}^n$ may be defined as $\operatorname{Proj}\operatorname{Sym}_{\mathbb{C}}(\mathbb{C}^{n+1})$. The inclusion of rings $\mathbb{C} = \operatorname{Sym}_{\mathbb{C}}(\mathbb{C}^{n+1})_0 \subset \operatorname{Sym}_{\mathbb{C}}(\mathbb{C}^{n+1})$ induces a structure map $\pi : \mathbb{P}^n \to \operatorname{Spec}\mathbb{C}$. The variety $\mathbb{P}^n$ comes equipped with a tautological line bundle $\mathcal{O}_{\mathbb{P}^N}(1)$, which is associated to the graded module $(\operatorname{Sym}_{\mathbb{C}}\mathbb{C}^{n+1})(1)$, and a tautological map

$$\mathbb{C}^{n+1} \otimes \mathcal{O}_{\mathbb{P}^N} = \pi^*(\mathbb{C}^{n+1}) \to \mathcal{O}_{\mathbb{P}^N}(1)$$

that induces an isomorphism on global sections.

Cheerful Fact 10.3.1. In an exactly parallel way, we may make a projective space bundle $\mathbb{P}_B(\mathcal{E})$ over a variety B from a vector bundle $\mathcal{E}$ on B by taking $\mathbb{P}_B(\mathcal{E}) = \operatorname{Proj}\operatorname{Sym}_{\mathcal{O}_B}(\mathcal{E})$. The inclusion of sheaves of rings $\mathcal{O}_B = (\operatorname{Sym}_{\mathcal{O}_B}(\mathcal{E}))_0 \hookrightarrow \operatorname{Sym}_{\mathcal{O}_B}(\mathcal{E})$ induces a structure map $\pi : \mathbb{P}_B(\mathcal{E}) \to B$. If $\mathcal{E}$ has rank $n+1$, then over any closed point $b \in B$ we have $\mathcal{E}_b \cong \mathbb{C}^{n+1}$, and so the fiber $\pi^{-1}(b)$ is $\mathbb{P}^N$. The restriction of $\mathcal{O}_{\mathbb{P}_B(\mathcal{E})}(1)$ to $\pi^{-1}(b)$ is $\mathbb{P}^N$ is $\mathcal{O}_{\mathbb{P}^N}(1)$.

The variety $\mathbb{P}_B(\mathcal{E})$ comes equipped with a tautological line bundle $\mathcal{O}_{\mathbb{P}_B}(\mathcal{E})(1)$, which is associated to the graded module $(\operatorname{Sym}_{\mathcal{O}_B(\mathcal{E})}(1)$, and a tautological map

$$\pi^*(\mathcal{E}) \to \mathcal{O}_{\mathbb{P}_B(\mathcal{E})}(1)$$

that induces an isomorphism on global sections. Furthermore,

$$\pi_*\mathcal{O}_{\mathbb{P}_B(\mathcal{E})}(p)) = \operatorname{Sym}^p(\mathcal{E})$$

for every p.

Thus the pair $(\mathbb{P}_B(\mathcal{E}), \mathcal{O}_{\mathbb{P}_B(\mathcal{E})}(1))$ determines $\mathcal{E}$; but $\mathbb{P}_B(\mathcal{E})$ alone determines $\mathcal{E}$ only up to twisting with a line bundle on B. For example, if $\mathcal{E}$ is itself a line bundle on B, then $\mathbb{P}_B(\mathcal{E}) \cong B$, but $\mathcal{O}_{\mathbb{P}_B(\mathcal{E})}(1)) \cong \mathcal{E}$.

Conversely, if $\pi : X \to B$ is a map whose fibers are isomorphic to $\mathbb{P}^N$, and if X carries a line bundle $\mathcal{L}$ whose restriction to each fiber of π is $\mathcal{O}_{\mathbb{P}^N}(1)$, then $X \cong \mathbb{P}_B(\mathcal{E})$ and $\mathcal{L} \cong \mathcal{O}_{\mathbb{P}_B(\mathcal{E})}(1)$, where $\mathcal{E} = \pi_*(\mathcal{L})$.

Finally, the Picard group, of line bundles on X is $\operatorname{Pic} X \cong \operatorname{Pic} B \oplus \mathbb{Z}h$, where h is the class of the tautological bundle, and the map $\operatorname{Pic} B \to \operatorname{Pic} X$ is pull-back by π.

In particular, since $\operatorname{Pic}(\mathbb{P}^1) = \mathbb{Z}$, we see that the divisor class group of a scroll $S(a_1, a_2)$ is freely generated by the class H of a hyperplane section and the class F of a ruling. The intersection form, is now easy to compute. If C, D are divisor classes on the scroll, we write $C \cdot D \in \mathbb{Z}$ for their intersection number.

Theorem 10.3.2. *Let $X = S(a_1, a_s) \subset \mathbb{P}^N$ be a scroll and let C_{a_i}, for $i = 1, 2$ be the directrices. The divisor class group of X is $\mathbb{Z}F \oplus \mathbb{Z}H$, where F is the class of a fiber of the structure map and H is the hyperplane section. The intersection form is given by*

$$\begin{array}{cc} \cdot & \begin{array}{cc} F & \quad H \end{array} \\ \begin{array}{c} F \\ H \end{array} & \begin{pmatrix} 0 & 1 \\ 1 & a_1 + a_2 \end{pmatrix} \end{array}$$

The canonical class of X is $K_X = -2H + (a + b - 2)F$.

Moreover $C_{a_i} = H - a_j F$, where $\{i, j\} = \{1, 2\}$, so that $F \cdot C_{a_i} = 1, H \cdot C_{a_i} = a_i$, and $C_{a_i}^2 = a_i - a_j$.

Proof. The values in the intersection matrix follow at once because any two fibers are disjoint straight lines, meeting a general hyperplane transversely in single points, and H^2 is the degree of $X \subset \mathbb{P}^N$.

Finally, if we choose a general hyperplane containing C_{a_1} then it meets C_{a_2} in a_2 points, so $H \cap X$ consists of C_{a_1} plus a_2 fibers, proving the formula $C_{a_i} = H - a_j F$. The self-intersection formulas follow.

Finally, the canonical class K_X must have the form $pH + qF$ for some integers p, q. By the adjunction formula, $(F + K_X).F = -2$, whence $p = -2$. But also $(C_a + K_X)C_a = -2$, yielding $q = a + b - 2$. ☐

Note that if $a_1 < a_2$, then $C_{a_1}^2 = a_1 - a_2$ is negative. We shall see that this is the only curve of negative self-intersection on X.

Our interest in scrolls in this book is primarily for the curves that lie on them. The following result tells us where to look:

Theorem 10.3.3. *Let $X = S(a_1, a_2)$ be a scroll of dimension 2 with $a_1 \leq a_2$, and let F, H denote the class of the ruling and the hyperplane section, respectively. There are reduced irreducible curves in the class $D = pH + qF$ if and only if one of the following holds:*

1. *$D \sim F$; that is, $p = 0, q = 1$; or*
2. *$D \sim C_{a_1}$; that is, $p = 1, q = -a_2$; or*
3. *$p \geq 1$ and $D \cdot C_{a_1} \geq 0$; that is, $q \geq -pa_1$.*

In case (3) the linear series $|D|$ is basepoint free, and thus in each case the class contains smooth curves.

Note that in case (1) we have $D^2 = 0$; in case (2) we have $D^2 = a_1 - a_2 \leq 0$ and in case (3) we have $D^2 > 0$. This result follows by identifying the global sections of line bundle $\mathcal{O}_X(D)$:

Theorem 10.3.4. *Suppose that D is a divisor on the scroll $X = S(a_1, a_2)$ with $a_1 \leq a_2$, and set $\mathcal{E} = \mathcal{O}_{\mathbb{P}^1}(a_1) \oplus \mathcal{O}_{\mathbb{P}^1}(a_1)$. If $D \sim pH + qF$, then*

$$\begin{aligned} H^0(\mathcal{O}_X(D)) &= H^0(\mathcal{O}_{\mathbb{P}^1}(q) \otimes \operatorname{Sym}^p \mathcal{E}) \\ &= \oplus_{0 \leq i \leq p} H^0\big(\mathcal{O}_{\mathbb{P}^1}(q + (p-i)a_1 + ia_2)\big). \end{aligned}$$

and $|D|$ is basepoint free iff every summand is nonzero. Thus, numerically,

$$h^0(\mathcal{O}_X(D)) = \sum_{\{i \ \mid \ q+(p-i)a_1+ia_2 \geq 0\}} 1 + (q + (p-i)a_1 + ia_2),$$

and $|D|$ is base point free iff $p \geq 0$ and $q \geq -pa_1$.

Proof. Let $\pi : X \to \mathbb{P}^1$ be the structure map of the projective bundle $X = \mathbb{P}_{\mathbb{P}^1}(\mathcal{E})$. We have $H^0(\mathcal{O}_X(pH + qF)) = H^0(\pi_*(\mathcal{O}_X(pH + qF)))$. Also, We may write $\mathcal{O}_X(pH + qF)$ as $\mathcal{O}_X(p) \otimes \pi^*\mathcal{O}_{\mathbb{P}^1}(q)$, so by the push-pull formula and Fact 10.3.1,

$$\begin{aligned} \pi_*(\mathcal{O}_X(pH + qF)) &= \pi_*(\mathcal{O}_X(p) \otimes \pi^*\mathcal{O}_{\mathbb{P}^1}(q)) \\ &= \pi_*(\mathcal{O}_X(p)) \otimes \mathcal{O}_{\mathbb{P}^1}(q) \\ &= \operatorname{Sym}^p(\mathcal{E}) \otimes \mathcal{O}_{\mathbb{P}^1}(q) \\ &= \big(\oplus_{0 \leq i \leq p} \mathcal{O}_{\mathbb{P}^1}((p-i)a_1 + ia_2)\big) \otimes \mathcal{O}_{\mathbb{P}^1}(q), \end{aligned}$$

and the first formula follows, and we see that every term $H^0(\big(\mathcal{O}_{\mathbb{P}^1}(q+(p-i)a_1 + ia_2)\big))$ is nonzero iff and only if $H^0(\mathcal{O}_{\mathbb{P}^1}(q + pa_1)$ is nonzero iff $q \geq -pa_1$.

To establish the condition for base-point freeness, note that if all the summands are nonzero then there are sections vanishing on C_{a_1} but not C_{a_2}, and vice versa, so the system is base point free. Conversely, if $q < -pa_1$, then

$$D \cdot C_{a_1} = (pH + qF) \cdot (H - a_2F) = p(a_1 + a_2) - pa_1 + q = pa_1 + q < 0.$$

so any effective divisor in the class of D must have a component in common with C_{a_1}. □

Exercise 10.3.5.
((keep this? sketch proof!))
Minimal degree varieties; as the varieties of given degree lying on the maximal number of quadrics.

We can easily compute the degrees and genera of curves that lie on scrolls:

Proposition 10.3.6. *Suppose that $D \sim pH + qF$ is a smooth irreducible curve on $S(a_1, a_2)$ as in Theorem 10.3.3.*

- *The degree of D is $p(a_1 + a_2) + q$.*
- *The genus of D is $\binom{p}{2}(a_1 + a_2) + (p-1)(q-1)$.*

Proof. The degree of D is $H \cdot D$, yielding the given formula. Let g be the genus of D. By the adjunction formula

$$\begin{aligned} 2g - 2 = \big((p-2)H + &(q + a_1 + a_2 - 2)F\big) \cdot (pH + qF) \\ &= (p^2 - p)(a_1 + a_2) + 2(pq - p - q) \end{aligned}$$

so $g = \binom{p}{2}(a_1 + a_2) + (p-1)(q-1)$ as required. □

Cheerful Fact 10.3.2. A general curve C of genus ≥ 22 does not lie on any 2-dimensional scroll.

Proof. Except when $D \sim C_{a_1}$, a rational curve of negative self-intersection, every nonsingular curve on X moves in a non-trivial linear series. However the moduli space of curves of genus ≥ 22 is of general type, and this implies in particular that there there is no nontrivial rational family of curves containing a general curve of genus ≥ 22. But if the linear series containing C had all nonsingular fibers isomorphic to C, then X would be birationally isomorphic to $C \times \mathbb{P}^1$, and thus not rational, a contradiction. □

10.4 Smooth curves on a 2-dimensional scroll

Prove there is one in a given class iff it meets the directrix non-negatively.

10.5 Automorphisms

(first compute the intersection form...)
((probably drop this section – seems we don't use it elsewhere.))

10.6 Appendix: Varieties of minimal degree

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On Varieties of Minimal Degree
(A Centennial Account)

DAVID EISENBUD AND JOE HARRIS

Abstract. This note contains a short tour through the folklore surrounding the rational normal scrolls, a general technique for finding such scrolls containing a given projective variety, and a new proof of the Del Pezzo–Bertini theorem classifying the varieties of minimal degree, which relies on a general description of the divisors on scrolls rather than on the usual enumeration of low-dimensional special cases and which works smoothly in all characteristics.

Introduction. Throughout, we work over an algebraically closed field k of arbitrary characteristic with subschemes $X \subset \mathbf{P}^r_k$. We say that X is a *variety* if it is reduced and irreducible, and that it is *nondegenerate* if it is not contained in a hyperplane. There is an elementary lower bound for the degree of such a variety:

PROPOSITION 0. *If $X \subset \mathbf{P}^r$ is a nondegenerate variety, then* $\deg X \geq 1 + \operatorname{codim} X$.

(PROOF. If $\operatorname{codim} X = 1$ the result is trivial. Else we project to $\mathbf{P}^{r-1}$ from a general point of X, reducing the degree by at least 1 and the codimension by 1, and are done by induction. $\square$)

We say that $X \subset \mathbf{P}^r$ is a *variety of minimal degree* if X is nondegenerate and $\deg X = 1 + \operatorname{codim} X$. One hundred years ago Del Pezzo (1886) gave a remarkable classification for surfaces of minimal degree, and Bertini (1907) showed how to deduce a similar classification for varieties of any dimension. Of course the case of codimension 1 is trivial, X being then a quadric hypersurface, classified by its dimension and that of its singular locus. In other cases we may phrase the result as:

THEOREM 1. *If $X \subset \mathbf{P}^r$ is a variety of minimal degree, then X is a cone over a smooth such variety. If X is smooth and* $\operatorname{codim} X > 1$, *then $X \subset \mathbf{P}^r$ is either a rational normal scroll or the Veronese surface $\mathbf{P}^2 \subset \mathbf{P}^5$.*

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(See §1 for the definition and some properties of rational normal scrolls.)

The purpose of this note is to give a short and direct proof of the Del Pezzo–Bertini theorem, valid in any characteristic. The proofs (Bertini (1907), Harris (1981), and Xambò (1981)) are all essentially similar: they treat first the cases of surfaces in general (which is also done in Nagata (1960) and Griffiths-Harris (1978)), and finally they reduce the case of arbitrary varieties to the case of surfaces, distinguishing according to whether the general 2-dimensional plane section of the given variety is a scroll or the Veronese surface. Instead, we base our discussion on the following general result (§2), which is useful in many other circumstances:

THEOREM 2. *Let $X \subset \mathbf{P}^r$ be a linearly normal variety, and $D \subset X$ a divisor. If D moves in a pencil $\{D_\lambda | \lambda \in \mathbf{P}^1\}$ of linearly equivalent divisors, then writing $\overline{D}_\lambda$ for the linear span of D_λ in $\mathbf{P}^r$, the variety*

$$S = \bigcup_\lambda \overline{D}_\lambda$$

is a rational normal scroll.

This allows us (in §3) to write an arbitrary variety X of minimal degree as a divisor on a scroll, and simple considerations on the geometry of scrolls then lead to the result.

1. Description of the varieties of minimal degree. We first explain some of the terms used in Theorems 1 and 2 above:

If $L \subset \mathbf{P}^{r+s+1}$ is a linear space of dimension s, $p_L: \mathbf{P}^{r+s+1} \to \mathbf{P}^r$ is the projection from L, and X is a variety in $\mathbf{P}^r$, then the cone over X is the closure of $p_L^{-1}X$. In equations, the cone is simply given by the same equations as X, written in the appropriate subset of the coordinates on $\mathbf{P}^{r+s+1}$. Thus a *cone in* $\mathbf{P}^r$ *over the Veronese surface* $\mathbf{P}^2 \hookrightarrow \mathbf{P}^5$ may be defined as a variety given, with respect to suitable coordinates $x_0, \ldots, x_r$, by the (prime) ideal of 2×2 minors of the generic symmetric matrix:

$$\begin{pmatrix} X_0 & X_1 & X_2 \\ X_1 & X_3 & X_4 \\ X_2 & X_4 & X_5 \end{pmatrix}$$

(It is easy to see that a cone over any variety of minimal degree has minimal degree; our definition of rational normal scroll is such that the cone over a rational normal scroll is another rational normal scroll.)

Note that the Veronese surface contains no lines—indeed, any curve that lies on it must have even degree, as one sees by pulling back to $\mathbf{P}^2$—and thus a cone over the Veronese surface cannot contain a linear space of codimension 1. We shall see that this property separates the varieties of minimal degree which are cones over the Veronese surface from those that are scrolls.

We now describe rational normal scrolls in the terms necessary for Theorem 1. In our proof of the theorem we reduce rapidly to the case where X is a divisor on a scroll, and we shall describe these as well.

A rational normal scroll is a cone over a smooth linearly normal variety fibered over $\mathbf{P}^1$ by linear spaces; in particular, a rational normal scroll contains a pencil of linear spaces of codimension 1 (and these are the only linearly normal varieties with this property, as will follow from Proposition 2.1, below).

To be more explicit, think of $\mathbf{P}^r$ as the space of 1-quotients of k^{r+1}, so that a d-plane in $\mathbf{P}^r$ corresponds to a $d+1$-quotient of k^{r+1}. A variety $X \subset \mathbf{P}^r$ with a map $\pi: X \to \mathbf{P}^1$ whose fibers are d-planes is thus the projectivization of a rank $d+1$ vector bundle on $\mathbf{P}^1$ which is a quotient of

$$k^{r+1} \otimes_k \mathcal{O}_{\mathbf{P}^1} = \mathcal{O}_{\mathbf{P}^1}^{r+1}.$$

Slightly more generally, let

$$\mathcal{E} = \bigoplus_0^d \mathcal{O}_{\mathbf{P}^1}(a_i)$$

be a vector bundle on $\mathbf{P}^1$, and assume

$$0 \le a_0 \le \cdots \le a_d, \quad \text{with } a_d > 0,$$

so that $\mathcal{E}$ is generated by $\sum a_i + d + 1$ global sections. Write $\mathbf{P}(\mathcal{E})$, or alternately $\mathbf{P}(a_0, \ldots, a_d)$, for the projectivized vector bundle

$$\mathbf{P}(\mathcal{E}) = \operatorname{Proj} \operatorname{Sym} \mathcal{E} \to \mathbf{P}^1$$

(whose points over $\lambda \in \mathbf{P}^1$ are quotients $\mathcal{E}_\lambda \to k(\lambda)$), and let $\mathcal{O}_{\mathbf{P}(\mathcal{E})}(1)$ be the tautological line bundle. Because the a_i are ≥ 0, $\mathcal{O}_{\mathbf{P}(\mathcal{E})}(1)$ is generated by its global sections (see the computation below) and defines a "tautological" map

$$\mathbf{P}(\mathcal{E}) \to \mathbf{P}^{\sum a_i + d}.$$

This map is birational because $a_d > 0$. We write $S(\mathcal{E})$ or $S(a_0, \ldots, a_d)$ for the image of this map, which, as we shall see, is a variety of dimension $d+1$ and degree $\sum a_i$, so that it is a variety of minimal degree. A *rational normal scroll* is simply one of the varieties $S(\mathcal{E})$. Note that $\mathcal{O}_{\mathbf{P}(\mathcal{E})}(1)$ induces $\mathcal{O}_{\mathbf{P}^d}(1)$ on each fiber $F \cong \mathbf{P}^d$ of $\mathbf{P}(\mathcal{E}) \to P^1$, so F is mapped isomorphically to a d-plane in $S(\mathcal{E})$.

The most familiar examples of rational normal scrolls are probably

(o) $\mathbf{P}^d$, which is $S(0, \ldots, 0, 1)$,

(i) the rational normal curve of degree a in $\mathbf{P}^a$, which is $S(a)$,

(ii) the cone over a plane conic, $S(0,2) \subset \mathbf{P}^3$,

(iii) the nonsingular quadric in $\mathbf{P}^3$, $S(1,1)$,

(iv) the projective plane blown up at one point, embedded as a surface of degree 3 in $\mathbf{P}^4$ by the series of conics in the plane passing through the point; this is $S(1,2)$.

There is a pretty geometric description of $S(a_0, \ldots, a_d)$ from which the name "scroll" derives, and from which the equivalence of the two definitions above may be deduced:

The projection

$$\mathcal{E} = \bigoplus_0^d \mathcal{O}(a_i) \to \mathcal{O}(a_i)$$

defines a section $\mathbf{P}^1 \cong \mathbf{P}(\mathcal{O}(a_i)) \hookrightarrow \mathbf{P}(\mathcal{E})$, and

$$\mathcal{O}_{\mathbf{P}(\mathcal{E})}(1)|_{\mathbf{P}(\mathcal{O}(a_i))} = \mathcal{O}_{\mathbf{P}(\mathcal{O}(a_i))}(1) = \mathcal{O}_{\mathbf{P}^1}(a_i),$$

so this section is mapped to a rational normal curve of degree a_i in the $\mathbf{P}^{a_i} \subset \mathbf{P}^{\sum a_i + d}$ corresponding to the quotient $H^0(\mathcal{E}) \to H^0(\mathcal{O}_{\mathbf{P}^1}(a_i))$. (Of course if $a_i = 0$, the "rational normal curve of degree a_i" is a point $\subset \mathbf{P}^0$!) Thus we may construct the rational normal scroll $S(a_0, \ldots, a_d) \subset \mathbf{P}^{\sum a_i + d}$ by considering the parametrized rational normal curves

$$\mathbf{P}^1 \xrightarrow{\phi_i} C_{a_i} \subset \mathbf{P}^{a_i} \subset \mathbf{P}^{\sum a_i + d}$$

corresponding to the decomposition

$$k^{\sum a_i + d + 1} = \bigoplus_0^d k^{a_i + 1},$$

and letting $S(a_0, \ldots, a_d)$ be the union over $\lambda \subset \mathbf{P}^1$ of the d-planes spanned by $\phi_0(\lambda), \ldots, \phi_d(\lambda)$. In particular, we see that the cone in $\mathbf{P}^{\sum a_i + d + s}$ over $S(a_0, \ldots, a_d)$ is

$$S(\underbrace{0, \ldots, 0}_{s}, a_0, \ldots, a_d).$$

Also, $S(a_0, \ldots, a_d)$ is nonsingular iff $(a_0, \ldots, a_d) = (0, \ldots, 0, 1)$ or $a_i > 0$ for all i.

We note that this description is convenient for giving the homogeneous ideal of $S(a_0, \ldots, a_d)$. As is well known, the homogeneous ideal of a rational normal curve $S(a) \subset \mathbf{P}^a$ may be written as the ideal of 2×2 minors

$$\det{}_2 \begin{pmatrix} X_0, X_1, \ldots, X_{a-1} \\ X_1, X_2, \ldots, X_a \end{pmatrix},$$

and this expression gives the parametrization sending $(s, t) \in \mathbf{P}^1$ to the point of $\mathbf{P}^a$ where the linear forms

$$sX_0 + tX_1, \ldots, sX_{a-1} + tX_a$$

all vanish. (This is s times the first row of the given matrix plus t times the second row.) It follows at once that $S(a_0, \ldots, a_d)$ is at least set-theoretically the locus where the minors of a matrix of the form

$$\begin{pmatrix} X_{0,0}X_{0,1}, \ldots, X_{0,a_0-1} & | & X_{1,0}, \ldots, X_{1,a_1-1} & | & \ldots X_{d,a_d-1} \\ & | & & | & \ldots \\ X_{0,1}, X_{0,2}, \ldots, X_{0,a_0} & | & X_{1,1}, \ldots, X_{1,a_1} & | & \ldots X_{d,a_d} \end{pmatrix}$$

all vanish. That these minors generate the whole homogeneous ideal follows easily as in the proof of Lemma 2.1 below.

The divisor class group of a projectivized vector bundle $\mathbf{P}(\mathcal{E})$ over $\mathbf{P}^1$ is easy to describe (Hartshorne (1977), Chapter II, exc. 7.9): Writing H for a divisor in the class determined by $\mathcal{O}_{\mathbf{P}(\mathcal{E})}(1)$, and F for the fiber of $\mathbf{P}(\mathcal{E}) \to \mathbf{P}^1$, the divisor class group may be written (confusing divisors and their classes systematically)

$$\mathbf{Z}H + \mathbf{Z}F.$$

Moreover, the chow ring is given by

$$\mathbf{Z}[F, H]/\left(F^2, H^{d+2}, H^{d+1}F, H^{d+1} - \left(\sum a_i\right) H^d F\right).$$

We shall only need a numerical part of this, giving the degree of a scroll:

$$\operatorname{degree} S(a_0, \dots, a_d) = H^{d+1} = \sum_0^d a_i.$$

The simplest way to understand this is perhaps from the geometric description given above: In $\mathbf{P}^{\sum_0^d a_i + d}$ we may take a hyperplane containing the natural copy of $\mathbf{P}^{\sum_1^d a_i + d - 1}$ and meeting $C_{a_0} \subset \mathbf{P}^{a_0}$ transversely. The hyperplane section is then the union of $S(a_1, \dots, a_d)$ with a_0 copies of F (which is embedded as a d-plane).

It is also easy to compute the cohomology of the line bundles on $\mathbf{P}(\mathcal{E})$. In particular, the tautological map

$$\pi^* \mathcal{E} \to \mathcal{O}_{\mathbf{P}(\mathcal{E})}(1)$$

induces for any integer a a map

$$\operatorname{Sym}_a \mathcal{E} = \pi_* \operatorname{Sym}_a \pi^* \mathcal{E} \to \pi_* \mathcal{O}_{\mathbf{P}\mathcal{E}}(a)$$

and thus for every a, b a map

$$\mathcal{O}_{\mathbf{P}^1}(b) \otimes \operatorname{Sym}_a \mathcal{E} \to \mathcal{O}_{\mathbf{P}^1}(b) \otimes \pi_* \mathcal{O}_{\mathbf{P}\mathcal{E}}(a) \cong \pi_*(\pi^* \mathcal{O}_{\mathbf{P}^1}(b) \otimes \mathcal{O}_{\mathbf{P}\mathcal{E}}(a)),$$

which is an isomorphism, as one easily checks locally. Since π is surjective, π_* induces an isomorphism on global sections, and we see that an element

$$\sigma \in H^0(\pi^* \mathcal{O}_{\mathbf{P}^1}(b) \otimes \mathcal{O}_{\mathbf{P}\mathcal{E}}(a))$$

may be represented as an element of

$$\begin{aligned} H^0(\mathcal{O}_{\mathbf{P}^1}(b) \otimes \operatorname{Sym}_a \mathcal{E}) &= H^0\left(\mathcal{O}_{\mathbf{P}^1}(b) \otimes \sum_{|I|=a} \mathcal{O}_{\mathbf{P}^1}\left(\sum_{i \in I} a_i\right)\right) \\ &= \sum_{|I|=a} H^0\left(\mathcal{O}_{\mathbf{P}}\left(b + \sum_{i \in I} a_i\right)\right), \end{aligned}$$

where the notation $\sum_{|I|=a}$ indicates summation over all collections I consisting of a elements (with repetitions) from $\{0, \dots, d\}$.

From this we may derive a useful representation of divisors in $\mathbf{P}\mathcal{E}$, generalizing the idea of "bihomogeneous forms" in the case of $\mathbf{P}(\mathcal{O}_{\mathbf{P}^1}^{d+1}) = \mathbf{P}^1 \times \mathbf{P}^d$. If we let

$$x_i \in H^0(\pi^* \mathcal{O}_{\mathbf{P}^1}(-a_i) \otimes \mathcal{O}_{\mathbf{P}\mathcal{E}}(1)) = H^0 \mathcal{E}(-a_i)$$

be an element corresponding to a generator of the ith summand

$$\mathcal{O}_{\mathbf{P}^1}(a_i - a_i) = \mathcal{O}_{\mathbf{P}^1} \subset \mathcal{E}(-a_i),$$

and write

$$x^I := \prod_{i\in I} x_i \in H^0\Big[(\operatorname{Sym}_a \mathcal{E})\Big(-\sum_{i\in I} a_i\Big)\Big]$$
$$= H^0\Big(\pi^*\mathcal{O}_{\mathbf{P}^1}\Big(-\sum_{i\in I} a_i\Big)\otimes \mathcal{O}_{\mathbf{P}\mathcal{E}}(a)\Big)$$

for the product, then we may represent σ conveniently as a "polynomial":

$$\sigma = \sum_{|I|=a} \alpha_I(s,t)x^I,$$

where s, t are homogeneous coordinates on $\mathbf{P}^1$ and where $\alpha_I(s,t)$ is a homogeneous form of degree

$$\deg \alpha_I(s,t) = b + \sum_{i\in|I|} a_i.$$

This representation is convenient because the "variables"

$$x_i \in H^0(\pi^*\mathcal{O}_{\mathbf{P}^1}(-a_i)\otimes \mathcal{O}_{\mathbf{P}\mathcal{E}}(1))$$

restrict to a basis of the linear forms on each fiber of $\mathbf{P}\mathcal{E} \to \mathbf{P}^1$, and the divisor D of σ meets the $\mathbf{P}^d \cong F_{(u,v)}$ over $(u,v)\in \mathbf{P}^1$ in the hypersurface with equation $\sum_{|I|=a}\alpha_I(u,v)x^I$.

In practice, we wish to use this idea on a Weil divisor X of a scroll $S(\mathcal{E})$. Since $S(\mathcal{E})$ is normal and $\mathbf{P}\mathcal{E}\to S\mathcal{E}$ is birational, we may do this by defining $\tilde{X}\subset \mathbf{P}\mathcal{E}$ to be the "strict transform" of X—that is, for an irreducible subvariety X of codimension 1, $\tilde{X}$ is the closure of the image in $\mathbf{P}\mathcal{E}$ of the complement, in X, of the fundamental locus of the inverse rational map, $S(\mathcal{E})\to \mathbf{P}\mathcal{E}$. Then $\tilde{X}$ occupies a well-defined divisor class on $\mathbf{P}\mathcal{E}$, and we may apply the above technique to it.

2. Rational normal scrolls in the wild. The proof of Theorem 2 rests on a technique of constructing scrolls from their determinantal equations, as follows:

We say that a map of k-vector spaces

$$\phi: U\otimes V \to W$$

is *nondegenerate* if $\phi(u\otimes v)\neq 0$ whenever $u, v\neq 0$, or equivalently if each map $\phi_u: u\otimes V\to W$ is a monomorphism. The typical example, for our purposes, comes from a (reduced, irreducible) variety X and a pair of line bundles $\mathcal{L}$, $\mathcal{M}$; if $U = H^0(\mathcal{L})$, $V = H^0(\mathcal{M})$, and $W = H^0(\mathcal{L}\otimes\mathcal{M})$, then the multiplication map is obviously nondegenerate in the above sense. In our application, X will be embedded linearly normally in $\mathbf{P}^r$ by $\mathcal{L}\otimes\mathcal{M}$, so we may identify $H^0(\mathcal{L}\otimes\mathcal{M})$ with $H^0(\mathcal{O}_{\mathbf{P}^r}(1))$.

In general, given any map

$$k^\gamma\otimes k^\delta \to H^0\mathcal{O}_{\mathbf{P}^r}(1),$$

we define an associated map of sheaves

$$A_\phi: \mathcal{O}^\delta_{\mathbf{P}^r}(-1)\to \mathcal{O}^\gamma_{\mathbf{P}^r}$$

by twisting the obvious map

$$k^{\delta} \otimes \mathcal{O}_{\mathbf{P}^r} \to k^{\gamma^*} \otimes \mathcal{O}_{\mathbf{P}^r}(1)$$

by $\mathcal{O}_{\mathbf{P}^r}(-1)$. Taking $\gamma = 2$, we have

LEMMA 2.1. *If $\phi: k^2 \otimes k^{\delta} \to H^0\mathcal{O}_{\mathbf{P}^r}(1)$ is a nondegenerate pairing, then the ideal of 2×2 minors $\det_2 A_\phi$ is prime, and $V(\det_2 A_\phi)$ is a rational normal scroll of degree δ.*

PROOF. If the image of ϕ is a proper subspace of $H^0\mathcal{O}_{\mathbf{P}^r}(1)$, then $V(\det_2 A_\phi)$ is a cone. Since the cone over a scroll is a scroll, we may by reducing r assume that ϕ is an epimorphism, so that the rank of A_ϕ never drops to 0 on $\mathbf{P}^r$. It follows that $\mathcal{L} = \operatorname{Coker} A_\phi$ is a line bundle on $S = V(\det_2 A_\phi)$, generated by the image of $V = k^{2^*}$. The linear series $(\mathcal{L}, V)$ defines a map $\pi: S \to \mathbf{P}^1$. If $(s,t) \in \mathbf{P}^1$, then the fiber F of π over (s,t) is the scheme defined by the vanishing of the composite map

$$\mathcal{O}^{\delta}_{\mathbf{P}^r}(-1) \to \mathcal{O}^2_{\mathbf{P}^r} \xrightarrow{(s,t)} \mathcal{O}_{\mathbf{P}^r};$$

and this scheme is, by our nondegeneracy hypothesis, given by the vanishing of δ linearly independent linear forms, so F is a plane of codimension δ. By the general formula for the maximum codimension of (any component of) a determinantal variety we have $\operatorname{codim} S \le \delta - 1$, so the map $S \to \mathbf{P}^1$ is onto, and since the fibers are smooth and irreducible, and the map is proper, S is smooth and irreducible of codimension $\delta - 1$.

Since $\det_2 A_\phi$ thus has height $\delta - 1$ in the homogeneous coordinate ring of $\mathbf{P}^r$, it is perfect, and in particular unmixed (Arbarello et al. (1984), Chapter II, 4.1; note that the characteristic 0 hypothesis there is irrelevant). Thus $\det_2 A_\phi$ is the entire homogeneous ideal of S, and since $\det_2 A_\phi$ is perfect, S is arithmetically Cohen-Macaulay, so in particular S is linearly normal.

The fibers of π, being linear spaces in $\mathbf{P}^r$, correspond to quotients of k^{r+1}, and this defines a vector bundle on $\mathbf{P}^1$ of rank $r - \delta + 1$ such that $S \to \mathbf{P}^1$ is the associated projective space bundle; thus S is a rational normal scroll as claimed. $\square$

REMARK. Using the same ideas, one sees that the height of $\det_2(A_\phi)$ is $\delta - 1$ iff the rank of ϕ_u never drops by more than 1; then $X = V(\det_2 A_\phi)$ is a "crown", that is, the union of a scroll of codimension $\delta - 1$ and some linear spaces of codimension $\delta - 1$ which intersect the scroll along linear spaces of codimension δ (fibers of π)—see Xambò (1981).

With this result in hand, it is easy to complete the proof of Theorem 2:

PROOF OF THEOREM 2. Let $k^2 \cong V \subset H^0\mathcal{O}_X(D)$ be the vector space of sections corresponding to the pencil D_λ, and let H be the hyperplane section of X. The natural multiplication map

$$V \otimes H^0\mathcal{O}_X(H - D) \to H^0\mathcal{O}_X(H) = H^0\mathcal{O}_{\mathbf{P}^r}(1)$$

is nondegenerate, and thus gives rise to a scroll S containing all the D_λ, and thus X. The linear space $\overline{D}_\lambda$ is the intersection of all the hyperplanes containing

D_λ, which correspond to elements of $H^0\mathcal{O}_X(H-D)$, so $\overline{D}_\lambda$ is the fiber over λ of $S \to \mathbf{P}^1$, as desired. $\square$

EXAMPLES. (i) Let C be a hyperelliptic (or elliptic) curve, $C \subset \mathbf{P}^r$ an embedding by a complete series of degree d. C is a divisor on the variety S which is the union of the secants corresponding to the $\mathfrak{g}^1_2$ on C (or, if C is elliptic, any $\mathfrak{g}^1_2$ on C). This variety is a rational normal scroll $S(\mathcal{E})$ and $\tilde{C} \sim 2H + (d-2r+2)F$ on $\mathbf{P}(\mathcal{E})$. More generally, a linearly normal curve $C \subset \mathbf{P}^r$ which possesses a $\mathfrak{g}^1_d$ lies on a scroll of dimension $\leq d$; if $C \subset \mathbf{P}^r$ is the canonical embedding, then this scroll is of dimension $\leq d-1$, so in particular the canonical image of any trigonal curve is a divisor on a 2-dimensional scroll $S(\mathcal{E})$, and $\tilde{C} \sim 3H + (4-g)F$ on $\mathbf{P}(\mathcal{E})$. See Schreyer (1986) for a study of canonical curves using this idea.

(ii) A $K3$ surface, embedded linearly normally in any projective space, is a divisor on a 3-dimensional scroll if it contains an elliptic cubic (which then moves in a nontrivial linear series). See for example Saint-Donat (1974).

3. The classification theorem. Before giving our proof of the Del Pezzo–Bertini Theorem, we record three elementary observations about projections:

(1) If X is a variety of minimal degree, then X is linearly normal. (*Proof*: If X were the isomorphic projection of a nondegenerate variety X' in $\mathbf{P}^{r+1}$, then X' would have degree less than that allowed by Proposition 0.)

(2) If $X \subset \mathbf{P}^r$ is a variety of minimal degree and $p \in X$, then the projection $\pi_p X \subset \mathbf{P}^{r-1}$ is a variety of minimal degree, the map $X - p \to \pi_p X$ is separable, and if p is singular then X is a cone with vertex p. (*Proof*: Indeed, $\pi_p X$ is obviously nondegenerate. If X is a cone with vertex p, the result is obvious. Else $\dim \pi_p X = \dim X$ but $\deg \pi_p X \leq \deg X - 1$. The inequality must actually be an equality by Proposition 0, which shows in particular that p is a nonsingular point, and $\pi_p: X - p \to \pi_p X$ is birational.)

(3) If $p \in X \subset \mathbf{P}^r$ is any point on any variety, E_X the exceptional fiber of the blow-up of p in X, and $E_{\mathbf{P}^r} \cong \mathbf{P}^{r-1}$ the exceptional fiber of the blow-up of $\mathbf{P}^r$ at p, then E_X is naturally embedded in $E_{\mathbf{P}^r}$, which is mapped isomorphically to $\mathbf{P}^{r-1}$ by the map induced by π_p. Thus $E_X \subset \pi_p(X) \subset \mathbf{P}^{r-1}$. In particular, if p is a nonsingular point on X, so that E_X is a linear subspace of $\mathbf{P}^{r-1}$, then the "image of p" under $\pi_p: X \to \pi_p(X) \subset \mathbf{P}^{r-1}$ is a linear subspace of $\mathbf{P}^{r-1}$ which is a divisor on $\pi_p(X)$. More naively, this is the image of the tangent plane to X at p.

In view of observation (3) it will be useful to begin with the following result, which "recognizes" scrolls:

PROPOSITION 3.1. *If $X \subset \mathbf{P}^r$ is a variety of minimal degree, and X contains a linear subspace of $\mathbf{P}^r$ as a subspace of codimension 1, then X is a scroll.*

PROOF. By Proposition 2.1 it suffices to show that X contains a pencil of linear divisors, though the given subspace itself may not move.

Let $F \subset X$ be the given linear subspace. We may assume (by projecting, if necessary) that X is smooth along F. Let $H \subset \mathbf{P}^r$ be a general hyperplane containing F, and let $S = H \cap X - F$. Let π_F be projection from F. We distinguish two cases:

Case 1. $\dim \pi_F(X) \geq 2$. By Bertini's Theorem and observation (2) above, S is then a reduced and irreducible variety, of degree and dimension one less than that of X. Thus by Proposition 0, S is degenerate in H, so $F = H \cdot X - S$ moves in (at least) a pencil of linear spaces, and we are done.

Case 2. $\dim \pi_F(X) = 1$. By observation (1), $\pi_F(X)$ is a curve of minimal degree, say of degree s in $\mathbf{P}^s$. Projecting $\pi_F(X)$ from $s-1$ general points on it gives a birational map to $\mathbf{P}^1$, so $\pi_F(X) \cong \mathbf{P}^1$. Further, the cone on S with vertex p is a union of s planes, the spans of F with the points of a general hyperplane section of $\pi_F(X)$, so S has s components. But $s = r - \dim F - 1 = \operatorname{codim} X = \deg X - 1 = \deg S$, so S is the union of s planes, and these are linearly equivalent to each other since the points of $\pi_F(X)$ are. Thus a component of S is a linear space moving in a pencil as desired. □

Proof of Theorem 1. Let $X \subset \mathbf{P}^r$ be a variety of minimal degree. We may assume that the codimension c of X is ≥ 2 and that X is not a cone. By Proposition 3.1 we may as well also assume that X contains no linear space of codimension 1, so that in particular the dimension d of X is ≥ 2, and we must prove that under these hypotheses X is the Veronese surface $\mathbf{P}^2 \subset \mathbf{P}^5$. In fact, it suffices to prove that $X \cong \mathbf{P}^2$; for the embedding of $\mathbf{P}^2$ by the complete series of curves of degree d gives a surface of degree d^2 and codimension

$$\binom{2+d}{2} - 3,$$

which is $< d^2 - 1$ for $d \geq 3$.

Let $p \in X$ be any point. By observation (3) and Proposition 3.1, $\pi_p(X)$ is a scroll, so the cone $S \subset \mathbf{P}^r$ with vertex p over $\pi_p(X)$ (or over X) is a scroll, say $S = S(\mathcal{E})$, with $\mathcal{E} = \bigoplus_0^d \mathcal{O}_{\mathbf{P}^1}(a_i)$ and $0 \leq a_0 \leq \cdots \leq a_d$. X is a divisor on S.

Consider the strict transform $\tilde{X} \subset \mathbf{P}(\mathcal{E})$ of X under the desingularization $\mathbf{P}(\mathcal{E}) \to S(\mathcal{E}) = S$, and let its divisor class be $aH - bF$. We will prove under the hypotheses above that $a = 2$ and X is a surface. (Along the way we will see numerically that $b = 4$, $(a_0, a_1, a_2) = (0, 1, 2)$, so $c = 3$ and $X \subset \mathbf{P}^5$ as befits the Veronese, but we will not use this directly.)

First, because the degree $c+1$ of X is 1 more than that of S, and on the other hand is $H^{d-1} \cdot (aH - bF)$, we get $b = (a-1)c - 1$.

To bound a, first note that X must meet every fiber of $\mathbf{P}\mathcal{E} \to \mathbf{P}^1$, so $aH - bF|_F = aH|_F > 0$, and $a \geq 1$. If a were 1, then $\tilde{X}$ would meet each fiber F in a linear space of dimension $d-1$. Since each fiber F is mapped isomorphically to a d-plane in $\mathbf{P}^r$ under $\mathbf{P}(\mathcal{E}) \to S(\mathcal{E})$, X would contain linear spaces of dimension $d-1$, contrary to our hypothesis. Thus $a \geq 2$.

As in §2, $\tilde{X}$ may be represented by an equation $g = 0$ with g of the form:

$$g = \sum_{|I|=a} \alpha_I(s,t) x^I,$$

with

$$\deg \alpha_I = \left(\sum_{i\in I} a_i\right) - b = \sum_{i\in I} a_i - (a-1)c + 1.$$

If the variable x_0 did not occur in g, then $\tilde{X}$ would meet each fiber F in a cone over the preimage of p, and X itself would be a cone contrary to hypothesis. But for x_0 to occur we must have

$$0 \le \deg \alpha_{0,d,\dots,d} = a_0 + (a-1)a_d - (a-1)c + 1.$$

Since S is a cone we have $a_0 = 0$, and we derive

$$(*) \qquad a_d \ge c - 1/(a-1).$$

If x_d occurred in every nonzero term of g, then for every fiber F, $\tilde{X} \cap F$ would contain the $(d-1)$-plane $x_d = 0$, and again X would contain a $(d-1)$-plane, contradicting our hypotheses. Thus

$$(**) \qquad 0 \le \deg \alpha_{d-1,d-1,\dots,d-1} = a a_{d-1} - (a-1)c + 1.$$

Now if $a \ge 3$, then $a_d = c$ by $(*)$; but $c = \deg X - 1 = \deg S = \sum_{i=0}^{d} a_i$, so this implies $a_{d-1} = 0$, and $(**)$ gives a contradiction. Thus $a = 2$ as claimed, and $a_d \le c - 1$. Condition $(*)$ now gives $a_d = c - 1$, so $a_{d-1} = 1$ and $a_0 = \cdots = a_{d-2} = 0$. Applying $(**)$ again we get $a_d = 1$ or $a_d = 2$.

In the first case $(a_0, \dots, a_d) = (0, \dots, 0, 1, 1)$, so S is a cone over $\mathbf{P}^1 \times \mathbf{P}^1 \subset \mathbf{P}^3$. A suitable hypersurface section of S will consist of the union of two planes F_1 and F_2, the cones over the rulings of $\mathbf{P}^1 \times \mathbf{P}^1$. Since each of these rulings sweeps out all of $\mathbf{P}^1 \times \mathbf{P}^1$, X must meet each of F_1 and F_2 in codimension 1. Because $c = 2$ we have $\deg X = 3$, so either $X \cap F_1$ or $X \cap F_2$ must be a linear space, contradicting our assumption on X.

We thus see that $a = 2$, $c = 3$, and $(a_0, \dots, a_d) = (0, \dots, 0, 1, 2)$. Under these circumstances the sum of the terms of g involving $x_0, \dots, x_{d-2}$ may be written

$$\left(\sum_0^{d-2} \alpha_{i,d} x_i\right) x_d,$$

with $\alpha_{i,d}$ constant. Thus if $d \ge 3$ the locus $g = 0$ in each fiber F is a cone with vertex the $(d-3)$-dimensional linear space given by

$$x_d = x_{d-1} = \sum_0^{d-2} \alpha_{i,d} x_i = 0.$$

Of course S is itself a cone with $(d-2)$-dimensional vertex L, say. The $(d-2)$-dimensional subspaces of the fibers F given by $x_d = x_{d-1} = 0$ are all mapped isomorphically to L under $\mathbf{P}(\mathcal{E}) \to S$, and the restrictions of the coordinates $x_0, \dots, x_{d-2}$ are all identified, and become coordinates on L. Thus X meets the image of each fiber in a cone with vertex given in L by $\sum_0^{d-2} \alpha_{i,d} x_i = 0$, so X is a cone, contradicting our assumption. This shows $d = 2$.

We have now shown that $a = 2$ and X is a surface. In this case, for every fiber $F \cong \mathbf{P}^2$ of $\mathbf{P}(\mathcal{E})$, $F \cap \tilde{X}$ is a conic, necessarily nonsingular since else X would contain a line. Thus $\tilde{X}$ is a rational ruled surface. But the preimage in X of p is a line, so $\tilde{X}$ is the blow-up of X at p, and is not a minimal surface. This is only possible if $\tilde{X} \cong \mathbf{P}(\mathcal{O}_{\mathbf{P}^1} \oplus \mathcal{O}_{\mathbf{P}^1}(1))$ and $X \cong \mathbf{P}^2$, as required. $\square$

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Chapter 11

Curves on Scrolls

11.1 Hyperelliptic curves

((maybe start a new Chapter?))

In our early encounters with curves, we frequently assumed that the curve we were considering was non-hyperelliptic, since the behavior of hyperelliptic curves is so atypical. In this section, we'll describe the geometry of hyperelliptic curves.

11.1.1 Basic models of hyperelliptic curves

((move this section to ch 2; add discussion of adjoints— perhaps as exercises?))

We start by establishing some basic facts about hyperelliptic curves. Many of these follow from general theorems like Riemann-Roch; but since they can be established by direct examination we will carry that out here.

Suppose C is a smooth, projective hyperelliptic curve of genus $g \geq 2$. By definition, C admits a degree 2 map $\pi : C \to \mathbb{P}^1$; and as we've observed (**??**) this map is unique.

By Riemann-Hurwitz,
((attibution?))
the map $\pi : C \to \mathbb{P}^1$ will have $2g + 2$ distinct simple branch points, say $\lambda_1, \dots, \lambda_{2g-2} \in \mathbb{P}^1$. An open subsect C° of C can then be realized as the

smooth projective completion of the affine curve given as

$$C^\circ = \left\{(x,y) \in \mathbb{A}^2 \;\mid\; y^2 = \prod_{i=1}^{2g+2}(x-\lambda_i)\right\}.$$

((if two of the λ_i coinncide, then the curve develops a singular point. Much of what we will do carries over to the singular case.))

((say the smooth model has 2 points at ∞.))
Note that if we simply take the closure of this locus in $\mathbb{P}^2$, the resulting curve will be highly singular at the point $[1,0,0]$, as can be seen either directly by making an appropriate change of variables, or by invoking the genus formula for plane curves: if the closure were smooth, it would have genus $\binom{2g+1}{2}$. We can, however, complete the curve simply in $\mathbb{P}^1 \times \mathbb{P}^1$, for example by setting
((this is a rabbit from a hat. Consider either saying that by the previous section, if there's an emb in P3 then its on P1 x P1 as a divisor of type 2,g+1; and then "finding" this embedding as below; or moving this page to the early place where hyperelliptic curves are first mentioned.))

$$y' = \frac{y}{\prod_{i=1}^{g+1}(x-\lambda_i)};$$

we can then write the equation of a still smaller open subset of C as

$$y'^2 \cdot \prod_{i=1}^{g+1}(x-\lambda_i) \;=\; \prod_{i=g+2}^{2g+2}(x-\lambda_i).$$

If we now take the closure of this locus in $\mathbb{P}^1 \times \mathbb{P}^1$, we get a curve of type $(2, g+1)$ on $\mathbb{P}^1 \times \mathbb{P}^1$; this curve is smooth, as can be seen again either directly in coordinates or by invoking the genus formula for curves on $\mathbb{P}^1 \times \mathbb{P}^1$. In other words,

$$C \;=\; V\Big(Y_0^2 \cdot \prod_{i=1}^{g+1}(X_1 - \lambda_i X_0) - Y_1^2 \cdot \prod_{i=g+2}^{2g+2}(X_1 - \lambda_i X_0)\Big)$$

Next, let's describe the space of regular differentials on C. For this, it's convenient to work with the affine model $C^\circ = V(f) \subset \mathbb{A}^2$, where

$$f(x,y) = y^2 - \prod_{i=1}^{2g-2}(x-\lambda_i).$$

We'll denote the two points at infinity—that is, the two points of $C \setminus C^\circ$—as p and q.

To start, consider the simple differential $dx \in \Omega_{C^\circ/k}$. This is clearly regular on C°, with zeros at the ramification points $r_i = (\lambda_i, 0)$. But it does not extend to a regular differential on all of C: it will have double poles at p and q, as can be seen either directly or by degree considerations: as we said, dx has $2g+2$ zeros, while the degree of K_C is $2g-2$, meaning that there must be poles at the points p and q.

To kill these poles, we can of course divide by x^2 (or any quadratic polynomial in x). But that just introduces new poles in the finite part C° of C. Instead, we want to multiply dx by a rational function with zeros at p and q, but *whose poles occur only at the points where dx has zeroes*—that is, the points r_i. A natural choice is simply the reciprocal of the partial derivative $f_y = \partial f/\partial y = 2y$, which vanishes exactly at the points r_i, and has correspondingly a pole of order $g+1$ at each of the points p and q (reason: the involution $y \to -y$ fixes C° and x), and exchanges the points p, q. In other words, the differential

$$\omega = \frac{dx}{f_y}$$

is regular, with divisor

$$(\omega) = (g-1)p + (g-1)q.$$

The remaining regular differentials on C are now easy to find: Since x has only a simple pole at the two points at infinity
((say why.))
we can multiply ω by any x^k with $k = 0, 1, \ldots, g-1$. Since this gives us g independent differentials, these form a basis for $H^0(K_C)$.

1) special linear series are mult g^1_2+basepoints. 2) Given an embedding, there's a union of lines. If the embedding is complete, we get a matrix...that defines the union of lines. Scrolls in all dimensions as unions of spans of divisors.

11.1.2 General embeddings of degree genus+3

It's a divisor on a quadric in $\mathbb{P}^3$ of type $(2, g+1)$

11.2 Trigonal curves

11.2.1 Trigonal curves lie on scrolls

The key to our analysis of linear systems on trigonal curves will be the fact that *trigonal curves lie on scrolls*; we'll start by establishing that fact.

There are two ways we might do this: concretely and abstractly. To start with the former, let C be a smooth, projective trigonal curve of genus $g > 2$. By the exercise below, C cannot be hyperelliptic, and so its canonical map embeds C as a canonical curve in $\mathbb{P}^{g-1}$.

Exercise 11.2.1. 1. Show that a curve of genus $g > 2$ cannot be both hyperelliptic and trigonal.

2. Show that a trigonal curve of genus $g > 4$ has a unique g^1_3.

Consider the divisors $\{D_\lambda\}_{\lambda \in \mathbb{P}^1}$ of the g^1_3 on C. By the geometric Riemann-Roch theorem, each consists of three colinear points; and hence any quadric hypersurface Q containing C will contain each of the lines L_λ spanned by these divisors.

Now, let $S \subset \mathbb{P}^{g-1}$ be the surface swept out by these lines. S is clearly an irreducible, nondegenerate surface in $\mathbb{P}^{g-1}$, lying on each of the $\binom{g-2}{2}$ quadrics containing the curve C. But we've seen that the maximum possible number of quadrics containing an irreducible, nondegenerate surface $T \subset \mathbb{P}^n$ is $\binom{n-1}{2}$, and any such surface lying on that many quadrics must be either a rational normal scroll or a Veronese surface. Since S is swept out by lines, and the Veronese surface contains no lines, we conclude that *the canonical model of a trigonal curve lies on a rational normal scroll.*

In fact, we can describe the scroll S directly: if we consider the product map

$$H^0(\mathcal{O}_C(D)) \otimes H^0(K_C(-D)) \to H^0(K),$$

the surface S is simply the rank 1 locus of the transpose map

$$H^0(K)^* \to \mathrm{Hom}\left((H^0(\mathcal{O}_C(D)), H^0(K_C(-D))^*\right).$$

There is another, more abstract way of describing the scroll S. Let $\pi : C \to \mathbb{P}^1$ be the map associated to the g^1_3 on C. We take the direct image $E = \pi_*\mathcal{O}_C$ of the structure sheaf of C; this is a vector bundle of rank 3 on $\mathbb{P}^1$. There is a natural inclusion of the structure sheaf $\mathcal{O}_{\mathbb{P}^1}$ in E, and if we let $F = E/\mathcal{O}_{\mathbb{P}^1}$ be the quotient, we have a natural embedding

$$C \hookrightarrow \mathbb{P}F;$$

the projectivization of $\mathbb{P}F$ is the scroll S.

11.2.2 Which scrolls?

There is a natural follow-up question: given that a trigonal curve C has a natural inclusion in a rational normal scroll, we can, "which one?" The following lemma gives the answer:

Lemma 11.2.2. *Let C be a trigonal curve of genus g, and let $S \cong \mathbb{F}_n$ be the scroll associated to C as above.*

1. *$n \equiv g$ mod(2); and*
2. *If C is general, then either $n = 0$ (if g is even) or $n = 1$ (if g is odd).*

11.2.3 The Maroni inviariant

Theorem 11.2.3. *There is a smooth canonical curve on a rational normal scroll S in $\mathbb{P}^{g-1}$ iff the self-intersection of the directrix on S is equivalent to $H - mR$ where $m \leq (2g-2)/3$.*

11.2.4 Special linear series on trigonal curves

In analyzing special linear series on a hyperelliptic curve, we made crucial use of the facts that the canonical image of a hyperelliptic curve is a rational normal curve, and that any collection of points on a rational normal curve $C \subset \mathbb{P}^n$ either are linearly independent or span $\mathbb{P}^n$. In a similar (though necessarily less complete) way, we can use the fact that the canonical image of a trigonal curve lies on a rational normal surface scroll to describe special linear series on it.

Theorem 11.2.4. *Let L be a special line bundle on a trigonal, non-hyperelliptic curve, and suppose that $h^0(L) \geq 2$ and $h^1(L) \geq 2$. Then either contains a g^1_3 or is contained in K minus a g^1_3. If $g \geq 5$ then both conditions must hold.*

((can we give a necessary and sufficient condition?))

Proof. Consider the canonical embedding of a trigonal curve C, and a general divisor D of a base-point free special linear series. Since D is special, it lies in. a hyperplane, and since D moves, the Geometric Riemann-Roch Theorem **??** shows that D spans a space of dimension $< \deg D - 1$.

First suppose that $g \geq 4$. Since C lies on a rational normal scroll, D lies on a hyperplane section C' of the scroll. If C; is irreducible, then it is a rational normal curve; but divisors on rational normal curves are always linearly independent. Thus C' must be reducible. By Lemma 11.2.5, C' consists of lines of the ruling together with a rational normal curve C'' of lower degree, and thus embedded in a lower-dimensional plane. Since the points of D are dependent, D must have at least 3 points on one of the rulings (and thus contains a g^1_3) or at least 3 points on C'', in which case it is contained in K minus a g^1_3.

If $g \geq 5$ then

In the case $g = 3$, C is a smooth plane curve, and the points of D must lie on a line. Since the lines through any point of C cut out a g^1_3, we see that D is contained in a g^1_3. ☐

Lemma 11.2.5. *Let $S = S_{a,b} \subset \mathbb{P}^n$ be a rational normal surface scroll. Any hyperplane section $H \cap S$ consists of the union of a rational normal curve E, which is a section of the scroll, and a union of lines of the ruling of the scroll.*

Note that the curve E must be a reduced component of $S \cap H$, but the lines L_i may coincide, i.e., may be non-reduced components of the intersection. In the following proof, we'll assume for clarity that the lines L_i are distinct (that is, $S \cap H$ is reduced); we leave it as an exercise to rewrite the proof to accommodate the remaining cases.

Proof. Let $F \in \mathrm{Pic}(S)$ be the class of a line of the ruling. Since $F^2 = 0$ and $H \cdot F = 1$, exactly one of the components of $S \cap H$ must have intersection number 1 with F; all other components must have intersection number 0 with F and so must be lines of the ruling.

It remains to show that the unique component E of $H \cap S$ having intersection number 1 with F is a rational normal curve. This can be seen directly, but there's a shortcut. Suppose that we have

$$S \cap H = E \cup L_1 + \cdots + L_k,$$

so that in particular $\deg(E) = n - 1 - k$. Since each of the lines L_i of the ruling must meet C, we have that

$$\begin{aligned} n - 1 &= \dim(\overline{S \cap H}) \\ &\leq \dim(\overline{E}) + k \\ &\leq (n - 1 - k) + k \\ &= n - 1. \end{aligned}$$

We conclude that $\dim(\overline{E}) = n - k - 1$, and hence that E is a rational normal curve. $\square$

Note that if $S = S_{a,b}$ with $a \leq b$, we must have either $0 \leq k \leq a$ or $k = b$: as soon as $k > a$, the span of the lines L_i will contain the directrix of the scroll, and so must consist of the union of the directrix with $n - 1 - a = b$ lines.

Now let C be a trigonal curve of genus $g \geq 5$, embedded in P^{g-1} as a canonical curve, and let S be the scroll containing C. We want to describe special linear series $\mathcal{D} = |D|$. If our linear series has base points, we can delete them; so we'll assume that $|D|$ and $|K - D|$ are base point free. Note that this implies that both $r(D) \geq 1$ and $r(K - D) \geq 1$. In addition, it follows by Bertini that a general divisor $D \in \mathcal{D}$ is reduced, that is, consists of distinct points $p_1, \ldots, p_d$.

Now, the first hypothesis, that $r(D) \geq 1$, says that the points $p_1, \ldots, p_d$ are linearly dependent. The second hypothesis, that $r(K - D) \geq 1$, says that the points p_i span a subspace of codimension at least 2 in $\mathbb{P}^{g-1}$ They therefore lie on at least a pencil of hyperplanes; let H be a general hyperplane containing D.

. is effective, says that the divisor D lies in a hyperplane section $C \cap H$; let H be a general such hyperplane. At the same time canonical image lies on a 2-dim scroll (non -subcanonical embedding only on 3-dim scrolls). embedding of a trigonal curve lies on the same scroll.Stratification of trigonal curves by Maroni invariants. Dimensions via automorphism groups of scrolls.

11.3 Castelnuovo's Theorem

(Statement only)
((we'll need the existence of smooth curves in given classes — base point freeness of certain divisor classes on the scroll. Theorem: bpf iff they meet both a,b rational normal curves positively. reference to Montreal? better to make a tex file of the essential bit and put it in, as appendix. or ACGH?))

Chapter 12

Hilbert Schemes I: Examples

In Chapter **??**, we looked at curves of low genus and described the linear systems on them; that is, their maps to (and in particular their embeddings in) projective space. In this chapter we'll ask a more refined question: can we describe the family of all such curves in projective space?

((Add a section on basics of the Hilbert scheme explaining why Hilbert schemes; the universal property; and the tangent space **I think this should go in Chapter 6—we should have a "cast of characters" section there, where we introduce all the moduli spaces we'll be dealing with))**

Denote by $\mathcal{H} = \mathcal{H}_{g,r,d}$ the Hilbert scheme parametrizing subschemes of $\mathbb{P}^r$ with Hilbert polynomial $p(m) = dm - g + 1$ (which includes smooth curves of degree d and genus g in $\mathbb{P}^r$), and by $\mathcal{H}^\circ \subset \mathcal{H}$ the open subset parametrizing smooth, irreducible, nondegenerate curves $C \subset \mathbb{P}^r$ (called the *restricted Hilbert scheme*).

Three basic questions about the schemes $\mathcal{H}^\circ$ are:

- Is $\mathcal{H}^\circ$ irreducible? and
- What is its dimension or dimensions?
- Where is it smooth, and where is it singular?

Of course, there are many more questions about the geometry of $\mathcal{H}^\circ$: for example, what is the closure $\overline{\mathcal{H}^\circ} \subset \mathcal{H}$ in the whole Hilbert scheme? (In other words, when is a subscheme $X \subset \mathbb{P}^r$ with Hilbert polynomial $dm - g + 1$ *smoothable*, in the sense that it is the flat limit of a family of smooth curves?) What is the Picard group of $\mathcal{H}^\circ$ or of its closure? We will for the most part not address these, though we will indicate the answers in special cases.

We'll limit ourselves in this chapter to looking at curves in $\mathbb{P}^3$. Most of the questions we raise in what follows could be asked, and many of them answered, in $\mathbb{P}^r$ for any $r \geq 3$, but for the most part the $r = 3$ case is enough to give us the flavor. We will start with curves of the lowest possible degree:

12.1 Degree 3

The smallest possible degree of an irreducible, nondegenerate curve $C \subset \mathbb{P}^3$ is 3. Any irreducible, nondegenerate curve $C \subset \mathbb{P}^3$ of degree 3 is a twisted cubic, so that in this case $\mathcal{H}^\circ$ is the parameter space for twisted cubics.

Proposition 12.1.1. *The open subset $\mathcal{H}^\circ$ of the Hilbert scheme $\mathcal{H}_{0,3,3}$ parametrizing twisted cubics is irreducible of dimension 12.*

Proof. There are in fact several ways of establishing this statement. To start with the simplest, let $C_0 \subset \mathbb{P}^3$ be any given twisted cubic, and consider the family of translates of C_0 by automorphisms $A \in \mathrm{PGL}_4$ of $\mathbb{P}^3$: that is, the family

$$\mathcal{C} = \{(A, p) \in \mathrm{PGL}_4 \times \mathbb{P}^3 \mid p \in A(C_0)\}.$$

Via the projection $\pi : \mathcal{C} \to \mathrm{PGL}_4$, this is a family of twisted cubics, and so it induces a map

$$\phi : \mathrm{PGL}_4 \to \mathcal{H}^\circ.$$

Since every twisted cubic is a translate of C_0, this is surjective, with fibers isomorphic to the stabilizer of C_0, that is, the subgroup of PGL_4 of automorphisms of $\mathbb{P}^3$ carrying C_0 to itself. By the discussion in Section **??**, every automorphism of C_0 is induced by an automorphism of $\mathbb{P}^3$, so the stabilizer is isomorphic to PGL_2 and thus has dimension 3. Since PGL_4 is irreducible of dimension 15, we conclude that $\mathcal{H}^\circ$ *is irreducible of dimension 12.* □

Exercise 12.1.2. Use an analogous argument to show that the restricted Hilbert scheme $\mathcal{H}^\circ \subset \mathcal{H}_{0,r,r}$ of rational normal curves $C \subset \mathbb{P}^r$ is irreducible of dimension $r^2 + 2r - 3$.

Second proof of Proposition 12.1.1

The argument above for Proposition 12.1.1 is based on a rather special fact, that all irreducible nondegenerate cubic curves $C \subset \mathbb{P}^3$ are translates of one another. There is another, less ad-hoc way of arriving at the conclusion above, called the method of *liaison*, or *linkage*, which we'll now describe. While it is more involved, it is more broadly applicable, at least in $\mathbb{P}^3$.

The idea behind this approach is the fact the intersection of any two distinct quadrics $Q, Q' \supset C$ containing a twisted cubic curve C has degree 4 and is unmixed; therefore it is the union of C and a line $L \subset \mathbb{P}^3$.

Conversely, suppose that $L \subset \mathbb{P}^3$ is any line and Q, Q' two general quadrics containing L; write the intersection $Q \cap Q'$ as a union $L \cup C$. Since smooth quadrics contain lines a general quadric containing L is smooth. The quadric Q' will intersect it in a curve of type $(2,2)$, so the curve C will have class $(2,1)$ or $(1,2)$. The quadrics Q' containing L cut out on Q the complete linear system of curves of type $(2,1)$, which has no base locus, so Bertini's theorem tells us that C will be smooth, so that the intersection $Q \cap Q' = L \cup C$ will be the union of L and a twisted cubic. This suggests that we set up an incidence correspondence: let $\mathbb{P}^9$ denote the projective space of quadrics in $\mathbb{P}^3$, and consider

$$\Phi = \{(C, L, Q, Q') \in \mathcal{H}^\circ \times \mathbb{G}(1,3) \times \mathbb{P}^9 \times \mathbb{P}^9 \mid Q \cap Q' = C \cup L\}.$$

We'll analyze Φ by considering the projection maps to $\mathcal{H}^\circ$ and $\mathbb{G}(1,3)$; that is, by looking at the diagram

$$\begin{tikzcd} & \Phi \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ \mathcal{H}^\circ & & \mathbb{G}(1,3) \end{tikzcd}$$

Consider first the projection map $\pi_2 : \Phi \to \mathbb{G}(1,3)$ on the second factor. By what we just said, the fiber over any point $L \in \mathbb{G}(1,3)$ is an open subset of $\mathbb{P}^6 \times \mathbb{P}^6$, where $\mathbb{P}^6$ is the space of quadrics containing L; it follows that Φ is irreducible of dimension $4 + 2 \times 6 = 16$. Going down the other side, we see that the map $\pi_1 : \Phi \to \mathcal{H}^\circ$ is surjective, with fiber over every curve C an open subsets of $\mathbb{P}^2 \times \mathbb{P}^2$, where $\mathbb{P}^2$ is the projective space of quadrics containing C; we conclude again that $\mathcal{H}^\circ$ *is irreducible of dimension 12.*

We'll see below several more instances of the application of liaison to the study of curves in $\mathbb{P}^3$. It should be said, though, that the method is largely limited to curves in $\mathbb{P}^3$ (and subvarieties $X \subset \mathbb{P}^r$ of codimension 2 in general); for example, you can't use it to do Exercise 12.1.2 for $r \geq 4$.

Third proof of Proposition 12.1.1

Yet another proof of Proposition 12.1.1 is based on a remarkable fact about twisted cubics, described in the next proposition; the application to $\mathcal{H}^\circ$ is carried out in the following exercise. In fact, the proposition here applies more generally to *rational normal curves*, and we'll state it in that generality.

Proposition 12.1.3. *If $p_1, \dots, p_{n+3} \in \mathbb{P}^n$ are any $n+3$ points in $\mathbb{P}^n$ in* linear general position*, that is, with no $n+1$ lying in a hyperplane, then there exists a unique rational normal curve $C \subset \mathbb{P}^n$ containing them.*

Proof. To start, we observe that there is an automorphism $\Phi : \mathbb{P}^n \to \mathbb{P}^n$ carrying the points $p_1, \dots, p_{n+1}$ to the coordinate points $[0, \dots, 0, 1, 0, \dots, 0] \in \mathbb{P}^n$; denote the images of the remaining two points p_{n+2} and p_{n+3} by $[\alpha_0, \dots, \alpha_n]$ and $[\beta_0, \dots, \beta_n]$. We consider maps $\mathbb{P}^1 \to \mathbb{P}^n$ given in terms of an inhomogeneous coordinate z on $\mathbb{P}^1$ by

$$z \mapsto \left[\frac{\alpha_0}{z-\nu_0}, \frac{\alpha_1}{z-\nu_1}, \dots, \frac{\alpha_n}{z-\nu_n}\right]$$

with $\nu_0, \dots, \nu_n$ any distinct scalars, and $\alpha_0, \dots, \alpha_n$ any nonzero scalars. Clearing denominators, we see that the image of such a map is a rational normal curve, and it passes through the $n+1$ coordinate points of $\mathbb{P}^n$, which are the images of the points $z = \nu_0, \dots, \nu_n \in \mathbb{P}^1$. Moreover, the image of the point $z = \infty$ at infinity is the point $[\alpha_0, \dots, \alpha_n]$; and we can adjust the values of $\nu_0, \dots, \nu_n$ so that the image of the point $z = 0$ is $[\beta_0, \dots, \beta_n]$. This proves existence; we'll leave uniqueness as the following exercise. □

Exercise 12.1.4. Show that if $C, C' \subset \mathbb{P}^n$ are two rational normal curves and $\#(C \cap C') \geq n+3$, then $C = C'$. (Hint: use induction on n.)

There is another way to prove Proposition 12.1.3 that may provide more insight (it actually produces the equations defining the rational normal curve through the points $p_1, \dots, p_{n+3}$); this is described in [Harris 1982].

There are also a number of further statements and open problems involving generalizations of this construction. For example, in the statement of Proposition 12.1.3, we can generalize the points $p_1, \dots, p_{n+3} \in \mathbb{P}^n$ to an arbitrary *curvilinear scheme* $\Gamma \subset \mathbb{P}^n$, where by curvilinear scheme we mean a 0-dimensional scheme with Zariski tangent space of dimension at most 1 at every point (equivalently, such that every irreducible component of Γ is isomorphic to $\operatorname{Spec} K[\epsilon]/(\epsilon^k)$ for some k). In this setting the condition of "linear general position" is generalized to the condition that for any hyperplane $H \subset \mathbb{P}^n$ we have $\deg(\Gamma \cap H) \leq n+1$; and it's shown in [Eisenbud and Harris 2016] that the statement of Proposition 12.1.3 holds in this greater generality.

For an open problem related to Proposition 12.1.3, let's return to $\mathbb{P}^3$ and suppose $\mathcal{H}^\circ$ is any component of the restricted Hilbert scheme parametrizing curves of degree d and genus g in $\mathbb{P}^3$; say the dimension $\dim \mathcal{H}^\circ = 2m$. A straightforward dimension count then shows that if $p_1, \dots, p_m \in \mathbb{P}^3$ are general points, then there will be a finite number of curves in this component containing the points p_i; Proposition 12.1.3 asserts that in case $\mathcal{H}^\circ$ parametrizes twisted cubics, that number is 1. The question is, are there any other components of the restricted Hilbert scheme for which the number is similarly 1, other than components parametrizing complete intersections of two surfaces of the same degree?

In any case, returning to the case $n = 3$, we see that if $p_1, \ldots, p_6 \in \mathbb{P}^3$ are any six points, with no four lying in a plane, then there is a unique twisted cubic containing all six; as promised, we can use this somewhat esoteric fact to deduce the dimension of the Hilbert scheme parametrizing twisted cubics.

Exercise 12.1.5. Consider the incidence correspondence

$$\Phi = \{(p_1, \ldots, p_6, C) \in (\mathbb{P}^3)^6 \times \mathcal{H}^\circ \ \mid p_1, \ldots, p_6 \in C\}.$$

Use the result above to show that $\mathcal{H}^\circ$ is irreducible of dimension 12. More generally, use Proposition 12.1.3 to give a second proof of Exercise 12.1.2.

12.1.1 Tangent spaces to Hilbert schemes

As we've said, our descriptions of Hilbert schemes of curves is primarily concerned with issues like the irreducibility and dimension of the restricted Hilbert scheme $\mathcal{H}^\circ$. Nonetheless, it is worth pointing out that we have at least one useful tool for answering questions about the smoothness or singularity of the restricted Hilbert scheme. In practice, it's very often the case that we can describe the Zariski tangent space $T_{[C]}\mathcal{H}^\circ$ to the Hilbert scheme at a point $[C] \in \mathcal{H}^\circ$, via the identification of $T_{[C]}\mathcal{H}^\circ$ with the space $H^0(\mathcal{N}_{C/\mathbb{P}^3})$ of global sections of the normal sheaf of C in $\mathbb{P}^3$, described in Section 6.6. In particular, we'll see in Section 13.4 below how to exhibit an everywhere nonreduced component of the restricted Hilbert scheme.

To illustrate how this may go, the following exercise gives a very simple and basic example.

Exercise 12.1.6. Let $C \cong \mathbb{P}^1 \subset \mathbb{P}^3$. Show that the normal bundle $\mathcal{N}_{C/\mathbb{P}^3} \cong \mathcal{O}_{\mathbb{P}^1}(5)^{\oplus 2}$; that is, the normal bundle of a twisted cubic is the direct sum of two line bundles of degree 5. Use this to prove that the restricted Hilbert scheme $\mathcal{H}^\circ$ of twisted cubics is everywhere smooth.

12.1.2 Extraneous components

Although $\mathcal{H}^\circ$ is open in the Hilbert scheme $\mathcal{H} = \mathcal{H}_{3m+1}(\mathbb{P}^3)$, its closure is not all of $\mathcal{H}$! There is a second irreducible component of $\mathcal{H}$, of dimension 15. This is an example of what is called an *extraneous component* of the Hilbert scheme; they are components of the Hilbert scheme whose general point does *not* correspond to a smooth, irreducible nondegenerate curve $C \subset \mathbb{P}^n$. They are the bane of anyone who works with Hilbert schemes; and while choosing to work just with the locus $\mathcal{H}^\circ \subset \mathcal{H}$ means that we won't be dealing with them directly, it's worth describing their behavior in at least the case of twisted cubics.

To start, observe that any plane cubic $C \subset \mathbb{P}^2 \subset \mathbb{P}^3$ has Hilbert polynomial $p(m) = 3m$. If $p \in \mathbb{P}^3 \setminus C$ is any point not on C, then, the union $C' = C \cup \{p\}$ is a subscheme of $\mathbb{P}^3$ with Hilbert polynomial $3m+1$, and so corresponds to a point of $\mathcal{H}$.

Now, let $\mathcal{H}' \subset \mathcal{H}$ be the open subset corresponding to unions $C' = C \cup \{p\}$ of a plane cubic and a point. By an argument analogous to the one given in [Eisenbud and Harris 2016] for plane conics, the Hilbert scheme $\mathcal{H}_{3m}$ is a $\mathbb{P}^9$-bundle over the dual projective space $(\mathbb{P}^3)^*$, and so in particular is irreducible of dimension 12; the locus $\mathcal{H}'$ is then an open subset of the product $\mathcal{H}_{3m} \times \mathbb{P}^3$, and so is irreducible of dimension 15.

Exercise 12.1.7. Show that the Hilbert scheme $\mathcal{H}_{3m+1}$ is indeed the union of the closures of the loci $\mathcal{H}^\circ$ and $\mathcal{H}'$ above (in other words, any subscheme of $\mathbb{P}^3$ with Hilbert polynomial $3m+1$ is either a flat limit of twisted cubics, or a flat limit of subschemes of the form $C \cup \{p\}$ with C a plane cubic).

Given this, we conclude that the Hilbert scheme $\mathcal{H}_{3m+1}$ consists of two irreducible components: one, the closure of the locus $\mathcal{H}^\circ$ of twisted cubics, which has dimension 12; and a second, the closure of $\mathcal{H}'$, of dimension 15.

One further question: given that the Hilbert scheme $\mathcal{H}_{3m+1}$ consists of two irreducible components, it's natural to ask what their intersection is. The answer is suggested by an example in [Eisenbud and Harris 2000, II.3.4], where we take a general twisted cubic $C \subset \mathbb{P}^3$ and apply the family of linear maps $A_t : \mathbb{P}^3 \to \mathbb{P}^3$ given by

$$A_t = \begin{pmatrix} t & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix};$$

we see there that the flat limit $\lim_{t\to 0} A_t(C)$ is a nodal plane cubic, with a spatial embedded point of multiplicity 1 at the node. In fact, the intersection of the two components is exactly the closure of this locus, as the following exercise asks you to show.

Exercise 12.1.8. Show that the locus Σ of schemes X consisting of a nodal plane cubic curve C with a spatial embedded point of multiplicity 1 at the node is dense in the intersection $\overline{\mathcal{H}^\circ} \cap \overline{\mathcal{H}'}$.

Extraneous components in general

While we'll largely ignore the extraneous components of the Hilbert schemes that we'll be dealing with here, it's worth taking a moment out and seeing how they arise, and how numerous they are.

It starts already in dimension 0, actually. Let $\mathcal{H} = \mathcal{H}_d(\mathbb{P}^n)$ be the Hilbert scheme of subschemes of $\mathbb{P}^n$ with Hilbert polynomial the constant d. We have an open subset $\mathcal{H}^\circ \subset \mathcal{H}$ whose points correspond to reduced d-tuples of points in $\mathbb{P}^n$, and this open subset is easy to describe: it's just the complement of the diagonal in the dth symmetric power of $\mathbb{P}^n$. The closure of this open set will be called the *principal component* of $\mathcal{H}$.

You might think this would be all of the Hilbert scheme $\mathcal{H}$, but as the name suggests, it's not in general. Iarrobino in [Iarrobino 1985] first proved for any $n \geq 3$ and any sufficiently large d the existence of components of $\mathcal{H}_d(\mathbb{P}^n)$ having dimension strictly larger than dn—in particular, whose general point corresponded to a nonreduced subscheme of $\mathbb{P}^n$. Other such examples have been found (ref?); in general, no one knows how many irreducible components the Hilbert scheme $\mathcal{H} = \mathcal{H}_d(\mathbb{P}^n)$ has, or what their dimensions might be.

And that in turn infects the Hilbert schemes of curves. For example, if we're looking at the Hilbert scheme $\mathcal{H}_{dm-g+1}$ parametrizing curves of degree d and genus g in $\mathbb{P}^3$, we'll have a component whose general point corresponds to a union of a plane curve of degree d and $\binom{d-1}{2} - g$ points; moreover, if Γ is any irreducible component of the Hilbert scheme of zero-dimensional subschemes of degree $\binom{d-1}{2} - g$ in $\mathbb{P}^3$, there'll be a component of $\mathcal{H}_d(\mathbb{P}^n)$ whose general point corresponds to a union of a plane curve of degree d and the subscheme corresponding to a general point of Γ. And of course we can replace the plane curves in this construction with any component of the Hilbert scheme of curves of degree d and genus $g' > g$; in addition, there may also be components of $\mathcal{H}_{dm-g+1}$ whose general point corresponds to a subscheme of $\mathbb{P}^3$ with an embedded point—we don't know (see the paper by Dawei Chen and Scott Nollet, at https://arxiv.org/abs/0911.2221).

Bottom line, it's a mess. For many g, d the Hilbert scheme $\mathcal{H}_{dm-g+1}(\mathbb{P}^3)$ has many components. In most cases no one knows how many, or what their dimensions are. For that reason, we'll henceforth focus exclusively on the restricted Hilbert scheme, and ignore the extraneous components as much as possible.

12.2 Linkage

As the second proof of Proposition 12.1.1 suggests, when the union of two curves C and D forms a complete intersection we can use this fact to relate the geometry of their respective Hilbert schemes. This is a technique we'll use repeatedly. One thing we need in order to apply it is a formula relating the genera of the curves C and D. This is one aspect of the general theory of *liaison*, or *linkage*, of curves in $\mathbb{P}^3$.

Theorem 12.2.1. *Let $C \subset \mathbb{P}^3$ be a purely 1-dimensionsional subscheme of degree c, and let $S = V(F)$ and $T = V(G)$ be surfaces of degrees s and t containing C and having no common component. If $D \subset \mathbb{P}^3$ is the subscheme defined by $\mathcal{I}_D = (F, G) : \mathcal{I}_C$ then D is purely one-dimensional and $\mathcal{I}_C = (F, G) :$*

$\mathcal{I}_D$. Furthermore, if we denote by d the degree of D, then we have $c + d = st$ and

$$p_a(C) - p_a(D) = \frac{s+t-4}{2}(c-d); \tag{12.1}$$

In words, the difference between the genera of C and D is proportional to the difference in their degrees, with constant of proportionality $(s + t - 4)/2$.

We will prove Theorem **??** in its full generality in Chapter 16, using a homological algebra argument. For now, we'll give a simple proof by intersection theory in a case sufficient for our needs in this chapter, and postpone the general proof to Chapter 16. For this, assume that C and $D \subset \mathbb{P}^3$ are smooth curves of degrees c and d with no common components. Let $S = V(F)$ and $T = V(G)$ be surfaces of degrees s and t respectively, such that that $C \cup D = S \cap T$ is a complete intersection, and assume in addition that S smooth. In this situation, Bézout's Theorem tells us that $c+d = st$; we want a formula relating the genera $g = p_a(C)$ and $h = p_a(D)$ of C and D.

To do this, we work in the Chow ring of S. By adjunction, the canonical divisor class of S is $K_S = (s - 4)H$, where H denotes the hyperplane class on S, so that by adjunction

$$2g - 2 = (C \cdot C) + (K_S \cdot C) = C \cdot C + (s - 4)c,$$

or in other words,

$$(C \cdot C) = 2g - 2 - (s - 4)c.$$

Next, since $C \cup D$ is a complete intersection of S with a surface of degree t, we have $C + D \sim tH$. Thus we have

$$(C \cdot D) = (C \cdot (tH - C)) = tc - (C \cdot C) = tc - 2g + 2 + (s - 4)c$$

and similarly

$$(D \cdot D) = (D \cdot (tH - C)) = td - tc + 2g - 2 - (s - 4)c.$$

Finally, we can apply the adjunction formula to D to arrive at

$$2h - 2 = (D \cdot D) + (K_S \cdot D) = (s - 4)d + td - tc + 2g - 2 - (s - 4)c.$$

Collecting terms, we can write this in the convenient form

$$h - g = \frac{s+t-4}{2}(d-c); \tag{12.2}$$

We will see this formula used repeatedly in this chapter, and as we indicated it will be discussed as part of the larger theory of liaison for space curves in Chapter 16. For now, you should just take a moment and reassure yourself that the right hand side of (12.2) is indeed an integer!

12.3 Degree 4

By Clifford's Theorem an irreducible nondegenerate curve of degree 4 in $\mathbb{P}^3$ must have genus 0 or 1; we consider these cases in turn.

12.3.1 Genus 0

We can deal with rational quartics by a slight variant of the first method we used to deal with twisted cubics. A rational curve of degree 4 is the image of a map $\phi_F : \mathbb{P}^1 \to \mathbb{P}^3$ given by a four-tuple $F = (F_0, F_1, F_2, F_3)$ with $F_i \in H^0(\mathcal{O}_{\mathbb{P}^1}(4))$. The space of all such four-tuples up to scalars is a projective space of dimension $4 \times 5 - 1 = 19$; let $U \subset \mathbb{P}^{19}$ be the open subset of four-tuples such that the map ϕ is a nondegenerate embedding. We then have a surjective map $\pi : U \to \mathcal{H}^\circ$, whose fiber over a point C is the space of maps with image C. Since any two such maps differ by an automorphism of $\mathbb{P}^1$—that is, an element of PGL_2—the fibers of π are three-dimensional; we conclude that $\mathcal{H}^\circ_{0,3,4}$ *is irreducible of dimension 16.*

The same analysis can be used on rational curves of any degree d: the space U of nondegenerate embeddings $\mathbb{P}^1 \to \mathbb{P}^3$ of degree d is an open subset of the projective space $\mathbb{P}^{4(d+1)-1}$ of four-tuples of homogeneous polynomials of degree d on $\mathbb{P}^1$ modulo scalars; and the fibers of the corresponding map $U \to \mathcal{H}^\circ_{dm+1}$ are copies of PGL_2. This yields the

Proposition 12.3.1. *The open set $\mathcal{H}^\circ \subset \mathcal{H}_{0,3,d}$ parametrizing smooth, irreducible nondegenerate rational curves $C \subset \mathbb{P}^3$ is irreducible of dimension $4d$.*

Exercise 12.3.2. Give an argument for Proposition 12.3.1 in case $d = 4$ using linkage.

One further remark. Following our discussion of twisted cubics, we were able to see in Exercise 12.1.6 that the restricted Hilbert scheme of twisted cubics is smooth by identifying the normal bundle of a twisted cubic and determining the dimension of its space of global sections. In fact, the same is true for rational curves of any degree, as the following exercise shows.

Exercise 12.3.3. Let $C \cong \mathbb{P}^1 \subset \mathbb{P}^3$ be a smooth rational curve of any degree d.

1. Show that $h^1(\mathcal{N}_{C/\mathbb{P}^3}) = 0$; that is, the normal bundle of C is nonspecial.
2. Using this, the Riemann-Roch formula for vector bundles on a curve and Proposition 12.3.1, show that the Hilbert scheme $\mathcal{H}$ is smooth at the point $[C]$.

We should point out that, in contrast to the case of twisted cubics, smooth rational curves in $\mathbb{P}^r$ of the same degree may have different normal bundles. This gives an interesting stratification of the restricted Hilbert scheme of rational curves; see [Coskun and Riedl 2017] for a discussion.

12.3.2 Genus 1

As we saw in Section **??**, a quartic curve $C \subset \mathbb{P}^3$ of genus 1 is the intersection of two quadric surfaces, and by Lasker's theorem, every quadric containing C is a linear combination of those two. Conversely, the intersection of two general quadrics in $\mathbb{P}^3$ is a quartic curve of genus 1. We can thus construct a family of quartics of genus 1: let $V = H^0(\mathcal{O}_{\mathbb{P}^3}(2))$ be the 10-dimensional vector space of homogeneous quadric polynomials in $\mathbb{P}^3$ and $G(2, V)$ the Grassmannian of 2-planes in V, and consider the incidence correspondence

$$\Gamma = \{(\Lambda, p) \in G(2, V) \times \mathbb{P}^3 \mid F(p) = 0 \; \forall \, F \in \Lambda\}.$$

The fiber of Γ over a point $\Lambda \in G(2, V)$ is thus the base locus of the pencil of quadrics represented by Λ; let $B \subset G(2, V)$ be the Zariski open subset over which the fiber is smooth, irreducible and nondegenerate of dimension 1. By the universal property of Hilbert schemes, the family $\pi_1 : \Gamma_B \to U$ induces a map $\phi : B \to \mathcal{H}^\circ$ that is one-to-one on points; it follows that the reduced subscheme of $\mathcal{H}^\circ$ is birational to an open subset of the Grassmannian $G(2, 10)$, and we conclude that $\mathcal{H}^\circ_{1,3,4}$ *is irreducible of dimension 16*. Exercise 12.3.4 shows that this map is actually an isomorphism.

Exercise 12.3.4. Let $C = Q \cap Q' \subset \mathbb{P}^3$ be a smooth curve of degree 4 and genus 1. Identify the normal bundle $\mathcal{N}_{C/\mathbb{P}^3}$ of C, and use this to conclude that $\mathcal{H}^\circ_{1,3,4}$ is itself reduced, and even smooth, and thus isomorphic to an open subset of the Grassmannian $G(2, 10)$.

The argument here—where we constructed a family $\mathcal{C} \to B$ of curves of given type, and then invoked the universal property of the Hilbert scheme to get a map $B \to \mathcal{H}$ is typical in analyses of Hilbert schemes. Here here are two slightly more general cases:

Exercise 12.3.5. Let $m \geq n > 0$ be two positive integers. Show that the locus $U_{n,m} \subset \mathcal{H}^\circ$ of curves $C \subset \mathbb{P}^3$ that are smooth complete intersections of surfaces of degrees n and m is an open subset of the Hilbert scheme.

Exercise 12.3.6. Consider the locus $U_{n,n} \subset \mathcal{H}^\circ$ of curves $C \subset \mathbb{P}^3$ that are smooth complete intersections of two surfaces of degrees n. Show that $U_{n,n}$ is isomorphic to an open subset of the Grassmannian $G(2, H^0(\mathcal{O}_{\mathbb{P}^3}(n))$.

12.4 Degree 5

Let $C \subset \mathbb{P}^3$ be a smooth, irreducible, nondegenerate quintic curve of genus g. By Clifford's theorem the bundle $\mathcal{O}_C(1)$ must be nonspecial, so by the Riemann-Roch theorem we must have $0 \leq g \leq 2$. We have already seen that the space

$\mathcal{H}^\circ_{5m+1}$ of rational quintic curves is irreducible of dimension 20. We will treat the case $g = 2$ in detail, and leave the case $g = 1$ as an exercise. This case will be covered in a different way in Section 12.6.

12.4.1 Genus 2

We have considered curves of genus 2 in Section **??**. To recap the analysis, let $C \subset \mathbb{P}^3$ be a smooth, irreducible, nondegenerate curve of degree 5 and genus 2. By the Riemann-Roch theorem, $h^0(\mathcal{O}_C(2)) = 10-2+1 = 9 < h^0(\mathcal{O}_{\mathbb{P}^3}(2)) = 10$ so the restriction map

$$H^0(\mathcal{O}_{\mathbb{P}^3}(2)) \to H^0(\mathcal{O}_C(2))$$

has a kernel. Since $\deg C = 5 > 2\times 2$, the curve C cannot lie on two independent quadrics; thus C lies on a unique quadric surface Q. Similarly, the restriction map

$$H^0(\mathcal{O}_{\mathbb{P}^3}(3)) \to H^0(\mathcal{O}_C(3))$$

has at least a 6-dimensional kernel; since cubics of the form LQ span only a 4-dimensional space, we see that C lies on a cubic surface S not containing Q. The intersection $Q \cap S$ has degree 6, and is thus the union of C and a line. If Q is smooth then, in terms of the isomorphism $Q \cong \mathbb{P}^1 \times \mathbb{P}^1$, we can say C is a curve of type $(2, 3)$ on the quadric Q. Note that conversely if $L \subset \mathbb{P}^3$ is a line and Q and $S \subset \mathbb{P}^3$ are general quadric and cubic surfaces containing L, and if we write

$$Q \cap S = L \cup C$$

then the curve C is a curve of type $(2, 3)$ on the quadric Q and hence, by the adjunction formula, a quintic of genus 2.

This suggests two ways of describing the family $\mathcal{H}^\circ \subset \mathcal{H}_{5m-1}$ of such curves. First, we can use the fact that C is linked to a line to make an incidence correspondence

$$\Psi = \{(C, L, Q, S) \in \mathcal{H}^\circ \times \mathbb{G}(1, 3) \times \mathbb{P}^9 \times \mathbb{P}^{19} \mid Q \cap S = C \cup L\},$$

where the $\mathbb{P}^9$ (respectively, $\mathbb{P}^{19}$) is the space of quadric (respectively, cubic) surfaces in $\mathbb{P}^3$. Given a line $L \in \mathbb{G}(1, 3)$, the space of quadrics containing L is a $\mathbb{P}^6$, and the space of cubics containing L is a $\mathbb{P}^{15}$; thus the fiber of the projection $\pi_2 : \Psi \to \mathbb{G}(1, 3)$ over L is an open subset of $\mathbb{P}^6 \times \mathbb{P}^{15}$, and we see that Ψ *is irreducible of dimension* $4 + 6 + 15 = 25$.

On the other hand, the fiber of Ψ over a point $C \in \mathcal{H}^\circ$ is an open subset of the $\mathbb{P}^5$ of cubics containing C; and we conclude that $\mathcal{H}^\circ$ *is irreducible of dimension* 20.

Exercise 12.4.1. Let $C \subset \mathbb{P}^3$ be a smooth curve of degree 5 and genus 2, and assume that the quadric surface Q containing C is smooth. From the exact sequence

$$0 \to \mathcal{N}_{C/Q} \to \mathcal{N}_{C/\mathbb{P}^3} \to \mathcal{N}_{Q/\mathbb{P}^3}|_C \to 0,$$

calculate $h^0()$ and deduce that $\mathcal{H}^\circ_{2,3,5}$ *is smooth at the point* $[C]$. Does this conclusion still hold if Q is singular?

Another, in some ways more direct, approach to describing the restricted Hilbert scheme $\mathcal{H}^\circ_{2,3,5}$ would be to use the fact that the quadric surface Q containing a quintic curve $C \subset \mathbb{P}^3$ of genus 2 is unique. We thus have a map

$$\mathcal{H}^\circ \to \mathbb{P}^9,$$

whose fiber over a point $Q \in \mathbb{P}^9$ is the space of quintic curves of genus 2 on Q.

The problem is, the space of quintic curves of genus 2 on a given quadric Q is not in general irreducible: for a general, and thus smooth quadric Q it consists of the disjoint union of the open subsets of smooth elements in the two linear series of curves of type $(2,3)$ and $(3,2)$ on Q, each of which is a $\mathbb{P}^{11}$. We can conclude immediately that $\mathcal{H}^\circ$ is of pure dimension 20; but to conclude that it is irreducible we need to verify that, in the family of all smooth quadric surfaces, the monodromy exchanges the two rulings.
((refer to the place—earlier—where monodromy is discussed, and say this follows from the irreducibility of an appropriately modified incidence correspondence. Do this example where the monodromy if first discussed, too.))
This is not hard: it amounts to the assertion that the family

$$\Gamma = \{(Q, L) \in \mathbb{P}^9 \times \mathbb{G}(1,3) \mid L \subset Q\}$$

is irreducible, which can be seen via projection on the second factor.

Exercise 12.4.2. Show that a smooth, irreducible, nondegenerate curve $C \subset \mathbb{P}^3$ of degree 5 and genus 1 is residual to a rational quartic in the complete intersection of two cubics, and use the result of subsection 12.3.1 to deduce that the space of genus 1 quintics is irreducible of dimension 20.

12.5 Degree 6

Again the Clifford and Riemann-Roch theorems suffice to compute the possible genera of a curve of degree 6. To start with, if the line bundle $\mathcal{O}_C(1)$ is nonspecial, then by the Riemann-Roch theorem we have $g \leq 3$. Suppose on the other hand that $\mathcal{O}_C(1)$ is special. Since $h^0(\mathcal{O}_C(1)) \geq 4$, we have equality in Clifford's theorem, and either C is hyperelliptic and $\mathcal{O}_C(1)$ is a multiple of the g^1_2 or C is a canonically embedded curve of genus 4. The first case cannot occur, since no special multiple of the hyperelliptic series of degree $\leq 2g-2$ can be very ample; thus C must be a canonical curve of genus 4. In sum, by applying Clifford's Theorem and the Riemann-Roch Theorem, we see that a smooth irreducible, nondegenerate curve of degree 6 in $\mathbb{P}^3$ has genus at most 4.

Exercise 12.5.1.
1. Show that all genera $g \leq 4$ do occur; that is, there exists a smooth irreducible, nondegenerate curve of degree 6 and genus g in $\mathbb{P}^3$ for all $g \leq 4$.
2. What is the largest possible genus of a smooth irreducible, nondegenerate curve $C \subset \mathbb{P}^3$ of degree $d = 7$? Can you do this with Clifford and Riemann-Roch, or do you need to invoke Castelnuovo?

The cases of genera 0, 1 and 2 are covered under Proposition 12.6.1, leaving us the cases $g = 3$ and 4. Both are well-handled by the Cartesian approach of describing their ideals.

12.5.1 Genus 4

As we've seen in Section**??** a canonical curve of genus 4 is the complete intersection of a (unique) quadric Q and a cubic surface S. We thus have a map

$$\alpha : \mathcal{H}^\circ \longrightarrow \mathbb{P}^9$$

sending a curve C to the quadric Q containing it. Moreover, the fibers of this map are open subsets of the projective space $\mathbb{P}V$, where V is the quotient

$$V = \frac{H^0(\mathcal{O}_{\mathbb{P}^3}(3))}{H^0(\mathcal{I}_{Q/\mathbb{P}^3}(3))}$$

of the space of all cubic polynomials modulo cubics containing Q. Since this vector space has dimension 16, the fibers of α are irreducible of dimension 15, and we deduce that *the space* $\mathcal{H}^\circ_{6m-3}$ *is irreducible of dimension 24.*

In fact, Exercise 12.3.6 can be generalized in this way to smooth complete intersections of surfaces of any degree:

Exercise 12.5.2. As before, let $U_{n,m} \subset \mathcal{H}^\circ$ be the locus of curves $C \subset \mathbb{P}^3$ that are smooth complete intersections of surfaces of degrees n and m. In case $m > n$, show that $U_{m,n}$ is isomorphic to an open subset of a projective bundle over the projective space $\mathbb{P}(H^0(\mathcal{O}_{\mathbb{P}^3}(n))) \cong \mathbb{P}^{\binom{n+3}{3}-1}$ of surfaces of degree n, with fiber over the point $[S] \in \mathbb{P}(H^0(\mathcal{O}_{\mathbb{P}^3}(n)))$ the projective space $\mathbb{P}(H^0(\mathcal{O}_{\mathbb{P}^3}(m))/H^0(\mathcal{I}_{S/\mathbb{P}^3}(m)) \cong \mathbb{P}^{\binom{m+3}{3}-\binom{m-n+3}{3}-1}$.

12.5.2 Genus 3

We leave this to the reader to complete as follows:

Exercise 12.5.3. Let C be a curve of degree 6 and genus 3, and assume that C does not lie on any quadric surface. Show that C is residual to a twisted cubic in the complete intersection of two cubic surfaces, and use this to deduce that the space of such curves is irreducible of dimension 24.

Exercise 12.5.4. Now let C again be a curve of degree 6 and genus 3, but now assume that C *does* lie on a quadric surface Q. Show that such a curve is a flat limit of curves of the type described in the last exercise, and conclude that $\mathcal{H}^\circ_{3,3,6}$ is irreducible of dimension 24. (Hint: Let L, Q and F denote a general linear form, a general quadratic form and a general cubic form, and consider the pencil of surfaces $S_t = V(tF + LQ) \subset \mathbb{P}^3$ specializing from the cubic surface $V(F)$ the to reducible cubic $V(LQ)$.)

12.6 Why $4d$?

The sharp-eyed reader will have noticed that, in every case analyzed so far, the Hilbert scheme parametrizing smooth curves of degree d and genus g in $\mathbb{P}^3$ has dimension $4d$. While this is not the case in general (we will see shortly an example where it fails), $4d$ is indeed the "expected dimension" from certain points of view. In the following subsections we'll describe two such computations. For the remainder of this section, we will step outside $\mathbb{P}^3$ and consider, more generally, the restricted Hilbert scheme $\mathcal{H}^\circ$ of smooth, irreducible, nondegenerate curves in $\mathbb{P}^r$.

12.6.1 Estimating $\dim \mathcal{H}^\circ$ by Brill-Noether

One method of estimating the dimension of $\mathcal{H}^\circ$ is a generalization of the proof of Proposition 12.3.1, with two additional wrinkles: First, since not all line bundles of degree d on a curve C of genus $g > 0$ are linearly equivalent, we must invoke the Picard variety $\operatorname{Pic}_d(C)$ parametrizing line bundles of degree d on a given curve C, discussed in Chapter 4. Second, since not all curves of genus $g > 0$ are isomorphic, we must involve the moduli space M_g parametrizing abstract curves of genus g, discussed in Chapter 6.

To begin with a simple example, let $\mathcal{H}^\circ$ again be the space of smooth, irreducible, nondegenerate curves $C \subset \mathbb{P}^3$ of degree 5 and genus 2. By the property of M_2 as a coarse moduli space, we get a map

$$\mu : \mathcal{H}^\circ \longrightarrow M_2.$$

To analyze the fiber $\Sigma_C = \mu^{-1}(C)$ of the map μ over a point $C \in M_2$ we first use the map

$$\nu : \Sigma_C \longrightarrow \operatorname{Pic}_5(C),$$

obtained by sending a point in Σ_C to the line bundle $\mathcal{O}_C(1)$. Proposition **??**, implies that any line bundle of degree 5 on a curve of genus 2 is very ample, so this map is surjective. Note that $h^0(\mathcal{L}) = 4$, so the linear series giving the embedding is complete. Thus, once we have specified the abstract curve C, and the line bundle $\mathcal{L} \in \operatorname{Pic}_5(C)$ the embedding is determined by giving a basis for $H^0(\mathcal{L})$, up to scalars. In other words, each fiber of ν is isomorphic to PGL_4. We can now work our way up from M_2:

- We know that M_2 is irreducible of dimension 3.
- It follows that the space of pairs $(C, \mathcal{L})$ with $C \in M_2$ a smooth curve of genus 2 and $\mathcal{L} \in \mathrm{Pic}_5(C)$ is irreducible of dimension $3 + 2 = 5$; and finally
- It follows that $\mathcal{H}^\circ$ is irreducible of dimension $5 + 15 = 20$.

In fact, this approach applies to a much wider range of examples: whenever $d \geq 2g + 1$ and $r \leq d - g$, we can look at the tower of spaces

$$\begin{tikzcd}
\mathcal{H}^\circ = \mathcal{H}^\circ_{dm-g+1}(\mathbb{P}^r) \arrow[d] \\
\mathcal{P}_{d,g} = \{(C, \mathcal{L}) \mid \mathcal{L} \in \mathrm{Pic}_d(C)\} \arrow[d] \\
M_g.
\end{tikzcd}$$

Exactly as in the special case $(d, g, r) = (5, 2, 3)$ above, we can work our way up the tower:

- M_g is irreducible of dimension $3g - 3$;
- it follows from the fact that the Picard variety is irreducible of dimension g that $\mathcal{P}_{d,g}$ is irreducible of dimension $3g - 3 + g = 4g - 3$; and finally
- since the fibers of $\mathcal{H}^\circ \to \mathcal{P}_{d,g}$ consist of $(r + 1)$-tuples of linearly independent sections of $\mathcal{L}$ (mod scalars), and $h^0(\mathcal{L}) = d - g + 1$, it follows that $\mathcal{H}^\circ$ is irreducible of dimension $\dim(\mathcal{P}_{d,g}) + (r + 1)(d - g + 1) = 4g - 3 + (r + 1)(d - g + 1) - 1$.

In sum, we have the

Proposition 12.6.1. *Whenever $d \geq 2g+1$, the space $\mathcal{H}^\circ$ of smooth, irreducible, nondegenerate curves $C \subset \mathbb{P}^r$ is either empty (if $d - g < r$) or irreducible of dimension $4g - 3 + (r + 1)(d - g + 1) - 1$; in particular, if $r = 3$, the dimension of $\mathcal{H}^\circ$ is $4d$.*

Exercise 12.6.2. By analyzing the geometry of linear series of degrees $2g - 1$ and $2g$ on a curve of genus g, extend Proposition 12.6.1 to the cases $d = 2g - 1$ and $2g$. What goes wrong if $d \leq 2g - 2$?

Proposition 12.6.1 gives a simple and clean answer to our basic questions about the dimension and irreducibility of the restricted Hilbert scheme $\mathcal{H}^\circ$ in case $d \geq 2g-1$. But what happens outside of this range? In fact, we can use Brill-Noether theory to modify this analysis to extend this beyond the range $d \geq 2g+1$.

Basically, what's different in general is that the map $\mathcal{H}^\circ \to \mathcal{P}_{d,g}$ is no longer dominant; rather, over a point $[C] \in M_g$, its image is open in the subvariety $W^r_d(C) \subset \mathrm{Pic}_d(C)$ parametrizing line bundles $\mathcal{L}$ on C of degree d with at least $r+1$ sections. Now, as long as the Brill-Noether number $\rho(d,g,r)$ is nonnegative, the Brill-Noether theorem tells us that for a general curve C, the variety $W^r_d(C)$ has dimension ρ, and (assuming $r \geq 3$) the general point of $W^r_d(C)$ corresponds to a very ample line bundle with exactly $r+1$ sections. In this situation, there is a unique component of $\mathcal{H}_0 \subset \mathcal{H}^\circ$ dominating M_g, and the map $\mathcal{H}^\circ \to \mathcal{P}_{d,g}$ carries this component to a subvariety $\mathcal{W}^r_d \subset \mathcal{P}_{d,g}$ of dimension $3g-3+\rho$. In sum, then, we have the basic theorem

Theorem 12.6.3. *Let g, d and r be any nonnegative integers, with Brill-Noether number $\rho(g,r,d) = g-(r+1)(g-d+r) \geq 0$. There is then a unique component $\mathcal{H}_0$ of the restricted Hilbert scheme $\mathcal{H}^\circ_{g,r,d}$ dominating the moduli space M_g; and this component has dimension*

$$\dim \mathcal{H}_0 = 3g-3+\rho+(r+1)^2-1 = 4g-3+(r+1)(d-g+1)-1.$$

The component $\mathcal{H}_0$ identified in Theorem 12.6.3 is called the *principal component* of the Hilbert scheme; there may be others as well, of possibly different dimension, and we do not know precisely for which d, g and r these occur. Finally, in case $\rho < 0$, the Brill-Noether theorem tells us only that there is no component of $\mathcal{H}^\circ_{g,r,d}$ dominating M_g; we'll discuss some of the outstanding questions in this range in Section 13.5 below.

12.6.2 Estimating $\dim \mathcal{H}^\circ$ by the Euler characteristic of the normal bundle

It is interesting to compare the estimate of $\dim \mathcal{H}^\circ$ above with what we get from deformation theory. Let $\mathcal{H}$ be a component of the scheme $\mathcal{H}^\circ$, with $C \subset \mathbb{P}^r$ a curve corresponding to a general point $[C]$ of $\mathcal{H}$.

We start with the idea that the dimension of the scheme $\mathcal{H}$ is approximated by the dimension of its Zariski tangent space $T_{[C]}\mathcal{H}$ at a general point $[C]$. In Section **??** we saw that the tangent space to $\mathcal{H}$ at $[C]$ is the space $H^0(\mathcal{N}_{C/\mathbb{P}^r})$ of global sections of the normal bundle $\mathcal{N} = \mathcal{N}_{C/\mathbb{P}^r}$. We can think of the dimension $h^0(\mathcal{N})$ as approximated by the Euler characteristic $\chi(\mathcal{N})$, with "error term" $h^1(\mathcal{N})$ coming from its first cohomology group.

Given these two approximations, we arrive at a number we can compute.

From the exact sequence

$$0 \to T_C \to T_{\mathbb{P}^r}|_C \to \mathcal{N} \to 0$$

we deduce that

$$\begin{aligned} c_1(\mathcal{N}) &= c_1(T_{\mathbb{P}^r}|_C) - c_1(T_C) \\ &= (r+1)d - (2-2g). \end{aligned}$$

Now we can apply the Riemann-Roch Theorem for vector bundles on curves ([Eisenbud and Harris 2016, Theorem ???]) to conclude that

$$\begin{aligned} \chi(\mathcal{N}) &= c_1(\mathcal{N}) - \operatorname{rank}(\mathcal{N})(g-1) \\ &= (r+1)d - (r-3)(g-1). \end{aligned}$$

Note that our two "estimates" are actually inequalities. But, unfortunately, they go in opposite directions: we have

$$\dim \mathcal{H} \le \dim T_{[C]}\mathcal{H},$$

but

$$\dim T_{[C]}\mathcal{H} \ge \chi(\mathcal{N}).$$

Nonetheless, one can show that if $C \subset \mathbb{P}^r$ is a smooth curve then the versal deformation space of $C \subset \mathbb{P}^r$ has dimension at least $\chi(\mathcal{N})$. If we consider the family of Picard varieties over the family of smooth curves in a neighborhood of C and we can deduce that for any component of $\mathcal{H}^\circ$ containing C we have

$$\dim \mathcal{H}^\circ \ \ge \ (r+1)d - (r-3)(g-1)$$

12.6.3 They're the same!

Proposition 12.6.1 suggests that the "expected dimension" of the restricted Hilbert scheme $\mathcal{H}^\circ$ of curves of degree d and genus g in $\mathbb{P}^r$ should be

$$h(g,r,d) := 4g - 3 + (r+1)(d-g+1) - 1.$$

But the calculation immediately above suggests it should be $(r+1)d - (r-3)(g-1)$. Which is it? The answer is both: they're the same number!

Chapter 13

Hilbert Schemes II: Counterexamples

In the preceding chapter, we described a number of examples of Hilbert schemes, and observed some patterns in their behavior: in each case the restricted Hilbert scheme $\mathcal{H}^\circ$ parametrizing smooth, irreducible and nondegenerate curves was irreducible of the "expected dimension" $h(g,r,d) := 4g-3+(r+1)(d-g+1)-1$. In fact, Theorem 12.6.3 tells us that these patterns persist, for those components of $\mathcal{H}^\circ$ dominating the moduli space M_g.

But what about other components of the Hilbert scheme—components with $\rho(g,r,d) < 0$, or for that matter components with $\rho(g,r,d) \geq 0$ that simply don't dominate M_g? In fact, none of the patterns we've observed so far hold in general, and the first thing we'll do in this chapter is to give some examples, culminating with Mumford's celebrated example of a component of the restricted Hilbert scheme that is everywhere non-reduced.

We will close the chapter by discussing some intriguing conjectures suggested by Brill-Noether theory and by observed behavior in small cases.

13.1 Degree 8

We start with an example of a component of the restricted Hilbert scheme $\mathcal{H}^\circ$ whose dimension is strictly greater than $h(g,r,d)$, the space $\mathcal{H}^\circ = \mathcal{H}^\circ_{9,3,8}$ of smooth, irreducible, nondegenerate curves of degree 8 and genus 9. Let C be such a curve, and consider the restriction map

$$\rho_2 : H^0(\mathcal{O}_{\mathbb{P}^3}(2)) \longrightarrow H^0(\mathcal{O}_C(2)).$$

The source of ρ_2 has dimension 10, but the Riemann-Roch Theorem

$$h^0(\mathcal{O}_C(2)) = \begin{cases} 9, & \text{if } \mathcal{O}_C(2) \cong K_C; \\ 8, & \text{if } \mathcal{O}_C(2) \not\cong K_C \end{cases}$$

admits two possibilities for the dimension of target of ρ_2. However, if $h^0(\mathcal{O}_C(2))$ were 8 then C would lie on two distinct quadrics Q and Q'. Since C is nondegenerate, it cannot lie on any irreducible quadrics; thus Q and Q' would have to be irreducible, which would violate Bézout's Theorem. We deduce that $\mathcal{O}_C(2) \cong K_C$, and thus that C lies on a unique quadric surface Q (which must be irreducible since C is irreducible and doesn't lie on a plane).

Similarly, C cannot lie on any cubic not containing Q. Moving on to quartics, we look again at the restriction map

$$\rho_4 : H^0(\mathcal{O}_{\mathbb{P}^3}(4)) \longrightarrow H^0(\mathcal{O}_C(4)).$$

The dimensions here are, respectively, 35 and $4 \cdot 8 - 9 + 1 = 24$; and we deduce that C lies on at least an 11-dimensional vector space of quartic surfaces. On the other hand, only a 10-dimensional vector subspace of these vanish on Q; and so we conclude that C *lies on a quartic surface not containing* Q. It follows from Bézout's Theorem that $C = Q \cap S$. By Lasker's Theorem, the ideal (Q, S) is saturated, so it is equal to the homogeneous ideal of C. Thus $\ker(\rho_4)$ has dimension exactly 11, and S is unique modulo quartics vanishing on Q.

From these facts it is easy to compute the dimension of $\mathcal{H}^\circ$. This is a special case of Exercise 12.5.2, but just to say it: associating to C the unique quadric on which it lies gives a map $\mathcal{H}^\circ \to \mathbb{P}^9$ with dense image, and each fiber is an open subset of the projective space $\mathbb{P}V$, where V is the 25-dimensional vector space

$$V = \frac{H^0(\mathcal{O}_{\mathbb{P}^3}(4))}{H^0(\mathcal{I}_{Q/\mathbb{P}^3}(4))}.$$

It follows that *the space* $\mathcal{H}^\circ_{8m-8}(\mathbb{P}^3)$ *is irreducible of dimension 33*—one larger than the "expected" $4d$.

13.2 Degree 9

For the next example, consider the space $\mathcal{H}^\circ = \mathcal{H}^\circ_{9m-9}(\mathbb{P}^3)$ of curves of degree 9 and genus 10. Once more, to describe such a curve C, we look to the restriction maps $\rho_m : H^0(\mathcal{O}_{\mathbb{P}^3}(m)) \longrightarrow H^0(\mathcal{O}_C(m))$. The Riemann-Roch Theorem tells us that

$$h^0(\mathcal{O}_C(2)) = \begin{cases} 10, & \text{if } \mathcal{O}_C(2) \cong K_C \text{ ("the first case,") and} \\ 9, & \text{if } \mathcal{O}_C(2) \not\cong K_C \text{ ("the second case.")} \end{cases}$$

Unlike the the situation in degree 8, both are possible; we'll analyze each.

1. Suppose first that C does not lie on any quadric surface (so that we are necessarily in the first case above), and consider the map $\rho_3 : H^0(\mathcal{O}_{\mathbb{P}^3}(3)) \to H^0(\mathcal{O}_C(3))$. By the Riemann-Roch Theorem, the dimension of the target is $3 \cdot 9 - 10 + 1 = 18$, from which we conclude that C lies on at least a pencil of cubic surfaces. Since C lies on no quadrics, all of these cubic surfaces must be irreducible, and it follows by Bézout's Theorem that the intersection of two such surfaces is exactly C. At this point, Lasker's Theorem assures us that C lies on exactly two cubics.

By Exercise 12.3.6, then, the space $\mathcal{H}_1^\circ$ of curves of this type is thus an open subset of the Grassmannian $G(2, 20)$ of pencils of cubic surfaces, which is irreducible of dimension 36.

2. Next, suppose that C does lie on a quadric surface $Q \subset \mathbb{P}^3$; let $\mathcal{H}_2^\circ \subset \mathcal{H}^\circ$ be the locus of such curves. In this case, we claim two things:

a. Q must be smooth; and

b. C must be a curve of type $(3, 6)$ on Q

For part (a), we claim that in fact *a smooth, irreducible nondegenerate curve C of degree 9 lying on a singular quadric must have genus 12.* We can see this by observing that Q must be a cone over a smooth conic curve, and so its blow-up at the vertex is the Hirzebruch surface $\mathbb{F}_2$, with the directrix $E \subset \mathbb{F}_2$ the exceptional divisor of the blowup, and a line L of the ruling of $\mathbb{F}_2$ the proper transform of a line lying on Q. The pullback to $\mathbb{F}_2$ of the hyperplane class has intersection number 1 with L and 0 with E, from which it follows that its class must be $H = 2L + E$

Now, the proper transform $\tilde{C}$ of C in $\mathbb{F}_2$ has intersection number 1 with E, since C passes through the vertex of Q and is smooth there; given this, and the fact that it has intersection number 9 with $H = 2L + E$, we can deduce that the class of $\tilde{C}$ is $9L + 4E$. Now, we know that $K_{\mathbb{F}_2} = -2E - 4L$; by adjunction we deduce that the genus of C is 12.

For the second part, once we know that Q is smooth, the genus formula on Q tells us immediately that C must be of type $(3, 6)$ or $(6, 3)$. Now, since the quadric Q containing C is unique, by Bézout, we have a map $\mathcal{H}_2^\circ \to \mathbb{P}^9$ associating to each curve C of this type the unique quadric containing it. The fiber of this map over a given quadric Q is the disjoint union of open subsets of the projective spaces $\mathbb{P}^{27}$ parametrizing curves of type $(3, 6)$ and $(6, 3)$ on Q, and we see that the locus $\mathcal{H}_2^\circ$ again has dimension 36.

Exercise 13.2.1. While the above argument does not prove that the locus $\mathcal{H}_2^\circ$ is irreducible (in the absence of a monodromy argument), we can see that it's irreducible via a liaison argument: we're saying that a curve C of the second type is residual to a union of three skew lines in the intersection of a quadric and a sextic curve. Carry out this argument to establish that $\mathcal{H}_2^\circ$ is indeed irreducible.

In sum, there are two types of smooth, irreducible, nondegenerate curves $C \subset \mathbb{P}^3$ of degree 9 and genus 10: type 1, which are complete intersections of two cubics; and type 2, which are curves of type $(3,6)$ on a quadric surface. Moreover, the family of curves of each type is irreducible of dimension 36; and we conclude that *the space* $\mathcal{H}^\circ_{9m-9}(\mathbb{P}^3)$ *is reducible, with two components of dimension 36.*

Exercise 13.2.2. In the preceding argument, we used a dimension count to conclude that a general curve of type 1 could not be a specialization of a curve of type 2, and vice versa. Prove these assertions directly: specifically, argue that

1. by upper-semicontinuity of $h^0(\mathcal{I}_{C/\mathbb{P}^3}(2))$, argue that a curve C not lying on a quadric cannot be the specialization of curves C_t lying on quadrics; and

2. show that for a general curve of type $(3,6)$ on a quadric, $K_C \not\cong \mathcal{O}_C(2)$, and deduce that a general curve of type 2 is not a specialization of curves of type 1.

Exercise 13.2.3. Let Σ_1 and $\Sigma_2 \subset \mathcal{H}^\circ_{9m-9}(\mathbb{P}^3)$ be the loci of curves of types 1 and 2 respectively.

1. What is the intersection of the closures of Σ_1 and Σ_2 in $\mathcal{H}^\circ_{9m-9}(\mathbb{P}^3)$?

2. What is the intersection of the closures of Σ_1 and Σ_2 in the whole Hilbert scheme $\mathcal{H}_{9m-9}(\mathbb{P}^3)$?

13.3 Special components in the nonspecial range

If we ignore the finer points of the Brill-Noether theorem and focus just on the statement about the dimension and irreducibility of the variety of linear series on a curve, we can express it in a simple form: according to Theorem 12.6.3 *Any component of the restricted Hilbert scheme* $\mathcal{H}^\circ$ *of curves of degree* d *and genus* g *that dominates the moduli space* M_g *has the expected dimension*

$$h(g,r,d) = 4g-3+(r+1)(d-g+1)-1 = (r+1)d-(r-3)(g-1)$$

as calculated in Section 12.6 above; and, in the Brill-Noether range (that is, when the Brill-Noether number $\rho(g,r,d) \geq 0$ *is nonnegative), there exists a unique such component.*

If we restrict further to the nonspecial range $d \geq g+r$, we don't need the ghosts of Brill or Noether to tell us this: if $\mathcal{L}$ is a general line bundle of degree d on a general curve C of genus g, and $V \subset H^0(\mathcal{L})$ a general $(r+1)$-dimensional subspace, the linear system V will embed the curve C as a nondegenerate curve of degree d in $\mathbb{P}^r$, and the curve obtained in this way will comprise an irreducible component of the restricted Hilbert scheme.

But that doesn't mean that there aren't other components of the restricted Hilbert scheme, even in the nonspecial range! In this section, we'll construct an example of this: a component of the restricted Hilbert scheme $\mathcal{H}^\circ_{g,r,d}$, with $d \geq g + r$, that does not dominate M_g and indeed has the wrong dimension.

For our example, we'll take $d = 28$, $g = 21$ and $r = 7$. Again, a general line bundle $\mathcal{L}$ of degree 28 on a general curve C of genus 21 will be very ample (we could invoke the Brill-Noether theorem for this, but it follows from the more elementary argument for Theorem **??**). Curves of genus 21 embedded in $\mathbb{P}^7$ in this way comprise a component $\mathcal{H}_0$ of the Hilbert scheme $\mathcal{H}^\circ_{21,7,28}$ having the expected dimension

$$h(21, 7, 28) = 4g - 3 + (r + 1)(d - g + 1) - 1 = 144.$$

But here's another way to construct a curve of degree 28 and genus 21 in $\mathbb{P}^7$, that will produce a larger family of such curves! To start with, let's restrict to the trigonal locus in M_{21}; that is, we'll assume the curve C is trigonal. (This immediately cuts down on our degrees of freedom, but we'll make up for it in the choice of linear system.)

We now want to look at the line bundle residual to 4 times the g^1_3 on C; that is, if $\mathcal{M}$ is the line bundle of degree 3 on C having two sections, we take $\mathcal{L} = K_C \otimes \mathcal{M}^{-4}$. We first need to calculate the dimension of the space of sections of $\mathcal{L}$, and to show that this bundle is in fact very ample; these will be special cases of the following lemma.

Lemma 13.3.1. *Let C be a general trigonal curve of genus g, $\mathcal{M}$ the line bundle of degree 3 on C having two sections, and $\mathcal{L} = K_C \otimes \mathcal{M}^{-l}$.*

1. *If $l \leq g/2$, then $h^0(\mathcal{L}) = g - 2l$; and*
2. *If $l \leq (g - 4)/2$, then $\mathcal{L}$ is very ample.*

Proof. Both statements follow from our description of the geometry of canonical models of trigonal curves, carried out in **??**. We observed there that a trigonal canonical curve lies on a rational normal scroll S, and that *if C is general, then the scroll S is balanced.* The linear system $|\mathcal{L}| = |K_C \otimes \mathcal{M}^{-l}|$ is then cut out by hyperplanes in $\mathbb{P}^{g-1}$ containing any l chosen lines from the ruling of S; and the first part follows from the fact that *on a balanced scroll $S \subset \mathbb{P}^r$, any $(r + 1)/2$ lines of the ruling are linearly independent.*

The second part follows similarly, when we observe that if $r \geq 5$, $S \subset \mathbb{P}^r$ is any balanced rational normal scroll, and $L \subset S$ any line of the ruling, then the projection $\pi_L : S \to \mathbb{P}^{r-2}$, while a priori only rational, in fact extends to a regular map on all of S, embedding S as a balanced scroll in $\mathbb{P}^{r-2}$. Restricting to any curve $C \subset S$, it follows that π_L gives an embedding of C in $\mathbb{P}^{r-2}$ as well. □

Getting back to our present example, what we see is that if C is a general trigonal curve of genus 21 with $g^1_3 = |\mathcal{M}|$, and $\mathcal{L} = K_C \otimes \mathcal{M}^{-4}$, then the line bundle $\mathcal{L}$ embeds C as a curve of degree $2g-2-12 = 28$ in $\mathbb{P}^{12}$. Now we consider the projection of the image curve in $\mathbb{P}^{12}$ to $\mathbb{P}^7$. The family of such projections is parametrized by an open subset of the Grassmannian $\mathbb{G}(4, 12)$, which has dimension 40. We thus have $2g + 1 = 43$ degrees of freedom in choosing the general trigonal curve C, and another 40 degrees of freedom in choosing the projection (that is, the subseries $g^7_{28} \subset |\mathcal{L}|$); together these determine the image curve $C \subset \mathbb{P}^7$ up to automorphisms of $\mathbb{P}^7$. In sum, we see that the family $\mathcal{H}_1$ of curves $C \subset \mathbb{P}^7$ described in this way has dimension

$$\dim \mathcal{H}_1 = 43 + 40 + 63 = 146.$$

In particular, $\mathcal{H}_1$ cannot be in the closure of $\mathcal{H}_0$. Thus, even though we are in the nonspecial range $d \geq g+r$, there is at least one other irreducible component of the restricted Hilbert scheme, which maps to a proper subvariety of M_g and has dimension strictly greater than the expected.

13.4 Degree 14: Mumford's example

In many of the analyses above, we've been able to use the identification of the tangent space to the Hilbert scheme $\mathcal{H}$ at a point $[C]$ with the space $H^0(\mathcal{N}_{C/\mathbb{P}^3})$ of global sections of the normal bundle of C to tell whether the Hilbert scheme was smooth or singular at the point $[C]$. What's more, in every case where we carried this out, the conclusion was that the restricted Hilbert scheme $\mathcal{H}^\circ$ at least was smooth.

Does this pattern persist? The answer is a resounding "no:" in this section, we'll analyze an example, first discovered by Mumford, of an entire irreducible component of $\mathcal{H}^\circ$ that is everywhere singular, that is, everywhere nonreduced.

The example is the Hilbert scheme $\mathcal{H}^\circ = \mathcal{H}^\circ_{24,3,14}$ parametrizing smooth, irreducible curves C of degree 14 and genus 24 in $\mathbb{P}^3$. We shall analyze this example in out usual way, and examine three irreducible components of $\mathcal{H}^\circ$, one of which will be the celebrated Mumford component.

We will begin as always by analyzing the possible degrees of generators of the ideal of C, for $C \subset \mathbb{P}^3$ a smooth, irreducible curve of degree 14 and genus 24. By applying the genus formula for plane curves and curves on quadrics we see that C cannot lie in a plane or on a quadric. By Bézout's Theorem, C cannot lie on both a cubic and a quartic hypersurface, though we shall see that both possibilities are realized.

For $m \geq 3$ let $\rho_m : H^0(\mathcal{O}_{\mathbb{P}^3}(m)) \longrightarrow H^0(\mathcal{O}_C(m))$ be the natural maps. We will proceed by computing the size of the kernel of ρ_m for $m \geq 3$.

For $m \geq 4$, the line bundle $\mathcal{O}_C(m)$ has degree $> 2g - 2 = 46$, so the Riemann-Roch Theorem gives an exact value of $h^0(\mathcal{O}_C(m))$. However, when

m	$h^0(\mathcal{O}_C(m))$	$h^0(\mathcal{O}_{\mathbb{P}^3}(m))$
3	19, 20 or 21	20
4	33	35
5	47	56
6	61	84

Table 13.1: Postulation table

$m = 3$ we have

$$h^0(\mathcal{O}_C(3)) = 42 - 24 + 1 + h^0(K_C(-3)).$$

Since $d - g + 1 = 14 - 24 + 1$ is negative, C is embedded in $\mathbb{P}^3$ by a special linear series, and it follows from Section 5.1.3 that C is not hyperelliptic. The special line bundle $K_C(-3)$ has degree $46 - 42 = 4$ so, by Clifford's Theorem in the non-hyperelliptic case, $h^0(K_C(-3)) \leq 2$. Thus $h^0(\mathcal{O}_C(3)) = 19, 20$ or 21.

The "postulation table" (13.1) collects the dimensions of the source and target of ρ_m for $m = 3, \ldots, 6$.

13.4.1 Case 1: C does not lie on a cubic surface

Proposition 13.4.1. *The locus $\mathcal{H}_1 \subset \mathcal{H}^\circ$ parameterizing curves not lying on a cubic surface is dense in an irreducible component of $\mathcal{H}^\circ$. It has dimension 56, and is generically smooth.*

The proof of this proposition will occupy us for several pages. Let C be curve in $\mathcal{H}_1$. Table 13.1 shows that C lies on at least two linearly independent quartic surfaces S and S'; and since C does not lie on any surface of smaller degree, neither can be reducible. It follows that the intersection $S \cap S'$ must consist of the union of the curve C and a curve D of degree 2. The linkage formula (12.2) says that

$$p_a(C) - p_a(D) = (14 - 2)\frac{4 + 4 - 4}{2} = 24,$$

so D has arithmetic genus 0. Note that the proof above of formula (12.2) requires that at least one of the quartic surfaces containing C is smooth, which we don't a priori know in this setting; to apply it we need to invoke the more general Theorem 16.2.2 from Chapter 16.

We can now invoke the following lemma:

Lemma 13.4.2. *A subscheme $D \subset \mathbb{P}^3$ of dimension 1, degree 2 and arithmetic genus 0 (that is, $\chi(\mathcal{O}_D) = 1$) is necessarily a plane conic; that is, the complete intersection of a plane and a quadric.*

We remark that the need to prove a lemma like this is one of the drawbacks of the method of liaison: even if we are a priori interested just in smooth, irreducible and nondegenerate curves in $\mathbb{P}^3$, applying liaison can lead to singular and/or nonreduced curves. There are some restrictions—by Theorem 16.2.2, for example, says that a curve residual to a pure-dimensional scheme in a complete intersection is pure dimensional. For the present case, knowing even this is unnecessary because a general curve in $\mathcal{H}_1$ lies on a smooth quartic surface.

Proof of Lemma 13.4.2. Let $H \subset \mathbb{P}^3$ be a general plane, and set $\Gamma = C \cap H$. This is a scheme of dimension 0 and degree 2 in $H \cong \mathbb{P}^2$, which is then either the union of two reduced points, or a single nonreduced point isomorphic to $\operatorname{Spec} k[\epsilon]/(\epsilon^2)$. Either way, we observe that the restriction map $H^0(\mathcal{O}_{\mathbb{P}^3}(m)) \to H^0(\mathcal{O}_\Gamma(m))$ is surjective for all $m \geq 1$, and hence the map $H^0(\mathcal{O}_C(m)) \to H^0(\mathcal{O}_\Gamma(m))$ is as well. It follows that

$$h^0(\mathcal{O}_C(m)) \geq h^0(\mathcal{O}_C(m-1)) + 2$$

for all $m \geq 1$; since we know by hypothesis that $h^0(\mathcal{O}_C(m)) = 2m + 1$ for m large, we may conclude that $h^0(\mathcal{O}_C(1)) \leq 3 < h^0(\mathcal{O}_{\mathbb{P}^3}(1))$—in other words, the scheme C must be contained in a plane. It is thus a plane conic, without embedded points since any embedded points would mean $p_a(C) < 0$. $\square$

Conversely, if C is any curve residual to a conic D in the complete intersection of two quartics, it must have degree 14 and genus 24, and by Bézout's Theorem it cannot lie on a cubic surface. We can thus compute the dimension of the family $\mathcal{H}_1$ of smooth curves of degree 14 and genus 24 not lying on a cubic surface via the incidence correspondence

$$\Phi = \{(C, D, S, S') \in \mathcal{H}^\circ \times \mathcal{H}_D \times \mathbb{P}^{34} \times \mathbb{P}^{34} \mid S \cap S' = C \cup D\}.$$

where $\mathcal{H}_D$ denotes the Hilbert scheme of plane conics. The Hilbert scheme $\mathcal{H}_D$ is irreducible of dimension 8 (this is a special case $m = 1$, $n = 2$ of Exercise 12.5.2); and for any conic $D = V(L, Q)$ given as the complete intersection of the plane $V(L)$ and the quadric $V(Q)$, Lasker's Theorem says that the homogeneous ideal of $D \subset \mathbb{P}^3$ is generated by L and Q; this allows us to see that the space of quartic surfaces containing D is a linear subspace of $\mathbb{P}^{34}$ of dimension 26. The fibers of Φ over $\mathcal{H}_D$ are thus open subsets of $\mathbb{P}^{25} \times \mathbb{P}^{25}$, and we deduce that Φ is irreducible of dimension 58.

Exercise 13.4.3. The general members of the family of quartic surfaces containing a smooth conic are themselves smooth.
((give a hint))

The general members of the family of quartic surfaces containing a smooth conic are themselves smooth, so we see from considering C, D as divisors on a smooth quartic, as in the derivation of the linkage formula, that $(C \cdot D) = 10$.

It follows that any quartic surface containing C must contain D as well and so, by Lasker's Theorem, must be a linear combination of S and S'. The fibers of Φ over its image in $\mathcal{H}_C$ are thus open subsets of $\mathbb{P}^1 \times \mathbb{P}^1$. The condition of not lying on a cubic surface is open, so $\mathcal{H}_1$ is dense in an irreducible component of $\mathcal{H}^\circ$ of dimension 56.

13.4.2 Tangent space calculations

It remains to show that $\mathcal{H}_1$ is generically smooth. To do this, we have to show that, at a general point $[C] \in \mathcal{H}_1$, the dimension of the Zariski tangent space $H^0(\mathcal{N}_{C/\mathbb{P}^3})$ is 56. Let S be a smooth quartic surface containing C, and consider the exact sequence

$$0 \to \mathcal{N}_{C/S} \to \mathcal{N}_{C/\mathbb{P}^3} \to \mathcal{N}_{S/\mathbb{P}^3}|_C \to 0. \tag{13.1}$$

The bundle $\mathcal{N}_{S/\mathbb{P}^3}|_C \cong \mathcal{O}_C(4)$, which is nonspecial; we have $h^0(\mathcal{O}_C(4)) = 33$ and $h^1(\mathcal{O}_C(4)) = 0$. By the adjunction formula applied to S we see that $K_S = \mathcal{O}_S$, and applying the formula again on S we see that $\mathcal{N}_{C/S} \cong K_C$. Thus $h^0(\mathcal{N}_{C/S}) = 24$ and $h^1(\mathcal{N}_{C/S}) = 1$.

From the long exact sequence in cohomology associated to the sequence (*) we see that there are two possibilities for the dimension of $H^0(\mathcal{N}_{C/\mathbb{P}^3})$: 56 and 57, depending on whether the map $H^0(\mathcal{N}_{C/\mathbb{P}^3}) \to H^0(\mathcal{N}_{S/\mathbb{P}^3}|_C)$ is surjective or of corank 1.

To settle this question, we need to invoke a basic fact about deformations of subschemes of a given scheme. For this discussion, let Z be an arbitrary fixed scheme, and $X \subset Y \subset Z$ a nested pair of subschemes. We can ask two questions:

1. Given a first-order deformation $\tilde{Y} \subset \operatorname{Spec} k[\epsilon]/(\epsilon^2) \times Z$ of Y in Z, does there exist a first-order deformation $\tilde{X} \subset \operatorname{Spec} k[\epsilon]/(\epsilon^2) \times Z$ of X contained in it? and
2. Given a first-order deformation $\tilde{X} \subset \operatorname{Spec} k[\epsilon]/(\epsilon^2) \times Z$ of X in Z, does there exist a first-order deformation $\tilde{Y} \subset \operatorname{Spec} k[\epsilon]/(\epsilon^2) \times Z$ of Y containing it?

The answer is a basic fact from deformation theory. Let α, β be the natural maps in the following diagram:

$$\begin{tikzcd} H^0(\mathcal{N}_{X/Z}) \arrow[r, "\alpha"] & H^0(\mathcal{N}_{Y/Z}|_X) \\ & H^0(\mathcal{N}_{Y/Z}). \arrow[u, "\beta"] \end{tikzcd}$$

Lemma 13.4.4. *The first-order deformation of X corresponding to the global section $\sigma \in H^0(\mathcal{N}_{X/Z})$ is contained in the first-order deformation of Y corresponding to the global section $\tau \in H^0(\mathcal{N}_{Y/Z})$ if and only if $\alpha(\sigma) = \beta(\tau)$. In particular, every first-order deformation of Y contains a first-order deformation of X if and only if $\mathrm{im}(\beta) \subset \mathrm{im}(\alpha)$.*

For a proof of this lemma, see Chapter 6 of [Eisenbud and Harris 2016].

We apply this construction to $Z = \mathbb{P}^3$, $Y = S \subset \mathbb{P}^3$ a smooth quartic surface, and $X = D \subset S$ a smooth plane conic curve. We start with the sequence

$$0 \to \mathcal{N}_{D/S} \to \mathcal{N}_{D/\mathbb{P}^3} \to \mathcal{N}_{S/\mathbb{P}^3}|_D \to 0.$$

Identifying D with $\mathbb{P}^1$, we have by adjunction that $\mathcal{N}_{D/S} \cong \mathcal{O}_{\mathbb{P}^1}(-2)$, and S being a quartic, we have $\mathcal{N}_{S/\mathbb{P}^3}|_D \cong \mathcal{O}_{\mathbb{P}^1}(8)$. Moreover, since D is the complete intersection of a quadric and a plane, we have $\mathcal{N}_{D/\mathbb{P}^3} \cong \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(4)$, so that the sequence above looks like

$$0 \to \mathcal{O}_{\mathbb{P}^1}(-2) \to \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(4) \to \mathcal{O}_{\mathbb{P}^1}(8) \to 0$$

Now, we know that $H^1(\mathcal{O}_{\mathbb{P}^1}(-2)) \neq 0$, while $H^1(\mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(4)) = 0$, so we conclude by Lemma 13.4.4 that the map $H^0(\mathcal{N}_{D/\mathbb{P}^3}) \to H^0(\mathcal{N}_{S/\mathbb{P}^3}|_D)$ cannot be surjective; in other words, there exist first-order deformations of S that contain no first-order deformation of D.

The same argument works if D is the union of two lines meeting at a point.

We need to introduce one more element into the argument, which is expressed in the following proposition.

Proposition 13.4.5. *Let S be a smooth quartic surface, and C and $D \subset S$ a pair of curves forming the complete intersection of S with another quartic surface S', with D a plane conic curve. A first-order deformation $\tilde{S}$ of S contains a first-order deformation of C if and only if it contains a first-order deformation of D.*

Proof. The key ingredient is the observation that $H^1(\mathcal{O}_S(D)) = H^1(\mathcal{O}_S(C)) = 0$. What this says is that a first-order deformation $\tilde{S}$ of S contains a first-order deformation of D if and only if it contains a first-order deformation of the line bundle $\mathcal{L} = \mathcal{O}_S(D)$; that is, if and only if there exists a line bundle $\tilde{\mathcal{L}}$ on $\tilde{S}$ such that $\mathcal{L}|_S \cong \mathcal{O}_S(D)$, and likewise for C. But the existence of a line bundle $\tilde{\mathcal{L}}$ on $\tilde{S}$ extending $\mathcal{O}_S(D)$ is equivalent to the existence of a line bundle $\tilde{\mathcal{M}}$ on $\tilde{S}$ extending $\mathcal{O}_S(C)$, since they're related by $\tilde{\mathcal{M}} = \mathcal{O}_{\tilde{S}}(4) \otimes \tilde{\mathcal{L}}$. $\square$

Now, going back to the exact sequence (13.1), we have shown that there exist first-order deformations of S that contain no first-order deformations of C; thus the sequence (13.1) is not exact on global sections, and hence the dimension $h^0(\mathcal{N}_{C/\mathbb{P}^3}) = 56$, showing that $\mathcal{H}_1$ is generically smooth.

13.4.3 What's going on here?

We should take a moment to give some background for the argument above. The basic idea is built on a striking fact about curves on surfaces in $\mathbb{P}^3$, called the *Noether-Lefschetz theorem.*

Theorem 13.4.6 (Noether-Lefschetz)**.** *If $S \subset \mathbb{P}^3$ is a very general surface of degree $d \geq 4$ in $\mathbb{P}^3$, and $C \subset S$ is any curve, then C is a complete intersection $S \cap T$ with S.*

Thus, for example, a very general quartic surface contains no lines, conics or twisted cubics—facts you can readily establish for yourself via a standard dimension count, as the following exercises suggest.

Exercise 13.4.7. Let $\mathbb{G}(1,3)$ be the Grassmannian of lines in $\mathbb{P}^3$, let $\mathbb{P}^{19}$ denote the space of quartic surfaces $S \subset \mathbb{P}^3$, and consider the incidence correspondence

$$\Gamma = \{(S, L) \in \mathbb{P}^{19} \times \mathbb{G}(1,3) \mid L \subset S\}$$

Calculate the dimension of Γ, and deduce in particular that the projection map $\Gamma \to \mathbb{P}^{19}$ cannot be dominant.

Exercise 13.4.8. In the preceding exercise, replace the Grassmannian $\mathbb{G}(1,3)$ with the restricted Hilbert schemes $\mathcal{H}^\circ$ parametrizing conics and twisted cubics, and carry out the analogous calculation to deduce that a general quartic surface $S \subset \mathbb{P}^3$ contains no conics or twisted cubics. What goes wrong when we replace $\mathcal{H}^\circ$ with the restricted Hilbert scheme of curves of higher degree?

In fact, calculations like the one suggested in these exercises were how Noether first came to propose Theorem 13.4.6; it was not until Lefschetz that a complete proof was given.

In these terms, we can identify the crucial ingredient in the proof of the generic smoothness part of Proposition 13.4.1 as a strengthened form of the Noether-Lefschetz Theorem:

Theorem 13.4.9 (Deformation Noether-Lefschetz)**.** *If $S \subset \mathbb{P}^3$ is a smooth surface of degree $d \geq 4$, and $C \subset S$ is any curve that is not a complete intersection with S, then there exists a first-order deformation $\tilde{S}$ of S that does not contain a first-order deformation of C.*

We will prove this by ad-hoc methods in the case of interest to us here; the proof of the general case, given in ****, uses Hodge theory.

13.4.4 Case 2: C lies on a cubic surface S

Now suppose that C is a smooth irreducible curve of degree 14 and genus 24 that *does* lie on a cubic surface S. Bézout's Theorem tells us that S is unique, and we will restrict ourselves to the open subset $\mathcal{H}_2 \subset \mathcal{H}^\circ \setminus \mathcal{H}_1$ where the surface S is smooth, which in fact is dense—see [Nasu 2008].

Bézout's Theorem tells us that C cannot lie on a quartic surface not containing S. If C lay on a quintic surface not containing S then C would be residual to a line in the complete intersection of S and the quintic, and the liaison formula **??** would tell us that

$$g(C) = (14-1)\frac{3+5-4}{2} = 26,$$

a contradiction, so C lies on no quintic surface.

On the other hand, Table 13.1 tells us that there is at least a $84 - 61 = 23$-dimensional vector space of sextic polynomials vanishing on C, only a 20-dimensional subspace of which can vanish on S. Thus there is a $\mathbb{P}^2$ of sextic surfaces containing C but not containing S, and, choosing one of them we can write

$$S \cap T = C \cup D$$

with T a sextic surface and D a curve of degree 4. The liaison formula tells us that

$$g(C) - g(D) = (14-4)\frac{3+6-4}{2} = 25,$$

so the arithmetic genus of D is -1. We will henceforth take T to be general among sextics containing C, so that D will be a general member of the (at least) 2-dimensional linear system cut on S by sextics containing C.

Proposition 13.4.10. *D must either be (a) the disjoint union of a line and a twisted cubic on S; or (b) a union of two disjoint conics on S.*

Exercise 13.4.11. (Guided exercise to prove this proposition: first, D cannot have multiple components; then, must be disconnected.)

Since neither of the cases described in Proposition 13.4.10 is a specialization of the other, we conclude that the locus $\mathcal{H}_2$ is the union of two disjoint loci $\mathcal{H}_2'$ and $\mathcal{H}_2''$ corresponding to these two cases. We consider these in turn.

Exercise 13.4.12. (Guided exercise to prove this AND deduce that $\mathcal{H}_2'$ and $\mathcal{H}_2''$ are irreducible, either by the incidence correspondences or by monodromy.)

Case $2'$: D is the disjoint union of a twisted cubic and a line

Proposition 13.4.13. *The locus $\mathcal{H}_2' \subset \mathcal{H}^\circ$ parameterizing curves C residual to the disjoint union of a line and a twisted cubic in the complete intersection of a sextic and a smooth cubic surface is an irreducible component of $\mathcal{H}^\circ$. It has dimension 56, and is generically smooth.*

Proof. Let $\mathcal{H}$ be the locus in the Hilbert scheme $\mathcal{H}_{-1,3,4}$ corresponding to disjoint unions of twisted cubics and lines, and consider the correspondence

$$\Phi = \{(C, D, S, T) \in \mathcal{H}_2' \times \mathcal{H} \times \mathbb{P}^{19} \times \mathbb{P}^{83} \mid S \cap T = C \cup D\}.$$

We have $\dim \mathcal{H} = 16$, and by Proposition 13.4.15 the fiber of Φ over a point $[D] \in \mathcal{H}$ is an open subset of the product $\mathbb{P}^5 \times \mathbb{P}^{37}$; so we see that Φ is irreducible of dimension 58. The fibers of Φ over $\mathcal{H}_2'$ are 2-dimensional, and we conclude that $\mathcal{H}_2'$ is irreducible of dimension 56.

Finally, we calculate the dimension of the Zariski tangent space $H^0(\mathcal{N}_{C/\mathbb{P}^3})$ to $\mathcal{H}_2'$ at a general point $[C]$. We do this, as before, by considering the exact sequence associated to the inclusion of C in S:

$$0 \to \mathcal{N}_{C/S} \to \mathcal{N}_{C/\mathbb{P}^3} \to \mathcal{N}_{S/\mathbb{P}^3}|_C \to 0$$

Here there is no ambiguity about the first term: by adjunction, the degree of the normal bundle of C in S is 60, which is greater than $2g(C) - 2 = 46$; so $h^1(\mathcal{N}_{C/S}) = 0$ and $h^0(\mathcal{N}_{C/S}) = 37$.

On the other hand, $\mathcal{N}_{S/\mathbb{P}^3}|_C \cong \mathcal{O}_C(3)$, and from Table 13.1, we see that $h^0(\mathcal{O}_C(3))$ can a priori be 19, 20 or 21. We will use the explicit description of C to show that, in this case, $h^0(\mathcal{O}_C(3)) = 19$.

For this purpose, let L and T denote the line component and the twisted cubic component of D respectively; and let H denote the hyperplane class on S. From the adjunction formula we can compute the self-intersection numbers of these curves on S as $(L \cdot L) = -1$ and $(T \cdot T) = 1$. Since $C \sim 6H - D$ on S, we have

$$(C \cdot L) = ((6H - L - T) \cdot L) = 7; \quad \text{and} \quad (C \cdot T) = ((6H - L - T) \cdot L) = 17$$

In other words, the curves L and T intersect C in divisors E_L and E_T of degrees 7 and 17 respectively. By Serre duality,

$$h^1(\mathcal{O}_C(3)) = h^0(K_C(-3))$$

and by adjunction,

$$K_C(-3) = K_S(C)(-3)|_C = \mathcal{O}_S(-H + 6H - D - 3H)|_C = \mathcal{O}_C(2)(-E_L - E_T).$$

Now, the quadrics in $\mathbb{P}^3$ cut out on C the complete linear series $|\mathcal{O}_C(2)|$,
((Could be proven by using the representation of a cubic surface as a blowup of the plane.))
so $h^1(\mathcal{O}_C(3))$ is the dimension of the space of quadratic polynomials vanishing on E_L and E_T. But E_L consists of seven points on the line L, so any quadric containing E_L contains L; and likewise since E_T has degree $17 > 2 \cdot 3$, any quadric containing E_T contains T. Since no quadric contains the disjoint union of a line and a twisted cubic, we conclude that $h^1(\mathcal{O}_C(3)) = 0$ and $h^0(\mathcal{O}_C(3)) = 19$.

Putting this all together, we conclude that $h^0(\mathcal{N}_{C/\mathbb{P}^3}) = 56$; so the component $\mathcal{H}_2'$ of the Hilbert scheme $\mathcal{H}^\circ$ is generically smooth of dimension 56. $\square$

Case $2''$: D is the disjoint union of two conics

Proposition 13.4.14. *The locus $\mathcal{H}_2'' \subset \mathcal{H}^\circ$ parameterizing curves C residual to the disjoint union of two conics in the complete intersection of a sextic and a smooth cubic surface is an irreducible component of $\mathcal{H}^\circ$. It has dimension 56, but is non-reduced: its tangent space at a generic point has dimension 57.*

Proof. The analysis this case follows the same path as the preceding until the very last step, where the residual curve D is the disjoint union of two conic curves rather than the disjoint union of a line and a twisted cubic. What difference does this make? Both the disjoint union of two conic curves and the disjoint union of a line and a twisted cubic are curves of degree 4 and arithmetic genus -1, so they both have Hilbert polynomial $p(m) = 4m + 2$. The difference is that they do not have the same Hilbert function, according to the following proposition:

Proposition 13.4.15. *Let E be the disjoint union of two conic curves in $\mathbb{P}^3$ and E' the disjoint union of a line and a twisted cubic. Let $h(m)$ and $h'(m)$ be their respective Hilbert functions, and $p(m) = 4m + 2$ their common Hilbert polynomial.*

1. *For all $m \neq 3$, we have $h(m) = h'(m)$; and both are equal to $p(m) = 4m+2$ for $m \geq 3$; but*
2. *$h(2) = 9$, while $h'(2) = 10$ (in other words, E lies on a unique quadric surface, while E' is not contained in any quadric surface).*

Proof. Let S be the homogeneous coordinate ring of $\mathbb{P}^3$, and let $I_E = I_{Q_1} \cap I_{Q_2}$ be the homogeneous ideal of E, where the I_{Q_i} are the homogeneous ideals of the two disjoint conics. Similarly, let $I_{E'} = I_L \cap I_T$ be the homogeneous ideal of E', where I_L is the homogeneous ideal of a line and I_T is the homogeneous ideal of a disjoint twisted cubic. We have exact sequences

$$
\begin{aligned}
0 \to & S/I_E \to S/I_{Q_1} \oplus S/I_{Q_2} \to S/(I_{Q_1} + I_{Q_2}) \to 0 \\
0 \to & S/I_{E'} \to S/I_L \oplus S/I_T \to S/(I_L + I_T) \to 0.
\end{aligned}
$$

Writing h_Q, h_L, h_T for the Hilbert functions of Q, L and T respectively, we have

$$
\begin{aligned}
h_Q(m) &= 2m + 1 \\
h_L(m) &= m + 1 \\
h_T(m) &= 3m + 1
\end{aligned}
$$

for all $m \geq 0$.

Because each of E, E' is a disjoint union, the rings $U := S/(I_{Q_1} + I_{Q_2})$ and $V := S/(I_L + I_T)$ have finite length. We claim that $U \cong k[x, y]/(q_1, q_2)$ is a complete intersection of 2 quadrics while $V \cong k[x, y]/(x^2, xy, y^2)$. It follows that the dimensions of the homogeneous components of U in degrees $0, 1, 2, 3\ldots$ are $1, 2, 1, 0\ldots$ while those of V are $1, 2, 0, 0\ldots$. Together with the computation above, this will prove the Proposition.

To analyze U, let write $I_{Q_i} = (\ell_i, q_i)$ where the ℓ_i are linear forms and the q_i are quadratic forms. Since $I_{Q_1} + I_{Q_2}$ has finite length, the four forms ℓ_1, ℓ_2, q_1, q_2 must be a regular sequence. Working modulo (ℓ_1, ℓ_2) we see that U is isomorphic to a complete intersection of 2 quadrics in 2 variables, as claimed.

To prove that V has the given Hilbert function, it suffices to show that the degree 2 part of V is 0. Since the Hilbert function of $S/I_L \oplus S/I_T$ is $4m + 2$, this is equivalent to showing that the degree 2 part of $S/I_{L\cup T}$ is 10-dimensional; that is, that no quadric vanishes on both L and T. Since T spans $\mathbb{P}^3$ and is irreducible, the quadric must be irreducible. By **** the residual $L \cup T$ to C is unmixed, and it follows that T is unmixed and spans $\mathbb{P}^3$.

We claim that if a line and a curve of degree 3 and genus 0 lie on any quadric, then they meet: If the quadric is smooth then T would have class $(1, 2)$ and the line would have to have class $(1, 0)$ or $(0, 1)$ both of which meet T. If the quadric is an irreducible cone, then we note that every curve meets every line on the cone. If T lies on the union of two planes then T has components in both planes and thus meets any line in one of them; and finally if T lies on a double plane, then the line would meet T_{red}. Thus $T \cup L$ cannot lie on a quadric, and we are done. □

To return to the proof of Proposition 13.4.14, let $\mathcal{H}$ now be the locus in the Hilbert scheme $\mathcal{H}_{-1,3,4}$ corresponding to disjoint unions of two conics, and consider the correspondence

$$\Phi = \{(C, D, S, T) \in \mathcal{H}_2'' \times \mathcal{H} \times \mathbb{P}^{19} \times \mathbb{P}^{83} \mid S \cap T = C \cup D\}.$$

Once more we have $\dim \mathcal{H} = 16$, and the fiber of Φ over a point $[D] \in \mathcal{H}$ is again an open subset of the product $\mathbb{P}^5 \times \mathbb{P}^{37}$ (unions of two disjoint conics imposes the same number of conditions on cubics and sextics as the disjoint union of a line and a twisted cubic); so we see that Φ is irreducible of dimension 58. The fibers of Φ over $\mathcal{H}''$ are 2-dimensional, and we conclude that $\mathcal{H}''$ is irreducible of dimension 56.

The calculation of the dimension of the Zariski tangent space $H^0(\mathcal{N}_{C/\mathbb{P}^3})$ to $\mathcal{H}''$ at a general point $[C]$ also proceeds as in the last case: we start with the exact sequence

$$0 \to \mathcal{N}_{C/S} \to \mathcal{N}_{C/\mathbb{P}^3} \to \mathcal{N}_{S/\mathbb{P}^3}|_C \to 0.$$

Again, the line bundle $\mathcal{N}_{C/S}$ has degree 60 and so is nonspecial with $h^1(\mathcal{N}_{C/S}) = 0$ and $h^0(\mathcal{N}_{C/S}) = 37$.

However, the determination of the cohomology of the third term, $\mathcal{N}_{S/\mathbb{P}^3}|_C \cong \mathcal{O}_C(3)$ is different. Let Q and Q' be the two conics comprising the residual curve D; and let H denote the hyperplane class on S. The planes P, P' spanned by Q and Q' respectively meet in a line L. Since L contains the scheme of length 4 of intersection with $Q \cup Q'$, it is contained in S. Thus the curves Q and Q' are linearly equivalent on S, so we can write the class of C on S as $6H - 2Q \sim 4H + 2L$.

Since $Q \cap Q' = \emptyset$ we have $Q \cdot Q = 0$; and since $C \sim 6H - 2Q$ on S, we have

$$(C \cdot Q) = ((6H - 2Q) \cdot Q) = 12.$$

In other words, the curves Q and Q' intersect C in divisors E_Q and $E_{Q'}$ of degree 12. As before, we can write

$$h^1(\mathcal{O}_C(3)) = h^0(K_C(-3)) = h^0(\mathcal{O}_C(2)(-E_Q - E_{Q'}))$$

and using again the completeness of the linear series cut out on C by quadrics, we see that $h^1(\mathcal{O}_C(3))$ *is the dimension of the space of quadratic polynomials vanishing on* E_Q *and* $E_{Q'}$; again, since $12 > 2 \cdot 2$, this is the same as the space of quadrics containing the two curves Q and Q'.

Here is where the stories diverge: we saw in Proposition 13.4.15 that whereas there is no quadric containing the disjoint union of a line and a twisted cubic, there is indeed a unique quadric containing the union of two given disjoint conics, namely, the union of the planes of the conics. Thus $h^1(\mathcal{O}_C(3)) = 1$ so $h^0(\mathcal{O}_C(3)) = 20$ and correspondingly $h^0(\mathcal{N}_{C/\mathbb{P}^3}) = 57$. □

What's going on here?

What accounts for the different behaviors of curves in cases $2'$ and $2''$? Here is one explanation:

To start, let C be a curve corresponding to a general point of $\mathcal{H}_2'$. As we've seen, we have

$$h^1(\mathcal{O}_C(3)) = 0 \quad \text{and} \quad h^0(\mathcal{O}_C(3)) = 19,$$

so we see already from Table 13.1 that C must lie on a cubic surface. Moreover, by upper-semicontinuity, the same is true of any deformation of C, and so in an étale neighborhood of $[C]$ the Hilbert scheme looks like a projective bundle over the space of cubic surfaces.

By contrast, if C is the curve corresponding to a general point of $\mathcal{H}_2''$, we have

$$h^1(\mathcal{O}_C(3)) = 1 \quad \text{and} \quad h^0(\mathcal{O}_C(3)) = 20.$$

In other words, C is not forced to lie on a cubic surface, it just chooses to do so! The "extra" section of the normal bundle corresponds to a first-order deformation of C that is not contained in any deformation of S. If we could extend these deformations to arbitrary order, we would arrive at a family of curves whose general member lay in the first component $\mathcal{H}_1$; but we know that a general point of $\mathcal{H}''$ is not in the closure of $\mathcal{H}_1$, and so *these deformations of* C *must be obstructed.*

One note: it may seem that the phenomenon described in this last example—a component of the Hilbert scheme that is everywhere nonreduced, even though the objects parametrized are perfectly nice smooth, irreducible curves in $\mathbb{P}^3$—represents a pathology, and indeed, it was first described by David Mumford, in a paper entitled "Pathologies"! But, as Ravi Vakil has shown, it is to be expected: Vakil shows that *every* complete local ring over an algebraically closed field, up to adding power series variables, occurs as the completion of the local ring of a Hilbert scheme of smooth curves—that is, in effect, every singularity is possible. (reference to Vakil's paper, and more precise statement of Ravi's theorem).

13.5 Open problems

13.5.1 Brill-Noether in low codimension

If we ignore the finer points of the Brill-Noether theorem and focus just on the statement about the dimension and irreducibility of the variety of linear series on a curve, we can express it in a simple form: according to Theorem 12.6.3 *Any component of the restricted Hilbert scheme $\mathcal{H}^\circ$ of curves of degree d and genus g that dominates the moduli space M_g has the expected dimension*

$$h(g, r, d) = 4g - 3 + (r + 1)(d - g + 1) - 1 = (r + 1)d - (r - 3)(g - 1)$$

as calculated in Section 12.6 above.

Now, we saw in Section 13.1 an example of a component of the Hilbert scheme violating this dimension estimate, and it's not hard to produce lots of similar examples: components of the Hilbert scheme that parametrize complete intersections, or more generally determinantal curves, have in general dimension larger than the Hilbert number $h(g, r, d)$, and the following exercise gives a way of generating many more.

Exercise 13.5.1. Let $\mathcal{H}^\circ$ be a component of the Hilbert scheme parametrizing curves of degree d and genus g in $\mathbb{P}^3$ that dominates the moduli space M_g. For $s, t \gg d$, let $\mathcal{K}^\circ$ be the family of smooth curves residual to a curve $C \in \mathcal{H}^\circ$ in a complete intersection of surfaces of degrees s and t.

1. Show that $\mathcal{K}^\circ$ is open and dense in a component of the Hilbert scheme of curves of degree $st - d$ and the appropriate genus.

2. Calculate the dimension of $\mathcal{K}^\circ$, and in particular show that it is strictly greater than $h(g, r, d)$.

So it may seem that the issue is settled: components of the Hilbert scheme dominating M_g have the expected dimension; others don't in general. But there is an observed phenomenon that suggests more may be true: that components of

$\mathcal{H}^\circ$ whose image in M_g have low codimension still have the expected dimension $h(g, r, d)$.

The cases with codimension ≤ 2 are already known: In [Eisenbud and Harris 1989], it is shown that if $\Sigma \subset M_g$ is any subvariety of codimension 1, then the curve C corresponding to a general point of Σ has no linear series with Brill-Noether number $\rho < -1$; and Edidin in [Edidin 1993] proves the analogous (and much harder) result for subvarieties of codimension 2. Indeed, looking over the examples we know of components of the Hilbert scheme whose dimension is strictly greater than the expected $h(g, r, d)$, there are none whose image in M_g has codimension less than $g - 4$. We could therefore make the conjecture:

Conjecture 13.5.2. *If $\mathcal{K} \subset \mathcal{H}^\circ_{d,g,r}$ is any component of a restricted Hilbert scheme, and the image of $\mathcal{K}$ in M_g has codimension $\leq g - 4$, then $\dim \mathcal{K} = h(g, r, d)$.*

13.5.2 Maximally special curves

Most of Brill-Noether theory, and the theory of linear systems on curves in general, centers on the behavior of linear series on a general curve. The opposite end of the spectrum is also interesting, and we may ask: How special a linear series on a special curve can be?

To make such a question precise, let $\tilde{M}^r_{g,d} \subset M_g$ be the closure of the image of the map $\phi : \mathcal{H}^\circ_{d,g,r} \to M_g$ sending a curve to its isomorphism class.

1. What is the smallest possible dimension of $\mathcal{H}^\circ_{d,g,r}$?

2. What is the smallest possible dimension of $\tilde{M}^r_{g,d}$?

3. Modifying the last question slightly, let $M^r_{g,d} \subset M_g$ be the closure of the locus of curves C that possess a g^r_d (in other words, we are dropping the condition that the g^r_d be very ample). We can ask what is the smallest possible dimension of $M^r_{g,d}$?

One might suppose that the most special curves, from the point of view of questions 2 and 3, are hyperelliptic curves but the locus in M_g of hyperelliptic curves has dimension $2g - 1$. What about smooth plane curves? That's better – in the sense that the locus in M_g of smooth plane curves has dimension asymptotic to g, as the following exercise will show – but there are still a lot of them.

Exercise 13.5.3. 1. Let $C \subset \mathbb{P}^2$ be a smooth plane curve of degree d. Show that the g^2_d cut by lines on C is unique; that is, $W^2_d(C)$ consists of one point.

2. Using this, find the dimension of the locus of smooth plane curves in M_g.

Can we do better? Well, in $\mathbb{P}^3$ we can consider the locus of smooth complete intersections of two surfaces of degree m. As we saw in Exercise 12.3.5, these comprise an open subset $\mathcal{H}^\circ_{ci}$ of the Hilbert scheme of curves of degree $d = m^2$, and genus g given by the relation

$$2g - 2 = \deg K_C = m^2(2m - 4),$$

or, asymptotically,

$$g \sim m^3.$$

Moreover, the dimension of this component of the Hilbert scheme is easy to compute, since as we saw in Exercise 12.3.6 that it is isomorphic to an open subset of the Grassmannian $G(2, \binom{m+3}{3})$, and so has dimension

$$2(\binom{m+3}{3} - 2) \sim \frac{m^3}{3}$$

Finally, we observe that if $C \subset \mathbb{P}^r$ is a complete intersection curve of genus $g > 1$, the canonical bundle K_C is a positive power of $\mathcal{O}_C(1)$, and by Lasker's Theorem C is linearly normal. In particular, for a given abstract curve C there are only finitely many embeddings of C in projective space $\mathbb{P}^r$ as a complete intersection, up to PGL_{r+1}; in other words, the fibers of $\mathcal{H}^\circ_{ci}$ over M_g have dimension $\dim(PGL_{r+1}) = r^2 + 2r$.

Thus, we have a sequence of components of the restricted Hilbert scheme $\mathcal{H}^\circ$ whose images in M_g have dimension tending asymptotically to $g/3$.

The following exercise suggests why we chose complete intersections of surfaces of the same degree.

Exercise 13.5.4. Consider the locus of curves $C \subset \mathbb{P}^3$ that are complete intersections of a quadric surface and a surface of degree m. Show that these comprise components of the restricted Hilbert scheme, and that their images in moduli have dimension asymptotically approaching g as $m \to \infty$.

More generally, we can consider complete intersections of $r-1$ hypersurfaces of degree m in $\mathbb{P}^r$; in a similar fashion we can calculate that their images in M_g have dimension asympotically approaching $2g/r!$ as $m \to \infty$, as we ask you to verify in the following exercise.

Exercise 13.5.5. Consider the locus $\mathcal{H}^\circ_{ci}$, in the Hilbert scheme $\mathcal{H}^\circ$, of smooth, irreducible, nondegenerate curves $C \subset \mathbb{P}^r$ that are complete intersections of $r-1$ hypersurfaces of degree m.

1. Show that $\mathcal{H}^\circ_{ci}$ is open in $\mathcal{H}^\circ$;
2. Calculate the dimension of $\mathcal{H}^\circ_{ci}$ (and observe that it is irreducible); and
3. Show that the dimension of the image of $\mathcal{H}^\circ_{ci}$ in M_g is asymptotically $2g/r!$ as $m \to \infty$

The question is, can we do better? For example, if we fix r, can we find a sequence of components $\mathcal{H}_n$ of restricted Hilbert schemes $\mathcal{H}^\circ_{g_n,r,d_n}$ of curves in $\mathbb{P}^r$ such that

$$\lim \frac{\dim \mathcal{H}_n}{g_n} \;=\; 0?$$

13.5.3 Rigid curves?

In the last section, we considered components of the restricted Hilbert scheme whose image in M_g was "as small as possible." Let's go now all the way to the extreme, and ask: is there a component of the restricted Hilbert scheme $\mathcal{H}^\circ_{g,r,d}$ whose image in M_g is a single point? Of course M_0 itself is a single point, so we exclude genus 0! We can give three flavors of this question, in order of ascending preposterousness.

1. First, we'll say a smooth, irreducible and nondegenerate curve $C \subset \mathbb{P}^r$ is *moduli rigid* if it lies in a component of the restricted Hilbert scheme whose image in M_g is just the point $[C] \in M_g$—in other words, if the linear series $|\mathcal{O}_C(1)|$ does not deform to any nearby curves.

2. Second, we say that such a curve is *rigid* if it lies in a component $\mathcal{H}^\circ$ of the restricted Hilbert scheme such that PGL_{r+1} acts transitively on $\mathcal{H}^\circ$. This is saying that C is moduli rigid, plus the line bundle $\mathcal{O}_C(1)$ does not deform to any other g^r_d on C.

3. Finally, we say that such a curve is *deformation rigid* if the curve $C \subset \mathbb{P}^r$ has no nontrivial infinitesimal deformations other than those induced by PGL_{r+1}—in other words, every global section of the normal bundle $\mathcal{N}_{C/\mathbb{P}^r}$ is the image of the restriction of a vector field on $\mathbb{P}^r$.

In truth, these are not so much questions as howls of frustration. The existence of irrational rigid curves seems outlandish; we don't know anyone who thinks there are such things. But then *why can't we prove that they don't exist?*

Chapter 14

Hilbert Schemes III: The rest of the Hilbert Scheem

In this section we will give the description of all of the Hilbert scheme of twisted cubics.

((mention Ellia-Hirschowitz-Mezzetti 1992: the number of components not bounded by a poly in d and g. Also, Joachim Jelisiejew has proved Murphy for points in A^{16} and he says his student showed that 13 points in A^{6} has an everywhere nonreduced component.))

Chapter 15

Plane Curves

**** David: This started out as a chapter in which we extend the constructions/theorems of the rest of the book to the case of singular curves. We since abandoned that goal as too difficult and open-ended, and my understanding is that we were going to replace it with a chapter on plane curves. So here's the chapter on plane curves, but I haven't changed the name of the file (and I commented out, rather than deleted, the stuff that was already written). ****

As anyone who has read the last chapter knows, describing curves in projective space $\mathbb{P}^3$ is difficult: the ideal of such a curve typically has three or more generators, which in turn have to satisfy certain syzygies; in consequence, even basic facts about them—for example, the dimension of the family of all curves of given degree and genus—remain very unclear. And of course, as you might expect we know even less about curves in $\mathbb{P}^n$ for $n > 3$.

The case of plane curves makes a striking contrast: a curve $C \subset \mathbb{P}^2$ is necessarily the zero locus of a single homogeneous polynomial, and conversely any homogeneous polynomial $F(X, Y, Z)$ defines a plane curve. There is a downside, however: while any smooth projective curve can be embedded in $\mathbb{P}^r$ for any $r \geq 3$, most curves cannot be embedded in the plane.

On the other hand, it is true that every curve can be birationally embedded in $\mathbb{P}^2$: we can embed C as a curve $\tilde{C} \subset \mathbb{P}^r$ in a higher-dimensional projective space and find a projection $\mathbb{P}^r \to \mathbb{P}^2$ that carries $\tilde{C}$ birationally onto its image. This is indeed how 19th century geometers typically described a curve, in the days before abstract varieties: as the normalization of a plane curve. (The points on the normalization were realized as valuations on the function field of C, essentially taking advantage of the fact that for smooth curves birational and biregular isomorphism are the same thing.) Much of the analysis of the geometry of the curve—for example, the description of the linear systems on the curve—was carried out in this setting.

Plane curves, in other words, occupied a central role in the development of the theory of algebraic curves; and there are still many aspects of the geometry of a curve that are best approached in this way. In this chapter, we'll describe some of the tools used to study curves via their plane models.

15.1 Our goal

There are many, many questions we can ask about the geometry of plane curves, and we'll touch on several of these in what follows. But in this chapter we'll focus for the most part on one basic problem, which we'll now describe.

We've been dealing since the opening chapters of this book with a basic construction: given a smooth projective curve C, and a divisor $D = \sum m_i p_i$ on C, we define the complete linear system $|D|$ to be the family of all effective divisors E on C with $E \sim D$. A question we have not yet addressed is a simple but fundamental one: if we are given the equations of C as embedded in some projective space, and the coordinates of the points p_i, can we explicitly and algorithmically determine the complete linear series $|D|$?

The answer is "yes," and the way we do it is by working with plane models of our curves: that is, we realize a given curve C as the normalization of a plane curve $C_0 \subset \mathbb{P}^2$ (for example, by a general projection of C to $\mathbb{P}^2$), and working in the plane.

Let's start by posing two "keynote" problems. In both, we'll be given the equation $F(X, Y, Z)$ of a smooth plane curve C_0, with normalization C; in the second, we'll be given in addition a divisor $D = \sum m_i p_i$ on C. We ask:

1. Find all regular differentials/sections of K_C; that is, write down a basis for $H^0(K_C)$; and

2. Describe the complete linear system $|D|$; that is, find all effective divisors E on C with $E \sim D$.

Note that this subsumes questions like, are two given divisors D and E linearly equivalent? And, is a given divisor D linearly equivalent to an effective divisor?

Our plan is to solve these problems in three stages of increasing generality. To start with, in the following section we'll solve these problems in case $C = C_0$ is a smooth plane curve. (As we've observed, this is a very limited form of our general problem—the vast majority of curves cannot be realized as smooth plane curves—but it will serve to establish our basic approach.) In Section 15.3 we'll do it for (the normalization of) a plane curve with nodes. This is a significant expansion by virtue of Theorem **??**, which says that every smooth projective curve is the normalization of a nodal plane curve.

Finally, in Section 15.4 we'll describe how to answer these questions algorithmically for the normalization of an arbitrary plane curve. This last extension is significant for practical reasons: many times a curve C is given to us as (the normalization of) a plane curve with singularities other than nodes, and while Theorem **??** assures us in principal that we can also realize C as the normalization of a nodal plane curve, it is almost always easier to work with the original plane model, as given.

15.2 Smooth plane curves

For this section, we'll we'll take $C \subset \mathbb{P}^2$ to be a smooth plane curve, given as the zero locus of a homogeneous polynomial $F(X,Y,Z)$ of degree d. We'll introduce affine coordinates $x = X/Z$ and $y = Y/Z$ on the affine open subset $U \cong \mathbb{A}^2$ given by $Z \neq 0$, and let $f(x,y) = F(X,Y,1)$ be the inhomogeneous form of F, so that $\tilde{C} = C \cap U$ is given as the zero locus $V(f) \subset \mathbb{A}^2$.

Finally, just for simplicity, we'll make a couple assumptions about the relation of C to the coordinates in $\mathbb{P}^2$:

1. We'll assume that the point $[0,1,0]$ (that is, the point at infinity in the vertical direction) does not lie on C; in other words, the projection $C \to \mathbb{P}^1$ given by $[X,Y,Z] \mapsto [X,Z]$ (or $(x,y) \mapsto x$ in affine coordinates) has degree d; and

2. We'll assume that the line L at infinity given by $Z = 0$ intersects C transversely in d distinct points $p_1, \dots, p_d$; by way of notation, we'll denote by H the divisor $H = p_1 + \cdots + p_d$ of intersection of C with L.

These conditions are clearly satisfied for a general choice of coordinate system, so they're not restrictive. Nor are they necessary: in Exercise **??** we'll see how to do everything in the absence of these assumptions, albeit with slightly more complicated notation.

Enough talk! Let's see how to solve the damn problem. We start by drastically scaling back our ambitions: instead of asking for a basis of $H^0(K)$, let's just see if we can write down a single rational 1-form on C. This is manageable: we can just take a regular 1-form on $\mathbb{A}^2$, such as dx, and restrict/pull back to C. In fact, this will be regular on the open subset $\tilde{C}$, but a change of coordinate calculation shows that it will have double poles at the points $p_1, \dots, p_d$ of $H = C \cap L$.

How do we get rid of the poles of dx? The natural thing to do would be simply to divide dx by a polynomial $h(x,y)$ of degree 2 or more, but there's a problem: $h(x,y)$ will vanish at points of $C \cap U$, potentially creating poles of the quotient dx/h. There is a solution to this: *choose a polynomial h that vanishes only at those points of $C \cap U$ where dx already has a zero*; hopefully, the zeroes of h will just cancel the zeroes of dx rather than creating new poles.

In fact, we have just the polynomial: we take

$$h(x,y) = \frac{\partial f}{\partial y}(x,y).$$

To see that this works, note that on C,

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy \equiv 0.$$

Since $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ have no common zeroes on $\tilde{C}$ by the hypothesis of smoothness, we see that at any point $p \in \tilde{C}$,

$$\mathrm{ord}_p(dx) = \mathrm{ord}_p(\frac{\partial f}{\partial y})$$

so in fact the quotient

$$\omega_0 = \frac{dx}{\partial f/\partial y}$$

is everywhere regular and nowhere 0 in $\tilde{C}$.

What about the points p_i? Well, the differential dx had poles of order 2 at the points p_i; and $\partial f/\partial y$, being a polynomial of degree $d-1$, will have poles of order $d-1$. We conclude that ω_0 has zeroes of order $d-3$ at the points p_i; in other words, the divisor

$$(\omega_0) = (d-3)D.$$

In particular, if $d \geq 3$ then ω_0 is a global regular differential on C.

This also says that we can afford to multiply ω_0 by any polynomial $g(x,y)$ of degree $d-3$ or less without introducing poles, so that

$$g\omega_0 = \frac{g(x,y)dx}{\partial f/\partial y}$$

is likewise a global regular differential, for g any polynomial of degree $\leq d-3$.

We have thus found a vector space of regular differentials, of dimension $\binom{d-1}{2}$. But at the same time, the degree of a differential like ω_0 is

$$\deg((\omega_0)) = (d-3)\deg(D) = d(d-3),$$

so that the genus of C is

$$\frac{d(d-3)}{2} + 1 = \binom{d-1}{2}.$$

In other words, we have found all the global regular differentials on C! We have

$$H^0(K_C) = \left\{ \frac{g(x,y)dx}{\partial f/\partial y} \mid \deg g \leq d-3 \right\};$$

or, equivalently, the space of regular differentials on C has basis $\{\omega_{i,j}\}_{i+j\leq d-3}$, where

$$\omega_{i,j} = \frac{x^i y^j dx}{\partial f/\partial y}$$

In fact, we could have done this all abstractly, without coordinates: by adjunction, we have

$$K_C = (K_{\mathbb{P}^2} \otimes \mathcal{O}_{\mathbb{P}^2}(d))|_C = \mathcal{O}_C(d-3),$$

and from the exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^2}(-3) \to \mathcal{O}_{\mathbb{P}^2}(d-3) \to \mathcal{O}_C(d-3) = K_C \to 0$$

and the vanishing of $H^1(\mathcal{O}_{\mathbb{P}^2}(-3))$, we see that the map on global sections

$$H^0(\mathcal{O}_{\mathbb{P}^2}(d-3)) \to H^0(K_C)$$

is surjective.

Exercise 15.2.1. Let C be a smooth plane curve of degree d. Show that C admits a map $C \to \mathbb{P}^1$ of degree $d-1$, but does not admit a map $C \to \mathbb{P}^1$ of degree $d-2$ or less.

15.2.1 Finding complete linear systems on smooth plane curves

What about our second problem, finding all effective divisors linearly equivalent to a given divisor D? To answer this, start by expressing D as the difference

$$D = E - F$$

of two effective divisors on C. Next, let $G(X, Y, Z)$ be a polynomial in the plane of any degree m vanishing on the divisor E, but not vanishing identically on C (taking m sufficiently large ensures the existence of such a polynomial), and let A be the divisor cut on C residually; that is, we write

$$(G) = E + A$$

as divisors on C. Now, let H be another polynomial of the same degree m as G, vanishing on $A+F$ but again not vanishing identically on C. (Since m is already chosen, there may not exist any such polynomial, which is as it should be: the original divisor D need not be linearly equivalent to any effective divisor.) Let D' be the divisor cut on C by H residual to $A + F$; that is, write

$$(H) = A + F + D'.$$

Now, the divisor of the rational function H/G is principal, so we have

$$0 \sim (H) - (G) = D' + F - E$$

or in other words, D' *is an effective divisor linearly equivalent to* D.

Note that in the simplest nontrivial case $d = 3$, we have reproduced the classic description of the group law on a plane cubic curve C. If we choose as origin on the curve C a point o, then to add two points p and $q \in C$ means to find the (unique) effective divisor of degree 1 linearly equivalent to $p+q-o$. In this situation, we can carry out the process described above with $m = 1$: draw the line L through the points p and q, and let $r \in C$ be the remaining point of intersection of L with C; then draw the line M though the points r and o, and let $s \in C$ be the remaining point of intersection of L with C. This is the classical construction of the group law.

We claim now that in fact we have found in this way *all* effective divisors $D' \sim D$. To see this, suppose D' is any effective divisor with $D' \sim D$. Carrying out the first step of the process as before, we arrive at a divisor A with

$$\mathcal{O}_C(A + F + D') = \mathcal{O}_C(A + F + D) = \mathcal{O}_C(m).$$

But from the exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^2}(m - d) \to \mathcal{O}_{\mathbb{P}^2}(m) \to \mathcal{O}_C(m)] \to 0$$

and the vanishing of $H^1(\mathcal{O}_{\mathbb{P}^2}(m - d))$, we have that every global section of $\mathcal{O}_C(m)$ is the restriction to C of a homogeneous polynomial of degree m on $\mathbb{P}^2$. Thus there is a polynomial H cutting out the divisor $A + F + D'$ on C, as claimed.

Note that if, in the process described, it turns out there is no polynomial H vanishing on $A + F + D$ but not vanishing identically on C, that simply means that $|D| = \emptyset$; that is, D *is not linearly equivalent to any effective divisor.* (It may not be obvious that the existence of such an H is independent of the choice of m or G, but the argument here shows it is.)

15.3 Nodal plane curves

As noted, smooth plane curves are very special among all curves. We now want to carry out the analyses above for a larger class of plane curves, curves with at most nodes as singularities. These are still special among all plane curves, but as we'll see in Section 15.3.2 below, every smooth curve is the normalization of a nodal plane curve, so that this will in theory allow us to answer the "keynote" questions above for an arbitrary smooth curve. (In the final section of this chapter, we'll indicate how the constructions here may be extended to an arbitrary plane curve.)

The set-up, in any event, is as follows: we have a nodal plane curve $C_0 \subset \mathbb{P}^2$, with normalization $\nu : C \to C_0$; or, equivalently, a smooth projective curve C and a birational embedding of C in $\mathbb{P}^2$ with image a nodal curve C_0. Our goals will be as before:

1. to write out explicitly all global regular 1-forms on C; and
2. given a divisor D on C, to determine $|D|$; that is, find all effective divisors linearly equivalent to D.

As before, we'll choose homogeneous coordinates $[X, Y, Z]$ on $\mathbb{P}^2$ so that the curve C_0 intersects the line $L = V(Z)$ at infinity transversely at points $p_1, \dots, p_d$ other than $[0, 1, 0]$ (meaning in particular that all the nodes of C_0 lie in the affine plane $U = \mathbb{P}^2 \setminus L$). In addition, we can assume that neither branch of C_0 at a node has vertical tangent. (As before, these conditions are not logically necessary; they serve only to keep the notation reasonably simple, and in any case are satisfied by a general choice of coordinates.) Let the nodes of C_0 be $q_1, \dots, q_\delta$, with $r_i, s_i \in C$ lying over q_i; we'll denote by Δ the divisor $\sum r_i + \sum s_i$ on C.

Let $F(X, Y, Z)$ be the homogeneous polynomial of degree d defining the curve C_0, and let $f(x, y) = F(x, y, 1)$ be the defining equation of the affine part $C_0 \cap U$ of C_0. We start by considering the rational differential $\nu^*(dx)$ on C. In the smooth case, we saw that this differential was regular and nonzero in the finite plane, but had poles of order 2 at the point of $C \cap L$; this followed from the equation

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy \equiv 0.$$

and the fact that $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ have no common zeroes on C_0. But now $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ *do* have common zeroes; specifically, the pullbacks $\nu^*(\frac{\partial f}{\partial x})$ and $\nu^*(\frac{\partial f}{\partial y})$ have simple zeroes at the points r_i and s_i. We conclude, accordingly, that the differential $\nu^* dx$ has double poles at the points p_i, and simple poles at the points r_i and s_i; proceeding as before, we see that for a polynomial $g(x, y)$ of degree $\leq d - 3$, the differential

$$\nu^*(\frac{g(x, y)dx}{\partial f / \partial y})$$

will be regular except for simple poles at the points r_i and s_i.

So, how do we get rid of these poles? There is one simple way: we require that g vanishes at the points q_i. We say in this case that g (and the curve defined by g) *satisfies the adjoint conditions.* (In the following section, we'll describe the adjoint conditions associated to an arbitrary singularity.) In any event, we see that

$$\left\{ \nu^* \frac{g(x, y)dx}{\partial f / \partial y} \mid \deg g \leq d - 3 \text{ and } g(q_i) = 0 \;\forall i \right\} \subset H^0(K_C).$$

Now, in the smooth case, we were able to compare dimensions to conclude that this inclusion was indeed an equality. We can do the same thing here: to begin with, we have seen that the rational 1-form $\omega = \nu^*(\frac{dx}{\partial f / \partial y})$ has zeroes of order $d - 3$ at the points $p_1, \dots, p_d$ and simple poles at the points r_i and s_i and is

otherwise regular and nonzero; in other words, if we set $H = p_1 + \cdots + p_d$, the divisor
$$(\omega) = (d-3)H - \Delta.$$
In particular, we see that
$$\deg((\omega)) = d(d-3) - 2\delta$$
and correspondingly
$$g(C) = \binom{d-1}{2} - \delta;$$
this is called the genus formula for plane curves.

On the other hand, the space of polynomials g of degree $\leq d-3$ vanishing at the points q_i has dimension at least $\binom{d-1}{2} - \delta$; we conclude from this that indeed
$$H^0(K_C) = \left\{ \nu^* \frac{g(x,y)dx}{\partial f/\partial y} \mid \deg g \leq d-3 \text{ and } g(q_i) = 0 \ \forall i \right\},$$
and as lagniappe we see also that *the nodes q_i of an irreducible nodal plane curve of degree d impose independent conditions on curves of degree $d-3$.*

In Exercise 15.2.1, we saw how to use the description of the canonical series on a smooth plane curve to determine its gonality. Now that we have an analogous description of the canonical series on (the normalization of) a nodal plane curve, we can deduce a similar statement about the gonality of such a curve. Here are the first two cases

Exercise 15.3.1. Let C_0 be a plane curve of degree d with one node and no other singularities, and let C be its normalization. Show that C admits a unique map $C \to \mathbb{P}^1$ of degree $d-2$, but does not admit a map $C \to \mathbb{P}^1$ of degree $d-3$ or less.

Exercise 15.3.2. Let C_0 be a plane curve of degree d with two nodes and no other singularities, and let C be its normalization. Show that C admits two maps $C \to \mathbb{P}^1$ of degree $d-2$, but does not admit a map $C \to \mathbb{P}^1$ of degree $d-3$ or less.

15.3.1 Linear series on a nodal curve

Next, we take up the second of our keynote problems in this setting: with $C \to C_0 \subset \mathbb{P}^2$ as above, given a divisor D on C, can we find the complete linear series $|D|$?

In fact we can, by a process analogous to what we did in the smooth case. We'll do this first in the case where $D = E - F$ is the difference of two effective divisors whose support is disjoint from the support $\{r_i, s_i\}$ of Δ; the general case is only notationally more complicated. To start, we find an integer m and a polynomial G vanishing on the divisor E *and at the nodes* $r_1, \dots, r_\delta$ *of* C_0, but not vanishing identically on C_0. We can then write the zero locus of G pulled back to C as

$$(\nu^* G) = E + \Delta + A,$$

as before. Once more, just for simplicity, let's assume that the support of A is disjoint from the support of Δ; this means just that the curve $V(G)$ is smooth at the points q_i and is not tangent to either of the branches of C_0 there (this can certainly be done if we take m large).

Next, we find polynomials H of the same degree m, vanishing at $A + F$ and at the points q_i but not on all of C_0. Let D' be the divisor cut on C by H residual to $E + \Delta + A$; that is, we write

$$(\nu^* H) = E + \Delta + A + D'.$$

Finally, since $\nu^*(G/H)$ is a rational function on C, we see that

$$E + \Delta + A = (\nu^* H) \sim (\nu^* G) = E + \Delta + A + D',$$

and we conclude that D' is an effective divisor linearly equivalent to D on C.

But, do we get in this way *all* effective divisors linearly equivalent to D on C? The answer is yes, but it's not immediate; it follows from the following proposition, known classically as *completeness of the adjoint series.*

Proposition 15.3.3. *If $C_0 \subset \mathbb{P}^2$ is a nodal plane curve and $\nu : C \to C_0$ its normalization, then the linear series cut on C by plane curves of degree m passing through the nodes is complete.*

Note that the solution to our problem follows from this proposition exactly as in the smooth case; that is, *every* effective divisor $D' \sim D$ on C is obtained in this way.

Proof. To prove Proposition 15.3.3, it will be helpful to introduce another surface: the blow-up $\pi : S \to \mathbb{P}^2$ of $\mathbb{P}^2$ at the points r_i. The proper transform on $C_0 \subset \mathbb{P}^2$ in S is the normalization of C_0, which we will again call C.

There are two divisor classes on S that will come up in our analysis: the pullback of the class of line in $\mathbb{P}^2$, which we'll denote H; and the sum of the exceptional divisors, which we'll call E. In these terms, we have

$$C \sim dH - 2E \quad \text{and} \quad K_S \sim -3H + E$$

(the first follows from the fact that C_0 has multiplicity 2 at each of the points q_i, the second from considering the pullback to S of a rational 2-form on $\mathbb{P}^2$). If A is a curve in $\mathbb{P}^2$ of degree m passing through the points q_i, we can associate to it the effective divisor $\pi^* A - E$; this gives us an isomorphism

$$H^0(\mathcal{I}_{\{q_1, \dots, q_\delta\}/\mathbb{P}^2}(m)) \cong H^0(\mathcal{O}_S(mH - E)).$$

In these terms we can describe the linear series cut on C by plane curves of degree m passing through the nodes of C_0 as the image of the map

$$H^0(\mathcal{O}_S(mH-E)) \to H^0(\mathcal{O}_C(mH-E)),$$

and the proposition amounts to the assertion that this map is surjective.

The obvious way to prove this is to view this map as part of the long exact cohomology sequence associated to the exact sequence of sheaves

$$0 \to \mathcal{O}_S(mH-E-C) = \mathcal{O}_S((m-d)H+E) \to \mathcal{O}_S(mH-E) \to \mathcal{O}_C(mH-E) \to 0,$$

from which we see that it will suffice to establish that $H^1(\mathcal{O}_S((m-d)H+E)) = 0$. To do this, we apply Serre duality, which says that $H^1(\mathcal{L}) \cong H^1(K_S \otimes \mathcal{L}^{-1})^*$; in this instance it tells us that

$$H^1(\mathcal{O}_S((m-d)H+E)) \cong H^1(\mathcal{O}_S((d-m-3)H))^*$$

Now, the line bundle $\mathcal{O}_S((d-m-3)H)$ is just the pullback to S of the bundle $\mathcal{O}_{\mathbb{P}^2}(d-m-3)$, which has vanishing H^1; thus the Proposition will follow from the

Lemma 15.3.4. *Let X be a smooth projective surface, and $\pi : Y \to X$ a blow-up. If $\mathcal{L}$ is any line bundle on X, then*

$$H^1(Y, \pi^*\mathcal{L}) = H^1(X, \mathcal{L}).$$

The lemma follows by applying the Leray spectral sequence, which relates the cohomology of $\mathcal{L}$ on Y to the cohomology of the direct image $\pi_*\pi^*\mathcal{L}$ (Leray is particularly simple in this setting, since all higher direct images are 0), and the observation that $\pi_*\pi^*\mathcal{L} \cong \mathcal{L}$. □

15.3.2 Existence of good projections

In this section, we want to verify the assertion made above that every smooth curve C is birational to a nodal plane curve $C_0 \subset \mathbb{P}^2$; we'll do this by first embedding C in $\mathbb{P}^n$, and then arguing that the projection $\pi_\Lambda : C \to \mathbb{P}^2$ from a general $(n-3)$-plane $\Lambda \subset \mathbb{P}^n$ is birational onto its image C_0 and that C_0 has only nodes as singularities. This is certainly plausible, but in fact a proof relies on an application of the uniform position lemma of Section **??**.

Proposition 15.3.5. *If $C \subset \mathbb{P}^n$ is a smooth curve in projective space, and $\Lambda \cong \mathbb{P}^{n-3} \subset \mathbb{P}^n$ a general $(n-3)$-plane, then the projection $\pi_\Lambda : C \to \mathbb{P}^2$ is birational onto its image, which will be a nodal curve.*

Proof. The basic idea here is to look at how the plane Λ intersects the *secant variety* of the curve $C \subset \mathbb{P}^n$. The secant variety consists of the union of the lines $\overline{q,r}$ joining pairs of distinct points $q, r \in C$, plus the tangent lines $\mathbb{T}_q(C)$;

altogether, these lines form a single family, parametrized by the symmetric square of C. The secant variety thus has dimension 3, so that a general $(n-3)$-plane Λ will meet it in finitely many points $p \in \mathbb{P}^n$. These will in turn correspond to the singularities of the image curve $C_0 \subset \mathbb{P}^2$: if $p \in \overline{q,r}$ lies on a secant line, then the projection π_Λ fails to be one-to-one at the image point $\pi_\Lambda(q) = \pi_\Lambda(r)$, while if p lies on a tangent line $\mathbb{T}_q(C)$ the differential of π_Λ will vanish at q. To prove the proposition, accordingly, we just have to say a little more about the intersection of Λ with the secant variety of C.

It is logically superfluous, but it will be much easier to visualize what's going on if we first reduce to the case $n = 3$. This is straightforward: if $\Gamma \subset \mathbb{P}^n$ is a general $(n-4)$-plane, then a general $(n-3)$-plane Λ containing Γ is a general $(n-3)$-plane in $\mathbb{P}^n$, so we can view the projection $\pi_\Lambda : C \to \mathbb{P}^2$ as the composition of the projection $\pi_\Gamma : C \to \mathbb{P}^3$ with the projection π_p of the image $\pi_\Gamma(C)$ from a general point $p \in \mathbb{P}^3$. Moreover, since a general $(n-4)$-plane $\Gamma \subset \mathbb{P}^n$ will be disjoint from the secant variety of $C \subset \mathbb{P}^n$, the projection $\pi_\Gamma : C \to \mathbb{P}^3$ will be an embedding; thus we can just start with a smooth curve $C \subset \mathbb{P}^3$ and project from a general point $p \in \mathbb{P}^3$.

Now, when we project $C \subset \mathbb{P}^3$ from a general point, we do expect to introduce singularities: since the family of secant lines to C is two-dimensional, we expect a finite number of them will contain a general point $p \in \mathbb{P}^3$. Indeed, we see from this naive dimension count that p can lie on at most a finite number of secant lines, proving that *the map π_p is birational onto its image*. We can say more: since there is only a 1-dimensional family of tangent lines to C, a general point $p \in \mathbb{P}^3$ will not lie on any tangent lines, we see as well that the differential of $\pi_p : C \to \mathbb{P}^2$ does not vanish; in other words, *the map π_p is an immersion*.

We now just have to make sure the singularities introduced are just nodes, and here is where we need to invoke the uniform position lemma. For example, suppose we want to prove that the map π_p is never three-to-one. This amounts to saying that a general point $p \in \mathbb{P}^3$ does not lie on any *trisecant line*; equivalently, that the family of trisecant lines to C has dimension at most one. Moreover, since the variety of secant lines is irreducible of dimension 2, this follows from the simple assertion that *not every secant line to $C \subset \mathbb{P}^3$ is a trisecant.*

At this point we'd encourage you to try to give an elementary argument for this seemingly obvious assertion. Whether you succeed or not, it does follow immediately from the uniform position lemma: this says that if $H \subset \mathbb{P}^3$ is a general plane, then the points of intersection $p_1, \dots, p_d \in C \cap H$ are in linear general position in H; that is, no three are collinear. The line joining any two is thus a secant line to C but not a trisecant line, which suffices to prove our assertion.

So now we know that the map pi_p is an immersion, and at most two-to-one everywhere; thus the image curve $C_0 \subset \mathbb{P}^2$ will have at most double points, and an analytic neighborhood of each double point will consist of two smooth branches. To complete the proof of Proposition 15.3.5 we have to show that those two branches have distinct tangent lines; that is, that if $q, r \in C$ are any two points collinear with p, then the images of the tangent lines $\mathbb{T}_q(C)$ and $\mathbb{T}_r(C)$ in $\mathbb{P}^2$ are distinct. But if in fact it were the case that $\pi_p(\mathbb{T}_q(C)) = \pi_p(\mathbb{T}_r(C))$ then the tangent lines $\mathbb{T}_q(C)$ and $\mathbb{T}_r(C)$ would necessarily intersect.

Classically, a secant line $\overline{q, r}$ was called a *stationary secant* if the tangent lines $\mathbb{T}_q(C)$ and $\mathbb{T}_r(C)$ met; our remaining goal now is to show that *a general point $p \in \mathbb{P}^3$ does not lie on any stationary secant.* Again, this follows from the assertion that the family of stationary secants has dimension at most 1; and again, since the family of all secant lines is irreducible of dimension 2, it will suffice to show that *not every secant line to C is a stationary secant.*

And now we are done, at least in characteristic 0. Simply, it can't be the case that for a general pair of points $q, r \in C$ we have $\mathbb{T}_q C \cap \mathbb{T}_r C \neq \emptyset$; otherwise the projection $\pi_{\mathbb{T}_q C} : C \to \mathbb{P}^1$ would have derivative everywhere 0, but be nonconstant. Thus the space of stationary secants is at most 1-dimensional, and a general $p \in \mathbb{P}^3$ will not lie on one. □

Note that the the last case in this argument relies on the hypothesis of characteristic 0. Indeed, in characteristic $p > 0$ it may not be the case that a general projection of a smooth curve to $\mathbb{P}^2$ is nodal—but it's still true that every curve is the normalization of a nodal plane curve: we have the slightly weaker

Proposition 15.3.6 (Proposition 15.3.5 in characteristic p)**.** *Let $C \subset \mathbb{P}^n$ be a smooth curve. If we re-embed C by a Veronese map of sufficiently high degree—that is, we let $\nu_m : \mathbb{P}^n \to \mathbb{P}^N$ be the mth Veronese map, and let $\tilde{C} = \nu_m(C)$ be image of C—then the projection of $\tilde{C}$ from a general $\mathbb{P}^{N-3}$ will be nodal.*

Exercise 15.3.7. In the setting of Proposition 15.3.6, for a general $\Lambda \cong \mathbb{P}^{N-3}$ show that

1. there do not exist three points $p, q, r \in \tilde{C}$ such that p, q, r and Λ are all contained in a $\mathbb{P}^{N-2}$; and
2. there does not exist a pair of points $p, q \in \tilde{C}$ such that p, q and Λ are contained in a $\mathbb{P}^{N-2}$ and $\mathbb{T}_p(\tilde{C}), \mathbb{T}_q(\tilde{C})$ and Λ are contained in a $\mathbb{P}^{N-1}$.

From this one can deduce Proposition 15.3.6.

Exercise 15.3.8. Let C_0 be a plane quartic curve with two nodes q_1, q_2; let $\nu : C \to C_0$ be its normalization, and let $o \in C$ be any point not lying over a node of C_0. By the genus formula, C has genus 1. Using the construction above, describe the group law on C with o as origin.

15.4 Arbitrary plane curves

Well, not exactly arbitrary: in this section, we'll deal with a plane curve C_0 that is assumed to be reduced and irreducible, with normalization $\nu : C \to C_0$.

The geometry of singular curves is a fascinating topic, from the local analysis of the singularities to the global questions involving linear series on singular curves. Indeed, it's remarkable how many of the constructions and theorems we've discussed in the realm of smooth curves can be extended to the world of singular curves, given the right definitions (and some restrictions on the type of singularities, such as the Gorenstein condition). But this is a topic beyond our ken, at least in this book; for us, the questions are about smooth curves, with singular curves appearing as a useful adjunct. The description, in the last section, of complete linear series on a smooth curve C, using a nodal plane model C_0 of C, is a perfect example.

There is, however, one fundamental invariant of a singular curve C that is both readily calculated and highly relevant in relating it to smooth curves: the *arithmetic genus* of C. This will come up in what we're going to do next, which is to describe linear series on a smooth curve C via a birational model as a plane curve with general singularities, and so we'll take a moment out here and introduce this notion.

15.4.1 Arithmetic genus and geometric genus

To start with, the arithmetic genus is a very broadly applicable: it is defined for an arbitrary one-dimensional scheme over a field. Recall that among the characterizations of the genus g of a smooth projective curve C there was one in terms of the Euler characteristic of the structure sheaf: we have $g = 1-\chi(\mathcal{O}_C)$. This is directly equivalent to the characterization in terms of the constant term of the Hilbert polynomial of any projective embedding.

And that is how we extend the notion of genus to arbitrary singular curves C_0: for any 1-dimensional scheme C_0 over a field, we define the *arithmetic genus* of C_0 to be $1-\chi(\mathcal{O}_{C_0})$. Very often (as, for example, right now!) we want also to deal at the same time with the genus of the normalization $\nu : C \to C_0$; to distinguish between these two notions of the genus of a singular curve, we call $1-\chi(\mathcal{O}_{C_0})$ the arithmetic genus of C_0 and denote it $p_a(C_0)$; the genus of the normalization is called the *geometric genus* and often denoted $g(C_0)$ or $p_g(C_0)$.

What's the difference? Well, we can relate the two notions via the map of sheaves

$$\mathcal{O}_{C_0} \to \nu_*\mathcal{O}_C.$$

This is injective; the cokernel sheaf will be a skyscraper sheaf supported exactly on the singular points of C_0. Denoting this sheaf by $\mathcal{F}$, we have an exact sequence

$$0 \to \mathcal{O}_{C_0} \to \nu_*\mathcal{O}_C \to \mathcal{F} \to 0.$$

Now, the normalization map $\nu : C \to C_0$ is finite, so that the higher direct images $R^i\nu_*\mathcal{O}_C = 0$ for $i > 0$; it follows from the Leray spectral sequence that

$\chi(\nu_*\mathcal{O}_C) = \chi(\mathcal{O}_C)$. We have, accordingly,

$$p_a(C_0) - g(C) = \chi(\mathcal{O}_C) - \chi(\mathcal{O}_{C_0}) = \chi(\mathcal{F}) = h^0(\mathcal{F});$$

in other words, the difference between the arithmetic and geometric genera of C_0 is the sum of the vector space dimensions of the stalks on $\mathcal{F}$; colloquially, it's the number of linear conditions a function f on C has to satisfy to be the pullback of a function from C_0. The length of the stalk of $\mathcal{F}$ at a particular singular point $p \in C_0$ is called the *δ-invariant of the singularity*; to rephrase the statement above in these terms, we have

$$p_a(C_0) - g(C) = \sum_{p \in (C_0)_{sing}} \delta_p$$

Happily, the δ invariant of a singularity is readily calculated. Here are some examples:

1. (nodes) If $p \in C_0$ is a node, with points $r, s \in C$ lying over it, the condition for a function f on C to descend is simply that $f(r) = f(s)$; this is one linear condition and accordingly $\delta_p = 1$.

2. (cusps) If $p \in C_0$ is a cusp, with $r \in C$ lying over it, the condition for a function f on C to descend is simply that the derivative $f'(r) = 0$; again, this is one linear condition and accordingly $\delta_p = 1$.

3. (tacnodes) Suppose now that $p \in C_0$ is a *tacnode*, that is, C_0 has two smooth branches at p simply tangent to one another. There will be two points $r, s \in C$ lying over it, and the condition for a function f on C to descend is that in terms of suitable local coordinates both $f(r) = f(s)$ and $f'(r) = f'(s)$. This represents two linear conditions and accordingly $\delta_p = 2$.

4. (planar triple points) Next up, consider an ordinary triple point $p \in C_0$ of a plane curve: that is, a singularity consisting of three smooth branches meeting pairwise transversely, such as the zero locus of $y^3 - x^3$. There will be three points $r, s, t \in C$ lying over p, and certainly a necessary condition for a function f on C to descend is that $f(r) = f(s) = f(t)$—two linear conditions. But there's a third, less obvious linear condition: in order for f to descend, the derivatives $f'(r), f'(s), f'(t)$ have to satisfy a linear condition—a reflection of the fact that a function on C_0 cannot vanish to order 2 on each of two branches without vanishing to order 2 along the third as well. Thus $\delta_p = 3$

5. (spatial triple points) We will be concerned in what follows only with planar singularities, but spatial triple points provide a useful contrast to the last example. A spatial triple point is a singularity consisting of three smooth branches, with linearly independent tangent lines, so that its

Zariski tangent space is 3-dimensional. An example would be the union of the three coordinate axes in $\mathbb{A}^3$.

In this case, in contrast to the last one, the condition that $f(r) = f(s) = f(t)$ is both necessary and sufficient for f to descend, and accordingly we have $\delta_p = 2$.

Exercise 15.4.1. Let $p \in C$ be a singular point of a reduced curve C. Show that if $\delta_p = 1$, then p must be either a node or a cusp.

15.4.2 Linear series on (the normalization of) a plane curve

We return now to our basic setting: we have a reduced and irreducible curve $C_0 \subset \mathbb{P}^2$ with normalization $\nu : C \to C_0$, and we want to extend our solution to the keynote problems—finding all regular differentials on C, and finding all divisors on C linearly equivalent to a given $D \in \mathrm{Div}(C)$—to this more general setting.

In this setting, if we simply try to mimic the analysis above in the nodal case we're led to introduce the *adjoint ideal* of each singularity (which is simply the maximal ideal of the point in the case of a node), in terms of which we have theorems analogous to the results obtained above in the nodal case.

So: let $C_0 \subset \mathbb{P}^2$ be a reduced and irreducible plane curve, with normalization $\nu : C \to C_0$. We focus for now on one singular point $q \in C_0$, with points $r_1, \dots, r_k \in C$ lying over q. If our goal is to describe the canonical series on C, we can start as we did in the previous two sections: by considering differentials of the form $g(x, y)\omega_0$, where

$$\omega_0 = \nu^* \frac{dx}{\partial f/\partial y},$$

and f is the defining equation of C_0 in an affine open containing q. As we saw in the nodal case, ω_0 will have poles at the points r_i; let m_i be the order of the pole of ω_0 at r_i. We can define the *adjoint ideal* of C_0 at q to be the ideal

$$A_q = \left\{ g \in \mathcal{O}_{\mathbb{P}^2,q} \mid \mathrm{ord}_{r_i}(\nu^* g) \geq m_i \ \forall i \right\}$$

In other words, A is the ideal of functions g such that $\nu^* \frac{g dx}{\partial f/\partial y}$ is regular at all the points r_i. We accordingly define the *adjoint ideal $\mathcal{I}_A$ of C_0* to be the product of A_q over all singular points $q \in C_0$; the *adjoint series of degree m* is then the linear series $H^0(\mathcal{I}_A(m))$.

In these terms, we can give the solution to our keynote problems much as we did in the case of plane curves with nodes. Specifically:

First, we can say that every global regular 1-form on the curve C is of the form

$$\frac{g(x, y)dx}{\partial f/\partial y},$$

with g in the adjoint ideal A, and of degree $d-3$ or less.

Second, if we are given a divisor $D = E - F$ on the curve C, we can find all effective divisors D' on C linearly equivalent to D exactly as we did in the previous case. We start by choosing a polynomial G of any degree m in the adjoint ideal and vanishing on E but not vanishing identically on C_0; we write

$$(G) = E + \Delta + A.$$

We then find all polynomials H of degree m in the adjoint ideal, vanishing on $A + F$ but not vanishing identically on C_0; writing

$$(H) = F + A + \Delta + D'$$

we arrive at an effective divisor D' on C linearly equivalent to D. Indeed, the analog of the theorem of completeness of the adjoint series tells us that we arrive at *every* effective divisor D' on C linearly equivalent to D in this way.

At this point it may seem that without an explicit description of the adjoint ideal we have merely slapped a label on our ignorance. But in fact, the adjoint ideal is relatively straightforward to find. To begin with, let's do some simple examples:

Example 15.4.2 (nodes and cusps)**.** We have already seen that in case q is a node of C_0, there are two points of C lying over it, and $m_1 = m_2 = 1$; the adjoint ideal is thus just the maximal ideal $\mathcal{I}_q$ at q. In the case of a cusp, for example the zero locus of $y^2 - x^3$, there is only one point $r = r_1$ of C lying over q, and the differential ω_0 vanishes to order $m_1 = 2$; since the pullback to C of any polynomial g vanishing at q will vanish to order at least two at r, and so again the adjoint ideal is again just the maximal ideal at q.

Example 15.4.3 (tacnodes)**.** Next, consider the case of a *tacnode*; that is, a singularity with two smooth branches simply tangent to one another, such as the zero locus of $y^2 - x^4$. In this case there are again two points of C lying over q, and a simple calculation shows that $m_1 = m_2 = 2$. The adjoint ideal is thus the ideal of functions vanishing at q and having derivative 0 in the direction of the common tangent line to the branches.

Example 15.4.4 (ordinary triple points)**.** In the case of an ordinary triple point—three smooth branches simply tangent to one another pairwise—there are three points of C lying over q, and we have $m_1 = m_2 = m_3 = 2$; the adjoint ideal is correspondingly just the square of the maximal ideal at q

Exercise 15.4.5. Find the adjoint ideals of the following plane curve singularities:

1. a *triple tacnode*: three smooth branches, pairwise simply tangent
2. a triple point with an infinitely near double point: three smooth branches, two of which are simply tangent, with the third transverse to both

3. a unibranch triple point, such as the zero locus of $y^3 - x^4$

In general, the adjoint ideal of an isolated plane curve singularity is something we can determine in practice; for example, here is a simply general description in case the individual branches of C_0 at p are each smooth:

Proposition 15.4.6. *Let $\nu : C \to C_0$ be the normalization of a plane curve C_0 and $p \in C_0$ a singular point. Denote the branches of C_0 at p by $B_1, \ldots, B_k$, and let r_i be the point in B_i lying over p. If the individual branches B_i of C_0 at p are each smooth, and we set*

$$m_i = \sum_{j \neq i} mult_p(B_i \cdot B_j$$

then the adjoint ideal of C_0 at p is simply the ideal of functions g such that $ord_{r_i}(\nu * g) \geq m_i$.

15.4.3 The conductor ideal

There is another ideal we can associate to an isolated curve singularity, called the *conductor ideal*. It's simple to define: if C_0 is a reduced curve and $\nu : C \to C_0$ its normalization, we can think of the direct image $\nu_* \mathcal{O}_C$ as a module over the structure sheaf $\mathcal{O}_{C_0}$; the conductor ideal is simply the annihilator of the quotient $\nu_* \mathcal{O}_C / \mathcal{O}_{C_0}$. In concrete terms, on any affine open $U \subset C_0$ this is the ideal of functions $g \in \mathcal{O}_{C_0}(U)$ such that for any function $h \in \mathcal{O}_C(\nu^{-1}(U))$ the product hg will be the pullback of a function on C_0.

For example, in the case of a node $q \in C_0$, with r_1, r_2 the points of C lying over q, we see that any function on C vanishing at both r_1 and r_2 is the pullback of a function on C_0; thus the conductor is simply the maximal ideal at q. Similarly, if $q \in C_0$ is a cusp, with $r \in C$ lying over it, a function f on C will descend to C_0—that is, be the pullback of a function on C_0—if it vanishes to order 2 at r; since the pullback to C of any function on C_0 vanishing at q vanishes to order at least 2 at r, the conductor ideal is just the maximal ideal at q.

The sharp-eyed reader will have noticed a coincidence here, and will not be completely surprised by the

Theorem 15.4.7. *For any plane curve singularity, the adjoint ideal and the conductor ideal coincide.*

This is a reflection of the fact that plane curve singularities are necessarily local complete intersections, and hence *Gorenstein*, about which we will hear more in the following chapter.

Chapter 16

Linkage of curves in $\mathbb{P}^3$

((DualityChapter should refer to a different chapter!))

16.1 Introduction

In this Chapter we will study invariants associated to a free resolution, or syzygies, of the homogeneous coordinate ring of a curve in projective space, with an emphasis on their relation to the complete intersections containing the curve—this is the theory of *linkage*. The theory is most powerful in the case of curves in $\mathbb{P}^3$, so we will concentrate on this case. Throughout this Chapter, the word *curve* will refer to a purely 1-dimensional projective scheme.

Recall that two curves in $\mathbb{P}^3$ without common components are *directly linked* if their union is a complete intersection. In this section we will study the generalization of this notion, and the equivalence relation it generates, to the case of arbitrary purely 1-dimensional subschemes of $\mathbb{P}^3$. A simple example is the linkage of a twisted cubic and one of its secant lines, which together form the complete intersection of two quadrics.

We have already used the relation of *linkage* in Chapter 12
((ref?))
in a special case case of smooth curves without common components. In this setting it is obvious that the relation is symmetric, and that the degrees of the two curves add up to the degree of the complete intersection. We showed in **??** that the genera of the two curves is related by the formula **??**.

Linkage was first studied extensively in [Halphén 1882] and taken up in the 1940's in [Apéry 1945] and [Gaeta 1952]. The subject was modernized and generalized in [Peskine and Szpiro 1974]. Hartshorne and his student Rao [Prabhakar Rao 1978/79] made decisive breakthroughs, showing that a simple invariant classifies curves up to linkage; and [Lazarsfeld and Rao 1983] explained how to describe a given linkage equivalence class. A thorough exposition of the subject in the general case can be found in the book [Migliore 1998]. Note that the linkage relation is often called by its French name, *liaison*.

Notation: We write $S := k[x_0, \dots, x_3]$ for the homogeneous coordinate ring of $\mathbb{P}^3$ and $\mathfrak{m} = (x_0, \dots, x_3)$ for its irrelevant ideal. If $X \subset \mathbb{P}^3$ is a subscheme we write I_X for the homogeneous ideal of X and $S_X := S/I_X$ for the homogeneous coordinate ring of X.
((except for the restriction to 3 dimensions, this notation should be standard in the book already...))
In this section, the word *curve* will mean a closed subscheme of pure dimension 1 in a projective space.

16.2 General definition and basic results

Here is the definition of (complete intersection) linkage:

Definition 16.2.1. If X and Y are curves of codimension c in a complete intersection scheme P then X and Y are *directly linked* if there exists a codiimension c complete intersection $Z \subset P$ containing $X \bigcup Y$ such that $I_X = I_Z : I_Y$. In this case we say that X is directly linked to Y by Z.

More generally, we say that X and Y are *evenly* (respectively oddly) linked if they are connected by an even (respectively odd) number of direct linkages.

The relation of direct linkage is symmetric in X and Y, and satisfies the same formulas for degree and genus as in the special case we treated in Theorem **??**:

Theorem 16.2.2. *The relation of direct linkage is symmetric. Moreover, if $X, Y \subset \mathbb{P}^3$ are purely 1-dimensional subschemes and X is linked to Y by the complete intersection Z of surfaces of degrees d_1, d_2, then*

1. *Y is linked to X by Z; that is, linkage is symmetric.*
2. $\deg X + \deg Y = \deg Z = d_1 d_2$.
3. *The arithmetic genera of X and Y are related by*

$$p_a(Y) - p_a(X) = \frac{(d_1 + d_2 - 4)}{2}(\deg Y - \deg X)$$

The proofs involve several important results from commutative algebra:

Theorem 16.2.3. *1. For any ideals G, I in a commutative Noetherian ring R, the associated primes of $J = G : I$ are are among the associated primes of G. Moreover, if G is unmixed (that is, all primary components have the* *same dimension) then the associated primes of J are precisely the associated primes of G whose primary components do not contain I. Moreover, the associated primes of $G : (G : I)$ are the primary components of G whose associated primes contain I.*

2. *(*symmetry*) If R is Gorenstein, G is a complete intersection in R, and $I \subset R$ is an ideal containing G, then $G : (G : I)$ is the intersection of the primary components of I that have the same codimension as I.*

3. *Under these hypotheses, the sum of the multiplicities of R/I and $R/(G : I)$ is the multiplicity of R/G.*

4. $\omega_{R/I} = \mathrm{Hom}(R/I, \omega_{R/G}) \cong (G : I)/G.$

In case $R = k[x_0, \ldots, x_n]$ and both I and G are graded, with $G = (f_1, \ldots f_c)$ the intersection of forms of degree $d_1 + \cdots + d_c$, then $\omega_{R/G} = R/G(\sum d_i - n - 1)$ as graded modules by Example 20.0.6 so, *as graded modules*,

$$\omega_{R/I} = \mathrm{Hom}(R/I, \omega_{R/G}) \cong (G : I)/G(\sum d_i - n - 1).$$

Proof of Theorem 16.2.3. 1) If $G = \bigcap Q_i$ is an irredundant primary decomposition of G then $G : I = \bigcap_i (Q_i : I)$. If $I \subseteq Q_i$, then $Q_i : I = R$, and this term can be omitted. If P_i is the associated prime of Q_i and $I \not\subseteq Q_i$ then $Q_i \subset Q_i : I \subset P_i$ (since P_i is the set of zerdivisors mod Q_i), so $\sqrt{Q_i : I} = P_i$. Furthermore, if $xy \in Q_i : I$ and $x \notin P_i$, then x is a nonzerodivisor mod Q_i, so from $xyI \subset Q_i$ we deduce $yI \subset Q_i$; that is, $y \in Q_i : I$. This shows that $Q_i : I$ is P_i-primary. Finally, if $I \not\subseteq P_i$, then I contains a nonzerodivisor mod Q_i, so $Q_i : I = Q_i$.

This proves that $G : I$ has a primary decomposition whose terms are primary to the associated primes of the primary components of G that do not contain I, so the associated primes of $G : I$ are among these. If some $(Q_i : I)$ were contained in the intersection of the others, then we would have $P_i^n \subset \cap_{j \neq i} P_j$, and P_i would be contained in one of the P_j with $j \neq i$, and this is impossible if all the P_i have the same codimension.

2) Since G is a complete intersection, it is unmixed, and it follows from part 1 that $G : I$ and $G : (G : I)$ are unmixed too. Further, the primary components of $G : I$ have the form $Q_i : I$, where the Q_i are the primary components of G that do not contain I. Now Q_i contains $Q_i : I$ if and only if $Q_i = Q_i : I$, and this happens if and only if $I \not\subseteq P_i$, the associated prime of Q_i. Since G is unmixed and $G \subseteq I$, this proves that the associated primes of $G : (G : I)$ are exactly the associated primes of G that are also associated primes of I.

Now suppose that if P is a minimal prime of G and $Q \subset I \subset P$, where Q is the P-primary component of G. Since P is minimal over I, it is an associated prime. Write Q' for the P-primary component of I. By the argument above, the P-primary component of $G : (G : I)$ is $Q : (Q : Q')$, and we must show that this is the same as Q'.

Since both Q' and $Q:(Q:Q')$ are P-primary, it suffices to prove this after localizing at P, so we may assume that R is a local ring, with $\dim R/Q = \dim R/Q' = 0$. Since R/Q is a localization of R/G it is again a complete intersection, and thus Gorenstein (more generally, it is true that the localization of any Gorenstein ring is Gorenstein, but we do not need this.) Furthermore, $Q:Q' = \mathrm{Hom}(R/Q', R/Q)$, and similarly $Q:(Q:Q') = \mathrm{Hom}(\mathrm{Hom}(R/Q', R/Q), R/Q) = R/Q'$ by duality.

3) Since

$$\begin{aligned}\mathrm{length}(R/G) &= \mathrm{length}(R/(G:I)) + \mathrm{length}((G:I)/G)\\ &= \mathrm{length}(R/(G:I) + \mathrm{length} R/I,\end{aligned}$$

we see from the associativity formula for multiplicity that when I has the same codimension as G, then the multiplicity of I plus that of $G:I$ is the multiplicity of G. In the graded case, this means that $\deg \mathrm{Proj}(R/I) + \deg \mathrm{Proj}(R/(G:I)) = \deg \mathrm{Proj}(R/G)$.

4)See Proposition 20.0.7. □

Proof of Theorem 16.2.2. By hypothesis, $X, Y \subset Z$ are unmixed of dim 1, with $\mathcal{I}_Y = (\mathcal{I}_Z : \mathcal{I}_X)$. By Part 1 of Lemma 16.2.3, we have $\mathcal{I}_X = (\mathcal{I}_Z : \mathcal{I}_Y)$ as well, proving symmetry.

1) This is the formula of part 3 of Lemma 16.2.3, interpreted in the graded case.

2) By Proposition 20.0.7

$$\begin{aligned}\omega_Y &= \mathcal{H}om_{\mathcal{O}_Z}(\mathcal{O}_X, \omega_Z) = \mathcal{H}om_{\mathcal{O}_Z}(\mathcal{O}_X, \mathcal{O}_Z(d_1 + d_2 - 4))\\ &= \frac{(\mathcal{I}_Z : \mathcal{I}_X)}{\mathcal{I}_Z}(d_1 + d_2 - 4).\end{aligned}$$

Thus there is an exact sequence

$$0 \to \omega_Y(-d_1 - d_2 + 4) \to \mathcal{O}_Z \to \mathcal{O}_X \to 0,$$

whence

$$\chi(\mathcal{O}_Z) = \chi(\omega_Y(-d_1 - d_2 + 4) + \chi(\mathcal{O}_X).$$

Applying the adjunction formula twice, and using the Riemann-Roch Theorem, together with the formula $\deg Z = \deg X + \deg Y$, we see that the arithmetic genus of Z is

$$\chi(\mathcal{O}_Z) = -\frac{(\deg X + \deg Y)(d_1 + d_2 - 4)}{2}.$$

Furthermore, by the Riemann-Roch Theorem **??**, $\chi(\mathcal{O}_X) = 1 - p_a(X)$ while

$$\begin{aligned}\chi(\omega_Y(-d_1 - d_2 + 4)) &= 2p_a(Y) - 2 + \deg(Y)(-d_1 - d_2 + 4) + 1 - p_a(Y)\\ &= p_a(Y) - 1 - \deg(Y)(d_1 + d_2 + 4).\end{aligned}$$

Putting this together we get

$$-\frac{(\deg X + \deg Y)(d_1 + d_2 - 4)}{2} = 1 - p_a(X) + p_a(Y) - 1 - \deg(Y)(d_1 + d_2 + 4)$$

and thus

$$p_a(Y) - p_a(X) = \frac{(\deg Y - \deg X)(d_1 + d_2 - 4)}{2}$$

as claimed. $\square$

16.3 The Hartshorne-Rao module

The main theorem on linkage of curves in $\mathbb{P}^3$ is due to Hartshorne and Rao [Prabhakar Rao 1978/79]. If X is a purely 1-dimensional projective scheme, then S_X is locally Cohen-Macaulay , and thus $H^1(\mathcal{I}_X(i)$ is nonzero for only finitely many values of $i \in \mathbb{Z}$, so the vector space

$$M(X) := H^1_*(\mathcal{I}_X) := \oplus_{d\in\mathbb{Z}} H^1(\mathcal{I}_X(d)),$$

which is a graded module over the homogeneous coordinate ring of $\mathbb{P}^n$, has finite length (equivalently, finite dimension as a vector space over the ground field.)
((put this into the local coho section?))
There are two other ways to look at $M(X)$ that are sometimes useful:

$$M(X) = H^1_{\mathfrak{m}}(S_X) = Ext^3(S_X, S(-4))^\vee.$$

The first of these equalities follows immediately from the exact sequence

$$0 \to I_X \to S \to \bigoplus_{d\in\mathbb{Z}} H^0(\mathcal{O}_X(d) \to H^1_{\mathfrak{m}}(S_X) \to 0$$

and the corresponding sequence in which $H^1_{\mathfrak{m}}(S_X)$ is replaced by $\oplus_{d\in\mathbb{Z}} H^1(\mathcal{I}_X(d))$. while the second is a special case of local duality for sheaves on $\mathbb{P}^3$; see Section 18.2

Theorem 16.3.1 (Hartshorne-Rao[Prabhakar Rao 1978/79])**.** *Write S for the homogeneous coordinate ring of $\mathbb{P}^3$, and suppose that $X, Y \subset \mathbb{P}^3$ are subschemes of pure dimension 1. If X, Y are directly linked by a complete intersection of surfaces of degree d_1, d_2 then, as graded S-modules,*

$$M(Y) \cong M(X)^\vee(-d_1 - d_2 + 4).$$

Moreover, X, Y are evenly linked if and only if

$$M(Y) \cong M(X)(t)$$

for some integer t.

Every graded module of finite length is isomorphic, up to a shift in grading, to the module $H^1_*(\mathcal{I}_X)$ for some smooth curve X.

Note that the module $M(X)$ appears in the exact sequence in cohomology coming from the surjection $\mathcal{O}_{\mathbb{P}^3} \to \mathcal{O}_X$:

$$0 \to H^0_*(\mathcal{I}_X) \to H^0_*(\mathcal{O}_{\mathbb{P}^3}) \to H^0_*(\mathcal{O}_X) \to H^1_*(\mathcal{I}_X) \to 0$$

Thus $M(X) = H^1_*(\mathcal{I}_X) = 0$ if and only if the linear series cut by hyperplanes of degree d is complete for all d, that is, X is arithmetically Cohen-Macaulay.

We will prove Hartshorne's half of the Hartshorne-Rao Theorem:

Theorem 16.3.2 (Hartshorne [Hartshorne 1977])**.** *If $X, Y \subset \mathbb{P}^3$ are purely 1-dimensional subschemes that are directly linked through the complete intersection Z, which is given by forms f_1, f_2 of degrees d_1, d_2 respectively then*

$$M(X)^\vee \cong M(Y)(d_1 + d_2 - 4)$$

where $M(X)^\vee$ denotes the vector space dual of $M(X)$ equipped with the natural module structure.

Proof. Let S be the homogeneous coordinate ring of $\mathbb{P}^3$. The ring S/I_X may not be Cohen-Macaulay, but because it is purely 1-dimensional, $X = \operatorname{Proj} S/I_X$ is locally Cohen-Macaulay of codimension 2, and thus its second syzygy sheafifies to a vector bundle $\mathcal{E}$ on $\mathbb{P}^3$, and we see that

$$M(X) = \oplus_{d\in\mathbb{Z}} H^1(\mathcal{I}_X(d)) \cong \oplus_{d\in\mathbb{Z}} H^2(\mathcal{E}(d)).$$

The natural surjection $S/(f_1, f_2) \to S/I_X$ lifts to a map of free resolutions, and sheafifying we get a diagram with exact rows and right-hand column:

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{E} \arrow[r] & F_1 \arrow[r] & \mathcal{O}_{\mathbb{P}^3} \arrow[r] & \mathcal{O}_X \arrow[r] & 0 \\
0 \arrow[r] & \mathcal{O}_{\mathbb{P}^3}(-d_1-d_2) \arrow[r] \arrow[u] & \mathcal{O}_{\mathbb{P}^3}(-d_1) \oplus \mathcal{O}_{\mathbb{P}^3}(-d_2) \arrow[r] \arrow[u] & \mathcal{O}_{\mathbb{P}^3} \arrow[r] \arrow[u, "1"] & \mathcal{O}_Z \arrow[r] \arrow[u] & 0 \\
 & & & & \mathcal{I}_X/(f_1, f_2) \arrow[u] & \\
 & & & & 0. \arrow[u] &
\end{tikzcd}$$

By Proposition 20.0.7, $I_X/(f_1, f_2) \cong \omega_{S/I_Y}(-d_1 - d_2 + 4)$, so the mapping cone of the map of complexes above has first homology $\omega_Y(-d_1 - d_2 - 4)$. Dropping the two copies of $\mathcal{O}_{\mathbb{P}^3}$ and the identity map between them, we get a locally free resolution

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^3}(-d_1 - d_2) \longrightarrow \mathcal{E} \oplus \mathcal{O}_{\mathbb{P}^3}(-d_1) \oplus \mathcal{O}_{\mathbb{P}^3}(-d_2) \longrightarrow F_1$$

of $\omega_Y(-d_1-d_2+4)$.

Let $R := \oplus_{d\in\mathbb{Z}} H^0(\mathcal{O}_Y(d))$. The natural map $S/I_Y \to R$ has cokernel $\oplus_{d\in\mathbb{Z}} H^1\mathcal{I}_Y(d) = M(Y)$, which has finite length. Thus

$$\omega_{S/I_Y} = \mathrm{Ext}^2_S(S/I_Y, \omega_S) = \mathrm{Ext}^2_S(R, \omega_S).$$

Moreover R is Cohen-Macaulay, so also $\mathrm{Ext}^2_S(\omega_{S/I_Y}, \omega_S) = R$. Dualizing the resolution above and sheafifying, we get an exact sequence of sheaves

$$0 \to F_1^* \to \mathcal{E}^* \oplus \mathcal{O}_{\mathbb{P}^3}(d_1) \oplus \mathcal{O}_{\mathbb{P}^3}(d_2) \to \mathcal{O}_{\mathbb{P}^3}(d_1+d_2) \to R(d_1+d_2) \to 0.$$

It follows that the image of $\mathcal{E}^* \oplus \mathcal{O}_{\mathbb{P}^3}(d_1) \oplus \mathcal{O}_{\mathbb{P}^3}(d_2)$ in $\mathcal{O}_{\mathbb{P}^3}(d_1+d_2)$ is $\mathcal{I}_Y(d_1+d_2)$. By Serre duality, $M(X) = \oplus_{d\in\mathbb{Z}} H^2(\mathcal{E}(d)) = \big(\oplus_{d\in\mathbb{Z}} H^1(\mathcal{E}^*(-d-4))\big)^\vee$, the dual over the ground field. From the above sequence we see that

$$M(X)^\vee \cong \oplus_{d\in\mathbb{Z}} H^1(\mathcal{E}^*(-d-4)) \oplus_{d\in\mathbb{Z}} = M(Y).$$

Thus $M(X)^\vee = M(Y)(d_1+d_2-4)$ as required. $\square$

There are sometimes large families of curves having a given Hartshorne-Rao module, sometimes very few. Here are three simple examples:

Example 16.3.3 (Two lines)**.** Let $X \subset \mathbb{P}^3$ be the union of two disjoint lines, L_1, L_2. Supposing that the lines are given by equations $x_0 = x_1 = 0$ and $x_2 = x_3 = 0$ respectively. Since $H^0_*(\mathcal{O}_X) \cong k[x_2, x_3] \times k[x_0, x_1]$ the exact sequence

$$(*)0 \longrightarrow I_X \longrightarrow S \xrightarrow{\text{restriction}} H^0_*(\mathcal{O}_X) \longrightarrow M(X) \longrightarrow 0,$$

where the map labeled restriction sends each variable to the variable with the same name shows that $M(X) = M(X)_0 = k$. We can see directly that X is linked in two steps to any other union of 2 disjoint lines L_1', L_2'. Indeed there are two lines K_1, K_2 (or possibly $K = K_1 = K_2$ as a double line) meeting each of the 4 lines L_1, L_2, L_1', L_2' in two points (or possibly 1 with multiplicity 2). (Proof: any three disjoint lines lie on a unique quadric, which must be smooth since the lines are disjoint, and the lines lie in the same ruling. The 4th line pierces that quadric in 2 points or is tangent to it; the two lines from the opposite ruling through those two points (or the double line in the case of tangency) meet all 4 lines.) The unions $Z = L_1 \cup L_2 \cup K_1 \cup K_2$ and $Z' = L_1' \cup L_2' \cup K_1 \cup K_2$ are each the complete intersection of 2 quadrics, each of which may be taken to be the union of two planes; for example

$$Z = \big(\overline{L_1, K_1} \cup \overline{L_2, K_2}\big) \bigcap \big(\overline{L_1, K_2} \cup \overline{L_2, K_1}\big)$$

((add a picture!))
It is not hard to show that any curve of type $(a, a+2)$ on a smooth quadric is also linked to X.

Example 16.3.4 (Three lines)**.** Let $X \subset \mathbb{P}^3$ be the union of 3 disjoint lines. Since a line imposes 3 conditions on a quadric to contain it, and since there is a 10-dimensional vector space of quadratic forms in 4 variables, X is contained in at least 1 quadric Q. Since no two of the lines can lie on a plane, Q is irreducible; and since any two lines on an irreducible singular quadric in $\mathbb{P}^3$ meet, X must be smooth. Recall that Q has two linear equivalence classes of lines, and lines from one class all meet the lines from the other class; thus the three lines are all linearly equivalent on Q.

Proposition 16.3.5 (Migliore)**.** *If $X' \subset \mathbb{P}^3$ is another union of 3 disjoint lines then X is linked to X' if and only if $X' \subset Q$ as well. Moreover, X is directly linked to X' if X' is in the opposite linear equivalence class, and evenly linked in two steps to X' if X' is in the same equivalence class.*

For more results in this direction, see [Migliore 1986] from which the argument below is taken.

Proof. First, if $X' \subset Q$ is in the opposite equivalence class as X, then $X + X' \sim 3H$ as divisors on Q, where H is the hyperplane section. Thus $X + X' = X \cup X'$ is the complete intersection of Q with a cubic surface, proving that X and X' are directly linked.

On the other hand, if $X' \subset Q$ is in the same equivalence class as X, then the union Y of three lines in the opposite equivalence class is linked to both X and X'.

From the exact sequences analogous to (*) in Example 16.3.3 we see that $M(X)_0 \cong k^2 \cong M(X)_1$, and $M(X)_d = 0$ for $d \neq 0, 1$. Each linear form ℓ on $\mathbb{P}^3$ induces a map $m_\ell : M(X)_0 \to M(X)_1$ by multiplication. Let

$$Q'(X) := \{\ell \mid m_\ell \text{ has nonzero kernel on } M(X)\} \subset \mathbb{P}^{3*};$$

where $\mathbb{P}^{3*}$ is the projective space of linear forms on $\mathbb{P}^3$, that is, the dual projective space to $\mathbb{P}^3$.

We next show that if X' is linked to X then $Q'(X) = Q'(X')$. By Hartshorne's theorem, if X' is linked to X then $M(X') \cong M(X)$ up to twist or $M(X') \cong M(X)^\vee$ up to twist. In the first case it is obvious that $Q'(X') = Q'(X)$ and this is also true in the second case, because the multiplication map

$$m_\ell : M(X')_0 \cong M(X)_1^\vee \cong k^2 \longrightarrow M(X')_1 \cong M(X)_0^\vee \cong k^2$$

is simply $m_\ell^\vee$.

It remains to show that $Q'(X)$ determines Q. We claim that Q' is the set of linear forms vanishing on one of the lines of Q. Let $L_\ell \subset \mathbb{P}^3$ be the hyperplane on which ℓ vanishies. Since a hyperplane meets Q in a plane conic, $\ell \in Q'$ iff $L_\ell \cap Q$ is a divisor of type $L + L' \subset Q$, where L, L' belong to opposite rulings. Thus Q' is the set of linear forms whose hyperplanes meet Q in singular curves, that is, the set of tangent hyperplanes, also known as the dual variety to Q. Since the dual of the dual is the original variety, the dual of Q' is Q.

Finally, we must show that

$$Q' = \{\ell \mid L_\ell \text{ contains a line of } Q\}.$$

First, suppose that L_ℓ contains one of the components L_1 of $X = L_1 \cup L_2 \cup L_3$. We may write

$$M(X)_0 = ke_1 \oplus ke_2 \oplus ke_3/k(e_1 + e_2 + e_3)$$

where e_i is a rational function that is nonzero on L_i and zero on L_j for $j \neq i$. It follows that $m_\ell(e_1) = 0$, so $\ell \in Q'$.

Next suppose L_ℓ does not contain any of the L_i. For any linear form ℓ there is an exact sequence of sheaves

$$0 \longrightarrow \mathcal{I}_{X/\mathbb{P}^3} \xrightarrow{\ell} \mathcal{I}_{X/\mathbb{P}^3}(1) \longrightarrow \mathcal{I}_{(X\cap L_{\ell'})/L_\ell}(1) \longrightarrow 0.$$

and thus, from the long exact sequence in cohomology,

$$0 \longrightarrow H^0(\mathcal{I}_{(X\cap L_\ell)/L_\ell}(1)) \longrightarrow H^1(\mathcal{I}_{X/\mathbb{P}^3}) \longrightarrow H^1(\mathcal{I}_{X/\mathbb{P}^3}(1));$$

that is,

$$H^0(\mathcal{I}_{(X\cap L_\ell)/L_\ell}(1)) \cong \ker m_\ell : M(X)_0 \to M(X)_1.$$

However, $H^0(\mathcal{I}_{(X\cap L_\ell)/L_\ell}(1)) \neq 0$ if and only if there is a linear form ℓ', not a multiple of ℓ, that vanishes on $X \cap L_\ell$; that is, if the three points of $X \cap L_\ell$ are colinear. Since Q is a quadric, this is the same as saying that $Q \cap L_\ell$ contains a line, completing the argument. ☐

For double lines not lying on an irreducibly quadric, see Example 16.8.3

16.4 Construction of curves with given Hartshorne-Rao module

Some of the main remaining results about linkage of curves in $\mathbb{P}^3$ depend on careful general position arguments, and we merely sketch them. In the following $S = k[x_0, x_1, x_2, x_3]$.

Theorem 16.4.1. *(Rao[Prabhakar Rao 1978/79] Let M be a graded S-module of finite length. There is a smooth curve in $\mathbb{P}^3$ with Hartshorne-Rao module $M(t)$ for some twist $t \in \mathbb{Z}$*

Sketch of Proof. Recall that the homogeneous coordinate ring S_X of X has resolution of the form

$$\mathbb{G} : 0 \longrightarrow F_3 \xrightarrow{A} F_2 \xrightarrow{\phi} F_1 \longrightarrow S$$

with $\operatorname{rank} F_1 = \operatorname{rank}\phi + 1$ ignoring shifts of the grading, we have $\operatorname{coker} A^* = \operatorname{Ext}^3_S(S_X, S) = M(X)^\vee$. The dual of $\mathbb{G}$ is not exact, but maps to the resolution $\mathbb{L}$ of $M(X)^\vee$. The dual of $\mathbb{L}$ is a resolution (of $M(X)$), and has the form

$$\mathbb{L}^* : 0 \longrightarrow F_3 \longrightarrow F_2 \xrightarrow{\psi} L_2 \longrightarrow L_1 \longrightarrow L_0.$$

It turns out that if we take a sufficiently general projection $p : Ł_2 \to L'_2$, with $\operatorname{rank} L'_2 = \operatorname{rank}\psi + 1$, then the cokernel of $\psi' = p \circ \psi$ is torsion free of rank 1. Thus this cokernel is equal to an ideal I, up to some twist, and we get a resolution of S/I of the the form

$$0 \longrightarrow F_3(t) \longrightarrow F_2(t) \xrightarrow{\psi} L'_2(t) \longrightarrow S$$

proving that $M(S/I) = M$. Possibly after twisting further, an application of Bertini's theorem shows that S/I will be the coordinate ring of a smooth curve. $\square$

It is nevertheless the case that *not* every twist of every module occurs as the Rao module of a curve, even when we allow the curve to be an arbitrary purely 1-dimension subscheme; see Corolllary 16.6.3 below.

16.5 Curves on a surface

For curves on a surface, the relation of even linkage reduces to that of linear equivalence:

Proposition 16.5.1. *Let S be a surface in $\mathbb{P}^3$, and let $X, Y \subset S$ be purely 1-dimensional schemes. The schemes X and Y are directly linked on S if and only if there is a rational function f on S such that the divisor f is $X - Y$. Thus X and Y are evenly linked if and only if X Y, and they are oddly linked if and only if $-X \sim Y$.*

Proof. Suppose first that X, Y are directly linked. After passing to an affine open set we have $(a) : I_X = I_Y$ for some nonzerodivisor a. Write $K(S)$ for the sheaf of rational functions on S, so that when restricted to an affine open set U we have

$$K(S)|_U = \{a/b \mid a, b \in \mathcal{O}_S(U), b \text{a nonzerdivisor}\}.$$

In this context the divisor $-X$ is the divisor associated to the fractional ideal $I^{-1} := \{q \in K(S) \mid qI \subset \mathcal{O}_S(U)\}$. Write Z for the divisor of a. Since I_X contains a nonzerodivisor, $a : I_X = aI_X^{-1}$; that is, $-X + Z = Y$, so indeed $-X$ is linearly equivalent to Y. Iterating this argument we see that if X, Y are evenly linked then they are linearly equivalent, and if oddly linked then $-X$ is linearly equivalent to Y, as claimed.

Conversely, suppose that $X \sim Y$. Passing to an open affine subset, this means that there are regular functions g, h such that $gI_X = hI_Y$. Since I_X and

I_Y contain nonzerodivisors on S, we may multiply and assume $h \in I_Y$. We know from **** that $(h : (h : I_Y)) = I_Y$, and it follows that

$$I_X = (g/h)(h : (h : I_Y)) = g : (h : I_Y)$$

as required. $\square$

16.6 Liaison Addition and Basic Double Links

Phillip Schwartau discovered a simple way to construct a curve Z whose Rao invariant $M(Z)$ is the direct sum of Rao invariants $M(X), M(Y)$ for given curves X, Y:

Proposition 16.6.1 (Liaison Addition)**.** *16.6.1 Let X, Y be purely 1-dimensional subschemes of $\mathbb{P}^3$, and let $f \in I_Y,\ g \in I_X$ be forms such that f, g is a regular sequence. The ideal $fI_X + gI_Y$ is unmixed of codimension 2 and the scheme Z it defines has Rao invariant*

$$M(Z) \cong M(X)(-\deg f) \oplus M(Y)(-\deg g).$$

Proof. We write $S = k[x_0, \dots, x_3]$ for the homogeneous coordinate ring of $\mathbb{P}^3$, with maximal homogeneous ideal $\mathfrak{m}$ and set $J = fI_X \oplus gI_Y$. Since

$$(fg) \subset fI_X \cap gI_Y \subset (f) \cap (g) = (fg)$$

we have in fact $(fg) = fI_X \cap gI_Y$ and thus an exact sequence

$$0 \to S/(fg) \longrightarrow S/fI_X \oplus S/gI_Y \longrightarrow S/J \longrightarrow 0,$$

from which we see that J has codimension 2. If $J \subset P \subset S$ is were an associated prime of J having codimension 3 in S, then localizing at P we would find $\operatorname{depth}(S/(fg)) \leq 1$; contradicting the fact that $(S/fg)_P$ is Cohen-Macaulay of dimension 2. Thus J is unmixed. Further, since S/fg is Cohen-Macaulay of dimension 3, we have $H^1_{\mathfrak{m}}(S/fg) = H^2_{\mathfrak{m}}(S/fg) = 0$ and thus

$$\begin{aligned} M(Z) = H^1_{\mathfrak{m}}(S/J) &= H^1_{\mathfrak{m}}(S/fI) \oplus H^1_{\mathfrak{m}}(S/gJ) \\ &= H^1_{\mathfrak{m}}(S/I)(-\deg f) \oplus H^1_{\mathfrak{m}}(S/J)(-\deg g) \\ &= M(X)(-\deg f) \oplus M(Y)(-\deg g) \end{aligned}$$

$\square$

In the case $Y = \emptyset, I_Y = S, f = 1$, the Hartshorne-Rao Theorem implies that $M(Z) = M(X)(-\deg g)$. In particular, every negative twist of a module that is the Hartshorne-Rao invariant of a curve is again the Hartshorne-Rao invariant of a curve.

This case was exploited by Lazarsfeld and Rao under the name *Basic double link*, and under a mild additional hypothesis the linking sequence can be made explicit:

Proposition 16.6.2 (Basic Double Links). *Let X be a purely 1-dimensional subschemes of $\mathbb{P}^3$, and let f, g be a regular sequence of forms, with $g \in I_X$. The ideal $fI_X + gS$ is unmixed of codimension 2 and in the even linkage class of I. Moreover, if $I_X + fS$ has codimension 3, then the scheme Z it defines is linked in two steps to I: for any $h \in I_X$ such that g, h is a regular sequence,*

$$fI_X + gS = (g, fh) : \big((g, h) : I_X\big).$$

Proof. By Theorem 16.6.1 the ideal $J := fI_X + gS$ is unmixed and has the same Hartshorne-Rao invariant as I_X, so by Theorem 16.3.1 it is evenly linked to I_X.

if $r(I_X) \subset (g, fh)$ so that $r \in (g, h) : I_X$, then $rJ \subset (g, fh)$, so $J \subset (g, fh) : \big((g, h) : I_X\big)$. Thus to prove the equality in the case when $I_X + fS$ has codimension 3, it suffices to do so after localizing at each of the associated primes of J. By Proposition 16.6.1, $J := fI_X + gS$ is unmixed of codimension 2, so it suffices to prove the equality after localizing at a codimension 2 prime P. By our hypothesis, either $f \notin P$ or $I_X \not\subset P$

If $f \notin P$ then $J_P = (I_X)_P$ and

$$\Big((g, fh) : \big((g, h) : I_X\big)\Big)_P = \Big((g, h) : \big((g, h) : I_X\big)\Big)_P = (I_X)_P$$

by the assumption that g, h is a regular sequence and the symmetry of linkage, Theorem 16.2.2.

On the other hand, if $I_X \not\subset P$ then after localizing the equality becomes

$$(g, f) = (g, fh) : (g, h)$$

which holds because g, h is a regular sequence. □

Corollary 16.6.3. *If $M = M(X) \neq 0$ for some purely 1-dimensional scheme $X \subset \mathbb{P}^3$, then $M_d \neq 0$ for some $d \geq -1$. Thus for any nonzero graded S-module M of finite length, there is a maximal integer d such that $M(d)$ occurs as a Rao module. Moreover, if M occurs as a Rao module, then for all $e \leq d$ the module $M(e)$ also occurs.*

Proof. If $M = M(X)$ then for $d \leq -1$ we have $M_d = H^0(\mathcal{O}_X(d)$. Since $\bigoplus_{d\in\mathbb{Z}} H^0(\mathcal{O}_X(d)$ is an S-module of depth ≥ 1, we have $M_d \geq M_{d-1}$ for all $d \leq -1$. In particular, if $M \neq 0$ and $M_d \neq 0$ for some $d \leq -2$, the $M_{-1} \neq 0$, and the conclusion follows. If Y is obtained from X by a basic double link with $\deg g = 1$, then $M(Y) = M(X)(-1)$. □

A much sharper result is given in [Martin-Deschamps and Perrin 1993]: $M_n = 0$ unless

$$g + 1 - ((d-2)(d-3)/2) \leq n \leq (d(d-3)/2) - g.$$

The main result of Lazarsfeld and Rao gives a description of a given linkage class:

Theorem 16.6.4 (Structure of a linkage class)**.** *([Lazarsfeld and Rao 1983]) Let $M = M(X)$ be the Rao module of a purely 1-dimensional subscheme, and suppose that $M(1)$ does not occur as a Rao module. All the curves Y that are evenly linked to X are obtained from X by a series of basic double links followed by a deformation.*

16.7 Arithmetically Cohen-Macaulay Curves

Before discussing the proof of Theorem 16.3.1, we examine the case $M(X) = 0$, which was first elucidated by Gaeta.

Theorem 16.7.1 ([Gaeta 1952])**.** *If X is a curve in $\mathbb{P}^3$ then X is in the (even and odd) linkage class of a complete intersection if and only if the homogeneous coordinate ring of X is Cohen-Macaulay. Moreover, if I_X can be generated by n elements, then X is linked to a complete intersection in $n-2$ steps.*

We first prove that even and odd linkage are the same in this case, and that any two complete intersection curves are evenly linked:

Lemma 16.7.2. *If f, g and f, h are regular sequences, then (f, g) and (f, h) are directly linked. Moreover, any two complete intersections are both evenly and oddly linked.*

Proof. Since f, g and f, h are regular sequences, so is f, gh. We claim that

$$(f, h) = (f, gh) : (f, g).$$

Indeed if $ag = bf + cgh \in (f, gh)$ then $(a - ch)g = bf$ so $a - ch \in (f)$, whence $a \in (f, h)$.

It follows that if m, n are independent linear forms, neither a divisor of f or g, then each consecutive pair in the sequence of complete intersections

$$(f, g), (fm, g), (fm, gn), (m, gn), (m, n)$$

are directly linked, so (f, g) is evenly linked to (m, n). The sequence

$$(f, g), (f, mg), (f, mng), (f, g)$$

shows that (f, g) is also oddly linked to itself, completing the proof. □

((can these things be done with geometric links?))

Before proving Theorem 16.7.1 we need one more result from commutative algebra the Hilbert-Burch theorem:

Theorem 16.7.3 (Hilbert-Burch[Burch 1967])**.** *Let A be a homogeneous $n \times (n-1)$ matrix of forms in $S := k[x_0, \dots, x_r]$ and Let $I := I_{n-1}(A)$ be the ideal generated by the $n-1 \times n-1$ minors of A.*

1. *If $I \neq S$ then* $\operatorname{codim} I \leq 2$.
2. *If I has codimension 2, then S/I is Cohen-Macaulay. Moreover, if Δ_i is the determinant of the matrix obtained from A by omitting the i-th column, then*
$$0 \longrightarrow S^{n-1} \xrightarrow{A} S^n \xrightarrow{\begin{pmatrix}\Delta_1 & -\Delta_1 & \dots & \pm\Delta_n\end{pmatrix}} S$$
is a resolution of S/I, and its dual is a resolution of $\omega_{S/I}$.
3. *Furthermore, every graded Cohen-Macaulay factor ring of S of codimension 2 arises in this way.*

Proof. If we augment A to an $n \times n$ matrix by repeating the i-th column, the determinant is zero. The product of the row of signed minors
$$\begin{pmatrix}\Delta_1 & -\Delta_2 & \dots & \pm\Delta_n\end{pmatrix}$$
with the i-th row of A is the Cauchy expansion of this determinant. Thus the give sequence of maps forms a complex. The fact that it is a resolution, and that its dual is a resolution, follows from a general result on finite free complexes, [Eisenbud 1995, Theorem *****]. This shows that $S/I = S/(\Delta_1, \dots \Delta_n)$ is Cohen-Macaulay.

Now suppose that S/I is a homogeneous factor ring of S that is Cohen-Macaulay and of codimension 2. By Theorem **??**, The minimal free resolution of S/I as an S-module has the form
$$\mathbb{F}: \quad 0 \longrightarrow S^m \xrightarrow{A} S^n \xrightarrow{B} S$$
where n is the minimal number of generators of I. Tensoring with the quotient field of S, we get a complex of vector spaces that is exact, so $m = n-1$, and we see that A is a homogeneous $n \times (n-1)$ matrix. Again by [Eisenbud 1995, Theorem *****], the ideal of $n-1 \times n-1$ minors of A has codimension 2, and the dual of the resolution is a resolution of ω_I. Write Δ for the row of signed minors of A. Both Δ^* and B^* can be regarded as the kernel of A^*, so $\Delta = uB$ for some unit, and we are done. $\square$

We remark that the complex $\mathbb{F}$ in the proof of Theorem 16.7.3 is a special case of the Eagon-Northcott complex, to be treated in the next chapter. The following Corollary is the corresponding special case of [Buchsbaum and Eisenbud 1977, Theorem ***].

Corollary 16.7.4. *Let A be a homogeneous $n \times (n-1)$ matrix of forms in $S := k[x_0, \dots, x_n]$ and Let $I := I_{n-1}(A)$ be the ideal generated by the $n-1 \times n-1$ minors of A. If the codimension of I is (at least) 2, then the annihilator of the cokernel of $A* : S^n \to S^{n-1}$ is exactly I.*

Proof. The dual $\mathbb{F}^*$ of the complex $\mathbb{F}$ in the proof of Theorem 16.7.3 is a resolution of coker A^*, and thus any element of s that anniliates the kernel induces a map of complexes that is homotopic to 0. Dualizing again, we see that it induces the zero map on S/I—that is, it lies in I. The same argument applied to F itself shows that any element of I annihilates coker A^*. $\square$

Proof of Gaeta's Theorem. From Theorem 16.3.2 if follows that if $X \subset \mathbb{P}^3$ is in the linkage class of a complete intersection then $M(X) = 0$, so the homogeneous coordinate ring of X is Cohen-Macaulay.

For the converse we prove a more general version: Suppose that $I \subset S = k[x_0, \ldots, x_r]$ is a homogeneous ideal of codimension 2, generated by n elements, such that S/I is Cohen-Macaulay; we will show that I can be linked in $n-2$ steps to a complete intersection. Let A be the presentation matrix of I so that, as in the Hilbert-Burch Theorem, A has n rows and $n-1$ columns, and I is equal to the ideal of $(n-1) \times (n-1)$ minors of A.

Replacing the generators of I by appropriate linear combinations, and making a corresponding change of generators of the module S^n in the complex $\mathbb{F}$ of Theorem 16.7.3, we may assume that the first two generators, which are the $(n-1) \times (n-1)$ subdeterminants Δ_1, Δ_2 of A omitting the first two rows, form a regular sequence.

We now compute the linked ideal $(\Delta_1, \Delta_2) : I$. Let A' be $(n-2) \times (n-1)$ matrix obtained from A by deleting the first two rows. We may interpret the columns of A', as generating the syzygies of $I/(\Delta_1, \Delta_2)$. By Corollary 16.7.4, the ideal I' generated by the $(n-2) \times (n-2)$ of A' is the annihilator of the module $I/(\Delta_1, \Delta_2)$; that is, $I' = (\Delta_1, \Delta_2) : I$ is directly linked to I. Moreover, I' has codimension 2 because the laplace expansions express the regular sequence Δ_1, Δ_2 in terms of these minors. By Theorem 16.7.3, S/I' is Cohen-Macaulay, and I' has $n-1$ generators, so we are done by induction. $\square$

16.8 The structure of an even linkage class

The structure present within a given (even) linkage class was illuminated in the work of Lazarsfeld and Rao[Lazarsfeld and Rao 1983], which proved a version of a conjecture of Harris; we close this chapter by sketching a result of their paper. But first, an elementary result:

Proposition 16.8.1. *Let M be a graded S-module of finite length. The set of $t \in \mathbb{Z}$ such that there is a curve in $\mathbb{P}^3$ with Rao module $M(t)$ is bounded below.*

We say that a curve X is *minimal* in its even linkage class if, for all curves Y in the even linkage class of X we have $M(Y) \cong M(X)(t)$ with $t \geq 0$.

Proof. If suffices to show that if $M = M(C)$ for some curve $C \subset \mathbb{P}^3$ then $\max\{n \mid M_n \neq 0\} \geq 0$. Let S_C be the homogeneous coordinate ring of C, and let $\tilde{S}_C =$

$\oplus_n H^0(\mathcal{O}_C(n))$. Since $(S_C)_n = 0$ for $n < 0$, we see that $M(C) = \tilde{S}_C/S_C$ agrees with $\tilde{S}_C$ in negative degrees. But depth $\tilde{S}_C \geq 2$, so $\dim(\tilde{S}_C)_n \geq \dim(\tilde{S}_C)_{n-1}$ for all $n < 0$, and since $M_n = 0$ for n sufficiently large, the conclusion follows. $\square$

Much sharper bounds are known; see for example [Martin-Deschamps and Perrin 1990].

Theorem 16.8.2 (Structure of Linkage)**.** *Any two minimal curves in an even linkage class are connected by a deformation. Every curve in an even linkage class is obtained from a minimal curve by a sequence of basic double links, followed by a deformation.*

Sketch of the proof.

$$0 \to F_1^* \to \mathcal{E}^* \oplus \mathcal{O}_{\mathbb{P}^3}(d_1) \oplus \mathcal{O}_{\mathbb{P}^3}(d_2) \to \mathcal{O}_{\mathbb{P}^3}(d_1+d_2) \to R(d_1+d_2) \to 0.$$

$\square$

Example 16.8.3. The minimal elements in a linkage class may be unique and may not be reduced. From the formula for the degree of a linked curve in Theorem 16.2.2 we see that any curve of degree 2 must be minimal in its linkage class, and can only be linked to another minimal curve in its class by the complete intersection of two quadrics.

Consider the double line C with ideal

$$I_C = (x_0^2,\ x_0x_1,\ x_1^2,\ x_0F_0(x_2,x_3) + x_1F_1(x_2,x_3),)$$

supported on the line C_{red} with ideal (x_0, x_1), where F_0, F_1 are relatively prime forms of degree $d \geq 1$. It is not hard to show that this is a curve of degree 2 with arithmetic genus $-d$.

We first claim that the free resolution of the homogeneous coordinate ring S_C over $S = k[x_0, \ldots, x_3]$ is

$$S \xleftarrow{\begin{pmatrix} x_0^2 & x_0x_1 & x_1^2 & x_0F + x_1G \end{pmatrix}} S^3(-2) \oplus S(-d-1)$$

$$\xleftarrow{\begin{pmatrix} 0 & -F & x_1 & 0 \\ F & -G & -x_0 & x_1 \\ G & 0 & 0 & -x_0 \\ -x_1 & x_0 & 0 & 0 \end{pmatrix}} S^2(-d-2) \oplus S^2(-3) \xleftarrow{\begin{pmatrix} x_0 \\ x_1 \\ F \\ G \end{pmatrix}} S(-d-3)$$

To prove that this is a resolution we use Theorem 18.7.1. It is easy to check that this sequence of maps forms a finite free complex. The only non-obvious fact needed to apply the Theorem is that the 3×3 minors of the middle matrix generate an ideal of codimension ≥ 2, and in fact it clearly contains (x_0^3, x_1^3).

From the resolution we see that

$$M(C) = \mathrm{Ext}^3(S_C, S(-4))^\vee = S(d-1)/JS(d-1),$$

where $J = (x_0, x_1, F, G)$. Since $S/(x_0, x_1, F_0, F_1)$ has socle in degree $2d-2$, we see that

$$M(C) = M(C)^\vee = S(d-1)/JS(d-1).$$

as well.

In earlier chapters we analyzed various families of curves in $\mathbb{P}^3$ by linking the curves to simpler curves. One of the main results of Lazarsfeld and Rao verified a conjecture of Joe Harris that this won't work for general curves of high genus. Using the Maximal Rank theorem of Eric Larson [Larson 2017], we can give a bound:

Theorem 16.8.4. *If C is a general smooth projective curve of large genus, or if C has genus ≥ 10 and is embedded in $\mathbb{P}^3$ by a general linear series, then C is minimal in its linkage class, and thus any Curve in the even linkage class of C is of at least as high genus and degree as C.*

The first version is proven in [Lazarsfeld and Rao 1983]. We prove the version with a general line bundle:

Proof. What Lazarsfeld and Rao actually prove is that if $e(C) := \max\{e \mid H^1(\mathcal{O}_C(e)) \neq 0\}$, and C lies on no surface o degree $\leq e+3$, then C is minimal in its even linkage class (if C lies on no surface of degree $\leq e+4$, then C is (up to automorphisms of $\mathbb{P}^3$) the only curve C' with Rao module $M(C') = M(C)$ is C itself.

Now suppose that C is a general curve of genus g, embedded in $\mathbb{P}^3$ as a non-degenerate curve, by a general line bundle of degree d. By Petri's Theorem **??**, the line bundle $\mathcal{O}_C(2)$ is nonspecial
((insert pf,))
so $e(C) = 1$. By the maximal rank theorem [Larson 2017], C lies on no surface of degree $e+3 = 4$ if and only if

$$4d - g + 1 = H^0(\mathcal{O}_C(4)) \geq H^0\ (\mathcal{O}_{\mathbb{P}^3}(4) = 35.$$

By the Brill-Noether Theorem **??**, $d \geq \lceil (3/4)g \rceil + 3$. For $g = 10$ we have $d \geq 11$, so $4d - g + 1 = 35$. and for $g > 10$ the difference $(4d - g + 1) - 35$ only grows. ☐

Chapter 17

Syzygies of canonical curves and curves of high degree

((This section needs to be revised slightly in light of the new appendix 18))

17.1 Introduction

In this Chapter we will study invariants associated to a free resolution, or syzygies, of the homogeneous coordinate ring of a curve in projective space, with an emphasis on their relation to the varieties (or schemes) containing the curve. We have two cases in mind: curves of (relatively) high degree, and canonical curves.

17.2 How syzygies can reflect geometry

One of the main ways in which syzygies can be seen to reflect the geometry of a scheme $C \subset \mathbb{P}^r$ depends on the possibility of factoring the line bundle $\mathcal{O}_C(1)$ as the tensor product of two bundles on C‘ with sections. Suppose for example that C is nondegenerate, so that $\mathcal{O}_C(1) = \mathcal{L}_1 \otimes \mathcal{L}_2$. Choose 2 independent global sections σ_1, σ_2 of $H^0(\mathcal{L}_1)$ and a basis $\tau_1, \dots, \tau_n$ of $H^0(\mathcal{L}_2)$. Set

$$l_{i,j} = \sigma_i \otimes \tau_j \in H^0(\mathcal{O}_C(1)) = H^0(\mathcal{O}_{\mathbb{P}^r}(1))$$

and consider the matrix

$$M = \begin{pmatrix} l_{1,1} & l_{1,2} & \dots & l_{1,n} \\ l_{2,1} & l_{2,2} & \dots & l_{2,n} \end{pmatrix},$$

which we think of as a matrix of linear forms.

We claim that the 2×2 minors $l_{1,j}l_{2,j'} - l_{1,j'}l_{2,j}$ are in the homogeneous ideal of I_C of C in $\mathbb{P}^r$. To see this, let $K(C)$ be the ring of rational functions on C

((have we made this definition somewhere? what's the notation?.)) Choosing identifications $\mathcal{L}_i \otimes K(C) \cong K(C)$ we see that the σ_i and the τ_j commute with each other as elements of $K(C) \otimes_{\mathcal{O}_X} K(C)$, and thus

$$\big(l_{1,j}l_{2,j'} - l_{1,j'}l_{2,j}\big)|_C = \sigma_1\tau_j\sigma_2\tau_j' - \sigma_1\tau_j'\sigma_2\tau_j = 0.$$

Example 17.2.1. The most familiar example is that of the twisted cubic. In this case the global sections $x_0 \dots x_3$ of $\mathcal{O}_C(1)$ may be identified with the forms $s^3, s^2t, st^2, t^3 \in k[s,t]$, and if $p \in C \cong \mathbb{P}^1$ then the multiplication of sections in the factorization $\mathcal{O}_C(1) = \mathcal{O}_C(p) \otimes \mathcal{O}_C(2p)$

$$\begin{array}{cc} & \begin{array}{ccc} s^2 & st & t^2 \end{array} \\ \begin{array}{c} s \\ t \end{array} & \begin{pmatrix} s^3 & s^2t & st^2 \\ s^2t & st^2 & t^3 \end{pmatrix} \end{array}$$

leads to the familiar matrix

$$\begin{pmatrix} x_0 & x_1 & x_2 \\ x_1 & x_2 & x_3 \end{pmatrix}.$$

We have $I_2(M) = I_C$, and the same idea works for the rational normal curve of any degree.

In case C is reduced and irreducible the matrix above has a special property: $K(X)$ is a domain, so no product of a nonzero section of $\mathcal{L}_1$ with a nonzero section of $\mathcal{L}_2$ can be zero. We can state this without any reference to C:

Definition 17.2.2. Let R be a commutative ring. A map $M : R^n \to R^m$ is 1-generic if the kernel of the corresponding map $R^n \otimes R^{m*} \to R$ contains no pure tensor $a \otimes b$. In more concrete terms, a matrix M is *1-generic* if there are no invertible matrices A, B such that AMB has some entry equal to 0.

By the material in Chapter **??**, the ideal $I_2(M)$ of a 1-generic matrix of linear forms is the homogeneous ideal of a rational normal scroll of codimension $n-1$ and degree n.

In the next section we will show that it has a free resolution of a special form called the Eagon-Northcott complex that is a subcomplex of the minimal free resolution of I_C. The presence of such a variety containing C or a subcomplex of this special form in the minimal free resolution of C is thus necessary for the factorization of the line bundle $\mathcal{O}_C(1)$ as above, and it is sometimes sufficient, as well.

17.3 The Eagon-Northcott Complex of a $2 \times n$ matrix of linear forms

The Eagon-Northcott complex is a complex of free modules associated to any matrix over any commutative ring. The most familiar special case is the Koszul complex, which one may think of as the Eagon-Northcott complex of a $1 \times n$ matrix, and even in the general case the Eagon-Northcott complex is in a sense built out of the Koszul complexes. A full treatment of the Eagon-Northcott complex and a whole family of related constructions can be found in [**?**, Appendix ***], and, from a more conceptual and general point of view, in [**?**]. Here we will only make use of the case of a matrix such as the one above, we will present a simplified account in that case only. Here is the result we need:

Theorem 17.3.1. *Let $S = k[x_0, \dots, x_r]$ be a polynomial ring, and let $M : F \to G$ be a homomorphism with $F = S^n(-1), G = S^2$. If M is 1-generic, then the minimal free resolution of $S/I_2(M)$ has the form:*

$$\begin{aligned} EN(M) := S \xleftarrow{\bigwedge^2 M} \bigwedge^2 F \xleftarrow{\delta_2} S^{2*} \otimes \bigwedge^3 F \xleftarrow{\delta_3} (\operatorname{Sym}^2 S^2)^* \otimes \bigwedge^4 F \\ \xleftarrow{\delta_4} \cdots \xleftarrow{\delta_{n-1}} (\operatorname{Sym}^{n-2} S^2)^* \otimes \bigwedge^n F \longleftarrow 0. \end{aligned}$$

From Chapter **** we know also that the ideal of minors defines a rational normal scroll.

Proof. We first show that $r \geq n$; more precisely, we show that the span of the entries of M has dimension $\geq n+1$. As noted above, to say that the $2 \times n$ matrix of linear forms M is 1-generic means that the kernel of the corresponding map $\phi : k^2 \otimes k^n \to S_1$ contains no pure tensors. In the projective space $\mathbb{P}^{2n-1} = \mathbb{P}(k^2 \otimes k^n)$ the pure tensors form a variety isomorphic to $\mathbb{P}^1 \times \mathbb{P}^{n-1}$, and thus of dimension n. Consequently the kernel of ϕ can have dimension at most $n-1$, whence the image of ϕ in $S_1 = k^{r+1}$ has dimension at least $2n-(n-1) = n+1$.

We begin the discussion of $EN(M)$ by defining the maps δ_i and and proving that the given sequence is indeed a complex—that is, consecutive maps compose to 0. For simplicity of notation, we choose a generator of $\wedge^2 S^2$ and identify it with S, which gives a sense to the map labeled $\bigwedge^2 M$.

Although it is not hard to do this directly, the dual maps

$$\partial_i : \operatorname{Sym}^{i-2} G \otimes \bigwedge^i F^* \longrightarrow \operatorname{Sym}^{i-1} G \otimes \bigwedge^{i+1} F^*$$

have a more familar-looking description, so we define these instead. Indeed, the map M corresponds to an element $\mu \in G \otimes F^*$. We may think of $\operatorname{Sym}^{i-2} G \otimes \bigwedge^i F^*$ as a (bigraded) component of the exterior algebra over $\operatorname{Sym} G$ of

$$\operatorname{Sym} G \otimes \bigwedge_S F^* = \bigwedge_{\operatorname{Sym} G} (\operatorname{Sym} G \otimes F^*).$$

We define ∂_i to be multiplication by μ in the sense of this exterior algebra. Since μ has degree 1 in this sense, its square is 0.

To show that $(\bigwedge^2 M)\circ\delta_2$ is zero, it is simplest to choose a matrix representing M. Direct computation using only the usual expansion of a determinant along a row shows that, up to sign, pure basis vector $e \otimes f_i \wedge f_j \wedge f_k$ of $G^* \otimes \bigwedge^3 F$ maps under the composition $(\bigwedge^2)M \circ \delta_2$ to the determinant of the 3×3 matrix obtained from M by repeating the row corresponding to e and the columns i, j, k. This determinant is 0 because it has a repeated row.

We next prove the split exactness of a complex of the form $EN(M')$ where M' is surjective, so that we may write $F = G \oplus F'$ and the map $M' : G \oplus F' \to G$ as projection on the first factor. Of course it suffices to prove the split exactness of the dual sequence, $EN(M')^*$:

$$EN(M')^* := S \xrightarrow{\bigwedge^2 M'^*} \bigwedge^2 F^* \xrightarrow{\partial_2} G \otimes \bigwedge^3 F^* \xrightarrow{\partial_3} \operatorname{Sym}^2 G \otimes \bigwedge^4 F^*$$
$$\xrightarrow{\partial_4} \cdots \xrightarrow{\partial_{n-1}} \operatorname{Sym}^{n-2} G \otimes \bigwedge^n F^* \longrightarrow 0.$$

In this case the proof is an exercise in multilinear algebra. We begin by proving split exactness at the positions $\operatorname{Sym}^i G \otimes \bigwedge^{i+2} F^*$ where $i \geq 1$.

The module $\operatorname{Sym}^i G \otimes \bigwedge^{i+2} F^*$ decomposes as

$$\operatorname{Sym}^i G \otimes \bigwedge^2 G^* \otimes \bigwedge^i F'^* \oplus$$
$$\operatorname{Sym}^i G \otimes G^* \otimes \bigwedge^{i+1} F'^* \oplus$$
$$\operatorname{Sym}^i G \otimes \bigwedge^{i+2} F'^*$$

Note that under our hypothesis, the element $\mu' \in G \otimes F^* = G \otimes G^* \oplus G \otimes F'^*$ has the form $(\mu_G, 0)$, where μ_G represents the identity map $G \to G$. Thus the complex $EN(M')^*$ is a direct sum over i of 3-term complexes of the form

$$\operatorname{Sym}^{i-1} G \xrightarrow{-\wedge\mu'} \operatorname{Sym}^i G \otimes G^* \xrightarrow{-\wedge\mu'} \operatorname{Sym}^{i+1} G \otimes \bigwedge^2 G^*$$

tensored with various $\bigwedge^j F'^*$, and it suffices to show that the former are split exact when $i \geq 0$. Now $\operatorname{Sym} G$ may be identified with $R := S[x, y]$, where x, y are a basis of S^2, and as such the sequences above may be identified with components of the Koszul complex of x, y over R,

$$0 \to R \longrightarrow R \otimes G \longrightarrow R \otimes \wedge^2 G$$

The only homology of this sequence is $R/(x, y)R$ at the right so if we replace $R \otimes \wedge^2 G \cong R$ by the ideal $(x, y)R$, this sequence is a split exact sequence of free S-modules. This is the desired result.

It remains to treat the beginning of the complex $EN(M')^*$,

$$S \xrightarrow{\bigwedge^2 M'^*} \bigwedge^2 F^* \xrightarrow{-\wedge\mu'} G \otimes \bigwedge^3 F^*$$

which, in our case, may be written:

$$S \xrightarrow{\bigwedge^2 M'^*} \bigwedge^2 G^* \;\oplus\; (G^* \otimes F'^*) \;\oplus\; \bigwedge^2 F'^* \xrightarrow{-\wedge\mu'}$$
$$G \otimes \bigwedge^2 G^* \otimes F'^* \;\oplus\; (G \otimes G^* \otimes \bigwedge^2 F'^*) \;\oplus\; G \otimes \bigwedge^3 F'^*$$

The map $\bigwedge^2 M'$ is the projection to $\bigwedge^2 G$ composed with the chosen isomorphism $\bigwedge^2 G \cong S$, and is thus a split monomorphism. To complete the argument, we must show that the map marked $- \wedge \mu'$ is a monomorphism on $(G^* \otimes F'^*) \oplus \bigwedge^2 F'^*$. But this map is the direct sum of the two maps

$$(G^* \xrightarrow{-\wedge\mu'} G \otimes \bigwedge^2 G^*) \otimes F'^*$$

and

$$(S \xrightarrow{\mu'} G \otimes G^*) \otimes \bigwedge^2 F'^*$$

which are evidently split monomorphisms. completing the proof of split exactness of $EN(M')^*$ and thus of $EN(M')$.

To go further we use a basic result, proven in a more general form (and with a slightly different statement) in [**?**, Theorem ***]. We make the convention that the codimension of the empty set is infinity.

Theorem 17.3.2. *Let $S = k[x_0, \dots, x_r]$, and*

$$\mathbb{F} : F_0 \xleftarrow{\phi_1} F_1 \longleftarrow \cdots \longleftarrow F_{n-1} \xleftarrow{\phi_n} F_n \longleftarrow 0$$

be a finite complex of free S-modules. Set

$$X_i = \{p \in \mathbb{A}^{n+1} \mid H_i(\mathbb{F} \otimes \kappa(p)) \neq 0\}$$

The complex $\mathbb{F}$ is acyclic *(that is, $H_i(\mathbb{F}) = 0$ for all $i > 0$) if and only if*

$$\operatorname{codim} X_i \geq i$$

for all $i > 0$. Moreover, $X_0 \supseteq X_1 \supseteq \cdots \supseteq X_n$ □

For example, Nakayama's Lemma implies that X_0 is the support of coker ϕ_1; thus X_0 is the set defined by the rank F_0-sized minors of ϕ_1. Similarly, and that X_n is the support of the cokernel of the dual of ϕ_n.

Also, if $n = 1$, the theorem simply says that a map $F_1 \to F_0$ is a monomorphism iff it becomes a monomorphism after tensoring with the field of rational functions K, which follows from the flatness of localization and the fact that F_1 is torsion-free, so that $F_1 \subset F_1 \otimes K$.

Cheerful Fact 17.3.1. Theorem 18.7.1 is true in this form over any Cohen-Macaulay ring; for more general rings, "codimension" must be replaced by "grade", as in the given reference. The Theorem can be generalized to case where the F_i are not free, but are sufficiently "like" free modules, too.

Conclusion of the proof of Theorem 17.3.1. Let $X_i \subset \mathbb{A}^{r+1}$ be the variety defined from the complex $EN(M)$ as in Theorem 18.7.1. Since $EN(M)$ becomes split exact after inverting any 2×2 minor of M X_i is contained in the closed set defined by $I_2(M)$, for all i. Thus if $I_2(M)$ has codimension $n-1$, then $EN(M)$ is acyclic. □

((START COMMENTED OUT MATERIAL))
We will also use a special case of the Auslander-Buchsbaum formula connecting projective dimension and depth:

Theorem 17.3.3. *If R is a regular local ring of dimension d, and M is a finitely generated R-module, then the projective dimension of M is $\leq d$ with equality only if M contains a submodule of finite length.*

Corollary 17.3.4. *If R is a regular local ring of dimension d, and M is a finitely generated R-module, then the codimension of an associated prime of A is at most the projective dimension of A.*

Proof of Corollary 17.3.4. Projective dimension can only decrease under localization, and the associated primes P of A are those for which A_P contains a submodule of finite length. □

With this and the multi-linear algebra above we can prove the basic acyclicity result for an Eagon-Northcott complex:

((the Theorem as now stated doesn't need the following. I've copied the short proof of acyclicity into the end of the proof above.))

Proposition 17.3.5. *Let $S = k[x_0, \dots, x_r]$ be a polynomial ring, and let $M : F \to G$ be a homomorphism with $F = S^n(-1), G = S^2$.*

$$S^n \cong F \xrightarrow{M} G \cong S^2$$

is a (not necessarily homogeneous) map of free S-modules. The Eagon-Northcott complex $EN(M)$ is acyclic if and only if $\operatorname{codim} I_2(M) \geq n-1$, in which case the dual complex is also acyclic and the associated primes of $I_2(M)$ are all minimal and of codimension $n-1$.

Proof of Proposition 17.3.5. Let $X_i \subset \mathbb{A}^{r+1}$ be the variety defined from the complex $EN(M)$ as in Theorem 18.7.1. Since $EN(M)$ becomes split exact after inverting any 2×2 minor of M X_i is contained in the closed set defined by $I_2(M)$, for all i. Thus if $I_2(M)$ has codimension $n-1$, then $EN(M)$ is acyclic.

In this case the projective dimension of $S/I_2(M)$ is $n-1$, so all the associated primes of $I_2(M)$ have codimension exactly $n-1$.

If $EN(M)$ is acylic then, by Theorem 18.7.1, the codimension of X_{n-1} is at least $n-1$. Thus to prove that the acyclicity of $EN(M)$ implies $\operatorname{codim} I_2(M) \geq n-1$ (and thus $\operatorname{codim} I_2(M))n-1$, it suffices to show that $X_{n-1} = X_0$ as algebraic sets.

To see this, note that the ideal of 2×2 minors of M. By definition, X_{n-1} is the set of points p where $\kappa(p)\otimes\delta_{n-1}$ is not an inclusion, or equivalently, that that

$$\kappa(p)\otimes F\otimes\operatorname{Sym}^{n-3} G \cong \kappa(p)\otimes \bigwedge^{n-1} F^*\otimes\operatorname{Sym}^{n-3} G \xrightarrow{\partial_{n-1}} \kappa(p)\otimes\bigwedge^{n} F^*\otimes\operatorname{Sym}^{n-2} G \cong \kappa(p)\otimes\operatorname{Sym}^{n-2} G$$

is not a split surjection, and it is easy to see that the composite map takes $a\otimes b$ to $\kappa(p)\otimes M(a)\cdot b$, so the cokernel is the $(n-2)$-nd symmetric power of the cokernel of the map $\kappa(p)\otimes M$. Thus X_n is equal to the support of the cokernel of M itself.

By Nakayama's Lemma, X_0 is the support of M; furthermore, the localization of $\operatorname{coker} M$ at p is 0 if and only if one of the 2×2 minors of M is a unit locally at p so X_0, so this is defined set-theoretically by $I_2(M)$.

It now follows from Theorem 18.7.1 that all the X_i are equal, so $EN(M)$ is acyclic if and only if $EN(M)^*$ is acyclic.

Since M is 1-generic the entries of of the second row of M are linearly independent, and since the dimension of the span of all the linear forms is at least $n+1$, some element in the first row is outside the span of the the elements in the second. After a permutation of columns we may assume that $l_{1,1}, l_{2,1}, l_{2,2}, \dots l_{2,n}$ are linearly independent, and we may take them to be a subset of the variables, say $x_0, \dots x_{n+1}$

We next show by induction on n that $I_2(M)$ is prime. In the case $n=2$ we have $I_2(M) = (x_0x_2 - x_1l_{1,2})$ which obviously does not factor.

Now suppose that $n > 2$, and let M' be the matrix M with the first column omitted. we know by induction that $I_2(M')$ is prime of codimension $n-2$. Since $I := I_2(M)$ does not have the maximal ideal as an associated prime, it is saturated. The ideal $I_2(M) + x_0$ properly contains $I_2(M')$ and thus has codimension $\geq n-1$ in S/x_0, whence we see that every component of $I_2(M)$ meets the open set $x_0 = 1$. Restricting to this open set □

The first non-trivial example of a finite free resolution is the Koszul complex on 3 variables, which is the minimal $S = k[x,y,z]$-free resolution of the module

$S/(x,y,z)$:

$$0 \to S(-3) \xrightarrow{\begin{pmatrix} x \\ y \\ z \end{pmatrix}} S^3(-2) \xrightarrow{\begin{pmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{pmatrix}} S^3(-1) \xrightarrow{\begin{pmatrix} x & y & z \end{pmatrix}} S$$

In fact this is the first example that Hilbert presented in his famous paper [].
((END COMMENTED OUT MATERIAL))

17.4 Canonical Curves

We follow the treatment in [**?**], and treats a more general situation than that of the images of smooth curves under the canonical embeddings.

We define a *canonical curve* in $\mathbb{P}^{g-1}$ to be a purely one-dimensional, nondegenerate closed subscheme such that

$$h^0(\mathcal{O}_C) = 1,\ h^0(\mathcal{O}_C(1) = g,\ \text{and}\ \omega_C = \mathcal{O}_C(1).$$

Note that the first of these hypotheses implies that C is connected, and the last shows that C is (locally) Gorenstein.

The first of these conditions is always satisfied when C is reduced and connected. The last two conditions imply that C is locally Gorenstein, and that C is embedded in $\mathbb{P}^{g-1}$ by g independent sections of the dualizing, line bundle.

We say that a canonical curve C has a *simple* $g-2$ secant if C contains $g-2$ smooth points spanning a $(g-3)$ dimensional plane Λ in $\mathbb{P}^{g-1}$ that meets C only in the $g-2$ points; equivalently, the hyperplanes containing Λ then intersect the curve in an additional base-point-free pencil. In characteristic 0, such secant planes always exist for reduced, irreducible curves. More generally:

Lemma 17.4.1. *If $C \subset \mathbb{P}^n$ is a reduced, irreducible, nondegenerate curve, and $m \leq n-2$, then the linear span $L := \overline{p_1, \dots, p_m}$ of m general points of C is a simple m-secant; that is, a plane of dimension $m-1$ such that $C \cap L = \{p_1, \dots, p_m\}$ scheme-theoretically.*

Proof. The plane L is contained in a hyperplane H, and since the points are general, we may take this to be a general hyperplane. By Bertini's Theorem, $C \cap H$ is reduced, so $C \cap L$ is also reduced. If $C \cap L$ had length $> m$, then by Theorem **??**
((in ch 8-BrillNoether))
every set of $m+1$ points of $C \cap H$ would be dependent, and the span of $C \cap H$ would thus have dimension $\leq m-1 < n-1$, and we could choose a hyperplane section $C \cap H'$ with more points than $C \cap H$, which is absurd. $\square$

Theorem 17.4.2 (Max Noether). *A canonical curve in* $\mathbb{P}^{g-1}$ *has degree* $2g-2$ *and arithmetic genus* g*. If the curve has a simple* $g-2$ *secant, then it is arithmetically Cohen-Macaulay; that is,* $\mathrm{H}^1(\mathcal{I}_{C/\mathbb{P}^{g-1}}(m)) = 0$ *for all* $m \in \mathbb{Z}$*.*

For a canonically embedded irreducible curve the simple $g-3$-dimensional $g-2$ secant planes Λ correspond to base-point-free pencils of degree $g = 2g - 2 - (g-2)$: Given Λ, the linear series of hyperplanes containing Λ intersects C in Λ plus the fibers of this pencil.
((I worry about the converse; why shouldn't the base locus of $K - g^1_g$ have a multiple point, or even contain a singular point?))
Conversely, given such a pencil, the plane is the span of the complement of a general member P of the pencil in $C \cap \overline{P}$, where $\overline{P}$ is the hyperplane that is the linear span of P.

Proof. The Hilbert polynomial $\chi_C(t) = h^0\mathcal{O}(t) - \mathrm{h}^1\mathcal{O}(t)$ of C has degree equal to $\dim C = 1$, so it is determined by two values.

We begin by showing that $\mathcal{O}(-m)$ has no global sections for $m > 0$. If D is a divisor equivalent to m times the hyperplane section, we have an exact sequence

$$0 \to \mathrm{H}^0(\mathcal{O}_C(-m)) \to \mathrm{H}^0(\mathcal{O}_C) \to \mathrm{H}^0(\mathcal{O}_D) \to \cdots .$$

By hypothesis, the vector space $\mathrm{H}^0\mathcal{O}_C$ is spanned by the constant functions, and these restrict non-trivially to $\mathcal{O}_D$, and $\mathrm{H}^0(\mathcal{O}_C(-m)) = 0$ as claimed.

Using the Riemann-Roch Theorem we can now compute the Hilbert function $\chi_C(m)$: We have

$$\chi_C(0) = h^0(\mathcal{O}_C) - h^1(\mathcal{O}_C) = h^0(\mathcal{O}_C) - h^0(\omega_C) = 1 - g.$$
$$\chi_C(1) = h^0(\mathcal{O}_C(1)) - h^1(\mathcal{O}_C(1)^* \otimes \omega_C) = h^0(\omega_C) - h^0(\mathcal{O}_C) = g - 1.$$

and we deduce $\chi_C(m) = (2g-2)m - g + 1$, whence we see that the degree of C is $2g-2$ and $\mathrm{p_a})(C) = g$ as claimed.

To show that C is arithmetically Cohen-Macaulay we use the sequence

$$\cdots \to \mathrm{H}^0(\mathcal{O}_{\mathbb{P}^n}(m)) \to \mathrm{H}^0(\mathcal{O}_C(m)) \to \mathrm{H}^1(\mathcal{I}_C(m)) \to \mathrm{H}^1(\mathcal{O}_{\mathbb{P}^n}(m)) \to \cdots .$$

Since $\mathrm{H}^0(\mathcal{O}_{\mathbb{P}^n}(m)) = 0$, it is enough to show that the natural map

$$\mathrm{H}^0(\mathcal{O}_{\mathbb{P}^n}(m)) \to \mathrm{H}^0(\mathcal{O}_C(m))$$

is surjective for all $m \in \mathbb{Z}$. For $m = 0, 1$ this is immediate from the hypothesis.

For $m < 0$ we must show $\mathrm{H}^0(\mathcal{O}_C(m)) = 0$. If D is a divisor equivalent to $-m$ times the hyperplane section, we have an exact sequence

$$0 \to \mathrm{H}^0(\mathcal{O}_C(m)) \to \mathrm{H}^0(\mathcal{O}_C) \to \mathrm{H}^0(\mathcal{O}_D) \to \cdots .$$

By hypothesis, the vector space $\mathrm{H}^0\mathcal{O}_C$ is spanned by the constant functions, and these restrict non-trivially to $\mathcal{O}_D$, so the kernel, $\mathrm{H}^0(\mathcal{O}_C(m))$, is 0 as claimed.

To prove surjectivity for $m \geq 2$ we use the remaining hypothesis, the existence of a simple $g-3$-dimensional $g-2$ secant plane Λ and an idea sometimes called the *base-point-free pencil trick*. Let $p_0, \dots p_{g-3}$ be the points in which Λ meets C. Since the p_i are linearly independent by hypothesis, we may choose homogeneous coordinates $x_i \in \mathrm{H}^0(\mathcal{O}_C(1))$ so that $x_i(p_j \neq 0$ if and only if $i = j$. It follows that the sections x_i^m of $\mathcal{O}_C(m)$ span $\mathrm{H}^0(\mathcal{O}_C(m)|_{\{p_0,\dots,p_{g-3}\}}$. Let $V \subset \mathrm{H}^0(\mathcal{O}_C(1))$ be the two-dimensional subspace of linear forms vanishing on Λ, and thus on the p_i.

For $m \geq 2$ there are maps of vector spaces

$$\wedge^2 V \otimes \mathrm{H}^0(\mathcal{O}_C(m-2)) \to V \otimes \mathrm{H}^0(\mathcal{O}_C(m-1)) \to \mathrm{H}^0(\mathcal{O}_C(m))$$

where the right hand map is multiplication and the left hand map sends $s_1 \wedge s_2 \otimes \sigma$ to $s_1\sigma - s_2\sigma$ for any local section σ. The sequence is exact because the sections s_1, s_2 that span V never vanish simultaneously except on the p_i, and has image consisting of sections that vanish on the points p_i

□

Corollary 17.4.3. *If $C \subset \mathbb{P}^{g-1}$ is a canonical curve with a simple $g-3$-secant, then the Hilbert function of the homogeneous coordinate ring S_C of C depends only on g, and is given by:*

$$\dim(S_C)_d = h^0(\mathcal{O}_C(d)) = \begin{cases} 0 & \text{if } d < 0 \\ 1 & \text{if } d = 0 \\ g & \text{if } d = 1 \\ (2n-1)g + 1 & \text{if } d > 1 \end{cases}$$

Proof. By Theorem 17.4.2 implies, in particular, that the homogeneous coordinate ring of C can be identified with $\oplus_{n\in\mathbb{Z}} \mathrm{H}^0\mathcal{O}_C(n)$. □

17.5 Betti tables of canonical curves

((We need to add the regularity of the canonical curve = 3; that plus Gorenstein, self-dual resolution, gives the shape of the Betti table in which you could look for invariants. Gorenstein comes from the $H^1 I$ computation. In the homol alg appendix we should explain the resolution duality, apply it here.))

17.6 Syzygies and the Clifford index

Corollary 17.4.3 implies that the dimension of the vector space of forms of degree d vanishing on a canonical curve is independent of the curve; for example, for $d = 2$ we get $\dim(I_C)_2 = \binom{g-2}{2}$ The next question one might ask is whether or not these quadric generate the ideal I_C, and (much) more generally, what is the Betti table of the homogeneous coordinate ring of C.

For example, when C is trigonal, with a g^1_3 defined by a line bundle $\mathcal{L}$, the complementary linear series, defined by $\omega_C \otimes \mathcal{L}^{-1}$ has $g-2$ sections, and we see from Theorem **** that C lies on the $\binom{g-2}{2}$ quadrics defined by the minors of a $2\times g-2$, 1-generic matrix of linear forms. The exactness of the Eagon-Northcott complex associated to this matrix shows that there are no relations of degree 0 on these minors – that is, they are linearly independent over the ground field. It follows that they generate the vector space of all quadrics containing C. But by **** the locus defined by the ideal of minors is a rational normal scroll of dimension 2, and thus the minors cannot generate I_C.

Furthermore, if $g = 6$ and C is isomorphic to a plane quintic curve, then the canonical series of the plane quintic is $5 - 3 = 2$ times the hyperplane series, and it follows that the canonical image of C lies on the Veronese surface in $\mathbb{P}^5$. Using Theorem *** again, we see that the Veronese is contained in (in fact, equal to) the intersection of the quadrics defined by the 2×2 minors of a generic symmetric matrix, coming from the multiplication map

$$\mathrm{H}^0(\mathcal{O}_{\mathbb{P}^1}(1)) \otimes \mathrm{H}^0(\mathcal{O}_{\mathbb{P}^1}(1)) \to \mathrm{H}^0(\mathcal{O}_{\mathbb{P}^1}(2)) = \mathrm{H}^0(\mathcal{O}_{\mathbb{P}^5}(1))$$

and there are $6 = \binom{g-2}{2}$ independent quadrics in this ideal. Again in this case, they cannot generate the ideal of the curve.

One might fear that this is the beginning of some long series of examples, but in fact it is not:

Theorem 17.6.1 (Petri). *The ideal of a canonical curve of genus ≥ 5 is generated by the $\binom{g-2}{2}$-dimensional space of quadrics it contains unless the curve is either trigonal or isomorphic to a plane quintic; in the latter cases, the ideal of the curve is generated by quadrics and cubics.*

For a modern treatment of Petri's Theorem in this level of generality see [**?**]; for a different treatment see [**?**]
((E. Arbarello and E. Sernesi, Petri's approach to the study of the ideal associated to a special divisor, Inventiones Math. 49 (1978), 99-119,))
and for

The two exceptions can be described simultaneously by using the Clifford index:

Definition 17.6.2. The Clifford index Cliff $\mathcal{L}$ of a line bundle $\mathcal{L}$ on a curve C is $d - 2r$, where $d := \deg \mathcal{L}$ and $r := h^0(\mathcal{L}) - 1$. The Clifford index Cliff C of a curve C of genus ≥ 2 is the minimum of the Clifford indices of special line bundles with at least 2 sections.

Cliffords Theorem **??** says that Cliff $C \geq 0$, and that Cliff $C = 0$ if and only if C is hyperelliptic. If C is not hyperelliptic, then it turns out that Cliff $C = 1$ if and only if C is either trigonal or isomorphic to a plane quintic. The Clifford index of any smooth curve of genus $g \geq 2$ is $\leq \lceil g/2 \rceil + 1$, with equality for a general curve, as one sees from the Brill-Noether Theorem **??**, and for "most" curves the line bundle $\mathcal{L}$ of maximal Clifford index has only 2 sections, though there is an infinite sequence of examples where this "Clifford dimension" is greater.

Moving to cubic forms, we see that $\dim(I_C)_3 = \binom{g+2}{3} - (5g-5)$. Comparing this number with the number of (possibly linearly dependent) cubics obtained by multiplying g linear forms and $\binom{g-2}{2}$ quadrics, we see that the ideal of the curve has at least

$$\binom{g-2}{2} - \binom{g+2}{3} - (5g-5)$$

independent syzygies of total degree 3 (that is, linear syzygies on the quadrics. For example when $g = 4$ so that $C \subset \mathbb{P}^3$ there is one quadric and 5 independent cubics, at most 4 of which are multiples of the quadric. Since the curve has degree $6 = 2 \times 3$, the ideal of the curve must be generated by the quadric and one cubic. When $g = 5$ there are genuinely two possibilities: the three quadrics in the ideal might be a complete intersection (then they generate the ideal), so the Betti table would be

$j\backslash i$	0	1	2
0	1	–	–
1	–	2	–
2	–	–	1

or the curve could be trigonal, in which case the 3 quadrics generate the ideal of a surface scroll F. In the latter case, the Eagon-Northcott complex resolves the homogeneous coordinate ring S_F of the scroll,

$$0 \to S^2(-3) \to S^3(-2) \to S \to S_F \to 0$$

which has Betti table

$j\backslash i$	0	1	2
0	1	–	–
1	–	3	2

and we see that there are 2 linear relations among the quadrics. Thus the minimal generators of I_C must include exactly 2 cubics as well as the 3 quadrics. Since the homogeneous ring of a canonical curve is Gorenstein, its minimal free resolution is symmetric, and this is enough for us to fill in its Betti table:

$j\backslash i$	0	1	2	3
0	1	–	–	–
1	–	3	2	–
2	–	2	3	–
3	–	–	–	1

Note that we can “see” the scroll reflected in the top two lines of the table.

From the analogue of the Hilbert-Burch Theorem for Gorenstein rings of codimension 3 one can show that the 5 generators can be written as the pfaffians of a skew symmetric 5×5 matrix whose entries are of degrees 1 and 2, in the following pattern (we give just the degrees, and put - in the places that are 0):

$$\begin{pmatrix} - & - & 1 & 1 & 1 \\ - & - & 1 & 1 & 1 \\ 1 & 1 & - & 2 & 2 \\ 1 & 1 & 2 & - & 2 \\ 1 & 1 & 2 & 2 & - \end{pmatrix}$$

Here the2×2 minors of the upper 2×3 block of linear forms generate the ideal of the scroll.

Applying this logic more generally we get the following result about the canonical embedding of curves with low degree maps to $\mathbb{P}^1$:

Theorem 17.6.3. *Let $C \subset \mathbb{P}^{g-1}$ be a reduced, irreducible canonical curve. If C has a line bundle $\mathcal{L}$ of degree d with $h^0(\mathcal{L}) = 2$ then there is a $2 \times g+1-d$ 1-generic matrix of linear forms whose minors define a scroll of codimension $g-d$ containing C; and thus an Eagon-Northcott complex of length $g-d$ is a quotient complex of the minimal free resolution of S_C. In particular, the Betti table of S_C is termwise $\geq$ that of the homogeneous coordinate ring of the scroll.*

Thus the existence of the g^1_d on C, together with the symmetry of the resolution of the Gorenstein ring S_C, implies that the Betti table of S_C has the form

$j\backslash i$	0	1	2	…	d-3	d-2	…	g-d-1	g-d	…	g-3	g-2
0	1	–	–	⋯	–	–	–	–	–	–	–	–
1	–	*	*	…	*	*	…	*	?	…	?	?
2	–	?	?	…	?	*	…	*	*	…	*	*
3	–	–	–	⋯	–	–	–	–	–	–	–	1

where we have assumed for illustration that $d-2 < g-d-1$. The places marked $-$ are definitely 0 and those marked * are definitely nonzero. The rows marked 0 and 1 contain the Betti table of the scroll.

We can summarize this by saying that if the curve C has a line bundle $\mathcal{L}$ of degree d with exactly 2 sections, and thus of Clifford index $c = d-2$ the row labeled 2 in the Betti diagram definitely has nonzero entries starting in the c-th place. As with the case of the plane quintics, above, one can make a similar argument for *any* line bundle of Clifford index c

Starting from such examples, Mark Green made a bold conjecture that was still open at the time this book was written:

Conjecture 17.6.4 (Green's Conjecture)**.** *If C is a smooth canonical curve of genus g and Clifford index d-2, then the entries marked with ? in the Betti table above are all 0.*

The conjecture was made for curves over a field of characteristic 0, and is known in many cases, though it is also known to fail in small finite characteristics. For example, it is true for generic curves of each Clifford index, and is true for *every* curve of Clifford index $c = \lceil g/2 \rceil + 1$, the maximal value. It is also true for plane curves, and in a number of other special cases. See **** for a survey.

17.6.1 Low genus canonical embeddings

Schreyer's table (include $g = 9$)?

Chapter 18

Appendix: Homological commutative algebra

18.1 Introduction: Homological commutative algebra

The groundwork for homological commutative algebra was laid by Arthur Cayley, David Hilbert, Frances Sowerby Macaulay, Wolfgang Gröbner and others, in the context of polynomial rings. The rings we encounter in studying projective geometry are mostly factor rings S/I where S is a polynomial ring, or localizations (and more rarely completions) of these. But after the work of Emmy Noether and her student Wolfgang Krull it was apparent that much of commutative algebra could be done axiomatically, without reference to a base polynomial ring, making the theory at once simpler and more powerful. With the work of Chevalley, Zariski, and Cohen it became clear that most of the basic properties of interest were best treated starting from the case of local rings.

With the work of Auslander, Buchsbaum and Serre, homological techniques became important. This means, roughly, focusing on modules over these rings (representation theory) and on complexes of modules, especially free resolutions. The groundwork for this extension had of course been laid by Cayley, Hilbert, Macaulay, Gröbner..., but always in the context of polynomial rings.

In this tradition, we will begin by describing homological properties of local rings. However, every statement can be transposed to the setting of a standard graded algebra, graded modules, and homogeneous ideals, where by a *standard graded algebra* we mean a positively graded algebra R over a field k such that $R_0 = k$ and R is generated as an algebra by the finite-dimensional vector space R_1. The analogue of the maximal ideal of a local ring is then the maximal homogeneous ideal (necessarily generated by R_1. We generally leave the translation to the reader. It is also possible to define a local ring to be a non-negatively graded ring R whose degree zero component is a local (ungraded) ring, and to develop the whole theory in this style, in parallel with Grothendieck's idea that it is best always to work with varieties over some base scheme. But that adds enough weight to otherwise simple arguments that we have not taken this path.

We return to the standard graded case in Chapter 18, and discuss the homological properties in terms of syzygies and Betti tables of R a graded module over a polynomial ring.

All rings in this chapter will be assumed Noetherian. To indicate that R is a local ring with maximal ideal $\mathfrak{m}$ and residue field $R/\mathfrak{m} = k$, we sometimes say: "Let $(R, \mathfrak{m}, k)$ be a local ring.") We denote by $\dim R$ the *Krull dimension* of R; that is, the maximum length of a chain of prime ideals in R.

18.2 Modules and sheaves: local and global cohomology

When we study projective varieties both graded modules and the sheaves associated to them play a role. As Serre explained in the category of coherent sheaves is the category of finitely generated graded modules *modulo* the subcategory of graded modules of finite length. One expression of this is the relationship between local and global cohomology, which we will explain in this section. For the sake of simplicity, we will stick with coherent sheaves on projective space.

Let $S = k[x_0, \dots, x_n]$, the homogeneous coordinate ring of $\mathbb{P}^n$, and write. Recall that the cohomology of the sheaf $\mathcal{M}$ associated to a graded S-module M is the cohomology of the Cech complex:

$$\mathcal{C}(M) : 0 \to \bigoplus_{0 \le i \le n}^{n} M_{x_i} \to \bigoplus_{0 \le i < j \le n} M_{x_i x_j} \to \cdots$$

where M_m denotes the localization of M at the powers of m, corresponding to the restriction of the sheaf to the open set where $m \neq 0$. The same formula works for a module over any finitely generated graded k-algebra R and the corresponding sheaf on $\operatorname{Proj} R$. The homology of this complex at the term

$$\bigoplus_{0 \le j_0 < j_1 < \cdots < j_i \le n} M_{x_{j_0} x_{j_1} \cdots x_{j_i}}$$

is

$$H^i_*(\mathcal{M}) := \bigoplus_{d \in \mathbb{Z}} H^i(\mathcal{M}(d)),$$

the sum of the i-th cohomology spaces of all twists of $\mathcal{M}$.

It is easy to see that we can add M itself to the left of this complex, making an augmented complex

$$\mathcal{C}'(M) : 0 \to M \to \bigoplus_{0 \le i \le n}^{n} M_{x_i} \to \bigoplus_{0 \le i < j \le n} M_{x_i x_j} \to \cdots$$

We define the homology of this complex to be the *local cohomology* $H_{\mathfrak{m}}(M)$ of M with respect to the ideal $\mathfrak{m} := (x_0, \ldots, x_n)$. It follows immediately that for $i \ge 1$ we have $H^i_*(\mathcal{M}) = H^{i+1}_{\mathfrak{m}}(M)$, and this holds degree by degree: the local cohomology module inherits a grading from M and the fact that the x_i are homogeneous and its homogeneous component of degree d is $H^i(\mathcal{M}(d))$. Thus the modules $H^i_{\mathfrak{m}}$ have geometric meaning for $i \ge 2$.

We can easily elucidate the meaning for $i = 0, 1$ as well: First, the elements of M that map to 0 in M_{x_j} are the elements annihilated by some power of x_j so $H^0_{\mathfrak{m}}(M)$ is the set of elements annihilated by some power of every x_j, or equivalently annihilated by some power of the maximal ideal. This is precisely the set of elements that go to 0 under the natural map $M \to H^0_*(\mathcal{M})$. In particuar, if M is the homogeneous coordinate ring $R = S/I$ of an algebraic set X, then I is the saturated ideal of X if and only if $H^0_{\mathfrak{m}}(R) = 0$.

We can interpret $H^0_{\mathfrak{m}}(M)$ in module-theoretic terms too: Write $(0_M : \mathfrak{m}^e)$ for the elements of M annihilated by $\mathfrak{m}^e$, and $H^0_{\mathfrak{m}}(M) = (0_M : \mathfrak{m}^\infty)$ for the union of all the $(0_M : \mathfrak{m}^e)$. Since M is Noetherian, we have $H^0_{\mathfrak{m}}(M) = (0_M : \mathfrak{m}^e)$ for some e. Since M is graded, this can be interpreted as the maximal submodule of M of finite length.

The discussion above shows that the sequence

$$0 \to H^0_{\mathfrak{m}}(M) \to M \to H^0_*(\mathcal{M})$$

is exact. But from the definitions it follows at once that we can extend this to an exact sequence

$$0 \to H^0_{\mathfrak{m}}(M) \to M \to H^0_*(\mathcal{M}) \to H^1_{\mathfrak{m}}(M) \to 0.$$

The module $H^1_{\mathfrak{m}}(M)$ also has an important interpretation: Consider again the case $M = R = S/I$, corresponding to a projective scheme X, and suppose for simplicity that $n \ge 2$, so that $H^1(\mathcal{O}_{\mathbb{P}^n}(d)) = 0$ for all d. In this case the long exact sequence in cohomology associated to

$$0 \to \mathcal{I}_X \to \mathcal{O}_{\mathbb{P}^n} \to \mathcal{O}_X \to 0,$$

begins

$$0 \to H^0_*(\mathcal{I}_X) \to H^0_*(\mathcal{O}_{\mathbb{P}^n}) \to H^0_*(\mathcal{O}_X) \to H^1_*(\mathcal{I}_X) \to 0.$$

Since the image of $H^0_*(\mathcal{O}_{\mathbb{P}^n}) \to H^0_*(\mathcal{O}_X)$ is the same as the image of $S/I \to H^0_*(\mathcal{O}_X)$ (both are equal to S_X), this shows that $H^1_{\mathfrak{m}}(S/I) = H^1_*(\mathcal{I}_X)$. This module will play an important role in the treatment of linkage, below.

The local cohomology modules are typically artinian, but not finitely generated. But there is an expression of the cohomology of a sheaf or module in terms of finitely generated modules. If $M = \oplus_i M_i$ is a graded module over a standard graded polynomial ring $S = k[x_0..x_n]$, and M has infinitely many nonzero components, then the dual $\mathrm{Hom}_k(M,) = \prod_i M_i$ is naturally a module over the power series ring $\hat{S} = k[[x_0..x_n]]$, and thus is never finitely generated over R. But we can also form the *graded dual*, $\mathrm{Hom}_{k,gr}(M, k) := \oplus_i \mathrm{Hom}_k(M_i,)$, which is again an S-module.

Theorem 18.2.1 (Local duality)**.** *Let $S = k[x_0..x_n]$ be a standard graded polynomial ring over k. If M is a finitely generated graded S-module, then*

$$H^i_{\mathfrak{m}} M = \mathrm{Hom}_{gr,k}(\mathrm{Ext}_S^{n+1-i}(M, S(-n-1), k).$$

18.3 Regular local rings and syzygies

Let $(R, \mathfrak{m}, k)$ be a local ring. By the Principal Ideal Theorem [?,], the maximal ideal $\mathfrak{m}$ cannot be generated by $< \dim R$ elements.

Definition 18.3.1. We say that R is *regular* if $\mathfrak{m}$ can be generated by $\dim R$ elements.

This deceptively simple property was first identified as important by Krull, and later recognized by Zariski as the appropriate algebraic expression of nonsingularity: A point p on a scheme X is called nonsingular if and only if the local ring $R = \mathcal{O}_{X,p}$ is *regular*. This is justified by the fact that if R is the local ring of a point p on a variety over an algebraically closed field, then the cotangent space to p is naturally identified with the k-vector space $\mathfrak{m}/\mathfrak{m}^2$, whose dimension is, by Nakayama's Lemma, the minimal number of generators of $\mathfrak{m}$.

The analogue of "regular" in the case of a standard graded algebra R is that R is isomorphic to the polynomial ring on a basis of R_1. Indeed, the localization of R at the maximal homogeneous ideal is regular in the local sense if and only if this condition is satisfied.

For the regularity of $\mathcal{O}_{X,p}$ to be a reasonable algebraic analogue of nonsingularity, it should of course imply that X is reduced and irreducible at p; that is, a regular local ring should be a domain. This was proven by Krull, before the work of Zariski:

Proposition 18.3.2. *If R is regular local ring is an integral domain; that is, 0 is a prime ideal.*

Proof. We do induction on the dimension. If $\dim R = 0$ then by definition $\mathfrak{m}$ is generated by 0 elements, so $R = k$, a field. If $\dim R > 0$ then by the prime avoidance theorem [?,] there is an element x not contained in trhe union of $\mathfrak{m}^2$ and the minimal primes of R. By the Principal Ideal Theorem, $R/(x)$ has dimension $\dim R - 1$ and the maximal ideal $\mathfrak{m}/(x)$ has $\dim R - 1$ generators, so $R/(x)$ is again regular.

By induction, (x) is a prime ideal of R that is not a minimal prime. If Q is a minimal prime contained in (x), then $q \in Q$ implies $q = q'x$ for some $q' \in R$, and since Q is prime, we have $q' \in Q$. Thus $Q = Qx$, and it follows from Nakayama's Lemma that $Q = 0$, so R is a domain. $\square$

This result has a consequence that leads to an important definition:

Corollary 18.3.3. *Let R be a regular local ring of dimension d. If $x_1, \ldots, x_d$ generate $\mathfrak{m}$, then x_{i+1} is a nonzerodivisor modulo $(x_1, \ldots, x_i)$ for every $i = 1, \ldots, n$*

Proof. Obvious, since $R/(x_1, \ldots, x_i)$ is again regular, and thus a domain, and $x_{i+1} \notin (x_1, \ldots, x_i)$. $\square$

We say that a sequence of elements in the maximal ideal of R that satisfies the condition of the Corollary is a *regular sequence*. It is convenient to extend the definition to modules:

Definition 18.3.4. Let R be a commutative ring, and let M be an R-module. A sequence of elements $x_1, \ldots, x_n \in R$ is called a *regular sequence on M*, or an *M-sequence*, if x_i is a nonzerodivisor on $M/(x_1, \ldots x_{i-1})M$ for all $i = 1, \ldots, n$, and $(x_1, \ldots, x_n)M \neq M$.

Note that if $(R, \mathfrak{m}, k)$ is a local ring, $(x_1, \ldots, x_n) \subset \mathfrak{m}$ and M is finitely generated, then he last condition is auttomatic from Nakayama's Lemma.

Recall that if M is a finitely generated R-module, then an *R-free resolution* of M is a sequence of free modules and maps

$$\mathbb{F}: \qquad F_0 \xleftarrow{d_1} F_1 \xleftarrow{d_2} F_2 \cdots,$$

an *augmentation* map $F_0 \xrightarrow{d_0} M$ such that the kernel of d_i is equal to the image of d_{i+1} for every i. We say that the resolution is *finite of length n* if $F_{n+1} = 0$ but $F_n \neq 0$.The resolution is called *minimal* if the $d_i(F_i) \subset \mathfrak{m}F_{i-1}$ for all i; it follows from Nakayama's Lemma that this is the case if and only if the rank of F_i is equal to the minimal number of generates of $\ker d_{i-1}$ for all i.

The minimal resolution of a module is a direct summand of any resolution; and it follows that any two minimal resolutions of a module are isomorphic [**?**, Theorem ***].

Example 18.3.5. The Koszul complex of a sequence $x_1, \ldots, x_n$: Consider first a single element $x = x_1 \in R$. We define the Koszul complex on x, denoted $\mathbb{K}(x; R)$, to be the complex

$$\mathbb{K}(x; R): \quad R \xleftarrow{x} R \longleftarrow 0.$$

This complex is a minimal free resolution of $R/(x)$ if and only if x is a nonzerodivisor contained in the maximal ideal of R. Observe that this is also the condition for the one element sequence x to be a regular sequence.

Next consider a pair of elements $x_1, x_2 \in R$. The Koszul complex on x_1, x_2 is the R-free complex

$$\mathbb{K}(x_1, x_2; R): \quad R \xleftarrow{\phi_1 = \begin{pmatrix} x_1 & x_2 \end{pmatrix}} R^2 \xleftarrow{\phi_2 = \begin{pmatrix} x_2 \\ -x_1 \end{pmatrix}} R \longleftarrow 0.$$

It is obvious that $\operatorname{coker} \phi_1 = R/(x_1, x_2)$. Also $\ker \phi_2$ is the annihilator of the ideal (x, y), and it follows from the theory of associated primes that this is 0 if and only if the ideal (x_1, x_2) contains a nonzerodivisor. For simplicity, let us assume that x_1 is a nonzerodivisor itself, although this is not actually necessary. The kernel of ϕ_1 obviously consists of the elements $(y_2, -y_1) \in R^2$ such that $y_2 x_1 = y_1 x_2$. Since we have assumed that x_1 is a nonzerodivisor, the element y_2 is uniquely determined by y_1 such that $y_1 x_2 \in (x_1)$, usually written $y_1 \in ((x_1) : x_2)$. Thus, given that x_1 is a nonzerodivisor, the kernel of ϕ_1 is equal to the image of ϕ_2 if and only if x_2 is a nonzerodivisor mod x_1; that is if and only if x_1, x_2 is a regular sequence.

Note that the right-hand term R^1 of $\mathbb{K}(x_1, x_2; R)$ is somehow naturally indexed by the pair of elements x_1, x_2; rather pedantically, we could write it as $\wedge^2(R^2)$. This has the advantage that ϕ_2 can be described as the result of extending ϕ_1 to be a degree -1 derivation of the exterior algebra: if we denote the basis elements of R^2 as e_1, e_2 so that $\phi_1(e_i) = x_i$, then $\phi_2(e_1 \wedge e_2) = \phi_1(e_1)e_2 - e_1\phi_1(e_2)$. Here the negative sign comes because we have commuted the derivation, of degree -1, with an element of odd degree, e_1. This leads us to rewrite the Koszul complex in the suggestive form:

$$\mathbb{K}(x_1, x_2; R): \quad \bigwedge^0 R^2 \xleftarrow{\phi_1 = \begin{pmatrix} x_1 & x_2 \end{pmatrix}} \bigwedge^1 R^2 \xleftarrow{\phi_2 = \begin{pmatrix} x_2 \\ -x_1 \end{pmatrix}} \bigwedge^2 R^2 \longleftarrow 0.$$

In general the Koszul complex of a sequence of elements $\mathbb{K}(x_1, \dots, x_n; R)$ is defined to be the exterior algebra of $R^n = \oplus_{i=1}^n Re_i$, with first differential

$$\bigwedge^0 R^n = R \xleftarrow{\phi_1 = \begin{pmatrix} x_1 & \cdots & x_n \end{pmatrix}} R^n = \bigwedge^1 (R^n)$$

and the other differentials defined to extend ϕ_1 to be a derivation of degree -1, so that

$$\phi_m(e_{i_1} \wedge \cdots \wedge e_{i_m}) = \sum_{j=1}^{m} (-1)^{j-1} x_{i_j} e_{i_1} \wedge \cdots \wedge \widehat{e_{i_j}} \wedge \cdots \wedge e_{i_m}).$$

It is easy to check that $\phi_{m-1}\phi_m = 0$ for all $m \geq 1$, so $\mathbb{K}(x_1, \dots, x_n; R)$ is a complex.

There is a surprisingly simple necessary and sufficient condition for $\mathbb{K}(x_1, \ldots, x_n; R)$ to be a minimal free resolution of coker $\phi_1 = R/(x_1, \ldots, x_n)$ [**?**,]:

Theorem 18.3.6. *If $(R, \mathfrak{m})$ is a local ring, and $x_1, \ldots x_n \in R$, then the Koszul complex $\mathbb{K}(x_1, \ldots, x_n; R)$ is a minimal free resolution (of $R/(x_1, \ldots, x_n)$) if and only if $x_1, \ldots x_n$ is a regular sequence in R.* □

This result also holds in the graded polynomial ring case, if we assume that the x_i are all of strictly positive degree. For a proof, see [**?**, Theorem 17.6].

Here is the homological characterization of regularity:

Theorem 18.3.7 (Auslander, Buchsbaum, Serre []). *The following conditions on a d-dimensional local Noetherian ring R with residue field k are equivalent:*

1. *R is regular.*
2. *Every finitely generated R-module has a finite free resolution.*
3. *Every finitely generated R-module has a free resolution of length at most d.*
4. *A minimal set of generators $x_1, \ldots, x_d$ of $\mathfrak{m}$ is a regular sequence; equivialently, the Koszul complex $\mathbb{K}(x_1, \ldots, x_d; R)$ is the minimal R-free resolution of k.*
5. $\operatorname{Ext}^i_R(k, M) = 0$ *for all $i > d$ and all finitely generated modules M.*
6. $\operatorname{Ext}^{d+1}_R(k, k) = 0$.

Perhaps the most interesting part of this is the implication 1) $\to$ 3), a vast extension of Hilbert's Syzygy Theorem. Given Theorem ***, and basic facts about the functor Tor, it is surprisingly easy to prove:

Proof that 1) $\to$ 3). Suppose that $(R, \mathfrak{m}, k)$ is a regular local ring M be a finitely generated R-module. Let $\mathbb{F}$ be a minimal free resolution of M, so that the differentials of the complex of vector spaces $k \otimes_R \mathbb{F}$ are all 0. It follows that the length of $\mathbb{F}$ is the maximal i such that

$$H_i(k \otimes_R \mathbb{F}) = \operatorname{Tor}_i^R(k, M) = 0.$$

However, we can compute $\operatorname{Tor}_i^R(k, M)$ using a resolution of k. By Corollary**??** and Theorem **??**, the Koszul complex of a minimal sequence of generators of $\mathfrak{m}$ is the minimal free resolution of k, and it has length d, so $\operatorname{Tor}_i^R(k, M) = 0$ for $i > d$ as required. □

The homological characterization of regularity enabled the proof of long-standing conjectures:

Theorem 18.3.8. *[?] If R is a regular local ring then:*

- *Every localization of R at a prime ideal is again a regular local ring*
- *R is a unique factorization domain*

18.4 Projective dimension

The first new invariant that we can read from the minimal S-free resolution of a module M is its length; that is, the number of nonzero maps, which is finite by the Syzygy Theorem. This is called the *projective dimension* of M as an S-module, written $pd_S M$. An older name, in some ways more suitable, was *homological codimension*; this is justified by the following results:

Proposition 18.4.1. *If M is a graded S-module then $\mathrm{pd}(M)$ is at least the codimension of the support of M.*

In case $\mathrm{pd}(M)$ is equal to the codimension of the support of M, we say that M is a Cohen-Macaulay S-module, or equivalently that the sheaf $\widetilde{M}$ is *arithmetically Cohen-Macaulay.* When $M = S_X$, the homogeneous coordinate ring of a projective scheme X, we say that X is itself is arithmetically Cohen-Macaulay. From the examples above we see that plane curves, and also the twisted cubic, are Cohen-Macaulay.

A famous result of Auslander and Buchsbaum clarifies the meaning of projective dimension. We define the *depth* of M to be the maximum length ℓ of a *regular sequence on* M; that is, a sequence $G_1, \dots, G_\ell$ of homogeneous forms of strictly positive degree such that

$$\begin{aligned} &G_1 \text{ is a nonzerodivisor on } M;\\ &G_2 \text{ is a nonzerodivisor on } M/G_1M;\\ &\quad\vdots \qquad\qquad\qquad\qquad \vdots\\ &G_\ell \text{ is a nonzerodivisor on } M/(G_1, \dots, G_{\ell-1})M. \end{aligned}$$

Theorem 18.4.2. *If M is a finitely generated graded module over the polynomial ring $S := \mathbb{C}[x_0, \dots, x_n]$, and M has depth ℓ, then the projective dimension of M is $n+1-\ell$.*

Suppose again that M is a finitely generated graded module. Every associated prime of M must then be homogeneous, and, since the set of zerodivisors on M is the union of all the associated primes, there is form G_1 of positive degree that is a nonzerodivisor on M if and only if the maximal ideal $\mathfrak{m}$ is not an associated prime of M, or equivalently M contains no element annihilated by $\mathfrak{m}$

((this uses prime avoidance too; probably should have a reference.

))
Since H^1_{gm} is the submodule of all elements of M annihilated by a power of $\mathfrak{m}$, we see that the projective dimension of M is $< n+1$ if and only $H^0_{\mathfrak{m}}(M) = 0$, or equivalently M is a submodule of $H^0_*(\widetilde{M})$.

Though this is not obvious from the definition, all maximal regular sequences on M have the same length, and if the depth of M is ℓ then a sequence of general linear forms of length ℓ is a regular sequence. This makes the depth easier to compute. Even better, the depth has an interpretation in terms of cohomology:

Theorem 18.4.3. *Let M be a finitely generated graded S-module. The depth of M is the smallest integer i such that $H^i_{\mathfrak{m}}(M) \neq 0$.*

Exercise 18.4.4. Prove Theorem 18.4.3 by induction on the length of a maximal regular sequence.

We can easily translate this into global cohomology in the case of a module of twisted global sections:

Theorem 18.4.5. *Suppose that $X \subset \mathbb{P}^n$ is a subscheme without 0-dimensional (isolated or embedded) components. The module $M = \oplus_{t\in\mathbb{Z}} H^0(\mathcal{O}_X(t))$ is finitely generated, and* depth M *is the smallest integer $\ell \geq 2$ such that $H^{\ell-1}(\mathcal{O}_X(t)) \neq 0$ for some t. The homogeneous coordinate ring of X is equal to M if $\oplus_{t\in\mathbb{Z}} H^1(\mathcal{I}_X(t)) =$ 0 and has depth exactly 1 otherwise.*

Proof. For the first statement, note that $H^\ell_{\mathfrak{m}} M = \oplus_t H^{\ell-1}(\mathcal{O}_X(t))$. For the second statement use the exact sequence ☐

18.5 Cohen-Macaulay rings

It is quite possible for a local ring $(R, \mathfrak{m}, k)$ of dimension d to contain a regular sequence of length d without being regular; an easy example is the 2-dimensional local ring

$$R = k[[x, y, z]]/(y^4 - x^3 z) \cong k[[s^4, s^3 t, t^4]]$$

In fact, we claim that z, x is such a regular sequence. Since the ring R is 2-dimensional, and the maximal ideal requires 3 generators x, y, z, the ring R is not regular.

Definition 18.5.1. A local ring $(R, \mathfrak{m}, k)$ of dimension d is said to be *Cohen-Macaulay* if $\mathfrak{m}$ contains a regular sequence of length d.

Note that every 0-dimensional (that is, Artinian) local ring is automatically Cohen-Macaulay.

The Cohen-Macaulay condition is made easier to check by the following important homological interpretation:

Theorem 18.5.2. *Let $(R, \mathfrak{m}, k)$ be a local ring, and let $I \subset \mathfrak{m}$ be an ideal. Let M be a finitely generated R-module. Every maximal M-sequence in I has the same length, called the* depth of I on R, *and this number is the smallest integer i such that $\mathrm{Ext}^i_R(R/I, M) \neq 0$. Moreover, if $x_1, \ldots, x_i$ is a maximal M-sequence in I then $\mathrm{Hom}_R(R/I, M/(x_1, \ldots, x_i)) = \mathrm{Ext}^i_R(R/I, M)$ is independent of the maximal regular sequence.*

Proof. Suppose that $x_1, \ldots, x_i \in \mathfrak{m}$ is a maximal regular sequence on M. We will show by induction on i that $\mathrm{Ext}^i_R(R/I, M) = \mathrm{Hom}(R/I, M/(x_1, \ldots, x_i) \neq 0$ and that $\mathrm{Ext}^j_R(R/I, M) = 0$ for $j < i$.

First suppose $i = 0$; that is, every element of I is a zero-divisor on M. This means that I is contained in the union of the finitely many associated primes of M. By the Prime Avoidance Lemma [?, ****] I is contained in a single associated prime of M, and thus I annihilates a nonzero element $m \in M$ of M, so that $\mathrm{Hom}(R/I, M)$ contains a nonzero homomorphism sending the class of 1 to m.

Next suppose that $i > 0$. Since $x_2, \ldots, x_i$ is a maximal regular sequence on $M/(x_1)M$ we see by induction. that $\mathrm{Ext}^{i-1}_R(R/I, M/x_1M) \neq 0$ and $\mathrm{Ext}^j_R(R/I, M/x_1M) = 0$ for $j < i - 1$. From the short exact sequence

$$0 \to M \xrightarrow{x_1} M \longrightarrow M/x_1M \to 0$$

we get a long exact sequence in Ext_R containing the terms

$$\begin{aligned} &\mathrm{Ext}^j_R(R/I, M) \xrightarrow{0} \mathrm{Ext}^j_R(R/I, M) \longrightarrow \mathrm{Ext}^j_R(R/I, M/x_1M) \longrightarrow \\ &\mathrm{Ext}^{j+1}_R(R/I, M) \xrightarrow{0} \mathrm{Ext}^{j+1}_R(R/I, M) \longrightarrow \cdots, \end{aligned}$$

where the maps marked 0 vanish because x_1 annihilates R/I; that is, we have short exact sequences

$$0 \to \mathrm{Ext}^j_R(R/I, M) \longrightarrow \mathrm{Ext}^j_R(R/I, M/x_1M) \longrightarrow \mathrm{Ext}^{j+1}_R(R/I, M) \to 0.$$

By induction, the middle term of this sequence vanishes for $j < i - 1$, so $\mathrm{Ext}^j_R(R/I, M) = 0$ for $j < i$ and

$$\mathrm{Ext}^i_R(R/I, M) \cong \mathrm{Ext}^{i-1}_R(R/I, M/x_1M) \cong \mathrm{Hom}(R/I, M/(x_1, \ldots x_i)) \neq 0$$

as required. □

Exercise 18.5.3. Use Theorem 18.5.2 to check that the ring

$$R = k[[s^4, s^3t, st^3, t^4]]$$

is *not* Cohen-Macaulay.

The Cohen-Macaulay property has a homological interpretation that we shall use:

Theorem 18.5.4. *Let $(R, \mathfrak{m}, k)$ be a local ring, and suppose that $S \to R$ is a map of local rings such that S is a regular local ring and R is a finitely generated S-module. The length of a minimal resolution of R as an S module is at least $\dim S - \dim R$; and it is equal to this value if and only if the ring R is Cohen-Macaulay.*

By Proposition 18.4.1, if $C \subset \mathbb{P}^n$ is 1-dimensional, then the projective dimension of S_C is at least $n-1$. But we can be much more precise. Recall that a curve $C \subset \mathbb{P}^n$ is said to be *projectively normal* if the homogeneous coordinate ring of C is integrally closed (which implies, in particular, that C is smooth).

Theorem 18.5.5. *Let $C \subset \mathbb{P}^n$ be a purely 1-dimensional subscheme. The projective dimension of the homogeneous coordinate ring S_C of C is*

$$pd_S S_C = \begin{cases} n-1 & \text{if } H^1(\mathcal{I}_C(t)) = 0 \text{ for all } t \in \mathbb{Z} \\ n & \text{otherwise.} \end{cases}.$$

Thus in the first case S_C is Cohen-Macaulay. In particular, if C is a smooth curve, the $pd_S(S_C) = n-1$ if and only if C is projectively normal.
((we used Serre's Criterion. Ref?))

Here is a version that gives a measure of how far S_C is from being Cohen-Macaulay:

Theorem 18.5.6. *Let $C \subset \mathbb{P}^n$ be a purely 1-dimensional subscheme, and let*

$$\mathbb{F} : F_0 \xleftarrow{d_1} F_1 \xleftarrow{d_2} F_2 \longleftarrow \cdots \longleftarrow F_{n-1} \xleftarrow{d_n} F_n \longleftarrow F_{n+1} \longleftarrow 0$$

be the minimal S-free resolution of the homogeneous coordinate ring of C. We have $F_{n+1} = 0$, and

$$\oplus_{t\in\mathbb{Z}}(H^1\mathcal{I}_C(t)) = \operatorname{Hom}_{\mathbb{C}}(\operatorname{Ext}^n(S_C, S(-n-1)), \mathbb{C})$$

which is sometimes called the Rao module *of C. Thus, up to a shift in grading, the Rao module of C is the vector space dual of the cokernel of the dual $d_n^* : F_{n-1}^* \to F_n$. This is a graded module of finite length.*

((Having introduced local coho, this is pretty much done.))

Theorem 18.5.7. *Let $C \subset \mathbb{P}^n$ be a curve (or more generally a purely 1-dimensional subscheme). The homogeneous coordinate ring $S_C = S/I_C$ of C is Cohen-Macaulay if and only if $H^1(\mathcal{I}_C(d)) = 0$ for all d.*

Proof. Except in the trivial case $n = 1$ we have $H^1(\mathcal{O}_{\mathbb{P}^n}(d))$ for all d, and since C is supposed purely 1-dimensional we have $H^0(\mathcal{O}_C(d)) = 0$ for $d < 0$, so from the exact sequence

$$0 \to \mathcal{I}_c \to \mathcal{O}_{\mathbb{P}^n} \to \mathcal{O}_C \to 0$$

we deduce that $H^1(\mathcal{O}_C(d)) = 0$ for all $d < 0$ in any case.

By Theorem 18.5.4, the projective dimension of S_C is at least $\dim S - \dim S_c = n - 1$.

Let $\mathfrak{m}$ be the maximal homogeneous ideal $(x_0, \dots, x_n)$ of the homogeneous coordinate ring S of $\mathbb{P}^n$. By the Auslander-Buchsbaum formula
((this should be first, and we should give the graded version too(x))
?? the projective dimension of S_C is $n+1$ (the dimension of S minus the depth of $\mathfrak{m}S_C$, that is, the length of a maximal regular sequence in S_C, which can be taken to be homogeneous.

If a finitely generated graded S-module M has depth > 0 (that is, $\mathfrak{m}$ contains a homogeneous nonzerdivisor on M) then clearly no nonzero element of M is annihilated by $\mathfrak{m}$. The converse of this statement is a consequence of the theory of primary decomposition. Further, writing $\tilde{M}$ for the associated coherent sheaf on $\mathbb{P}^n$, we have a map $M \to \oplus_{d \in \mathbb{Z}} H^0(\tilde{M}(d))$ whose kernel is precisely the set of elements annihilated by some power of $\mathfrak{m}$.

If J is the homogeneous ideal of any scheme, then by definition J is saturated; that is, no element is annihilated by $\mathfrak{m}$, or equivalently $\mathfrak{m}$ is not an associated prime ideal. Thus S/I_C has projective dimension $\leq n = \dim S - 1$. To simplify the notation, if $\mathcal{F}$ is a coherent sheaf, then we write $H^i_*(\mathcal{F})$ for $\bigoplus_i H^i(\mathcal{F}(i))$.

Suppose now that $f \in S_C$ is a nonzerodivisor of degree d and let H be the hypersurface in $\mathbb{P}^n$ that it defines. By Theorem *** we must decide whether S_C/fS_C contains a non-zerodivisor, that is, whether S_C/fS_C is saturated, or, equivalently, whether the map

$$\alpha : S_C/fS_C \to H^0_*(\widetilde{S_C/fS_C}) = H^0_*(\mathcal{O}_{H\cap C}).$$

is an injection.

The diagram below has exact rows and columns,

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & S_C \arrow[r, "f"] \arrow[d] & S_C(d) \arrow[r] \arrow[d] & S_C/(f)(d) \arrow[r] \arrow[d, "\alpha"'] & 0 \\
0 \arrow[r] & H^0_*(\mathcal{O}_C) \arrow[r, "f"] \arrow[d] & H^0_*(\mathcal{O}_C(d)) \arrow[r] \arrow[d] & H^0_*(\mathcal{O}_{H\cap C}(d)) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & H^1_*(\mathcal{I}_C) \arrow[r, "f"] \arrow[d] & H^1_*(\mathcal{I}_C(d)) \arrow[r] \arrow[d] & H^1_*(\mathcal{I}_{H\cap C}(d)) & \\
 & 0 & 0 & &
\end{tikzcd}$$

and it follows from a diagram chase (the "snake lemma") that the kernel of α

is the same as the kernel of

$$H^1_*(\mathcal{I}_C) \xrightarrow{f} H^1_*(\mathcal{I}_C(d)).$$

By Serre's vanishing theorem, $H^1_*(\mathcal{I}_C)$ is zero in high degree, and since multiplication by f raises the degree, its kernel is 0 if and only if $H^1_*(\mathcal{I}_C)$ is zero, completing the proof. $\square$

18.6 Gorenstein rings and duality

Intermediate between the class of Cohen-Macaulay rings and the class of regular rings is the class of Gorenstein rings. Roughly speaking, they are the rings for which duality is the simplest. As we shall see, all complete intersections are Gorenstein, a fact that will be central to our study of linkage, below.

Definition 18.6.1. A local ring $(R, \mathfrak{m}, k)$ of dimension d is said to be *Gorenstein* if it is Cohen-Macaulay and $\mathrm{Ext}^d_R(k, R) = k$. A (not necessarily) local ring is Gorenstein if all its localizations are Gorenstein.

If $(R, \mathfrak{m}, k)$ is regular then the resolution of k is the Koszul complex of any set of d generators of $\mathfrak{m}$, and we see directly that $\mathrm{Ext}^d_R(k, R) = k$, so R is Gorenstein.

Proposition 18.6.2. *If $(R, \mathfrak{m}, k)$ is Cohen-Macaulay, and $x_1, \dots x_s \in R$ is a regular sequence, then R is Gorenstein if and only if $R/(x_1, \dots x_s)$ is Gorenstein.*

Proof. Since every regular sequence in R is part of a maximal regular sequence, it suffices to prove the result when $s = d$, the dimension of R so that $\overline{R} = R/(x_1, \dots x_d)$ is Artinian. Since R is Cohen-Macaulay, the smallest i such that $\mathrm{Ext}^i_R(k, R) \neq 0$ is d, so by Lemma 18.6.3, we see that

$$\mathrm{Ext}^d_R(k, R) = \mathrm{Ext}^0_R(k, \overline{R}) = Hom_{\overline{R}}(k, \overline{R})$$

proving the Proposition. $\square$

Lemma 18.6.3. *If $(R, \mathfrak{m}, k)$ is a local ring, and $x \in \mathfrak{m}$ is a nonzerodivisor on N that annihilates M, and i is the smallest index such that $\mathrm{Ext}^i_R(M, N) \neq 0$, then*

$$\mathrm{Ext}^i_R(M, N) = \mathrm{Ext}^{i-1}_R(M, N/xN).$$

for all i.

Proof. The element x annihilates all the $Ext^j_R(M, N)$ because it annihilates M. The short exact sequence $0 \to N \xrightarrow{x} N \longrightarrow N/xN \to 0$ gives rise to a long exact sequence containing the terms and

$$0 = Ext^{i-1}_R(M, N) \longrightarrow Ext^{i-1}_R(M, N/xN) \longrightarrow Ext^i_R(M, N) \xrightarrow{0} \cdots.$$

$\square$

The algebraic version of the canonical module of a scheme is usually called the *dualizing module*:

Definition 18.6.4. Let $(R, \mathfrak{m}, k)$ be a local Cohen-Macaulay ring of dimension d. A dualizing module for R is a Cohen-Macaulay R module with $\dim M = d$ such that $\mathrm{Ext}^d_R(M, k) \cong k$.

Proposition 18.6.5. *Let R be a local Cohen-Macaulay ring. Any two canonical modules for R are isomorphic. Moreover, if $R = S/I$, with S regular, then $\mathrm{Ext}_S^{\operatorname{codim} R}(R, S)$ is a canonical module for R.*

Proof. ****** □

We shall see that $(R, \mathfrak{m}, k)$ is Gorenstein if and only if it is Cohen-Macaulay and R itself is a dualizing module. We begin with the 0-dimensional case, where we can identify the dualizing module with the injective hull of the residue field. Recall that E can be characterized as a module E containing a copy of k such that k is *essential* in E; that is, every nonzero submodule of E meets k; and E maximal with this property in the sense that if $E \subsetneq E'$, then k is not essential in E. Such a module always exists, by Zorn's lemma, and it is not difficult to show that it it unique up to isomorphism. Except for rings of dimension 0, it is never finitely generated.

Theorem 18.6.6. *If $(R, \mathfrak{m}, k)$ is a local ring of dimension 0, and E is the injective hull of k, then $(-)^\vee := \mathrm{Hom}_R(-, E)$ is a perfect duality on modules of finite length. That is, $(-)^\vee$ is a contravariant equivalence of categories. Moreover, for any module M of finite length we have*

1. length $M^\vee$ = length M*; in particular, E is a module of finite length* = length R*.*
2. *The natural map $\nu_M : M \to M^{\vee\vee}$ is an isomorphism.*
3. *The ring R is Gorenstein if and only if R has a unique minimal nonzero ideal.*
4. *The ring R is Gorenstein if and only if $R \cong E$.*

Proof. First, since E is injective the functor $Hom_R(-, E)$ is exact. Since k is essential in E, the largest submodule of E annihilated by $\mathfrak{m}$ must be k itself.

We prove both (1) and (2) by induction on the length of M. If length$M = 1$, then $M = k$. The previous remark shows that $k^\vee = k$, so $k^{\vee\vee} = k$, proving (1). Choosing a generator ϕ of $k^\vee$ and a generator α of k, we see that ν_k takes α to the map sending ϕ to $\phi(\alpha) \neq 0$. Since k is a simple module, $\nu_k : k \to k$ is a monomorphism, and thus an isomorphism, as requireed.

Now suppose by induction that (1) and (2) are true for all modules M' of length at most j, and that M is a module of length $j + 1$.

Any minimal nonzero submodule of M is isomorphic to k, so we may choose an exact sequence $0 \to k \to M \to M' \to 0$. Applying $(-)^\vee$ we get an exact sequence $0 \to M'^\vee \to M^\vee \to k^\vee \to 0$, proving (1) for M, and a diagram

$$\begin{tikzcd} 0 \arrow[r] & k \arrow[r] \arrow[d, "\nu_k"'] & M \arrow[r] \arrow[d, "\nu_M"'] & M' \arrow[r] \arrow[d, "\nu_{M'}"'] & 0 \\ 0 \arrow[r] & k^{\vee\vee} \arrow[r] & M^{\vee\vee} \arrow[r] & M'^{\vee\vee} \arrow[r] & 0 \end{tikzcd}$$

with exact rows, proving (2) for M, and completing the induction.

To prove (3) we note that if R has a unique minimal ideal $I \cong k$ if and only if $\mathrm{Hom}_R(k, R) = k$. Since any 0-dimensional ring is Cohen-Macaulay, this is equivalent to the Gorenstein property.

Finally we prove (4): Since E has unique miminal ideal in any case, we see that $R \cong E$ implies that R is Gorenstein. Conversely, suppose that $I \cong k$ is the unique minimal nonzero ideal of R. The unique map $I \hookrightarrow E$ extends to a map $\phi : R \to E$. If $\ker \phi$ were nonzero it would contain I, so ϕ is a monomorphism. Moreover, $\mathrm{length} R = \mathrm{length} R^\vee = E$, so ϕ is surjective as well. □

Cheerful Fact 18.6.1. Conditions (2) and (4) of Theorem 18.6.6 have extensions to the higher dimensional case, though we will not need to use them:

Theorem 18.6.7. *Suppose that $(R, \mathfrak{m}, k)$ is a d-dimensional local Noetherian ring R. the following conditions are equivalent:*

1. *R is Gorenstein.*
2. *R has finite injective dimension (equivalently, injective dimension d) as an R-module.*
3. *R is a Cohen-Macaulay ring and the functor $\mathrm{Hom}_R(-, R)$ is a perfect duality on the category of maximal Cohen-Macaulay modules.*

□

Note that the equivalence of 1) and s) in Theorem 18.6.7 implies that the localization of a Gorenstein ring is Gorenstein, something not obvious from the definition.

((The following proof is incomplete))

Proof. Suppose first that $d = 0$ and R is Gorenstein, so that $\mathrm{Hom}_R(k, R) = k$ —that is, R has a unique minimal submodule N, necessarily $\cong k$.

Let E be the R-injective hull of k. The inclusion $N \subset R$ induces an inclusion $R \subset E$. Since $\mathrm{Hom}_R(-, E)$ is an exact functor, and $\mathrm{Hom}_R(k, E) \cong k$, it follows by induction on the length of a finitely generated module M that the length of M is equal to the length of $\mathrm{Hom}_R(M, E)$. Thus $E = \mathrm{Hom}_R(R, E)$ has the same length as R, so R and E coincide. Thus $\mathrm{Ext}^i_R(M, R)$ vanishes for all $i > 0$ and all R-modules M. This shows that 1) implies 2) and 3) in this case.s

Next suppose that $d = 0$ and $\mathrm{Ext}^1_R(k, R) = 0$. So 2) implies 3) in this case.

To show that 3) implies 1), let N be the largest submodule of R that is annihilated by $\mathfrak{m}$, so that $N \cong k^s$ for some s. We must show that $s = 1$. The vanishing of $\mathrm{Ext}^1_R(R/N, R)$ shows that the map we get a short exact sequence

$$\mathrm{Hom}_R(R, R) \to \mathrm{Hom}_R(N, R) \cong N = k^s$$

is surjective. But this map factors through $R \to R/\mathfrak{m} = k$, so $s = 1$, so R is Gorenstein.

Now we do induction on d, and we may suppose $d > 0$. Suppose first that there is a nonzerodivisor $x \in \mathfrak{m}$.

((the following para is almost right; the resolution over R/x is the mapping cone, ... – the conditions 2,3 refer to Exts over different rings))

From the exact sequences $(*)$ of Lemma 18.6.3 we see that each of the three conditions of the Theorem for $R/(x)$ is equivalent to the corresponding condition for R. By induction, the three conditions are equivalent for $R/(x)$, so they are equivalent for R.

If R is Gorenstein then it is Cohen-Macaulay by definition, and since $d > 0$, $\mathfrak{m}$ automatically contains a nonzerodivisor. Thus, to conclude the proof, it suffices to show that $\mathrm{Ext}^{d+1}_R(k, R) = 0$ implies that R contains a nonzerodivisor.

In the contrary case, the prime ideal $\mathfrak{m}$ is an associated prime of 0; this means that there is a submodule N of R isomorphic to k. $\square$

((Need to add: somewhere: S regular, $R = S/I$ is Gorenstein iff

$$\omega_R := \mathrm{Ext}_S^{\mathrm{codim}\, R}(R, S) \cong R$$

up to shift.))

18.7 What Makes a Complex Exact

Given that the resolution of any module over a polynomial ring $S = k[x_1, \ldots, x_n]$ (or regular local ring) has length bounded by n, the resolution of an S-module that is a k-th syzygy—that is, the image of the k-th map in a resolution—must have length $\leq n - k$. What is special about the presentation matrix of such a module? – put differently, what is it about the k-th map in a resolution that is different than the first map?

One answer to this question was given by David Hilbert in his orginal paper [] and another can be given in terms of Gröbner bases and initial ideals []. Here we give a third answer, one that lends itself better to conceptual proofs, such as the proof of exactness of the Eagon-Northcott complex resolving the ideal of minors of a matrix given in Section **??**. We give the result in a special case; in its general form in [] it applies to finite free complexes over any ring.

Theorem 18.7.1. *Let*

$$\mathbf{F}: \quad 0 \longrightarrow F_m \xrightarrow{\phi_m} \cdots \xrightarrow{\phi_2} F_1 \xrightarrow{\phi_1} F_0$$

be a finite complex of finitely generated free modules over a polynomial ring or, more generally, a Cohen-Macaulay ring. The following conditions are equivalent:

1. *The complex is exact (and thus a resolution of* $\operatorname{coker}\phi_1$*);*
2. *The ranks* r_i *of the maps* ϕ_i *satisfy* $r_i + r_{i+1} = \operatorname{rank} F_i$ *and the* $r_i \times r_i$ *minors of a matrix representation of* ϕ_i *generate an ideal of codimension* $\geq i$.
3. *For every prime ideal* P *of codimension* c*, the localized truncated complex*

$$(\mathbf{F}_{\geq c})_P: \quad 0 \longrightarrow (F_m)_P \xrightarrow{(\phi_m)_P} \cdots \xrightarrow{(\phi_{c+1})_P} (F_c)_P$$

is split exact (and thus is a free resolution of a projective module.)

The following easy Proposition plays a role in the proof:

Proposition 18.7.2. *Let* $\phi : F \to G$ *be a map of finitely generated free modules over a local ring* R*, and let* $I_r(\phi)$ *denote the ideal generated by the* $r \times r$ *minors of a matrix representing* ϕ*. The cokernel of* ϕ *is free if and only if, for some integer* r *we have* $I_r = R$ *and* $I_{r+1} = 0$.

The proof that exactness implies items 2,3 is easy to summarize: if $\mathbf{F}$ is a free resolution of an S-module M and P is a prime of codimension c then $(\mathbf{F})_P$ is a free resolution of M_P over S_P, a regular ring of dimension c.Thus the kernel of $(\phi_{c-1})_P$ must be projective (actually, free) over S_P, and $(\mathbf{F}_{\geq c})_P$ is a free resolution of this module, and thus split exact. Taking P to be a minimal prime (that is, if S is a domain, the prime 0) gives the condition on ranks in item 1), while the condition on codimension follows because the cokernel of each $(\phi_i)_P$ is projective for $i \geq c$.

For a detailed treatment and the opposite implications, we refer to [**?**].

Chapter 19

Appendix: Syzygies and Geometry

19.1 Betti tables

In this Chapter we will return to the case of standard graded rings R; that is, R is an algebra over the field $k := R_0$, and is generated by the finite dimensional vector space R_1. In particular, R is a homomorphic image of $S = k[R_1]$, and thus R is Noetherian. Throughout this section, S will denote a standard graded polynomial ring.

A module over any ring can be specified by its generators and relations. But Hilbert saw that there was something to be gained by studying the relation on the relations, and the relations on those, and so on—in short, the syzygies of the module. In the case of a graded module M over a graded ring R, we may take a minimal free resolution

$$0 \longleftarrow M \longleftarrow F_0 \xleftarrow{d_1} F_1 \xleftarrow{d_2} \cdots$$

that is graded, with differential d of degree 0. The i-th syzygy module of M is by definition the cokernel of d_{i+1}. Recall that $R(-a)$ denotes a rank one free graded module generated in degree a (so that $R(-a)_0 = R_a$ and more generally $R(a)_b = R_{a+b}$). We may identify each F_i with a direct sum of the form $\bigoplus_j R(-a_j)^{\beta_{i,j}}$ in such a way that the homomorphisms d_i are homogeneous of degree 0. Theorem **??** holds in this case as well, so that the numbers $\beta_{i,j}$ depend only on the module M, as is also obvious from the formula $\operatorname{Tor}_i(M, k) = \bigoplus_j k(-a_j)^{\beta_{i,j}}$. We sometimes refer to $\beta_{i,j}$ informally as the *number of i-th syzygies of degree j*; more properly, it is the number of minimal generators of degree j required by the module of i-th syzygies of M.

For convenience, the graded Betti numbers are usually displayed in a compact *Betti table*, with the nonzero $\beta_{i,j}$ appearing in the i-th column and the

$(j-i)$-th row: [1]

$j \backslash i$	0	1	2	
0	$\beta_{0,0}$	$\beta_{1,1}$	$\beta_{2,2}$	$\cdots$
1	$\beta_{0,1}$	$\beta_{1,2}$	$\beta_{2,3}$	$\cdots$
2	$\beta_{0,2}$	$\beta_{1,3}$	$\beta_{2,4}$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\ldots$	

For the sake of simplicity we usually replace some 0 entries with $-$. We sometimes speak of the *Betti table of a module M*, by which we mean the Betti table of the minimal free resolution of M. Similarly, when we speak of the Betti table of a variety $X \subset \mathbb{P}^n$, we mean the Betti table of the minimal free resolution of the homogeneous coordinate ring S_X of X, as a module over the homogeneous coordinate ring of $\mathbb{P}^n$.

The rows of the betti table correspond to *linear strands* of the resolution—sequences of the free summands where the maps between them are represented by matrices of linear forms. The length of the linear strands plays a central role in Green's conjecture.

19.2 Which tables are Betti tables of minimal resolutions?

Here are some ideas that can help with the computation of Betti tables. The fact that minimal free resolutions are constructed by successively choosing minimal sets of generators for the kernel of a previously computed map leads to this:

Proposition 19.2.1. *Supoose that $\beta_{i,j}$ are the graded Betti numbers of an S-module. If $\beta_{i_0,i_0+j} = 0$ for all $j \le j_0$ then $\beta_{i,i+j} = 0$ for all $i \ge i_0,\ j \le j_0$ as illustrated in Figure 19.1.*

Note that $\beta_{i,j}$ and $\beta_{i+1,j+1}$ are consecutive elements of the same row of the Betti table, so the Proposition says that a zero entry in the i-position of all the rows of the table at and above a certain spot implies that all successive entries of those rows are zero as well.

Proof. The table represents the summands of a minimal free resolution, and minimality means that the maps in the resolution are represented by matrices whose entries have strictly positive degree. Thus if F_i has no summand isomorphic to $S(-j)$ for $j \le j_0$ it follows that there can be no summand of F_{i+1} isomorphis to $S(-j-1)$ for the same range of j. $\square$

[1] The reason for the initially non-intuitive choice $j-i$ instead of j is that, for the resolution $\mathbb{F}$ to be minimal, it is necessary that if $\beta_{i,j} \neq 0$, then some $\beta_{i-1,k} \neq 0$ for some $k < j$. Thus shift by $-j$ makes the diagram more compact.) This useful convention goes back to the development of the program Macaulay by Dave Bayer and Mike Stillman in the 1980s. In those days of small displays, efficiency was even more important than now.

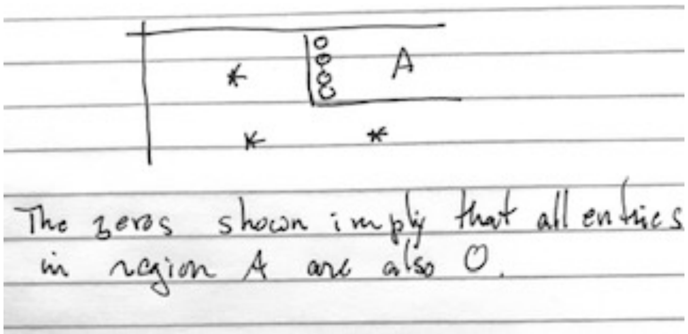

Figure 19.1: The zeros shown imply that all entries in region A are also 0.

Hilbert's original application of the Syzygy Theorem was to compute the Hilbert function of a module as a (finite!) alternating sum of the (easy to compute) Hilbert functions of the free modules in the resolution. If the graded Betti numbers of M are $\{\beta_{i,j}\}$, then clearly

$$\begin{aligned} H_M(t) &= \sum_i (-1)^i \sum_j \beta_{i,j} H_{S(-j-i)} \\ &= \sum_i (-1)^i \sum_j \beta_{i,j} \binom{n-j+t}{n} \end{aligned}$$

Reversing the order of summation, we note that $H_M(t)$ can be computed from the Betti table as the alternating sum, with appropriate binomial coefficients, of the t-th anti-diagonal of the Betti table.

Suppose now that the Hilbert function $H_M(t)$ of a finitely generated graded module M is known, and that we are interested in understanding the Betti table of the module's minimal resolution. To simplify the notation, set $m_t := H_M(t)$. Since the Hilbert function of the residue field k has values $(1, 0, 0, \dots)$, and the S-free resolution of k is the Koszul complex $\mathbb{K}$, with Betti table

$j\backslash i$	0	1	2	
0	1	$n+1$	$\binom{n+1}{2}$	$\dots,$
1	–	–	–	$\dots$

we can produce a (generally not finitely generated) module

$$M' := \bigoplus_{-N}^{\infty} k(-t)^{m_t}$$

with the same Hilbert function, and resolution a corresponding direct sum of shifted Koszul complexes. A priori this tells us nothing about the Betti table of M. However, there is a flat deformation $\mathcal{M}$ over $k[z]$ of the module M to the trivial module $\oplus_t k(-t)^{m_t}$ obtained by pulling back M along the map

$$k[x_0 \dots, x_n] \to k[x_0, \dots, x_n]: \quad x_i \mapsto zx_i$$

and taking the flat limit at $z = 0$. The Betti table of the fiber of $\mathcal{M}$ over $z = 0$ is thus

$$B_0 = \begin{array}{r|cccc} j\backslash i & 0 & 1 & 2 & \\ \hline \text{t-1} & \cdots & \cdots & \cdots & \cdots \\ \text{t} & m_t & m_t(n+1) & m_t\binom{n+1}{2} & \dots \\ \text{t+1} & m_{t+1} & m_{t+1}(n+1) & m_{t+1}\binom{n+1}{2} & \dots \\ \text{t+2} & \cdots & \cdots & \cdots & \cdots \end{array}$$

Such a flat deformation corresponds to a deformation of resolutions, as well, and one may deduce:

Proposition 19.2.2. *[?] With notation as above, the Betti table B of M is obtained from the array B_0 by successively cancelling pairs of terms along the anti-diagonals; that is B is derived from B_0 by repeated moves of the form*

$$\begin{array}{r|cccc} j\backslash i & j-1 & j & j+1 & \\ \hline & \cdots & \cdots & \cdots & \cdots \\ t & \dots & \dots & b & \dots \\ t+1 & \dots & c & \dots & \dots \\ & \cdots & \cdots & \cdots & \cdots \end{array} \quad \mapsto \quad \begin{array}{r|cccc} j\backslash i & j-1 & j & j+1 & \\ \hline & \cdots & \cdots & \cdots & \cdots \\ t & \dots & \dots & b-1 & \dots \\ t+1 & \dots & c-1 & \dots & \dots \\ & \cdots & \cdots & \cdots & \cdots \end{array}$$

19.3 Examples of Betti Tables

Some examples will help absorb these ideas.

Example 19.3.1. As a first example, let X be a set of three non-collinear points in $\mathbb{P}^2$, say $X = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$. Since no linear form vanishes on X, the Hilbert function of S_X begins $1, 3$. Since S_X is reduced, the multiplication by a general linear form on S_X is a monomorphism, so the Hilbert function is non-decreasing and since $\deg X = 3$ it must continue $1, 3, 3, 3, \dots$. The linear form $\ell := x_0+x_1+x_2$ does not vanish on any of the points, so it is a nonzerodivisor on S_X. By Proposition 19.3.4, the Betti table of X is the same as that of $S_X/(\ell)$, regarded as a module over $\overline{S} := k[x_0, x_1]$. The Hilbert function of $S_X/(\ell)$ is the first difference

$$1 - 0,\ 3 - 1,\ 3 - 3, \dots = 1, 2, 0, \dots$$

of the Hilbert function of S_X. Thus the Betti table of S_X is the same as that of the square of the maximal ideal in $\overline{S}$. By Proposition 19.2.2 if must be obtained from the table

$j\backslash i$	0	1	2	3
0	1	2	1	
1	2	4	2	–

by successive cancellations along anti-diagonals. Since I_X does not contain a linear form, we see from Proposition 19.2.1 that the first row must become $1, 0, 0$ after cancellation, and thus the second row becomes $0, 3, 2$; that is, the Betti table of X is

$j\backslash i$	0	1	2
0	1	–	–
1	–	3	2
2	–	–	–

The fact that the 3 and the 2 are on the same line is reflects the fact that the 2 syzygies of the ideal $(x, y)^2$ are linear; this is a linear strand of the resolution.

To check all this, we observe that I_X indeed contains the three quadrics x_0x_1, x_0x_2, x_1x_2. Since $\ell_i(\ell_j\ell_k) = \ell_j(\ell_k\ell_i) = \ell_k(\ell_i\ell_j)$ we also see the two linear syzygies among these forms.

Since we already know the Betti table, we see that the resolution of S_X must be

$$0 \longleftarrow S/I \longleftarrow S \xleftarrow{\begin{pmatrix}\ell_2\ell_3 & \ell_1\ell_3 & \ell_1\ell_2\end{pmatrix}} S(-1)^3 \xleftarrow{\begin{pmatrix}\ell_1 & 0\\ -\ell_2 & \ell_2\\ 0 & -\ell_3\end{pmatrix}} S(-2)^2 \longleftarrow 0$$

Exercise 19.3.2. Prove that the given complex is a resolution using Theorem 18.7.1.

Note that the ideal I can be written as the ideal of 2×2 minors of the matrix

$$\begin{pmatrix}\ell_1 & 0\\ -\ell_2 & \ell_2\\ 0 & -\ell_3\end{pmatrix}.$$

The resolution above is thus a special case of the Eagon-Northcott complex, described in Chapter **??**

Example 19.3.3. Next, consider the twisted cubic curve $C \subset \mathbb{P}^3$, the image of $\mathbb{P}^1$ under the linear series $(s^3, s^2t, st^2, t^3) \subset H^0(\mathcal{O}_{\mathbb{P}^1}(3))$. Write $S = k[x_0, \dots, x_3]$ for the homogeneous coordinate ring of $\mathbb{P}^3$ and $S_C = S/I_C$ for the homogeneous coordinate ring of C.

As we saw in Chapter ****, the ideal of forms vanishing on the twisted cubic is generated by three quadrics q_1, q_2, q_3 that are the 2×2 minors of the

$$\phi := \begin{pmatrix}x_0 & x_1\\ x_1 & x_2\\ x_2 & x_3\end{pmatrix}.$$

As discussed in Chapter **??** the S-free resolution of S_C is an Eagon-Northcott complex, in this this case

$$0 \longrightarrow S^2(-3) \xrightarrow{\phi} S^3(-2) \xrightarrow{\begin{pmatrix}q_1 & q_2 & q_3\end{pmatrix}} S.$$

Thus the Betti table of the twisted cubic is the same as that of the 3 non-collinear points in $\mathbb{P}^2$.

To understand this from another point of view, note that since I_C is a prime ideal, x_3 (or any variable) is a nonzerodivisor on S_C. Thus we may apply the following result:

Proposition 19.3.4. *Let R be a local or standard graded polynomial ring, and let M be an R-module with minimal R-free resolution $\mathbb{F}$. If $f \in R$ is a nonzerodivisor on both R and M, then the minimal $\overline{R} := R/(f)$-free resolution of $M/fM = \overline{R} \otimes_R M$ is $\overline{R} \otimes_R \mathbb{F}$, and the Hilbert function of M/fM is the first difference of that of M:*

$$H_{M/fM}(t) = H_M(t) - H_M(t-1).$$

Proof. The homology of $\overline{R} \otimes_R \mathbb{F}$ is $\mathrm{Tor}^R(\overline{R}, M)$, which can also be computed as the homology of the tensor product of M with the R-free resolution of $R/(f)$. Since f is a nonzerodivisor on R, this is

$$0 \to R(-1) \xrightarrow{f} R \longrightarrow R/(f) \to 0.$$

It follows that the homology of $\overline{R} \otimes_R \mathbb{F}$ is the same as the homology of

$$(0 \to R(-1) \xrightarrow{f} R) \otimes_R M = 0 \to M(-1) \xrightarrow{f} M.$$

Since f is a nonzerodivisor on M, we see that $H_i(\overline{R} \otimes_R \mathbb{F}) = 0$ for $i > 0$, showing that $\overline{R} \otimes_R \mathbb{F}$ is a free resolution of $M/(f)M$. The minimality is obvious. The formula for the Hilbert functions follows from the same exact sequences. $\square$

Proposition 19.3.4 shows that the minimal free resolution of C has the same Betti table as the minimal free resolution of $S_C/(\ell) = S/(I_c + (\ell))$ for any linear form ℓ and this corresponds to the scheme that is the intersection of C with the hyperplane $\ell = 0$. For sufficiently general ℓ (or even for $\ell = x_0 - x_3$)) this consists of three non-colinear points in the plane.

However, this is not quite all that needs to be said: certainly $I_C + (x_3)$ is an ideal defining the hyperplane section $x_3 = 0$ of C, but to conclude from Proposition 19.3.4 that the Betti table of C is the same as that of the 3 noncolinear points, we need to know that $I_C + (x_3)$ is saturated.

By the Auslander-Buchsbaum formula, the depth of S_C is 2—that is, S_C is a Cohen-Macaulay ring. It follows that the depth of $S_C/(x_3)$ is 1, and this shows that $I_c + (x_3)$ is a saturated ideal completing the circle of ideas.

Exercise 19.3.5. Let C be the nonsingular rational quartic $\mathbb{P}^3$, the image of $\mathbb{P}^1$ under the linear series $(s^4, s^3t, st^3, t^4) \subset H^0(\mathcal{O}_{\mathbb{P}^1}(3))$. Show that the general hyperplane section of C is the intersection of two quadrics, while the ideal of C has just one quadric generator. Conclude that that the Betti table of C is not the same as the Betti table of the hyperplane section, and thus that S_C is not Cohen-Macaulay. (in fact, the Betti table of S_C is

$j\backslash i$	0	1	2	3
0	1	–	–	–
1	–	1	–	–,
2	–	3	4	1

from which we see that the projective dimension of S_C is 3, so by the Auslander-Buchsbaum theorem the depth is only 1.)

Example 19.3.6. Consider a set X of 7 points in linearly general position $\mathbb{P}^3$. Castelnuovo's Theorem **??** shows that any 6 points in linearly general position lie on a unique twisted cubic, so the first distinction one might make among sets of 7 points is whether the 7th point lies on the twisted cubic too.

As we saw in **??**, any set of $2n+1$ points in linearly general position in $\mathbb{P}^n$ imposes $2n+1$ conditions on quadrics, so in any case I_X requires exactly 3 quadratic minimal generators, q_1, q_2, q_3 and the Hilbert function of S_X begins $1, 4, 7, ?$. Since S_X is reduced, the Hilbert function is non-decreasing, and since the eventual value must be $\deg X = 7$, the function must be $1, 4, 7, 7, \dots$, and the ideal contains $20 - 7 = 13$ cubics. From this information we can work out the whole Betti table, as follows.

Since any set of points has Cohen-Macaulay homogeneous coordinate ring we can apply Proposition **??** to say that the Betti table of $R := S_X/(x_3)$ is the same as that of S_X. Write $R = \overline{S}/I$, where $\overline{S} = k[x_0, x_1, x_2]$. The Hilbert function of R is determined by the exact sequence

$$0 \longrightarrow S_X(-1) \xrightarrow{x_3} S_X \longrightarrow R \longrightarrow 0$$

as the first difference function of that of S_X, namely

$$1-0, 4-1, 7-4, 7-7, \dots = 1, 3, 3, 0, \dots;$$

that is, I is generated (not minimally) by the 3 quadrics and a total of 10 cubics. By Propositions 19.3.4 and 19.2.2, the Betti table of X must be obtained by successive cancellations from the table

$j\backslash i$	0	1	2	3
0	1	3	3	1
1	3	9	9	3
2	3	9	9	3
3	–	–	–	–

Moreover, since X is not contained in a hyperplane, the zero-th row, after cancellation, must become $0 \mid\ 1\ \ 0\ \ 0\ \ 0$, so we can begin by making this cancellation. We also know that I_X has exactly 3 quadratic generators, necessitating another cancellation, and we see that the actual Betti table of X must be obtained by cancelling from

(1)

$j\backslash i$	0	1	2	3
0	1	0	0	0
1	0	3	8	3
2	0	9	9	3
3	–	–	–	–

Now three general quadrics define a complete intersection of 8 points, and it follows that in the case of 7 general points q_1, q_2, q_3 form a complete intersection with resolution the Koszul complex, having Betti table:

$j\backslash i$	0	1	2	3
0	1	–	–	–
1	–	3	–	–.
2	–	–	3	–
3	–	–	–	1

In this case there are no linear relations among q_1, q_2, q_3, so the 8 and 3 in row 1 of table (1) must cancel completely, giving

$j\backslash i$	0	1	2	3
0	1	0	0	0
1	0	3	0	0
2	0	1	6	3
3	–	–	–	–

Since no further cancellation is possible, this must be the Betti table of a set X of 7 general points.

On the other hand, if the points lie on a twisted cubic, then q_1, q_2, q_3 generate the ideal of the twisted cubic, and as we see from the Betti table in Example **??**, these have 2 linear relations, which are themselves independent. It follows that row 1 of the Betti table of X must cancel to become $1 \mid \ 0 \quad 3 \quad 2 \quad 0$. Thus the Betti table of X in this case is

$j\backslash i$	0	1	2	3
0	1	0	0	0
1	0	3	2	0
2	0	7	9	3
3	–	–	–	–

Thus the Betti tables detect whether the points lie on a twisted cubic curve.

19.4 Interpreting homological ring theory through Betti tables

Suppose that $R = S/I$ is a factor ring of codimension c in S and that

$$\mathbb{F} : 0 \longrightarrow F_m \longrightarrow \cdots \longrightarrow F_0$$

is its minimal free resolution. Theorem 18.5.2 Shows that the dual of $\mathbb{F}$, which computes $\mathrm{Ext}_S(R, S)$, is exact for the c steps beginning with F_0^*, but has homology at F_c^*:

$$\mathbb{F} : 0 \longrightarrow F_0^* \longrightarrow \cdots \longrightarrow F_c^* \longrightarrow \cdots .$$

On the other hand, the Auslander-Buchsbaum formula shows that the projective dimension of R as an S-module is $\dim S - \operatorname{depth} R$. Taking into account that $\omega_S = S(-n-1)$, this proves:

Proposition 19.4.1. *Let $R = S/I$ be a graded factor ring of the polynomial ring S. If I has codimension c, then R is Cohen-Macaulay if and only if the projective dimension of R is c, the smallest possible value. In this case the dual of the minimal S-free resolution of R is the minimal S-free resolution of $\omega_R := Ext^c(R, S(-n-1))$.*

The Betti table of $\mathbb{F}^*$ is obtained from that of $\mathbb{F}$ by reversing it left-to-right and top-to-bottom, adjusting the row indices appropriately; for example if $\mathbb{F}$ is the resolution of the twisted cubic, (Example **??**), then the Betti table of $\mathbb{F}^*$, the resolution of the module associated to the sheaf $\omega_C(4)$, is

$j\backslash i$	0	1	2	3
-3	2	3	0	0
-2	0	0	1	0
-1	–	–	–	–

The restriction on the tables that can occur as Betti tables of resolutions given in Proposition **??** is not symmetric under this operation (the other restrictions in Section 19.2 are). Dualizing it, we get a new restriction on the Betti table of a Cohen-Macaulay factor ring:

Corollary 19.4.2. *Suppose that $R = S/I$ is a Cohen-Macaulay graded ring with minimal free resolution $\mathbb{F}$ having graded Betti numbers $\beta_{i,j}$. If, for some index i_0 we have $\beta_{i,j} = 0$ for all $j \geq i_0 + j_0$, then $\beta_{i,j} = 0$ for all i, j with $i \leq i_0$ and $j \geq i + j_0$.*

We can also characterize the condition for S/I to be Gorenstein:

Proposition 19.4.3. *Suppose that $R = S/I$ is a graded Cohen-Macaulay ring of codimension c, with minimal S-free resolution $\mathbb{F}$. The following conditions are equivalent:*

1. *R is Gorenstein.*

2. *$F_c \cong S(-d)$ for some integer d.*

3. *The Betti table of R is symmetric; it is equal up to shift to the Betti table of $\mathbb{F}^*$.*

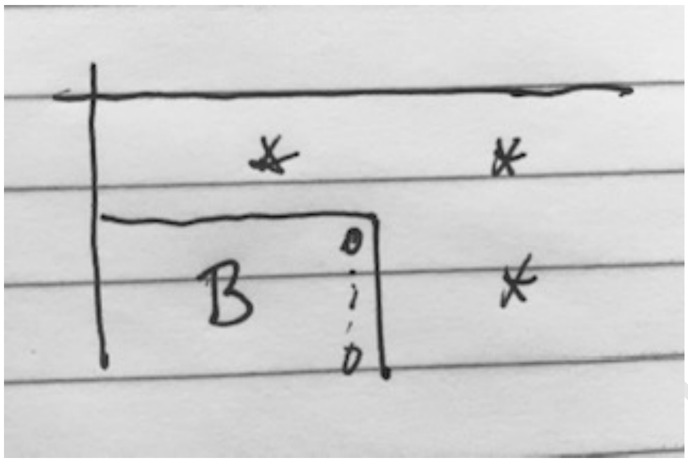

Figure 19.2: In the resolution of a Cohen-Macaulay module, the zeros shown imply that the entries in region B are zero.

Even the shifts in this duality can be eliminated if we take not $\mathbb{F}^*$ but $\mathrm{Hom}(\mathbb{F}, \omega_S)$ with $\omega_S = S(-n-1)[-n]$, that is, a rank one free module generated in degree $n+1$ whose homological degree is n.

Proof. □

19.5 Regularity of modules and sheaves

The number of the last nonzero column of the Betti table of the resolution of a graded S-module M is the projective dimension of M. The number of the last row is also an important invariant called the (Castelnuovo-Mumford) *regularity* of M. More formally,

Definition 19.5.1. Let $t_i(M) = \max\{j \mid \mathrm{Tor}_i^S(k, M)_j \neq 0\}$. The regularity of M is

$$\max_{i \geq 0} t_i(M) - i.$$

As defined above, the regularity of M is obviously an upper bound for the degrees of a minimal generating set of M, which is simply $t_0(M)$. One reason the regularity is important is that there is a different expression for the regularity in terms that do not seem to involve the generators directly:

Theorem 19.5.2. *Let $s_i(M) = \max\{j \mid H^i_{\mathfrak{m}}(M)_j \neq 0\}$. The regularity of M is equal to*

$$\max_{i \geq 0} t_i(M) + i.$$

Note the change of sign; see **** for a proof using local duality. The notion of regularity was introduced by Mumford in the study of sheaves, where it has a slightly simpler expression:

Definition 19.5.3. A $\mathcal{F}$ be a sheaf on $\mathbb{P}^n$ is *m-regular* if $H^i(\mathcal{F}(m-i) = 0$ for all $i > 0$. The *regularity* of $\mathcal{F}$ is the minimal m for which $\mathcal{F}$ is m-regular.

Note that the regularity of the sheaf $\tilde{M}$ associated to a graded S-module M ignores $H^0_{\mathfrak{m}}(M)$ and $H^1_{\mathfrak{m}}(M)$; but otherwise the definitions are related by the translation from local to global cohomology and, indeed, if depth $M \geq 2$ then the regularity of M is equal to the the regularity of the associated sheaf $\tilde{M}$.

Most applications of regularity involve the following result[**?**]:

Theorem 19.5.4. *Suppose that $\mathcal{F}$ is a sheaf on $\mathbb{P}^n$. If $\mathcal{F}$ is m-regular, then $\mathcal{F}$ is m'-regular for all $m' \geq m$, and $\mathcal{F}(m)$ is generated by its global sections.*

The regularity of a projective variety $X \subset \mathbb{P}^n$ is defined to be the regularity of the ideal sheaf of X. A striking result of Gruson-Lazarsfeld-Peskine [**?**] relates this to more elementary notions:

Theorem 19.5.5. *Let C be a reduced, irreducible nondegenerate curve of degree d in $\mathbb{P}^n$. The regularity of C is $\leq d-n+2$, with equality if and only if C is smooth and rational and either $d = n$ or $d = n+1$ or C has a $d-n+3$-secant line.*

To see the relevance of the last condition, note that if C has a $d-n+3$-secant line any function of degree $d-n+2$ that vanishes on C vanishes on the secant line as well; so the ideal of I_C of C must require generators of degree at least $d-n+1$. Since the regularity is an upper bound for the degrees of the minimal generators of the ideal of C, it follows from the Theorem that I_C is generated by forms of degree $\leq d-n+1$, and we see that the Theorem is sharp in this case. A curve of degree d with a $d-n+2$-secant is rational, since projection from the line is an isomorphism with $\mathbb{P}^1$.

For a treatment of these ideas, see for example [Eisenbud 2005].

((Revised to here 12-5-2020))

19.6 Maximal cancellation and the minimal resolution conjecture

Voisin's Theorem **??** (Green's conjecture) for canonical embeddings of general curves may be interpreted as saying that all possible cancellations actually occur in that case. Larson's Theorem **??**

((where should Larson's theorem go?))
is statement of the same kind for general linear series of given degree and dimension on general curves of given genus.

It would thus seem reasonable to conjecture that such cancellations happen more generally. However, this fails in some cases. Perhaps the simplest example is that of the general projection into $\mathbb{P}^6$ of an elliptic normal curve of degree 9, whose ideal has Betti table

Perhaps the simplest example is that of a general elliptic curve of degree 9 in $\mathbb{P}^6$ has Betti table

```
2:  12 24 12  . . .
3:   . 2  16 20 8 1
```

Such examples were discovered, even for linearly normal curves, by Green and Lazarsfeld [GREEN and LAZARSFELD 1988] and Schreyer (unpublished).

In 1993 Lorenzini formulated the conjecture that maximal cancellation would hold for the Betti tables of general sets of points. One can of course derive examples of sets of points that do not have this property by taking hyperplane sections of the curves above, but these are not general sets of points. However, even for general points, the conjecture fails. The first example, discovered by Schreyer, is that of 11 general points in $\mathbb{P}^6$, which has Betti table

```
2:  17 46 45  5  . .
3:   .  . 1  25 18 4
```

See @article MR1894365, AUTHOR = Eisenbud, David and Popescu, Sorin and Schreyer, Frank-Olaf and Walter, Charles, TITLE = Exterior algebra methods for the minimal resolution conjecture, JOURNAL = Duke Math. J., FJOURNAL = Duke Mathematical Journal, VOLUME = 112, YEAR = 2002, NUMBER = 2, PAGES = 379–395, ISSN = 0012-7094, MRCLASS = 13D02 (14M05 15A75), MRNUMBER = 1894365, MRREVIEWER = Martin Kreuzer, DOI = 10.1215/S0012-9074-02-11226-5, URL = https://doi-org.libproxy.berkeley.edu/10.1215/S0012-9074-02-11226-5, for what seems still to be the state of knowledge about counterexamples for general sets of points, as of this writing.

Any set of points $X \subset \mathbb{P}^n$ defines a Cohen-Macaulay ring, so the Betti table will not change if we pass to $A := S_X/\ell$ for a general linear form ℓ, now regarded as a module over a factor ring in n variables. Since X imposes independent conditions on forms of every degree, it is easy to show that, if the lowest degree of a generator of the ideal defining A is $d-1$, then the ideal contains the d-th power of the ideal of all the variables. But again, this is not a general ideal of that type. It seems that there is no known counterexample to the "minimal resolution conjecture" for ideals of the form

$$J_V := (x_1, \dots, x_n)^d + V \subset k[x_1, \dots, x_n]$$

for a general linear subspace V of the forms of degree $d+1$; more formally, this would say:

Conjecture 19.6.1 (0-dimensional version of the minimal resolution conjecture)**.** *In the minimal free resolution of an ideal J_V as above maximal cancellation holds; that is, there is no index i such that both $\beta_{i,d+i}$ and $\beta_{i+1,d+i}$ are both nonzero.*

19.7 Boij-Soederberg Theory

There are further, more subtle restrictions on the form of a Betti table. Since the Betti table of the direct sum of two modules is the sum of the Betti tables of the modules, the set of Betti tables of graded modules over a polynomial ring in $n+1$ variables forms a submonoid of $\mathbb{Z}^{n+2} \times \mathbb{Z}^{\infty}$. This monoid is not saturated; for example,

$$B := \begin{array}{c|cccc} j\backslash i & 0 & 1 & 2 & 3 \\ \hline 0 & 1 & 2 & 0 & 0 \\ 1 & - & - & 2 & 1 \end{array}$$

is not the Betti table of a module, because the top row indicates a map $S^2(-1) \to S$, and such a map must have a linear kernel $S^1(-2)$. However, the minimal resolution of the cokernel of a generic 2×4 matrix has Betti table

$$2B := \begin{array}{c|cccc} j\backslash i & 0 & 1 & 2 & 3 \\ \hline 0 & 2 & 4 & - & -. \\ 1 & - & - & 4 & 2 \end{array}$$

Finding generators for the monoid of Betti tables is an open problem except in a few small cases (see for example [**?**]. But the cone generated by the Betti tables is known: it is locally simplicial, and there is a simple formula for the extremal rays, as well as a greedy algorithm for writing any Betti table as a sum of rational multiples of the extremal rays. This is the subject of Boij-Soederberg theory. See [Eisenbud and Schreyer 2009] for more details.

19.8 Embeddings of high degree

((This section is a digression; could go nearly anywhere))

There is a certain uniformity of behavior of embedded by "complete linear series associated to line bundles that are sufficiently positive". Often "sufficiently positive" for a line bundle $\mathcal{L}$ is is taken to mean that $\mathcal{L}\otimes\omega_X^{-1}$ is a sufficiently high multiple of an ample line bundle. See [Lazarsfeld 2004] for more information. Here is a simple avatar of such results:

Cheerful Fact 19.8.1. Let $X \subset \mathbb{P}^n$ be a variety of dimension d, and let $X' \subset \mathbb{P}^N = \mathbb{P}(H^0(\mathcal{O}_X(r))$ be the image of X under the r-th Veronese mapping. For all $r \gg 0$ the ideal I_X is generated in degree 2, and the Betti table of I_X has nonzero entries at most in the $d+2$ rows $2, 3, \ldots, d+3$. Moreover, if X is a set of points in general position, then the Betti table of X has at most 2 nonzero rows.

Exercise 19.8.1. Prove the part of the the statement above proven by Mumford in [**?**]:

1. The ideal of the r-th Veronese image of $\mathbb{P}^n$ in $\mathbb{P}^{\binom{n+r}{n}-1}$ is generated by the 2×2 minors of the matrix associated to the multiplication map
$$H^0(\mathcal{O}_{\mathbb{P}^n}(1) \otimes H^0(\mathcal{O}_{\mathbb{P}^n}(r-1) \to H^0(\mathcal{O}_{\mathbb{P}^n}(r)$$
(see Theorem **??** for this construction).

2. if $r \gg 0$ then the homogeneous ideal of $X' \subset \mathbb{P}^N$ is generated by the ideal of the Veronese image of $\mathbb{P}^n$ together with linear forms.

Chapter 20

Appendix: Dualizing sheaves

((Put the linkage computation of the dualizing sheaf in here explicitly!))

Other than the structure sheaf, the most important line bundle on a smooth variety X over $\mathbb{C}$ is the top exterior power of the complex cotangent bundle, usually called the canonical line bundle or canonical sheaf ω_X.

In the case of curves we have many times used it's key property, that if $\mathcal{F}$ is a line bundle on the smooth curve C, then $H^0(\mathcal{F}^{-1} \otimes \omega_C)$ is the vector space dual of $H^1(\mathcal{F})$. Because $\mathcal{F}$ is locally free, we may rewrite the formula in the attractively symmetric form:

$$\mathrm{Hom}_C(\mathcal{F}, \omega_C) \cong \mathrm{Hom}_k(H^1(\mathcal{F}), k).$$

The reward for writing the formula this way is that, in this form, it holds for any coherent sheaf $\mathcal{F}$:

Proposition 20.0.1. *Let C be a smooth curve. There are natural isomorphisms*

$$\eta_{\mathcal{F}} : \mathrm{Hom}(\mathcal{F}, \omega_C) \Longrightarrow \mathrm{Hom}_k(H^1(\mathcal{F}), k).$$

for any coherent sheaf $\mathcal{F}$ on C.

Proof. Let $\mathcal{F}'$ be the torsion subsheaf of $\mathcal{F}$, a sheaf of finite support. And let $\mathcal{F}'' = \mathcal{F}/\mathcal{F}'$. Since $\mathcal{F}'$ is locally free, the sequence is locally split (in fact it is globally split too, but we don't need this.)

Since ω_C is a line bundle,

$$\mathcal{H}om_{\mathbb{P}^1}(\mathcal{F}, \omega_C) = \mathcal{H}om_{\mathbb{P}^1}(\mathcal{F}', \omega_C) \oplus \mathcal{H}om_{\mathbb{P}^1}(\mathcal{F}'', \omega_C) = \mathcal{H}om_{\mathbb{P}^1}(\mathcal{F}', \omega_C)$$

and $H^1(\mathcal{F}) = H^1(\mathcal{F}'')$ because $H^1(\mathcal{F}') = 0$, so the duality formula for arbitrary coherent sheaves follows from the case of line bundles. $\square$

An equivalent formulation can be made using a *residue isomorphism* $\eta : H^1\omega \to k$. When C is smooth over $\mathbb{C}$, then regarding elements of $H^1(\omega_C)$ as rational differential forms modulo linear equivalence, we may take η to be the classical "sum of the residues" map of complex analysis. Of course, given the natural isomorphisms $\eta_{\mathcal{F}}$ above, we can take

$$\eta := \eta_{\omega_C}(1_{\omega_C}.$$

Conversely,an isomorphism η, determines, for every $\mathcal{F}$ a map

$$\eta_{\mathcal{F}} : \mathrm{Hom}_C(\mathcal{F}, \omega_C) \to \mathrm{Hom}_k(H^1(\mathcal{F}), k).$$

sending a homomorphism $\alpha \in \mathrm{Hom}_C(\mathcal{F}, \omega_C)$ to $\eta \circ H^1(\alpha)$. In particular, we see that η itself corresponds to the identity map of ω_C

Grothendieck extended these ideas to all pure-dimensional projective schemes and beyond. Here is the basic definition:

Definition 20.0.2. If X is a purely r-dimensional projective scheme over the field k, we say that a coherent sheaf ω on X, together with a linear functional $\eta : H^r(\omega) \to k$ is *dualizing* if the map

$$f_\eta : H^0(\mathcal{H}\mathrm{om}_X(\mathcal{F}, \omega)) \to \mathrm{Hom}_k(H^r(\mathcal{F}), k).$$

defined as above is an isomorphism for every coherent sheaf $\mathcal{F}$.

Proposition 20.0.3. *Any two dualizing pairs (ω_X, η) and (ω'_X, η'), on a scheme X are canonically isomorphic.*

Proof. We will show that there is a unique isomorphism $g : \omega \to \omega'$ making the diagram

$$\begin{tikzcd} H^1(\omega) \arrow[r, "g"] \arrow[dr, "\eta"'] & H^1(\omega') \arrow[d, "\eta'"] \\ & k \end{tikzcd}$$

commute.

The duality property of ω' yields

$$\mathrm{Hom}_X(\omega, \omega') = \mathrm{Hom}(H^1(\omega), k).$$

Let $g : \omega \to \omega'$ be the map corresponding under this isomorphism to η'. It follows from the relation of η' to the duality isomorphism, that $\eta' g = \eta$. Similarly, we get a map $g' : \omega' \to \omega$ such that $\eta g = \eta'$, and it also follows that $\eta g g' = f_\eta(gg') = \eta$, so that $gg' = 1_{\omega'}$. Similarly, $g'g = 1_\omega$, and we are done. $\square$

We often abuse the terminology, and say simply that ω is a *dualizing sheaf* or a *canonical sheaf* on X. If X is reduced and connected, so that $H^0(\mathcal{H}\mathrm{om}(\mathcal{O}_X, \mathcal{O}_X)) = k$, then $H^r(\omega) \cong k$, whence η is, in any case, unique up to a nonzero scalar.

Of course it is far from obvious that such a dualizing sheaf will exist on an arbitrary pure-dimensional scheme, and in general there is no such sheaf! However, dualizing sheaves do exist on any pure-dimensional scheme that is embeddable in a smooth scheme, and thus, in particular, they exist on any projective scheme.

On a smooth projective variety, Serre duality shows that we can choose the dualizing sheaf to be the top exterior power of the sheaf of differential forms, as already explained. To understand how dualizing sheaves are constructed in general, we must abandon the idea that the the canonical sheaf of X must "come from" differential forms on X. For example, consider the ring $R = k[x,y,z]/x^2$ and the scheme $X = \mathrm{Proj}\, R$, a double line in $\mathbb{P}^2$. Writing $d : \mathcal{O}_X \to \Omega_{\mathcal{O}_X/k}$ for the universal derivation, we have $0 = d(x^2) = 2xd(x)$ so (at least in characteristic $\neq 2$), $dx = 0$. Thus

$$\Omega_{\mathcal{O}_X/k} = \Omega_{\mathcal{O}_{X_{\mathrm{red}}}/k} = \omega_{X_{\mathrm{red}}};$$

that is, the differentials do not "see" the nilpotent part of the structure sheaf at all. Furthermore, $h^0(\mathcal{O}_X(1)) = 3$ (as would be the case with a smooth conic in the plane) while $h^1(\Omega_X(-1)) = h^1(\Omega_{X_{red}}(-1)) = h^1\mathcal{O}_{\mathbb{P}^1}(-3) = 2$ so Serre duality would fail if we took $\omega_X = \Omega_X$ as we would do for smooth curves.

In general, a dualizing sheaf on a scheme X can be constructed by comparing X with a variety Y that already has a dualizing sheaf, such as $Y = \mathbb{P}^r$. To understand the motivation behind the construction, consider first the situation where $\iota : X \subset Y$ is a closed immersion of smooth varieties, and suppose thta X has dimension d and codimension c in Y. In this case the conormal bundle of X in Y is by definition the sheaf $\mathcal{I}/\mathcal{I}^2$, where $\mathcal{I}$ is the ideal sheaf of X in Y. If $p \in X$ then because X is smooth, the kernel of the map of local rings $\mathcal{O}_{Y,p} \xrightarrow{\iota^*} \mathcal{O}_{X,p}$ is generated by a subset of a set of minimal generators of the maximal ideal $\mathfrak{m}_{Y,p} \subset \mathcal{O}_{Y,p}$, and is thus a complete intersection. It follows that the left-most term of the right exact sequence

$$\mathcal{I}/\mathcal{I}^2 \to \iota^*(\Omega_Y) \to \Omega_X \to 0$$

is a vector bundle on X whose rank is $c = \mathrm{rank}\, \Omega_Y - \mathrm{rank}\, \Omega_X$, so the sequence is exact on the left as well. All the terms are vector bundles on X, and thus the sequence is locally split. It follows that

$$\omega_X = \wedge^d \Omega_X = \wedge^{c+d} \iota^*(\Omega_Y) \otimes \wedge^c(\mathcal{I}/\mathcal{I}^2)^\vee = \mathcal{H}\mathrm{om}(\wedge^c(\mathcal{I}/\mathcal{I}^2), \omega_Y).$$

The next step is to recognize that this expression for ω_X can be interpreted as saying, always in the case $X \subset Y$ is smooth of codimension c, that

$$\omega_X = \mathcal{E}xt^c_{\mathcal{O}_Y}(\mathcal{O}_X, \omega_Y).$$

To motivate this formula, consider just the simple case where X is a complete intersection of hypersurfaces of degrees d_i in $Y = \mathbb{P}^n$. In this case the Koszul complex

$$0 \to \wedge^c(\oplus_i \mathcal{O}(-d_i)) \xrightarrow{\phi_c} \cdots \to \oplus_i \mathcal{O}(-d_i) \xrightarrow{\phi_1} \mathcal{O}_{Y,p} \to \mathcal{O}_{X,p} \to 0.$$

Thus $\mathcal{E}xt^c_{\mathcal{O}_Y}(\mathcal{O}_X, \omega_Y) = \mathcal{H}\mathrm{om}(\operatorname{coker} \phi_c^\vee, \omega_Y)$ and $\operatorname{coker} \phi_c^\vee$ may be canonically identified with $\wedge^c(\mathcal{I}/\mathcal{I}^2)$.

This computation suggests the bold idea that the dualizing module of a closed subscheme $X \subset Y$ of pure codimension c can be computed by the formula

$$\omega_X := \mathcal{E}xt^c_Y(\mathcal{O}_X, \omega_Y),$$

or, still more generally, that given any finite morphism $\pi : X \to Y$ we have

$$\omega_X := \mathcal{E}xt^c_Y(\mathcal{O}_X, \omega_Y),$$

Where we can give the sheaf on the right the unique structure of a sheaf on X such that $\pi_*\mathcal{E}xt^c_Y(\mathcal{O}_X, \omega_Y) = \mathcal{E}xt^c_Y(\pi_*\mathcal{O}_X, \omega_Y)$, as explained below in the case $c = 0$.

The truth of this assertion implies a web of theorems proving that the sheaf $\mathcal{E}xt^c_Y(\mathcal{O}_X, \pi^*\omega_Y)$ is independent of π; and moreover that such sheaves satisfy some form of Serre duality. For all this, see the book **??**.

We now explain the construction above in the one case we will need for studying the linkage of curves in $\mathbb{P}^3$.

Theorem 20.0.4. *Let $\pi : X \to Y$ is a finite morphism of a purely 1-dimensional schemes, and suppose that ω_Y, η_y is a dualizing pair on Y. Let $\omega := \mathcal{H}\mathrm{om}(\pi_*\mathcal{O}_X, \omega_Y)$ regarded as a sheaf on X. There are natural isomorphisms*

$$\eta_{\mathcal{F}} : \mathrm{Hom}_X(\mathcal{F}, \omega) \to \mathrm{Hom}(H^1(\mathcal{F}), k),$$

and thus ω, together with $\eta = \eta_\omega(1_\omega)$ is a dualizing pair for X.

Note that one possible choice of $\pi : X \to Y$ in the theorem would be a Noether normalization, that is, a finite map to $X \to \mathbb{P}^1$; in this form, at least when X is smooth, it is the Riemann-Hurwitz formula
((did we decide on this name?))
Another is the inclusion of X into another curve, perhaps a complete intersection curve, and this is the one we need for linkage:

Corollary 20.0.5. *Let $X \subset Y \subset \mathbb{P}^n$ be closed, purely 1-dimensional schemes. If $\omega_Y = \mathcal{O}_Y(d)$ for some integer d, then*

$$\omega_X = \frac{\mathcal{I}_Y : \mathcal{I}_X}{\mathcal{I}_Y}(d).$$

Proof. By Theorem 20.0.4 we have

$$\omega_X = \mathcal{H}\mathrm{om}(\mathcal{O}_X, \omega_Y) = \mathcal{H}\mathrm{om}(\mathcal{O}_X, \mathcal{O}_Y)(d).$$

Clearly any section of $\mathcal{I}_Y : \mathcal{I}_X$ on on open set U gives rise by multiplication to a map $\mathcal{O}_X(U) \to \mathcal{O}_Y(U)$, and the sections of $\mathcal{I}_Y$ give the zero map, so there a natural mapping $\frac{\mathcal{I}_Y:\mathcal{I}_X}{\mathcal{I}_Y} \to \mathcal{H}\mathrm{om}(\mathcal{O}_X, \mathcal{O}_Y)$, and locally this is an isomorphism because every map from $\mathcal{O}_X$ is determined by the image of the global section 1. ☐

Proof of Theorem 20.0.4. The sheaf $\omega := \mathcal{H}\mathrm{om}(\pi_*\mathcal{O}_X, \omega_Y)$ which is, a priori a sheaf on Y, has the structure of a sheaf on X specified by the property that

$$\pi_*(\omega) = \mathcal{H}\mathrm{om}(\pi_*\mathcal{O}_X, \omega_Y).$$

as a sheaf on Y.

To see that there is such a sheaf, take an open affine cover $\{U_i\}$ of Y and pull it back to an open affine cover $\{V_i = \pi^{-1}(U_i)\}$ of X. Because π is finite, the restriction of $\pi_*\mathcal{O}_X$ to U_i is naturally isomorphic to $\mathcal{O}_{V_i}$, regarded as an $\mathcal{O}_{U_i}$-module, and thus the restriction of $\mathcal{H}\mathrm{om}(\pi_*\mathcal{O}_X, \omega_Y)$ to U_i is $Hom_{U_i}(\mathcal{O}_{V_i}, \omega_Y|_{U_i})$, which is naturally a module over $\mathcal{O}_{V_i}$. This gives $\omega_X := \mathcal{H}\mathrm{om}(\pi_*\mathcal{O}_X, \omega_Y)$ the structure of a sheaf on X, and it is obvious from the construction that this has the desired pushforward.

Because π is finite we have

$$\begin{aligned}
\mathrm{Hom}(H^1(\omega), k) &= \mathrm{Hom}(H^1(\pi_*\omega), k)\\
&= \mathrm{Hom}(H^1(\mathcal{H}\mathrm{om}(\pi_*\mathcal{O}_X, \omega_Y)), k)\\
&\cong \mathrm{Hom}(\mathcal{H}om(\pi_*\mathcal{O}_X, \omega_Y), \omega_Y)
\end{aligned}$$

by the dualizing property of ω_Y.

It now suffices to show that there exist natural isomorphisms $\mathrm{Hom}_X(\mathcal{F}, \omega) \cong \mathrm{Hom}_k(H^1(\mathcal{F}), k)$. Because π is finite, the cohomology of a sheaf on X is the same as the cohomology of its pushforward. In view of the construction of ω, and the fact that ω_Y is a dualizing sheaf for Y, it suffices to show that there is a natural isomorphism

$$\phi : \pi_*\,\mathcal{H}\mathrm{om}_X(\mathcal{F}, \mathcal{H}\mathrm{om}_Y(\mathcal{O}_X, \omega_Y)) \to \mathcal{H}\mathrm{om}_Y(\pi_*\mathcal{F}, \omega_Y).$$

Passing to an affine open set $U_i \subset Y$ and its preimage $V_i \subset X$ as in the definition of ω, the left hand side becomes

$$\mathrm{Hom}_{V_i}(\mathcal{F}|_{V_i}, \mathrm{Hom}_{U_i}(\mathcal{O}_X|V_i, \omega_Y|_{U_i}))$$

where $\mathcal{O}_X|V_i$ is considered a U_i-modules via the structure map $\mathcal{O}_{U_i} \to \mathcal{O}_{V_i}$. Note that $\mathcal{F}|_{V_i} = \pi_*\mathcal{F}|_{U_i}$. We define ϕ to be the map sending an element a in the left hand side to

$$\phi(a) \in \mathcal{H}\mathrm{om}_{U_i}(\pi_*\mathcal{F}|_{U_i}, \omega_Y|_{U_i}) \qquad \phi(a) : t \mapsto a(t)(1).$$

It is easy to check that this is a natural isomorphism. ☐

Cheerful Fact 20.0.1 (Dualizing sheaves in higher dimension)**.** There are two important extensions of Theorem 20.0.4:

First, the proof given for curves above actually works for a purely r-dimensional projective scheme X over k if we replace the occurrences of H^1 by H^r, and shows that every such scheme has a dualizing sheaf ω. However, the isomorphisms

$$H^i(\mathcal{H}\mathrm{om}(\mathcal{F}, \omega)) \cong \mathrm{Hom}_k(H^{r-i}(\mathcal{F}), k)$$

hold for all coherent $\mathcal{F}$ *if and only if* X is Cohen-Macaulay. In our situation this condition means that, if $\pi : X \to \mathbb{P}^r$ is a finite map, then $\pi_* \mathcal{O}_X$ is locally free.

Example 20.0.6 (Adjunction formula)**.** Prove directly that if ω_Y is a dualizing sheaf on a surface Y and X is a Cartier divisor on Y, then $\omega := \mathcal{O}_X \otimes_Y \omega_Y(X))$ is a dualizing sheaf for X. Use this to show by induction that if

$$X = \bigcap_{i=1}^{c} H_1 \cap \cdots \cap H_n$$

is a complete intersection in $\mathbb{P}^r$ of hypersurfaces of degrees $\deg H_i = d_i$, then

$$\omega_X = \mathcal{O}_X(\sum_{i=1}^{c} d_i - r - 1).$$

Here is a special case that will be important to us:

Proposition 20.0.7. *Suppose that $C \subset \mathbb{P}^n$ is a purely 1-dimensional scheme. If $(f_1, \dots, f_{n-1}) \subset I_C$ is a regular sequence of forms of degrees $d_1 \dots, d_r$ defining the scheme $X \supset C$, then*

$$\omega_C \cong \widetilde{(\mathcal{I}_X : \mathcal{I}_C)}(q),$$

where $q = \sum_{i=1}^{n-1} d_i) - r - 1$.

((both the prop and the proof should be done for affine cones, then localized.))

Proof. By Theorem **??** we know that $\omega_X = \mathcal{O}_X(q)$. Further, the map $C \to X$ is finite, so

$$\omega_C = Hom(\mathcal{O}_C, \omega_X) = \mathrm{Hom}(\mathcal{O}_C, \mathcal{O}_X)(q).$$

But $\mathrm{Hom}(\mathcal{O}_C, \mathcal{O}_X) = (\mathcal{I}_X : \mathcal{I}_C)$, completing the proof. □

Theorem 20.0.8. *R is Gorenstein if and only if R is Cohen-Macaulay and $\omega_R \cong R$ (up to twist in the graded case)*

Proof. □

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