Geometry of Families of Curves

Fall 2012, taught by Joe Harris.

Contents

1 Parameter Spaces

We'll be interested in studying parameter spaces describing families of objects we're interested in. Examples include:

- {subschemes, or subvarieties, of $\mathbb{P}^n$ with a given dimension, degree, and Hilbert polynomial}
- {sheaves or vector bundles on a given X}
- {schemes with given numerical invariants} modulo isomorphism

etc. Our goal is to identify these spaces with parts of a variety or scheme in a natural way. As a simple example, the set of hypersurfaces of degree d in $\mathbb{P}^n$ can be identified with $\mathbb{P}^N = \mathbb{P}H^0(\mathcal{O}_{\mathbb{P}^n}(d))$.

But in general, we need to specify what "natural" means. In the language of varieties, we mean this: Given a family of objects

$$\begin{tikzcd} \mathfrak{X} \arrow[r, hook] \arrow[dr] & B \times \mathbb{P}^n \arrow[d] \\ & B \end{tikzcd} \tag{1.1}$$

with fibers schemes $X_b \subseteq \mathbb{P}^n$ of specified Hilbert polynomial p, we get a map

$$B \xrightarrow{\phi_{\mathfrak{X}}} \mathcal{H}_p = \{\text{subschemes with Hilbert polynomial } p\}. \tag{1.2}$$

We want to make $\mathcal{H}_p$ into a variety such that $\phi_{\mathfrak{X}}$ is regular.

In the world of schemes: A family means a flat family

$$\begin{tikzcd} \mathfrak{X} \arrow[r, hook] \arrow[dr, "\pi"'] & B \times \mathbb{P}^n \arrow[d] \\ & B \end{tikzcd} \tag{1.3}$$

We want that for all flat families $\mathfrak{X}$ with Hilbert polynomial p, we have a map $\phi_\pi : B \to \mathcal{H}_p$ commuting with base change. Observe that if $B = \operatorname{Spec} \mathbb{C}$, then we get a bijection between points of $\mathcal{H}_p$ and subschemes of $\mathbb{P}^n$ with Hilbert polynomial p. If B is arbitrary and B' is a point, then $\phi_\pi : b \mapsto [X_b]$.

Given p, we have a functor {schemes} $\to$ {sets} given by

$$B \mapsto \{\mathfrak{X} \subseteq B \times \mathbb{P}^n \text{ flat over } B \text{ with Hilbert polynomial } p\}. \tag{1.4}$$

We say $\mathcal{H}_p$ is a fine parameter space if we have an isomorphism of functors

$$\{\text{families over } B\} \leftrightarrow \operatorname{Hom}(B, \mathcal{H}_p). \tag{1.5}$$

Theorem 1.1. *There exists a fine parameter space $\mathcal{H}_{p,n}$ for subschemes of $\mathbb{P}^n$ with Hilbert polynomial p.*

As an example, the set of twisted cubics is given by maps $\mathbb{P}^1 \hookrightarrow \mathbb{P}^3$ defined by $[f_0 : f_1 : f_2 : f_3]$. But the corresponding morphism isn't injective, for we could change the f_i by an automorphism of $\mathbb{P}^1$. And maps from a single curve isn't useful in high genus.

Here is an alternate approach: $C = V(Q_1, Q_2, Q_3)$ and each quadric has specified coefficients. C is determined by $\operatorname{span}(Q_1, Q_2, Q_3)$. We associate to C the vector space $H^0(\mathcal{I}_{C,\mathbb{P}^3}(2))$, which we also denote by $I(C)_2 \subseteq S_2$, where S is the graded ring $K[W, X, Y, Z]$. This vector space is a point in $G(3, 10)$. We obtain a map of sets {twisted cubics} $\to G(3, 10)$. The problem is that the image is not closed: we can deform twisted cubics to get a nodal plane cubic with an embedded point at the node.

Here's another fact about fine parameter spaces: there exists a universal family, obtained from $\mathbf{1} : \mathcal{H} \to \mathcal{H}$. This family is a

$$\begin{tikzcd} \mathcal{C} \arrow[r, hook] \arrow[dr] & \mathcal{H} \times \mathbb{P}^n \arrow[d] \\ & \mathcal{H} \end{tikzcd} \tag{1.6}$$

such that for every family

$$\begin{tikzcd} \mathfrak{X} \arrow[r, hook] \arrow[dr, "\pi"'] & B \times \mathbb{P}^n \arrow[d] \\ & B \end{tikzcd} \tag{1.7}$$

we have $\mathfrak{X} = \mathcal{C} \times_{\phi_\pi} B$.

Goal: understand the geometry of parameter spaces, and use this to answer questions about the schemes in question.

Examples of questions we could answer with the help of parameter spaces are:

- Given 12 lines $\ell_1, \dots, \ell_{12} \subseteq \mathbb{P}^3$, how many twisted cubics meet all 12?
- Do there exist nonconstant families of twisted cubics over a complete base?

2 Construction of the Hilbert Scheme

Idea: associate to any $X \subseteq \mathbb{P}^n$ having Hilbert polynomial p its ideal $I(X)_m \subseteq S_m$, a point in $G(-, S_m)$. But can such an m always exist? It turns out it does. In fact it is required if we want $\mathcal{H}$ to exist and be of finite type.

Theorem 2.1. *Given p, there exists m_0 such that for every $m \geq m_0$ and for every $X \subseteq \mathbb{P}^n$ with Hilbert polynomial p, $\dim I(X)_m = \binom{m+n}{n} - p(m)$, and $I(X)_m$ determines $I(X)$ up to saturation.*

(Relevant background: flat limits, Geometry of Schemes II.3; cohomology and base change, 3264 6.7)

For $\Sigma = \{$subschemes X of $\mathbb{P}^n$ with Hilbert polynomial $p\}$, we have an embedding

$$\Sigma \hookrightarrow G\left(\binom{m+n}{n} - p(m), \binom{m+n}{n}\right), \qquad X \mapsto I(X)_m \subseteq S_m. \tag{2.1}$$

We need to know that the image has an algebraic structure.

Theorem 2.2. *The image is closed and has the structure of a scheme $\mathcal{H}$ satisfying the property of being a fine parameter space.*

To write down equations for the image, consider the map $\Lambda \otimes S_k \xrightarrow{\mu_k} S_{m+k}$. We want $\operatorname{rank}(\mu_k) \leq \binom{m+n}{n} - p(m+k)$. For U the universal bundle, this gives a determinental variety.

(Relevant text for next part: Chapter of of Moduli of Curves, Chapter 1 of Curves in Projective Space)

Recall $\mathcal{H} = \mathcal{H}_{p,n} = \{$subsets $X \subseteq \mathbb{P}^n$ with Hilbert polynomial $p\}$. We have to show that given

$$\begin{tikzcd} \mathfrak{X} \arrow[r, hook] \arrow[dr] & B \times \mathbb{P}^n \arrow[d] \\ & B \end{tikzcd} \tag{2.2}$$

flat with fibers of Hilbert polynomial p, we get a map $B \to \mathcal{H}$ with $b \mapsto [I(X_b)_m \subseteq S_m] \in G$. The problem is with nonreduced points; this is where cohomology and base change is used.

3 Geometry of Hilbert Schemes

We will first consider the punctual Hilbert schemes

$$\mathcal{H}_{d,n} = \{\text{subschemes of dimension 0 and degree d contained in } \mathbb{P}^n\}. \tag{3.1}$$

$\mathcal{H}$ contains the subset $\mathcal{H}^0$ of reduced schemes, which is equal to $((\mathbb{P}^n)^d \backslash \Delta)/S^d$, the set of distinct d-tuples of points.

Observation: if $n \geq 3$ and $d >> 0$, then $\mathcal{H}$ has other, larger, components. As an example, for $n = 3$, look at subschemes $\Gamma \subseteq \mathbb{P}^n$ supported at a single point, and such that $\mathfrak{m}_p^{k+1} \subseteq \mathcal{I}_p \subseteq \mathfrak{m}_p^k$. $\mathfrak{m}_p^k$ has codimension $\binom{k+2}{3}$, while $\mathfrak{m}_p^{k+1}$ has codimension $\binom{k+3}{3}$. The $\mathcal{I}_\Gamma$ are of the form $I = (\mathfrak{m}^{k+1}, V)$ for $V \subseteq \mathfrak{m}^k/\mathfrak{m}^{k+1}$, with $d = \deg V(I) = \binom{k+2}{3} + \operatorname{codim} V$. Choose $\operatorname{codim} V \sim \frac{1}{2}\binom{k+2}{2} \sim \frac{1}{4}k^2$. The dimension of the family of such schemes is $\sim \frac{1}{16}k^4$, but $d \sim \frac{1}{6}k^3$. So $\frac{1}{16}k^4 >> 3d$ for d large, implying $\mathcal{H}$ has a component other than $\overline{\mathcal{H}^0}$.

On the other hand, for $n = 1$, $\mathcal{H} = \overline{\mathcal{H}^0} = \mathbb{P}^d$, and for $n = 2$, $\mathcal{H} = \overline{\mathcal{H}^0}$ is smooth.

Now we'll look at Hilbert schemes of curves. As an example,

$$\mathcal{H} = \mathcal{H}_{3m+1,3} \supseteq \{\text{twisted cubics}\} \supseteq \mathcal{H}_t. \tag{3.2}$$

We will see that $\mathcal{H}_t$ is irreducible of dimension 12; let $\mathcal{H}^0 = \overline{\mathcal{H}_t}$.

Proposition 3.1. *$\mathcal{H} = \mathcal{H}^0 \cup \mathcal{H}^1$, where $\mathcal{H}^1 = I(C : C = C_0 \cup \{p\})$ for C_0 a plane cubic and $p \in \mathbb{P}^3 \setminus C_0$ a point.*

$\mathcal{H}^1$ is irreducible of dimension 15, because the C_0's form a $\mathbb{P}^9$-bundle over $(\mathbb{P}^3)^\vee$, and then the C's are fibered by the p's.

Lemma 3.2. *If $C \subseteq \mathbb{P}^3$ is of dimension 1 and degree 3, then $p_a(C) \leq 1$, with equality if and only if C is a plane cubic.*

Proof. Look at a general plane section $\Gamma = C \cap H$. Then letting h denote the Hilbert function, $h_C(m) \geq h_C(m-1) + h_\Gamma(m)$, because of the exact sequence

$$0 \to \mathcal{O}_C(m-1) \to \mathcal{O}_C(m) \to \mathcal{O}_\Gamma(m) \to 0. \tag{3.3}$$

If Γ lies on a line, then $h_\Gamma(m) = 1, 2, 3, 3, \ldots$, otherwise $h_\Gamma(m) = 1, 3, 3, 3, \ldots$. Therefore $h_C(m) \geq 1, \underbrace{3, 6, 9, 12, \ldots}_{3m}$, implying $p_a(C) \leq 1$. □

As a challenge problem, take $C_0 = V(Z, Y^4, Y^3X, Y^3W)$, a triple line with a planar embedded point. Is $C_0 \in \mathcal{H}^0$?

4 Generalizations of the Hilbert Scheme Construction

- If $Z \subseteq \mathbb{P}^n$ is a closed subscheme, we can construct $\mathcal{H}_{p,Z}$, the parameter space for subschemes of Z with Hilbert polynomial p. Consider those X with $I(X) \supseteq I(Z)$, so we get a sub-Grassmannian.

- Given X and Y, we can parameterize $\{f : X \to Y\}$ as

$$\{\Gamma : X \times Y \hookrightarrow \mathbb{P}^n : p_1 : X \to X \text{ an isomorphism}\}, \tag{4.1}$$

  a quasi-projective scheme. (We might want to restrict the behavior of the embedded graph to get finite type.)

- We can consider the Hilbert scheme of nested pairs $X \subseteq Y \subseteq \mathbb{P}^n$, as a subscheme of a flag manifold.

- $\{X, Y, f : X \to Y\}$ can be thought of as a pair using the above graph construction.

- Relative Hilbert scheme: Given $X \xrightarrow{\pi} Y$, we can consider $\mathcal{H}_{p,X/Y}$ parameterizing subschemes of X contained in a fiber of Y.

As an example, this can be used as an aid to answer the question of how many nonconstant maps $f : C \to D$ there can be, where C, D are smooth curves of genus $g, h \geq 2$. Riemann-Hurwitz bounds the degree, so we can then fit the maps in a quasi-projective variety of finite type. This shows existence of a uniform bound in terms of g and h.

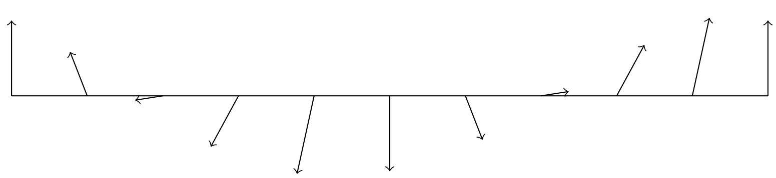
5 Extraneous Components

Recall the situation with $\mathcal{H} = \mathcal{H}_{3m+1,3} = \mathcal{H}_0 \cup \mathcal{H}_1$, where $\mathcal{H}_0$ is the closure of the locus of twisted cubics, while $\mathcal{H}_1$ is the closure of the locus of $C \cup p$ for C a plane cubic and $p \in \mathbb{P}^3 \setminus C$.

Fact. $\mathcal{H}_0 \cap \mathcal{H}_1$ is the closure of the locus of nodal plane cubics with a spatial embedded point at the node.

In general, $\mathcal{H} = \mathcal{H}_{dm-g+1,n}$ will have many "extraneous components" (ones which don't contain the smooth irreducible nondegenerate curves). For example, there are components with general member $C \cup \Gamma$ where C is a plane curve of degree d and Γ is 0-dimensional of degree $\binom{d-1}{2} - g$.

Another example of extraneous components is given by ribbons. Take $\ell \subseteq \mathbb{P}^3$ a line. X will be a subscheme of degree 2 supported on ℓ. For example, consider a normal vector field to a line, and then can pick a ribbon with corresponding planar Zariski tangent spaces.



(5.1)

Then $I(X) = (X^2, XY, Y^2, F(Z,W)X + G(Z,W)Y)$ for F, G homogeneous of degree m. We have $g(X) = -m$, so in particular, for $m \geq 2$, X cannot be a limit of reduced curves.

From now on, we will usually restrict to the non-extraneous components. The restricted Hilbert scheme $\mathcal{H}_0$ is a union of all of the irreducible components of $\mathcal{H}$ whose general point corresponds to a reduced, irreducible, nondegenerate curve. However, extraneous components cannot necessarily be avoided.

As an example, we would like to know if $\mathcal{H}_0 \subseteq \mathcal{H}_{3m+1,3}$ is smooth. We can calculate the Zariski tangent space to $\mathcal{H}$ at a given point $[C]$, so that $\mathcal{H}_0 \setminus (\mathcal{H}_0 \cap \mathcal{H}_1)$ has tangent space of dimension 12, so that $\mathcal{H}$ is smooth at p. On $\mathcal{H}_0 \cap \mathcal{H}_1$, though, the tangent space to $\mathcal{H}$ has dimension 16; it is not clear whether this locus is smooth in $\mathcal{H}_0$. (It turns out to be.)

6 Basic Properties of Reduced Hilbert Schemes

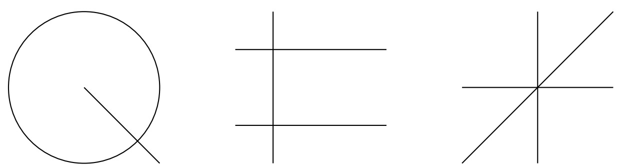
Let $\mathcal{H} = \mathcal{H}_{dm-g+1,n}$ (for now, $n = 3$), and let $\mathcal{H}^0$ be the restricted Hilbert scheme. We first look at the example of $d = 3, g = 0$ (twisted cubics), and illustrate various methods of working with Hilbert schemes:

- The parametric approach: consider the scheme

$$\Phi = \{(F_0, F_1, F_2, F_3) : F_i \text{ a basis for } H^0(\mathcal{O}_{\mathbb{P}^1}(3))\}/\text{scalars} \overset{o}{\hookrightarrow} \mathbb{P}^{15}. \tag{6.1}$$

  So Φ is irreducible of dimension 15. We have a map $\Phi \to \mathcal{H}^0$ with fibers isomorphic to PGL_2, which is 3-dimensional, so $\mathcal{H}^0$ is irreducible of dimension 12.

  Examples of curves in $\mathcal{H}^0$: the curves



(6.2)

  all lie in $\mathcal{H}$: the first two lie on smooth quadrics of type $(2,1)$, and the third is a limit of curves in the second family. What about $I(X^2, XY, Y^2)$?

- Examination of varieties containing the curves: Consider

$$\Sigma = \{(Q, C) : C \subseteq Q\} \subseteq U \times \widetilde{\mathcal{H}}, \tag{6.3}$$

  where $\widetilde{\mathcal{H}}$ is the locus of twisted cubics, and $U \subseteq \mathbb{P}^9$ the locus of smooth quadrics. Σ projects to U and $\widetilde{\mathcal{H}}$.

  To find the fibers of $\Sigma \to U$: the twisted cubics on a fixed quadric Q are curves of type $(2,1)$ and $(1,2)$, so the fibers are open in $\mathbb{P}^5 \amalg \mathbb{P}^5$. Here $\mathbb{P}^5$ is the projectivization of (quadratics $\otimes$ linears). So Σ has dimension 14.

  The fibers of $\Sigma \to \widetilde{\mathcal{H}}$ have dimension 2, since each C lies on a $\mathbb{P}^2$ of quadrics, and the smooth ones are an open subset, so $\mathcal{H}^0$ has dimension 12. However, we haven't proven irreducibility, since this method shows that Σ, and therefore $\widetilde{\mathcal{H}}$, has at most 2 components.

  Actually Σ is irreducible, since $(2,1)$ and $(1,2)$ are interchangeable upon varying quadrics.

- Linkage: Say S, T are two surfaces of degrees s, t, and $S \cap T = C \cup D$, curves of degrees c, d and genera g, h. We'll assume S is smooth (this isn't necessary). Then $K_S \sim (s-4)H$ for H the hyperplane section, and

$$2g - 2 = C.C + K_S.C = C^2 + (s-4)c \implies C^2 = 2g - 2 - (s-4)c. \tag{6.4}$$

We also have

$$C.D = C.(tH - C) = tc - C^2 = tc - (s-4)c - (2g-2). \tag{6.5}$$

Finally, the last intersection is given by

$$D^2 = D.(tH - C) = td - tc - (s-4)c + 2g - 2. \tag{6.6}$$

By adjunction,

$$2h - 2 = D^2 + K.D = D^2 + (s-4)d = (d-c)(s+t-4) + 2g - 2. \tag{6.7}$$

In other words, we obtain

$$h - g = (d-c) \cdot \frac{s+t-4}{2}. \tag{6.8}$$

(Also $d = st - c$, of course.)

Now consider

$$\begin{tikzcd} & & \widetilde{\mathcal{H}} \\ \{(Q, Q', C, \ell) : Q \cap Q' = C \cup \ell\} \arrow[r, equal] & \Sigma \arrow[ur] \arrow[dr] & \\ & & \mathbb{G}(1,3) \arrow[r, equal] & \mathcal{H}_{m+1,3} \end{tikzcd} \tag{6.9}$$

First, $\mathbb{G}(1,3)$ is irreducible of dimension 4. The fibers of $\Sigma \to \mathbb{G}(1,3)$ are open subsets of $\mathbb{P}^6 \times \mathbb{P}^6$, where $\mathbb{P}^6$ is the locus of quadrics containing a fixed ℓ, so Σ is irreducible of dimension 16. Finally, the fibers of $\Sigma \to \widetilde{\mathcal{H}}$ are open in $\mathbb{P}^2 \times \mathbb{P}^2$, so $\mathcal{H}^0$ is irreducible of dimension 12.

A harder example is $d = 6, g = 3$. For $C \subseteq \mathbb{P}^3$ of degree 6 and genus 3, we have

$$0 \to H^0(\mathcal{I}_C(2)) \to \underbrace{H^0(\mathcal{O}_{\mathbb{P}^3}(2)}_{10} \to \underbrace{H^0(\mathcal{O}_C(2))}_{12-3+1=10} \tag{6.10}$$

so $H^0(\mathcal{I}_C(2))$ may be trivial. In degree 3, though, we have 20 and 16, so C lies on a $\mathbb{P}^3$ of cubic surfaces. The residual curve is a twisted cubic.

We'll now look at some general families of examples.

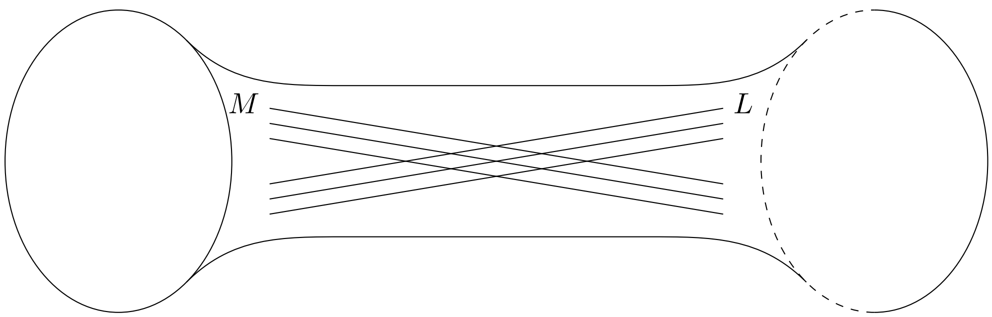
- $g = 0$ and d, n arbitrary: Look at $\mathcal{H}^0 \subseteq \mathcal{H}_{md+1,n}$, and consider

$$\begin{aligned}
\Psi = \{(F_0, \ldots, F_n) : F_i \in H^0(\mathcal{O}_{\mathbb{P}^1}(d)), \quad & \qquad (6.11)\\
F_i \text{ linearly independent, no common zeros, very ample}\}/\text{scalars} \quad & \qquad (6.12)\\
\overset{o}{\hookrightarrow} \mathbb{P}(H^0(\mathcal{O}_{\mathbb{P}^1}(d))^{\oplus(n+1)}) \quad & \qquad (6.13)\\
= \mathbb{P}^{(d+1)(n+1)-1}. \quad & \qquad (6.14)
\end{aligned}$$

Fix Me Formatting is awful right now. (1)

We have a natural map $\Psi \to \mathcal{H}^0$ with fibers isomorphic to PGL_2 of dimension 3, so $\mathcal{H}^0$ is irreducible of dimension $(d+1)(n+1) - 4$.

- Curves on (smooth) quadrics: (Curves on a quadric cone are limits of curves on smooth quadrics.) If $Q \subseteq \mathbb{P}^3$ is a smooth quadric, then $Q \cong \mathbb{P}^1 \times \mathbb{P}^1$, with two rulings by lines L, M.



(6.15)

$C \subseteq Q$ has type (a, b) if $C \sim aL + bM$. That is, C is the zero locus of a bihomogeneous polynomial of bidegree (a, b). If C has type (a, b), then $\deg C = a + b$. To get the genus, use adjunction: $K_Q = -2L - 2M$, as a sum of pullbacks of the $K_{\mathbb{P}^1}$'s; alternatively, it is $-2H$ for H the hyperplane class in $\mathbb{P}^3$. This gives

$$g(C) = \frac{C.C + K_Q.C}{2} + 1 = \frac{2ab - a - b}{2} + 1 = (a-1)(b-1). \tag{6.16}$$

Now use the incidence correspondence

$$\begin{tikzcd}
 & U \subseteq \mathbb{P}^9 \text{ open} \\
\Phi = \left\{ (Q, C) : C \underset{(a,b)}{\subseteq} Q \right\} \arrow[ur] \arrow[dr] & \\
 & \mathcal{H}^0
\end{tikzcd} \tag{6.17}$$

The fibers of $\Phi \to U$ have dimension $(a+1)(b+1)-1$, and there are two connected components if $a \neq b$ (the fibers are irreducible if $a = b$). So Φ has dimension $(a+1)(b+1)+8$. If $a+b \geq 5$, then $\Phi \to \mathcal{H}^0$ has finite fibers, so $\dim \mathcal{H}^0 = (a+1)(b+1) + 8$.

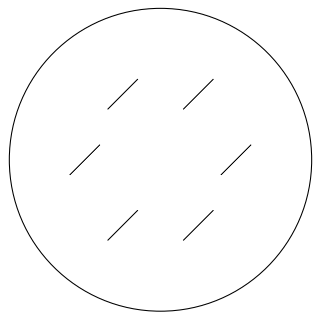
But does $\mathcal{H}^0$ has one or two components? We want to show that the monodromy on quadrics is transitive:

Lemma 6.1. *The monodromy exchanges the two rulings, so* Φ*, and therefore* $\mathcal{H}^0$*, is irreducible.*

Finally, if $a, b \geq 3$, then this is an entire component of the Hilbert scheme, but this is not the case if $a = 1, 2$ and $b \geq 4$.

6.1 Digression on Picard Groups of Surfaces and Monodromy

If S is a smooth cubic surface, then $S \cong Bl_{\{p_1,\ldots,p_6\}}\mathbb{P}^2$.



(6.18)

So we have

$$\operatorname{Pic} S = \mathbb{Z}\ell \oplus \mathbb{Z}e_1 \oplus \cdots \oplus \mathbb{Z}e_6, \tag{6.19}$$

with $\ell^2 = 1, e_i^2 = -1$, and all other intersection pairings 0. Now $Bl(\mathbb{P}^2) \hookrightarrow S \subseteq \mathbb{P}^3$ is given by the linear system $3\ell - e_1 - \cdots - e_6$, and

$$K_S = \mathcal{O}_S(-1) = -3\ell + e_1 + \cdots + e_6. \tag{6.20}$$

The monodromy statement is that the monodromy acts on $\operatorname{Pic} S$ as the full symmetry group of the lattice $\operatorname{Pic} S$ preserving K_S. This group has order $72 \cdot 6! = 51840$.

On the other hand, a very general surface $S \subseteq \mathbb{P}^3$ of degree $m \geq 4$ has $\operatorname{Pic} S = \mathbb{Z}$. The locus where $\operatorname{Pic} S$ has rank at least 2 is a countable union of subvarieties.

6.2 Linkage Example

Recall the $d = 6, g = 3$ example. $C \subseteq \mathbb{P}^3$ may not lie on a quadric, but does lie on a cubic. If C lies on a quadric, then the type is $(2, 4)$. Otherwise, C lies on a cubic. It turns out that a general such cubic is smooth, and the residual curve of an intersection of two cubics is a twisted cubic.

In the first case, we get a 23-dimensional family of curves. In the second case, $S \cap S' = C \cup T$ for T a twisted cubic, $T^2 = 1, T.K_S = -3, T.C = 8$. Now consider the space

$$\begin{tikzcd}
 & \{T\}\ (\dim 12) \\
\Sigma = \{(S, S', C, T)\} \arrow[ur] \arrow[dr] & \\
 & \mathcal{H}^0
\end{tikzcd} \tag{6.21}$$

For the fibers of $\Sigma \to \{T\}$, we have a map

$$H^0(\mathcal{O}_{\mathbb{P}^3}(3)) \twoheadrightarrow H^0(\mathcal{O}_T(3)) = \underbrace{H^0(\mathcal{O}_{\mathbb{P}^1}(9))}_{10}, \tag{6.22}$$

so the fiber dimension is $9 + 9 = 18$, implying Σ is irreducible of dimension 30. To get the fibers of $\Sigma \to \mathcal{H}^0$, check that C lies on exactly four dimensions of cubics, so the fiber dimension is $3 + 3 = 6$, showing $\mathcal{H}^0$ for the second case is irreducible of dimension 24.

Finally, we claim the curves in the first case are limits of those in the second case, so $\mathcal{H}^0$ is irreducible.

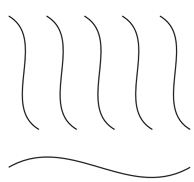
7 General Theory of Hilbert Schemes

(A reference on the theorem on cohomology and base change is: Hartshorne, III 12.9 (?), or 3264 6.7.

For flat families, refer to Geometry of Schemes I 3, or (in connection with Hilbert schemes) 3264 8.)

Problem: Given a family of sheaves parameterized by B:

$$\begin{tikzcd}
\mathfrak{X} \arrow[d] \\
B
\end{tikzcd} \tag{7.1}$$

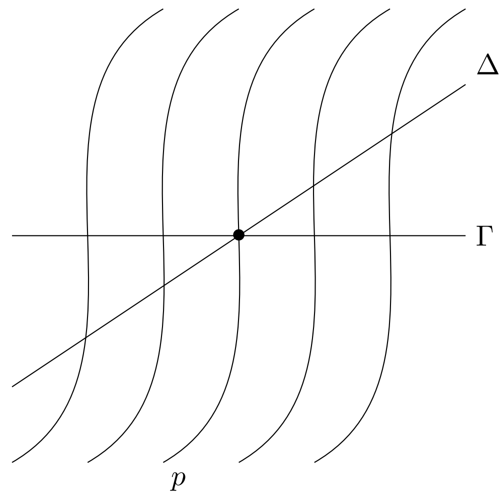


that is, a sheaf $\mathcal{F}$ on $\mathfrak{X}$ which is flat over B, let $X_b = \pi^{-1}(b)$ and $\mathcal{F}_b = \mathcal{F}|_{X_b}$.

Question: Do the spaces $H^0(\mathcal{F}_b)$ fit together to form the fibers of a vector bundle or coherent sheaf?

This turns out to not be the case, but there is a "best possible approximation" $\mathcal{E} = \pi_*\mathcal{F}$. We have maps $\phi_b : (\pi_*\mathcal{F})_b \to H^0(\mathcal{F}_b)$. Similarly for $H^i(\mathcal{F}_b)$, we have $R^i\pi_*\mathcal{F}$.

Examples: for E an elliptic curve and $p \in E$, let $\mathfrak{X} = E \times E \to E$, let Δ be the diagonal and $\Gamma = E \times \{p\}$.



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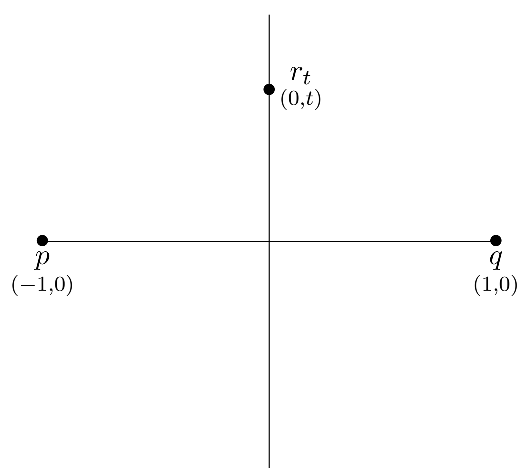
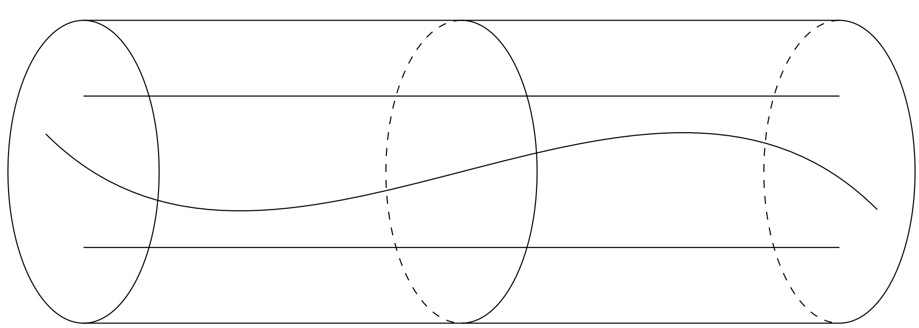
Let $\mathcal{F} = \mathcal{O}_{E\times E}(\Delta - \Gamma)$. This captures line bundles of degree 0 on E.

As another example, consider



(7.3)

Here $\mathfrak{X} = \mathbb{A}^1_t \times \mathbb{P}^2$ and $\Gamma = \mathbb{A}^1 \times \{p, q\} \cup \{(t, r_t)\}$.



(7.4)

Here $\mathcal{F} = \mathcal{I}_\Gamma(1)$.

Basic facts:

- The $h^i(\mathcal{F}_b)$ are upper semicontinuous.
- In the open set $U \subseteq B$ where $h^0(\mathcal{F}_b)$ is constant, $(\pi_*\mathcal{F})|_U$ is locally free and ϕ_b is an isomorphism for every $b \in U$.
- Where the $h^i(\mathcal{F}_b)$ jump, they jump in adjacent pairs.
- When $h^i(\mathcal{F}_b)$ and $h^{i+1}(\mathcal{F}_b)$ both jump, the jump is reflected in $R^{i+1}\pi_*\mathcal{F}$ but not $R^i\pi_*\mathcal{F}$.

(For the first example, $\pi_*\mathcal{F} = 0$ but $R^1\pi_*\mathcal{F}$ is a skyscraper sheaf supported at p.)

Idea/mnemonic: There is a complex of locally free sheaves

$$0 \to K^0 \to K^1 \to K^2 \to \cdots \tag{7.5}$$

whose cohomology sheaves are the $R^i\pi_*\mathcal{F}$. The kernel sheaves don't recognize the drop in rank, but the cokernel sheaves do.

This relates to base change by

$$\begin{tikzcd} \mathcal{F} \arrow[d, dash] & \mathcal{F}' \arrow[d, dash] \\ \mathfrak{X} \arrow[d] & \mathfrak{X} \times_B B' \arrow[l] \arrow[d] \\ B & B' \arrow[l] \end{tikzcd} \tag{7.6}$$

and identifying $\pi_*\mathcal{F}'$. ϕ_b occurs for $B' = \{b\}$.

This result is used to show that Hilbert schemes as constructed give a fine parameter space. A map $B \to G(k, V)$ is equivalent to inclusions

$$\begin{tikzcd} \mathcal{E} \arrow[rr, "\phi"] \arrow[dr] & & V \otimes \mathcal{O}_B \arrow[dl] \\ & B & \end{tikzcd} \tag{7.7}$$

where $\mathcal{E}$ is locally free of rank $\dim V - k$. So given

$$\begin{tikzcd} \mathfrak{X} \arrow[rr, hook] \arrow[dr] & & B \times \mathbb{P}^n \arrow[dl, "\pi"] \\ & B & \end{tikzcd} \tag{7.8}$$

we get $\mathcal{I}_{\mathfrak{X}}(m) \hookrightarrow \mathcal{O}_{B\times\mathbb{P}^n}(m)$, then take direct images $\pi_*\mathcal{I}_{\mathfrak{X}(m)} \to S_m \otimes \mathcal{O}_B$.

8 The Tangent Space of a Hilbert Scheme

The basic fact we'll use is that $\mathcal{H} = \mathcal{H}_{p,n}$ has the universal property of being a fine parameter space: for each scheme B, there is a natural bijection

$$\{\ \mathfrak{X} \subseteq B \times \mathbb{P}^n \text{ flat over } B \text{ with Hilbert polynomial } p\} \leftrightarrow \operatorname{Mor}(B, \mathcal{H}).$$

Apply this to $B = \operatorname{Spec} \mathbb{C}[\epsilon]/(\epsilon^2)$, which we denote by $\mathbb{I}$. For any scheme Z and point $p \in Z$, we have

$$T_p Z = \{f : \mathbb{I} \to Z : f(\operatorname{Spec} \mathbb{C}) = p\}. \tag{8.1}$$

So for every $X \subseteq \mathbb{P}^n$, we have

$$T_{[X]}\mathcal{H} = \{\mathfrak{X} \subseteq \mathbb{I} \times \mathbb{P}^n \text{ flat over } \mathbb{I} \text{ such that } \mathfrak{X} \cap (\operatorname{Spec} \mathbb{C} \times \mathbb{P}^n) = X\}. \tag{8.2}$$

We call these first-order deformations of X.

To describe first-order deformations of $X \subseteq \mathbb{P}^n$, do this locally (in $\mathbb{A}^n \subseteq \mathbb{P}^n$). Suppose $\mathfrak{X} \subseteq \mathbb{I} \times \mathbb{A}^n$ such that $\mathfrak{X} \cap (\operatorname{Spec} \mathbb{C} \times \mathbb{A}^n) = X$. Then $\mathfrak{X} = V(\widetilde{I})$ for $\widetilde{I} \subseteq \mathbb{C}[x_1, \ldots, x_n, \epsilon]/(\epsilon^2)$.

Write $\widetilde{I} = \{(f_\alpha + \epsilon g_\alpha)\}$ for $f_\alpha, g_\alpha \in \mathbb{C}[x_1, \ldots, x_n]$. The condition that $\mathfrak{X} \cap (\operatorname{Spec} \mathbb{C} \times \mathbb{A}^n) = X$ means that

$$\{f \in \mathbb{C}[x_1, \ldots, x_n] : f + \epsilon g \in \widetilde{I} \text{ for some } g\} = I(X). \tag{8.3}$$

For flatness, if a module M over $\mathbb{C}[\epsilon]/(\epsilon^2)$ is flat, then it preserves exactness of

$$0 \to (\epsilon) \hookrightarrow \mathbb{C}[\epsilon]/(\epsilon^2) \to \mathbb{C} \to 0. \tag{8.4}$$

So $\mathbb{C}[x_1, \ldots, x_n, \epsilon]/\widetilde{I}$ is flat if and only if $\epsilon g \in \widetilde{I} \implies g \in I(X)$. So for every $f \in I(X)$, there exists $g \in \mathbb{C}[x_1, \ldots, x_n]$ such that $f + \epsilon g \in \widetilde{I}$, and such a g is unique $\pmod{I(X)}$. This means $\mathfrak{X}$ flat over $\mathbb{I}$ gives a homomorphism $I(X) \to \mathbb{C}[x_1, \ldots, x_n]/I(X)$ by $f \mapsto g$ such that $f + \epsilon g \in \widetilde{I}$.

We can patch these to get a global version: a first order deformation of $X \subseteq \mathbb{P}^n$ is a map $\mathcal{I}_X \to \mathcal{O}_{\mathbb{P}^n}/\mathcal{I}_X = \mathcal{O}_X$, so is a global section of $\operatorname{Hom}(\mathcal{I}_X, \mathcal{O}_X) = \mathcal{N}_{X/\mathbb{P}^n}$. $\mathcal{N}$ is called the normal sheaf; when X is smooth, $\mathcal{N}_p = T_p\mathbb{P}^n/T_pX$.

We conclude that $T_{[X]}\mathcal{H} = H^0(\mathcal{N}_{X/\mathbb{P}^n})$.

For example, if $X \subseteq Z \subseteq \mathbb{P}^n$ are both smooth, we have an exact sequence

$$0 \to \mathcal{N}_{X/Z} \to \mathcal{N}_{X/\mathbb{P}^n} \to \mathcal{N}_{Z/\mathbb{P}^n}|_X \to 0. \tag{8.5}$$

As an application, if $a, b \geq 3$, then the locus of smooth curves of type (a, b) on smooth quadric surfaces is open in the appropriate $\mathcal{H}$. Suppose $C \subseteq Q \subseteq \mathbb{P}^3$ is a smooth curve of type (a, b) on a smooth quadric. We have $Q \cong \mathbb{P}^1 \times \mathbb{P}^1$. Write $\mathcal{O}_Q(k, \ell) = \pi_1^*\mathcal{O}_{\mathbb{P}^1}(k) \otimes \pi_2^*\mathcal{O}_{\mathbb{P}^1}(\ell)$. So $\mathcal{O}_Q(C) = \mathcal{O}_Q(a, b)$. Now use the exact sequence

$$
\begin{tikzcd}
0 \arrow[r] & \mathcal{N}_{C/Q} \arrow[r] \arrow[d, equal] & \mathcal{N}_{C/\mathbb{P}^3} \arrow[r] & \mathcal{N}_{Q/\mathbb{P}^3}|C \arrow[r] \arrow[d, equal] & 0 \\
 & \mathcal{O}_C(a,b) & & \mathcal{O}_C(2) = \mathcal{O}_C(2,2) &
\end{tikzcd} \tag{8.6}
$$

We need to show

$$\dim\{\text{curves of type } (a,b) \text{ on smooth quadrics}\} = \dim T_{[C]}\mathcal{H} = h^0(\mathcal{N}_{C/\mathbb{P}^3}).$$

The left hand side has been computed to be $(a+1)(b+1) - 1 + 9$. Subclaims:

1. $h^1(\mathcal{O}_C(a,b)) = 0$.
2. $h^0(\mathcal{O}_C(a,b)) = (a+1)(b+1) - 1$.
3. $h^0(\mathcal{O}_C(2,2)) = 9$.

For claim 1, we have the exact sequence

$$0 \to \mathcal{O}_Q \to \mathcal{O}_Q(a,b) \xrightarrow{\text{res}} \mathcal{O}_C(a,b) \to 0. \tag{8.7}$$

By Kunneth, $h^1(\mathcal{O}_Q(a,b)) = 0$ if $a, b \geq -1$. Alternatively, use Riemann-Roch. Also $h^2(\mathcal{O}_Q) = 0$, so $h^1(\mathcal{O}_C(a,b)) = 0$. And since $h^1(\mathcal{O}_Q) = 0$, the above sequence is exact over global sections, so

$$h^0(\mathcal{O}_C(a,b)) = h^0(\mathcal{O}_Q(a,b)) - h^0(\mathcal{O}_Q) = (a+1)(b+1) - 1, \tag{8.8}$$

showing claim 2.

For claim 3, use the exact sequence

$$0 \to \mathcal{O}_Q(2-a, 2-b) \to \mathcal{O}_Q(2,2) \to \mathcal{O}_C(2,2) \to 0. \tag{8.9}$$

Now $h^1(\mathcal{O}_Q(k,\ell)) = 0$ for $k, \ell \leq -1$, so

$$h^0(\mathcal{O}_C(2,2)) = h^0(\mathcal{O}_Q(2,2)) = 9. \tag{8.10}$$

9 Mumford's Example

Here we encounter a situation where a component of the restricted Hilbert scheme $\mathcal{H}^0$ is everywhere nonreduced. Consider $d = 14, g = 24$, and write $\mathcal{H} = \mathcal{H}^0_{14m-23,3}$.

Suppose $C \subseteq \mathbb{P}^3$ is smooth of degree 14 and genus 24. Consider the restriction $H^0(\mathcal{O}_{\mathbb{P}^3}(m)) \to H^0(\mathcal{O}_C(m))$. C can't lie on a quadric (this is easy to check). Now check higher degrees.

We have $\deg K_C = 46$, while for $m \geq 4$, $\mathcal{O}_C(m)$ has degree at least 56. So in this case, the divisor is non-special, and Riemann-Roch implies $h^0(\mathcal{O}_C(m)) = h^0(\mathcal{O}_{\mathbb{P}^3}(m) - 23$. But for $m = 3$, $h^0(\mathcal{O}_C(3))$ equals 19 or 20. So either C lies on a cubic or C doesn't.

We check the second case first. Suppose C doesn't lie on a cubic. Then C lies on two independent quartics S, S'. Write $S \cap S' = C \cup Q$. By linkage, Q has degree 2 and genus 0, so Q is a plane conic. Consider $\Phi = \{(C, Q)\}$ mapping to $\{C\}$ and $\{Q\}$. For the fibers of $\Phi \to \{Q\}$, the fibers are open in $\mathbb{P}^{25} \times \mathbb{P}^{25}$, since the quartics containing a given conic have codimension 9, or dimension 25. This means the fiber dimension is 50. $\{Q\}$ has dimension 8, so Φ is irreducible of dimension 58. $\Phi \to \{C\}$ has fiber dimension 2, so the locus of C not lying on a cubic is irreducible of dimension 56. (We need to know that $H^0(\mathcal{O}_{\mathbb{P}^3}(4)) \to H^0(\mathcal{O}_C(4))$ is surjective; that is, $h^1(\mathcal{I}_C(4)) = 0$.)

Now consider the case of C lying on a cubic S. We'll assume S is smooth. C obviously can't lie on a quartic, and we can check that C can't lie on a quintic by linkage. But C must lie on a new sextic surface. Call such s sextic T. Now $S \cap T = C \cup C'$, where C' has degree 4 and genus -1. C' could be either a union of two disjoint conics or a union of a line and a twisted cubic. (It is also possible for C' to be nonreduced, but this won't occur generically.) We will conclude that $\mathcal{H}$ has three components:

- $\mathcal{H}_0$: C not lying on a cubic.
- $\mathcal{H}_1$: C lying on a cubic, and residual to two conics.
- $\mathcal{H}_2$: C lying on a cubic, and residual to a line and a twisted cubic.

The "pathological" component is $\mathcal{H}_1$, but we'll also look at $\mathcal{H}_2$ to see how it differs from $\mathcal{H}_1$.

A conic Q on S has $Q \sim H - L$ for H the hyperplane class. Given two conics Q, Q', they have the same residual line, so $Q, Q' \sim H - L$. Now on S, $C \sim 6H - Q - Q' \sim 4H + 2L$. We then have

$$h^0(\mathcal{O}_C(3)) = 19 + h^1(\mathcal{O}_C(3)) = 19 + h^0(K_C(-3)) = 19 + h^0(\mathcal{O}_C(2L)) \tag{9.1}$$

(since $K_C = (C + K_S)|_C = \mathcal{O}_C(3H + 2L)$), which is at least 20.

Consider $\Sigma = \{(C, C')\}$ mapping to $\{C\}$ and $\{C'\}$. Two conics impose independent conditions on cubics, so the space of cubics containing C' has dimension $19 - 7 - 7 = 5$. Meanwhile, the space of sextics containing C' is $83 - 13 - 13 = 57$. $\{C'\}$ has dimension 16, so Σ is irreducible of dimension 78. $\Sigma \to \{C\}$ has fiber dimension 23, so $\mathcal{H}_1$ is irreducible of dimension 56. In particular, $\mathcal{H}_1$ is not a limit of curves in $\mathcal{H}_0$.

Suppose C is smooth and $C \sim 4H + 2L \subseteq S \subseteq \mathbb{P}^3$. Then we have

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{N}_{C/S} \arrow[r] \arrow[d, equal] & \mathcal{N}_{C/\mathbb{P}^3} \arrow[r] & \mathcal{N}_{S/\mathbb{P}^3}|_C \arrow[r] \arrow[d, equal] & 0 \\ & \mathcal{O}_C(4H+2L) & & \mathcal{O}_C(3) & \end{tikzcd} \tag{9.2}$$

Now $K_S = \mathcal{O}_S(-1), K_C = (C + K_S)|_C, C^2 = K_C + \mathcal{O}_C(1)$, of degree $46 + 14 = 60$, so $\mathcal{O}_C(4H+2L)$ is nonspecial. The above exact sequence is exact on global sections, and $h^0(\mathcal{O}_C(4H+2L)) = 37$ and $h^0(\mathcal{O}_C(3)) = 20$, so $\dim T_C\mathcal{H} = 57$.

In the $\mathcal{H}_2$ case, we can check that $h^0(\mathcal{O}_C(3)) = 19$.

10 Generalizing These Techniques

A first approximation to the dimension of $\mathcal{H}$ at C is the dimension of $T_C\mathcal{H}$, which equals $h^0(\mathcal{N}_{C/\mathbb{P}^r})$. This can be further approximated by $\chi(\mathcal{N}_{C/\mathbb{P}^r})$. The Euler characteristic can actually be calculated: $\mathcal{N} = \mathcal{N}_{C/\mathbb{P}^r}$ is a vector bundle of rank $r-1$, and we have the exact sequence

$$0 \to T_C \to T_{\mathbb{P}^r}|_C \to \mathcal{N}_{C/\mathbb{P}^r} \to 0 \tag{10.1}$$

with the tangent bundles having degrees $2-2g$ and $(r+1)d$, respectively, so $c_1(\mathcal{N}) = (r+1)d + 2g-2$. Riemann-Roch implies

$$\chi(N) = (r+1)d + 2g - 2 - (r-1)(g-1) = (r+1)d - (r-3)(g-1). \tag{10.2}$$

This quantity will be called $h(d,g,r)$, the expected dimension of $\mathcal{H}$. In fact, a deformation theory argument can be used to show that every component of $\mathcal{H}^0$ has dimension at least $h(d,g,r)$.

Remark. Castelnuovo implies $g \leq \pi(d,r) \sim \frac{d^2}{2(r-1)}$. In high degree and genus and $r \geq 4$, this lower bound is useless.

We say $C \subseteq \mathbb{P}^r$ is rigid if its deformations in $\mathbb{P}^r$ all arise from automorphisms of $\mathbb{P}^r$; that is, PGL_{r+1} acting on the component of $\mathcal{H}$ containing C has a dense orbit. Question: Do there exist rigid curves other than rational normal curves?

10.1 Generalizing the Parametric Approach

Here is an assertion for now: there exists a variety $\mathfrak{M}_g$ parameterizing smooth abstract curves of genus g up to isomorphism, and $\mathfrak{M}_g$ is irreducible of dimension $3g-3$ for $g \geq 2$.

To parameterize curves in $\mathcal{H}$, start with a given curve in $\mathfrak{M}_g$, then specify a map to $\mathbb{P}^r$ by a line bundle of degree d and an $(r+1)$-tuple of sections. Let

$$\mathcal{P}_{d,g} = \{(C,L) : L \in \operatorname{Pic}^d C\} \to \mathfrak{M}_g = \{C\}. \tag{10.3}$$

The fiber dimension is g, so $\mathcal{P}_{d,g}$ is irreducible of dimension $4g-3$. Also let

$$\mathfrak{g}^r_d = \{(C,L,V) : V^{r+1} \subseteq H_0(L)\} \to \mathcal{P}_{d,g}. \tag{10.4}$$

Finally, $\mathcal{H} \to \mathfrak{g}^r_d$ with fibers isomorphic to PGL_{r+1}, having dimension r^2+2r.

The problem is to find the fibers of the map $\mathfrak{g}^r_d \to \mathcal{P}_{d,g}$. Brill-Noether theory provides a answer:

$$\dim \mathfrak{g}^r_d \geq \dim \mathcal{P}_{d,g} - (r+1)(g-d+r). \tag{10.5}$$

This implies $\dim \mathcal{H} \geq 4g - 3 - (r+1)(d-g+r) + r^2 + 2r$, which exactly equals $h(d,g,r)$. Furthermore, Brill-Noether implies any component of $\mathcal{H}$ that dominates $\mathfrak{M}_g$ has dimension $h(d,g,r)$.

Conjecture 10.1. *This is true for any component of $\mathcal{H}$ whose image in $\mathfrak{M}_g$ has codimension less than $g-4$.*

10.2 The Theory of Liaison (Linkage)

We say that C, D are linked if $C \cup D$ is a complete intersection. If C is locally Cohen-Macaulay (CM) in $\mathbb{P}^3$, and S, T surfaces containing C with no common component, we define the residual curve D in $S \cap T$ as $D = V(\mathrm{Ann}(I_C/I_{S\cap T}))$. Liaison is the equivalence relation generated by $C \sim D$ when C, D are linked.

Theorem 10.2 (Gaeta)**.** *C is linked to a complete intersection if and only if C is arithmetically CM. That is, $H^0(\mathcal{O}_{\mathbb{P}^3}(m)) \to H^0(\mathcal{O}_C(m))$ is surjective for every m. Equivalently, $H^1(\mathcal{I}_C(m)) = 0$ for every m.*

Suppose $S \cap T = C \cup D$, with C Cartier on S. Then use the sequence

$$0 \to \mathcal{I}_{S/\mathbb{P}^3}(m) \to \mathcal{I}_{C/\mathbb{P}^3}(m) \to \mathcal{I}_{C/S}(m) \to 0. \tag{10.6}$$

We have $\mathcal{I}_{S/\mathbb{P}^3}(m) = \mathcal{O}_{\mathbb{P}^3}(m-s)$, and $h^1(\mathcal{O}_{\mathbb{P}^3}(m-s)) = h^2(\mathcal{O}_{\mathbb{P}^3}(m-s)) = 0$, so

$$\begin{aligned}
H^1(\mathcal{I}_{C/\mathbb{P}^3}(m)) &= H^1(\mathcal{I}_{C/S}(m)) && (10.7)\\
&= H^1(\mathcal{O}_S(m)(-C)) && (10.8)\\
&= H^1(K_S(m-s+4)(-C)) && (10.9)\\
&= H^1(\mathcal{O}_S(s+t-4-m)(-D))^* && (10.10)\\
&= H^1(\mathcal{I}_{D/\mathbb{P}^3}(s+t-4-m))^*. && (10.11)
\end{aligned}$$

So the property of being arithmetically CM (or not) is preserved by liaison.

More generally, define

$$M_C = \bigoplus_m H^1(\mathcal{I}_{C/\mathbb{P}^3}(m)) = \bigoplus_m H^0(\mathcal{O}_C(m))/(\text{restrictions of polynomials of degree } m). \tag{10.12}$$

Then M_C is a finite module over the ring $S = \mathbb{C}[X, Y, Z, W]$. Then $M_{\mathbb{C}}$ is a liaison invariant up to possible twisting and dualizing. M_C is called the Hartshorne-Rao module associated to C.

An example of non-arithmetically CM curves: If $C = L \cup L'$ is a union of two skew lines, then

$$H^1(\mathcal{I}_C(m)) = \begin{cases} \mathbb{C} & m = 0 \\ 0 & \text{otherwise} \end{cases}, \tag{10.13}$$

so $M_C = \mathbb{C}_0$. This is linked to a rational quartic curve by the intersection of a quadric and a cubic. For such a curve, $M_C = \mathbb{C}_1$.

Theorem 10.3 (Hartshorne-Rao)**.** *M defines a bijection between finite graded modules over S and liaison classes of $C \subseteq \mathbb{P}^3$.*

As an example, take C to be the union of three pairwise skew lines. Then $(M_C)_0 = \mathbb{C}^2$, $(M_C)_1 = \mathbb{C}^2$, and all other graded pieces are zero. The module structure corresponds to a map $H^0(\mathcal{O}_{\mathbb{P}^3}(1)) \to$

$\mathrm{Hom}(\mathbb{C}^2, \mathbb{C}^2)$. The locus where the determinant is zero is the dual of $Q \supseteq L_1 \cup L_2 \cup L_3$. So two triplets of skew lines are linked if and only if they lie on the same quadric.

Recall that sextic curves of genus 3 are linked to twisted cubics in a complete intersection of two cubics. So $\mathcal{H}_{6m-2,3}$ is unirational, and in particular, $\mathfrak{M}_3$ unirational. Of course, unirationality of $\mathfrak{M}_3$ could have also been deduced from the plane quartic characterization.

11 Deformation Theory

References for this section are the book by Hartshorne and the paper by Vistoli.

Suppose $X \subseteq \mathbb{P}^n$ is any scheme. A deformation of X is an étale germ of a pointed scheme (B, b) and a subscheme $\mathfrak{X} \subset \mathbb{P}^n$, flat over B ,with $\mathfrak{X} \cap (\{b\} \times \mathbb{P}^n) = X$. so $(\mathfrak{X}', B', b')$ and $(\mathfrak{X}, B, b)$ are equivalent if they agree on an étale neighborhood of the marked point.

Here are some other settings for deformations:

- $f : X \to \mathbb{P}^n$ with X fixed: a deformation of f is a germ of $\widetilde{f} : B \times X \to B$ with $\widetilde{f}|_{\{b\} \times X} = X$.
- For E a vector bundle on X, a deformation of E is $\mathcal{E}$ on $B \times X$ with $\mathcal{E}|_{\{b\} \times X} \cong E$.
- For X an abstract scheme, a deformation is $\mathfrak{X} \to B$ along with $\phi : \mathfrak{X}_b \xrightarrow{\sim} X$.

Our goal is to describe a versal deformation space for X; that is, a deformation $\mathfrak{X} \to \Delta$ with $X \to 0$ such that every deformation of X is a pullback of this one: given $\mathcal{B} \to B$ with $X \to b$, there exists $\phi : B \to \Delta$ with $b \mapsto 0$. If we have uniqueness for first order deformations, meaning $T_0\Delta$ is the space of first order deformations, we say that $\mathfrak{X}/\Delta$ is miniversal.

To approach this ,we first describe the space of first order deformations. For $X \subseteq \mathbb{P}^n$ a subscheme, this is $H^0(\mathcal{N}_{X/\mathbb{P}^n})$. For $f : X \to \mathbb{P}^n$, we get $H^0(f^*T_{\mathbb{P}^n})$. For E, it is $H^1(\mathcal{E}nd(E))$. In particular, if E is a line bundle, this space is $H^1(\mathcal{O}_X)$. Finally, for abstract X, if X is smooth, this space is $H^1(T_X)$.

The next issue is to see which first order deformations we can extend. Given a first order deformation $\mathfrak{X}$, we want to determine whether there exists

$$
\begin{tikzcd}
X \arrow[r, hook] \arrow[d] & \mathfrak{X} \arrow[r, hook, dotted, "\overset{?}{\exists}"] \arrow[d] & \mathfrak{X}' \arrow[d] \\
\operatorname{Spec}\mathbb{C} \arrow[r, hook] & \operatorname{Spec}\mathbb{C}[\epsilon]/(\epsilon^2) \arrow[r, hook] & \operatorname{Spec}\mathbb{C}[\epsilon]/(\epsilon^3)
\end{tikzcd}
\tag{11.1}
$$

Let $\mathrm{Def}(X)$ denote the space of first order deformations of X. We have an addition in $\mathrm{Def}(X)$ because if $\mathbb{A}^2_{x,y}$, if $X_1 = V(x, y^2)$ and $X_2 = V(x^2, y)$, then $X_1 \cup X_2 = V(x^2, xy, y^2)$, then restrict to the sum of the two tangent vectors. This can be generalized since the union of two tangent vectors contains all tangent vectors. So $\mathrm{Def}(X)$ is a vector space.

Now introduce a second vector space $\mathrm{Obs}(X)$, the "obstruction space". We have a map $\phi_2 : \mathrm{Obs} \to \mathrm{Sym}^2(\mathrm{Def}^*)$, the space of homogeneous quadratic polynomials on Def, such that $\eta \in \mathrm{Def}$ extends to second order if and only if $\phi_2(\tau)(\eta) = 0$ for every $\tau \in \mathrm{Obs}$.

For $X \subseteq \mathbb{P}^n$, we have $\mathrm{Obs} = H^1(\mathcal{N}_{X/\mathbb{P}^n})$. For f, it is given by $H^1(f^*T_{\mathbb{P}^n})$. For E, it is $H^2(\mathcal{E}nd(E))$. And for X abstract, it is $H^2(T_X)$.

We get a sequence of maps $\phi_3 : \ker\phi_2 \to \mathrm{Sym}^3(\mathrm{Def}^*)/(\mathrm{im}\,\phi_2)$, and more generally, $\phi_k : \ker\phi_{k-1} \to \mathrm{Sym}^k(\mathrm{Def}^*)/(\mathrm{im}\,\phi_2, \dots, \mathrm{im}\,\phi_{k-1})$. The images generate a homogeneous ideal $I \subseteq \mathrm{Sym}^*(\mathrm{Def}^*)$, and $\eta \in \mathrm{Def}(X)$ is infinitely extendable if and only if $\eta \in V(I)$.

Theorem 11.1 (Artin Approximation)**.** *Suppose we are given $\mathfrak{X}_k \to \operatorname{Spec}\mathbb{C}[\epsilon]/(\epsilon^k)$ with $\mathfrak{X}_k \cap \operatorname{Spec}\mathbb{C}[\epsilon]/(\epsilon^{k-1}) = \mathfrak{X}_{k-1}$. Then for each k, there exists an actual deformation over a smooth curve that agrees with the $\mathfrak{X}_k$.*

In particular, if $\mathfrak{X} \to \Delta$ is miniversal, then the tangent cone to Δ at 0 is $V(I)$. This implies

$$\dim\Delta = \dim V(I) \geq \dim\mathrm{Def} - \dim\mathrm{Obs}. \tag{11.2}$$

So if $X \subseteq \mathbb{P}^n$ is a curve, we get

$$\dim_X \mathcal{H} \geq h^0(\mathcal{N}) - h^1(\mathcal{N}) = \chi(\mathcal{N}). \tag{11.3}$$

As a special case, if $\mathrm{Obs} = 0$, then Δ is smooth and $\dim\Delta = \dim\mathrm{Def}$. The problem is that the ϕ_k are too difficult to get a handle on.

As an example, recall $\mathcal{H} = \mathcal{H}_{3m+1,3}$ the Hilbert scheme parameterizing twisted cubics. We have $\mathcal{H} = \mathcal{H}_0 \cup \mathcal{H}_1$, where $\mathcal{H}_0$ is the closure of the locus of twisted cubics and $\mathcal{H}_1$ is the closure of the locus of (plane cubic plus point)'s. Along the locus of twisted cubics (and in $\mathcal{H}_0 \setminus \mathcal{H}_1$), we have $h^0(\mathcal{N}) = 12$ and $h^1(\mathcal{N}) = 0$, so for $C \in \mathcal{H}_0 \setminus \mathcal{H}_1$, C is a smooth point. But for $C \in \mathcal{H}_0 \cap \mathcal{H}_1$, we have $h^0(\mathcal{N}_{C/\mathbb{P}^3}) = 16$ and $h^1(\mathcal{N}_{C/\mathbb{P}^3}) = 4$. We conclude that $\dim T_C\mathcal{H} = 16$.

We want to know whether $\mathcal{H}_0$ is smooth. Of course $\mathcal{H}$ isn't, but only because the intersection necessitates it. To resolve this, Piene and Schlesinger calculated ϕ_2 for $C \in \mathcal{H}_0 \cap \mathcal{H}_1$ and found (in terms of $x_1, \dots, x_{16}$ for Def)

$$\mathrm{im}\,\phi_2 = \langle x_1x_2, x_1x_3, x_1x_4, x_1x_5\rangle. \tag{11.4}$$

So the tangent cone to $\mathcal{H}$ is $V(x_1) \cap V(x_2, x_3, x_4, x_5)$ This implies the two components are both smooth.

Remark. It is possible for Obs to be nonzero but all ϕ_k to be zero. For example, $C = S \cap T \subseteq \mathbb{P}^3$. Then $K_C = \mathcal{O}_C(s+t-4)$ and $\mathcal{N}_{C/\mathbb{P}^3} = \mathcal{N}_{C/S} \oplus \mathcal{N}_{C/T} = \mathcal{O}(t) \oplus \mathcal{O}(s)$. If either of s or t is at least 4, then $h^1(\mathcal{N}) \neq 0$. Meanwhile, if $s = t$, then $\mathcal{H}^0$ is open in $G(2, H^0(\mathcal{O}_{\mathbb{P}^3}(s)))$. The dimension of the Grassmannian is $2\binom{s+3}{3} - 4$. But $h^0(\mathcal{N}_{C/\mathbb{P}^3}) = 2h^0(\mathcal{O}_C(s)) = 2(\binom{s+3}{3} - 2)$. So no obstructions exist.

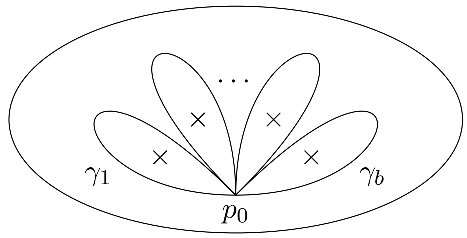
12 Hurwitz Spaces

Fix d and g, and let $\mathfrak{H}^0_{d,g}$ be the space of simply branched covers $f : C \to \mathbb{P}^1$ of degree d (simply branched means that the branch divisor is reduced of degree $b = 2d+2g-2$, so C must be smooth).

We have a map $\mathfrak{H}^0_{d,g} \xrightarrow{\pi} \mathbb{P}^b \setminus \Delta$, corresponding to the distinct set of b points in $\mathbb{P}^1$.

Claim. This map is finite, and can give $\mathfrak{H}$ a structure of a variety such that the map is étale.

To find the preimage of a set of branch points, we find a cover of $\mathbb{P}^1$ with b points removed:



(12.1)

Label the points over p_0 in C by $p_1, \ldots, p_b$. We then get $\sigma_i \in S_d$ by the monodromy along γ_i. Simply branched forces the σ_i to be transpositions. Connectedness implys the monodromy group generated by the σ_i is transitive. We also have $\prod_i \sigma_i = \mathbf{1}$. So a fiber of π is given by

$$\left\{(\sigma_1, \ldots, \sigma_b) : \sigma_i \text{ transpositions}, \langle \sigma_1, \ldots, \sigma_b \rangle = S_d, \prod \sigma_i = \mathbf{1}\right\} / (\text{simultaneous conjugation}). \tag{12.2}$$

We find that $\mathfrak{H}^0_{d,g}$ is smooth of dimension $2d + 2g - 2$. Observe that this equals $h(d, g, 1)$.

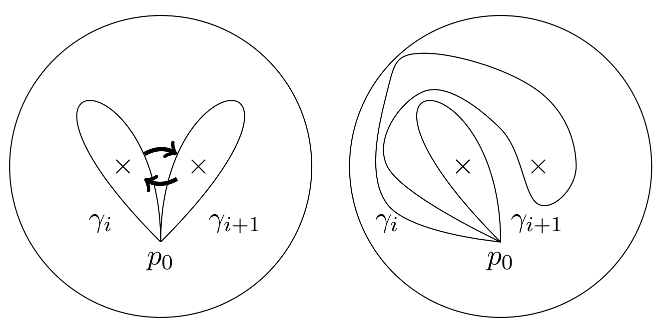
We can now understand abstract curves C using $\mathfrak{H}^0_{d,g}$:

$$\begin{tikzcd} & \mathbb{P}^b \setminus \Delta \\ \mathfrak{H}^0_{d,g} \arrow[ur, "\pi"] \arrow[dr, "\phi"'] & \\ & \mathfrak{M}_g \end{tikzcd} \tag{12.3}$$

If $d \geq 2g + 1$ (and we can improve this), then ϕ is surjective.

Theorem 12.1 (Clebsch)**.** *$\mathfrak{H}^0_{d,g}$ is connected, hence irreducible. Therefore $\mathfrak{M}_g$ is irreducible.*

To prove this, we want to determine the monodromy of π.



(12.4)

So $(\sigma_1, \ldots, \sigma_b) \rightsquigarrow (\sigma_1, \ldots, \sigma_{i-1}, \sigma_{i+1}, \sigma_{i+1}^{-1} \sigma_i \sigma_{i+1}, \sigma_{i+2}, \ldots, \sigma_b)$.

($\mathfrak{H}^0_{d,g}$ is called the "small Hurwitz space"; it isn't compact. Other conventions may be to mod out by $\mathrm{Aut}(\mathbb{P}^1)$, or to mark the branch points.)

For $\pi : \mathfrak{H}^0_{d,g} \to \mathbb{P}^1 \setminus \Delta$, $\deg \pi$ is called the Hurwitz number.

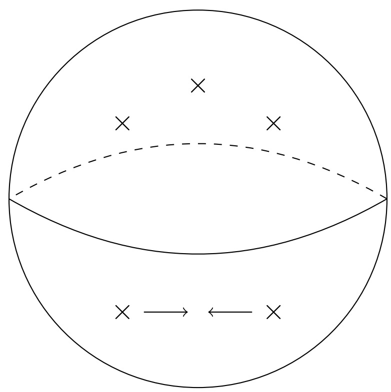
Claim. Given any $(\sigma_1, \dots, \sigma_b)$ satisfying the appropriate conditions, then afer applying the above transformation, we may arrive at the normal form

$$\underbrace{(1\ 2), (1\ 2), \dots, (1\ 2)}_{2g+2}, (2\ 3), (2\ 3), (3\ 4), (3\ 4), \dots (d-1\ d), (d-1\ d). \tag{12.5}$$

(We can get rid of high numbers by $(a\ b), (a\ b) \mapsto -, (a\ b)(a\ c) \mapsto (a\ c), (b\ c)$.)

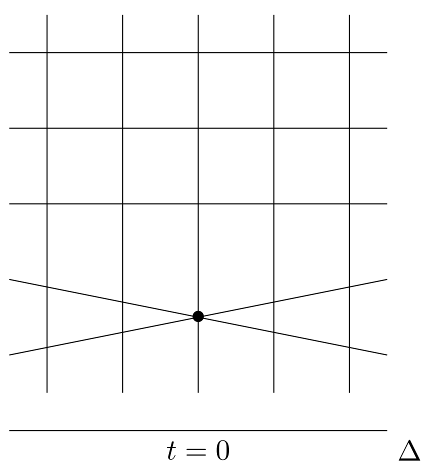
12.1 Aspects About the Geometry of $\mathfrak{H}^0_{d,g}$

If you try to compactify by letting the branch points come together, the curve C can be very singular - we have little control over the result. Here is a better approach: as the branch points come together,



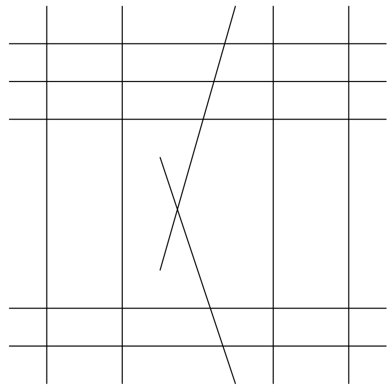
(12.6)

the family of curves and sections looks like



(12.7)

We can then blow up at the intersection point to obtain



E (12.8)

Compactify $\mathfrak{H}^0_{d,g}$ to the space $\mathfrak{H}^{d,g}$ of admissible covers: covers of nodal curves of arithmetic genus 0. The branch points still never come together.

We will also consider the Maroni stratification and divisor class theory.

13 Severi Varieties

Our goal here is to parameterize the set of maps $f : C \to \mathbb{P}^2$ with C smooth of genus g and f of degree d. For now, we'll focus on the image curves. Let $\mathbb{P}^N$ denote the set of plane curves of degree d (specifically $N = \binom{d+2}{2} - 1$), and consider $V_{d,g}$ parameterizing reduced, irreducible plane curves of degree d and geometric genus g. We have $V_{d,g} \subseteq \overline{V}_{d,g} \subseteq \mathbb{P}^N$. Inside $V_{d,g}$, there is $U_{d,g}$ consisting of nodal curves. Equivalently, write $U^{d,\delta}$ for curves having δ nodes, where $\delta = \binom{d-1}{2} - g$.

Basic facts about Severi varieties:

- $U_{d,g}$ is smooth of dimension $N - \delta = 3d + g - 1 = h(d, g, 2)$.
- $U_{d,g}$ is dense in $V_{d,g}$.
- $U_{d,g}$ is irreducible.

We'll be able to prove the first assertion. First consider the case $\delta = 1$. Let

$$\Phi = \{(C, p) : C \text{ singular at } p\} \subseteq \mathbb{P}^N \times \mathbb{P}^2. \tag{13.1}$$

Then Φ has natural maps to $V = \overline{V}^{d,1}$ and to $\mathbb{P}^2$. We can write

$$\Phi = \left\{ (a_{ij}, x, y) : \underbrace{\sum a_{ij} x^i y^j}_{F} = \underbrace{\sum i a_{ij} x^{i-1} j}_{G} = \underbrace{\sum j a_{ij} x^i y^{j-1}}_{H} = 0 \right\}. \tag{13.2}$$

Our first claim is that if C has a node at p, then C is smooth at (C, p). To do this, we need to find an invertible 3×3 submatrix of the Jacobian. We'll consider the submatrix

$$\begin{pmatrix} \frac{\partial F}{\partial x} & \frac{\partial G}{\partial x} & \frac{\partial H}{\partial x} \\ \frac{\partial F}{\partial y} & \frac{\partial G}{\partial y} & \frac{\partial H}{\partial y} \\ \frac{\partial F}{\partial a_{00}} & \frac{\partial G}{\partial a_{00}} & \frac{\partial H}{\partial a_{00}} \end{pmatrix}. \tag{13.3}$$

Take $p = (0, 0)$, so that $a_{00} = a_{10} = a_{01} = 0$. The above matrix becomes

$$\begin{pmatrix} 0 & a_{20} & a_{11} \\ 0 & a_{11} & a_{02} \\ 1 & 0 & 0 \end{pmatrix}. \tag{13.4}$$

Because C has a node at p, $\det \left(\begin{smallmatrix} a_{20} & a_{11} \\ a_{11} & a_{02} \end{smallmatrix} \right) \neq 0$. We conclude that Φ is smooth of dimension $N - 1$ at (C, p). In addition, this lets us conclude that $\pi : \Phi \to \mathbb{P}^N$ is an immersion at (C, p), and the image of the tangent space is $a_{00} = 0$. Hence if C is a plane curve with a node at p and no other singularities, then V is smooth at C with tangent hyperplane $\{B : p \in B\}$.

Now consider the case of arbitrary δ. Now suppose C has nodes at $p_1, \ldots, p_\delta$ and no other singularities. There are then δ points of Φ lying over C. We have $\binom{d-1}{2} - \delta = g(C) = h^0(K_{\widetilde{C}})$.

If $\widetilde{C} \to C = V(f)$, then $\omega = g(x, y) \frac{dx}{\partial f / \partial y}$ is a regular differential on $\widetilde{C}$ if and only if g has degree at most $d - 3$ and $g(p_\alpha) = 0$ for $\alpha = 1, \ldots, \delta$. Now $H^0(K_{\widetilde{C}})$ is the space of polynomials of degree at most $d - 3$ vanishing at $p_1, \ldots, p_\delta$, having dimension at least $\binom{d-1}{2} - \delta$. We force equality, so $p_1, \ldots, p_\delta$ impose independent conditions on polynomials of degree $d-3$, so a fortiori on polynomials of degree d. This implies $\overline{V}^{d,1}$ has normal crossings at C. Now in a neighborhood of C, $V^{d,\delta}$ is smooth of codimension δ in $\mathbb{P}^N$ and has tangent space $\{B : p_1, \ldots, p_\delta\}$ at C.

14 Final Remarks Regarding Hilbert Schemes

Let $\widetilde{H}_{d,g,r}$ denote the Hurwitz space of simply branched covers if $r = 1$, the Severi variety of nodal curves if $r = 2$, and the Hilbert scheme of smooth curves otherwise. The expected dimension is $h(d, g, r)$. For $r = 1, 2$, this scheme is always of the expected dimension, always smooth, and always irreducible.

We would like to compactify these spaces. For $r \geq 2$, there is an immediate compactification, but it is ugly. For $r = 1$, there is a nice compactification of admissible covers. One problem is to find a better compactification if $r \geq 2$. (Even Kontsevich spaces have issues.)

A conjecture is that $\operatorname{Pic} \widetilde{H}_{d,g,r} \otimes \mathbb{Q} = 0$ for $r = 1, 2$. This is possibly also true for $r \geq 3$ if d is large and for components of H dominating moduli.

15 Moduli Spaces

WE would hope to have a fine moduli space $\mathfrak{M}_g$ for smooth projective curves of genus g. That is, we want a scheme $\mathfrak{M}_g$ and, for every B, a natural bijection between families of curve of genus g over B and morphisms from B to $\mathfrak{M}_g$. Natural means commuting with base change. The property of being a fine moduli space would imply an isomorphism of functors $\mathrm{Sch} \to \mathrm{Set}, F \cong \mathrm{Mor}(-, \mathfrak{M}_g)$ for $F : B \to \{\text{families}\}$.

We'll first look at the case $g = 1$. Given C of genus 1 and $p \in C$, $|2p|$ give a map $C \to \mathbb{P}^1$ which is a double cover branched at four points. After sending three of the points to $0, 1, \infty$ by an automorphism of $\mathbb{P}^1$, we obtain the equation

$$C_\lambda : y^2 = x(x-1)(x-\lambda). \tag{15.1}$$

A different choice of these points to send takes C_λ to $C_{\lambda'}$ for

$$\lambda' \in \left\{ \lambda, \frac{1}{\lambda}, 1-\lambda, \frac{1}{1-\lambda}, \frac{\lambda-1}{\lambda}, \frac{\lambda}{\lambda-1} \right\}. \tag{15.2}$$

S_3 acts on $U = \mathbb{P}^1 \setminus \{0, 1, \infty\}$, and set $j(\lambda) = 256\frac{(\lambda^2-\lambda+1)^3}{(\lambda^2(\lambda-1)^2}$, a rational function invariant under S_3, so we get a map

$$\begin{tikzcd} U \arrow[rr, "j"] \arrow[dr] & & \mathbb{A}^1_j \\ & U/S_3 \arrow[ur, "\sim"'] & \end{tikzcd} \tag{15.3}$$

Take $\mathfrak{M}_1 = \mathbb{A}^1_j$, but $\mathfrak{M}_1$ is not a fine moduli space! We do have that $\mathfrak{M}_1(\mathbb{C})$ is in biejction with isomorphism classes of genus 1 curves, and also for a family $\mathcal{C} \to B$ of curves of genus 1, we get a map $B \to \mathfrak{M}_1$ via $y^2 = (x-a)(x-b)(x-c)(x-d)$ for $a, b, c, d \in \mathcal{O}_B(\Delta)$ (for Δ some étale open set), or $y^2 = x(x-1)(x-\lambda)$ for $\lambda \in \mathcal{O}_B(\Delta)$, so we get a map $\Delta \to U \xrightarrow{j} \mathbb{A}^1_j$.

We have obtained a natural transformation ϕ of functors $F \to \mathrm{Mor}(-, \mathfrak{M}_1)$, which is a bijection on $B = \operatorname{Spec} \mathbb{C}$, but it is not an ismorphism of functors.

A problem is that j is triply ramified over $j = 0$. So if j is the j-function of an actual family, then all zeros of j must have order divisible by 4. Also all zeros of $j - 1728$ have order divisible by 2. Finally, there's another (global) obstruction to a map to $\mathfrak{M}_1$ to come from a family.

The above shows the natural transformation isn't surjective. It's not injective, either: choose E of genus 1 with a marked point. Let ι be multiplication by -1. Also let $B' \to B$ be an unramified double cover and consider $(B' \times E)/(\tau\iota) \to B'/(\tau) = B$. This is a nontrivial family of genus 1 curves over B whose associated map $B \to \mathfrak{M}_1$ is constant.

Suppose $F : \mathrm{Sch}/\mathbb{C} \to \mathrm{Set}$ is a moduli functor. Then a scheme $\mathfrak{M}$ is a Deligne-Mumford moduli space for F if:

- there exist natural transformations $\phi : F \to \mathrm{Mor}(-, \mathfrak{M})$
- $\phi_{\operatorname{Spec} \mathbb{C}}$ is a bijection
- ϕ is an isomorphism up to finite covers. This last point means:
  - Given $f : B \to \mathfrak{M}$, there is a finite cover $i : B' \to B$ such that $f \circ i \in \phi_{B'}$.
  - Given $\mathcal{C} \to B$ and $\mathcal{D} \to B$ such that $\phi(\mathcal{C}) = \phi(\mathcal{D})$, then there is a finite cover B' such that $\mathcal{C} \times_B B'$ is isomorphic to $\mathcal{D} \times_B B'$ over B.

Theorem 15.1 (Deligne-Mumford, '69)**.** *There exists a Deligne-Mumford moduli space $\mathfrak{M}_g$ for smooth curves of genus g.*

What we would like to know about $\mathfrak{M}_g$:

- Basic properties: $\mathfrak{M}_g$ is irreducible of dimension $3g-3$ (if $g \geq 2$).
- Can generalize to $\mathfrak{M}_{g,n}$, a Deligne-Mumford moduli space for n-pointed curves (curves with n distant ordered marked points)
- How do we handle the failure of $\mathfrak{M}_g$ to be a fine moduli space?
- How is $\mathfrak{M}_g$ constructed? Furthermore, how do we compactify $\mathfrak{M}_g$?

A family of n-pointed curves over B is $\mathcal{C} \to B$ with sections $\sigma_1, \ldots, \sigma_n$ having disjoint images.

To establish basic properties of $\mathfrak{M}_g$, we look at the map $\mathfrak{H}^0_{d,g} \xrightarrow{\pi} \mathfrak{M}_g$. If $d >> g$ (specifically $d \geq 2g+1$), then π is surjective.

Given C of genus g, $\pi^{-1}(C)$ is an open subset of the set of rational functions f on C of degree d. To choose such a rational function, choose $D = (f)_\infty$ (there is a d-dimensional family of choices), and then choose $f \in \mathcal{L}(D)$ (Riemann-Roch implies the dimension is $d-g+1$). So

$$\dim \pi^{-1}(C) = 2d - g + 1 \implies \dim \mathfrak{M}_g = (2d+2g-2) - (2d-g+1) = 3g-3. \tag{15.4}$$

Remark. We also have $\dim \mathfrak{M}_{g,n} = 3g-3+n$, as long as $\operatorname{Aut}(C, p_1, \ldots, p_n)$ is finite. That is, $g \geq 2$, or $g = 1$ and $n \geq 1$ or $g = 0$ and $n \geq 3$,

To deal with the fact that $\mathfrak{M}_g$ is not a fine moduli space (and there is not a universal family), there are three things that we can do:

- restrict to $\mathfrak{M}^0_g$, the subset of automorphism-free curves, an open subset whose complement has codimension $g-2$. $\mathfrak{M}^0_g$ is a fine moduli space for automorphism-free curves.
- rigidify: for example consider curves with level structure, such as

  $$\{(C, \sigma_1, \ldots, \sigma_{2g})\} \text{ where } \sigma_i \text{ form a symplectic basis for } H^1(C, \mathbb{Z}/m).$$

  If $m \geq 3$, then $(C, \sigma_1, \ldots, \sigma_{2g})$ has no automorphisms. (We could also look at partial level structure.) There exists a fine moduli space $\mathfrak{M}^{[m]}_g$ for such objects, and this is a finite cover of $\mathfrak{M}_g$.
- Use stacks: enlarge the category of schemes to make $\mathfrak{M}_g$ a fine moduli stack. (We won't go into detail here.)

Compactification of $\mathfrak{M}_g$

We want a projective $\overline{\mathfrak{M}_g}$ containing $\mathfrak{M}_g$ as an open subset, sch that $\overline{\mathfrak{M}_g}$ is a moduli space for some larger class of objects.

Recall the valuative criterion for properness: for Δ a disc (or Spec of a DVR), Δ^* the punctured disc or generic fiber, then X is proper if and only if for all maps $\Delta^* \to X$, there exists a unique extension $\Delta \to X$.

Sow we want that for every family $\mathcal{C}^* \to \Delta^*$ of smooth curves, then after a finite base change $\Delta^* \to \Delta^*$, there is a unique extension to a family $\mathcal{C} \to \Delta$ of curves of this enlarged class.

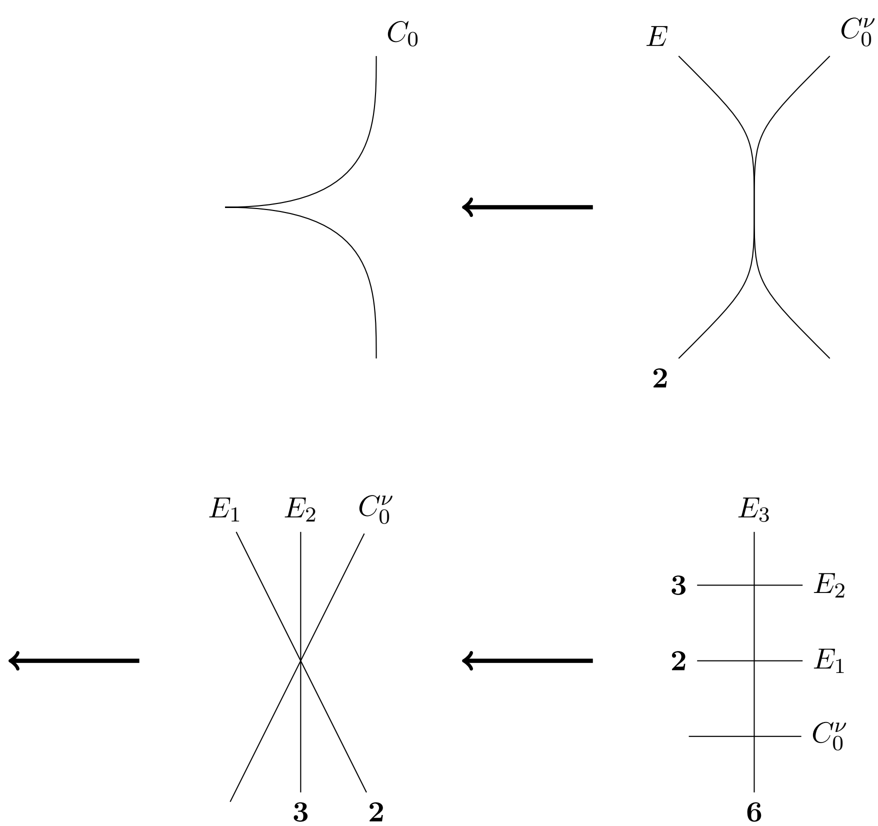
A connected curve C is stable if C has only nodes as singularities and $\mathrm{Aut}(C)$ is finite.

Theorem 15.2. *There exists a Deligne-Mumford moduli space $\overline{\mathfrak{M}_g}$ for the class of stable curves of arithmetic genus g, and $\overline{\mathfrak{M}_g}$ is projective.*

To do I think I missed this lecture. Try to fill it in. (2)

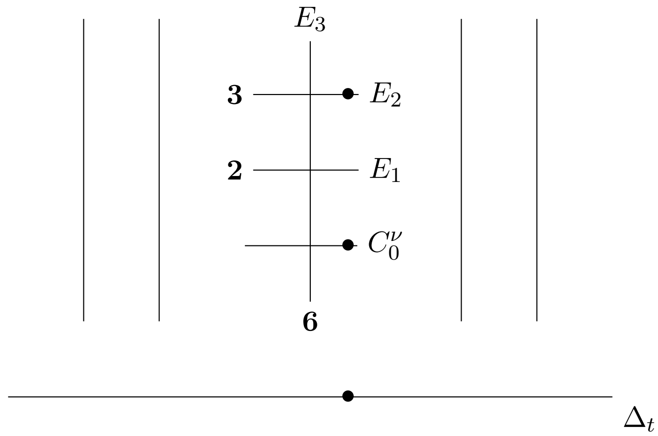
16 Examples of Stable Reduction

Suppose we have $\mathcal{C} \to \Delta$ with $\mathcal{C}$ smooth and C_0 having a cusp. After blowing up the cusp, we obtain a tacnode. Blowing up the tacnode gives E_1 and C_0^ν intersecting E_2 at some point, then blow up the triple point.



(16.1)

However, the special fiber is now highly nonreduced. The next step involves base change and normalization.



(16.2)

If $p \in D \subseteq C_0$ with D appearing in C_0 with multiplicity 1, then (x, y, t) is given by $y = t$. Introduce s with $s^2 = t$, then we get $(x, y, s) : y = s^2$. This is a smooth branched cover of $\mathcal{C}$ which is branched along D.

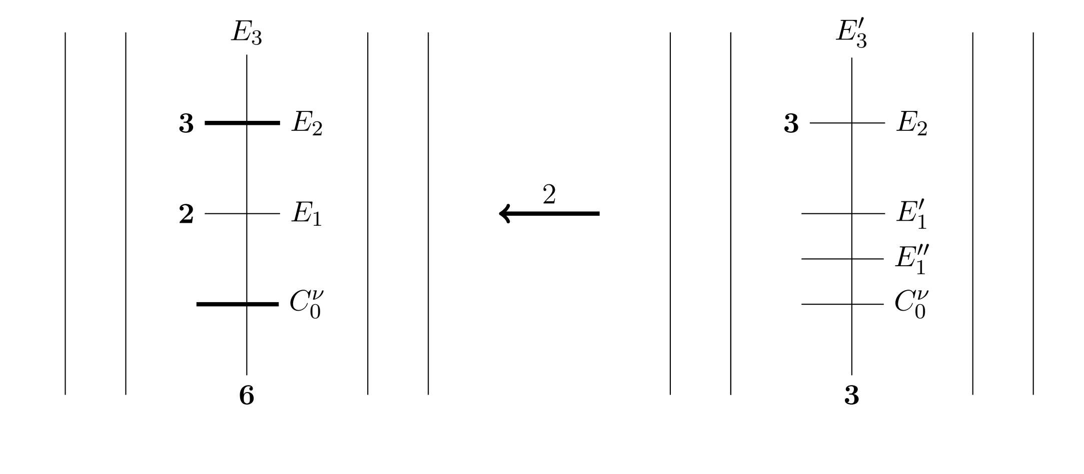
If D appears in C_0 with multiplicity 2, we have $t = y^2$. After a base change, we get $s^2 = y^2$, and then have two sheets intersecting transversally along D. After normalizing, we get an unbranched double cover.

If the multiplicity is 3, then we get $t = y^3$, so $s^2 = y^3$ (a family of cusps), and then normalizing gives a smooth surface branched along D.

In summary, the effect of base change of order 2 and normalization is to replace $\mathcal{C}$ by a double cover of $\mathcal{C}$ branched along the components of $\mathcal{C}$ with odd multiplicity. If D has odd multiplicity in C_0, then the multiplicity is the same in the new family. But if D has even multiplicity in C_0, then the preimage is a double cover of D, so the multiplicity is halved.

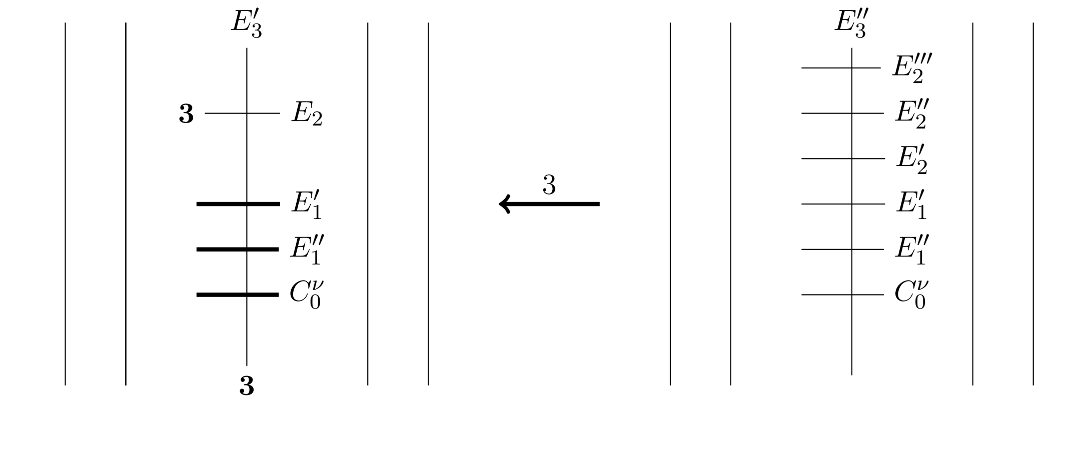
This statement generalizes to a base change of order p for p a prime ($t = s^p$).

Applying base changes gives



(16.3)

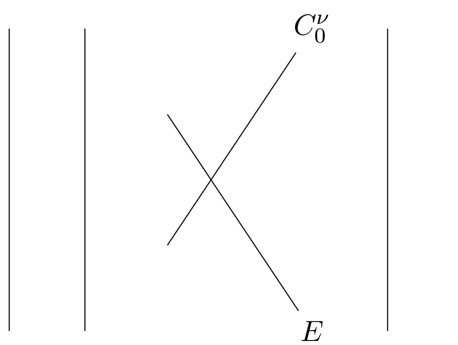
E_3' is a double cover of E_3 branched at 2 points, so is rational. An unramified double cover of $\mathbb{P}^1$ is two disjoint $\mathbb{P}^1$'s.



(16.4)

E_3'' is a cyclic triple cover of E_3' branched at three points, so is isomorphic to $y^3 = x^3 - 1$, an elliptic curve with $j = 0$. This is true because $\mathcal{C}$ is smooth.

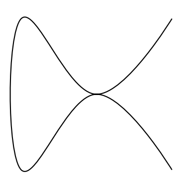
The final step is to blow down E_2', E_2'', E_1', E_1''. We arrive at the stable reduction



(16.5)

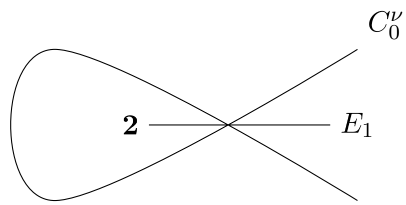
We say that E is an elliptic tail.

Next consider an example where $\mathcal{C}$ is smooth and C_0 has a tacnode.



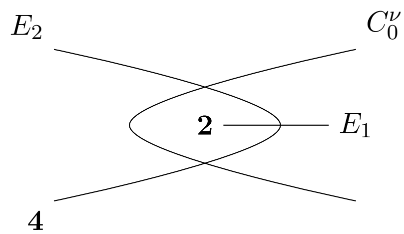
(16.6)

To resolve this, first blow up:



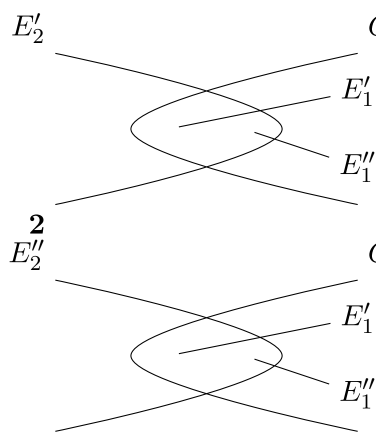
(16.7)

Now E_1 intersects C_0^ν at the node, so blow up again:



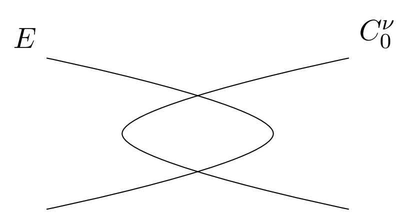
(16.8)

Now perform two base changes of order 2 and normalization:



(16.9)

Finally, blow down the rational tails:

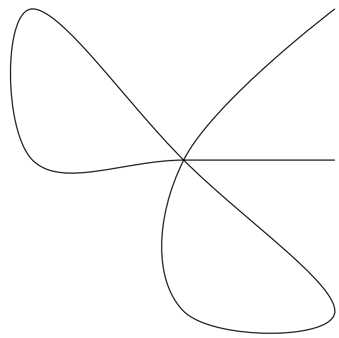


(16.10)

We have $E_2' \cong \mathbb{P}^1$, E_2'' an elliptic curve, and $E = E_2''$, called an elliptic bridge. Look at the branch points of $E \to \mathbb{P}^1$: we have $j(E) = 1728$ if $\mathcal{C}$ is smooth.

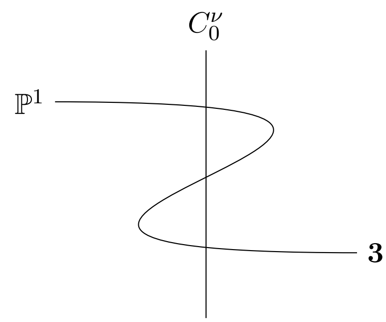
To do Figure this one out. (3)

We'll also look at a triple point.



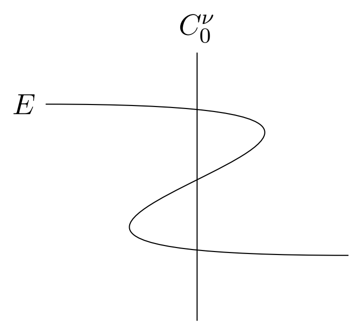
(16.11)

Blowing up gives



(16.12)

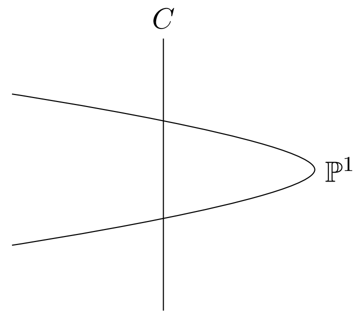
Finally, base change and normalization gives us the stable reduction



(16.13)

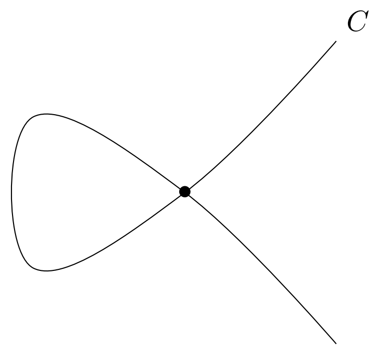
with $j(E) = 0$.

Suppose we ended up with



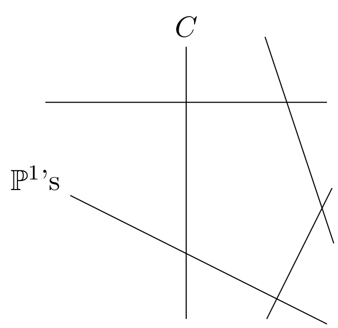
(16.14)

a semistable curve. We can still blow down because the $\mathbb{P}^1$ has self-intersection number -2. We end up with



(16.15)

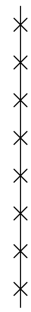
with local equation $xy = t^2$. More generally, $xy = t^k$ arises from



(16.16)

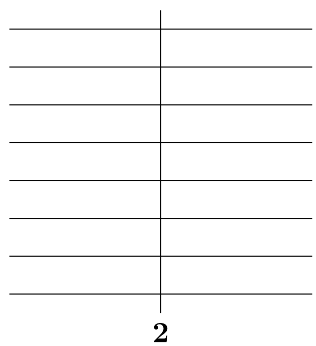
We also need to consider uniqueness of stable limits.

Finally, what if the curve is nonreduced? The last example we'll look at is a family of plane quartics specializing to a double conic. That is, we have a quadric $Q(X,Y,Z)$ and a quartic $F(X,Y,Z)$, and we'll consider the family $V(Q^2 + tF) \subseteq \Delta \times \mathbb{P}^2$.



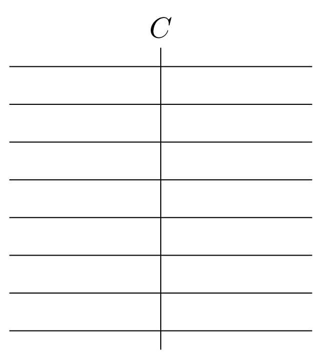
(16.17)

The total space is not smooth; it's singular at $t = 0$ and $Q = F = 0$ (8 points). Blowing up the total space gives



(16.18)

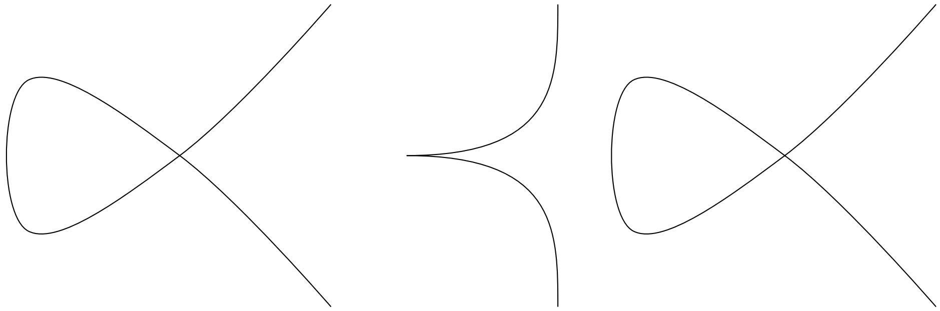
with 8 $\mathbb{P}^1$'s as tails. Now we have a smooth total space and set-theoretic normal crossings. We can proceed as before:



(16.19)

C is a double cover of $V(Q) \cong \mathbb{P}^1$ branched at 8 points. So it is a hyperelliptic curve of genus 3. Blowing down, we get the curve C as the special fiber of the stable reduction. Observe that C is a hyperelliptic limit of quartic curves.

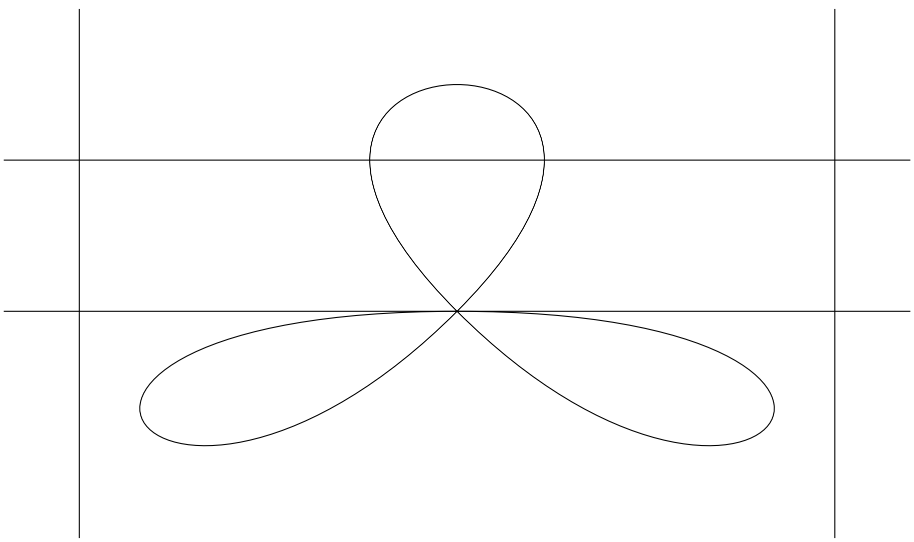
If the general fiber is singular:



(16.20)

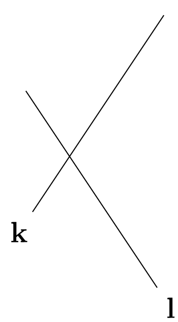
then normalize the general fiber

Fix Me Is this picture actually correct? (4)



(16.21)

Finally, in a situation such as

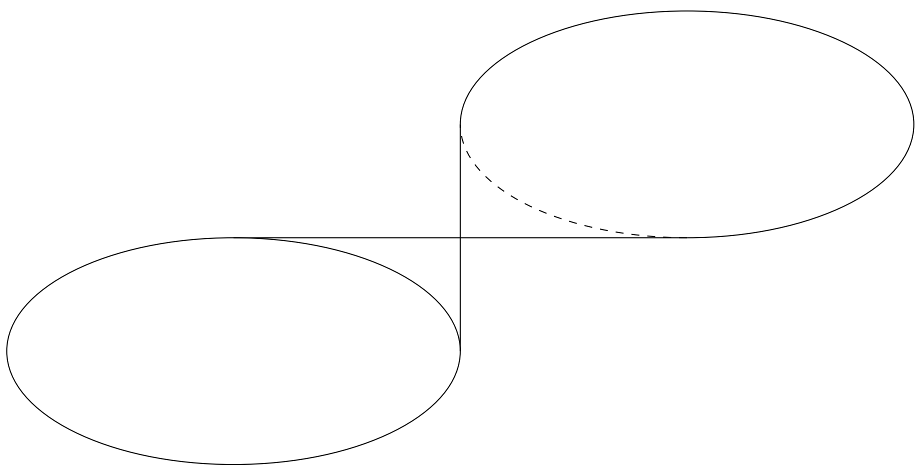


(16.22)

if $p \nmid k\ell$, then we have $z^p = x^k y^\ell$ after base change. We get a singularity, which we would need to resolve.

If $\mathcal{C} \to \Delta$ is a 1-parameter family of nodal curves and $p \in C_0$ is a node of C_0, then there exist local coordinates on $\mathcal{C}$ near p given by $\mathcal{C} = V(xy - t^k)$ for some k. For $k \geq 2$, this is called an A_{k-1} singularity.

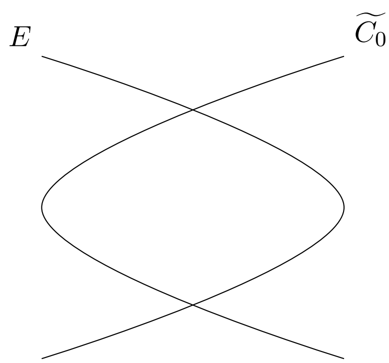
For $k = 2$:



(16.23)

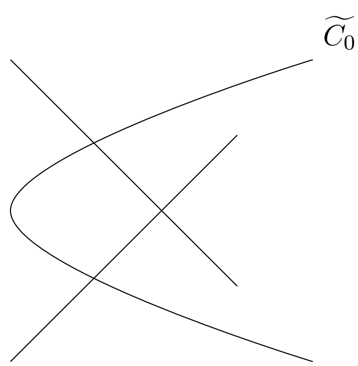
Fix Me Make this look better? (It will be tough!) (5)

Blowing up gives



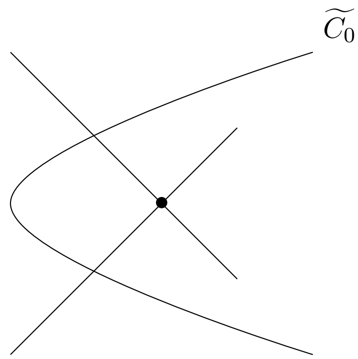
(16.24)

For $k = 3$: we have $xy = t^3$. The tangent cone is a union of two lines. Blowing up gives



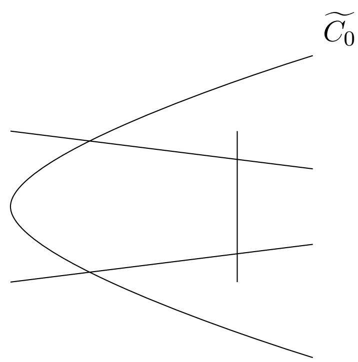
(16.25)

For $k \geq 4$, blowing up gives



(16.26)

This has local equation $xy = t^{k-2}$, so we can repeat. We'll end up with



(16.27)

with $k-1$ $\mathbb{P}^1$'s. This curve isn't stable.

16.1 Variants of Stable Reduction

We say C is semistable if C is nodal and every smooth rational component of C meets the rest of C in at least two points.

Theorem 16.1. *Given any family $\mathcal{C} \to \Delta$ with general fiber smooth, then after a finite base change, there exists $\mathcal{C}' \to \Delta$ with all fibers semistable and $\mathcal{C}'$ smooth.*

Given a family

$$\begin{tikzcd} \mathcal{C} \arrow[rr, hook] \arrow[dr] & & \Delta \times \mathbb{P}^n \arrow[dl] \\ & \Delta & \end{tikzcd} \tag{16.28}$$

with general fiber smooth:

Theorem 16.2. *After a finite base change, there exists*

$$\begin{tikzcd} \mathcal{C}' \arrow[rr, hook] \arrow[dr] & & \mathcal{C} \arrow[dl] \\ & \Delta & \end{tikzcd} \tag{16.29}$$

such that all fibers of $\mathcal{C}' \to \Delta$ are nodal.

(Also observe that $\mathcal{C}' \to \mathcal{C}$, so this extends a family of curves in $\mathbb{P}^n$.)

17 Geometry of Singular Curves

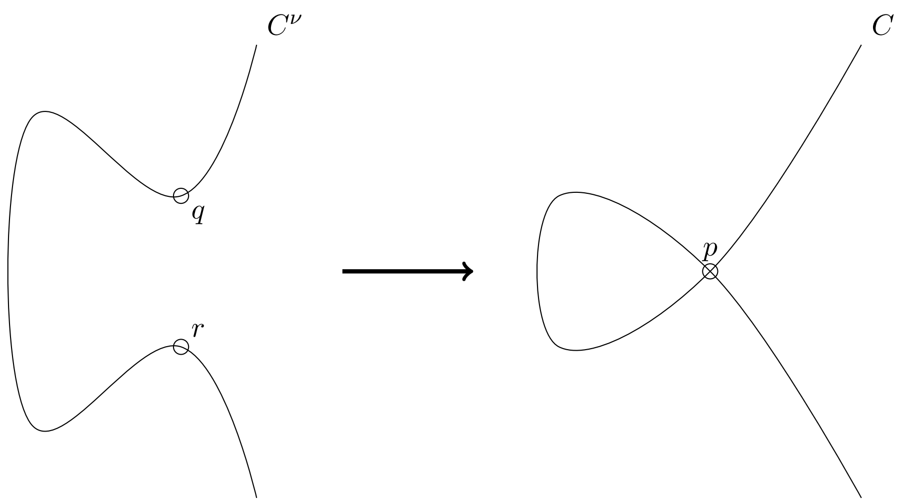
17.1 The δ-invariant of a Singularity

Suppose C is reduced with an isolated singularity at p. Let C^ν be the normalization of C at p. We have a map $C^\nu \xrightarrow{\pi} C$, and want to compare the arithmetic genera: that is, compare $p_a(C^\nu)$ with $p_a(C)$. To do this, we look at the sequence

$$0 \to \mathcal{O}_C \to \pi_* \mathcal{O}_{C^\nu} \to \mathcal{F}_p \to 0. \tag{17.1}$$

$\mathcal{F}_p$ is a skyscraper sheaf supported at p, which depends on the singularity at p.

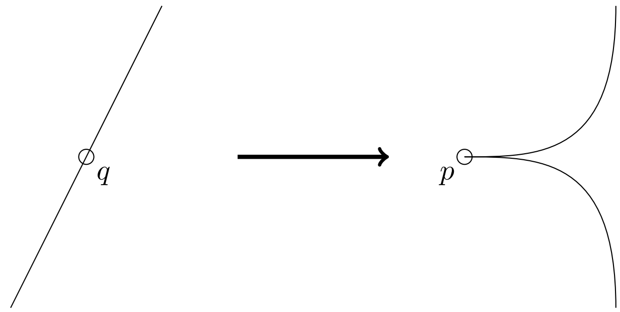
For p a node: $\mathcal{F}_p$ has rank 1.



(17.2)

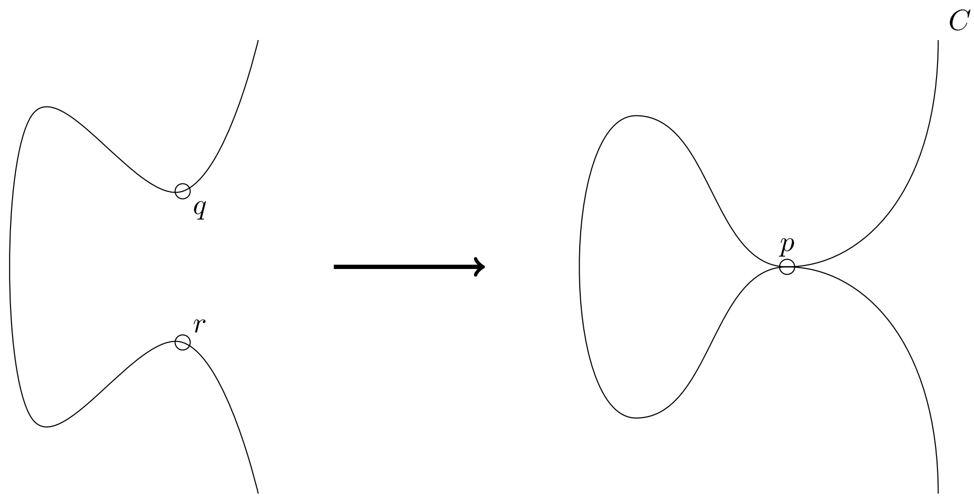
In the above picture, f defined near q, r descends to C if and only if $f(q) = f(r)$.

For p a cusp, the rank is 1. We need $f'(q) = 0$.



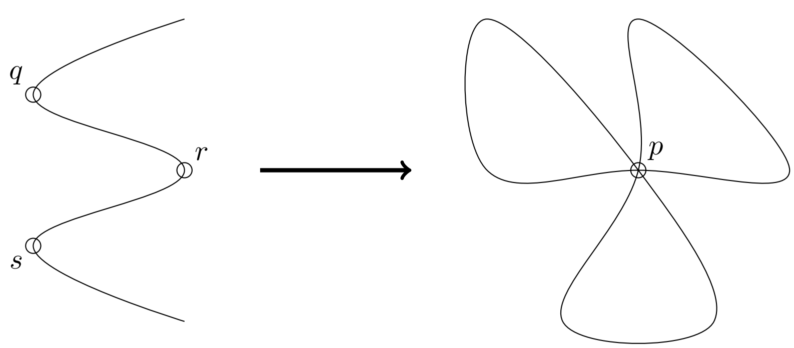
(17.3)

For p a tacnode, the rank is 2. We need $f(q) = f(r)$ and $f'(q) = f'(r)$.



(17.4)

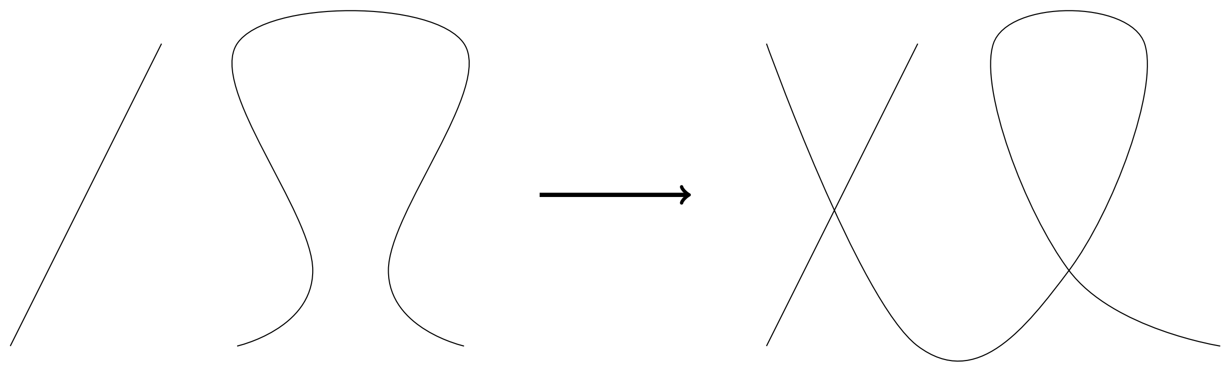
Now suppose p is a planar triple point. Then we need $f(q) = f(r) = f(s)$, but also a linear relation between $f'(q), f'(r), f'(s)$ because the dimension of the tangent space to C at p has dimension 2. So the rank of $\mathcal{F}_p$ is 3. (It would be 2 for a spatial triple point.)



(17.5)

Set $\delta(p)$ to be the rank of $\mathcal{F}_p$ (that is, $h^0(\mathcal{F}_p)$). Then $\chi(\pi_*\mathcal{O}_{C^\nu}) = \chi(\mathcal{O}_C) + \chi(\mathcal{F}_p)$. Since π_* is finite, it preserves cohomology, so we obtain $\chi(\pi_*\mathcal{O}_{C^\nu}) = \chi(\mathcal{O}_{C^\nu}) = 1 - p_a(\mathcal{O}_{C^\nu})$. Also $\chi(\mathcal{O}_C) = 1 - p_a(\mathcal{O}_C)$. We conclude that $p_a(C) = p_a(C^\nu) + \delta$.

Now suppose C is nodal and connected, with δ nodes. Write $C = \bigcup_{i=1}^k C_i$ with C_i nodal. Then $C^\nu = \coprod C_i^\nu$.



(17.6)

Then $p_a(C) = p_a(C^\nu) + \delta$. Also, $\chi(\mathcal{O}_{C \amalg D}) = \chi(\mathcal{O}_C) + \chi(\mathcal{O}_D)$, so $p_a(C \amalg D) = p_a(C) + p_a(D) - 1$. This implies

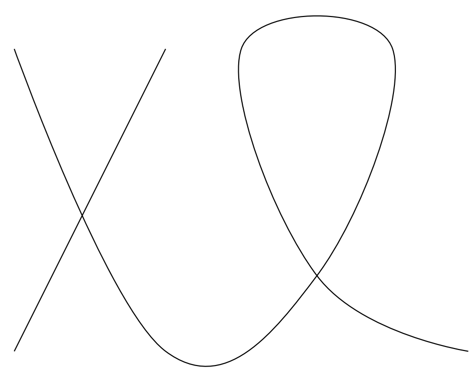
$$p_a(C) = p_a(C^\nu) + \delta = \left(\sum p_a(C_i^\nu)\right) - k + 1 + \delta = \left(\sum g(C_i)\right) - k + 1 + \delta. \tag{17.7}$$

Therefore

$$\sum g(C_i) = p_a(C) + k - 1 - \delta \tag{17.8}$$

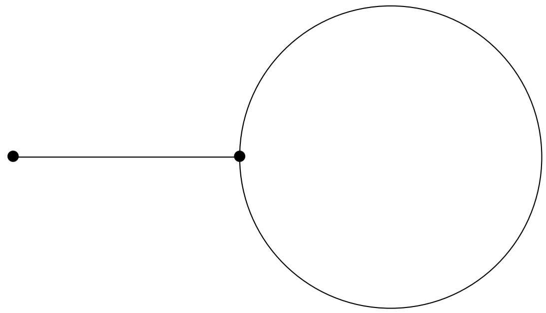
which is less than or equal to $p_a(C)$.

17.2 The Dual Graph of a Nodal Curve



(17.9)

Given C, we construct a weighted graph Γ_C with irreducible components of C corresponding to vertices of Γ_C and nodes of C corresponding to edges of Γ_C. For example, the dual graph of the above curve is



(17.10)

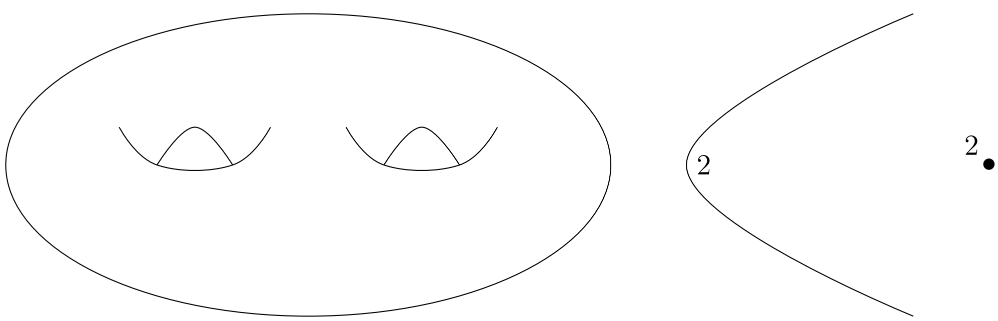
Make each vertex have weight equal to the genus of each component.

We can stratify $\overline{\mathfrak{M}_g}$ by the dual graph: set

$$\mathfrak{M}_g^\Gamma = \{C \in \overline{\mathfrak{M}_g} : \Gamma_C = \Gamma\} \subset \overline{\mathfrak{M}_g}, \tag{17.11}$$

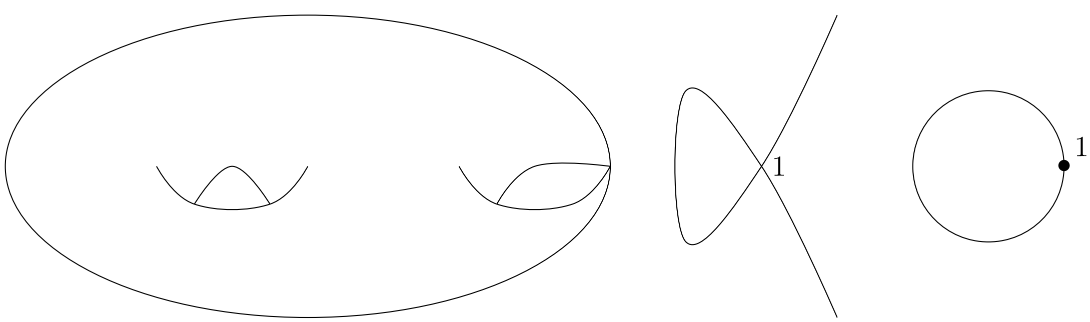
a locally closed subset.

As an example, take $g = 2$. The open stratum is



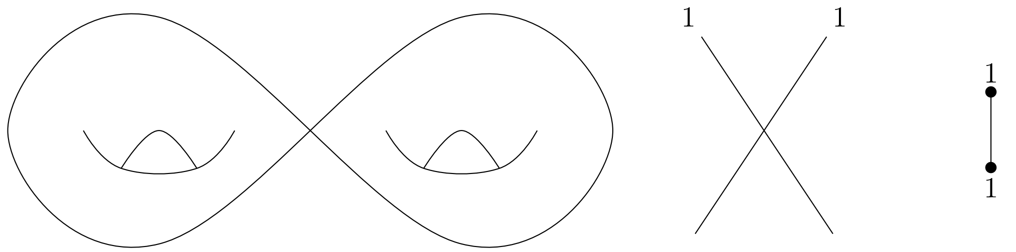
(17.12)

The strata with 1 node are



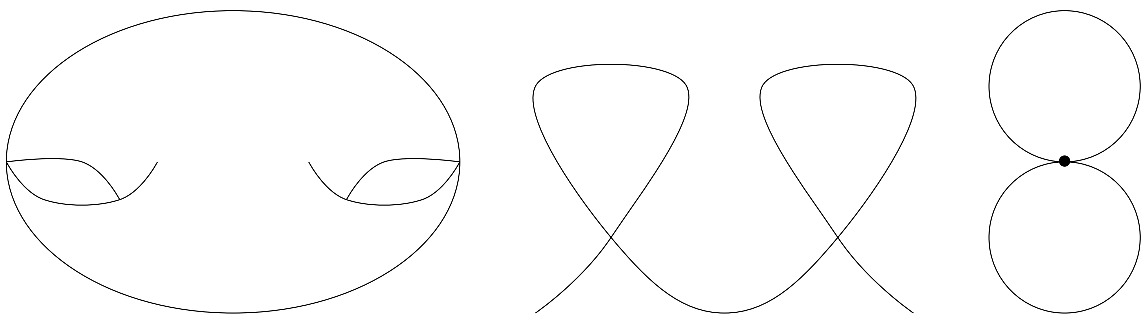
(17.13)

and



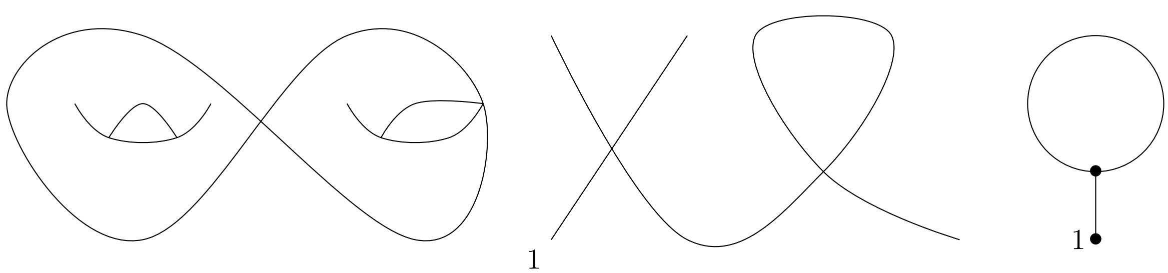
(17.14)

Those with 2 nodes are



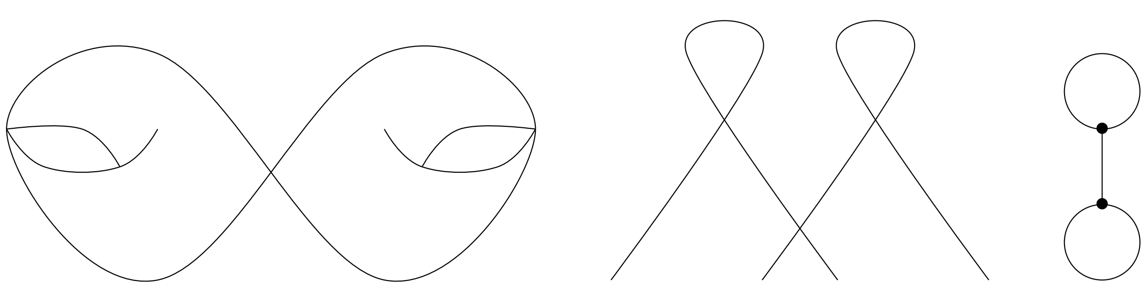
(17.15)

and



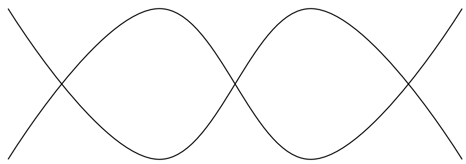
(17.16)

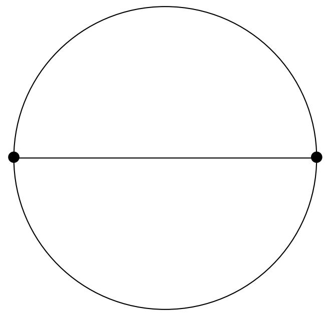
Those with 3 nodes are



(17.17)

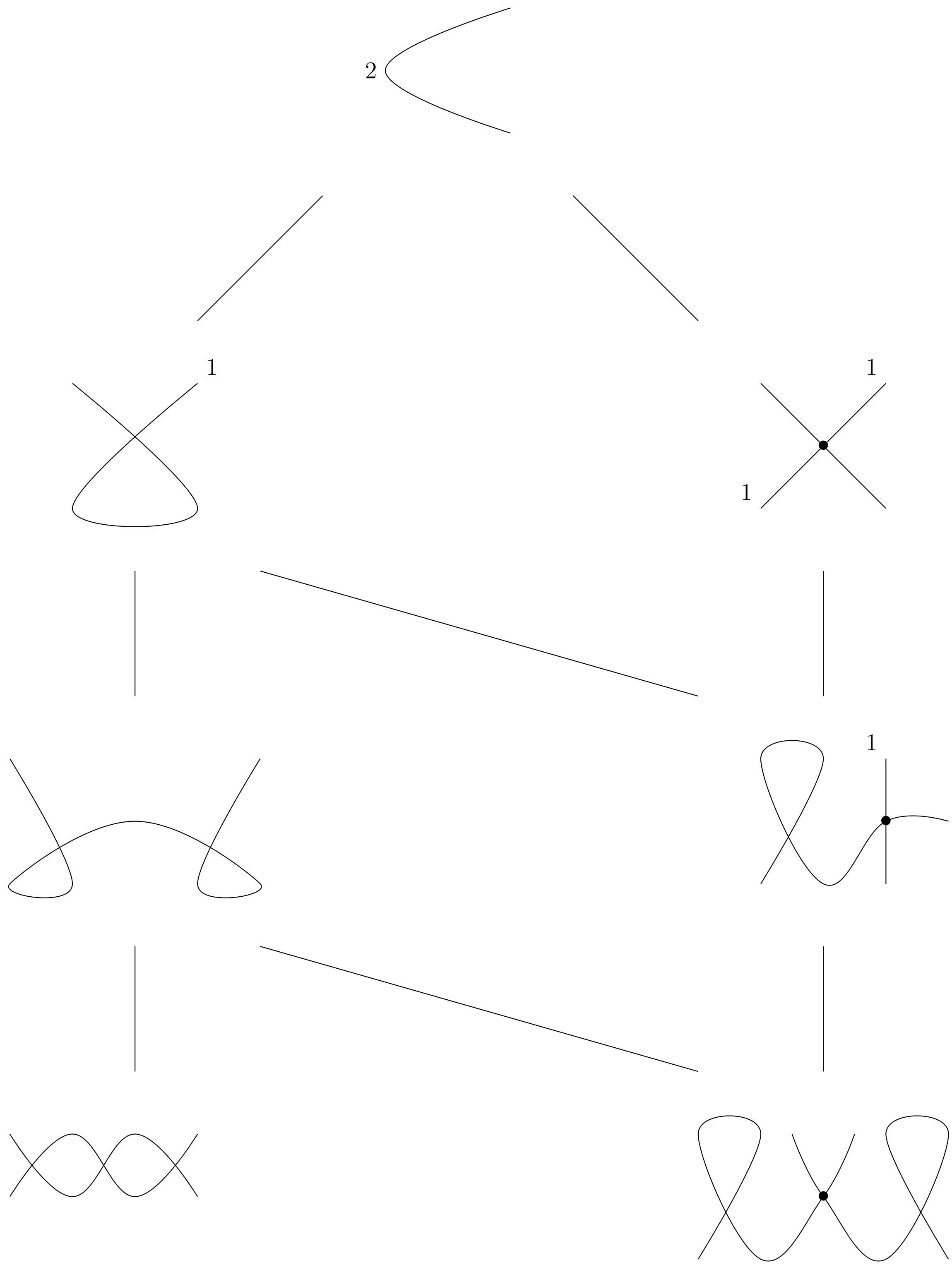
and (the surface below consists of two spheres intersecting in three points)





(17.18)

The specialization of strata is below. The marked points are disconnecting nodes; these allow us to constrain the possible limits of families of stable curves. The possibilities not already obstructed do occur:



(17.19)

Now we determine the dimension of the stratum of type Γ. Say n_i is the number of points on C_i^ν lying over nodes. Then the number of parameters is

$$\sum_{i=1}^{k}(3g_i - 3 - n_i) \tag{17.20}$$

but we have

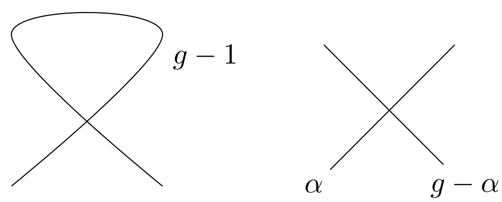
$$\sum_{i=1}^{k} n_i = 2\delta \tag{17.21}$$

so the number of parameters equals

$$3\left(\sum_{i=1}^{k} g_i\right) = 3k + 2\delta = 3g + 3k - 3 - 3\delta - 3k + 2\delta = 3g - 3\delta. \tag{17.22}$$

We conclude that the codimension of a stratum of curve with δ nodes is δ.

As a special case, the codimension 1 strata correspond to curves with a single node:



(17.23)

The corresponding $\mathfrak{M}_g^\Gamma$ are called Δ_0 and Δ_α for $\alpha = 1, \ldots, \lfloor \frac{g}{2} \rfloor$.

17.3 Planar Curve Singularities

(C, p) planar means $\dim T_pC \leq 2$, so C is locally embeddable in a smooth surface. A deformation of C is

$$\begin{tikzcd} C \arrow[r, equal, "\sim"] & C_0 \arrow[r, hook] \arrow[d] & \mathcal{C} \arrow[d] \\ & 0 \arrow[r, hook] & \Delta \end{tikzcd} \tag{17.24}$$

modulo isomorphism over étale neighborhoods of $0 \in \Delta$; deformations of (C, p) are similar, but modulo isomorphism of étale neighborhoods of p.

Theorem 17.1 (Reference: Hartshorne, Deformation Theory)**.** *Suppose $C \subseteq \mathbb{A}^2$ is reduced, $p = (0,0)$, and $C = V(f)$ with $f(x,y) \in \mathbb{C}[x,y]$. Introduce the Jacobian ideal*

$$J = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) \subseteq \mathbb{C}[x,y]_{(x,y)} = \mathcal{O}_{p,\mathbb{A}^2}. \tag{17.25}$$

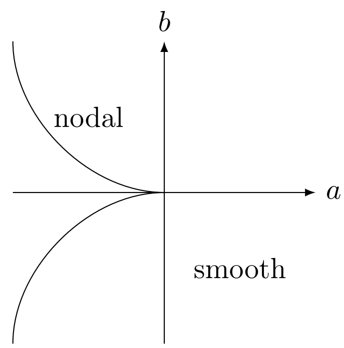
(J is an ideal of finite index.) Then the miniversal deformation of (C, p) is simply

$$V(f + t_1 f_1 + \cdots + t_k f_k) \twoheadrightarrow \Delta_{t_1,\ldots,t_k} \tag{17.26}$$

where f_i is a basis for $\mathcal{O}_p/J$.

Here are some examples:

- For a node, $f(x, y) = xy$ so $J = (x, y)$, with versal deformation $V(xy - t)$. So any family $\mathcal{C} \to B$ with C_0 nodal at p is locally $V(xy - \alpha)$ for $\alpha \in \mathcal{O}_{0,0}$. α is either 0 or locally t^k. So in any family of nodal curves $\mathcal{C} \to B$, the locus of nodal fibers has codimension at most 1 in B. If we have equality and B is reduced, then we have a Cartier divisor.
- For a cusp, say $V(y^2 - x^3)$, $\mathcal{C} = V(y^2 - x^3 - ax - b)$ is a versal deformation space $\mathcal{C} \to \Delta_{a,b}$. The fibers of $\mathcal{C}/\Delta$ are: the fiber over $(0, 0)$ is a cusp, the general fiber is smooth, and the fiber in the zero locus of $4a^3 + 27b^2$ is nodal.



(17.27)

We get a map $\Delta_{a,b} \setminus \{(0, 0)\} \to \overline{\mathfrak{M}_g}$, which can be resolved by 3 blow-ups.

- For a tacnode, $V(y^2 - x^4)$, we have $\mathcal{C} = V(y^2 - x^4 - ax^2 - bx - c)$. For the fibers of $\mathcal{C}/\Delta$, we can have two nodes, a cusp, a single node, or a smooth curve. The cases of two nodes and a cusp are eac curves in $\Delta_{a,b,c}$, while the nodal case is the discriminant surface.
- For a triple point, $V(y^3 - x^3)$, $\mathcal{C} = V(y^3 - x^3 - axy - bx - cy - d)$ is a family over $\Delta_{a,b,c,d}$.

18 Dualizing Sheaves of Curves

We will extend the notion of the canonical bundle to possibly singular curves. We want Riemann-Roch, and we want families.

Suppose C is a reduced curve, and $C^\nu \xrightarrow{\pi} C$ is its normalization. If $p \in C$ is a singular point, and U is a neighborhood of p, then

$$\omega_C(U) = \left\{ \text{meromorphic differentials } \varphi \text{ on } \pi^{-1}(U) : \forall f \in \mathcal{O}_C(U), p \in C, \sum_{q \in \pi^{-1}(p)} \mathrm{Res}_q(f\varphi) = 0 \right\}. \tag{18.1}$$

As an example, suppose $p \in C$ is a node. φ can have at most simple poles at points lying over the node, otherwise find a function vanishing to order 1 on one branch and arbitrarily high order on the other branch; then the sum of the residues can't be zero. Also, φ must have opposite residues at these points.

If $C = (y^2 - x^2)$, then $\frac{dx}{y}$ is such a differential. (We have $t \mapsto (t, t)$ and $t \mapsto (t, -t)$.)

Next, suppose $p \in C$ is a cusp, such as $y^2 = x^3$. Then the map is $t \mapsto (t^2, t^3)$. Now a differential can have a double pole with residue zero, but ho higher order poles. We can take $\frac{dt}{t^2}$, or $\frac{dx}{y}$ again.

For $p \in C$ a tacnode, $y^2 = x^4$: $t \mapsto (t, t^2)$ and $t \mapsto (t, -t^2)$. Again, we can have double poles but no triple poles. A valid differential is $(\frac{dt}{t^2}, -\frac{dt}{t^2})$, or $\frac{dx}{y}$. Another one is $\frac{x\,dx}{y}$.

We observe that in these three cases:

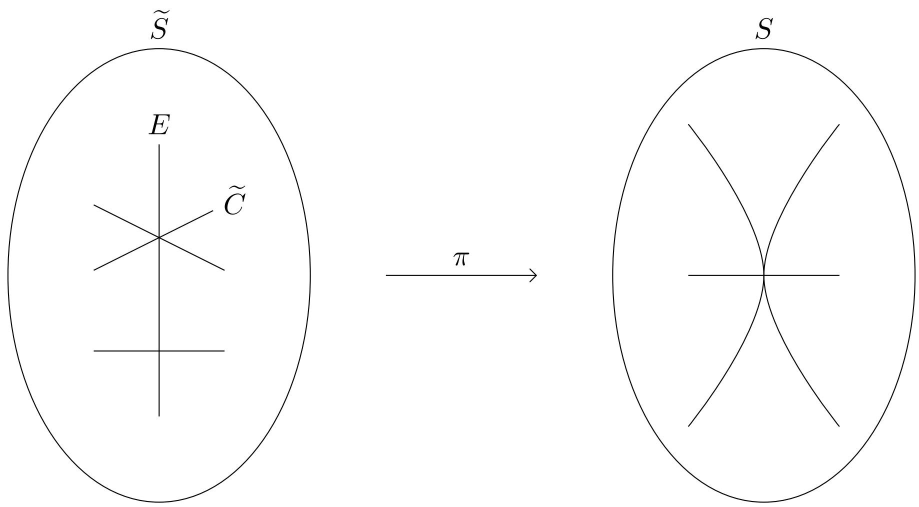
- ω is locally free of rank 1.
- $\deg \omega = \deg K_{C^\nu} + 2\delta = 2g(C) - 2$. Here g means arithmetic genus.
- $h^0(\omega) = h^0(C^\nu) + \delta = g(C)$.

These hold whenever C has planar singularities, or more generally for local complete intersection singularities, or even Gorenstein singularities.

An example of a non-Gorenstein singularity is a spatial triple point. Then for U near the singularity, $\omega_C(U)$ consists of rational differentials on $\pi^{-1}(U)$ with simple poles at three points and residues adding to 0. This fails the first two observations.

For $C \subseteq S$ a smooth surface, if C is a smooth curve, then $\omega_C = K_S(C)|_C$. This formula applies even if C is not smooth, as long as C is a Cartier divisor.

Suppose $p \in C$ is a point of multiplicity m. Let $\widetilde{S}$ be the blow-up of S at p, and $\widetilde{C}$ be the proper transform of C in $\widetilde{S}$.



(18.2)

$\widetilde{C} \sim \pi^*(C) - mE$, and $K_{\widetilde{S}} = \pi^*(K_S) + E$. We have $E^2 = -1$ and $E \cdot \pi^*(D) = 0$ for every divisor D of S. So

$$2g(\widetilde{C})-2=\widetilde{C}.\widetilde{C}+K_{\widetilde{S}}.\widetilde{C}=C.C-m^2+C.K_S+m=2g(C)-2-m(m-1). \tag{18.3}$$

We conclude $g(\widetilde{C})=g(C)-\binom{m}{2}$. For example, δ of a tacnode with a smooth component intersecting the node is $\binom{3}{2}+\binom{2}{2}=4$.

To determine sections of ω_C, relate to $\omega_{\widetilde{C}}$ by blowing up.

19 Kontsevich Spaces

Suppose X is a projective variety over $\mathbb{C}$, and $\beta \in H_2(X;\mathbb{Z})$. Then

$$\overline{\mathfrak{M}_g}(X,\beta)=\{f:C\to X : C \text{ nodal, } \operatorname{Aut}(f) \text{ finite, } f_*([C])=\beta\}. \tag{19.1}$$

Here $\operatorname{Aut}(f)$ finite means that for $C_0 \subseteq C$ any smooth rational component such that $f|_{C_0}$ is constant, we must have that $C_0 \cap \overline{C \setminus C_0}$ consists of at least three points.

Remark. Taking X to be a point recovers $\overline{\mathfrak{M}_g}$. More generally, for any X, $\beta=0$ gives $\overline{\mathfrak{M}_g}\times X$.

As a first example, consider $\overline{\mathfrak{M}_0}(\mathbb{P}^2,2)$, the Kontsevich space of "conic curves". The Hilbert scheme of conics is $\mathbb{P}^5$. But the double line is not a Kontsevich stable map. So if $\mathbb{P}^1 \xrightarrow{f_t} \mathbb{P}^2$ is given by $f_t:[F_t,G_t,H_t]$ of smooth conics specializing to a double line, the specialization is not 2-1 onto a line, We can associate a pair of branch points. So we get a map $\overline{\mathfrak{M}_0}(\mathbb{P}^2,2)\to\mathbb{P}^5$ which is an ismorphism over the complement of the surface $S\subseteq\mathbb{P}^5$ of double lines, and blows up S: the fiber over a double line is ismorphic to $\mathbb{P}^2$.

Think of $\overline{\mathfrak{M}_g}(X,\beta)$ as an alternative compactification of the open $U\subseteq\mathcal{H}(X)$ parameterizing smooth curves of genus g and class β, We want to find a relationship between $\overline{\mathfrak{M}_g}(X,\beta)$ and Hilbert schemes.

If $X=\mathbb{P}^n$ and $\beta=d\cdot\ell$, write $\overline{\mathfrak{M}_g}(\mathbb{P}^n,d)$ compactifying curves of degree d and genus g in $\mathbb{P}^n$.

As a next example, we'll consider $\overline{\mathfrak{M}_0}(\mathbb{P}^3,2)$, If $f:C\to\mathbb{P}^3$ is Kontsevich stable, then f must be finite. Either the domain is $\mathbb{P}^1$ or a union of two $\mathbb{P}^1$'s. When we get a 2:1 map to a double line in the limit, we also get two branch points on the double line as a result.

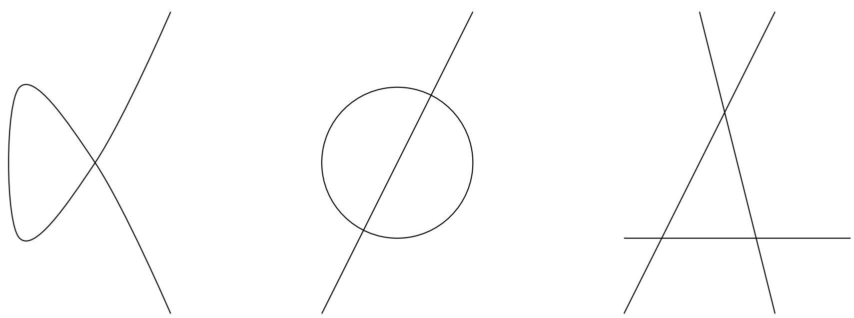
But we don't get a regular map to the Hilbert scheme! In $\mathcal{H}_{2m+1}(\mathbb{P}^3)$, every point is a subscheme $C\subseteq\mathbb{P}^3$ lying on a unique plane. In the Kontsevich limit, we only see a 2:1 map onto a line, and don't see a particular plane.

We can introduce the Chow variety $\mathcal{C}_{2,1}$ of cycles of degree 2 and dimension 1 in $\mathbb{P}^3$. We get a map $\overline{\mathfrak{M}_0}(\mathbb{P}^3,2)\to\mathcal{C}_{2,1}$ taking a map to its image (with multiplicities indicated). We also have a map $\mathcal{H}_{2m+1}(\mathbb{P}^3)\to\mathcal{C}_{2,1}$ similarly. In fact, they are related birationally.

$$
\begin{tikzcd}
 & \Gamma \arrow[dl] \arrow[dr] & \\
\overline{\mathfrak{M}_0}(\mathbb{P}^3, 2) \arrow[rr, dashed, leftrightarrow] \arrow[dr] & & \mathcal{H}_{2m+1}(\mathbb{P}^3) \arrow[dl] \\
 & \mathcal{C}_{2,1} &
\end{tikzcd}
\tag{19.2}
$$

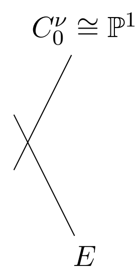
If we blow up the space of double lines, we get the space Γ.

As a third example, $\overline{\mathfrak{M}_1}(\mathbb{P}^2, 3)$ includes the locus of smooth plane cubics, equal to $U \subseteq \mathbb{P}^9 = \mathcal{H}_{3,m}(\mathbb{P}^2)$. We have a map $\overline{\mathfrak{M}_1}(\mathbb{P}^2, 3) \to \mathcal{H}_{3,m}(\mathbb{P}^2) \cong \mathbb{P}^9$. In the cases



(19.3)

the resulting map is Kontsevich stable, so the map is locally an isomorphism. In the cuspidal case, stable reduction gives

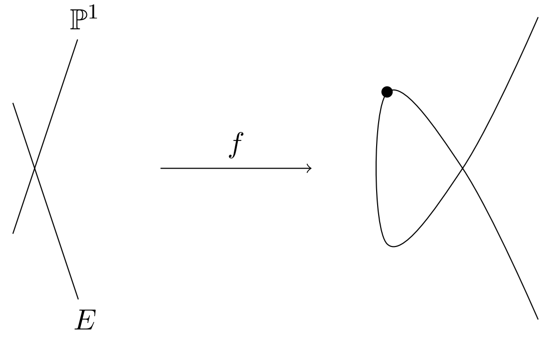


(19.4)

(for E an elliptic tail). We obtain a map from this curve to the cuspidal curve taking E to a point. This is a triple blow-up followed by a double blow-down. In the case of a triple line, we arrive at an elliptic curve mapping to a line with four branch points. This map is complicated!

Fact. $\overline{\mathfrak{M}_1}(\mathbb{P}^2, 3)$ has three irreducible components!

Consider stable maps

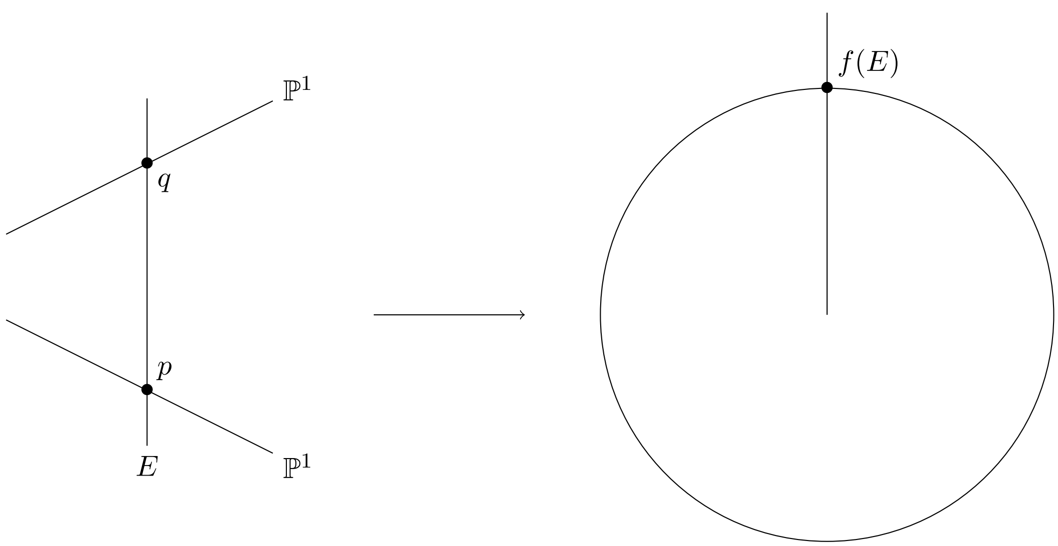


(19.5)

collapsing E to any point. Let's count parameters: the image curve can be any singular cubic (8 parameters), and we specify where E maps to (1 parameter), and the j-invariant of E can be anything (1 parameter). We obtain a 10-dimensional family, larger than U.

Conversely, singular curves can't specialize to smooth curves, so we get two different components (it's an exercise to show that they're not both specializations of a larger one).

The third component is given by



(19.6)

The image curve has $5 + 2 = 7$ parameters, and we need to specify (E, p, q) (2 parameters), so we get a 9-dimensional family which is a separate component.

We've just seen that for plane cubics, $\mathcal{H} \cong \mathbb{P}^9$, but $\overline{\mathfrak{M}_1}(\mathbb{P}^2, 3)$ has extraneous components. On the other hand,for twisted cubics, $\overline{\mathfrak{M}_0}(\mathbb{P}^3, 3)$ is irreducible, but $\mathcal{H}_{3m+1}(\mathbb{P}^3)$ has extraneous components.

Here is a problem: we can't tell when a point in either $\mathcal{H}$ or $\overline{\mathfrak{M}_g}(X, \beta)$ lies in the closure of the curves we're interested in. As an example (we don't know the answer): what stable maps $f : C \to \mathbb{P}^2$ of degree d and genus g lie in the closure of the Severi variety?

A question from Bjorn Poonen: let C be a general hyperelliptic curve of genus $g \geq 2$ with p a Weierstrass point. Does there exist $q \in C$ and $m > 2$ with $mq \sim mp$ other than q another Weierstrass point?

Here are some ideas:

- If (C, p) has such a q, try to find a deformation of (C, p) losing q.
- Look at the monodromy on torsion points. If it's transitive, it's enough to show that there aren't $m^{2g} - 1$ such points.

20 Diaz's Theorem

We would like to know what the largest dimensional complete subvariety of $\mathfrak{M}_g$ is. We could also ask a similar question, requiring that the variety pass through a general point on $\mathfrak{M}_g$.

Here is what we know:

- There exist complete curves in $\mathfrak{M}_g$ through a general point (in fact, through any finite set of points). This follows from the Satake compactification of $\mathfrak{M}_g$. $\mathfrak{M}_g^s$ is a projective variety with $\mathfrak{M}_g$ as an open subset, but it is highly singular at the boundary and is not a moduli space. A virtue is that for $g \geq 3$, $\mathfrak{M}_g^s \setminus \mathfrak{M}_g$ has codimension 2.
- There exist complete families of large dimension when $g >> 0$. Here is a variant of Kodaira's construction: Start with a curve C_0 of genus h, and consider those C for which there exists $f : C \to C_0$ of degree 3, branched at only one point. This is contained in $\mathfrak{M}_g$ for $g = 3h - 1$, and is a complete curve. (It's a finite covering space of C_0.)

  Iterating this, given $\Sigma \subseteq \mathfrak{M}_g$ complete of dimension m, we get $\Sigma' \subseteq \mathfrak{M}_{3g-1}$ complete of dimension $m + 1$, where Σ' is the set of C such that there exists $f : C \to C_0$ of degree 3 branched at one point, and $C_0 \in \Sigma$.

Theorem 20.1 (Diaz)**.** *If $\Sigma \subseteq \mathfrak{M}_g$ is complete of dimension m, then $m \leq g - 2$.*

Remark. The largest dimension m of complete $\Sigma \subseteq \mathfrak{M}_g$ has $\log_3 g < m \leq g - 2$. We also don't know if there exists a complete surface $\Sigma \subseteq \mathfrak{M}_g$ through a general point.

Here is Arbarello's original proposed method of proof: we stratify $\mathfrak{M}_g$. Let

$$\Gamma_d \subseteq \mathfrak{M}_g = \{C : \exists p \in C \text{ with } r(dp) \geq 1\} \tag{20.1}$$

and $\widetilde{\Gamma}_d = \Gamma_d \setminus \Gamma_{d-1}$. Alternatively, Γ_d is the set of C for which there exists $f : C \to \mathbb{P}^1$ of degree d which is totally ramified at a point $p \in C$.

Arbarello proposed that $\widetilde{\Gamma}_d$ contains no complete curves. Now we have

$$\widetilde{\Gamma}_2 = \Gamma_2 \subseteq \Gamma_3 \subseteq \Gamma_4 \subseteq \cdots \subseteq \Gamma_g = \mathfrak{M}_g. \tag{20.2}$$

(We have $\Gamma_g = \mathfrak{M}_g$ since every curve has a Weierstrass point.)

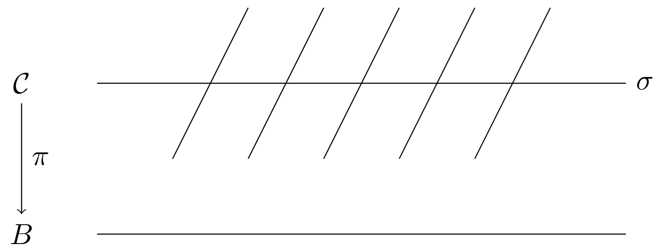
If $C \to \mathbb{P}^1$ is of degree d with a point of total ramification, then the number of other branch points is $(2g - 2 + 2d) - (d - 1)$, so

$$\dim \Gamma_d = \# \text{ branch points} - \dim PGL_2 = (2g + 2d - 2) - (d - 1) + 1 - 3 = 2g + d - 3. \tag{20.3}$$

So Γ_d is a hypersurface in Γ_{d+1}.

The proposed method of proving that $\widetilde{\Gamma}_d$ has no complete curves: given $B \subseteq \widetilde{\Gamma}_d$ complete, then after base change, we can assume:

- B is smooth
- There exists a universal family $\mathcal{C} \to B$
- There exists a section $\sigma : B \to \mathcal{C}$ with $r(d\sigma(b)) = 1$ for every $b \in B$ and $r((d-1)\sigma(b)) = 0$ for every $b \in B$.

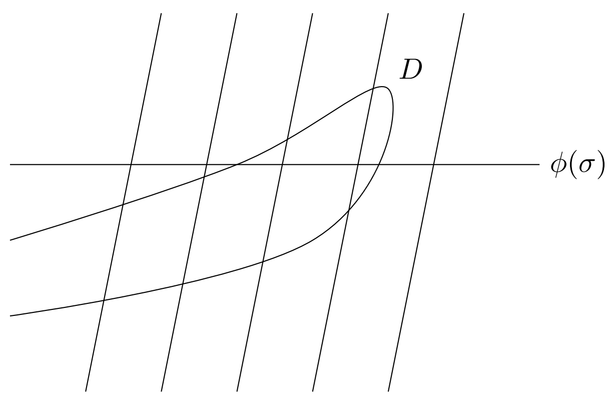


(20.4)

Look at $E = \pi_* \mathcal{O}_C(d\sigma)$, a vector bundle of rank 2 on B. It is basepoint free, so we get

$$\begin{tikzcd} \mathcal{C} \arrow[rr, "\phi"] \arrow[dr, "\pi"'] & & \mathbb{P}E^* \arrow[dl, "\mathbb{P}^1\text{-bundle}"] \\ & B & \end{tikzcd} \tag{20.5}$$

Letting D be the rest of the branch locus of ϕ, we get

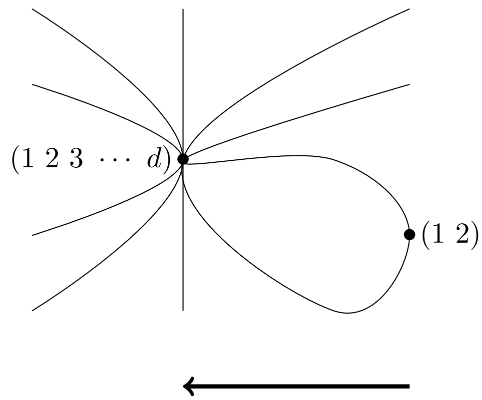


(20.6)

What happens if D meets $\phi(\sigma)$? We get a singular point of the cover. For example,

$$(1\ 2)(1\ 2\ 3 \cdots d) = (2 \cdots d), \tag{20.7}$$

so we get a point with total ramification index $d-2$. This requires a singularity (node). This forces a base point.



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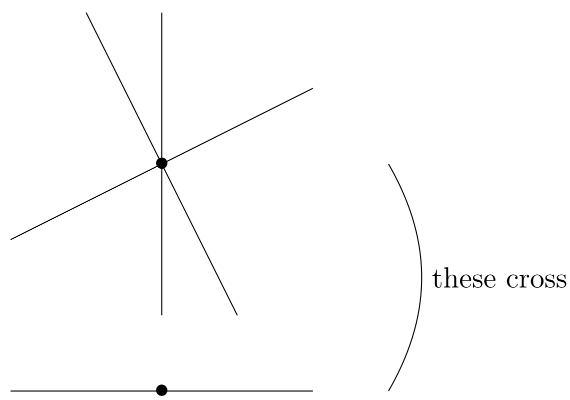
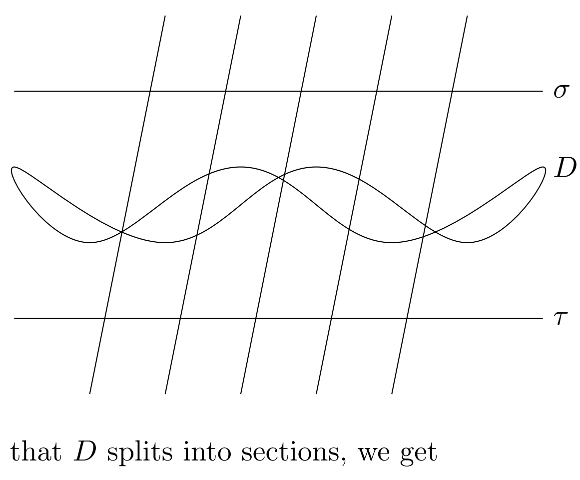


(20.9)

Here is the problem with this argument: D does not have to meet $\phi(\sigma)$. For there exist $\mathbb{P}^1$ bundles over B with a section σ and a disjoint curve D. Indeed, we don't know whether $\widetilde{\Gamma}_d$ can contain a complete curve.

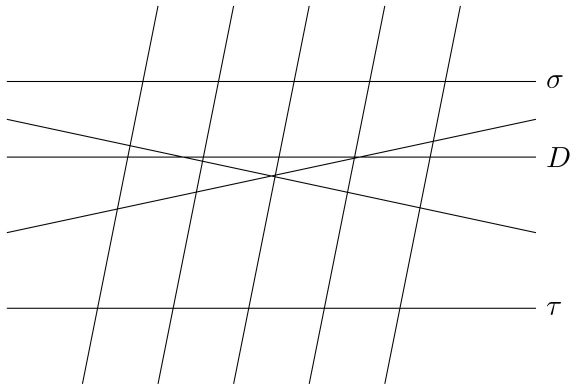
But there do not exist $\mathbb{P}^1$ bundles over B with disjoint sections σ, τ and a curve D disjoint from both of them, except fo the trivial bundle and constant sections.

To see this, suppose we had



(20.10)

After a base change so that D splits into sections, we get



(20.11)

This forces D, σ, τ to be the union of at least three constant sections of trivial $\mathbb{P}^1$-bundles, since a $\mathbb{P}^1$-bundle with at least 3 disjoint sections is trivial.

Diaz's solution is to look at a different stratification

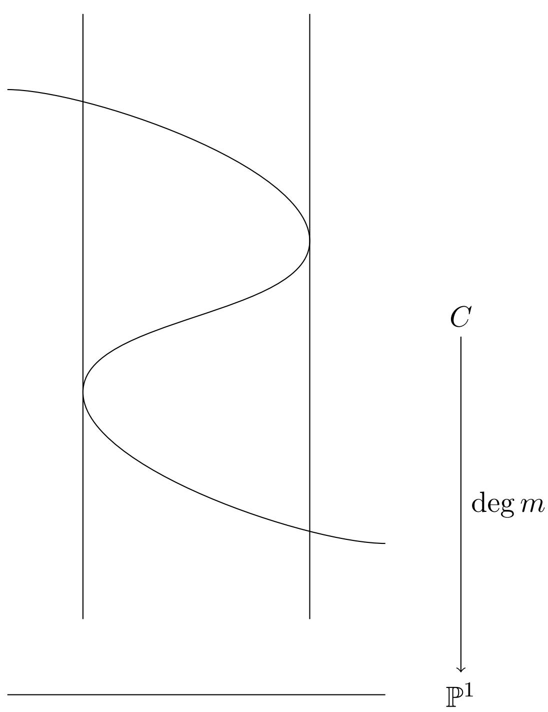
$$\Delta_d = \{C : \exists\, f : C \to \mathbb{P}^1 \text{ with } \deg f \leq g \text{ and } \# f^{-1}(0, \infty) \leq d\}. \tag{20.12}$$

Then we have

$$\Delta_2 \subseteq \Delta_3 \subseteq \cdots \subseteq \Delta_g = \mathfrak{M}_g. \tag{20.13}$$

($\Delta_g = \mathfrak{M}_g$ because, for example, we can take 0 to be a Weierstrass point and ∞ any other branch point of a function totally ramified of degree d at 0.)

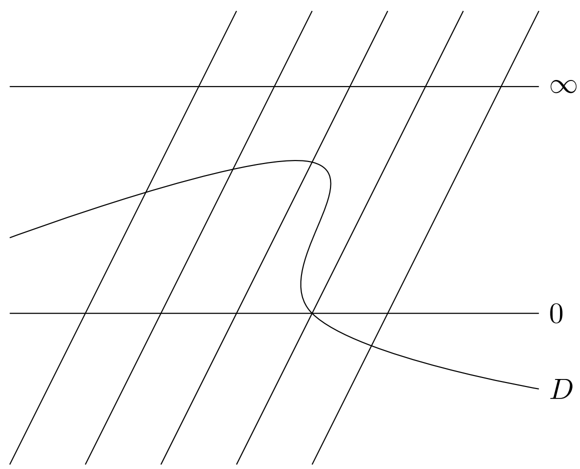
Now Δ_2 is the set of C for which there exists $f : C \to \mathbb{P}^1$ totally ramified at two points. We have a component of Δ_2 for each possible degree (up to g). Notice that Δ_g has pure dimension $2g-3+d$.



(20.14)

We have $2g+2m-2$ branch points in total. There are $2m-d$ over $0, \infty$, so the number of other branch points is $2g+2m-2-(2m-d)$. We get $2g+d$ distinct branch points, so the dimension is $2g-3+d$.

Now suppose $B \subseteq \widetilde{\Delta}_d$ is complete. Then we have



(20.15)

with D of degree $2g+d-2$ over B. By the above observation, D must meet either the 0 or ∞ section. Say $C_t \xrightarrow{f_t} \mathbb{P}^1$ is branched at 0 and $\alpha(t)$. The monodromy over 0 is $\sigma \in S_d$, while the monodromy over $\alpha(t)$ is $(1\ 2)$ after appropriate labelling. If 1 and 2 belong to different orbits of σ, then C_0 is smooth, but $\#f_0^{-1}(0) < \#f_t^{-1}(0)$. We end up in Δ_{d-1}. If 1 and 2 belong to the same orbit, then C_0 is singular.

21 Unirationality of the Moduli Space in Small Genus

One important question is whether we can write down a general curve of genus g. In other words, does there exist a family $\mathcal{C} \to B$ of curves of genus g, with B open in an affine (or projective) space, with the induced map $\phi : B \to \mathfrak{M}_g$ dominant?

This is possible in low genus. We've already seen the genus 1 case.

- $g = 2$: All curves are hyperelliptic, so of the form

$$y^2 = \prod_{i=1}^{6} (x - \lambda_i). \qquad (21.1)$$

  All curves of genus 2 are expressible by choosing λ_i appropriately.

- $g = 3$: The non-hyperelliptic curves are plane quartics, so consider

$$\sum_{i+j \leq 4} a_{ij} x^i y^j = 0 \qquad (21.2)$$

  and let the a_{ij} vary, avoiding the hypersurface of singular quartics. The resulting image is dense in $\mathfrak{M}_3$.

- $g = 4$: $f_2(x, y, z) = g_3(x, y, z) = 0$ for general f_2, g_3 gives a complete intersection; the resulting image is dominant in $\mathfrak{M}_4$.

- $g = 5$: In general, the canonical curve is an intersection of three quadrics, so take three quadrics in $\mathbb{P}^4$.

- $g = 6$: $C \xhookrightarrow{\phi_K} \mathbb{P}^5$ is of degree 10, but not a complete intersection, so we need another method.

  Brill-Noether theory says that a general C of genus 6 can be birationally embedded in $\mathbb{P}^2$ as a sextic with four nodes. Furthermore, a general curve will have no three nodes collinear. Now we can take the four points to be

$$p_1 = [1, 0, 0], p_2 = [0, 1, 0], p_3 = [0, 0, 1], p_4 = [1, 1, 1]. \tag{21.3}$$

  So let V be the vector space of homogeneous $F(X, Y, Z)$ of degree 6 vanishing to order at least 2 at $p_1, \ldots, p_4$. We have an open subset $U \subseteq V$ consisting of those F which are nodal at $p_1, \ldots, p_4$ and smooth otherwise. Then $U \to \mathfrak{M}_6$ is dominant.

- $g = 7$: Again, Brill-Noether theory implies that a general C can be embedded in $\mathbb{P}^2$ as a curve of degree 7 and 8 nodes. This means the Severi variety $V_{7,7} = V^{7,8}$ dominates $\mathfrak{M}_7$.

  Consider $V^{7,8} \to (\mathbb{P}^2)_8$, the set of 8-tuples of points in $\mathbb{P}^2$ (specification of the nodes). The general fiber is $\mathbb{P}^{\binom{9}{2}-1-3\cdot 8} = \mathbb{P}^{11}$. This means that $V^{7,8}$ is birationally a $\mathbb{P}^{11}$-bundle over $(\mathbb{P}^2)_8$. Labelling the nodes would replace $(\mathbb{P}^2)_8$ with $(\mathbb{P}^2)^8$. This shows that $V^{7,8}$ is rational. Specifically, if

$$\Delta = \{(p, \Gamma) \in \mathbb{P}^2 \times (\mathbb{P}^2)^8 : p \in \Gamma\}, \tag{21.4}$$

  then $V^{7,8}$ is birational to

$$\mathbb{P}\Big((\pi_2)_*(\pi_1^*\mathcal{O}_{\mathbb{P}^2}(7) \otimes \mathcal{I}_\Delta^2)\Big). \tag{21.5}$$

  Therefore there exists $U \subseteq V$ with $U \hookrightarrow \mathbb{A}^{27}$ and a family $\mathcal{C} \to U$ such that the map $U \to \mathfrak{M}_7$ is dominant.

- $g = 8$: Brill-Noether implies we can birationally embed a general curve in $\mathbb{P}^2$ as a degree 8 curve with 13 nodes. Now we have $V_{8,8} \to (\mathbb{P}^2)^{13}$ with general fiber isomorphic to $\mathbb{P}^{\binom{10}{2}-1-3\cdot 13} = \mathbb{P}^5$, so the Severi variety is generically a $\mathbb{P}^5$-bundle over $(\mathbb{P}^2)^{13}$. Again the Severi variety is rational.

- $g = 9$; embed as a octic with 12 nodes, so $V_{8,9} \to (\mathbb{P}^2)^{12}$ with general fiber $\mathbb{P}^8$.

- $g = 10$: take $d = 9$ and $\delta = 18$. Then $V_{9,10} \to (\mathbb{P}^2)^{18}$, with general fiber $\mathbb{P}^{\binom{11}{2}-1-3\cdot 18} = \mathbb{P}^0$. So $V_{9,10}$ is actually birational to $(\mathbb{P}^2)^{18}$.

- $g = 11$: if we try the same method, we will take $d = 10, \delta = 25$, but then $V_{10,11}$ does not dominate $(\mathbb{P}^2)^{25}$! This means the 25 nodes must be in special position.

For the cases $g = 11, 12, 13, 14$, Sernesi, Ran, and Chang have used embeddings into $\mathbb{P}^3$ to show that the Hilbert scheme was rational.

$\phi : B \to \mathfrak{M}_g$ dominant with B rational implies $\mathfrak{M}_g$ is unirational. In particular, its plurigenera $h^0(K^m_{\overline{\mathfrak{M}_g}})$ must be zero.

(Note: the genus 6-10 arguments actually had some holes. These were filled in by Arbarello and Cornalba.)

One consequence of the above work for $g \le 10$ (14): the set of curves defined over $\mathbb{Q}$ is dense in $\mathfrak{M}_g$. This is not known for $g \ge 15$.

The existence of a family is equivalent to unirationality of $\mathfrak{M}_g$. (The other implication holds since we can restrict to the automorphism-free locus, so that the map to $\mathfrak{M}_g$ comes from a family.)

If X is smooth and projective, then these exists a dominant rational map $\mathbb{P}^n \dashrightarrow X$ only if $h^0(K_X^m) = 0$ for every $m > 0$. (We can replace "smooth" by "having only canonical singularities", which is satisfied by $\mathfrak{M}_g$.) So we want to know whether there exists an effective pluricanonical divisor on $\overline{\mathfrak{M}_g}$. To do this, we study the divisor class theory of $\overline{\mathfrak{M}_g}$, enough to identify some of $\mathrm{Pic}(\overline{\mathfrak{M}_g}), K_X$, and classes of effective divisors.

22 Divisors of the (Compactification of the) Moduli Space

Divisors of $\overline{\mathfrak{M}_g}$, or more generally subvarieties, come from geometric conditions. Here are some examples:

- The locus Δ of singular curves.
- Curves with special Weierstrass points: at least two give divisors.
- Curves with a semicanonical pencil (that is, there exists $\mathcal{L}$ with $\mathcal{L}^2 = K_C$ and $h^0(\mathcal{L}) \ge 2$) gives a divisor.
- Brill-Noether loci: If $\rho = \rho(r,d) = g - (r+1)(g-d+r) < 0$, we can look at the set of C for which C has a g^r_d. If $\rho = -1$, this gives a divisor.
- More generally, the set of C for which there exist $p_1, \ldots, p_k$ satisfying

$$h^0\left(\sum_i m_i p_i\right) \ge r \tag{22.1}$$

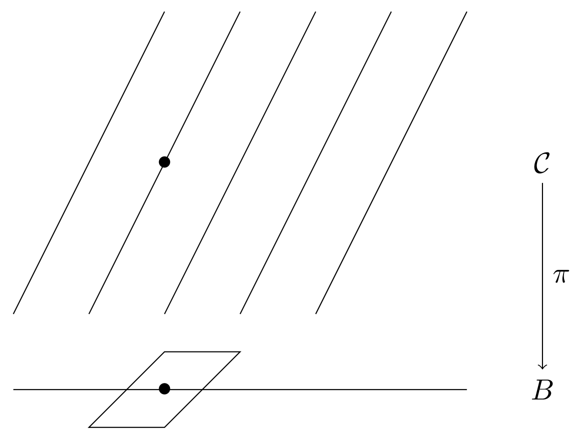
  works.

Now suppose X is any variety, and $\mathcal{L}$ is a line bundle on X. For every map $\phi : B \to C$ with B a curve, we get a line bundle $\mathcal{L}_\phi = \phi^*\mathcal{L}$ on B. If $B' \xrightarrow{\alpha} B \xrightarrow{\phi} X$, then $\mathcal{L}_{\phi\circ\alpha} = \alpha^*\mathcal{L}_\phi$. Conversely, the association $\phi \mapsto \mathcal{L}_\phi$ with the naturality condition determines $\mathcal{L}$.

So a line bundle on $\overline{\mathfrak{M}_g}$ associates to every one-parameter family $\mathcal{C} \xrightarrow{\pi} B$ of stable curves a divisor class $\mathcal{L}_C$ on B, such that for every $B' \xrightarrow{\alpha} B$, we have $\mathcal{L}_{\mathcal{C}\times_B B'} = \alpha^*\mathcal{L}_C$. In this way, we can view a divisor class on $\overline{\mathfrak{M}_g}$ as a gadget that associates to a family of stable curves over B a divisor class on B, which is compatible with base change.

Now we need to determine the variation of a family. Here are some methods:

- Look at the Hodge bundle of $\mathcal{C} \xrightarrow{\pi} B$. This bundle is given by $E = \pi_*(\omega_{\mathcal{C}/B})$, a vector bundle of rank g on B.



(22.2)

We get a divisor class $\lambda = c_1(E)$ on B.

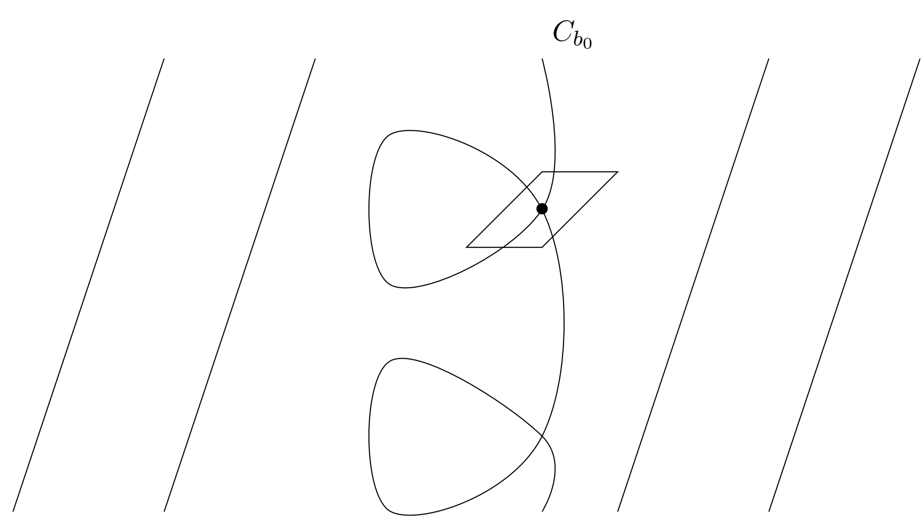
- Use the self-intersection number to determine $\kappa = \pi_*(c_1(\omega_{\mathcal{C}/B})^2)$.

Given $\mathcal{C} \to B$, we'll want to determine the intersection number of Δ with the image in $\overline{\mathfrak{M}_g}$ associated to this family. We'll need a better understanding of Δ to determine the scheme-theoretic intersection.

Recall that if p is a node of a curve C, then $\mathrm{Def}(C, p) = \Delta_t$, with versal family $xy - t$. So around $[C] \in \overline{\mathfrak{M}_g}$, the divisor Δ is locally (t). If C is stable with nodes $p_1, \ldots, p_\delta$, then we have

$$\mathrm{Def}(C) \to \prod \mathrm{Def}(C, p_i) = \prod \Delta_{t_i} \implies \Delta = \left(\prod t_i\right). \tag{22.3}$$

We conclude that if $\mathcal{C} \to B$ is a 1-parameter family of stable curves and $\phi : B \to \overline{\mathfrak{M}_g}$ is the associated map, and $b_0 \in B$ is such that C_{b_0} has nodes $p_1, \ldots, p_\delta$ with the local equation of $\mathcal{C}$ at p_i being $xy - t^{m_i}$,



(22.4)

then we have

$$\mathrm{mult}_{b_0}(\phi^*\Delta) = \sum m_i. \tag{22.5}$$

As an example, we'll look at a general pencil of plane quartics. Take F, G to be two general quartic polynomials in $\mathbb{P}^2$ and look at

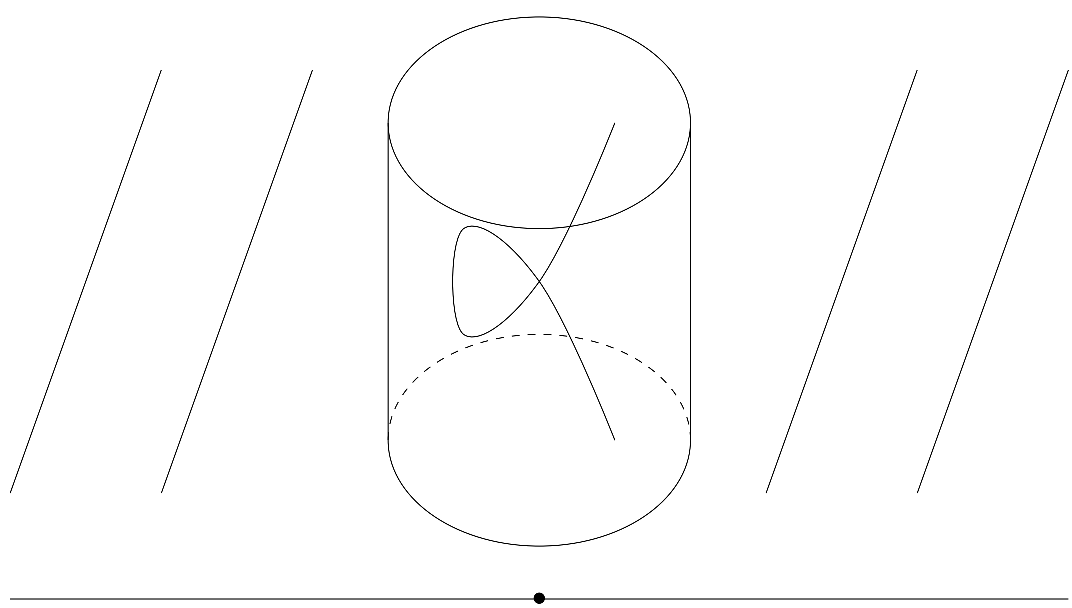
$$\mathcal{C} = V(t_0F + t_1G) \subseteq \mathbb{P}^1_t \times \mathbb{P}^2. \tag{22.6}$$

Observe that all fibers C_t are stable, since the locus of quartics having a non-nodal singularity has codimension 2. (We can further assume that all singular curves are irreducible with 1 node.) We would like to find the degrees of δ, λ, κ on $\mathbb{P}^1$.

Remark. $\mathcal{C}$ is a general divisor of type $(1, 4)$, so is smooth by Bertini.

δ: Since $\mathcal{C}$ is smooth, each multiplicity is 1, implying that $\deg \delta$ equals the number of nodes of singular fibers, which in turn equals the number of singular fibers.

Use Riemann-Hurwitz: given $\mathfrak{X} \xrightarrow{\pi} B$ with B a smooth curve and $\mathfrak{X}$ smooth of dimension n, let F be the general fiber and let $\Gamma = \{b_1, \dots, b_\delta\}$ be the points at which the singular fibers lie over.

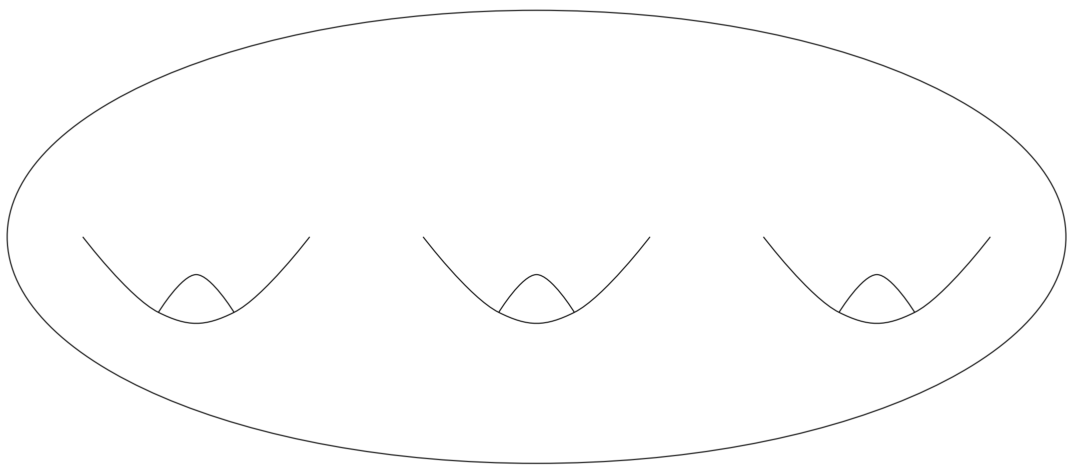


(22.7)

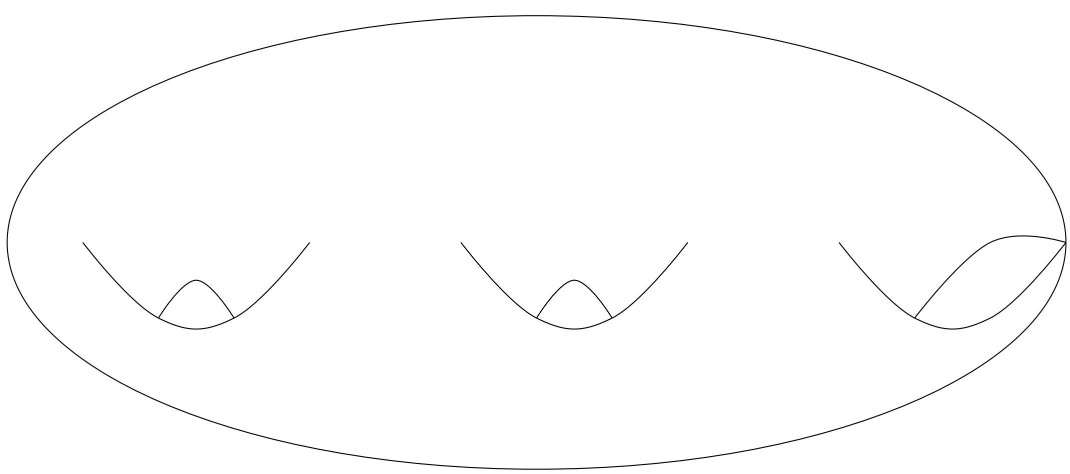
Write $\mathfrak{X} = \pi^{-1}(B \setminus \Gamma) \cup X_{b_1} \cup \cdots \cup X_{b_\delta}$. Each X_{b_i} has a tubular neighborhood, which after removing X_{b_i}, gives a S^1-bundle, therefore of Euler characteristic 0. We obtain

$$
\begin{aligned}
\chi(\mathfrak{X}) &= \chi(\pi^{-1}(B \setminus \Gamma)) + \sum \chi(X_{b_i}) && (22.8)\\
&= \chi(B \setminus \Gamma)\chi(F) + \sum \chi(X_{b_i}) && (22.9)\\
&= \chi(B)\chi(F) - \delta\chi(F) + \sum_{i=1}^{\delta} \chi(X_{b_i}) && (22.10)\\
&= \chi(B)\chi(F) + \sum_{b \in B} \Big(\chi(X_b) - \chi(F)\Big). && (22.11)
\end{aligned}
$$

In our situation, we have δ singular fibers.



$$F : \chi = -4 \tag{22.12}$$



$$\chi = -2 - 1 = -3 \tag{22.13}$$

We have $\chi(X_b) - \chi(F) = 1$ for every nodal fiber, so the number of singular fibers equals $\chi(\mathcal{C}) - \chi(B)\chi(F)$. Now $\chi(\mathcal{C}) = \chi(\mathrm{Bl}_{V(F,G)}\mathbb{P}^2) = 3 + 16 = 19$ since $V(F,G)$ consists of 16 points. Also $\chi(B) = 2$ and $\chi(F) = -4$, so there are 27 singular fibers.

Here is an alternative method that only works in $\mathbb{P}^2$. The locus of singular fibers is

$$V\left(t_0\frac{\partial F}{\partial x} + t_1\frac{\partial G}{\partial x}, t_0\frac{\partial F}{\partial y} + t_1\frac{\partial G}{\partial y}, t_0\frac{\partial F}{\partial z} + t_1\frac{\partial G}{\partial z}\right). \tag{22.14}$$

κ: for $Z \subseteq \mathbb{P}^1 \times \mathbb{P}^2$, write $\mathcal{O}_Z(m,n)$ for $(\pi_1^*\mathcal{O}_{\mathbb{P}^1}(m) \otimes \pi_2^*\mathcal{O}_{\mathbb{P}^2}(n))|_Z$. We have $K_{\mathbb{P}^1\times\mathbb{P}^2} = \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^2}(-2,-3)$ so $K_C = K_{\mathbb{P}^1\times\mathbb{P}^2}(\mathcal{C})|_{\mathcal{C}} = \mathcal{O}_{\mathcal{C}}(-1,1)$. This means $\omega_{\mathcal{C}/\mathbb{P}^1} = K_{\mathcal{C}} \otimes \pi_1^* K_{\mathbb{P}^1}^* = \mathcal{O}_C(1,1)$.

So $\deg\kappa$ equals the triple intersection of divisors on $\mathbb{P}^1 \times \mathbb{P}^2$ of bidegrees $(1,4), (1,1), (1,1)$. Using the ring $A(\mathbb{P}^1 \times \mathbb{P}^2) = \mathbb{Z}[\alpha,\beta]/(\alpha^2,\beta^3)$, we obtain

$$c_1^2(\omega_{\mathcal{C}/\mathbb{P}^1}) = (\alpha + 4\beta)(\alpha+\beta)(\alpha+\beta) = 9\alpha\beta^2. \tag{22.15}$$

Therefore $\deg\kappa = 9$.

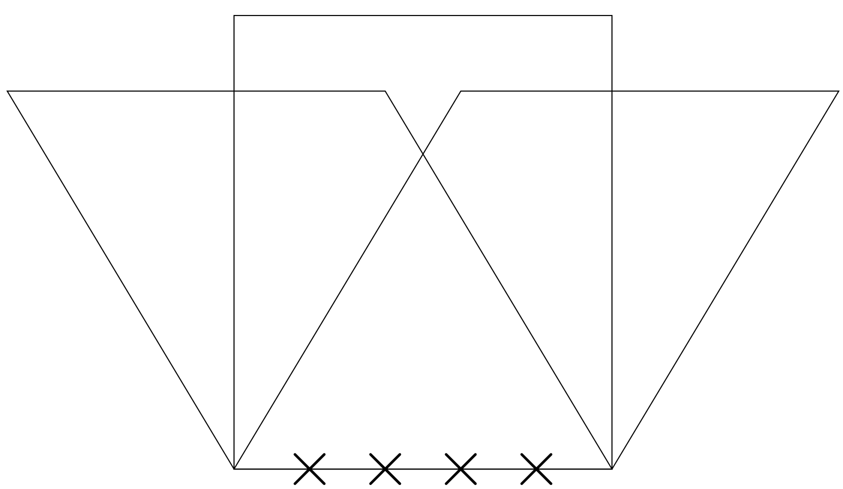
λ: We have

$$E = (\pi_1)_*\omega_{\mathcal{C}/\mathbb{P}^1} = (\pi_1)_*\mathcal{O}_C(1,1) = \mathcal{O}_{\mathbb{P}^1} \otimes (\pi_1)_*\mathcal{O}_C(0,1) \tag{22.16}$$

and $(\pi_1)_*\mathcal{O}_{\mathcal{C}}(0,1)$ is the trivial bundle of rank 3, so $E = \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus 3}$ implying $\deg c_1(E) = 3$.

Remark. δ can be broken up into $\delta = \delta_1 + \cdots + \delta_{\lfloor g/2\rfloor}$, where δ_i is the contribution to the ith boundary component of $\overline{\mathfrak{M}_g}$, using disconnecting nodes. We usually don't do this, though.

For our next example, let $S \subseteq \mathbb{P}^3$ be a smooth quartic surface and $\mathcal{C} \xrightarrow{\pi} \mathbb{P}^1$ be a general pencil of plane sections.



(22.17)

Let Γ be the intersection of the surface with the base line of the pencil. Then

$$
\begin{tikzcd}
\mathcal{C} \arrow[r, equal] & \mathrm{Bl}_\Gamma S \arrow[r, hook] & \mathbb{P}^1 \times S \arrow[r, "\alpha"] \arrow[d, "\pi"] & S \\
& & \mathbb{P}^1 &
\end{tikzcd}
\tag{22.18}
$$

We can use genericity to show that every fiber is either smooth or irreducible with one node. (Or just look at general S.)

For the degree of δ, use Riemann-Hurwitz again. We have $\chi(B) = 2$ and $\chi(F) = -4$. It turns out that $\chi(S) = 24$ (S is a K3 surface), so $\chi(\mathcal{C}) = 28$. Therefore $\deg \delta = \chi(\mathcal{C}) - \chi(B)\chi(F) = 36$.

Alternatively, for $S \subseteq \mathbb{P}^3$, consider the Gauss map $S \xrightarrow{[\frac{\partial f}{\partial x}, \ldots]} S^* \subseteq (\mathbb{P}^3)^*$. We want $\#(L \cap S^*)$ for L a line. This equals $\deg S^*$.

Now introduce divisor classes on $\mathcal{C} \subseteq \mathbb{P}^1 \times S$: let $\eta = \pi^* \mathcal{O}_{\mathbb{P}^1}(1)$ and $\zeta = \alpha^* \mathcal{O}_S(1)$. Note that $\mathcal{C} \sim \eta + \zeta$ in $A(\mathbb{P}^1 \times S)$. η is the class of a fiber, so $\eta^2 = 0$. We have $\eta \cdot \zeta = 4$ since a plane section with meet a fiber (a quartic curve) in four points. Finally, $\zeta^2 = 4$ as the intersection of S with a general line. Now

$$
K_{\mathbb{P}^1 \times \mathbb{P}^3} = -2\eta - 4\zeta \implies K_{\mathbb{P}^1 \times S} = -2\eta \implies K_{\mathcal{C}} = (K_{\mathbb{P}^1 \times S} + [\mathcal{C}])|_{\mathcal{C}} = -\eta + \zeta \implies \omega_{\mathcal{C}/\mathbb{P}^1} = \eta + \zeta.
\tag{22.19}
$$

We get $\kappa = (\eta + \zeta)^2 = 12$ as a result.

As for λ,

$$
\pi_* \omega_{\mathcal{C}/\mathbb{P}^1} = \pi_*(\pi^* \mathcal{O}_{\mathbb{P}^1}(1) \otimes \alpha^* \mathcal{O}_S(1)) = \mathcal{O}_{\mathbb{P}^1}(1) \otimes \pi_* \alpha_S^{\mathcal{O}}(1).
\tag{22.20}
$$

$\pi_* \alpha^* \mathcal{O}_S(1)$ is a bundle of rank 1 with fiber over $t \in \mathbb{P}^1$ equal to $H^0(\mathcal{O}_{C_t}(1)) = H^0(\mathcal{O}_{H_t}(1))$. We have an exact sequence

$$
0 \to \mathcal{O}_{\mathbb{P}^1}(-1) \to \mathcal{O}_{\mathbb{P}^1}^{\oplus 4} \to \pi_* \alpha^* \mathcal{O}_S(1) \to 0
\tag{22.21}
$$

so $c_1 = 1$, implying λ has degree 4.

Theorem 22.1. *1. $\operatorname{Pic} \overline{\mathfrak{M}_g} \otimes \mathbb{Q}$ is generated by $\lambda, \kappa, \delta_0, \ldots, \delta_{\lfloor g/2 \rfloor}$.*

2. If $g \geq 3$, these satisfy the unique relation $12\lambda = \kappa + \delta$.

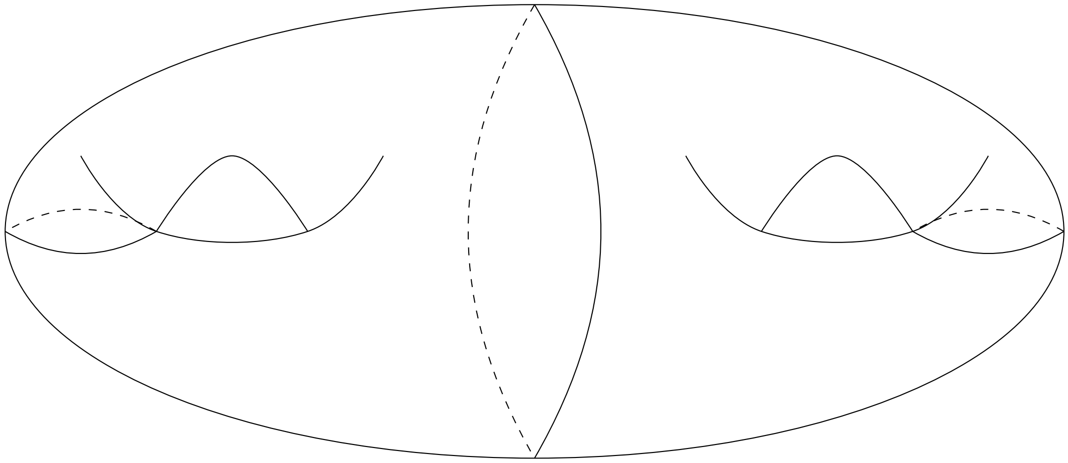
3. $\alpha\lambda - \beta\delta$ is ample if and only if $\alpha > 11\beta > 0$.

4. $K_{\overline{\mathfrak{M}}_g} = 13\lambda - 2\delta$ (at least for the moduli stack).

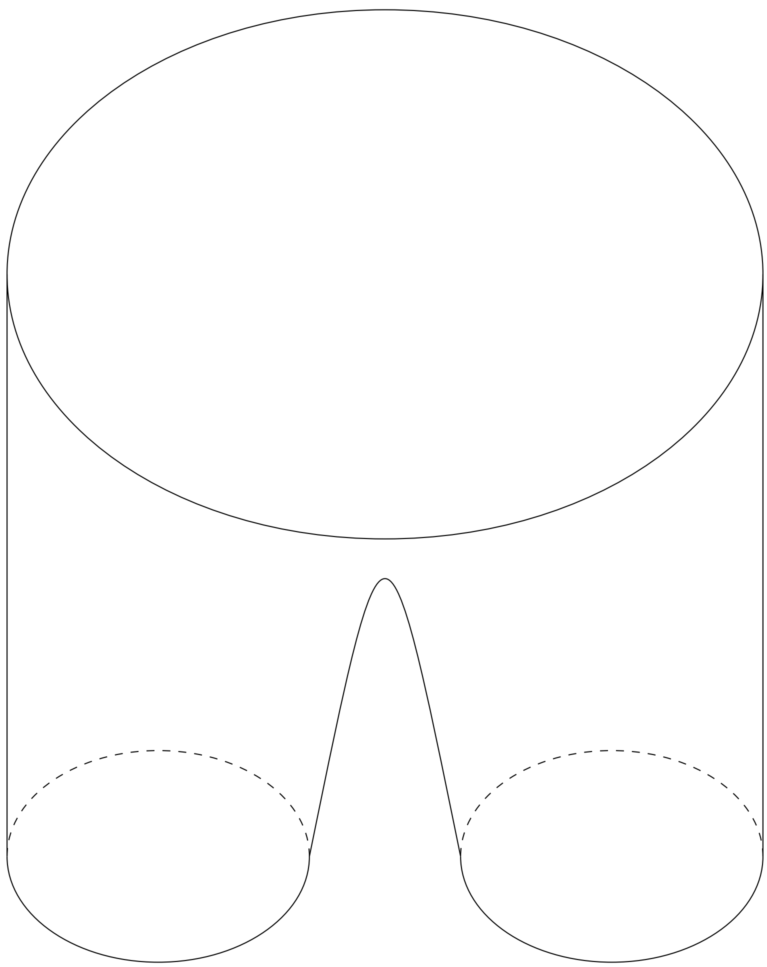
We will prove parts 2 and 4. 3 uses a technique not yet introduced, and 1 is very hard.

(Reference of how to get better understanding of stacks: Mumford, Picard groups of moduli problems.)

Part 1 was proved by Harer, using topological methods. This uses the description of $\mathfrak{M}_g$ via Teichmüller space:



(22.22)



(22.23)

A Riemann surface with a pair-of-pants decomposition is specified by $2g-2$ pieces with various gluings ($6g-6$ real parameters in total). We can recover $\mathfrak{M}_g = \tau_g/\Gamma_g$, where τ_g is the space above (which is contractible in $\mathbb{C}^{3g-3}$), and Γ_g is the mapping class group, which leaves the surface alone, but changes Teichmüller space. Harer tried to describe Γ_g.

More recently, Arbarello and Cornalba gave a simpler proof, but one which was still heavily based on topology and complex analysis.

Part 3 uses the notion of stability. This can be looked up in Moduli of Curves.

Parts 2 and 4 are applications of Grothendieck-Riemann-Roch.

After Serre duality, Riemann-Roch could be formulated as follows: for $\mathcal{L}$ a line bundle on C, $\chi(\mathcal{L}) = \underbrace{c_1(\mathcal{L})}_{d} - \frac{1}{2}\underbrace{c_1(T_C)}_{2g-2}$. More generally, if $\mathcal{F}$ is any coherent sheaf on C, then $\chi(\mathcal{F}) = c_1(\mathcal{F}) - \frac{1}{2} \cdot \operatorname{rank}(\mathcal{F}) \cdot c_1(T_C)$.

Now if S is a surface and $\mathcal{L}$ is a line bundle, then

$$\chi(\mathcal{L}) = \frac{c_1^2(\mathcal{L}) + c_1(T_S)c_1(\mathcal{L})}{2} + \frac{c_1^2(T_S) + c_2(T_S)}{12}. \tag{22.24}$$

Hirzebruch generalized this further. Let X be a smooth variety of dimension n and $\mathcal{F}$ a sheaf on X having rank r. Then $c(\mathcal{F}) = \prod(1+\alpha_i)$ where $c_k(\mathcal{F})$ is the kth elementary symmetric polynomial of the α_i. Now define $ch = \sum e^{\alpha_i}$. The homogeneous terms are symmetric, so expressible in terms of the c_i:

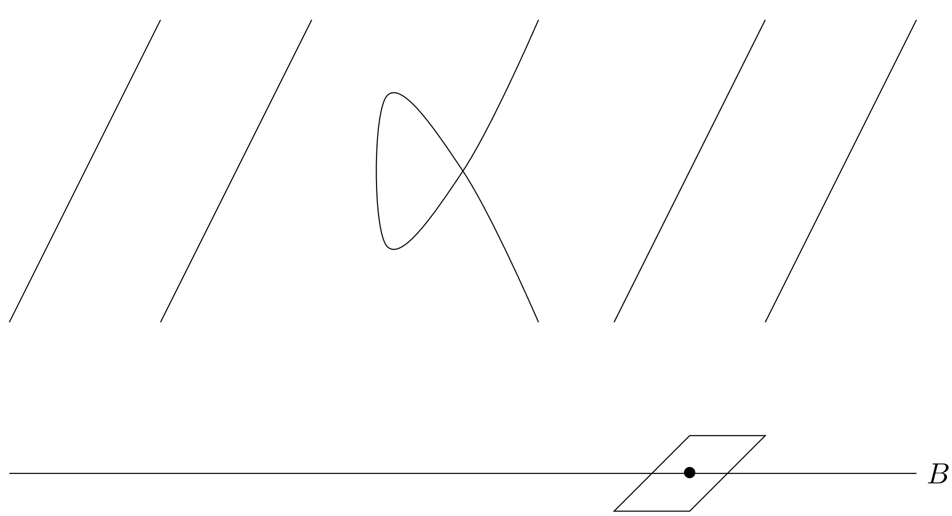
$$ch(\mathcal{F}) = \operatorname{rank}(\mathcal{F}) + c_1(\mathcal{F}) + \frac{c_1^2(\mathcal{F}) - 2c_2(\mathcal{F})}{2} + \cdots \tag{22.25}$$

Also define the Todd class

$$Td(\mathcal{F}) = \prod \frac{\alpha_i}{1 - e^{-\alpha_i}} = 1 + \frac{c_1(\mathcal{F})}{1} + \frac{c_1^2(\mathcal{F}) + c_2(\mathcal{F})}{12} + \cdots \tag{22.26}$$

Hirzebruch-Riemann-Roch states that if $\mathcal{F}$ is a coherent sheaf on X, then $\chi(\mathcal{F}) = [ch(\mathcal{F}) \cdot Td(T_x)]_n$.

Grothendieck put this in the relative setting. Suppose we have $\mathfrak{X} \xrightarrow{\pi} B$ and $\mathcal{F}$ a coherent sheaf on $\mathfrak{X}$.



(22.27)

Then we have

$$\sum_i (-1)^i ch(R^i \pi_* \mathcal{F}) = \pi_* \left(ch(\mathcal{F}) \cdot \frac{Td(T_{\mathfrak{X}})}{Td(T_B)} \right). \tag{22.28}$$

Given a family $\mathcal{C} \to B$ of stable curves (smooth for simplicity), write ω for $c_1(\omega_{\mathcal{C}/B})$. We want $\lambda = c_1(\pi_* \omega_{\mathcal{C}/B})$. By GRR, $R^1 \pi_* \omega_{\mathcal{C}/B}$ is the structure sheaf, so we end up with

$$c_1(\pi_* \omega_{\mathcal{C}/B}) = \pi_* \left(\left(1 + \omega + \frac{1}{2}\omega^2 + \cdots \right) \left(1 - \frac{1}{2}\omega + \frac{1}{12}\omega^2 + \cdots \right) \right) \tag{22.29}$$

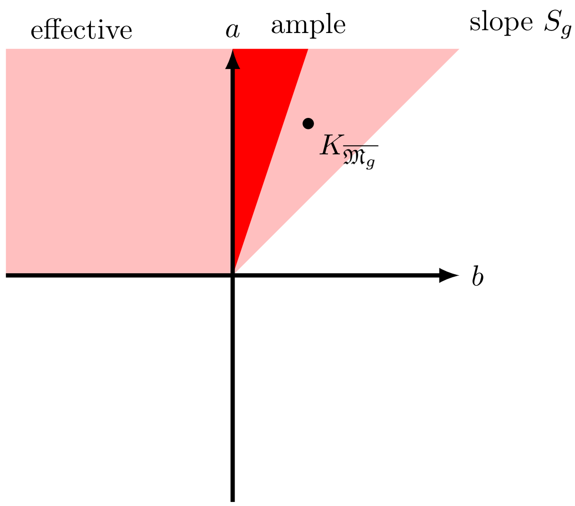
$$= \pi_* \left(1 + \frac{\omega}{2} + \frac{\omega^2}{12} + \cdots \right) \tag{22.30}$$

$$= g - 1 + \frac{\kappa}{12}. \tag{22.31}$$

If there are singular fibers, we get a contribution of δ from c_2 of the relative tangent bundle. This proves part 2.

To understand the canonical sheaf, if $C \in \mathfrak{M}_g$ is automorphism-free, then $T_C(\mathfrak{M}_g) = H^1(T_C)$. Suppose $\mathcal{C} \xrightarrow{\pi} \mathfrak{M}_g^0$ is the universal family of smooth automorphism-free curves. Then $T(\mathfrak{M}_g) = R^1\pi_*(T_{\mathcal{C}/\mathfrak{M}_g^0})$ so $T^*(\mathfrak{M}_g) = \pi_*(\omega^2_{\mathcal{C}/\mathfrak{M}_g})$ by Kodaira-Serre duality. So $K_{\mathfrak{M}_g} = \pi_*(ch(\omega^2) \cdot Td(\omega^*))$, and we obtain $K_{\mathfrak{M}_g} = 13\lambda$. (We can make a correction of $13\lambda - 2\delta$ for nodal curves.)

Recall the ample cone of $a\lambda - b\delta$. We want to determine the effective cone:



(22.32)

Observe that if $S_g < \frac{13}{2}$, then $\overline{\mathfrak{M}_g}$ is of general type, so not unirational. If $S_g > \frac{13}{2}$, then $h^0(\mathfrak{M}_g^k) = 0$ for every $k > 0$, so $\overline{\mathfrak{M}_g}$ has negative Kodaira dimension.

We won't take care of the issue that $\overline{\mathfrak{M}_g}$ is not smooth, but rather has canonical singularities. (To do this, analyze the local geometry of $\overline{\mathfrak{M}_g}$.)

To show S_g has low slope, we need to exhibit effective divisors of $\overline{\mathfrak{M}_g}$ and calculate their class.

We might look at W, the set of C with a special Weierstrass point $p \in C$ such that $h^0(\mathcal{O}_C((g-1)p)) \geq 2$. It turns out that the slope of $[W]$ is $9 + O(\frac{1}{g})$. This isn't low enough.

Another thing we might try is the set T of curves with a semicanonical pencil. This time the slope of $[T]$ equals $8 + O(\frac{1}{g})$, better but still not enough.

What does work is to look at the Brill-Noether divisor $B \subseteq \overline{\mathfrak{M}_g}$, equal to the set of C with a g^r_d having $\rho = -1$.

As an example, consider C of genus $g = 2k + 1$ with a pencil of degree $k + 1$; that is, genus 3 hyperelliptic curves. The result of calculation is that the slope of $[B]$ is $6 + \frac{12}{g+1}$. Hence $\overline{\mathfrak{M}_g}$ is of general type for $g \geq 24$.

(Actually, this doesn't always work. If $g = p - 1$, then such B doesn't exist. Then the Petri divisor can be used. We won't go over this.)

To do this, we first argue that $[B]$ is a linear combination of the form $a\lambda - b_0\delta_0 - \cdots - b_{\lfloor g/2 \rfloor}\delta_{\lfloor g/2 \rfloor}$, and then calculate a and the b_i by intersecting with test curves.

An example of a test curve is a pencil on a general K3 surface. A polarized K3 surface is S with a (birational) embedding $S \hookrightarrow \mathbb{P}^g$ of degree $2g - 2$. For $H \cong \mathbb{P}^{g-1}$ a hyperplane section of $\mathbb{P}^g$, $S \cap H = C$ a canonical curve. Taking a pencil of hyperplane sections of S, we get a family of stable curves of genus g

$$
\begin{tikzcd}
\widetilde{S} \arrow[r, equal] \arrow[d, "\pi"] & \mathrm{Bl}_\Gamma S \arrow[r, "\alpha"] & S \\
\mathbb{P}^1
\end{tikzcd}
\tag{22.33}
$$

Here $\Gamma = S \cap \mathbb{P}^{g-2}$, consisting of $2g - 2$ points.

We have $\chi(\widetilde{S}) = \chi(S) + 2g - 2 = 2g + 22$. But also $\delta = \chi(\widetilde{S}) - 2\chi(C) = 6g + 18$. Now $\omega_{\widetilde{S}/\mathbb{P}^1} = \pi^*\mathcal{O}_{\mathbb{P}^1}(1) \otimes \alpha^*\mathcal{O}_{\mathbb{P}^g}(1)$, so $E = \pi_*\omega_{\widetilde{S}/\mathbb{P}^1} = \mathcal{O}_{\mathbb{P}^1}(1) \otimes \pi_*(\alpha^*\mathcal{O}_{\mathbb{P}^g}(1))$, where $\pi_*(\alpha^*\mathcal{O}_{\mathbb{P}^g}(1))$ is a quotient of a trivial bundle of rank $g + 1$ by $\mathcal{O}_{\mathbb{P}^1}(-1)$. We conclude that $\lambda = \deg E = g + 1$.

On the other hand, the degree of $[B]$ on this family (that is, the number of fibers of the family that have Brill-Noether number -1) is zero. By geometric Riemann-Roch, no hyperplane section has a linear series with $\rho = -1$, because S does not contain many points with enough dependence relations.

We conclude that the slope of $[B]$ is $\frac{6g+18}{g+1} = 6 + \frac{12}{g+1}$.

Remark. This is only a heuristic. We need to calculate coefficients of higher boundary components.

To do...

- ☐ 1 (p. 9): **Fix Me** Formatting is awful right now.
- ☐ 2 (p. 27): I think I missed this lecture. Try to fill it in.
- ☐ 3 (p. 30): Figure this one out.
- ☐ 4 (p. 34): **Fix Me** Is this picture actually correct?
- ☐ 5 (p. 35): **Fix Me** Make this look better? (It will be tough!)