Contents

Chapter 2

The Riemann–Roch theorem

2A. How many sections?

To study curves via their maps to projective spaces, we want to estimate the dimension of the space of global sections of an invertible sheaf $\mathcal{L}$. The beginning of the story is the Riemann–Roch theorem.

Though we would like to be able to compute $h^0(\mathcal{L})$, it is much easier to compute the Euler characteristic indexed

$$\chi(\mathcal{L}) := \sum_{i\geq 0}(-1)^i h^i(\mathcal{L}).$$

This computes $h^0(\mathcal{L})$ itself in many cases, by virtue of the following result:

Theorem 2.1 (Serre–Grothendieck vanishing theorem). *If $\mathcal{F}$ is a coherent sheaf on a projective scheme X of dimension n, then for any i, the vector space $H^i(\mathcal{F})$ is finite-dimensional, and is 0 if $i > n$. Moreover, if $X \subset \mathbb{P}^m$ then for $d \gg 0$, $\mathcal{F}(d)$ is generated by its global sections and $H^i(\mathcal{F}(d)) = 0$ for all $i > 0$.* □

Proof. This is a combination of theorems due to Grothendieck and Serre. See [?, Theorems III.2.7 and III.5.2], and also [?] for a reasonably concrete proof. □

A shortcoming of this vanishing theorem is the lack of a bound on the number d needed to achieve the second assertion. For smooth curves and invertible sheaves this is corrected by Theorem 2.16, which gives a bound in terms of the genus and the degree.

One immediate consequence of Theorem 2.1 is that on a smooth variety the groups of invertible sheaves and divisor classes are the same:

Corollary 2.2. *If X is a projective variety that is nonsingular in codimension* 1*, every invertible sheaf $\mathcal{L}$ on X is of the form $\mathcal{L} = \mathcal{O}_C(D)$ for some Cartier divisor D on X. Thus if X is a smooth projective variety the map* div *is an isomorphism from the group of invertible sheaves to the group of divisor classes.*

Proof. Let $H \subset \mathbb{P}^r$ be a general hyperplane, and E the divisor of intersection of C with H. We know that for $n \gg 0$, $\mathcal{L}(n)$ has sections; and if F is the divisor of zeroes of one such section, we have

$$\mathcal{L} = \mathcal{O}_C(F - nE).$$

If X is smooth, then, since a regular local ring is a unique factorization domain, every codimension-one subvariety is defined locally by a single nonzerodivisor, and thus corresponds to a Cartier divisor. This implies that div is surjective. Furthermore any isomorphism between invertible sheaves is defined by multiplication with a global rational function, so that invertible sheaves defining linearly equivalent divisors are isomorphic. Thus div is injective as well. □

indexed

(de): this proof is pretty swift, and the result fundamental. Let's give a [redacted]artshorne citation too.

Riemann–Roch without duality. It follows from Theorem 2.1 that on any scheme $X \subset \mathbb{P}^r$ we have $\chi(\mathcal{L}(d)) = h^0(\mathcal{L}(d))$ for large d, and that $\chi(\mathcal{L}) = h^0(\mathcal{L}) - h^1(\mathcal{L})$ in the case of a curve.

Theorem 2.3 (easy Riemann–Roch). *If C is a smooth projective curve, and $\mathcal{L}$ is an invertible sheaf on C, then $\chi(\mathcal{L}) = \deg \mathcal{L} + \chi(\mathcal{O}_C)$.*

Proof. The result is tautological if $\mathcal{L} = \mathcal{O}_C$. Every invertible sheaf on C has the form $\mathcal{L} = \mathcal{O}_C(D)$ for some divisor D. If $p \in C$, then writing $\kappa(p)$ for the structure sheaf of the subscheme $p \in C$, the long exact sequence in cohomology associated to the short exact sequence

$$0 \to \mathcal{L}(-p) \to \mathcal{L} \to \mathcal{L} \otimes \kappa(p) \to 0$$

together with the isomorphism $\mathcal{L} \otimes \kappa(p) \cong \kappa(p)$ and the vanishing of higher cohomology of a sheaf with zero-dimensional support allows us to compute

$$\chi(\mathcal{L}) = \chi(\mathcal{L}(-p)) + \chi(\kappa(p)) = \chi(\mathcal{L}(-p)) + 1.$$

Since every divisor on C can be reached by adding and subtracting points, this suffices. □

Since the Euler characteristic of a sheaf is well-behaved, we can extend the result of Theorem 2.3 to invertible sheaves on any one-dimensional scheme C, by defining $\deg \mathcal{L} := \chi(\mathcal{L}) - \chi(\mathcal{O}_C)$. We will use this definition to express the self-intersection of a divisor on a surface in Section 2F.

We can make the Riemann–Roch theorem still more useful by understanding the error term $h^1(\mathcal{L})$. This requires the canonical divisor and Serre duality, to which we now turn.

2B. The most interesting linear series

The most important vector bundles on a manifold are the tangent and cotangent bundles. For reasons that will become clear, the focus in algebraic geometry is on the cotangent bundle or, equivalently, the sheaf of differential 1-forms. On a smooth curve C the *canonical sheaf* is the sheaf of differentials, which is an invertible sheaf; on a smooth variety of dimension n we define the canonical sheaf to be the n-th exterior power of the sheaf of differentials. A section of ω_C is thus a differential form, and the class of the divisor of such a form is usually denoted K_C. indexed

Cheerful Fact 2.4. Canonical sheaves are defined for any projective scheme; see Definition 16.6. They are usually called *dualizing sheaves* in that generality. The condition for the dualizing sheaf to be an invertible sheaf is that the scheme is (locally) Gorenstein, something that is true, for example, for any subscheme of $\mathbb{P}^r$ that is locally a complete intersection (see Section 16F).

On projective space we can compute the canonical sheaf directly; other computations of the canonical sheaf will usually reduce to this central case.

Theorem 2.5. *The canonical sheaf of $\mathbb{P}^r$ is $\mathcal{O}_{\mathbb{P}^r}(-r-1)$.* indexed

Proof. Let $x_0, \dots, x_r$ be the projective coordinates on $\mathbb{P}^r$ and let $U = \mathbb{P}^r \setminus H$ be the affine open set where $x_0 \neq 0$. Thus $U \cong \mathbb{A}^r$ with coordinates $z_1 := x_1/x_0$, $\dots, z_r := x_r/x_0$. The space of r-dimensional differential forms on U is spanned by $d(x_1/x_0) \wedge \cdots \wedge d(x_r/x_0)$, which is regular everywhere in U. In view of the formula

$$d\frac{x_i}{x_0} = \frac{x_0\,dx_i - x_i\,dx_0}{x_0^2}$$

we get

$$d\frac{x_1}{x_0} \wedge \cdots \wedge d\frac{x_r}{x_0} = \frac{dx_1 \wedge \cdots \wedge dx_r}{x_0^r} - \sum_{i=1}^{r} x_i \frac{dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_r}{x_0^{r+1}}$$

which has a pole of order $r+1$ along the locus H defined by x_0. Thus the divisor of this differential form is $-(r+1)H$, and this is the canonical class. $\square$

Cheerful Fact 2.6. A different derivation: there is a short exact sequence of sheaves, called the Euler sequence [?, II.8]:

Ch. II → II (or else Chapter II, §8 if you feel it's safer)

$$0 \to \Omega_{\mathbb{P}^r} \to \mathcal{O}_{\mathbb{P}^r}^{r+1}(-1) \to \mathcal{O}_{\mathbb{P}^r} \to 0.$$

Summing over all twists, and taking global sections, that is, applying H^0_*, we see that $H^0_*(\Omega_{\mathbb{P}^r})$ fits into an exact sequence:

$$0 \to H^0_*(\Omega_{\mathbb{P}^r}) \to S^{r+1}(-1) \xrightarrow{\delta_1} S \to \mathbb{C} \to 0,$$

where S is the homogeneous coordinate ring of $\mathbb{P}^r$ and δ_1 sends the i-th basis vector of $S^{r+1}(-1)$ to the i-th variable of S; that is, $H^0_*(\Omega_{\mathbb{P}^r})$ is the second syzygy of the residue field $\mathbb{C}$ of S. We can extend this sequence to the Koszul complex that is the free resolution of $\mathbb{C}$, [?, §17.5]:

$$0 \to S(-r-1) \xrightarrow{\delta_{r+1}} \bigwedge^r S^{r+1}(-r) \xrightarrow{\delta_r} \cdots \to S^{r+1}(-1) \to S \to \mathbb{C} \to 0.$$

For each i, the i-th exterior power of the map $H^0_*(\Omega_{\mathbb{P}^r}) \to S^{r+1}(-1)$ is an inclusion, and represents $\bigwedge^i(\Omega_{\mathbb{P}^r})$ as the sheaf associated to the graded module that is the $(i+1)$-st syzygy of $\mathbb{C}$. In particular, the canonical module $\omega_{\mathbb{P}^r} = \bigwedge^r(\Omega_{\mathbb{P}^r})$ is the sheaf associated to the $(r+1)$-st syzygy, $S(-r-1)$.

For more on syzygies, see Chapter 18.

The most important invariant of a smooth curve can be defined in terms of the canonical sheaf:

Definition 2.7. If C is an irreducible smooth curve we define the genus $g(C)$ to be the dimension of $H^0(\omega_C)$.

Computations of the canonical sheaf on a variety usually involve comparing the variety to a variety whose canonical sheaf is already known. The most useful results of this type are the *adjunction formula* and *Hurwitz's theorem*.

The adjunction formula. In the simplest case, the adjunction formula says that the canonical divisors of a smooth plane curve C of degree d are the intersections of C with curves of degree $d-3$ (see Figure 2.1). More generally, for a divisor X on a smooth variety Y, it says that the canonical sheaf on X is $\omega_Y(X)|_X$. This is an immediate consequence of the still more general formula below because the normal bundle of X is $\mathcal{O}_Y(X)$.

In general, the adjunction formula describes the difference between the canonical divisor of a subscheme and the restriction of the canonical divisor from the ambient variety. If $X \subset Y$ we define the *conormal sheaf* of X in Y to be $\mathcal{I}_{X/Y}/\mathcal{I}^2_{X/Y}$, and the *normal sheaf* of X in Y to be its dual,

$$\mathcal{N}_{X/Y} = \mathcal{H}om(\mathcal{I}_{X/Y}/\mathcal{I}^2_{X/Y}, \mathcal{O}_Y).$$

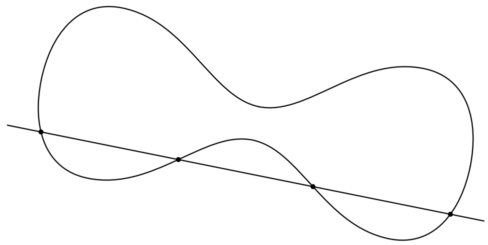

Figure 2.1. On a smooth plane quartic, the canonical divisors are its intersections with lines.

If X and Y are smooth, X is locally a complete intersection in Y, so $\mathcal{I}_{X/Y}/\mathcal{I}^2_{X/Y}$ is a vector bundle on X of rank equal to the codimension, $\dim Y - \dim X$. When, in addition, the codimension is 1, so that X is a divisor and $\mathcal{I}_X = \mathcal{O}_Y(-X)$, we get

$$\mathcal{N}_{X/Y} = \mathcal{O}_X(X).$$

Proposition 2.8 (adjunction formula). *Let $X \subset Y$ a smooth subscheme of codimension c in a smooth variety Y, and let K_Y be the canonical class of Y. The canonical class K_X of X is*

$$\omega_X = \bigwedge^c \mathcal{N}_{X/Y} \otimes \omega_Y.$$

In particular, when X is a divisor, K_X is the restriction to X of the divisor $K_Y + X$ on Y.

Proof. Because X is locally a complete intersection in Y there is an exact sequence of sheaves

$$0 \to \mathcal{I}_{X/Y}/\mathcal{I}^2_{X/Y} \to \Omega_Y|_X \to \Omega_X \to 0,$$

where Ω_X is the sheaf of differential forms on X (see [?, Proposition 16.3]), and $\mathcal{I}_{X/Y}|_X = \mathcal{O}_Y(-X)|_X = \mathcal{O}_X(-X)$. The proposition follows by taking top exterior powers, as in Lemma 2.9. $\square$

Lemma 2.9. *If*

$$0 \to \mathcal{E} \to \mathcal{F} \to \mathcal{G} \to 0$$

is a short exact sequence of locally free sheaves of ranks e, f, g on a scheme X, then there is a natural isomorphism

$$\bigwedge^e \mathcal{E} \otimes \bigwedge^g \mathcal{G} \to \bigwedge^f \mathcal{F}.$$

Proof of Lemma 2.9. We may define a map $\bigwedge^e \mathcal{E} \otimes \bigwedge^g \mathcal{G} \to \bigwedge^f \mathcal{F}$ in terms of local sections as

$$(\epsilon_1 \wedge \cdots \wedge \epsilon_e) \otimes (\gamma_1 \wedge \cdots \wedge \gamma_g) \mapsto \epsilon_1 \wedge \cdots \wedge \epsilon_e \wedge \gamma_1 \wedge \cdots \wedge \gamma_g.$$

This is globally well-defined because changing one of the γ_i by a local section of $\mathcal{E}$ would not change the exterior product. To check that the map is an isomorphism, it is enough to show that this is true locally.

Because $\mathcal{G}$ is locally free, there is a covering of X by open sets U so that the sequence

$$0 \to \mathcal{E}|_U \to \mathcal{F}|_U \to \mathcal{G}|_U \to 0$$

is a split exact sequence of free modules, $\mathcal{F}|_U = \mathcal{E}|_U \oplus \mathcal{G}|_U$. It follows that

$$\bigwedge^f \mathcal{F}|_U = \bigoplus_{i+j=f} \bigwedge^i \mathcal{E}|_U \otimes \bigwedge^j \mathcal{G}|_U.$$

In our case all the exterior powers of $\mathcal{E}$ vanish above the e-th, and all the exterior powers of $\mathcal{G}$ vanish above the g-th, so

$$\bigwedge^f \mathcal{F}|_U = \bigwedge^e \mathcal{E}|_U \otimes \bigwedge^g \mathcal{G}|_U,$$

with isomorphism given as above. □

Corollary 2.10. *If $C \subset \mathbb{P}^2$ is a smooth plane curve of degree d, then $\omega_C = \mathcal{O}_C(d-3)$; more generally, if $X \subset \mathbb{P}^r$ is a smooth complete intersection of hypersurfaces of degrees $d_1, \dots, d_c$ in $\mathbb{P}^r$ then $\omega_X = \mathcal{O}_X(\sum d_i - r - 1)$.*

Proof. Since $\mathcal{N}_{X/Y} = \bigoplus_{i=1}^{c} \mathcal{O}_X(d_i)$, the result follows from Theorem 2.8. □

Hurwitz's theorem. Given a (nonconstant) morphism $f : C \to X$ of smooth projective curves, the Riemann–Hurwitz formula computes the canonical sheaf C in terms of that of X and the local geometry of f. To do this we define the *ramification index*of f at $p \in C$, denoted $\mathrm{ram}(f, p)$, by the formula indexed

$$f^{-1}(q) = \sum_{\substack{p \in C \\ f(p)=q}} (\mathrm{ram}(f, p) + 1) \cdot p$$

for any point $q \in X$.

Proposition 2.11. *If $f : C \to X$ is a (nonconstant) morphism of smooth projective curves, there are only finitely many points $p \in C$ such that $\mathrm{ram}(f, p) > 0$.*

In light of this result we define the *ramification divisor* of f to be the divisor

$$R = \sum_{p \in C} \mathrm{ram}(f, p) \cdot p \ \in \ \mathrm{Div}(C).$$

and the *branch divisor* to be

$$B = \sum_{q \in X} \Big(\sum_{p \in f^{-1}(q)} \mathrm{ram}(f, p) \Big) \cdot q \ \in \ \mathrm{Div}(X).$$

Note that R and B have the same degree, which is $\sum_{p \in C} \mathrm{ram}(f, p)$.

Proof. The result follows from the separability of the map of fields of rational functions, $K(X) \to K(C)$, which holds because we are in characteristic 0 (in characteristic p the Frobenius map provides a counterexample). A proof using separability is given in [?, Section IV.2]. Here is an analytic version: indexed

In terms of local parameters z on C around p and w on X around $f(p)$, we can write the morphism as $z \mapsto w = z^m$ for some integer $m > 0$; that is, if w is a local parameter on X and z is a local parameter in the source, then the map

$$\mathbb{C}\{\{w\}\} \cong \widehat{\mathcal{O}}_{X,f(p)} \xrightarrow{\hat{f}^*} \widehat{\mathcal{O}}_{C,p} \cong \mathbb{C}\{\{z\}\}$$

of convergent power series rings induced by f^* sends w to uz^m, where u is a power series with nonvanishing constant term. In this case $\operatorname{ram}(f,p) = m-1$. These power series expansions are valid in a neighborhood of p, and the derivative of f vanishes at the ramification points in this neighborhood. Since the zeros of a nonconstant analytic function are isolated, the ramification points are isolated. Since C is compact in the classical topology, there are only finitely many. □

Hurwitz's theorem describes the difference between the canonical divisor of C and the pullback of the canonical divisor of X.

Theorem 2.12 (Hurwitz's theorem). [?, Proposition IV.2.3] *If $f : C \to X$ is a nonconstant morphism of smooth curves, with ramification divisor R, then*

$$K_C = f^*(K_X) + R,$$

or equivalently $\omega_C = (f^*\omega_X)(R)$.

Proof. Let B be the branch divisor of f. Choose a rational 1-form ω on X, and let $\eta = f^*(\omega)$ be its pullback to C. Since we have the freedom to multiply by any rational function on X, we can arrange for the zeroes and poles of ω to avoid B, so that ω is regular and nonzero at each branch point. (Actually the calculation goes through even without this assumption, albeit with more complicated notation.)

With this arrangement, for every zero of ω of multiplicity m we have exactly d zeroes of η, each with multiplicity m; and likewise for the poles of ω. At a point p where (locally) f has the form $z \mapsto w = z^e$ and $\omega = dw$, we have $\eta = z^{e-1}dz$; that is, η has a zero of multiplicity $\operatorname{ram}(f,p)$ at p. Thus the divisor K_C of η is $K_C = f^*(K_X) + R$. □

Example 2.13. Let $C \subset \mathbb{P}^2$ be a smooth plane curve and let p be a point of $\mathbb{P}^2$ not on C. Suppose that the coordinates on $\mathbb{P}^2$ are chosen so that the ideal sheaf of p is generated by the vector space of linear forms $W = \langle x_0, x_1\rangle$. The linear series $(\mathcal{O}_C(1), W)$ defines the projection of C from p to $\mathbb{P}^1$, a map of degree $d = \deg C$ (see Figure 2.2).

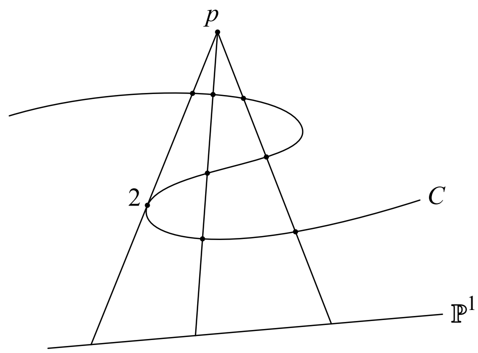

Figure 2.2. Projection of a plane cubic from a general point p to $\mathbb{P}^1$ is a three-to-one map.

The canonical sheaf of $\mathbb{P}^1$ has degree -2, so by Hurwitz's theorem K_C has degree $-2d + \deg R$, where R is the ramification divisor. We may choose coordinates so that none of the branch points lie on the line $x_0 = 0$. Taking this to be the line at infinity, we may compute R after passing to the affine open set $x_0 \neq 0$, where the projection map is given by the function $z = x_1/x_0$. Suppose that C is defined, in this open set, by the equation $f(x, y) = 0$. A point $q \in C$ is a ramification point if the tangent line to C at q passes through p, that is, if dx and

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$$

are linearly dependent. Since C is smooth, $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ cannot vanish simultaneously, this happens if and only if $\partial f/\partial y$ vanishes at q. The intersection of C with the curve defined by $\partial f/\partial y = 0$ has degree $d(d-1)$ by Bézout's theorem, so the degree of the ramification divisor R is $d(d-1)$. Thus the degree of the canonical divisor on C is $\deg K_C = -2d + d(d-1) = d(d-3)$, which is in accord with Corollary 2.10.

Example 2.14. Let $V = H^0(\mathcal{O}_{\mathbb{P}^1}(d))$ be the vector space of homogeneous polynomials of degree d in two variables. In the projectivization $\mathbb{P}(V^*) \cong \mathbb{P}^d$, let Δ be the locus of polynomials with a repeated factor. Since Δ is defined by the vanishing of the discriminant, it is a hypersurface. What is its degree?

To answer this, we intersect Δ with a general line; the degree of Δ is the degree of the intersection. Let $W \subset V$ be a general 2-dimensional linear subspace, that is, a general pencil of forms of degree d on $\mathbb{P}^1$. The linear series $\mathcal{W} = (\mathcal{O}_{\mathbb{P}^1}, W)$ defines a morphism $\phi_{\mathcal{W}} : \mathbb{P}^1 \to \mathbb{P}(W) \cong \mathbb{P}^1$ and the fiber over the point of $\mathbb{P}(W)$ corresponding to a form f of degree d is the divisor $\{f = 0\} \subset \mathbb{P}^1$. Thus the intersection of Δ with the line is the locus of polynomials in W with a multiple root; that is, the branch locus of $\phi_{\mathcal{W}}$, where we would count an m-fold root $m-1$ times if there were multiple roots. By Hurwitz's formula, the degree

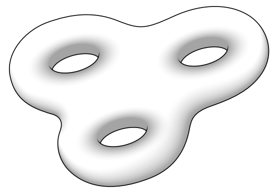

Figure 2.3. A Riemann surface of genus 3.

of the branch locus B of a degree d morphism from $\mathbb{P}^1$ to $\mathbb{P}^1$ is

$$\deg B = \deg \omega_{\mathbb{P}^1} - d \deg \omega_{\mathbb{P}^1} = 2d - 2.$$

Thus $\deg \Delta = 2d - 2$.

2C. Riemann–Roch with duality

We now return to the task of understanding $h^0(\mathcal{L})$ for an invertible sheaf $\mathcal{L}$ on a smooth curve. Since $\chi(\mathcal{L}) = h^0(\mathcal{L}) - h^1(\mathcal{L})$ is easier to compute, we would like to understand $h^1(\mathcal{L})$ in a more concrete way. The key is duality:

Theorem 2.15 (Serre duality). *If C is a smooth curve and D is a divisor on C, then*

$$H^1(D) = H^0(K_C - D)^* := \mathrm{Hom}_{\mathbb{C}}(H^0(K_C - D), \mathbb{C}),$$

and thus $h^1(D) = h^0(K_C - D)$.

For proofs see [?, Theorem III.5.2 and III.7.6].

For example we see that if C is a smooth connected curve then $h^1(\mathcal{O}_C) = h^0(K_C) = g(C)$ and thus $\chi(\mathcal{O}_C) = 1 - g(C)$. Using this we can recast Theorem 2.3 in the more useful form:

Theorem 2.16 (Riemann–Roch). *If D is any divisor on C, then*

$$h^0(D) - h^0(K_C - D) = \deg D - g(C) + 1.$$

In particular $\deg K_C = 2g(C) - 2$.

Proof. Combine Theorem 2.3 with Theorem 2.15. For the second statement, apply the formula with $D = K_C$. □

See Sections 16E and Theorem 16.24 for the corresponding results on singular curves.

We can now explain the relationship between the genus of a smooth curve, as we have defined it and the topological genus, the "number of holes" in the Riemann surface (Figure 2.3):

Cheerful Fact 2.17. (Hodge theory) The sole topological invariant of a smooth projective curve C, viewed as an analytic space, is its genus. As a manifold it is a compact, oriented surface, and its genus is half the rank of its first singular cohomology, $H^1(C;\mathbb{C})$, which is equal to its first de Rham cohomology. Breaking up the de Rham cohomology of any smooth projective complex variety X in terms of holomorphic and antiholomorphic differential forms we get the *Hodge decomposition*

$$H^i(X,\mathbb{C}) = H^i_{\text{de Rham}}(X) = \bigoplus_{j=0}^{i} H^j(\textstyle\bigwedge^{i-j}\Omega_X).$$

For a smooth curve C, this says in particular that

$$H^1(C;\mathbb{C}) = H^0(\omega_C) \oplus H^1(\mathcal{O}_C) = H^0(\omega_C) \oplus (H^0(\omega_C))^\vee,$$

so $h^0(\omega_C)$ is half the rank of the middle singular cohomology group, justifying the name "genus". For details, see [?, p. 116].

A divisor E of negative degree satisfies the equation $H^0(E) = 0$, so we get the form of the Riemann–Roch theorem originally proved by Riemann: indexed

Corollary 2.18. *For any divisor D of degree d we have*

$$h^0(D) \geq d - g + 1,$$

with equality if $d > 2g - 2$.

It was Gustav Roch, a student of Riemann's, who supplied the correction term $h^0(K_C - D)$ for divisors of lower degree. The dimension $h^0(K_C - D) = h^1(D)$ was called the *superabundance* of D: the "expected" number of sections was $d - g + 1$, and $h^1(\mathcal{L})$ reflected how much larger the actual number was. indexed indexed

Corollary 2.18 and Proposition 1.11 together show that all high degree divisors come from hyperplane sections in suitable embeddings; and unlike the general vanishing theorems, they give a bound on the degree necessary for vanishing of cohomology and for global generation:

Corollary 2.19. *Let D be a divisor of degree d on a smooth, connected projective curve of genus g.*

(1) *If $d > 2g - 2$ then $H^1(\mathcal{O}_C(D)) = 0$.*
(2) *If $d \geq 2g$ then $\mathcal{O}_C(D)$ is generated by global sections; that is, the complete linear series $|D|$ is basepoint free; and $\mathcal{O}_C(D)$ is very ample unless $D = K_C + E$ for some divisor E of degree 2.*
(3) *If $d \geq 2g + 1$ then $\mathcal{O}_C(D)$ is very ample; that is, the associated morphism $\phi_D : C \to \mathbb{P}^{d-g}$ is an embedding, and D is the preimage of the intersection of C with a hyperplane in $\mathbb{P}^{d-g}$.*

Proof. If $d > 2g - 2$ then $K - D$ has negative degree, and thus

$$h^1(D) = h^0(K - D) = 0.$$

The last two parts follow from the Riemann–Roch theorem and Proposition 1.11 ☐

Since the complement of a hyperplane in projective space is an affine space, we get an affine embedding result too:

Corollary 2.20. *If C is any smooth projective curve and $\Gamma \subset C$ a nonempty finite subset then $C \setminus \Gamma$ is affine (that is, isomorphic to a closed subscheme of an affine space).*

Proof. Let D be the divisor defined by Γ. By Corollary 2.19 a high multiple of D is very ample, and gives an embedding $\phi : C \to \mathbb{P}^n$ such that the preimage of the intersection of C with some hyperplane H is a multiple of D. It follows that $C \setminus \Gamma$ is embedded in $\mathbb{A}^n = \mathbb{P}^n \setminus H$. ☐

We can use Corollary 2.18 to determine the Hilbert polynomial of a projective curve. To do this, let $C \subset \mathbb{P}^r$ be a smooth curve of degree d and genus g, and consider the exact sequence of sheaves

$$0 \to \mathcal{I}_{C/\mathbb{P}^r}(m) \to \mathcal{O}_{\mathbb{P}^r}(m) \to \mathcal{O}_C(m) \to 0$$

and the corresponding exact sequence

$$H^0(\mathcal{O}_{\mathbb{P}^r}(m)) \xrightarrow{\rho_m} H^0(\mathcal{O}_C(m)) \to H^1(\mathcal{I}_{C/\mathbb{P}^r}(m)) \to 0.$$

The *Hilbert function* $h_C(m)$ of C is defined in terms of the homogeneous coordinate ring R_C of C by

$$h_C(m) = \dim_{\mathbb{C}}(R_C)_m = \operatorname{rank} \rho_m,$$

where $(R_C)_m$ is the degree m component of the homogeneous coordinate ring of C in $\mathbb{P}^n$.

By Theorem 2.1 we have $H^1(\mathcal{I}_{C/\mathbb{P}^r}(m)) = 0$ for large m so, for large m, $h_C(m) = h^0(\mathcal{O}_C(m))$. Using Theorem 2.1 again, we see that, for large m,

$$h^0(\mathcal{O}_C(m)) = \chi(\mathcal{O}_C(m)).$$

Finally, by the Riemann-Roch theorem,

$$\chi(\mathcal{O}_C(m)) = dm - g + 1,$$

so, for large m, the Hilbert function $h_C(m) = dm - g + 1$ is in agreement with the Hilbert polynomial $p_C(m) := \chi(\mathcal{O}_C(m))$.

More generally, we define the *arithmetic genus*:

Definition 2.21. If $C \subset \mathbb{P}^n$ is a 1-dimensional projective scheme with Hilbert polynomial $p_C(m) = \chi(\mathcal{O}_C(m))$, the *arithmetic genus* of $p_a(C)$ is $1 - \chi(\mathcal{O}_C) = 1 - p_C(0)$. If C is reduced and irreducible, then the *geometric genus* $g(C)$ is the genus of the normalization of C.

We see from the Riemann–Roch theorem that if C is smooth and connected, then $p_a(C) = g(C) = h^0(\omega_C)$, the genus of C. We will see that for reduced and irreducible curves $p_a(C) \geq g(C)$, with equality only when C is smooth. For some examples with curves that are not reduced and irreducible, see Exercise 2-8.

The Riemann–Roch theorem and Serre duality have extensions to arbitrary coherent sheaves in place of invertible sheaves and to singular curves, which we will explain in Chapter 16.

Divisors D for which $h^0(K_C - D) > 0$ are called *special divisors*. The existence or nonexistence of divisors D with given $h^0(D)$ and $h^1(D)$ often serves to distinguish one curve from another, and will be an important part of our study.

Residues. The Riemann–Roch theorem is so central to the study of curves that it is worth understanding from another point of view. We remarked at the beginning of Chapter 1 that a smooth projective curve over $\mathbb{C}$ is the same thing as a compact Riemann surface. We will briefly adopt the complex analytic viewpoint, and give an explanation of a special case of Theorem 2.16.

If $D = \sum a_i p_i$ is an effective divisor on a compact Riemann surface X then we write $L(D)$ for the vector space of meromorphic functions on X with poles of order at most a_i at p_i.

Theorem 2.22. *Let X be a compact Riemann surface of genus g, and let D be an effective divisor of degree d on X. Suppose that $K - D$ is also effective for some canonical divisor K. The dimension of $L(D)$ is $d - g + 1 + \dim_{\mathbb{C}} L(K - D)$.*

Because meromorphic functions on a Riemann surface are rational functions on the corresponding algebraic curve, and $L(D) = H^0(\mathcal{O}_C(D))$, the assertion is equivalent to Theorem 2.16.

Proof. Recall that the *residue* of a meromorphic 1-form ϕ at a point p on X is defined by an integral: choose a disc $\Delta \subset X$ containing p and in which ϕ is holomorphic except for its pole at p. The residue $\mathrm{Res}_p(\phi)$ is $\frac{1}{2\pi i}$ times the integral of ϕ along the boundary of Δ. If z is a local coordinate on Δ zero at p and we write the differential ϕ as indexed

$$\phi = \sum_{i=-n}^{\infty} a_i z^i \, dz$$

then by Cauchy's formula, the residue of ϕ at p is the coefficient a_{-1}.

Proposition 2.23. *If ϕ is a meromorphic differential on a compact Riemann surface X, then the sum of the residues of ϕ at all the poles of ϕ is* 0.

Proof. Apply Stokes' theorem to the complement of the union of small discs indexed
around each of the poles of ϕ. □

Let $D = \sum a_i p_i$ be an effective divisor on X, and set $d = \sum a_i = \deg D$. Locally, a function with a pole of order at most n at p may be written in terms of a local coordinate z at p as $\sum_{i=-n}^{\infty} a_i z^i$; the sum $\sum_{i=-n}^{-1} a_i z^i$ is called its *polar part*. By the maximum principle, a meromorphic function in $L(D)$ is determined, up to the addition of a constant, by its polar parts at the points p_i. Thus we have $\dim L(D) \le d + 1$.

When is a given collection $c_1, \dots, c_d \in \mathbb{C}[z^{-1}]$ the polar parts of a global meromorphic function f on X? A necessary condition is that if $\phi \in L(K)$ is a holomorphic differential on X, then

$$\sum \mathrm{Res}_{p_i}(f \cdot \phi) = 0.$$

This gives g linear relations on the c_i. However, if ϕ is a holomorphic differential vanishing at all the points p_i then the corresponding relation is trivial. Thus the number of linearly independent relations on the polar parts of f is actually $g - \dim L(K - D)$; and we arrive at an inequality

$$\dim L(D) \le d + 1 - g + \dim L(K - D).$$

This is a priori only an inequality. But we can apply the same logic to an effective divisor $K - D$, and we see that

$$\begin{aligned}\dim L(K - D) &\le \deg(K - D) + 1 - g + L(K - (K - D)) \\ &= 2g - 2 - d + 1 - g + L(D) \\ &= g - d - 1 + L(D).\end{aligned}$$

Adding the two inequalities we have

$$L(D) + L(K - D) \le L(D) + L(K - D).$$

Since the sum of the inequalities is an equality, we conclude that each inequality is also an equality; this is the Riemann–Roch formula in our special case. □

Arithmetic genus and geometric genus. Throughout this book, we will be primarily concerned with the geometry of smooth curves. Of course singular curves will arise — for example, as images of smooth curves under morphisms that are not embeddings. At least when C is a 1-dimensional variety (that is, a reduced irreducible 1-dimensional scheme) we can regard C as the image of a smooth curve in an optimal way:

Proposition 2.24. *If C_0 is a projective variety of dimension 1 then the* normalization $\nu : C \to C_0$ *of C_0 is a birational morphism from a smooth curve C. The curve C is again projective, and the pair (C, ν) is unique up to isomorphism. In particular, every birational map of smooth curves is an isomorphism.*

Proof. We use the result that the normalization (= integral closure) of a domain finitely generated over a field is again finitely generated, and nonsingular in codimension 1 [?, Theorem 4.14 and 11.5]. Thus, starting with a projective embedding of C we normalize the homogeneous coordinate ring of C. The resulting ring may have generators in many degrees, but a suitable Veronese subring will be generated in a single degree, and thus is the homogeneous coordinate ring of a smooth projective curve.

Localization commutes with normalization, and any map from a normal ring to a domain factors uniquely through any normalization. Therefore the normalization constructed above is independent of the choices. □

A different procedure for finding a smooth curve birational to a given curve is explained in Exercises 2-13 to 2-15.

Still assuming that C_0 is reduced and irreducible, we can relate the arithmetic and geometric genera of C_0 using the map of sheaves

$$\mathcal{O}_{C_0} \to \nu_* \mathcal{O}_C.$$

Since the normalization map of rings is injective and finite, this map is injective. The cokernel $\mathcal{F}$ is a coherent sheaf supported on the singular points of C_0, and is thus finite over $\mathbb{C}$. The definition implies that there is a short exact sequence

$$0 \to \mathcal{O}_{C_0} \to \nu_* \mathcal{O}_C \to \mathcal{F} \to 0.$$

Proposition 2.25. *Suppose that C_0 is a reduced, irreducible curve and let $\nu : C \to C_0$ be its normalization. Let $\mathcal{F} = \nu_* \mathcal{O}_C / \mathcal{O}_{C_0}$. If we set $\delta(C_0) := h^0(\mathcal{F})$, then*

$$p_a(C_0) - g(C) = \delta(C_0)$$

Proof. Since the normalization map $\nu : C \to C_0$ is finite, the direct images $R^i \nu_* \mathcal{O}_C$ vanish for $i > 0$, so the Leray spectral sequence (see 2.26 just below) computes $\chi(\nu_* \mathcal{O}_C) = \chi(\mathcal{O}_C)$. Thus

$$p_a(C_0) - g(C) = \chi(\mathcal{O}_C) - \chi(\mathcal{O}_{C_0}) = \chi(\mathcal{F}) = h^0(\mathcal{F})$$

since the support of $\mathcal{F}$ is finite. □

The operation of normalization localizes, so we can understand $\mathcal{F}$ by looking at it locally. Writing R for the affine ring of some affine subset $U \subset C_0$ we see that $\mathcal{O}_C|_U$ is the integral closure $\bar{R}$ of R, so $\mathcal{F}|_U$ is $\bar{R}/R$. Thus the annihilator of $\mathcal{F}|_U$, called the *conductor* $\mathfrak{f}_{\bar{R}/R}$, is an ideal of both $\bar{R}$ and R, and correspondingly $\nu^{-1}(\mathfrak{f}_{C/C_0})$ is also an ideal sheaf in $\mathcal{O}_C$.

Informally, we may say that $\delta(C_0)$ is the number of linear conditions a locally defined function f on C has to satisfy to be the pullback of a function from C_0. The length of the stalk of $\mathcal{F}$ at a particular singular point $p \in C_0$ is called the *δ-invariant* δ_p of the singularity. Thus

$$p_a(C_0) - g(C) = \sum_{p \in (C_0)_{\rm sing}} \delta_p.$$

Cheerful Fact 2.26. In general, the Leray spectral sequence addresses the situation of a morphism $f : X \to Y$ of varieties or schemes, and a coherent sheaf $\mathcal{G}$ on the source X; it says in this circumstance that there is a spectral sequence indexed

$$H^p(R^q f_*(\mathcal{G})) \Longrightarrow H^{p+q}(\mathcal{G}).$$

(This is a special case of the spectral sequence for the derived functors of a composite functor (H^0 composed with f_*); see [?, II.4.17.1] or [?, Section III.7] for proofs.)

In the simplest cases the δ invariant is easy to compute:

(1) A *node* of a curve C_0 is a point p such that an analytic neighborhood of p in C_0 consists of two smooth arcs intersecting transversely at p (See Figure 2.4, left.) Equivalently, the completion of the local ring $\mathcal{O}_{C_0,p}$ is isomorphic to $k[[x, y]]/(xy)$. If $p \in C_0$ is a node, its preimage in the normalization C of C_0 consists of two points $r, s \in C$. The condition for a function f on C to descend to C_0 — that is, to be the pullback of a function on C_0 — is that $f(r) = f(s)$; this is one linear condition, so $\delta_p = 1$. indexed

(2) A *cusp* (strictly speaking, an *ordinary cusp*) of a curve C_0 is a point p such that an analytic neighborhood of p in C_0 is given by the equation $y^2 = x^3$. (See Figure 2.4, second from left.) If $p \in C_0$ is an ordinary cusp then its preimage in the normalization C of C_0 will consist of one point $r \in C$. The condition for a function f on C is that the derivative $f'(r) = 0$; again, this is one linear condition, so $\delta_p = 1$. indexed

We will give more examples at the end of Chapter 15.

2D. The canonical morphism

We now return to the world of smooth curves.

The canonical sheaf on $\mathbb{P}^1$ has negative degree, so $|K_{\mathbb{P}^1}| = \emptyset$. If C is a curve of genus 1 then the canonical sheaf has a nonzero global section, and since the sheaf has degree $2g - 2 = 0$ this is nowhere vanishing, whence $\omega_C = \mathcal{O}_C$,

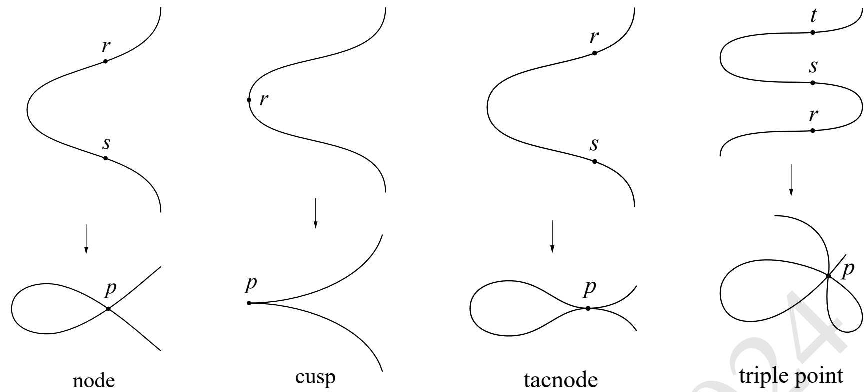

Figure 2.4. Simple planar curve singularities.

and $K_C = 0$. Thus in studying the canonical series we restrict our attention to curves C of genus $g \geq 2$.

Theorem 2.27. *Let C be a smooth curve of genus ≥ 2.*

(1) *$|K_C|$ is basepoint free.*

(2) *$|K_C|$ is very ample if and only if C admits no map of degree 2 to $\mathbb{P}^1$.*

A curve of genus ≥ 2 is said to be *hyperelliptic* if there exists a morphism $f : C \to \mathbb{P}^1$ of degree 2. It is easy to describe this in terms of linear series: indexed

Lemma 2.28. *Let C be a smooth, projective curve of genus $g \geq 2$. If C has an invertible sheaf $\mathcal{L}$ of degree ≤ 2 with two independent sections, then $\mathcal{L}$ has degree 2 and $|\mathcal{L}|$ defines a morphism of degree 2 to $\mathbb{P}^1$, so C is hyperelliptic. In particular, if $g(C) = 2$ then the canonical series $|K_C|$ defines a 2-to-1 morphism to $\mathbb{P}^1$, so C is hyperelliptic.*

Proof. Since $C \not\cong \mathbb{P}^1$, Theorem 1.7 shows that C cannot have a 1-dimensional linear series of degree < 2. Thus $\mathcal{L}$ has degree exactly 2, and no basepoints, so it defines a morphism of degree 2 to $\mathbb{P}^1$ as claimed. $\square$

Proof of Theorem 2.27. A point p is a basepoint of $|K_C|$ if and only if

$$h^0(K_C - p) = h^0(K_C) = g.$$

By the Riemann–Roch theorem, this is equivalent to $h^0(p) = 2$, which would imply that $C \cong \mathbb{P}^1$ by Theorem 1.7. Thus K_C has no basepoints.

By Proposition 1.11 we must show that for any two points $p, q \in C$ we have

$$h^0(K_C(-p - q)) = h^0(K_C) - 2 = g - 2.$$

Applying the Riemann–Roch theorem we see that this fails if and only if $h^0(\mathcal{O}_C(p + q)) \geq 2$ for some $p, q \in C$. By Lemma 2.28, this implies that C is hyperelliptic. Conversely, if C is hyperelliptic then for some divisor $D = p + q$ we have $h^0(D) = 2$, whence $h^0(K - p - q) = h^0(K) - 1$ by the Riemann–Roch formula. $\square$

The image of the canonical morphism of a nonhyperelliptic curve of genus $g > 2$ is called a *canonical curve,* and (for a nonhyperelliptic curve) we speak of the image as being *canonically embedded*. indexed

Geometric Riemann–Roch. There is a useful way of expressing the Riemann–Roch formula in terms of the geometry of the canonical map. Suppose $C \subset \mathbb{P}^r$ is a smooth curve and D an effective divisor on C, thought of as a subscheme of C. We define the *span* $\overline{D}$ of D to be the intersection of the hyperplanes $H \subset \mathbb{P}^r$ containing D. More generally, if $\phi : C \to \mathbb{P}^r$ is a map and D an effective divisor on C, we define the span $\overline{\phi(D)} \subset \mathbb{P}^r$ to be the intersection indexed

$$\overline{\phi(D)} := \bigcap_{\substack{H\subset\mathbb{P}^r \text{ a hyperplane} \\ D\subset\phi^{-1}(H)}} H.$$

Now suppose C is a smooth curve of genus $g \geq 2$ and $\phi_K : C \to \mathbb{P}^{g-1}$ is the canonical morphism. If D is an effective divisor on C of degree d, then since the hyperplanes in $\mathbb{P}^{g-1}$ containing $\phi_K(D)$ correspond (up to scalars) to sections of K_C vanishing on D, we see that

$$h^0(K_C - D) = \operatorname{codim} \overline{\phi_K(D)} \subset \mathbb{P}^{g-1} = g - 1 - \dim \overline{\phi_K(D)}.$$

Applying the Riemann–Roch theorem we obtain the *geometric Riemann–Roch theorem*:

Corollary 2.29. *If D is a divisor on a smooth curve C of genus ≥ 2 then*

$$r(D) = d - 1 - \dim \overline{\phi_K(D)}.$$

In particular, if $D = \sum_{i=1}^{d} p_i$ is a sum of distinct points, then the dimension of the linear series $|D|$ is equal to the number of linear relations among the images of the p_i on the canonical image of C. Thus, for example, a divisor given as the sum $D = p + q + r$ of three points will move in a pencil if and only if the three points are collinear on the canonical model.

In general, the condition that a form of degree d vanishes at a given point p in $\mathbb{P}^r$ is expressed by one homogeneous linear equation on the coefficients of the form, obtained by evaluating the variables at the coordinates of p. Thus the condition of vanishing on a set Γ of γ points is given by γ linear equations and the same is true when Γ is a finite subscheme of length γ, as one sees from the exact sequence

$$0 \to H^0(\mathcal{I}_\Gamma(d)) \to H^0(\mathcal{O}_{\mathbb{P}^r}(d)) \xrightarrow{ev} H^0(\mathcal{O}_\Gamma(d)) \to 0,$$

bearing in mind that

$$H^0(\mathcal{O}_\Gamma(d)) \cong H^0(\mathcal{O}_\Gamma) \cong \mathbb{C}^\gamma.$$

However, the linear equations may be dependent; that is, the map marked *ev* above may have rank $< \gamma$. We will say that the points of Γ *fail to impose independent conditions on hypersurfaces of degree d* by the amount equal to $\gamma - \operatorname{rank}(ev)$.

With this terminology, the geometric Riemann–Roch theorem says that the dimension of the complete linear series $|D|$ on a smooth curve C is the amount by which the points of D fail to impose independent conditions on canonical divisors, and the geometric version simply translates this into hyperplanes in the canonical embedding of C. The Cayley–Bacharach–Macaulay theorem 4.5 is a useful special case.

Remark 2.30. For a singular curve C_0 to have a canonical morphism it is necessary first of all that its canonical sheaf ω_{C_0} be invertible; when this is the case, we say that C_0 is (locally) Gorenstein, and then the theory above can be applied. See for example [?] and [?] for examples.

Linear series on a hyperelliptic curve. We can describe the canonical map of a hyperelliptic curve — and indeed all its special linear series — quite precisely:

Corollary 2.31. *Let C be a smooth hyperelliptic curve of genus $g \geq 2$. The curve C admits a unique degree 2 morphism $\pi : C \to \mathbb{P}^1$, and the canonical map $\phi_K : C \to \mathbb{P}^{g-1}$ is the composition of π with the Veronese map of degree $g-1$ from $\mathbb{P}^1$ to the rational normal curve of degree $g-1$; in particular, every canonical divisor is a sum of $g-1$ fibers of π and vice versa.*

Proof. If D is a general fiber of a degree 2 map $\pi : C \to \mathbb{P}^1$ so that $D = p + q$ and $r(D) = 1$, then by the Riemann–Roch theorem or its geometric version, D imposes only one condition on sections of K_C; that is, $\phi_K(p) = \phi_K(q)$. Consequently the degree of the canonical morphism ϕ_K is at least 2, and the image is thus a nondegenerate curve in $\mathbb{P}^{g-1}$ of degree $\leq (2g-2)/2 = g-1$. By Theorem 1.7, the image of ϕ_K is the rational normal curve of degree $g-1$ and every fiber of ϕ_K is a divisor linearly equivalent to D. It follows that π is determined by K, so it is unique. Since every canonical divisor is the pullback of a hyperplane section of the rational normal curve, every canonical divisor is a sum of $g-1$ fibers of π. □

We can use these ideas to analyze all special divisors.

Corollary 2.32. *Let C be a smooth hyperelliptic curve with map $\pi : C \to \mathbb{P}^1$ of* indexed
degree 2. Every special divisor D of degree d on C is a sum of s fibers of π and $d-2s$ points $p_1, \dots, p_{d-2s}$ with distinct images under the canonical map; and in

this case we have $r(D) = s$ and the points p_i are base points of the linear series $|D|$. Thus no divisor on a hyperelliptic curve is both special and very ample.

Proof. Suppose that D contains the sum E of s fibers of π and no more; and write $D = E + D'$, with $\deg D' = d - 2s$.

If $\phi_K : C \to \mathbb{P}^{g-1}$ is the canonical map, then $\phi_K(D')$ consists of $d - 2s$ distinct points, while $\phi_K(E)$ consists of s points. By Corollary 1.9 the span of $\phi_K(D)$ has dimension $\min\{g-1, d-s-1\}$, so $r(D) = d-1-\min\{g-1, d-s-1\} = \max\{s, d-g\}$. Since D is a special divisor, $r(D) > d - g$, so $r(D) = s$. Since $r(E) = s$, we see that E is basepoint free, so D' is the base locus of D. □

2E. Clifford's theorem

While the Riemann–Roch theorem gives a lower bound for the dimension of a linear series, $r(\mathcal{L}) := h^0(\mathcal{L}) - 1 \geq \deg \mathcal{L} - g$, Clifford's theorem gives an upper bound. If $\deg \mathcal{L} > 2g - 2$, then the Riemann–Roch inequality becomes an equality, so it is enough to treat the case $\deg \mathcal{L} \leq 2g - 2$. The bound is actually a corollary of the Riemann–Roch theorem.

Corollary 2.33. *Let C be a curve of genus g and $\mathcal{L}$ a line bundle of degree $d \leq 2g - 2$. Then*

$$r(\mathcal{L}) \leq \frac{d}{2}.$$

Proof. If $\mathcal{L}$ is nonspecial then, since $g \geq d/2+1$, we have $r(\mathcal{L}) = d-g+1 \leq d/2$. Otherwise we have

$$r(K_C \otimes \mathcal{L}^{-1}) = r(\mathcal{L}) + g - d - 1$$

by the Riemann–Roch theorem, and so by Proposition 1.10

$$g = r(K_C) + 1 \geq r(\mathcal{L}) + r(K_C \otimes \mathcal{L}^{-1}) + 1 = 2r(\mathcal{L}) + g - d;$$

hence $r(\mathcal{L}) \leq d/2$. □

The usual statement of Clifford's theorem includes a description of when equality can occur:

Theorem 2.34. *Let C be a curve of genus g and $\mathcal{L}$ a line bundle of degree $d \leq 2g - 2$. If*

$$r(\mathcal{L}) = \frac{d}{2},$$

the largest possible value, then either

(1) *$d = 0$ and $\mathcal{L} = \mathcal{O}_C$;*

(2) *$d = 2g - 2$ and $\mathcal{L} = K_C$; or*

(3) *C is hyperelliptic, and $|\mathcal{L}|$ is a multiple of the g^1_2 on C.*

IN PROGRESS

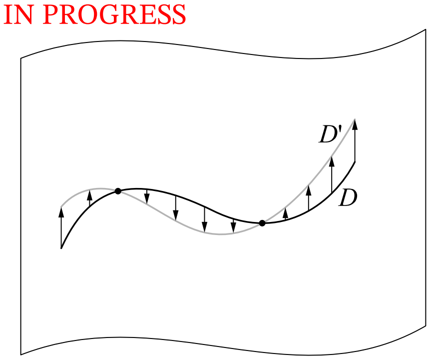

Figure 2.5. The result, D', of moving D infinitesimally along a normal vector field meets D twice.

From Corollary 2.31 and Corollary 2.29 we see that the equality does indeed hold in each of the cases enumerated. See Corollary 10.13 for the converse and [?, IV.5.4] for a different proof.

2F. Curves on surfaces

We will often analyze curves on a smooth surface. Compared to the theory of linear series on curves, there is a new element: the intersection pairing. We refer to [?, Chapter V] and [?, Chapter I] for proofs of the unproven statements in this section. We suppose for this section that S is a smooth projective surface, and write $\mathrm{Pic}(S)$ for the group of invertible sheaves on S.

The intersection pairing. When two codimension-1 subschemes D, E on S meet transversely we define $D \cdot E$ to be the number of points in which they meet. It is less obvious how one might define an intersection number $D \cdot E$ in more general cases, including the case $D = E$. However a codimension-1 subvariety of S has a fundamental class in $H^2(X, \mathbb{Z})$, and the product just defined agrees with the cup product pairing

$$H^2(S,\mathbb{Z}) \times H^2(S,\mathbb{Z}) \to H^4(S,\mathbb{Z}) \cong \mathbb{Z},$$

showing that it is possible. For example, the self-intersection $D \cdot D$ can be realized geometrically by choosing a normal vector field σ on D and using it to push a copy D' of D infinitesimally away from D; then one can count the points where σ vanishes (see Figure 2.5. Here one must take orientations into account, and the result is that $D \cdot D$ may be negative. In the previous case (and when σ is complex analytic) the multiplicities are all positive because a complex subvariety is canonically oriented using the orientation of $\mathbb{C}$ itself.

This pairing can be realized algebraically as follows:

First, if D and E have no common components, then the *intersection multiplicity* $m_S(D, E, p)$ at $p \in S$ is defined as the vector space dimension of the local ring $\mathcal{O}_{S,p}/(\mathcal{I}_{D,p} + \mathcal{I}_{E,p})$, and

$$D \cdot E = \sum_p m_S(D, E, p).$$

Quite generally, the intersection product can be computed using the (algebraic) Euler characteristic. This works even for Cartier divisors on a singular surface: Setting $\mathcal{L} := \mathcal{O}_S(D)$ and $\mathcal{M} := \mathcal{O}_S(E)$ to simplify the notation, we have indexed

$$D \cdot E = \chi(\mathcal{O}_S) - \chi(\mathcal{L}^{-1}) - \chi(\mathcal{M}^{-1}) + \chi(\mathcal{L}^{-1} \otimes \mathcal{M}^{-1}).$$

Theorem 2.35. *The pairing* $(D, E) \mapsto (D \cdot E)$ *is the unique bilinear map* $\operatorname{Pic}(S) \times \operatorname{Pic}(S) \to \mathbb{Z}$ *extending the case of intersections of two curves meeting transversely at smooth points of* S*.*

An important special case is that of the self-intersection $D \cdot D = D^2$. The general formula reduces immediately to the case of an effective divisor, and in that case it resembles the C^∞ construction above:

Corollary 2.36. *If* D *is a codimension* 1 *subscheme of* S*, then the self-intersection* $D \cdot D$ *is the degree of the normal bundle* $\mathcal{N}_{D/S} := \mathcal{O}_D(D)$*.*

Proof. Substituting $\mathcal{O}_S(-D)$ for both $\mathcal{L}^{-1}$ and $\mathcal{M}^{-1}$ we have the exact sequences

$$0 \to \mathcal{O}_S(-D) \to \mathcal{O}_S \to \mathcal{O}_D \to 0$$
$$0 \to \mathcal{O}_S(-2D) \to \mathcal{O}_S(-D) \to \mathcal{O}_D(-D) \to 0;$$

the Riemann–Roch theorem then gives

$$\chi(\mathcal{O}_S) - \chi(\mathcal{O}_S(-D)) = \chi(\mathcal{O}_D) = -p_a(D) + 1,$$
$$\chi(\mathcal{O}_S(-D)) - \chi(\mathcal{O}_S(-2D)) = \chi(\mathcal{O}_D(-D)) = -\deg \mathcal{O}_D(D) - p_a(D) + 1.$$

Subtracting the second of these equations from the first we see that $D \cdot D = \deg \mathcal{O}_D(D)$. ☐

Using the intersection pairing we can turn the adjunction formula of Proposition 2.8 into a numerical formula:

Theorem 2.37 (adjunction formula). *If* C *is a Cartier divisor on a smooth surface* S *then*

$$p_a(C) = \frac{(K_X + C) \cdot C}{2} + 1.$$

The Riemann–Roch theorem for smooth surfaces. Often we wish to compute the dimension of the space of sections of an invertible sheaf, but as with the case of curves, the Euler characteristic is more accessible:

Theorem 2.38 (Riemann–Roch for surfaces). *Let $\mathcal{L}$ be an invertible sheaf on a smooth surface S. The Euler characteristic $\chi(D) := h^0(\mathcal{L}) - h^1(\mathcal{L}) + h^2(\mathcal{L})$, where $\mathcal{L} = \mathcal{O}_S(D)$, is given by*

$$\chi(D) = \chi(\mathcal{O}_S) + \frac{(D - K_S) \cdot D}{2} + 1.$$

Blowups of smooth surfaces. It is useful to know what happens under mappings of surfaces, particularly the case of the mapping corresponding to blowing up a point.

Theorem 2.39. *If $\pi : X \to Y$ is a birational map of smooth surfaces, then the pullback map on divisors preserves the intersection pairing. If X is the blowup of Y at a point p with exceptional divisor $E = \pi^{-1}(p)$, then:*

(1) $\operatorname{Pic} X = \pi^*(\operatorname{Pic} Y) \oplus \mathbb{Z}E$.

(2) *The canonical class on X is given by $K_X = \pi^*(K_Y) + E$.*

(3) *The intersection pairing on* $\operatorname{Pic} X$ *is given by:*
   - $\pi^*(D) \cdot \pi^*(D') = D \cdot D'$ *for all* $D, D' \in \operatorname{Pic} Y$.
   - $\pi^*(D) \cdot E = 0$ *for all* $D \in \operatorname{Pic} Y$.
   - $E \cdot E = -1$.
   - $K_X \cdot E = -1$.
   - *If C is a curve that has an m-fold point at p then $\pi^{-1}(C)$ contains E with multiplicity m, so that the proper transform $\tilde{C}$ — that is, the closure in X of the preimage of $C \setminus \{p\}$ — has class*

$$\tilde{C} \sim \pi^* C - mE.$$

Using these facts, we can compare the adjunction formula applied to a curve C on a smooth surface S to the formula applied to the proper transform $\tilde{C} \subset \tilde{S}$: we have

$$\tilde{C} \cdot \tilde{C} = C \cdot C - m^2 \quad \text{and} \quad \tilde{C} \cdot K_{\tilde{S}} = C \cdot K_S + m.$$

Combining these, we arrive at the formula

$$p_a(\tilde{C}) = p_a(C) - \binom{m}{2},$$

and thus the δ-invariant of $p \in C$ is $\binom{m}{2}$ plus the sum of the δ-invariants of all the points of $\tilde{C}$ in the preimage of p. One can resolve the curve singularity by repeatedly blowing up in this way and thus compute the δ-invariant of any singularity as a sum of binomial coefficients.

In the special case where $\tilde{C}$ is smooth — for example, if the point $p \in C$ is an *ordinary m-fold point*, consisting of m smooth branches with distinct tangent lines — we conclude that the δ-invariant of the point p is $\binom{m}{2}$.

Blowups occur frequently in the theory of surfaces, and are characterized by *Castelnuovo's theorem* [?, Theorem V.5.7].

Theorem 2.40. *If $E \subset X$ is a curve on a smooth projective surface X and that $E^2 = E \cdot K_X = -1$ then E can be "blown down" in the sense that X is the blowup of a smooth surface Y at a point $p \in Y$, and E is the exceptional divisor.*

2G. Quadrics in $\mathbb{P}^3$ and the curves they contain

We will frequently be interested in curves on a quadric surface in $\mathbb{P}^3$, and we can describe these curves and their intersections very concretely.

The classification of quadrics. The form defining a quadric $Q \subset \mathbb{P}^3$ can be written in suitable coordinates x_i as

$$q = \sum_{i=0}^{3} x_i^2,$$

where $r \leq 4$ is the *rank* of Q. (The corresponding statement is also true in $\mathbb{P}^r$, with $m \leq r+1$.) Equivalently, this is the rank of the symmetric matrix M representing the bilinear form $\frac{1}{2}(q(x+y) - q(x) - q(y))$.

(1) A quadric of rank 1 is a double plane.

(2) A quadric of rank 2 is the union of two distinct planes.

(3) A quadric Q of rank 3 is a cone over a smooth plane conic. The cone point is called the vertex. Every line on such a quadric passes through the vertex. The Picard group of Q is $\mathbb{Z}$ [?, Exercise II.6.5]. indexed

(4) A quadric Q of rank 4 is smooth and is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^3$, embedded by the Segre embedding. Every line on such a quadric has the form $\mathbb{P}^1 \times \{p\}$ or $\{p\} \times \mathbb{P}^1$; the two families of such lines are called *rulings*. indexed
The Picard group of Q is $\mathbb{Z} \times \mathbb{Z}$, generated by the lines of the rulings. [?, Example II.6.1].

(5) Quadrics in $\mathbb{P}^r$: Inside the space $\mathbb{P}(\mathcal{O}_{\mathbb{P}^r}(2)) = \mathbb{P}^{\binom{r+2}{2}-1}$ the space of singular quadrics is a hypersurface of degree $r+1$, called the *discriminant*, which is defined by the determinant of a generic symmetric $(r+1) \times (r+1)$ matrix M. More generally, the rank k locus is defined by the $(k+1) \times (k+1)$ minors of M.

It follows that an irreducible nondegenerate curve cannot be contained in a quadric of rank 1 or 2.

Some classes of curves on quadrics.

Example 2.41. Let C be a reduced curve on a quadric Q of rank 3 in $\mathbb{P}^3$. If C has even degree $2m$ then C is a complete intersection of the quadric with a hypersurface of degree m. Therefore if the curve is smooth it cannot contain the vertex of the quadric. The genus of C is $(m-1)^2$.

If C has odd degree $2m+1$, then the union of C with a line on the quadric has even degree, so if L_1, L_2 are lines on the quadric, then

$$C \cup L_1 = Q \cap F_1 \quad \text{and} \quad C \cup L_2 = Q \cap F_2$$

are complete intersections of Q with forms of degree $m+1$, and the homogeneous ideal of C is generated by the equations of Q, F_1 and F_2. The genus of C is $m(m-1)$. The curve C contains the vertex, and is a Weil divisor, not a Cartier divisor. These things follow from the description of the desingularization of Q in Corollary 17.25.

Example 2.42. If C is a reduced curve on a quadric Q of rank 4, then in terms of the isomorphism $Q \cong \mathbb{P}^1 \times \mathbb{P}^1$ we can express the divisor class of C as $(a,b) \in \operatorname{Pic}(Q) = \mathbb{Z} \oplus \mathbb{Z}$. The intersection form on Q is determined by the fact that the lines of a ruling are all linearly equivalent, so that $(1,0)^2 = (0,1)^2 = 0$ and $(1,0)\cdot(0,1) = 1$; thus $(a,b)\cdot(d,e) = ae + bd$. The hyperplane section has class $(1,1)$, and the canonical class is $(K_{\mathbb{P}^3} + Q)\mid_Q = (-2,-2)$ by the adjunction formula.

Thus if C has class (a,b) then the degree of C is $a+b$, while the genus of C is $(a-1)(b-1)$. Note that if $a = b$, the curve is a complete intersection, and if $b = a+1$ then the curve is residual to a line in a complete intersection of the quadric with a surface of degree b. In these cases we get the same genus and degree as we would for a curve on a rank 3 quadric.

The cohomology of $\mathcal{O}_Q(a,b)$ is determined by the Künneth formula (or the Leray spectral sequence) as

$$H^i(\mathcal{O}_Q(a,b)) = \bigoplus_{i=s+t} H^s(\mathcal{O}_{\mathbb{P}^1}(a)) \otimes H^t(\mathcal{O}_{\mathbb{P}^1}(b)).$$

The degrees of the generators of the homogeneous ideal of C can be determined from the exact sequence

$$0 \to \mathcal{I}_{Q/\mathbb{P}^3} \to \mathcal{I}_{C/\mathbb{P}^3} \to \mathcal{I}_{C/Q} \to 0.$$

Since $\mathcal{I}_{C/Q}(m) = (m-a, m-b)$ we see that, supposing $0 < a \le b$ and $1 < b$ there are no generators other than the equation q of Q having degree $< b$, and there are $b-a+1$ generators of degree exactly b; with q these are the minimal generators of the homogeneous ideal (Exercise 2-12).

2H. Exercises

Exercise 2-1. (1) The degree of a zero-dimensional subscheme $X \subset \mathbb{P}^r$ is by definition the value of the Hilbert polynomial of X, which is a constant. Show that this is the sum of the lengths of the components of X. (Hint: use the Serre–Grothendieck vanishing theorem.) indexed

(2) The degree of a projective subscheme $X \subset \mathbb{P}^r$ of dimension n is defined to be the degree of the 0-dimensional scheme that is the intersection of X with a general plane of degree $r - n$. Prove that this is the leading coefficient of the Hilbert polynomial, multiplied by $n!$. (Hint: compare the Hilbert polynomial of X with that of a hyperplane section of X.)

(3) If $C \subset \mathbb{P}^r$ is a smooth curve of genus g and degree d, show that the Hilbert polynomial of C is $H_C(m) = dm - g + 1$. (Hint: use the Riemann–Roch formula together with the Serre–Grothendieck vanishing theorem.)

Exercise 2-2. If $C \subset \mathbb{P}^r$ is a 1-dimensional variety with normalization $\phi : \widetilde{C} \to C$, and $X \subset \mathbb{P}^r$ is any subscheme that does not contain C we define the of intersection $\mathrm{mult}(C, X; p)$ at the point $p \in X \cap C$ to be the sum of the lengths of the finite scheme $\phi^{-1}(X)$ at the points of $\phi^{-1}(p)$.

(1) Show that C is singular at p if and only if $\mathrm{mult}(C, X; p) \geq 2$ for all X containing p. Show further that if $H \subset \mathbb{P}^r$ is a hyperplane then $\deg C = \sum_{p \in C} \mathrm{mult}(C, H; p)$.

(2) Show that the degree of the image of C under projection from p is the degree of C minus $\mathrm{mult}(C, H; p)$ for a general hyperplane H.

Hint: for the degree formula note that $\sum_{p \in C} \mathrm{mult}(C, H; p)p$ is the divisor associated to the morphism $\widetilde{C} \to \mathbb{P}^r$ with image C.

In the following series of exercises, we will work with the smooth projective curve birational to a possibly singular affine curve $C^\circ := V(f(x, y)) \subset \mathbb{A}^2$; this is the unique smooth projective curve containing the normalization of C° as a Zariski dense open subset. indexed

Exercise 2-3. Let C be the smooth projective curve birational to the affine plane curve $C^\circ := V(y^3 + x^3 - 1)$, and let $\pi : C \to \mathbb{P}^1$ be the map given by the rational function x.

(1) Show that the closure $\overline{C^\circ}$ of C° in $\mathbb{P}^2$ is smooth (i.e., $C = \overline{C^\circ}$).

(2) Find the branch points and ramification points of π, and deduce that the genus of C is 1.

(3) For any two points $p, q \in C$ find the complete linear series $|p + q|$. (Hint: let r be the third point of intersection of the line $\overline{pq}$ with the closure of C°, and consider the pencil of lines through r.)

(4) Find the unique map $\eta : C \to \mathbb{P}^1$ of degree 2 such that $\eta((1,0)) = \eta((0,1))$, and determine the ramification points of η.

(5) Show that C is isomorphic to the smooth projective curve associated to the affine plane curve $y^2 + x^3 = 1$.

For the next three exercises, let C° be the affine plane curve given as the zero locus of $y^2 - x^6 + 1$, and let C be the corresponding smooth projective curve. The map $C^\circ \to \mathbb{A}^1$ given by the projection $(x, y) \mapsto x$ extends to a map $\pi : C \to \mathbb{P}^1$, expressing C as a 2-sheeted cover of $\mathbb{P}^1$ branched over the points $1, \zeta, \dots, \zeta^5$, where ζ is any primitive sixth root of unity. In addition, let p and $q \in C$ be the two points lying over the point $\infty \in \mathbb{P}^1$.

Exercise 2-4. With C° and C as above show that the map $C^\circ \to \mathbb{A}^1$ given by the projection $(x, y) \mapsto x$ extends to a map $\pi : C \to \mathbb{P}^1$, expressing C as a 2-sheeted cover of $\mathbb{P}^1$ branched over the points $1, \zeta, \dots, \zeta^5$, where ζ is any primitive sixth root of unity. Show that there are two distinct points p and $q \in C$ lying over the point $\infty \in \mathbb{P}^1$, so that C is unramified over ∞.

What is the genus of C?

Exercise 2-5. With C as in Exercise 2-4:

(1) Let r_α be the ramification point over ζ^α. Show that

$$p + q \sim 2r_\alpha \quad \text{and} \quad \sum_{\alpha=0}^{5} r_\alpha \sim 3p + 3q.$$

(2) Find the vector space $H^0(\mathcal{O}_C(D))$ where $D = r_0 + r_2 + r_4$, and find the (unique) divisor E on C such that $E + r_1 \sim r_0 + r_2 + r_4$.

Exercise 2-6. With C as in Exercise 2-4: Let D be the divisor $D = p + q + r_0 + r_3$

(1) Find the vector space $H^0(\mathcal{O}_C(D))$.

(2) Describe the map $\phi_{|D|} : C \to \mathbb{P}^2$.

(3) Find the equation of the image curve $\phi_{|D|}(C) \subset \mathbb{P}^2$, and describe its singularities.

Hint: for the first part, find rational functions in x and y whose pullback to C has poles along $D = p + q + r_0 + r_3$ but nowhere else. For the last part, observe that the equation of the image corresponds to the kernel of the map $\mathrm{Sym}^{\upsilon} H^0(\mathcal{O}_C(D)) \to H^0(\mathcal{O}_C(4D))$.

Exercise 2-7. Let C be the smooth projective curve associated to the affine curve $y^3 = x^5 - 1$. The map $\pi : C \to \mathbb{P}^1$ given by the function x expresses C as a cyclic, 3-sheeted cover of $\mathbb{P}^1$, branched over the fifth roots of unity and the point at ∞. By way of notation, if we take $\eta = e^{2\pi i/5}$ a primitive fifth root of unity, we'll denote by r_α the point $(\eta^\alpha, 0) \in C$ lying over η^α, and by p the point lying over $\infty \in \mathbb{P}^1$.

(1) Verify that there is indeed a unique point $p \in C$ lying over $\infty \in \mathbb{P}^1$, and the map has ramification index 2 at p.

(2) Show that the genus of C is 4.

(3) Establish the linear equivalences

$$3p \sim 3r_\alpha \quad \text{and} \quad r_1 + \cdots + r_5 \sim 5p.$$

(4) Find a basis for the space $H^0(K_C)$ of regular differentials on C.

(5) Show that C is not hyperelliptic.

(6) Describe the canonical map $\phi_K : C \to \mathbb{P}^3$ and find the equations of the image.

(7) Let D be the divisor $D = r_1 + \cdots + r_5$. Show that $h^0(K_C(-D)) = 1$; deduce that $r(D) = 2$, and find a basis for $H^0(\mathcal{O}_C(D))$.

(8) If $E = 3p$, show that $r(E) = 1$; that $|E|$ is the unique g^1_3 on C and that $2E \sim K$.

Exercise 2-8. (1) Show that the arithmetic genus of the disjoint union of two lines is -1.

(2) Let $L \subset Q \subset \mathbb{P}^3$ be a line on a smooth quadric surface in $\mathbb{P}^3$. Show that the divisor $3L$, regarded as a 1-dimensional scheme, has $p_a(3L) = -2$.

(3) Compare these results with the result of simply applying the adjunction formula.

Exercise 2-9. Show that a curve of genus $g \geq 3$ cannot be simultaneously hyperelliptic and a three-sheeted cover of $\mathbb{P}^1$. See Proposition 9.5 for a generalization.

Exercise 2-10. Show that normalization of the affine curve

$$C = V(xy(x-y)) \subset \mathbb{A}^2$$

is the disjoint union of three affine lines (that is, $\operatorname{Spec}(\mathbb{C}[x] \times \mathbb{C}[y] \times \mathbb{C}[z])$. Compute the linear conditions on the values and derivatives of three polynomial functions f, g, h defined on these three lines that they "descend" to give a regular function on the planar triple point. indexed

Exercise 2-11. Let $p \in C$ be a singular point of a reduced curve C. Show that if $\delta_p = 1$, then p must be either a node or an ordinary cusp. Hint: first show that there are at most two points in the normalization of C lying over p.

Exercise 2-12. Suppose that $C \subset Q \subset \mathbb{P}^3$ is a curve on a smooth quadric Q, and that C lies in the class (a, b) with $0 \leq a \leq b$.

(1) Compute the generators of the homogeneous ideal of C in the case $a = 0$; and in the case $(a, b) = (1, 1)$ not treated in Example 2.42.

(2) Prove the assertion made at the end of Example 2.42 about the generation of the homogeneous ideal by forms of degrees 2 and b.

The following three exercises give a proof (in arbitrary characteristic) of resolution of singularities for curves; that is, every reduced curve is birational to a smooth curve.

Exercise 2-13. Let $C_0 \subset \mathbb{P}^2$ be an irreducible and reduced plane curve of degree d. Show that for large m, the m-th Veronese map $\mathbb{P}^2 \hookrightarrow \mathbb{P}^{\binom{m+2}{2}-1}$ embeds C_0 as a curve C of degree md spanning a linear space $\mathbb{P}^N$ of dimension

$$N = md - \binom{d-1}{2}.$$

Hint: observe that the linear forms in $\mathbb{P}^{\binom{m+2}{2}-1}$ vanishing on the image of C_0 correspond to elements of the kernel of the map $H^0(\mathcal{O}_{\mathbb{P}^2}(m)) \to H^0(\mathcal{O}_C(m))$.

Exercise 2-14. We define the *multiplicity* $m_p(C)$ of a point $p \in C$ on a curve $C \subset \mathbb{P}^r$ to be the degree of the component of the intersection $H \cap C$, where $H \subset \mathbb{P}^r$ is a general hyperplane through p. (The Show that if the projection $\pi_p : C \to \mathbb{P}^{r-1}$ from p is birational onto its image, then the degree of the image curve $C_0 = \pi_p(C) \subset \mathbb{P}^{r-1}$ is $\deg C_0 = \deg C - m_p(C)$, and more generally if $\pi_p : C \to \mathbb{P}^{r-1}$ has degree k onto its image, then

$$\deg C_0 = \frac{\deg C - m_p(C)}{k}.$$

Note that the definition of multiplicity is trickier for varieties of dimension greater than 1; see [?].

Exercise 2-15. Returning to the situation of Exercise 2-13, suppose we take the curve $C \subset \mathbb{P}^N$ and project it from a singular point, then do the same to its image $C_1 \subset \mathbb{P}^{N-1}$ and continue in this fashion to produce a sequence of curves $C_n \subset \mathbb{P}^{N-n}$. Combining the last two exercises with the statement of Corollary 1.8, show that

(1) all the projection maps are birational onto their images; and
(2) for some n, the process terminates; that is, the image curve C_n is smooth.