Curves

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1 pre-requisites and conventions

Basic results used in this section: Bézout, Riemann-Roch, Lasker (aka AF+BG), Clifford, Adjunction.

Let's explicitly allow things like $\mathrm{H}^0(D)$ where D is a divisor, as well as $\mathrm{H}^0(\mathcal{O}(D))$, but be careful not to mix the two too much.

Would it be more confusing or less to use the same letter for a polynomial vanishing on C and the surface it defines?

2 Personalities

The subject of algebraic curves abounds with examples amenable to explicit construction and analysis. In this chapter, we will survey the basic geometry and embeddings of the curves of genus 0 to 6. Our knowledge of the geometry of curves becomes increasingly less complete as the genus increases, and 6, as we shall see, is a natural turning point.

2.1 Curves of genus 0

2.2 Curves of genus 1

2.3 Curves of genus 2

2.4 Curves of genus 3

2.5 Curves of genus 4

As in the case of curves of genus 3, the study of curves of genus 4 bifurcates immediately into two cases: hyperelliptic and non-hyperelliptic; again, we will study the geometry of hyperelliptic curves in Chapter **??** and focus here on the nonhyperelliptic case.

In genus 4 we have a question that the elementary theory based on the Riemann-Roch formula cannot answer: are nonhyperelliptic curves of genus 4 expressible as three-sheeted covers of $\mathbb{P}^1$? The answer will emerge from our analysis in Proposition 2.2 below.

Let C be a non-hyperelliptic curve of genus 4. We start by considering the canonical map $\phi_K : C \hookrightarrow \mathbb{P}^3$, which embeds C as a curve of degree 6 in $\mathbb{P}^3$. We identify C with its image, and investigate the homogeneous ideal $I = I_C$ of equations it satisfies. As in previous cases we may try to answer this by considering the restriction maps
((replaced K_C^m with mK_C.))

$$r_m : \mathrm{H}^0(\mathcal{O}_{\mathbb{P}^3}(m)) \rightarrow \mathrm{H}^0(\mathcal{O}_C(m)) = \mathrm{H}^0(mK_C).$$

For $m = 1$, this is by construction an isomorphism; that is, the image of C is non-degenerate (not contained in any plane).

For $m = 2$ we know that $\mathrm{h}^0(\mathcal{O}_{\mathbb{P}^3}(2)) = \binom{5}{3} = 10$, while by the Riemann-Roch Theorem we have

$$\mathrm{h}^0(\mathcal{O}_C(2)) = 12 - 4 + 1 = 9.$$

This shows that the curve $C \subset \mathbb{P}^3$ must lie on at least one quadric surface Q. The quadric Q must be irreducible, since any any reducible and/or non-reduced quadric must be a union of planes, and thus cannot contain an irreducible non-degenerate curve. If $Q' \neq Q$ is any other quadric then, by Bézout's Theorem, $Q \cap Q'$ is a curve of degree 4 and thus could not contain C. From this we see that Q is unique, and it follows that r_2 is surjective.

What about cubics? Again we consider the restriction map

$$r_3 : \mathrm{H}^0(\mathcal{O}_{\mathbb{P}^3}(3)) \;\to\; \mathrm{H}^0(\mathcal{O}_C(3)) = \mathrm{H}^0(3K_C).$$

The space $\mathrm{H}^0(\mathcal{O}_{\mathbb{P}^3}(3))$ has dimension $\binom{6}{3} = 20$, while the Riemann-Roch Theorem shows that

$$\mathrm{h}^0(\mathcal{O}_C(3)) = 18 - 4 + 1 = 15.$$

It follows that the ideal of C contains at least a 5-dimensional vector space of cubic polynomials. We can get a 4-dimensional subspace as products of the unique quadratic polynomial F vanishing on C with linear forms—these define the cubic surfaces containing Q. Since $5 > 4$ we conclude that the curve C lies on at least one cubic surface S not containing Q. Bézout's Theorem shows that the curve $Q \cap S$ has degree 6; thus it must be equal to C.

Let $G = 0$ be the cubic form defining the surface S. By Lasker's Theorem the ideal (F, G) is unmixed, and thus is equal to the homogeneous ideal of C. Putting this together, we have proven the first statement of the following result:

Theorem 2.1. *The canonical model of any nonhyperelliptic curve of genus 4 is a complete intersection of a quadric $Q = V(F)$ and a cubic surface $S = V(G)$. Conversely, any smooth curve that is the intersection of a quadric and a cubic surface in $\mathbb{P}^3$ is the canonical model of a nonhyperelliptic curve of genus 4.*

Proof. Let $C = Q \cap S$ with Q a quadric and S a cubic. Applying the Adjunction Formula to $Q \subset \mathbb{P}^3$ we get

$$\omega_Q = (\omega_{\mathbb{P}^3} \otimes \mathcal{O}_{\mathbb{P}^3}(2))|_Q = \mathcal{O}_Q(-4+2) = \mathcal{O}_Q(-2).$$

Applying it again to C on Q, and noting that $\emptyset_Q(C) = \emptyset_Q(3)$, we get

$$\omega_C = ((\omega_Q \otimes \mathcal{O}_3(3))|_C = \mathcal{O}_C(-2+3) = \mathcal{O}_C(1)$$

as required. $\square$

We can now answer the question we asked at the outset, whether a nonhyperelliptic curve of genus 4 can be expressed as a three-sheeted cover of $\mathbb{P}^1$. This amounts to asking if there are any divisors D on C of degree 3 with $r(D) \geq 1$; since we can take D to be a general fiber of a map $\pi : C \to \mathbb{P}^1$, we can for simplicity assume $D = p + q + r$ is the sum of three distinct points.

By the geometric Riemann-Roch theorem, a divisor $D = p + q + r$ on a canonical curve $C \subset \mathbb{P}^{g-1}$ has $r(D) \geq 1$ if and only if the three points $p, q, r \in C$ are colinear. If three points $p, q, r \in C$ lie on a line $L \subset \mathbb{P}^3$ then the quadric Q would meet L in at least three points, and hence would contain L. Conversely, if L is a line contained in Q, then the divisor $D = C \cap L = S \cap L$ on C has degree 3. Thus we can answer our question in terms of the family of lines contained in Q.

Any smooth quadric is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$, and contains two families of lines, or *rulings*. On the other hand, any singular quadric is a cone over a plane conic, and thus has just one ruling. By the argument above, the pencils of divisors on C cut out by the lines of these rulings are the g^1_3s on C. This proves:

Proposition 2.2. *A nonhyperelliptic curve of genus 4 may be expressed as a 3-sheeted cover of* $\mathbb{P}^1$ *in either one or two ways, depending on whether the unique quadric containing the canonical model of the curve is singular or smooth.*

A curve expressible as a 3-sheeted cover of $\mathbb{P}^1$ is called *trigonal*; by the analyses of the preceding sections, we have shown that *every curve of genus* $g \leq 4$ *is either hyperelliptic or trigonal.*

We can also describe plane models of nonhyperelliptic curves C of genus 4. By Clifford's Theorem a nonhyperelliptic curve of genus 4 cannot have a g^2_4; in particular, there cannot be a plane model of degree 4.

If now D is a divisor of degree 5 with $r(D) = 2$ then, by the Riemann-Roch Theorem, $\mathrm{h0}(K - D) = 1$; that is, D is of the form $K - p$ for some point $p \in C$. There are thus two cases:

((revised by DE to here, June 17))

((This analysis either needs more explanation, or a preliminary section about singularities of plane curves. Also the geometry of

projections (“same tangent line”) is usually NOT understood by beginning students, and needs explication here.))

1. If C has two g^1_3s—that is, if the quadric Q containing the canonical model of C is smooth—then projection π of the canonical curve from a point $p \in C$ will be an embedding, except at the other points of intersection of C with the two lines $L_1, L_2 \subset Q$ passing through p. If L_i is transverse to C, the two other points of $L_i \cap C$ will map to a node of the image curve (the tangent spaces to Q at the two other points of $L_i \cap C$ will be distinct, so the image curve $\pi(C)$ will have a node); if L_i meets C tangentially at one point other than p, the image curve will have a cusp. In sum, then, the plane quintic model of C will have two singularities, each either a node or a cusp.

2. If C has one g^1_3—that is, if the quadric Q containing the canonical model of C is a cone—then projection π of the canonical curve from a point $p \in C$ will again be an embedding, except at the other points of intersection of C with the single line $L \subset Q$ passing through p. In this case, however, the tangent planes to Q at points of L will all be the same, so that if L is transverse to C, the two other points of $L \cap C$ will map to a *tacnode* of the image curve; if L meets C tangentially at one point other than p, the image curve will have a *ramphoid cusp.*

2.6 Curves of genus 5

We consider now nonhyperelliptic curves of genus 5. There are now two questions that cannot be answered by simple application of the Riemann-Roch Theorem:

1. Is C expressible as a 3-sheeted cover of $\mathbb{P}^1$? In other words, does C have a g^1_3?; and

2. Is C expressible as a 4-sheeted cover of $\mathbb{P}^1$? In other words, does C have a g^1_4?

As we’ll see, all other questions about the existence or nonexistence of linear series on C can be answered by the Riemann-Roch Theorem.

As in the preceding case, the answers can be found through an investigation of the geometry of the canonical model $C \subset \mathbb{P}^4$ of C. This is an octic curve in $\mathbb{P}^4$, and as before the first question to ask is what sort of polynomial equations define C. We start with quadrics, by considering the restriction map

$$r_2 : \mathrm{H}^0(\mathcal{O}_{\mathbb{P}^4}(2)) \ \rightarrow \ \mathrm{H}^0(\mathcal{O}_C(2)).$$

On the left, we have the space of homogeneous quadratic polynomials on $\mathbb{P}^4$, which has dimension $\binom{6}{4} = 15$, while by the Riemann-Roch Theorem the target is a vector space of dimension

$$2 \cdot 8 - 5 + 1 = 12.$$

We deduce that C *lies on at least 3 independent quadrics*. (We will see in the course of the following analysis that it is exactly 3; that is, r_2 is surjective.)

The question now is, what is the intersection of the quadrics containing C? There are two possibilities:

1. If the intersection $Q_1 \cap Q_2 \cap Q_3$ of three of the quadrics is 1-dimensional, then by Bezout it must equal C; that is, C is a complete intersection of three quadrics. (In this case, Noether's AF+BG theorem tells us that there are no more quadrics containing C.)

2. If the intersection of the quadrics containing C is 2-dimensional, some further analysis is required; we'll carry this out following a description of the first case.

For the first case, suppose that the canonical curve $C \subset \mathbb{P}^4$ is the complete intersection of three quadrics. We can answer the first of our two questions immediately: C cannot contain three colinear points (or, more generally, a divisor of degree 3 contained in a line), because then the line L containing them would necessarily lie on each of the quadrics, meaning that C could not be their complete intersection.

What about g^1_4s? Again invoking the geometric Riemann-Roch Theorem, a divisor of degree 4 moving in a pencil lies in a 2-plane; so the question is, does $C \subset \mathbb{P}^4$ contain four coplanar points? Suppose first that it does: say $D = p_1 + \cdots + p_4 \subset C$ is contained in a 2-plane Λ. In that case, consider the restriction map

$$\mathrm{H}^0(\mathcal{I}_{C/\mathbb{P}^4}(2)) \ \rightarrow \ \mathrm{H}^0(\mathcal{I}_{D/\Lambda}(2)).$$

By hypothesis, the left hand space is 3-dimensional; but since D is not contained in a line, it must impose independent conditions on quadrics, so that the right hand space is 2-dimensional. It follows that Λ *must be contained in one of the quadrics* Q *containing* C; note in particular that such a quadric Q is necessarily singular.

Conversely, suppose that $Q \subset \mathbb{P}^4$ is a singular quadric containing C. Q is a cone over a quadric $\overline{Q}$ in $\mathbb{P}^3$, and just as $\overline{Q}$ has either one or two rulings by lines, Q will be swept out by one or two families of 2-planes, each parametrized by $\mathbb{P}^1$. Now say $\Lambda \subset Q$ is such a 2-plane. If Q' and Q'' are "the other two quadrics" containing C, we can write

$$\Lambda \cap C = \Lambda \cap Q' \cap Q'',$$

from which we see that $D = \Lambda \cap C$ is a divisor of degree 4 on C, and so has $r(D) = 1$ by the geometric Riemann-Roch Theorem. Thus, the rulings of singular quadrics containing C cut out on C pencils of degree 4; and every pencil of degree 4 on C arises in this way.

Does C lie on singular quadrics? You bet: there is a $\mathbb{P}^2$ of quadrics containing C—a 2-plane in the space $\mathbb{P}^{14}$ of quadrics in $\mathbb{P}^4$—and its intersection with the discriminant hypersurface in $\mathbb{P}^{14}$ will be a plane quintic curve B. (Note that by Bertini not every quadric containing C can be singular in this case.) So C does indeed have a g^1_4, and is expressible as a 4-sheeted cover of $\mathbb{P}^1$; in fact we can summarize our analysis in the

Proposition 2.3. *Let $C \subset \mathbb{P}^4$ be a canonical curve, and assume C is the complete intersection of three quadrics in $\mathbb{P}^4$. Then C may be expressed as a 4-sheeted cover of $\mathbb{P}^1$ in a one-dimensional family of ways; indeed, there is a map from the set of g^1_4s on C to a plane quintic curve B, whose fibers have cardinality 1 or 2.*

Of course, we can go further and ask about the geometry of the plane curve B and how it relates to the geometry of C; a fairly exhaustive list of possibilities is given in [**?**] [ACGH]. But that's enough for now.

Let's turn our attention to the second possibility above: that the canonical curve $C \subset \mathbb{P}^4$ is not a complete intersection; that is, the intersection of the quadrics containing C is two-dimensional.

2.7 Curves of genus 6