Contents

Chapter 8

Moduli of curves

In the preceding chapter, we described the *Hilbert scheme*, a fine moduli space for curves in projective space. In this chapter we will discuss the second moduli space central to the theory of algebraic curves: M_g, which parametrizes isomorphism classes of smooth projective curves of genus g. As we'll see, M_g is not a fine moduli space, but it comes close.

To describe the situation, we will start with the case of curves of genus 1, where everything can be made explicit.

8A. Curves of genus 1

Let C be a smooth curve of genus 1. Any invertible sheaf of degree 2 on C can be written as $\mathcal{O}_C(2p)$, and defines a morphism to $\mathbb{P}^1$ with 4 distinct branch points. Since the automorphism group of C is transitive, these 4 points in $\mathbb{P}^1$ are independent of the choice of p, and are well-defined up to an automorphism of $\mathbb{P}^1$. As explained in Section 6B, this means that every such curve C can be realized as the completion of an affine curve

$$y^2 = f(x)$$

where f is a quartic polynomial with distinct roots:

$$f(x) = \prod_{i=1}^{4}(x - \lambda_i).$$

Thus we would like to define M_1 to be the set of 4-tuples of distinct points $\{\lambda_1, \dots, \lambda_4\}$ of $\mathbb{P}^1$ modulo the action of $\mathrm{Aut}(\mathbb{P}^1) = \mathrm{PGL}_2$.

As we will explain in the next sections, quotients by infinite groups can behave badly, but in this case we can compute the quotient in a much simpler way: There is a unique automorphism of $\mathbb{P}^1$ carrying the three points $\lambda_1, \lambda_2, \lambda_3$ to the points $0, 1$ and $\infty \in \mathbb{P}^1$ respectively, so that we can write C as the zero locus of

$$y^2 = x(x-1)(x-\lambda)$$

for some complex number $\lambda \in \mathbb{P}^1 \setminus \{0, 1, \infty\}$; we'll call this curve C_λ. This expression is not unique, since if we reordered the original four points λ_i, we might arrive at a different value of λ; for example, if we exchanged 0 and ∞ and fixed 1, λ would be replaced by $1/\lambda$. Thus the symmetric group S_4 acts on the set $\mathbb{A}^1 \setminus \{0, 1, \infty\}$ and one can show that the orbit of λ under this action is

$$\left\{\lambda, 1-\lambda, \frac{1}{\lambda}, \frac{1}{1-\lambda}, \frac{\lambda-1}{\lambda}, \frac{\lambda}{\lambda-1}\right\}.$$

There are 6 points in the orbit rather than 24 because the Klein 4-group $K = \mathbb{Z}/2 \times \mathbb{Z}/2 \subset S_4$ of fixed-point-free involutions acts trivially, so what we really have is an action of $S_4/K \cong S_3$.

Since S_3 is finite and $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ is a normal affine curve, the quotient space by the action is again a normal affine curve whose points are in one-to-one correspondence with the orbits, and thus with the set of curves of genus 1. By Lüroth's theorem (Theorem 3.2), the quotient is rational, meaning that the field of rational functions on the quotient — that is, the subfield of $\mathbb{C}(\lambda)$ invariant under the action of S_3 — is of the form $\mathbb{C}(j)$ for some rational function $j(\lambda)$ of degree 6. Of course, there are many possible generators of the field of rational functions on the quotient; one that works is

$$(*) \qquad j(\lambda) := 256\frac{(\lambda^2-\lambda+1)^3}{\lambda^2(\lambda-1)^2},$$

known as the *j-function*. As λ varies in $\mathbb{P}^1 \setminus \{0, 1, \infty\}$, the values of $j(\lambda)$ range over all of $\mathbb{A}^1$.

Summarizing, we have proven:

Theorem 8.1. *The set of isomorphism classes of smooth projective curves of genus 1 is in bijection with the points of the affine line $M_1 \cong \mathbb{A}^1$. The bijection maps the curve defined by $y^2 = x(x-1)(x-\lambda)$ to $j(\lambda) \in \mathbb{A}^1$.*

M_1 is a coarse moduli space. As we will see in Exercises 8-3 and 8-4, M_1 is not a fine moduli space, but it comes close in two senses.

Proposition 8.2. *To any family $\pi : \mathcal{C} \to B$ of smooth projective curves of genus 1 over a reduced base B we can associate a natural morphism of schemes*

$$\phi : B \to M_1$$

whose value at any point $b \in B$ is the j-invariant of the corresponding fiber C_b.

Proof. To start, we will work locally in B: for a given $b_0 \in B$, we will choose a suitably small neighborhood U of $b_0 \in B$ and restrict ourselves to the preimage $\mathcal{C}_U = \pi^{-1}(U)$. The first thing to do is to express the curves C_b in our family as 2-sheeted covers of $\mathbb{P}^1$, which is to say we want to choose an invertible sheaf on $\mathcal{C}_U$ having degree 2 on each fiber C_b. Since we're working locally in B, we can find a section $\rho : U \to \mathcal{C}_U$ of $\pi : \mathcal{C} \to B$. If we let $D = \rho(U) \subset \mathcal{C}_U$ be the image, then we can take our invertible sheaf to be $\mathcal{L} := \mathcal{O}_{\mathcal{C}_U}(2D)$.

Next, we use the following result, which is a special case of the theorem on cohomology and base change (see [?, Appendix, Theorems B.5 and B.9] or [?, Theorem 12.11].)

Theorem 8.3 (cohomology and base change). *If $f : X \to Y$ is a morphism and $\mathcal{F}$ is a coherent sheaf on X such that $H^1(\mathcal{F}|_{f^{-1}(y)}) = 0$ for all $y \in Y$, then $h^0(\mathcal{F}|_{f^{-1}(y)})$ is a constant function of y, and $f_*(\mathcal{F})$ is a vector bundle of this rank.* □

This result implies that the direct image $\mathcal{E} := \pi_*(\mathcal{O}_{\mathcal{C}_U}(2D))$ is locally free of rank 2, and we get a morphism $\mathcal{C}_U \to \mathbb{P}(\mathcal{E})$ expressing each curve C_b as a 2-sheeted cover of the corresponding fiber $\mathbb{P}(\mathcal{E}_b)$. Again, since we are working locally in B, we can trivialize the bundle $\mathcal{E}$, so that we get a diagram

$$\begin{tikzcd} \mathcal{C}_U \arrow[rr] \arrow[dr] & & U \times \mathbb{P}^1 \arrow[dl] \\ & U & \end{tikzcd}$$

Once more restricting to a smaller neighborhood U if necessary, we can write the family $\mathcal{C}_U \to U$ as the locus

$$y^2 = \prod_1^4 (x - \lambda_i),$$

where the λ_i are regular functions on U. The j-function of the λ_i yields a map $U \to M_1$; since the value of the j-function at a point is determined by the isomorphism type of the fiber over this point, these maps agree on overlaps to give the desired morphism $B \to M_1$. □

The good news. The second way in which M_1 comes close to being a fine moduli space is seen in the next result:

Proposition 8.4. *Let B be a reduced scheme.*

(1) *If $j : B \to \mathbb{A}^1$ is any regular function on B, then there exists a finite cover $\alpha : B' \to B$ such that $j \circ \alpha$ is the j-function of a family of curves of genus* 1 *on B'.*

(2) *If $\pi : \mathcal{C} \to B$ and $\eta : \mathcal{D} \to B$ are two families of curves of genus* 1 *with the same associated j-function, then there exists a finite cover $\alpha : B' \to B$ and an isomorphism $\mathcal{C} \times_B B' \cong \mathcal{D} \times_B B'$ such that the diagram*

$$\begin{tikzcd} \mathcal{C} \times_B B' \arrow[rr] \arrow[dr] & & \mathcal{D} \times_B B' \arrow[dl] \\ & B' & \end{tikzcd}$$

commutes.

Proof. For the first of these assertions, let

$$B' := \{(b, \lambda) \in B \times (\mathbb{A}^1 \setminus \{0, 1\}) \mid j(b) = j(\lambda)\},$$

where $j(\lambda)$ is as given by formula ($*$) on page 150. We have already described a family of curves of genus 1 over the *λ-line* $\mathbb{A}^1 \setminus \{0, 1\}$; the pullback to B' is the desired family.

For the second half, we want to do something similar. Specifically, we want to choose sections $\sigma : B \to \mathcal{C}$ and $\tau : B \to \mathcal{D}$ and take

$$B' := \{(b, \phi) \mid b \in B,\ \phi : C_b \xrightarrow{\cong} D_b \text{ and } \phi(\sigma(b)) = \tau(b)\};$$

as a set, B' is the set of isomorphisms of between corresponding fibers in the two families. By Corollary 5.15, B' is a finite cover of B and when we pull back the two families to B' we have a tautological isomorphism between them. The only issue is how to give B' an appropriate scheme structure, and for this we can use the Isom scheme described at the end of Section 7D. □

Thus, M_1 is not a fine moduli space for smooth curves of genus 1, but it is the next best thing: we don't get a bijection between families of curves of genus 1 over a given base B and maps $j : B \to M_1$; but we do get a map from the former to the latter with "finite kernel and cokernel".

Compactifying M_1. A natural question to ask is, if every value of $j \in \mathbb{A}^1$ corresponds to an isomorphism class of curves C_j of genus 1, what happens to the curves C_j as j goes to ∞? Equivalently, what happens to the curve C_λ given as the double cover

$$y^2 = x(x-1)(x-\lambda)$$

when λ approaches 0, 1 or ∞ — the other branch points of the double cover? The answer is seen from the equation: when two branch points of a double cover of smooth curves coalesce the limiting curve has a node (Figure 8.1). In fact, there is a unique isomorphism class of irreducible curves of arithmetic genus 1 having a node; it's represented by the curve defined by $y^2 = x^2(x-1)$.

The upshot is that if we enlarge the original class of curves parametrized by M_1 — smooth projective curves of genus 1 — to the slightly larger class of

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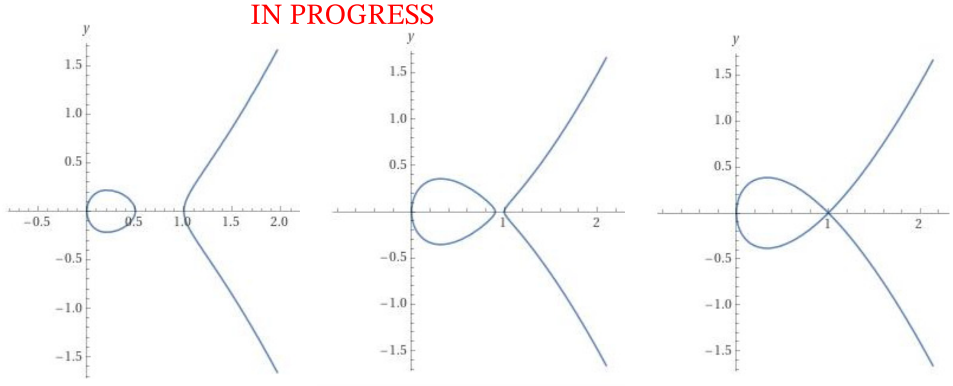


$\lambda = 0.5$ $\lambda = 0.9$ $\lambda = 1$

Figure 8.1. A curve of genus 1 degenerating to a rational curve with a node in the family $y^2 = x(x-1)(x-\lambda)$.

irreducible nodal projective curves of arithmetic genus 1, we still have a coarse moduli space $\overline{M}_1$ for this slightly larger class of objects. This enlarged moduli space is obtained by adding one point "at ∞" to the existing space $M_1 \cong \mathbb{A}^1$ to form $\overline{M}_1 \cong \mathbb{P}^1$.

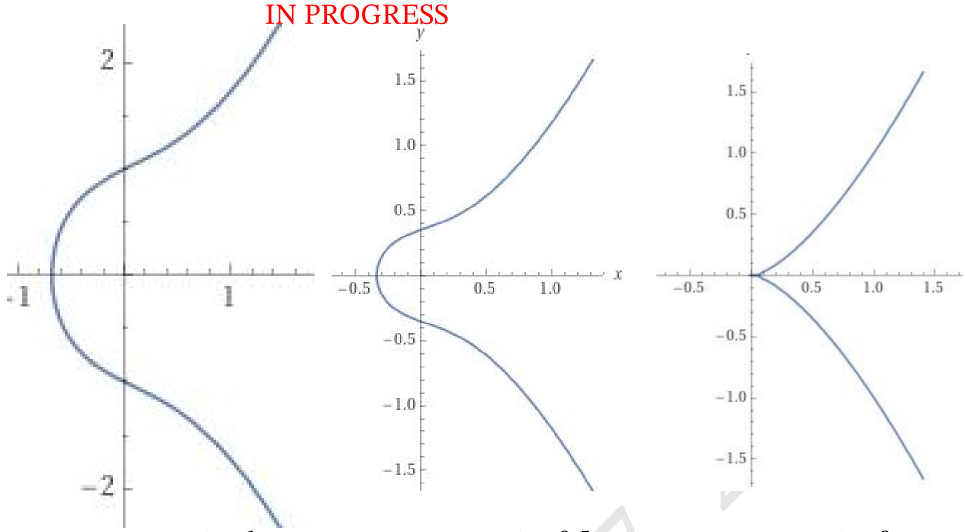
This is an example of what is called a *modular compactification.* There is no precise definition, but if we have a class of objects parametrized by a (noncompact) moduli space M we may be able enlarge the class of objects to be parametrized, with the result that the moduli space $\overline{M}$ of the larger class is compact.

Modular compactifications of a given moduli problem may or may not exist. It's sometimes a tricky problem to find a suitable class of objects to parametrize: if we don't add enough additional isomorphism classes, not every 1-parameter family of objects in our original class will have a limit in the larger class, meaning the enlarged moduli space will still not be compact; if we add too many, 1-parameter families may have more than one possible limit, meaning the enlarged space won't be separated. For example, in the family of curves C_t given as

$$C_t = V(y^2 - x^3 - t^2x - t^3),$$

the j-function is constant when $t \neq 0$, but the limiting curve C_0 has a cusp (Figure 8.2). This shows that we could not have added cuspidal curves to M_1.

When modular compactifications do exist, they are extremely valuable for the study of both the space M and of the objects parametrized by M: compactness allows us to apply the techniques of modern algebraic geometry to the space $\overline{M}$, while the fact that it is still a moduli space gives us a handle on its geometry. In the following section, we will describe a modular compactification of M_g. The objects parametrized are called *stable curves.*



$t = 1$ $t = 0.5$ $t = 0$

Figure 8.2. A curve of genus 1 degenerating to a cuspidal curve in the family $C_t = V(y^2 - x^3 - t^2x - t^3)$.

Getting back to the moduli space $\overline{M}_1$, if we have a family where $j(\lambda)$ has a pole, we would like to say that the limit of the curves in the family is an irreducible nodal curve, but this is not necessarily true! For example, the limit of the curves

$$y^2 = x(x - t)(tx - 1)$$

as $t \to 0$ is reducible, with two components meeting in two points, 0 and ∞. What is true is that a process called *semistable reduction* shows that after a base change and a birational modification of the family around the pole we can replace the family with one where the singular fiber is indeed an irreducible nodal curve (Figure 8.3). See [?] for a description of this process in general, and several explicit examples.

8B. Higher genus

The idea is analogous to the one used for genus 1 curves: to construct a moduli space, first parametrize curves with a choice of some additional structure, such as a map to projective space, and then mod out by the choices made. For any smooth projective curve C of genus $g \geq 2$, the tricanonical linear series $|3K_C|$ is very ample; it embeds C as a curve of degree $6g - 6$ in $\mathbb{P}^{5g-6}$. Thus we have a way of realizing a given abstract curve C as a curve in projective space, unique up to automorphisms of $\mathbb{P}^{5g-6}$.

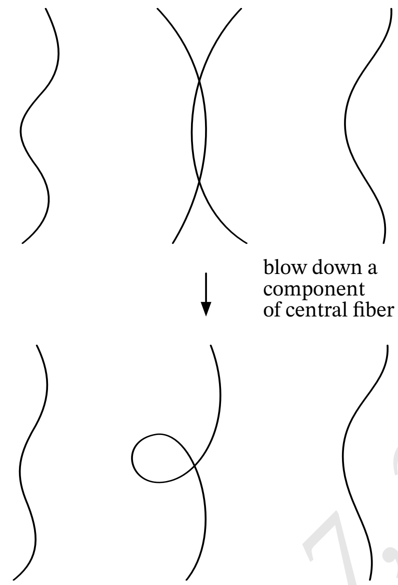


Figure 8.3. In this case a birational modification of the total space of the family changes the unstable reducible curve to a stable curve.

We claim next that the set of smooth, tricanonically embedded curves is a locally closed subset X of the Hilbert scheme $\mathrm{Hilb}_{(6g-6)m+1-g}(\mathbb{P}^{5g-6})$ parametrizing curves of genus g and degree $6g-6$ in $\mathbb{P}^{5g-6}$. By Lemma 7.10, the set of points in the base over which the curves are smooth is open. Let

$$\mathrm{Hilb}^{\circ} = \mathrm{Hilb}^{\circ}_{(6g-6)m+1-g}(\mathbb{P}^{5g-6}) \subset \mathrm{Hilb}_{(6g-6)m+1-g}(\mathbb{P}^{5g-6})$$

be this open set.

Next, on the universal family $\mathcal{C} \subset \mathrm{Hilb}^{\circ} \times \mathbb{P}^{5g-6}$, we have two families of invertible sheaves: we have the pullback of $\mathcal{O}_{\mathbb{P}^{5g-6}}(1)$; and we have the cube K^3 of the dualizing sheaf. Each gives rise to a section of the relative Picard variety over Hilb°, and the locus where they agree is thus a closed subset $X \subset \mathrm{Hilb}^{\circ}$.

The group PGL_{5g-5} of automorphisms of $\mathbb{P}^{5g-6}$ acts on the variety X and its orbits are the isomorphism classes of smooth curves of genus g; thus, we might hope to realize the moduli space M_g as the quotient of X by PGL_{5g-5}. But here things go awry in a hurry: unlike the case of an action of a finite group on a variety, the orbit spaces of infinite groups are often not algebraic varieties. (Think of the action of $\mathbb{C}^*$ on $\mathbb{C}$ by multiplication.) What is needed is a tool to extract the "best possible approximation" to a quotient. Happily, David Mumford created a tool that does exactly this: *geometric invariant theory* (GIT). To see how GIT can be used in this setting to produce the space M_g, see the wonderful introduction in [?] (linked from the book's AMS website) or the more technical version in [?]. it would be better to give a URL in the bib

Theorem 8.5 (Mumford). *The space of orbits of* PGL_{5g-5} *acting on the subset of the Hilbert scheme representing tricanonical curves has the structure of an algebraic variety* M_g *which is a* coarse moduli space *in the following sense:*

(1) *Given any flat family* $Y \to B$ *of smooth curves of genus g there is a morphism of schemes* $B \to M_g$ *sending each closed point* $p \in B$ *to the point of* M_g *representing the fiber* Y_b*.*

(2) *These maps form a natural transformation from the functor* $G(-)$ *of families of smooth curves to the functor* $\mathrm{Mor}_{\mathrm{schemes}}(-, M_g)$ *through which any natural transformation* $G \to \mathrm{Mor}_{\mathrm{schemes}}(-, M')$ *factors.*

The power of the theory of the moduli space of curves was greatly increased when compactifications of the space (there are many interesting ones) were introduced. One of these, the compactification of $M_1 = \mathbb{A}^1$ to $\overline{M}_1 = \mathbb{P}^1$ by adding a nodal curve, has already been mentioned. This has the desirable properties that the subset added to M_1 is a divisor; and the compactification is *modular* in the sense that the point added corresponds to a curve almost of the same type as the curves in M_1.

There are two reasons why a compactification is important:

First, the great majority of the techniques that algebraic geometers have developed for dealing with varieties apply directly only to projective varieties. For example, the Satake compactification is a projective variety containing M_g in such a way that the complement — usually referred to as the boundary — has codimension 2. Taking successive hyperplane sections that pass through a given point but don't meet the boundary, we see that for $g \geq 2$ there is a complete one-dimensional family of *smooth* curves containing any smooth curve of genus ≥ 2.

Often, though, we can learn the most from a compactification where the added boundary is a divisor, and this is the case for the Deligne–Mumford compactification $\overline{M}_g$, described below, introduced in the groundbreaking 1969 paper [?]. A central example of how this is used is given in Section 8D, where we take up the question, "can we write down a general curve of genus g?"

To describe this compactification, we first explain some of the language and results of geometric invariant theory.

Stable, semistable, unstable. Given a quasiprojective variety $X \subset \mathbb{P}^N$ and a group $G \subset \mathrm{PGL}_{N+1}$ that carries X into itself, we wish to construct as good a map as possible from the set of orbits to a projective space. Whatever map we take, the closure of the image will correspond to a graded ring. We want to preserve as much of the structure of the orbit space as possible, and on an open affine cover this means finding as many functions as possible that are invariant on the orbits. Thus it is natural to take the ring of invariants of the homogeneous coordinate ring A of the closure of X as the homogeneous coordinate ring of the closure of the image of X.

The first difficulty is that the elements of A are not functions on X, so G may not even act on A. However, it is possible to lift the action of G to an action on A of the slightly larger group, SL_{N+1}, a process called *linearization*. The kernel of the map $\mathrm{SL}_{N+1} \to \mathrm{PGL}_{N+1}$ consists of diagonal matrices of finite order dividing $N+1$, and the choice of a linearization amounts to a choice of a character of this abelian group. However, the choice doesn't matter, since the kernel acts trivially on forms of degree a multiple of $N+1$, and thus the action of PGL_{N+1} itself extends to an action on the homogeneous coordinate ring of the $(N+1)$-st Veronese embedding. Another way to say this is to introduce the cone $\bar{X} \subset \mathbb{A}^{N+1}$ over X; a linearization amounts to an action of SL_{N+1} on $\bar{X}$.

The second difficulty in this program is that the ring of invariants of an infinite group may not be finitely generated, so it may not correspond to a projective variety. Hilbert showed that if $G = \mathrm{SL}_{N+1}$, then the ring of invariants is finitely generated. Since Hilbert's time this result has been extended to the class of *linearly reductive* groups — see [?]. Thus the subring $A^G \subset A$ of invariant elements is finitely generated over the ground field.

The third difficulty is that the points of $\mathrm{Proj}(A^G)$, usually denoted $X/\!\!/G$, are generally not in one-to-one correspondence with the orbits of G on X!

Geometric invariant theory explains the relationship of $X/\!\!/G$ to the set of orbits. To do this, it performs a sort of triage on the points of X (or their orbits), dividing them into three classes: stable, semistable and unstable. The theory also provides tools for determining this stratification.

(1) *Stable points*. These are the points whose orbits in $\mathbb{A}^{N+1}$ are closed. They comprise an open subset $X^{\text{stable}} \subset X$, and the points of an open subset of $X/\!\!/G$ correspond one-to-one to the stable orbits, that is, an open subset that is set-theoretically X^{stable}/G. In general, this set may be empty, but in the case of the action of PGL_3 on the $\mathbb{P}^9$ of plane cubics, the stable points are the smooth plane cubics, and the quotient is the affine j-line.

(2) *Strictly semistable points*. These are the points p such that there exists an invariant form not vanishing at p. Together with the stable points, comprise a larger open subset $X^{\text{semistable}} \subset X$, called the *semistable* locus. Two semistable points p, q map to the same point in $X/\!\!/G$ if and only if $\overline{Gp} \cap \overline{Gq} \cap X^{\text{semistable}} \neq \emptyset$. In the example of the action of PGL_3 on $\mathbb{P}^9$, the semistable locus contains the orbits of smooth and nodal plane cubics; that is, smooth cubics together with the three orbits consisting of irreducible cubics with a node, unions of lines and conics meeting transversely, and triangles. In the quotient, these last three orbits correspond to just one

additional point, and this quotient is the compactification of the affine line to the projective line obtained by adding one point.

(3) *Unstable orbits*. These are the points p on which all invariant polynomials vanish, so that the induced map $\operatorname{Proj} A \to \operatorname{Proj}(A^G)$ is not even defined at p. Thus unstable points do not correspond to any points of $X/\!\!/G$; in fact, they cannot be included in any topologically separated quotient of an open subset of X defined in this way, though there may be other compactifications, coming from other representations of M_g as $X'/\!\!/G'$; see [?].

8C. Stable curves

The compactification $\overline{M}_g$ is also a *modular* compactification in the sense that the points of the boundary correspond to slightly more general objects of the same type as the points of M_g.

Definition 8.6. A reduced irreducible connected curve is *stable* if it has at most nodes as singularities and if every smooth rational component meets the other components at least three times (Figure 8.4).

The last phrase of the definition could be replaced by the equivalent condition that the automorphism group of C is finite.

These are stable points in the Hilbert scheme of tricanonical embeddings in the sense of geometric invariant theory, and the result is that M_g has a modular compactification that is a projective variety:

Theorem 8.7 (properties of $\overline{M}_g$ [?; ?]).

(1) *$\overline{M}_g$ is a projective variety.*

(2) *The points of $\overline{M}_g$ correspond one-to-one to isomorphism classes of smooth curves.*

(3) *For every family $\mathcal{C} \to B$ of stable curves there is a morphism of schemes $B \to M_g$ carrying each closed point $b \in B$ to the point representing the*

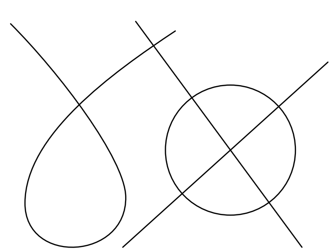


Figure 8.4. A stable curve.

isomorphism class of the fiber of $\mathcal{C}$ over b. These maps form a natural transformation from the functor $G(-)$ of families of stable curves to the functor

$$\mathrm{Mor}_{\mathrm{schemes}}(-, \overline{M}_g)$$

through which any natural transformation $G \to \mathrm{Mor}_{\mathrm{schemes}}(-, M')$ factors.

The deepest theorems about M_g have been proven using the divisor class group of $\overline{M}_g$, and many of the divisors that play a role are actually supported on the complement $\overline{M}_g \smallsetminus M_g$, often called the *boundary*.

Cheerful Fact 8.8. *For $g \geq 1$ the boundary $\overline{M}_g \smallsetminus M_g$ is the union of $1 + \lfloor g/2 \rfloor$ divisors whose generic points are*

(1) *irreducible nodal curves of geometric genus $g - 1$ and*
(2) *for $i = 1, \dots, \lfloor g/2 \rfloor$ the union of two smooth curves $C_i \cup C_{g-i}$ of genera i and $g - i$ meeting in a point.*

We will not prove either of Theorems 8.5 and 8.7. For an introduction to the proofs, with references, see [?].

How we deal with the fact that $\overline{M}_g$ is not fine. The fact that $\overline{M}_g$ is not a fine moduli space — and that correspondingly there does not exist a universal family of curves over it — is unquestionably a nuisance. Nonetheless, there are ways of dealing with the situation. The first step is to identify the cause of the problem, which is that some curves have nontrivial automorphisms. There are three ways to proceed:

(1) *Kill the automorphisms.* The idea here is to add additional structure to the objects parametrized, so as to eliminate automorphisms. Here is an example of such a construction. We saw in Chapter 5 that on a smooth projective curve C of genus g, the collection of invertible sheaves $\mathcal{L}$ with $\mathcal{L}^m \cong \mathcal{O}_C$ forms a group isomorphic to $(\mathbb{Z}/m)^{2g}$. We define a *curve with level m structure* to be such a curve, together with a choice of 2g generators $\mathcal{L}_1, \dots, \mathcal{L}_{2g}$ for this group. On every curve C of genus ≥ 2 an automorphism fixing all line bundles of order $m \geq 3$ is trivial, and there does exist a fine moduli space $M_g[m]$ for curves with level m structure; this space is a finite cover of M_g. Thus, while a universal family does not exist over M_g, one does exist over a finite cover of M_g, and this is sufficient for many purposes.
(2) *Ignore the automorphisms.* Here we use a basic fact, which we'll establish in Section 13C: in M_g, the locus $A \subset M_g$ of curves that do have automorphisms other than the identity has codimension $g - 2$. If we restrict to the complement $M_g^\circ = M_g \smallsetminus A$, accordingly, there does exist a universal family, and again this is sufficient for many purposes; for example, if $g \geq 4$

then a divisor class on M_g is determined by its restriction to M_g°, so we can just work over that open set.

(3) *Embrace the automorphisms*. We mentioned above that there does not exist a fine moduli space for curves of genus g in the category of schemes. But there is a larger category, called *stacks,* in which a fine moduli space does exist. This solution to the problem, pioneered by Deligne and Mumford, has many advantages but involves a substantial investment in mastering the technical issues; readers who wish to pursue this avenue may consult [?], [?], or the forthcoming book "Stacks and Moduli" by Jarod Alper.

8D. Can one write down a general curve of genus g?

We have made a fuss over the value of compactifying M_g to a projective variety. To see an example of the usefulness of $\overline{M}_g$, we'll take up a question we've raised before: Can one write down a general curve of genus g? More precisely, does there exist a family of curves depending freely on parameters that includes all the curves in an open subset of M_g, as the equation $y^2 = x(x-1)(x-\lambda)$ represents general curves of genus 1? Still more precisely, we say that a variety is *unirational* if it admits a dominant morphism from an open subset of $\mathbb{A}^n$. Our question is: Is M_g unirational?

We have produced families with free parameters in genera 2 and 3. Essentially the same approach works in genera 4 and 5; in each case a general canonical curve is a complete intersection, so that if we take the coefficients of its defining polynomials to be general scalars we have a general curve.

This method breaks down when we get to genus 6, where a canonical curve is not a complete intersection. But it's close enough: as discussed in Chapter 12, a general canonical curve of genus 6 is the intersection of a smooth del Pezzo surface $S \subset \mathbb{P}^5$ with a quadric hypersurface Q; since all smooth del Pezzo surfaces in $\mathbb{P}^5$ are isomorphic, we can just fix one such surface S and let Q be a general quadric.

It gets harder as the genus increases. Already genus 7 calls for a different approach. Here we want to argue that, by Brill–Noether theory, a general curve of genus 7 can be realized as (the normalization of) a plane septic curve with 8 nodes $p_1, \dots, p_8 \in \mathbb{P}^2$. Conversely, if $p_1, \dots, p_8 \in \mathbb{P}^2$ are general points then having nodes at the points p_i imposes $3 \times 8 = 24$ independent conditions on the $\mathbb{P}^{35}$ of curves of degree 7, so that we would expect that the septic curves double at the p_i form a linear series, parametrized by a projective space $\mathbb{P}^{11}$.

This suggests that we consider the space

$$\Sigma := \{(p_1, \dots, p_8, C) \in (\mathbb{P}^2)^8 \times \mathbb{P}^{35} \mid C \text{ is singular at } p_1, \dots, p_8\}$$

With a little work, we can see that there is a unique component Σ° of Σ dominating $(\mathbb{P}^2)^8$, which is a $\mathbb{P}^{11}$-bundle over an open subset of $(\mathbb{P}^2)^8$ and hence rational; this component dominates M_7, showing that M_7 is unirational.

A similar approach works through genus 10, and Severi conjectured that it would be possible to do something similar for all genera. The approach through plane curves, however, fails in genus 11: by the Brill–Noether theorem, the smallest degree of a planar embedding of a general curve of genus 11 is 10; by Theorem 12.7 (itself a consequence of the Brill–Noether theorem), such a curve has $\binom{9}{2} - 11 = 25$ nodes. But $3 \times 25 > 65$, the dimension of the space of plane curves of degree 10. Thus, if we introduce the analog of the incidence correspondence we used in the case of genus 7 — that is,

$$\Sigma := \{(p_1, \dots, p_{25}, C) \in (\mathbb{P}^2)^{25} \times \mathbb{P}^{65} \mid C \text{ is singular at } p_1, \dots, p_{25}\}$$

then the projection $\Sigma \to (\mathbb{P}^2)^{25}$ is not dominant, and we have no idea if Σ is rational. Ad hoc (and much more difficult) arguments have been given in genera 11, 12, 13 and 14, but so far no-one can go further in producing general curves; in genus 15 it is only known that any two general curves can be connected by a chain of rational curves that passes through the locus of irreducible nodal curves in $\overline{M}_g$ [?]. In genera 15 and 16 Chang and Ran showed the weaker statement that $\overline{M}_g$ has no pluricanonical divisors

However the issue is resolved for all genera ≥ 22. Surprisingly, this depends (in the current state of our knowledge) on an understanding of the complement $\overline{M}_g \setminus M_g$ and its image in the divisor class group of $\overline{M}_g$. The starting point is the fact that a smooth n-dimensional projective variety X with an effective pluricanonical canonical divisor — that is, a nonzero section of the sheaf $\omega_X^{\otimes p}$ for some $p > 0$ — cannot be unirational: if there were a dominant rational map $\mathbb{P}^n \to X$, we could pull this section back to get an effective pluricanonical divisor on $\mathbb{P}^n$, which doesn't exist because the canonical divisor on $\mathbb{P}^n$ has negative degree. At the same time, we can analyze the divisor class theory of the space $\overline{M}_g$ and for large g exhibit an effective pluricanonical divisor on M_g by using components of $\overline{M}_g \setminus M_g$. The upshot is this:

Theorem 8.9 [?; ?; ?; ?]. *For all $g \geq 22$, M_g is not unirational.*

In each case, what is actually proven is the stronger but more technical statement that $\overline{M}_g$ has *general type*. This line of argument requires a great deal of work; the interested reader can find more details, plus a guide to the literature, in [?].

8E. Hurwitz spaces

Hurwitz spaces are spaces parametrizing branched covers. They are fascinating objects; we know quite a bit about their geometry but there is much that is unknown as well. In this discussion, we'll stick to the simplest case, that of the *small Hurwitz spaces*, parametrizing simply branched covers of $\mathbb{P}^1$.

To start with the definition: the small Hurwitz space $\mathrm{Hur}^\circ_{g,d}$ parametrizes pairs (C, f) where C is a smooth curve of genus g and $f : C \to \mathbb{P}^1$ a map of degree d with simple branching; that is,

$$\mathrm{Hur}^\circ_{g,d} = \left\{(C, f) \mid C \in M_g \text{ and } f : C \to \mathbb{P}^1 \text{ simply branched of degree } d\right\}.$$

There are two natural maps from the Hurwitz space to other spaces. First, we can "project on the first factor;" that is, simply forget the map f to arrive at a map $\pi : \mathrm{Hur}^\circ_{g,d} \to M_g$. Secondly, we can associate to a point $(C, f) \in \mathrm{Hur}^\circ_{g,d}$ the branch divisor $B \subset \mathbb{P}^1$, which is an unordered b-tuple of distinct points in $\mathbb{P}^1$, which we can think of as a point in the b-th symmetric product $(\mathbb{P}^1)_b \cong \mathbb{P}^b$. We thus have a diagram

$$\begin{tikzcd} & \mathrm{Hur}^\circ_{g,d} \arrow[dl, "\pi"'] \arrow[dr, "\beta"] & \\ M_g & & U \subset \mathbb{P}^b \end{tikzcd}$$

where $U \subset \mathbb{P}^b$ is the complement of the hypersurface in $\mathbb{P}^b$ where at least 2 of the b points are equal, called the discriminant hypersurface. Thus the Hurwitz space is positioned between an object U we understand relatively well, and an object M_g about which we would like to know more; this accounts for the historical importance of Hurwitz spaces. We'll now illustrate how this can be exploited.

To begin with, by the analysis in Section 6B, we see that *the map β is a covering space*: for any reduced divisor $B \subset \mathbb{P}^1$ there are a finite number of simply branched covers of $\mathbb{P}^1$ with branch divisor B; and as we vary the points of B locally we can deform the cover along with them. This allows us to give the Hurwitz space $\mathrm{Hur}^\circ_{g,d}$ the structure of a smooth variety, and also tells us that

$$\dim(\mathrm{Hur}^\circ_{g,d}) = b = 2d + 2g - 2.$$

The dimension of M_g. Next, we look at the projection $\pi : \mathrm{Hur}^\circ_{g,d} \to M_g$. To start, let's assume d is large relative to g; $d \geq g + 1$ suffices, but you can take d as large as you like; taking $d > 2g$ may make the argument simpler.

Proposition 8.10. *If $d \geq g + 1$, the map $\pi : \mathrm{Hur}^\circ_{g,d} \to M_g$ is surjective, with fibers of dimension $2d - g + 1$.*

Proof. The question is, how many simply branched maps $f : C \to \mathbb{P}^1$ of degree d are there for a given curve C? To begin with, the $g+1$ theorem (Theorem 5-9) tells us that there are some, whence we see that π is surjective.

We can compute the dimension of the fibers, too. To specify a map $f : C \to \mathbb{P}^1$, we can start by choosing a divisor $D \in C_d$, which will be the divisor $f^{-1}(\infty)$;

this can be a general divisor of degree d on C. Second, we choose a divisor E which will be $f^{-1}(0)$; this can be a general member of the linear system $|D|$, which has dimension $d-g$. Finally, specifying $f^{-1}(\infty)$ and $f^{-1}(0)$ determines the map f up to scalar multiplication on $\mathbb{P}^1$; adding up the degrees of freedom, we see that the fibers of π have dimension

$$d + (d-g) + 1 = 2d - g + 1. \qquad \square$$

Finally, we conclude that if $g \geq 2$ then

$$\dim(M_g) = (2d + 2g - 2) - (2d - g + 1) = 3g - 3.$$

We can use this in turn to analyze the cases of smaller d. As a basic application, note that the group PGL_2 of automorphisms of $\mathbb{P}^1$ acts on the Hurwitz space: given $\varphi \in \mathrm{PGL}_2$, we can send (C, f) to $(C, \varphi \circ f)$. Moreover, the orbits of this action lie in fibers of the projection $\pi : \mathrm{Hur}^\circ_{g,d} \to M_g$, meaning that the fibers of π have dimension at least 3.

Corollary 8.11. *If $d < \lceil \frac{g}{2} \rceil + 1$, then a general curve C of genus g does not admit a map of degree d to $\mathbb{P}^1$.*

This is one-half of the case $r = 1$ of the Brill–Noether theorem, about which we will say much more later.

Irreducibility of M_g. Another important application is the original proof of the irreducibility of M_g. Hurwitz [?] analyzed the monodromy of the map $\beta : \mathrm{Hur}^\circ_{g,d} \to U \subset \mathbb{P}^b$, which describes what happens when you let the branch points of a cover wander around in U before coming back to their original locations. He proved that the monodromy is transitive, and hence that the Hurwitz space $\mathrm{Hur}^\circ_{g,d}$ is irreducible; since $\mathrm{Hur}^\circ_{g,d}$ dominates M_g for d large, he deduced that M_g must be irreducible as well.

Hurwitz's argument illustrates a fundamental point: in practice, moduli spaces of curves "with extra structure," such as a map to projective space, are often easier to work with, and provide a useful tool for understanding the geometry of abstract moduli spaces. Given an abstract curve C of genus g, it's hard without developing a fair amount of deformation theory, to show that C varies in a nontrivial family. But if C is expressed as a branched cover, we can find such families just by varying the branch points.

There are many open problems connected with the Hurwitz scheme; here are a few:

(1) A compactification of the Hurwitz scheme by *admissible covers* (allowing both source and target of the covering to be reducible in a controlled way) is known [?], but the boundary is very complicated, and it would be interesting to find a simpler one.

(2) It is conjectured that the Picard group of the Hurwitz scheme is torsion; see [?], where the conjecture is proved for $g \leq 5$, and [?] for the case $d > g-1$.

(3) There is active work and many open problems around computing the *Hurwitz numbers*, that is, the number of curves having maps to $\mathbb{P}^1$ with specified degree and branching; see for example [?] and [?].

8F. The Severi variety

Despite having been studied for so long, many questions about plane curves remain open — for example: which ones degenerate into which others, and in what way. All plane curves of degree d have the same Hilbert function, and thus the same arithmetic genus $\binom{d-1}{2}$, but since curves of degree d can have different sorts and numbers of singularities, they can have geometric genera from 0 to $\binom{d-1}{2}$. In this section we will explore the subset of (reduced, irreducible) curves of degree d with a fixed geometric genus. We will focus on the open set consisting of nodal curves (those with only nodes as singularities), and compute its dimension.

Let $\mathbb{P}^N := \mathbb{P}^{\binom{d+2}{2}-1}$ be the projective space parametrizing plane curves of degree d. Within $\mathbb{P}^N$ the set of reduced irreducible curves is open — it is the complement of the union of the images of the maps

$$\mathbb{P}^{\binom{d_1+2}{2}-1} \times \mathbb{P}^{\binom{d_2+2}{2}-1} \to \mathbb{P}^N$$

with $d_1 + d_2 = d$ given by multiplication of forms.

Definition 8.12. The *Severi variety* $V_{d,g} \subset \overline{V}_{d,g}$ is the locus of irreducible plane curves of degree d with $\delta = \binom{d-1}{2} - g$ nodes and no other singularities. This is a locally closed subset of $\mathbb{P}^N$. (Reason: having only nodes as singularities is an open condition; having at least a certain number of them is a closed condition.) It is sometimes called the *small Severi variety*, since we are excluding curves with more complicated singularities.

We will see that in a neighborhood of $V_{d,g}$, the closure $\overline{V}_{d,g}$ is well behaved; but away from this, even the singularities of $\overline{V}_{d,g}$ are not well understood. It is an interesting open problem to find a simpler partial compactification of $V_{d,g}$.

Cheerful Fact 8.13. Corollary 8.17 says that the variety $V_{d,g}$ is smooth. In 1921 F. Severi gave an incorrect proof that $V_{d,g}$ is connected, and thus irreducible. A correct proof was finally given in [?].

Local geometry of the Severi variety. We first consider the universal singular point

$$\Phi := \{(C,p) \in \mathbb{P}^N \times \mathbb{P}^2 \mid p \in C_{\text{sing}}\}$$

and its image $\Delta \subset \mathbb{P}^N$, the *discriminant variety*.

Proposition 8.14. *Φ is smooth and irreducible of dimension $N-1$, and the discriminant Δ is a hypersurface in $\mathbb{P}^N$.*

Proof. Projection on the second factor expresses Φ as a $\mathbb{P}^{N-3}$-bundle over $\mathbb{P}^2$. Explicitly, if $[X,Y,Z]$ are homogeneous coordinates on $\mathbb{P}^2$, and $\{a_{i,j,k} \mid i+j+k=d\}$ are homogeneous coordinates on $\mathbb{P}^N$, then the universal curve

$$\mathbb{C} := \{(C,p) \in \mathbb{P}^N \times \mathbb{P}^2 \mid p \in C\}$$

is given as the zero locus of the single bihomogeneous polynomial

$$F([a_{i,j,k}],[X,Y,Z]) = \sum a_{i,j,k} X^i Y^j Z^k$$

of bidegree $(1,d)$; and the universal singular point is the common zero locus of the three partial derivatives $\partial F/\partial X$, $\partial F/\partial Y$ and $\partial F/\partial Z$.

The set of forms F that define curves singular at a given point is defined by 3 independent linear conditions, and since the set of points is 2-dimensional, the set Δ of singular forms has dimension $N-1$. □

We next compute the differential of the map $\pi : \Phi \to \mathbb{P}^N$:

Lemma 8.15. *Suppose that $(C,p) \in \Phi$, with p a node of C. The differential*

$$d\pi : T_{(C,p)}\Phi \to T_C\mathbb{P}^N$$

is injective, with image the hyperplane $H_p \subset \mathbb{P}^N$ of plane curves containing the point p.

Thus, if p is a node of C and the only singularity of C, then Δ is smooth at C; and more generally the image of a small analytic neighborhood of $(C,p) \in \Phi$ is smooth, and we can identify its tangent space at p with the hyperplane H_p.

Proof. We will prove this using affine coordinates on $\mathbb{P}^2$ and $\mathbb{P}^N$. Changing coordinates if necessary, we may assume that the point $[1,0,0]$ is not in C, and that the point p is $[0,0,1]$. Let $x = X/Z$ and $y = Y/Z$ be coordinates on the affine plane $Z \neq 0$ and write the polynomial $F(x,y,1)$ above as

$$f(x,y) = \sum_{i+j \le d} a_{i,j} x^i y^j,$$

with $a_{d,0}$ normalized to 1.

Let g, h be the two partial derivatives of f:

$$g(x,y) := \frac{\partial f}{\partial x} = \sum_{i+j\le d} i a_{i,j} x^{i-1} y^j$$

$$h(x,y) := \frac{\partial f}{\partial y} = \sum_{i+j\le d} j a_{i,j} i x^i y^{j-1}.$$

The functions f, g and h are local defining equations for Φ; we consider their partial derivatives with respect to x, y and $a_{0,0}$, evaluated at the point (C, p), as in the table:

	f	g	h
$\partial/\partial x$	0	$a_{2,0}$	$a_{1,1}$
$\partial/\partial y$	0	$a_{1,1}$	$a_{0,2}$
$\partial/\partial a_{0,0}$	1	0	0

The fact that p is a node of C (and not a more complicated singularity) implies that the upper right 2×2 submatrix is nonsingular, which shows that the differential $d\pi$ is injective, and its image is the hyperplane $a_{0,0} = 0$ in $\mathbb{P}^N$, which is exactly the hyperplane of curves containing p. $\square$

Lemma 8.16. *The nodes q_i of an irreducible nodal plane curve C of degree d impose independent conditions on curves of degree $d-3$, and hence on curves of any degree $m \ge d-3$.*

Proof. We will prove in Chapter 15 that the g sections of the canonical sheaf on the normalization $\widetilde{C}$ of C are the preimages of the sections of $\mathcal{O}_C(d-3)$ that vanish at the nodes. On the other hand, $h^0(\mathcal{O}_C(d-3)) = \binom{d-1}{2}$, and the difference is exactly the number of nodes. $\square$

Corollary 8.17. *If C is a nodal curve of degree d and geometric genus $g = \binom{d-1}{2} - \delta$, then in a neighborhood of $C \in \mathbb{P}^N$ the discriminant hypersurface of all singular curves consists of δ smooth sheets, meeting transversely, and hence $V_{d,g}$ is smooth.*

In a neighborhood of $C \in \mathbb{P}^N$ the variety $\overline{V}_{d,g'}$ with $g' = \binom{d-1}{2} - \delta' > g$ is the union of $\binom{\delta}{\delta'}$ smooth branches, each of dimension $N - \delta'$, corresponding bijectively with subsets of $\{p_1, \ldots, p_\delta\}$ of cardinality δ'. changed $p_{\delta'}$ to p_δ

Figure 8.5 shows the case $\delta = 2$, $\delta' = 1$.

Proof. Lemma 8.15 shows that in an analytic neighborhood of $C \in \mathbb{P}^N$ the discriminant hypersurface Δ consists of δ smooth sheets, each corresponding to one node, and Lemma 8.16 implies that the tangent spaces to these sheets are linearly independent. $\square$

Corollary 8.18. *The Severi variety $V_{d,g}$ has pure dimension $N - \delta$, where*

$$\delta = \binom{d-1}{2} - g.$$

In Section 19K, we give a heuristic calculation of the "expected dimension" $h(g, r, d)$ of the variety parametrizing curves of degree d and genus g in $\mathbb{P}^r$:

$$h(g, r, d) := 4g - 3 + (r + 1)(d - g + 1) - 1.$$

The actual dimension of the restricted Hilbert scheme may be quite different. But Corollary 8.18 shows that in case $r = 2$ (as in the case of $r = 1$), the actual dimension is always the expected.

8G. Exercises

Exercise 8-1. Consider the action of G_m on $\mathbb{P}^3$ given in coordinates by

$$t : (x_0, x_1, x_2, x_3) \mapsto (tx_0,\ tx_1,\ t^{-1}x_2,\ t^{-1}x_3)$$

for $t \in G_m = \mathbb{C}^*$.

(1) Show that the ring of forms in $\mathbb{C}[x_0, \dots, x_3]$ that are invariant is generated by

$$x_0x_3,\ x_0x_2,\ x_1x_3,\ x_1x_2$$

and thus $\mathbb{P}^3 /\!\!/ G_m \cong \mathbb{P}^1 \times \mathbb{P}^1$.

(2) Show that the unstable locus for this action is the union of the two lines $x_0 = x_1 = 0$ and $x_2 = x_3 = 0$.

(3) Show that the orbits of G_m are the points on the unstable lines and, for each point p not on an unstable line, a copy of $\mathbb{P}^1 \smallsetminus \{0, \infty\} \cong G_m$ whose closure is the unique line containing p and meeting both unstable lines.

Exercise 8-2. Consider the action of G_m on $\mathbb{P}^3$ given in coordinates by

$$t : (x_0, x_1, x_2, x_3) \mapsto (tx_0,\ tx_1,\ tx_2,\ t^{-1}x_3)$$

for $t \in G_m = \mathbb{C}^*$.

(1) Show that the ring of forms in $\mathbb{C}[x_0, \dots, x_3]$ that are invariant is generated by forms

$$F(x_0, x_1, x_2)x_3$$

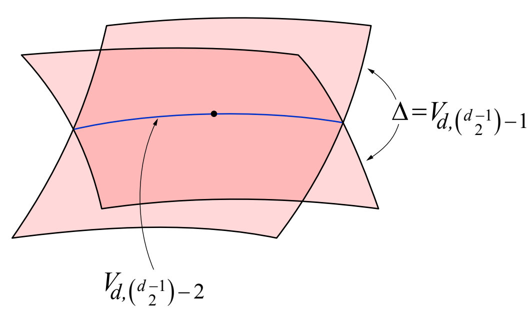


Figure 8.5. Near the point corresponding to a plane curve with 2 nodes, $V_{d,\binom{d-1}{2}-2}$ is the transverse intersection of two smooth hypersurfaces.

where F is a cubic form on $\mathbb{P}^2$, and thus $\mathbb{P}^3 /\!/ G_m \cong \mathbb{P}^2$, with the embedding given by the third Veronese map.

(2) Show that the unstable locus for this action is the union of the point $x_0 = x_1 = x_2 = 0$ and the plane $x_3 = 0$.

(3) Show that the orbits of G_m are the points on the components of the unstable locus and, for each point p that is not unstable, a copy of $\mathbb{P}^1 \setminus \{0, \infty\} \cong G_m$ whose closure is the unique line containing p and the unstable point. Thus the quotient map is the composition of the linear projection from the unstable point with the 3-uple embedding.

Exercise 8-3. Show from the explicit formula for the j-function on page 150 that if $j : B \to M_1 = \mathbb{A}^1$ is a map associated to a family $\mathcal{C} \to B$ of curves of genus 1, then every zero of the j-function has multiplicity divisible by 3, and conclude that some maps $B \to M_1$ do not correspond to families of curves; in particular there is no universal family over M_1, and thus M_1 is not a fine moduli space for curves of genus 1. There is a similar problem at $j(\lambda) = 1728$.

Exercise 8-4. In Exercise 8-3 we saw a local obstruction to the existence of a universal family over M_1. There is also a global obstruction, coming from the fact that some genus 1 curves have extra automorphisms. Show that there is a "tautological" family over the punctured j-line $L := \mathbb{A}^1 \setminus \{0, 1728\}$ — that is, a family $\mathcal{X} \to L$ whose fiber over t has j-invariant t; but show that this family is not universal as follows:

Let B be any curve of genus 1 and $\tau : B \to B$ a translation of order 2, and let E be a fixed elliptic curve (that is, a curve of genus 1 with a chosen point, so that we may identify the points of E with an abelian group). Let $\mathcal{X} \to L$ be the family $E \times B$ modulo the equivalence relation $(e, b) \sim (-e, \tau(b))$. The projection to B/τ has all fibers isomorphic to $E/(\pm) \cong E$. But the family is not isomorphic to the trivial family $E \times B/\tau \to B/\tau$.

Hint: show that the canonical bundle $K_{\mathcal{X}}$ is nontrivial.