A Chapter from "Practical Curves"

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Contents

Chapter 1

Hilbert Schemes II: Counterexamples

examplesChapter

In the preceding chapter, we described a number of examples of Hilbert schemes, and observed some patterns in their behavior: in each case the restricted Hilbert scheme $\mathcal{H}^\circ$ parametrizing smooth, irreducible and nondegenerate curves was irreducible of the "expected dimension" $h(g,r,d) := 4g-3+(r+1)(d-g+1)-1$. In fact, Theorem principal component ?? tells us that these patterns persist, for those components of $\mathcal{H}^\circ$ dominating the moduli space M_g.

But what about other components of the Hilbert scheme—components with $\rho(g,r,d) < 0$, or for that matter components with $\rho(g,r,d) \geq 0$ that simply don't dominate M_g? In fact, none of the patterns we've observed so far hold in general, and the first thing we'll do in this chapter is to give some examples, culminating with Mumford's celebrated example of a component of the restricted Hilbert scheme that is everywhere non-reduced.

We will close the chapter by discussing some intriguing conjectures suggested by Brill-Noether theory and by observed behavior in small cases.

1.1 Degree 8

egree 8 section

We start with an example of a component of the restricted Hilbert scheme $\mathcal{H}^\circ$ whose dimension is strictly greater than $h(g,r,d)$, the space $\mathcal{H}^\circ = \mathcal{H}^\circ_{9,3,8}$ of smooth, irreducible, nondegenerate curves of degree 8 and genus 9. Let C

be such a curve, and consider the restriction map

$$\rho_2 : H^0(\mathcal{O}_{\mathbb{P}^3}(2)) \longrightarrow H^0(\mathcal{O}_C(2)).$$

The source of ρ_2 has dimension 10, but the Riemann-Roch Theorem

$$h^0(\mathcal{O}_C(2)) = \begin{cases} 9, & \text{if } \mathcal{O}_C(2) \cong K_C; \\ 8, & \text{if } \mathcal{O}_C(2) \ncong K_C \end{cases}$$

admits two possibilities for the dimension of target of ρ_2. However, if $h^0(\mathcal{O}_C(2))$ were 8 then C would lie on two distinct quadrics Q and Q'. Since C is non-degenerate, it cannot lie on any irreducible quadrics; thus Q and Q' would have to be irreducible, which would violate Bézout's Theorem. We deduce that $\mathcal{O}_C(2) \cong K_C$, and thus that C lies on a unique quadric surface Q (which must be irreducible since C is irreducible and doesn't lie on a plane).

Similarly, C cannot lie on any cubic not containing Q. Moving on to quartics, we look again at the restriction map

$$\rho_4 : H^0(\mathcal{O}_{\mathbb{P}^3}(4)) \longrightarrow H^0(\mathcal{O}_C(4)).$$

The dimensions here are, respectively, 35 and $4 \cdot 8 - 9 + 1 = 24$; and we deduce that C lies on at least an 11-dimensional vector space of quartic surfaces. On the other hand, only a 10-dimensional vector subspace of these vanish on Q; and so we conclude that C *lies on a quartic surface not containing* Q. It follows from Bézout's Theorem that $C = Q \cap S$. By Lasker's Theorem, the ideal (Q, S) is saturated, so it is equal to the homogeneous ideal of C. Thus $\ker(\rho_4)$ has dimension exactly 11, and S is unique modulo quartics vanishing on Q.

From these facts it is easy to compute the dimension of $\mathcal{H}^\circ$. This is a special case of Exercise second complete intersection exercise ??, but just to say it: associating to C the unique quadric on which it lies gives a map $\mathcal{H}^\circ \to \mathbb{P}^9$ with dense image, and each fiber is an open subset of the projective space $\mathbb{P}V$, where V is the 25-dimensional vector space

$$V = \frac{H^0(\mathcal{O}_{\mathbb{P}^3}(4))}{H^0(\mathcal{I}_{Q/\mathbb{P}^3}(4))}.$$

It follows that *the space* $\mathcal{H}^\circ_{8m-8}(\mathbb{P}^3)$ *is irreducible of dimension 33*—one larger than the "expected" $4d$.

1.2 Degree 9

For the next example, consider the space $\mathcal{H}^\circ = \mathcal{H}^\circ_{9m-9}(\mathbb{P}^3)$ of curves of degree 9 and genus 10. Once more, to describe such a curve C, we look to the restriction maps $\rho_m : H^0(\mathcal{O}_{\mathbb{P}^3}(m)) \longrightarrow H^0(\mathcal{O}_C(m))$. The Riemann-Roch Theorem tells us that

$$h^0(\mathcal{O}_C(2)) = \begin{cases} 10, & \text{if } \mathcal{O}_C(2) \cong K_C \text{ (“the first case,”) and} \\ 9, & \text{if } \mathcal{O}_C(2) \not\cong K_C \text{ (“the second case.”)} \end{cases}$$

Unlike the the situation in degree 8, both are possible; we'll analyze each.

1. Suppose first that C does not lie on any quadric surface (so that we are necessarily in the first case above), and consider the map $\rho_3 : H^0(\mathcal{O}_{\mathbb{P}^3}(3)) \to H^0(\mathcal{O}_C(3))$. By the Riemann-Roch Theorem, the dimension of the target is $3 \cdot 9 - 10 + 1 = 18$, from which we conclude that C lies on at least a pencil of cubic surfaces. Since C lies on no quadrics, all of these cubic surfaces must be irreducible, and it follows by Bézout's Theorem that the intersection of two such surfaces is exactly C. At this point, Lasker's Theorem assures us that C lies on exactly two cubics.

`first complete intersection exercise`

By Exercise **??**, then, the space $\mathcal{H}^\circ_1$ of curves of this type is thus an open subset of the Grassmannian $G(2, 20)$ of pencils of cubic surfaces, which is irreducible of dimension 36.

2. Next, suppose that C does lie on a quadric surface $Q \subset \mathbb{P}^3$; let $\mathcal{H}^\circ_2 \subset \mathcal{H}^\circ$ be the locus of such curves. In this case, we claim two things:

a. Q must be smooth; and

b. C must be a curve of type $(3, 6)$ on Q

For part (a), we claim that in fact *a smooth, irreducible nondegenerate curve C of degree 9 lying on a singular quadric must have genus 12.* We can see this by observing that Q must be a cone over a smooth conic curve, and so its blow-up at the vertex is the Hirzebruch surface $\mathbb{F}_2$, with the directrix $E \subset \mathbb{F}_2$ the exceptional divisor of the blowup, and a line L of the ruling of $\mathbb{F}_2$ the proper transform of a line lying on Q. The pullback to $\mathbb{F}_2$ of the hyperplane class has intersection number 1 with L and 0 with E, from which it follows that its class must be $H = 2L + E$

Now, the proper transform $\tilde{C}$ of C in $\mathbb{F}_2$ has intersection number 1 with E, since C passes through the vertex of Q and is smooth there; given this, and the fact that it has intersection number 9 with $H = 2L + E$, we can deduce that the class of $\tilde{C}$ is $9L + 4E$. Now, we know that $K_{\mathbb{F}_2} = -2E - 4L$; by adjunction we deduce that the genus of C is 12.

For the second part, once we know that Q is smooth, the genus formula on Q tells us immediately that C must be of type $(3, 6)$ or $(6, 3)$. Now, since the quadric Q containing C is unique, by Bézout, we have a map $\mathcal{H}_2^\circ \to \mathbb{P}^9$ associating to each curve C of this type the unique quadric containing it. The fiber of this map over a given quadric Q is the disjoint union of open subsets of the projective spaces $\mathbb{P}^{27}$ parametrizing curves of type $(3, 6)$ and $(6, 3)$ on Q, and we see that the locus $\mathcal{H}_2^\circ$ again has dimension 36.

Exercise 1.2.1. While the above argument does not prove that the locus $\mathcal{H}_2^\circ$ is irreducible (in the absence of a monodromy argument), we can see that it's irreducible via a liaison argument: we're saying that a curve C of the second type is residual to a union of three skew lines in the intersection of a quadric and a sextic curve. Carry out this argument to establish that $\mathcal{H}_2^\circ$ is indeed irreducible.

In sum, there are two types of smooth, irreducible, nondegenerate curves $C \subset \mathbb{P}^3$ of degree 9 and genus 10: type 1, which are complete intersections of two cubics; and type 2, which are curves of type $(3, 6)$ on a quadric surface. Moreover, the family of curves of each type is irreducible of dimension 36; and we conclude that *the space $\mathcal{H}_{9m-9}^\circ(\mathbb{P}^3)$ is reducible, with two components of dimension 36.*

Exercise 1.2.2. In the preceding argument, we used a dimension count to conclude that a general curve of type 1 could not be a specialization of a curve of type 2, and vice versa. Prove these assertions directly: specifically, argue that

1. by upper-semicontinuity of $h^0(\mathcal{I}_{C/\mathbb{P}^3}(2))$, argue that a curve C not lying on a quadric cannot be the specialization of curves C_t lying on quadrics; and

2. show that for a general curve of type $(3, 6)$ on a quadric, $K_C \ncong \mathcal{O}_C(2)$, and deduce that a general curve of type 2 is not a specialization of curves of type 1.

Exercise 1.2.3. Let Σ_1 and $\Sigma_2 \subset \mathcal{H}^\circ_{9m-9}(\mathbb{P}^3)$ be the loci of curves of types 1 and 2 respectively.

1. What is the intersection of the closures of Σ_1 and Σ_2 in $\mathcal{H}^\circ_{9m-9}(\mathbb{P}^3)$?
2. What is the intersection of the closures of Σ_1 and Σ_2 in the whole Hilbert scheme $\mathcal{H}_{9m-9}(\mathbb{P}^3)$?

1.3 Special components in the nonspecial range

If we ignore the finer points of the Brill-Noether theorem and focus just on the statement about the dimension and irreducibility of the variety of linear series on a curve, we can express it in a simple form: according to Theorem ?? principal component *Any component of the restricted Hilbert scheme $\mathcal{H}^\circ$ of curves of degree d and genus g that dominates the moduli space M_g has the expected dimension*

$$h(g, r, d) = 4g - 3 + (r + 1)(d - g + 1) - 1 = (r + 1)d - (r - 3)(g - 1)$$

as calculated in Section estimating dim hilb ?? *above; and, in the Brill-Noether range (that is, when the Brill-Noether number $\rho(g, r, d) \geq 0$ is nonnegative), there exists a unique such component.*

If we restrict further to the nonspecial range $d \geq g + r$, we don't need the ghosts of Brill or Noether to tell us this: if $\mathcal{L}$ is a general line bundle of degree d on a general curve C of genus g, and $V \subset H^0(\mathcal{L})$ a general $(r+1)$-dimensional subspace, the linear system V will embed the curve C as a nondegenerate curve of degree d in $\mathbb{P}^r$, and the curve obtained in this way will comprise an irreducible component of the restricted Hilbert scheme.

But that doesn't mean that there aren't other components of the restricted Hilbert scheme, even in the nonspecial range! In this section, we'll construct an example of this: a component of the restricted Hilbert scheme $\mathcal{H}^\circ_{g,r,d}$, with $d \geq g + r$, that does not dominate M_g and indeed has the wrong dimension.

For our example, we'll take $d = 28$, $g = 21$ and $r = 7$. Again, a general line bundle $\mathcal{L}$ of degree 28 on a general curve C of genus 21 will be very ample (we could invoke the Brill-Noether theorem for this, but it follows from the more elementary argument for Theorem degree g+3 very ample ??). Curves of genus 21 embedded in $\mathbb{P}^7$ in this way comprise a component $\mathcal{H}_0$ of the Hilbert scheme $\mathcal{H}^{\circ}_{21,7,28}$ having the expected dimension

$$h(21,7,28) = 4g - 3 + (r+1)(d-g+1) - 1 = 144.$$

But here's another way to construct a curve of degree 28 and genus 21 in $\mathbb{P}^7$, that will produce a larger family of such curves! To start with, let's restrict to the trigonal locus in M_{21}; that is, we'll assume the curve C is trigonal. (This immediately cuts down on our degrees of freedom, but we'll make up for it in the choice of linear system.)

We now want to look at the line bundle residual to 4 times the g^1_3 on C; that is, if $\mathcal{M}$ is the line bundle of degree 3 on C having two sections, we take $\mathcal{L} = K_C \otimes \mathcal{M}^{-4}$. We first need to calculate the dimension of the space of sections of $\mathcal{L}$, and to show that this bundle is in fact very ample; these will be special cases of the following lemma.

Lemma 1.3.1. *Let C be a general trigonal curve of genus g, $\mathcal{M}$ the line bundle of degree 3 on C having two sections, and $\mathcal{L} = K_C \otimes \mathcal{M}^{-l}$.*

1. *If $l \leq g/2$, then $h^0(\mathcal{L}) = g - 2l$; and*
2. *If $l \leq (g-4)/2$, then $\mathcal{L}$ is very ample.*

Proof. Both statements follow from our description of the geometry of canonical models of trigonal curves, carried out in trigonal canonical models ??. We observed there that a trigonal canonical curve lies on a rational normal scroll S, and that *if C is general, then the scroll S is balanced.* The linear system $|\mathcal{L}| = |K_C \otimes \mathcal{M}^{-l}|$ is then cut out by hyperplanes in $\mathbb{P}^{g-1}$ containing any l chosen lines from the ruling of S; and the first part follows from the fact that *on a balanced scroll $S \subset \mathbb{P}^r$, any $(r+1)/2$ lines of the ruling are linearly independent.*

The second part follows similarly, when we observe that if $r \geq 5$, $S \subset \mathbb{P}^r$ is any balanced rational normal scroll, and $L \subset S$ any line of the ruling, then the projection $\pi_L : S \to \mathbb{P}^{r-2}$, while a priori only rational, in fact extends to a regular map on all of S, embedding S as a balanced scroll in $\mathbb{P}^{r-2}$. Restricting to any curve $C \subset S$, it follows that π_L gives an embedding of C in $\mathbb{P}^{r-2}$ as well. □

Getting back to our present example, what we see is that if C is a general trigonal curve of genus 21 with $g^1_3 = |\mathcal{M}|$, and $\mathcal{L} = K_C \otimes \mathcal{M}^{-4}$, then the line bundle $\mathcal{L}$ embeds C as a curve of degree $2g-2-12=28$ in $\mathbb{P}^{12}$. Now we consider the projection of the image curve in $\mathbb{P}^{12}$ to $\mathbb{P}^7$. The family of such projections is parametrized by an open subset of the Grassmannian $\mathbb{G}(4,12)$, which has dimension 40. We thus have $2g+1=43$ degrees of freedom in choosing the general trigonal curve C, and another 40 degrees of freedom in choosing the projection (that is, the subseries $g^7_{28} \subset |\mathcal{L}|$); together these determine the image curve $C \subset \mathbb{P}^7$ up to automorphisms of $\mathbb{P}^7$. In sum, we see that the family $\mathcal{H}_1$ of curves $C \subset \mathbb{P}^7$ described in this way has dimension

$$\dim \mathcal{H}_1 = 43 + 40 + 63 = 146.$$

In particular, $\mathcal{H}_1$ cannot be in the closure of $\mathcal{H}_0$. Thus, even though we are in the nonspecial range $d \geq g+r$, there is at least one other irreducible component of the restricted Hilbert scheme, which maps to a proper subvariety of M_g and has dimension strictly greater than the expected.

1.4 Degree 14: Mumford's example

mumford example

In many of the analyses above, we've been able to use the identification of the tangent space to the Hilbert scheme $\mathcal{H}$ at a point $[C]$ with the space $H^0(\mathcal{N}_{C/\mathbb{P}^3})$ of global sections of the normal bundle of C to tell whether the Hilbert scheme was smooth or singular at the point $[C]$. What's more, in every case where we carried this out, the conclusion was that the restricted Hilbert scheme $\mathcal{H}^\circ$ at least was smooth.

Does this pattern persist? The answer is a resounding "no:" in this section, we'll analyze an example, first discovered by Mumford, of an entire irreducible component of $\mathcal{H}^\circ$ that is everywhere singular, that is, everywhere nonreduced.

The example is the Hilbert scheme $\mathcal{H}^\circ = \mathcal{H}^\circ_{24,3,14}$ parametrizing smooth, irreducible curves C of degree 14 and genus 24 in $\mathbb{P}^3$. We shall analyze this example in out usual way, and examine three irreducible components of $\mathcal{H}^\circ$, one of which will be the celebrated Mumford component.

We will begin as always by analyzing the possible degrees of generators of the ideal of C, for $C \subset \mathbb{P}^3$ a smooth, irreducible curve of degree 14 and genus 24. By applying the genus formula for plane curves and curves on quadrics we see that C cannot lie in a plane or on a quadric. By Bézout's Theorem, ostulation table

m	$h^0(\mathcal{O}_C(m))$	$h^0(\mathcal{O}_{\mathbb{P}^3}(m))$
3	19, 20 or 21	20
4	33	35
5	47	56
6	61	84

Table 1.1: Postulation table postulation table

C cannot lie on both a cubic and a quartic hypersurface, though we shall see that both possibilities are realized.

For $m \geq 3$ let $\rho_m : H^0(\mathcal{O}_{\mathbb{P}^3}(m)) \longrightarrow H^0(\mathcal{O}_C(m))$ be the natural maps. We will proceed by computing the size of the kernel of ρ_m for $m \geq 3$.

For $m \geq 4$, the line bundle $\mathcal{O}_C(m)$ has degree $> 2g - 2 = 46$, so the Riemann-Roch Theorem gives an exact value of $h^0(\mathcal{O}_C(m))$. However, when $m = 3$ we have

$$h^0(\mathcal{O}_C(3)) = 42 - 24 + 1 + h^0(K_C(-3)).$$

Since $d - g + 1 = 14 - 24 + 1$ is negative, C is embedded in $\mathbb{P}^3$ by a special linear series, and it follows from Section hyperelliptic special ?? that C is not hyperelliptic. The special line bundle $K_C(-3)$ has degree $46 - 42 = 4$ so, by Clifford's Theorem in the non-hyperelliptic case, $h^0(K_C(-3)) \leq 2$. Thus $h^0(\mathcal{O}_C(3)) = 19, 20$ or 21.

The "postulation table" (postulation table 1.1) collects the dimensions of the source and target of ρ_m for $m = 3, \ldots, 6$.

1.4.1 Case 1: C does not lie on a cubic surface

mford example H1

Proposition 1.4.1. *The locus $\mathcal{H}_1 \subset \mathcal{H}^\circ$ parameterizing curves not lying on a cubic surface is dense in an irreducible component of $\mathcal{H}^\circ$. It has dimension 56, and is generically smooth.*

The proof of this proposition will occupy us for several pages. Let C be curve in $\mathcal{H}_1$. Table postulation table 1.1 shows that C lies on at least two linearly independent quartic surfaces S and S'; and since C does not lie on any surface of smaller degree, neither can be reducible. It follows that the intersection $S \cap S'$ must consist of the union of the curve C and a curve D of degree 2. The linkage formula (linked genus formula ??) says that

$$p_a(C) - p_a(D) = (14-2)\frac{4+4-4}{2} = 24,$$

so D has arithmetic genus 0. Note that the proof above of formula (linked genus formula ??) requires that at least one of the quartic surfaces containing C is smooth, which we don't a priori know in this setting; to apply it we need to invoke the more general Theorem justification of general linkage ?? from Chapter linkage Chapter ??.

We can now invoke the following lemma:

conics **Lemma 1.4.2.** *A subscheme $D \subset \mathbb{P}^3$ of dimension 1, degree 2 and arithmetic genus 0 (that is, $\chi(\mathcal{O}_D) = 1$) is necessarily a plane conic; that is, the complete intersection of a plane and a quadric.*

We remark that the need to prove a lemma like this is one of the drawbacks of the method of liaison: even if we are a priori interested just in smooth, irreducible and nondegenerate curves in $\mathbb{P}^3$, applying liaison can lead to singular and/or nonreduced curves. There are some restrictions—by Theorem justification of general linkage ??, for example, says that a curve residual to a pure-dimensional scheme in a complete intersection is pure dimensional. For the present case, knowing even this is unnecessary because a general curve in $\mathcal{H}_1$ lies on a smooth quartic surface.

Proof of Lemma conics *1.4.2.* Let $H \subset \mathbb{P}^3$ be a general plane, and set $\Gamma = C \cap H$. This is a scheme of dimension 0 and degree 2 in $H \cong \mathbb{P}^2$, which is then either the union of two reduced points, or a single nonreduced point isomorphic to $\operatorname{Spec} k[\epsilon]/(\epsilon^2)$. Either way, we observe that the restriction map $H^0(\mathcal{O}_{\mathbb{P}^3}(m)) \to H^0(\mathcal{O}_\Gamma(m))$ is surjective for all $m \geq 1$, and hence the map $H^0(\mathcal{O}_C(m)) \to H^0(\mathcal{O}_\Gamma(m))$ is as well. It follows that

$$h^0(\mathcal{O}_C(m)) \geq h^0(\mathcal{O}_C(m-1)) + 2$$

for all $m \geq 1$; since we know by hypothesis that $h^0(\mathcal{O}_C(m)) = 2m+1$ for m large, we may conclude that $h^0(\mathcal{O}_C(1)) \leq 3 < h^0(\mathcal{O}_{\mathbb{P}^3}(1))$—in other words, the scheme C must be contained in a plane. It is thus a plane conic, without embedded points since any embedded points would mean $p_a(C) < 0$. □

Conversely, if C is any curve residual to a conic D in the complete intersection of two quartics, it must have degree 14 and genus 24, and by Bézout's Theorem it cannot lie on a cubic surface. We can thus compute the dimension of the family $\mathcal{H}_1$ of smooth curves of degree 14 and genus 24 not lying on a cubic surface via the incidence correspondence

$$\Phi = \{(C, D, S, S') \in \mathcal{H}^\circ \times \mathcal{H}_D \times \mathbb{P}^{34} \times \mathbb{P}^{34} \mid S \cap S' = C \cup D\}.$$

where $\mathcal{H}_D$ denotes the Hilbert scheme of plane conics. The Hilbert scheme $\mathcal{H}_D$ is irreducible of dimension 8 (this is a special case $m = 1$, $n = 2$ of Exercise second complete intersection exercise ??); and for any conic $D = V(L, Q)$ given as the complete intersection of the plane $V(L)$ and the quadric $V(Q)$, Lasker's Theorem says that the homogeneous ideal of $D \subset \mathbb{P}^3$ is generated by L and Q; this allows us to see that the space of quartic surfaces containing D is a linear subspace of $\mathbb{P}^{34}$ of dimension 26. The fibers of Φ over $\mathcal{H}_D$ are thus open subsets of $\mathbb{P}^{25} \times \mathbb{P}^{25}$, and we deduce that Φ is irreducible of dimension 58.

Exercise 1.4.3. The general members of the family of quartic surfaces containing a smooth conic are themselves smooth.
((give a hint))

The general members of the family of quartic surfaces containing a smooth conic are themselves smooth, so we see from considering C, D as divisors on a smooth quartic, as in the derivation of the linkage formula, that $(C \cdot D) = 10$. It follows that any quartic surface containing C must contain D as well and so, by Lasker's Theorem, must be a linear combination of S and S'. The fibers of Φ over its image in $\mathcal{H}_C$ are thus open subsets of $\mathbb{P}^1 \times \mathbb{P}^1$. The condition of not lying on a cubic surface is open, so $\mathcal{H}_1$ is dense in an irreducible component of $\mathcal{H}^\circ$ of dimension 56.

1.4.2 Tangent space calculations

It remains to show that $\mathcal{H}_1$ is generically smooth. To do this, we have to show that, at a general point $[C] \in \mathcal{H}_1$, the dimension of the Zariski tangent space $H^0(\mathcal{N}_{C/\mathbb{P}^3})$ is 56. Let S be a smooth quartic surface containing C, and consider the exact sequence

bundle sequence (1.1)
$$0 \to \mathcal{N}_{C/S} \to \mathcal{N}_{C/\mathbb{P}^3} \to \mathcal{N}_{S/\mathbb{P}^3}|_C \to 0.$$

The bundle $\mathcal{N}_{S/\mathbb{P}^3}|_C \cong \mathcal{O}_C(4)$, which is nonspecial; we have $h^0(\mathcal{O}_C(4)) = 33$ and $h^1(\mathcal{O}_C(4)) = 0$. By the adjunction formula applied to S we see that $K_S = \mathcal{O}_S$, and applying the formula again on S we see that $\mathcal{N}_{C/S} \cong K_C$. Thus $h^0(\mathcal{N}_{C/S}) = 24$ and $h^1(\mathcal{N}_{C/S}) = 1$.

From the long exact sequence in cohomology associated to the sequence (*) we see that there are two possibilities for the dimension of $H^0(\mathcal{N}_{C/\mathbb{P}^3})$: 56 and 57, depending on whether the map $H^0(\mathcal{N}_{C/\mathbb{P}^3}) \to H^0(\mathcal{N}_{S/\mathbb{P}^3}|_C)$ is surjective or of corank 1.

To settle this question, we need to invoke a basic fact about deformations of subschemes of a given scheme. For this discussion, let Z be an arbitrary fixed scheme, and $X \subset Y \subset Z$ a nested pair of subschemes. We can ask two questions:

1. Given a first-order deformation $\tilde{Y} \subset \operatorname{Spec} k[\epsilon]/(\epsilon^2) \times Z$ of Y in Z, does there exist a first-order deformation $\tilde{X} \subset \operatorname{Spec} k[\epsilon]/(\epsilon^2) \times Z$ of X contained in it? and

2. Given a first-order deformation $\tilde{X} \subset \operatorname{Spec} k[\epsilon]/(\epsilon^2) \times Z$ of X in Z, does there exist a first-order deformation $\tilde{Y} \subset \operatorname{Spec} k[\epsilon]/(\epsilon^2) \times Z$ of Y containing it?

The answer is a basic fact from deformation theory. Let α, β be the natural maps in the following diagram:

$$\begin{tikzcd} H^0(\mathcal{N}_{X/Z}) \arrow[r, "\alpha"] & H^0(\mathcal{N}_{Y/Z}|_X) \\ & H^0(\mathcal{N}_{Y/Z}). \arrow[u, "\beta"] \end{tikzcd}$$

and deformation

Lemma 1.4.4. *The first-order deformation of X corresponding to the global section $\sigma \in H^0(\mathcal{N}_{X/Z})$ is contained in the first-order deformation of Y corresponding to the global section $\tau \in H^0(\mathcal{N}_{Y/Z})$ if and only if $\alpha(\sigma) = \beta(\tau)$. In particular, every first-order deformation of Y contains a first-order deformation of X if and only if* $\operatorname{im}(\beta) \subset \operatorname{im}(\alpha)$.

For a proof of this lemma, see Chapter 6 of [?].

We apply this construction to $Z = \mathbb{P}^3$, $Y = S \subset \mathbb{P}^3$ a smooth quartic surface, and $X = D \subset S$ a smooth plane conic curve. We start with the sequence

$$0 \to \mathcal{N}_{D/S} \to \mathcal{N}_{D/\mathbb{P}^3} \to \mathcal{N}_{S/\mathbb{P}^3}|_D \to 0.$$

Identifying D with $\mathbb{P}^1$, we have by adjunction that $\mathcal{N}_{D/S} \cong \mathcal{O}_{\mathbb{P}^1}(-2)$, and S being a quartic, we have $\mathcal{N}_{S/\mathbb{P}^3}|_D \cong \mathcal{O}_{\mathbb{P}^1}(8)$. Moreover, since D is the complete intersection of a quadric and a plane, we have $\mathcal{N}_{D/\mathbb{P}^3} \cong \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(4)$, so that the sequence above looks like

$$0 \to \mathcal{O}_{\mathbb{P}^1}(-2) \to \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(4) \to \mathcal{O}_{\mathbb{P}^1}(8) \to 0$$

Now, we know that $H^1(\mathcal{O}_{\mathbb{P}^1}(-2)) \neq 0$, while $H^1(\mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(4)) = 0$, so we conclude by Lemma 1.4.4 that the map $H^0(\mathcal{N}_{D/\mathbb{P}^3}) \to H^0(\mathcal{N}_{S/\mathbb{P}^3}|_D)$ cannot be surjective; in other words, there exist first-order deformations of S that contain no first-order deformation of D.

The same argument works if D is the union of two lines meeting at a point.

We need to introduce one more element into the argument, which is expressed in the following proposition.

Proposition 1.4.5. *Let S be a smooth quartic surface, and C and $D \subset S$ a pair of curves forming the complete intersection of S with another quartic surface S', with D a plane conic curve. A first-order deformation $\tilde{S}$ of S contains a first-order deformation of C if and only if it contains a first-order deformation of D.*

Proof. The key ingredient is the observation that $H^1(\mathcal{O}_S(D)) = H^1(\mathcal{O}_S(C)) = 0$. What this says is that a first-order deformation $\tilde{S}$ of S contains a first-order deformation of D if and only if it contains a first-order deformation of the line bundle $\mathcal{L} = \mathcal{O}_S(D)$; that is, if and only if there exists a line bundle $\tilde{\mathcal{L}}$ on $\tilde{S}$ such that $\mathcal{L}|_S \cong \mathcal{O}_S(D)$, and likewise for C. But the existence of a line bundle $\tilde{\mathcal{L}}$ on $\tilde{S}$ extending $\mathcal{O}_S(D)$ is equivalent to the existence of a line bundle $\tilde{\mathcal{M}}$ on $\tilde{S}$ extending $\mathcal{O}_S(C)$, since they're related by $\tilde{\mathcal{M}} = \mathcal{O}_{\tilde{S}}(4) \otimes \tilde{\mathcal{L}}$. □

Now, going back to the exact sequence (1.1), we have shown that there exist first-order deformations of S that contain no first-order deformations of C; thus the sequence (1.1) is not exact on global sections, and hence the dimension $h^0(\mathcal{N}_{C/\mathbb{P}^3}) = 56$, showing that $\mathcal{H}_1$ is generically smooth.

1.4.3 What's going on here?

We should take a moment to give some background for the argument above. The basic idea is built on a striking fact about curves on surfaces in $\mathbb{P}^3$, called the *Noether-Lefschetz theorem.*

ether Lefschetz **Theorem 1.4.6** (Noether-Lefschetz)**.** *If $S \subset \mathbb{P}^3$ is a very general surface of degree $d \geq 4$ in $\mathbb{P}^3$, and $C \subset S$ is any curve, then C is a complete intersection $S \cap T$ with S.*

Thus, for example, a very general quartic surface contains no lines, conics or twisted cubics—facts you can readily establish for yourself via a standard dimension count, as the following exercises suggest.

Exercise 1.4.7. Let $\mathbb{G}(1,3)$ be the Grassmannian of lines in $\mathbb{P}^3$, let $\mathbb{P}^{19}$ denote the space of quartic surfaces $S \subset \mathbb{P}^3$, and consider the incidence correspondence

$$\Gamma = \{(S, L) \in \mathbb{P}^{19} \times \mathbb{G}(1,3) \mid L \subset S\}$$

Calculate the dimension of Γ, and deduce in particular that the projection map $\Gamma \to \mathbb{P}^{19}$ cannot be dominant.

Exercise 1.4.8. In the preceding exercise, replace the Grassmannian $\mathbb{G}(1,3)$ with the restricted Hilbert schemes $\mathcal{H}^\circ$ parametrizing conics and twisted cubics, and carry out the analogous calculation to deduce that a general quartic surface $S \subset \mathbb{P}^3$ contains no conics or twisted cubics. What goes wrong when we replace $\mathcal{H}^\circ$ with the restricted Hilbert scheme of curves of higher degree?

In fact, calculations like the one suggested in these exercises were how Noether first came to propose Theorem Noether Lefschetz 1.4.6; it was not until Lefschetz that a complete proof was given.

In these terms, we can identify the crucial ingredient in the proof of the generic smoothness part of Proposition mumford example H1 1.4.1 as a strengthened form of the Noether-Lefschetz Theorem:

Theorem 1.4.9 (Deformation Noether-Lefschetz)**.** *If $S \subset \mathbb{P}^3$ is a smooth surface of degree $d \geq 4$, and $C \subset S$ is any curve that is not a complete intersection with S, then there exists a first-order deformation $\tilde{S}$ of S that does not contain a first-order deformation of C.*

We will prove this by ad-hoc methods in the case of interest to us here; the proof of the general case, given in ****, uses Hodge theory.

1.4.4 Case 2: C lies on a cubic surface S

Now suppose that C is a smooth irreducible curve of degree 14 and genus 24 that *does* lie on a cubic surface S. Bézout's Theorem tells us that S is unique, and we will restrict ourselves to the open subset $\mathcal{H}_2 \subset \mathcal{H}^\circ \setminus \mathcal{H}_1$ where the surface S is smooth, which in fact is dense—see [?].

Bézout's Theorem tells us that C cannot lie on a quartic surface not containing S. If C lay on a quintic surface not containing S then C would be residual to a line in the complete intersection of S and the quintic, and the liaison formula **??** would tell us that

$$g(C) = (14-1)\frac{3+5-4}{2} = 26,$$

a contradiction, so C lies on no quintic surface.

postulation table

On the other hand, Table 1.1 tells us that there is at least a $84-61=23$-dimensional vector space of sextic polynomials vanishing on C, only a 20-dimensional subspace of which can vanish on S. Thus there is a $\mathbb{P}^2$ of sextic surfaces containing C but not containing S, and, choosing one of them we can write

$$S \cap T = C \cup D$$

with T a sextic surface and D a curve of degree 4. The liaison formula tells us that

$$g(C) - g(D) = (14-4)\frac{3+6-4}{2} = 25,$$

so the arithmetic genus of D is -1. We will henceforth take T to be general among sextics containing C, so that D will be a general member of the (at least) 2-dimensional linear system cut on S by sextics containing C.

2a,b **Proposition 1.4.10.** *D must either be (a) the disjoint union of a line and a twisted cubic on S; or (b) a union of two disjoint conics on S.*

character of D **Exercise 1.4.11.** (Guided exercise to prove this proposition: first, D cannot have multiple components; then, must be disconnected.)

Since neither of the cases described in Proposition 2a,b 1.4.10 is a specialization of the other, we conclude that the locus $\mathcal{H}_2$ is the union of two disjoint loci $\mathcal{H}_2'$ and $\mathcal{H}_2''$ corresponding to these two cases. We consider these in turn.

Exercise 1.4.12. (Guided exercise to prove this AND deduce that $\mathcal{H}_2'$ and $\mathcal{H}_2''$ are irreducible, either by the incidence correspondences or by monodromy.)

Case $2'$: D is the disjoint union of a twisted cubic and a line

Proposition 1.4.13. *The locus $\mathcal{H}_2' \subset \mathcal{H}^\circ$ parameterizing curves C residual to the disjoint union of a line and a twisted cubic in the complete intersection of a sextic and a smooth cubic surface is an irreducible component of $\mathcal{H}^\circ$. It has dimension 56, and is generically smooth.*

Proof. Let $\mathcal{H}$ be the locus in the Hilbert scheme $\mathcal{H}_{-1,3,4}$ corresponding to disjoint unions of twisted cubics and lines, and consider the correspondence

$$\Phi = \{(C, D, S, T) \in \mathcal{H}_2' \times \mathcal{H} \times \mathbb{P}^{19} \times \mathbb{P}^{83} \mid S \cap T = C \cup D\}.$$

We have $\dim \mathcal{H} = 16$, and by Proposition quartic curve postulation 1.4.15 the fiber of Φ over a point $[D] \in \mathcal{H}$ is an open subset of the product $\mathbb{P}^5 \times \mathbb{P}^{37}$; so we see that Φ is irreducible of dimension 58. The fibers of Φ over $\mathcal{H}_2'$ are 2-dimensional, and we conclude that $\mathcal{H}_2'$ is irreducible of dimension 56.

Finally, we calculate the dimension of the Zariski tangent space $H^0(\mathcal{N}_{C/\mathbb{P}^3})$ to $\mathcal{H}_2'$ at a general point $[C]$. We do this, as before, by considering the exact sequence associated to the inclusion of C in S:

$$0 \to \mathcal{N}_{C/S} \to \mathcal{N}_{C/\mathbb{P}^3} \to \mathcal{N}_{S/\mathbb{P}^3}|_C \to 0$$

Here there is no ambiguity about the first term: by adjunction, the degree of the normal bundle of C in S is 60, which is greater than $2g(C) - 2 = 46$; so $h^1(\mathcal{N}_{C/S}) = 0$ and $h^0(\mathcal{N}_{C/S}) = 37$.

On the other hand, $\mathcal{N}_{S/\mathbb{P}^3}|_C \cong \mathcal{O}_C(3)$, and from Table postulation table 1.1, we see that $h^0(\mathcal{O}_C(3))$ can a priori be 19, 20 or 21. We will use the explicit description of C to show that, in this case, $h^0(\mathcal{O}_C(3)) = 19$.

For this purpose, let L and T denote the line component and the twisted cubic component of D respectively; and let H denote the hyperplane class on S. From the adjunction formula we can compute the self-intersection numbers of these curves on S as $(L \cdot L) = -1$ and $(T \cdot T) = 1$. Since $C \sim 6H - D$ on S, we have

$$(C \cdot L) = ((6H - L - T) \cdot L) = 7; \quad \text{and} \quad (C \cdot T) = ((6H - L - T) \cdot L) = 17$$

In other words, the curves L and T intersect C in divisors E_L and E_T of degrees 7 and 17 respectively. By Serre duality,

$$h^1(\mathcal{O}_C(3)) = h^0(K_C(-3))$$

and by adjunction,

$$K_C(-3) = K_S(C)(-3)|_C = \mathcal{O}_S(-H + 6H - D - 3H)|_C = \mathcal{O}_C(2)(-E_L - E_T).$$

Now, the quadrics in $\mathbb{P}^3$ cut out on C the complete linear series $|\mathcal{O}_C(2)|$,
((Could be proven by using the representation of a cubic surface as a blowup of the plane.))
so $h^1(\mathcal{O}_C(3))$ is the dimension of the space of quadratic polynomials vanishing on E_L and E_T. But E_L consists of seven points on the line L, so any quadric containing E_L contains L; and likewise since E_T has degree $17 > 2 \cdot 3$, any quadric containing E_T contains T. Since no quadric contains the disjoint union of a line and a twisted cubic, we conclude that $h^1(\mathcal{O}_C(3)) = 0$ and $h^0(\mathcal{O}_C(3)) = 19$.

Putting this all together, we conclude that $h^0(\mathcal{N}_{C/\mathbb{P}^3}) = 56$; so the component $\mathcal{H}_2'$ of the Hilbert scheme $\mathcal{H}^\circ$ is generically smooth of dimension 56. $\square$

Case $2''$: D is the disjoint union of two conics

umford component

Proposition 1.4.14. *The locus $\mathcal{H}_2'' \subset \mathcal{H}^\circ$ parameterizing curves C residual to the disjoint union of two conics in the complete intersection of a sextic and a smooth cubic surface is an irreducible component of $\mathcal{H}^\circ$. It has dimension 56, but is non-reduced: its tangent space at a generic point has dimension 57.*

Proof. The analysis this case follows the same path as the preceding until the very last step, where the residual curve D is the disjoint union of two conic curves rather than the disjoint union of a line and a twisted cubic. What difference does this make? Both the disjoint union of two conic curves and the disjoint union of a line and a twisted cubic are curves of degree 4 and arithmetic genus -1, so they both have Hilbert polynomial $p(m) = 4m + 2$. The difference is that they do not have the same Hilbert function, according to the following proposition:

rve postulation

Proposition 1.4.15. *Let E be the disjoint union of two conic curves in $\mathbb{P}^3$ and E' the disjoint union of a line and a twisted cubic. Let $h(m)$ and $h'(m)$ be their respective Hilbert functions, and $p(m) = 4m+2$ their common Hilbert polynomial.*

1. *For all $m \neq 3$, we have $h(m) = h'(m)$; and both are equal to $p(m) = 4m + 2$ for $m \geq 3$; but*
2. *$h(2) = 9$, while $h'(2) = 10$ (in other words, E lies on a unique quadric surface, while E' is not contained in any quadric surface).*

Proof. Let S be the homogeneous coordinate ring of $\mathbb{P}^3$, and let $I_E = I_{Q_1} \cap I_{Q_2}$ be the homogeneous ideal of E, where the I_{Q_i} are the homogeneous ideals of the two disjoint conics. Similarly, let $I_{E'} = I_L \cap I_T$ be the homogeneous ideal of E', where I_L is the homogeneous ideal of a line and I_T is the homogeneous ideal of a disjoint twisted cubic. We have exact sequences

$$\begin{aligned} 0 \to &S/I_E \to S/I_{Q_1} \oplus S/I_{Q_2} \to S/(I_{Q_1} + I_{Q_2}) \to 0 \\ 0 \to &S/I_{E'} \to S/I_L \oplus S/I_T \to S/(I_L + I_T) \to 0. \end{aligned}$$

Writing h_Q, h_L, h_T for the Hilbert functions of Q, L and T respectively, we have

$$\begin{aligned} h_Q(m) &= 2m + 1 \\ h_L(m) &= m + 1 \\ h_T(m) &= 3m + 1 \end{aligned}$$

for all $m \geq 0$.

Because each of E, E' is a disjoint union, the rings $U := S/(I_{Q_1} + I_{Q_2})$ and $V := S/(I_L + I_T)$ have finite length. We claim that $U \cong k[x, y]/(q_1, q_2)$ is a complete intersection of 2 quadrics while $V \cong k[x, y]/(x^2, xy, y^2)$. It follows that the dimensions of the homogeneous components of U in degrees $0, 1, 2, 3 \ldots$ are $1, 2, 1, 0 \ldots$ while those of V are $1, 2, 0, 0 \ldots$. Together with the computation above, this will prove the Proposition.

To analyze U, let write $I_{Q_i} = (\ell_i, q_i)$ where the ℓ_i are linear forms and the q_i are quadratic forms. Since $I_{Q_1} + I_{Q_2}$ has finite length, the four forms ℓ_1, ℓ_2, q_1, q_2 must be a regular sequence. Working modulo (ℓ_1, ℓ_2) we see that U is isomorphic to a complete intersection of 2 quadrics in 2 variables, as claimed.

To prove that V has the given Hilbert function, it suffices to show that the degree 2 part of V is 0. Since the Hilbert function of $S/I_L \oplus S/I_T$ is $4m + 2$, this is equivalent to showing that the degree 2 part of $S/I_{L \cup T}$ is 10-dimensional; that is, that no quadric vanishes on both L and T. Since T spans $\mathbb{P}^3$ and is irreducible, the quadric must be irreducible. By **** the residual $L \cup T$ to C is unmixed, and it follows that T is unmixed and spans $\mathbb{P}^3$.

We claim that if a line and a curve of degree 3 and genus 0 lie on any quadric, then they meet: If the quadric is smooth then T would have class $(1, 2)$ and the line would have to have class $(1, 0)$ or $(0, 1)$ both of which meet T. If the quadric is an irreducible cone, then we note that every curve meets every line on the cone. If T lies on the union of two planes then T has components in both planes and thus meets any line in one of them; and finally if T lies on a double plane, then the line would meet T_{red}. Thus $T \cup L$ cannot lie on a quadric, and we are done. □

To return to the proof of Proposition mumford component 1.4.14, let $\mathcal{H}$ now be the locus in the Hilbert scheme $\mathcal{H}_{-1,3,4}$ corresponding to disjoint unions of two conics, and consider the correspondence

$$\Phi = \{(C, D, S, T) \in \mathcal{H}_2'' \times \mathcal{H} \times \mathbb{P}^{19} \times \mathbb{P}^{83} \mid S \cap T = C \cup D\}.$$

Once more we have $\dim \mathcal{H} = 16$, and the fiber of Φ over a point $[D] \in \mathcal{H}$ is again an open subset of the product $\mathbb{P}^5 \times \mathbb{P}^{37}$ (unions of two disjoint conics imposes the same number of conditions on cubics and sextics as the disjoint union of a line and a twisted cubic); so we see that Φ is irreducible of dimension 58. The fibers of Φ over $\mathcal{H}''$ are 2-dimensional, and we conclude that $\mathcal{H}''$ is irreducible of dimension 56.

The calculation of the dimension of the Zariski tangent space $H^0(\mathcal{N}_{C/\mathbb{P}^3})$ to $\mathcal{H}''$ at a general point $[C]$ also proceeds as in the last case: we start with the exact sequence

$$0 \to \mathcal{N}_{C/S} \to \mathcal{N}_{C/\mathbb{P}^3} \to \mathcal{N}_{S/\mathbb{P}^3}|_C \to 0.$$

Again, the line bundle $\mathcal{N}_{C/S}$ has degree 60 and so is nonspecial with $h^1(\mathcal{N}_{C/S}) = 0$ and $h^0(\mathcal{N}_{C/S}) = 37$.

However, the determination of the cohomology of the third term, $\mathcal{N}_{S/\mathbb{P}^3}|_C \cong \mathcal{O}_C(3)$ is different. Let Q and Q' be the two conics comprising the residual curve D; and let H denote the hyperplane class on S. The planes P, P' spanned by Q and Q' respectively meet in a line L. Since L contains the scheme of length 4 of intersection with $Q \cup Q'$, it is contained in S. Thus the curves Q and Q' are linearly equivalent on S, so we can write the class of C on S as $6H - 2Q \sim 4H + 2L$.

Since $Q \cap Q' = \emptyset$ we have $Q \cdot Q = 0$; and since $C \sim 6H - 2Q$ on S, we have

$$(C \cdot Q) = ((6H - 2Q) \cdot Q) = 12.$$

In other words, the curves Q and Q' intersect C in divisors E_Q and $E_{Q'}$ of degree 12. As before, we can write

$$h^1(\mathcal{O}_C(3)) = h^0(K_C(-3)) = h^0(\mathcal{O}_C(2)(-E_Q - E_{Q'}))$$

and using again the completeness of the linear series cut out on C by quadrics, we see that $h^1(\mathcal{O}_C(3))$ *is the dimension of the space of quadratic polynomials vanishing on E_Q and $E_{Q'}$*; again, since $12 > 2 \cdot 2$, this is the same as the space of quadrics containing the two curves Q and Q'.

quartic curve postulation

Here is where the stories diverge: we saw in Proposition 1.4.15 that whereas there is no quadric containing the disjoint union of a line and a twisted cubic, there is indeed a unique quadric containing the union of two given disjoint conics, namely, the union of the planes of the conics. Thus $h^1(\mathcal{O}_C(3)) = 1$ so $h^0(\mathcal{O}_C(3)) = 20$ and correspondingly $h^0(\mathcal{N}_{C/\mathbb{P}^3}) = 57$. $\square$

What's going on here?

What accounts for the different behaviors of curves in cases $2'$ and $2''$? Here is one explanation:

To start, let C be a curve corresponding to a general point of $\mathcal{H}'_2$. As we've seen, we have

$$h^1(\mathcal{O}_C(3)) = 0 \quad \text{and} \quad h^0(\mathcal{O}_C(3)) = 19,$$

postulation table

so we see already from Table 1.1 that C must lie on a cubic surface. Moreover, by upper-semicontinuity, the same is true of any deformation of C, and so in an étale neighborhood of $[C]$ the Hilbert scheme looks like a projective bundle over the space of cubic surfaces.

By contrast, if C is the curve corresponding to a general point of $\mathcal{H}_2''$, we have

$$h^1(\mathcal{O}_C(3)) = 1 \quad \text{and} \quad h^0(\mathcal{O}_C(3)) = 20.$$

In other words, C is not forced to lie on a cubic surface, it just chooses to do so! The "extra" section of the normal bundle corresponds to a first-order deformation of C that is not contained in any deformation of S. If we could extend these deformations to arbitrary order, we would arrive at a family of curves whose general member lay in the first component $\mathcal{H}_1$; but we know that a general point of $\mathcal{H}''$ is not in the closure of $\mathcal{H}_1$, and so *these deformations of C must be obstructed.*

One note: it may seem that the phenomenon described in this last example—a component of the Hilbert scheme that is everywhere nonreduced, even though the objects parametrized are perfectly nice smooth, irreducible curves in $\mathbb{P}^3$—represents a pathology, and indeed, it was first described by David Mumford, in a paper entitled "Pathologies"! But, as Ravi Vakil has shown, it is to be expected: Vakil shows that *every* complete local ring over an algebraically closed field, up to adding power series variables, occurs as the completion of the local ring of a Hilbert scheme of smooth curves—that is, in effect, every singularity is possible. (reference to Vakil's paper, and more precise statement of Ravi's theorem).

1.5 Open problems

open problems

1.5.1 Brill-Noether in low codimension

If we ignore the finer points of the Brill-Noether theorem and focus just on the statement about the dimension and irreducibility of the variety of linear series on a curve, we can express it in a simple form: according to Theorem ?? principal component *Any component of the restricted Hilbert scheme $\mathcal{H}^\circ$ of curves of degree d and genus g that dominates the moduli space M_g has the expected dimension*

$$h(g, r, d) = 4g - 3 + (r + 1)(d - g + 1) - 1 = (r + 1)d - (r - 3)(g - 1)$$

as calculated in Section ?? estimating dim hilb *above.*

degree 8 section

Now, we saw in Section 1.1 an example of a component of the Hilbert scheme violating this dimension estimate, and it's not hard to produce lots of similar examples: components of the Hilbert scheme that parametrize complete intersections, or more generally determinantal curves, have in general dimension larger than the Hilbert number $h(g, r, d)$, and the following exercise gives a way of generating many more.

Exercise 1.5.1. Let $\mathcal{H}^\circ$ be a component of the Hilbert scheme parametrizing curves of degree d and genus g in $\mathbb{P}^3$ that dominates the moduli space M_g. For $s, t \gg d$, let $\mathcal{K}^\circ$ be the family of smooth curves residual to a curve $C \in \mathcal{H}^\circ$ in a complete intersection of surfaces of degrees s and t.

1. Show that $\mathcal{K}^\circ$ is open and dense in a component of the Hilbert scheme of curves of degree $st - d$ and the appropriate genus.

2. Calculate the dimension of $\mathcal{K}^\circ$, and in particular show that it is strictly greater than $h(g, r, d)$.

So it may seem that the issue is settled: components of the Hilbert scheme dominating M_g have the expected dimension; others don't in general. But there is an observed phenomenon that suggests more may be true: that components of $\mathcal{H}^\circ$ whose image in M_g have low codimension still have the expected dimension $h(g, r, d)$.

The cases with codimension ≤ 2 are already known: In [**?**], it is shown that if $\Sigma \subset M_g$ is any subvariety of codimension 1, then the curve C corresponding to a general point of Σ has no linear series with Brill-Noether number $\rho < -1$; and Edidin in [**?**] proves the analogous (and much harder) result for subvarieties of codimension 2. Indeed, looking over the examples we know of components of the Hilbert scheme whose dimension is strictly greater than the expected $h(g, r, d)$, there are none whose image in M_g has codimension less than $g - 4$. We could therefore make the conjecture:

Conjecture 1.5.2. *If $\mathcal{K} \subset \mathcal{H}^\circ_{d,g,r}$ is any component of a restricted Hilbert scheme, and the image of $\mathcal{K}$ in M_g has codimension $\leq g - 4$, then* $\dim \mathcal{K} = h(g, r, d)$.

1.5.2 Maximally special curves

Most of Brill-Noether theory, and the theory of linear systems on curves in general, centers on the behavior of linear series on a general curve. The opposite end of the spectrum is also interesting, and we may ask: How special a linear series on a special curve can be?

To make such a question precise, let $\tilde{M}^r_{g,d} \subset M_g$ be the closure of the image of the map $\phi : \mathcal{H}^\circ_{d,g,r} \to M_g$ sending a curve to its isomorphism class.

1. What is the smallest possible dimension of $\mathcal{H}^\circ_{d,g,r}$?

2. What is the smallest possible dimension of $\tilde{M}^r_{g,d}$?

3. Modifying the last question slightly, let $M^r_{g,d} \subset M_g$ be the closure of the locus of curves C that possess a g^r_d (in other words, we are dropping the condition that the g^r_d be very ample). We can ask what is the smallest possible dimension of $M^r_{g,d}$?

One might suppose that the most special curves, from the point of view of questions 2 and 3, are hyperelliptic curves but the locus in M_g of hyperelliptic curves has dimension $2g-1$. What about smooth plane curves? That's better – in the sense that the locus in M_g of smooth plane curves has dimension asymptotic to g, as the following exercise will show – but there are still a lot of them.

Exercise 1.5.3. 1. Let $C \subset \mathbb{P}^2$ be a smooth plane curve of degree d. Show that the g^2_d cut by lines on C is unique; that is, $W^2_d(C)$ consists of one point.

2. Using this, find the dimension of the locus of smooth plane curves in M_g.

Can we do better? Well, in $\mathbb{P}^3$ we can consider the locus of smooth complete intersections of two surfaces of degree m. As we saw in Exercise ??, these comprise an open subset $\mathcal{H}^\circ_{ci}$ of the Hilbert scheme of curves of degree $d = m^2$, and genus g given by the relation

complete intersection open

$$2g - 2 = \deg K_C = m^2(2m - 4),$$

or, asymptotically,

$$g \sim m^3.$$

Moreover, the dimension of this component of the Hilbert scheme is easy to compute, since as we saw in Exercise first complete intersection exercise ?? that it is isomorphic to an open subset of the Grassmannian $G(2, \binom{m+3}{3})$, and so has dimension

$$2(\binom{m+3}{3} - 2) \sim \frac{m^3}{3}$$

Finally, we observe that if $C \subset \mathbb{P}^r$ is a complete intersection curve of genus $g > 1$, the canonical bundle K_C is a positive power of $\mathcal{O}_C(1)$, and by Lasker's Theorem C is linearly normal. In particular, for a given abstract curve C there are only finitely many embeddings of C in projective space $\mathbb{P}^r$ as a complete intersection, up to PGL_{r+1}; in other words, the fibers of $\mathcal{H}^\circ_{ci}$ over M_g have dimension $\dim(PGL_{r+1}) = r^2 + 2r$.

Thus, we have a sequence of components of the restricted Hilbert scheme $\mathcal{H}^\circ$ whose images in M_g have dimension tending asymptotically to $g/3$.

The following exercise suggests why we chose complete intersections of surfaces of the same degree.

Exercise 1.5.4. Consider the locus of curves $C \subset \mathbb{P}^3$ that are complete intersections of a quadric surface and a surface of degree m. Show that these comprise components of the restricted Hilbert scheme, and that their images in moduli have dimension asymptotically approaching g as $m \to \infty$.

More generally, we can consider complete intersections of $r-1$ hypersurfaces of degree m in $\mathbb{P}^r$; in a similar fashion we can calculate that their images in M_g have dimension asympotically approaching $2g/r!$ as $m \to \infty$, as we ask you to verify in the following exercise.

Exercise 1.5.5. Consider the locus $\mathcal{H}^\circ_{ci}$, in the Hilbert scheme $\mathcal{H}^\circ$, of smooth, irreducible, nondegenerate curves $C \subset \mathbb{P}^r$ that are complete intersections of $r-1$ hypersurfaces of degree m.

1. Show that $\mathcal{H}^\circ_{ci}$ is open in $\mathcal{H}^\circ$;

2. Calculate the dimension of $\mathcal{H}^\circ_{ci}$ (and observe that it is irreducible); and

3. Show that the dimension of the image of $\mathcal{H}^\circ_{ci}$ in M_g is asymptotically $2g/r!$ as $m \to \infty$

The question is, can we do better? For example, if we fix r, can we find a sequence of components $\mathcal{H}_n$ of restricted Hilbert schemes $\mathcal{H}^\circ_{g_n,r,d_n}$ of curves in $\mathbb{P}^r$ such that

$$\lim \frac{\dim \mathcal{H}_n}{g_n} \;=\; 0?$$

1.5.3 Rigid curves?

In the last section, we considered components of the restricted Hilbert scheme whose image in M_g was "as small as possible." Let's go now all the way to the extreme, and ask: is there a component of the restricted Hilbert scheme $\mathcal{H}^\circ_{g,r,d}$ whose image in M_g is a single point? Of course M_0 itself is a single point, so we exclude genus 0! We can give three flavors of this question, in order of ascending preposterousness.

1. First, we'll say a smooth, irreducible and nondegenerate curve $C \subset \mathbb{P}^r$ is *moduli rigid* if it lies in a component of the restricted Hilbert scheme whose image in M_g is just the point $[C] \in M_g$—in other words, if the linear series $|\mathcal{O}_C(1)|$ does not deform to any nearby curves.

2. Second, we say that such a curve is *rigid* if it lies in a component $\mathcal{H}^\circ$ of the restricted Hilbert scheme such that PGL_{r+1} acts transitively on $\mathcal{H}^\circ$. This is saying that C is moduli rigid, plus the line bundle $\mathcal{O}_C(1)$ does not deform to any other g^r_d on C.

3. Finally, we say that such a curve is *deformation rigid* if the curve $C \subset \mathbb{P}^r$ has no nontrivial infinitesimal deformations other than those induced by PGL_{r+1}—in other words, every global section of the normal bundle $\mathcal{N}_{C/\mathbb{P}^r}$ is the image of the restriction of a vector field on $\mathbb{P}^r$.

In truth, these are not so much questions as howls of frustration. The existence of irrational rigid curves seems outlandish; we don't know anyone who thinks there are such things. But then *why can't we prove that they don't exist?*