Personalities of Curves

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Contents

Chapter 1

Hyperelliptic curves and curves of genus 2 and 3

2 and 3 chapter

1.1 Hyperelliptic Curves

In the world of curves, hyperelliptic curves are outliers: they behave differently from other curves, and the techniques used to analyze them are different from the techniques used for more general curves. Many theorems about curves contain the hypothesis "non-hyperelliptic," with the corresponding result for hyperelliptic curves arrived at directly by ad hoc methods. Because the methods of this section will not be used in other cases, it could be skipped in first reading

There will be a further discussion of hyperelliptic curves in Chapter ScrollsChapter ??, focussing on the algebra and geometry of their projective embeddings; the analysis here will cover most of the questions we'll be asking about curves in general in the next four chapters.

1.1.1 The equation of a hyperelliptic curve

By definition, a hyperelliptic curve C is one admitting a degree two map $\pi : C \to \mathbb{P}^1$. Because the degree is only 2, each point in $\mathbb{P}^1$ has either two distinct preimages, or one point of simple ramification. there can be no higher ramification, so at all but finitely many points $p \in C$ the map π is

a local isomorphism ("local" here in the complex analytic/classical or étale topology, not the Zariski topology!); at any other point $p \in C$, the map is given in terms of local analytic coordinates on C and $\mathbb{P}^1$ simply by $z \mapsto z^2$. In particular, both the ramification divisor and the branch divisor
((where are these defined?))
are reduced. Thus by the Riemann-Hurwitz formula there are exactly $2g+2$ branch points $q_1, \dots, q_{2g+2} \in \mathbb{P}^1$. These points determine the curve:

liptic existence

Theorem 1.1.1. *There is a unique smooth projective hyperelliptic curve C expressible as a 2-sheeted cover of $\mathbb{P}^1$ branched over any set of $2g+2$ distinct points.*

We can easily construct such a curve, postponing for a moment the uniqueness: If the coordinate of the point $p_i \in \mathbb{P}^1$ is λ_i, it is the smooth projective model of the affine curve

$$C^\circ = \left\{(x,y) \in \mathbb{A}^2 \mid y^2 = \prod_{i=1}^{2g+2}(x-\lambda_i)\right\}.$$

Note that we're choosing a coordinate x on $\mathbb{P}^1$ with the point $x = \infty$ at infinity not among the q_i, so that the pre-image of $\infty \in \mathbb{P}^1$ is two points $r, s \in C$. Concretely, we see that as $x \to \infty$, the ratio $y^2/x^{2g+2} \to 1$, so that

$$\lim_{x\to\infty} \frac{y}{x^{g+1}} = \pm 1;$$

the two possible values of this limit correspond to the two points $r, s \in C$.

It's worth pointing out that C is *not* simply the closure of the affine curve $C^\circ \subset \mathbb{A}^2$ in either $\mathbb{P}^2$ or $\mathbb{P}^1 \times \mathbb{P}^1$: as you can see from a direct examination of the equation, each of these closures will be singular at the (unique) point at infinity.

hyperelliptic existence

Completion of the proof of Theorem **??**. The proof (in characteristic 0) of uniqueness follows from elementary algebraic topology:

First, a punctured 2-disk has fundamental group $\mathbb{Z}$ and the unique n-sheeted covering is again a punctured disk; regarding these disks as neighborhoods of the origin in $\mathbb{C}^2$, the covering map can be taken to be $x \mapsto x^n$. This map can of course be extended (by the same formula) to a map analytic also at the origin, with ramification index (by definition) $n-1$.

Now suppose that $\Gamma = \{p_1, \ldots, p_d\}$ is the desired branch divisor. Globally, if γ_i is a small loop around p_i then the abelianization of the fundamental group π of the d-times punctured sphere

$$S' := \mathbb{P}^1 \setminus \Gamma$$

is its first homology group,

$$H := H_1(S', \mathbb{Z}) = \frac{\oplus \mathbb{Z} \cdot \gamma_i}{\mathbb{Z} \cdot \sum_i [\gamma_i]}$$

((Insert "lollipop picture".))
Since $\mathbb{Z}/2$ is abelian, a degree 2 unramified covering of S' corresponds to a map $H \to \mathbb{Z}/2$, and this map must send $2\gamma_i$ to 0 for $i = 1 \ldots d$. There is such a map if and only if d is even, and in this case the map is unique.

Summarizing: there is, a unique degree 2 topological covering $C' \to \mathbb{P}^1 \setminus \Gamma$ by a surface C' that extends to a ramified covering of $\rho : C \to \mathbb{P}^1$, simply ramified over the points of Γ, as long as the number of ramification points is even.

A triangulation of $\mathbb{P}^1$ with V vertices including the points of Γ, E edges, and F triangles must have

$$V - E + F = \chi_{\text{top}}(S^2) = 2.$$

It lifts to a triangulation of C with $2V - d$ vertices, $2E$ edges, and $2F$ faces, so

$$\chi_{\text{top}}(C) = 2V - d - 2E + 2F = 4 - d,$$

so if $d = 2g + 2$ then $\chi_{\text{top}}(C) = 2 - 2g$, so C is a surface of genus g.

Though given as a topological surface, the map ρ is a local homeomorphism at every point not in the preimage of Γ, so C inherits a unique complex structure from the requirement that ρ be holomorphic; thus C is actually a smooth algebraic curve of genus g.
((we had better say topological and algebraic genus are the same in the intro.))

$\square$

Exercise 1.1.2. In the case $g = 1$, show that the closure $\overline{C^\circ}$ of $C^\circ \subset \mathbb{A}^2$ in either $\mathbb{P}^2$ or $\mathbb{P}^1 \times \mathbb{P}^1$ consists of the union of C° with one additional point, with that point a tacnode of $\overline{C^\circ}$ in either case.

It is also possible to give a projective model of the hyperelliptic curve C with given branch divisor: if we divide the points $q_1, \dots, q_{2g+2} \in \mathbb{P}^1$ into two sets of $g+1$—say, for example, $q_1, \dots, q_{g+1}$ and $q_{g+2}, \dots, q_{2g+2}$—then C is the closure in $\mathbb{P}^1 \times \mathbb{P}^1$ of the locus

$$\left\{(x,y) \in \mathbb{A}^2 \mid y^2 \prod_{i=1}^{g+1}(x-\lambda_i) = \prod_{i=g+2}^{2g+2}(x-\lambda_i)\right\};$$

in projective coordinates, this is

$$C = \left\{(X,Y) \in \mathbb{P}^1 \times \mathbb{P}^1 \mid Y_1^2 \prod_{i=1}^{g+1}(X_1-\lambda_i X_0) = Y_0^2 \prod_{i=g+2}^{2g+2}(X_1-\lambda_i X_0)\right\}.$$

(No local analysis is needed to see that $C \subset \mathbb{P}^1 \times \mathbb{P}^1$ is smooth: it is a curve of bidegree $(2, g+1)$ in $\mathbb{P}^1 \times \mathbb{P}^1$, and the genus formula tells us that such a curve has arithmetic genus g.)

1.1.2 Differentials on a hyperelliptic curve

We can give a very concrete description of the differentials, and thus the canonical linear series, on a hyperelliptic curve C by working with the affine model $C^\circ = V(f) \subset \mathbb{A}^2$, where

$$f(x,y) = y^2 - \prod_{i=1}^{2g-2}(x-\lambda_i).$$

We will again denote the two points at infinity—that is, the two points of $C \setminus C^\circ$ by r and s; for convenience, we'll denote the divisor $r+s$ by D.

To start, consider the simple differential dx on C. (Technically, we should write this as $\pi^* dx$, since we mean the pullback to C of the differential dx on $\mathbb{P}^1$, but for simplicity of notation we'll suppress the π^*.) The function x is regular on C°, and is a local parameter over points other than the λ_i; but since $2dy = \sum_i$
((add formula))
with zeros at the ramification points $q_i = (\lambda_i, 0)$. But it does not extend to a regular differential on all of C: it will have double poles at r and s. This can be seen directly: the differential dx extends to a rational differential on

$\mathbb{P}^1$, and in terms of the local coordinate $w = 1/x$ around the point $x = \infty$ on $\mathbb{P}^1$, we have

$$dx = d\left(\frac{1}{w}\right) = \frac{-dw}{w^2}$$

so dx has a double pole at the point at ∞; since the map π is a local isomorphism near r and s the pullback of dx to C likewise has double poles at the points r and s.

We could also see that dx must have poles by degree considerations: as we said, dx has $2g+2$ zeros and no poles in C°, while the degree of K_C is $2g-2$, meaning that there must be a total of four poles at the points r and s. In any event, we have an expression for the canonical divisor class on C: denoting by $R = q_1 + \cdots + q_{2g+2}$ the sum of the ramifications points of π, we have

$$K_C \sim (dx) \sim R - 2D;$$

this is a case of the Riemann-Hurwitz formula above.

So, given that dx has poles at r and s, how do we find regular differentials on C? One thing to do would be simply to divide by x^2 (or any quadratic polynomial in x) to kill the poles. But that just introduces new poles in the finite part C° of C. Instead, we want to multiply dx by a rational function with zeros at p and q, but *whose poles occur only at the points where dx has zeroes*—that is, the points q_i. A natural choice is simply the reciprocal of the partial derivative $f_y = \partial f/\partial y = 2y$, which vanishes exactly at the points r_i, and has correspondingly a pole of order $g+1$ at each of the points r and s (reason: the involution $y \to -y$ fixes C° and x, and exchanges the points p, q). In other words, the differential

$$\omega = \frac{dx}{f_y}$$

is regular, with divisor

$$(\omega) = (g-1)r + (g-1)s = (g-1)D.$$

The remaining regular differentials on C are now easy to find: Since x has only a simple pole at the two points at infinity we can multiply ω by any x^k with $k = 0, 1, \ldots, g-1$. Since this gives us g independent differentials, these form a basis for $H^0(K_C)$.

1.1.3 The canonical map of a hyperelliptic curve

elliptic special

Given that a basis for $H^0(K_C)$ is given by

$$H^0(K_C) = \langle \omega, x\omega, \ldots, x^{g-1}\omega \rangle,$$

we see that the canonical map $\phi : C \to \mathbb{P}^{g-1}$ is given by $[1, x, \ldots, x^{g-1}]$. In other words, the canonical map ϕ is simply the composition of the map $\pi : C \to \mathbb{P}^1$ with the Veronese embedding $\mathbb{P}^1 \hookrightarrow \mathbb{P}^{g-1}$ of $\mathbb{P}^1$ into $\mathbb{P}^{g-1}$ as a rational normal curve of degree $g - 1$.

Note that as a consequence of this fact, we see that *a hyperelliptic curve C has a* unique *linear series g^1_2 of degree 2 and dimension 1*, that is, a unique map of degree 2 to $\mathbb{P}^1$. Finally, we can give an explicit description of special linear series on a hyperelliptic curve: if $D = \sum p_i$ is any effective divisor on C, we can pair up points p_i that are conjugate under the involution ι exchanging sheets of the degree 2 map $C \to \mathbb{P}^1$; each conjugate pair is a divisor of the unique g^1_2 on C, and so we can write

$$D \sim r \cdot g^1_2 + q_1 + \cdots + q_{d-2r},$$

where no two of the points q_i are conjugate under ι. Now the geometric form of the Riemann-Roch formula tells us that the dimension $r(D)$ of the complete linear series $|D|$ is exactly r, so that in fact

$$|D| = |r \cdot g^1_2| + +q_1 + \cdots + q_{d-2r};$$

that is, the points q_i are base points of the linear series D.

One key observation is that, according to this analysis, *no special linear series on a hyperelliptic curve can be very ample*; the map associated to any special series factors through the degree 2 map $C \to \mathbb{P}^1$. This is in marked contrast to the case of non-hyperelliptic curves, for which the embeddings of minimal degree in projective space are given by special linear series.

1.2 Curves of genus 2

Canonical map to $\mathbb{P}^1$. Embedding in $\mathbb{P}^3$ as $(2, 3)$ on a quadric, via any degree 5 line bundle. Ideal is 1 quadric, 2 cubics. Plane model of degree 4 with node or cusp.

Representations as double covers of $\mathbb{P}^1$

As with curves of genus 1, there are no nontrivial linear series of degree 0 or 1 on a curve of genus 2; the first positive-dimensional linear series occurs in degree 2. Unlike the case of genus 1, however, this series is unique: by Riemann-Roch, if D is any divisor of degree 2 on a curve C of genus 2, we have

$$h^0(D) = 1 + h^1(D) = 1 + h^0(K - D);$$

since $K - D$ has degree 0, this says that $h^0(D) > 1$ if and only if $D = K$, in which case $|D| = |K|$ is the canonical g^1_2 on C.

The canonical series gives a map $\phi_K : C \to \mathbb{P}^1$ expressing C as a double cover of $\mathbb{P}^1$; as in the case of genus 1, this means we can realize C as the smooth projective compactification of the affine curve given by

$$y^2 = x(x-1)(x-\alpha)(x-\beta)(x-\gamma)$$

for some triple $\alpha, \beta, \gamma \in \mathbb{C}$ distinct from each other and from 0 and 1. This representation shows us that the moduli space M_2 is the space of 6-tuples of distinct points in $\mathbb{P}^1$ modulo the action of PGL_2. This tells us immediately that M_2 is irreducible of dimension 3; with a fair amount of additional work, we can also use this to describe the coordinate ring of M_2 (** ??).

Embeddings in $\mathbb{P}^3$

For line bundles L of degree $d \geq 3$ on C, Riemann-Roch tells us simply that $h^0(D) = d - 1$; if we want to embed our curve C in projective space, accordingly, we had better take $d \geq 5$. Conversely, Corollary (??) tells us that any line bundle of degree 5 on C is very ample, so we'll consider first the embeddings of C given by those. degree 2g+1 embedding

So: for the following, let L be any line bundle of degree 5 on our curve C, and $\phi_L : C \to \mathbb{P}^3$ the embedding given by the complete linear system $|L|$. By a mild abuse of language, we'll also denote the image $\phi_L(C) \subset \mathbb{P}^3$ by C.

The first question to ask is once more, what degree surfaces in $\mathbb{P}^3$ contain the curve C? We start with degree 2, where we consider the restriction map

$$H^0(\mathcal{O}_{\mathbb{P}^3}(2)) \to H^0(\mathcal{O}_C(2)) = H^0(L^2).$$

The space on the left has dimension 10 as always; on the right, Riemann-Roch tells us that $h^0(L^2) = 2 \cdot 5 - 2 + 1 = 9$. It follows that C must lie on a quadric surface Q; and by Bezout that Q is unique (since C can't lie on a union of planes, any quadric containing C must be irreducible; if there were more than one such, Bezout would imply that $\deg(C) \leq 4$).

We might ask at this point: is Q smooth or a quadric cone? The answer depends on the choice of line bundle L:

Proposition 1.2.1. *Let $C \subset \mathbb{P}^3$ be a smooth curve of degree 5 and genus 2 and $Q \subset \mathbb{P}^3$ the unique quadric containing C. If $L = \mathcal{O}_C(1) \in \operatorname{Pic}^5(C)$, then Q is singular if and only if we have*

$$L \cong K^2(p)$$

for some point $p \in C$; in this case, the point p is the vertex of Q.

Note that there is a 2-parameter family of line bundles of degree 5 on C, of which a one-dimensional subfamily are of the form $K^2(p)$, conforming to our naive expectation that "in general" Q should be smooth, and that it should become singular in codimension 1.

Proof. First, suppose that the line bundle $L \cong K^2(p)$ for some $p \in C$. Then $L(-p) \cong K^2$, meaning that the map $\pi : C \to \mathbb{P}^2$ given by projection from p is the map $\phi_{K^2} : C \to \mathbb{P}^2$ given by the square of the canonical bundle.

What does this map look like? □

Whether the quadric Q is smooth or not, we can describe a minimal set of generators of the homogeneous ideal $I(C) \subset \mathbb{C}[x_0, x_1, x_2, x_3]$ similarly. First, we look at the restriction map

$$H^0(\mathcal{O}_{\mathbb{P}^3}(3)) \to H^0(\mathcal{O}_C(3));$$

since the dimensions of these spaces are 20 and $15 - 2 + 1 = 14$ respectively, we see that vector space of cubics vanishing on C has dimension at least 6. Four of these are already accounted for: we can take the defining equation of Q and multiply it by any of the linear forms on $\mathbb{P}^3$; we conclude, accordingly, that *there are at least two cubics vanishing on C linearly independent modulo those vanishing on Q.*

In fact, we can prove the existence of these cubics geometrically, and show that there are no more than 2 linearly independent modulo the ideal of Q. Suppose first that Q is smooth, so that C is a curve of type $(2, 3)$ on Q. In that case, if $L \subset Q$ is any line of the first ruling, the sum $C + L$ is the complete intersection of Q with a cubic S_L, unique modulo the ideal of Q; conversely, if S is any cubic containing C but not containing S, the intersection $S \cap Q$ will be the union of C and a line L of the first ruling; thus, mod $I(Q)$, $S = S_L$. A similar argument applies in case Q is a cone, and L is any line of the (unique) ruling of Q.

Exercise 1.2.2. Show that for any pair of lines L, L' of the appropriate ruling of Q, the three polynomials Q, S_L and $S_{L'}$ generate the homogeneous ideal $I(C)$. Find relations among them. Write out the minimal resolution of $I(C)$.

Projective normality III

Theorem 1.2.3. *Let C be a smooth (is reduced, irreducible enough?) curve of arithmetic genus g, and let $\mathcal{L}$ be a line bundle on C of degree $\geq 2g + 1$. The image of C under the complete linear series $|\mathcal{L}|$ is projectively normal (when C is singular, aritmetically Cohen-Macaulay).*

Proof. The line bundle $\mathcal{L}$ is very ample by ??. Thus it suffices We must show that the multiplication map $H^0(\mathcal{L}) \otimes H^0(\mathcal{L}^m) \to H^0(\mathcal{L}^{m+1})$ is surjective for all $m \geq 1$. For $m = 1$ do it by number of quadrics, uniform position. For m¿1 the bpf pencil trick. □

1.3 Curves of genus 3

If C be a smooth projective curve of genus 3. The is an immediate bifurcation into two cases, hyperelliptic and non-hyperelliptic curves; we will discuss hyperelliptic curves of any genus in Section ??, and so for the following we'll assume C is nonhyperellitic. By our general theorem ??, this means that the canonical map $\phi_K : C \to \mathbb{P}^2$ embeds C as a smooth plane quartic curve; and conversely, by adjunction any smooth plane of degree 4 has genus 3 and is canonical (that is, $\mathcal{O}_C(1) \cong K_C$).

((maybe a reference to the plane curve chapter for differentials etc?))

Note that this gives us a way to determine the dimension of the moduli space M_3 of smooth curves of genus 3: if $\mathbb{P}^{14}$ is the space of all plane quartic curves, and $U \subset \mathbb{P}^{14}$ the open subset corresponding to smooth curves, we have a dominant map $U \to M_3$ whose fibers are isomorphic to the 8-dimensional affine group PGL_3. (Actually, the fiber over a point $[C] \in M_3$ is isomorphic to the quotient of PGL_3 by the automorphism group of C; but since $Aut(C)$ is finite this is still 8-dimensional.) We conclude, therefore, that

$$\dim M_3 = 14 - 8 = 6.$$

What about other linear series on C, and the corresponding models of C? To start with, by hypothesis C has no g^1_2s; that is, it is not expressible as a 2-sheeted cover of $\mathbb{P}^1$. On the other hand, it is expressible as a 3-sheeted cover: if $L \in \mathrm{Pic}^3(C)$ is a line bundle of degree 3, by Riemann-Roch we have

$$h^0(L) = \begin{cases} 2, & \text{if } L \cong K - p \text{ for some point } p \in C\text{; and} \\ 1 & \text{otherwise.} \end{cases}$$

There is thus a 1-dimensional family of representations of C as a 3-sheeted cover of $\mathbb{P}^1$. In fact, these are plainly visible from the canonical model: the degree 3 map $\phi_{K-p} : C \to \mathbb{P}^1$ is just the composition of the canonical embedding $\phi_K : C \to \mathbb{P}^2$ with the projection from the point p.

There are of course other representations of C as the normalization of a plane curve. By Riemann-Roch, C will have no g^2_3s and the canonical series is the only g^2_4, but there are plenty of models as plane quintic curves: by Proposition ??, if L is any line bundle of degree 5, the linear series $|L|$ will be a base-point-free g^2_5 as long as L is not of the form $K + p$, so that ϕ_L maps C birationally onto a plane quintic curve $C_0 \subset \mathbb{P}^2$. But these can also be described geometrically in terms of the canonical model: any such line bundle L is of the form $2K - p - q - r$ for some trio of points $p, q, r \in C$ that are not colinear in the canonical model, and we see correspondingly that C_0 is obtained from the canonical model of C by applying a Cremona transform with respect to the points p, q and r.

We can also embed C in $\mathbb{P}^3$ as a smooth sextic curve by Proposition ??; in fact, a line bundle $L \in \mathrm{Pic}^6(C)$ of degree 6 will be very ample if and only if it is not of the form $K + p + q$ for any $p, q \in C$. One cheerful fact in this connection is that these curves are determinantal:

Exercise 1.3.1. Let $C \subset \mathbb{P}^3$ be a smooth non-hyperelliptic curve of degree 3 and genus 6. Show that there exists a 3×4 matrix M of linear forms on $\mathbb{P}^3$ such that

$$C = \{p \in \mathbb{P}^3 \mid \operatorname{rank}(M(p)) \leq 2\}.$$