Personalities of Curves

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Chapter 1

Plane Curves

neCurvesChapter

**** David: This started out as a chapter in which we extend the constructions/theorems of the rest of the book to the case of singular curves. We since abandoned that goal as too difficult and open-ended, and my understanding is that we were going to replace it with a chapter on plane curves. So here's the chapter on plane curves, but I haven't changed the name of the file (and I commented out, rather than deleted, the stuff that was already written). ****

As anyone who has read the last chapter knows, describing curves in projective space $\mathbb{P}^3$ is difficult: the ideal of such a curve typically has three or more generators, which in turn have to satisfy certain syzygies; in consequence, even basic facts about them—for example, the dimension of the family of all curves of given degree and genus—remain very unclear. And of course, as you might expect we know even less about curves in $\mathbb{P}^n$ for $n > 3$.

The case of plane curves makes a striking contrast: a curve $C \subset \mathbb{P}^2$ is necessarily the zero locus of a single homogeneous polynomial, and conversely any homogeneous polynomial $F(X, Y, Z)$ defines a plane curve. There is a downside, however: while any smooth projective curve can be embedded in $\mathbb{P}^r$ for any $r \geq 3$, most curves cannot be embedded in the plane.

On the other hand, it is true that every curve can be birationally embedded in $\mathbb{P}^2$: we can embed C as a curve $\tilde{C} \subset \mathbb{P}^r$ in a higher-dimensional projective space and find a projection $\mathbb{P}^r \to \mathbb{P}^2$ that carries $\tilde{C}$ birationally onto its image. This is indeed how 19th century geometers typically described a curve, in the days before abstract varieties: as the normalization of a plane curve. (The points on the normalization were realized as valuations on the function field of C, essentially taking advantage of the fact that for smooth curves birational and biregular isomorphism are the same thing.) Much of the analysis of the geometry of the curve—for example, the description of the linear systems on the curve—was carried out in this setting.

Plane curves, in other words, occupied a central role in the development of the theory of algebraic curves; and there are still many aspects of the geometry of a curve that are best approached in this way. In this chapter, we'll describe some of the tools used to study curves via their plane models.

1.1 Our goal

There are many, many questions we can ask about the geometry of plane curves, and we'll touch on several of these in what follows. But in this chapter we'll focus for the most part on one basic problem, which we'll now describe.

We've been dealing since the opening chapters of this book with a basic construction: given a smooth projective curve C, and a divisor $D = \sum m_i p_i$ on C, we define the complete linear system $|D|$ to be the family of all effective divisors E on C with $E \sim D$. A question we have not yet addressed is a simple but fundamental one: if we are given the equations of C as embedded in some projective space, and the coordinates of the points p_i, can we explicitly and algorithmically determine the complete linear series $|D|$?

The answer is "yes," and the way we do it is by working with plane models of our curves: that is, we realize a given curve C as the normalization of a plane curve $C_0 \subset \mathbb{P}^2$ (for example, by a general projection of C to $\mathbb{P}^2$), and working in the plane.

Let's start by posing two "keynote" problems. In both, we'll be given the equation $F(X, Y, Z)$ of a smooth plane curve C_0, with normalization C; in the second, we'll be given in addition a divisor $D = \sum m_i p_i$ on C. We ask:

1. Find all regular differentials/sections of K_C; that is, write down a basis for $H^0(K_C)$; and

2. Describe the complete linear system $|D|$; that is, find all effective divisors E on C with $E \sim D$.

Note that this subsumes questions like, are two given divisors D and E linearly equivalent? And, is a given divisor D linearly equivalent to an effective divisor?

Our plan is to solve these problems in three stages of increasing generality. To start with, in the following section we'll solve these problems in case $C = C_0$ is a smooth plane curve. (As we've observed, this is a very limited form of our general problem—the vast majority of curves cannot be realized as smooth plane curves—but it will serve to establish our basic approach.) In Section 1.3 we'll do it for (the normalization of) a plane curve with nodes. This is a significant expansion by virtue of Theorem ??, which says that every smooth projective curve is the normalization of a nodal plane curve.

Finally, in Section 1.4 we'll describe how to answer these questions algorithmically for the normalization of an arbitrary plane curve. This last extension is significant for practical reasons: many times a curve C is given to us as (the normalization of) a plane curve with singularities other than nodes, and while Theorem ?? assures us in principal that we can also realize C as the normalization of a nodal plane curve, it is almost always easier to work with the original plane model, as given.

1.2 Smooth plane curves

For this section, we'll we'll take $C \subset \mathbb{P}^2$ to be a smooth plane curve, given as the zero locus of a homogeneous polynomial $F(X, Y, Z)$ of degree d. We'll introduce affine coordinates $x = X/Z$ and $y = Y/Z$ on the affine open subset $U \cong \mathbb{A}^2$ given by $Z \neq 0$, and let $f(x, y) = F(X, Y, 1)$ be the inhomogeneous form of F, so that $\tilde{C} = C \cap U$ is given as the zero locus $V(f) \subset \mathbb{A}^2$.

Finally, just for simplicity, we'll make a couple assumptions about the relation of C to the coordinates in $\mathbb{P}^2$:

1. We'll assume that the point $[0, 1, 0]$ (that is, the point at infinity in the vertical direction) does not lie on C; in other words, the projection $C \to \mathbb{P}^1$ given by $[X, Y, Z] \mapsto [X, Z]$ (or $(x, y) \mapsto x$ in affine coordinates) has degree d; and

2. We'll assume that the line L at infinity given by $Z = 0$ intersects C transversely in d distinct points $p_1, \dots, p_d$; by way of notation, we'll denote by H the divisor $H = p_1 + \cdots + p_d$ of intersection of C with L.

These conditions are clearly satisfied for a general choice of coordinate system, so they're not restrictive. Nor are they necessary: in Exercise ?? **** we'll see how to do everything in the absence of these assumptions, albeit with slightly more complicated notation.

Enough talk! Let's see how to solve the damn problem. We start by drastically scaling back our ambitions: instead of asking for a basis of $H^0(K)$, let's just see if we can write down a single rational 1-form on C. This is manageable: we can just take a regular 1-form on $\mathbb{A}^2$, such as dx, and restrict/pull back to C. In fact, this will be regular on the open subset $\tilde{C}$, but a change of coordinate calculation shows that it will have double poles at the points $p_1, \dots, p_d$ of $H = C \cap L$.

How do we get rid of the poles of dx? The natural thing to do would be simply to divide dx by a polynomial $h(x, y)$ of degree 2 or more, but there's a problem: $h(x, y)$ will vanish at points of $C \cap U$, potentially creating poles of the quotient dx/h. There is a solution to this: *choose a polynomial h that vanishes only at those points of $C \cap U$ where dx already has a zero*; hopefully, the zeroes of h will just cancel the zeroes of dx rather than creating new poles.

In fact, we have just the polynomial: we take

$$h(x, y) = \frac{\partial f}{\partial y}(x, y).$$

To see that this works, note that on C,

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy \equiv 0.$$

Since $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ have no common zeroes on $\tilde{C}$ by the hypothesis of smoothness, we see that at any point $p \in \tilde{C}$,

$$\operatorname{ord}_p(dx) = \operatorname{ord}_p\left(\frac{\partial f}{\partial y}\right)$$

so in fact the quotient

$$\omega_0 = \frac{dx}{\partial f / \partial y}$$

is everywhere regular and nowhere 0 in $\tilde{C}$.

What about the points p_i? Well, the differential dx had poles of order 2 at the points p_i; and $\partial f/\partial y$, being a polynomial of degree $d-1$, will have poles of order $d-1$. We conclude that ω_0 has zeroes of order $d-3$ at the points p_i; in other words, the divisor

$$(\omega_0) = (d-3)D.$$

In particular, if $d \geq 3$ then ω_0 is a global regular differential on C.

This also says that we can afford to multiply ω_0 by any polynomial $g(x, y)$ of degree $d-3$ or less without introducing poles, so that

$$g\omega_0 = \frac{g(x, y)dx}{\partial f/\partial y}$$

is likewise a global regular differential, for g any polynomial of degree $\leq d-3$.

We have thus found a vector space of regular differentials, of dimension $\binom{d-1}{2}$. But at the same time, the degree of a differential like ω_0 is

$$\deg((\omega_0)) = (d-3)\deg(D) = d(d-3),$$

so that the genus of C is

$$\frac{d(d-3)}{2} + 1 = \binom{d-1}{2}.$$

In other words, we have found all the global regular differentials on C! We have

$$H^0(K_C) = \left\{ \frac{g(x, y)dx}{\partial f/\partial y} \mid \deg g \leq d-3 \right\};$$

or, equivalently, the space of regular differentials on C has basis $\{\omega_{i,j}\}_{i+j \leq d-3}$, where

$$\omega_{i,j} = \frac{x^i y^j dx}{\partial f/\partial y}$$

In fact, we could have done this all abstractly, without coordinates: by adjunction, we have

$$K_C = (K_{\mathbb{P}^2} \otimes \mathcal{O}_{\mathbb{P}^2}(d))|_C = \mathcal{O}_C(d-3),$$

and from the exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^2}(-3) \to \mathcal{O}_{\mathbb{P}^2}(d-3) \to \mathcal{O}_C(d-3) = K_C \to 0$$

and the vanishing of $H^1(\mathcal{O}_{\mathbb{P}^2}(-3))$, we see that the map on global sections

$$H^0(\mathcal{O}_{\mathbb{P}^2}(d-3)) \to H^0(K_C)$$

is surjective.

ooth plane curve

Exercise 1.2.1. Let C be a smooth plane curve of degree d. Show that C admits a map $C \to \mathbb{P}^1$ of degree $d-1$, but does not admit a map $C \to \mathbb{P}^1$ of degree $d-2$ or less.

1.2.1 Finding complete linear systems on smooth plane curves

What about our second problem, finding all effective divisors linearly equivalent to a given divisor D? To answer this, start by expressing D as the difference

$$D = E - F$$

of two effective divisors on C. Next, let $G(X, Y, Z)$ be a polynomial in the plane of any degree m vanishing on the divisor E, but not vanishing identically on C (taking m sufficiently large ensures the existence of such a polynomial), and let A be the divisor cut on C residually; that is, we write

$$(G) = E + A$$

as divisors on C. Now, let H be another polynomial of the same degree m as G, vanishing on $A + F$ but again not vanishing identically on C. (Since m is already chosen, there may not exist any such polynomial, which is as it should be: the original divisor D need not be linearly equivalent to any effective divisor.) Let D' be the divisor cut on C by H residual to $A + F$; that is, write

$$(H) = A + F + D'.$$

Now, the divisor of the rational function H/G is principal, so we have

$$0 \sim (H) - (G) = D' + F - E$$

or in other words, *D' is an effective divisor linearly equivalent to D.*

Note that in the simplest nontrivial case $d = 3$, we have reproduced the classic description of the group law on a plane cubic curve C. If we choose as origin on the curve C a point o, then to add two points p and $q \in C$ means to find the (unique) effective divisor of degree 1 linearly equivalent to $p+q-o$. In this situation, we can carry out the process described above with $m = 1$: draw the line L through the points p and q, and let $r \in C$ be the remaining point of intersection of L with C; then draw the line M though the points r and o, and let $s \in C$ be the remaining point of intersection of L with C. This is the classical construction of the group law.

We claim now that in fact we have found in this way *all* effective divisors $D' \sim D$. To see this, suppose D' is any effective divisor with $D' \sim D$. Carrying out the first step of the process as before, we arrive at a divisor A with

$$\mathcal{O}_C(A + F + D') = \mathcal{O}_C(A + F + D) = \mathcal{O}_C(m).$$

But from the exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^2}(m-d) \to \mathcal{O}_{\mathbb{P}^2}(m) \to \mathcal{O}_C(m)] \to 0$$

and the vanishing of $H^1(\mathcal{O}_{\mathbb{P}^2}(m-d))$, we have that every global section of $\mathcal{O}_C(m)$ is the restriction to C of a homogeneous polynomial of degree m on $\mathbb{P}^2$. Thus there is a polynomial H cutting out the divisor $A + F + D'$ on C, as claimed.

Note that if, in the process described, it turns out there is no polynomial H vanishing on $A + F + D$ but not vanishing identically on C, that simply means that $|D| = \emptyset$; that is, *D is not linearly equivalent to any effective divisor.* (It may not be obvious that the existence of such an H is independent of the choice of m or G, but the argument here shows it is.)

1.3 Nodal plane curves

al plane curves

As noted, smooth plane curves are very special among all curves. We now want to carry out the analyses above for a larger class of plane curves, curves with at most nodes as singularities. These are still special among all plane curves, but as we'll see in Section 1.3.2 below, every smooth curve is the normalization of a nodal plane curve, so that this will in theory allow us to

projection section

answer the "keynote" questions above for an arbitrary smooth curve. (In the final section of this chapter, we'll indicate how the constructions here may be extended to an arbitrary plane curve.)

The set-up, in any event, is as follows: we have a nodal plane curve $C_0 \subset \mathbb{P}^2$, with normalization $\nu : C \to C_0$; or, equivalently, a smooth projective curve C and a birational embedding of C in $\mathbb{P}^2$ with image a nodal curve C_0. Our goals will be as before:

1. to write out explicitly all global regular 1-forms on C; and
2. given a divisor D on C, to determine $|D|$; that is, find all effective divisors linearly equivalent to D.

As before, we'll choose homogeneous coordinates $[X, Y, Z]$ on $\mathbb{P}^2$ so that the curve C_0 intersects the line $L = V(Z)$ at infinity transversely at points $p_1, \dots, p_d$ other than $[0, 1, 0]$ (meaning in particular that all the nodes of C_0 lie in the affine plane $U = \mathbb{P}^2 \setminus L$). In addition, we can assume that neither branch of C_0 at a node has vertical tangent. (As before, these conditions are not logically necessary; they serve only to keep the notation reasonably simple, and in any case are satisfied by a general choice of coordinates.) Let the nodes of C_0 be $q_1, \dots, q_\delta$, with $r_i, s_i \in C$ lying over q_i; we'll denote by Δ the divisor $\sum r_i + \sum s_i$ on C.

Let $F(X, Y, Z)$ be the homogeneous polynomial of degree d defining the curve C_0, and let $f(x, y) = F(x, y, 1)$ be the defining equation of the affine part $C_0 \cap U$ of C_0. We start by considering the rational differential $\nu^*(dx)$ on C. In the smooth case, we saw that this differential was regular and nonzero in the finite plane, but had poles of order 2 at the point of $C \cap L$; this followed from the equation

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy \equiv 0.$$

and the fact that $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ have no common zeroes on C_0. But now $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ *do* have common zeroes; specifically, the pullbacks $\nu^*(\frac{\partial f}{\partial x})$ and $\nu^*(\frac{\partial f}{\partial y})$ have simple zeroes at the points r_i and s_i. We conclude, accordingly, that the differential $\nu^* dx$ has double poles at the points p_i, and simple poles at the points r_i and s_i; proceeding as before, we see that for a polynomial $g(x, y)$ of degree $\leq d - 3$, the differential

$$\nu^*\left(\frac{g(x, y)dx}{\partial f / \partial y}\right)$$

will be regular except for simple poles at the points r_i and s_i.

So, how do we get rid of these poles? There is one simple way: we require that g vanishes at the points q_i. We say in this case that g (and the curve defined by g) *satisfies the adjoint conditions*. (In the following section, we'll describe the adjoint conditions associated to an arbitrary singularity.) In any event, we see that

$$\left\{\nu^* \frac{g(x,y)dx}{\partial f/\partial y} \mid \deg g \leq d-3 \text{ and } g(q_i) = 0 \ \forall i\right\} \subset H^0(K_C).$$

Now, in the smooth case, we were able to compare dimensions to conclude that this inclusion was indeed an equality. We can do the same thing here: to begin with, we have seen that the rational 1-form $\omega = \nu^*(\frac{dx}{\partial f/\partial y})$ has zeroes of order $d-3$ at the points $p_1, \dots, p_d$ and simple poles at the points r_i and s_i and is otherwise regular and nonzero; in other words, if we set $H = p_1 + \cdots + p_d$, the divisor

$$(\omega) = (d-3)H - \Delta.$$

In particular, we see that

$$\deg((\omega)) = d(d-3) - 2\delta$$

and correspondingly

$$g(C) = \binom{d-1}{2} - \delta;$$

this is called the genus formula for plane curves.

On the other hand, the space of polynomials g of degree $\leq d-3$ vanishing at the points q_i has dimension at least $\binom{d-1}{2} - \delta$; we conclude from this that indeed

$$H^0(K_C) = \left\{\nu^* \frac{g(x,y)dx}{\partial f/\partial y} \mid \deg g \leq d-3 \text{ and } g(q_i) = 0 \ \forall i\right\},$$

and as lagniappe we see also that *the nodes q_i of an irreducible nodal plane curve of degree d impose independent conditions on curves of degree $d-3$.*

gonality of smooth plane curve

In Exercise 1.2.1, we saw how to use the description of the canonical series on a smooth plane curve to determine its gonality. Now that we have an analogous description of the canonical series on (the normalization of) a nodal plane curve, we can deduce a similar statement about the gonality of such a curve. Here are the first two cases

Exercise 1.3.1. Let C_0 be a plane curve of degree d with one node and no other singularities, and let C be its normalization. Show that C admits a unique map $C \to \mathbb{P}^1$ of degree $d-2$, but does not admit a map $C \to \mathbb{P}^1$ of degree $d-3$ or less.

Exercise 1.3.2. Let C_0 be a plane curve of degree d with two nodes and no other singularities, and let C be its normalization. Show that C admits two maps $C \to \mathbb{P}^1$ of degree $d-2$, but does not admit a map $C \to \mathbb{P}^1$ of degree $d-3$ or less.

1.3.1 Linear series on a nodal curve

Next, we take up the second of our keynote problems in this setting: with $C \to C_0 \subset \mathbb{P}^2$ as above, given a divisor D on C, can we find the complete linear series $|D|$?

In fact we can, by a process analogous to what we did in the smooth case. We'll do this first in the case where $D = E - F$ is the difference of two effective divisors whose support is disjoint from the support $\{r_i, s_i\}$ of Δ; the general case is only notationally more complicated. To start, we find an integer m and a polynomial G vanishing on the divisor E *and at the nodes* $r_1, \dots, r_\delta$ *of* C_0, but not vanishing identically on C_0. We can then write the zero locus of G pulled back to C as

$$(\nu^* G) = E + \Delta + A,$$

as before. Once more, just for simplicity, let's assume that the support of A is disjoint from the support of Δ; this means just that the curve $V(G)$ is smooth at the points q_i and is not tangent to either of the branches of C_0 there (this can certainly be done if we take m large).

Next, we find polynomials H of the same degree m, vanishing at $A + F$ and at the points q_i but not on all of C_0. Let D' be the divisor cut on C by H residual to $E + \Delta + A$; that is, we write

$$(\nu^* H) = E + \Delta + A + D'.$$

Finally, since $\nu^*(G/H)$ is a rational function on C, we see that

$$E + \Delta + A = (\nu^* H) \sim (\nu^* G) = E + \Delta + A + D',$$

and we conclude that D' is an effective divisor linearly equivalent to D on C.

But, do we get in this way *all* effective divisors linearly equivalent to D on C? The answer is yes, but it's not immediate; it follows from the following proposition, known classically as *completeness of the adjoint series.*

nt completeness **Proposition 1.3.3.** *If $C_0 \subset \mathbb{P}^2$ is a nodal plane curve and $\nu : C \to C_0$ its normalization, then the linear series cut on C by plane curves of degree m passing through the nodes is complete.*

Note that the solution to our problem follows from this proposition exactly as in the smooth case; that is, *every* effective divisor $D' \sim D$ on C is obtained in this way.

adjoint completeness

Proof. To prove Proposition 1.3.3, it will be helpful to introduce another surface: the blow-up $\pi : S \to \mathbb{P}^2$ of $\mathbb{P}^2$ at the points r_i. The proper transform on $C_0 \subset \mathbb{P}^2$ in S is the normalization of C_0, which we will again call C.

There are two divisor classes on S that will come up in our analysis: the pullback of the class of line in $\mathbb{P}^2$, which we'll denote H; and the sum of the exceptional divisors, which we'll call E. In these terms, we have

$$C \sim dH - 2E \quad \text{and} \quad K_S \sim -3H + E$$

(the first follows from the fact that C_0 has multiplicity 2 at each of the points q_i, the second from considering the pullback to S of a rational 2-form on $\mathbb{P}^2$). If A is a curve in $\mathbb{P}^2$ of degree m passing through the points q_i, we can associate to it the effective divisor $\pi^*A - E$; this gives us an isomorphism

$$H^0(\mathcal{I}_{\{q_1,\dots,q_\delta\}/\mathbb{P}^2}(m)) \cong H^0(\mathcal{O}_S(mH - E)).$$

In these terms we can describe the linear series cut on C by plane curves of degree m passing through the nodes of C_0 as the image of the map

$$H^0(\mathcal{O}_S(mH - E)) \to H^0(\mathcal{O}_C(mH - E)),$$

and the proposition amounts to the assertion that this map is surjective.

The obvious way to prove this is to view this map as part of the long exact cohomology sequence associated to the exact sequence of sheaves

$$0 \to \mathcal{O}_S(mH-E-C) = \mathcal{O}_S((m-d)H+E) \to \mathcal{O}_S(mH-E) \to \mathcal{O}_C(mH-E) \to 0,$$

from which we see that it will suffice to establish that $H^1(\mathcal{O}_S((m-d)H+E))=0$. To do this, we apply Serre duality, which says that $H^1(\mathcal{L}) \cong H^1(K_S \otimes \mathcal{L}^{-1})^*$; in this instance it tells us that

$$H^1(\mathcal{O}_S((m-d)H+E)) \cong H^1(\mathcal{O}_S((d-m-3)H))^*$$

Now, the line bundle $\mathcal{O}_S((d-m-3)H)$ is just the pullback to S of the bundle $\mathcal{O}_{\mathbb{P}^2}(d-m-3)$, which has vanishing H^1; thus the Proposition will follow from the

Lemma 1.3.4. *Let X be a smooth projective surface, and $\pi : Y \to X$ a blow-up. If $\mathcal{L}$ is any line bundle on X, then*

$$H^1(Y, \pi^*\mathcal{L}) = H^1(X, \mathcal{L}).$$

The lemma follows by applying the Leray spectral sequence, which relates the cohomology of $\mathcal{L}$ on Y to the cohomology of the direct image $\pi_*\pi^*\mathcal{L}$ (Leray is particularly simple in this setting, since all higher direct images are 0), and the observation that $\pi_*\pi^*\mathcal{L} \cong \mathcal{L}$. □

ojection section

1.3.2 Existence of good projections

In this section, we want to verify the assertion made above that every smooth curve C is birational to a nodal plane curve $C_0 \subset \mathbb{P}^2$; we'll do this by first embedding C in $\mathbb{P}^n$, and then arguing that the projection $\pi_\Lambda : C \to \mathbb{P}^2$ from a general $(n-3)$-plane $\Lambda \subset \mathbb{P}^n$ is birational onto its image C_0 and that C_0 has only nodes as singularities. This is certainly plausible, but in fact a proof relies on an application of the uniform position lemma of Section ****??.

nodal projection

Proposition 1.3.5. *If $C \subset \mathbb{P}^n$ is a smooth curve in projective space, and $\Lambda \cong \mathbb{P}^{n-3} \subset \mathbb{P}^n$ a general $(n-3)$-plane, then the projection $\pi_\Lambda : C \to \mathbb{P}^2$ is birational onto its image, which will be a nodal curve.*

Proof. The basic idea here is to look at how the plane Λ intersects the *secant variety* of the curve $C \subset \mathbb{P}^n$. The secant variety consists of the union of the lines $\overline{q,r}$ joining pairs of distinct points $q, r \in C$, plus the tangent lines $\mathbb{T}_q(C)$; altogether, these lines form a single family, parametrized by the symmetric square of C. The secant variety thus has dimension 3, so that a general $(n-3)$-plane Λ will meet it in finitely many points $p \in \mathbb{P}^n$. These will in turn correspond to the singularities of the image curve $C_0 \subset \mathbb{P}^2$: if $p \in \overline{q,r}$ lies on a secant line, then the projection π_Λ fails to be one-to-one at the image point $\pi_\Lambda(q) = \pi_\Lambda(r)$, while if p lies on a tangent line $\mathbb{T}_q(C)$ the differential of π_Λ will vanish at q. To prove the proposition, accordingly, we just have to say a little more about the intersection of Λ with the secant variety of C.

It is logically superfluous, but it will be much easier to visualize what's going on if we first reduce to the case $n = 3$. This is straightforward: if $\Gamma \subset \mathbb{P}^n$ is a general $(n-4)$-plane, then a general $(n-3)$-plane Λ containing Γ is a general $(n-3)$-plane in $\mathbb{P}^n$, so we can view the projection $\pi_\Lambda : C \to \mathbb{P}^2$ as the composition of the projection $\pi_\Gamma : C \to \mathbb{P}^3$ with the projection π_p of the image $\pi_\Gamma(C)$ from a general point $p \in \mathbb{P}^3$. Moreover, since a general $(n-4)$-plane $\Gamma \subset \mathbb{P}^n$ will be disjoint from the secant variety of $C \subset \mathbb{P}^n$, the projection $\pi_\Gamma : C \to \mathbb{P}^3$ will be an embedding; thus we can just start with a smooth curve $C \subset \mathbb{P}^3$ and project from a general point $p \in \mathbb{P}^3$.

Now, when we project $C \subset \mathbb{P}^3$ from a general point, we do expect to introduce singularities: since the family of secant lines to C is two-dimensional, we expect a finite number of them will contain a general point $p \in \mathbb{P}^3$. Indeed, we see from this naive dimension count that p can lie on at most a finite number of secant lines, proving that *the map π_p is birational onto its image*. We can say more: since there is only a 1-dimensional family of tangent lines to C, a general point $p \in \mathbb{P}^3$ will not lie on any tangent lines, we see as well that the differential of $\pi_p : C \to \mathbb{P}^2$ does not vanish; in other words, *the map π_p is an immersion*.

We now just have to make sure the singularities introduced are just nodes, and here is where we need to invoke the uniform position lemma. For example, suppose we want to prove that the map π_p is never three-to-one. This amounts to saying that a general point $p \in \mathbb{P}^3$ does not lie on any *trisecant line*; equivalently, that the family of trisecant lines to C has dimension at most one. Moreover, since the variety of secant lines is irreducible of dimension 2, this follows from the simple assertion that *not every secant line to $C \subset \mathbb{P}^3$ is a trisecant.*

At this point we'd encourage you to try to give an elementary argument for this seemingly obvious assertion. Whether you succeed or not, it does follow immediately from the uniform position lemma: this says that if $H \subset \mathbb{P}^3$ is a general plane, then the points of intersection $p_1, \dots, p_d \in C \cap H$ are in linear general position in H; that is, no three are collinear. The line joining any two is thus a secant line to C but not a trisecant line, which suffices to prove our assertion.

So now we know that the map pi_p is an immersion, and at most two-to-one everywhere; thus the image curve $C_0 \subset \mathbb{P}^2$ will have at most double points, and an analytic neighborhood of each double point will consist of two smooth branches. To complete the proof of Proposition nodal projection 1.3.5 we have to show that those two branches have distinct tangent lines; that is, that if $q, r \in C$ are any two points collinear with p, then the images of the tangent lines $\mathbb{T}_q(C)$ and $\mathbb{T}_r(C)$ in $\mathbb{P}^2$ are distinct. But if in fact it were the case that $\pi_p(\mathbb{T}_q(C)) = \pi_p(\mathbb{T}_r(C))$ then the tangent lines $\mathbb{T}_q(C)$ and $\mathbb{T}_r(C)$ would necessarily intersect.

Classically, a secant line $\overline{q,r}$ was called a *stationary secant* if the tangent lines $\mathbb{T}_q(C)$ and $\mathbb{T}_r(C)$ met; our remaining goal now is to show that *a general point $p \in \mathbb{P}^3$ does not lie on any stationary secant.* Again, this follows from the assertion that the family of stationary secants has dimension at most 1; and again, since the family of all secant lines is irreducible of dimension 2, it will suffice to show that *not every secant line to C is a stationary secant.*

And now we are done, at least in characteristic 0. Simply, it can't be the case that for a general pair of points $q, r \in C$ we have $\mathbb{T}_q C \cap \mathbb{T}_r C \neq \emptyset$; otherwise the projection $\pi_{\mathbb{T}_q C} : C \to \mathbb{P}^1$ would have derivative everywhere 0, but be nonconstant. Thus the space of stationary secants is at most 1-dimensional, and a general $p \in \mathbb{P}^3$ will not lie on one. □

Note that the the last case in this argument relies on the hypothesis of characteristic 0. Indeed, in characteristic $p > 0$ it may not be the case that a general projection of a smooth curve to $\mathbb{P}^2$ is nodal—but it's still true that every curve is the normalization of a nodal plane curve: we have the slightly weaker

acteristic nodes **Proposition 1.3.6** (Proposition nodal projection 1.3.5 in characteristic p). *Let $C \subset \mathbb{P}^n$ be a smooth curve. If we re-embed C by a Veronese map of sufficiently high degree—that is, we let $\nu_m : \mathbb{P}^n \to \mathbb{P}^N$ be the mth Veronese map, and let $\tilde{C} = \nu_m(C)$ be image of C—then the projection of $\tilde{C}$ from a general $\mathbb{P}^{N-3}$ will be nodal.*

Exercise 1.3.7. In the setting of Proposition positive characteristic nodes 1.3.6, for a general $\Lambda \cong \mathbb{P}^{N-3}$ show that

1. there do not exist three points $p, q, r \in \tilde{C}$ such that p, q, r and Λ are all contained in a $\mathbb{P}^{N-2}$; and

2. there does not exist a pair of points $p, q \in \tilde{C}$ such that p, q and Λ are contained in a $\mathbb{P}^{N-2}$ and $\mathbb{T}_p(\tilde{C}), \mathbb{T}_q(\tilde{C})$ and Λ are contained in a $\mathbb{P}^{N-1}$.

From this one can deduce Proposition positive characteristic nodes 1.3.6.

Exercise 1.3.8. Let C_0 be a plane quartic curve with two nodes q_1, q_2; let $\nu : C \to C_0$ be its normalization, and let $o \in C$ be any point not lying over a node of C_0. By the genus formula, C has genus 1. Using the construction above, describe the group law on C with o as origin.

1.4 Arbitrary plane curves

ry plane curves

Well, not exactly arbitrary: in this section, we'll deal with a plane curve C_0 that is assumed to be reduced and irreducible, with normalization $\nu : C \to C_0$.

The geometry of singular curves is a fascinating topic, from the local analysis of the singularities to the global questions involving linear series on singular curves. Indeed, it's remarkable how many of the constructions and theorems we've discussed in the realm of smooth curves can be extended to the world of singular curves, given the right definitions (and some restrictions on the type of singularities, such as the Gorenstein condition). But this is a topic beyond our ken, at least in this book; for us, the questions are about smooth curves, with singular curves appearing as a useful adjunct. The description, in the last section, of complete linear series on a smooth curve C, using a nodal plane model C_0 of C, is a perfect example.

There is, however, one fundamental invariant of a singular curve C that is both readily calculated and highly relevant in relating it to smooth curves: the *arithmetic genus* of C. This will come up in what we're going to do next, which is to describe linear series on a smooth curve C via a birational model as a plane curve with general singularities, and so we'll take a moment out here and introduce this notion.

1.4.1 Arithmetic genus and geometric genus

To start with, the arithmetic genus is a very broadly applicable: it is defined for an arbitrary one-dimensional scheme over a field. Recall that among the characterizations of the genus g of a smooth projective curve C there was one in terms of the Euler characteristic of the structure sheaf: we have $g = 1 - \chi(\mathcal{O}_C)$. This is directly equivalent to the characterization in terms of the constant term of the Hilbert polynomial of any projective embedding.

And that is how we extend the notion of genus to arbitrary singular curves C_0: for any 1-dimensional scheme C_0 over a field, we define the *arithmetic genus* of C_0 to be $1 - \chi(\mathcal{O}_{C_0})$. Very often (as, for example, right now!) we want also to deal at the same time with the genus of the normalization $\nu : C \to C_0$; to distinguish between these two notions of the genus of a singular curve, we call $1 - \chi(\mathcal{O}_{C_0})$ the arithmetic genus of C_0 and denote it $p_a(C_0)$; the genus of the normalization is called the *geometric genus* and often denoted $g(C_0)$ or $p_g(C_0)$.

What's the difference? Well, we can relate the two notions via the map of sheaves

$$\mathcal{O}_{C_0} \to \nu_*\mathcal{O}_C.$$

This is injective; the cokernel sheaf will be a skyscraper sheaf supported exactly on the singular points of C_0. Denoting this sheaf by $\mathcal{F}$, we have an exact sequence

$$0 \to \mathcal{O}_{C_0} \to \nu_*\mathcal{O}_C \to \mathcal{F} \to 0.$$

Now, the normalization map $\nu : C \to C_0$ is finite, so that the higher direct images $R^i\nu_*\mathcal{O}_C = 0$ for $i > 0$; it follows from the Leray spectral sequence that $\chi(\nu_*\mathcal{O}_C) = \chi(\mathcal{O}_C)$. We have, accordingly,

$$p_a(C_0) - g(C) = \chi(\mathcal{O}_C) - \chi(\mathcal{O}_{C_0}) = \chi(\mathcal{F}) = h^0(\mathcal{F});$$

in other words, the difference between the arithmetic and geometric genera of C_0 is the sum of the vector space dimensions of the stalks on $\mathcal{F}$; colloquially, it's the number of linear conditions a function f on C has to satisfy to be the pullback of a function from C_0. The length of the stalk of $\mathcal{F}$ at a particular singular point $p \in C_0$ is called the *δ-invariant of the singularity*; to rephrase the statement above in these terms, we have

$$p_a(C_0) - g(C) = \sum_{p \in (C_0)_{sing}} \delta_p$$

Happily, the δ invariant of a singularity is readily calculated. Here are some examples:

1. (nodes) If $p \in C_0$ is a node, with points $r, s \in C$ lying over it, the condition for a function f on C to descend is simply that $f(r) = f(s)$; this is one linear condition and accordingly $\delta_p = 1$.

2. (cusps) If $p \in C_0$ is a cusp, with $r \in C$ lying over it, the condition for a function f on C to descend is simply that the derivative $f'(r) = 0$; again, this is one linear condition and accordingly $\delta_p = 1$.

3. (tacnodes) Suppose now that $p \in C_0$ is a *tacnode*, that is, C_0 has two smooth branches at p simply tangent to one another. There will be two points $r, s \in C$ lying over it, and the condition for a function f on C to descend is that in terms of suitable local coordinates both $f(r) = f(s)$ and $f'(r) = f'(s)$. This represents two linear conditions and accordingly $\delta_p = 2$.

4. (planar triple points) Next up, consider an ordinary triple point $p \in C_0$ of a plane curve: that is, a singularity consisting of three smooth branches meeting pairwise transversely, such as the zero locus of $y^3 - x^3$. There will be three points $r, s, t \in C$ lying over p, and certainly a necessary condition for a function f on C to descend is that $f(r) = f(s) = f(t)$—two linear conditions. But there's a third, less obvious linear condition: in order for f to descend, the derivatives $f'(r), f'(s), f'(t)$ have to satisfy a linear condition—a reflection of the fact that a function on C_0 cannot vanish to order 2 on each of two branches without vanishing to order 2 along the third as well. Thus $\delta_p = 3$

5. (spatial triple points) We will be concerned in what follows only with planar singularities, but spatial triple points provide a useful contrast to the last example. A spatial triple point is a singularity consisting of three smooth branches, with linearly independent tangent lines, so that its Zariski tangent space is 3-dimensional. An example would be the union of the three coordinate axes in $\mathbb{A}^3$.

   In this case, in contrast to the last one, the condition that $f(r) = f(s) = f(t)$ is both necessary and sufficient for f to descend, and accordingly we have $\delta_p = 2$.

Exercise 1.4.1. Let $p \in C$ be a singular point of a reduced curve C. Show that if $\delta_p = 1$, then p must be either a node or a cusp.

1.4.2 Linear series on (the normalization of) a plane curve

We return now to our basic setting: we have a reduced and irreducible curve $C_0 \subset \mathbb{P}^2$ with normalization $\nu : C \to C_0$, and we want to extend our solution to the keynote problems—finding all regular differentials on C, and finding all divisors on C linearly equivalent to a given $D \in \mathrm{Div}(C)$—to this more general setting.

In this setting, if we simply try to mimic the analysis above in the nodal case we're led to introduce the *adjoint ideal* of each singularity (which is simply the maximal ideal of the point in the case of a node), in terms of which we have theorems analogous to the results obtained above in the nodal case.

So: let $C_0 \subset \mathbb{P}^2$ be a reduced and irreducible plane curve, with normalization $\nu : C \to C_0$. We focus for now on one singular point $q \in C_0$, with points $r_1, \dots, r_k \in C$ lying over q. If our goal is to describe the canonical series on C, we can start as we did in the previous two sections: by considering differentials of the form $g(x, y)\omega_0$, where

$$\omega_0 = \nu^* \frac{dx}{\partial f/\partial y},$$

and f is the defining equation of C_0 in an affine open containing q. As we saw in the nodal case, ω_0 will have poles at the points r_i; let m_i be the order of the pole of ω_0 at r_i. We can define the *adjoint ideal* of C_0 at q to be the ideal

$$A_q = \{g \in \mathcal{O}_{\mathbb{P}^2,q} \mid \mathrm{ord}_{r_i}(\nu^* g) \geq m_i \ \forall i\}$$

In other words, A is the ideal of functions g such that $\nu^* \frac{g dx}{\partial f/\partial y}$ is regular at all the points r_i. We accordingly define the *adjoint ideal* $\mathcal{I}_A$ *of* C_0 to be the product of A_q over all singular points $q \in C_0$; the *adjoint series of degree* m is then the linear series $H^0(\mathcal{I}_A(m))$.

In these terms, we can give the solution to our keynote problems much as we did in the case of plane curves with nodes. Specifically:

First, we can say that every global regular 1-form on the curve C is of the form

$$\frac{g(x,y)dx}{\partial f/\partial y},$$

with g in the adjoint ideal A, and of degree $d-3$ or less.

Second, if we are given a divisor $D = E - F$ on the curve C, we can find all effective divisors D' on C linearly equivalent to D exactly as we did in the previous case. We start by choosing a polynomial G of any degree m in the adjoint ideal and vanishing on E but not vanishing identically on C_0; we write

$$(G) = E + \Delta + A.$$

We then find all polynomials H of degree m in the adjoint ideal, vanishing on $A + F$ but not vanishing identically on C_0; writing

$$(H) = F + A + \Delta + D'$$

we arrive at an effective divisor D' on C linearly equivalent to D. Indeed, the analog of the theorem of completeness of the adjoint series tells us that we arrive at *every* effective divisor D' on C linearly equivalent to D in this way.

At this point it may seem that without an explicit description of the adjoint ideal we have merely slapped a label on our ignorance. But in fact, the adjoint ideal is relatively straightforward to find. To begin with, let's do some simple examples:

Example 1.4.2 (nodes and cusps)**.** We have already seen that in case q is a node of C_0, there are two points of C lying over it, and $m_1 = m_2 = 1$; the adjoint ideal is thus just the maximal ideal $\mathcal{I}_q$ at q. In the case of a cusp, for example the zero locus of $y^2 - x^3$, there is only one point $r = r_1$ of C lying over q, and the differential ω_0 vanishes to order $m_1 = 2$; since the pullback to C of any polynomial g vanishing at q will vanish to order at least two at r, and so again the adjoint ideal is again just the maximal ideal at q.

Example 1.4.3 (tacnodes)**.** Next, consider the case of a *tacnode*; that is, a singularity with two smooth branches simply tangent to one another, such as the zero locus of $y^2 - x^4$. In this case there are again two points of C lying over q, and a simple calculation shows that $m_1 = m_2 = 2$. The adjoint ideal is thus the ideal of functions vanishing at q and having derivative 0 in the direction of the common tangent line to the branches.

Example 1.4.4 (ordinary triple points)**.** In the case of an ordinary triple point—three smooth branches simply tangent to one another pairwise—there are three points of C lying over q, and we have $m_1 = m_2 = m_3 = 2$; the adjoint ideal is correspondingly just the square of the maximal ideal at q

Exercise 1.4.5. Find the adjoint ideals of the following plane curve singularities:

1. a *triple tacnode*: three smooth branches, pairwise simply tangent
2. a triple point with an infinitely near double point: three smooth branches, two of which are simply tangent, with the third transverse to both
3. a unibranch triple point, such as the zero locus of $y^3 - x^4$

In general, the adjoint ideal of an isolated plane curve singularity is something we can determine in practice; for example, here is a simply general description in case the individual branches of C_0 at p are each smooth:

Proposition 1.4.6. *Let $\nu : C \to C_0$ be the normalization of a plane curve C_0 and $p \in C_0$ a singular point. Denote the branches of C_0 at p by $B_1, \dots, B_k$, and let r_i be the point in B_i lying over p. If the individual branches B_i of C_0 at p are each smooth, and we set*

$$m_i = \sum_{j \neq i} mult_p(B_i \cdot B_j$$

then the adjoint ideal of C_0 at p is simply the ideal of functions g such that $ord_{r_i}(\nu * g) \geq m_i$.

1.4.3 The conductor ideal

There is another ideal we can associate to an isolated curve singularity, called the *conductor ideal*. It's simple to define: if C_0 is a reduced curve and $\nu : C \to C_0$ its normalization, we can think of the direct image $\nu_* \mathcal{O}_C$ as a module over the structure sheaf $\mathcal{O}_{C_0}$; the conductor ideal is simply the annihilator of the quotient $\nu_* \mathcal{O}_C / \mathcal{O}_{C_0}$. In concrete terms, on any affine open $U \subset C_0$ this is the ideal of functions $g \in \mathcal{O}_{C_0}(U)$ such that for any function $h \in \mathcal{O}_C(\nu^{-1}(U))$ the product hg will be the pullback of a function on C_0.

For example, in the case of a node $q \in C_0$, with r_1, r_2 the points of C lying over q, we see that any function on C vanishing at both r_1 and r_2 is the pullback of a function on C_0; thus the conductor is simply the maximal ideal at q. Similarly, if $q \in C_0$ is a cusp, with $r \in C$ lying over it, a function f on C will descend to C_0—that is, be the pullback of a function on C_0—if it vanishes to order 2 at r; since the pullback to C of any function on C_0 vanishing at q vanishes to order at least 2 at r, the conductor ideal is just the maximal ideal at q.

The sharp-eyed reader will have noticed a coincidence here, and will not be completely surprised by the

Theorem 1.4.7. *For any plane curve singularity, the adjoint ideal and the conductor ideal coincide.*

This is a reflection of the fact that plane curve singularities are necessarily local complete intersections, and hence *Gorenstein*, about which we will hear more in the following chapter.