

GORENSTEIN IDEALS OF HEIGHT 3

David A. Buchsbaum and David Eisenbud

PREFACE

This paper is an early version of "Algebra structures for finite free resolutions, and some structure theorems for ideals of codimension 3" that appeared in the American Journal of Mathematics (99). The work was actually done in the Spring of 1973. The printed version is considerably more complete; so the reader may well ask why this preliminary version should now appear.

The main theorem in this paper is quite simple: Gorenstein ideals of height 3 are always generated by Pfaffians of a suitable matrix. In this form, it has had reasonably wide application, mostly in providing a convenient class of examples on which conjectures could be formulated or checked. But the proof given in our final version was based on technical multilinear algebra of a sort which is unpleasant to read, whereas, in this early version, the technique was suppressed to a certain extent to allow us to give a more relaxed exposition.

Several people who liked the naive approach of this early version have suggested that it should, finally, appear too, and these notes seem a plausible format in which that can happen.

Several people have done further work related to this paper that seems to us quite significant. Among these, Avramov has shown that there is a real obstructions to the associativity of the algebra structure on a resolution [Am. J. Math. 103 (1981) 1-31] and Kustin and Miller have shown that it vanishes for resolutions of Gorenstein ideals of codimension 4 [Math. Z. 173 (1980) 171-184], and, in a series of preprints, have worked out a piece of the corresponding structure theory for codimension 4 Gorenstein ideals.

Received February 1, 1982

Both authors were partially supported by the NSF during the preparation of this paper.

§0. Introduction

In [SER, Prop 3], Serre proved that if R is a regular local ring, and if I is an ideal of R of height 2 such that R/I is Gorenstein, then I can be generated by two elements. Serre also remarked that the corresponding statement for height 3 ideals is false. The purpose of this paper is to prove the structure theorem that does hold for the Gorenstein ideals of height 3, namely that these ideals are characterized as the ideals of Pfaffians of certain alternating matrices (Theorem 4; see below for definitions). From this it follows, for example, that the minimal number of generators of such an ideal must be odd. Fundamental to our proof is the fact that a minimal free resolution of any cyclic R -module can be equipped with the structure of a (not necessarily associative) graded commutative algebra.

Our structure theorem is modelled on the theorem of Hilbert that characterizes the ideals I of height 2 in a regular local ring R such that R/I is Macaulay, as the ideals generated by the $n \times (n - 1)$ minors of certain matrices. In its modern form, this theorem actually shows that ideals of homological dimension 1 in any noetherian ring can be similarly characterized. [Ref. HIL, BUR, B-E 3]. (The same proofs also yield a global version; the result is generally known by the slogan "Macaulay subschemes of co-dimension 2 of a smooth scheme are determinantal", and attributed variously.) Like the result of Hilbert, our structure theorem admits an extension to all local rings, which we will describe below, though we have not been able to globalize it.

We will follow Bourbaki's terminology in saying that the height of an ideal I of a noetherian ring R is the minimum of the dimensions of the local rings R_P where P runs through the prime ideals containing I . The grade of I is the length of a maximal R -sequence contained in I (Bourbaki calls this the

I -depth of R).

We will say that the ideal I in the local noetherian ring R is Gorenstein of grade k if

a) grade $I = k = \text{hd } R/I$ (where hd denotes homological dimension).

b) $\text{Ext}_R^k(R/I, R) \cong R/I$.

As is easily seen, this is equivalent to the requirement that the minimal free resolution of R/I have the form

$\underline{F}: 0 \rightarrow R \xrightarrow{\phi_k} F_{k-1} \rightarrow \dots \rightarrow F_1 \rightarrow R$ and that $\underline{F}^* = \text{Hom}_R(\underline{F}, R)$ be a resolution of $\text{Coker}(\phi_k^*)$. It is a theorem of Serre's [BASS, TH. 5.1] that an ideal I of a regular local ring R is Gorenstein in this sense if and only if the factor ring R/I is a Gorenstein ring. Our structure theorem actually characterizes the Gorenstein ideals of grade 3 in any local ring.

The structure of Gorenstein ideals of grade ≥ 4 remains a mystery. Of course, any complete intersection - that is, any ideal I which can be generated by (grade I) elements - is Gorenstein. It is known [B-E 2 and KUN] that a Gorenstein ideal of grade k in a local ring cannot be minimally generated by $k+1$ elements.

As was mentioned above, our structure theorem implies that any Gorenstein ideal of grade 3 has an odd number of generators. It is easy to show, using that theorem, that every odd number $2n+1 \geq 3$ is actually achieved (see Section 5). It follows that for any k and $n \geq 0$ there are Gorenstein ideals of depth k that are minimally generated by $k+2n$ ($n \geq 0$) generators.

It is known that if ϕ is a square $n \times n$ matrix of indeterminates, then the ideal generated by the $p \times p$ minors of ϕ is a Gorenstein ideal of grade $(n-p+1)^2$, minimally generated by $\binom{n}{p}^2$ elements ([G-N] for $p=n-1$, [SVA] in general).

Thus, for example, it is known that the minimum number of generators of a Gorenstein ideal of grade 4 may be any even number ≥ 4 , or any perfect square. 5 is known to be impossible, but there are examples, due to Hochster, Kunz and Herzog, with 7, 11 and 13 generators. It seems reasonable to guess that there should be Gorenstein ideals of grade 4 minimally generated by any number of generators ≥ 6 .

The structure theorem presented here can be used to "lift" (in the sense of Grothendieck's lifting problem - see [B-E-O]) four generator ideals of height 3 in a 3-dimensional Gorenstein ring and Gorenstein ideals of grade 3 in any ring.

We will now describe the contents of this paper.

In the first section we show that any free resolution of a cyclic module has a graded commutative algebra structure, for which the differential of the resolution is a derivation (the algebra is only homotopy-associative), and we deduce the fact we will need about the algebra structure on a resolution of a Gorenstein ideal:

"multiplication into the top degree" induces an isomorphism between the resolution and its dual. Sections 2 and 3 review the tools needed for the main theorem, and contain no proofs. Section 2 is devoted to a criterion for the exactness of a complex [B-E 1], while Section 3 is devoted to some facts about linear algebra which were probably familiar to Cayley [GAY].

The fourth section is devoted to the structure theorem itself. The theorem is applied in Section 5 to give two classes of examples of Gorenstein ideals of height 3.

We wish to express our thanks to S. MacLane, who clarified the material in Section 1 for us.

We have recently learned that Watanabe [WAT] has independently proved that Gorenstein ideals of grade three are minimally generated by an odd number of elements, and conjectured that this pheno-

menon should be connected with skew-symmetry, as we have shown.

§1. Commutative algebra structures on resolutions

Let R be a commutative ring, and let $\underline{P}: \dots P_2 \xrightarrow{d} P_1 \xrightarrow{d} R$ be a projective resolution, with $P_0 = R$. We define the symmetric square $S_2(\underline{P})$ to be

$$S_2(\underline{P}) = (\underline{P} \otimes \underline{P}) / M$$

where M is the graded submodule of $\underline{P} \otimes \underline{P}$ generated by

$\{f \otimes g - (-1)^{(\deg f)(\deg g)} g \otimes f | f, g \text{ homogeneous elements of } \underline{P}\}.$

Of course $\underline{P} \otimes \underline{P}$ is not only a graded module, it is a complex with differential

$$d(f \otimes g) = d f \otimes g + (-1)^{\deg f} f \otimes dg.$$

It is easy to check that $d(M) \subseteq M$, so $S_2(\underline{P})$ inherits the structure of a complex from $\underline{P} \otimes \underline{P}$. Moreover, each component of $S_2(\underline{P})$ is a projective R -module since for each k we have

$$S_2(\underline{P})_k \cong (\bigcap_{\substack{i+j=k \\ i < j}} P_i \otimes P_j) + T_k$$

where

$$T_k = \begin{cases} 0 & \text{if } k \text{ is odd,} \\ \bigwedge^{k/2} P_{k/2} & \text{if } k \text{ is of the form } 4n+2, \\ S_2(P_{k/2}) & \text{if } k \text{ is of the form } 4n. \end{cases}$$

Thus $S_2(\underline{P})$ is a complex of projective R -modules, and is isomorphic to \underline{P} in degrees 0 and 1. By the comparison theorem for projective complexes, there exists a map of complexes $S_2(\underline{P}) \xrightarrow{\phi} \underline{P}$ which extends the equality in degrees 0 and 1, and the map ϕ may clearly be chosen so that the restriction of ϕ to

$P_0 \otimes P_k = R \otimes P_k \subseteq S_2(\underline{P})_k$ is the canonical isomorphism
 $R \otimes P_k \rightarrow P_k$. If we write $f \cdot g = \phi(\overline{f \otimes g})$, where $f, g \in \underline{P}$, and
 $\overline{f \otimes g}$ is the image, in $S_2(\underline{P})$, of $f \otimes g$, then the conditions
 that the map

must satisfy are easily seen to be equivalent to the statement
 that \cdot makes \underline{P} into a (non-associative) graded commutative
 differential algebra, with differential d ,

and with structure map $R \rightarrow \underline{P}$ given by the inclusion into degree 0.
 Furthermore, it follows from the homotopy uniqueness of ϕ that
 this algebra is homotopy associative. If $P_i = 0$ for $i \geq 4$, it
 is easily seen that \underline{P} is actually associative.

We now specialize, and suppose that R is a local noetherian
 ring, that I is a Gorenstein ideal of R of depth n , and that

$$\underline{F}: 0 \longrightarrow F_n \xrightarrow{d_n} F_{n-1} \cdots F_1 \xrightarrow{d_1} R$$

is a minimal free resolution of R/I , equipped with a multiplication
 $\cdot : \underline{F} \otimes \underline{F} \rightarrow \underline{F}$, as above. Since I is Gorenstein, we may
 make the identification $F_n = R$, so that, for each $k \leq n$,
 the map $\cdot : F_k \otimes F_{n-k} \rightarrow F_n = R$ induces a map

$$s_k : F_k \rightarrow F_{n-k}^*$$

Theorem 1. For each $k \leq n$, s_k is an isomorphism.

Remark: Let K be the residue field of R . The multiplication
 we have defined on \underline{F} induces the usual algebra structure on
 $\text{Tor}_k^R(R/I, K)$. Thus our theorem extends the result of [A-G] who
 proved that if R is regular, and R/I is Gorenstein, then
 $\text{Tor}_k^R(R/I, K)$ is a "Poincaré algebra" - that is, the pairings

$$\text{Tor}_k^R(R/I, K) \otimes \text{Tor}_{n-k}^R(R/I, K) \longrightarrow \text{Tor}_n^R(R/I, K)$$

are perfect (where $n = \text{depth } I = \text{hd}_R R/I$). Of course, if R is regular, our theorem can be deduced immediately from this one, but our proof is considerably simpler.

Proof of Theorem 1: We claim that the following diagram commutes up to sign:

$$\begin{array}{ccccccc}
 \underline{F}: 0 \rightarrow & F_n & \xrightarrow{d_n} & F_{n-1} & \rightarrow \cdots \rightarrow & F_2 & \xrightarrow{d_2} & F_1 & \xrightarrow{d_1} & R \\
 & \parallel & & s_{n-1} & & \parallel & & s_2 & & \parallel & & s_1 & & \parallel \\
 & & d_1^* & \downarrow & & & & d_{n-1}^* & \downarrow & & & & d_n^* & \downarrow & \\
 \underline{F}^*: 0 \rightarrow & R & \longrightarrow & F_1^* & \longrightarrow \cdots \longrightarrow & F_{n-2}^* & \longrightarrow & F_{n-1}^* & \longrightarrow & F_n
 \end{array}$$

For, if $f \in F_k$, $g \in F_{n-k+1}$, then $f \cdot g \in F_{n+1} = 0$, so

$$0 = d_{n+1}(f \cdot g) = d_k(f) \cdot g + (-1)^k f \cdot d_{n-k+1}(g),$$

$$\text{or } d_k(f) \cdot g = \pm f \cdot d_{n-k+1}(g).$$

Consequently,

$$\begin{aligned}
 (s_{k-1} d_k(f))(g) &= d_k(f) \cdot g \\
 &= \pm f \cdot d_{n-k+1}(g) \\
 &= s_k(f)(d_{n-k+1}(g)) \\
 &= d_{n-k+1}^*(s_k(f))(g).
 \end{aligned}$$

Thus $s_{k-1} d_k = \pm d_{n-k+1}^* s_k$, as required.

Now, since I is Gorenstein, both \underline{F} and \underline{F}^* are minimal free resolutions of R/I and it follows that any map of complexes $\underline{F} \rightarrow \underline{F}^*$ which extends an isomorphism in degree 0 must be an isomorphism. Thus each s_k is an isomorphism.

§2. How to prove that a complex is exact

The proof of our main theorem on Gorenstein rings hinges on showing that a certain complex constructed from an alternating matrix is exact. In this section we will review the technique that we will use to prove this exactness, and state one further necessary result on complexes.

Let R be any commutative ring, and suppose $f:A \rightarrow B$ is a map of R -modules. We define the rank of f to be the largest integer k such that the k^{th} exterior power $\bigwedge^k f \neq 0$. For any integer $j \geq 0$, the map $\bigwedge^j f: \bigwedge^j A \rightarrow \bigwedge^j B$ induces a map $\bigwedge^j A \times (\bigwedge^j B)^* \rightarrow R$, where $*$ denotes $\text{Hom}_R(-, R)$, as usual. We will associate to f an ideal $I(f) \subseteq R$ by defining

$$I(f) = \text{Image}(\bigwedge^k A \times (\bigwedge^k B)^* \rightarrow R) \quad \text{for } k = \text{rank } f.$$

If A and B are free modules with given bases, and f corresponds to a matrix ϕ , then $\text{rank } f$ is just the size of the largest nonvanishing minor of ϕ , and $I(f)$ is the ideal generated by all the $(\text{rank } f) \times (\text{rank}(f))$ minors of ϕ . ($I(f)$ may thus be viewed as the smallest non-zero Fitting ideal of $\text{Coker } f$.)

If I is an ideal of R , we will write \sqrt{I} for the radical of I .

Now let R be a noetherian ring, and let

$$(*) \quad 0 \rightarrow F_n \xrightarrow{f_n} F_{n-1} \rightarrow \dots \rightarrow F_1 \xrightarrow{f_1} F_0$$

be a complex of finitely generated free R -modules. In this paper, we will use the following two results:

Theorem 2.1. The complex $(*)$ is exact if and only if, for all $k \geq 1$,

1) $\text{rank } f_k + \text{rank } f_{k+1} = \text{rank } F_k$ and

2) grade $I(f_k) \geq k$.

Proposition 2.2. Suppose (*) is exact. Then for each k ,

$$\sqrt{I(f_{k+1})} > \sqrt{I(f_k)}.$$

Theorem 2.1 is a special case of the main theorem of [B-E-1], and Proposition 2.2 occurs as Theorem 2.1 a) in [B-E-3].

§3. A little linear algebra

In this section we will briefly review the results on Pfaffians that we will need. For a modern, basis-free treatment of these matters, which also works for projective modules, see [B-E-4]. An exposition which uses bases, but which contains more, may be found in [HEY].

Let R be a commutative ring, and let F be a finitely generated free R -module. A map $f: F \rightarrow F^*$ is said to be alternating if with respect to some (and therefore with respect to any) basis and dual basis of F and F^* , the matrix of f is skew-symmetric, and all its diagonal elements are 0.

Choose a basis of F , and identify f and the matrix of f with respect to the chosen basis of F and the dual basis of F^* . Suppose f is alternating. Then

3.1) If $\text{rk } F = n$ is even, there exists an element $Pf(f) \in R$, called the Pfaffian of f , which is a polynomial function of the entries of f , such that $\det(f) = (Pf(f))^2$.

Let ϕ_{ij} be the $(i,j)^{\text{th}}$ entry of f with respect to the chosen bases. For $1 \leq i \neq j \leq n$, let f_{ij} be the matrix obtained from f by deleting the i th and j th rows and columns. Suppose that n is even. Then for any i , $1 \leq i \leq n$, the Pfaffian of f can be com-

ted from the formula:

$$3.2) \quad Pf(f) = \sum_{j \neq i} (-1)^j \phi_{ij} Pf(f_{ij}) ,$$

(see [BOU, Exer. 5, p. 86]).

Given any free summand G of F , generated by an even number $2k$ of the basis elements e_i of F , we say that the Pfaffian of the composite map

$$G \xrightarrow{\text{inclusion}} F \xrightarrow{\text{(inclusion)}^*} F^* \xrightarrow{\text{*}} G^*$$

is "a Pfaffian of f of order $2k$ ". We denote by $Pf_{2k}(f)$ the ideal of R which is generated by all the Pfaffians of f of order $2k$. If $2k > n$, we set $Pf_{2k}(f) = 0$. We will denote by $I_t(f)$ the ideal of R generated by all the minors of f of order t .

We now have, for each k ,

$$3.3) \quad \begin{cases} I_{2k-1}(f) \subseteq Pf_{2k}(f), \\ \vdots \\ Pf_{2k}(f) \supseteq I_{2k}(f) \supseteq (Pf_{2k}(f))^N \text{ where } N = \binom{n}{2k} + 1 . \end{cases}$$

Remark: There are in fact more precise formulae which express the minors of f in terms of Pfaffians of f . See [B-E-4] or [HEY] for details.

3.4) It follows from 3.3) that if f is alternating, and $\text{rk } f = 2k-1$ is an odd number, then $I(f) = I_{2k-1}(f)$ is nilpotent.

Now suppose that n is odd. Then we have

$$3.5) \quad \text{rk}(f) \leq n-1 \text{ and } I_{n-1}(f) = (Pf_{n-1}(f))^2 .$$

§4. The Structure Theorem

Throughout this section, R will denote a noetherian local ring with maximal ideal J .

Theorem 4. 1) Let $n \geq 3$ be an odd integer, and let F be a free R -module of rank n . Let $f:F \rightarrow F^*$ be an alternating map of rank $n-1$ whose image is contained in JF^* . If $Pf_{n-1}(f)$ has grade 3, then $Pf_{n-1}(f)$ is Gorenstein, and the minimal number of generators of $Pf_{n-1}(f)$ is n .

2) Every Gorenstein ideal of R of grade 3 arises as in 1).

Corollary: The minimal number of generators of a Gorenstein ideal of grade 3 is odd.

Proof: This follows from part 2) of the theorem, since only odd n are considered in part 1).

Remarks: 1) In the proof of part 1), we will construct a complex whose existence shows that if F is a free module of odd rank n , and $f:F \rightarrow F^*$ is an alternating map of rank $n-1$, then

$$\text{grade } (Pf_{n-1}(f)) \leq 3.$$

2) One further question that can be raised in connection with Theorem 4 is the uniqueness question: what can be said about two maps f and f' , both satisfying the conditions of part 1) of the theorem, and having

$$Pf_{n-1}(f) = Pf_{n-1}(f') ?$$

In case R is complete and 2 is a unit in R , the most desirable conclusion holds: there exists an automorphism $a:F \rightarrow F$ such that

$$f' = a^*fa.$$

If R is not complete, we have not been able to show the existence of such an a ; but neither have we been able to find a counterexample.

Proof: 1) Choose a basis for F , a dual basis for F^* , and let ϕ be the matrix of f with respect to these bases. Let γ_i be the Pfaffian of the alternating matrix obtained from ϕ by deleting the i th row and the i th column, and let $g:R \rightarrow F$ be the map whose matrix, in terms of the given basis of F is

$$\gamma = \begin{pmatrix} \gamma_1 \\ -\gamma_2 \\ \gamma_3 \\ -\gamma_4 \\ \vdots \\ \gamma_n \end{pmatrix}$$

Then, by virtue of the formula for the expansion of Pfaffians along a row (i.e., formula 3.2), the i th coefficient of $\phi\gamma$ is the Pfaffian of the $n+1$ by $n+1$ matrix obtained from ϕ by repeating the i th row and the i th column; thus $\phi\gamma = 0$.

Since $\phi^* = -\phi$, we also have

$$\gamma^*\phi = -\gamma^*\phi^* = -(\phi\gamma)^* = 0$$

so that

$$(*) \quad 0 \rightarrow R \xrightarrow{g} F \xrightarrow{f} F^* \xrightarrow{g^*} R$$

is a complex. We will prove that this complex is exact. Since $\text{Coker } g^* = R/(Pf_{n-1}(f))$, this will show that $Pf_{n-1}(f)$ is a Gorenstein ideal.

To prove that $(*)$ is exact, we invoke Theorem 2.1. The condition on the ranks of the maps is easily checked. For we have assumed

that $\text{rank } f = n - 1$, and by 3.5) the square of the ideal generated by the entries of \mathbf{f} is $I_{n-1}(f) \neq 0$, so $\text{rank } g = 1$.

To check the second condition of Theorem 2.1 is even easier. For we have $I(g^*) = I(g) = Pf_{n-1}(f)$, and $I(f) = (Pf_{n-1}(f))^2$, which has the same grade as $Pf_{n-1}(f)$. Since we have assumed that $Pf_{n-1}(f)$ has grade 3, condition 2 of Theorem 2.1 is also satisfied, so that $(*)$ is exact.

2) Next, suppose that I is a Gorenstein ideal of grade 3, and let

$$(***) \quad 0 \rightarrow F_3 \xrightarrow{f_3} F_2 \xrightarrow{f_2} F_1 \xrightarrow{f_1} R \rightarrow R/I \rightarrow 0$$

be a minimal free resolution of R/I . We have seen in Section 1 that $(***)$ has the structure of a differential graded commutative algebra. By Theorem 1 the multiplication in this algebra gives a perfect pairing $F_2 \otimes F_1 \rightarrow F_3$. If we identify F_3 and R , then this pairing yields an identification of F_1 with F_2^* . We will show that with respect to this identification, f_2 is alternating.

Let $\{e_i\}_{i=1}^n$ be a basis of F_2 , and let $\{\varepsilon_i\}_{i=1}^n$ be the dual basis of F_2^* . Let $\phi = \{\phi_{ij}\}$ be the matrix of f_2 with respect to the basis $\{e_i\}$ and $\{\varepsilon_i\}$. We want to show that $\phi_{ij} = -\phi_{ji}$ and that $\phi_{ii} = 0$. But $\phi_{ij} = (f_2(e_j)) \cdot e_i$, where we regard $f_2(e_j)$ as a functional on F_2 . Of course, we may regard this last term as a product with respect to the algebra structure of $(***)$ and write

$$\phi_{ij} = (f_2(e_j)) \cdot e_i \in F_3 = R.$$

Applying f_3 , and using the formula for the differentiation of a product, we get

$$f_3(\phi_{ij}) = f_1 f_2(e_j) \cdot e_i + f_2(e_j) \cdot f_2(e_i) = f_2(e_j) \cdot f_2(e_i),$$

since $f_1 f_2 = 0$. But we know from Section 1 that the product in (**) is strictly skew-commutative. Since $f_2(e_j)$ and $f_2(e_i)$ both have degree 1, we see that

$$\begin{aligned} f_3(\phi_{ij}) &= -f_3(\phi_{ji}) \quad \text{and} \\ f_3(\phi_{ii}) &= 0. \end{aligned}$$

Since f_3 is a monomorphism, it follows that

$$\phi_{ij} = -\phi_{ji} \quad \text{and} \quad \phi_{ii} = 0.$$

Thus f_2 is alternating.

We now show that $Pf_{n-1}(f_2) = I$. First of all, we have rank $f_2 = n - 1$ by Theorem 2.1, and depth $I(f_2) \geq 3$ by Proposition 2.2. Consequently, if we view f_2 as a map from F_2 to $F_1 = F_2^*$, then f_2 will satisfy all the conditions of the first part of the theorem. As in the proof of the first part of the theorem, we can construct a map $g: R \rightarrow F_2$ so that

$$0 \rightarrow R \xrightarrow{g} F_2 \xrightarrow{f_2} F_2^* \xrightarrow{g^*} R$$

is exact, and so that $I(g) = Pf_{n-1}(f)$. Since both g and f_1^* are kernels of $f_2^* = -f_2$, we see that there must exist a unit $u \in R$ such that $f_1^* = gu$, $f_1 = ug^*$, and $I = Pf_{n-1}(f)$. This concludes the proof.

55. Some examples of Gorenstein ideals

a) Generic Gorenstein ideals of height 3.

Let k be a commutative ring, and let $G_n(k)$ be a generic alternating $(2n + 1) \times (2n + 1)$ matrix over

$$k[\{x_{ij}\}_{1 \leq i < j \leq 2n+1}] = R_n(k):$$

$$G_n(k) = \begin{pmatrix} 0 & x_{12} & \cdots & x_{12n+1} \\ -x_{12} & 0 & & \cdot \\ \cdot & \cdot & \ddots & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & & \cdot & x_{2n-2n+1} \\ -x_{12n+1} & \cdots & -x_{2n-2n+1} & 0 \end{pmatrix}$$

Proposition 5.1. Let k be a field. For every $n \geq 1$, Pfaffian $(G_n(k))$ is a Gorenstein ideal of height 3 in $R_n(k)$, so that $R_n(k)/Pf_{2n}(G_n(k))$ is a Gorenstein ring. Moreover, $R_n(k)/Pf_{2n}(G_n(k))$ is a normal domain, non-singular in codimension 6.

Proof. By 3.5) and the fact that $G_n(k)$ is "generic", we have rank $G_n(k) = 2n$. Thus part 1 of Theorem 4 shows that $Pf_{2n}(G_n(k))$ is a Gorenstein ideal if it has grade 3. Since $R_n(k)$ is a Macaulay ring, grade $(Pf_{2n}(G_n(k))) = \text{height } (Pf_{2n}(G_n(k)))$. However, it is known [ROO, p. 196] that

$$\text{height } (Pf_{2n-2\ell}(G_n(k))) = (\ell + 1)(2\ell + 3).$$

In particular, $\text{height } (Pf_{2n}(G_n(k))) = 3$. Thus we have the first assertion of Proposition 5.1.

To prove that $S = R_n(k)/Pf_{2n}(G_n(k))$ is a normal domain, we will use the Krull-Serre characterization [MAT, Th. 39] and verify the criteria S_2 and R_1 . Since S is Gorenstein and thus, in particular, Macaulay, we need only show that R_1 holds for S ; that is, that every localization of S at a prime of height 1 is regular. What we will actually see is that if P is any prime not containing $I = Pf_{2n-2}(G_n(k))/Pf_{2n}(G_n(k))$, then S_P is regular.

Since height $(\text{Pf}_{2n-2}(G_n(k))) = 10$, the height of I is 7 so this will show that S is non-singular in codimension 6 as well.

Let A be any $(2n-2) \times (2n-2)$ Pfaffian of $G_n(k)$ and let \underline{U} be the multiplicatively closed set generated by A . It is not difficult to see that $R_n(k)_{\underline{U}}/\text{Pf}_{2n}(G_n(k))$ is isomorphic to $R_n(k)_{\underline{U}}/\text{Pf}_2(G_1(k))$. For, assume that A is the $(2n-2) \times (2n-2)$ Pfaffian of the submatrix of $G_n(k)$ obtained by omitting the first three columns and rows. Since A is invertible in $R_n(k)_{\underline{U}}$, $G_n(k)$ may be transformed over $R_n(k)_{\underline{U}}$ to

$$\begin{pmatrix} 0 & y_{12} & y_{13} & & \\ -y_{12} & 0 & y_{23} & & 0 \\ -y_{13} & -y_{23} & 0 & & \\ & 0 & & \begin{array}{|c|c|} \hline 0 & 1 \\ \hline -1 & 0 \\ \hline \end{array} & \\ \end{pmatrix}$$

where the y_{ij} are polynomials of the form

$$y_{ij} = x_{ij} + (\text{higher order terms}).$$

We therefore see that $R_n(k)_{\underline{U}}/\text{Pf}_{2n}(G_n(k))$ is a localization of a polynomial ring. Thus if P is a prime of S not containing all the Pfaffians of order $2n-2$ of $G_n(k)$, then one of these Pfaffians is invertible in S_P so that S_P is a localization of a regular ring. This shows that S is a normal ring. Since $\text{Pf}_{2n}(G_n(k))$ is a homogeneous ideal, $\text{Spec } S$ is connected, so S is a normal domain, non-singular in codimension 6.

b) As was mentioned in the introduction, this investigation began from a desire to know about the Gorenstein ideals of height (= depth) 3 in regular local rings. In particular, we may

consider rings of the form $k[[X, Y, Z]]$, where k is a field and X, Y, Z are indeterminates. By part 2) of Theorem 4.1, any depth 3 Gorenstein ideal in this ring must have as its minimal number of generators an odd number ≥ 3 . We now give an example to show that all these odd numbers do occur.

Let $n \geq 3$ be odd, and let $R = k[[X, Y, Z]]$. Let H_n be the $n \times n$ matrix of the form:

$$H_n = \begin{pmatrix} 0 & X & 0 & \dots & \dots & 0 & Z \\ -X & 0 & Y & \dots & \dots & Z & 0 \\ 0 & -Y & 0 & \dots & & \cdot & \cdot \\ \cdot & & & \cdot & & \cdot & \cdot \\ \cdot & & & \cdot & Z & \cdot & \cdot \\ \cdot & & & \cdot & 0 & \cdot & \cdot \\ \cdot & & & \cdot & -Z & \dots & \cdot \\ \cdot & & & \cdot & & & \cdot \\ \cdot & & & \cdot & & X & 0 \\ 0 & -Z & & & -X & 0 & Y \\ -Z & 0 & \dots & \dots & 0 & -Y & 0 \end{pmatrix}$$

Proposition 5.2. $Pf_{n-1}(H_n)$ is a Gorenstein ideal of height 3 in R .

Proof. An easy computation shows that $Pf_{n-1}(H_n)$ contains powers of X, Y, Z . Thus $\text{height } (Pf_{n-1}(H_n)) = \text{depth } (Pf_{n-1}(H_n)) = 3$, and an application of part 1 of Theorem 4.1 finishes the proof.

References

- [BAS] Bass, H.: On the ubiquity of Gorenstein rings; Math. Zeitschr. 82 (1963), 8-28.
- [BOU] Bourbaki, N.: Algèbre, Ch. I-III; Hermann, Paris (1970).
- [B-E-0] Buchsbaum, D. and Eisenbud, D.: Lifting Modules and a Theorem on Finite Free Resolutions; Proceedings of a conference on Ring Theory, R. Gordon Ed., Acad. Press (1972).
- [B-E-1] -----; What makes a complex exact? J. Alg. 25, (1973), 259-268.
- [B-E-2] -----; Remarks on ideals and resolutions; Symposia Math. XI, (1973).
- [B-E-3] -----; Some structure theorems for finite free resolutions; Advances in Math. 12 (1974), 84-139.
- [B-E-4] -----; Algebra structures for finite free resolutions and some structure theorems for ideals of codimension 3. Am. J. Math. 99 (1977), 447-485.
- [BUR] Burch, L.: On ideals of finite homological dimension in local rings; Proc. Cam. Phil. Soc. 64 (1968), 941-946.
- [CAY] Cayley, A.: Recherches ultérieures sur les déterminants gauches; J. Reine Angew. Math. (1854).
- [A-G] Avramov, L. and Golod, E.: On the homology algebra of the Koszul complex of a local Gorenstein ring. Math. Notes 9 (1971), 30-32.

- [G-N] Gulliksen, T. and Negård, O.: Un complexe résolvant pour certains idéaux déterminantiels; C. R. Acad. Sci.; Paris, t. 274; (1972), 16-18.
- [HEY] Heymans, P.: Pfaffians and skew-symmetric matrices; Proc. Lond. Math. Soc. 19 (1969), 730-768.
- [HIL] Hilbert, D.: Über die Theorie der Algebraischen Formen; Math. Ann. 36 (1890), 43-534.
- [KUN] Kunz, E.: Almost complete intersections are not Gorenstein rings; J. of Alg. 28 (1974), 111-115.
- [MAT] Matsumura, H.: Commutative Algebra; W. Benjamin; 1970.
- [ROO] Room, T.: The Geometry of Determinantal Loci. Cambridge U. Press., 1938.
- [SER] Serre, J.-P.: Sur les modules projectifs; Séminaire Dubreil; 1960/61.
- [SVA] Svanes, T.: Coherent Cohomology of Schubert Subschemes of Flag Schemes and Applications. - M. I. T. Thesis.
- [WAT] Watanabe, J.: A note on Gorenstein rings of embedding codimension 3; Nagoya Math. J. Vol. 50 (1973), 227-232.

David A. Buchsbaum
 David Eisenbud
 Brandeis University
 Department of Mathematics
 Waltham, Massachusetts 02254
 U.S.A.