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The Kodaira dimension of the moduli space of curves of genus $\geqq 23$

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Summary. We use the theory of limit linear series to prove:

Theorem A. *The moduli space of curves of genus g is of general type for all $g \geqq 24$, and has Kodaira dimension $\geqq 1$ for $g=23$.*

Along the way to this result we give a strong version of the Brill-Noether Theorem, we study some general conditions on divisors of special curves in the moduli space, and we compute the genus of the variety of special line bundles W^r_d on a sufficiently general curve in case W^r_d is itself a curve.

Introduction

The proof of Theorem A given here completes the program started in Harris and Mumford [1982] and Harris [1984], where the result was proved for odd $g \geqq 25$ and even $g \geqq 40$. The use of limit series in place of admissable covers not only allows us to treat the additional cases, but also gives a substantially simpler treatment of the cases already covered.

Recall that by the theorem of Harer [1983] the rational divisor class group of the moduli space $\bar{\mathscr{M}}_g$ of stable genus g curves is generated by classes λ (the first

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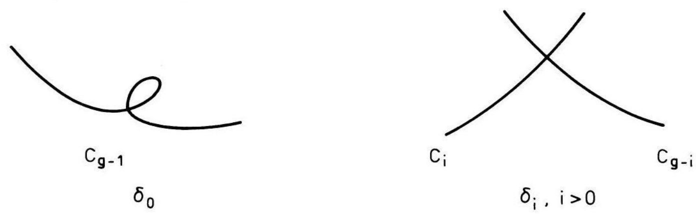
chern class of the push-forward to $\bar{\mathcal{M}}_g$ of the relative dualizing sheaf on the "universal curve"), and δ_i $(i=0,\ldots,[g/2])$ where δ_i represents the divisor whose generic point is given in Fig. 1.



C_i a curve of geometric genus i

The following criterion is verified in the first two sections of Harris and Mumford [1982] (see the introduction to Harris [1984] for an outline of the argument):

Criterion. $\mathcal{M}_g$ is of general type if there exists an effective divisor D on $\bar{\mathcal{M}}_g$ of class

$$a\lambda - \sum_{i=0}^{[g/2]} b_i\delta_i$$

with

$$\frac{a}{b_i} < \tfrac{13}{2} \quad \text{for all } i$$

and

$$\frac{a}{b_1} < \tfrac{13}{3}.$$

Further, $\mathcal{M}_g$ has Kodaira dimension $\geqq 1$ if there are two divisors with distinct support in $\mathcal{M}_g$ satisfying the weaker inequalities

$$\frac{a}{b_i} \leqq \tfrac{13}{2} \quad \text{for all } i,$$

$$\frac{a}{b_1} \leqq \tfrac{13}{3}.$$

The divisor class on $\bar{\mathcal{M}}_g$ that we will use with this criterion depends on the nature of g:

i) If $g+1$ is composite we may write

$$g=(r+1)(s-1)-1, \qquad s\geqq 3,$$

and set

$$d:=rs-1$$

for some integers $r,s>0$. We consider the union D_s^r of the codimension 1 components in $\bar{\mathcal{M}}_g$ of the closure of the locus of curves in $\mathcal{M}_g$ which possess a linear series of degree $d=rs-1$ and projective dimension r (this is the case of Brill-Noether number $\varrho=-1$).

In Sect. 4 we compute the class of this divisor, up to a positive rational multiple, for all r, s:

Theorem 1. *If* $g=(r+1)(s-1)-1$ *with* $s\geqq 3$, *then the class of* D_s^r *on* $\bar{\mathcal{M}}_g$ *is given, for some rational number* $c>0$, *by:*

$$D_s^r = c\left((g+e)\lambda - \frac{g+1}{6}\delta_0 - \sum_{i=1}^{[g/2]} i(g-i)\delta_i\right).$$

ii) If on the other hand $g+1>2$ is not composite then, in particular g is even. More generally if g is composite we may write

$$g=(r+1)(s-1)$$

and set

$$d=rs$$

for some integers $r, s>0$. We consider the union E_s^r of the codimension 1 components in $\bar{\mathcal{M}}_g$ of the closure of the locus in $\mathcal{M}_g$ of curves C which possess a linear series $L=(\mathcal{L}, V)$ of degree $d=rs$ and dimension r "violating the Petri condition"; that is, such that the product map

$$\mu_0: V\otimes H^0(C, K_C\otimes\mathcal{L}^{-1})\to H^0(C, K)$$

is not injective. (This is the case of Brill-Noether number $\varrho=0$, and E_s^r is essentially the closure of the branch locus of the generically finite covering of $\mathcal{M}_g$ by the family of all g_s^r's.)

In Sect. 5 we compute the class of this divisor, but only in case $r=1$, corresponding to the case where g is even. We obtain rather messy formulae for its class from which we deduce:

Theorem 2. *If* $g=2(d-1)$ *then the class of* E_d^1 *on* $\bar{\mathcal{M}}_g$ *is given by*

$$E_d^1 = c\left(e\lambda - \sum_0^{g/2} f_i\delta_i\right).$$

Where

i)
$$c=2\frac{(2d-4)!}{d!(d-2)!},$$
$$e=6d^2+d-6,$$
$$f_0=d(d-1),$$
$$f_1=(2d-3)(3d-2),$$
$$f_2=3(d-2)(4d-3)$$

ii) for $e\leqq i\leqq g/2$,

$$f_i>f_{i-1}.$$

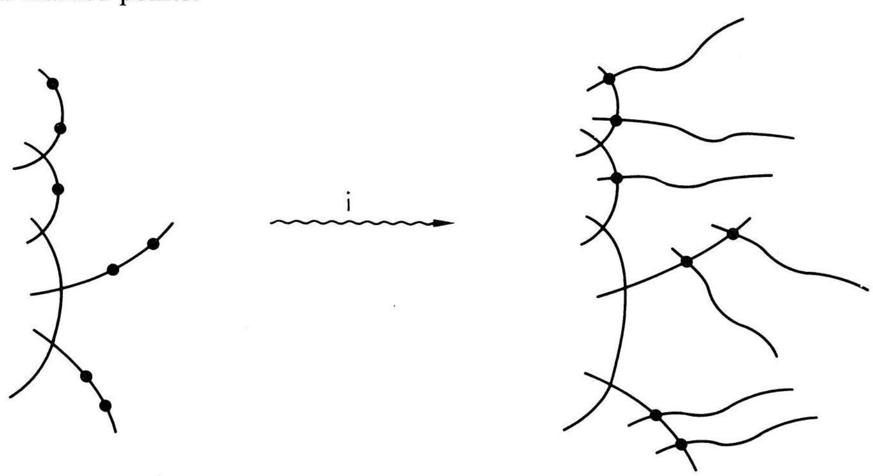
We see that for $g \geqq 28$ and even, the divisor $E^1_{(g/2)+1}$ satisfies the general type criterion. For $g \geqq 24$ and odd, the divisor $D^1_{(g+1)/2}$ satisfies the general type criterion (as noted already in Harris and Mumford [1982]). For $g=24$ D^4_6 and for $g=26$ D^2_{10} satisfy the general type criterion. Finally, for $g=23$, we note that D^1_{13} and D^2_9 both satisfy the weaker inequalities necessary to check the criterion for Kodaira dimension $\geqq 1$. At the end of Sect. 4 we prove:

Proposition 3. *The divisors D^1_{13} and D^2_9 in $\bar{\mathscr{M}}_{23}$ have distinct support.*

These results complete the proof of Theorem A.

Both Theorems 1 and 2 are deduced by considering the pullbacks of D^r_s and E^1_d to smaller spaces, as follows.

Let P_g be the moduli space of stable g-pointed rational curves as considered by Knudsen [1983] (or see Mumford and Harris [1982] p. 57), and let $i: P_g \to \bar{\mathscr{M}}_g$ be the map obtained by attaching g copies of a fixed pointed elliptic curve, at each of the marked points:



Similarly, let $\bar{\mathscr{M}}_{2,1}$ be the moduli space of stable pointed curves of genus 2, and let $j: \bar{\mathscr{M}}_{2,1} \to \bar{\mathscr{M}}_g$ be the map obtained by attaching a fixed general smooth pointed curve of genus $g-2$ at the marked point.

It seems to be rather common that loci of "special" curves in $\bar{\mathscr{M}}_g$ meet $j(\bar{\mathscr{M}}_{2,1})$ only along the closure $W \subset \bar{\mathscr{M}}_{2,1}$ of the locus of Weierstrass points in $\mathscr{M}_{2,1}$. This is the case both for the D^r_s and the E^1_d. Section 2 is devoted to the conditions on a divisor that this implies.

Similarly, the curves in $i(P_g)$ seem to be rather general, and Sect. 3 is devoted to the conditions that must be satisfied by a divisor missing $i(P_g)$, as do the D^r_s, but unfortunately not the E^1_d. Using these two conditions the proof of the relation in Theorem 1 is easy, and this is done in Sect. 4 along with the limit-series constructions necessary to prove that the constant c in Theorem 1 is >0, and to prove Proposition 3.

Although the E^1_d are not disjoint from $i(P_g)$, we are able to compute the restriction of $i^*E^1_d$ to various curves in P_g. This, with the result of Sect. 2 and a computation in one more family of curves, is enough to prove Theorem 2, (and even to compute the coefficients there explicitly). This proof is quite complex, and is outlined more completely at the beginning of Sect. 5.

Section 6, which may be read independently of the rest of this paper, contains a short proof of the following result, which is used in Sect. 5 and seems to have independent interest. Suppose that r, d, and g are such that $\pi = g - (r+1)(g-d+r) = 1$, and let C be a curve of genus g. We write $W_d^r \subset \mathrm{Pic}^d(C)$ for the subvariety consisting of line bundles $\mathscr{L}$ of degree d on C such that $h^0(\mathscr{L}) \geqq r+1$. If, as in the case for general C, W_d^r is a curve, we obtain:

Theorem 4. *The genus h of W_d^r is*

$$h = 1 + \frac{g-d+r}{g-d+2r+1}(r+1) \prod_{i=0}^{r} \frac{i!}{(g-d+r+i)!} \cdot g!$$

This result, which follows from results of Harris and Tu [1984] generalizes a result of Kempf [1985].

Theorem 1 in the special case $r=1$ is proved by Sects. 3–6 of Harris and Mumford [1982], whence the restriction there to odd $g \geqq 25$. The idea of using the divisor E_d^1 as in Theorem 2 is already enunciated in Harris [1984], and part i) of Theorem 2 is stated without proof there; but part ii) was not proved at that time, so that a less efficient class of divisors was used, leading to the bound $g \geqq 40$ instead of $g \geqq 28$.

The necessary basis of the computation of the divisors D_s^r and E_d^1 is a Brill-Noether theorem that goes beyond simply stating that general curves are outside such loci; one needs actually to know specific families of curves outside the loci, and to have some grip on the linear series on these curves and their possible ramification indices. Such a "strong Brill-Noether Theorem" is the object of Sect. 1; a large family of (mostly reducible) pointed curves, including those in $i(P_g)$ and $j(\bar{\mathscr{M}}_{2,1} - W)$ is given which behave like general curves from the point of view of dimensions of families of special divisors and existence of such divisors with prescribed ramification. We take this as an opportunity to briefly review the mechanism of limit series, introduced in our [1986], which will be used extensively here.

In contrast to the results presented above, it is presently known that $\mathscr{M}_g$ is unirational (and in some cases rational or stably rational) for $g \leqq 13$, while the range $14 \leqq g \leqq 22$ remains completely mysterious [Igusa (1960), Kollar and Schreyer (1984), Sernesi (1981), Chang and Ran (1984)]. Shepherd-Barron (unpublished) has recently proved that $\mathscr{M}_6$ is rational.)

A technical point deserves note: since we are only concerned with relations of given classes in $\mathrm{Pic}\,\bar{\mathscr{M}}_g \otimes \mathbb{Q}$, we may and will work throughout with the corresponding classes in $\mathrm{Pic}_{fun}(\bar{\mathscr{M}}_g)$; that is essentially, we will usually check them by examining relations that hold for various families of curves. A discussion of the relationship of these Picard groups may be found in Harris and Mumford [1982], p. 50ff. Note also that we will generally use the same notation for a divisor on $\mathscr{M}_g$, its pull back to a given family of curves, and the classes these represent, when this causes no confusion.

We are grateful to Richard Stanley for discussions leading to the Schubert calculus calculations (concealed) in the proof of Proposition (1.2).

1. Basic ideas and Brill-Noether Theorems

Central to this paper is what might be called Brill-Noether theory: that is, the theory of what linear series exist, and with what kind of ramification, on general curves. Most of our work will be done with families of reducible curves, and we therefore need to know something about which of these "special" curves are "sufficiently general." The results that we need are somewhat stronger than those in the literature, so we provide brief proofs.

We begin with some terminology for notions which will play a fundamental role in this paper. References are our [(1983a), (1983b), and (1986)].

Curves in this paper will always be complex, complete, reduced, connected, and will have at most ordinary nodes as singularities. If C is a curve, the *genus* $g(C)$ will be the arithmetic genus.

The *dual graph* of a curve consists of one vertex for each irreducible component, one edge for each node. A curve is *tree-like* if, after deleting edges leading from a node to itself the dual graph becomes a tree; it is of *compact type* if the dual graph actually is a tree.

A linear series of dimension r and degree d, or $\mathfrak{g}^r_d$, on a irreducible curve C is a pair $L=(\mathscr{L}, V)$ where $\mathscr{L}$ is a line bundle of degree d and $V \subset H^0(\mathscr{L})$ is a subspace of vectorspace dimension $r+1$. If we wish to allow $\mathscr{L}$ to be a torsion-free sheaf we will speak of *generalized linear series*.

If $p \in C$ is a smooth point then we define the *vanishing sequence*

$$a^L(p): 0 \leqq a^L_0(p) < \ldots < a^L_r(p) \leqq d$$

to be the sequence of distinct orders of vanishing of sections in V at p, and we define the *ramification sequence*

$$\alpha^L(p): 0 \leqq \alpha^L_0(p) \leqq \ldots \leqq \alpha^L_p(p) \leqq d-r$$

by $\alpha^L_i(p) = a^L_i(p) - i$.

The notion of linear series makes sense but is not so useful for reducible curves. Instead, if C is a tree-like curve we define a *crude limit* $\mathfrak{g}^r_d$ on C to be a collection

$$L = \{L_Y \text{ a } \mathfrak{g}^r_d \text{ on } Y\}_{Y \text{ a component of } C}$$

satisfying the *compatibility condition*:

If components Y and Z of C meet at p then

$$\alpha^{L_Y}_i(p) + \alpha^{L_Z}_{r-i}(p) \geqq d-r;$$

if equality holds everywhere, we say L is *refined* or simply that L is a *limit* $\mathfrak{g}^r_d$, or *limit series*. We apply the adjective *generalized* if we wish to allow the L_Y to be generalized linear series. If $D \subset C$ is a (connected) union of components of C, then the *D-aspect* of L will be the collection $L_D = \{L_Y\}_{Y \subset D}$.

If q is a smooth point of a tree-like curve C, lying on a component Y, say, and if L is a (generalized, crude) limit $\mathfrak{g}^r_d$ on C, then we set $\alpha^L(p) = \alpha^{L_Y}(p)$.

If C is a treelike curve, $q_1, \ldots, q_n \in C$ are smooth points, and $\alpha^1, \ldots, \alpha^n$ are Schubert indices of type r, d – that is, sequences of integers

$$\alpha^i: 0 \leqq \alpha^i_0 \leqq \ldots \leqq \alpha^i_r \leqq d-r,$$

– then we define

$$G_d^r(C, (q_1, \alpha^1), \ldots, (q_n, \alpha^n))$$

to be the scheme of all generalized limit series L on C satisfying

$$\alpha_j^L(q_i) \geqq \alpha_j^i .$$

If $g(C) = g$ then the "expected" dimension of this scheme is

$$\varrho(g, r, d, \alpha^1, \ldots, \alpha^n) := (r+1)(d-r) - rg - \sum_{i,j} \alpha_j^i .$$

These definitions may be extended to certain families of curves with sections, and we will sometimes use them in the extended form; see our [1986].

If L is a generalized $\mathfrak{g}_d^r$ on C, it will be convenient to set

$$\varrho(L, q_1, \ldots, q_n) := \varrho(g, r, d, \alpha^L(q_1), \ldots, \alpha^L(q_n)) .$$

A convenient feature of this definition is the *additivity of* ϱ: If L is a crude limit series on a curve $C = \bigcup_i Y_i$ of compact type, $\{p_{ij}\}$ are the nodes of C on Y_i, then

$$\varrho(L) \leqq \sum_i \varrho(L_{Y_i}, \{p_{i,j}\}) ,$$

with equality iff L is a limit series. This follows from the Plücker formula [our (1983a), Sect. 1].

Let L_Y be a generalized $\mathfrak{g}_d^r$ on an irreducible curve Y, and let $p_1, \ldots, p_n$ be smooth points of Y. We say that L_Y is *dimensionally proper with respect to* $p_1, \ldots, p_n$ if, letting

$$\pi : \tilde{Y} \to B ,$$
$$\tilde{p}_i : B \to \tilde{Y}$$

be the versal deformation of $(Y, p_1, \ldots, p_n)$, we have

$$\dim G_d^r(\tilde{Y}/B, \{(\tilde{p}_i, \alpha^{L_Y}(p_i))\}_{1,\ldots,n}) = \dim B + \varrho(L_Y, p_1, \ldots, p_n) .$$

A crude generalized limit series is dimensionally proper if each of its aspects is.

Limit series do arise as limits: if C_t is a 1-parameter family of smooth (or more generally tree-like) curves degenerating in a flat proper family to the tree-like curve C_0, and if L_t is a family of (generalized crude limit) $\mathfrak{g}_d^r$'s on C_t, for $t \neq 0$, then there is a well defined generalized crude limit $\mathfrak{g}_d^r$ on C_0 which arises as the limit of the L_t; see our [1986].

Conversely, our smoothing result, Corollary 3.7 of [1986], says that if a limit $\mathfrak{g}_d^r L$ on a curve C of compact type is dimensionally proper with respect to smooth points $p_1, \ldots, p_n \in C$, then it can be "smoothed" to a family of (ordinary) $\mathfrak{g}_d^r$'s on a family B of n-pointed smooth curves near $(C, p_1, \ldots, p_n)$, maintaining ramification at the n points exactly like that of L at the p_i, the family B having codimension $-\varrho(L, p_1, \ldots, p_n)$ in $\mathcal{M}_{g,n}$.

Together, these results say that one can find the closure, in the family of tree-like curves, of a divisor such as D_s^r or E_d^1 on $\mathcal{M}_g$, by considering the existence or properties of limit series on tree-like curves; this is what will be done in later sections.

We can now state our results on existence and dimensions of families of limit series with specified ramification conditions on sufficiently general tree-like curves.

(1.1) **Theorem.** *Let C be a tree-like curve and suppose that each irreducible component Y, and the points $p_1, \ldots, p_s \in Y$ where Y meets other components of C satisfy:*

a) *if $g(Y)=1$ then $s=1$;*

b) *if $g(Y)=2$, then $s=1$ and p_1 is not a Weierstrass point of Y;*

c) *if $g(Y)\geqq 3$, then $(Y, p_1, p_2, \ldots, p_s)$ is a general s-pointed curve.*

If $q_1, \ldots, q_t \in C$ are general points, or arbitrary smooth points of C on smooth rational components then, for any ramification sequences α^i,

$$\dim G_d^r(C, (q_1, \alpha^1), \ldots, (q_t, \alpha^t)) = \varrho(g(C), r, d, \alpha^1, \ldots, \alpha^t),$$

and the generalized series which are not ordinary series form a subvariety of strictly smaller dimension. In particular, $\varrho(L)\geqq 0$ for any generalized crude limit series on C.

Remarks. 1) Since conditions a), b), c) are open, this theorem and our smoothing result show that any limit series on a tree-like curve as in the theorem can be smoothed, preserving the ramification at $q_1, \ldots, q_t$.

2) If (C, p) is a general pointed curve (say smooth, or somewhat more generally) then the techniques of our (1983a) [or indeed of the original paper of Griffiths and Harris (1980)] can be used to show that $G_d^r(C, (p, \alpha))$ is always equidimensional and reduced [the proof given in our (1983a) can be simplified by replacing Sects. 7, 8 of that paper with the results and ideas of our (1987c)]. This result will be used in Sect. 5a.

3) Note that if C is irreducible and (C, q_1) satisfies the condition on Y, p_1 in 1), 2), 3), then the theorem implies the corresponding statement for (C, q_1); for we can simply add a smooth rational component through q_1 and apply the theorem to a point q_1' on it. In any case the proof to be given includes a direct proof of this consequence.

Proof. Letting any q_i which are initially located on components of genus >0 degenerate to the node lying on that component, or first attaching a rational curve and then making such a degeneration, we may reduce to the case where all the q_i are located on rational components.

We may stratify $G_d^r(C, \{(q_i, \alpha^i)\})$ by the ramification conditions satisfied by the aspects at nodes where distinct components meet, and reduce in this way to checking the result one component Y at a time.

If $g(Y)=0$, the result follows easily from the "Plücker formula"; see our (1983a), Theorem 2.3.

If $g(Y)=1$ or 2 we need prove the result only with respect to a single smooth point q $(=p_1)\in Y$. If the underlying torsion free sheaf $\mathscr{L}$ of a generalized crude limit series L is not locally free at a node x of Y, we may "partially normalize" Y at that node; this raises the degree of L by 1 and lowers the genus of Y by 1, so ϱ drops and we are done by induction. Thus such series form a lower dimensional subscheme, and we may assume that $\mathscr{L}$ is locally free.

Dropping the section in L that vanishes to lowest order at p and removing base points at p alternately, we may reduce to the case where L is special and has no base point at p. This makes the cases $g(Y) \leqq 2$ trivial.

Finally, if $g(Y) > 2$ we let Y degenerate to a curve of compact type consisting of a tree of rational curves with elliptic tails attached, and the result follows from the case above of curves whose components have genus $\leqq 1$; cf. our (1983b).

Theorem (1.1) shows for example that on a general curve of genus g the ramification of a $\mathfrak{g}_d^r$ L at general point p must satisfy

$$\varrho(\mathrm{g}, \mathrm{r}, \mathrm{d}) \geqq \mathrm{w}^L(p) := \sum_i \alpha_i^L(p),$$

but easy examples show that this condition is not sufficient. However, by using limit series it is not hard to give a necessary and sufficient condition. For any integer n we write $(n)_+ = \max(0, n)$ for the *positive part* of n.

(1.2) **Proposition.** *A general pointed curve (C, q) of genus g possesses a $\mathfrak{g}_d^r$ with ramification sequence $\alpha = (\alpha_0, \ldots, \alpha_r)$ at q iff*

$$\sum_0^r (\alpha_i + g - d + r)_+ \leqq g.$$

Remarks. 1) One checks that the condition $\varrho(g, r, d) \geqq w(\alpha) := \sum \alpha_i$ may be expressed as

$$\sum_0^r (\alpha_i + g - d + r) \leqq g.$$

In particular, if $\varrho(g, r, d, \alpha) = 0$ then the condition of the Proposition becomes $\alpha_0 + g - d + r \geqq 0$.

2) The proposition can easily be adapted to the case of many points p_i and ramification conditions α^i; it becomes the existence of an α as in the proposition such that the Schubert cycle $\sigma_\alpha \supset \Pi \sigma_{\alpha^i}$ in the sense of our (1987c). See that paper for related ideas.

Before the proof, we need one more piece of notation. There is a significant relation between ramification indices of $\mathfrak{g}_d^r$'s and Schubert cycles of $r+1$-planes in $d+1$-space, which will be exploited in the proof. Our notation for the indices of the Schubert cycles is the same as for the ramification indices, which we sometimes refer to as Schubert indices. If

$$\alpha : 0 \leqq \alpha_0 \leqq \ldots \leqq \alpha_r \leqq d - r,$$

then we write

$$\sigma_a \subset \mathrm{Grass}(r+1, d+1)$$

for the variety of $r+1$-planes meeting the $d-i+1-\alpha_i$ plane of a fixed flag in dimension $\geqq r-i+1$, for $i = 0, \ldots, r$; this notation is that of Griffiths and Harris (1978), for example, with the order of the α_i reversed.

Proof. It is enough to show that the Proposition holds for limit series on an arbitrary pointed curve (C_0, q) where C_0 is a curve of compact type consisting of a tree of rational curves with g elliptic tails attached, and q is a smooth point of C lying on some rational component; indeed, since every such series is dimensionally proper by Theorem (1.1), it can be smoothed by our (1986) Corollary 3.7 to a nearby general pointed curve, so this shows the sufficiency of the condition, while the necessity follows by letting the general curve degenerate to (C_0, q).

The existence of a limit $\mathfrak{g}_d^r$ on (C_0, q) with the desired ramification may be handled by a Schubert calculus argument: reasoning essentially as in our (1983b) we must show that the given condition is equivalent to the condition

$$\sigma_\alpha \cdot (\sigma_{0, \underbrace{1, \ldots, 1}_{r}})^g \neq 0$$

in the cohomology ring of the Grassmann variety of $r+1$-planes in $d+1$-space. This last condition is equivalent to the condition that there exist $\alpha' \geqq \alpha$ (termwise order) with $w(\alpha') = (r+1)(d-r) - rg$ such that

$$\sigma_{\alpha'} \cdot (\sigma_{0,1,\ldots,1})^g \neq 0.$$

Using induction on g with the "Littlewood-Richardson rule" [Fulton (1984), p. 265] one sees that this final condition is equivalent to $\alpha'_0 \geqq d-r-g$, so such an α' exists if and only if the weight of the index

$$\alpha'' = (\ldots, (\alpha_i - (g-d+r))_+ + g - d + r, \ldots),$$

is $\leqq (r+1)(d-r) - rg$, a condition equivalent to the one given in the Proposition. [The combinatorial part of this is a special case of the "Snapper-Liebler-Vitali-Lam Theorem" – see Hazewinkel and Martin (1983).]

We finish this section with a special result of Brill-Noether type, which will be used below in Sect. 5c.

(1.3) **Proposition.** *Let (C, p) be a general pointed curve of genus $g-1$ and let $\bar{C} = C/q \sim p$ be the nodal curve obtained by identifying p with an arbitrary point q on C. If $g = 2k$, so that $\varrho(g, 1, k+1) = 0$, then $\bar{C}$ possess only finitely many $\mathfrak{g}_{k+1}^1$'s.*

Proof. By Theorem (1.1) every $\mathfrak{g}_{k+1}^1$ on C is base-point free. We must show that C has only a finite number of $\mathfrak{g}_{k+1}^1$'s with p and q in the same fiber of the map to $\mathbb{P}^1$. Suppose, on the contrary, that for each p in some open set of C, there were points $q_1^{(p)}, \ldots, q_m^{(p)}$ which were in the same fiber as p with respect to infinitely many $\mathfrak{g}_{k+1}^1$'s in some component of $G_{k+1}^1(C)$. Since every $\mathfrak{g}_{k+1}^1$ on C is complete, and is thus determined by its fiber through p, this subset is a proper subset of the fiber and varies continuously with general p. If we let $\pi: C \to \mathbb{P}^1$ be a fixed general $\mathfrak{g}_{k+1}^1$, and G the monodromy group of the fiber, it follows that $\{q_1(p), \ldots, q_m(p)\}$ is preserved by the subgroup of G stabilizing p.

Of course every curve of genus g is a $k+1$-sheeted covering of $\mathbb{P}^1$, ramified over a branch divisor of degree $6k$; the general such covering, and thus π, will have distinct branch points, over each of which is just one ramification point, which is simple. It follows that the monodromy of π is generated by transpositions. Since C is irreducible, the monodromy is also transitive, and is thus the full symmetric group, contradicting the existence of the points $q_1(p), \ldots, q_m(p)$ above. □

Remark. The natural generalization of this result seems to be to say that the family of $\mathfrak{g}_d^r$'s on $\bar{C}$ is of lower dimension than the family of $\mathfrak{g}_d^r$'s on C in the case where the latter family is positive dimensional. This is of course far from giving the Brill-Noether estimate if $r>1$. Also, even in the case where $r=1$, if $\varrho(g,1,d)=-1$, there will clearly be points q for which at least one $\mathfrak{g}_d^1$ from C descends to $\bar{C}$.

2. Divisors on $\mathcal{M}_g$ that restrict to the locus of Weierstrass points in $\bar{\mathcal{M}}_{2,1}$

Consider the map $j:\bar{\mathcal{M}}_{2,1}\to\bar{\mathcal{M}}_g$ as in the introduction.

Theorem 2.1. *Let D be a divisor on $\bar{\mathcal{M}}_g$, and suppose*

$$D\sim a\lambda-\sum_{i=0}^{[g/2]} b_i\delta_i.$$

If j^*D is supported on W, then

$$a=5b_1-2b_2,$$

$$b_0=\frac{b_1}{2}-\frac{b_2}{6}.$$

Further, if we write $j^*D=qW$ for some (rational) number q, then $b_2=3q$.

The theorem will be deduced from a computation of W in terms of the following classes:

δ_0,δ_1: These are the classes of the divisors whose generic points are given in Fig. 2.1a, b (whose notation we shall use throughout this section). They are of course pullbacks of the classes of the same name in $\bar{\mathcal{M}}_2$.

a) generic point of δ_0 b) generic point of δ_1

E, E_1, E_2 elliptic curves

Fig. 2.1

ω: This is the class obtained, in any family $\mathscr{C}\xrightarrow{\pi}B$ of stable genus 2 curves with section $\sigma:B\to\mathscr{C}$ whose image is in the smooth locus, as

$$\omega=\sigma^*\omega_{\mathscr{C}/B}$$

where $\omega_{\mathscr{C}/B}$ is the relative dualizing sheaf.

These classes form a basis of $(\mathrm{Pic}\,\bar{\mathcal{M}}_{2,1})\otimes\mathbb{Q}$, but we shall not need this fact.

Another class that naturally enters our computation is the class λ obtained (in the setting above) as $\lambda := c_1(\pi_* \omega_{\mathscr{C}/B}) = \Lambda^2 \pi_* \omega_{\mathscr{C}/B}$, the "Hodge bundle". Since λ is also pulled back from $\bar{\mathscr{M}}_2$ we have, as in $\bar{\mathscr{M}}_2$ [see for example Mumford (1983)]

$$\lambda = \tfrac{1}{10}\delta_0 + \tfrac{1}{5}\delta_1 .$$

We have:

(2.2) **Theorem.** *W irreducible, and its class in $\bar{\mathscr{M}}_{2,1}$ is given by*

$$W = 3\omega - \lambda - \delta_1 = 3\omega - \tfrac{1}{10}\delta_0 - \tfrac{6}{5}\delta_1 .$$

Proof of Theorem (2.2). W is defined as the closure of $W \cap \mathscr{M}_{2,1}$, the locus of Weierstrass points on smooth genus 2 curves. By the usual construction of curves of genus 2, the monodromy of $W \to \mathscr{M}_{2,1}$ is transitive, and thus W is irreducible.

A smooth point of a smooth curve is a Weierstrass point if it is a ramification point of the canonical series. On the other hand, our theory of limit canonical series shows that a smooth point s on a reducible curve $D = E_1 \cup_p E_2$, $s \in E_2 - \{p\}$ as in Fig. 2.1b), is limit of Weierstrass points on nearby smooth curves if and only if s is a ramification point of the series $\omega_D|_{E_2}(p)$; that is, if s is a ramification point of the E_2-aspect of the (unique) limit canonical series on D [see our (1987b), Sect. 2, 3].

It is enough to prove the relation on divisor classes given in the theorem after restricting to families

$$\pi : \mathscr{C} \to B , \qquad \sigma : B \to \mathscr{C}$$

of stable pointed genus 2 curves for which B is a complete smooth curve. Further, we may harmlessly assume that B avoids any codimension 2 phenomena in $\bar{\mathscr{M}}_{2,1}$ which would be inconvenient. Thus we may assume that all the singular fibers C_b of $\mathscr{C}/B$ are of the forms given in Fig. 2.1.

The condition that $\sigma(b)$ be a Weierstrass point on C_b, for any fiber, may be expressed as a degeneracy condition on the matrix giving the Taylor expansion of the sections in the canonical (or aspect of the limit canonical) series. These matrices fit together into a map of bundles over B defined as follows:

Let $\omega_{\mathrm{lim}} = \omega_{\mathscr{C}/B}\left(-\sum_b E_{2,b}\right)$ where the sum runs over $b \in \delta_1|_B$, that is points of B whose fibers are as in Fig. 2.1b), and $E_{2,b}$ is the component of C_b containing $\sigma(b)$, so that if $E_{1,b}$ is the other component of such a curve,

$$\omega_{\mathrm{lim}}|_{C_b} = \omega_{C_b} \quad \text{if } C_b \text{ is irreducible}$$

while if $b \in \delta_1|_B$,

$$\omega_{\mathrm{lim}}|_{C_b}|_{E_{2,b}} = \mathscr{O}_{E_{2,b}}(2p) ,$$

the E_2-aspect of the limit canonical series, and

$$\omega_{\mathrm{lim}}|_{C_b}|_{E_{1,b}} = \mathscr{O}_{E_{1,b}} .$$

Let

$$\mathscr{E} = \pi_* \omega_{\mathrm{lim}} ;$$

since $h^0(\omega_{\mathrm{lim}}|_{C_b}) = 2$ for all b, $\mathscr{E}$ is a rank 2 vectorbundle on B.

Next, let $\Sigma=\sigma(B)$, be the section, $\mathscr{I}$ its ideal sheaf, and

$$\mathscr{F}=\pi_*(\omega_{\lim}\otimes\mathcal{O}_C/\mathscr{I}^2).$$

It is easy to see that $\mathscr{F}$ is also a vector bundle of rank 2, and in fact we have an exact sequence

$$0\to\sigma^*\omega_{\lim}\otimes\omega\to\mathscr{F}\to\sigma^*\omega_{\lim}\to 0,$$

where we have written for the line bundle $\sigma^*(\mathscr{I}/\mathscr{I}^2)=\sigma^*(\omega_{\mathscr{C}/B})$. We have a natural map

$$\omega_{\lim}\to\omega_{\lim}\otimes\mathcal{O}_{\mathscr{C}}/\mathscr{I}^2$$

which induces an "evaluation map"

$$\phi:\mathscr{E}\to\mathscr{F},$$

whose degeneracy locus is W. Thus $W=c_1\mathscr{F}-c_1\mathscr{E}$, and it remains to compute $c_1\mathscr{E}$ and $c_1\mathscr{F}$.

For $\mathscr{F}$ this is immediate: we obviously have $\sigma^*\omega_{\lim}=\omega(-\delta_1)$, so from the exact sequence above

$$c_1(\mathscr{F})=3\omega-2\delta_1.$$

To evaluate $c_1\mathscr{E}$ we use the sequence

$$0\to\omega_{\lim}\to\omega_{\mathscr{C}/B}\to\sum{}^{u}{}_{\mathscr{C}/B}|_{E_{2,b}}\to 0,$$

which pushes forward to an exact sequence

$$0\to\mathscr{E}\to\pi_*\omega_{\mathscr{C}/B}\to\mathcal{O}_{\delta_1 B},$$

since with notation as in Fig. 2.1

$$\omega_{\mathscr{C}/B}|_{E_{2,b}}=\mathcal{O}_{E_{2,b}}(p)$$

and

$$H^0(\mathcal{O}_{E_{2,b}}(p))\cong\mathbb{C}.$$

We claim that for each $b\in\delta_1|_B$, the map

$$\pi_*\omega_{\mathscr{C}/B}\to H^0(\omega_{\mathscr{C}/B}|_{E_{2,b}})=\mathbb{C}$$

is onto – that is that a nonzero section of $\omega_{\mathscr{C}/B}|_{E_{2,b}}$ extends to a neighborhood of C_b in $\mathscr{C}$. Indeed, this is clear, since $\pi_*\omega_{\mathscr{C}/B}$ is a vectorbundle with fiber over b

$$H^0(W_{\mathscr{C}/B}|_{E_{1,b}})\oplus H^0(W_{\mathscr{C}/B}|_{E_{2,b}}).$$

Thus we get an exact sequence

$$0\to\mathscr{E}\to\pi_*\omega_{\mathscr{C}/B}\to\mathcal{O}_{\delta_1}|_B\to 0,$$

and $c_1\mathscr{E}=\lambda-\delta_1$, whence the desired relation

$$W=c_1\mathscr{F}-c_1\mathscr{E}=3\omega-\lambda-\delta_1. \quad \square$$

Proof of Theorem (2.1). We must compute the pull-backs to $\bar{\mathcal{M}}_{2,1}$ of the divisors λ, δ_i. Evidently

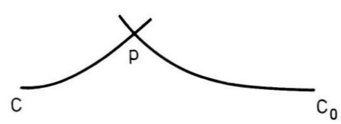
$$j^*\delta_0 = \delta_0,$$

$$j^*\delta_1 = \delta_1,$$

and

$$j^*\delta_i = 0 \quad \text{for} \quad i \geqq 3.$$

Further, if $C \cup_p C_0$ is a curve of compact type



then $H^0(\omega_{C \cup C_0})$ is naturally the direct sum $H^0(\omega_C) \oplus H^0(\omega_{C_0})$, so

$$j^*\lambda = \lambda = \tfrac{1}{10}\delta_0 + \tfrac{1}{5}\delta_1.$$

Finally, consider $j^*(\delta_2)$. On any family $\pi: \mathscr{C} \to B$, $\sigma: B \to \mathscr{C}$ of stable pointed genus 2 curves $j^*\delta_2|_B$ is the pull back of the normal bundle $\mathcal{O}_{\delta_2}(\delta_2)$, which may be identified with the normal bundle of the section σ. By the adjunction formula, we thus get

$$j^*(\delta_2) = -\omega.$$

Applying Theorem (2.2) and these formulas we get the desired result. □

3. Divisors on $\mathcal{M}_g$ that miss P_g

Consider the map $i: P_g \to \bar{\mathcal{M}}_g$ as in the introduction.

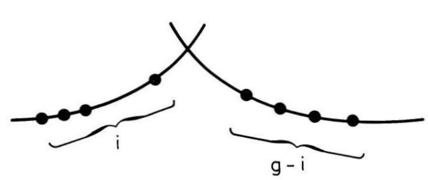
(3.1) **Theorem.** *Let D be a divisor on $\bar{\mathcal{M}}_g$, and set*

$$D = a\lambda - \sum_{0}^{[g/2]} b_i \delta_i.$$

If $i^*D = 0$ then

$$b_i = \frac{i(g-i)}{g-1} b_1 \quad \textit{for} \quad i = 2, \ldots, [g/2].$$

Proof. We must find relations on the classes $i^*\lambda$ and $i^*\delta_i$. We will write them in terms of the following classes ε_i. For $i = 2, \ldots, [g/2]$, we take ε_i to be the class of the divisor which is the closure in P_g of the set of 2-component curves with exactly i of the g marked points on one of the components:



These divisors are contracted to distinct lower-dimensional subvarieties under the birational map $P_g \to \mathbb{P}^{g-3}$, so they are independent.

On any family $\pi: \mathscr{C} \to B$ of curves of genus g formed by attaching fixed elliptic tails to curves in P_g at the marked points the vector bundle $\pi_* \omega_{\mathscr{C}/B}$ is trivial, so $i^*\lambda = 0$.

Since $i(P_g)$ misses δ_0 we have $i^*\delta_0 = 0$. On the other hand, for $i \geqq 2$ we have $i^*\delta_i = \varepsilon_i$, so it remains to compute $i^*\delta_1$.

We will show that

$$i^*\delta_1 = -\sum_{i=2}^{[g/2]} \frac{i(g-i)}{(g-1)} \varepsilon_i,$$

from which Theorem (3.1) follows easily.

To do this it is enough to check the equation after restricting to families

$$\pi: \mathscr{C} \to B,$$

$$\sigma_1, \ldots, \sigma_g : B \to \mathscr{C}$$

of stable rational g-pointed curves, where B is a smooth curve missing any inconvenient codimension-2 loci in P_g, and transverse to relevant codimension-1 loci in P_g, so that $\mathscr{C}$ is a smooth surface. Let $\mathscr{C}' \to B$ be the family obtained by attaching a copy of $B \times E_i$ along σ_i and $B \times p_i$ for some fixed pointed elliptic curves (E_i, p_i). The family $\mathscr{C}' \to B$ lies in the g-fold locus of the divisor with normal crossings δ_1, and $i^*\delta_1$ is thus the sum of the pull-backs of the normal bundles to the branches.

The branch of δ_1 at a fiber C'_b of $\mathscr{C}'$ corresponding to the node $\delta_i(b)$ has normal bundle at C'_b equal to $T_{\sigma_i(b), C_b} \otimes T_{p_i, E}$. Thus it pulls back to the bundle on B which is the normal bundle to the section $\sigma_i(B)$; we may rewrite this as $\pi_*(\sigma_i(B))^2$, and we have

$$i^*\delta_1 = \pi_* \sum \sigma_i(B)^2 .$$

Avoiding a codimension 2 locus of P_g, we may assume that all the fibers of $\mathscr{C}$ have $\leqq$ two components. Of course the general fiber of $\mathscr{C}$ is smooth, and we may contract the component of each fiber meeting the smaller number of sections (either component if both components meet $g/2$ sections) to obtain a $\mathbb{P}^1$-bundle $\bar{\pi}: \bar{\mathscr{C}} \to B$ with g sections $\bar{\sigma}_i : B \to \bar{\mathscr{C}}$. These sections meet transversely in groups of i over points of ε_i, and are otherwise disjoint. Clearly $\bar{\pi}_* \sum \bar{\sigma}_i(B)^2 = \pi_* \sum \sigma_i(B)^2 + \sum_i^{[g/2]} i\varepsilon_i$.

On any $\mathbb{P}^1$-bundle the difference of two sections is a linear combination of fibers, and thus has self-intersection 0, so

$$\sigma_i(B)^2 + \sigma_j(B)^2 = 2\bar{\sigma}_i(B) \cdot \bar{\sigma}_j(B) .$$

Summing over $i<j$ we get

$$\begin{aligned}(g-1)\bar{\pi}_* \sum_1^g \bar{\sigma}_i(B)^2 &= 2\bar{\pi}_* \sum_{i<j} \bar{\sigma}_i(B) \cdot \bar{\sigma}_j(B) \\ &= \sum_{i=2}^{[g/2]} i(i-1)\varepsilon_i .\end{aligned}$$

Putting this together we have

$$i^*\delta_1 = \sum_{2}^{[g/2]} \frac{i(i-1)}{g-1}\varepsilon_i - \sum i\varepsilon_i$$
$$= -\sum_{2}^{[g/2]} \frac{i(g-i)}{g-1}\varepsilon_i,$$

as required. □

4. The case where $g+1$ is composite: the divisors D^r_s

In this section we will prove Theorem 1 and Proposition 3. We begin with:

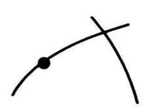
(4.1) **Proposition.** *If* $g=(r+1)(s-1)$, $s \geqq 3$, *then the divisor* $D^r_s \subset \bar{\mathcal{M}}_g$ *does not meet either* $i(P_g)$ *or* $j(\bar{\mathcal{M}}_{2,1}-W)$.

The conclusions of Theorem 1, except for the positivity of c, follow from this by Theorems (2.1) and (3.1).

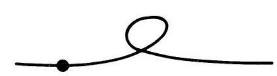
Proof. The tree-like curves in D^r_s are limits of smooth curves possessing certain linear series with negative ϱ, so they all possess generalized crude limit series with negative ϱ.

The curves in $i(P_g)$ are all of compact type, so tree-like, and possess no series with negative ϱ, whence the first statement.

For the same reason, D^r_s cannot contain any tree-like curve in $j(\bar{\mathcal{M}}_{2,1}-W)$. But the generic points of the boundary components of $\bar{\mathcal{M}}_{2,1}$ are



and



where all components are geometrically elliptic curves, and these are tree like curves. Thus the locus of non-tree-like curves is of codimension >1 in $j(\bar{\mathcal{M}}_{2,1})$, and the intersection of $j(\bar{\mathcal{M}}_{2,1}-W)$ with a divisor, were it nonempty, could not consist only of non-tree-like curves. □

To complete the proof of Theorem 1 we will show that if (Y,p) is a curve of genus 2 with Weierstrass point p and (Z,p) is a general pointed curve of genus $g-2$ with $g=(r+1)(s-1)-1$, then the curve $C=Y\cup_p Z$ possesses a smoothable limit $\mathfrak{g}^r_{rs-1}$ which extends to a codimension 1 family of nearby smooth curves. This will show that $C \in D^r_s$, so $j^*D^r_s>0$; by Theorem (2.1) $c\cdot[2(g-2)|>0$, whence $c>0$ as required.

We will construct the desired limit series L aspect by aspect; leaving the easy case $r=1$, $s=3$ to the reader, we may assume $r\geqq 2$ or $s\geqq 4$, so that $rs-r-3\geqq 0$.

On Y we take the aspect L_Y to be $|(r+2)p|+(rs-r-3)p$; that is, $L_Y=(\mathcal{L}_Y, V_Y)$ with $\mathcal{L}_Y=\mathcal{O}_Y((rs-1)p)$ and V_Y is the image of the natural map

$$H^0(\mathcal{O}_Y((r+2)p))\to H^0(\mathcal{L}_Y).$$

One computes easily that

$$\alpha^{L_Y}(p)=(rs-r-3,\dots,rs-r-3,\ rs-r-2,\ rs-r-1),$$
$$\varrho(L_Y,p)=-1,$$

and L_Y is dimensionally proper with respect to p.

On Z: By the Brill-Noether theory of Sect. 1 there are finitely many (dimensionally proper) $\mathfrak{g}^r_{rs-1}$'s on Z with ramification sequence $(0,1,2,\dots,2)$ at p. We may take any of these to be L_Z.

We have $\varrho(L)=-1$ by additivity. Since both the aspects of L constructed above are dimensionally proper, L smooths to a codimension 1 family by our (1986) Corollary 3.7. Thus $Y\cup_p Z=j(Y)\in D^r_s$, as required

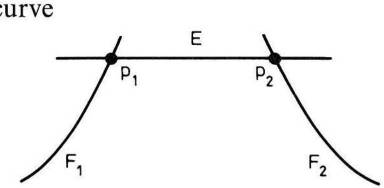
Remark. The constant c of Theorem 1 can obviously be computed from the number n such that $j^*D^r_s=nW$. Presumably n is just the number of points in

$$G^r_{rs-1}(Z,(p,(0,1,2,\dots,2))),$$

which can be computed through the Schubert calculus.

Proof of Proposition 3. We first note that $D^1_{13}\cap\mathcal{M}_{23}$, which is by definition the divisorial part of the subscheme of smooth curves of genus 23 possessing a $\mathfrak{g}^1_{12}$, actually contains all such curves because of the irreducibility of the Hurwitz scheme [Fulton (1969)]. Thus to show that $D^1_{13}\not\subset\operatorname{supp}D^2_9$ it suffices to exhibit a smooth curve possessing a $\mathfrak{g}^1_{12}$ but no $\mathfrak{g}^2_{17}$, or, by our usual smoothing result, even a curve C of compact type of genus 23 possessing a dimensionally proper limit $\mathfrak{g}^1_{12}$ but no crude limit $\mathfrak{g}^2_{17}$.

We take for C the curve



where (F_i,p_i) are general pointed curves of genus 11, E is an elliptic curve, and p_1,p_2 differ by primitive 12-torsion in $\operatorname{Pic}E$; that is, $12(p_1-p_2)\sim 0$, but $n(p_1-p_2)\nsim 0$ for $n<12$.

To exhibit the limit $\mathfrak{g}^1_{12}$ L on C aspect by aspect, take L_{F_i} to be the complete series on F_i

$$L_{F_i}=|12p_i|,$$

and take L_E to be the pencil spanned by $12p_1$ and $12p_2$ on E. One checks easily that the limit series compatibility conditions are satisfied and that the series is dimensionally proper.

On the other hand, we claim that C has no $\mathfrak{g}^2_{17}$. Indeed, suppose that M were a $\mathfrak{g}^2_{17}$. Since (F_i,p_i) are general, we have $\varrho(M_{F_i},p_i)\geqq 0$. Since $\varrho(23,2,17)=-1$ we must by additivity have $\varrho(M_E,p_1,p_2)<0$.

To estimate $\varrho(M_E,p_1,p_2)$ note that there are for dimension reasons sections σ_i in V_E vanishing on the divisors

$$a_i^{M_E}(p_1)p_1+a_{2-i}^{M_E}(p_2)p_2,$$

so

$$(*) \qquad a_i^{M_E}(p_1)+a_{2-i}^{M_E}(p_2) \leqq 17\,.$$

Summing these, and transforming to ramification indices, we get the weight estimate

$$w^{M_E}(p_1)+w^{M_E}(p_2) \leqq 45\,.$$

With $\varrho(1,2,17)=43$ this gives

$$\varrho(M_E, p_1, p_2) \geqq -2\,.$$

Further, for $\varrho(M_E, p_1, p_2)<0$ it is necessary that equality hold in $(*)$ for at least two of the three values $i=0,1,2$. Let σ_{i_1} and σ_{i_2} be the two sections thus specified, with $i_1<i_2$, say. We must have

$$(\sigma_{i_j})=a_{i_j}^{M_E}(p_1)p_1+a_{i_j}^{M_E}(p_2)p_2\,,$$

and in particular the divisors (σ_{i_j}) are distinct. Subtracting, we obtain an equation $0 \sim (\sigma_{i_1})-(\sigma_{i_2})=a(p_2-p_1)$ in $\operatorm{Pic} E$, for some integer $a \neq 0$, and our hypothesis forces $a=12$. Thus we may write

$$(**) \qquad \begin{aligned}(\sigma_{i_1})&=D+12p_2\,,\\ (\sigma_{i_2})&=D+12p_1\end{aligned}$$

for some effective divisor D of degree 5 supported on p_1 and p_2.

If $\varrho(M_E, p_1, p_2)=-2$, then the inequality $(*)$ would be an equality for all three values. If $i_3>i_1$ then the above procedure applied to σ_{i_1} and σ_{i_3} would yield the contradiction

$$\sigma_{i_3}=D+12p_1=\sigma_{i_2}\,,$$

and similarly in the other cases. Thus $\varrho(M_E, p_1, p_2)=-1$, and we must have $\varrho(F_i, p_i)=0$.

By Proposition (1.3) we must have

$$\sum_j (\alpha_j^{M_{F_i}}(p_i)-4)_+ \leqq g\,.$$

Since $\varrho(M_{F_i}, p_i)=0$ we have also

$$\sum_j (\alpha_j^{M_{F_i}}(p_i)-4)=g\,,$$

from which we deduce $\alpha_j^{M_F}(p_i) \geqq 4$ for each i,j. By the compatibility conditions, $\alpha_j^{M_E}(p_i) \leqq 11$ for each i,j, so $a_2^{M_E}(p_i) \leqq 13$. This contradicts at least one of the equations $(**)$ since the divisor D has degree $\geqq 5$, so that M cannot exist. □

5. The case where g is even: the divisors E_d^1

Let $g=2d-2$, and set

$$E_d^1=a\lambda-\sum_0^{d-1} b_i\delta_i\,.$$

In this section we will compute the coefficients a and b_i, proving Theorem 2. The result is trivial (and irrelevant) if $g \leqq 2$, so we assume $g \geqq 4$. To simplify formulas we set $k = d-1$.

The computation takes the following form:

a) We consider the pullback of E_d^1 under the map $j: \bar{\mathcal{M}}_{2,1} \to \bar{\mathcal{M}}_g$, and show that $j^* E_d^1$ is supported on the Weierstrass point locus W. From Theorem (2.1) we get

$$(5.1) \qquad \begin{aligned} a &= 5b_1 - 2b_2, \\ b_0 &= \frac{b_1}{2} - \frac{b_2}{6}. \end{aligned}$$

b) By restricting to a certain family of curves $C_j \subset P_g$ for $1 \leqq j \leqq k-1$ we obtain relations R_j giving the second differences of the b_i:

$$R_j: b_j - 2b_{j+1} + b_{j+2} = -2b_1 + b_2 + 2\frac{(2k-2)!}{k!(k-1)!} - \begin{cases} 2\dfrac{(2l)!(2k-2-2l)!}{(l+1)!l!(k-l)!(k+l-1)!} & \text{if } \; j = 2l \text{ is even} \\ 0 & \text{if } j \text{ is odd}, \end{cases}$$

(where b_{k+1} is interpreted as $= b_{k-1}$), which gives the following formulas for $b_2, \ldots, b_k$ in terms of b_1:

$$(5.2) \qquad b_2 = \frac{4k-4}{2k-1} b_1 - 2\frac{(2k-1)!}{(k-2)!(k+1)!}$$

(5.3) For $3 \leqq i \leqq k = d-1$

$$b_i = -(i-2)ib_1 + \frac{(i-1)i}{2} b_2 + (i-2)(i-1)\frac{(2k-2)!}{k!(k-1)!} - \sum_{l=1}^{[(i-2)/2]} 2(i-1-2l)\frac{(2l)!(2k-2-2l)!}{(l+1)!l!(k-l)!(k-l+1)!}.$$

The formulas (5.1)–(5.3) express all the coefficients in terms of b_1. However, because (5.2) and (5.3) are inhomogeneous, this is not enough to check the criterion given in the introduction. We thus need one further relation:

c) We consider the 1-dimensional family of curves obtained by identifying a variable point on a fixed general curve of genus $g-1$ with a fixed, general point. Restricting E_d^1 to this family, we get a relation (5.7) from which, with (5.1) and (5.2), we deduce that

$$(5.4) \qquad \begin{aligned} a &= c(6k^2 + 13k + 1), \\ b_0 &= ck(k+1), \\ b_1 &= c(2k-1)(3k+1), \\ b_2 &= 3c(k-1)(4k+1) \end{aligned}$$

where
$$c=2\frac{(2k-2)!}{(k+1)!(k-1)!};$$
this is part i) of Theorem 2.

In particular, we see that R_j gives
$$\begin{aligned}b_j-2b_{j+1}+b_{j+2} &\leqq -2b_1+b_2+2\frac{(2k-2)!}{k!(k-1)!}\\ &\leqq -12k\frac{(2k-2)!}{(k+1)!(k-1)!}\\ &<0,\end{aligned}$$
so the sequence of b_i is convex. But since b_{k+1} is taken to $be=b_{k-1}$, R_{k-1} gives $b_k>b_{k-1}$, and part ii) of Theorem 2 follows.

We now begin the proof.

a) Restriction to $\bar{\mathcal{M}}_{2,1}$

(5.5) **Proposition.** *$j^*E_d^1$ is supported on W.*

Proof. Let $\bar{\mathcal{M}}_{2,1}^0$ be the result of removing from $\bar{\mathcal{M}}_{2,1}-W$ the codimension 2 loci of reducible stable pointed curves (Y,p) with any of the following properties:

– Y is not treelike;

– Y has >2 components;

– Y is reducible and has a singular component.

– Y has two smooth components, $Y=E_1\cup_{p'}E_2$ with $p\in E_1$, say, and $3(p-p')\sim 0$ on E_1 (note that the curves of this type with $2(p-p')\sim 0$ are in W, and are thus already excluded).

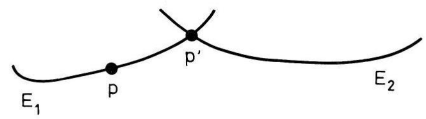
Since E_d^1 is by definition a divisor, it suffices to show that $j(\bar{\mathcal{M}}_{2,1}^0)\cap E_d^1=\emptyset$.

Let (Z,p) be a general pointed curve of genus $g-2$. For any ramification index β, the Brill-Noether theorems show that $G_d^r(Z,(p,\beta))$ is either empty or reduced and of pure dimension $\varrho(g-2,r,d,\beta)$ (see Sect. 1).

Let $Y\in\bar{\mathcal{M}}_{2,1}^0$. By Theorem (1.1) (see Remark 3 following it), if
$$\alpha: 0\leqq\alpha_0\leqq\alpha_1\leqq d-1$$
is a ramification sequence with $\varrho(2,1,d,\alpha)=0$, then every generalized crude limit $\mathfrak{g}_d^1$ on Y with ramification $\geqq\alpha$ at p is in fact a limit $\mathfrak{g}_d^1$, having ramification exactly α at p, and the number of these is finite. We will show that the number of these is independent of $(Y,p)\in\bar{\mathcal{M}}_{2,1}^0$. This implies on the one hand by the additivity of ϱ (Sect. 1) that there are only finitely many limit $\mathfrak{g}_d^1$'s on $U\cup_p Y$, so that these are all smoothable to all nearby smooth curves, and on the other hand, since the locus of $Z\cup_p Y$'s has in its closure the reducible "general curves" of our (1986), we see that the number of limit $\mathfrak{g}_d^1$'s on a $Z\cup_p Y$ is the same as for the general curve, so $Z\cup_p Y$ is not in E_d^1.

Removing α_0 base points at p and replacing d by $d'=d-\alpha_0$ we reduce to the case $\alpha_0=0$ and
$$\alpha_1=\varrho(2,1,d')=2d'-4,$$
so the limit series we are looking for have a section vanishing to order $2d'-3$ at p. Thus $2d'-3\leqq d'$, or $d'\leqq 3$.

If $d'=3$, $2d'-3=d'$, so if p lies on the component Y_1 of Y, and the Y_1-aspect of L is $(\mathscr{L}_{Y_1}, V_{Y_1})$, then $\mathscr{L}_{Y_1}=\mathcal{O}_{Y_1}(3p)$. If Y_1 is irreducible then $h^0\mathscr{L}_{Y_1}=2$, and L is the (unique) complete series $|3p|$. If on the other hand Y is reducible, then by our hypothesis (Y, p) has the form

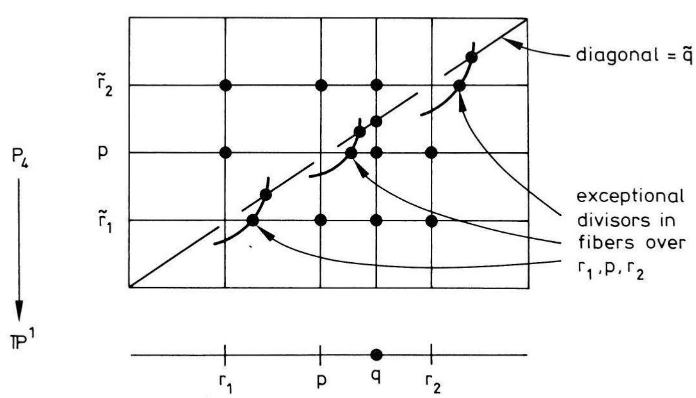


with E_1 and E_2 elliptic. Since $3(p-p')\not\sim 0$ on E_1 one checks easily that there is only one (crude) limit $\mathfrak{g}_3^1$ with a section vanishing three times at p: its E_1 aspect is the linear span of $3p$ and $2p'+p''$, where $2p'+p''\sim 3p$, and its E_2-aspect is $|2p'|+p'$. Since p is not a Weierstrass point, $2(p-p')\not\sim 0$, $p''\neq p$ and the vanishing sequence at p is $(0,2)$ as desired.

If $d'=2$ the desired statement is that there is a unique (crude) limit $\mathfrak{g}_2^1$ on a curve such as Y; this is well-known in the irreducible case and easy in the reducible one [the curve $E_1\cup_p E_2$ is "aresidually generic" in the language of our (1987b)]. This series is unramified at p, as required, precisely because p is not a Weierstrass point.

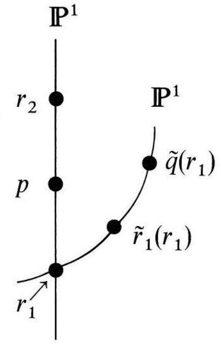
b) Restriction to the curves $C_j \subset P_g$

To construct the curves C_j for $1\leqq j\leqq g/2$ consider first the family of stable 4-pointed rational curves $P_4\to\mathbb{P}^1$, which is obtained by choosing arbitrary fixed points $r_1, r_2, p\in\mathbb{P}^1$ and blowing up $\mathbb{P}^1\times\mathbb{P}^1$ at the three points where the diagonal meets $r_1\times\mathbb{P}^1$, $r_2\times\mathbb{P}^1$, and $p\times\mathbb{P}^1$ respectively.



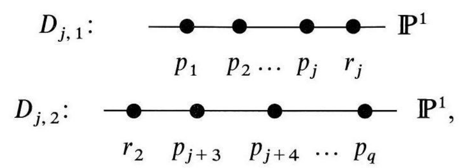
We write $\tilde{r}_i$, $\tilde{p}$ for the sections corresponding to the r_i and p, and we write $\tilde{q}$ for the strict transform of the diagonal. Thus the fiber over a $q\neq r_1, r_2, p$ is a copy of $\mathbb{P}^1$

with four marked points r_1, r_2, p, and q, while the fiber over r_1 has the form



and similarly for r_2 and p.

Let $D_{j,1}$ and $D_{j,2}$ be the pointed smooth rational curves



and let

$$\mathscr{C}_j \to B_j \cong \mathbb{P}^1$$

by the family obtained by attaching $D_{j,1} \times \mathbb{P}^1$ and $D_{j,2} \times \mathbb{P}^1$ to P_4 along $\tilde{r}_1$ and $\tilde{r}_2$ (for $j=1$ we drop $D_{j,1}$ to keep the curves stable).

We easily have $i^{-1}\delta_{j+1} = r_1 + r_2$ on B_j, and $i^{-1}\delta_2 = p$, since $i(B_j)$ meets δ_{j+1} and δ_2 transversely at the corresponding points.

On the other hand, $i(B_j)$ lies entirely in branches of $\delta_{\alpha_1}, \delta_{j+2}$ and the first chern classes of the normal bundles of these divisors pull back respectively to the normal bundles in $\mathscr{C}_j$ of the sections $\tilde{r}_1, \tilde{r}_2$, respectively, which are $-1 \in \mathbb{Z} = \operatorname{Pic} B_j$. Similarly, $i^*(\delta_1) = -2$, corresponding to the normal bundles of $\tilde{p}$ and $\tilde{q}$. Thus

$$i^*(E_d^1)|_{B_j} = 2b_1 - b_2 + b_j - 2b_{j+1} + b_{j+2},$$

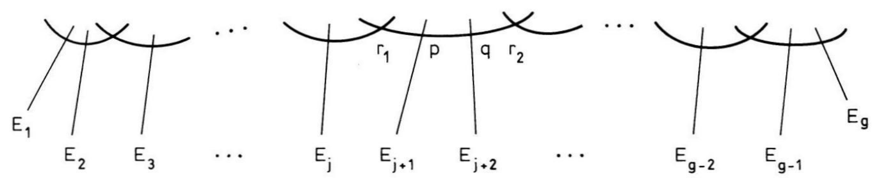
or

$$b_j - 2b_{j+1} + b_{j+2} = -2b_1 + b_2 + i^*(E_d^1)|B_j.$$

To obtain the relation R_j we need simply compute the degree of $(i^*E_d^1)|_{B_j}$, or what comes to the same thing, determine the degree of the covering $E_d^1|_{i(B_j)} \to i(B_j)$.

E_d^1 is defined as the closure of $E_d^1 \cap \mathscr{M}_g$. Since $i(B_j) \subset i(P_g)$ lies in the locus of curves of compact type, which satisfy the Brill-Noether theorem, and all the curves in $i(P_g)$ have the right number of limit $\mathfrak{g}_d^1$'s, counted with multiplicity, [our (1983a) and (1987a)], we may compute this degree as the degree of the ramification divisor of the finite covering $G_d^1(\mathscr{C}_j/B_j) \to B_j$ of B_j by the family of limit series on the fibers of $\mathscr{C}_j \to \mathrm{B}_j$.

The groundwork for this computation is laid in our (1987a), where the ramification points of the map $G_d^r(\mathscr{C}) \to B$ are computed whenever $\varrho(g,d,r)=0$ and $\mathscr{C} \to B$ is a family of curves of the form



in which q varies, the E_i are elliptic, and all the other components are rational (the replacement of

by $D_{j,1}$ as in our present context does not essentially change the argument, and similarly with $D_{j,2}$).

What is shown in (1987a), adapted to our situation, is:

1) Every (crude) limit series in $G_d^1(\mathscr{C}_j/B_j)$ is refined

2) The connected components of $G_d^1(\mathscr{C}_j/B_j)$ are specified by the $D_{j,1}$ and $D_{j,2}$ aspects of the linear series; in fact by the ramification sequences at r_1, r_2 of the aspect on the corresponding fibers of $\mathscr{C}/B$; and each connected component is irreducible, consisting either of two sheets over B with two ramification points, or just one sheet.

3) The components having two sheets are exactly those whose C_q-aspects have ramification sequences $(\alpha_0(r_i), \alpha_1(r_i))$ at r_i satisfying

$$\alpha_0(r_1)+\alpha_1(r_1)=j,$$
$$\alpha_0(r_2)+\alpha_1(r_2)=2k-j-2,$$

and

$$\alpha_0(r_2)=k-\alpha_1(r_1)-1$$
$$\alpha_1(r_2)=k-\alpha_0(r_1)-1,$$

and the condition that the series obtained by removing base points has degree $e=k+1-\alpha_0(r_1)-\alpha_0(r_2)>2$. [Our (1987a), Theorems 1.2 and 1.3.]

Thus the desired degree is simply the number of limit $\mathfrak{g}_d^1$'s on $\mathscr{C}_{j,q}$, for general q, with C_q-aspect satisfying 3.

We now count these. It will be convenient to express the quantities α_i, β_i in terms of the single number $i=\alpha_0(r_1)$. We have then

$$\alpha_0(r_1)=i,$$
$$\alpha_1(r_1)=j-i,$$
$$\alpha_0(r_2)=k-j+i-1,$$
$$\alpha_1(r_2)=k-i-1.$$

The number of $\mathfrak{g}^1_{k+1}$'s on $D_{j,1}$ with ramification $(k-\alpha_1(r_1), k-\alpha_0(r_1))$ at r_1 is by the Schubert calculus with our (1986) or by Harris and Mumford (1982) Sect. 5, Theorem B.

$$\begin{aligned}&[(k-\alpha_0(r_1))-(k-\alpha_1(r_1))+1]\cdot\frac{j!}{(\alpha_1(r_1)+1)!(j-\alpha_0(r_1))!}\\&=(j+1-2i)\frac{j!}{(j-i+1)!i!}\end{aligned}$$

and the number of $\mathfrak{g}^1_{k+1}$'s on $D_{j,2}$ with ramification $(k-\alpha_1(r_2), k-\alpha_0(r_2))$ at r_2 is similarly

$$(j+1-2i)\frac{(2k-2-j)!}{(k-i)!(k+i-j-1)!}.$$

Further, we have $e=j-2i+2>2$, so $0\leqq i<j/2$. Thus the total degree of the divisor E^1_d on our curve $\mathbb{P}^1$ is

$$2\sum_{i=0}^{(j-1)/2}(j+1-2i)^2\frac{j!(2k-2-j)!}{(j+1-i)!(k-i)!(k+i-j-1)!}.$$

Now, observing that the quantity under the summation sign is invariant under the substitution $i\rightsquigarrow j+1-i$, and zero when $i=j+1-i$, we may distinguish two cases: first, when j is odd, we may write this quantity as

$$\begin{aligned}\deg E^1_d&=\sum_{i=0}^{j+1}(j+1-2i)^2\frac{j!(2k-2-j)!}{(j+1-i)!i!(k-i)!(k+i-j-1)!}\\&=\sum_{i=0}^{j+1}\left[\binom{j}{i}\binom{2k-2-j}{k-i-1}-\binom{j}{i-1}\binom{2k-2-j}{k-i-1}\right.\\&\qquad\left.-\binom{j}{i}\binom{2k-2-j}{k-i}+\binom{j}{i-1}\binom{2k-2-j}{k-i}\right]\\&=2\left[\binom{2k-2}{k-1}-\binom{2k-2}{k-2}\right]\\&=2\frac{(2k-2)!}{k!(k-1)!}.\end{aligned}$$

On the other hand, when $j=2l$ is even, the original sum $2\sum_{i=0}^{(j-1)/2}$ lacks the term corresponding to $i=j/2=l$ that appears twice in the sum $\sum_{i=0}^{j+1}$; thus

$$\deg i^*(E^1_d)|_{B_j}=2\frac{(2k-2)!}{k!(k-1)!}-2\frac{(2l)!(2k-2-2l)!}{(l+1)!l!(k-l)!(k+l-1)!}.$$

This gives the sought-for relation R_j.

We may obtain a relation on b_1 and b_2 alone from this by forming the linear combination

$$2R_1+2R_2+\ldots+2R_{k-2}+R_{k-1};$$

indeed, this yields:

$$\begin{aligned}
&(4k-4)b_1-(2k-1)b_2\\
&\quad=(2k-3)\frac{2(2k-2)!}{k!(k-1)!}-2\sum_{l=1}^{(k-2)/2}2\frac{(2l)!(2k-2-2l)!}{(l+1)!l!(k-l)!(k-l-1)!}\\
&\quad-\begin{cases}2\dfrac{(2l)!(2k-2-2l)!}{(l+1)!l!(k-l)!(k-l-1)!} & \text{if } k=2l+1 \text{ is odd}\\ 0 & \text{if not}.\end{cases}
\end{aligned}$$

This may be more conveniently written if we express the last two terms as the sum

$$\sum_{l=1}^{k-2}2\frac{(2l)!(2k-2-2l)!}{(l+1)!l!(k-l)!(k-l-1)!}$$

and then add $4\cdot\dfrac{(2k-2)!}{k!(k-1)!}$ to this sum, extending the range of summation to between $l=0$ and $l=k-1$. We have

$$(4k-4)b_1-(2k-1)b_2=(2k-1)\frac{2(2k-2)!}{k!(k-1)!}-\sum_{l=0}^{k-1}2\frac{(2l)!(2k-2-2l)!}{(l+1)!l!(k-l)!(k-l-1)!}.$$

The point of this rearrangement is that the sum on the right can now be evaluated: since $(2l)!/l!(l+1)!$ is the l^{th} Catalan number c_l, and these satisfy the recursion relation

$$c_n=\sum_{i+j=n-1}c_ic_j,$$

[see Comtet (1974) pp. 52–53] we have

$$\begin{aligned}
(4k-4)b_1-(2k-1)b_2&=2(2k-1)\frac{(2k-2)!}{k!(k-1)!}-2\frac{(2k)!}{k!(k+1)!}\\
&=\frac{(2k-1)!}{(k-1)!(k+1)!}(2(k+1)-4)\\
&=2\frac{(2k-1)!}{(k-2)!(k+1)!}
\end{aligned}$$

which is (5.2).

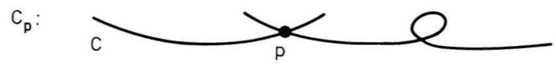
On the other hand, (5.3) is simply the combination

$$(i-2)R_1+(i-3)R_2+\ldots+2R_{i-3}+R_{i-2}.$$

c) A family in δ_0

Let (C, p) be a general pointed curve of genus $g-1=2k-1$ and let $\mathscr{C}\to C$ be the family of stable curves obtained by identifying a variable point $q\in C$ with p; thus $\mathscr{C}$ is the blow-up of $C\times C$ at $p\times p$ modulo the identification of the diagonal with the proper transform of $\{p\}\times C$. The fiber C_q of $\mathscr{C}$ over $q\neq p$ is

and the fiber over p is



the union of C and a nodal rational curve.

The curve $\mathscr{C}\hookrightarrow\bar{\mathscr{M}}_g$ meets the boundary component Δ_1 once, transversely, at p; it is disjoint from Δ_α for $\alpha>1$. It lies in Δ_0 with normal bundle $N_{\Delta_0/\bar{\mathscr{M}}_g}|_{\mathscr{C}}$ isomorphic to the tensor produce of the normal bundles of the proper transforms of Δ and $\{p\}\times C$ in $\mathscr{C}$ thus

$$\deg_C\Delta_0=(1-2(2k-1))+(-1)=2-4k.$$

The Hodge bundle on C being trivial, we have $\deg_C\lambda=0$. It follows that

$$(4k-2)b_0-b_1=\deg E_d^1|_C \tag{5.6}$$

and it remains to determine this degree.

To do this, we first observe that we may extend the covering $\mathscr{G}_{k+1}^1\to\mathscr{M}_{2k}$ to a map $\bar{\mathscr{G}}_{k+1}^1\to\bar{\mathscr{M}}_{2k}$, proper and finite in a neighborhood of the locus of curves $\{C_q\}$, by letting $\bar{\mathscr{G}}_{k+1}^1$ be the space of limit $\mathfrak{g}_{k+1}^1$'s on tree like curves – that is, over C_1 for $q\neq p$, simply a $\mathfrak{g}_{k+1}^1$ on the curve C_q, and over C_{p_1} a $\mathfrak{g}_{k+1}^1$ on C with ramification of order 2 at p. That this is proper over $\{C_q\}$ follows from the fact that the limit of a family of $\mathfrak{g}_{k+1}^1$'s on stable curves tending to C_q is either a limit $\mathfrak{g}_{k+1}^1$ on C_q, or a limit $\mathfrak{g}_k^1$ on a partial normalization of C_1 – but C, being general of genus $2k-1$, possesses no $\mathfrak{g}_k^1$'s, so the latter cannot occur. That it is finite follows from the fact that $G_{k+1}^1(C_q)$ is finite for every q by Theorem 1.1.

Since the fiber of $\bar{\mathscr{G}}_{k+1}^1$ over the point C_q is the locus of $\mathfrak{g}_{k+1}^1$'s L on C such that $h^0(L(-p-q))>0$, we consider the curve $\Sigma=C_{k+1}^1\cap X_p$ of divisors D of degree $k+1$ on C moving in a pencil and such that $D-p\geqq 0$ [so $\Sigma\cong W_{k+1}^1(C)$], and the incidence correspondence

$$\Phi=\{(q,D): D-p-q\geqq 0\}\subset C\times\Sigma\subset C\times C_d.$$

Note that the sum $\Phi+\Gamma$, where $\Gamma=\{p\}\times\Sigma$, is just the restriction to $C\times\Sigma$ of the universal divisor D on $C\times C_d$. The projection maps $\pi_1:\Phi\to C$ and $\pi_2:\Phi\to\Sigma$ express Φ as, respectively, a $\dfrac{(2k)!}{k!(k+1)!}$ sheeted cover of C and a k-sheeted cover of Σ; of course, it is the degree of the branch divisor of π, that we want to determine.

By Theorem 4 of the introduction the genus $g(\Sigma)$ of Σ satisfies

$$2g(\Sigma)-2=(2k-2)\frac{(2k)!}{k!(k+1)!},$$

so we may compute the genus of Φ from the number of branch points of $\Phi\to\Sigma$. This number is the number of divisors $D\in\Sigma\subset C^1_{k+1}$, moving in a pencil, containing p, and containing a double point q other than p, that is, such that $D-p$ lies in the diagonal Δ in C^1_k. Using the results of Arbarello et al. (1984), p. 326 or Harris (1984) we see that the locus $\{D-p: D\in\Sigma\}\subset C^1_k$ has class $\frac{\theta^{k-1}}{(k-1)!}-\frac{\theta^{k-2}x}{(k-2)!}$, while under the diagonal map $\phi: C\times C_{k-2}\to C_k$ sending (q,E) to $2q+E$ we have

$$\phi^*\theta^{k-1}=;\ 4(k-1)\frac{2(k-1)}{k!},$$

$$\phi^*\theta^{k-2}x=(4k-6)\frac{(2k-1)!}{(k+1)!}.$$

We conclude that the number of branch points of $\pi_2:\Phi\to\Sigma$ is

$$(4k-4)\frac{(2k-1)!}{k!(k-1)!}-(4k-6)\frac{(2k-1)!}{(k+1)!(k-2)!}=10\frac{(2k-1)!}{(k+1)!(k-2)!}.$$

By Riemann-Hurwitz, the degree of the canonical bundle of Φ is

$$\begin{aligned}2g(\Phi)-2&=(k)(2g(\Sigma)-2)+10\frac{(2k-1)!}{(k+1)!(k-2)!}\\&=2(k-1)k\frac{(2k)!}{k!(k+1)!}+5(k-1)\frac{(2k)!}{(k+1)!k!}\\&=(2k+5)(k-1)\frac{(2k)!}{k!(k+1)!}\\&(4k+10)\frac{(2k-1)!}{(k-2)!(k+1)!}.\end{aligned}$$

The number of branch points of the first projection $\pi_1:\Phi\to C$ is thus

$$(4k+10)\frac{(2k-1)!}{(k-2)!(k+1)!}-(2(2k-1)-2)\frac{(2k)!}{k!(k+1)!}=(4k+2)\frac{(2k-1)!}{(k-2)!(k+1)!}$$

We conclude by (5.6) that

(5.7) $$(4k-2)b_0-b_1=(4k+2)\frac{(2k-1)!}{(k-2)!(k+1)!}.$$

With (5.1) and (5.2), this gives four linear relations on a, b_0, b_1, b_2. One checks that they are non-degenerate, and by solving them one obtains the formulas (5.4).

6. The genus of a curve of special divisors

Proof of Theorem 4. We use the "determinantal adjunction formula" of Harris and Tu (1984), Sect. 3. Let M be a smooth variety of dimension $(m-k)(n-k)+1$, and let $Z\subset M$ be a curve which is the rank k-locus of a bundle map $\phi:\mathscr{E}\to\mathscr{F}$ with rank $(\mathscr{E})=m$, rank $\mathscr{F}=n$.

Set $c_i=c_i(\mathscr{F}-\mathscr{E})$,

$$\Delta=\det\begin{vmatrix} c_{n-k} & c_{n-k+1} & \cdots & c_{n-k+(m-k-1)} \\ c_{n-k-1} & c_{n-k} & & \vdots \\ \vdots & & \ddots & \\ c_{n-k-(m-k-1)} & & \cdots & c_{n-k} \end{vmatrix}$$

and, raising each index in the top row of the matrix giving Δ,

$$\Delta_1=\det\begin{vmatrix} c_{n-k+1} & c_{n-k+2} & \cdots & c_{n-k+1+(m-k-1)} \\ c_{n-k-1} & c_{n-k} & & c_{n-k+(m-k)} \\ \vdots & & \ddots & \vdots \\ c_{n-k-(m-k-l)} & & & c_{n-k} \end{vmatrix}.$$

The formula is

$$c_1(Z)=(c_1(M)-(m-k)c_1)\cdot\Delta+(m-n)\Delta_1.$$

[The proof given by Harris and Tu works only for smooth determinantal curves; but the result extends to the general case because of the existence of a complex of Schur functors resolving the determinantal ideals. Alternately, one may use that $W_d^r(C)$ is a smooth curve for generic C, and that $W_d^r(C_0)$ is a flat specialization of this for any C_0 such that $W_d^r(C_0)$ is 1-dimensional.]

In the setting of Theorem 4 we may take D to be a fixed divisor of large degree e on C and then consider the bundles $\mathscr{E},\mathscr{F}$ on $\mathrm{Pic}^d(C)$ whose fibers at a line bundle $\mathscr{L}$ of degree d are naturally:

$$\mathscr{E}_{\mathscr{L}}=H^0(C,\mathscr{L}(D)),$$
$$\mathscr{F}_{\mathscr{L}}=H^0(C,\mathscr{O}_D\otimes\mathscr{L}(D)).$$

These bundles have chern classes

$$C(\mathscr{F})=1,$$
$$C(\mathscr{E})=\exp(-\theta),$$

so

$$C(\mathscr{F}-\mathscr{E})=\exp(\theta),$$

[see Arbarello et al. (1984) Chap. VII, Sect. 5 for a precise definition of $\mathscr{E}$ and $\mathscr{F}$ and the computation of chern classes].

Thus

$$\Delta=\prod_{i=0}^{r}\frac{i!}{(g-d+r+i)!}\theta^{g-1},$$

and

$$\Delta_1 = \frac{(r+1)!}{(g-d+2r+1)!} \prod_{i=0}^{r-1} \frac{i!}{(g-d+r+i)!}.$$

Of course $c_1(M)=0$, and the desired formula now follows by direct calculation. □

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