

# **RING THEORY AND ALGEBRA III**

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**edited by**

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## INTRODUCTION TO ALGEBRAS WITH STRAIGHTENING LAWS

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§1. Introduction In this paper I will survey some recent joint work with Corrado DeConcini and Claudio Procesi. A more complete treatment, with full details, will appear elsewhere [DEP-2].

An algebra with straightening law (abbreviated ASL) on a poset  $H$  is an algebra admitting a presentation with generators indexed by  $H$  and relations having particularly nice properties. A principal example, due to Doubilet-Rota-Stein ([DRS]), is that of a polynomial ring  $A$  over a ring  $R$  on  $pq$  variables  $X_{ij}$ , thought of as the entries of a generic  $p \times q$  matrix  $X = (X_{ij})$ , where  $H$  is the set of minors of all orders of  $X$  (see Example (2) in Section 2). The advantage of taking this large set of generators for  $A$  (instead of the more obvious set of generators consisting only of the  $1 \times 1$  minors - the  $X_{ij}$ ) is that determinantal ideals will be generated by subsets of the set of generators of  $A$ , instead of by vast sums of monomials in the generators of  $A$ , so that computation with determinantal ideals is facilitated. The price is, of course, that the relations become more complicated; the statement

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that  $A$  is an algebra with straightening law on  $H$  indicates that the complication is, in fact, not too bad.

There are many other naturally occurring examples of algebras with straightening law, some of them described in Section 2 below. However, the definition is in another sense very limiting; many problems for algebras with straightening laws can be reduced to corresponding problems for the extremely simple "discrete" algebras of Example (0) in Section 2 below.

Some steps in this reduction procedure are explained in Section 3 below; one actually produces a special sort of deformation (normally flat in most of the examples) whose general fiber  $B$  is the given algebra with straightening law and whose special fiber is discrete.

In Section 4, we expound a different side of the philosophy that  $H$  determines many of the properties of any ASL on  $H$ : we give a simple direct argument showing that any ASL on a particularly nice kind of poset - here called "wonderful" - is Cohen-Macaulay. Though the restriction on  $H$  involved is strong, the conditions still include (homogeneous) coordinate rings of Schubert cycles, and determinantal or Pfaffian varieties. This proof, which represents an axiomatization of Musili's proof of the arithmetic Cohen-Macaulayness of the Schubert cycles [M], was really the starting point of our investigation.

I would like to describe last a hypothetical connection between three interesting classes of algebras: those with rational singularities, those which are F-pure in the sense of Hochster and Roberts [H-Ro] and those which have the structure of an ASL on a wonderful (or perhaps,

more generally, a "Cohen-Macaulay") poset. At the moment, the most striking connection is that the known examples of wonderful algebras with straightening law are known to have rational singularities (when they are integral domains), and, in some cases, to be F-pure. Further, a recent result of Goto and Watanabe ([GW]) shows that in a reasonable setting, all complete 1-dimensional F-pure algebras are the completions of algebras with straightening laws (or, have complete straightening laws, in a reasonable sense). Further, recent work of Hochster and Huneke shows that an ASL is F-pure whenever it is a Gorenstein domain.

I suppose one reasonable conjecture is that every ASL  $A$  on a wonderful poset is F-pure (or of F-pure type in characteristic 0, in a strong sense) and is normal, with rational singularities, whenever  $A$  is a domain; but the situation is still very obscure.

§2. Definitions and Examples All rings and algebras to be considered here will be commutative and have units.

Suppose  $A$  is a ring and  $H$ , a subset of  $A$ , is a partially ordered set called a poset. A standard monomial is a product of the form  $\alpha_1 \dots \alpha_k$  where  $\alpha_1 \leq \alpha_2 \leq \dots \leq \alpha_k$ .

Now let  $R$  be a ring,  $A$  an  $R$ -algebra,  $H$  a finite poset contained in  $A$  which generates  $A$  as an  $R$ -algebra. Then  $A$  is an algebra with straightening law (on  $H$ , over  $R$ ) if

(ASL-1) The algebra  $A$  is a free  $R$ -module whose basis is the set of standard monomials.

(ASL-2) If  $\alpha$  and  $\beta$  in  $H$  are incomparable and if

$$\alpha\beta = \sum r_i \gamma_{i1} \gamma_{i2} \cdots \gamma_{ik_i}$$

where  $0 \neq r_i$  is in  $R$  and  $\gamma_{i1} \leq \gamma_{i2} \leq \cdots$

is the unique expression for  $\alpha\beta$  in  $A$  as a linear combination of standard monomials, then  $\gamma_{i1} \leq \alpha, \beta$  for every  $i$ .

The right-hand side of the relation in (ASL-2) is allowed to be the empty sum ( $= 0$ ), but we require  $k_i > 0$  in each term that appears.

The relations mentioned in (ASL-2) are called the straightening relations. From Theorem 3.4 below it easily follows that they generate all the relations.

We will abbreviate the phrase "algebra with straightening law" to ASL.

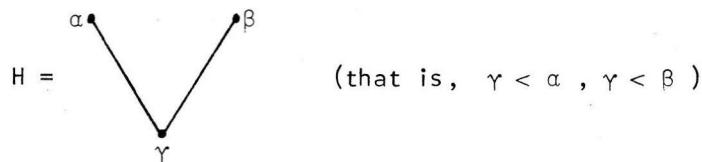
Examples We fix a commutative ring  $R$ , which might as well be taken to be a field or the ring of integers.

(0) The discrete ASL. Let  $H$  be any finite poset. The polynomial ring on the elements of  $H$ , modulo the relations  $\alpha\beta = 0$  whenever  $\alpha$  and  $\beta$  in  $H$  are incomparable, is called the discrete ASL on  $H$ , written  $R[H]$ . (Actually, any polynomial ring modulo an ideal generated by square-free monomials is ASL.)

If  $H$  is a chain, i.e., totally ordered set, then  $R[H]$  is simply the polynomial ring on the elements of  $H$ ; this is the only ASL possible if  $H$  is a chain.

If  $H$  is a clutter, that is, no two elements are comparable, containing  $n$  elements, then  $R[H] = R[x_1, \dots, x_n]/(x_i x_j \mid i \neq j)$  is the coordinate ring of the variety given by the union of the  $n$  coordinate lines in  $n$ -space. In this case,  $\dim R[H] = \dim R + 1$ ; these are the only algebras with straightening law of relative dimension 1.

- (1) To see a simple nontrivial family of examples, consider the poset



By IASL-2) and the remark following the definition, any ASL on  $H$  must have the form  $A = R[\alpha, \beta, \gamma]/(f)$ , where  $f$  has the form  $f(\alpha, \beta, \gamma) = \alpha\beta - \gamma g(\alpha, \beta, \gamma)$  and  $g$  is a linear combination of monomials  $\gamma^i \alpha^j$  and  $\gamma^i \beta^j$ . If

$$g = g_1(\beta, \gamma) + \alpha g_2(\gamma)$$

where  $g_1$  and  $g_2$  are arbitrary polynomials in the variables specified, then  $A$  is ASL. To check (ASL-1), one orders the monomials  $\alpha^i \beta^j$  lexicographically and then proves by induction in this order (which has descending chain condition) that  $\alpha^i \beta^j$  is in the span of the standard monomials. This implies that the standard monomials span  $A$ , while linear independence may be checked in the ring

$$(A/\gamma A) \oplus (\gamma A/\gamma^2 A) \oplus \dots$$

which is the discrete ASL on  $H$ .

On the other hand, if  $f = \alpha\beta - \gamma\alpha^2 - \gamma\beta^2$ , then  $A$  can be shown not to be ASL: One shows that  $\alpha\beta^2$  is not in the span of the standard monomials (I am indebted to Alan Adler for help with this). This leads one to ask, "If  $\alpha\beta - \gamma(h_1(\beta, \gamma) + h_2(\alpha, \gamma))$  is the straightening relation of a straightening law on  $H$  over a domain  $R$ , then must  $h_1$  or  $h_2$  be linear in  $\alpha$  and  $\beta$ ?"

(2) One of the motivating examples in the theory, in a simple special case, is the following: Let  $A = R[x_{11}x_{12}x_{21}x_{22}]$  and consider  $\bar{A} = A/(\Delta)$ , where  $\Delta = x_{11}x_{22} - x_{12}x_{21}$  is the determinant of the matrix  $\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}$ . Using the relation  $\Delta = 0$ , it is clear that  $\bar{A}$  is spanned by monomials not involving the product  $x_{12}x_{21}$ , and it is easy to show (for example, by calculating the dimension of the space of forms of each degree that are multiples of  $\Delta$  in  $A$ ) that these monomials are linearly independent. If we order the  $x_{ij}$  "following the rows and columns of the matrix," thus:

$$x_{11} \leq x_{12}$$

$$\wedge \quad \quad \wedge$$

$$x_{21} \leq x_{22}$$

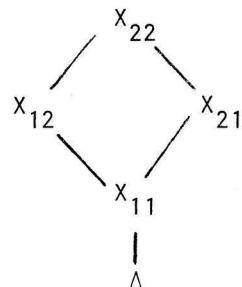
then, we see that  $x_{12}, x_{21}$  form the only pair of incomparable elements, so the monomials forming a basis of  $\bar{A}$  are exactly the standard monomials. Moreover,  $x_{11} \leq x_{12}$  and  $x_{21} \leq x_{22}$ , so the relation  $x_{12}x_{21} = x_{11}x_{22}$

satisfies (ASL-2), and we see that  $\tilde{A}$  with the poset of  $x_{ij}$  is an ASL.

Let us see if we can extend this to give an ASL structure for  $A$ , the polynomial ring. We can think of the  $x_{ij}$ , with the same partial order as before, as elements of  $A$ ; clearly  $A$  will have as basis the set consisting of the standard monomials in the  $x_{ij}$  multiplied by arbitrary powers of  $\Delta$ . We can partially order the set  $\{\Delta, x_{11}, x_{12}, x_{21}, x_{22}\}$  (which generates  $A$  as an  $R$ -algebra!) to make these products standard by making  $\Delta \leq$  all the  $x_{ij}$ , or by making  $\Delta \geq$  all the  $x_{ij}$ . But in order to make the relation

$$x_{12}x_{21} = x_{11}x_{22} + \Delta$$

conform to (ASL-2), we must make  $\Delta \leq$  the  $x_{ij}$ . Accordingly,  $A$  is an ASL on the poset



This idea can be generalized to matrices and minors of arbitrary size.

If

$$X = \begin{pmatrix} x_{11} & \cdots & x_{1q} \\ \vdots & & \vdots \\ x_{p1} & \cdots & x_{pq} \end{pmatrix}.$$

is a  $p \times q$  matrix of indeterminants over  $R$ , then one can specify a  $k \times k$  minor (= subdeterminant) by specifying the row indices  $i_1, \dots, i_k$  and column indices  $j_1, \dots, j_k$  that are involved; we write  $(i_1, \dots, i_k \mid j_1, \dots, j_k)$  in  $A = R[x_{11}, \dots, x_{pq}]$  for the associated minor, regarded as a polynomial. We partially order the set  $H$  of minors by setting

$$(i_1, \dots, i_k \mid j_1, \dots, j_k) \leq (i'_1, \dots, i'_\ell \mid j'_1, \dots, j'_\ell)$$

if and only if  $k \geq \ell$  and  $i_t \leq i'_t$ ,  $j_t \leq j'_t$  for  $t = 1, \dots, \ell$ .

It can be shown that  $A$  is an ASL on  $H$ . If  $I_K \subset A$  is the ideal generated by the  $k \times k$  minors of  $X$ , then  $A/I_K$  inherits a straightening law on the subposet of minors of order  $< k$ . This example was first worked out in [DRS] where the word "straightening" seems first to have been used in this context. For a faster but still combinatorial treatment see [DKR]; to see everything done by linear algebra, see [DEP-1] (which examines this example in depth) or [ABW] and [Stein] where, more generally, "Schur functors" are treated.

(3) Ideals generated by square-free monomials. Let  $\tilde{A} = R[x_0, \dots, x_n]$  be the polynomial ring on  $n+1$  indeterminants over  $R$ , and let  $A = \tilde{A}/I$ , where  $I$  is an ideal generated by some square-free monomials in the  $x_i$ .

Associated to  $I$  is a simplicial complex  $\Delta_I$  with vertices  $0, 1, \dots, n$  ( $=$  set of subsets, called faces) of  $\{0, \dots, n\}$  such that a subset of a face is a face, by the rule that  $\{i_1, \dots, i_k\}$  is a face if and only if  $x_{i_1} \cdot \dots \cdot x_{i_k}$  are not in  $I$ . For details on this construction, see [R] or [H]. We regard the set  $\{\Delta\}$  of faces of  $\Delta$  as a poset by reverse inclusion; that is,  $f \leq g$  if and only if  $f \supseteq g$  as subsets of  $\{0, \dots, n\}$ . If we identify the face  $f = \{i_1, \dots, i_k\}$  with the monomial  $x_{i_1} \cdot \dots \cdot x_{i_k}$ , then  $A$  becomes an ASL on the poset  $\{\Delta\}$ . To check this, note first that  $A$  has a basis consisting of those monomials which are products of powers of elements in some maximal face  $f$  of  $\Delta$ , and that any such monomial can be written uniquely in the form

$$f_1^{n_1} f_2^{n_2} \dots f_m^{n_m}, \text{ with } f \supset f_1 \supset \dots \supset f_m$$

faces of  $\Delta$ . This proves (ASL-1). For (ASL-2), it suffices to note that the relations on  $A$  may be written as

$$fg = \begin{cases} 0 & \text{if } f, g \text{ are not contained in a common face of } \Delta \\ f \cup g & \text{if } f \cup g \text{ is a face of } \Delta \end{cases}.$$

We note in passing that this example generalizes that of (0); the complex  $\Delta$  associated to the discrete algebra on  $H$  is just the set of chains ( $=$  totally ordered subsets) of  $H$ .

What about the rings  $\tilde{A}/J$ , when  $J$  is generated by an arbitrary (finite) set of monomials? We will show in the next section that every ASL is reduced, so  $\tilde{A}/J$  is not an ASL if  $J$  is not generated by

square-free monomials. However,  $\tilde{A}/J$  is not far from an ASL: as J. Weyman has remarked (this will appear in his Brandeis thesis),  $\tilde{A}/J$  may be written as

$$A/J = (\tilde{A}/I)/(f_1, \dots, f_m) ,$$

where  $\tilde{A}$  is a larger polynomial ring, I is an ideal generated by square-free monomials and  $f_1, \dots, f_m$  is a regular sequence in  $\tilde{A}/I$ , each  $f_i$  being a difference of two variables. An example will suffice to illustrate the technique:

$$R[x_1x_2x_3]/(x_1x_2^2, x_2x_3^2) = \frac{R[x_1, x_2, x_3, y, z_1, z_2]/(x_1x_2y, x_2x_3z_1z_2)}{(x_2 - y, z_1 - z_2, x_3 - z_1)} .$$

We next mention some classes of examples related to that of (2).

In all of these, the partially ordered sets that arise are subposets of the following (which also contains a copy of the poset in Example (2)):

Let  $2^{\{n\}}$  be the set of all subsets of  $\{1, 2, \dots, n\}$ . We write

$[i_1, \dots, i_k]$  for a  $k$  element subset with  $i_1 < \dots < i_k$ , and we set  $[i_1, \dots, i_k] \leq [j_1, \dots, j_\ell]$  if and only if  $k \geq \ell$  and  $i_s \leq j_s$  for  $s = 1, \dots, \ell$ . We write  $\{d\}^n$  for the subposet of  $2^{\{n\}}$  containing the subsets of  $d$  elements.

(4) The homogeneous coordinate ring of the Grassmann variety and the Schubert cycles. The homogeneous coordinate ring  $A$  of the Grassmann variety  $G(d, n)$  of  $d$ -dimensional planes in  $n$ -space (if  $R$  is not a field, one reads "free module" for space and "summand" for

plane) may be regarded as a subring of  $R[X_{11} \dots X_{dn}]$ , the polynomial ring in  $d_n$  variables, generated by the  $d \times d$  minors (called Plücker coordinates) of the  $d \times n$  matrix

$$X = \begin{pmatrix} X_{11} & \cdots & X_{1n} \\ \vdots & & \vdots \\ X_{d1} & \cdots & X_{dn} \end{pmatrix}.$$

We identify  $[i_1, \dots, i_d]$  in  $\binom{n}{d}$  with the Plücker coordinate which is the determinant of the  $d \times d$  submatrix of  $X$  consisting of columns  $i_1, \dots, i_d$ .

This construction makes  $A$  into an ASL on  $\binom{n}{d}$ , a fact which was essentially proved by Hodge [Ho]; see [L] or [DEP-1] for a modern treatment. The fact that  $A$  is ASL on  $\binom{n}{d}$  can be deduced from the example given in (2). On the other hand, the polynomial ring on  $pq$  variables is the coordinate ring of the affine open subset  $[p+1, \dots, p+q] = 1$  in  $G(q, p+q)$ , and the straightening law of Doubilet-Rota-Stein on this polynomial ring mentioned in (2) can be deduced from that of  $A$ , as is done in [DEP-1]; it turns out that the poset of minors of a  $p \times q$  matrix is naturally isomorphic to  $\binom{p+q}{p}$  with the element  $[p+1, \dots, p+q]$  omitted.

The Schubert cycles are subvarieties of  $G(d, n)$  which are formed as follows: Let  $V$  be a vector space of dimension  $n$ , and let  $V = V_n \supset V_{n-1} \supset \cdots \supset V_1$  be a complete flag of subspaces (that is,  $\dim V_i = i$ ). Then, if  $a_1 < \cdots < a_d$ , the Schubert variety  $\Omega_{a_1, \dots, a_d}$  is defined to be the subvariety of  $d$ -dimensional subspaces  $W \subset V$  such

that  $\dim(W \cap V_i) \geq i$ . The ideal  $I_{a_1, \dots, a_d}$  which defines  $\Omega_{a_1, \dots, a_d}$  is generated in the ring  $A$  of (3) by the Plücker coordinates that are  $\not\in [n - a_d + 1, \dots, n - a_1 + 1]$ ; it follows easily that the homogeneous coordinate ring of  $\Omega_{a_1, \dots, a_d}$  is ASL on the subposet of  $\{^n_d\}$  consisting of all Plücker coordinates that are  $\geq [n - a_d + 1, \dots, n - a_1 + 1]$ .

(5) The flag variety. The variety  $F_{n, n_1, n_2, \dots, n_k}$  of flag  $V \supset V_{n_1} \supset V_{n_2} \supset \dots \supset V_{n_k}$ , where  $V_i$  is a subspace of dimension  $i$ , has as multihomogeneous coordinate ring the subring  $A$  of  $k[X_{11}, \dots, X_{nn}]$  generated by the  $n_i \times n_i$  minors of the first  $n_i$  rows of the generic  $n \times n$  matrix

$$X = \begin{pmatrix} X_{11} & \cdots & X_{1n} \\ \vdots & & \vdots \\ X_{n1} & \cdots & X_{nn} \end{pmatrix},$$

for  $i = 0, \dots, n$  (here we take  $n_0 = n$ ). We identify  $[i_1, \dots, i_{n_k}]$  in  $2^{\{n\}}$  with the  $n_k \times n_k$  minor of  $X$  involving the first  $n_k$  rows and columns numbered  $i_1, \dots, i_{n_k}$ . With this set of generators,  $A$  becomes an ASL. This is best proved by using the explicit form of the straightening law on the polynomial ring  $R[X_{ij}]$ . See, for example, [DEP-1] for the technique.

(5) Pfaffian varieties. Let  $X_{ij}$ , for  $1 \leq i < j \leq n$ , be indeterminates

and let

$$X = \begin{pmatrix} 0 & x_{12} & \cdots & x_{1n} \\ -x_{12} & 0 & \cdots & \cdot \\ \vdots & \vdots & \ddots & \cdot \\ -x_{1n} & \cdot & \cdot & 0 \end{pmatrix}$$

be the generic alternating  $n \times n$  matrix over  $\mathbb{R}$ . If we identify  $[i_1, \dots, i_{2k}]$  in  $2^{\{n\}}$  with the pfaffian (the square-root of the determinant) of the alternating submatrix of  $X$  involving rows and columns  $i_1, \dots, i_{2k}$ , regarded as an element of the polynomial ring  $A = \mathbb{R}[x_{12}, \dots, x_{n-1,n}]$ , then  $A$  becomes an ASL. Writing  $Pf_{2k}$  for the ideal generated by all the  $2k \times 2k$  pfaffians of  $X$  we get, as in Example (2), an induced straightening law on  $A/Pf_{2k}$ , whose poset consists of the elements  $[i_1, \dots, i_{2\ell}]$  in  $2^{\{n\}}$  for all  $\ell < k$ .

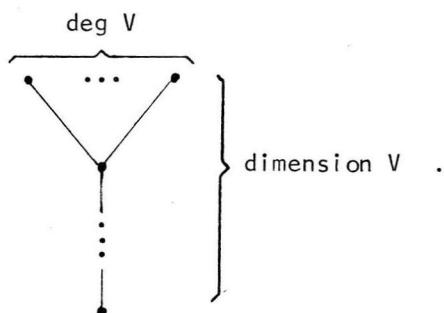
The reader may note that Examples (2), (4), (5) are associated to the special linear groups, while (6) is associated to the orthogonal groups in the following way: if  $G$  is a semisimple algebraic group and  $P$  is a parabolic subgroup (= subgroup containing a maximal solvable subgroup), then  $G/P$  is in a natural way a projective variety, and the Grassmannian and Flag varieties are of this type, the Schubert cycles are distinguished subvarieties, and the polynomial rings and determinantal varieties are affine open subsets of these. The Pfaffian varieties of (6) are similarly related to the "Grassmann variety" of isotropic subspaces with respect to a symmetric bilinear form, which is

of the form  $O(2n)/P$ , where  $P$  is the stabilizer of a maximal isotropic subspace. (If we take the form to have matrix

$$\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix},$$

where  $I$  is an  $n \times n$  identity matrix, then an isotropic  $n$ -dimensional subspace in the open set of  $n$ -dimensional subspaces whose projection into the first  $n$  coordinates is still isotropic, is represented as inclusion  $R^n \rightarrow R^{2n}$  with matrix of the form  $[I \mid A]$ , where  $A$  is an arbitrary alternating  $n \times n$  matrix.)

(7) Subvarieties of minimal degree in  $\mathbb{P}^n$ . Suppose  $R$  is a field. A subvariety  $V \subset \mathbb{P}^n$  is called nondegenerate if it is contained in no hyperplane. It is known that a nondegenerate subvariety  $V$  satisfies  $\text{degree}(V) \geq \text{codimension}(V) + 1$ , and that equality holds if and only if  $V$  is one of a short list of subvarieties (Bertini); in this case,  $V$  is said to have minimal degree. One can show that the homogeneous coordinate ring of a subvariety of minimal degree is an ASL on an  $n$ -element poset of the form



§3. Some Results on ASL's Throughout this section we fix a ring  $R$  and a finite poset  $H$ .

The main result on ASL's is that, in some sense, the discrete ASL on  $H$  is the worst behaved of any ASL on  $H$ . This is useful because there is a rather complete and simple theory of algebras given by ideals of square-free monomials as in Example (3), and they are in many ways very well-behaved; their properties can be read off from the combinatorics of the associated simplicial complex, or, in the case of a discrete algebra  $R[H]$ , from the combinatorics of  $H$ .

We will here examine only the main step in this direction and then explain some consequences.

We fix an ASL  $A$ , and consider multiplicative filtrations  $I = \{I_j\}$  of  $A$  by ideals, (that is, filtrations  $A = I_0 \supset I_1 \supset \dots$  satisfying  $I_p I_q \subset I_{pq}$ ). We define the order of a standard monomial as follows: For  $\alpha$  in  $H$ , we define  $\text{ord}_I(\alpha) = \sup\{j \mid \alpha \text{ in } I_j\}$ . If  $M = \alpha_1 \dots \alpha_k$  is a standard monomial, we define

$$\text{ord}_I(M) = \sum_1^k \text{ord}_I(\alpha_j).$$

We will say that  $I$  is standard if, for each  $j$ ,  $I_j$  is spanned by the standard monomials of order  $\geq j$ . (In this case,  $\text{ord}_I$  as defined above coincides with the usual  $\text{ord}_I(\alpha) = \sup\{j \mid \alpha \text{ in } I_j\}$ .

If  $I$  is any multiplicative filtration of  $A$ , the Rees algebra of  $I$  is defined by

$$A = \dots \oplus At^k \oplus \dots \oplus At \oplus A \oplus I_1 t^{-1} \oplus \dots \oplus I_k t^{-k} \oplus \dots.$$

This algebra is in  $A[t, t^{-1}]$ . Note that  $A = At^0$  is a subring of  $A$  in a natural way and  $A$  is an  $R[t]$ -algebra.

The main result is the next theorem.

Theorem 3.1 If  $A$  is an ASL on  $H$  over  $R$ , then  $A$  is an ASL on  $H$  over  $R[t]$ , when the embedding  $H \rightarrow A$  is given by  $\alpha \mapsto \tilde{\alpha} = t^{-\text{ord}_I(\alpha)} \alpha$ , where  $\alpha$  is in  $A$ .

Corollary 3.2 If  $I$  is a standard filtration of  $A$ , then the associated graded ring  $\text{gr}_I(A)$  is an ASL on the  $H$ , where the embedding  $H \rightarrow \text{gr}_I(A)$  is given by (for  $\alpha$  in  $H \subset A$ )

$$\alpha \mapsto \tilde{\alpha} \text{ in } I_{\text{ord}_I(\alpha)}/I_{1+\text{ord}_I(\alpha)}$$

Proof of the Corollary  $\text{gr}_I(A) = A/tA = A \otimes_{R[t]} R[t]/(t)$  and straightening laws are obviously preserved under base-change.

Proof of Theorem 3.1 We must check that the ASL axioms are satisfied by  $A$  and  $\{\tilde{\alpha}\}_{\alpha \in H}$ .

(ASL-1): Linear independence is obvious from the fact that  $A \subset A[t, t^{-1}]$ , since the standard monomials in the elements  $\alpha$  in  $H$  are  $R[t, t^{-1}]$ -linearly independent in  $A[t, t^{-1}]$ . To see that the standard monomials in the  $\tilde{\alpha}$  span  $A$ , suppose that  $x$  is in  $I_m$ .

If we express  $x$  as a linear combination of standard monomials

$$x = \sum r_i \gamma_{i1} \dots \gamma_{ik_i}, \quad 0 \neq r_i \text{ in } R,$$

and set  $m_i = \sum_j \text{ord}_I(\gamma_{ij})$ , then by the standardness of  $I$ , we have

$m_i \geq m$  for each  $i$ . Thus, we may write

$$\begin{aligned} xt^{-m} &= \sum_i t^{-m} r_i \gamma_{i1} \dots \gamma_{ik_i} \\ &= \sum_i t^{m_i - m} r_i \tilde{\gamma}_{i1} \dots \tilde{\gamma}_{ik_i}, \end{aligned}$$

as required. This formula also provides a proof of (ASL-2): if  $x = \alpha\beta$  with  $\alpha, \beta$  incomparable, then  $x$  is in  $\text{Ind}(I(\alpha) + \text{ord}_I(\beta))$ , so we may express  $\tilde{\alpha}\tilde{\beta} = t^{-(\text{ord}_I(\alpha) + \text{ord}_I(\beta))} x$  in the appropriate way.

One sort of application of Theorem 3.1 is to allow that proof of many facts about algebras with straightening laws by a reduction to the discrete case: to explain this we need a measure of the "indiscreteness" of an algebra. To this end we define a subset  $\text{Ind}(A) \subset H$  which consists of all the elements  $\gamma_{ij}$  that occur on the right-hand side of straightening relations. Note that if  $\alpha$  is minimal in  $\text{Ind}(A)$  and  $\beta$  is incomparable to  $\alpha$ , then  $\alpha\beta = 0$ : thus,  $\alpha^n$  times a standard monomial is either again standard or 0. This is the basis of the following result.

Theorem 3.3 Let  $\alpha$  be a minimal element of  $\text{Ind}(A)$ , and set

$$I = \{I_n = (\alpha^n)\}. \text{ Then}$$

(1)  $I$  is a standard filtration.

(2)  $\text{Ind}(\text{gr}_I A) \subsetneq \text{Ind } A$ .

Proof Part (1) follows from the remarks just preceding the theorem.

Part (2) is deduced as follows: we have seen that the straightening relations in the Rees algebra  $A$  associated to  $I$  have the form

$$\tilde{\delta}\tilde{\varepsilon} = \sum r_i t^{m_i - \text{ord}_I \delta} \tilde{\gamma}_{i1} \dots \tilde{\gamma}_{ik_i},$$

when  $\delta\varepsilon = \sum r_i \gamma_{i1} \dots \gamma_{ik_i}$  was the corresponding straightening relation in  $A$ , and  $m_i = \sum_j \text{ord}_I \gamma_{ij}$ . Since  $\text{gr}_I(A)$  is obtained from this by setting  $t = 0$ , it will suffice to prove that  $\tilde{\alpha}$  never occurs on the right-hand side without being multiplied by  $t$ , or that  $\alpha$  never occurs simultaneously on both sides of a straightening relation in  $A$ . However, if  $\delta$  or  $\varepsilon$  is  $\alpha$ , then  $\delta\varepsilon = 0$  since  $\delta$  and  $\varepsilon$  are incomparable. This finishes the proof.

As a first consequence, we have the following theorem.

Theorem 3.4 If  $A$  is ASL on  $H$ , then any sum of monomials in  $H$  can be expressed as a sum of standard monomials by repeated applications of the following procedure: choose some nonstandard monomial that appears and in it locate the product of a pair of incomparable elements; replace the product of this pair by its expression in terms of standard monomials.

We will not prove Theorem 3.2 here; note that it is in any case easy if  $A$  is positively graded and the elements of  $H$  are homogeneous, as

in most of our examples. Instead we turn to some consequences.

Corollary 3.5 (1) If  $M = \beta_1 \dots \beta_k$  is a nonstandard monomial in  $A$ , then if

$$M = \sum r_i \gamma_{i1} \dots \gamma_{ik},$$

(with  $r_i$  in  $R$ , and  $\gamma_{i1} < \gamma_{i2} < \dots$ )

is the unique expression for  $M$  as a linear combination of standard monomials, then

$$\gamma_{i1} < \beta_j \text{ for all } i, j.$$

(2) The straightening relations generate the ideal of all relations on the set of generators  $H$  and thus give a presentation of  $A$ .

(3) If  $K \subset H$  is a poset ideal (that is,  $\beta \leq \alpha$  and  $\alpha$  in  $H$  implies  $\beta$  in  $H$ ), then the ideal  $I(K)$  generated by the elements of  $K$  is spanned by the standard monomials it contains, and  $A/I(K)$  is an ASL over  $R$  on  $H - K$ .

Proof (1) and (2) are clear from the theorem and (3) is an immediate consequence of (2).

Note Part (3) above is the proof that the straightening laws on the polynomial ring or the homogeneous coordinate ring of the Grassmannian in the examples of Section 2 induce straightening laws on the determinantal or Pfaffian varieties or Schubert cycles.

Using Theorem 3.3, one also has the next result.

Corollary 3.6 Let  $A$  be an algebra with straightening law on  $H$  over  $R$ . Then

- (1) If  $R$  is reduced, then  $A$  is too.
- (2) If  $\alpha_1 \dots \alpha_n$  are a maximal clutter meeting every maximal chain in  $H$  and  $r_i$  in  $R$  are nonzerodivisors, then  $\sum r_i \alpha_i$  is a nonzerodivisor in  $H$ .
- (3)  $\dim(A) = \dim(H) + \dim(R) + 1$  where  $\dim(H)$  is the maximal length  $d$  of a chain  $\alpha_0 > \dots > \alpha_d$  in  $H$ .

Proof Observing that  $\bigcap (\alpha^n) = 0$  for a minimal element  $\alpha$  of  $\text{Ind}(A)$ , all these things may be reduced to the case of the discrete algebra  $R[H]$ , for which they are well-known (and easy).

Another result of this sort gives an explicit system of parameters; it was suggested to us by R. Stanley.

For  $\alpha$  in  $H$ , we define  $\text{ht}(\alpha)$  to be the maximal length  $d$  of a sequence of elements  $\alpha > \alpha_1 > \dots > \alpha_d$  of  $H$  descending from  $\alpha$ .

Proposition 3.7 Let  $M$  contained in  $A$  be the ideal generated by all the elements  $\alpha$  in  $H$ . Then,  $\text{ht}(M) = 1 + \dim(H)$  and if  $x_i = \sum_{\text{ht}\alpha=i} \alpha$ , then  $M$  is nilpotent modulo  $(x_0, \dots, x_d)$ .

Proof If  $\alpha$  is an element of height  $i$  in  $H$ , then  $\alpha x_i$  equals  $\alpha^2 + (\text{a linear combination of standard monomials of heights } < i)$ .

Thus, modulo  $(x_1, \dots, x_d)$ , every product of two elements of height

$\leq i$  is congruent to a linear combination of standard monomials beginning with elements of height  $< i$ . Similarly, a monomial with  $2^k$  factors of height  $\leq i$  is congruent to a sum of monomials each of which has at least  $2^{k-1}$  factors of height  $k - 1$ . By induction, every product of  $2^{d+1}$  elements of  $H$  is congruent to 0.

§4. Cohen-Macaulay Algebras and Wonderful Posets Again, we fix an ASL  $A$  on a poset  $H$  over the ring  $R$  and consider the question of the length of a maximal regular sequence in  $M$ , the ideal generated by the elements of  $H$ . This number is called depth( $M$ ), and we have

$$\text{depth}(M) \leq \text{height}(M) = 1 + \dim(H).$$

We are interested in conditions under which equality holds.

From the induction procedure explained in Section 3, one can show that  $\text{depth}(M) \geq$  the depth of the corresponding ideal in  $R[H]$ , the discrete algebra and, in this case,  $[R]$  and  $[H]$  contain exact conditions for  $\text{depth} = \text{height}$ . However, we can give a simpler condition on  $H$  which includes all the examples (except (0) and (3), naturally) of Section 2, and which yields a stronger result.

An element  $\beta$  in  $H$  is a cover of  $\alpha$  in  $H$  if  $\beta > \alpha$  and no element lies strictly between  $\beta$  and  $\alpha$ .

The poset  $H$  is called wonderful (or locally-semimodular) if in the poset  $H \cup \{\infty\} \cup \{-\infty\}$  obtained by adjoining greatest and least elements (if they are not already present), the following condition

holds: if  $\beta_1, \beta_2 < \gamma$  are covers of an element  $\alpha$ , then there is an element  $\beta < \gamma$  which covers both  $\beta_1$  and  $\beta_2$ .

As in Section 3, we write  $x_i = \sum_{ht(\alpha)=i} \alpha$ .

Theorem 4.1 If  $H$  is a wonderful poset of dimension  $d$ , then

$x_0, \dots, x_d$  form a regular sequence in  $A$ ; in particular, in this case,  $\text{height}(M) = \text{depth}(M)$ .

Corollary 4.2 Suppose that  $A$  is positively graded,  $A_0 = R$  is Cohen-Macaulay, and the elements of  $H$  are homogeneous. If  $H$  is wonderful, then  $A$  is Cohen-Macaulay. In particular, the algebras of Examples (1), (2), (4), (5), (6) and (7) are Cohen-Macaulay.

Note: The Cohen-Macaulayness of locally semimodular posets is deduced combinatorially in [B].

Proof of (4.2) This criterion for Cohen-Macaulayness may be found, for example, in [H-Ra]. It is easy to see that  $2^{\{n\}}$  and  $\binom{n}{d}$  and the various subsets used in the examples are all wonderful. For example, for  $\binom{n}{d}$ , a cover of  $[i_1, \dots, i_d]$  is obtained by replacing  $i_j$  by  $i_j + 1$  (supposing this is  $< i_{j+1}$ ). Two distinct covers are thus obtained by raising two distinct indices, and their common cover, demanded by the definition of wonderfulness, is obtained by raising both.

To prove the theorem, we need some elementary combinatorial properties of wonderful posets. First, if  $\alpha_1, \dots, \alpha_k$  are in  $H$ , we define the poset ideal  $K \subset H$  cogenerated by  $\alpha_1, \dots, \alpha_k$  to be the

subset of all  $\beta$  such that  $\beta \neq \alpha_i$  for all  $i$ .

Lemma 4.3 Let  $H$  be a wonderful poset.

- (1) If  $I$  is an ideal in  $H$  and if for any two minimal elements  $\alpha_1, \alpha_2$  in  $H - I$  with  $\alpha_1, \alpha_2 < \gamma$ , there is a common cover  $\beta < \gamma$  for  $\alpha_1, \alpha_2$ , then  $H - I$  is wonderful.
- (2) Any maximal chain in  $H$  has length equal to  $\dim(H)$ .
- (3) If  $\alpha_1, \dots, \alpha_k$  are the minimal elements of  $H$  and if  $I \subset H$  is the ideal cogenerated by  $\alpha$  while  $J \subset H$  is the ideal cogenerated by  $\alpha_2, \dots, \alpha_k$ , then  $I \cap J = \emptyset$  and  $H - I$ ,  $H - J$  and  $H - (I \cup J)$  are wonderful, the last having dimension  $\dim(H) - 1$ .

Proof The proofs (in the order given) are easy: we prove (2) by way of example, using induction on the number of elements in  $H$  to prove that the lengths of two maximal chains  $\alpha_0 < \alpha_1 < \dots$  and  $\beta_0 < \beta_1 < \dots$  are equal. We may assume  $\alpha_0 \neq \beta_0$ ; else pass to  $H -$  (the ideal cogenerated by  $\alpha_1$ ). Since  $\alpha_0$  and  $\beta_0$  both cover  $-\infty$ , they have a common cover  $\gamma_1$ . Let  $\gamma_1 < \gamma_2 < \dots$  be a maximal chain ascending from  $\gamma_1$ . Passing to  $H -$  (the ideal cogenerated by  $\alpha_0$ ), we see that  $\alpha_0 < \alpha_1 < \dots$  and  $\alpha_0 < \gamma_1 < \dots$  have the same length, namely one more than the length of  $\gamma_1 < \gamma_2 < \dots$ . Repeating the argument with  $\beta_0$  in place of  $\alpha_0$ , we are done.

Proof of 4.1 We do induction on the number of elements of  $H$ . If  $H$  has a unique minimal element  $\alpha$ , then  $x_0 = \alpha$  is a nonzerodivisor

by Corollary 3.6(2), so we can factor it out and finish by induction, using Corollary 3.5(3) and Lemma 4.3(1).

Now suppose  $H$  has minimal elements  $\alpha_1, \dots, \alpha_k$  with  $k \geq 2$ .

We write  $I, J$  contained in  $H$  for the ideals defined as in Lemma 4.3(3), and we write  $\bar{I}, \bar{J}$  for the ideals of  $A$  generated by the elements of  $I$  and  $J$ , respectively. By induction and Corollary 3.5(3), it follows that  $x_0, \dots, x_d$  is a regular sequence on  $A/\bar{I}$  and on  $A/\bar{J}$ , and that  $x_1, \dots, x_d$  is a regular sequence on  $A/(\bar{I} + \bar{J})$ , while  $(x_0 A) \subset (\bar{I} + \bar{J})$ . Further,  $C = \bar{I} \cap \bar{J}$  by Corollary 3.5(3), so we have a short exact sequence

$$0 \rightarrow A \rightarrow A/\bar{I} \oplus A/\bar{J} \rightarrow A/\bar{I} + \bar{J} \rightarrow 0.$$

The theorem now follows from the next, well-known, lemma (which was also used by Musili).

Lemma 4.4 Suppose  $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$  is a short exact sequence of modules, and that  $x_0, \dots, x_d$  are ring elements such that  $x_0 C = 0$ . If  $x_0, \dots, x_d$  is a regular sequence on  $B$  and  $x_1, \dots, x_d$  is a regular sequence on  $C$ , then  $x_0, \dots, x_d$  is a regular sequence on  $A$ .

Proof Since  $A \subseteq B$ ,  $x_0$  is a nonzerodivisor in  $H$ . Modulo  $x_0$  we obtain the exact sequence

$$0 \rightarrow C \rightarrow A/x_0 A \rightarrow B/x_0 B \rightarrow C \rightarrow 0.$$

Let  $K = \ker(B/x_0 B \rightarrow C)$ . Since  $x_1, \dots, x_d$  is a regular sequence on

$B/x_0B$  and on  $C$ , it is regular on  $K$  as well, and repeating the argument, on  $A/x_0A$ .

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