LIFTING MODULES AND A THEOREM

ON FINITE FREE RESOLUTIONS*

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Summary

The purpose of this paper is to expound some results connected with the lifting problem of Grothendieck, and to record a new theorem about finite free resolutions over commutative noetherian rings which has proved useful in connection with the lifting problem. The result on free resolutions, a statement of which can be found at the end of section 1, has some other interesting applications; it yields Burch's theorem on the structure of cyclic modules of homological dimension two, and also the main results of [7]. In particular, it yields a new proof that any regular local ring is a unique factorization domain.

1. Discussion of the lifting problem, and statement of the main result

Throughout this paper, rings will be commutative and will have units.

The lifting problem of Grothendieck poses the following question. Suppose R is a regular local ring with maximal ideal M, and $x \in M - M^2$. Set $S = R/(x)$, and let B be a finitely generated S-module. Does there exist an R-module A such that

(1) The element x is not a zero divisor on A; and

(2) $A/xA \cong B$?

We shall call any module A satisfying these conditions a *lifting* of B to R.

The reason for Grothendieck's interest in the lifting problem was that an affirmative answer to the question it raises would yield a proof of a conjecture of Serre on multiplicities [9, p. 2]. Serre's conjecture is that if B and C are modules over a regular local ring S such that $\mathrm{Tor}_i^S(B, C)$ has finite length for all i, then $\Sigma_{i=0}^\infty (-1)^i \ell_s(\mathrm{Tor}_i^S(B, C)) \geqslant 0$, where $\ell_s(\mathrm{Tor}_i^S(B, C))$ is the length of an S-composition series for $\mathrm{Tor}_i^S(B, C)$. In [9], the problem is reduced to the case where S is complete, and B and C are cyclic. Also, the conjecture is proved in the case where S is equicharacteristic or unramified.

A positive solution to the lifting problem would allow one to reduce the general (complete regular local) case of Serre's conjecture to the unramified case. For, if S is a complete regular local ring, then by the Cohen structure theory [8, p. 106], there is an unramified regular local ring R with maximal ideal M and an element $x \in M - M^2$ such that $S \cong R/(x)$. If A is a lifting to R of the S-module B, then

$$\mathrm{Tor}_i^R(A, C) \cong \mathrm{Tor}_i^S(B, C).$$

To see this, note that the fact that x is a non-zero divisor on A implies that $\mathrm{Tor}_i^R(A, S) = 0$ for all $i > 0$. Thus if $P \to A$ is an R-free resolution of A, then $S \otimes_R P \to S \otimes_R A \cong B$ is an S-free resolution of B, so the two sides of the desired formula are the homology, respectively, of $P \otimes_R (S \otimes_S C)$ and $(P \otimes_R S) \otimes_S C$.

Thus $\Sigma_{i=0}^\infty (-1)^i \ell_s(\mathrm{Tor}_i^S(B, C)) = \Sigma_{i=0}^\infty (-1)^i \ell_R(\mathrm{Tor}_i^R(A, C)) \geqslant 0$ by Serre's result for the unramified case.

We note in passing that because of Serre's reduction of the conjecture to the cyclic case, it would be sufficient to be able to lift cyclic modules.

We now sketch a few of the known results on lifting. Let R be a ring, $S = R/(x)$, where x is a non-zero divisor in R. Most theorems about lifting take advantage of the well-known result (lemma 3.1 of this paper) which says that lifting an S-module B to R is the same thing as "lifting" an S-projective resolution of B to an R-projective complex. From this it follows easily that any S-module of free dimension less than or equal to 1 can be lifted, and that any cyclic module $S/(s_1 \cdots s_n)$, where $s_1 \cdots s_n$ forms an R-sequence can be lifted; see section 3.

We now specialize to the case where R and S are local, and the module to be lifted is cyclic, say of the form S/I. Then if S/I has homological dimension 2, S/I is liftable. Moreover, if I is generated by at most 3 elements, then S/I is liftable. The first of these results is a simple consequence of a structure theorem for cyclic modules of homological dimension 2 which is due (in this generality) to Burch [3, Theorem], though a special case of it goes back to Hilbert [4, pp. 239-240 in the Ges. Abh., V. 2]. (See [5, p. 148, ex. 8] for a slick, short proof of Burch's theorem that works even in the non-noetherian case.) We will now describe Burch's theorem.

Let S be a local ring, $I = (s_1 \cdots s_n)$ an ideal of S such that S/I has homological dimension 1. The free resolution of S/I will have the form

$$0 \longrightarrow S^{n-1} \xrightarrow{\phi_2} S^n \xrightarrow{\phi_1} S \longrightarrow S/I \longrightarrow 0.$$

Suppose that with respect to the canonical bases of S^{n-1}, S^n, and S, ϕ_2 has the form (ϕ_{ij}) and ϕ_1 has the form $(s_1, \cdots, s_n)$. We will write $[\phi_2; i]$ for the minor obtained from the matrix (ϕ_{ij}) by omitting the ith row. We are now ready to state Burch's theorem.

Theorem. There exists a non−zero divisor $s \in S$ such that $s \cdot [\phi_2; i] = s_i$ for $i = 1, \cdots, n$.

The result on lifting cyclic modules S/I where S/I has finite homological dimension and I is generated by three elements follows from a broad generalization of Burch's theorem which gives information about the form of a finite free resolution of any length.

We need some notation. We will say that a map of finitely generated free $S-$modules $\phi: F \rightarrow G$ has rank r if $\Lambda^r \phi \neq 0$ and $\Lambda^{r+1} \phi = 0$. Equivalently, ϕ has rank r if ϕ has a non−zero minor of order r, but every minor of order $r + 1$ is 0. If we choose a basis for F indexed by a set μ and a basis for G indexed by v, we will write $\phi: S^\mu \rightarrow S^v$, and we will identify ϕ and its matrix with respect to the canonical bases of S^μ and S^v.

For the minor of ϕ with columns $\mu' \subset \mu$ and rows $v' \subset v$, we will write $[\phi; \mu', v']$. Of course this notation only makes sense when $\#\mu'$ (the number of elements in μ') is the same as $\#v'$, and in this case, $[\phi; \mu', v'] \in S$. If $\mu \supseteq \mu'$, we write $\mu - \mu'$ for the complement of μ' in μ.

Recall that the depth of an ideal $I \subseteq S$ is defined to be the length of a maximal $S-$sequence contained in I.

Theorem 1. Let S be a noetherian ring, and let

$$0 \longrightarrow S^{\mu(n)} \xrightarrow[\phi_n]{} S^{\mu(n-1)} \xrightarrow[\phi_{n-1}]{} \cdots \xrightarrow[\phi_1]{} S^{\mu(0)}$$
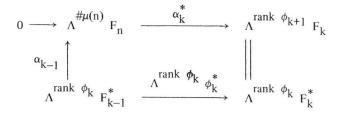
be an exact sequence of finitely generated free $S-$modules with chosen bases. Then, for each $0 \leqslant k \leqslant n-1$ and $\mu \subset \mu_k$ with $\#\mu = \text{rank } \phi_k$, there is an element $\alpha(k, \mu)$ such that

(1) if $\mu \subseteq \mu(k)$ and $v \subseteq \mu(k-1)$ with $\#\mu = \#v = \text{rank } \phi_k$, then,
$[\phi_k : \mu, v] = \alpha(k, \mu)\alpha(k-1, \mu(k-1)-v)$,

(2) if $\mu \subseteq \mu(n-1)$ and $\#\mu = \text{rank } \phi_{n-1}$, then $\alpha(n-1, \mu) = [\phi_n ; \mu(n), \mu(n-1)-\mu]$, and

(3) for each k, the depth of the ideal generated by $\{\alpha(k, \mu) | \mu \subseteq \mu(k)$ and $\#\mu = \text{rank } \phi_k\}$ is at least $k + 1$.

Remarks.

(1) Note that rank $\phi_k + \text{rank } \phi_{k-1} = \#\mu_{k-1}$, so parts (1) and (2) of the theorem make sense.

(2) There is also an "intrinsic" version of the theorem: if we let $F_k = R^{\mu(k)}$, its statement depends only on a choice of basis for $\Lambda^{\mu(k)} F_k$. Its conclusion asserts, instead of the existence of elements $\alpha(k, \mu)$, the existence of maps α_k making the following diagrams commute

$$
\begin{array}{ccc}
0 \longrightarrow \Lambda^{\#\mu(n)} F_n & \xrightarrow{\alpha_k^*} & \Lambda^{\text{rank } \phi_{k+1}} F_k \\
\alpha_{k-1} \Big\uparrow & & \Big\| \\
\Lambda^{\text{rank } \phi_k} F_{k-1}^* & \xrightarrow{\Lambda^{\text{rank } \phi_k} \phi_k^*} & \Lambda^{\text{rank } \phi_k} F_k^*
\end{array}
$$

where the isomorphism on the right is that given by the exterior product with a chosen basis of $\Lambda^\lambda F_k$ where $\lambda = \text{rank } F_k$; see [1, A, III, §11].

(3) Burch's Theorem is the special case of Theorem 1 in which $n = 2$ and $\#\mu(0) = 1$.

(4) It is easy to see that if Cok ϕ_1 is annihilated by a non−zero divisor of S, then rank $\phi_1 = \#\mu(0)$, so there will be a non−zero divisor $\alpha(0, \mu(0)) \in S$. This is precisely MacRae's $G(\text{Cok } \phi_1)$ [7, p. 159]. Our result thus gives a new (and more direct) derivation of many of the results in [7]. In particu− lar, we obtain a new proof of the well−known theorem that a regular local ring is a unique factorization domain. This proof is detailed in the next section of this paper.

(5) The proof of Theorem 1, along with the proof that any cyclic module S/I such that I is generated by three elements and S/I has finite homological dimension can be lifted, will appear elsewhere. In this paper we will do a special case of the liftability theorem − the case where S/I has homological dimension 3 and I has only 3 relations. This is contained in section four of this paper; the lifting technique to be used there is discussed in section three.

2. Some consequences of Theorem 1

Our first application is to the proof of a result proved for domains in MacRae's paper [6, Cor. 4.4] and extended, although not explicitly, to all noetherian rings in [7]. We abbreviate projective dimension to pd.

Theorem 2. Let S be a noetherian ring, $I = (s_1, s_2)$ an ideal generated by 2 elements. If pd $S/I < \infty$, then pd $S/I \leqslant 2$.

Proof. It suffices to prove that pd S/I is locally at most 2, so we may assume that S is local, so that S/I has a finite free resolution of the form

$$0 \to S^{\mu(n)} \to S^{\mu(n-1)} \to \ldots \to S^2 \xrightarrow[\phi_1]{} S \to S/I \to 0$$

where 2 is to be thought of as the index set with elements $\{1, 2\}$, and where the matrix representing ϕ_1 is (s_1, s_2).

Of course we may assume that pd $S/I \geqslant 2$, and that the above resolution is minimal.

We have $[\phi_1 ; i, 1] = s_i$, $i = 1, 2$, so (2) of Theorem 1 tells us that there are elements $\alpha = \alpha(0, 1)$ and $\alpha_i = \alpha(1, i)$, such that $\alpha \alpha_i = s_i$, $i = 1, 2$. Moreover, (3) of Theorem 1 tells us that the depth of the ideal (α_1, α_2) is at least 2, so (α_1, α_2) forms an S–sequence (if one of the α_i were a unit, s_1, s_2 would not form a minimal generating set for I). Part (3) of Theorem 1 also tells us that α is a non–zero divisor, so $(\alpha_1, \alpha_2) \cong (s_1, s_2)$ as modules. Since α_1, α_2 is an S–sequence, pd $(\alpha_1, \alpha_2) = 1$. Consequently pd $(s_1, s_2) = 1$, and pd $S/I = 2$, as required.

Corollary 1. Every regular local ring is a unique factorization domain.

Proof. Let S be a regular local ring. We will show that the intersection of principal ideals (s_1) and (s_2) in S is again a principal ideal; from this, unique factorization follows at once. Since S is local, projective ideals are principal. It is easy to check that the obvious sequence $0 \to (s_1) \cap (s_2) \to (s_1) \oplus (s_2) \to (s_1, s_2) \to 0$ is exact. But by Theorem 2, $\text{pd}(s_1, s_2) = \max(0, \text{pd } S/(s_1, s_2) - 1) \leqslant 1$, so $(s_1) \cap (s_2)$ is projective.

We will now state the special case of Theorem 1 that we will need for the lifting result to be proved in the next section.

Lemma 1. Suppose

$$0 \to S^1 \xrightarrow[\phi_3]{} S^\mu \xrightarrow[\phi_2]{} S^\mu \xrightarrow[\phi_1]{} S$$

is exact, where $\mu = \{\ 1,2,3\ \}$. Suppose that the matrices of ϕ_3, ϕ_2, ϕ_1 with respect to the given bases are

$$\phi_3 = \begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \end{pmatrix},$$

$$\phi_2 = \begin{pmatrix} \phi_{11} & \phi_{12} & \phi_{13} \\ \phi_{21} & \phi_{22} & \phi_{23} \\ \phi_{31} & \phi_{32} & \phi_{33} \end{pmatrix},$$

$$\phi_1 = (s_1 \ \ s_2 \ \ s_3);$$

then there exist α, α_1, α_2, α_3 such that α is a non–zero divisor, depth $(\alpha_1, \alpha_2, \alpha_3) \geqslant 2$, $s_i = \alpha\alpha_i$, $i = 1,2,3$, and $[\phi \ ; \mu - i, \ \mu - j] = \sigma_i\alpha_j$, $i,j = 1,2,3$. Moreover, since depth$(\sigma_1, \sigma_2, \sigma_3)$ is at least 3, σ_1, σ_2, σ_3 form an S–sequence.

3.   A technique for lifting

Our approach to lifting is based on the following lemma.

Lemma 2. Let R be a ring, $x \in R$, and $S = R/(x)$. Let B be an S–module, and let

$$\mathfrak{F} : F_2 \xrightarrow{\phi_2} F_1 \xrightarrow{\phi_1} F_0$$

be an exact sequence of S–modules with cok $\phi_1 \cong B$. Suppose that

$$\Gamma : G_2 \xrightarrow{\psi_2} G_1 \xrightarrow{\psi_1} G_0$$
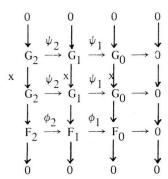
is a complex of R–modules such that

(1)   The element x is a non–zero divisor on each G_i,

(2) $G_i \otimes_R S = F_i$, and

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(3) $\psi_i \otimes_R S = \phi_i$.

Then, coker ψ_1 is a lifting of B to R.

Proof.   Conditions (1) and (2) imply that for each i the sequence $0 \to G_i \xrightarrow{X} G_i \to F_i \to 0$ is exact. Using condition (3) we obtain a commutative diagram with exact columns:

$$
\begin{array}{ccccccc}
0 & & 0 & & 0 & & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
G_2 & \xrightarrow{\psi_2} & G_1 & \xrightarrow{\psi_1} & G_0 & \longrightarrow & 0 \\
x \downarrow & & x \downarrow & & x \downarrow & & \downarrow \\
G_2 & \xrightarrow{\psi_2} & G_1 & \xrightarrow{\psi_1} & G_0 & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
F_2 & \xrightarrow{\phi_2} & F_1 & \xrightarrow{\phi_1} & F_0 & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & & 0 & & 0 & & 0
\end{array}
$$

Thus we obtain an exact sequence in homology $0 = H_1(\mathfrak{F}) \to H_0(\Gamma) \xrightarrow{x} H_0(\Gamma) \to H_0(\mathfrak{F}) \cong B \to 0$. But $H_0(\Gamma) = \mathrm{cok}(\psi_1)$, so $\mathrm{cok}(\psi_1)$ is a lifting of B as required.

Remarks.

(1) If we assume $x \in \mathrm{Rad}\, R$ and take $\mathfrak{F}'$ to be a complete resolution of B by S–modules, and Γ' a complex of R modules which reduces, modulo x, to $\mathfrak{F}'$, then the same argument, together with Nakayama's Lemma, shows that Γ' is exact.

(2) A sort of converse to Lemma 2 is also easy: Suppose x is a non–zero divisor in R, $S = R/(x)$. If A is a lifting to R of the S–module B, and if Γ is a projective R–resolution of A, then $\Gamma \otimes_R S$ is a projective resolution of B. Thus the problem of lifting a module from S to R is equivalent to the problem of finding an R–projective complex which reduces to a given S–projective resolution.

(3) Lemma 2 suggests a stronger form of the lifting problem: If ϕ_1 and ϕ_2 are matrices over S such that $\phi_1 \cdot \phi_2 = 0$, are there matrices over R which reduce modulo x to ϕ_1 and ϕ_2 and whose composite is 0? Of course, the answer is in general "no", but if S is a discrete valuation ring, every matrix over S is diagonalizable, so the answer is "yes". The question seems at least reasonable if R and S are regular local rings.

We now turn to two easy applications of Lemma 2. Recall that a sequence $s_1 \cdots s_n$ of elements of S is said to be a generalized S-sequence if the Koszul complex

$$K(S;s_1,\cdots,s_n) : \cdots \to \Lambda^3 R^n \to \Lambda^2 R^n \to R^n \xrightarrow{(s_1 \cdots s_n)} R$$

is exact.

Corollary 2. With the notation of Lemma 2, suppose x is a non-zero divisor on R and $B = S/(s_1,\cdots,s_n)$, where $s_1,\cdots,s_n$ form a generalized S-sequence. Then B can be lifted to R.

Proof. Lift the s_i arbitrarily to elements r_i of R. The Koszul complex $K(R;r_1,\cdots,r_n)$ reduces modulo x to $K(S;s_1,\cdots,s_n)$. By Lemma 2, $R/(r_1,\cdots,r_n)$ is a lifting.

Another class of modules which can be lifted for a similar reason is the class of cokernels of matrices whose minors generate an ideal of "large" depth. Explicitly, suppose that S is a noetherian ring, $\phi : S^m \to S^n$ a map of rank n of free S-modules. Suppose that the $n \times n$ minors of ϕ generate an ideal of depth $m - n + 1$ (which is the maximum possible depth). Then $\operatorname{cok} \phi$ can be lifted. An equivalent condition to the largeness of depth of the ideal above is that $\operatorname{Ext}^k(\operatorname{cok} \phi, S) = 0$, $0 \leqslant k \leqslant m - n$. The reason that this works is that under these hypotheses one can write down a free resolution for $\operatorname{cok} \phi$ in terms of the matrix ϕ itself, much as in the case of the Koszul complex. This resolution is called a generalized Koszul complex (see [2, Theorem 2.4] for details). Since there is a similar complex resolving $\operatorname{cok} \Lambda^n \phi$, modules of this form can also be lifted.

As a last result in this direction, we mention that since one can also explicitly write down a complex resolving modules of the form S/I where I is generated by any set of monomials in an S-sequence, these modules can also be lifted. For details of this complex, see [10, section IV].

Finally, Lemma 2 shows that any module of projective dimension 1 can be lifted.

Corollary 3. With notation as in Lemma 2, suppose x is a non-zero divisor on R and $\operatorname{pd}_S B = 1$. Then B can be lifted to R.

Proof. Let $0 \to F_1 \xrightarrow{\phi} F_0 \to B \to 0$ be an S-free resolution of B. Lift F_1 and F_0 to free R-modules G_1 and G_0. Since the canonical map $\operatorname{Hom}_R(G_1, G_0) \to \operatorname{Hom}_S(F_1, F_0)$ is onto, we may choose a map $G_1 \xrightarrow{\psi} G_0$ which reduces to ϕ modulo x. Lemma 2 now applies to the complex $0 \to G_1 \xrightarrow{\psi} G_0$,

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showing that coker ϕ is a lifting of B.

4. Ideals with 3 generators and 3 relations

As we mentioned in the introduction, Theorem 1 gives enough information about the structure of finite free resolutions to allow the lifting of any cyclic modules S/I where I is generated by three elements $I = (s_1, s_2, s_3)$. In this section we will exhibit a simple special case of the techniques involved, namely, the case in which there are just three relations on the elements (s_1, s_2, s_3).

Theorem 3. Suppose R is a noetherian ring, $x \in R$ a non–zero divisor, $S = R/(x)$. Suppose that $I = (s_1, s_2, s_3)$ is an ideal of S, and that S/I has a free resolution of the form

$$(*) \qquad 0 \to S \xrightarrow{\phi_3} S^\mu \xrightarrow{\phi_2} S^\mu \xrightarrow{\phi_1} S \to S/I$$

where $\mu = \{1,2,3\}$. Then S/I can be lifted to R.

Proof. We will adopt the notations of Lemma 1 of Section 2. Because $\mathrm{coker}(\phi_2)$ is torsion free, the sequence

$$0 \to S \xrightarrow[\phi_3]{} S^\mu \xrightarrow[\phi_2]{} S^\mu \xrightarrow[(\alpha_1, \alpha_2, \alpha_3)]{} S$$

is exact. If this sequence could be lifted to a complex of free R–modules

$$0 \to R \xrightarrow[\psi_3]{} R^\mu \xrightarrow[\psi_2]{} R^\mu \xrightarrow[(\beta_1, \beta_2, \beta_3)]{} R,$$

then if β is any element of R which reduces modulo x to α, it is clear that the sequence

$$0 \to R \xrightarrow[\psi_3]{} R^\mu \xrightarrow[\psi_2]{} R^\mu \xrightarrow[(\beta\beta_1, \beta\beta_2, \beta\beta_3)]{} R$$

is a lifting of (*). Thus it suffices to treat the case $\alpha = 1$, $\alpha_i = s_i$, $i = 1,2,3$, so that $\phi_1 = (\alpha_1, \alpha_2, \alpha_3)$.

Since the Koszul complex associated with ϕ_1 is a free complex, and since (*) is exact, there is a map $\gamma : \Lambda^2 S^\mu \to S^\mu$ making the following diagram commutative:

$$\Lambda^2 S^\mu \xrightarrow{\delta} S^\mu \xrightarrow{\phi_1} S \to S/I$$

$$\gamma \downarrow$$

$$S^\mu \xrightarrow[\phi_2]{} S^\mu \xrightarrow{\phi_1} S \to S/I \ ,$$

where δ is the boundary map of the Koszul complex associated to ϕ_1. If we take ϵ_1, ϵ_2, ϵ_3 to be the canonical basis of S^μ, then we may write

$$\gamma(\epsilon_{i_1} \wedge \epsilon_{i_2}) = \sum_j \gamma_j^{i_1 i_2} \epsilon_j,$$

and the commutativity yields

$$\phi_2(\gamma(\epsilon_{i_1} \wedge \epsilon_{i_2})) = \sum_k (\sum_j \gamma_j^{i_1 i_2} \phi_{kj}) \epsilon_k$$

$$= \delta(\epsilon_{i_1} \wedge \epsilon_{i_2})$$

$$= \alpha_{i_1} \epsilon_{i_2} - \alpha_{i_2} \epsilon_{i_1}$$

for all $i_1 < i_2$.

On the other hand, by virtue of Lemma 2,

$$\sigma_j \alpha_{i_1} \epsilon_{i_2} - \sigma_j \alpha_{i_2} \epsilon_{i_1} = [\phi_2 ; \mu - j, \mu - i_1] \epsilon_{i_2} - [\phi_2 ; \mu - j, \mu - i_2] \epsilon_{i_1}$$

$$= \phi_2(\sum_{k \neq j} \pm [\phi_2 ; \omega, \lambda] \epsilon_k)$$

where $\omega = \mu - \{ j, k \}$ and $\lambda = \mu - \{ i_1, i_2 \}$. Thus $\sigma_j \gamma(\epsilon_{i_1} \wedge \epsilon_{i_2}) - \sum_{k \neq j} \pm$ $[\phi_2 ; \omega, \lambda] \epsilon_k$ is in ker ϕ_2.

Since (*) is exact, we may write

$$\sigma_j \gamma(\epsilon_{i_1} \wedge \epsilon_{i_2}) - \sum_{k \neq j} \pm [\phi_2 ; \omega, \lambda] \epsilon_k = \phi_2(s_j^{i_1 i_2}) = \sum_k s_j^{i_1 i_2} \sigma_k \epsilon_k \ ,$$

for some $s_j^{i_1 i_2} \in S$.

Now the coefficient of ϵ_j in $\sum_{k \neq j} \pm [\phi_2 ; \omega, \lambda] \epsilon_k$ is 0, so

$$\sigma_j \gamma_j^{i_1 i_2} = \sigma_j s_j^{i_1 i_2} .$$

Since by Lemma 2, σ_j is part of an S-sequence, it is a non-zero divisor, so

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$$\gamma_j^{i_1 i_2} = s_j^{i_1 i_2}.$$

This gives

$$[\phi_2 ; \omega,\lambda] = \phi_{\omega,\lambda} = \gamma_k^{i_1 i_2} \sigma_j - \gamma_j^{i_1 i_2} \sigma_k.$$

We have now expressed the elements ϕ_{ij} of ϕ_2 in terms of ϕ_3 and γ. But the elements α_i of ϕ_1 are expressible in terms of ϕ_2 and γ, since $\delta = \phi_2 \gamma$. Thus both ϕ_1 and ϕ_2 may be expressed in terms of ϕ_3 and γ. In fact, if we form the matrix

(**)
$$\begin{pmatrix} \sigma_1 & \sigma_2 & \sigma_3 \\ \gamma_1^{12} & \gamma_2^{12} & \gamma_3^{12} \\ \gamma_1^{13} & \gamma_2^{13} & \gamma_3^{13} \\ \gamma_1^{23} & \gamma_2^{23} & \gamma_3^{23} \end{pmatrix},$$

then it is easy to verify that the element ϕ_{ij} is $\pm$ the 2×2 minor of (**) obtained by omitting the ith column and the rows involving $\gamma_k^{i_1 i_2}$ with $i_1 = j$ or $i_2 = j$. The element α_i is $\pm$ the 3×3 minor of (**) obtained by omitting the row involving $a_k^{i_1 i_2}$ with $i \neq i_1$ or i_2.

It is now easy to lift ϕ_2 and ϕ_1 to matrices $\widetilde{\phi}_2 = (\widetilde{\phi}_{ij})$ and $\widetilde{\phi}_1 = (\widetilde{\alpha}_1, \widetilde{\alpha}_2, \widetilde{\alpha}_3)$ such that $\widetilde{\phi}_1 \widetilde{\phi}_2 = 0$; this will complete the lifting, by virtue of Lemma 2.

Simply find elements $\widetilde{\sigma}_i$ and $\widetilde{\gamma}_k^{ij}$ in R which reduce modulo x to σ_i and γ_k^{ij}. Form the matrix

(**)~
$$\begin{pmatrix} \widetilde{\sigma}_1 & \widetilde{\sigma}_2 & \widetilde{\sigma}_3 \\ \widetilde{\gamma}_1^{12} & \widetilde{\gamma}_2^{12} & \widetilde{\gamma}_3^{12} \\ \widetilde{\gamma}_1^{13} & \widetilde{\gamma}_2^{13} & \widetilde{\gamma}_3^{13} \\ \widetilde{\gamma}_1^{23} & \widetilde{\gamma}_2^{23} & \widetilde{\gamma}_3^{23} \end{pmatrix}$$

Now define the elements $\widetilde{\phi}_{ij}$ and $\widetilde{\alpha}_i$ of $\widetilde{\phi}_2$ and $\widetilde{\phi}_1$ to be the minors formed from (**)~ in the same way that we formed minors of (**) to represent ϕ_{ij} and α_i, so that $\widetilde{\phi}_{ij}$ and $\widetilde{\alpha}_i$ will reduce, modulo x, to ϕ_{ij} and α_i respectively.

It is now easy to check by direct computation that $\widetilde{\phi}_1 \widetilde{\phi}_2 = 0$.

Note Added in Proof. The computation involved in the proof of Theorem 3 can be considerably shortened by comparing (*) with the dual of the Koszul complex associated to ϕ_3^* and applying Theorem 1 to get α's for this Koszul complex. This trick eliminates the use of the Koszul complex of ϕ_1.

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