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Modules Over Dedekind Prime Rings

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A *Dedekind prime ring* [18] is an hereditary Noetherian prime ring[1] which is a maximal order in its quotient ring, or equivalently [5] an hereditary Noetherian prime ring with no proper idempotent two-sided ideals. The object of this paper is to give a structure theory for finitely generated modules over a Dedekind prime ring. The similarity of our results to the commutative case [12] is in striking contrast to the situation for hereditary Noetherian prime rings in general (see Section 4). Specifically, we prove that a finitely generated module over a Dedekind ring is a direct sum of a projective module and an Artinian torsion module; that the projective part is a direct sum of a free module and a right ideal; that the right ideal can be generated by two elements, one chosen (almost) at random; that the torsion part is a direct sum of cyclic modules with nonzero annihilators and a cyclic module no subquotient of which has a nonzero annihilator; (In [5], we prove that this last result holds for a larger class of hereditary Noetherian prime rings.); and that a finitely generated projective module and an essential submodule can be decomposed simultaneously as direct sums of right ideals.

One large class of Dedekind rings is the class of maximal orders in central simple algebras over commutative Dedekind domains. All these rings are *bounded*, that is, each of their essential one-sided ideals contains a nonzero two-sided ideal. From our point of view, bounded Dedekind prime rings are practically commutative, for in the bounded case, all our results are

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[1] Conditions assumed on rings will always be assumed to hold on both sides; for example, a Noetherian ring satisfies the ascending chain condition both on left and on right ideals.

obtainable by straightforward modifications of the commutative proofs, and are more or less well known. Our results are new for Dedekind prime rings which are not bounded. Some examples of this kind are described in Section 4.

We now outline the paper. Section 1 is devoted to various results on hereditary Noetherian prime rings which will be useful in the remainder of this paper. Most of these are well-known, an exception being Theorem 1.3, which says that such a ring satisfies a restricted minimum condition. Sections 2, 3, and 4 detail the results about Dedekind prime rings outlined above. Section 2 deals with the splitting of a finitely generated module and with the projective summand, Section 3 with the torsion summand, and Section 4 with examples. In addition, it is proved in Section 2 that a non-finitely generated projective module is free and, in Section 3, that any two essential right ideals of a Dedekind prime ring are in a certain sense coprime (Corollary 3.8). The latter result enables us to apply the theory of Chevalley [4] to arbitrary Dedekind prime rings, and some consequences of this are outlined in Section 5.

For simplicity in the statements of our results, we will assume throughout that the Dedekind prime rings with which we deal are *not* simple Artinian rings.

1. Hereditary Prime Rings

We begin with some results on arbitrary hereditary Noetherian prime rings which will be useful in the sequel. The reader who is interested only in domains may skip this section except for Lemma 1.2 and Theorem 1.3. By virtue of Goldie's theorem [7] any Noetherian prime ring is a right and left order in a simple Artinian quotient ring; we digress for a moment to sketch some rather technical properties of an order R in a simple Artinian quotient ring Q.

A right ideal U of R is *uniform* if any two nonzero submodules of U have a nonzero intersection. It is easy to show that if U is a uniform right ideal of R then UQ is a minimal right ideal of Q. Also, if M is a minimal right ideal of Q, then $M \cap R$ is a maximal uniform right ideal of R. Since M is the right annihilator in Q of an element of Q, it follows that any maximal uniform right ideal is the right annihilator in R of some element of R.

Of course, Q is an n by n matrix ring over a division ring for some n, and any direct sum of n minimal right ideals inside Q is equal to Q. Thus we see that a right ideal of R is an essential submodule of R if and only if it contains a direct sum of n uniform right ideals. (Recall that, by [7] Theorem 4.8, a right ideal is essential in R if, and only if, it contains a regular element.) In general, if the direct sum of k uniform right ideals is essential in a right ideal I, we say that I has *uniform dimension* k.

Any uniform right ideal contains a copy of any other. For, if U and V are uniform right ideals, then $UV \neq 0$ since R is prime. But VQ is a minimal right ideal of Q so that, for any $q \in Q$, $qVQ = 0$ or $qx \neq 0$ for each nonzero $x \in VQ$. Since there is an element $u \in U$ such that $uV \neq 0$, we see that $U \supseteq uV \cong V$. One consequence of this is that, if n is the uniform dimension of R, then an arbitrary direct sum of n uniform right ideals is isomorphic to an essential right ideal.

LEMMA 1.1. *A Noetherian prime ring which contains a minimal right ideal is simple Artinian.*

Proof. The minimal right ideal is certainly uniform. Since an isomorphic copy of every other uniform right ideal is contained in it, every uniform right ideal is minimal. Now there is a finite direct sum of uniform right ideals which is an essential right ideal and so contains a regular element, whence it contains a right ideal isomorphic to R. Thus R is a finite direct sum of simple right modules, all isomorphic. Hence R is a simple Artinian ring. ▮

Let R again be an order in a simple Artinian ring Q. A right R-submodule I of Q is called a *fractional right R-ideal* if there is a unit $q \in Q$ such that qI is an essential right ideal of R. It is clear that any homomorphism from I to another fractional right R-ideal can be extended to an endomorphism of Q and hence may be regarded as left multiplication by some element of Q. Since I contains a unit of Q (qI contains a regular element of R) this extension is unique. Thus

$$I^* = \{q \in Q \mid qI \subseteq R\} \cong \operatorname{Hom}_R(I, R)$$
$$O_l(I) = \{q \in Q \mid qI \subseteq I\} \cong \operatorname{Hom}_R(I, I).$$

I^* is clearly a fractional left R-ideal. We will have occasion to use the following easy generalization of [3] Proposition 3.2, p. 132. For more details see [18], Section 1.

LEMMA 1.2. *Let R be an order in a simple Artinian ring Q and let I be a fractional right R-ideal. Then I is projective if, and only if, $II^* = O_l(I)$, and in this case I and I^* are finitely generated R-modules.*

We now shift our attention to hereditary Noetherian prime rings. The next theorem is due to Webber [20].

THEOREM 1.3. *Let R be an hereditary Noetherian prime ring which is not simple Artinian, and let $J \subseteq I$ be right ideals of R. Then I/J is an Artinian R-module if, and only if, J is an essential submodule of I.*

Proof. $\Rightarrow$. If J is not essential in I, there is a right ideal $K \subseteq I$ such that $J \oplus K \subseteq I$. If I/J were Artinian, K would contain a minimal right ideal, a contradiction, by Lemma 1.1, of the assumption that R is not simple Artinian.

$\Leftarrow$. Suppose J is essential in I. There is a right ideal H such that $I \oplus H$ is essential in R; and then $J \oplus H$ is essential in R too and $I \oplus H/J \oplus H \simeq I/J$. Hence we may assume that J and I are essential in R to begin with. Suppose $I \supseteq I_1 \supseteq I_2 \supseteq \cdots \supseteq J$ is a descending chain of essential right ideals. Evidently $I^* \subseteq I_1{}^* \subseteq \cdots \subseteq J^*$ is an ascending chain of submodules of the finitely generated left module J^* (Lemma 1.2). Hence the chain terminates.

Set $I_i^{**} = \{q \in Q \mid I_i{}^* q \subseteq R\}$. Obviously $I_i \subseteq I_i^{**}$. On the other hand, $I_i = I_i R \supseteq I_i I_i{}^* I_i^{**} \supseteq I_i^{**}$, since $I_i I_i{}^* = O_l(I) \ni 1$. Thus the original chain terminates. ∎

In Section 2 we will require the following two results concerned with uniform right ideals.

LEMMA 1.4. *Let R be an hereditary Noetherian prime ring. Then any projective module is a direct sum of uniform right ideals.*

Proof. In [3] Theorem 5.3, p. 13 (or see [12]) it is shown that if $R \simeq \coprod_{i=1}^{n} I_i$ is a direct sum of right ideals, then any projective module has the form $\coprod_{k \in K} J_k$, where each J_k is contained in some I_i. Hence it suffices to show that R is a finite direct sum of uniform right ideals. But we have already remarked that each maximal uniform right ideal U is the right annihilator of some element $r \in R$. Since rR is projective, the exact sequence $U \rightarrowtail R \twoheadrightarrow rR$ splits. Iterating the argument, we see that R is a direct sum of uniform right ideals. ∎

COROLLARY 1.5. *Let R be an hereditary Noetherian prime ring, I and I' two right ideals of the same uniform dimension. Then I is isomorphic to an essential submodule of I'.*

Proof. It suffices to note that I and I' are direct sums of uniform right ideals of the same length, and that each uniform right ideal is isomorphic to an essential submodule of every other uniform right ideal. ∎

2. THE SPLITTING OF FINITELY GENERATED MODULES, AND THE STRUCTURE OF THE PROJECTIVE SUMMAND

In this section, we describe the splitting of a finitely generated module over an hereditary Noetherian prime ring into a direct sum of a torsion module and a projective module, as established by Levy [14]. We then go on to the structure of projective modules over a Dedekind prime ring. Along the way we produce an easy proof that every right ideal may be generated by two elements.

Let R be an hereditary prime ring with quotient ring Q, A a right R-module. Following Levy, we say that $a \in A$ is a *torsion element* if there is a regular element $r \in R$ such that $ar = 0$. Since, by Goldie's theorem, R satisfies the Ore condition, the set of torsion elements of A is a submodule $t(A) \subseteq A$. $A/t(A)$ is evidently torsion free (has no torsion elements). From Theorem 1.3 it is evident that a finitely generated module is a torsion module if and only if it is Artinian.

As is shown in [14], any finitely generated torsion free module N is a submodule of a free module. For, since N is torsion free, N is a submodule of the right Q module $N \otimes_R Q$, which, as a finitely generated right Q-module, is a submodule of a free Q-module of rank n, say. A generating set for N may thus be thought of as a finite set of n-tuples of elements of Q. Multiplying by a common denominator, we easily see that N is a submodule of a free R-module. Thus a finitely generated torsion free module is projective.

THEOREM 2.1. *Let R be an hereditary Noetherian prime ring, and let A be a finitely generated right R-module. Then $A/t(A)$ is projective and $A \cong t(A) \oplus A/t(A)$.*

Proof. We have already shown that $A/t(A)$ is projective, so the exact sequence $t(A) \rightarrowtail A \twoheadrightarrow A/t(A)$ splits. This yields $A \cong t(A) \oplus A/t(A)$ as required. ∎

For the remainder of this section, we turn our attention to the structure of the projective summand $A/t(A)$. In Section 3 we take up the torsion summand. All our results after this point are proved for Dedekind prime rings; in fact, most of them fail for arbitrary hereditary Noetherian prime rings. Some examples of this failure are given in Section 4.

For the proof of the next theorem, we require the fact that every right ideal I of a Dedekind prime ring R is a generator for the category of right R-modules, or, equivalently, $I^*I = R$, where $I^* = \{q \in Q \mid qI \in R\}$. Since R is Dedekind, every essential right ideal J is *invertible*, that is, $J^*J = R$ and $JJ^* = O_l(J)$. (We will write $J^* = J^{-1}$ in this case.) Now $I^*I = T$ is a two-sided ideal and, since R is a prime ring, T is essential and thus invertible, so $T^{-1}I^*I = R$. From this equation, $T^{-1}I^* \subseteq I^*$, and hence $I^*I = R$.

The following two theorems generalize similar theorems of Webber [20] for the case of a simple hereditary domain. An improved result in the direction of Theorem 2.2 will be given in Corollary 3.8.

THEOREM 2.2. *Let R be a Dedekind prime ring, and let I and K be right* *ideals of the same uniform dimension. Let J be any right ideal. Then there is a right ideal L, essential in J, such that*

$$I \oplus J \cong K \oplus L.$$

Proof. By Corollary 1.5, we can assume that $I \subseteq K$, and that K/I is Artinian. We will prove the theorem by induction on the length n of a composition series for K/I. If $n = 0$, take $L = J$. If $n = 1$, K/I is simple. Since J is a generator for the category of right R-modules, there is a nonzero map $J \to K/I$ which must be an epimorphism, since K/I is simple. Choose L to be the kernel of this map, so that $J/L \cong K/I$. By Schanuel's Lemma [13], p. 169, $I \oplus J \cong K \oplus L$ as required. L is essential in J by Theorem 1.3.

Suppose $n > 1$, and let K' be a right ideal such that $K \supset K' \supset I$ and K'/I is simple. By induction, there is a right ideal L' essential in J, such that $I \oplus J \cong K' \oplus L'$. Also, since K/K' has length $n - 1$, there is a right ideal L essential in L' with $K' \oplus L' \cong K \oplus L$. Thus $I \oplus J \cong K \oplus L$, and L is essential in J. ∎

Note 2.3. Let K be any essential right ideal of R. By Theorem 2.2 with $I = J = R$, there is a right ideal L such that $R \oplus R \cong K \oplus L$. Thus, since every right ideal is a direct summand of an essential right ideal, we see that every right ideal of R may be generated by two elements. See also Corollary 3.5.

As an easy consequence of Theorem 2.2, we now obtain the structure of a projective module over a Dedekind prime ring.

THEOREM 2.4. *Let R be a Dedekind prime ring, and let A be a projective right R-module. Then:*

(i) *If A is finitely generated, then $A \cong R \oplus \cdots \oplus R \oplus I$ for some right ideal I of R.*

(ii) *If A is not finitely generated, then A is free.*

Proof. (i) As was noted in Section 1, A is isomorphic to a direct sum of uniform right ideals of R. Moreover, if the uniform dimension of R is n, then any direct sum of $k \leqslant n$ uniform right ideals is isomorphic to a right ideal of R, and if $k = n$, the right ideal is essential. Thus, grouping the uniform summands of A in groups of n, $A \cong I_1 \oplus \cdots \oplus I_t \oplus J$, where $I_1, \ldots, I_n$ are essential right ideals and J is a nonzero right ideal. By Theorem 2.2, $I_1 \oplus I_2 \cong R \oplus L_1$, $L_1 \oplus I_3 \cong R \oplus L_2$, and thus

$$I_1 \oplus \cdots \oplus I_t \cong R \oplus \cdots \oplus R \oplus L_{t-1},$$

where $L_1, \ldots, L_{t-1}$ are essential right ideals of R. Finally $L_{t-1} \oplus J \cong R \oplus I$ where I is some right ideal of R. Hence $A \cong R \oplus \cdots \oplus R \oplus I$ as required.

(ii) In this case, by grouping the uniform summands of A in groups of n, we see that A is isomorphic to a direct sum of essential right ideals. Clearly, it suffices to show that a countable sum of essential right ideals is free. Our proof follows an arrangement due to L. Levy of Kaplansky's commutative proof [12].

Let $A = I_1 \oplus I_2 \oplus I_3 \oplus I_4 \oplus \cdots$, where the I_k are essential right ideals of R. We know, by Theorem 2.2, that, given an essential right ideal I, there is an essential right ideal which we call I', such that $I \oplus I' \cong R \oplus R$. Note that $I_1 \oplus I_2 \cong R \oplus J_1$, for some essential right ideal J_1. Then $I_3 \oplus I_4 \cong J_1' \oplus J_2$ for some right ideal J_2, and so on. Thus we can write:

$$\begin{aligned} A &\cong (I_1 \oplus I_2) \oplus (I_3 \oplus I_4) \oplus (I_5 \oplus I_6) \oplus \cdots \\ &\cong (R \oplus J_1) \oplus (J_1' \oplus J_2) \oplus (J_2' \oplus J_3) \oplus \cdots \\ &\cong R \oplus (J_1 \oplus J_1') \oplus (J_2 \oplus J_2') \oplus \cdots. \end{aligned}$$

Hence A is an ascending union of free modules, each a summand of the next, so A is free. ∎

For a maximal order R in a central simple algebra, Swan shows in [19] that if $F \oplus A \cong F \oplus B$ where F is a free module of finite rank and A, B are essential right ideals of R, then $R \oplus A \cong R \oplus B$. It would be interesting to know whether this is true for Dedekind prime rings.

3. The Structure of the Torsion Summand

It is well-known that ideals in a commutative Dedekind domain D can be generated by two elements, the first of which may be an arbitrary nonzero element of the ideal, that a finitely generated torsion module over the domain is a direct sum of cyclics, and that a finitely generated projective module and an essential submodule can be decomposed simultaneously as direct sums of right ideals. The theory is facilitated by the fact that a finitely generated torsion module has a nonzero annihilator T and may be considered as a module over the Artinian principal ideal ring D/T. The same is true of a bounded Dedekind prime ring and the commutative proofs extend readily to this case. These results can also be proved for simple Dedekind rings, although with more difficulty. Thus the main result of this section is Theorem 3.9 which says, in effect, that in dealing with a finitely generated torsion module over a Dedekind prime ring R, it suffices to have control over both the case when R is bounded and the case when R is simple. First, however, we prove the "one generator at random" theorem (Theorem 3.3 and Corollary 3.6); that is, we prove that any Artinian subquotient of R is cyclic. The reader will notice that effectively we attack this by first dealing separately with the case when the subquotient has no annihilator (as in a simple ring) and with the case when it has an annihilator (as in a bounded ring). Throughout this section, we will maintain the convention that R is a Dedekind prime ring and Q is its quotient ring.

We start by isolating a lemma which we will have several occasions to use. First, two definitions. A module is *completely faithful* if every submodule of each of its factor modules is faithful. A module is *unfaithful* if it has a nonzero right annihilator which then, of course, is a two-sided ideal.

LEMMA 3.1. *Suppose that A is a module of finite length and that C is a submodule such that A/C is cyclic. Then*

(a) *If C is simple and $C \rightarrowtail A \twoheadrightarrow A/C$ does not split, then A is cyclic.*

(b) *If C is completely faithful, then A is cyclic.*

Proof. (a) We claim that if $a \in A$ is such that $a + C$ generates A/C, then $aR = A$. Clearly aR contains a representative of each coset of A modulo C. If $aR \not\supseteq C$, then aR is mapped isomorphically onto A/C, and the inverse of this isomorphism splits $C \rightarrowtail A \twoheadrightarrow A/C$. Hence if this sequence does not split, $aR \supseteq C$ and $aR = A$.

(b) The proof is by induction on the length of C. For $C = 0$, the result is trivial. Otherwise, choose a simple module $S \subseteq C$. By the induction hypothesis A/S is cyclic. If $S \rightarrowtail A \twoheadrightarrow A/S$ does not split, A is cyclic by part (a).

On the other hand, suppose $A \cong S \oplus A/S$. Since S is simple, the annihilators of nonzero elements of S are the maximal right ideals M such that $R/M \cong S$. The intersection of all these maximal right ideals is the annihilator of S. Since C is assumed to be completely faithful, S is faithful, so this intersection is 0. Now A/S is cyclic and has finite length, so we may write $A/S \cong R/K$ for some nonzero right ideal K. We see that there is a maximal right ideal M such that $R/M \cong S$ and $M \not\supseteq K$. Then

$$\begin{aligned} R/(M \cap K) &\cong [K/(M \cap K)] \oplus M/(M \cap K) \\ &\cong [(M + K)/M] \oplus [(M + K)/K] \\ &\cong (R/M) \oplus (R/K) \cong S \oplus A/S, \end{aligned}$$

and the left side of this formula is clearly cyclic. ∎

The next theorem is easily seen (Corollary 3.6) to be equivalent to the statement that any right ideal of R may be generated by two elements, the first of which may be chosen at random. By way of introduction, we remark that this is not too unreasonable, since R is closely associated with various completions which are even principal ideal rings. Since the heart of Theorem 3.3 is a special case (Lemma 3.2) established by considering the completion, we will digress to sketch its properties. The details are given in [8].

Given a Dedekind prime ring R and a nonzero two-sided ideal T, the completion of R at T (that is, the completion of R in the topology given by taking the powers of T as basic neighborhoods of 0) is given by

$$\hat{R}_T = \varprojlim R/T^n.$$

Denote the operation of completing in this topology by $\hat{}$.

In the commutative case, $\hat{R}$ corresponds to the ring obtained by first localizing at T and then completing with respect to the T-adic topology. As in the commutative case, $\hat{R}$ is a principal ideal ring. (Roughly speaking, this happens because the R/T^n are all principal ideal rings.) It is easy to see that $\hat{R}/\hat{T} \cong R/T$, and it is also true that, for any right ideal I of R, $I/IT \cong \hat{I}/\hat{I}\hat{T}$ as an $R/T \cong \hat{R}/\hat{T}$-module. Since $\hat{I}$ is a principal right ideal of $\hat{R}$, I/IT is a cyclic module. This sketches the proof of the following result ([8], Lemma 3.5).

Lemma 3.2. *If I is any right ideal of R and T is a proper two-sided ideal of R then I/IT is a cyclic R-module.*

We now turn to the proof of the theorem.

Theorem 3.3. *Let $J \subseteq I$ be right ideals of R such that J is an essential submodule of I. Then I/J is a cyclic module.*

Proof. The proof is by induction on length(I/J), which is finite by Theorem 1.3. The case $n = 1$ is trivial. For $n > 1$, choose a simple submodule S of I/J and write

$$S \rightarrowtail I/J \twoheadrightarrow I/J'. \tag{*}$$

By the induction hypothesis, I/J' is cyclic and then, by Lemma 3.1, I/J is cyclic unless S is unfaithful and (*) splits. Since S was arbitrary, we conclude that if I/J is not cyclic, it is a finite direct sum of unfaithful simple modules, $I/J = \coprod_i S_i^{(1)} \oplus \coprod_i S_i^{(2)} \oplus \cdots$, where all the $S_i^{(j)}$ are simple and $S_i^{(j)} \cong S_k^{(j)}$ for every i, j, k. It is enough to show that each $\coprod_i S_i^{(j)}$ is cyclic; for, if $R/K_j \cong \coprod_i S_i^{(j)}$, then $R/\bigcap_j K_j \cong I/J$. (It is clearly contained in I/J and they both have the same length.) Hence, we may assume that all the $S_i^{(j)}$ are isomorphic; we drop the superscript. Let T be the annihilator of S_i. Then

T is a nonzero two-sided ideal and, by Lemma 3.2, I/IT is cyclic. But T annihilates I/J; that is, $J \supseteq IT$. So $I/J \cong (I/IT)/(J/IT)$ is cyclic too. ∎

COROLLARY 3.4. *Any submodule B of an Artinian cyclic module A is cyclic.*

Proof. $A \cong R/K$ for some essential right ideal K. Therefore $B \cong L/K$ for some right ideal $L \supseteq K$. ∎

COROLLARY 3.5. *If I is an essential right ideal of R and $i \in I$ is a regular element, then $I = iR + i'R$ for some element $i' \in I$.*

Proof. Take $J = iR$ in the theorem. ∎

To extend this corollary to an arbitrary right ideal I we note that, if $I \oplus K$ is essential, then $I \oplus K$ contains a regular element $i + k$. Evidently $iR \oplus kR$ is essential and so iR is an essential submodule of I. In fact, since an essential right ideal is generated by its regular elements ([17], Theorem 5.5) I is generated by elements such as i. The next corollary says that any one of these may be chosen as the first of two generators of I.

COROLLARY 3.6. *If I is any right ideal of R and i is any element of I such that iR is essential in I, then $I = iR + i'R$ for some element $i' \in I$.*

This corollary is of interest in the following situation. Let S be a prime principal ideal ring. By [6], S is a full matrix ring over a domain D. An example of Swan ([19], the ring Λ, p. 57.) shows that D need not be a principal ideal domain. However, by [18] Theorem 4.5, D is a (noncommutative) Dedekind domain, and so every right ideal of D may be generated by two elements, the first chosen at random.

COROLLARY 3.7. If I, J, K are right ideals of R with K essential in R and J essential in I, then there is a right ideal $L \subseteq K$ such that $I/J \cong K/L$.

Proof. As in the proof of Theorem 1.3, we can assume that I and J are essential right ideals of R. Then $IK^{-1} \supseteq JK^{-1}$ are fractional right ideals of $O_l(K) = KK^{-1}$. Therefore there is a unit $q \in Q$, the quotient ring of R, such that $qIK^{-1} \subseteq O_l(K)$. Then $qIK^{-1}/qJK^{-1} \cong IK^{-1}/JK^{-1}$. However, by [18] Lemma 3.1, $O_l(K)$ is itself a Dedekind prime ring and so, by Theorem 3.3, $IK^{-1}/JK^{-1} \cong KK^{-1}/H$ where H is an essential right ideal of $O_l(K)$. Now right multiplication by K preserves isomorphisms (since K is a progenerator this actually induces an equivalence of $\mathrm{Mod}(O_l(K))$ and $\mathrm{Mod}(R)$—see [2].) Therefore $I/J = IK^{-1}K/JK^{-1}K \cong KK^{-1}K/HK = K/HK$. Set $HK = L$. ∎

Applying Schanuel's lemma ([13], p. 169) we see that $I \oplus L \cong J \oplus K$. Thus we obtain a version of Theorem 2.2 except that there K is not required to be essential. However, it can be seen that K must be essential in Corollary 3.7. For, otherwise, choose a right ideal K' such that $K \oplus K'$ is essential and let T be a nonzero two-sided ideal of R. Then ([8], Lemma 3.5)

$$(K \oplus K')/(K \oplus K')\, T \cong R/T \cong K/KT \oplus K'/K'T$$

and so length$(K/KT) <$ length(R/T). On the other hand, if $K/L \cong R/T$ for some L, it must be that $KT \subseteq L$ and so length$(K/KT) \geqslant$ length(R/T), a contradiction.

As a final corollary to Theorem 3.3 we give the following result which will be used in Section 5.

Corollary 3.8. *Given an ordered pair I, J of essential right ideals, there is a unit q in the quotient ring Q such that $qI + J = R$.*

Proof. By Corollary 3.7, there is a homomorphism of I onto R/J. Since I is projective this factors as $I \xrightarrow{\varphi} R \twoheadrightarrow R/J$ say. Hence $\varphi(I) + J = R$. However, every homomorphism of I into R is a left multiplication by some element $q' \in Q$. Also J, being essential, contains a regular element, and it follows, by [17] Theorem 5.5, that there is a regular element q in $q' + J$. Then q satisfies the requirements. ∎

We now return to the central theme of this section, the structure of finitely generated torsion modules. In fact, their structure will be deduced as an easy consequence of the following theorem.

Theorem 3.9. *A finitely generated torsion module over R is the direct sum of a completely faithful module and an unfaithful module.*

As a first step in its proof, we prove the following much weaker result.

Lemma 3.10. *Let A be a finitely generated torsion module over R. Then A is an extension of a completely faithful module by an unfaithful module.*

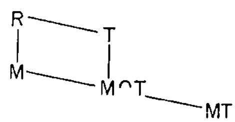
Proof. Assume for the moment that every faithful finitely generated torsion module has a faithful simple submodule. Let B be any finitely generated torsion module, C a maximal completely faithful submodule of B. Then, if B/C is faithful, it must have a faithful simple submodule B'/C and clearly B' is completely faithful and larger than C, a contradiction.

Thus it suffices to prove that every finitely generated torsion module A has a faithful simple submodule. If A is completely faithful, we are done. Otherwise, choose a composition series $0 = A_0 \subset A_1 \subset \cdots \subset A_n = A$, and let k be the smallest integer such that A_{k+1}/A_k is faithful. If $a \in A_{k+1}$ generates A_{k+1} modulo A_k, then following [15], p. 35, we consider this picture.

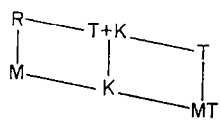
$$\begin{tikzcd}
& A_{k+1} = A_k + aR \arrow[dl, dash] \arrow[dr, dash] & \\
A_k \arrow[dr, dash] & & aR \arrow[dl, dash] \\
& A_k \cap aR = B \arrow[d, dash] & \\
& 0 &
\end{tikzcd}$$

Now $A_{k+1}/A_k \simeq aR/(A_k \cap aR)$. It is clear that the cyclic module $aR \subseteq A$ is an extension of an unfaithful module by a faithful simple module. Hence, without loss of generality, we may assume that A is cyclic and has an unfaithful submodule B such that A/B is faithful and simple. We will prove that $A \simeq B \oplus A/B$.

Since A is cyclic, we may write $A \simeq R/K$ for some right ideal K. Let $M \supseteq K$ be the maximal right ideal corresponding to B. Since B is unfaithful by hypothesis, $\operatorname{ann}(B) = \operatorname{ann}(M/K) \neq 0$. Let T be any two-sided ideal contained in $\operatorname{ann} B$. We first prove that $R/MT \simeq M/MT \oplus T/MT$. For this, it suffices to show that $M + T = R$ and $M \cap T = MT$. The former is easy; M is a maximal right ideal and cannot contain T since R/M is faithful by hypothesis. To prove the latter consider this diagram.



Since $R/M \simeq T/M \cap T$, $\text{length}(T/M \cap T) = 1$. But length $T/MT =$ length $R/M = 1$, since T is invertible. Hence $MT = M \cap T$. Thus we consider this picture.



Since M/K is annihilated by T, $K \supseteq MT$. It now follows that $M + (T + K) = R$ and $M \cap T + K) = K$ so that

$$A \simeq R/K \simeq M/K \oplus (T + K)/K \simeq B \oplus A/B. \quad \blacksquare$$

Proof of Theorem 3.9. By Lemma 3.10, any finitely generated torsion module A is an extension of a completely faithful module C by an unfaithful module U. We will show that $\operatorname{Ext}_R{}^1(U, C) = 0$.

Let T be a nonzero ideal, $T \subseteq \operatorname{ann} U$. Then U is an R/T module, and so there is an exact sequence $L \rightarrowtail \coprod_X R/T \twoheadrightarrow U$ for some index set X. Applying the long exact sequence in Ext we get

$$\operatorname{Hom}_R(L, C) \to \operatorname{Ext}_R{}^1(U, C) \to \prod_X \operatorname{Ext}_R{}^1(R/T, C) \text{ exact.}$$

But $\operatorname{Hom}_R(L, C) = 0$ since every element of L is annihilated by a nonzero two-sided ideal and no element of C is. Thus it suffices to show that $\operatorname{Ext}_R{}^1(R/T, C) = 0$.

Now suppose $C \rightarrowtail A \twoheadrightarrow R/T$ is exact. By Lemma 3.1 (b), A is cyclic and we may write $A \cong R/K$ with $R \supset T \supset K$, $T/K \cong C$. If in the adjoining picture, $L = R$ and $T \cap KT^{-1} = K$ then $A \cong R/K \cong R/T \oplus T/K \cong R/T \oplus C$ and the proof is complete.

$$
\begin{tikzcd}[column sep=small]
& R \arrow[d, dash] & \\
& L = T+KT^{-1}\ (=R) \arrow[dl, dash] \arrow[dr, dash] & \\
T \arrow[dr, dash] & & KT^{-1} \arrow[dl, dash] \\
& T \cap KT^{-1}\ (=K) \arrow[d, dash] & \\
& K &
\end{tikzcd}
$$

Assume $T + KT^{-1} = L \subset R$. Then $T \supset LT \supseteq KT^{-1}T = K$ and $T^2 \subseteq LT$ so T annihilates T/LT, contradicting the hypothesis that T/K is completely faithful. To see that $T \cap KT^{-1} = K$ we note that it contains K and that $T/(T \cap KT^{-1}) \cong T + KT^{-1}/KT^{-1} = R/KT^{-1}$. But length $(R/KT^{-1}) =$ length(T/K) since right multiplication by T and T^{-1} induces a one-to-one correspondence of right ideals between R and KT^{-1} with right ideals between T and K. Hence $T \cap KT^{-1} = K$. ▮

The argument above actually proves more than is claimed. Call a module U *locally unfaithful* if the annihilator of each element of U contains a nonzero two-sided ideal of R. Then the argument shows that $\mathrm{Ext}_R{}^1(U, C) = 0$ for any completely faithful module C of finite length.

As an easy consequence of Theorem 3.9 we have

THEOREM 3.11. *Every finitely generated torsion R-module A is a direct sum of cyclic modules each of which is either unfaithful or completely faithful.*

Proof. By Theorem 3.9, A is a direct sum of an unfaithful module and a completely faithful module. By Lemma 3.1 (b), the completely faithful module is cyclic; and by [10] Theorem 43 and the fact that factor rings of R are Artinian principal ideal rings, the unfaithful module is a direct sum of cyclics also. ▮

By [10], p. 46, each indecomposable unfaithful cyclic module has a unique composition series. Whether something similar is true of indecomposable completely faithful cyclics is an open question.

As another application of Theorems 3.2 and 3.9, we prove a simultaneous decomposition theorem for torsion free modules over a noncommutative Dedekind prime ring. The commutative case of this result was established by Steinitz. For a proof see Chevalley [4]. Our proof closely follows a proof by Hunter [9] for the case of a bounded Dedekind prime ring.

THEOREM 3.12. *Let $P' \subset P$ be finitely generated projective right modules over a Dedekind prime ring R, such that P/P' is a torsion module. Then there is a decomposition $P = \coprod_{k=1}^{n} I_k$, such that I_k is a right ideal of R, and such that $P' = \coprod_{k=1}^{n} P' \cap I_k$.*

First we prove

LEMMA 3.13. *Let A be a finitely generated torsion module over a Dedekind prime ring R. Then A has a maximal cyclic submodule which is also a direct summand.*

Proof. By Theorem 3.9, we may write $A \cong B \oplus C$, where C is completely faithful and B has nonzero annihilator T. By [9], Theorem 6 (or a close reading of [10], pp. 78–79) the R/T module B has a maximal cyclic submodule B' which is also a direct summand of B. By Lemma 3.1 (b), $C \oplus B'$ is a cyclic submodule of A. It is a maximal cyclic submodule since a larger one would have a larger, cyclic projection into B. Moreover, it is a direct summand, as required. ∎

Proof of Theorem 3.12. Let $p \in P$ be such that $\bar{p} \in P/P'$ generates a maximal cyclic submodule of P/P' which is a direct summand. Since P is projective, we may regard it as embedded in a free Q-module. It is clear that $P/(pQ \cap P)$ is torsion free, and hence by Theorem 2.1, $P/(pQ \cap P)$ is projective, and $P = (pQ \cap P) \oplus D$. Let $\pi : P \to P/P'$ be the projection map. We wish first to establish that $\pi(pQ \cap P) = \bar{p}R$.

Clearly $\pi(pQ \cap P) \supseteq \bar{p}R$, so it suffices to show that $\pi(pQ \cap P) \cong pQ \cap P/pQ \cap P'$ is cyclic. Now pQ is a homomorphic image of Q as a right Q module; thus, since Q is simple Artinian, we may regard pQ as a right ideal of Q. Clearly $pQ \cap P$ is then a finitely generated R-submodule of Q, so it is isomorphic to a right ideal of R. Since $pQ \cap P/pQ \cap P'$ is contained in an Artinian module, it is itself Artinian, and so by Theorem 1.3, $pQ \cap P'$ is an essential submodule of $pQ \cap P$. Thus by Theorem 3.3, $pQ \cap P/pQ \cap P'$ is cyclic, as desired, and $\pi(pQ \cap P) = \bar{p}R$.

Let f be the composite map $D \xrightarrow{\pi|D} P/P' \to \bar{p}R$, where the last map is the projection onto the direct summand $\bar{p}R$. If $f = 0$, then we are done. For in that case, if $p' \in P'$, with $p' = (p_1', p_2') \in (pQ \cap P) \oplus D$, $\pi(p') = 0$ implies $\pi(p_1') = \pi(p_2') = 0$, so $P' = pQ \cap P' \oplus D \cap P'$, and an induction on the rank of P finishes the proof.

If $f \neq 0$ we can use the projectivity of D to lift f to a map g making the following diagram commutative.

$$\begin{tikzcd} & & D \arrow[d, "f"] \arrow[lld, dashed, "g"'] \\ pQ\cap P \arrow[r] & P/P' \arrow[r] & \bar{p}R \end{tikzcd}$$

If now we take $D' = (1 - g)(D)$, we see that $P = (pQ \cap P) \oplus D'$ and that the composite map $D' \xrightarrow{\pi|D'} P/P' \to \bar{p}R$ is 0. This reduces the problem to the case with which we have just dealt.

4. Examples

We now give some examples in order to illustrate these results and place them in perspective. As mentioned in the introduction, maximal orders in central simple algebras over a commutative Dedekind domain provide a range of examples of bounded Dedekind prime rings (see [1]). The rings we describe next are noncommutative Dedekind domains which fall outside this range. Then, by [18] Theorem 4.5, any ring Morita equivalent to one of these will be a Dedekind prime ring, again outside this range.

(i) *Differential Polynomials*

Let E be a field of characteristic zero.

(a) Let $D = E(y)[x]$ be the ring of noncommutative polynomials in x over the field of rational functions $E(y)$ subject to $xy - yx = 1$. Then D is a simple principal ideal domain ([10], p. 30, Ex. 4).

(b) Let $D = E[y, x]$ be the ring of noncommutative polynomials in x and y over E subject to $xy - yx = 1$. Then D is a simple hereditary Noetherian domain [16].

(ii) *Skew Polynomials*

Let E be any field with an automorphism σ.

(a) Let $D = E[x; \sigma]$ be the ring of polynomials in x subject to $fx = xf^\sigma$, $f \in E$. Then D is a principal ideal domain ([10], p. 29). If σ is of infinite order, D is primitive but not simple ([11], p.22)

(b) Let $D = E[x, x^{-1}; \sigma]$ be the ring of polynomials in x and x^{-1} subject to $fx = xf^\sigma, f \in E$. Then D is a principal ideal domain and, if σ is of infinite order, D is simple ([11], p. 211).

Next we consider the results of Sections 2, 3 and ask whether they hold for an arbitrary hereditary Noetherian prime ring. It is easy to see that the complete theory cannot be extended in this direction. For example, let D be a commutative Dedekind domain with a unique maximal ideal dD. Then the ring

$$R = \begin{pmatrix} D & D \\ dD & D \end{pmatrix}$$

is known (see [18], pp. 262–3) to be a bounded hereditary Noetherian prime ring. The factor ring by the ideal

$$\begin{pmatrix} dD & D \\ d^2D & dD \end{pmatrix}$$

is isomorphic to $\begin{pmatrix} F & 0 \\ F & F \end{pmatrix}$ where $F = D/dD$. Since this ring is not a principal ideal ring (e.g. the right ideal $\begin{pmatrix} F & 0 \\ F & 0 \end{pmatrix}$ is not principal) the "one generator at random" property (Theorem 3.3) does not hold for R. Nor is a factor ring by an invertible ideal a principal ideal ring. For example, $\begin{pmatrix} dD & dD \\ d^2D & dD \end{pmatrix}$ is an invertible ideal of R, but $\bar{R} = R/\begin{pmatrix} dD & dD \\ d^2D & dD \end{pmatrix}$ has as a homomorphic image the ring $R/\begin{pmatrix} dD & D \\ d^2D & dD \end{pmatrix}$, which is not a principal ideal ring, as was shown above. (Note: $\bar{R}$ is not isomorphic to the ring $\begin{pmatrix} F & F \\ F & F \end{pmatrix}$.)

This ring R has idempotent ideals, e.g., $T = \begin{pmatrix} dD & D \\ dD & D \end{pmatrix}$. Since T is not a generator for the category of right modules, neither is $T \oplus T$. However, for any right ideal I, $R \oplus I$ is a generator. Thus $T \oplus T$ is not isomorphic to any module of that form and so the theory of projective modules does not extend to hereditary Noetherian prime rings.

As regards the splitting of torsion modules the interested reader is referred to [5], Sections 3 and 5.

5. Chevalley's Arithmetic Theory

In the monograph [4], Chevalley considers (in our terminology) a noncommutative Dedekind domain D satisfying the condition that, given any ordered pair I, J of nonzero right ideals, there is an element f in the quotient division ring F of D such that $I + fJ = D$. Since, as we have shown in Corollary 3.8 this condition is automatically satisfied by a noncommutative Dedekind domain D, the theory of [4] can be applied to D.

Let F_n be the n by n matrix ring over F. Then D_n is a Dedekind prime ring (see [18]) and is an order in F_n. Chevalley characterizes all the Dedekind prime rings R of the form $R = O_l(L)$, for some fractional right D_n ideal L. Call such a subring R of F_n a Dedekind prime ring over D. By [18], every Dedekind prime ring is such a ring R, for some D. This, together with [4] yields

Theorem 5.1. *Let R be a Dedekind prime ring. Then*

$$R \cong \begin{pmatrix} D & \cdots & D & I^{-1} \\ \vdots & & \vdots & \\ D & \cdots & D & I^{-1} \\ I & \cdots & I & O_l(I) \end{pmatrix}$$

where D is a noncommutative Dedekind domain, and I is a right ideal of D.

We note that this theorem may also be obtained as an easy consequence of Theorem 2.3 and [18] Theorem 4.5.

The final theorem of the relevant section of [4] gives a characterization of Dedekind prime rings over D very similar to that of maximal orders over a commutative Dedekind domain. It reads as follows:

THEOREM. *A subring R of F_n is a Dedekind prime ring over D if, and only if,*

(i) $1 \in R$

(ii) RD[2] *is a finitely generated D-module, and*

(iii) *R is maximal subject to satisfying* (i) *and* (*ii*).

It is not difficult to see that some alteration is required. Consider (i) (a) of Section 4, $D = E(y)[x]$. Let $I = xD$, and consider $R = xDx^{-1} = O_2(xD)$. Then using the theorem in the case $n = 1$, RD should be a finitely generated right D-module. But $RD = xDx^{-1}D \cong Dx^{-1}D$, and this would make $Dx^{-1}D$ a fractional right D-ideal. Note that we can equally well write $D = E(y)[-x]$ where $y(-x) - (-x)y = 1$. This symmetry shows that $Dx^{-1}D = D(-x)^{-1}D$ must also be a fractional left D-ideal. But D, being simple, has only one fractional two-sided ideal, namely itself. Since $Dx^{-1}D \neq D$, RD cannot be finitely generated.

A minor alteration of the theorem reads as follows:

THEOREM 5.2. *Let D be a noncommutative domain, F its quotient division ring. Then a subring R of F_n is a Dedekind prime ring over D if, and only if,*

(i) $1 \in R$

(ii) *There is a nonzero element $d \in D$ such that RdD is a finitely generated right D-module, and*

(iii) *R is maximal subject to satisfying* (i) *and* (ii).

Proof. $\Rightarrow$: Let R be a Dedekind prime ring over D. Then by [18],

$$R = \begin{pmatrix} I_1 I_1^{-1} & \cdots & I_1 I_n^{-1} \\ \vdots & & \vdots \\ I_n I_1^{-1} & \cdots & I_n I_n^{-1} \end{pmatrix}$$

where $I_1, \ldots, I_n$ are fractional right D-ideals, and so are finitely generated. Choose $d \in D$ such that $I_i^{-1} d \subseteq D$ for $i = 1, \ldots, n$. Then

$$RdD \subseteq \begin{pmatrix} I_1 & \cdots & I_1 \\ \vdots & & \\ I_n & \cdots & I_n \end{pmatrix}$$

[2] There is a printer's error at this point in [4].

which is clearly a finitely generated right D-module. Finally, to check (iii), let $R' \supseteq R$, and suppose there is a nonzero $d' \in D$ such that $R'd'D$ is a finitely generated right D-module. Then $Rd'D_n$ must be a finitely generated right D_n-module. This, together with the fact that $Rd'D_n$ contains a regular element d', makes it plain that $Rd'D_n$ is a fractional right D_n-ideal. However, since D_n is a Dedekind prime ring over D, so too is $O_l(Rd'D_n)$. But $O_l(Rd'D_n) \supseteq R' \supseteq R$, and R, being a Dedekind prime ring, is a maximal order. Therefore, $O_l(Rd'D_n) = R' = R$.

$\Leftarrow$: Conversely, given that RdD is a finitely generated right D-module, it follows that RdD_n is finitely generated over D_n and, as above, is a fractional right D_n-ideal. Thus $O_l(RdD_n) = S$, say, is a Dedekind prime ring over D. Clearly $S \supseteq R$ and $RdD_n = S(RdD_n) = SdD_n$. Therefore, SdD, being a D-submodule of RdD_n, must be finitely generated. Thus S satisfies (i) and (ii) and so $S = R$. This proves that R is a Dedekind prime ring over D. ∎

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