Comment. Math. Helvetici 62 (1987) 169–184 0010–2571/87/020169–16$01.50 + 0.20/0

On the non-vanishing of cotangent cohomology

LUCHEZAR AVRAMOV and STEPHEN HALPERIN

To E. S. Golod on his fiftieth birthday

1. Introduction

In this paper rings are always commutative, noetherian with identity. With a ring homomorphism $f: R \to S$ there is associated its *cotangent complex* $\mathbb{L}_{S|R}$, defined uniquely up to isomorphism in the derived category of the category of S-modules. Introduced in dimensions ≤ 1 by Grothendieck for the study of deformation theory, it was extended to dimension 2 by Lichtenbaum–Schlessinger and subsequently constructed in all dimensions by André [1] and Quillen [15]. The elegance and flexibility of their approach, which makes essential use of simplicial methods, is accompanied by a considerable computational complexity, so that information on the higher cotangent cohomology groups $T^i(S \mid R, M) \overset{\text{def}}{=} H^i \operatorname{Hom}_S (\mathbb{L}_{S|R}, M)$ has remained very scarce, except for the cases where they are known to vanish for all S-modules M.

Indeed, the vanishing of $T^i(S \mid R, -)$ in low dimensions has been shown to characterize important classes of maps, and to display remarkable rigidity properties, as illustrated in the Vanishing Theorem below. For the precise statement we need the following terminology, generally in accord with [16; esp. Exposé VIII].

Recall that S has *flat dimension* $<n$ over R ($\operatorname{fd}_R S < n$) if $\operatorname{Tor}_i^R(S, -) = 0$, $i \geq n$ or, equivalently, if S admits an R-flat resolution F_* with $F_i = 0$, $i \geq n$. Similarly, $\mathbb{L}_{S|R}$ has *projective dimension* $<n$ ($\operatorname{pd}_S \mathbb{L}_{S|R} < n$) if $T^i(S \mid R, -) = 0$, $i \geq n$ or, equivalently if $\mathbb{L}_{S|R}$ is isomorphic in the derived category to a complex P_* of projective S-modules with $P_i = 0$, $i \geq n$.

The ring homomorphism $f: R \to S$ is called *smooth* if it is flat, with geometrically regular fibres. (Departing from [16] we do not insist that S be a finitely generated R-algebra.) More generally, f is *smoothable* if it can be factored as $R \xrightarrow{g} Q \xrightarrow{h} S$ with g smooth and h surjective, and so if S is a finitely generated R-algebra then f is smoothable.

Finally, a smoothable ring homomorphism $f: R \to S$ is said to be a locally complete intersection (l.c.i.) map if for each prime ideal $\mathfrak{q}$ of S the ideal $(\ker h)_{q'}$ is generated by a $Q_{\mathfrak{q}'}$ regular sequence – here $\mathfrak{q}' = h^{-1}(\mathfrak{q})$. It is known ([16; Exposé VIII, (1.3)] or the Vanishing Theorem below) that this condition does not depend on the choice of factorization of f. Evidently the properties of being smooth, or l.c.i., localize.

We can now state the Vanishing Theorem, essentially due to Grothendieck for (1) and to Lichtenbaum–Schlessinger for (2), but proved in the generality stated below by André [1; Supplément, Théorème 30 and Chap. VI, Théorème 25]. (See also Quillen [15; Theorems 5.4 and 5.5].)

VANISHING THEOREM. (1) *For a homomorphism f the following are equivalent*:

(i) *f is smooth*;
(ii) $\mathrm{pd}_S\, \mathbb{L}_{S|R} < 1$;
(iii) $T^1(S \mid R, -) = 0$.

(2) *For a smoothable homomorphism f the following are equivalent*:

(i) *f is l.c.i.*;
(ii) $\mathrm{pd}_S\, \mathbb{L}_{S|R} < 2$;
(iii) $T^2(S \mid R, -) = 0$.

In relation to these results Quillen ([15; p. 76]) sets the following:

"*Unsolved Problem.* Characterize the morphisms . . . of finite type, for which the cotangent complex is of finite projective dimension. What computations we have been able to make show this is rare and support the following conjectures:

CONJECTURE 1. If $\mathrm{pd}_S\, \mathbb{L}_{S|R} < \infty$, then $\mathrm{pd}_S\, \mathbb{L}_{S|R} < 3$.

CONJECTURE 2. If $\mathrm{pd}_S\, \mathbb{L}_{S|R} < \infty$ and $\mathrm{fd}_R\, S < \infty$, then f is a l.c.i. homomorphism."

(The original labels the conjectures (5.6) and (5.7) and uses slightly different notation.)

Conjecture 1 is proved, when R is local and $S = k$ is its residue field, in case $\mathrm{char}\,(k) = 0$ by combining [15, (7.4)] with the main result of [12], and in general in [5]. In no other case has either conjecture been verified.

Our first theorem establishes, in characteristic zero, a result which is stronger than the claim of Conjecture 2.

THEOREM A. *Let $f: R \to S$ be a smoothable homomorphism such that $\mathrm{fd}_R\,(S) < \infty$, and suppose S contains as a subring the field $\mathbb{Q}$ of rational numbers.*

If $\mathbf{q}$ *is a prime ideal of* S, *at which* f *is not l.c.i. then there exists an integer* N *such that* $T^i(S \mid R, S/\mathbf{q}) \neq 0$ *for* $i \geq N$.

The proof of Theorem A is reduced by localizing to the study of local homomorphisms $f:(R, \mathbf{m}, k) \to (S, \mathbf{n}, l)$ of local rings in which we need only consider the behaviour of f at $\mathbf{n}$. Thus we replace the conditions of smoothable (resp. l.c.i. map) by *factorizable* (resp. *l.c.i. map at* $\mathbf{n}$). A local homomorphism $f:(R, \mathbf{m}, k) \to (S, \mathbf{n}, l)$ is factorizable if it decomposes as $R \xrightarrow{g} Q \xrightarrow{h} S$ with Q flat over R, $Q/\mathbf{m}Q$ a regular local ring and h surjective; and f is l.c.i. at $\mathbf{n}$ if f is factorizable with $\ker h$ generated by a Q-regular sequence. If the maximal-ideal-adic completion $\hat{f}: \hat{R} \to \hat{S}$ is factorizable then f is called *formally factorizable*.

In this setup we get a stronger result (Theorem B below) than is needed for Theorem A with, in particular, no restriction on the characteristic of k. For the statement we recall (cf. §2 for more details) that the local homomorphism, f, determines a class of augmented DG Γ-algebras F^f: the *homotopy fibre* of f. These in turn yield a class of naturally isomorphic graded Lie algebras, $\pi^*(F^f)$.

If f is surjective and char $k = 0$ then $\pi^{i+1}(F^f) = T^i(S \mid R, k)$ – cf. §6 – and this provides the connection to Theorem A and the reason for the restriction there to characteristic zero.

THEOREM B. *Let* $f:(R, \mathbf{m}, k) \to (S, \mathbf{n}, l)$ *be a factorizable* (*resp. formally factorizable*) *local homomorphism such that* $\mathrm{fd}_R(S) < \infty$. *Then either* f (*resp.* $\hat{f}$) *is an l.c.i. map at* $\mathbf{n}$, *or else there is an integer* N *for which* $\pi^i(F^f) \neq 0$, $i \geq N$.

The preceding results show how the structure of the map f is reflected by the (non-)vanishing of certain homotopy invariants, hence – ultimately – in terms of numerical data. However, they are reduced from the following structural statement, which really lies at the heart of the matter.

THEOREM C. *Let* $f:(R, \mathbf{m}, k) \to (S, \mathbf{n}, l)$ *be a factorizable* (*resp. formally factorizable*) *local homomorphism such that* $\mathrm{fd}_R(S) < \infty$. *Then either* f (*resp.* $\hat{f}$) *is an l.c.i. map at* $\mathbf{n}$, *or else there are elements* $\alpha \in \pi^2(F^f)$, $\beta \in \pi^*(F^f)$ *such that*

$$(\mathrm{ad}\,\alpha)^n \beta \neq 0, \qquad n \geq 0.$$

The last two theorems specialize to yield new homotopical characterizations of local rings S which are complete intersections; i.e., which have the property that in some Cohen presentation of the n-adic completion $\hat{S}$ as $R/\mathbf{a}$, with R regular local, the ideal $\mathbf{a}$ is generated by an R-regular sequence. Recall that the integer

$e_i(S) = \dim_k \pi^i(S)$ is called the i-th deviation of S, and that it can be formally defined from the identity of power series in $\mathbb{Z}[[t]] : \sum_{i\geq 0} \dim_k \mathrm{Tor}_i^R(k, k)\ t^i = \prod_{i>0} (1-(-t)^i)^{(-1)^{i+1}} e_i(S)$.

We then have

THEOREM D. *The following conditions on the local ring S are equivalent:*

(i) *S is a complete intersection;*
(ii) *$e_i(S) = 0$ for $i \geq 3$;*
(iii) *$e_i(S) = 0$ for i sufficiently large;*
(iv) *$e_i(S) = 0$ for infinitely many values of i;*
(v) *$\pi^{\geq 2}(S)$ is an abelian Lie algebra;*
(vi) *$\pi^*(S)$ is nilpotent;*
(vii) *$\pi^*(S)$ is Engel; i.e. there exists an integer n such that* $(\mathrm{ad}\ \alpha)^n = 0$ *for all* $\alpha \in \pi^*(S)$;
(viii) *each $\alpha \in \pi^2(S)$ acts locally nilpotently on $\pi^*(S)$.*

Of course, (i)$\Leftrightarrow$(ii) is not new, being the Tate–Assmus characterization of local complete intersections (cf. [13]), and (iii)$\Leftrightarrow$(i) is the main result of [12]: these statements, included for completeness, also follow easily during the course of our proof. That (v) implies (i) was announced in [3]. The equivalence of (i) with the remaining conditions was proved and announced by the authors ([7]) at the conference held at the C.I.R.M. of Luminy in June, 1982. We would also like to point out that one of the ingredients in the proof of Theorem C is inspired by an idea of Felix and Halperin ([9; Theorem 10.4]).

Let us finally note that the implication (iv)$\Rightarrow$(i) may be restated as the assertion that $e_i(S) \neq 0$ for all but finitely many i's, when S *is not* a complete intersection. For connected graded algebras over a field of characteristic zero, this was established earlier by Felix and Thomas [10; (IV.5)]. The motivation behind these results is Conjecture C_3 from [3], which amplifies an early remark from [13; p. 154], and asserts that the vanishing of a single $e_i(S)$ already implies S is a complete intersection.

2. Homotopy Lie algebras, and the homotopy fibre

All differential objects are graded non-negatively below with derivative of degree -1. Consider the category of differential graded algebras with divided powers (= DG Γ-algebras; for definitions see, e.g., [13]), augmented onto a field. Morphisms preserve all the structure and, in particular, commute with the augmentation. Thus local homomorphisms of local rings are in the category, the rings being treated as concentrated in degree zero. A morphism which induces an isomorphism of homology is called a *quasi-isomorphism*, and is denoted by $\xrightarrow{\simeq}$.

If $R \to k$ is a DG Γ-algebra then the augmentation factors as $R \to R\langle X\rangle \to k$, where the first map is an inclusion of DG Γ-algebras obtained by adjunction of a set X of free anticommuting Γ-variables, while the second is a quasi-isomorphism sending X to zero. Then $H(k \otimes_R R\langle X\rangle)$ is a graded Γ-algebra independent of the choices involved.

Recall that Moore [14] (but see also [11] and [8]) has extended the functor Tor to the differential category. Here we have a natural identification ([8])

$$\operatorname{Tor}^R(k, k) = H(k \otimes_R R\langle X\rangle)$$

so that $\operatorname{Tor}^R(k, k)$ inherits the structure of a Γ-algebra.

The graded vector space obtained from $\operatorname{Tor}^R_+(k, k)$ by dividing the linear span of products xy and divided powers $\gamma^i(z)$ $(x, y, z \in \operatorname{Tor}^R_+(k, k)$, $\deg z$ even, $i \geq 2)$ is denoted $\pi_*(R)$. When $H(R)$ is piecewise noetherian (H_0 is noetherian and each H_i is a finitely generated $H_0(R)$-module) each $\pi_i(R)$ is finite dimensional over k. In this case its graded dual, $\pi_*(R)^\vee$, is naturally a graded Lie algebra, called the *homotopy Lie algebra* of R, and written $\pi^*(R)$. For more details the reader is referred to [4] and [8]. *Henceforth all DG Γ-algebras will be supposed piecewise noetherian.*

(2.1) EXAMPLES. (1) If R is a regular local ring, the Koszul complex L on a minimal set of generators of its maximal ideal is a minimal R-free resolution of k, hence $\operatorname{Tor}^R(k, k) = H(k \otimes_R L) = k \otimes_R L = \bigwedge \operatorname{Tor}^R_1(k, k)$, and $\pi^*(R) = \pi^1(R)$. It is well known that this characterizes regularity: cf. e.g. [13].

(2) Let $k\langle \bar{a}_1, \ldots, \bar{a}_m\rangle$ be the exterior algebra on generators of degree 1 over a field k. Let U denote the DG Γ-algebra $k\langle \bar{a}_1, \ldots, \bar{a}_m, y_1, \ldots, y_m\rangle$ with the y_i's variables of degree 2 and the differential defined by $dy_i = \bar{a}_i$, $1 \leq i \leq m$. It is classical that $H(U) = k$. Hence $\operatorname{Tor}^{k\langle \bar{a}_1, \ldots, \bar{a}_m\rangle}(k, k) = H(k \otimes_{k\langle \bar{a}_1, \ldots, \bar{a}_m\rangle} U) = k\langle y_1, \ldots, y_m\rangle$, so that $\pi^*(k\langle \bar{a}_1, \ldots, \bar{a}_m\rangle) = \pi^2(k\langle \bar{a}_1, \ldots, \bar{a}_m\rangle) = (ky_1 \oplus \cdots \oplus ky_m)^\vee$.

If $f:(R \to k) \to (S \to l)$ is a homomorphism of augmented DG Γ-algebras, a natural map $f^*: \pi^*(S) \to \pi^*(R) \otimes_k l$ is defined, which is a homomorphism of graded Lie algebras over l. When f is a quasi-isomorphism, so that in particular $k = l$, f^* is an isomorphism.

Suppose now that $f:(R \to k) \to (S \to l)$ is an arbitrary morphism of augmented DG Γ-algebras and let $R \to R\langle X\rangle \to k$ be as above. Then the augmented DG Γ-algebra

$$F^f = S \otimes_R R\langle X\rangle \to l$$

is uniquely determined up to quasi-isomorphism ([4], [8]). It is called the *homotopy fibre* of f.

On the other hand we can also factor f as $R \to R\langle Y\rangle \xrightarrow{\phi} S$ with ϕ a quasi-isomorphism. The natural maps

$$S \otimes_R R\langle X\rangle \leftarrow R\langle X\rangle \otimes_R R\langle Y\rangle \to k \otimes_R R\langle Y\rangle \tag{2.2}$$

are then quasi-isomorphisms ([4], [8]) of augmented DG Γ-algebras. It follows that F^f is uniquely determined up to quasi-isomorphism and quasi-isomorphic to $k\langle Y\rangle = k \otimes_R R\langle Y\rangle$.

There is a natural identification

$$H(F^f) = \mathrm{Tor}^R(S, k),$$

and the observations above imply that the homotopy Lie algebra $\pi^*(F^f)$ is also an invariant of f, independent of the choice of F^f. (The identifications between two choices of homotopy fibre are themselves uniquely determined up to homotopy, but we need not go into details here, and use the symbol F^f to denote ubiquitously any one of the DG Γ-algbras in (2.2).)

The canonical inclusion $S \hookrightarrow S \otimes_R R\langle X\rangle$ is a map of DG Γ-algebras, which will be denoted $j: S \to F^f$. The importance of the notion of homotopy fibre is apparent from the next result:

(2.3) FACT [4; (2.7)]. If $\mathrm{Tor}_i^R(S, k) = 0$ for i sufficiently large, there are for every $n > 0$, six-term exact sequences of vector spaces over l

$$0 \longrightarrow \pi^{2n-1}(F^f) \xrightarrow{j^*} \pi^{2n-1}(S) \xrightarrow{f^*} \pi^{2n-1}(R) \otimes_k l$$
$$\longrightarrow \pi^{2n}(F^f) \xrightarrow{j^*} \pi^{2n}(S) \xrightarrow{f^*} \pi^{2n}(R) \otimes_k l \longrightarrow 0$$

Finally, we review the notion of minimal models, as introduced in this context by Avramov (cf, e.g. [4]); they are our main computational technique. Let $\varepsilon: R \to k$ be a DG Γ-algebra. A *free extension* of R is a DG algebra of the form $R[Y]$ obtained from R by the adjunction of a set Y of free anticommuting variables (exterior in odd degrees, polynomial in even degrees) augmented by ε in R and 0 on Y, and such that $R \hookrightarrow R[Y]$ is a DG algebra morphism. The free extension is called *minimal* if in $k[Y] = k \otimes_R R[Y]$ the image of the differential is in the square of the (maximal) ideal generated by Y.

Now observe that $k[Y]$ is the direct sum of the subspaces $k^m[Y]$ linearly spanned by the elements of the form $y_{i_1} \cdot \cdots \cdot y_{i_m}$ ($y_{i_a} \in Y$). Thus if $R[Y]$ is a minimal free extension we may write the differential in $k[Y]$ as an "infinite" sum:

$$d = d_2 + d_3 + \cdots$$

with d_m a derivation taking Y *to* $k^m[Y]$.

In this case $(d_2)^2 = 0$. Hence (cf. [4]) $(k[Y], d_2)$ determines a graded Lie algebra L_Y^* defined as follows. Set $\bar{Y} = k^1[Y]$ – it is the graded space with Y as basis – and set $L_Y^i = \mathrm{Hom}_k(\bar{Y}_{i-1}, k)$. Interpret the elements of $k^2[Y]$ as bilinear functions in L_Y^* via

$$\langle y_1 \cdot y_2; \alpha, \beta \rangle = \langle y_1, \beta \rangle \langle y_2, \alpha \rangle + (-1)^{\deg y_2 \deg y_1} \langle y_1, \alpha \rangle \langle y_2, \beta \rangle,$$
$$y_i \in Y, \ \alpha, \beta \in L_Y^*.$$

Then the Lie bracket in L_Y^* is defined by

$$\langle y, [\alpha, \beta] \rangle = (-1)^{\deg \beta} \langle d_2 y; \alpha, \beta \rangle.$$

In particular, suppose $f: R \to S$ is a morphism of augmented DG Γ-algebras such that $R_0 \to S_0$ is surjective map of local rings. Then ([4]) f factors as $R \to R[Y] \to S$ in which the first morphism is a minimal free extension and the second induces an isomorphism of homology. This is called a *minimal model* for f. Write $V = R[Y]$ and let $V(i)$ be the sub DGA generated over R by the variables in Y of degree $\leq i$. The following properties are then obtained ([4]):

(2.4.1) Y is concentrated in degrees ≥ 1.

(2.4.2) Y has finitely many variables of any given degree.

(2.4.3) If $Y_i = \{y_1, \ldots, y_s\}$ then $dy_1, \ldots, dy_s \in V(i-1)$ represent a basis of $k \otimes_R H_{i-1}(V(i-1))$.

On the other hand factor f as $R \to R\langle X \rangle \xrightarrow{m} S$ with m a surjective quasi-isomorphism and use induction on Y to get a quasi-isomorphism $R[Y] \to R\langle X \rangle$ reducing to the identity in R. This yields (see e.g. [8]) a quasi-isomorphism $k[Y] = k \otimes_R R[Y] \xrightarrow{\simeq} k \otimes_R R\langle X \rangle = F^f$ (cf. (2.2)). There follows

(2.5) FACT. $\mathrm{Tor}^R(S, k) \cong H(k[Y])$ as graded algebras.

Furthermore if W is any DG Γ-algebra with $W_0 = k$ and if $k[Y] \xrightarrow{\simeq} W$ is a minimal model for the inclusion $k \hookrightarrow W$ then

(2.6) FACT ([4: (4.2)]). There is a graded Lie algebra isomorphism $L_Y^* \cong \pi^*(W)$.

In particular in the situation above we have

(2.6)′ FACT. $L_Y^* \cong \pi^*(F^f)$.

Specialize now to the case that $R = R_0$, $S = S_0$, so that f is a surjective homomorphism of local rings. Put $\mathbf{a} = \ker f$; then

$$\mathbf{a} \otimes_R k \cong \mathrm{Tor}_1^R(S, k) \cong k[Y]_1 = \bar{Y}_1 \cong [\pi^2(F^f)]^\vee. \tag{2.7}$$

Indeed, since $k[Y]$ is minimal $H_1(k[Y]) = k[Y]_1$, and so (2.7) simply restates (2.5) and (2.6)′ in dimension one.

Now $V(1)$ is the Koszul complex on a minimal set of generators of $\mathbf{a}$, while (by minimality) we have that in $k[Y]$ the differential is zero in Y_1 and Y_2 and maps Y_3 to $\bar{Y}_1 \cdot \bar{Y}_1$. Combining these facts with (2.4.3), (2.5) and (2.6)′ yields

$$k \otimes_R H_1(E) \cong \bar{Y}_2 \cong \mathrm{Tor}_2^R(S, k)/[\mathrm{Tor}_1^R(S, k)]^2 \cong [\pi^3(F^f)]^\vee. \tag{2.8}$$

3. Factorizable local homomorphisms

Let $f: R, \mathbf{m}, k) \to (S, \mathbf{n}, l)$ be a local homomorphism of local rings. Observe that if f is smoothable it is factorizable, and if it is factorizable it is formally factorizable. Moreover [6; (4.2)] if l is separable over k (via f) then f is formally factorizable.

(3.1) LEMMA. *There is a natural augmented* DG *Γ-algebra morphism $f': F^f \to F^{\hat{f}}$ which induces an isomorphism $(f')^*: \pi^*(F^{\hat{f}}) \xrightarrow{\cong} \pi^*(F^f)$; moreover, $fd_R(S) = fd_{\hat{R}}(\hat{S})$.*

Proof. Let $U = R\langle X\rangle \xrightarrow{\simeq} k$ be as in §2; then $\hat{U} = \hat{R} \otimes_R U = \hat{R}\langle X\rangle \xrightarrow{\simeq} k$ by flatness of completions. Let f' be the morphism

$$F^f = S \otimes_R U \to \hat{S} \otimes_R U = \hat{S} \otimes_{\hat{R}} \hat{U} = F^{\hat{f}}.$$

Now factor the augmentation $F^f \to l$ as $F^f \to F^f\langle Z\rangle \xrightarrow{\simeq} l$. Then $F^{f'} = F^{\hat{f}} \otimes_{F^f} F^f\langle Z\rangle = \hat{S} \otimes_S F^f\langle Z\rangle$ and so by flatness of completions $F^{f'} \xrightarrow{\simeq} 1$. In particular $\mathrm{Tor}_+^{F^f}(F^{\hat{f}}, l) = 0$ and (2.3) implies that $(f')^*$ is an isomorphism.

The equality of flat dimensions follows from the classical $\mathrm{Tor}^{\hat{R}}(\hat{S}, k) = \hat{S} \otimes_S \mathrm{Tor}^R(S, k)$ by faithful flatness of completions. ■

(3.2) LEMMA. *If f is factorizable (as $R \xrightarrow{g} Q \xrightarrow{h} S$), then there is a natural augmented* DG *Γ-algebra morphism $g' : F^f \to F^h$ such that $(g')^* : \pi^*(F^h) \to \pi^*(F^f)$ is an isomorphism in degrees >2. Moreover $fd_R(S) \le fd_Q(S) = pd_Q(S) \le fd_R(S) + edim(Q/\mathbf{m}Q)$, edim denoting the minimal number of generators of the maximal ideal.*

Proof. Let $U = R\langle X\rangle \xrightarrow{\simeq} k$ and let $Q\langle b_1, \ldots, b_r\rangle$ be the Koszul complex on elements $a_i = db_i$ which project to a minimal set of generators of the maximal ideal of $Q/\mathbf{m}Q$. Set $Z = X \cup \{b_i\}$; then $U \otimes_R Q\langle b_1, \ldots, b_r\rangle$ has the form $Q\langle Z\rangle$. Because Q is R-flat and $Q/\mathbf{m}Q$ is regular, the composite

$$\begin{aligned} Q\langle Z\rangle &= R\langle X\rangle \otimes_R Q\langle b_1, \ldots, b_r\rangle \xrightarrow{\simeq} k \otimes_R Q\langle b_1, \ldots, b_r\rangle \\ &= (Q/\mathbf{m}Q)\langle b_1, \ldots, b_r\rangle \xrightarrow{\simeq} l \end{aligned}$$

is a quasi-isomorphism.

Thus $F^h = S \otimes_Q Q\langle Z\rangle = (S \otimes_R R\langle X\rangle)\langle b_1, \ldots, b_r\rangle = F^f\langle b_1, \ldots, b_r\rangle$. Using (2.2) we have that the homotopy fibre of the inclusion $g' : F^f \to F^h$ is just $F^{g'} = l \otimes_{F^f} F^h = (l\langle b_1, \ldots, b_r\rangle, 0)$. In particular, the long exact sequence (2.3) is valid for g'. Since (2.1.2) $\pi^*(F^{g'}) = \pi^2(F^{g'})$ the assertion on $(g')^*$ follows.

Finally, since $\mathrm{Tor}^Q(S, l) = H(S \otimes_Q Q\langle X\rangle) = H(S\langle X\rangle\langle a_1, \ldots, a_r\rangle)$, it is the limit of a spectral sequence starting at $\mathrm{Tor}^R(S, k)\langle b_1, \ldots, b_r\rangle$ and this implies $\mathrm{fd}_Q(S) \le \mathrm{fd}_R(S) + r = \mathrm{fd}_R(S) + \mathrm{edim}(Q/\mathbf{m}Q)$. Since S is a f.g. Q-module $\mathrm{pd}_Q(S) = \mathrm{fd}_Q(S)$, and the standard $\mathrm{fd}_R(S) \le \mathrm{fd}_R(Q) + \mathrm{fd}_Q(S)$ becomes $\mathrm{fd}_R(S) \le \mathrm{fd}_Q(S)$ because Q is R-flat. ■

Next we give several characterizations of l.c.i. maps, for most of which there seems to be no published reference, outside of the special case of regular R with f surjective. Even in this situation our treatment allows for some shortening of existing proofs, although the ideas involved claim no originality.

(3.3) PROPOSITION. *Assume f is factorizable (as $h \circ g$). The following are then equivalent:*

(i) Ker h *is generated by a Q-regular sequence.*

(ii) $\pi^i(F^f) = 0$ *for* $i \geq 3$;

(iii) $\pi^3(F^f) = 0$;

(iv) $\pi^4(F^f) = 0$ *and* $\mathrm{fd}_R\, S < \infty$.

If furthermore f is surjective, and E denotes the Koszul complex on a set $a_1, \ldots, a_r$ of generators of $\ker f$, *they are also equivalent to each of:*

(i)′ $\ker f$ *is generated by R-regular sequence*;

(ii)′ $\pi^*(F^f) = \pi^2(F^f)$;

(v) $\mathrm{Tor}^R(S, k) = \bigwedge_k \mathrm{Tor}_1^R(S, k)$;

(vi) $\mathrm{Tor}_2^R(S, k) = (\mathrm{Tor}_1^R(S, k))^2$;

(vii) $H_1(E)$ *is a free S-module and* $\mathrm{pd}_R\, S < \infty$;

(viii) *Whenever $\{a_i\}$ is a minimal set of generators of* $\ker f$ *then* $H_1(E) = 0$.

Proof. We suppose f surjective and prove the equivalence of (i)′, (ii)′ and (iii)–(viii): once this is done, the Proposition follows from (3.2).

The implication (viii)$\Rightarrow$(i)′ being standard, we proceed along the following scheme:

$$\begin{tikzcd}
\text{(i)}' \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \text{(v)} \arrow[r, Rightarrow] & \text{(vi)} \arrow[d, Rightarrow] \\
\text{(ii)}' \arrow[d, Rightarrow] & & \text{(iii)} \arrow[d, Rightarrow] \\
\text{(iv)} \arrow[r, Rightarrow] & \text{(vii)} \arrow[r, Rightarrow] & \text{(viii)}
\end{tikzcd}$$

(i)′$\Rightarrow$(ii)′, (v). Suppose the $\{a_i\}$ are a minimal set of generators. Then (i)′ implies that f extends to a quasi-isomorphism $E \xrightarrow{\simeq} S$. This identifies (cf. (2.5)) $F^f = (k \otimes_R E, 0) = (\bigwedge_k (x_1, \ldots, x_r), 0) = \mathrm{Tor}^R(S, k)$. Now (v) is immediate and (ii)′ follows from (2.1.1).

(ii)′$\Rightarrow$(iv). Let $V = R[Y] \to S$ be a minimal model for f. In view of (2.6)′ and (2.4), (ii)′ implies that Y is concentrated in degree 1 with finitely many variables $x_1, \ldots, x_n$, so that $F^f = (\bigwedge_k (x_1, \ldots, x_n), 0)$. This gives (iv) – cf. (2.1.2).

(iv)$\Rightarrow$(vii). We may suppose the $\{a_i\}$ are a minimal set of generators for $\ker f$. Let $V = R[Y] \xrightarrow{\simeq} S$ be a minimal model for f. Then by (2.6)′, (iv) implies $Y_3 = \phi$. From (2.4.3) we deduce $H_2(V(2)) = 0$.

Write $Y_2 = \{y_1, \ldots, y_s\}$ and $Y_1 = \{x_1, \ldots, x_r\}$, with $dx_i = a_i$. Thus $\{dy_i\}$ represents a minimal set of generators for $H_1(E)$ and we need to show that if $\sum c_i\, dy_i$ represents zero in $H_1(E)$ ($c_i \in R$) then each $c_i \in \ker f$.

But in this case for some $w \in E_2$, $\sum c_i y_i - w$ is a cycle, and hence a boundary in $V(2)$. The map $R \to S$, $x_i \to 0$, $y_i \to y_i$ is a homomorphism $V(2) \to S[Y_2]$ mapping boundaries to zero. Thus $\sum f(c_i)y_i = 0$.

(vii)$\Rightarrow$(viii). This is a result of Gulliksen: [13; (1.4.9)].

(v)$\Rightarrow$(vi)$\Rightarrow$(iii)$\Rightarrow$(viii). These implications are obvious, or follow at once from (2.8). ■

4. Proof of Theorem C

In view of (3.1) and (3.2) we may suppose f surjective. We further suppose f is not l.c.i. at $\mathbf{n}$, and will (eventually) establish the existence of α and β as promised. The proof is broken up into two propositions.

(4.1) PROPOSITION. *There is a minimal extension $k[Z_{\geq 1}]$ of k with finitely many variables in each degree, and an element $z \in Z_1$ such that*

(i) *$H_*(k[Z])$ is finite dimensional.*

(ii) *$H_i(k[Z]/(z)) \neq 0$, $i \geq 0$.*

(iii) *L_Z^* is a graded subalgebra of $\pi^*(F^f)$.*

(4.2) PROPOSITION. *Let $k[Z_{\geq 1}]$, $z \in Z_1$ be as in* (4.1). *Let $\alpha \in L_Z^2$ satisfy $\langle z, \alpha \rangle = 1$. Then for some $\beta \in L_Z^*$,*

$$(\mathrm{ad}\,\alpha)^n \beta \neq 0, \qquad n \geq 0.$$

Proof of (4.1). Let $a_1, \ldots, a_r$ be a minimal set of generators of $\ker f$ chosen so that $a_1, \ldots, a_m$ is a regular sequence of maximal length in $\ker f$ – this is possible by the prime avoidance theorem. Since f is not l.c.i. at $\mathbf{n}$, $\ker f/(a_1, \ldots, a_m)$ is non-zero and every element in it is a zero divisor. Thus a celebrated theorem of Auslander–Buchsbaum gives $\mathrm{Tor}_i^{R/(a_1,\ldots,a_m)}(S, k) \neq 0$, all i.

On the other hand, our hypothesis $\mathrm{fd}_R(S) < \infty$ implies that $\mathrm{Tor}^R(S, k)$ is finite dimensional. There is thus an integer $n < r$ such that

$$\mathrm{Tor}^{R/(a_1,\ldots,a_{n-1})}(S, k) \text{ is finite dimensional} \tag{4.3}$$

and

$$\mathrm{Tor}_i^{R/(a_1,\ldots,a_n)}(S, k) \neq 0, \qquad i \geq 0. \tag{4.4}$$

Next, as in §2 we get a commutative DG algebra diagram

$$
\begin{tikzcd}
 & R[Y] \arrow[dr, "\simeq"] \arrow[dd, "\simeq"', "\Phi"] & \\
R \arrow[ur] \arrow[dr] & & S \\
 & R\langle X\rangle \arrow[ur, "\simeq"'] &
\end{tikzcd}
$$

in which the upper row is a minimal model for f. Moreover we may take $R[Y_1] \overset{\Phi}{=} R\langle X_1\rangle$ as the Koszul complex $(R\langle x_1, \ldots, x_r\rangle;\ dx_i = a_i)$.

Put $U = R[x_1, \ldots, x_{n-1}] = R\langle x_1, \ldots, x_{n-1}\rangle$, and consider $k \otimes_U R[Y]$; it is a minimal extension of k of the form $k[Z] = k[x_n, \ldots, x_r]\ [Y_{\geq 2}]$. Now $k[Z]$ has finitely many variables in each degree (all of degree ≥ 1) by (2.4.1) and (2.4.2). We set $z = x_n$ and verify (i)–(iii).

Put $\mathbf{a} = (a_1, \ldots, a_{n-1})$. Define $\psi : U \xrightarrow{\simeq} R/\mathbf{a}$ to be the projection in R and zero in the x_i; it is a quasi-isomorphism because $a_1, \ldots, a_{n-1}$ is a regular sequence. Now $R[Y] \xrightarrow{\simeq} S$ factors as $R[Y] \xrightarrow{\simeq} R/\mathbf{a} \otimes_U R[Y] \xrightarrow{\simeq} S$ in which the first arrow is a quasi-isomorphism because ψ is and $R[Y]$ is U-free (forgetting differentials). Thus $R/\mathbf{a} \otimes_U R[Y]$ is an $R/\mathbf{a}$-free resolution of S and so

$$H(k[Z]) = H(k \otimes_{R/\mathbf{a}} R/\mathbf{a} \otimes_U R[Y]) = \mathrm{Tor}^{R/\mathbf{a}}(S, k).$$

Now (i) follows from (4.3).

In the same way (ii) follows from (4.4) with U replaced by $R[x_1, \ldots, x_n]$ and $\mathbf{a}$ replaced by $(a_1, \ldots, a_n)$ in the argument.

Finally, by construction (cf. §2) $k[Y] = k \otimes_R R[Y]$ is a minimal model for F^f. Hence by (2.6)$'$, $\pi^*(F^f) \cong L_Y^*$. On the other hand, the projection $k[Y] \to k[Z]$ is a DG algebra map, and so the dual inclusion $L_Z^* \to L_Y^*$ is a Lie algebra map. This proves (iii). ■

Proof of (4.2). At the cost of replacing Z_1 by a new basis of $k[Z_1]$ we may suppose $Z_1 = z \cup W_1$ with $\langle W_1, \alpha\rangle = 0$. Put $W_i = Z_i$, $i \geq 2$ and write (forgetting differentials) $k[Z] = k[z] \otimes_k k[W]$; the first factor is the exterior algebra on z. The projection $k[Z] \to k[Z]/(z)$ restricts to an isomorphism $k[W] \xrightarrow{\cong} k[Z]/(z)$ of algebras, and hence endows $k[W]$ with a differential, $\bar{d}$.

The full differential, d, in $k[Z]$ then sastisfies

$$d(1 \otimes \Phi) = 1 \otimes \bar{d}\Phi + z \otimes \theta\Phi, \tag{4.5}$$

and from $d^2=0$ it follows that $\mathcal{O}$ is a derivation of $(k[W], \bar{d})$ of degree -2. The minimality of d implies that $\mathcal{O}=\sum_{i\geq 1}\mathcal{O}_i$ with $\mathcal{O}_i$ a derivation and $\operatorname{Im}\mathcal{O}_i\subset k^{\geq i}[W]$. Moreover $L_W^*\subset L_Z^*$ is an ideal of codimension 1.

We now assume (4.2) fails, and deduce a contradiction. Let $(C^*, \delta)=(k[W]_+, d)^\vee$ be the graded dual. An increasing filtration $0\subset F_1\subset\cdots\subset F_i\subset\cdots$ of (C^*, δ) is defined as follows: F_i consists of the linear functions vanishing in $k^{>i}[W]=k[W]_+\cdot\cdots\cdot k[W]_+$ ($i+1$ factors). Thus F_1 is the kernel of the reduced diagonal $C^*\to C^*\otimes_k C^*$ dual to the multiplication in $k[W]_+$. From the definition of L_W^* we have $L_W^{p+1}=F_1^p$.

Now consider $\mathcal{O}_1^\vee: C^*\to C^*$. It is filtration preserving, and its restriction to F_1 coincides up to sign with ad α, as follows from (4.5) and the definition in §2 of the Lie bracket. Since we suppose (4.2) to fail we have

$$(\mathcal{O}_1^\vee)^{n(w)}w=0, \qquad w\in F_1.$$

Next, since $\mathcal{O}_1$ is a derivation, $\mathcal{O}_1^\vee$ is a co-derivation with respect to the codiagonal in C^*. It follows easily from this that

$$(\mathcal{O}_1^\vee)^{n(w)}w=0, \qquad w\in C^*.$$

Finally, since the $\mathcal{O}_i^\vee$, $i\geq 2$, are filtration *decreasing* it follows that

$$(\mathcal{O}^\vee)^{n(w)}w=0, \qquad w\in C^*. \tag{4.6}$$

On the other hand, a short exact sequence of differential spaces is given by

$$0\to k[W]\xrightarrow{\lambda} k[Z]\to k[W]\to 0,$$

with $\lambda(w)=(-1)^{\deg w}z\otimes w$, $w\in k[W]$. This leads to a long exact homology sequence, which dualizes to

$$\xleftarrow{\partial} H^i(C^*)\longleftarrow H_{i+1}(k[Z])^\vee\longleftarrow H^{i+1}(C^*)\xleftarrow{\partial} H^{i-1}(C^*)\longleftarrow.$$

In particular (4.1) (i) implies that $\ker\partial$ is finite dimensional and so for some i, ∂ is injective in $H^j(C^*)$, $j\geq i$. But a simple calculation using (4.5) identifies $\partial=H(\mathcal{O})^\vee=H(\mathcal{O}^\vee)$. In view of (4.6), $\partial^{n(w)}w=0$, $w\in H(C^*)$ and hence $H^{\geq i}(C^*)=0$. But $H(C^*)=H(k[W], \bar{d})^\vee=H(k[Z]/(z))^\vee$, so that (4.1) (ii) is contradicted. ∎

5. Proof of Theorem B

In view of Theorem C and (3.1) and (3.2) we need only establish the following:

(5.1) PROPOSITION. *Let $f:(E, \mathbf{m}, k) \to (S, \mathbf{n}, k)$ be a surjective local homomorphism, such that $\pi^*(F^f)$ contains elements α of degree 2 and β of degree $q \geq 3$ with $[\alpha, \beta] \neq 0$. Then $\pi^{q+1}(F^f) \neq 0$.*

Proof. Let $V = R[Y] \xrightarrow{\simeq} S$ be a minimal model for f. If $\pi^{q+1}(F^f) = 0$ then (cf. (2.6)') $Y_q = \phi$. Thus for $x \in V_{q+1}$,

$$dx = \sum c_{ij} y_i z_j + w$$

where $Y_1 = \{y_1, \ldots, y_r\}$, $Y_{q-1} = \{z_1, \ldots, z_s\}$, $c_{ij} \in R$ and $w \in V(q-2)$.

The coefficient of z_j in $d^2x(=0)$ is $\sum_i c_{ij}\, dy_i$. Thus for each j, $\sum_i c_{ij}\, dy_i = 0$. The minimality of the dy_i as generators of $\ker f$ now implies $c_{ij} \in \mathbf{m}$ for all i, j. It follows that in $k \otimes_R R[Y]$ the differential maps Y_{q+1} into $k[Y_{\leq q-2}]$. The formula in §2 for the Lie bracket now shows that $[\pi^2(F^f), \pi^q(F^f)] = 0$. ■

6. Proof of Theorem A

The Jacobi–Zariski exact sequence [1; (5.1)] for the maps $R \xrightarrow{g} Q \xrightarrow{h} S$ in the definition of a smoothable homomorphism has the form:

$$T^{i-1}(Q \mid R, S/\mathbf{q}) \to T^i(S \mid Q, S/\mathbf{q}) \to T^i(S \mid R, S/\mathbf{q}) \to T^i(Q \mid R, S/\mathbf{q}),$$

Since g is smooth, the Vanishing Theorem of Section 1 shows the long exact sequence (essentially) reduces to isomorphisms $T^i(S \mid Q, S/\mathbf{q}) \cong T^i(S \mid R, S/\mathbf{q})$ for $i \geq 2$, so we can assume f is surjective.

Then, by [1; (4.59) and (5.27)], one has canonical isomorphisms $(T^i(S \mid R, S/\mathbf{q}))_{\mathbf{q}} \cong T^i(S_{\mathbf{q}} \mid R_{f^{-1}(\mathbf{q})}, (S/\mathbf{q})_{\mathbf{q}})$ for $i \geq 0$, so that it suffices to show $T^i(S \mid R, k) \neq 0$ for $i \geq N$, when R is local with residue field k.

Let $V = R[Y]$ be a minimal model for f. Since $\mathbb{Q} \subset k$ it follows from Quillen [15; (9.5)], that $(S \otimes_R V)_+/(S \otimes_R V)_+^2$ is canonically isomorphic to $\mathbb{L}_{S|R}$ in the derived category, hence:

$$\begin{aligned} T^i(S \mid R, k) &= H^i \operatorname{Hom}_S((S \otimes_R V)_+/(S \otimes_R V)_+^2, k) \\ &= H^i \operatorname{Hom}_k(k[Y]_+/k[Y]_+ \cdot k[Y]_+, k) \\ &= (k[Y]_+/k[Y]_+ \cdot k[Y]_+)^\vee \end{aligned}$$

the last equality holding because of minimality. But now (2.6)′ yields $T^i(S \mid R, k) \cong \pi^{i+1}(F^f)$ and Theorem B applies. ∎

7. Proof of Theorem D

Let $f: R \to \hat{S}$ be a Cohen presentation of the completion of S. One can assume $\operatorname{edim} R = \operatorname{edim} S$, i.e. $\ker f \subset \mathbf{m}^2$. If L is the Koszul complex on a minimal set of generators of $\mathbf{m}$, then the fibre of f – equal to $S \otimes_R L$ – is naturally identified with the Koszul complex $K^{\hat{S}}$ on a minimal system of generators of $\mathbf{n}$. It can for all purposes be replaced by its quasi-isomorphic K^S. Furthermore, the exact sequence (2.3), which exists since pd $< \infty$ is automatic when R is regular, reduces by (2.1.1) to an isomorphism of graded Lie algebras $\pi^*(K^{\hat{S}}) \cong \pi^{\geq 2}(\hat{S})$, whence $\pi^*(K^s) \cong \pi^{\geq 2}(S)$.

After these remarks, one can apply (3.3) in order to get (i) ⇒ (ii) in Theorem D, use the triviality of (ii) ⇒ (iii) ⇒ (iv), and come back to (i) by means of Theorem B. The implications (ii) ⇒ (v) ⇒ (vii) ⇒ (viii) and (ii) ⇒ (vi) ⇒ (viii) are simply a matter of definitions, and (viii) ⇒ (i) by Theorem C.

REFERENCES

[1] M. André, *Homologie des algèbres commutatives*, Springer, Berlin, 1974.

[2] L. Avramov, *Differential graded models for local rings*, RIMS Kolyurku *446*, Kyoto Univ., 1981, 80–88.

[3] L. Avramov, *Free Lie subalgebras of the cohomology of local rings*, Trans. Amer. Math. Soc. *270* (1982), 589–608.

[4] L. Avramov, *Local algebra and rational homotopy*, Astérisque *113–114* (1984), 15–43.

[5] L. Avramov, *Local rings of finite simplicial dimension*, Bull. Amer. Math. Soc. (N.S.) *10* (1984), 289–291.

[6] L. Avramov, H. B. Foxby and S. Halperin, *Descent and ascent of local properties along homomorphisms of finite flat dimension*, J. Pure Appl. Algebra *38* (1985), 167–186.

[7] L. Avramov and S. Halperin, *On the structure of the homotopy Lie algebra of a local ring*, Astérisque *113–114* (1984), 153–155.

[8] L. Avramov and S. Halperin, Through the looking glass: A dictionary between rational homotopy and local algebra, Proceedings of the Conference "Algebra, algebraic topology, and

their Interaction" (Stockholm, 1983), Lecture Notes in Mathematics no. 1183, Springer Verlag, (1986) 1–27.

[9] Y. FELIX and S. HALPERIN, *Rational L.-S. category and its applications,* Trans. Amer. Math. Soc. *273* (1982), 1–37.

[10] Y. FELIX and J. C. THOMAS, *The radius of convergence of Poincaré series of loop spaces,* Invent. Math. *68* (1982), 257–274.

[11] V. K. A. M. GUGENHEIM and J. P. MAY, *On the Theory and Application of Differential Torsion Products,* Memoirs of the Amer. Math. Soc. *142* (1974).

[12] T. H. GULLIKSEN, *A homological characterization of local complete intersections,* Compositio Math. *23* (1971), 251–255.

[13] T. H. GULLIKSEN and G. LEVIN, *Homology of local rings,* Queen's Paper in Pure and Appl. Math. No. *20*, Queen's University, Kingston, Ont., 1969.

[14] J. C. MOORE, Algebre homologique et homologie des espaces classifiants, *Séminaire H. Cartan,* Ecole Normale Supérieure, 1959–60, Exposé 7, Secrétariat Math., Paris, 1961.

[15] D. QUILLEN, *On the (co-)homology of commutative rings,* Proc. Symp. Pure Math. *17*, 65–87, Amer. Math. Soc., Providence, R.I., 1970.

[16] SGA 6: *Théorie des intersections et théorème de Riemann–Roch,* Lecture Notes in Math. *225*, Springer, Berlin, 1971.

University of Sofia
Institute of Mathematics
P.O. Box 373, 1090 Sofia
Bulgaria

University of Toronto,
Scarborough College,
Physical Sciences Division,
Scarborough, Canada, M1C 1A4

Received April 23, 1986