A COMPARISON OF DG ALGEBRA RESOLUTIONS WITH PRIME RESIDUAL CHARACTERISTIC

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Abstract. In this article we fix a prime integer p and compare certain dg algebra resolutions over a local ring whose residue field has characteristic p. Namely, we show that given a closed surjective map between such algebras there is a precise description for the minimal model in terms of the acyclic closure, and that the latter is a quotient of the former. A first application is that the homotopy Lie algebra of a closed surjective map with residual characteristic p is abelian. We also use these calculations to show deviations enjoy rigidity properties which detect the (quasi-)complete intersection property.

Introduction

Differential graded (abbreviated to dg) algebra resolutions are a fundamental and effective tool for acquiring ring-theoretic information in local algebra. Two such resolutions are the acyclic closure and minimal model of a local ring map. When the source of the map contains the rationals $\mathbb{Q}$, it is well known that these resolutions coincide; in general, they can differ drastically.

Let $\varphi\colon R \to S$ be a surjective map of local rings with common residue field k. In [20], Tate introduced a dg algebra resolution of S over R obtained by successively adjoining a graded set of exterior and divided power variables Y called an acyclic closure of φ, denoted $R\langle Y\rangle$; see 1.9 for a precise definition. If polynomial variables are used in lieu of divided power variables one obtains the minimal models which were imported to local algebra from rational homotopy theory by Avramov [1]; cf. 1.3. We write $R[X]$ for a minimal model for φ where X is the set of exterior and polynomial variables adjoined. The main result in this article applies when an acyclic closure of φ is minimal as a complex of R-modules; such maps are called closed and were in introduced in [9].

Theorem A. *Let $\varphi\colon R \to S$ be a surjective map of local rings whose residue field k has characteristic $p > 0$. If φ is closed, then its acyclic closure $R\langle Y\rangle$ is a quotient of its minimal model $R[X]$, and there is an exact sequence of graded k-spaces*

$$0 \to \bigoplus_{i=1}^{\infty}(kY_{\text{even}}^{(p^i)} \oplus \Sigma kY_{\text{even}}^{(p^i)}) \to kX \to kY \to 0\,,$$

where Σ is the usual suspension functor of a graded object and $Y_{\text{even}}^{(p^i)}$ consists of all divided power monomials $y^{(p^i)}$ for $y \in Y_{\text{even}}$.

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Closed maps arise frequently. For example, large and quasi-complete intersection homomorphisms are broad, well-studied classes of ring maps, and each belong to the class of closed maps; see 1.13 for more details. The conclusion of Theorem A holds under the more technical assumption that φ is *weakly-closed*, as defined in Definition 1.10; cf. Remark 5.6.

As discussed above, $R[X] \cong R\langle Y\rangle$ when R contains $\mathbb{Q}$. The exact sequence in Theorem A supplies a precise measurement of the difference between a minimal model and acyclic closure in certain cases when R has positive residual characteristic. Furthermore, Theorem A follows from a technical refinement in Theorem 3.6 in which $R[X]$ and a comparison map $R[X] \twoheadrightarrow R\langle Y\rangle$ are described explicitly in terms of $R\langle Y\rangle$.

Another consequence of Theorem 3.6 is an exact computation of the homotopy Lie algebra of φ, denoted $\pi(F^\varphi)$, for closed maps; see 4.1. The homotopy Lie algebra—also adopted from rational homotopy theory to local algebra by Avramov [1]—is a graded Lie algebra naturally associated to a local homomorphism. Properties of $\pi(F^\varphi)$ relate to those of φ, such as whether φ is complete intersection [7] or Golod [2], making $\pi(F^\varphi)$ a useful computational tool. For example, it was recently used by Briggs to settle a long-standing conjecture of Vasconcelos [12].

Theorem B. *If φ is closed with prime residual characteristic $p > 0$, then $\pi(F^\varphi)$ is an abelian Lie algebra with k-basis dual to*

$$\Sigma Y \cup \bigcup_{i=1}^{\infty} \left(\Sigma Y_{\text{even}}^{(p^i)} \cup \Sigma^2 Y_{\text{even}}^{(p^i)} \right) .$$

Furthermore if $p > 2$, then $\pi(F^\varphi)$ also has trivial reduced square.

Theorem B should be contrasted with [7, Theorem C] which shows, regardless of the residual characteristic, that when φ has finite projective dimension the only instance $\pi(F^\varphi)$ is abelian is when φ is complete intersection. Hence, without the assumption φ has finite projective dimension, Theorem B provides many examples for which $\pi(F^\varphi)$ is abelian and φ is not complete intersection; cf. Remark 4.5.

Theorem B follows from the stronger result Theorem 4.3. From the latter we also calculate the homotopy Lie algebra of a quasi-complete intersection map, which exhibits the following trichotomy:

Theorem C. *Let φ be a quasi-complete intersection map with residue field k, and set $U = \pi^2(F^\varphi)$ and $V = \pi^3(F^\varphi)$.*

1. *If the characteristic of k is zero, then $\pi(F^\varphi) = U \oplus V$ and has trivial Lie bracket and reduced square.*
2. *If the characteristic of k is $p > 0$, then*

$$\pi(F^\varphi) \cong U \oplus V \oplus \bigoplus_{t=1}^{\infty} \Sigma^{-2p^t+2} V \oplus \bigoplus_{t=1}^{\infty} \Sigma^{-2p^t+1} V$$

has trivial Lie bracket;

   (a) *If $p > 2$, then $\pi(F^\varphi)$ has trivial reduced square.*
   (b) *If $p = 2$, then the reduced square on $\bigoplus_{t=1}^{\infty} \Sigma^{-2p^t+2} V$ is determined by*

$$\Sigma^{-2^t+2} v \mapsto \Sigma^{-2^{t+1}+1} v\,.$$

and the reduced square operation is trivial otherwise.

Suitably interpreted, a converse to Theorem C holds when φ is assumed to be weakly-closed; see Corollary 5.4. An additional application of the results above is given in Corollary 5.3 which establishes a characterization of the complete intersection property in terms of the eventual vanishing of the deviations of φ, complementing similar results in [3, 7, 9]; see the discussion preceding Corollary 5.3 for more details.

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1. Minimal models and acyclic closures

Throughout $\varphi\colon R \to S$ is a surjective map of commutative noetherian local rings with common residue field k, and A will be a dg R-algebra; in this article all dg algebras will be nonnegatively graded and strictly graded-commutative.

1.1. Suppose A_0 is the local ring $(R, \mathfrak{m})$. If $\partial(A_1) \subseteq \mathfrak{m}$, then

$$\mathfrak{m}_A \coloneqq \mathfrak{m} \oplus A_1 \oplus A_2 \oplus \dots$$

is the maximal dg ideal of A. This gives rise to the complex of k-spaces

$$\operatorname{ind} A \coloneqq \frac{\mathfrak{m}_A}{\mathfrak{m} + \mathfrak{m}_A^2}$$

which is called the indecomposable complex of A.

1.2. We write $A[X]$ to denote a semifree extension of A on the graded set of variables $X = X_1, X_2, X_3, \dots$ as defined in [4, Section 2.1]. That is, each variable of X_i has homological degree i: When i is even X_i consists of polynomial variables and when i is odd X_i consists of exterior variables. In particular, $A[X]^\natural$ is the free strictly graded-commutative A-algebra on X where $(-)^\natural$ forgets the differential of a graded object. Note that when $A = R$, the underlying graded k-space of $\operatorname{ind}(R[X])$ is kX; cf. 1.1.

1.3. A minimal model for φ is a factorization of φ as $R \to R[X] \xrightarrow{\simeq} S$ where $\operatorname{ind}(R[X])$ has zero differential. Note that $\operatorname{ind}(R[X])$ has zero differential if and only if the differential of $R[X]$ is decomposable in the sense that $\partial(R[X])$ is contained in $\mathfrak{m}_R R[X] + (X)^2$. By [4, Section 7.2], minimal models always exist and are unique up to an isomorphism of dg algebras. We will slightly abuse terminology and refer to $R[X]$ itself as being the minimal model for φ.

1.4. Fix a semifree extension $A[X]$ of A. It is well known that $A[X]$ enjoys the following lifting property: Given a dg A-algebra map $f : A[X] \to C$ and a surjective quasi-isomorphism of dg A-algebras $g : B \xrightarrow{\simeq} C$, then there exists a dg A-algebra map $\tilde{f} : A[X] \to B$, extending f, with $g\tilde{f} = f$ that is unique up to A-linear homotopy; see, for example, [4, Proposition 2.1.9]. As a consequence, whenever $R[X]$ is a minimal model of φ and $B \xrightarrow{\simeq} S$ is a dg R-algebra resolution, there exists a quasi-isomorphism $R[X] \xrightarrow{\simeq} B$ compatible with the augmentation maps to S.

Our analysis in Section 3 relies on the following general lifting property for semifree extensions; its proof is implicitly contained in the proof of [4, Proposition 2.1.9].

Lemma 1.5. *Let $\alpha : A \to B$ be a morphism of dg algebras, and Z a set of cycles in A whose image $\alpha(Z)$ is a set of boundaries in B. If $\{b_z : z \in Z\}$ is a collection of elements of B with $\alpha(z) = \partial(b_z)$, then there exists a unique map of dg algebras, extending α,*

$$\tilde{\alpha} : A[X \mid \partial(x_z) = z] \to B \quad \textit{with} \quad \tilde{\alpha}(x_z) = b_z$$

where $X = \{x_z : |x_z| = |z| + 1\}_{z \in Z}$. □

Discussion 1.6. Let $A[x]$ be a semifree extension of A on a single variable of even degree x, and suppose n is a positive integer. The element $\partial(x)x^{n-1}$ is always a cycle in $A[x]$. It is a boundary if and only if n is invertible in A_0, in which case the image of $\frac{1}{n}x^n$ under the differential is $\partial(x)x^{n-1}$. Tate [20] realized that the dependence on the invertiblility of n can be eliminated by adjoining x as a *divided power variable*, as recalled in the sequel.

1.7. We write $A\langle Y\rangle$ for a semifree extension obtained by adjoining a graded set of divided power variables $Y = Y_1, Y_2, Y_3, \ldots$ to A; see [4, Section 6.1] as well as [14, Chapter 1]. We refer to the variables of Y as Γ-variables, and to $A\langle Y\rangle$ as a semifree Γ-extension of A. Analogous to 1.2, each Γ-variable of Y_i has homological degree i with Y_i consisting of divided power variables when i is even, and exterior variables when i is odd. Recall that when $y \in Y_{\text{even}}$, its set of divided powers $\{y^{(i)} : |y^{(i)}| = |y|i\}_{i \geqslant 0}$ with $y^{(0)} = 1$ and $y^{(1)} = y$ satisfy the equalities

$$y^{(n)}y^{(m)} = \binom{n+m}{n} y^{(n+m)} \quad \text{and} \quad \partial(y^{(n)}) = \partial(y)y^{(n-1)}\,.$$

These fundamental equations are crucial to the analysis in the sequel.

1.8. Consider a semifree Γ-extension $A\langle Y\rangle$. We may well-order Y first by homological degree and then by ordering each set Y_n. Associated to $A\langle Y\rangle$, with this choice of ordering, is a canonical A-linear basis called the *normal* Γ*-monomials*; this basis consists of 1 together with the set of terms of the form

$$y_{\lambda_1}^{(i_1)} \cdots y_{\lambda_n}^{(i_n)} \tag{1}$$

with $y_{\lambda_1} < \ldots < y_{\lambda_n}$ and each i_j is a positive integer, along with the additional constraint that $i_j = 1$ when y_{λ_j} is of odd degree; see [4, Section 6] for further details.

Suppose A is the local ring R with maximal ideal $\mathfrak{m}$ and residue field k. Let $\mathfrak{m}^{(2)}_{R\langle Y\rangle}$ denote the ideal of $R\langle Y\rangle$ generated by all normal Γ-monomials (1) with $i_1 + \ldots + i_n \geqslant 2$. The complex of Γ-indecomposables of $A\langle Y\rangle$ is the complex of k-spaces

$$\Gamma\text{-ind}R\langle Y\rangle \coloneqq \frac{\mathfrak{m}_{R\langle Y\rangle}}{\mathfrak{m}R\langle Y\rangle + \mathfrak{m}^{(2)}_{R\langle Y\rangle}}\,;$$

it is clear that as a graded k-vector space $\Gamma\text{-ind}R\langle Y\rangle$ is simply kY.

1.9. An acyclic closure for φ is a factorization of φ as $R \to R\langle Y\rangle \xrightarrow{\simeq} S$ where $\Gamma\text{-ind}(R\langle Y\rangle)$ has zero differential. Acyclic closures may be constructed inductively by adjoining Γ-variables to minimally kill cycles generating the homology modules $\mathrm{H}_n(R\langle Y_{\leqslant n}\rangle)$, as originally described by Tate [20] (see also [4, 14]). Uniqueness, up to an isomorphism of dg Γ-algebras, is contained in [14, Theorem 1.9.5]. As with minimal models, by a slight abuse of terminology $R\langle Y\rangle$ is referred to as an acyclic closure for φ.

Definition 1.10. We say φ is *weakly-closed* provided that an acyclic closure of φ has decomposable differential; that is, φ admits an acyclic closure with

$$\partial(R\langle Y\rangle) \subseteq \mathfrak{m}_R R\langle Y\rangle + (Y)^2\,.$$

Remark 1.11. In analogy to Definition 1.10, the containment

$$\partial(R\langle Y\rangle) \subseteq \mathfrak{m}_R R\langle Y\rangle + \mathfrak{m}_{R\langle Y\rangle}^{(2)}$$

is called Γ-decomposability of the differential. It implies that the differential of Γ-ind$R\langle Y\rangle$ is zero, and is equivalent to $R\langle Y\rangle$ being an acyclic closure. For a comparison of this notion with that of Definition 1.10 see Lemma 2.6.

Remark 1.12. As any two acyclic closures of φ are isomorphic as local dg algebras (in fact, as local dg Γ-algebras) the property of φ being weakly-closed is independent of choice of acyclic closure. If $\mathbb{Q} \subseteq R$, then φ is trivially weakly-closed. Nontrivial examples of such maps—regardless of characteristic considerations—are closed maps, defined in the sequel. See also the discussion in Remark 5.6.

1.13. Following [10, 1.3], we say φ is closed if its acyclic closure $R\langle Y\rangle$ is minimal as a complex of R-modules. That is, $\partial(R\langle Y\rangle) \subseteq \mathfrak{m}_R R\langle Y\rangle$. Clearly any closed homomorphism is weakly-closed. Examples of closed morphisms are plentiful:

(1) The augmentation map to the residue field is closed. This is the content of a celebrated theorem of Gulliksen and Schoeller [13, 19]

(2) More generally, any large homomorphism—that is, φ satisfying that the induced map on Tor algebras $\operatorname{Tor}^R(k,k) \to \operatorname{Tor}^S(k,k)$ is surjective—is closed by a theorem of Avramov and Rahbar-Rochandel; see [16, Theorem 2.5]. A prominent example of a family of large homomorphisms is provided by algebra retracts, in the sense that there exists a ring map from S to R which is a left inverse of φ.

(3) Quasi-complete intersection maps, defined and studied by Avramov, Henriques and Şega, are closed; cf. [8, 1.6]. Recall φ is quasi-complete intersection if in the acyclic closure $R\langle Y\rangle$ of φ we have $Y_i = \emptyset$ for all $i \geqslant 3$.

2. Divided powers in prime residual characteristic

This section contains various facts involving divided power algebras and binomial coefficients that are needed for our analysis in Section 3. Foundational material on divided power algebras is due to Roby [18], which includes some of the formulas in this section; we provide a self-contained treatment below. Throughout this section p will be a fixed prime number.

2.1. Recall that every integer n may be written uniquely in its base p expansion as

$$n_0p^0 + n_1p^1 + \cdots + n_lp^l$$

where each n_i is a nonnegative integer strictly less than p; in this case we write $n = [n_0n_1 \dots n_l]_p$. We use $\equiv_p$ to denote equivalence of integers modulo p.

In the notation above, we recall the following classical theorem of Lucas [17]: *If $n = [n_0 \dots n_l]_p$ and $m = [m_0 \dots m_l]_p$, then*

$$\binom{n}{m} \equiv_p \binom{n_0}{m_0} \cdots \binom{n_l}{m_l}.$$

In applying this theorem, note the convention that $\binom{n}{m} = 0$ whenever $m > n$.

Notation 2.2. Let y be a divided power variable over a ring R. Using 1.7 it follows easily that for any positive integer partition of $n = n_0 + n_1 + \ldots + n_l$, there is the equality

$$y^{(n_0)} \cdots y^{(n_l)} = \binom{n}{n_0, n_1, \ldots, n_l} y^{(n)} \quad \text{where} \quad \binom{n}{n_0, n_1, \ldots, n_l} = \frac{n!}{n_0! n_1! \cdots n_l!}\,.$$

Define the following nonnegative integers

$$b_{t,m} = \binom{tp^m}{\underbrace{p^m, p^m, \ldots, p^m}_{t\text{-times}}} \quad \text{and} \quad c_n \coloneqq \binom{n}{n_0, n_1 p, \ldots, n_l p^l}$$

for each $m, n, t \in \mathbb{N}$, where $n = [n_0 \ldots n_l]_p$. These arise as the coefficients in the following multiplications:

$$(y^{(p^m)})^t = b_{t,m} y^{(tp^m)}$$
$$y^{(n_0)} y^{(n_1 p)} \cdots y^{(n_l p^l)} = c_n y^{(n)} \quad \text{where} \quad n = [n_0 \ldots n_l]_p\,.$$

The coefficient $b_{p-1,m}$ arises sufficiently often that we will denote it b_m for simplicity.

Lemma 2.3. *Adopting Notation 2.2, there are equivalences*

$$b_{t,m} \equiv_p t! \qquad b_m \equiv_p -1 \qquad c_n \equiv_p 1$$

for any nonnegative integers m and n and for $t < p$. In particular, if R is a local $\mathbb{Z}_{(p)}$-algebra, then $b_{t,m}$, b_m, and c_n are units in R.

Proof. For the first equivalence

$$b_{t,m} = \binom{p^m}{p^m}\binom{2p^m}{p^m} \cdots \binom{tp^m}{p^m} \equiv_p t!$$

where the equivalence follows from Lucas' theorem, recalled in 2.1. Wilson's theorem [15] says $(p-1)! \equiv_p -1$, giving the second equivalence.

For the third equivalence, we induct on l; the base case, when $l = 0$, is trivial. Next assume $l > 0$ and set $n' = [n_0 \ldots n_{l-1}]_p$. Observe that

$$c_n = c_{n'} \binom{n}{n_l p^l} \equiv_p 1$$

where the first equality uses $n = n' + n_l p^l$ and the equivalence holds by Lucas' theorem and induction. □

Remark 2.4. The precise formulas for the coefficients specified in Notation 2.2 are primarily needed in Section 3 to handle the case when R has mixed characteristic. When R itself has characteristic p, the arguments in the proof of Lemma 2.3 show that $b_m = -1$ and $c_n = 1$. That is,

$$(y^{(p^m)})^{p-1} = -y^{((p-1)p^m)} \quad \text{and} \quad y^{(n)} = y^{(n_0)} y^{(n_1 p)} \cdots y^{(n_l p^l)}$$

where $n = [n_0 \ldots n_l]_p$.

Notation 2.5. Suppose $(R, \mathfrak{m}, k)$ is local and k has characteristic $p > 0$. For any semifree Γ-extension $R\langle Y\rangle$ of R, we let $kY^{(p^\infty)}$ denote the subcomplex of $\operatorname{ind} R\langle Y\rangle$ consisting of cycles of the form $y^{(p^i)}$ with $y \in Y_{\text{even}}$ and $i > 0$.

Lemma 2.6. *Suppose $(R, \mathfrak{m}, k)$ is local and k has characteristic $p > 0$. For any semifree Γ-extension $R\langle Y\rangle$ of R, there is an exact sequence of complexes of k-spaces*

$$0 \to kY^{(p^\infty)} \to \mathrm{ind} R\langle Y\rangle \to \Gamma\text{-}\mathrm{ind} R\langle Y\rangle \to 0\,.$$

Proof. Set $A = R\langle Y\rangle$. The containment of subcomplexes

$$\mathfrak{m} + \mathfrak{m}_A^2 = \mathfrak{m}A + \mathfrak{m}_A^2 \subseteq \mathfrak{m}A + \mathfrak{m}_A^{(2)} \tag{2}$$

induces a surjection $\mathrm{ind}A \to \Gamma\text{-}\mathrm{ind}A$. It suffices to examine when $y^{(n)}$ is zero in $\mathrm{ind}A$ with $y \in Y_{\mathrm{even}}$, and $n > 1$ with $n = [n_0 \dots n_l]_p$.

When n is not a power of p, there are two cases. First if at least two n_i are nonzero, then using Notation 2.2 and Lemma 2.3, we obtain the equality

$$y^{(n)} = c_n^{-1} y^{(n_0)} \dots y^{(n_l p^l)}\,.$$

The second case is that $n_i = 0$ for $l > i$ and $n_l > 1$. Here

$$y^{(n)} = y^{(n_l p^l)} = b_{n_l,l}^{-1} (y^{(p^l)})^{n_l}$$

where the second equality again uses Notation 2.2 and Lemma 2.3. Hence, in either case, we have shown $y^{(n)}$ is zero in $\mathrm{ind}A$.

Now we show that when n is a power of p, say p^i, then $y^{(n)}$ is not zero in $\mathrm{ind}A$. Assuming to contrary, in $A \otimes_R k$ one can write $y^{(n)}$ as a k-linear combination of Γ-monomials of the form

$$y^{(i_1)} \cdots y^{(i_t)}$$

with $i_1 + \ldots + i_t = n$ and $t > 1$. Set $m = i_1 + \cdots + i_{t-1}$. Using Lemma 2.3, we have

$$y^{(i_1)} \cdots y^{(i_t)} = \binom{m}{n_1, \dots, n_{t-1}} \binom{n}{i_t} y^{(p^i)}$$

and since $t \geqslant 1$, it follows that $i_t < n$. Observe that $\binom{n}{i_t} \equiv_p 0$ by Lucas' theorem. As a consequence $y^{(i_1)} \cdots y^{(i_t)}$ equals zero in $A \otimes k$. □

Remark 2.7. A simple consequence of Lemma 2.6, that is easy to see independently, is that decomposability of the differential of $R\langle Y\rangle$ implies Γ-decomposability of its differential. The main content of Lemma 2.6 is that the converse holds if no summand of the form $y^{(p^i)}$, for y in Y_{even} and positive integer i, appears as a summand in the differential of any element of $R\langle Y\rangle$.

3. Comparison map

Throughout, we fix a surjective homomorphism of commutative local noetherian rings $\varphi\colon R \to S$ with common residue field k. Let $R[X] \xrightarrow{\simeq} S$ be a minimal model for φ and let $R\langle Y\rangle \xrightarrow{\simeq} S$ denote an acyclic closure for φ.

3.1. By 1.4, there exists a quasi-isomorphism of dg R-algebras, which is unique up to homotopy $R[X] \xrightarrow{\simeq} R\langle Y\rangle$ compatible with the augmentations to S. In this section we construct an explicit representative of this map, which will denoted by γ^φ and referred to as the *comparison map of* φ. As γ^φ is a morphism of local dg algebras, along with an analogous containment to (2), the comparison map induces a map on complexes of k-spaces

$$\mathrm{ind}(\gamma^\varphi)\colon \mathrm{ind}(R[X]) \to \Gamma\text{-}\mathrm{ind}(R\langle Y\rangle)\,.$$

Remark 3.2. When the characteristic of k is further assumed to be zero, it is then standard that $R[X]$ and $R\langle Y\rangle$ are isomorphic, and easy to see that γ^φ is itself an isomorphism of dg algebras which induces the isomorphism $\operatorname{ind}(\gamma^\varphi)$. The goal of this section is to provide insight on the situation when k has prime characteristic $p>0$, which we assume for the remainder of the section.

In this section, we prove Theorem A which is recast, in the notation set above, in Corollary 3.3. It is an immediate consequence of Theorem 3.6 presented at the end of the section. Recall that for a graded object V, the i-fold suspension of V is $\Sigma^i V$ where $(\Sigma^i V)_j = V_{j-i}$.

Corollary 3.3. *Suppose $\varphi\colon R\to S$ is a surjective map of local rings whose common residue field k has characteristic $p>0$. If φ is weakly-closed, then the comparison map γ^φ is surjective and induces the following exact sequence of graded k-spaces*

$$0\to kY^{(p^\infty)}\oplus \Sigma kY^{(p^\infty)}\to \operatorname{ind}(R[X])\xrightarrow{\operatorname{ind}(\gamma^\varphi)}\Gamma\text{-}\operatorname{ind}(R\langle Y\rangle)\to 0. \quad \square$$

Construction 3.4. Let A be a dg R-algebra and $z\in A_{\mathrm{odd}}$ be a cycle. By Lemma 1.5, there is a canonical map of dg A-algebras

$$\gamma\colon A[x\,|\,\partial x = z]\to A\langle y\,|\,\partial y = z\rangle$$

completely determined by $x\mapsto y$. We define a semifree extension

$$A[x]\hookrightarrow A[\{x_i, x'_{i+1}\}_{i\geqslant 0}]$$

and a map $\gamma_z(A) : A[\{x_i,x'_{i+1}\}_{i\geqslant 0}]\to A\langle y\rangle$ of dg A-algebras extending γ. Recall the coefficients $b_{t,m}$ and c_n defined in Notation 2.2. For notational convenience, set

$$d_i = c_{p^i-1}\prod_{j=0}^{i-1}(b_j c_{p^j-1}^{p-1})^{p^{i-1-j}}\,. \tag{3}$$

Set $x_0 \coloneqq x$ and by induction assume we have constructed $A[\{x_i,x'_{i+1}\}_{0\leqslant i\leqslant n}]$, denoted $A(n)$, with differential determined by

$$\partial(x_i) = zx_0^{p-1}x_1^{p-1}\dots x_{i-1}^{p-1} \quad\text{and}\quad \partial(x'_i) = x_{i-1}^p - px_i,$$

and a dg algebra map $\gamma_z(n): A(n)\to A\langle y\rangle$ determined by $\gamma_z(n)(x_0)=y$ and for all $i\geqslant 1$:

$$\gamma_z(n)(x_i) = d_i y^{(p^i)} \quad\text{and}\quad \gamma_z(n)(x'_i)=0\,.$$

By Lemma 2.3, each d_i is a product of units and hence is a unit of R. A tedious direct calculation, using Lemma 2.3, yields

$$\gamma_z(n)(zx_0^{p-1}\cdots x_n^{p-1}) = d_{n+1}\partial(y^{(p^{n+1})})\,.$$

Also, as z has odd degree, it follows that $zx_0^{p-1}x_1^{p-1}\dots x_n^{p-1}$ is a cycle of degree $(|x|p^{n+1}-1)$. Therefore by Lemma 1.5, there is a unique dg A-algebra map

$$A(n)[x_{n+1}\,|\,\partial x_{n+1} = zx_0^{p-1}x_1^{p-1}\dots x_n^{p-1}]\to A\langle y\rangle \tag{4}$$

extending $\gamma_z(n)$ with $x_{n+1}\mapsto d_{n+1}y^{(p^{n+1})}$.

Under this extension the image of $x_n^p - px_{n+1}$ is a cycle of degree $|x|p^{n+1}$. Using the definitions of d_n and c_i, the image of this cycle under (4) is

$$\begin{aligned} d_n^p(y^{(p^n)})^p - pd_{n+1}y^{(p^{n+1})} &= \left(b_n d_n^p \binom{p^{n+1}}{p^n} - pd_{n+1}\right) y^{(p^{n+1})} \\ &= \left(\binom{p^{n+1}}{p^n} \frac{c_{p^n-1}}{c_{p^{n+1}-1}} - p\right) d_{n+1} y^{(p^{n+1})} \\ &= 0\,. \end{aligned}$$

Hence $\gamma_z(n)$ can be further extended, again applying Lemma 1.5, to a dg A-algebra map $\gamma_z(n+1) : A(n+1) \to A\langle y\rangle$ where

$$A(n+1) \coloneqq A[\{x_i, x'_{i+1}\}_{0\leqslant i\leqslant n+1} \mid \partial x_{i+1} = zx_0^{p-1}\dots x_i^{p-1},\ \partial x'_{i+1} = x_i^p - px_{i+1}]$$

with $\gamma_z(n+1)(x_i) = d_i y^{(p^i)}$ and $\gamma_z(n+1)(x'_i) = 0$ for each i, which completes the induction.

Now taking the colimit of the maps $\gamma_z(n)$, we obtain the dg A-algebra map $\gamma_z(A) : A[\{x_i, x'_{i+1}\}_{i\geqslant 0}] \to A\langle y\rangle$ extending γ with

$$\gamma_z(A)(x_i) = d_i y^{(p^i)} \quad \text{and} \quad \gamma_z(A)(x'_i) = 0\,.$$

Lemma 3.5. *In the notation of Construction 3.4, the dg A-algebra map*

$$\gamma_z(A) : A[\{x_i, x'_{i+1}\}_{i\geqslant 0}] \to A\langle y\rangle$$

is a quasi-isomorphism.

Proof. As the regular element $x_i^p - px_{i+1}$ is the boundary of the corresponding x'_{i+1}, by iteratively applying [20, Theorem 3] it follows that the canonical map

$$A[\{x_i, x'_{i+1}\}_{i\geqslant 0}] \xrightarrow{\simeq} A[\{x_i\}_{i\geqslant 0}]/(\{x_i^p - px_{i+1}\}_{i\geqslant 0})$$

is a quasi-isomorphism. Next observe that $\gamma_z(A)$ factors through the canonically induced map

$$\overline{\gamma} : A[\{x_i\}_{i\geqslant 0}]/(\{x_i^p - px_{i+1}\}_{i\geqslant 0}) \to A\langle y\rangle$$

with $x_i \mapsto d_i y^{(p^i)}$ for each $i \geqslant 0$; cf. Construction 3.4(3) for the definition of d_i which is defined in terms of the coefficients introduced in Notation 2.2.

Let n be a nonnegative integer and write $n = [n_0 n_1 \dots n_l]_p$. Observe that

$$\begin{aligned} \overline{\gamma}(x_0^{n_0} x_1^{n_1} \cdots x_l^{n_l}) &= \prod_{i=1}^{l} (d_i y^{(p^i)})^{n_i} \\ &= \prod_{i=1}^{l} d_i^{n_i} b_{n_i,i} y^{(n_i p^i)} \\ &= \left(\prod_{i=1}^{l} d_i^{n_i} b_{n_i,i}\right) c_n y^{(n)} \end{aligned}$$

the first equality holds as $\overline{\gamma}$ is an algebra map and the other equalities use Notation 2.2. Also, an A-linear basis for $A[\{x_i\}_{i\geqslant 0}]/(\{x_i^p - px_{i+1}\}_{i\geqslant 0})$ is

$$\{x_0^{n_0} x_1^{n_1} \cdots x_l^{n_l} : 0 \leqslant n_i < p\}$$

and it is standard that an A-linear basis for $A\langle y\rangle$ is $\{y^{(n)}\}_{n\geqslant 0}$, the divided powers of y. By Lemma 2.3, the nonnegative integers

$$\left(\prod_{i=1}^{l} d_i^{n_i} b_{n_i,i}\right) c_n$$

are units in R and so $\overline{\gamma}$ is an isomorphism of dg A-algebras, which finishes the proof of the lemma once recalling $\gamma_z(A)$ factors as

$$A[\{x_i, x'_{i+1}\}_{i\geqslant 0}] \xrightarrow{\simeq} A[\{x_i\}_{i\geqslant 0}]/(\{x_i^p - px_{i+1}\}_{i\geqslant 0}) \xrightarrow{\overline{\gamma}} A\langle y\rangle. \qquad \square$$

Theorem 3.6. *Suppose $\varphi\colon R\to S$ is a surjective morphism of local rings of residual characteristic $p>0$. If φ is weakly-closed, then the minimal model $R[X]$ for φ has the form*

$$R[X(0), X(1), X'(1), X(2), X'(2), \dots]$$

where

(1) $X(0)=\{x_0(y) : y\in Y\}$ *with* $|x_0(y)|=|y|$,
(2) *for* $i\geqslant 1$, $X(i)=\{x_i(y) : y\in Y_{\text{even}}\}$ *with* $|x_i(y)|=|y|p^i$,
(3) *for* $i\geqslant 1$, $X'(i)=\{x'_i(y) : y\in Y_{\text{even}}\}$ *with* $|x'_i(y)|=|y|p^i+1$,

and there exists a surjective quasi-isomorphism $\gamma^\varphi\colon R[X]\to R\langle Y\rangle$ determined by the formulas

$$\gamma^\varphi(x_i(y)) = d_i y^{(p^i)} \quad \text{and} \quad \gamma^\varphi(x'_i(y))=0$$

where d_i is the unit defined in Construction 3.4(3). Furthermore, the differential of $R[X]$ is given by

$$\partial(x_i(y)) = \tilde{z}\prod_{j=0}^{i-1} x_j(y)^{p-1} \quad \text{and} \quad \partial(x'_i(y)) = x_{i-1}(y)^p - px_i(y)$$

where $\tilde{z}$ is a cycle lifting the cycle $\partial(y)$ along γ^φ.

Proof. Assume, by induction, we have constructed a surjective quasi-isomorphism of dg R-algebras

$$\gamma(n) : R[X(0)_{\leqslant n}, X(1)_{\leqslant n}, X'(1)_{\leqslant n}, \dots] \to R\langle Y_{\leqslant n}\rangle$$

with the desired properties for some $n\geqslant 0$, and let $A(n)$ denote the source of this map.

Let z be a cycle in $R\langle Y_{\leqslant n}\rangle$ of homological degree n. Since $\gamma(n)$ is a surjective quasi-isomorphism, there exists a cycle $\tilde{z}$ in $A(n)$ with $\gamma(n)(\tilde{z})=z$. Furthermore, by Remark 2.7 the assumption that φ is weakly-closed guarantees no summand of z is of the form $y^{(p^i)}$ for $y\in Y_{\text{even}}$, which implies that $\tilde{z}$ can be chosen to be in $\mathfrak{m}_R A(n) + \mathfrak{m}^2_{A(n)}$.

When n is even, $\gamma(n)$ extends to the quasi-isomorphism of dg R-algebras

$$A(n)\langle x \mid \partial x = \tilde{z}\rangle \xrightarrow{\simeq} R\langle Y_{\leqslant n}\rangle\langle y \mid \partial y = z\rangle;$$

cf. [4, Lemma 7.2.10]. As x is an exterior variable, $A(n)[x \mid \partial x = \tilde{z}]$ and $A(n)\langle x\rangle$ coincide.

Now assume n is odd. In this case we obtain a surjective quasi-isomorphism

$$A(n)[\{x_i, x'_{i+1}\}_{i\geqslant 0}] \xrightarrow{\simeq} A(n)\langle y \mid \partial y = \tilde{z}\rangle, \tag{5}$$

with $\partial x_i = \tilde{z} x_0^{p-1} \dots x_{i-1}^{p-1}$ and $\partial x'_{i+1} = x_i^{p-1} - p x_{i+1}$, using Lemma 3.5. Furthermore, another application of [4, Lemma 7.2.10], extends $\gamma(n)$ to a surjective quasi-isomorphism

$$A(n)\langle y \mid \partial y = \tilde{z}\rangle \xrightarrow{\simeq} R\langle Y_{\leqslant n}\rangle\langle y \mid \partial y = z\rangle. \tag{6}$$

Composing the surjective quasi-isomorphisms from (5) and (6) yield another one

$$A(n)[\{x_i, x'_{i+1}\}_{i\geqslant 0}] \xrightarrow{\simeq} R\langle Y_{\leqslant n}\rangle\langle y \mid \partial y = z\rangle.$$

Repeating this for each cycle of degree n, extends $\gamma(n)$ to a surjective quasi-isomorphism

$$\gamma(n+1) : R[X(0)_{\leqslant n+1}, X(1)_{\leqslant n+1}, X'(1)_{\leqslant n+1}, \dots] \to R\langle Y_{\leqslant n+1}\rangle.$$

Taking the colimit of these maps yields the desired surjective quasi-isomorphism

$$\gamma^{\varphi} \colon R[X(0), X(1), X'(1), X(2), X'(2), \dots] \xrightarrow{\simeq} R\langle Y\rangle$$

satisfying all of the desired properties. □

4. The homotopy Lie algebra

Fix a surjective map $\varphi\colon R \to S$ of local rings with common residue field k. First we recall the homotopy Lie algebra introduced by Avramov; its structure reflects interesting ring-theoretic properties of φ; suitable references include [1, 4, 6, 7].

4.1. Let $R[X] \xrightarrow{\simeq} S$ be a minimal model for φ, and let $k[X]$ denote $k \otimes_R R[X]$. Also let kX^n be the k-linear space generated by all monomials in X of degree n; the space $kX^1 = kX$ is canonically isomorphic to $\operatorname{ind} R[X]$, and so these will be naturally identified. The assumption that $R[X]$ has decomposable differential yields the decomposition

$$\partial^{k[X]} = \partial^{[2]} + \partial^{[3]} + \dots$$

with $\partial^{[i]}|_{kX} : kX \to kX^i$. The equality $\partial^{[2]}\partial^{[2]} = 0$ defines a graded Lie algebra structure on $(\Sigma kX)^*$ as recalled below. In what follows, $(-)^*$ denotes k-linear duality, and we equip $(\Sigma kX)^*$ with the dual basis of functionals Σx^* where

$$\Sigma x^*(\Sigma x') = \begin{cases} 1 & x = x' \\ 0 & x \neq x' \end{cases}$$

for $x, x' \in X$.

As a graded k-space the homotopy Lie algebra of φ, denoted $\pi(F^{\varphi})$, is $(\Sigma kX)^*$. The Lie structure on $\pi(F^{\varphi})$ is defined using $\partial^{[2]}$ along with a fixed well-ordering of X, as usual first ordered by homological degree. Namely, writing

$$\partial^{[2]}(x_l) = \sum_{i<j} q_{ij}^l x_i x_j + \sum_i q_i^l x_i^2\,,$$

the Lie bracket and reduced square on $(\Sigma kX)^*$ are defined, with $j > i$, as

$$[\Sigma x_j^*, \Sigma x_i^*] = \sum_l q_{ij}^l \Sigma x_l^* \quad \text{and} \quad (\Sigma x_i^*)^{[2]} = -\sum_l q_i^l \Sigma x_l^*\,.$$

Remark 4.2. There are two functors involved in the formation of the homotopy Lie algebra of a surjective local homomorphism, which explains our choice of notation above. Namely, the first functor F associates to the surjective local homomorphism $\varphi\colon R \to S$ the (derived) fiber of the map

$$F^{\varphi} := k \otimes^{\mathsf{L}}_R S \simeq k[X]$$

in the notation of 4.1. The functor π is naturally defined on the category of semifree dg k-algebras with decomposable differential as discussed in 4.1; the target category is that of graded Lie algebras over k of finite type.

One can also regard π as the functor which associates to a semifree dg k-algebra with decomposable differential the homology of the dg Lie algebra of Γ-derivations of an acyclic closure of k over A; this is the perspective taken in [4, Chapter 10], though this will be less relevant for what follows.

We now arrive at one of the main applications of Theorem 3.6 which shows the homotopy Lie algebra of a non-complete intersection map always has an infinitely generated abelian Lie subalgebra.

Theorem 4.3. *Let $\varphi\colon R \to S$ be a surjective local map with prime residual characteristic $p > 0$ and let $R\langle Y\rangle$ be an acyclic closure of φ. If φ is weakly-closed, then there is an isomorphism of graded Lie algebras*

$$\pi(F^{\varphi}) \cong L \times L^{\infty} \quad \textit{with} \quad [\pi(F^{\varphi}), L^{\infty}] = 0$$

where $L \cong (\Sigma kY)^*$ and $L^{\infty} \cong (\Sigma kY^{(p^{\infty})} \oplus \Sigma^2 kY^{(p^{\infty})})^*$ as k-spaces. Furthermore, when $p > 2$ the reduced square on L^{∞} is trivial. When $p = 2$, the reduced square of $(\Sigma y^{(2^i)})^* \in L^{\infty}$ is given by $(\Sigma^2 y^{(2^{i+1})})^*$ and trivial otherwise.

Proof. Take $R[X] = R[X(0), X(1), X'(1), \dots]$ as described in Theorem 3.6 to be a minimal model of φ. Define the k-subspaces of $\pi(F^{\varphi})$:

$$L := (\Sigma kX(0))^* \quad \text{and} \quad L^{\infty} := \left(\bigoplus_{i=1}^{\infty} \Sigma kX(i) \oplus \Sigma^2 kX'(i)\right)^* .$$

The asserted isomorphisms on k-spaces are induced by the obvious isomorphisms

$$kY \cong kX(0) \quad \text{and} \quad kY^{(p^{\infty})} \cong \bigoplus_{i=1}^{\infty} kX(i)$$

given by $y \mapsto x_0(y)$ for each $y \in Y$ and $y^{(p^j)} \mapsto x_j(y)$ for each $y \in Y_{\text{even}}$ and any $j \geqslant 1$, respectively.

When $p \neq 2$, a direct calculation using Theorem 3.6 shows that for $i > 0$ no element of $X(i)$ or $X'(i)$ appears as a summand of any element in the image of $\partial^{[2]}$ and that $\partial^{[2]}(X(i) \cup X'(i)) = 0$. Using these facts and 4.1 to compute the Lie bracket and reduced square on $\pi(F^{\varphi})$ it follows that L and L^{∞} are Lie subalgebras of $\pi(F^{\varphi})$ with the property $[\pi(F^{\varphi}), L^{\infty}] = 0$ with trivial reduced square on L^{∞}.

Finally, when $p = 2$, a similar calculation together with the equality

$$\partial^{[2]}(x'_{i+1}(y)) = x_i(y)^2$$

yields the last conclusion. □

The next corollary, along with Theorem 4.3, establishes Theorem C from the Introduction.

Corollary 4.4. *If φ is closed with prime residual characteristic $p > 0$, then $\pi(F^{\varphi})$ is an abelian Lie algebra. Furthermore, $\pi(F^{\varphi})$ is abelian as a restricted Lie algebra whenever $p > 2$.*

Proof. The assumption that φ is closed guarantees $\partial^{[2]}$ vanishes on any variable belonging to $X(0)$ in the notation of Corollary 3.3. Hence the Lie algebra L in Theorem 4.3 is an abelian restricted Lie subalgebra of $\pi(F^\varphi)$. $\square$

Remark 4.5. When φ is a map of finite projective dimension—in the sense that S has finite projective dimension over R—the only case that $\pi(F^\varphi)$ is abelian is when φ is complete intersection; cf. [7, Theorem C]. When the residual characteristic is prime (and the map is not necessarily of finite projective dimension), Corollary 4.4 provides a wide class of maps for which $\pi(F^\varphi)$ is abelian; see the examples in 1.13. A consequence is that there cannot be a direct analog to [7, Theorem C] for detecting the quasi-complete intersection property in terms of an abelian Lie algebra structure of $\pi(F^\varphi)$.

5. Vanishing of deviations and closing remarks

Let $\varphi\colon R \to S$ be a surjective local map, and let k denote the common residue field of R and S.

5.1. The numbers $\varepsilon_i(\varphi) := \dim_k \pi^i(F^\varphi)$ are called the *deviations of* φ. They are encoded in the Poincaré series

$$P(t) = \sum_{i\geqslant 0} \dim_k(\operatorname{Tor}_i^{k\otimes^{\mathsf{L}}_S R}(k,k))t^i$$

of k over the derived fiber $k \otimes^{\mathsf{L}}_R S$ according to the equality

$$P(t) = \frac{\prod_{i=1}^\infty (1+t^{2i-1})^{\varepsilon_{2i-1}(\varphi)}}{\prod_{i=1}^\infty (1-t^{2i})^{\varepsilon_{2i}(\varphi)}}\,.$$

Furthermore, upon fixing a minimal model $R[X]$ for φ, we see that the number of variables adjoined in X_i is exactly $\varepsilon_{i+1}(\varphi)$. These naturally generalize the deviations of a local ring; a standard reference for the latter is [4, Section 7], and see the references contained in *loc. cit.*

The rigidity and vanishing of the deviations of a local homomorphism have been well-studied; see for example, [3, 7, 9, 11]. The corollaries in this section add to the rigidity results in the case $\varphi\colon R \to S$ is a weakly-closed surjective local map with residual characteristic $p > 0$.

5.2. When φ is a weakly-closed surjective map and R has residual characteristic $p > 0$, with acyclic closure $R\langle Y\rangle$, Corollary 3.3 shows that the deviations of φ are completely determined by the numbers $\Gamma\text{-}\varepsilon_i(\varphi) := \dim_k(\Gamma\text{-ind}_{i-1}(R\langle Y\rangle))$; the latter value is exactly the number of Γ-variables adjoined in Y_{i-1} and referred to as the i^{th} Γ-deviation of φ. In particular, we have

$$\varepsilon_i(\varphi) = \begin{cases} \displaystyle\sum_{s=0}^{t} \Gamma\text{-}\varepsilon_{2jp^s+1}(\varphi) & i = 2jp^t+1 \\ \displaystyle\Gamma\text{-}\varepsilon_{2jp^t+2}(\varphi) + \sum_{s=0}^{t-1} \Gamma\text{-}\varepsilon_{2jp^s+1}(\varphi) & i = 2jp^t+2 \\ \Gamma\text{-}\varepsilon_i(\varphi) & \text{otherwise} \end{cases}$$

Furthermore, since the first steps in the inductive constructions of an acyclic closure and a minimal model coincide, the equalities below always hold

$$\varepsilon_2(\varphi) = \Gamma\text{-}\varepsilon_2(\varphi) \quad\text{and}\quad \varepsilon_3(\varphi) = \Gamma\text{-}\varepsilon_3(\varphi)\,.$$

It is known that the eventual vanishing of the deviations of a surjective local map is equivalent to the map being complete intersection when it is a map of finite projective dimension, or more generally, when the map has finite weak category; see [7, Theorem C] for the former, and [3, Section 3] for the latter (as well as [9, Theorem 5.4]). The numeric relations listed in 5.2 establishes the equivalence for a completely different class of surjective homomorphisms:

Corollary 5.3. *If $\varphi\colon R \to S$ is a weakly-closed surjective local map with residual characteristic $p > 0$, then the following are equivalent:*

(1) φ is complete intersection;
(2) $\varepsilon_i(\varphi) = 0$ for all $i \gg 0$;
(3) $\varepsilon_i(\varphi) = 0$ for $i = 2p^t + 1$ or $i = 2p^t + 2$ for some $t \geqslant 1$.

Our last main result, another immediate consequence of 5.2, shows that rigidity of certain deviations detects the quasi-complete intersection property in positive characteristic among weakly-closed maps; this should be compared with [8, Theorem 5.3] and [11, Theorem 33] which characterizes the quasi-complete intersection property in terms of the functorial map $\pi(F^\rho) \to \pi(F^{\varphi\rho})$ where $\rho\colon Q \to R$ is a Cohen presentation of R.

Corollary 5.4. *Let $\varphi\colon R \to S$ be a weakly-closed surjective local map with residual characteristic $p > 0$. Then φ is a quasi-complete intersection homomorphism if and only if there are equalities*

$$\varepsilon_i(\varphi) = \begin{cases} \varepsilon_3(\varphi) & \text{if } i = 2p^t + 1 \text{ or } i = 2p^t + 2 \\ 0 & \text{otherwise} \end{cases}$$

for all $i \geqslant 4$ and $t \geqslant 1$.

We end this article with some closing remarks.

Remark 5.5. The results in the present article were stated for surjective homomorphisms; they naturally extend—and can be deduced from the surjective case—to the setting of (arbitrary) local homomorphisms using the theory of Cohen factorizations introduced in [5]. We leave these deductions to the interested reader.

Remark 5.6. Recall by Remark 2.7, φ is weakly-closed provided ∂y does not contain a nonzero summand from $kY^{(p^\infty)}$ for each $y \in Y$ where $R\langle Y\rangle$ is an acyclic closure of φ; this is needed in the core result Theorem 3.6. The examples of such maps listed in 1.13 are closed, which is *a priori* stronger than requiring φ be weakly-closed. An example of a weakly-closed map that is *not* closed, when k has characteristic p, is unknown to the authors.

Remark 5.7. The Lie algebra L in Theorem 4.3 is intrinsic to the map φ. Namely, as a k-space L is $(\Sigma\Gamma\text{-ind}R\langle Y\rangle)^*$ where $R\langle Y\rangle$ is an acyclic closure of φ. If the characteristic of k is different from 2 or 3, without the assumption φ is weakly-closed, then L acquires a Lie algebra structure by adjusting the formulas for the bracket and reduced square in 4.1. An investigation of this Lie algebra is reserved for another time as the development of this Lie algebra is not entirely relevant to the content in this work.

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