$X = \{ d \text{ general points in } \mathbb{P}^2 \}$

Write $\quad d = \binom{s+t+1}{2} + s \qquad$ UNIQUE

then the Betti table is

	0	1	
$s+t-1$	-	$t+1$	$t-s$
$s+t$	-	-	s

if $t \geq s$

	0	1	
$s+t-1$	-	$t+1$	-
$s+t$	-	$s-t$	s

if $s \geq t$

1) if $t > s$ $\quad \varphi = \underbrace{\left[\begin{array}{c|c} \overbrace{\quad 1 \quad}^{t-s} & \overbrace{\; 2 \;}^{s} \end{array}\right]}_{t} \Big\} \, t+1 \qquad$ possible tecnique : divisors

* if $s=0$ $\quad \varphi$ is linear entries $\quad d = \binom{t+1}{2}$ the only case that is a binomial coefficient (SOL: HU)

2) if $t = s$ $\quad \varphi$ is $t \times t+1$ quadratic entries

3) if $t < s$

$$\varphi = \underbrace{\left[\begin{array}{c} 1 \\ \hline 2 \end{array}\right]}_{s} \begin{array}{l} \} \, s-t \\ \} \, t+1 \end{array} \Big\} \, s+1$$

possible tecnique: generalization of HU

Analysis:

$d=3 \qquad 3 = \binom{3}{2} \qquad s=0 \qquad t=2 \qquad$ linear type since $\mu(I) = 3$

$d=4 \qquad 3 = \binom{3}{2}+1 \qquad s=1 \qquad t=1 \qquad$ CI is again linear type

$d=5$ $\quad 5=\binom{3}{2}+2$ $\quad s=2$ $\quad t=0$ $\quad$ linear type

$d=6$ $\quad 6=\binom{4}{2}$ $\quad s=0$ $\quad t=3$ $\quad$ by MU $\quad \mathcal{J}=(\mathcal{L}, I_3(B))$

$(1,1)^3 \quad (0,3)$ $\quad$ bidegrees of gens $\mathcal{J}$

$d=7$ $\quad 7=\binom{4}{2}+1$ $\quad s=1$ $\quad t=2$ $\quad$ linear type

$d=8$ $\quad 8=\binom{4}{2}+2$ $\quad s=2$ $\quad t=1$ $\quad$ linear type

* $\boxed{d=9}$ $\quad 9=\binom{4}{2}+3$ $\quad s=3$ $\quad t=0$

We use MU

$$\varphi=\begin{bmatrix} 1 \\ \\ *** \end{bmatrix}_2$$

the assumption in MU is

* $\left\{\begin{array}{l} \exists\ \varphi' = s-2 \times s \quad \text{the last } s-2 \text{ block of } \varphi \\ \text{with} \\ I_{s-2}(\varphi') = I_1(\varphi)^{s-2} \end{array}\right.$

In our case this simply says that after row operations one row generates the maximal ideal. This should follow since the points are general.

by MU : the min gens of $\mathcal{J}$ are the 3 def eqs of $Sym(I)$
plus one eq given by $I_3(B)$
(B is a 3×3 matrix)

If we give degrees $(y_1\ y_2\ y_3\ y_4)$, then the honest
$(5,1)^3 \quad (12,3)$

bidegree of the gens of $\mathcal{J}$ is 5^3, 12

$d=10 \qquad 10=\binom{5}{2} \qquad s=0, \ t=4 \qquad$ solved by MU

$(1,1)^4 \quad (0,3)^4 \qquad \mathcal{J}=(\mathcal{L}, I_3(B))$

$d=11 \qquad 11=\binom{5}{2}+1 \qquad s=1 \qquad t=3 \qquad$ solved by Boswell-Mukundan

$$\begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & 2 \\ 1 & 1 & 2 \\ 1 & 1 & 2 \end{bmatrix}$$

NB: we cannot use MU since one row will not generate the maximal ideal which is $I_1(\varphi)$

$\mathcal{J}=(\mathcal{L}, I_3(B_2(\varphi)))$

$$\underset{3\times3}{B_1(\varphi)} = \begin{bmatrix} (0,1) & (0,1) & (1,1) \\ (0,1) & (0,1) & (1,1) \\ (0,1) & (0,1) & (1,1) \end{bmatrix} \qquad \det \quad (1,3)$$

$$\underset{3\times4}{B_2(\varphi)} = \left[\begin{array}{c|c} B_1(\varphi) & \begin{matrix}(0,3)\\(0,3)\\(0,3)\end{matrix} \end{array}\right] \qquad (1,3) \quad (0,5) \quad (1,5)^2$$

hence $\underbrace{(1,1)^2 \quad (2,1) \quad (1,3) \quad (0,5)}_{\text{bidegrees of min gens}} \quad (1,5)^2$ not needed

$d=12$ $\quad 12 = \binom{5}{2} + 2 \quad s=2 \quad t=2$, linear type

* $\boxed{d=13}$ $\quad 13 = \binom{5}{2} + 3 \quad s=3 \quad t=1$
*

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{bmatrix}$$

Again **if one row generates the maximal ideal we can use MU [**should follow from the genericity of the pts]

$\Rightarrow \quad J = (\mathcal{L}, I_3(B))$
honest bidegree
$(6,1)^3 \quad (15,3)$

$d=14$ $\quad 14 = \binom{5}{2} + 4 \quad s=4 \quad t=0$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 \end{bmatrix}$$

assumption of MU

2×2 minors of a general sublock of the matrix consisting of the first 3 rows

has to be m^2 the assumption should be satisfied because of the genericity of the pts

$$J = (\mathcal{L}, I_3(B))$$

$(6,1)^4 \qquad (15,3)^4$

$d=15 \qquad 15 = \binom{6}{2} \qquad s=0 \qquad t=5 \qquad \text{MU} \qquad (1,1)^5 \qquad (0,3)^{10}$

$d=16^{*} \qquad 16 = \binom{6}{2}+1 \qquad s=1 \qquad t=4$ we use the theorem 1.3 of BM

$$\left[\begin{array}{ccc:c} \multicolumn{3}{c}{\overbrace{\hphantom{1\;1\;1}}^{\varphi'}} & \\ 1 & 1 & 1 & 2 \\ 1 & 1 & 1 & 2 \\ 1 & 1 & 1 & 2 \\ 1 & 1 & 1 & 2 \\ 1 & 1 & 1 & 2 \end{array}\right]$$

that says

$$JA = g_{x_3^2} K^{(2)}$$

We need to compute K and show that $K \overset{?}{=} K^{(2)}$

We consider the ring

$$A = \frac{k[x_1, x_2, x_3, y_1, \ldots, y_5]}{\left(\underset{\substack{\shortparallel \\ \underline{x} \cdot B(\varphi')}}{\underline{y} \cdot \varphi'},\; I_3(B(\varphi')) \right)}$$

this is a residual $\cap$ in this case a Link

$$I_1(\underline{x}(B(\varphi'))) : (\underline{x})$$

The ideal K in A is $K = (x_3, I_2(B'))A$

we can assume $B' = \begin{bmatrix} y_1 & y_2 & y_4 \\ y_2 & y_3 & y_5 \end{bmatrix}$

indeed $B' = B(\overline{\varphi^1})$ to obtain it we first we go modulo x_3 and then compute the Jacobian dual of the linear part.

Since $\overline{\varphi}_1$ is a matrix in 2 variables we can pass to the normal form hence

$$B(\overline{\varphi^1}) = \begin{bmatrix} y_1 & y_2 & y_4 \\ y_2 & y_3 & y_5 \end{bmatrix}$$

$$\Rightarrow K = \left(x_3, I_2 \begin{pmatrix} y_1 & y_2 & y_4 \\ y_2 & y_3 & y_5 \end{pmatrix} \right) A$$

prove $K^2 = K^{(2)}$

and $K^2 = (x_3^2, x_3\Delta_1, x_3\Delta_2, x_3\Delta_3, (\Delta_1, \Delta_2, \Delta_3)^2) A$

$$JA = \frac{K^2}{x^3} \cdot g$$

which has as generators

$(2,1)$ $(1,3)^3$ $(0,5)^6$

plus the 4 from A

$(1,1)^3$ $(0,3)$

In general we can do

$\boxed{s=0}$ $d = \binom{t+1}{2}$ by MU

$\mathcal{J} = \mathcal{L} + I_3(B)$ $(1,1)^t$ $(0,3)^{\binom{t}{3}}$

$\boxed{s=1}$ $d = \binom{t+2}{2} + 1$ by BS | NOT TRUE

$$\overbrace{\left[\begin{array}{c|c} \underbrace{\quad\vdots\quad}_{\varphi'} & \begin{matrix} 2 \\ \vdots \\ 2 \end{matrix} \end{array}\right]}^{t-1}$$

$(1,1)^{t-1}$ $(0,3)^{\binom{t-1}{3}}$ def eqs of A

$K^2 \begin{cases} (2,1) \\ (1,3)^{\binom{t-1}{2}} \end{cases}$ $(0,5)^{\binom{t+2}{2}-1}$ g from K^2

$[y_1, \dots, y_{t+1}]\begin{bmatrix} \varphi' \end{bmatrix}$ modulo x_3

$$\begin{bmatrix} x_1 & & & \\ x_2 & x_1 & & \\ \vdots & x_2 & & \\ & \vdots & & x_1 \\ & & & x_2 \end{bmatrix}$$

$$B(\overline{\varphi'}) = \left[\begin{array}{cccc|c} y_1 & y_2 & \cdots & y_{t-1} & y_t \\ y_2 & y_3 & & y_t & y_{t+1} \end{array}\right] \quad K = (x_3, I_2(B(\overline{\varphi'})))$$

$2 \times t$ $(0,4)^{\binom{t-1}{2}}$

$$K^2 = (x_3^2, \underbrace{x_3(\Delta_1 \dots \Delta_{\binom{t-1}{2}})}_{(2,1)}, \underbrace{(\Delta_1, \dots, \Delta_{\binom{t-1}{2}})^2}_{(1,3)^{\binom{t-1}{2}}})$$

$(0,5)^{\binom{t+2}{2}-1}$

$g \dfrac{K^3}{x_3^2}$

$K^{(2)} = ?$

$\binom{\binom{t-1}{2}+1}{2}$, $\binom{6}{2} = 15$

for $t=4,5$ this work but for $t=6,7$ does not

$K^2 \neq K^{(2)}$

the gens are in degree $(e,4)$
what are they?

$\boxed{t=6}$ $y_1, \dots, y_7, x_1, x_2, x_3$

Let $K = (x_3, \Delta_1, \dots, \Delta_{10})$

in this case

(1) $KA^{(2)} = (x_3^2, x_3(\Delta_1, \dots, \Delta_{10}),$

$(\Delta_1, \dots, \Delta_{10})^2 : (y_1, \dots, y_7))$

and indeed

(2) $KA^{(2)} = (\Delta_1, \dots, \Delta_{10})^{(2)}$

which has 21 generators

$(2,0)$ $(1,2)^{10}$ $(0,3)^{10}$

(3) $KA^{(2)} = KA^2 : (y_1, \dots, y_7)$

Summary

$X = \{ d \text{ general points in } \mathbb{P}^2 \}$

Write $\quad d = \binom{s+t+1}{2} + s \qquad$ UNIQUE

then the HB matrix is

1) if $t \geq s$ $\quad \varphi = \underbrace{\left[\begin{array}{c|c} \overbrace{\quad 1 \quad}^{t-s} & \overbrace{\;2\;}^{s} \end{array}\right]}_{t} \Big\} t+1$

2) if $t \leq s$ $\quad \varphi = \begin{array}{r} s-t \{ \\ t+1 \{ \end{array} \underbrace{\left[\begin{array}{c} 1 \\ \hline 2 \end{array}\right]}_{s} \Big\} s+1$

We know:

- $\underbrace{s=0, \quad t \leq 2 < s}_{MU}$, $\quad \underbrace{\max\{t,s\} \leq 2}_{\text{linear type}}$, $\quad \underbrace{s=1 \text{ and } t \leq 5}_{BM}$ (gen of KPU)

$\mathcal{J} = (\mathcal{L}, I_3(B))$

$\mathcal{J}A = \dfrac{gK^2}{x_3^2} A \qquad A = \dfrac{k[\underline{x},\underline{y}]}{(\underline{x}B(\varphi') : \underline{x})}$

$K = (x_3, I_2(\overline{B})) \qquad \; (\underline{x}B(\varphi'), I_3(B(\varphi')))$ (with $=$ above it)

- $\max\{t,s\} = 3$ $\qquad$ by Jouanolou

$\underbrace{\text{Sol of } \max\{t,s\} = 3,}_{\Downarrow \; I \text{ has } \mathcal{F}_0}$ $\quad$ we may assume $3 = t \geq s$

$\text{Sym}(I)$ CI

$\Rightarrow \quad \mathcal{A}_i = \underline{\text{Hom}}_{T}\left(\text{Sym}(I)_{(\delta - i, *)}, T(-3) \right)$, $\quad T = k[\underline{y}]$

For $s=2$ $\Rightarrow$ $\mathcal{A}_0$ this is the only eq. of $\mathcal{J}(I)$ by KPU in bidegree $(0,8)$

$\mathcal{A}_1$

Northcott ideal

$\mathcal{A}_2 \Rightarrow \mathcal{L}: m = (g_1, g_2, g_3):(x_1,x_2,x_3) \overset{?}{=} I_3(B)$ in bidegree $(2,3)$

$\mathcal{A}_i = 0 \quad i \geq 3$

$\mathcal{A}_0$ can be obtained with the resultant of g_1, g_2, g_3

For $\mathcal{A}_1$ one uses Jouanolou and the Morley forms to make the $\cong$ explicit

$\mathcal{A}_1$ is the dual of $Sym(I)_{(1,*)}$

$$\psi: T(-1) \xrightarrow{\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}} Tx_1 \oplus Tx_2 \oplus Tx_3 \to Sym(I)_{(1,*)}$$

the first column of B because we can assume $\varphi_1 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ 0 \end{bmatrix}$

and $\mathcal{A}_1 = \ker \psi^\vee$

$$\to T^3 \xrightarrow{[y_1 y_2 y_3]} T(1)$$

Koszul relations

$\Rightarrow \mathcal{A}_1 \quad (1,4)^3$

For $s=3$ $\mathcal{A}_0$ eq of special fiber $(0,12)$ by KPU

$\mathcal{A}_1$

$\left.\begin{matrix} \mathcal{A}_2 \\ \mathcal{A}_3 \end{matrix}\right\}$ given by linkage use mapping cone gives $(2,3)^3$

A_1 we need to use J again

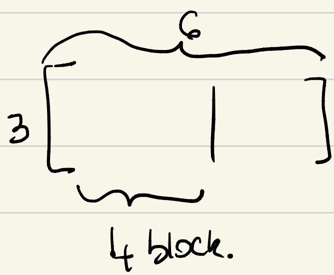
$$T^3(-1) \xrightarrow{B} T^6 \to \mathrm{Sym}_{(2,-)} \to 0$$

wrt the basis of m^2 $[x_1^2 \; x_1x_2 \; x_2^2 \ldots x_3^2]$

$A_1 = \ker B^\vee$

which is given by the BR relations 15 in degree $(1,3)$ but then we have the shift $(1,6)^{15}$

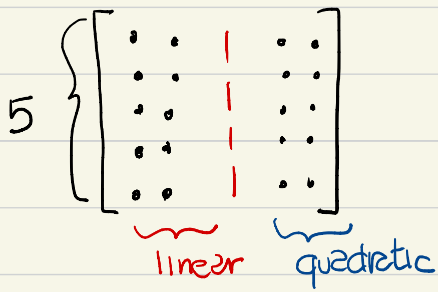
$15 = \binom{6}{4}$



Every block gives us one relation and there are 15 ways to choose a block.

First case not cover by thms is $d=23$

$d=23 = \binom{7}{2} + 2 \qquad s=2 \qquad t=4$



$(1,1)^2 \qquad (2,1)^2$

$(1,3)^2 \qquad \boxed{(2,3)^2}$

$(1,1)^2 \quad (2,1)^2$ Sym

$(1,3)^3 \quad 2$

$(1,4)^4$

$(0,5)^3$

$(0,6)^4$

$(0,8)^5$

$$\left[\begin{array}{cc|cc} (0,1) & (0,1) & (1,1) & (1,1) \\ (0,1) & (0,1) & (1,1) & (1,1) \\ (0,1) & (0,1) & (1,1) & (1,1) \end{array}\right]$$

Jacobian dual gives $(1,3)^2$ and $(2,3)^2$

We need to find $(1,3)$ and $(1,4)^4$ then we could generate all J using Jacobian duals:

$(1,1)^2 \quad x_1, l_2$

$(2,1)^2 \quad l_3\, l_4$ } eqs of $\mathrm{Sym}(I)$

$(1,3)^3 \quad t_1 = \mathrm{Jac}(l_1, l_2, l_3) \quad t_2 = \mathrm{Jac}(l_1, l_2, l_4) \quad t_3$ unknown

$(1,4)^4 \quad p_1 = \mathrm{Jac}(x_1^2, x_1x_2, x_2^2, x_3;\ x_1 l_1, x_2 l_1, l_3, l_4)$

$P_2 = \mathrm{Jac}(x_1^2, x_1x_2, x_2^2, x_3;\ x_1\ell_2, x_2\ell_2, \ell_3, \ell_4)$

$P_3 = \mathrm{Jac}(x_1^2, x_1x_2, x_2^2, x_3;\ x_1\ell_1, x_2\ell_2, \ell_3, \ell_4)$

$P_1 = \mathrm{Jac}(x_1, x_2^2, x_2x_3, x_3^2;\ x_2\ell_1, x_3\ell_1, \ell_3, \ell_4)$

$(0,5)^3$ $\quad q_1 = \mathrm{Jac}(\ell_1, \ell_2, t_1)$

$q_2 = \mathrm{Jac}(\ell_1, \ell_2, t_2)$

$q_3 = \mathrm{Jac}(\ell_1, \ell_2, t_3)$

$(0,6)^4$ $\quad \mathrm{Jac}(\ell_1, \ell_2, P_1) \sim \mathrm{Res}(1,2,3,6,7,8)$

$\mathrm{Jac}(\ell_1, \ell_2, P_2) \sim \mathrm{Res}(3,4,5,6,7,8)$

$\mathrm{Jac}(\ell_1, \ell_2, P_3) \sim \mathrm{Res}(1,3,5,6,7,8)$

$\mathrm{Jac}(\ell_1, \ell_2, P_4) \sim \mathrm{Res}(1,4,5,6,7,8)$

$(0,8)^5$ $\quad \mathrm{Jac}(\ell_1, t_3, P_1)$

$\mathrm{Jac}(\ell_1, t_3, P_2)$

$\mathrm{Jac}(\ell_2, t_3, P_2)$

$\mathrm{Jac}(\ell_2, t_3, P_3)$

$\mathrm{Jac}(\ell_1, t_3, P_4)$

Also $I_6(\mathrm{Jac}(x_1^2, x_1x_2, x_2^2, x_1x_3, x_2x_3, x_3^2;\ x_1\ell_1, x_2\ell_1, x_3\ell_1, x_1\ell_2, x_2\ell_2, x_3\ell_2, \ell_3, \ell_4))$ generates the degree $(0,6)$-part of the Rees ideal.

If one knows $J_{(0,5)}$ one gets t_3: write a general element of $J_{(0,5)}$ is $\Delta_1\alpha_1 + \Delta_2\alpha_2 + \Delta_3\alpha_3$ where Δ_i are the signed minors of $Jac(l_1, l_2)$ then $t_3 = \sum \Delta_i x_i$

Next open case after 23 is 29:

$\boxed{29}$ $t=6$ $s=1$ open

$\boxed{30}$ $t=5$ $s=2$ open

$\boxed{31}$ $t=4$ $s=3$ open

$\boxed{32}$ $t=3$ $s=4$ open

We look at 29 and 37 ($t=7$) which are similar $s=1$

By HB we know that $J \cong K^{(2)}$ but $K^{(2)} \neq K^2$

The situation is as follow:

29 points $\quad k[x_1,x_2,x_3,y_1,\dots,y_7]$

$$\underset{\text{HB}}{\Psi} = \left[\underbrace{\varphi}_{5} \;\Big|\; 2\right] 7$$

φ 7×5 linear matrix

$\varphi = \varphi' + x_3 \underset{\text{matrix of constants}}{\varphi''}$ $\qquad$ φ' is a linear matrix in x_1, x_2

it is general so

$$\begin{array}{c} \begin{array}{cc} 0 & 2 \end{array} \\ \left[\begin{array}{ccc|c} x_1 & & & \\ x_2 & x_1 & & \\ & x_2 & & \end{array}\right] \\ \begin{array}{cc} \underbrace{\qquad}_{3} & \underbrace{\quad}_{2} \end{array} \end{array}$$

$$B(\varphi') = B' = \begin{bmatrix} y_1 & y_2 & y_3 & y_5 & y_6 \\ y_2 & y_3 & y_4 & y_6 & y_7 \end{bmatrix}$$

$$\Rightarrow B = B(\varphi) = \begin{bmatrix} y_1 & y_2 & y_3 & y_5 & y_6 \\ y_2 & y_3 & y_4 & y_6 & y_7 \\ \hline \ell_1 & \ell_2 & \ell_3 & \ell_4 & \ell_5 \end{bmatrix}$$

$\ell_1, \dots, \ell_5$ general linear forms in the y's

$$A = \frac{k[\underline{x}, \underline{y}]}{(\underline{x} \cdot B : \underline{x})}$$

A is defined by the residual intersection $\underline{x} \cdot B : \underline{x} = \underline{y} \cdot \varphi : \underline{x}$ of a c.i. (x_1, x_2, x_3)

$= (\underline{x}\cdot B, I_3(B))$ $\quad$ A is a normal domain

we consider the ideal K which generates the divisor class group of A, $\quad K = (x_1, x_2, x_3)^{-1}$

and

$$JA \cong K^{(2)} \quad \text{more precisely } JA = \frac{g}{x_3^2}\cdot K^{(2)} A$$

where $K = (x_3, I_2(B'))$

<u>CONJECTURE 1</u> : $K^{(2)} = (\underbrace{x_3^2, \; x_3 I_2(B')}_{\text{set}}, \underbrace{(I_2(B'))^2 : (y_1, \dots, y_7)}_{\text{set of } I_2(B')^2})$

<u>CONJECTURE 3</u> : By conj 1 we only have to look at the special fiber ring

$k[y]/I_3(B)$ $\quad$ we need to compute $(I_2(B'))^2$

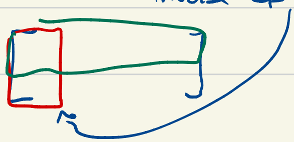
Any t:

$t=3 \begin{bmatrix} y_1 & y_3 \\ y_2 & y_4 \\ l_1 & l_2 \end{bmatrix}$ $\quad t=4 \begin{bmatrix} y_1 & y_2 & y_4 \\ y_2 & y_3 & y_5 \\ l_1 & l_2 & l_3 \end{bmatrix}$ $\quad t=5 \begin{bmatrix} y_1 & y_2 & y_4 & y_5 \\ y_2 & y_3 & y_5 & y_6 \\ l_1 & l_2 & l_3 & l_4 \end{bmatrix}$

$I_2(B')^{(2)} \overset{?}{=} I_2(B')^2$ why?

Now $\quad I_2(B') = ((y_1 y_3 - y_2^2), I_3(B)) : I_2 \begin{bmatrix} y_1 & y_2 \\ y_2 & y_3 \\ l_1 & l_2 \end{bmatrix}$

see also craig for this. In A $I_2(B')$ is the inverse of



If B is all generic

then $I_2(B')^k = I_2(B')^{(k)}$

for all k for the positive power of the divisor class group is clear by CRAP for the negative IT MUST BE IN THE LITERATURE.

Now for $t \leq 5$ it must specialize b/c some ideal must be small one

for $t \geq 6$ it does not we need to compute $K^{(2)}$.

for $t=6$ we have $(0,3)^{10}$

for $t=7$ " $(0,3)^{24}$

for $t=8$ " $(0,3)^{42}$

for $t=9$ " $(0,2)^4 (0,3)^{24}$

for $t=10$ we have $(0,2)^9$ so they are all minimal generators of K but we don't know which ones

for $t=11$

for $t=6$ we could find one of the elts the minor $\begin{vmatrix} y_1 & y_2 & y_3 \\ y_2 & y_3 & y_4 \\ y_5 & y_6 & y_7 \end{vmatrix}$ this is the only det that seems to be a minor