MATH 244 Tutorial Answers (Tut 3)

2019

1. (a) Torricelli says need velocity v of a particle falling height h. Newton's 2^{nd} law says

$$\text{force} = ma = mg \qquad m = \text{mass}, \quad g = 9.81 \text{ m/s}^2$$

i.e. $\dfrac{dv}{dt} = g \quad \Rightarrow \quad v = gt + v_0$

and for a particle starting at zero velocity, at time zero, $v_0 = 0$ so $\boxed{v = gt}$

We need V in terms of height fallen, so consider

$$v = \frac{dz}{dt} = gt \quad \Rightarrow \quad z = \frac{1}{2} g t^2 + z_0$$

and start z at $z = 0$ when $t = 0$ so

$$z = \frac{1}{2} g t^2 \quad \text{or} \quad t = \sqrt{2 \frac{z}{g}}$$

So the time to fall $z = h$ is $t = \sqrt{2 \cdot h/g}$

and hence $V = gt = g\sqrt{\dfrac{2h}{g}} = \sqrt{2gh}$.

(b) outflow is through an area a m^2 so the amount of outflow in m^3/s is $av = a\sqrt{2gh}$ and it is a loss rate.

The rate of volume change in the tank similarly is a velocity times the cross-sectional area A, and the relevant velocity is $\dfrac{dh}{dt}$, the <u>upwards</u> speed of the liquid in the tank at the surface.

Hence, since v is positive but h must be decreasing with time, the rate of volume change in the tank must equal minus the rate liquid is leaving the tank, i.e.

$$A(h)\,\frac{dh}{dt} = -a\sqrt{2gh}\,.$$

(c) $h = 5$, $r_{tank} = 2$ so A is constant, $A = \pi \cdot 4$

and $a = \pi\,(0.2)^2 = 0.04\pi$ is outlet area.

Hence

$$A\,\frac{dh}{dt} = -a\sqrt{2gh} \qquad g = 9.8$$

$$\Rightarrow \quad 4\pi\, h' = -0.04\pi\sqrt{19.6\,h}$$

$$\Rightarrow \quad \frac{dh}{\sqrt{h}} = -0.01\sqrt{19.6}\,dt$$

integrate both sides: $\displaystyle\int h^{-\frac{1}{2}}\,dh = -0.01\sqrt{19.6}\int dt$

$$\Rightarrow \quad 2h^{\frac{1}{2}} = -0.01\sqrt{19.6}\;t + c$$

when $t=0$, $h=5$ so $2\sqrt{5} = c$ and

$$2\sqrt{h} = 2\sqrt{5} - 0.01\sqrt{19.6}\;t$$

This reaches $h=0$ when the RHS is zero, i.e. when

$$t = \frac{2\sqrt{5}}{0.01\sqrt{19.6}} \qquad (\approx 101 \text{ seconds})$$

2. Continuous compounding is described by the differential equation derived in class,

$$\frac{dS}{dt} = rS$$ where r is the rate of return or interest rate.

If payments are made, the derivation is modified a little: Let $S(t)$ be the amount owing, with payments made at a rate k. Then

$$S(t+\Delta t) = S(t) + \underbrace{r\Delta t\, S(t)}_{\substack{\text{interest due} \\ \text{in time } \Delta t}} - \underbrace{k\Delta t}_{\substack{\text{amount} \\ \text{paid off} \\ \text{in time} \\ \Delta t}}$$

$$\Rightarrow \quad \frac{S(t+\Delta t) - S(t)}{\Delta t} = rS - k$$

and taking limits as $\Delta t \to 0$ gives the continuous case:

$$S'(t) = rS - k$$

The general solution is $S = Ae^{rt} + \frac{k}{r}$.

The initial loan amount is \$8000 (using $r = 0.1$ and time is in years)

$$8000 = A + 10k \quad \Rightarrow \quad A = 8000 - 10k$$

and $$S(t) = (8000 - 10k)e^{0.1t} + 10k$$

Payoff in 3 years means $S(3) = 0$, i.e.

$$0 = (8000 - 10k)e^{0.1 * 3} + 10k$$

$$= 8000 \times e^{0.3} - 10ke^{0.3} + 10k$$

$$\Rightarrow \quad k(10e^{0.3} - 10) = e^{0.3} \times 8000$$

$$\Rightarrow \quad k = \frac{e^{0.3}}{10(e^{0.3} - 1)} \times 8000$$

$$\approx 0.385 \times 8{,}000$$

$\approx \$3{,}086.64$ to pay each year

Total payment made is then $3k \approx \$9{,}259.92$
less original loan $8{,}000$
$\Rightarrow$ total interest paid is $\$1{,}259{\cdot}92$

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3. (a) in standard form, the DE is

$$y' + \frac{1}{t(t-4)}y = 0 \quad , \quad y(2) = 1$$

The coefficient function $\frac{1}{t(t-4)}$ is discontinuous (and undefined in fact) at $t = 0$, $t = 4$.

Hence any interval excluding $t = 0, 4$ is O.K.
The initial value is specified at $t = 2$,

so the largest interval on which a unique solution exists for this IVP is $t \in (0, 4)$.

3 (b) in standard form,

$$y' + \frac{2t}{4-t^2} y = \frac{3t^2}{4-t^2} \quad , \quad y(-3) = 1$$

so the coefficient functions are continuous everywhere except at $t = \pm 2$.

Since the initial value is set at $t = -3$, the largest interval on which the IVP has a unique solution is $t \in (-\infty, -2)$.

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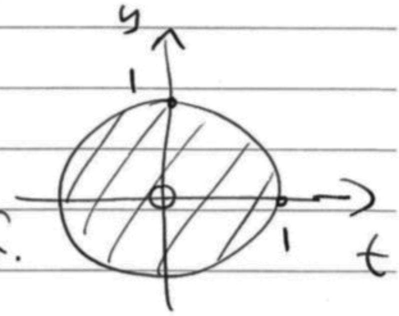
4. $$y' = \sqrt{1 - t^2 - y^2} = f(t, y)$$

Theorem 2.4.2 requires f and $\frac{\partial f}{\partial y}$ to be continuous. f is continuous provided the argument of the $\sqrt{\ }$ is > 0, ie

$$1 - t^2 - y^2 > 0$$
$$\Rightarrow \quad t^2 + y^2 < 1$$

i.e. anywhere inside the unit circle in the (t, y) plane is OK for f.



$$\frac{\partial f}{\partial y} = \frac{-2y}{2\sqrt{1 - t^2 - y^2}} = -\frac{y}{\sqrt{1 - t^2 - y^2}}$$

and $\frac{\partial f}{\partial y}$ is also real and continuous provided t, y lie inside the unit circle.

Hence the hypotheses of Theorem 2.4.2 are satisfied inside the unit circle (centered on origin) in the t, y - plane

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5. (a) $n=0$: $y' + py = q$

multiply by integrating factor $\mu = e^{\int p\,dt}$

$\Rightarrow$

$$\frac{d}{dt}(\mu y) = q\mu$$

$\Rightarrow$ $\mu y = \int q\mu\,dt$ (+c optional)

$\Rightarrow$ $y = \frac{1}{\mu}\int q(t)\mu(t)\,dt$ (+c is optional; not really needed unless one writes down an antiderivative of $q\mu$.)

$\underline{n=1}$: $y' + py = qy$

$\Rightarrow$ $y' + (p+q)y = 0$

$\Rightarrow$ $\mu = e^{\int p+q\,dt}$ (or separate variables is OK too)

$\Rightarrow$

$$\frac{d}{dt}(\mu y) = 0$$

$\Rightarrow$ $\mu y = c$ $\Rightarrow$ $y = \frac{c}{e^{\int (p+q)\,dt}}$

5 (b) $V = y^{1-n} \Rightarrow \dfrac{dV}{dt} = (1-n)\, y^{-n} \dfrac{dy}{dt}$

$$\Rightarrow \frac{dy}{dt} = \left(\frac{y^n}{1-n}\right)\frac{dV}{dt}$$

and the DE becomes

$$\left(\frac{y^n}{1-n}\right)\frac{dv}{dt} + py = q\,y^n$$

and we must eliminate all y's from this. Hence dividing through by y^n gives

$$\left(\frac{1}{1-n}\right)\frac{dv}{dt} + p\,y^{1-n} = q \qquad , y \neq 0$$

i.e.

$$\boxed{\frac{dV}{dt} + (1-n)\,pV = (1-n)\,q}$$

a linear first-order DE in $v(t)$.

(c) $\dfrac{dy}{dt} = (\Gamma\cos t + T)\,y - y^3$ so $n=3$

let $V = y^{1-n} = y^{1-3} = y^{-2}$; $\dfrac{dV}{dt} = -2y^{-3}\dfrac{dy}{dt}$

and the DE becomes

$$-\frac{y^3}{2}\frac{dV}{dt} = (\Gamma\cos t + T)\,y - y^3$$

$\Rightarrow$

$$\frac{dV}{dt} = -2\,(\Gamma\cos t + T)\,y^{-2} + 2$$

$$= -2\left(\Gamma\cos t + T\right)v + 2$$

i.e. $$\frac{dv}{dt} + 2\left(\Gamma\cos t + T\right)v = +2$$

$$\mu = \exp\left(\int\left(\Gamma\cos t + T\right)dt\right) = \exp\left(\Gamma\sin t + Tt\right)$$

and $$\frac{d}{dt}\left(\mu v\right) = 2\mu$$

$$\Rightarrow \quad \mu v = 2\int e^{\Gamma\sin t}\, e^{Tt}\, dt$$

$$\Rightarrow \quad v(t) = 2e^{-\Gamma\sin t - Tt}\underbrace{\int e^{\Gamma\sin t + Tt}\, dt}_{\text{don't try to simplify this...}}$$

(+C is optional in here)