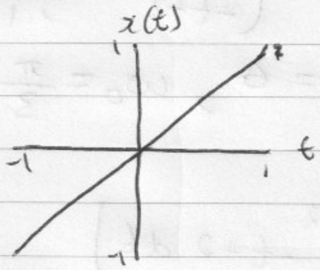
~~ECE22~~ ECEN220 - Assignment 4 Solutions.

3.22

(a)



$$x(t)=t \quad , \quad -1<t<1$$

$$T_0=2 \quad , \quad \omega_0=\pi$$

$$a_0=\frac{1}{2}\int_{-1}^{1} t\,dt=\frac{1}{4}\left[t^2\right]_{-1}^{1}=0$$

$$a_k=\frac{1}{2}\int_{-1}^{1} t e^{-jk\omega_0 t}\,dt$$

(I.B.P) Integration by parts - Tanzalin method

Google this! Very useful!

$$\begin{array}{ccc} & D & I \\ + & t & e^{-jk\omega_0 t} \\ - & 1 & -\frac{1}{jk\omega_0}e^{-jk\omega_0 t} \\ + & 0 & -\frac{1}{k^2\omega_0^2}e^{-jk\omega_0 t} \end{array}$$

So,
$$a_k=\frac{1}{2}\left[-\frac{t}{jk\omega_0}e^{-jk\omega_0 t}+\frac{1}{k^2\omega_0^2}e^{-jk\omega_0 t}\right]_{-1}^{1}$$

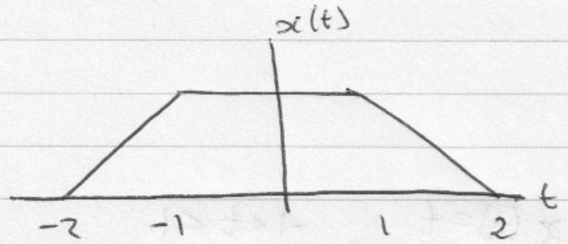
$$=\frac{1}{2}\left(-\frac{1}{jk\omega_0}e^{-jk\omega_0}+\frac{1}{k^2\omega_0^2}e^{-jk\omega_0 t}-\frac{1}{jk\omega_0}e^{jk\omega_0}-\frac{1}{k^2\omega_0^2}e^{jk\omega_0}\right)$$

$$\cancel{=\frac{\cos(\omega_0 k)}{jk\omega_0}}=-\frac{\cos(\omega_0 k)}{jk\omega_0}\underbrace{-j\frac{\sin(\omega_0 k)}{k^2\omega_0^2}}_{=0 \text{ since } \omega_0=\pi.}$$

$$=\frac{j(-1)^k}{k\pi}$$

So
$$a_k=\begin{cases}0 & , \quad k=0\\ \frac{j(-1)^k}{k\pi} & , \quad \text{otherwise}\end{cases}$$

(b)



$$x(t) = \begin{cases} t+2 & , -2<t<-1 \\ 1 & , -1<t<1 \\ -t+2 & , 1<t<2 \end{cases}$$

$$T_0 = 6, \quad \omega_0 = \frac{\pi}{3}$$

$$a_0 = \frac{1}{6}\left(\int_{-2}^{-1} t+2 \, dt + \int_{-1}^{1} dt + \int_{1}^{2} -t+2 \, dt\right)$$

$$= \frac{1}{6}\left(\left[\frac{t^2}{2}+2t\right]_{-2}^{-1} + \left[t\right]_{-1}^{1} + \left[-\frac{t^2}{2}+2t\right]_{1}^{2}\right)$$

$$= \frac{1}{6}\left(\frac{1}{2} - 2 - \frac{2^2}{2} + 4 + 1 + 1 - \frac{2^2}{2} + 4 - \frac{1}{2} - 2\right)$$

$$= \frac{1}{6}\left(1 - 4 - 4 + 4 + 2 + 4\right)$$

$$= \frac{1}{2}$$

For a_k, integrate each integrand seperately, then sum together ~~and~~ and multiplying by $\frac{1}{T_0}$.

$$\int_{-2}^{-1} (t+2)e^{-jk\omega_0 t} \, dt$$

	D	I
+	$t+2$	$e^{-jk\omega_0 t}$
−	1	$-\frac{1}{jk\omega_0} e^{-jk\omega_0 t}$
+	0	$-\frac{1}{k^2\omega_0^2} e^{-jk\omega_0 t}$

$$= \left[-\frac{t+2}{jk\omega_0} e^{-jk\omega_0 t} + \frac{1}{k^2\omega_0^2} e^{-jk\omega_0 t}\right]_{-2}^{-1}$$

$$= -\frac{1}{jk\omega_0} e^{jk\omega_0} + \frac{1}{k^2\omega_0^2} e^{jk\omega_0} + 0 - \frac{1}{k^2\omega_0^2} e^{j2k\omega_0}$$

$$\int_{-1}^{1} e^{-jk\omega_0 t} \, dt = \left[-\frac{1}{jk\omega_0} e^{-jk\omega_0 t}\right]_{-1}^{1} =$$

$$= -\frac{1}{jk\omega_0} e^{-jk\omega_0} + \frac{1}{jk\omega_0} e^{jk\omega_0}$$

$$\cancel{= \frac{2}{k\omega_0} \sin(\omega_0 k)}$$

$$\int_1^2 (-t+2)e^{-jk\omega_0 t}\,dt$$

$\vdots$ I.B.P

$$= \frac{1}{jk\omega_0}e^{-jk\omega_0} + \frac{1}{k^2\omega_0^2}e^{-jk\omega_0} - \frac{1}{k^2\omega_0^2}e^{-2jk\omega_0}$$

Summing each term yields

$$\sum_i \int_{\ell_i} = \frac{1}{k^2\omega_0^2}\left(e^{jk\omega_0} + e^{-jk\omega_0}\right) - \frac{1}{k^2\omega_0^2}\left(e^{2jk\omega_0} + e^{-2jk\omega_0}\right)$$

$$= \frac{2}{k^2\omega_0^2}\left(\cos(\omega_0 k) - \cos(2k\omega_0)\right)$$

$$= \frac{\cancel{2}4}{k^2\omega_0^2}\sin\left(\frac{3\omega_0 k}{2}\right)\sin\left(\frac{k\omega_0}{2}\right)$$ Trig identity - product sum.

$$= \frac{36}{k^2\pi^2}\sin\left(\frac{\pi k}{2}\right)\sin\left(\frac{k\pi}{6}\right)$$ Substitute $\omega_0 = \pi$

Finally,

$$a_k = \frac{1}{6}\left(\sum_i \int_{\ell_i}\right)$$

$$= \begin{cases} \frac{6}{\pi^2 k^2}\sin\left(\frac{\pi k}{2}\right)\sin\left(\frac{k\pi}{6}\right), & k \text{ is odd} \\ 0, & k \text{ is even.} \end{cases}$$

So,

$$a_k = \begin{cases} \frac{1}{2}, & k = 0 \\ 0, & k \text{ is even} \\ \frac{6}{\pi^2 k^2}\sin\left(\frac{\pi k}{2}\right)\sin\left(\frac{\pi k}{6}\right), & k \text{ is odd} \end{cases}$$

3.24 $T_0 = 2$, $\omega_0 = \pi$, $x(t) = \begin{cases} t & , 0 \le t \le 1 \\ 2-t & , 1 \le t \le 2. \end{cases}$

(a)

$$a_0 = \frac{1}{2}\int_0^1 t\,dt + \frac{1}{2}\int_1^2 2-t\,dt$$

$$= \frac{1}{4}\left[t^2\right]_0^1 + \frac{1}{2}\left[2t - \frac{1}{2}t^2\right]_1^2$$

$$= \frac{1}{4} + \frac{1}{2}\left(4 - \frac{2^2}{2} - 2 + \frac{1}{2}\right)$$

$$= \frac{1}{4} + 2 - \frac{1}{2} - 2 + \frac{1}{2}$$

$$\cancel{= \frac{1}{4} - \frac{2}{2} + \frac{1}{2}}$$

$$= \frac{1}{2}$$

(b)

$$\frac{dx(t)}{dt} = \begin{cases} 1 & , 0 \le t \le 1 \\ -1 & , 1 \le t \le 2 \end{cases}$$

so

$$b_0 = \frac{1}{2}\int_0^1 dt - \frac{1}{2}\int_1^2 dt = 0$$

and

$$b_k = \frac{1}{2}\int_0^1 e^{-jk\omega_0 t}dt - \frac{1}{2}\int_1^2 e^{-jk\omega_0 t}\,dt$$

$$= \frac{1}{2}\left(\left[-\frac{1}{jk\omega_0}e^{-jk\omega_0 t}\right]_0^1 + \left[\frac{1}{jk\omega_0}e^{-jk\omega_0 t}\right]_1^2\right)$$

$$= \frac{1}{2}\left(-\frac{1}{jk\omega_0}e^{-jk\omega_0} + \frac{1}{jk\omega_0} + \frac{1}{jk\omega_0}e^{-2jk\omega_0} - \frac{1}{jk\omega_0}e^{-jk\omega_0}\right)$$

$$= \frac{1}{2}\left(-\frac{2}{jk\omega_0}e^{-jk\omega_0} + \frac{1}{jk\omega_0} + \frac{1}{jk\omega_0}e^{-2jk\omega_0}\right)$$

$$= \frac{1}{jk\pi}\left(1 - e^{-jk\pi}\right)$$

sub. $\omega_0 = \pi$, so $e^{-2jk\pi} = 1$

(c) Using the FS properties table,

$$\frac{dx(t)}{dt} \xleftrightarrow{FS} b_k = jk\pi \underbrace{a_k}_{\text{FS of } x(t)}$$

So

$$a_k = \frac{1}{jk\pi} b_k$$

$$= -\frac{1}{\pi^2 k^2}\left(1 - e^{-jk\pi}\right)$$

3.25

(a) $\cos(4\pi t) = \dfrac{e^{j4\pi t} + e^{-j4\pi t}}{2}$, $T_0 = \frac{1}{2}$, $\omega_0 = 4\pi$

We can find a_k by direct comparisan to the ~~Inverse~~ Fourier series. representation of the signal

$$\frac{1}{2} e^{j4\pi t} = a_k e^{j4\pi k t} \longrightarrow a_1 = \frac{1}{2}$$

$$\frac{1}{2} e^{-j4\pi t} = a_k e^{j4\pi k t} \longrightarrow a_{-1} = \frac{1}{2}$$

(b) $\sin(4\pi t) = \dfrac{e^{j4\pi t} - e^{-j4\pi t}}{2j}$, $T_0 = \frac{1}{2}$, $\omega_0 = 4\pi$

Agan by direct comparisan,

$$\frac{1}{2j} e^{j4\pi t} = b_k e^{j4\pi t k} \longrightarrow b_1 = \frac{1}{2j}$$

$$-\frac{1}{2j} e^{-j4\pi t} = b_k e^{j4\pi t k} \longrightarrow b_{-1} = -\frac{1}{2j}$$

$$b_{-1}^* = \frac{1}{2j}$$

(c) Using the multiplication property,

$$x(t)y(t) \xleftrightarrow{FS} \sum_{l=-\infty}^{\infty} a_l b_{k-l} = c_k$$

Note that we only have to test C_k for a small range of k as everything will be 0. Turns out that.

$$C_2 = \sum_\ell \alpha_\ell b_{2-\ell} = \alpha_1 b_1 = \frac{1}{4j}$$

$$C_{-2} = \sum_\ell \alpha_\ell b_{-2-\ell} = \alpha_{-1} b_{-1} = -\frac{1}{4j} \rightarrow C_{-2}^* = \frac{1}{4j}$$

(d) By direct expansion using the product-sum trig identity,

$$\begin{aligned} z(t) &= \sin(4\pi t)\cos(4\pi t) \\ &= \frac{1}{2}\sin(8\pi t) \\ &= \frac{1}{4j} e^{j8t\pi} - \frac{1}{4j} e^{-j8t\pi} \end{aligned}$$

By direct comparison to the Fourier series representation,

$$\left.\begin{aligned} \frac{1}{4j} e^{j8t\pi} = C_k e^{jk4\pi t} &\longrightarrow C_2 = \frac{1}{4j} \\ -\frac{1}{4j} e^{-j8\pi t} = C_k e^{jk4\pi t} &\longrightarrow C_{-2} = -\frac{1}{4j} \\ & \quad\; C_{-2}^* = \frac{1}{4j} \end{aligned}\right\} \text{Matches the coefficients in part (c)}$$

3.26

(a) Using the FS properties table, if $x(t)$ is real, then $a_k = a_{-k}^*$.

~~Let $k = [1, -1]$, then~~

$$\left.\begin{aligned} a_1 &= \tfrac{1}{2}j \\ a_{-1}^* &= -\tfrac{1}{2}j \end{aligned}\right] a_k \neq a_{-k}^*$$

Therefore $x(t)$ is not real

(b) If $x(t)$ is even, then $a_k = a_{-k}$. Using the example given in (a), we can see that this is true. Therefore $x(t)$ is even.

(c)

$$\frac{dx(t)}{d\cancel{x}t} \xleftrightarrow{FS} b_k = jk\omega_0 a_k = jk\frac{2\pi}{T_0} a_k$$

So

$$b_k = \begin{cases} 0 & , \quad k = 0 \\ -k\left(\frac{1}{2}\right)^{|k|}\left(\frac{2\pi}{T_0}\right) & , \quad \text{otherwise.} \end{cases}$$

It can be seen that $b_1 \neq b_{-1}$. Therefore $\frac{dx(t)}{dt}$ is not even.