Common Laplace Transforms

$$
\begin{aligned}
\delta(t) &\Longleftrightarrow 1 \\
\delta^{(n)}(t) &\Longleftrightarrow s^n \\
\mathrm{u}(t) &\Longleftrightarrow \frac{1}{s} \\
\mathrm{e}^{-at}\,\mathrm{u}(t) &\Longleftrightarrow \frac{1}{s+a} \\
t\mathrm{e}^{-at}\,\mathrm{u}(t) &\Longleftrightarrow \frac{1}{(s+a)^2} \\
\frac{t^n}{n!}\mathrm{e}^{-at}\,\mathrm{u}(t) &\Longleftrightarrow \frac{1}{(s+a)^{n+1}} \\
\sin(\omega t)\,\mathrm{u}(t) &\Longleftrightarrow \frac{\omega}{s^2+\omega^2} \\
\cos(\omega t)\,\mathrm{u}(t) &\Longleftrightarrow \frac{s}{s^2+\omega^2} \\
\mathrm{e}^{-at}\sin(\omega t)\,\mathrm{u}(t) &\Longleftrightarrow \frac{\omega}{(s+a)^2+\omega^2} \\
\mathrm{e}^{-at}\cos(\omega t)\,\mathrm{u}(t) &\Longleftrightarrow \frac{s+a}{(s+a)^2+\omega^2} \\
\frac{\omega_\mathrm{n}}{\sqrt{1-\zeta^2}}\mathrm{e}^{-\zeta\omega_\mathrm{n}t}\sin\left(\omega_\mathrm{n}\sqrt{1-\zeta^2}\,t\right)\,\mathrm{u}(t) &\Longleftrightarrow \frac{\omega_\mathrm{n}^2}{s^2+2\zeta\omega_\mathrm{n}s+\omega_\mathrm{n}^2}
\end{aligned}
$$

Properties of the Laplace Transform

$$\mathcal{L}\left\{f(t)\right\} := \int_{0^-}^{\infty} f(t)\mathrm{e}^{-st}\,\mathrm{d}t \qquad \mathcal{L}^{-1}\left\{F(s)\right\} := \frac{1}{2\pi\mathrm{j}}\int_{c-\mathrm{j}\infty}^{c+\mathrm{j}\infty} F(s)\mathrm{e}^{st}\,\mathrm{d}s$$

Definition:	$f(t) \Longleftrightarrow F(s)$
Linearity:	$af(t)+bg(t) \Longleftrightarrow aF(s)+bG(s)$
t-scaling	$f(ct) \Longleftrightarrow \frac{1}{\lvert c\rvert}F\left(\frac{s}{c}\right)$
t-shifting: s-shifting:	$f(t-t_0)\mathrm{u}(t-t_0) \Longleftrightarrow e^{-st_0}F(s)$ $\mathrm{e}^{-s_0t}f(t) \Longleftrightarrow F(s+s_0)$
Differentiation in t: Integration in t:	$f'(t) \Longleftrightarrow sF(s)-f(0)$ $f''(t) \Longleftrightarrow s^2F(s)-sf(0)-f'(0)$ $f^{(k)} \Longleftrightarrow s^kF(s)-s^{k-1}f(0)-s^{k-2}f'(0)\ldots-f^{(k-1)}(0)$ $\int_0^t f(\tau)\,\mathrm{d}\tau \Longleftrightarrow \frac{1}{s}F(s)$
Differentiation in s: Integration in s:	$tf(t) \Longleftrightarrow -F'(s)$ $\frac{f(t)}{t} \Longleftrightarrow \int_s^\infty F(\tilde{s})\,\mathrm{d}\tilde{s}$
Convolution:	$f(t)*g(t) \Longleftrightarrow F(s)G(s)$ $f(t)g(t) \Longleftrightarrow \frac{1}{2\pi\mathrm{j}}F(s)*G(s)$
Periodicity	$f(t) \Longleftrightarrow F_1(s)\times\frac{1}{1-\mathrm{e}^{-sp}}$ for $f_1(t)$ one cycle of $f(t)$ with period p.
Initial value theorem: Final value theorem:	$f(0^+) = \lim_{s\to\infty} sF(s)$ $\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$

(for $a, b, t_0, s_0 \in \mathbb{R}, c \in \mathbb{R}_{++}$).

Partial Fractions Expansion

If a partial fraction expansion of $Y(s)$ includes terms $\dfrac{A_m}{(s-a)^m} + \dfrac{A_{m-1}}{(s-a)^{m-1}} + \ldots + \dfrac{A_1}{s-a}$, then the coefficients of factors having multiplicity $m > 1$ are given by the following expressions, where $k \neq m$.

$$
\begin{aligned}
A_m &= \lim_{s \to a} (s-a)^m Y(s) \\
A_k &= \frac{1}{(m-k)!} \lim_{s \to a} \frac{\mathrm{d}^{m-k}}{\mathrm{d}s^{m-k}} (s-a)^m Y(s)
\end{aligned}
$$

Trigonometric Identities

$$
\sin(\theta \pm \phi) = \sin\theta\cos\phi \pm \cos\theta\sin\phi \quad \Longrightarrow \quad \begin{cases} \sin(\theta + \frac{\pi}{2}) & = \cos(\theta) \\ \sin(\theta - \frac{\pi}{2}) & = -\cos(\theta) \end{cases}
$$

$$
\cos(\theta \pm \phi) = \cos\theta\cos\phi \mp \sin\theta\sin\phi \quad \Longrightarrow \quad \begin{cases} \cos(\theta + \frac{\pi}{2}) & = -\sin(\theta) \\ \cos(\theta - \frac{\pi}{2}) & = \sin(\theta) \end{cases}
$$

$$
\begin{aligned}
\sin(2\theta) &= 2\sin\theta\cos\theta \\
\cos(2\theta) &= \cos^2\theta - \sin^2\theta \\
&= 2\cos^2\theta - 1 \\
&= 1 - 2\sin(\theta)
\end{aligned}
$$

First Order Systems

For a first order system with transfer function $G(s) = \dfrac{1}{s+a}$,

Rise time (10–90%) is $t_{\mathrm{r}} = 2.2\tau$

Settling time (to 2%) is $t_{\mathrm{s}} = 4\tau$

Second Order Systems

The following relationships hold for an underdamped second order system having transfer function

$$G(s) = \frac{\omega_{\mathrm{n}}^2}{s^2 + 2\zeta\omega_{\mathrm{n}} s + \omega_{\mathrm{n}}^2} \quad .$$

The time taken to reach the peak value is $t_{\mathrm{peak}} = \dfrac{\pi}{\omega_{\mathrm{n}}\sqrt{1-\zeta^2}}$.

The percentage overshoot is related to damping ratio by

$$\begin{aligned} \%OS &= 100 \times \exp\left(-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}\right) \\ \implies \zeta &= \frac{-\ln(\%OS/100)}{\sqrt{\pi^2 + \ln^2(\%OS/100)}} \end{aligned}$$

Settling time (to ±2%) is $t_{\mathrm{s}} = \dfrac{4}{\zeta\omega_{\mathrm{n}}} = 4\tau$.

Phase margin - ζ relation.

$$\begin{aligned} \mathsf{PM} &= \arctan \frac{2\zeta}{\sqrt{-2\zeta^2 + \sqrt{1+4\zeta^4}}} \\ &\approx 100\zeta \quad \text{for } \zeta \leq 0.6 \end{aligned}$$

The frequency response has a peak magnitude of $M_{\mathrm{p}} = \dfrac{1}{2\zeta\sqrt{1-\zeta^2}}$, at $\omega_{\mathrm{p}} = \omega_{\mathrm{n}}\sqrt{1-2\zeta^2}$.

Steady State Errors

The following hold for a unity-gain, negative-feedback system with transfer function $G(s)$.

$$e_{\text{step}}(\infty) = \frac{1}{1+K_{\text{p}}} := \frac{1}{1+\lim_{s\to 0} G(s)}$$

$$e_{\text{ramp}}(\infty) = \frac{1}{K_{\text{v}}} := \frac{1}{\lim_{s\to 0} sG(s)}$$

$$e_{\text{parabola}}(\infty) = \frac{1}{K_{\text{a}}} := \frac{1}{\lim_{s\to 0} s^2 G(s)}$$

Compensator Topologies

Proportional Compensator

$$C(s) = K_{\text{p}}$$

Proportional-Integral (PI) Compensator

$$C(s) = K_{\text{p}} \frac{s+\omega_{\text{b}}}{s} \equiv K_{\text{p}} \frac{\frac{s}{\omega_{\text{b}}}+1}{\frac{s}{\omega_{\text{b}}}}$$

Lag Compensator

$$C(s) = K_{\text{p}} \frac{s+\omega_{\text{b}}}{s+\frac{\omega_{\text{b}}}{\alpha}} \equiv K_{\text{p}}\alpha \frac{\frac{s}{\omega_{\text{b}}}+1}{\frac{\alpha s}{\omega_{\text{b}}}+1}, \quad \text{where } \alpha > 1.$$

Proportional-Derivative (PD) Compensator

$$C(s) = K_{\text{p}} \left(\frac{s}{\omega_{\text{b}}} + 1 \right)$$

Lead Compensator

$$C(s) = \frac{K_{\text{p}}}{\alpha} \frac{s+\omega_{\text{b}}}{s+\frac{\omega_{\text{b}}}{\alpha}} \equiv K_{\text{p}} \frac{\frac{s}{\omega_{\text{b}}}+1}{\frac{\alpha s}{\omega_{\text{b}}}+1}, \quad \text{where } \alpha < 1.$$

The maximum phase lead of $\phi_{\max} = \arcsin\left(\frac{1-\alpha}{1+\alpha}\right)$ occurs at a frequency $\omega = \frac{\omega_{\text{b}}}{\sqrt{\alpha}}$. Consequently,

$$\alpha = \frac{1-\sin(\phi_{\max})}{1+\sin(\phi_{\max})}$$