1. a) $x' = \begin{pmatrix} 10 & -5 \\ 8 & -12 \end{pmatrix} x$

e-val: $\det(A - rI) = 0$, $\det\begin{pmatrix} 10-r & -5 \\ 8 & -12-r \end{pmatrix} = 0$

$\Rightarrow (10-r)(-12-r) + 40 \Rightarrow r^2 + 2r - 80 \Rightarrow (r+10)(r-8)$

$r = \{-10, 8\}$

e-vect: $r = 8$

$(A - 8I)\hat{v} = \hat{0} \Rightarrow \begin{pmatrix} 10-8 & -5 \\ 8 & -12-8 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\Rightarrow \begin{pmatrix} 2 & -5 \\ 8 & -20 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$2v_1 - 5v_2 = 0$, $t = 2v_1$, $t = 5v$ $\rightarrow \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = t\begin{pmatrix} \frac{1}{2} \\ \frac{1}{5} \end{pmatrix} \Rightarrow \hat{v} = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{5} \end{pmatrix} = \begin{pmatrix} 5 \\ 2 \end{pmatrix}$

x4 $8v_1 - 20v_2 = 0$

$\hat{x}_1(t) = \begin{bmatrix} 5 \\ 2 \end{bmatrix} e^{8t}$

$r = -10$, $(A + 10I)\hat{v} = \hat{0} \Rightarrow \begin{pmatrix} 20 & -5 \\ 8 & -2 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$8v_1 - 2v_2 = 0 \Rightarrow 4v_1 - v_2 = 0$: $t = v_2$, $t = 4v_1$

$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = t\begin{pmatrix} \frac{1}{4} \\ 1 \end{pmatrix}$, $\hat{v} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$ $\therefore \hat{x}_2(t) = \begin{bmatrix} 1 \\ 4 \end{bmatrix} e^{-10t}$

check w/ $W(x_1, x_2)$ from FSS:

$W(\hat{x}_1, \hat{x}_2)(0) = \det\begin{bmatrix} 5e^{8t} & e^{-10t} \\ 2e^{8t} & 4e^{-10t} \end{bmatrix} = \det\begin{pmatrix} 5 & 1 \\ 2 & 4 \end{pmatrix} = 20 - 2 \neq 0$ ✓

$\therefore$ the FSS solves for all t

in general: $\hat{x}(t) = c_1\begin{pmatrix} 5 \\ 2 \end{pmatrix} e^{8t} + c_2\begin{pmatrix} 1 \\ 4 \end{pmatrix} e^{-10t}$

$\hat{x}(t) = \begin{pmatrix} c_1 \cdot 5e^{8t} + c_2 e^{-10t} \\ c_1 \cdot 2e^{8t} + c_2 \cdot 4e^{-10t} \end{pmatrix}$

1. b) $x' = \begin{pmatrix} 12 & -9 \\ 4 & 0 \end{pmatrix} x$

e-val: $\det(A - rI) = 0$, $\det\begin{pmatrix} 12-r & -9 \\ 4 & -r \end{pmatrix} = 0$

$(12-r)(-r) + 36 = 0 \Rightarrow r^2 - 12r + 36 \Rightarrow (r-6)(r-6)$

$r = (6)^{\times 2}$

e-vect: $(A - 6I)\hat{v} = \hat{0} \Rightarrow \begin{pmatrix} 6 & -9 \\ 4 & -6 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$4v_1 - 6v_2 = 0$, $t = 4v_1$, $t = 6v_1$ $\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = t\begin{pmatrix} \frac{1}{4} \\ \frac{1}{6} \end{pmatrix}$ $\bar{v} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$

$\hat{x}_1(t) = \begin{bmatrix} 3 \\ 2 \end{bmatrix} e^{6t}$

$(A - 6I)\hat{\omega} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$, $\begin{pmatrix} 6 & -9 \\ 4 & -6 \end{pmatrix}\begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$

$4\omega_1 - 6\omega_2 = 2$, $\omega_1 = 2$, $\omega_2 = 1$, $\hat{\omega} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$\hat{x}_2(t) = \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{6t} + \begin{bmatrix} 3 \\ 2 \end{bmatrix} t e^{6t}$

check w/ $W(x_1, x_2)$ from FSS

$W(\tilde{x}_1, \tilde{x}_2)(0) = \det\begin{pmatrix} 3 & 2 \\ 2 & 1 \end{pmatrix} \Rightarrow 3 - 4 \neq 0$

$\therefore$ FSS solves for all t

in general: $\tilde{x}(t) = c_1\begin{pmatrix} 3 \\ 2 \end{pmatrix} e^{6t} + c_2\left[\begin{pmatrix} 3 \\ 2 \end{pmatrix} t e^{6t} + \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{6t}\right]$

1. c) $x' = \begin{pmatrix} 1 & -8 \\ 1 & -3 \end{pmatrix} x$

e-val: $\det(A - rI) = 0$, $\det\begin{pmatrix} 1-r & -8 \\ 1 & -3-r \end{pmatrix} = 0$

$\Rightarrow (1-r)(-3-r)+8)$, r^2+2r+5

$r = \frac{-2 \pm \sqrt{-16}}{2} = -1 \pm 2i = r_{1,2}$

e-vect: $(A - (-1-2i)I)\hat{v} = \bar{0}$

$\Rightarrow \begin{pmatrix} 1-(-1-2i) & -8 \\ 1 & -3-(-1-2i) \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 2+2i & -8 \\ 1 & -2+2i \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$(2+2i)v_1 - 8v_2 = 0$; choose $v_1 = 8$, $v_2 = (2+2i)$

$\bar{v} = \begin{pmatrix} 8 \\ 2+2i \end{pmatrix}$

$\bar{x}(t) = \begin{pmatrix} 8 \\ 2+2i \end{pmatrix} e^{(-1-2i)t} = \begin{pmatrix} 8 \\ 2+2i \end{pmatrix} e^{-t} e^{-2it} = \begin{bmatrix} 8 \\ 2+2i \end{bmatrix} e^{-t}(\cos(2t) - i\sin(2t))$

$\bar{x}(t) = \left(\begin{bmatrix} 8 \\ 2 \end{bmatrix} + i\begin{bmatrix} 0 \\ 2 \end{bmatrix}\right) e^{-t}(\cos(2t) - i\sin(2t))$

check lin w/ FSS

$W(0) = \det\begin{pmatrix} 8 & 0 \\ 2 & 2 \end{pmatrix} = 16 - 0 \neq 0$ ✓

so: $\bar{x}_1 = \left(\begin{pmatrix} 8 \\ 2 \end{pmatrix} \cos 2t - \begin{pmatrix} 0 \\ 2 \end{pmatrix} \sin 2t\right) \cdot e^{-t}$

$\bar{x}_2 = \left(\begin{pmatrix} 8 \\ 2 \end{pmatrix} \sin 2t - \begin{pmatrix} 0 \\ 2 \end{pmatrix} \cos 2t\right) \cdot e^{-t}$

in general: $x(t) = c_1 \bar{x}_1 + c_2 \bar{x}_2$

2. $x' = \begin{pmatrix} 10 & -5 \\ 8 & -12 \end{pmatrix} x + \begin{pmatrix} e^t \\ 0 \end{pmatrix}$, $F(t) = \begin{pmatrix} e^{2t} & 2e^{6t} \\ e^{2t} & e^{6t} \end{pmatrix}$

$x_p = F(t)\tilde{u}(t)$: $\tilde{u}(t) = \int F(t)^{-1} \tilde{g}(t)\, dt$

$$\det(F(t)) = e^{2t} e^{6t} - 2e^{6t} e^{2t}$$
$$= e^{8t} - 2e^{8t} = -e^{8t}$$

$$F(t)^{-1} \tilde{g}(t) = \frac{1}{-e^{8t}} \begin{pmatrix} e^{6t} & -2e^{6t} \\ -e^{2t} & e^{2t} \end{pmatrix} \begin{pmatrix} e^t \\ 0 \end{pmatrix}$$
$$= \frac{1}{-e^{8t}} \begin{pmatrix} e^{7t} \\ -e^{3t} \end{pmatrix} = \begin{pmatrix} -\frac{1}{e^t} \\ \frac{1}{e^{5t}} \end{pmatrix} = \begin{pmatrix} -e^{-t} \\ e^{-5t} \end{pmatrix}$$

$$\tilde{u}(t) = \int \begin{pmatrix} -e^{-t} \\ e^{-5t} \end{pmatrix} = \begin{pmatrix} e^{-t} \\ -\frac{1}{5} e^{-5t} \end{pmatrix}$$

$$x_p = F(t)\tilde{u}(t) = \begin{pmatrix} e^{2t} & 2e^{6t} \\ e^{2t} & e^{6t} \end{pmatrix} \begin{pmatrix} e^{-t} \\ -\frac{1}{5} e^{-5t} \end{pmatrix}$$

$$x_p = \begin{pmatrix} e^t - \frac{2}{5} e^t \\ e^t - \frac{1}{5} e^t \end{pmatrix} = \begin{pmatrix} \frac{3}{5} e^t \\ \frac{4}{5} e^t \end{pmatrix}$$

generally: $\tilde{x}(t) = F(t)\tilde{c} + x_p(t)$ for $c \in \mathbb{R}^2$

$$\tilde{x}(t) = \begin{pmatrix} e^{2t} & 2e^{6t} \\ e^{2t} & e^{6t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} + \begin{pmatrix} \frac{3}{5} e^t \\ \frac{4}{5} e^t \end{pmatrix}$$