ECEN203
Analogue Circuits and Systems
Section 1: DC Circuit Analysis

2019

Overview

- Kirchhoff's Laws
- Mesh Analysis
- Nodal Analysis
- Circuit Theorems
- Superposition

Revision

- Voltage, V: energy/charge, (V)
- Current, I: charge/second, (C/s)
- Charge measured in Coulombs (C) -1C = charge on 6.241509×10^{18} electrons

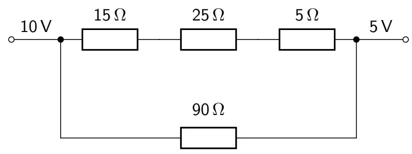
Series and Parallel Resistances

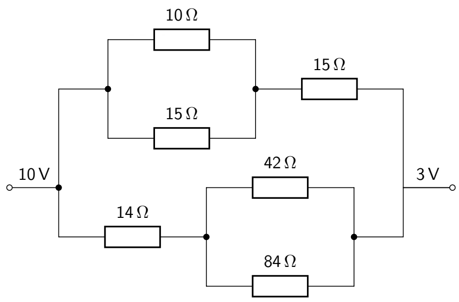
- Ohm's Law: $v = iR$
- Power: $P = vi = i^2R = V^2/R$ (W or J/s)
- Series Resistance: $R_\mathsf{S} = \sum_i R_i$ ($R_\mathsf{S} > R_i$)
- Parallel Resistance: $\frac{1}{R_\mathsf{P}} = \sum_i \frac{1}{R_i}$ ($R_\mathsf{P} < R_i$)
- Two parallel resistances: $R_\mathsf{P} = \frac{R_1 R_2}{R_1 + R_2}$

Conductance

- $G = \frac{1}{R}$
- Series conductances: $\frac{1}{G_\mathsf{S}} = \sum_i \frac{1}{G_i}$
- Parallel conductances: $G_\mathsf{P} = \sum_i G_i$

Revision - Find R_{TOT}, P_{TOT}





?

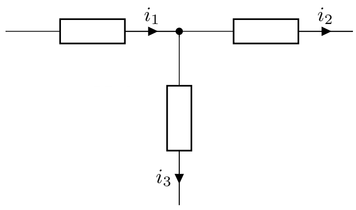
$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$
$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$$
$$\nabla \cdot \mathbf{D} = \rho$$
$$\nabla \cdot \mathbf{B} = 0$$

Maxwell's Equations

$\oint_C \mathbf{E} \cdot d\boldsymbol{\ell} = -\frac{d\Phi}{dt}$	Faraday's Law
$\oint_C \mathbf{H} \cdot d\boldsymbol{\ell} = I + \int \frac{\partial \mathbf{D}}{\partial t} \cdot ds$	Ampère's Law
$\oint_S \mathbf{D} \cdot ds = Q$	Gauss' Law
$\oint_S \mathbf{B} \cdot ds = 0$	No magnetic monopole

Kirchhoff's Current Law

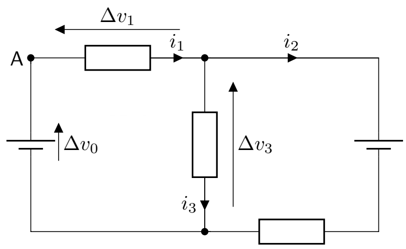
- Current Law (KCL): $\sum$ currents in $= \sum$ currents out
- OR (if we label all currents as inwards or outwards) $\sum$ currents $= 0$



$$i_1 = i_2 + i_3$$

Kirchhoff's Voltage Law

- Voltage Law (KVL): $\sum$ voltages in a loop $= 0$



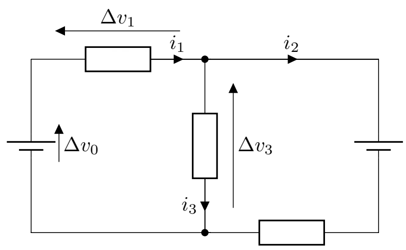
$$-\Delta v_1 - \Delta v_3 + \Delta v_0 = 0$$

Kirchhoff's Laws

- Why do Kirchhoff's Laws help us to find currents in a circuit?
- For m currents & n junctions:
  - KVL produces $m - n + 1$ independent equations
  - KCL produces $n - 1$ independent equations
- Total m equations for m unknown currents
- In practice we always need far fewer than this

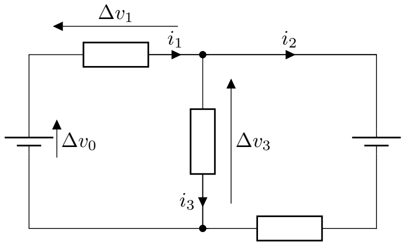
KVL Polarity: Resistor

- Loop traversing a resistor:
  - In *same* direction as current: $\Delta v = -iR$
  - In *opposite* direction to current: $\Delta v = +iR$



KVL Polarity: Voltage Source

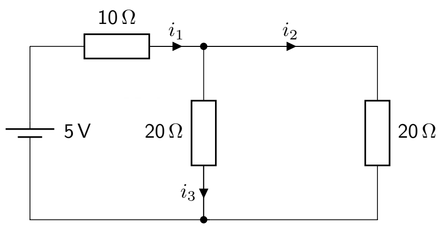
- Loop traversing a voltage source:
  - from -ve to +ve: $\Delta v > 0$
  - from +ve to -ve: $\Delta v < 0$



KVL Polarity: Negatives

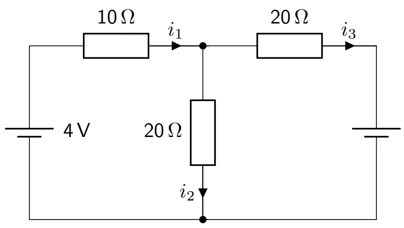
- If the solved i is -ve, the actual current is in the opposite direction to the initial arrow
- If the solved v is -ve, the actual voltage is in the opposite direction to the initial arrow

KVL+KCL Example: Find i_1, i_2 & i_3

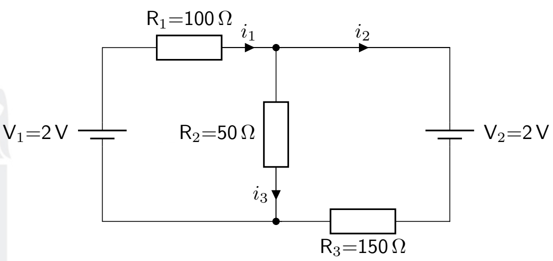


KVL+KCL Example 2

- Find the currents in each of the three resistors
- Find the power delivered by each source and dissipated by each resistor



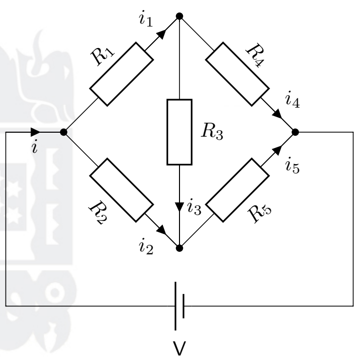
KVL+KCL Example 3



- $n = 2$ junctions: $n - 1$ eqns
- $m = 3$ currents, $m - n + 1$ eqns

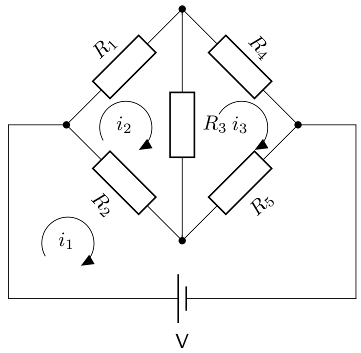
A better way...

- Using KCL & KVL to find currents:
- 6 equations, 6 unknowns



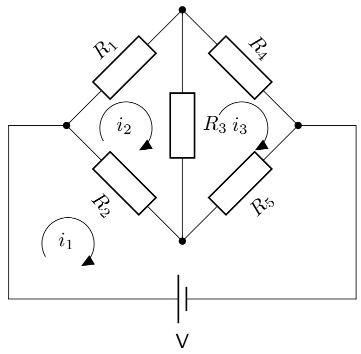
4 junctions give 3 equations:
$i_1 = i_3 + i_4$
$i_2 + i_3 = i_5$
$i = i_4 + i_5 = i_1 + i_2$
Loops give 3 equations:
lower: $V - i_2R_2 - i_5R_5 = 0$
left: $-i_1R_1 - i_3R_3 + i_2R_2 = 0$
right: $i_3R_3 - i_4R_4 + i_5R_6 = 0$

Mesh Analysis (I)



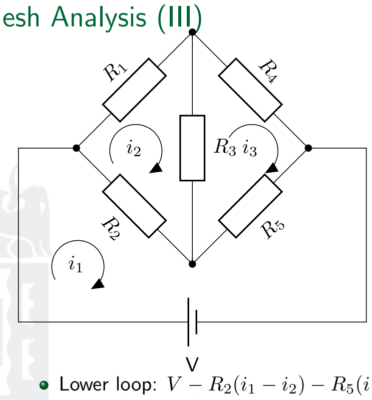
- Assign 1 current/loop
- Apply KVL around each loop, taking into account *all* currents

Mesh Analysis (II)



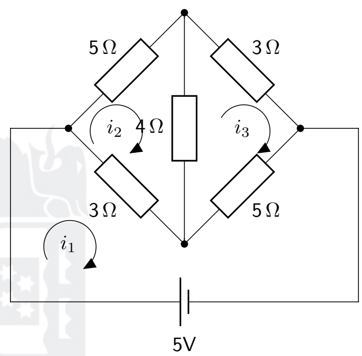
- 3 equations in 3 unknowns
- Junction equations (KCL) are “hidden”
- Actual current is sum of nominal currents, e.g., i in R_2 is $i_1 - i_2$.

Mesh Analysis (III)



- Lower loop: $V - R_2(i_1 - i_2) - R_5(i_1 - i_3) = 0$
  $V - i_1(R_2 + R_5) + i_2R_2 + i_3R_5 = 0$
- LH loop: $-R_1i_2 - R_3(i_2 - i_3) - R_2(i_2 - i_1) = 0$
  $i_1R_2 - i_2(R_1 + R_2 + R_3) + i_3R_3 = 0$
- RH loop: $-R_3(i_3 - i_2) - R_4i_3 - R_5(i_3 - i_1) = 0$
  $i_1R_5 + i_2R_3 - i_3(R_3 + R_4 + R_5) = 0$

Mesh Analysis (IV)

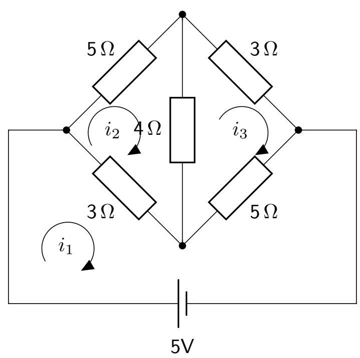


- Lower loop:
  $5 - i_1(3 + 5) + i_23 + i_35 = 0$
- LH loop:
  $i_13 - i_2(5 + 4 + 3) + i_34 = 0$
- RH loop:
  $i_15 + i_24 - i_3(4 + 3 + 5) = 0$

$$\begin{bmatrix} -8 & 3 & 5 \\ 3 & -12 & 4 \\ 5 & 4 & -12 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \\ i_3 \end{bmatrix} = \begin{bmatrix} -5 \\ 0 \\ 0 \end{bmatrix} \quad \Rightarrow \quad \begin{aligned} i_1 &= 1.290\,\text{A} \\ i_2 &= 0.564\,\text{A} \\ i_3 &= 0.726\,\text{A} \end{aligned}$$

Mesh Analysis: Example

- Find *i* through 4 Ω resistor

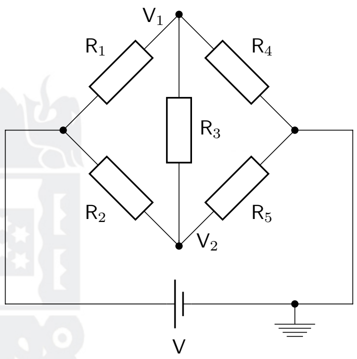


$i_3 - i_2 = 0.726 - 0.565 = 0.161$ A in the direction of i_3 (upwards)

Nodal Analysis

- Assign a reference node
- Use KCL at each subsequent node where the voltage is unknown
- Form and solve equations for voltages

Nodal Analysis Example (I)



- KCL @ V_1

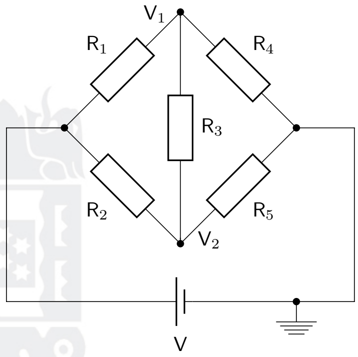
$$\frac{V - v_1}{R_1} + \frac{v_2 - v_1}{R_3} + \frac{-v_1}{R_4} = 0$$

- KCL @ V_2:

$$\frac{V - v_2}{R_2} + \frac{v_1 - v_2}{R_3} + \frac{-v_2}{R_5} = 0$$

Mesh Analysis Example (II)

$R_1 = R_5 = 2\,\Omega$, $R_2 = R_4 = 4\,\Omega$, $R_3 = 3\,\Omega$, $V =$5 V



$$\frac{5 - v_1}{2} + \frac{v_2 - v_1}{3} + \frac{-v_1}{4} = 0$$

$$\frac{5 - v_2}{4} + \frac{v_1 - v_2}{3} + \frac{-v_2}{2} = 0$$

$$\begin{bmatrix} \frac{-1}{2} + \frac{-1}{3} + \frac{-1}{4} & \frac{1}{3} \\ \frac{1}{3} & \frac{-1}{2} + \frac{-1}{3} + \frac{-1}{4} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} -\frac{5}{2} \\ -\frac{5}{4} \end{bmatrix}$$

Nodal Analysis Example (III)

$$\begin{bmatrix} -13 & 4 \\ 4 & -13 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \begin{bmatrix} -30 \\ -15 \end{bmatrix}$$

- 2 Eqns. in 2 unknowns
- Solve by substitution, computer, or Cramer's Rule

Solving matrix equations

- In Mesh / Nodal Analysis, often have to solve $Ax = b$, where
  - A is an $n \times n$ matrix of coefficients
  - x is a vector of unknown variables, e.g., $i_1, \ldots, i_3$ or v_1, v_2
  - b is a vector of known voltages or currents

Matrix Equations

$$\begin{bmatrix} -13 & 4 \\ 4 & -13 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \begin{bmatrix} -30 \\ -15 \end{bmatrix}$$

$$\begin{bmatrix} -8 & 3 & 5 \\ 3 & -12 & 4 \\ 5 & 4 & -12 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \\ i_3 \end{bmatrix} = \begin{bmatrix} -5 \\ 0 \\ 0 \end{bmatrix}$$

$$Ax = b$$

Cramer's Rule Example

- Cramer's Rule says we can find x_i by replacing the ith column of A with b and calculating the normalised determinant of the resulting matrix, A_i:

$$A : \begin{bmatrix} -8 & 3 & 5 \\ 3 & -12 & 4 \\ 5 & 4 & -12 \end{bmatrix}$$

$$\Rightarrow \quad A_1 : \begin{bmatrix} \begin{bmatrix} -5 \\ 0 \\ 0 \end{bmatrix} & \begin{matrix} 3 \\ -12 \\ 4 \end{matrix} & \begin{matrix} 5 \\ 4 \\ -12 \end{matrix} \end{bmatrix}$$

$$\Rightarrow \quad A_2 : \begin{bmatrix} \begin{matrix} -8 \\ 3 \\ 5 \end{matrix} & \begin{bmatrix} -5 \\ 0 \\ 0 \end{bmatrix} & \begin{matrix} 5 \\ 4 \\ -12 \end{matrix} \end{bmatrix}$$

$$\Rightarrow \quad A_3 : \begin{bmatrix} \begin{matrix} -8 \\ 3 \\ 5 \end{matrix} & \begin{matrix} 3 \\ -12 \\ 4 \end{matrix} & \begin{bmatrix} -5 \\ 0 \\ 0 \end{bmatrix} \end{bmatrix}$$

Cramer's Rule

$$x_i = \frac{|A_i|}{|A|}$$

Example

- Find $|A|$, then find i_1, i_2, i_3

$$\begin{bmatrix} -8 & 3 & 5 \\ 3 & -12 & 4 \\ 5 & 4 & -12 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \\ i_3 \end{bmatrix} = \begin{bmatrix} -5 \\ 0 \\ 0 \end{bmatrix}$$

Example

- Find $|A|$, then find v_1 and v_2

$$\begin{bmatrix} -13 & 4 \\ 4 & -13 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \begin{bmatrix} -30 \\ -15 \end{bmatrix}$$

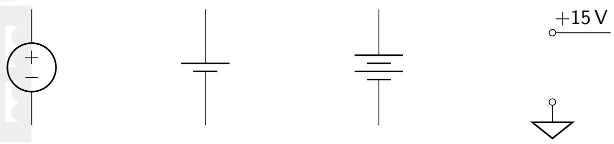
Cramer's Rule

- Some (equivalent) conditions for valid results:
  - Need as many equations as unknowns
  - Need a unique solution for results to be valid
  - A must be square (i.e., $n \times n$)

Voltage Source

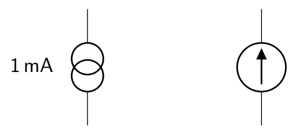
Voltage Source

- Ideally, maintains fixed voltage across its terminals *regardless of load*
- i.e., an ideal voltage source has 0 resistance
- e.g., behaves like an *ideal* battery



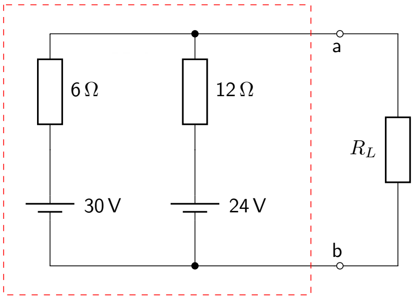
Current Source

- Ideally, provides a constant current *regardless of load*
- i.e. ideal current source has ∞ resistance.
- Practically – very complicated: Requires lots of semiconducting elements
- See more of this in ECEN303, ECEN403



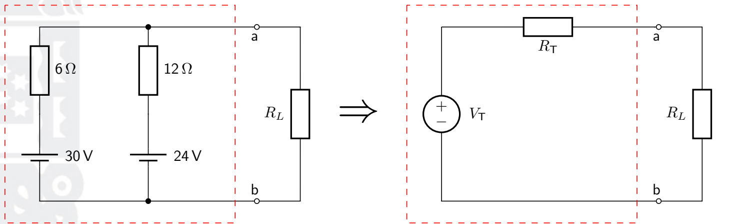
Circuit Theorems

- A common theme of this course is that complicated circuits can be reduced to much simpler ones.
- e.g., What does the circuit in the red box look like to the load, R_L?



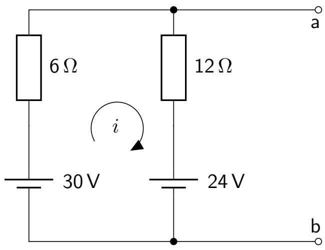
Thevenin's Theorem

- We can replace a two terminal network of resistors and energy sources with a *voltage* source V_{T} and a *series* resistance, R_{T}:
- V_{T} = open-cct voltage of the network
- $R_{\mathrm{T}} = V_{\mathrm{OC}}/I_{\mathrm{SC}}$



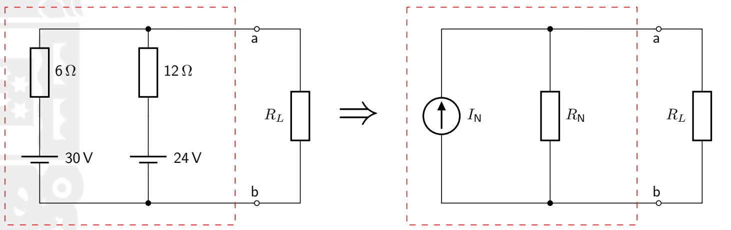
Thevenin's Theorem Example

- What is the Thevenin equivalent of this circuit?



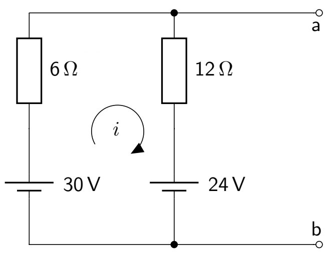
Norton's Theorem

- We can replace any two terminal network of resistors and energy sources with a *current* source and a *parallel* resistance, R_{N}:
- I_{N} = short-circuit current of the network
- $R_{\mathrm{N}} = V_{\mathrm{OC}}/I_{\mathrm{SC}}$



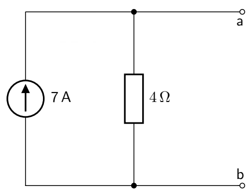
Norton's Theorem Example

- What is the Norton equivalent of this circuit?



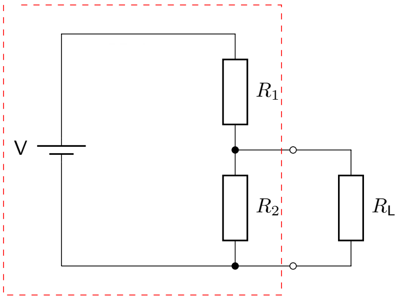
Norton's Theorem Example (II)

- Example: What is the Norton equivalent of this circuit?
- Answer:



Example: equivalent of voltage divider

- Find the Thevenin and Norton equivalents of a voltage divider
  - i.e. what is the equiv. cct inside the box?
  - Remove load
  - Find V_{OC}
  - Find I_{SC}

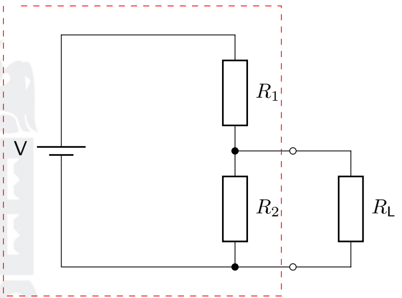


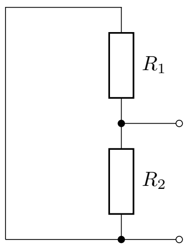
Source Resistance

- Another way to find R_{T} or R_{N} is to measure the resistance "looking into" the network, with sources zeroed, i.e.,:
  - Voltage sources shorted out ($V = 0$)
  - Current sources open-circuited ($I = 0$)

Source Resistance

- Remove load
- Short voltage source

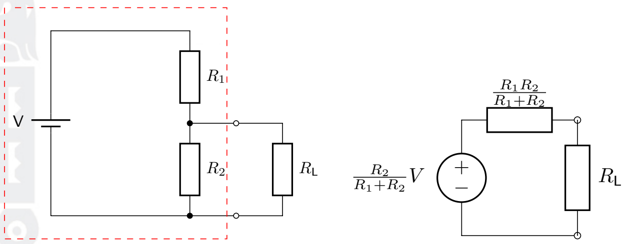




$$R_{\mathrm{T}} = R_{\mathrm{N}} = \frac{R_1 R_2}{R_1 + R_2}$$

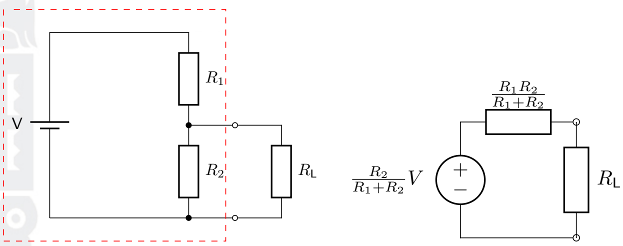
Circuit Loading

- Previous example shows we can replace a voltage divider with a Thevenin equivalent V_{T} and series resistance, R_{T}.
- What happens when a load is connected?



Circuit Loading

- R_{L} & R_{T} form new voltage divider, so $V_{\mathrm{L}} < V_{\mathrm{T}}$
- Voltage drop, $V_{\mathrm{L}} - V_{\mathrm{T}}$ depends entirely on the ratio of R_{L} to $R_1 \| R_2$.
- If $R_{\mathrm{L}} \ll R_1 \| R_2$, then $V_{\mathrm{L}} \ll V_{\mathrm{T}}$

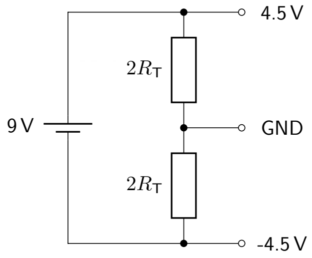


Circuit Loading

- Practically, we can think of R_{T} as the internal resistance of the source
- We say RL loads the source
- We can reduce loading by using smaller R_1, R_2 (stiff voltage divider – will see this in ECEN204)
- But: this leads to larger currents and more resistive heating

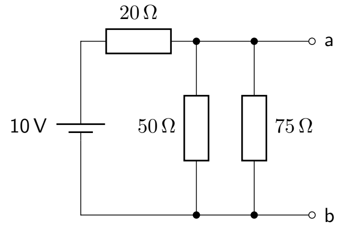
Circuit Loading Example

- Design a power supply based on a 9V battery for an audio amplifier
- How big should R_T be?



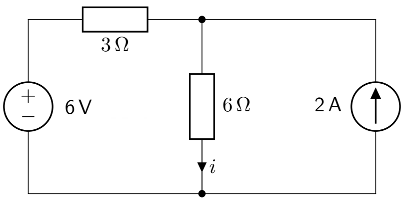
Another example

- Find the Thevenin equivalent circuit



- $R_L = 100\,\Omega$ is connected between a & b.
- What is V_L?
- What is P_L (power dissipated across R_L)?

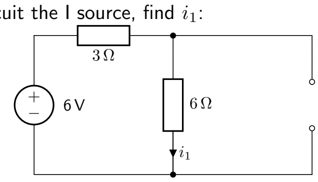
Superposition



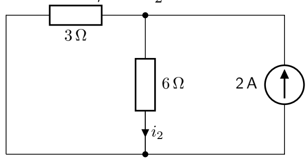
- Previously, had to use KCL + KVL around two loops to find i
- Alternatively, can consider effect of sources independently:

Superposition (II)

- Open-circuit the I source, find i_1:



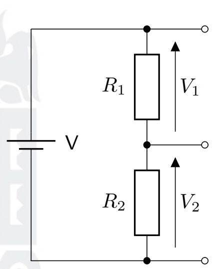
- Short-circuit V source, find i_2:

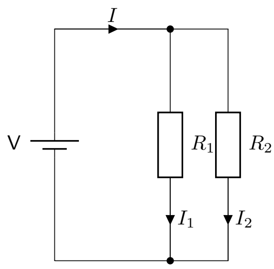


- By superposition, $i = i_1 + i_2$

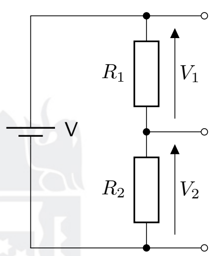
(Aside:)Current Divider

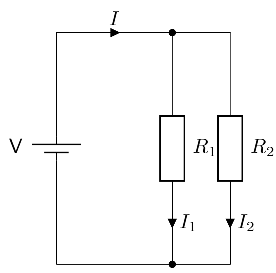
- Consider the similarities between these two circuits:





Current Divider (II)





$$\frac{V_1}{V_2} = \frac{R_1}{R_2}$$
$$V = V_1 + V_2$$
$$V_1 = V\frac{R_1}{R_1 + R_2}$$

$$\frac{I_1}{I_2} = \frac{G_1}{G_2} = \frac{R_2}{R_1}$$
$$I = I_1 + I_2$$
$$I_1 = I\frac{G_1}{G_1 + G_2} = I\frac{R_2}{R_1 + R_2}$$

Summary

- Kirchhoff's Laws
- Mesh Analysis
- Nodal Analysis
- Cramer's Rule
- Circuit Theorems: $R_{\mathsf{R}} = V_{\mathsf{OC}}/I_{\mathsf{SC}}$
- V & I Sources
- Superposition