School of Mathematics and Statistics

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ENGR 222 **Assignment 3** **Due: Thursday 01 April 11:59pm**

1 Multiple integrals

(a) Evaluate the integral of $f(x, y, z) = xyz$ over the region

$$G = \{(x, y, z) : xy \leq z \leq 1\,,\ 0 \leq x \leq y\,,\ 0 \leq y \leq 1\}\,.$$

(b) Using spherical coordinates, determine the integral of $f(x, y, z) = x$ over the region G described by the inequalities $x, y, z \geq 0$ and $x^2 + y^2 + z^2 \leq 1$.
Hint: Observe that G consists of one eighth of a ball/sphere.

(c) Calculate the integral of $f(x, y) = y^{-2}e^{-x}$ over the region

$$R = \{(x, y) : x \in [0, \infty]\,,\ y \in [2, \infty]\}\,.$$

(d) Determine the centroid of the two dimensional object described in polar coordinates by

$$R = \{(r, \theta) : 0 \leq r \leq \theta\,,\ \theta \in [0, 2\pi]\}\,.$$

Hint: evaluate any required integrals using polar coordinates, and you may use one or more of the following integrals if needed:

$$\int_0^{2\pi} \theta^2 \sin(\theta)\, d\theta = -4\pi^2\,, \qquad \int_0^{2\pi} \theta^2 \cos(\theta)\, d\theta = 4\pi\,,$$
$$\int_0^{2\pi} \theta^3 \sin(\theta)\, d\theta = 12\pi - 8\pi^3\,, \qquad \int_0^{2\pi} \theta^3 \cos(\theta)\, d\theta = 12\pi^2\,.$$

2 Vector fields

(a) Calculate the divergence of the vector field $\mathbf{F} = x^2y^3z^4\mathbf{i} - xyz\mathbf{j} + (x + y + z)\mathbf{k}$.

(b) Calculate the curl of the vector field $\mathbf{F} = x^2y^3z^4\mathbf{i} - xyz\mathbf{j} + (x + y + z)\mathbf{k}$.

(c) Determine the gradient field of $\phi(x, y, z) = xz^2 + \sin(y)e^x$.

(d) Calculate the Laplacian of $\phi = xz^2 + \sin(y)e^x$ (i.e. $\nabla \cdot \nabla\phi$).

3 Line integrals

(a) Calculate the value of the line integral $\int_C f\, ds$ where

$$f(x,y,z) = \frac{y}{x}e^z$$

and C is described by

$$(x,y,z) = (2t, t^2, \ln(t)) \quad \text{for} \quad t \in [1,4]\,.$$

Hint: any square root terms can be eliminated by completing the square.

(b) Calculate the value of the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ where

$$\mathbf{F}(x,y,z) = x\mathbf{i} - e^z\mathbf{j} + y\mathbf{k}$$

and C is again described by

$$(x,y,z) = (2t, t^2, \ln(t)) \quad \text{for} \quad t \in [1,4]\,.$$

(c) Calculate the value of the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ where $\mathbf{F}$ is the gradient field of

$$\phi = \cos(x\sin(ye^z))$$

and C is described by the vector-valued function

$$\mathbf{r}(t) = \pi\cos(\pi t/2)\mathbf{i} + \left(\frac{\pi}{2} + \sin(8\pi t)\right)\mathbf{j} + t(1-t)\mathbf{k} \quad \text{for} \quad t \in [0,1]\,.$$

Hint: do not attempt to calculate the line integral directly!

(d) Confirm that the vector field

$$\mathbf{F}(x,y) = -2xe^{-x^2}\sin(y)\mathbf{i} + (1 + e^{-x^2}\cos(y))\mathbf{j}$$

is conservative via the conservative field test, then determine the potential function of $\mathbf{F}$. Comment on what the result of $\oint_C \mathbf{F} \cdot d\mathbf{r}$ is expected to be for any closed curve C.

4 Lab question:

This question will involve some Python coding, please include relevant code as part of your submission (copy and paste the code and any plots, or use a screenshot if needed). If you're not sure what needs to be included please ask. You should not need to include more than 100 lines of code, and ideally only around half of that.

(a) In lab notebook 3 we examined the family of arc length parameterised curves

$$x(s) = \int_0^s \cos(\theta(u))\, du\,, \qquad y(s) = \int_0^s \sin(\theta(u))\, du\,.$$

For the following two questions, consider the function $\theta(u) = \pi \sin(\ln(1 + u^2))$.

(i) Produce a plot of the resulting curve over the interval $s \in [0, 10]$. What are the coordinates of the point at $s = 10$?

(ii) Estimate the curvature at $s = 5$ using numerical differentiation.
(Hint: Use the finite difference approach from lab notebook 2, pick a couple of values of h you think will be near the optimum choice and check that the results are similar, up to at least 4 decimal places.)

(b) Consider the following data describing points lying on a parameterised curve:

```
ti = [0.0, 0.6 , 1.1 , 1.5 , 1.8 , 2.1 , 2.3 , 2.5 , 2.8 , 3.2 ]
xi = [0.0,-0.44,-0.69,-0.63,-0.62,-0.55,-0.63,-0.67,-0.44,-0.10]
yi = [0.0,-0.15,-0.41,-0.15,-0.11,-0.31,-0.20,-0.15,-0.23,-0.21]
zi = [0.0, 0.36, 0.11, 0.09, 0.16, 0.06,-0.04, 0.12, 0.01, 0.05]
```

Hint: you should be able to copy/paste the above data directly into your code. Some minor edits might be required depending on your specific pdf viewer.

(i) Construct a smooth parameterised curve through these points using the function "UnivariateSpline". What are the coordinates of the curve when $t = 2$.

(ii) Determine the unit tangent, principal unit normal and binormal vectors at $t = 2$.
Hint: utilise the code that calculated these vectors towards the end of lab notebook 4.

(c) (i) Using the function "dblquad" from scipy.integrate, evaluate the integral of

$$f(x, y) = \cos(x)e^y\,,$$

over the non-rectangular region

$$R = \{(x, y) : x^2 \le y \le 10 + \sin(x)\,,\ x \in [-3, 3]\}\,.$$

(ii) Using the function "tplquad" from scipy.integrate, evaluate the integral of

$$f(x, y, z) = \frac{4}{1 + x^2 + y^2 + z^2}\,,$$

over the unit ball centred at the origin, i.e. $G = \{(x, y, z) : x^2 + y^2 + z^2 \le 1\}$.
Hint: you need to start by coming up with a suitable description of the region G.