Math 2444 Ass 7 Daniel Eisen 300424349

1. $\mathcal{L}\left\{\int_0^x \tau^3 e^{x-\tau} d\tau\right\}$: then: $\mathcal{L}(f*g)(s) = \mathcal{L}\{f\}\cdot\mathcal{L}\{g\}$

defined: $(f*g) = \int_0^x f(\tau)g(x-\tau)\,d\tau$

so: $\mathcal{L}\int_0^x \tau^3 e^{x-t} d\tau = \mathcal{L}(t^3 * e^t) = \mathcal{L}(t^3)\cdot\mathcal{L}(e^t)$

$= \frac{6}{s^4}\cdot\frac{1}{s-1} = \frac{6}{s^5-s^4}$

2. proof on commutivity (w/ change of var)

commutivity states $f*g = g*f$; $(f*g)t = \int_0^t f(\tau)g(t-\tau)\,d\tau$

• change var: $\beta = t-\tau$, $\therefore d\beta = -d\tau$

so now $(f*g)(t) = \int_t^0 f(t-\beta)g(\beta)(-1)\,d\beta$

$= \int_0^t g(\beta)f(t-\beta)\,d\beta$ ← defines $(g*f)(t)$

So we can conclude $(f*g) = (g*f)$

3. $\mathcal{L}\{x^n f(x)\} = (-1)^n \frac{d^n}{ds^n}\mathcal{L}\{f(x)\}$

solve: $xy'' - y' - 2x^2 \Leftrightarrow \mathcal{L}\{xy''\} - \mathcal{L}\{y'\} = \mathcal{L}\{2x^2\}$

$(y(0) = y'(0) = 0)$

using derivative form: gives $-\frac{d}{ds}\left(\mathcal{L}\{y''\}\right) - \mathcal{L}\{y'\} = \frac{4}{s^3}$

$= -\frac{d}{ds}\left(s^2 Y\right) - sY = 4/s^3$

$= 2sY - sY = 4/s^2 \quad = Y(-3s) = 4/s^3, \; Y = \frac{-4}{3s^{\cancel{3}4}}$

$y = \mathcal{L}^{-1}\left\{\frac{-4}{3s^4}\right\}$

4. a) $x''' - x = \phi$: $y_1 = x$ $y_2 = x'$ $y_3 = x''$

$y_1' = y_2$

$y_2' = y_3$

$y_3' = x''' = x + 0 = y_1 + 0$

$$\begin{bmatrix} y_1' \\ y_2' \\ y_3' \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\hat{x}'(t) = A \cdot \hat{x}(t) + \hat{g}(t)$ ✓ and $\hat{g}(t) = \hat{\phi}$ ✓

so (a) is linear & homogeneous.

b) $x'' + tx' + (t^2-1)x = 0$

$y_1 = x$ $y_2 = x'$; $y_2' = x''$

$y_1' = y_2 (+0)$

$y_2' = -ty_2 - (t^2-1)x = (1-t^2)y_1 - ty_2 (+0)$

$$\begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ (1-t^2) & -t \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\hat{x}'(t) = A \cdot \hat{x}(t) + \hat{g}(t)$ ✓ ; $\hat{g}(t) = \hat{0}$ ✓

so (b) is linear & homogeneous.

5) $x' = x - y + z + t - 1$
$y' = 2x + y - z - 3t^2$
$z' = x + y + z + t^2 - 2$

$$\Rightarrow \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & -1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} t-1 \\ -3t^2 \\ t^2-2 \end{bmatrix}$$

6) $$\vec{x}'(t) = \begin{bmatrix} 4 & 2 \\ -1 & 3 \end{bmatrix} \vec{x}(t) + \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t}$$

$$\Rightarrow x' = 4x + 2y + e^{-t}$$
$$y' = -x + 3y - e^{-t}$$