TE WHARE WĀNANGA O TE ŪPOKO O TE IKA A MĀUI

VICTORIA

UNIVERSITY OF WELLINGTON

EXAMINATIONS – 2018

TRIMESTER 1

MATH 244

MODELLING WITH DIFFERENTIAL EQUATIONS

Time Allowed: THREE HOURS

CLOSED BOOK

Permitted materials: Silent non-programmable calculators or silent programmable calculators with their memories cleared are permitted.
Printed foreign language to English dictionaries are permitted.
Formula sheets are provided with this exam.
No other material is permitted.

Instructions: The exam will be marked out of a total of 100 marks.

Answer in the appropriate spaces if possible — if you write your answer elsewhere, make it clear where your answer can be found.

Please use the blank reverse sides of pages for any extra space you need, for working or for answers. There is also a blank page at the end that may be used.

If you answer all six questions, they will all be marked, and you will be credited with the five best marks.

For marking use only

1	/20
2	/20
3	/20
4	/20
5	/20
6	/20
best 5	/100

1. **(20 marks)**

(a) Consider the differential equation

$$y'' + 4y' + 13y = 0\ .$$

i. **(4 marks)** Find a fundamental set of solutions $y_1(x), y_2(x)$ for this differential equation.

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ii. **(4 marks)** Show that the set you obtained in part (i) is a fundamental set by calculating the value of the Wronskian at $x = 0$.

/ 8

(Question 1 continued on next page)

(Question 1 continued)

iii. **(2 marks)** Hence write down the general solution to this differential equation.

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iv. **(3 marks)** Hence find the solution to the initial value problem $y'' + 4y' + 13y = 0$, $y(0) = 0$, $y'(0) = 1$. Describe how this solution behaves as $x \to +\infty$.

(b) Consider the homogeneous differential equation

$$y'' + p(x)y' + q(x)y = 0\,,$$

where $p(x)$ and $q(x)$ are given continuous functions for all x in any open interval **I** that contains the point $x = 0$.

i. **(5 marks)** Write down the unique solution to the initial value problem that consists of this differential equation together with the initial conditions $y(0) = 0$, $y'(0) = 0$.

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ii. **(2 marks)** Over what interval is it guaranteed that your solution exists?

/ 12

2. **(20 marks)**

(a) Consider the differential equation

$$\left(\frac{y}{x}+6x\right)dx+(\ln x-2)dy=0\,,\quad x>0\,.$$

i. **(2 marks)** Show that it is exact.

ii. **(5 marks)** Find the general solution.

iii. **(3 marks)** Find the solution explicitly if the initial condition is $y(1)=1$.

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(b) Consider the homogeneous linear differential equation

$$(x-1)y'' - xy' + y = 0$$

i. **(1 mark)** Show that $y_1(x) = e^x$ is a solution.

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ii. **(6 marks)**

Use the method of reduction of order to find another solution.

/ 7

(c) **(3 marks)** Prove Abel's Theorem, which states: the Wronskian of a pair of solutions y_1, y_2 of the differential equation $y'' + p(t)y' + q(t)y = 0$ is given by $W[y_1, y_2](t) = c \exp\left(-\int p(t)\,dt\right)$.
Hint: calculate W', eliminate second derivatives to show that $W' = -p(t)W$, then solve for W.

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3. **(20 marks)**

(a) **(5 marks)** State and prove the formula for $\mathcal{L}(f')(s)$.

/ 5

(b) **(5 marks)**

State and prove the first translation theorem, which describes $\mathcal{L}(e^{ax}f(x))(s)$ in terms of $\mathcal{L}(f)$, and use it to compute $\mathcal{L}(x^4e^{3x})(s)$.

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(c) **(10 marks)**

Calculate the folowing inverse Laplace transforms:

(i) $\mathcal{L}^{-1}\left(\frac{1}{s^2-4}\right)$ (ii) $\mathcal{L}^{-1}\left(\frac{1}{s^2+4s+5}\right)$ (iii) $\mathcal{L}^{-1}\left(\frac{s}{s^2+2s+5}\right)$.

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4. **(20 marks)**

(a) **(10 marks)** Solve the following initial-value problem using Laplace transforms:

$$y'' - 2y' + 2y = e^x, \quad y(0) = y'(0) = 0.$$

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(b) **(6 marks)** Define the convolution $f * g$ of two functions f, g on $[0, \infty)$, and state the convolution theorem that describes $\mathcal{L}(f * g)$. Deduce that

$$\mathcal{L}\Big(\int_0^x f(t)\,dt\Big)(s) = \frac{1}{s}\mathcal{L}(f)(s).$$

/ 6

(c) **(4 marks)** Compute the Laplace transform of the function

$$f(x) = \begin{cases} 0 & \text{for } x < 0 \\ x & \text{for } 0 \leq x \leq 1 \\ 1 & \text{for } x > 1. \end{cases}$$

/ 4

5. **(20 marks)**

(a) **(10 marks)** Solve the initial-value problem

$$y' = 1 + 2e^x - \int_0^x y(t)\,dt, \qquad y(0) = 0.$$

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(b) **(10 marks)** Solve the following initial value problem using Laplace transforms:

$$y'' - 2y' + 2y = e^x, \qquad y(0) = y'(0) = 0.$$

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6. **(20 marks)**

(a) **(6 marks)** Find two solutions of the system

$$\begin{pmatrix} y' \\ z' \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} y \\ z \end{pmatrix},$$

and prove using results from the lectures that your solutions are the columns of a fundamental matrix for the system.

/ 6

(b) **(4 marks)** Write the following differential equations as first order systems, and say whether each system is linear and/or homogeneous:

$$\text{(i) } y'' = y' + y + 1, \qquad \text{(ii) } y'' = xy' + x^2y.$$

/ 4

(c) **(10 marks)** Find the general solution of the system

$$\begin{pmatrix} y' \\ z' \end{pmatrix} = \begin{pmatrix} -1 & 3 \\ -3 & 5 \end{pmatrix} \begin{pmatrix} y \\ z \end{pmatrix}.$$

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End of Questions

Use this spare page and the other side for rough working if needed.

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DIFFERENTIAL EQUATIONS FORMULA SHEET

Trigonometric identities

- $\sin^2 x + \cos^2 x = 1$
- $\sin 2x = 2 \sin x \cos x$
- $\cos 2x = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$
- $\sec^2 x = 1 + \tan^2 x$
- $\csc^2 x = 1 + \cot^2 x$
- $\sin x \cos y = \frac{1}{2}\left[\sin(x-y) + \sin(x+y)\right]$
- $\sin x \sin y = \frac{1}{2}\left[\cos(x-y) - \cos(x+y)\right]$
- $\cos x \cos y = \frac{1}{2}\left[\cos(x-y) + \cos(x+y)\right]$
- if $t = \tan\frac{x}{2}$ then $\sin x = \frac{2t}{1+t^2}$, $\cos x = \frac{1-t^2}{1+t^2}$, and $\tan x = \frac{2t}{1-t^2}$

Hyperbolic identities

- $\cosh^2 x - \sinh^2 x = 1$
- $\sinh 2x = 2 \sinh x \cosh x$
- $\cosh 2x = \cosh^2 x + \sinh^2 x = 2\cosh^2 x - 1 = 2\sinh^2 x + 1$
- $\operatorname{sech}^2 x = 1 - \tanh^2 x$
- $\operatorname{csch}^2 x = \coth^2 x - 1$
- $\sinh x \cosh y = \frac{1}{2}\left[\sinh(x+y) + \sinh(x-y)\right]$
- $\sinh x \sinh y = \frac{1}{2}\left[\cosh(x+y) - \cosh(x-y)\right]$
- $\cosh x \cosh y = \frac{1}{2}\left[\cosh(x+y) + \cosh(x-y)\right]$

Derivatives of trigonometric and hyperbolic functions

$$\frac{d}{dx}[\sin x] = \cos x$$

$$\frac{d}{dx}[\cos x] = -\sin x$$

$$\frac{d}{dx}[\tan x] = \sec^2 x$$

$$\frac{d}{dx}[\cot x] = -\csc^2 x$$

$$\frac{d}{dx}[\sec x] = \sec x \tan x$$

$$\frac{d}{dx}[\csc x] = -\csc x \cot x$$

$$\frac{d}{dx}[\tan^{-1} x] = \frac{1}{1+x^2}$$

$$\frac{d}{dx}[\sinh x] = \cosh x$$

$$\frac{d}{dx}[\cosh x] = \sinh x$$

$$\frac{d}{dx}[\tanh x] = \text{sech}^2 x$$

$$\frac{d}{dx}[\coth x] = -\text{csch}^2 x$$

$$\frac{d}{dx}[\text{sech} x] = -\text{sech} x \tanh x$$

$$\frac{d}{dx}[\text{csch} x] = -\text{csch} x \coth x$$

Exponents and logarithms

- $y = a^x \quad \Leftrightarrow \quad x = \log_a y$ and $y = e^x \quad \Leftrightarrow \quad x = \ln y$
- $\dfrac{d}{dx}[a^x] = a^x \cdot \ln a$ and $\frac{d}{dx}[\log_a x] = \frac{1}{x \ln a}$

Matrices

Let $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$.

- $A^{-1} = \dfrac{1}{\det A} \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}$.
- Cramer's Rule: the solution of $A\mathbf{x} = \mathbf{b}$ is given by $x_1 = \dfrac{\det A_1}{\det A}$, $x_2 = \dfrac{\det A_2}{\det A}$

  where $A_1 = \begin{pmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{pmatrix}$, $A_2 = \begin{pmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{pmatrix}$.

Sums and series

- $\displaystyle\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$

- $\displaystyle\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$

- $\displaystyle\sum_{k=1}^{n} k^3 = \left[\frac{n(n+1)}{2}\right]^2$

- Geometric progression: $\displaystyle\sum_{k=0}^{n-1} r^k = 1 + r + r^2 + \cdots + r^{n-1} = \frac{1-r^n}{1-r}, \quad r \neq 1$

  Geometric series: $\displaystyle\sum_{k=0}^{\infty} r^k = \frac{1}{1-r}$ when $|r| < 1$

- Maclaurin series: $\displaystyle\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k$

- Taylor series: $\displaystyle\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!} (x-a)^k$

- Fourier series for $f(x)$ on $[-\pi, \pi]$: $\displaystyle\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$

  where $\displaystyle a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos nx \, dx, \quad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx \, dx.$

Transforms

- Laplace transform: $\mathcal{L}\{f(x)\} = \int_0^\infty f(x)e^{-sx}\,dt = F(s)$

$\mathcal{L}\{1\} = \dfrac{1}{s}$

$\mathcal{L}\{t^n\} = \dfrac{n!}{s^{n+1}}$, $\quad n$ a positive integer

$\mathcal{L}\{e^{ax}\} = \dfrac{1}{s-a}$

$\mathcal{L}\{\cos ax\} = \dfrac{s}{s^2+a^2}$

$\mathcal{L}\{\sin ax\} = \dfrac{a}{s^2+a^2}$

$\mathcal{L}\{f'(x)\} = sF(s) - f(0)$

$\mathcal{L}\{f^{(n)}(x\} = s^nF(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - sf^{(n-2)}(0) - f^{(n-1)}(0)$

- Fourier transform: $\mathcal{F}\{f(x)\} = \int_{-\infty}^\infty f(x)e^{i\alpha x}\,dt = F(\alpha)$

$\mathcal{F}\{f'(x)\} = -i\alpha F(\alpha)$

Inverse Fourier transform: $\mathcal{F}^{-1}\{F(\alpha)\} = \dfrac{1}{2\pi}\int_{-\infty}^\infty F(\alpha)e^{-i\alpha x}\,d\alpha = f(x)$.