MATH 244 Assignment 4 Daniel Eisen 300447549

1. DE: $y' = y(y-1)(y-2)$

0: unstable 1: stable 2: unstable

2. a) $\underbrace{(\sin y - y\sin x)}_{M(x,y)}dx + \underbrace{(\cos x + x\cos y - y)}_{N(x,y)}dy = 0$

: Exact if $\partial M/\partial y = \partial N/\partial x$ $(M_y = N_x)$

if so, solution is $\psi(x,y) = c$: $\psi_x = M$ $\psi_y = N$

$M_y = \frac{\partial}{\partial y}(\sin y - y\sin x) = \underline{\cos y - \sin x}$

$N_x = \frac{\partial}{\partial x}(\cos x + x\cos y) = -\sin x + \cos y = \underline{\cos y - \sin x}$

$M_y = N_x$ ✓ $\therefore$ 2a is exact and solveable.

$\psi?$: $\frac{\partial \psi}{\partial x} = M = \sin y - y\sin x$

$\psi(x,y) = \int(\sin y - y\sin x) = \int \sin y - \int y\sin x = \sin y\int 1 - y\int \sin x$

$\psi = x\sin y + y\cos x + h(y)$

$\psi_y = N = \frac{\partial \psi}{\partial y}(x\sin y + y\cos x + h(y))$

$= x\cos y + \cos x + h'(y)$

$N = x\cos x + \cos x - y$ $\therefore$ $h'y = -y$

$hy = -y^2/2 + c$

$\psi = x\sin y + y\cos x - \frac{y^2}{2} = C$

2b) $x\frac{dy}{dx} = 2xe^{x} - y + 6x$ std: $\underbrace{(2xe^{x} - y + 6x^2)}_{M(x,y)}dx \underbrace{-x}_{N(x,y)}dy = 0$

// Exact if $M_y = N_x$
if so solvable as $\psi(x,y) = C$

$M_y = -1$, $N_x = -1$ $\therefore$ ~~eq~~ $M_y = N_x$ ✓ exact

2b Solving: ψ? : $\dfrac{\partial \psi}{\partial x} = M = 2xe^x - y + 6x^2$

$$\psi = \int 2xe^x - y + 6x^2 \, dx$$

$$= -y\int 1\,dx + 6\int x^2\,dx + \left[2\int xe^x\,dx\right] \qquad u = x \quad u' = 1$$

$$= -yx + 2x^3 + 2\left[xe^x - \int 1e^x\,dx\right] \qquad v' = e^x \quad v = e^x$$

$$\psi = -yx + 2x^3 + 2(xe^x - e^x) + h(y)$$

then $\psi_y = N \Big| = \dfrac{\partial}{\partial y}\left(-yx + 2x^3 + 2(xe^x - e^x)\right) + h'(y)$

$$\psi_y = -x + h'(y)$$

$$N = -x \qquad \therefore \; h'(y) = 0 \qquad h(y) = 0$$

(or any other constant)

$$\psi = -yx + 2x^3 + 2(xe^x - e^x) = C$$

3. Integrating factor for exactness.

DE: $(2y^2+3x)\,dx + (2xy)\,dy = 0$

$\quad\quad\quad\; M(x,y) \quad\quad\quad\quad N(x,y)$

$\left[\ \mu(x) : \text{if } \frac{M_y - N_x}{N} \text{ is } f(x) \text{ only then } \ln|\mu(x)| = \int \frac{M_y - N_x}{N}\,dx\ \right]$

$\frac{M_y - N_x}{N} = \frac{4y - 2y}{2xy} = \frac{2y}{2xy} = \frac{1}{x}$ is a $f(x)$ only ✓

so: $\ln|\mu(x)| = \int \frac{1}{x}\,dx$, $\mu(x) = e^{\ln|x|} = x = \mu(x)$

multiply through: $\mu \cdot DE \rightarrow (2xy^2 + 3x^2)\,dx + (2x^2y)\,dy = 0$

$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad M \quad\quad\quad\quad\quad n$

$M_y = \frac{\partial}{\partial y}(2xy^2) = 4xy$ ✓

$N_x = \frac{\partial}{\partial x}(2x^2y) = 4xy$ now Exact.

Solve: $\psi ? : \frac{\partial \psi}{\partial x} = M = 2xy^2 + 3x^2$

$\psi = \int 2xy^2 + 3x^2\,dx = 2y^2 \int x\,dx + 3\int x^2\,dx = 2y^2[x^2/2] + 3[x^3/3]$

$\psi = x^3 + x^2y^2 + h(y)$

3 cont'd... $\psi = x^3 + x^2y^2 + h(y)$

check $\psi_y = N$: $\frac{\partial}{\partial y}(x^3 + x^2y^2) + h'(y)$

$\psi_y = 2x^2y + h'(y)$

$N = 2x^2y + 0$ $\therefore h'y = 0$ $h(y) = 0$

$$\psi = x^3 + x^2y^2 = C$$

4. Not Done :(

5. a) $y'' - 36y = 0$

$r^2 - 36 = 0 \Rightarrow (r-6)(r+6) = 0$

$r = (6, -6) \Rightarrow y_1 = e^{6t} \quad y_2 = e^{-6t}$

general: $y = c_1 e^{6t} + c_2 e^{-6t}$

b) $y'' - 36y' = 0$

$r^2 - 36r = 0 \Rightarrow r(r-36) = 0$

$r = (0, 36) \Rightarrow y_1 = e^{0t} = 1 \quad y_2 = e^{36t}$

general: $y = c_1 + c_2 e^{36t}$

c) $y'' - 3y' + 2y = 0$

$r^2 - 3r + 2 = 0 \rightarrow (r-1)(r-2) = 0$

$r = 1, 2 \Rightarrow y_1 = e^{t} \quad y_2 = e^{2t}$

general: $y = c_1 e^{t} + c_2 e^{2t}$

6a) $ty'' - y' = 0$: $ax^2y'' + bxy' + cy = 0$ assume solution in form $\underline{t^r = y}$

$\to t(t^r)'' - (t^r)' = 0$

$= trt^{r-2}(r-1) - rt^{r-1} = 0$ $\quad (tt^{r-2} = t^{1+r-2})$

$= rt^{r-1}(r-1) - rt^{r-1} = 0$

$= r^2t^{r-1} - rt^{r-1} - rt^{r-1} = r^2t^{r-1} - 2rt^{r-1} = 0$

$= rt^{r-1}(r-2)$ $\quad \therefore (r_1 = 0,\ r_2 = 2)$ $\quad \left[\begin{array}{l} y_{1,2} = t^0, t^2 = 1, \\ y'_{1,2} = 0, 2t \end{array}\right.$

as $r_1 \neq r_2$

$y = c_1 t^{r_1} + c_2 t^{r_2}$

$y = c_1 t^0 + c_2 t^2$

$y = c_1 + c_2 t^2$ on $I\ (-\infty, \infty)$

6b) $y = c_1 + c_2 t^2$ $\quad y(1) = 0$

$y' = 2c_2 t$ $\quad y'(1) = 1$

$y(1) = c_1 + c_2 1^2 = 0$, $\quad c_1 = -\frac{1}{2}$

$y'(1) = 2c_2 1 = 1$, $\quad c_2 = \frac{1}{2}$

6c) the DE in std form is $y'' - \frac{1}{t} y' = 0$

ie with $p = -\frac{1}{t}$

So by 3.2.1 as p is not continuous on I including $t=0$ solutions cannot be found for $y(0)$ or $y'(0)$

By 3.2.3, the Wronskian $y_1 y_2' - y_1' y_2 = 2t - 0t^2$ which is non zero for all t's excluding $t=0$. Thus c_1, c_2 are findable for all other ~~soluti~~ IVPs $(t \neq 0)$

7. DE: $y'' + (\tan x)y = e^x$, $y(0)=1$, $y'(0)=0$

Theorem 3.2.1 states $y''+py'+qy=g(t)$ has a unique solution $\phi(t)$ throughout the interval I, where in I p & q & g are continuous.

In the above DE, $p=\tan x$ & $\tan x$ is discontinuous at $x = \pm\pi/2$

$\therefore$ $I = (-\pi/2, \pi/2)$

8. for $y'' + p(t)y' + q(t)y = 0$ can $y = t^4$ be a solution on I includes $t = 0$

a solution $y = 0$ exists for all t: $(t = (-\infty, \infty))$
3.2.1 states (as p, q, g are continuous) that
$\phi(t)$ is a unique solution in I
$\therefore$ $y = t^4$ cannot be a solution on I.

9a) DE: $y''-10y'+25y=0$

note: $y(t) = \underbrace{y_c(t)}_{\text{solut. to homogen}} + \overbrace{y_p(t)}^{\text{solution to non-homogen.}}$ use undetermined coeff.

$r^2-10r+25=0$

$(r-5)(r-5)=0$

$r=(5,5)$

solution: $y_c(t) = c_1 e^{5t} + c_2 t e^{5t}$

b) DE: $2y''+2y'+y=0$

$2r^2+2r+1=1$

$r = \frac{-2}{4} \pm \frac{\sqrt{-4}}{4} = -\frac{1}{2} \pm \frac{i}{2}$

$r_{1,2} = -\frac{1}{2} \pm i/2$

$y_c(t) = c_1 e^{-\frac{t}{2}} \sin(t/2) + c_2 e^{-t/2} \cos(t/2)$

10a) $y(t) = y_c(t) + y_p(t)$: $DE = y'' - 10y' + 25y = e^t$

$y_c(t) = c_1 e^{5t} + c_2 t e^{5t}$

find $y_p(t)$ using undetermined coefficients...

try: $Ae^t \Rightarrow Ae^t - 10Ae^t + 25Ae^t = e^t$

$16Ae^t = e^t$

$A = \frac{1}{16}$

$y_p(t) = \frac{1}{16} e^t$

general: $y(t) = c_1 e^{5t} + c_2 t e^{5t} + \frac{e^t}{16}$

10b) $2y'' + 2y' + y = t$ $y_c(t) = c_1 e^{-t/2}\sin(t/2) + c_2 e^{-t/2}\cos(t/2)$

$Y = At^2 + Bt + C$

$Y' = 2At + B$

$Y'' = 2A$

sub: $\underset{3}{4A} + \underset{2}{4At} + \underset{3}{2B} + \underset{1}{At^2} + \underset{2}{Bt} + \underset{3}{C} = t$

t^2: $A = 0$ t^1: $4A + B = 1 \therefore B = 1$ t^0: $4A + 2B + C = 0 \therefore C = -2$

$Y_p = t - 2$ in general: $y(t) = c_1 e^{-t/2}\sin(t/2) + c_2 e^{-t/2}\cos(t/2) + t - 2$

11. $t^2y'' + 3ty' + y = 0$, $y_1 = \frac{1}{t} = t^{-1}$

$y_2 = vy_1$, $y_2 = t^{-1}v$ $\quad y_2' = -t^{-2}v + t^{-1}v'$

$y_2'' = 2t^{-3}v - t^{-2}v' - t^{-2}v' + t^{-1}v''$

$= 2t^{-3} - 2t^{-2}v' + t^{-1}v''$

~~sub:~~ $\cancel{t^2(2t^{-3}}$

sub: $t^2\left(\frac{2v}{t^3} - \frac{2v'}{t^2} + \frac{v''}{t}\right) + 3t\left(\frac{-v}{t^2} + \frac{v'}{t}\right) + \frac{v}{t} = 0$

$= \cancel{\frac{2v}{t}} - 2v' + tv'' - \cancel{\frac{3v}{t}} + 3v' + \cancel{\frac{v}{t}} = 0$

$= v' + tv'' = 0 \Rightarrow tv'' = -1v'$

$\frac{v''}{v'} = \frac{-1}{t}$ $\quad$ integrate $\rightarrow \ln|v'| = -\ln|t| + c$

$= \ln(|t|^{-1}) + c$

$|v'| = A|t|^{-1}$: $A = e^c$

$v'(t) = \pm At^{-1}$ $\quad$ althoug $v'(t) = 0$ gives a solution so...

$v'(t) = Bt^{-1}$: $B \in \mathbb{R}$

$v(t) = B\int \frac{1}{t}\,dt$

$v(t) = B\ln|t| + D$, $y_2 = \frac{B\ln|t|}{t} + \frac{D}{t}$

y_{2p} : $B = 1$ e $D = 0$ $\quad y_2 = \frac{1}{t}\ln|t|$