ENGR 222
Assignment 1 Submission

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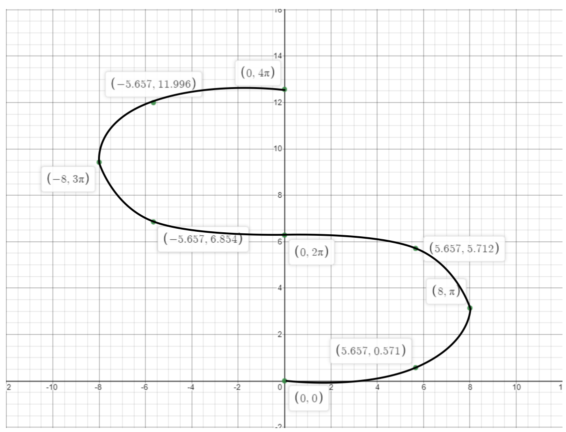
1. Consider the parametric equations: $(f(t), g(t)) = (8sin(t), 2t - sin(2t))$

(a)

$$(f(t), g(t)) = [(0,0), (5.657, 0.571), (8, \pi), (5.657, 5.712), (0, 2\pi), \\ (-5.657, 6.854), (-8, 3\pi), (-5.657, 11.996), (0, 4\pi)]$$



(b) Tangent vector:

$$(f'(t),\ g'(t)) = (8cos(t), 2 - 2cos(2t))$$
$$t = \pi/6\ :\ (f'(t),\ g'(t)) = (6.928203, 1)$$

Unit tangent vector:

$$\frac{(f'(t),\ g'(t))}{\|(f'(t),\ g'(t))\|} = \frac{(6.928203, 1)}{\sqrt{6.928203^2 + 1}} = \left(\frac{6.928203}{7}, \frac{1}{7}\right)$$

(c) Tangent Line equation (normalised):

$$= (x, y) + t \cdot \frac{(f'(t),\ g'(t))}{\|(f'(t),\ g'(t))\|}$$
$$= (4, 0.181) + t\left(\frac{6.928203}{7}, \frac{1}{7}\right)$$
$$= \left(\frac{6.928203t}{7} + 4, \frac{t}{7} + 0.181)\right)$$

(d) Normal line:

$$= \left(f(\frac{\pi}{6}) - tg'(\frac{\pi}{6}), g(\frac{\pi}{6}) + tf'(\frac{\pi}{6})\right)$$
$$= (4 - t, 0.181 + 6.928203t)$$

(e) Arc length:

$$\begin{aligned}
L &= \int_0^{2\pi} \sqrt{f'(t)^2 + g'(t)^2} dt \\
&= \int_0^{2\pi} \sqrt{(8\cos(t))^2 + (2 - 2\cos(2t))^2} dt \\
&= \int_0^{2\pi} \sqrt{(32\cos(2t) + 32) + \left(4 + 4\cos(2t)^2 - 8\cos(2t)\right)} dt \\
&= \int_0^{2\pi} \sqrt{4\cos(2t)^2 + 24\cos(2t) + 36} dt \\
&= \int_0^{2\pi} \sqrt{4\left(\cos(2t)^2 + 6\cos(2t) + 9\right)} dt \\
&= \int_0^{2\pi} \sqrt{4(\cos(2t) + 3)^2} dt \\
&= 2\int_0^{2\pi} (\cos(2t) + 3)\, dt \\
&= sin(2t) + 6t\Big|_0^{2\pi} \\
&= (0 + 12\pi) - (0 + 0) \\
&= 12\pi \approx 37.6991118431
\end{aligned}$$

2. Consider the curve described by the vector valued function:

$$\mathbf{r}(t) = \frac{e^{2t} - 2t}{4}\mathbf{i} + e^t\mathbf{j}$$

(a)

$$\begin{aligned}
\mathbf{r}(t)\cdot\mathbf{j} &= 2 \\
e^t &= 2 \\
t &= ln(2) \approx 0.693147 \\
\mathbf{r}(t = ln(2)) &= (0.653, 2)
\end{aligned}$$

(b)

$$\begin{aligned}
\mathbf{r}'(t) &= \frac{2e^{2t} - 2}{4}\mathbf{i} + e^t\mathbf{j} \\
\|\mathbf{r}'(t)\| &= \sqrt{\left(\frac{2e^{2t} - 2}{4}\right)^2 + (e^t)^2} = \sqrt{\frac{1}{4}(e^{4t} + 2e^{2t} + 1)} \\
&= \sqrt{\frac{1}{4}(e^{2t} + 1)^2} = \frac{e^{2t} + 1}{2} \\
\mathbf{T}(t) &= \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} \\
&= \frac{\frac{2e^{2t}-2}{4}\mathbf{i} + e^t\mathbf{j}}{\frac{e^{2t}+1}{2}} \\
&= \left(\frac{\frac{1}{4}\left(2e^{2t} - 2\right)}{\frac{1}{2}\left(e^{2t} + 1\right)}, \frac{e^t}{\frac{1}{2}\left(e^{2t} + 1\right)}\right) \\
&= \left(\frac{e^{2t} - 1}{e^{2t} + 1}, \frac{2e^t}{e^{2t} + 1}\right) \\
\mathbf{T}(t) &= \frac{(e^{2t} - 1)\mathbf{i} + 2e^t\mathbf{j}}{e^{2t} + 1}
\end{aligned}$$

(c)

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|}$$

$$\begin{aligned}\mathbf{T}'(t) &= \left(\frac{2e^{2t}\left(e^{2t}+1\right) - 2e^{2t}\left(e^{2t}-1\right)}{\left(e^{2t}+1\right)^2}, \frac{2e^{t}\left(e^{2t}+1\right) - 2e^{t}\cdot 2e^{2t}}{\left(e^{2t}+1\right)^2}\right) \\ &= \frac{4e^{2t}}{\left(e^{2t}+1\right)^2}\mathbf{i} + \frac{2e^{t}-2e^{3t}}{\left(e^{2t}+1\right)^2}\mathbf{j} \equiv \operatorname{sech}(t)^2\,\mathbf{i} + (-\operatorname{sech}(t)\tanh(t))\mathbf{j}\end{aligned}$$

$$\left\|\mathbf{T}'(t)\right\| = \sqrt{\operatorname{sech}(t)^4 + (-\operatorname{sech}(t)\tanh(t))^2}$$

$$\mathbf{N}(t) = \frac{\operatorname{sech}(t)^2\,\mathbf{i} + (-\operatorname{sech}(t)\tanh(t))\mathbf{j}}{\sqrt{\operatorname{sech}(t)^4 + (-\operatorname{sech}(t)\tanh(t))^2}}$$

(d)

$$\begin{aligned}\kappa(t) &= \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|} \\ &= \frac{2\sqrt{\operatorname{sech}(t)^4 + (-\operatorname{sech}(t)\tanh(t))^2}}{e^{2t}+1}\end{aligned}$$

(e)

$$\begin{aligned}L &= \int_0^3 \|\mathbf{r}'(t)\|dt \\ &= \int_0^3 \frac{e^{2t}+1}{2}dt = \frac{1}{2}\int_0^3 e^{2t}+1dt \\ &= \frac{1}{4}\left|e^{2t}+2t\right|_0^3 = \frac{1}{4}\left(\left(e^6+6\right) - \left(e^0+0\right)\right) \\ &= \frac{e^6+5}{4} \approx 102.107198373\end{aligned}$$

3. Quick questions:

(a)

$$(x, y, z) = (3t, -2+t, 7-5t) : \ t_0 = 0$$

$$\begin{aligned}\mathbf{r}'(t) &= (3, 1, -5) \\ \|\mathbf{r}'(t)\| &= \sqrt{3^2+1+(-5)^2} = \sqrt{35} = 5.91608 \\ s &= \int_{t_0}^{t} \|\mathbf{r}'(t)\|\ du \\ &= \int_0^t \sqrt{35}\ du \\ s &= \sqrt{35}\ t \\ t &= \frac{s}{\sqrt{35}}\end{aligned}$$

$$\mathbf{r}(s) = \left(\frac{3s}{\sqrt{35}}, -2+\frac{s}{\sqrt{35}}, 7-\frac{5s}{\sqrt{35}}\right)$$

(b)

$$\begin{aligned}
\mathbf{r}(t) &= (-3+5cos(t))\mathbf{i} + (2-5sin(t))\mathbf{j} \ : \ t_0 = 0 \\
\mathbf{r}'(t) &= (-5sin(t))\mathbf{i} + (-5cos(t))\mathbf{j} \\
\|\mathbf{r}'(t)\| &= \sqrt{(-5sin(t))^2 + (-5cos(t))^2} \\
&= \sqrt{25\,(sin(t)^2 + \cos(t)^2)} \\
&= \sqrt{25}\sqrt{1} = 5
\end{aligned}$$

$$\begin{aligned}
s &= \int_0^t 5 \ du = 5t \\
t &= \frac{s}{5} \\
\mathbf{r}(s) &= (-3+5cos(\frac{s}{5}))\mathbf{i} + (2-5sin(\frac{s}{5}))\mathbf{j}
\end{aligned}$$

(c)

$$\begin{aligned}
\mathbf{r}(t) &= \sqrt{2}cos(t)\mathbf{i} + sin(t)\mathbf{j} + sin(t)\mathbf{k} \\
\mathbf{T}(t) &= \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} \\
\mathbf{r}'(t) &= -\sqrt{2}sin(t)\mathbf{i} + cos(t)\mathbf{j} + cos(t)\mathbf{k} \\
\mathbf{r}'(\pi/3) &= \frac{-\sqrt{6}}{2}\mathbf{i} + 0.5\mathbf{j} + 0.5\mathbf{k} \\
\|\mathbf{r}'(t)\| &= \sqrt{\left(-\frac{1}{2}\sqrt{6}\right)^2 + 0.5^2 + 0.5^2} = \sqrt{2} \\
\mathbf{T}(t) &= \frac{-0.5\sqrt{6}\mathbf{i} + 0.5\mathbf{j} + 0.5\mathbf{k}}{\sqrt{2}}
\end{aligned}$$

(d)

$$\begin{aligned}
\mathbf{B}(t) &= \frac{\mathbf{r}'(t) \times \mathbf{r}''(t)}{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|} \\
\mathbf{r}(t) &= t\mathbf{i} - t^3\mathbf{j} + t^2\mathbf{k} \\
\mathbf{r}'(t) &= 1\mathbf{i} - 3t^2\mathbf{j} + 2t\mathbf{k} \\
\mathbf{r}'(1) &= 1\mathbf{i} - 3\mathbf{j} + 2\mathbf{k} \\
\mathbf{r}''(t) &= 0\mathbf{i} - 6t\mathbf{j} + 2\mathbf{k} \\
\mathbf{r}''(1) &= 0\mathbf{i} - 6\mathbf{j} + 2\mathbf{k}
\end{aligned}$$

$$\begin{aligned}
\mathbf{r}'(t) \times \mathbf{r}''(t) &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -3 & 2 \\ 0 & -6 & 2 \end{vmatrix} \\
&= \mathbf{i}(2(-3) - 2(-6)) - \mathbf{j}(2(1) - 0(2)) + \mathbf{k}(1(-6) - 0(-3)) \\
&= 6\mathbf{i} - 2\mathbf{j} - 6\mathbf{k} \\
\|\mathbf{r}'(t) \times \mathbf{r}''(t)\| &= \sqrt{6^2 + (-2)^2 + (-6)^2} = \sqrt{76}
\end{aligned}$$

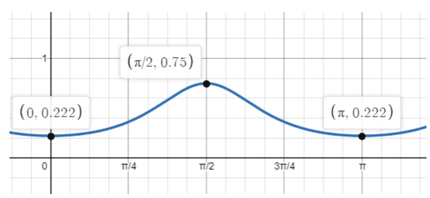
$$\mathbf{B}(t) = \frac{6\mathbf{i} - 2\mathbf{j} - 6\mathbf{k}}{\sqrt{76}}$$

(e)

$$\mathbf{r}(t) = (2 + 3\sin(t))\mathbf{i} + (1 + 2cos(t))\mathbf{j}$$
$$\kappa(t) = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3}$$

$$\mathbf{r}'(t) = 3cos(t)\mathbf{i} - 2sin(t)\mathbf{j}$$
$$\mathbf{r}''(t) = -3sin(t)\mathbf{i} - 2cos(t)\mathbf{j}$$
$$\|\mathbf{r}'(t)\| = \sqrt{(3cos(t))^2 + (-2sin(t))^2} = \sqrt{9\cos(t)^2 + 4\sin(t)^2}$$
$$= \sqrt{\left(\frac{9 + 9\cos(2t)}{2}\right) + (2 - 2\cos(2t))} = \sqrt{\frac{13 + 5\cos(2t)}{2}}$$
$$\|\mathbf{r}'(t)\|^3 = \left(\sqrt{\frac{13 + 5\cos(2t)}{2}}\right)^3 = \left(\frac{13 + 5\cos(2t)}{2}\right)^{\frac{3}{2}}$$

$$\mathbf{r}'(t) \times \mathbf{r}''(t) = \begin{vmatrix} \mathbf{i} & \mathbf{j} \\ 3cos(t) & -2sin(t) \\ -3sin(t) & -2cos(t) \end{vmatrix}$$
$$= (-2cos(t)3cos(t) - (-2sin(t)(-3sin(t))))\mathbf{k}$$
$$= -6\cos(t)^2 - 6\sin(t)^2\mathbf{k} = -6\mathbf{k}$$
$$\|\mathbf{r}'(t) \times \mathbf{r}''(t)\| = \sqrt{-6^2} = 6$$



$$\kappa(t) = \frac{6}{\left(\frac{13+5\cos(2t)}{2}\right)^{\frac{3}{2}}}$$
$$cos(2t) \Rightarrow min \ @ \ t = 0, max \ @ \ t = \pi/2$$
$$min = 0.222 \ max = 0.75$$

4. Suppose a roller coaster path described by:

$$\mathbf{r}(t) = \frac{1}{5}t(20 - t)\mathbf{i} + \frac{1}{50}t^2(20 - t)\mathbf{j} + \frac{1}{50}t(10 - t)(20 - t)\mathbf{k} \ : \ t \in [0, 20]$$

(a)
$$\mathbf{v}(t) = \mathbf{r}'(t) = \left(\frac{20 - 2t}{5}\right)\mathbf{i} + \left(\frac{40t - 3t^2}{50}\right)\mathbf{j} + \left(\frac{3t^2 - 60t + 200}{50}\right)\mathbf{k}$$

(b)
$$\mathbf{v}(5) = 2\mathbf{i} + 2.5\mathbf{j} - 0.5\mathbf{k}$$
$$v = \|\mathbf{v}(5)\| = \sqrt{2^2 + 2.5^2 + (-0.5)^2} = \sqrt{10.5} = 3.24037$$

(c)
$$\mathbf{a}(t) = \mathbf{v}'(t) = \mathbf{r}''(t) = -\frac{2}{5}\mathbf{i} + \left(\frac{40 - 6t}{50}\right)\mathbf{j} + \left(\frac{6t - 60}{50}\right)\mathbf{k}$$

(d)

$$\begin{aligned}
\kappa(t) &= \frac{\|\mathbf{v}(t)\times\mathbf{a}(t)\|}{\|\mathbf{v}(t)\|^3} \\
\mathbf{v}(5) &= 2\mathbf{i} + 2.5\mathbf{j} - 0.5\mathbf{k} \\
\|\mathbf{v}(5)\|^3 &= \left(\sqrt{10.5}\right)^3 = 34.023889 \\
\mathbf{a}(5) &= -0.4\mathbf{i} + 0.2\mathbf{j} - 0.6\mathbf{k}
\end{aligned}$$

$$\begin{aligned}
\mathbf{v}(5)\times\mathbf{a}(5) &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 2.5 & -0.5 \\ -0.4 & 0.2 & -0.6 \end{vmatrix} \\
&= (2.5(-0.6) - 0.2(-0.5))\mathbf{i} - (2(-0.6) - (-0.4)(-0.5))\mathbf{j} + (2(0.2) - 2.5(-0.4))\mathbf{k} \\
&= -1.4\mathbf{i} + 1.4\mathbf{j} + 1.4\mathbf{k} \\
\|\mathbf{v}(5)\times\mathbf{a}(5)\| &= \sqrt{(-1.4)^2 + 1.4^2 + 1.4^2} = \sqrt{5.88} = 2.424871 \\
\kappa(5) &= \frac{\sqrt{5.88}}{\sqrt{10.5}^3} = 0.07127
\end{aligned}$$

(e)

$$\mathbf{v}(5) = 2\mathbf{i} + 2.5\mathbf{j} - 0.5\mathbf{k} \qquad \mathbf{a}(5) = -0.4\mathbf{i} + 0.2\mathbf{j} - 0.6\mathbf{k}$$

$$\begin{aligned}
\|\mathbf{v}(5)\times\mathbf{a}(5)\| &= \sqrt{5.88} \\
\|\mathbf{v}(5)\| &= \sqrt{10.5} \\
a_N &= \frac{\|\mathbf{v}(5)\times\mathbf{a}(5)\|}{\|\mathbf{v}(5)\|} = \frac{\sqrt{5.88}}{\sqrt{10.5}} = \sqrt{5.88/10.5} = \sqrt{0.56} = 0.748331 \\
\|\mathbf{a}(5)\| &= \sqrt{(-0.4)^2 + 0.2^2 + (-0.6)^2} = \sqrt{0.56} = 0.748331 \\
\|\mathbf{a}(5)\| &= a_N
\end{aligned}$$

From this it can be inferred there is no tangential acceleration.