TE WHARE WĀNANGA O TE ŪPOKO O TE IKA A MĀUI

VICTORIA

UNIVERSITY OF WELLINGTON

EXAMINATIONS – 2017

TRIMESTER 1

MATH 244

MODELLING WITH DIFFERENTIAL EQUATIONS

Time Allowed: THREE HOURS

CLOSED BOOK

Permitted materials: Silent non-programmable calculators or silent programmable calculators with their memories cleared are permitted.

Printed foreign languages to English dictionaries are permitted.

Formula sheets are attached to the end of this exam script.

No other material is permitted.

Instructions: The exam will be marked out of a total of 100 marks.

Answer in the appropriate boxes if possible — if you write your answer elsewhere, make it clear where your answer can be found.

Please use the blank reverse sides of pages for any extra space you need, for working or for answers. There is also a blank page between the questions and the formula sheets, for working or extra answer space.

If you answer all six questions, they will all be marked, and you will be credited with the five best marks.

For marking use only

1	/20
2	/20
3	/20
4	/20
5	/20
6	/20
best 5	/100

Student ID: ....

1. **(20 marks)**

(a) Consider the differential equation

$$y'' + 2y' + 2y = 0 \, .$$

i. **(4 marks)** Find a fundamental set of solutions $y_1(x)$, $y_2(x)$, for this differential equation.

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ii. **(4 marks)** Show that the set you obtained in part (i) is a fundamental set, by calculating the value of the Wronskian at $x = 0$.

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(Question 1 continued on next page)

(Question 1 continued)

iii. **(2 marks)** Hence write down the general solution to this differential equation.

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iv. **(3 marks)** Hence find the solution to the initial value problem $y'' + 2y' + 5y = 0$, $y(0) = 0$, $y'(0) = 1$. Describe how this solution behaves, as $x \to +\infty$.

(b) Consider the homogeneous differential equation

$$y'' + p(x)y' + q(x)y = 0\,,$$

where $p(x)$ and $q(x)$ are continuous functions for all x in an open interval $\mathbf{I}$ which contains the point $x = 0$.

i. **(2 marks)** Over what interval is it guaranteed that there exists a unique solution $y = \phi(x)$ to this differential equation?

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ii. **(5 marks)** Is it possible that $y = x^2$ is a solution for such a differential equation for a suitable choice of the (continuous) functions $p(x)$ and $q(x)$? Explain.

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2. **(20 marks)**

(a) Consider the constant coefficient homogeneous linear differential equation

$$ay'' + by' + cy = 0$$

with the property that $b^2 - 4ac$ is zero, so that the auxiliary equation only has one root $r = r_1 = -b/2a$, where r_1 is a real number. This provides one solution, $y_1 = e^{r_1 t}$.

Use the method of reduction of order, substituting the trial solution $y = v(t)\, e^{r_1 t}$, to show that another solution to this differential equation is $y_2 = te^{r_1 t}$.

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(b) Consider the nonhomogeneous differential equation $y'' + 4y' + 4y = e^t$.

i. **(2 marks)** Write down the associated homogeneous differential equation.

ii. **(4 marks)** Find the general solution to the associated homogeneous differential equation.

iii. **(3 marks)** Find a particular solution to the original nonhomogeneous differential equation.

iv. **(1 mark)** Write down the general solution to the original nonhomogeneous differential equation.

v. **(1 mark)** Write down the solution to the original nonhomogeneous differential equation with initial values $y(0) = 1$, $y'(0) = 0$.

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3. **(20 marks)**

(a) **(3 marks)** Find the inverse Laplace transform

$$\mathcal{L}^{-1}\left\{\frac{s+2}{s^2+3}\right\}$$

(b) **(1 mark)** State the Derivative Theorem, i.e.

$$\mathcal{L}\{f'(x)\} = ?$$

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(c) **(6 marks)**

i. **(3 marks)** Prove the First Translation Theorem, i.e. show that

$$\mathcal{L}\{e^{ax}f(x)\}(s) = F(s-a), \quad \text{where} \quad F(s) = \mathcal{L}\{f(x)\}(s).$$

ii. **(3 marks)** Use the First Translation Theorem to compute $\mathcal{L}\{x^4 e^{3x}\}$.

/6

(d) **(10 marks)** Solve the initial value problem

$$y' + y = 3xe^{-2x} \qquad \text{subject to the initial conditions } y(0) = 0.$$

/10

4. **(20 marks)**

(a) **(7 marks)**

i. **(4 marks)** Define the convolution of two functions and state the Convolution Theorem.

ii. **(3 marks)** Use the Convolution theorem to show

$$\mathcal{L}\left\{\int_0^x f(\tau)d\tau\right\} = \frac{F(s)}{s}$$

where $F(s) = \mathcal{L}\{f(x)\}(s)$.

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(b) **(3 marks)** Let

$$f(x) = \begin{cases} 0 & x \leq 0 \\ x & 0 \leq x \leq 1 \\ 1 & 1 \leq x \end{cases}$$

Use the 2^{nd} Translation Theorem to compute the Laplace transform of $f(x)$.

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(c) **(10 marks)** Solve the integro-differential equation

$$\frac{dy}{dx} = 1 + \sin(x) - \int_0^x 4y(\tau)d\tau$$

with initial condition $y(0) = 0$.

/10

5. **(20 marks)**

(a) **(4 marks)** Show that

$$\int_{-\pi}^{\pi} \cos^2(mx)dx = \pi$$

for any $m \in \mathbb{N} = \{1, 2, 3, \ldots\}$.

(b) **(10 marks)** Consider the function

$$f(x) = \pi - x \quad \text{on } (0, \pi).$$

i. **(2 marks)** Extend $f(x)$ to an *odd* function $f_{\text{odd}}(x)$ with period 2π

ii. **(2 marks)** Sketch the graph of the function you obtain, on the interval $[-3\pi, 3\pi]$.

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iii. **(6 marks)** Compute the Fourier series of the periodic function $f_{\text{odd}}(x)$ with period $T = 2\pi$ that you obtained in (i).

/6

(c) **(6 marks)** Find the general solution of the system

$$y'' + \omega^2 y = f_{\text{odd}}(x),$$

where $f_{\text{odd}}(x)$ is the odd function with period 2π constructed in (b.i) above, and ω is not an integer multiple of π.

/6

6. **(20 marks)**

(a) **(12 marks)**

i. **(6 marks)** Find the fundamental set of solutions to the system

$$\mathbf{x}' = \begin{pmatrix} 1 & 1 \\ -1 & 3 \end{pmatrix} \mathbf{x}$$

/6

ii. **(2 marks)** Define the Wronskian of a system of equations.

iii. **(4 marks)** Use the Wronskian to show you have found a fundamental set of solutions to the system in part (i).

/6

(b) **(4 marks)** For each of the equations below, state whether the resulting sytem is (a) linear or not and (b) homogeneous or not.

i. **(2 marks)**

$$\frac{d^2y}{dx^2} = \frac{y-1}{3} + \frac{dy}{dx}$$

ii. **(2 marks)**

$$\frac{d^2x}{dt^2} = tx + t^2$$

(c) **(4 marks)** Write the system of differential equations

$$x' - 7x = 3y$$
$$y' + \frac{x}{2} = 0$$

in matrix form.

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End of Questions

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Differential Equations Formula Sheet

Trigonometric identities

- $\sin^2 x + \cos^2 x = 1$
- $\sin 2x = 2\sin x \cos x$
- $\cos 2x = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$
- $\sec^2 x = 1 + \tan^2 x$
- $\csc^2 x = 1 + \cot^2 x$
- $\sin x \cos y = \frac{1}{2}\left[\sin(x-y) + \sin(x+y)\right]$
- $\sin x \sin y = \frac{1}{2}\left[\cos(x-y) - \cos(x+y)\right]$
- $\cos x \cos y = \frac{1}{2}\left[\cos(x-y) + \cos(x+y)\right]$
- if $t = \tan\frac{x}{2}$ then $\sin x = \frac{2t}{1+t^2}$, $\cos x = \frac{1-t^2}{1+t^2}$, and $\tan x = \frac{2t}{1-t^2}$

Hyperbolic identities

- $\cosh^2 x - \sinh^2 x = 1$
- $\sinh 2x = 2\sinh x \cosh x$
- $\cosh 2x = \cosh^2 x + \sinh^2 x = 2\cosh^2 x - 1 = 2\sinh^2 x + 1$
- $\operatorname{sech}^2 x = 1 - \tanh^2 x$
- $\operatorname{csch}^2 x = \coth^2 x - 1$
- $\sinh x \cosh y = \frac{1}{2}\left[\sinh(x+y) + \sinh(x-y)\right]$
- $\sinh x \sinh y = \frac{1}{2}\left[\cosh(x+y) - \cosh(x-y)\right]$
- $\cosh x \cosh y = \frac{1}{2}\left[\cosh(x+y) + \cosh(x-y)\right]$

Derivatives of trigonometric and hyperbolic functions

$$\frac{d}{dx}[\sin x] = \cos x \qquad \frac{d}{dx}[\sinh x] = \cosh x$$

$$\frac{d}{dx}[\cos x] = -\sin x \qquad \frac{d}{dx}[\cosh x] = \sinh x$$

$$\frac{d}{dx}[\tan x] = \sec^2 x \qquad \frac{d}{dx}[\tanh x] = \mathrm{sech}^2 x$$

$$\frac{d}{dx}[\cot x] = -\csc^2 x \qquad \frac{d}{dx}[\coth x] = -\mathrm{csch}^2 x$$

$$\frac{d}{dx}[\sec x] = \sec x \tan x \qquad \frac{d}{dx}[\mathrm{sech} x] = -\mathrm{sech} x \tanh x$$

$$\frac{d}{dx}[\csc x] = -\csc x \cot x \qquad \frac{d}{dx}[\mathrm{csch} x] = -\mathrm{csch} x \coth x$$

Exponents and logarithms

- $y = a^x \quad \Leftrightarrow \quad x = \log_a y$

  A special case: $y = e^x \quad \Leftrightarrow \quad x = \ln y$
- $\frac{d}{dx}[a^x] = a^x \cdot \ln a$
- $\frac{d}{dx}[\log_a x] = \frac{1}{x \ln a}$

Matrices

Let $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$.

- $A^{-1} = \frac{1}{\det A}\begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}$.
- Cramer's Rule: the solution of $A\mathbf{x} = \mathbf{b}$ is given by $x_1 = \frac{\det A_1}{\det A}$, $x_2 = \frac{\det A_2}{\det A}$

  where $A_1 = \begin{pmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{pmatrix}$, $A_2 = \begin{pmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{pmatrix}$.

continued over ...

Sums and series

- $\displaystyle\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$

- $\displaystyle\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$

- $\displaystyle\sum_{k=1}^{n} k^3 = \left[\frac{n(n+1)}{2}\right]^2$

- Geometric progression: $\displaystyle\sum_{k=0}^{n-1} r^k = 1 + r + r^2 + \cdots + r^{n-1} = \frac{1-r^n}{1-r}, \quad r \neq 1$

  Geometric series: $\displaystyle\sum_{k=0}^{\infty} r^k = \frac{1}{1-r}$ when $|r| < 1$

- Maclaurin series: $\displaystyle\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k$

- Taylor series: $\displaystyle\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!} (x-a)^k$

- Fourier series for $f(x)$ on $[-\pi, \pi]$: $\displaystyle\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$

  where $\displaystyle a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx, \quad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\, dx.$

Transforms

- Laplace transform: $\mathcal{L}\{f(x)\} = \int_0^\infty f(x)e^{-sx}\,dt = F(s)$

$$\mathcal{L}\{1\} = \frac{1}{s}$$

$$\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}, \quad n \text{ a positive integer}$$

$$\mathcal{L}\{e^{ax}\} = \frac{1}{s-a}$$

$$\mathcal{L}\{\cos ax\} = \frac{s}{s^2+a^2}$$

$$\mathcal{L}\{\sin ax\} = \frac{a}{s^2+a^2}$$

$$\mathcal{L}\{f'(x)\} = sF(s) - f(0)$$

$$\mathcal{L}\{f^{(n)}(x\} = s^nF(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - sf^{(n-2)}(0) - f^{(n-1)}(0)$$

- Fourier transform: $\mathcal{F}\{f(x)\} = \int_{-\infty}^\infty f(x)e^{i\alpha x}\,dt = F(\alpha)$

$$\mathcal{F}\{f'(x)\} = -i\alpha F(\alpha)$$

Inverse Fourier transform: $\mathcal{F}^{-1}\{F(\alpha)\} = \dfrac{1}{2\pi}\displaystyle\int_{-\infty}^\infty F(\alpha)e^{-i\alpha x}\,d\alpha = f(x).$