MATH244 Test

During lecture, 11 April 2019

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Please use the spaces provided in this test booklet next to the questions, to give your answers.
Attempt all FOUR questions on pages 2–7. You have 45 minutes.
The value of each question is shown in square brackets.
You may use page 8 for rough working, and the reverse sides of all pages.
Please show all working.
Silent calculators may be used, provided you have cleared any alpha-numeric memory.
Please indicate where you have used a calculator.
A table of formulae is provided with this Test.

Page totals, for marking use only

Page	Mark	max
p.2	$2\frac{1}{2}$	3
p.3	4	4
p.4	1	2
p.5	4	4
p.6	$2\frac{1}{2}$	4
p.7	2	3
Total	16	**20**

1. Consider the differential equation

$$(x-1)y' + y = e^x \ .$$

(a) [1]

Without solving, state the largest interval on which a unique solution with initial condition $y(0) = 1$ exists.

Std: $y' + \frac{y}{x-1} = \frac{e^x}{x-1}$, ie not defined at $x-1=0$, or $x=1$

So, the 2 interval are $I_1 = [-\infty, 1)$ & $I_2 = (1, \infty^+)$

~~I~~ $x=0$ exists in $I_1 = [-\infty, 1]$

$\frac{1}{2}$

(b) [2]

Find the general solution to the above differential equation, in explicit form.

Std: $y' + \frac{1}{x-1} y = \frac{e^x}{x-1}$

$\mu = e^{\int \frac{1}{x-1}}$ $\mu = e^{\ln(x-1)}$ $\mu = x-1$

$\frac{d}{dx}(y(x-1)) = e^x$

$y(x-1) = \int e^x \, dx$

$y = \frac{e^x + C}{x-1}$

2

$2\frac{1}{2}$ / 3

(c) [1]

If the initial condition is $y(0) = 1$, solve the resulting initial value problem for $(x-1)y' + y = e^x$.

$\frac{e^0 + C}{0 - 1} = 1$, $\frac{1+C}{-1} = 1$

$C = -2$ ✓

$y = \frac{e^x - 2}{x-1}$ ✓

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2. (a) [3]

Solve the separable initial value problem:

$$y' = 2(1+y^2)t \quad y(0) = 0\,.$$

$\frac{dy}{1+y^2} = 2t$, $\Rightarrow \int \frac{1}{1+y^2}\,dy = 2\int t\,dt$

$\tan^{-1}(y) = t^2 + c$ ✓

$y = \tan(t^2 + c)$ ✓

$y(0) = 0 = \tan(0^2 + c)$ ✓

$\tan(0) = 0$, $c = 0$

$\cancel{\ln|1+y^2| = t^2 + c}$

$\cancel{1+y^2 = e^{t^2}e^{c} = A}$

$\cancel{y^2 = Ae^{t^2} - 1 \Rightarrow y = \sqrt{Ae^{t^2} - 1}}$

$\cancel{y(0) = 0,\ \sqrt{Ae^0 - 1} = 0}$

$\cancel{\sqrt{A-1} = 0 \quad (\sqrt{0} = 0)}$

$\cancel{\therefore A = 1}$

$\cancel{\text{So } y = \sqrt{e^{t^2} - 1}}$

$\therefore y = \tan(t^2)$ ✓

4 / 4

tan is undefined at $\tan(\frac{\pi}{2})$, so $t^2 = \frac{\pi}{2}$ $t = \sqrt{\frac{\pi}{2}}$

$I = (-\sqrt{\frac{\pi}{2}}, \sqrt{\frac{\pi}{2}})$ ✓

(b) [1]

Over what interval does your solution exist?

~~e^{t^2} is always >1, ie t^2 is never negative.~~

~~$\therefore$ $e^{t^2}-1$ is always positive.. so~~

~~$\sqrt{e^{t^2}-1}$ is always Real & define~~

~~$I = [-\infty, \infty]$~~

1

(c) [1]

Compare your result with the result of Theorem 2.4.2 on the existence and uniqueness of solutions to differential equations of the form $y' = f(t, y)$. 0

1 / 2

3. Consider the labelled list of DEs:

(a) $y' = 2y - 1$

(b) $y' = 2 + y$

(c) $y' = y - 2$

(d) $y' = y(y+3)$

(e) $y' = y(3-y)$

(f) $y' = 1 + 2y$

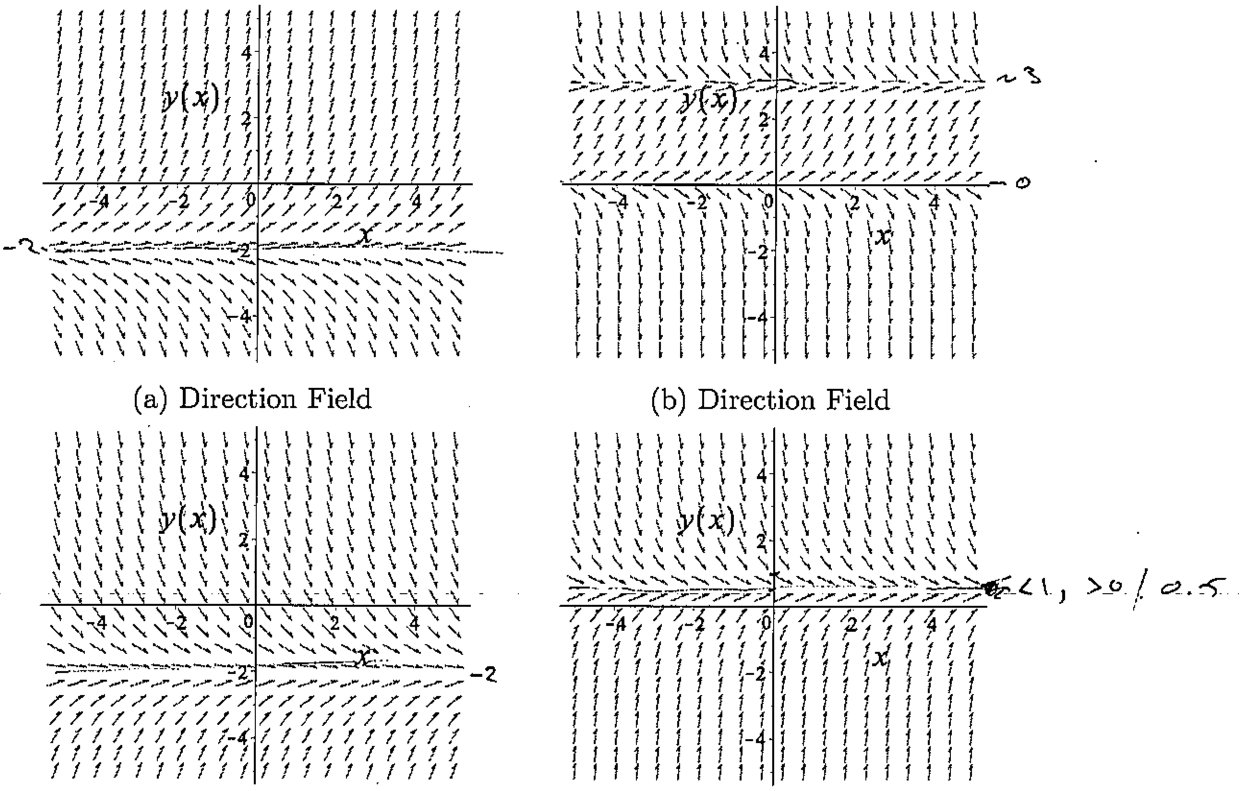
(g) $y' = -2 - y$

(h) $y' = y(y-3)$

(i) $y' = 1 - 2y$

(j) $y' = 2 - y$

(a) What label DE produced the direction field in Fig (1a)?	(b)	✓	[1]
(b) What label DE produced the direction field in Fig (1b)?	(e).	✓	[1]
(c) What label DE produced the direction field in Fig (1c)?	(g)	✓	[1]
(d) What label DE produced the direction field in Fig (1d)?	(i)	✓	[1]



(a) Direction Field

(b) Direction Field

(c) Direction Field

(d) Direction Field

Figure 1: Direction Fields for Test Q3

4 / 4

4. (a) Use an integrating factor to solve the differential equation [2]

$$\frac{dy}{dx} + 2xy = x \, .$$

$\mu = e^{\int 2x\,dx} = e^{x^2}$

$\hookrightarrow \frac{d}{dx}\left(ye^{x^2}\right) = xe^{x^2}$ $\quad ye^{x^2} = \int xe^{x^2}$

using integ by parts: $\int uv' = uv - \int u'v$, $u = x$, $u' = 1$, $v' = e^{x^2}$

sub $u = x^2$ $\quad du = 2x$

$\frac{1}{2}\int e^u\,du$

$ye^{x^2} = \frac{1}{2}e^{x^2} + c$ $\qquad \frac{1}{2}e^{x^2}$

$y = \frac{1}{2} + \frac{c}{e^{x^2}} = C_1$

$y = \frac{1}{2} + C_1$ ✗ or $y = \frac{1}{2} + \cancel{A}\,Ce^{-x^2}$ ✓

$1\frac{1}{2}$

(b) [2]

Find the general solution in explicit form, for the separable DE

$$y' = y^{1/3}$$

$y^{-1/3}$

$\int \frac{1}{y^{1/3}}\,dy = \int 1\,dt$

✗ $\ln(y^{1/3}) = t + c$

$\sqrt[3]{y} = e^{t}\left(e^{c}\right) = A$

$y = \left(Ae^{t}\right)^3$

1

$2\frac{1}{2}$ / 4

(c) [2]

Solve the IVP

$$y' = y^{1/3}\,, \quad y(1) = 1$$

$y = (Ae^{t})^3$

$(Ae^{1})^3 = 1 \quad , \quad Ae^{1} = 1 \qquad A = \frac{1}{e^{1}}$ ✓

$A \approx 0.368$

IVP; $y = (0.368\,e^{t})^3$ ✓ (c)

2

(d) [1]

Show that if $f(t, y) = y^{1/3}$, then $\partial f/\partial y$ is not continuous at $y = 0$. What conclusions can you draw, from Theorem 2.4.2 on the existence and uniqueness of solutions to $y' = f(t, y)$?

$\frac{\partial f}{\partial y}(0.368e^{t})^3 = \phi$

0

2 / 3

****End of Questions****

Use this page and the other side for rough working if needed.