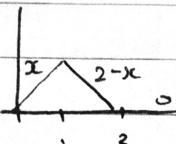


$$1. f(x) \begin{cases} 0 & 0 \leq x \leq 1 \\ 1 & 1 < x \leq 2 \\ 0 & x > 2 \end{cases}$$



$$\begin{aligned} f(x) &= x \cdot u(x) + ((2-x)-x)u(x-1) - (2-x)u(x-2) \\ &= xu(x) + (2-2x)u(x-1) - (2-x)u(x-2) \\ &= xu(x) - 2(x-1)u(x-1) + (x-2)u(x-2) \end{aligned}$$

$$\mathcal{L}f(s) = \mathcal{L}(x)u(x) - 2\mathcal{L}(x-1)u(x-1) + \mathcal{L}(x-2)u(x-2)$$

2nd Translation:

$$\Rightarrow e^{0s} \mathcal{L}(x)(s) - 2e^{-s} \mathcal{L}(x)(s) + e^{-2s} \mathcal{L}(x)(s)$$

$$\mathcal{L}(x)(s) = \int_0^\infty xe^{-sx} dx \Rightarrow -\frac{xe^{-sx}}{s} \Big|_0^\infty + \frac{1}{s} \int_0^\infty e^{-sx}$$

$$= -\frac{xe^{-sx}}{s} - \frac{e^{-sx}}{s^2} \Big|_0^\infty = 0 - \left(0 - \frac{1}{s^2}\right) = \frac{1}{s^2}$$

$$\text{sub in: } \mathcal{L}f(x)(s) = \frac{1}{s^2} - 2e^{-s} \frac{1}{s^2} + e^{-2s} \frac{1}{s^2}$$

$$= \frac{1}{s^2} \left( 1 - 2e^{-s} + e^{-2s} \right)$$

$$2a) \mathcal{L}(\cos ax)(s) = \int_0^{\infty} \cos ax \cdot e^{-sx} dx$$

$u = \cos ax \quad u' = -a \sin ax$   
 $v = e^{-sx} \quad v' = -e^{-sx} / s$

$$\text{IBP: } \left[ -\frac{e^{-sx}}{s} \cdot \cos ax \right]_0^{\infty} - \frac{a}{s} \int_0^{\infty} \sin ax \cdot e^{-sx}$$

$$\Rightarrow \frac{1}{s} - \frac{a}{s} \left( \left[ -\frac{\sin ax \cdot e^{-sx}}{s} \right]_0^{\infty} + \frac{a}{s} \int_0^{\infty} \cos ax e^{-sx} \right)$$

$$\frac{1}{s} + \frac{a}{s^2} \left( \left[ \sin ax \cdot e^{-sx} \right]_0^{\infty} - a \mathcal{L}(\cos ax)(s) \right)$$

$$\frac{1}{s} + \frac{a}{s^2} (0 - a \mathcal{L}(\cos ax)(s)) = \frac{1}{s} - \frac{a^2}{s^2} \mathcal{L}(\cos ax)(s)$$

$$\mathcal{L}(f)(s) + \frac{a^2}{s^2} \mathcal{L}(f)(s) = \frac{1}{s}, \quad \mathcal{L}(f)(s) = \frac{1}{s(1 + \frac{a^2}{s^2})} = \frac{1}{s + \frac{a^2}{s}} = \frac{s}{s^2 + a^2}$$

$$2b) \quad \mathcal{L}(\sin ax) = \frac{a}{s^2 + a^2}, \quad \mathcal{L}(f'(s)) = s\mathcal{L}(f(s)) - f(0)$$

$$f' = \cos ax \quad f = \frac{1}{a} \sin ax$$

$$\mathcal{L}(\cos ax) = \frac{s}{a} \mathcal{L}(\sin ax) - \sin(0)$$

$$= \frac{s}{a} \cdot \frac{a}{s^2 + a^2} - 0 = \frac{s}{s^2 + a^2}$$

$$2c) \quad \cos(2x) = 2\cos^2(x) - 1$$

$$\frac{1}{2}\cos(2x) + \frac{1}{2} = \cos^2(x)$$

$$\therefore \mathcal{L}(\cos^2 x) = \frac{1}{2} \mathcal{L}\{\cos(2x)\} + \mathcal{L}\left\{\frac{1}{2}\right\}$$

$$= \frac{1}{2} \cdot \frac{s}{s^2 + 4} + \frac{s}{2} \Rightarrow \frac{s}{2s^2 + 8} + \frac{s}{2}$$

$$3a) \quad \mathcal{L}^{-1}\left(\frac{2}{s} - \frac{1}{s^3}\right)^2, \quad F(s) = \left(\frac{2}{s} - \frac{1}{s^3}\right)\left(\frac{2}{s} - \frac{1}{s^3}\right) = \frac{1}{s^4} - \frac{4}{s^4} + \frac{4}{s^2}$$

$$= \mathcal{L}^{-1}\left(\frac{1}{s^6}\right) - \mathcal{L}^{-1}\left(\frac{4}{s^4}\right) + \mathcal{L}^{-1}\left(\frac{4}{s^2}\right) \quad \text{use } x^n \Rightarrow \frac{n!}{s^{n+1}}$$

$$\Rightarrow \frac{1}{5!}x^5 - \frac{4}{3!}x^3 + \frac{4}{1!}x \Rightarrow \mathcal{L}(F(s)) = \frac{x^5}{120} - \frac{4x^3}{6} + 4x$$

$$3b) \mathcal{L}^{-1}\left\{\frac{s+1}{s^2+2}\right\}, F(s) = \frac{s}{s^2+2} + \frac{1}{s^2+2}$$

$$= \frac{s}{s^2+2} + \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{2}}{s^2+2}$$

$$\therefore \mathcal{L}^{-1}F(s) = \cos(x\sqrt{2}) + \frac{1}{\sqrt{2}} \sin(x\sqrt{2})$$

$$3c) \quad \mathcal{L}^{-1} \left( \frac{s}{(s+2)(s^2+4)} \right) \quad F(s) = \frac{A}{s+2} + \frac{Bs+C}{s^2+4}$$

cover up ~~A~~:  $\frac{s}{s^2+4} \Big|_{s=2}, \quad A = \frac{-1}{4}$

$$S = -\frac{1}{4}(s^2+4) + (Bs+C)(s+2)$$

$$S = -\frac{s^2}{4} - 1 + Bs^2 + 2Bs + Cs + 2C$$

$$S: 1 = 2B + C$$

$$1: 0 = \cancel{2B} + \cancel{C}$$

$$-1 + 2C$$

$$C = \frac{1}{2}, \quad B = \frac{1}{4}, \quad A = -\frac{1}{4}$$

$$-\frac{1}{4} \left( \frac{1}{s+2} \right) + \frac{\frac{1}{4}s + \frac{1}{2}}{s^2+4} = \frac{1}{4} \left( \frac{s}{s^2+4} + \frac{2}{s^2+4} \right) - \frac{1}{4} \left( \frac{1}{s+2} \right)$$

$$\Rightarrow \frac{1}{4} (\cos 2x + \sin 2x - e^{-2x})$$

$$4. \quad y' + 6y = e^{4x}, \quad y(0) = 2 \quad : \quad Y = \mathcal{L}(y)(s)$$

$$sY - 2 + 6Y = \frac{1}{s-4}, \quad Y(s+6) = \frac{1}{s-4} + 2$$

$$Y = \frac{1}{(s-4)(s+6)} + \frac{2}{s+6}$$

$$\frac{1}{(s-4)(s+6)} = \frac{A}{s-4} + \frac{B}{s+6}, \quad \text{cover up: } A = \frac{1}{4+6} = \frac{1}{10}$$

$$Y = \frac{1}{10(s-4)} - \frac{1}{10(s+6)} + \frac{2}{s+6}$$

$$B = \frac{1}{-6-4} = -\frac{1}{10}$$

$$y = \mathcal{L}^{-1}(Y) = \frac{e^{4x}}{10} - \frac{e^{-6x}}{10} + 2e^{-4x} = y \quad 5'$$

$$5. \quad \mathcal{L}(x^3 e^{2x})(s) = \mathcal{L}(x^3)(s-2) = \frac{6}{(s-2)^4}$$

$$y'' - 4y' + 4y = x^3 e^{2x}, \quad y(0) = y'(0) = \emptyset \quad : \quad Y = \mathcal{L}(y)(s)$$

$$s^2 Y - 4s Y + 4Y = \frac{6}{(s-2)^4} \Rightarrow Y(s^2 - 4s + 4) = \frac{6}{(s-2)^4}$$

$$\Rightarrow Y = \frac{6}{(s^2 - 4s + 4)(s-2)^4} = \frac{6}{(s-2)^2(s-2)^4} = \frac{6}{(s-2)^6}$$

$$\mathcal{L}^{-1}(Y) : \quad x^5 \Leftrightarrow \mathcal{L}^{-1}\left(\frac{120}{s^6}\right) \quad \therefore \frac{1}{20}x^5 \Leftrightarrow \mathcal{L}^{-1}\left(\frac{6}{s^6}\right)$$

$$\text{so } \mathcal{L}^{-1}\left\{\frac{6}{(s-2)^6}\right\} \Rightarrow \text{1st Translat.} \Rightarrow \frac{e^{2x} x^5}{20} = y$$

$$6. \quad y'' - y' = e^{-x} \cos x, \quad y(0) = y'(0) = 0 \quad : \quad Y = \mathcal{L}(f)(s)$$

$$s^2Y - sY = \mathcal{L}(e^{-x} \cos x) = \mathcal{L}(\cos x)(s+1)$$

$$Y(s^2 - s) = \frac{s+1}{(s+1)^2 + 1}$$

$$Y = \frac{s+1}{(s^2+s)((s+1)^2+1)} = \frac{s+1}{s(s-1)((s+1)^2+1)}$$

$$\Rightarrow \frac{A}{s} + \frac{B}{s-1} + \frac{Cs+D}{s^2+2s+2}$$

$$\begin{aligned} s+1 &= As^3 + As^2 - 2A + Bs^3 + 2Bs^2 + 2Bs + Cs^3 - Cs^2 + Ds^2 - Ds \\ &= As^3 + Bs^3 + Cs^3 + As^2 + 2Bs^2 - Cs^2 + Ds^2 + 2Bs - Ds - 2A \end{aligned}$$

$$S^3: \quad 0 = A + B + C$$

$$\Rightarrow \frac{1}{2} = B + C$$

$$1 = 3B + D \Rightarrow 2 = 5B$$

$$S^2: \quad 0 = A + 2B - C + D$$

$$+ \frac{1}{2} = 2B - C + D$$

$$1 = 2B - D$$

$$S^1: \quad 1 = 2B - D$$

$$-1 = 2B - D$$

$$S^0: \quad 1 = -2A \quad : \quad A = \frac{-1}{2}$$

$$B = \frac{2}{5}$$

$$\left| \begin{array}{cccc} A & B & C & D \\ -\frac{1}{2} & \frac{2}{5} & \frac{1}{10} & -\frac{1}{5} \end{array} \right|$$

$$y = -\frac{1}{2s} + \frac{2}{5(s-1)} + \frac{s-2}{10(s^2+2s+2)}$$

$$= \left( \frac{1}{2} \cdot \frac{1}{s} \right) + \left( \frac{2}{5} \cdot \frac{1}{s-1} \right) + \left( \frac{1}{10} \cdot \frac{s-2}{(s+1)^2+1} \right)$$

$$y(s) = -\frac{1}{2} \frac{1}{s} + \frac{2}{5} \frac{1}{s-1} + \frac{1}{10} \frac{s+1}{(s+1)^2+1} - \frac{1}{10} \cdot \frac{3}{s+1^2+1}$$

$$y = -\frac{1}{2} + \frac{2e^x}{5} + \frac{1}{10} e^x \cdot \cos x - \frac{3}{10} e^x \sin x$$

$$y = \frac{2e^x}{5} + \frac{e^{-x}}{10} (\cos x - 3 \sin x) - \frac{1}{2}$$