1. Discrete Time Fourier Series **(20 marks)**

   A sensor system in a soft drink bottling plant monitors the level of syrup in a vat. Ten probes placed at different levels in the vat are polled one after the other and return '1' if they are covered by the liquid and '0' if they are dry. Once the tenth probe is polled the system goes back to the first probe, then the second, and so on. The data gathered from these probes when the vat is 80% full is represented by a periodic signal, $x_1[n]$, with

$$x_1[n] = \begin{cases} 1, & 0 \le n \le 7 \\ 0, & 8 \le n \le 9 \end{cases} \tag{1}$$

   (a) Calculate the period of $x_1[n]$. **(2 marks)**

   (b) Rather than store $x_1[n]$ directly, the system stores the Fourier series coefficients, a_k of $x_1[n]$. Derive an expression for a_k for the specific case of equation (1). **(8 marks)**

   (c) To guard against data corruption the sensor system monitors not only the values of $x_1[n]$, but also the rate at which the values in $x_1[n]$ change. This rate can be expressed as

$$g[n] = x_1[n] - x_1[n-1]. \tag{2}$$

   Sketch $g[n]$ when the vat is 80% full. **(5 marks)**

   (d) Use the properties of the discrete time Fourier series to derive an expression for the Fourier series coefficients of $g[n]$, b_k, in terms of a_k. **(5 marks)**

2. More Discrete Time Fourier Series **(20 marks)**

   In a different part of the soft drink bottling plant a monitoring system records the number of soft drink cartons (in thousands) processed each hour. The raw data, $x_2[n]$, is converted to a Fourier series, c_k, and transmitted to headquarters once every six hours. As the communications link to headquarters is not always reliable, summary statistics about c_k are also transmitted.

   During a thunderstorm the c_k data transmitted by the communications link are corrupted. Fortunately, the following statistical data are known or recovered:

   1. $x_2[n]$ is periodic with period $N = 6$
   2. $\sum_{n=0}^{5} x_2[n] = 18$
   3. $\sum_{n=4}^{9} (-1)^n x_2[n] = 6$
   4. The power per period in $x_2[n]$, $P = \frac{1}{N} \sum_{n=<N>} |x[n]|^2$ is 10.

   (a) Determine the DC value, c_0 of the Fourier series c_k. **(5 marks)**

   (b) Based on the fact that $\sum_{n=4}^{9} (-1)^n x[n] = 6$, determine one other coefficient in c_k. **(6 marks)**

   (c) Based on the power per period, determine the remaining coefficients for c_k. **(5 marks)**

   (d) Based on the coefficients you recovered for c_k, plot one period of $x[n]$. **(4 marks)**

3. Discrete-Time LTI System **(28 marks)**

Consider the discrete-time LTI system $y[n] - \frac{1}{6}y[n-1] - \frac{1}{6}y[n-2] = x[n]$

In order to obtain full credit, you must explain each step.

(a) Is this a first or second-order LTI system? (Explain your answer.) **(2 marks)**

(b) Find the frequency response of the system. **(6 marks)**

(c) Find the impulse response of the system. **(12 marks)**

(d) Is this system causal? (Explain your answer.) **(2 marks)**

(e) Find the response of this system to $x[n] = 2\delta[n] - \delta[n-1]$. **(6 marks)**

4. Frequency characterisation **(12 marks)**

An LTI system is said to have *phase lead* at a particular frequency $\omega = \omega_0$ if $\sphericalangle H(j\omega_0) > 0$. The terminology stems from the fact that if $e^{j\omega t}$ is the input to this system, then the phase of the output will exceed, or lead, the phase of the input. Similarly, if $\sphericalangle H(j\omega_0) < 0$, the system is said to have *phase lag* at this frequency. Not that the system with frequency response $\frac{1}{1+j\omega\tau}$ has phase lag for all $\omega > 0$, while the system with frequency response $1 + j\omega\tau$ has phase lead for $\omega > 0$.

In order to obtain full credit, you must explain each step.

(a) Using straight-line approximations, sketch the Bode magnitude plot for the following two systems i) $\frac{1+(j\omega/10)}{1+10j\omega}$ and ii) $\frac{1+10j\omega}{1+(j\omega/10)}$. **(6 marks)**

(b) Which one amplifies signals at certain frequencies? **(2 marks)**

(c) Determine the phase of each system. Which has phase lead and which has phase lag? (Note that $\arctan(x)$ is positive for $x > 0$ and negative for $x < 0$.) **(4 marks)**

5. Sampling **(22 marks)**

You have a digital signal processor system consisting of an analogue to digital convertor, a digital signal processor (DSP) chip and a digital to analogue convertor. To reduce cost, the analogue to digital convertor has no anti-aliasing lowpass filter at its input. The digital to analogue convertor includes a correctly designed lowpass filter. The sample rate is 8 kHz.

In order to obtain full credit, you must explain each step.

(a) What is the maximum possible frequency that this processor can correctly sample and reproduce at the output of the digital to analogue convertor? If a signal with half this frequency is sampled, what is the corresponding frequency in the digital domain? **(3 marks)**

(b) Suppose that a real sinusoidal signal with a frequency of 5 kHz was input to the digital signal processing system. Draw a frequency-domain picture of the spectrum of the sampled signal over the frequency range minus 16 kHz to 16 kHz showing all frequency components. What would be the frequency at the output of the digital processor system? **(10 marks)**

(c) Suppose you have a sinusoidal input frequency of 1 kHz. You need to increase the sample rate from 8 kHz to 16 kHz. Show with pictures both in the time and frequency domains how you would do this. Use a frequency scale appropriate for the problem. **(9 marks)**

6. Z transform **(18 marks)**

Consider the discrete signal $x[n] = \left(\frac{1}{2}\right)^n u[n] + (1.5)^n u[-n-1]$

(a) Draw a sketch of this signal. **(3 marks)**

(b) Derive the Z transform. What are the values of the poles? **(10 marks)**

(c) What is the region of convergence? Does the signal have a discrete Fourier transform? Why? **(5 marks)**