Differential Equations Formula Sheet

Trigonometric identities

- $\sin^2 x + \cos^2 x = 1$
- $\sin 2x = 2\sin x \cos x$
- $\cos 2x = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$
- $\sec^2 x = 1 + \tan^2 x$
- $\csc^2 x = 1 + \cot^2 x$
- $\sin x \cos y = \dfrac{1}{2}\left[\sin(x-y) + \sin(x+y)\right]$
- $\sin x \sin y = \dfrac{1}{2}\left[\cos(x-y) - \cos(x+y)\right]$
- $\cos x \cos y = \dfrac{1}{2}\left[\cos(x-y) + \cos(x+y)\right]$
- if $t = \tan\dfrac{x}{2}$ then $\sin x = \dfrac{2t}{1+t^2}$, $\quad \cos x = \dfrac{1-t^2}{1+t^2}$, $\quad$ and $\tan x = \dfrac{2t}{1-t^2}$

Hyperbolic identities

- $\cosh^2 x - \sinh^2 x = 1$
- $\sinh 2x = 2\sinh x \cosh x$
- $\cosh 2x = \cosh^2 x + \sinh^2 x = 2\cosh^2 x - 1 = 2\sinh^2 x + 1$
- $\operatorname{sech}^2 x = 1 - \tanh^2 x$
- $\operatorname{csch}^2 x = \coth^2 x - 1$
- $\sinh x \cosh y = \dfrac{1}{2}\left[\sinh(x+y) + \sinh(x-y)\right]$
- $\sinh x \sinh y = \dfrac{1}{2}\left[\cosh(x+y) - \cosh(x-y)\right]$
- $\cosh x \cosh y = \dfrac{1}{2}\left[\cosh(x+y) + \cosh(x-y)\right]$

Derivatives of trigonometric and hyperbolic functions

$$\frac{d}{dx}[\sin x] = \cos x$$

$$\frac{d}{dx}[\cos x] = -\sin x$$

$$\frac{d}{dx}[\tan x] = \sec^2 x$$

$$\frac{d}{dx}[\cot x] = -\csc^2 x$$

$$\frac{d}{dx}[\sec x] = \sec x\ \tan x$$

$$\frac{d}{dx}[\csc x] = -\csc x\ \cot x$$

$$\frac{d}{dx}[\tan^{-1} x] = \frac{1}{1+x^2}$$

$$\frac{d}{dx}[\sinh x] = \cosh x$$

$$\frac{d}{dx}[\cosh x] = \sinh x$$

$$\frac{d}{dx}[\tanh x] = \mathrm{sech}^2 x$$

$$\frac{d}{dx}[\coth x] = -\mathrm{csch}^2 x$$

$$\frac{d}{dx}[\mathrm{sech} x] = -\mathrm{sech} x\ \tanh x$$

$$\frac{d}{dx}[\mathrm{csch} x] = -\mathrm{csch} x\ \coth x$$

Exponents and logarithms

- $y = a^x \quad \Leftrightarrow \quad x = \log_a y$ and $y = e^x \quad \Leftrightarrow \quad x = \ln y$
- $\dfrac{d}{dx}[a^x] = a^x \cdot \ln a$ and $\frac{d}{dx}[\log_a x] = \frac{1}{x \ln a}$

Matrices

Let $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$.

- $A^{-1} = \dfrac{1}{\det A} \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}$.
- Cramer's Rule: the solution of $A\mathbf{x} = \mathbf{b}$ is given by $x_1 = \dfrac{\det A_1}{\det A}$, $x_2 = \dfrac{\det A_2}{\det A}$

  where $A_1 = \begin{pmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{pmatrix}$, $A_2 = \begin{pmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{pmatrix}$.

Sums and series

- $\displaystyle\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$
- $\displaystyle\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$
- $\displaystyle\sum_{k=1}^{n} k^3 = \left[\frac{n(n+1)}{2}\right]^2$
- Geometric progression: $\displaystyle\sum_{k=0}^{n-1} r^k = 1 + r + r^2 + \cdots + r^{n-1} = \frac{1-r^n}{1-r}, \quad r \neq 1$

  Geometric series: $\displaystyle\sum_{k=0}^{\infty} r^k = \frac{1}{1-r}$ when $|r| < 1$
- Maclaurin series: $\displaystyle\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k$
- Taylor series: $\displaystyle\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!} (x-a)^k$
- Fourier series for $f(x)$ on $[-\pi, \pi]$: $\displaystyle\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$

  where $\displaystyle a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx, \quad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\, dx.$

Transforms

- Laplace transform: $\mathcal{L}\{f(x)\} = \int_0^\infty f(x)e^{-sx}\,dt = F(s)$

  $\mathcal{L}\{1\} = \dfrac{1}{s}$

  $\mathcal{L}\{t^n\} = \dfrac{n!}{s^{n+1}}, \quad n$ a positive integer

  $\mathcal{L}\{e^{ax}\} = \dfrac{1}{s-a}$

  $\mathcal{L}\{\cos ax\} = \dfrac{s}{s^2+a^2}$

  $\mathcal{L}\{\sin ax\} = \dfrac{a}{s^2+a^2}$

  $\mathcal{L}\{f'(x)\} = sF(s) - f(0)$

  $\mathcal{L}\{f^{(n)}(x\} = s^nF(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - sf^{(n-2)}(0) - f^{(n-1)}(0)$

- Fourier transform: $\mathcal{F}\{f(x)\} = \int_{-\infty}^\infty f(x)e^{i\alpha x}\,dt = F(\alpha)$

  $\mathcal{F}\{f'(x)\} = -i\alpha F(\alpha)$

  Inverse Fourier transform: $\mathcal{F}^{-1}\{F(\alpha)\} = \dfrac{1}{2\pi}\int_{-\infty}^\infty F(\alpha)e^{-i\alpha x}\,d\alpha = f(x).$