Assignment 3

Due: 23:59 Wednesday 3 April, online

Modelling with DEs, Existence & Uniqueness. Boyce & DiPrima, Ch 2 sections 2.3, 2.4

1. In determining the integrating factor $\exp(\int p(t)\,dt)$ for the DE

$$y' + p(t)y = q(t)$$

   we do not use a constant of integration in the evaluation of $\int p(t)\,dt$. Explain or show why using $\int p(t)\,dt + c$ has no effect on the solution.

2. Suppose a tank containing a certain liquid has an outlet near the bottom. Let $h(t)$ be the height of the surface of the liquid above the outlet at time t. Torricelli's principle says that the fluid outflow velocity v at the outlet is equal to the velocity of a particle falling freely under gravity $g = 9.8$ m s^{-2} with no drag, from the height h.

   (a) Show that $v = \sqrt{2gh}$.

   (b) By equating the rate of outflow to the rate of change of the volume of liquid in the tank, show that $h(t)$ satisfies the eqn

$$A(h)\frac{dh}{dt} = -a\sqrt{2gh}$$

   where $A(h)$ is the area of the cross-section of the tank at height h, and a is the area of the outlet.

   (c) Consider a water tank in the form of a right circular cylinder with tank height 3 m above the outlet. Tank radius is 1 m, and the radius of the circular outlet is 0.1 m. If the tank is initially full, calculate how long it takes to drain the tank down to the level of the outlet.

3. A person with no capital invests k dollars per year at an annual rate of return r. Assume that investments are made continuously and that the return is compounded continuously.

   (a) Determine the sum $S(t)$ accumulated at any time t.

   (b) If $r = 7.5\%$, determine k so that one million dollars will be available in forty years.

   (c) If $k = \$2000$ a year, determine the rate of return required to have one million dollars in forty years.

4. Determine an interval, without actually solving the IVP, in which a solution of the IVP is sure to exist:

   (a) $(t-3)y' + (\ln t)y = 2t\ , \quad y(1) = 2$

   (b) $y' + (\tan t)y = \sin t\ , \quad y(\pi) = 0$

5. State where in the ty-plane the hypotheses of Theorem 2.4.2 are satisfied, for the DE

$$y' = \frac{t-y}{2t+3y}$$

6. Using the substitution discussed in question 5 of the Tutorial, solve the Bernoulli equation

$$t^2\frac{dy}{dt} + 2ty = y^3\ , \quad t > 0$$

7. One morning it started snowing in Wellington at a heavy and steady rate. A snowplow started out at noon, going 2 miles the first hour and 1 mile the second hour. What time did it start snowing?

   This is the Snowplow Problem of R.P. Agnew (Ralph Palmer Agnew, *Differential Equations*, McGraw-Hill, New York, 1942, 1960). You may assume that the speed of the plow is inversely proportional to the depth of the snow. Obtain a differential equation for the distance travelled by the snowplow, that you can solve. The answer is 11:23am — see if you can get this.

Tutorial 3 Exercises: 28 March– 2 April

1. Suppose a tank containing a certain liquid has an outlet near the bottom. Let $h(t)$ be the height of the surface of the liquid above the outlet at time t. Torricelli's principle says that the fluid outflow velocity v at the outlet is equal to the velocity of a particle falling freely under gravity $g = 9.8$ m s^{-2} with no drag, from the height h.

   (a) Show that $v = \sqrt{2gh}$. You can use Newton's second law.

   (b) By equating the rate of outflow to the rate of change of the volume of liquid in the tank, show that $h(t)$ satisfies the eqn

   $$A(h)\frac{dh}{dt} = -a\sqrt{2gh}$$

   where $A(h)$ is the area of the cross-section of the tank at height h, and a is the area of the outlet.

   (c) Consider a water tank in the form of a right circular cylinder with tank height 5 m above the outlet. Tank radius is 2 m, and the radius of the circular outlet is 0.2 m. If the tank is initially full, calculate how long it takes to drain the tank down to the level of the outlet.

2. A graduate borrows \$8000 to buy a car. The lender charges interest at an annual rate of 10%. Assuming the interest is compounded continuously, and the graduate makes continuous payments at a constant annual rate k, determine the required rate k to have the loan paid off in three years. Also calculate the total amount of interest paid.

3. Determine an interval, without actually solving the IVP, in which a solution of the IVP is sure to exist:

   (a) $t(t-4)y' + y = 0\ , \quad y(2) = 1$

   (b) $(4-t^2)y' + 2ty = 3t^2\ , \quad y(-3) = 1$

4. State where in the ty-plane the hypotheses of Theorem 2.4.2 are satisfied, for the DE

   $$y' = (1 - t^2 - y^2)^{1/2}$$

5. **Bernoulli Equations**

Bernoulli equations take the form

$$y' + p(t)y = q(t)y^n$$

and are named after Jakob Bernoulli.

(a) Solve Bernoulli's equation for the cases $n = 0$ and $n = 1$.

(b) Show that if $n \neq 0$, 1 then the substitution $v = y^{1-n}$ changes Bernoulli's equation to a first-order linear equation.

(c) Using the substitution discussed above, solve the Bernoulli equation

$$\frac{dy}{dt} = (\Gamma \cos t + T)y - y^3$$

where Γ and T are constants. [Note: do not try to simplify the integral of the integrating factor $\int \mu dt$ that arises in this DE.]