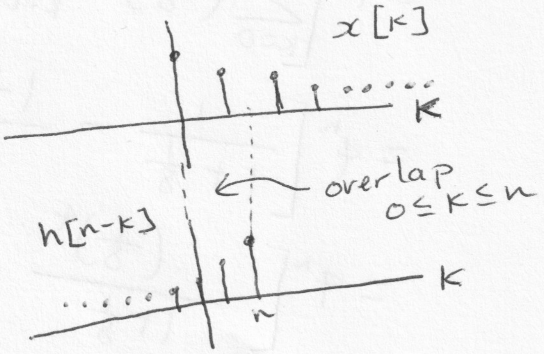
ECEN220 - Assignment 2 Solutions

2.21

(a) $x[n] = \alpha^n, \quad n \geq 0 \xrightarrow[n \to k]{\text{change}} x[k] = \alpha^k \quad k \geq 0$

$h[n] = \beta^n, \quad n \geq 0 \xrightarrow[n \to n-k]{\text{change}} h[n-k] = \beta^{n-k} \quad n-k \geq 0$

$k \leq n$



So,

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

$$= \sum_{k=0}^{n} \alpha^k \beta^{n-k}$$

$$= \beta^n \sum_{k=0}^{n} \left(\frac{\alpha}{\beta}\right)^k$$

$$= \beta^n \frac{1-\left(\frac{\alpha}{\beta}\right)^{n+1}}{1-\frac{\alpha}{\beta}}$$

$$= \beta^n \frac{\beta - \beta\left(\frac{\alpha}{\beta}\right)^{n+1}}{\beta-\alpha}$$

$$= \frac{\beta^{n+1}-\alpha^{n+1}}{\beta-\alpha} \quad \text{for } n \geq 0 = \frac{\beta^{n+1}-\alpha^{n+1}}{\beta-\alpha}u[n]$$

(b) $x[n] = \alpha^n \quad n \geq 0$

$h[n] = \alpha^n \quad n \geq 0$

using ~~the result from~~ 2.2(a) but let $\beta = \alpha$, so

$$y[n] = \sum_{k=0}^{n} \alpha^k \alpha^{n-k}$$

$$= \alpha^n \sum_{k=0}^{n} 1$$

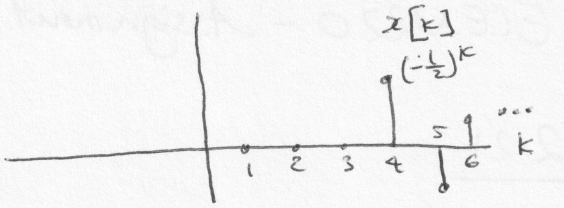
$$= (n+1)\alpha^n \quad \text{for } n \geq 0 = (n+1)\alpha^n u[n]$$

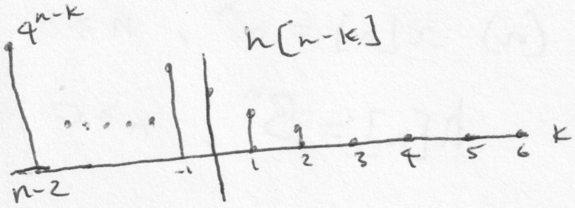
(c) $x[n] = \left(-\frac{1}{2}\right)^n$ for $n \geq 4 \xrightarrow[n \to k]{\text{change}} x[k] = \left(-\frac{1}{2}\right)^k$ for $k \geq 4$

$h[n] = 4^n$ for $n \leq 2 \xrightarrow[n \to n-k]{\text{change}} h[n-k] = 4^{n-k}$ for $n-k \leq 2$

$n-2 \leq k$

Case 1: Overlap on range $4 \leq k \leq \infty$





$$y[n] = \sum_{k=4}^{\infty} \left(-\frac{1}{2}\right)^k 4^{n-k}$$

$$= 4^n \sum_{k=4}^{\infty} \left(-\frac{1}{8}\right)^k$$

$$= 4^n \left[\sum_{k=0}^{\infty} \left(-\frac{1}{8}\right)^k - \sum_{k=0}^{3} \left(-\frac{1}{8}\right)^k \right]$$

$$= 4^n \left[\frac{1}{1+\frac{1}{8}} - \frac{1-\left(-\frac{1}{8}\right)^4}{1+\frac{1}{8}} \right]$$

$$= 4^n \left[\frac{\left(\frac{1}{8}\right)^4}{1+\frac{1}{8}} \right]$$

$$= \left(\frac{8}{9}\right)\left(\frac{1}{8}\right)^4 4^n \quad \text{for} \quad n \leq 6$$

Case 2: Overlap on range $n-2 \leq k \leq \infty$

$$y[n] = \sum_{k=n-2}^{\infty} \left(-\frac{1}{2}\right)^k 4^{n-k}$$

$$= 4^n \sum_{k=n-2}^{\infty} \left(-\frac{1}{8}\right)^k$$

Let $r = k-n+2$, then

$$y[n] = 4^n \sum_{r=0}^{\infty} \left(-\frac{1}{8}\right)^{r+n-2}$$

$$= \left(-\frac{1}{8}\right)^{-2}\left(-\frac{1}{2}\right)^n \sum_{r=0}^{\infty} \left(-\frac{1}{8}\right)^r$$

$$= 8^2 \left(-\frac{1}{2}\right)^n \frac{1}{1+\frac{1}{8}}$$

$$= \frac{8^3}{9}\left(-\frac{1}{2}\right)^n \quad \text{for} \quad n \geq 6$$

$$\therefore\ y[n] = \begin{cases} \left(\frac{8}{9}\right)\left(\frac{1}{8}\right)^4 4^n, & n \leq 6 \\ \frac{8^3}{9}\left(-\frac{1}{2}\right)^n, & n \geq 6 \end{cases}$$

Notice that y[n] have identical valus at $n=6$

2.22

(a) $x(t) = e^{-\alpha t}, t \geq 0 \xrightarrow[t \to \tau]{\text{change}} x(\tau) = e^{-\alpha \tau}, \tau \geq 0$

$h(t) = e^{-\beta t}, \quad t \geq 0 \xrightarrow[t \to t-\tau]{\text{change}} h(t-\tau) = e^{-(t-\tau)\beta}, \quad t-\tau \geq 0$

$t \geq \tau$

So overlap at $0 \leq \tau \leq t$,

$$y(t) = \int_0^t e^{-\alpha\tau} e^{-(t-\tau)\beta} \, d\tau$$

$$= e^{-t\beta} \int_0^t e^{-\alpha\tau} e^{\beta\tau} \, d\tau$$

$$= e^{-\beta t} \int_0^t e^{-\tau(\alpha-\beta)} \, d\tau$$

$$= e^{-\beta t} \left[-\frac{1}{\alpha-\beta} e^{-\tau(\alpha-\beta)} \right]_0^t$$

$$= e^{-\beta t} \left[-\frac{1}{\alpha-\beta} e^{-t(\alpha-\beta)} + \frac{1}{\alpha-\beta} \right]$$

$$= -\frac{e^{-\beta t} e^{-t(\alpha-\beta)}}{\alpha-\beta} + \frac{e^{-\beta t}}{\alpha-\beta}$$

$$= -\frac{e^{-\alpha t}}{\alpha-\beta} + \frac{e^{-\beta t}}{\alpha-\beta}$$

$$= \frac{e^{-\alpha t} - e^{-\beta t}}{\beta-\alpha} \quad \text{for } t \geq 0, \ \alpha \neq \beta$$

If $\alpha = \beta$, then

$$y(t) = \int_0^t e^{-\alpha\tau} e^{-(t-\tau)\alpha} \, d\tau$$

$$= \int_0^t e^{-\alpha t} \, d\tau$$

$$= e^{-\alpha t} \left[\tau \right]_0^t$$

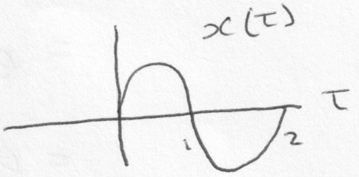
$$= t e^{-\alpha t} \quad \text{for } t \geq 0, \ \alpha = \beta$$

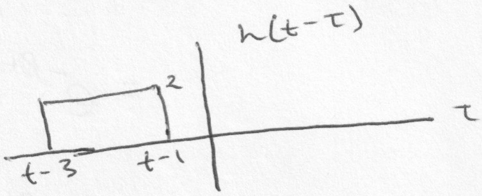
$$\therefore \quad y(t) = \begin{cases} \dfrac{e^{-\alpha t} - e^{-\beta t}}{\beta - \alpha} u(t) & , \quad \alpha \neq \beta \\ t e^{-\alpha t} u(t) & , \quad \alpha = \beta \end{cases}$$

(c) $x(t) = \sin(\pi t)$, $0 \le t \le 2 \xrightarrow[t \to \tau]{\text{change}} x(\tau) = \sin(\pi \tau)$, $0 \le \tau \le 2$

$h(t) = 2$, $1 \le t \le 3 \xrightarrow[t \to t-\tau]{\text{change}} h(t-\tau) = 2 \quad 1 \le t - \tau \le 3$

$t - 3 \le \tau \le t - 1$





Case 1: $t - 1 \le 0$

$y(t) = 0$ for $t \le 1$ as no overlap

Case 2: Overlap on range $0 \le \tau \le t-1$

So, $y(t) = \displaystyle\int_0^{t-1} 2\sin(\pi \tau)\, d\tau$

$= \dfrac{2}{\pi}\left[-\cos(\pi \tau)\right]_0^{t-1}$

$= \dfrac{2}{\pi}\left[1 - \cos(\pi(t-1))\right]$ for $1 \le t \le 3$

Case 3: Ovelap on range $t-3 \le \tau \le 2$

So, $y(t) = \displaystyle\int_{t-3}^{2} 2\sin(\pi \tau)\, d\tau$

$= \dfrac{2}{\pi}\left[-\cos(\pi \tau)\right]_{t-3}^{2}$

$= \dfrac{2}{\pi}\left[\cos(\pi(t-3)) - 1\right]$ for $3 \le t \le 5$

Case 5: $t \ge 5$, $y(t) = 0$ as no overlap

$$\therefore\ y(t)=\begin{cases} 0 & , \ t<1 \\ \frac{2}{\pi}\left[1-\cos(\pi(t-1))\right] & , \ 1\le t\le 3 \\ \frac{2}{\pi}\left[\cos(\pi(t-3))-1\right] & , \ 3\le t\le 5 \\ 0 & , \ t>5 \end{cases}$$

(d) $x(t)=at+b, \ \forall t$

$h(t)=h_1(t)-\frac{1}{3}\delta(t-2)$ where $h_1(t)=\begin{cases}\frac{4}{3} & , \ 0\le t\le 1 \\ 0 & , \ \text{otherwise.}\end{cases}$

~~The convolution~~ is

$\cancel{y(t)=\int_{-\infty}^{\infty}}$

$x(\tau)=a\tau+b, \ \forall\tau$

$h(t-\tau)=h_1(t-\tau)-\frac{1}{3}\delta(t-\tau-2)$ where $h_1(t-\tau)=\begin{cases}\frac{4}{3} & , \ t-1\le\tau\le t \\ 0 & , \ \text{otherwise}\end{cases}$

So the convolution is

$$y(t)=\int_{-\infty}^{\infty}x(\tau)h(t-\tau)\,d\tau$$

$$=\int_{-\infty}^{\infty}x(\tau)\left[h_1(t-\tau)-\frac{1}{3}\delta(t-\tau-2)\right]d\tau$$

$$=\int_{-\infty}^{\infty}x(\tau)h_1(t-\tau)\,d\tau-\int_{-\infty}^{\infty}\frac{1}{3}x(\tau)\delta(t-\tau-2)\,d\tau$$

$$=\int_{-\infty}^{\infty}x(t)h_1(t-\tau)\,d\tau-\frac{1}{3}x(t-2)$$

$$\int_{t-1}^{t}\frac{4}{3}(a\tau+b)\,d\tau=\int_{t-1}^{t}\frac{4}{3}a\tau\,d\tau+\int_{t-1}^{t}\frac{4b}{3}\,d\tau$$

$$=\frac{4}{3}\left[\frac{1}{2}a\tau^2\Big|_{t-1}^{t}+b\tau\Big|_{t-1}^{t}\right]$$

$$=\frac{4}{3}\left[\frac{1}{2}at^2-\frac{1}{2}a(t-1)^2+bt-b(t-1)\right]$$

Finally,

$$y(t) = \frac{4}{3}\left[\frac{1}{2}at^2 - \frac{1}{2}a(t-1)^2 + bt - b(t-1)\right] - \frac{1}{3}x(t-2)$$

$$= \frac{4}{3}\left[\frac{1}{2}at^2 - \frac{1}{2}a(t-1)^2 + bt - b(t-1)\right] - \frac{1}{3}\left[a(t-2)+b\right]$$

$\vdots$ algebraic simplification.

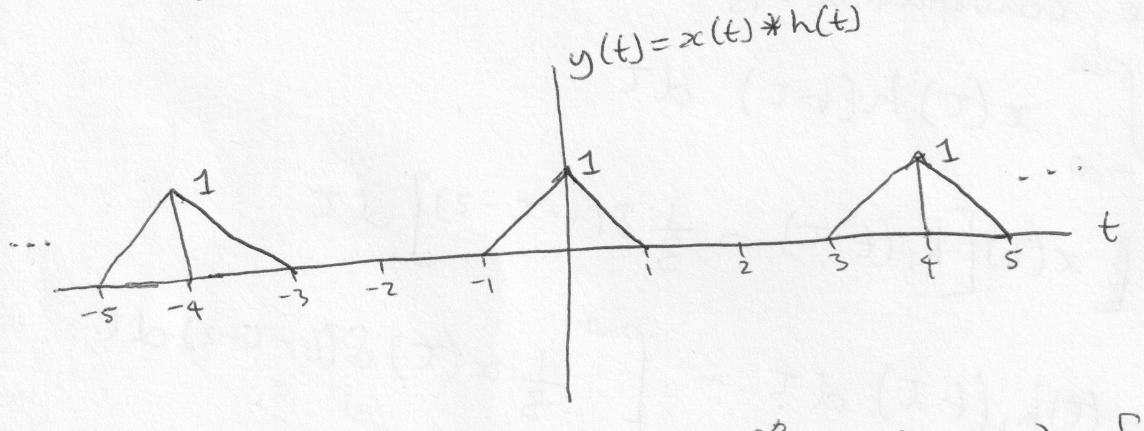
$$= at+b$$

$= x(t)$ is the result of the convolution

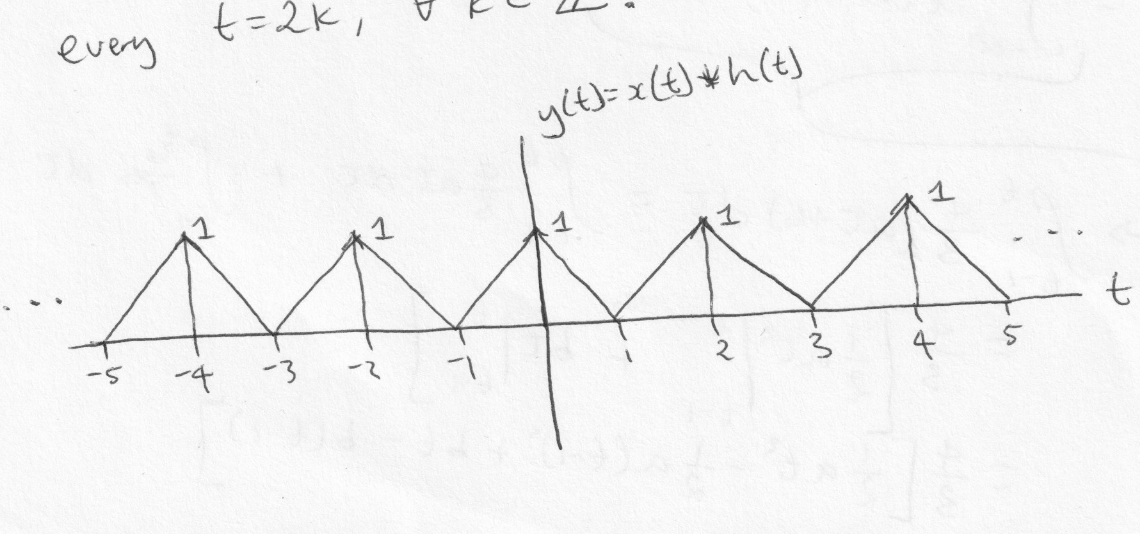
Note: Graphs not drawn to scale!

2.23: Note: Graphs not drawn to scale!

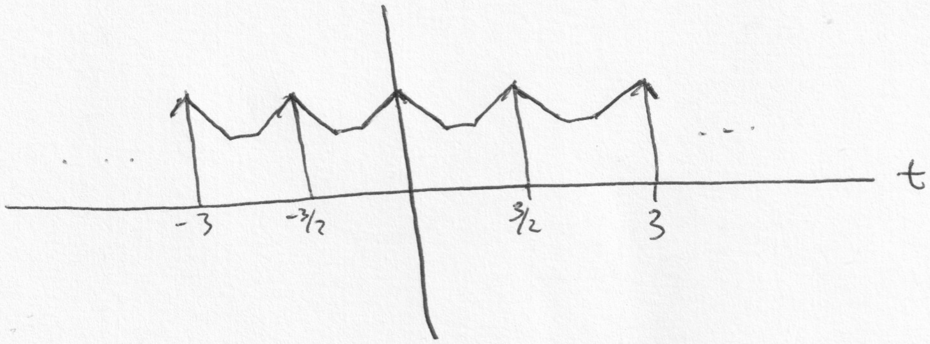
(a) With $T=4$, $x(t) = \sum_{k=-\infty}^{\infty} \delta(t-4k)$. There will be ~~an~~ a $\delta(\dots)$ at every $t=4k$, $\forall k \in \mathbb{Z}$.



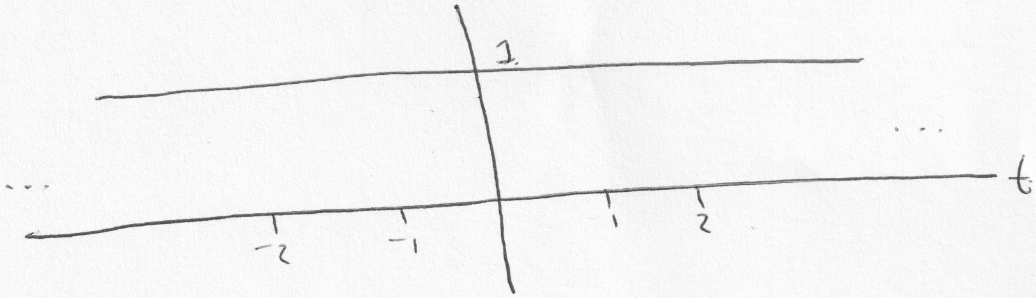
(b) With $T=2$, $x(t) = \sum_{k=-\infty}^{\infty} \delta(t-2k)$. $\delta(\dots)$ at every $t=2k$, $\forall k \in \mathbb{Z}$.



(c) With $T = \frac{3}{2}$, $x(t) = \sum_{k=-\infty}^{\infty} \delta(t - \frac{3}{2}k)$. $\delta(\ldots)$ at every $t = \frac{3}{2}k$, $\forall k \in \mathbb{Z}$.



(d) With $T=1$, $x(t) = \sum_{k=-\infty}^{\infty} \delta(t-k)$. $\delta(\ldots)$ at every $t = k$, $\forall k \in \mathbb{Z}$.



2.27

$$A_y = \int_{-\infty}^{\infty} y(t)\, dt$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x(\tau) h(t-\tau)\, d\tau\, dt \qquad \text{def. of conv.}$$

$$= \int_{-\infty}^{\infty} x(\tau) \left[\int_{-\infty}^{\infty} h(t-\tau)\, dt \right] d\tau \qquad \text{change of integration order}$$

$$= \int_{-\infty}^{\infty} x(\tau) A_h\, d\tau \qquad \text{def of continuous time signal } A_v.$$

$$= A_h \int_{-\infty}^{\infty} x(\tau)\, d\tau$$

$$= A_x A_h$$