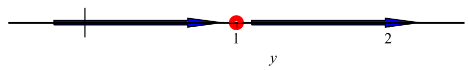
Assignment 4 Tutorial Answers

Tutorials held: 9, 10 April and 29 April - 7 May

1. *What can you say about the stability of the equilibrium point $y = 1$ for the DE $y' = (y-1)^2$?*

   Solution: For the DE $y' = (y-1)^2$, the phase diagram looks like this:

   and the equilibrium point at $y = 1$ is neither stable nor unstable as solutions with initial condition $y(0) = y_0 < 1$ converge while those with $y_0 > 0$ diverge to infinity. This type is sometimes called **semistable**.

2. *Determine if each of these differential equations is exact, and if so, solve it. Make sure you put the DE into standard differential form first.*

   Solution:

   (a) $(5y - 2x)y' - 2y = 0 \Rightarrow -2y\,dx + (5y-2x)\,dy = 0$
   Here, $M_y = -2 = N_x$ so it is exact. To solve, we know that $M = f_x$ and $N = f_y$ for some function $f(x, y)$. hence, by partial integration:

   $$f(x,y) = \int (-2y)\,dx = -2xy + h(y).$$

   Differentiating w.r.t. y:

   $$f_y = -2x + h'(y) = N = 5y - 2x$$

   so that $h'(y) = 5y$ and therefore, integrating gives $h(y) = \frac{5}{2}y^2$ (and we introduce the constant of integration through the solution). Therefore the general solution to the DE is

   $$f(x,y) = -2xy + \frac{5}{2}y^2 = C.$$

   (b) $\left(\frac{1}{t} + \frac{1}{t^2} - \frac{y}{t^2+y^2}\right) dt + \left(ye^y + \frac{t}{t^2+y^2}\right) , dy = 0$

Here we have:

$$M_y = \frac{y^2 - t^2}{(t^2+y^2)^2} = N_t$$

so the DE is exact. By partial integration:

$$f(t,y) = \int \left(\frac{1}{t} + \frac{1}{t^2} - \frac{y}{t^2+y^2} \right) dt = \ln|t| - \frac{1}{t} - \tan^{-1}\frac{t}{y} + h(y)$$

and then

$$f_y = \frac{t}{t^2+y^2} + h'(y) \Rightarrow h'(y) = ye^y$$

Integrating by parts, as in Q1(a):

$$h(y) = (y-1)e^y$$

and finally, the general solution to the DE is:

$$\ln|t| - \frac{1}{t} - \tan^{-1}\frac{t}{y} + (y-1)e^y = C.$$

3. *Make the differential equation exact by first finding an integrating factor μ, then solve:*

$$y(x+y+1)\,dx + (x+2y)\,dy = 0.$$

Solution: We know there will be an integrating factor $\mu(x)$ if $(M_y - N_x)/N$ is a function of x only. In fact:

$$\frac{M_y - N_x}{N} = \frac{x+2y+1-1}{x+2y} = 1$$

so the condition is satisfied. Indeed, the integrating factor should be

$$\mu = e^{\int 1\,dx} = e^x.$$

So we expect that

$$e^x y(x+y+1)\,dx + e^x(x+2y)\,dy = 0$$

is exact. We proceed as before,

$$\psi(x,y) = \int e^x y(x+y+1)\,dx = y(xe^x - e^x) + y(y+1)e^x + h(y) = (x+y)ye^x + h(y).$$

Differentiating w.r.t. y:

$$(x+2y)e^x + h'(y) = (x+2y)e^x \Rightarrow h'(y) = 0$$

so that $h(y)$ can be taken to be 0 and we obtain the solution:

$$(x+y)ye^x = C.$$

4. *Make the differential equation exact by first finding an integrating factor μ, and then solve it:*

$$y' = e^{2x} + y - 1.$$

Solution: After writing the DE in the standard form

$$1 - y - e^{2x} + y' = 0\ ,$$

we have $M = 1 - y - e^{2x}$ and $N = 1$. Then $M_y = -1$, and $N_x = 0$, so that

$$\frac{M_y - N_x}{N} = -1$$

which can be regarded as a function of x only (the constant function). The resulting equation for the integrating factor μ is

$$\int \frac{d\mu}{\mu} = \int -dx$$

giving

$$\mu = e^{-x}$$

and after multiplying the DE through by μ we obtain

$$e^{-x}(1-y) - e^{x} + e^{-x}y' = 0\ ,$$

so that now

$$M_y = -e^{-x} \quad N_x = -e^{-x}$$

so the DE is now exact. Then

$$\psi = \int M dx = (y-1)e^{-x} - e^{x} + h(y)$$

where h is an arbitrary function of y. To find h, we require

$$\phi_y = N$$

that is,

$$e^{-x} + h'(y) = e^{-x}$$

so that $h = 0$ and the solution to our DE is given implicitly by

$$(y-1)e^{-x} - e^{x} = c\ .$$

5 (a) $y'' - y' - 6y = 0$ put $y = e^{rt}$, $\Rightarrow$

$r^2 - r - 6 = 0 \Rightarrow (r-3)(r+2) = 0$

$\Rightarrow r = 3, -2 \Rightarrow y_1 = e^{3t} \quad y_2 = e^{-2t}$

and the general solution is $y = c_1 e^{3t} + c_2 e^{-2t}$.

(b) $y'' + y' = 0 \Rightarrow r^2 + r = 0 \Rightarrow r = 0$ or -1

$\Rightarrow y = c_1 e^0 + c_2 e^{-t} = c_1 + c_2 e^{-t}$

is the general solution.

(c) $y'' - 9y = 0 \Rightarrow r^2 - 9 = 0 \Rightarrow r = \pm 3$

$\Rightarrow y = c_1 e^{3t} + c_2 e^{-3t}$ is the general solution.

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6. $(x-2)y'' + 3y = x$, $y(0) = 0$, $y'(0) = 1$

in standard form, the DE is $y'' + \left(\frac{3}{x-2}\right) y = \frac{x}{x-2}$

so the coefficient functions are continuous everywhere except at $x = 2$. Since the initial conditions are specified at $x = 0$, the largest interval on which a unique solution exists is $(-\infty, 2)$.

7. No, since $y(0) = 0$ and $y'(0) = 2t\sin t^2$
$= 0$,

and Theorem 3.2.1 assures us that $\boxed{y = 0}$ is the <u>unique</u> solution to this IVP under the stated conditions (with zero $y(0)$, $y'(0)$).

8 (a) $y'' - 4y' + 4y = 0 \Rightarrow r^2 - 4r + 4 = 0$

$\Rightarrow r = 2$, twice

$\Rightarrow y = c_1 e^{2t} + c_2 t e^{2t}$

(b) $y'' + y' + y = 0 \Rightarrow r^2 + r + 1 = 0$

$\Rightarrow r = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$

$\Rightarrow y = c_1 e^{-\frac{t}{2}} \sin\left(\frac{\sqrt{3}t}{2}\right) + c_2 e^{-t/2} \cos\left(\frac{\sqrt{3}t}{2}\right)$

9. (a) $y'' - 4y' + 4y = e^t$

In Q 3(a) we found that the associated homogeneous DE has general solution

$$y = c_1 e^{2t} + c_2 t e^{2t}.$$

We now have to find any particular solution that gives the RHS, e^t, of our nonhomogeneous DE. We try $Y = Ae^t$. Substituting this into our DE gives

$$Ae^t - 4Ae^t + 4Ae^t = e^t \quad \Rightarrow \quad A = 1$$

so our general solution to the full nonhomogeneous DE is

$$y = c_1 e^{2t} + c_2 t e^{2t} + e^t .$$

(b) $y'' + y' + y = t^2$.

We seek a particular solution that is of the form

$$Y = At^2 + Bt + C.$$

$$\Rightarrow Y' = 2At + B, \quad Y'' = 2A.$$

Substituting into the DE gives

$$2A + 2At + B + At^2 + Bt + C = t^2$$

Equating coefficients of t^2 gives $A = 1$.
Equating coefficients of t gives $2A + B = 0$

$$\Rightarrow B = -2A = -2$$

Equating coefficients of t^0 (or setting $t=0$) gives $C = -B - 2A$
$= -2 + 2$
$= 0$

so that $Y = t^2 - 2t$
and noting our result in Q 3(b), the general solution is

$$y = c_1 e^{-\frac{t}{2}} \sin\left(\frac{\sqrt{3}}{2}t\right) + c_2 e^{-\frac{t}{2}} \cos\left(\frac{\sqrt{3}}{2}t\right) + t^2 - 2t .$$

10: $y_1 = t$ is a solution of $t^2 y'' + 2ty' - 2y = 0$.
So we substitute $y = v(t)\,t$ into the DE, seeking a second (independent) solution: then

$y' = v't + v$, $y'' = v''t + 2v'$, and the DE gives

$$t^2(v''t + 2v') + 2t(v't + v) - 2vt = 0$$

$$\Rightarrow v''t^3 + v'(4t^2) + v(2t - 2t) = 0$$

$$\Rightarrow v''t^3 + 4t^2 v' = 0$$ [which is first order in $w = v'$]

$$\Rightarrow w't + 4w = 0$$ provided $t \neq 0$

$$\Rightarrow \int \frac{dw}{w} = -4\int \frac{dt}{t}$$ (separating variables)

$$\Rightarrow \ln|w| = -4\ln|t| + c$$, c arbitrary

$$\Rightarrow w = Ae^{\ln(t^4)} = At^4$$, A arbitrary

$$\Rightarrow v = A\int t^{-4}\,dt = \frac{A\,t^{-3}}{-3} + c_2$$

$$\Rightarrow v = c_1 t^{-3} + c_2$$

$$\Rightarrow y = vt = c_1 t^{-2} + c_2 t.$$

We already know that $c_2 t$ is a solution.
Our second solution then is

$$\boxed{y_2 = t^{-2}}.$$

[We are not asked for it, but the general solution is

$$y = c_1 t^{-2} + c_2 t$$]