ENGR 222
Assignment 2 Submission

Daniel Eisen : 300447549

March 23, 2021

1. **Multivariate Function**

$$f(x,y) = -2x^3 + 3x^2y + 2y^3 - 9y + 5$$

(a)

$$f_x = -6x^2 + 6xy$$
$$f_y = 3x^2 + 6y^2 - 9$$

(b)

$$f_{xy} = 6x$$
$$f_{xx} = -12x + 6y$$
$$f_{yy} = 12y$$

(c)

$$\begin{aligned} f_x &= -6x^2 + 6xy = 0 \\ f_y &= 3x^2 + 6y^2 - 9 = 0 \\ &\text{by inspection } (x = y = 1, -1) \\ &for\ x = 0, \\ f_x &= 0 \\ f_y &= 6y^2 - 9 = 0 \\ \therefore\ y &= \sqrt{9/6} = \sqrt{\frac{3}{2}} \\ &for\ y = 0: \\ f_x &= -6x^2 = 0 \\ f_y &= 3x^2 - 9 = 0 \\ &\text{no x} \\ \text{critical points} &\Rightarrow [(1,1), (-1,-1), (0, \sqrt{\frac{3}{2}})] \end{aligned}$$

(d) Second Partials test:

$$D = f_{xx}(0, \sqrt{\tfrac{3}{2}}) \cdot f_{yy}(0, \sqrt{\tfrac{3}{2}}) - f_{xy}^2(0, \sqrt{\tfrac{3}{2}})$$

$$f_{xx} = -12x + 6y, f_{yy} = 12y, f_{xy} = 6x$$

$$\begin{aligned} D &= (-12(0) + 6\left(\sqrt{\tfrac{3}{2}}\right))(12\left(\sqrt{\tfrac{3}{2}}\right)) - (6(0))^2 \\ &= (0 + 3\sqrt{6})(6\sqrt{6}) - 0 \\ &= 108 \end{aligned}$$

$D > 0$ and $f_{xx} > 0$ therefore, this critical point is a local minimum.

2. **Quick questions**

(a) $f(x,y,z) = e^x cos(y)(1-z)^2,\ \mathbf{u} = (0.36, 0.48, 0.8)$

$$D_\mathbf{u} = f_x u_1 + f_y u_2 + f_z u_3$$

$$\begin{aligned} f_x &= e^x cos(y)(1-z)^2 \\ f_x(0,0,0) &= 1 \times 1 \times 1 = 1 \\ f_y &= -e^x sin(y)(1-z)^2 \\ f_y(0,0,0) &= -1 \times 0 \times 1 = 0 \\ f_z &= 2e^x cos(y)(z-1) \\ f_z(0,0,0) &= 2 \times 1 \times -1 = -2 \end{aligned}$$

$$D_\mathbf{u} = 1(0.36) + 0(0.48) + -2(0.8) = -1.24$$

(b) $f(x,y,z) = (1+x)(1-y^2)(1-z)^2,\ \mathbf{p} = (1,2,3)$

$$\begin{aligned} L(x,y,z) &= f(x_0,y_0,z_0) + f_x(x_0,y_0,z_0)(x-x_0) \\ &+ f_y(x_0,y_0,z_0)(y-y_0) \\ &+ f_z(x_0,y_0,z_0)(z-z_0) \end{aligned}$$

$$f(\mathbf{p}) = (1+1)(1-2^2)(1-3)^2 = -24$$

$$\begin{aligned} f_x &= (1-y^2)(1-z)^2 \\ f_x(\mathbf{p}) &= (1-2^2)(1-3)^2 = -12 \\ f_y &= (1+x)(-2y)(1-z)^2 \\ f_y(\mathbf{p}) &= (1+1)(-2(2))(1-3)^2 = -32 \\ f_z &= 2(1+x)(1-y^2)(z-1) \\ f_z(\mathbf{p}) &= 2(1+1)(1-2^2)(3-1) = -24 \end{aligned}$$

$$\begin{aligned} L(\mathbf{p}) &= -24 + (-12)(x-1) + (-32)(y-2) + (-24)(z-3) \\ &= 124 - 12x - 32y - 24z \end{aligned}$$

(c) $f(x,y) = e^{-x^2-y^2} = e^{-x^2}e^{-y^2},\ \mathbf{p} = (1,1)$

$$L(x,y) = f(\mathbf{p}) + f_x(\mathbf{p})(x-x_0) + f_y(\mathbf{p})(y-y_0)$$
$$p_2(x,y) = L(x,y) + \frac{1}{2}\left[(x-x_0)^2 f_{xx}(\mathbf{p}) + 2(x-x_0)(y-y_0)f_{xy}(\mathbf{p}) + (y-y_0)^2 f_{yy}(\mathbf{p})\right]$$

$$\begin{aligned} f_x &= -2xe^{-x^2}e^{-y^2} \\ &= -2xe^{-x^2-y^2} \\ f_y &= -2ye^{-x^2}e^{-y^2} \\ &= -2ye^{-x^2-y^2} \end{aligned}$$

$$\begin{aligned} f_{xx} &= e^{-y^2}(-2(e^{-x^2}) + -2x(-2xe^{-x^2})) \\ &= (4x^2-2)e^{-x^2-y^2} \\ f_{yy} &= (4y^2-2)e^{-x^2-y^2} \\ f_{xy} &= -2xe^{-x^2}(-2ye^{-y^2}) \\ &= 4xye^{-x^2-y^2} \end{aligned}$$

$$\begin{aligned} f(\mathbf{p}) &= e^{-1^2-1^2} = e^{-2} \\ f_x(\mathbf{p}) &= -2e^{-1^2-1^2} = -2e^{-2} \\ f_y(\mathbf{p}) &= -2e^{-1^2-1^2} = -2e^{-2} \\ f_{xx}(\mathbf{p}) &= (4(1^2)-2)e^{-1^2-1^2} = 2e^{-2} \\ f_{yy}(\mathbf{p}) &= (4(1^2)-2)e^{-1^2-1^2} = 2e^{-2} \\ f_{xy}(\mathbf{p}) &= 4e^{-1^2-1^2} = 4e^{-2} \end{aligned}$$

$$\begin{aligned} L(\mathbf{p}) &= e^{-2} + -2e^{-2}(x-1) + -2e^{-2}(y-1) = (5-2x-2y)e^{-2} \\ p_2(\mathbf{p}) &= (5-2x-2y)e^{-2} + \frac{1}{2}\left[(x-1)^2 2e^{-2} + (x-1)(y-1)8e^{-2} + (y-1)^2 2e^{-2}\right] \\ &= (5-2x-2y)e^{-2} + \left((x-1)^2 + 4(x-1)(y-1)e^{-2} + (y-1)^2\right)e^{-2} \\ &= \left(x^2 + y^2 + 4xy - 8x - 8y + 11\right)e^{-2} \end{aligned}$$

(d) $f(x,y) = x^3 + y^3 - 4x - 2y + 1,\ (x(t), y(t)) = (t^3 - 2t, t^2)$

$$\nabla f(x,y) = f_x(x,y)\mathbf{i} + f_y(x,y)\mathbf{j}$$

$$\begin{aligned} f_x &= 3x^2 - 4 \\ f_y &= 3y^2 - 2 \\ \nabla f(x,y) &= (3x^2-4)\mathbf{i} + (3y^2-2)\mathbf{j} \end{aligned}$$

$$\begin{aligned} (x(1), y(1)) &= (-1, 1) \\ \nabla f(-1,1) &= (3(-1^2) - 4) = -1\mathbf{i} + (3(1^2) - 2)\mathbf{j} \\ &= -\mathbf{i} + \mathbf{j} \end{aligned}$$

(e) $z = x^2 + xy - y^4,\ \mathbf{p} = (2,1)$

$$\begin{aligned} &\text{find z at}(2,1): \\ z &= 2^2 + 2 - 1 = 5 \\ F(x,y,z) &= z - x^2 - xy + y^4 = 0,\ \mathbf{p} = (2,1,5) \\ \nabla F(x,y,z) &= (-2x-y)\mathbf{i} + (4y^3 - x)\mathbf{j} + \mathbf{k} \\ \nabla F(\mathbf{p}) &= (-2(2)-1)\mathbf{i} + (4(1^3)-2)_\mathrm{j} + \mathbf{k} \\ &= -5\mathbf{i} + 2_\mathrm{j} + \mathbf{k} \\ \text{tangent plane} &= \nabla F(\mathbf{p}) \cdot (\mathbf{v} - \mathbf{p}) = 0\ :\ \mathbf{v} = (x,y,z) \\ &= -5(x-2) + 2(y-1) + (z-5) = 0 \\ &= -5x + 2y + z = -3 \text{ or } z = 5x - 2y - 3 \end{aligned}$$

3. **Double integrals**

(a) $e^{-x}cos(y)$

$$\begin{aligned} &\int_{\pi/2}^{-\pi/2}\int_0^2 e^{-x}cos(y)\,dx\,dy \\ &= \int_{\pi/2}^{-\pi/2} cos(y)\int_0^2 e^{-x}\,dx\,dy \\ &= \int_{\pi/2}^{-\pi/2} cos(y)\Big| - e^{-x}\Big|_{x=0}^{x=2} dy \\ &= \int_{\pi/2}^{-\pi/2} cos(y)\left(-e^{-2} - -e^{-0}\right) dy \\ &= \int_{\pi/2}^{-\pi/2} (1-e^{-2})cos(y)\,dy \\ &= (1-e^{-2})\Big|sin(y)\Big|_{y=\pi/2}^{y=-\pi/2} \\ &= (1-e^{-2})(sin(\pi/2) - sin(-\pi/2)) \\ &= 2(1-e^{-2}) = 2 - 2e^{-2} \end{aligned}$$

(b) $f(x,y) = sin(x+y),\ R: x,y \geq 0, x+y \leq \pi$

$$\begin{aligned} &\int_0^\pi\int_0^{\pi-y} sin(x+y)\,dx\,dy \\ &= \int_0^\pi \Big| - cos(x+y)\Big|_{x=0}^{x=\pi-y} dy \\ &= \int_0^\pi (-cos(\pi-y+y) + cos(0+y))\,dy \\ &= \int_0^\pi (-cos(\pi) + cos(y))\,dy \\ &= \int_0^\pi 1 + cos(y)\,dy = \Big|y + sin(y)\Big|_0^\pi = (\pi + sin(\pi) - 0 - sin(0)) \\ &= \pi \end{aligned}$$

(c)

$$\begin{aligned} |R| &= \int_0^5\int_{e^{y/3}}^{10+sin(y)} 1\,dx\,dy \\ &= \int_0^5\int_{e^{y/3}}^{10+sin(y)} 1\,dx\,dy \\ &= \int_0^5 \Big|x\Big|_{e^{y/3}}^{10+sin(y)} dy \\ &= \int_0^5 \Big|x\Big|_{e^{y/3}}^{10+sin(y)} dy \\ &= \int_0^5 10 + sin(y) - e^{y/3}\,dy \\ &= \Big|10y - cos(y) - 3e^{y/3}\Big|_0^5 dy \\ &= (50 - cos(5) - 3e^{5/3}) - (0 - cos(0) - 3e^0) \\ &\approx 37.83286.. \end{aligned}$$

(d) $f(x,y) = 3y - 2x,\ R = \{(x,y) : 0 \leq y \leq 4 - x^2, x \in [-2,2]\}$

$$\mu = \frac{1}{|R|}\int\int_R f(x,y)\,dA$$

$$\begin{aligned} |R| &= \int_{-2}^{2}\int_0^{4-x^2} 1\,dy\,dx \\ &= \int_{-2}^{2}\Big|y\Big|_{y=0}^{y=4-x^2} dx \\ &= \int_{-2}^{2} 4 - x^2\,dx \\ &= \Big|4x - \frac{x^3}{3}\Big|_{x=-2}^{x=2} \\ &= (4(2) - \frac{2^3}{3}) - (4(-2) - \frac{(-2)^3}{3}) = \frac{32}{3} \end{aligned}$$

$$\begin{aligned} &\int_{-2}^{2}\int_0^{4-x^2} 3y - 2x\,dy\,dx \\ &\int_{-2}^{2}\Big|\frac{3}{2}y^2 - 2xy\Big|_{y=0}^{y=4-x^2} dx \\ &\int_{-2}^{2}\frac{3}{2}(4-x^2)^2 - 2x(4-x^2)\,dx \\ &\int_{-2}^{2}\frac{3x^4}{2} + 2x^3 - 12x^2 - 8x + 24\,dx \\ &\Big|\frac{3x^5}{10} + \frac{2x^4}{4} - \frac{12x^3}{3} - \frac{8x^2}{2} + 24x\Big|_{x=-2}^{x=2} \\ &= 256/5 = 51.2 \end{aligned}$$

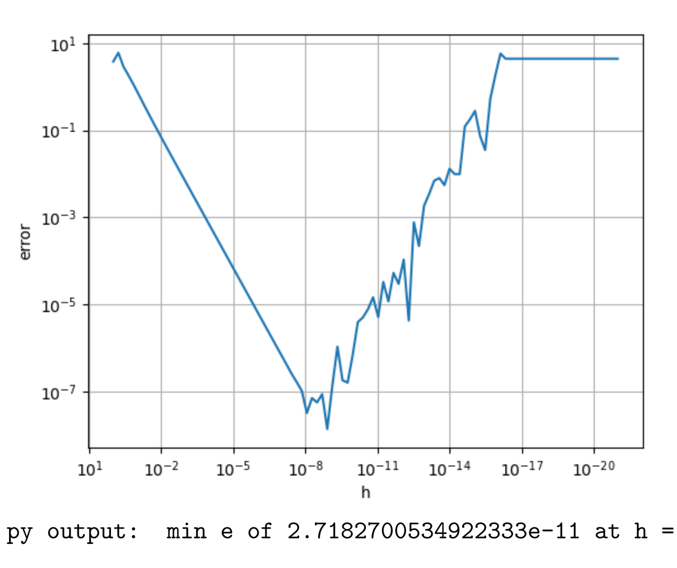
$$\mu = \frac{256/5}{32/3} = 4.8$$

(e) $z = \sqrt{9 - x^2},\ R = \{(x,y) : 0 \leq y \leq x, x \in [0,3]\}$

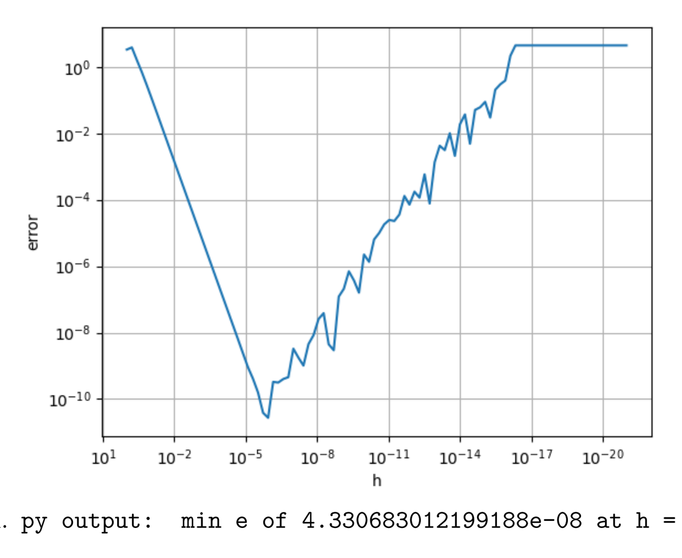
$$\begin{aligned} \text{surface area} &= \int\int_R \sqrt{(f_x)^2 + (f_y)^2 + 1}\,dA \\ &= \int_0^3\int_0^x \sqrt{\left(-\frac{x}{\sqrt{9-x^2}}\right)^2 + 0^2 + 1}\,dy\,dx \\ &= \int_0^3\int_0^x \frac{3}{\sqrt{9-x^2}}\,dy\,dx \\ &= \int_0^3 \Big|\frac{3y}{\sqrt{9-x^2}}\Big|_{y=0}^{y=x} dx \\ &= \int_0^3 \frac{3x}{\sqrt{9-x^2}}\,dx \\ &= \Big| - 3\sqrt{9-x^2}\Big|_{x=0}^{x=3} \\ &= (-3\sqrt{0}) - -3\sqrt{9} = 9 \end{aligned}$$

4. **Lab question**

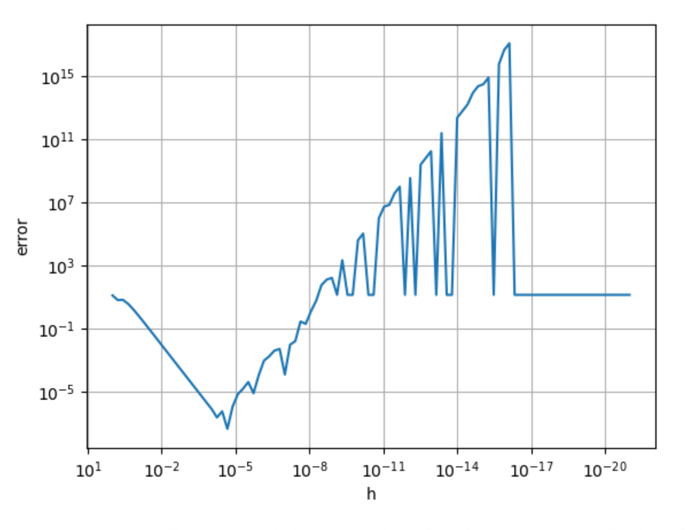
(a) i. py output: `min e of 1.358351831015625e-08 at h = 1.232846739442064e-09`



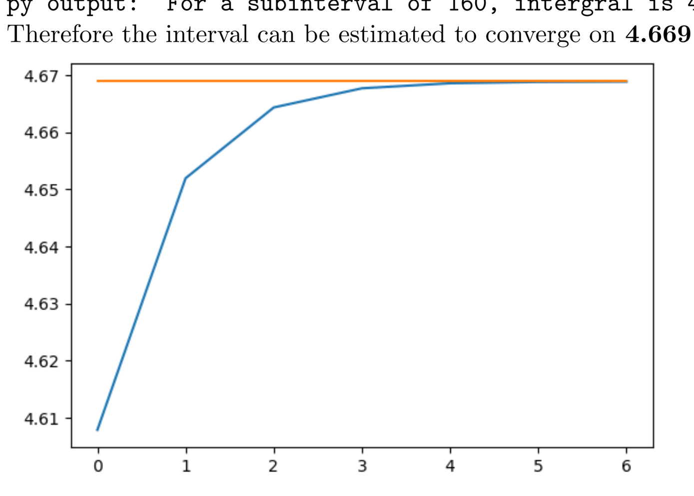
ii. py output: `min e of 2.7182700534922333e-11 at h = 1.1497569953977361e-06`



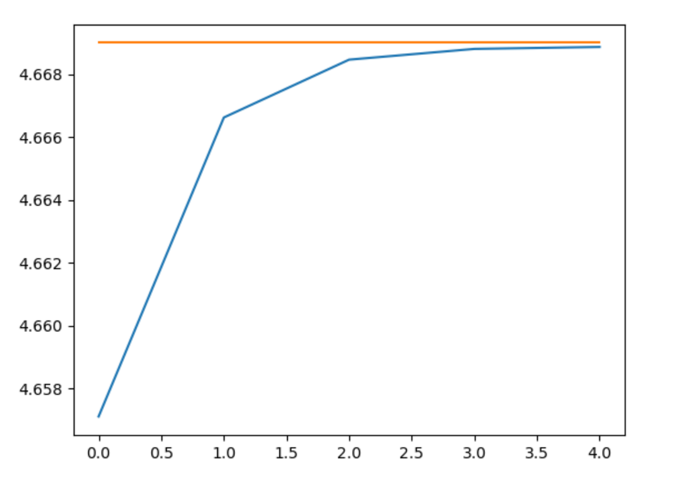
iii. py output: `min e of 4.330683012199188e-08 at h = 2.1544346900318827e-05`



(b) i.
```
py output:  For a subinterval of 10, intergral is 4.607872940794085
py output:  For a subinterval of 20, intergral is 4.651934278080257
py output:  For a subinterval of 40, intergral is 4.664330479310646
py output:  For a subinterval of 80, intergral is 4.667686103287532
py output:  For a subinterval of 160, intergral is 4.668571606382887
```
Therefore the interval can be estimated to converge on **4.669**



ii.
```
py output:  For a subinterval of 10, intergral is 4.657102287466295
py output:  For a subinterval of 20, intergral is 4.666621390508981
py output:  For a subinterval of 40, intergral is 4.668462546387442
py output:  For a subinterval of 80, intergral is 4.668804644613161
py output:  For a subinterval of 160, intergral is 4.668866774081337
```
Giving the speed of conversion, a comfortable approximation is 4dp after 5 iterations: **6.6689**



iii.
```
py output:  output at upperlimit:  10 = 4.668880328350932
py output:  output at upperlimit:  100 = 11.875967391881685
py output:  output at upperlimit:  1000 = 11.999999998713985
```
This appears to be converging on 12
```
py output:  To positive infinity:  12.000000000094914
```
With some floating point error, this confirms 12.

```python
import numpy as np
import matplotlib.pyplot as plt
from scipy.integrate import quad


def q4ai():
    f = lambda x:np.exp(np.cos(np.pi*x**2))
    x_0 = 1/(np.sqrt(2))
    h = 0.1**np.linspace(0, 21, 100)
    dfdx = (f(x_0+h)-f(x_0))/h #1st estimation

    error = np.abs(dfdx-(-np.sqrt(2)*np.pi))

    print(f'Min e of {np.min(error)} at h = {h[np.argmin(error)]}')
    plt.loglog(h,error)
    plt.gca().invert_xaxis()
    plt.xlabel('h')
    plt.ylabel('error')
    plt.grid()
    plt.show()


def q4aii():
    f = lambda x:np.exp(np.cos(np.pi*x**2))
    x_0 = 1/(np.sqrt(2))
    h = 0.1**np.linspace(0, 21, 100)
    dfdx = (f(x_0+h)-f(x_0-h))/(2*h) #2nd estimation

    error = np.abs(dfdx-(-np.sqrt(2)*np.pi))

    print(f'Min e of {np.min(error)} at h = {h[np.argmin(error)]}')
    plt.loglog(h,error)
    plt.gca().invert_xaxis()
    plt.xlabel('h')
    plt.ylabel('error')
    plt.grid()
    plt.show()

def q4aiii():
    f = lambda x:np.exp(np.cos(np.pi*x**2))
    x_0 = 1/(np.sqrt(2))
    h = 0.1**np.linspace(0, 21, 100)

    d2fdx2 = (f(x_0+h)-2*f(x_0)+f(x_0-h))/(h**2)

    error = np.abs(d2fdx2-(2*np.pi*(np.pi-1)))

    print(f'Min e of {np.min(error)} at h = {h[np.argmin(error)]}')
    plt.loglog(h,error)
    plt.gca().invert_xaxis()
    plt.xlabel('h')
    plt.ylabel('error')
    plt.grid()
    plt.show()

def q4bi():
    a = 0
    b = 10
    subs = [10,20,40,80,160,320,640]
    traps = []
    for n in subs:
        f = lambda x:x*np.exp(-np.sqrt(x))
        x = np.linspace(a,b,n+1)
        h = (b-a)/n
        y = f(x)
        trapezoidal_rule = (y[1:]+y[:-1]).sum()*h/2
        traps += [trapezoidal_rule]
        print(f"For a subinterval of {n}, intergral is {trapezoidal_rule}")
    plt.plot(traps)
    plt.plot(np.ones(len(traps))*4.669)
    plt.show()

def q4bii():
    a = 0
    b = 10
    subs = [10,20,40,80,160]
    simps = []
    for n in subs:
        f = lambda x:x*np.exp(-np.sqrt(x))
        x = np.linspace(a,b,n+1)
        h = (b-a)/n
        y = f(x)
        simpsons_rule = h/3*(y[0]+y[-1])+4*h/3*y[1::2].sum()+2*h/3*y[2:-1:2].sum()
        simps += [simpsons_rule]
        print(f"For a subinterval of {n}, intergral is {simpsons_rule}")
    plt.plot(simps)
    plt.plot(np.ones(len(simps))*4.669)
    plt.show()

def q4biii():
    f = lambda x:x*np.exp(-np.sqrt(x))
    upper_lims = [10,100,1000]
    ints = []
    for b in upper_lims:
        result = quad(f,0.0,b)
        print(f'output at upperlimit: {b} = {result[0]}')

    result = quad(f,0.0,np.inf)
    print(f"To positive infinity: {result[0]}")
```