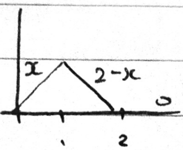
Math 2444 Ass 5 Daniel Eizen 300447749

1. $f(x) \begin{cases} & 0 \le x \le 1 \\ & 1 \le x \le 2 \\ & x > 2 \end{cases}$



$$\begin{aligned} f(x) &= x \cdot u(x) + ((2-x)-x)u(x-1) - (2-x)u(x-2) \\ &= xu(x) + (2-2x)u(x-1) - (2-x)u(x-2) \\ &= xu(x) - 2(x-1)u(x-1) + (x-2)u(x-2) \end{aligned}$$

$$\mathcal{L}fs = \mathcal{L}(x)u(x) - 2\mathcal{L}(x-1)u(x-1) + \mathcal{L}(x-2)u(x-2)$$

2nd Translation

$$\Rightarrow e^{0s}\mathcal{L}(x)(s) - 2e^{-s}\mathcal{L}(x)(s) + e^{-2s}\mathcal{L}(x)(s)$$

$$\mathcal{L}(x)(s) = \int_0^\infty xe^{-sx}\,dx \Rightarrow -\frac{xe^{-sx}}{s}\Big]_0^\infty + \frac{1}{s}\int_0^\infty e^{-sx}$$

$$= -\frac{xe^{-sx}}{s} - \frac{e^{-sx}}{s^2}\Big]_0^\infty = 0 - \left(0 - \frac{1}{s^2}\right)$$

$$= \frac{1}{s^2}$$

sub in: $\mathcal{L}f(x)(s) = \frac{1}{s^2} - 2e^{-s}\frac{1}{s^2} + e^{-2s}\frac{1}{s^2}$

$$= \frac{1}{s^2}\left(1 - 2e^{-s} + e^{-2s}\right)$$

2a) $\mathcal{L}(\cos ax)(s) = \int_0^\infty \cos ax \cdot e^{-sx}\,dx$ $\qquad u = \cos ax \quad u' = -a\sin ax$

$\qquad v' = e^{-sx} \quad v = -e^{-sx}/s$

IBP: $\left.\dfrac{-e^{-sx}}{s}\cos ax\right]_0^\infty - \dfrac{a}{s}\int_0^\infty \sin ax \cdot e^{-sx}$

$\Rightarrow \dfrac{1}{s} - \dfrac{a}{s}\left(\left.-\dfrac{\sin ax \cdot e^{-sx}}{s}\right]_0^\infty + \dfrac{a}{s}\int_0^\infty \cos ax\, e^{-sx}\right)$

$\dfrac{1}{s} + \dfrac{a}{s^2}\left(\left.\sin ax \cdot e^{-sx}\right]_0^\infty - a\,\mathcal{L}(\cos ax)(s)\right)$

$\dfrac{1}{s} + \dfrac{a}{s^2}\left(0 - a\,\mathcal{L}(\cos ax)(s)\right) = \dfrac{1}{s} - \dfrac{a^2}{s^2}\mathcal{L}(\cos ax)(s)$

$\mathcal{L}(f)(s) + \dfrac{a^2}{s^2}\mathcal{L}(f)(s) = \dfrac{1}{s}$, $\mathcal{L}(f)(s) = \dfrac{1}{s\left(1+\frac{a^2}{s^2}\right)} = \dfrac{1}{s+\frac{a^2}{s}} = \dfrac{s}{s^2+a^2}$ ✓

2b) $\mathcal{L}(\sin ax) = \dfrac{a}{s^2+a^2}$, $\mathcal{L}(f')(s) = s\mathcal{L}(f)(s) - f(0)$

$f' = \cos ax \qquad f = \dfrac{1}{a}\sin ax$

$$\mathcal{L}\cos ax = \frac{s}{a}\mathcal{L}\sin ax - \sin(0)$$

$$= \frac{s}{a}\cdot\frac{a}{s^2+a^2} - 0 = \frac{s}{s^2+a^2}$$ ✓

2c) $\cos(2x) = 2\cos^2(x) - 1$

$\dfrac{1}{2}\cos(2x) + \dfrac{1}{2} = \cos^2(x)$

$$\therefore \mathcal{L}(\cos^2 x) = \frac{1}{2}\mathcal{L}\{\cos(2x)\} + \mathcal{L}\left\{\frac{1}{2}\right\}$$

$$= \frac{1}{2}\cdot\frac{s}{s^2+4} + \frac{1}{2s} \Rightarrow \frac{s}{2s^2+8} + \frac{1}{2s}$$

3a) $\mathcal{L}^{-1}\left(\dfrac{2}{s} - \dfrac{1}{s^3}\right)^2$, $F(s) = \left(\dfrac{2}{s} - \dfrac{1}{s^3}\right)\left(\dfrac{2}{s} - \dfrac{1}{s^3}\right) = \dfrac{1}{s^6} - \dfrac{4}{s^4} + \dfrac{4}{s^2}$

$$= \mathcal{L}^{-1}\left(\frac{1}{s^6}\right) - \mathcal{L}^{-1}\left(\frac{4}{s^4}\right) + \mathcal{L}^{-1}\left(\frac{4}{s^2}\right)$$ use.. $x^n \Rightarrow \dfrac{n!}{s^{n+1}}$

$$\Rightarrow \frac{1}{5!}(x^5) - \frac{4}{3!}(x^3) + \frac{4}{1!}(x) \Rightarrow \mathcal{L}^{-1}F(s) = \frac{x^5}{120} - \frac{4x^3}{6} + 4x$$

3b) $\mathcal{L}^{-1}\left\{\frac{s+1}{s^2+2}\right\}$, $F(s) = \frac{s}{s^2+2} + \frac{1}{s^2+2}$

$$= \frac{s}{s^2+2} + \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{2}}{s^2+2}$$

$$\therefore \mathcal{L}^{-1}F(s) = \cos(x\sqrt{2}) + \frac{1}{\sqrt{2}}\sin(x\sqrt{2})$$

120 6

3c) $\mathcal{L}^{-1}\left(\frac{s}{(s+2)(s^2+4)}\right)$ $F(s) = \frac{A}{s+2} + \frac{Bs+C}{s^2+4}$

cover up ~~A~~: for A $\left.\frac{s}{s^2+4}\right|_{s=-2}$, $A = \frac{-1}{4}$

$s = -\frac{1}{4}(s^2+4) + (Bs+C)(s+2)$ $\quad s: 1 = 2B + C$

$s = -\frac{s^2}{4} - 1 + Bs^2 + 2Bs + Cs + 2c$ $\quad 1: 0 = \cancel{2B+C}$ $-1+2c$

$c = \frac{1}{2}$, $B = \frac{1}{4}$, $A = -\frac{1}{4}$

$-\frac{1}{4}\left(\frac{1}{s+2}\right) + \frac{1/4s + 1/2}{s^2+4} = \frac{1}{4}\left(\frac{s}{s^2+4} + \frac{2}{s^2+4}\right) - \frac{1}{4}\left(\frac{1}{s+2}\right)$

$\Rightarrow \frac{1}{4}\left(\cos 2x + \sin 2x - e^{-2x}\right)$ ✓

4. $y' + 6y = e^{4x}$, $y(0) = 2$: $Y = \mathcal{L}(y)(s)$

$sY - 2 + 6Y = \frac{1}{s-4}$, $Y(s+6) = \frac{1}{s-4} + 2$

$$Y = \frac{1}{(s-4)(s+6)} + \frac{2}{s+6}$$

$\frac{1}{(s-4)(s+6)} = \frac{A}{s-4} + \frac{B}{s+6}$, cover up: $A = \frac{1}{4+6} = \frac{1}{10}$

$B = \frac{1}{-6-4} = -\frac{1}{10}$

$$Y = \frac{1}{10(s-4)} - \frac{1}{10(s+6)} + \frac{2}{s+6}$$

$y = \mathcal{L}^{-1}(Y) = \frac{e^{4x}}{10} - \frac{e^{-6x}}{10} + 2e^{-6x} = y$ 3'

5. $\mathcal{L}(x^3 e^{2x})(s) = \mathcal{L}(x^3)(s-2) = \frac{6}{(s-2)^4}$

$y'' - 4y' + 4y = x^3 e^{2x}$, $y(0) = y'(0) = 0$: $Y = \mathcal{L}(y)(s)$

$s^2 Y - 4sY + 4Y = \frac{6}{(s-2)^4} \Rightarrow Y(s^2 - 4s + 4) = \frac{6}{(s-2)^4}$

$\Rightarrow Y = 6/(s^2-4s+4)(s-2)^4 = 6/(s-2)^2(s-2)^4 = 6/(s-2)^6$

$\mathcal{L}^{-1}(Y)$: $x^5 \Leftrightarrow \mathcal{L}^{-1}\left(\frac{120}{s^6}\right) \therefore \frac{1}{20}x^5 \Leftrightarrow \mathcal{L}^{-1}\left(\frac{6}{s^6}\right)$

so $\mathcal{L}^{-1}\left\{\frac{6}{(s-2)^6}\right\} \Rightarrow$ 1st Translation $\Rightarrow \frac{e^{2x}x^5}{20} = y$

6. $y''-y'=e^{-x}\cos x, \quad y(0)=y'(0)=\phi \quad : Y=\mathcal{L}(f)(s)$

$$s^2Y-sY=\mathcal{L}(e^{-x}\cos x)=\mathcal{L}(\cos x)(s+1)$$

$$Y(s^2-s)=\frac{s+1}{(s+1)^2+1}$$

$$Y=\frac{s+1}{(s^2+s)((s+1)^2+1)}=\frac{s+1}{s(s-1)((s+1)^2+1)}$$

$$\Rightarrow \frac{A}{s}+\frac{B}{s-1}+\frac{Cs+D}{s^2+2s+2}$$

$$s+1=As^3+As^2-2A+Bs^3+2Bs^2+2Bs+Cs^3-Cs^2+Ds^2-Ds$$
$$=As^3+Bs^3+Cs^3/+As^2+2Bs^2-Cs^2+Ds^2/+2Bs-Ds/-2A$$

$s^3: \quad 0=A+B+C$

$s^2: \quad 0=A+2B-C+D$

$s^1: \quad 1=2B-D$

$s^0: \quad 1=-2A \quad \therefore A=\frac{-1}{2}$

$\Rightarrow \frac{1}{2}=B+C$

$+\frac{1}{2}=2B-C+D$

$1=2B-D$

$\Rightarrow 1=3B+D \Rightarrow 2=5B$

$1=2B-D$

$B=\frac{2}{5}$

$C=\frac{1}{2}-\frac{2}{5}=\frac{1}{10}=C$

$D=2\left(\frac{2}{5}\right)-1=\frac{-1}{5}=D$

A	B	C	D
$-\frac{1}{2}$	$\frac{2}{5}$	$\frac{1}{10}$	$-\frac{1}{5}$

$$Y=-\frac{1}{2s}+\frac{2}{5(s-1)}+\frac{s-2}{10(s^2+2s+2)}$$

$$=\left(-\frac{1}{2}\cdot\frac{1}{s}\right)+\left(\frac{2}{5}\cdot\frac{1}{s-1}\right)+\left(\frac{1}{10}\cdot\frac{s-2}{(s+1)^2+1}\right)$$

$$Y_{(s)}=-\frac{1}{2}\frac{1}{s}+\frac{2}{5}\frac{1}{s-1}+\frac{1}{10}\frac{s+1}{(s+1)^2+1}-\frac{1}{10}\cdot\frac{3}{(s+1)^2+1}$$

$$y=-\frac{1}{2}+\frac{2e^x}{5}+\frac{1}{10}e^{-x}\cos x-\frac{3}{10}e^{-x}\sin x$$

$$y=\frac{2e^x}{5}+\frac{e^{-x}}{10}(\cos x-3\sin x)-\frac{1}{2}$$