

ENGR 222

Assignment 3 Submission

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1. Multiple Integrals

(a) $f(x, y, z) = xyz$

$$G = \{(x, y, z) : xy \leq z \leq 1, 0 \leq x \leq y, 0 \leq y \leq 1\}$$

$$\begin{aligned} \iiint_G f(x, y, z) dV &= \int_0^1 \int_0^y \int_{xy}^1 xyz \, dz \, dx \, dy \\ &= \int_0^1 \int_0^y xy \left| \frac{z^2}{2} \right|_{z=xy}^{z=1} dx \, dy \\ &= \int_0^1 \int_0^y xy \left(\frac{1}{2} - \frac{x^2 y^2}{2} \right) dx \, dy \\ &= \int_0^1 \int_0^y \frac{1}{2} (xy - x^3 y^3) dx \, dy \\ &= \int_0^1 \frac{1}{2} \left| \frac{x^2 y}{2} - \frac{x^4 y^3}{4} \right|_{x=0}^{x=y} dy \\ &= \int_0^1 \frac{1}{2} \left(\frac{y^3}{2} - \frac{y^7}{4} \right) dy = \int_0^1 \frac{y^3}{4} - \frac{y^7}{8} dy \\ &= \left| \frac{y^4}{16} - \frac{y^8}{64} \right|_{y=0}^{y=1} = \frac{1}{16} - \frac{1}{64} = \frac{3}{64} \end{aligned}$$

(b) Spherical Coordinates: $f(x, y, z) = x$

$$G = \{x, y, z \geq 0, x^2 + y^2 + z^2 \leq 1\}$$

To find the spherical region bounds, picture the region as an eight of the unit sphere in the all positive octant.

$$f(r, \theta, \phi) = r \cos(\theta) \sin(\phi)$$

$$G = \{(r, \theta, \phi) : r \in [0, 1], \theta \in [0, \pi/2], \phi \in [0, \pi/2]\}$$

$$\begin{aligned} \iiint_G f(r, \theta, \phi) dV &= \int_0^1 \int_0^{\pi/2} \int_0^{\pi/2} r \cos(\theta) \sin(\phi) d\phi d\theta dr \\ &= \int_0^1 r dr \int_0^{\pi/2} \cos(\theta) d\theta \int_0^{\pi/2} \sin(\phi) d\phi \\ &= \frac{r^2}{2} \Big|_{r=1}^{r=0} \times \sin(\theta) \Big|_{\theta=0}^{\theta=\pi/2} \times -\cos(\phi) \Big|_{\phi=0}^{\phi=\pi/2} \\ &= \frac{1}{2} \times 1 \times 1 = \frac{1}{2} \end{aligned}$$

(c) $f(x, y) = y^{-2} e^{-x}$

$$R = \{(x, y) : x \in [0 : \infty], y \in [2, \infty]\}$$

$$\begin{aligned} \iint_R f(x, y) dA &= \int_2^\infty \int_0^\infty y^{-2} e^{-x} dx dy \\ &= \int_2^\infty y^{-2} dy \int_0^\infty e^{-x} dx \\ &= \left[-\frac{1}{y} \right]_2^\infty \times [-e^{-x}]_0^\infty \\ &= \left(0 - -\frac{1}{2} \right) \times (0 - -1) = \frac{1}{2} \end{aligned}$$

(d) Centroid: $R = \{(r, \theta) : 0 \leq r \leq \theta, \theta \in [0, 2\pi]\}$

$$x_c = \frac{1}{|R|} \iint_R r^2 \cos(\theta) dr d\theta$$

$$y_c = \frac{1}{|R|} \iint_R r^2 \sin(\theta) dr d\theta$$

$$\begin{aligned} |R| &= \frac{1}{2} \int_0^{2\pi} \theta^2 d\theta \\ &= \left[\frac{\theta^3}{3} \right]_0^{2\pi} = \frac{4\pi^3}{3} \end{aligned}$$

$$x_c = \frac{3}{4\pi^3} \int_0^{2\pi} \int_0^\theta r^2 \cos(\theta) dr d\theta$$

$$= \frac{3}{4\pi^3} \int_0^{2\pi} \left[\frac{r^3}{3} \cos(\theta) \right]_0^\theta d\theta$$

$$= \frac{1}{4\pi^3} \int_0^{2\pi} \theta^3 \cos(\theta) d\theta$$

$$\text{from given : } = \frac{12\pi^2}{4\pi^3} = \frac{3}{\pi}$$

$$y_c = \frac{3}{4\pi^3} \int_0^{2\pi} \int_0^\theta r^2 \sin(\theta) dr d\theta$$

$$\dots$$

$$= \frac{1}{4\pi^3} \int_0^{2\pi} \theta^3 \sin(\theta) d\theta$$

$$\text{from given : } = \frac{12\pi - 8\pi^3}{4\pi^3} = \frac{3}{\pi^2} - 2$$

$$\text{centroid} = \left(\frac{3}{\pi}, \frac{3}{\pi^2} - 2 \right)$$

2. Vector Fields

(a) Divergence: $\mathbf{F} = x^2 y^3 z^4 \mathbf{i} - xyz \mathbf{j} + (x + y + z) \mathbf{k}$

$$\text{div } \mathbf{F} = f_x + g_y + h_z$$

$$= 2xy^3z^4 - yz + 1$$

(b) Curl: $\mathbf{F} = x^2 y^3 z^4 \mathbf{i} - xyz \mathbf{j} + (x + y + z) \mathbf{k}$

$$\text{curl } \mathbf{F} = (h_y - g_z) \mathbf{i} - (f_z - h_x) \mathbf{j} + (g_x - f_y) \mathbf{k}$$

$$= (1 + xy) \mathbf{i} - (4x^2 y^3 z^3 - 1) \mathbf{j} + (-yz - 3x^2 y^2 z^4) \mathbf{k}$$

(c) Gradient field: $\phi(x, y, z) = xz^2 + \sin(y)e^x$

$$\nabla \phi = \phi_x \mathbf{i} + \phi_y \mathbf{j} + \phi_z \mathbf{k}$$

$$= (z^2 + \sin(y)e^x) \mathbf{i} + (\cos(y)e^x) \mathbf{j} + (2xz) \mathbf{k}$$

(d) Laplacian: $\phi(x, y, z) = xz^2 + \sin(y)e^x$ (i.e $\nabla \cdot \nabla \phi$)

$$\nabla_\phi^2 = \phi_{xx} + \phi_{yy} + \phi_{zz}$$

$$= \sin(y)e^x - \sin(y)e^x + 2x$$

$$= 2x$$

3. Line Integrals

(a) Calculate $\int_C f \, ds$

$$f(x, y, z) = \frac{y}{x} e^z$$

$$C : (x, y, z) = (2t, t^2, \ln(t)) \text{ for } t \in [1, 4]$$

$$\begin{aligned} \int_C \frac{y}{x} e^z ds &= \int_1^4 \frac{t^2}{2t} e^{\ln(t)} \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2} dt \\ &= \int_1^4 \frac{t^3}{2t} \sqrt{(2)^2 + (2t)^2 + \left(\frac{1}{t}\right)^2} dt \\ &= \int_1^4 \frac{t^2}{2} \sqrt{4 + 4t^2 + \frac{1}{t^2}} dt \\ &= \int_1^4 \frac{t^2}{2} \sqrt{\frac{4t^4 + 4t^2 + 1}{t^2}} dt = \int_1^4 \frac{t^2}{2} \sqrt{\frac{(2t^2 + 1)^2}{t^2}} dt \\ &= \int_1^4 \frac{t^2}{2} \frac{2t^2 + 1}{t} dt = \int_1^4 \frac{2t^3 + t}{2} dt \\ &= \int_1^4 t^3 + \frac{t}{2} dt = \left. \frac{t^4}{4} + \frac{t^2}{2} \right|_1^4 \\ &= (4^4 + 4^2)/4 - (1^4 + 1^2)/4 = 67.5 \end{aligned}$$

(b) Calculate $\int_C \mathbf{F} \cdot d\mathbf{r}$

$$\mathbf{F}(x, y, z) = x\mathbf{i} - e^z \mathbf{j} + y\mathbf{k}$$

$$C : \mathbf{r}(t) = 2t\mathbf{i} + t^2 \mathbf{j} + \ln(t) \mathbf{k} \text{ for } t \in [1, 4]$$

$$\mathbf{r}'(t) = 2\mathbf{i} + 2t\mathbf{j} + \frac{1}{t} \mathbf{k}$$

$$\begin{aligned} \int_C \mathbf{F} \cdot d\mathbf{r} &= \int_1^4 (2ti - e^{\ln(t)} \mathbf{j} + t^2 \mathbf{k}) \cdot (2\mathbf{i} + 2t\mathbf{j} + \frac{1}{t} \mathbf{k}) dt \\ &= \int_1^4 (2(2t) - t(2t) + \frac{1}{t}(t^2)) dt = \int_1^4 (5t - 2t^2) dt \\ &= \left. \frac{5t^2}{2} - \frac{2t^3}{3} \right|_1^4 = (5(4^2)/2 - 2(4^3)/3) - (5(1^2)/2 - 2(1^3)/3) = -4.5 \end{aligned}$$

(c) Calculate $\int_C \mathbf{F} \cdot d\mathbf{r}$

$$\phi = \cos(x \sin(ye^z))$$

$$\mathbf{F} = \nabla \phi$$

$$\int_C \nabla \phi \cdot d\mathbf{r} = \phi(x_1, y_1, z_1) - \phi(x_0, y_0, z_0)$$

$$\mathbf{r}(t) = \pi \cos(\pi t/2) \mathbf{i} + \left(\frac{\pi}{2} + \sin(8\pi t) \right) \mathbf{j} + t(1-t) \mathbf{k} \text{ for } t \in [0, 1]$$

$$\mathbf{r}(0) = \pi \cos(0) \mathbf{i} + \left(\frac{\pi}{2} + \sin(0) \right) \mathbf{j} + 0 \mathbf{k}$$

$$= \pi \mathbf{i} + \frac{\pi}{2} \mathbf{j} + 0 \mathbf{k}$$

$$\mathbf{r}(1) = \pi \cos(\pi/2) \mathbf{i} + \left(\frac{\pi}{2} + \sin(8\pi) \right) \mathbf{j} + 0 \mathbf{k}$$

$$= 0 \mathbf{i} + \frac{\pi}{2} \mathbf{j} + 0 \mathbf{k}$$

$$\begin{aligned} \int_C \nabla \phi \cdot d\mathbf{r} &= \phi(0, \pi/2, 0) - \phi(\pi, \pi/2, 0) \\ &= \cos(0 \times \sin((\pi/2)e^0)) - \cos(\pi \times \sin((\pi/2)e^0)) \\ &= 1 - -1 = 2 \end{aligned}$$

(d)

$$\mathbf{F}(x, y) = -2xe^{-x^2} \sin(y) \mathbf{i} + (1 + e^{-x^2} \cos(y)) \mathbf{j}$$

$$f_y = -2xe^{-x^2} \cos(y)$$

$$g_x = -2xe^{-x^2} \cos(y)$$

\therefore conservative

$$\phi_x = f \therefore \phi = \int -2xe^{-x^2} \sin(y) dx = e^{-x^2} \sin(y) + k(y)$$

$$g = 1 + e^{-x^2} \cos(y)$$

$$\phi_y = e^{-x^2} \cos(y) + k'(y)$$

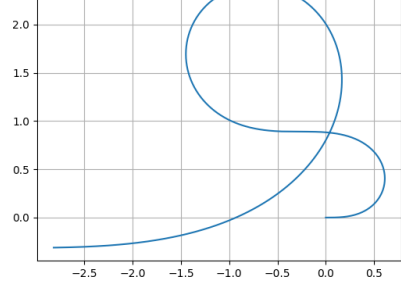
$$\therefore k'(y) = 1, k(y) = \int 1 dy = y + C$$

$$\phi = e^{-x^2} \sin(y) + y + C$$

For any closed curve C, the $\oint \mathbf{F} \cdot d\mathbf{r} = \iint_R (g_x - f_y) dA$, so as \mathbf{F} is conservative, thus the integrand is 0.

4. Lab Questions

(a) i. $(x(10), y(10)) = (-2.8179467555071627, -0.31125999010827476)$



ii. $h=0.0001$: $K(s=5) = 1.2001138830544655$

$h=0.001$: $K(s=5) = 1.200113871589581$

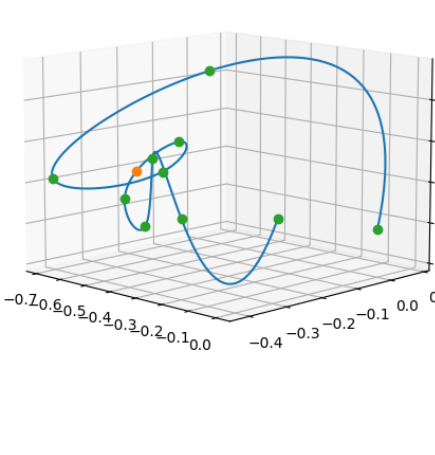
$h=0.003$: $K(s=5) = 1.200113789245757$

$h=0.005$: $K(s=5) = 1.2001135935941987$

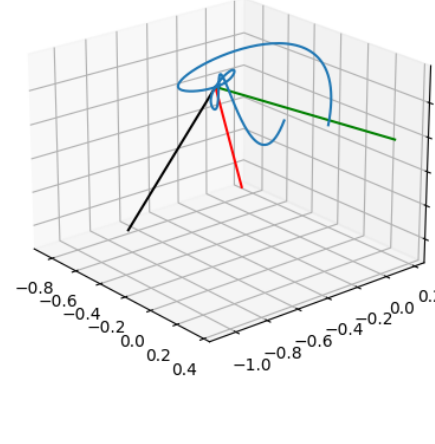
$h=0.01$: $K(s=5) = 1.200112748505328$

$$K(s=5) \approx 1.20011$$

(b) i. $t = 2$: $(x, y, z) = (-0.5580567444088435, -0.2720113135900402, 0.11995195426460033)$



ii. unit tangent, principal unit normal and binormal vectors at $t = 2$.



(c) i. result: 8906.117634354589, error: 6.960997495135904e-05

ii. result: 10.787064853079258, error: 1.1976047768013373e-13

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import numpy as np
from numpy.core.function_base import linspace
from scipy.integrate import cumulative_trapezoid, dblquad, tplquad, quad
import matplotlib.pyplot as plt
from scipy.interpolate import UnivariateSpline
from mpl_toolkits.mplot3d import Axes3D

def q4ai():
    s = linspace(0.0, 10.0, 1000)

    def f(u): return np.cos(np.pi*np.sin(np.log(1.0+u**2)))
    def g(u): return np.sin(np.pi*np.sin(np.log(1.0+u**2)))
    x = cumulative_trapezoid(f(s), s, initial=0)
    y = cumulative_trapezoid(g(s), s, initial=0)

    print(f"(x(10), y(10)) = ({x[-1]}, {y[-1]})")

    plt.plot(x, y)
    plt.grid()
    plt.gca().set_aspect(1.0)
    plt.show()

def q4aai():
    s = linspace(0.0, 10.0, 1000)
    def theta(s): return np.pi*np.sin(np.log(1.0+s**2))
    s_0 = 5

    for h in [1e-4, 1e-3, 3e-3, 5e-3, 10e-3]:
        theta_prime = (theta(s_0+h)-theta(s_0-h))/(2*h) # 2nd estimation

        print(f"h={h}: K(s=5) = {np.abs(theta_prime)}")

def q4bi():
    ti = [0.0, 0.6, 1.1, 1.5, 1.8, 2.1, 2.3, 2.5, 2.8, 3.2]
    xi = [0.0, -0.44, -0.69, -0.63, -0.62, -0.55, -0.63, -0.67, -0.44, -0.10]
    yi = [0.0, -0.15, -0.41, -0.15, -0.11, -0.31, -0.20, -0.15, -0.23, -0.21]
    zi = [0.0, 0.36, 0.11, 0.09, 0.16, 0.06, -0.04, 0.12, 0.01, 0.05]

    f = UnivariateSpline(ti, xi, s=0)
    g = UnivariateSpline(ti, yi, s=0)
    h = UnivariateSpline(ti, zi, s=0)
    t = np.linspace(ti[0], ti[-1], 200)

    print(f"t = 2: (x,y,z) = ({f(2)}, {g(2)}, {h(2)})")

    fig = plt.figure()
    ax = fig.add_subplot(111, projection='3d')
    ax.plot(f(t), g(t), h(t))
    ax.plot(f(2), g(2), h(2), 'o')
    ax.plot(xi, yi, zi, 'o')
    plt.show()

def q4bii():
    ti = [0.0, 0.6, 1.1, 1.5, 1.8, 2.1, 2.3, 2.5, 2.8, 3.2]
    xi = [0.0, -0.44, -0.69, -0.63, -0.62, -0.55, -0.63, -0.67, -0.44, -0.10]
    yi = [0.0, -0.15, -0.41, -0.15, -0.11, -0.31, -0.20, -0.15, -0.23, -0.21]
    zi = [0.0, 0.36, 0.11, 0.09, 0.16, 0.06, -0.04, 0.12, 0.01, 0.05]

    f = UnivariateSpline(ti, xi, s=0)
    g = UnivariateSpline(ti, yi, s=0)
    h = UnivariateSpline(ti, zi, s=0)

    t = np.linspace(ti[0], ti[-1], 200)

    def r(t): return np.array([f(t), g(t), h(t)]).T
    dfdt = f.derivative()
    dgdt = g.derivative()
    dhdt = h.derivative()
    def v(t): return np.array([dfdt(t), dgdt(t), dhdt(t)]).T
    d2fdt2 = dfdt.derivative()
    d2gdt2 = dgdt.derivative()
    d2hdt2 = dhdt.derivative()
    def a(t): return np.array([d2fdt2(t), d2gdt2(t), d2hdt2(t)]).T

    fig = plt.figure()
    ax = fig.add_subplot(111, projection='3d')
    rk = r(2)
    vk = v(2)
    ak = a(2)
    Tk = vk/np.linalg.norm(vk)
    Nk = ak*np.dot(vk, vk)-vk*np.dot(ak, vk)
    Nk /= np.linalg.norm(Nk)
    Bk = np.cross(Tk, Nk)
    ax.plot([rk[0], rk[0]+Tk[0]], [rk[1], rk[1]+Tk[1]],
            [rk[2], rk[2]+Tk[2]], 'k-') # plot a unit tangent
    ax.plot([rk[0], rk[0]+Nk[0]], [rk[1], rk[1]+Nk[1]],
            [rk[2], rk[2]+Nk[2]], 'r-') # plot the principal unit normal
    ax.plot([rk[0], rk[0]+Bk[0]], [rk[1], rk[1]+Bk[1]],
            [rk[2], rk[2]+Bk[2]], 'g-') # plot the binormal
    ax.plot(f(t), g(t), h(t)) # plot the smooth curve
    plt.show()

def q4ci():
    def f(y, x): return np.cos(x)*np.exp(y)
    def g1(x): return x**2
    def g2(x): return 10*np.sin(x)

    du_int = dblquad(f, -3, 3, g1, g2)
    print(f"result: {du_int[0]}, error: {du_int[1]}")

def q4cii():
    def f(x, y, z): return (4 / (1 + x**2 + y**2 + z**2))

    def F(r, t, p):
        x = r*np.cos(t)*np.sin(p)
        y = r*np.sin(t)*np.sin(p)
        z = r*np.cos(p)
        return f(x, y, z)*r**2*np.sin(p)

    tri_int = tplquad(F, 0, np.pi, 0, 2*np.pi, 0, 1)
    print(f"result: {tri_int[0]}, error: {tri_int[1]}")

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