TE WHARE WĀNANGA O TE ŪPOKO O TE IKA A MĀUI

VICTORIA

UNIVERSITY OF WELLINGTON

EXAMINATIONS – 2016

TRIMESTER 1

MATH 244

MODELLING WITH DIFFERENTIAL EQUATIONS

Time Allowed: THREE HOURS

CLOSED BOOK

Permitted materials: Silent non-programmable calculators or silent programmable calculators with their memories cleared are permitted.

Printed foreign languages to English dictionaries are permitted.

Formula sheets are attached to the end of this exam script.

No other material is permitted.

Instructions: The exam will be marked out of a total of 100 marks.

Answer in the appropriate boxes if possible — if you write your answer elsewhere, make it clear where your answer can be found.

Please use the blank reverse sides of pages for any extra space you need, for working or for answers. There is also a blank page between the questions and the formula sheets, for working or extra answer space.

If you answer all six questions, they will all be marked, and you will be credited with the five best marks.

For marking use only

1	/20
2	/20
3	/20
4	/20
5	/20
6	/20
best 5	/100

Student ID: ....

Question 1. [20 marks]

(a) Consider the differential equation

$$y'' - 6y' + 9y = 0\,.$$

(i) [4 marks] Find a fundamental set of solutions for this differential equation.

(ii) [4 marks] Show that the set you obtained in part (i) is a fundamental set, by calculating the value of the Wronskian at $x = 0$.

/ 8

(iii) [2 marks] Hence write down the general solution to this differential equation.

(iv) [2 marks] Hence find the solution to the initial value problem $y'' - 6y' + 9y = 0$, $y(0) = 0, y'(0) = 3$.

(b) Consider the homogeneous differential equation

$$y'' + p(x)y' + q(x)y = 0\,,$$

where $p(x)$ and $q(x)$ are continuous functions for all x in the open interval **I** which contains the point $x = 1$.

(i) [2 marks] Over what interval is it guaranteed that there exists a unique solution $y = \phi(x)$ to this differential equation?

(ii) [6 marks] Is it possible that $y = (x-1)^3$ is a solution for such a differential equation for a suitable choice of the (continuous) functions $p(x)$ and $q(x)$? Explain.

/ 12

Student ID: ....

Question 2. [20 marks]

(a) Consider the nonhomogeneous differential equation $y'' + y = \sin(t)$.

(i) [2 marks] Write down the associated homogeneous differential equation.

(ii) [4 marks] Find the general solution to the associated homogeneous differential equation.

(iii) [4 marks] Find a particular solution to the original nonhomogeneous differential equation.

(iv) [1 mark] Write down the general solution to the original nonhomogeneous differential equation.

/ 11

(b) Consider the differential equation $2t^2y'' + 3ty' - y = 0\,,\ \ t > 0.$

(i) [2 marks] Show that $y_1 = 1/t$ is a solution.

(ii) [6 marks] Use reduction of order to find a second solution y_2.

(iii) [1 mark] Hence write down the general solution.

/ 9

Student ID: ....

Question 3. [20 marks]

(a) [3 marks] Compute the Laplace transform of $f(x) = 1 - x$ by evaluating the necessary improper integral.

(b) [4 marks] Prove the Derivative Theorem, i.e. show that

$$\mathcal{L}\{f'(x)\} = s \cdot \mathcal{L}\{f(x)\} - f(0)$$

(c) [3 marks] Use the First Translation Theorem to find

$$\mathcal{L}\left\{e^{3x} \sin 2x\right\}$$

/ 10

(d) [10 marks] Use the Laplace transform to solve

$$y'' - y = e^{2x}$$

under initial conditions $y(0) = y'(0) = 0$.

/ 10

Student ID: ....

Question 4. [20 marks]

(a) [8 marks]

(i) [5 marks] Give the definition of the convolution of two functions and state the Convolution Theorem.

(ii) [3 marks] Use the Convolution theorem to compute

$$\mathcal{L}\{\cos(x) * (x^3 + 2x)\}$$

(b) [12 marks]

(i) [3 marks] Rewrite

$$f(x) = \begin{cases} 1 & 0 \leq x < 2 \\ 3 & 2 \leq x \end{cases}$$

using the unit step function and shifts.

/ 11

(ii) [3 marks] State the Second Translation Theorem.

(iii) [6 marks] Solve the differential equation

$$y' + 3y = f(x)$$

where $f(x)$ is as in (i) above and the initial condition is $y(0) = 0$.

/ 9

Question 5. [20 marks]

(a) [4 marks] Show that

$$\int_{-\pi}^{\pi} \sin(mx)\cos(nx)dx = 0$$

for all $m, n \in \{1, 2, \ldots\}$.

(b) [10 marks] Consider the function

$$f(x) = 2 - x \quad \text{on } [0, 1].$$

(i) [2 marks] Extend $f(x)$ to an *even* function with period 2

(ii) [2 marks] Sketch the graph of the function you obtain, on the interval $[-3, 3]$.

/ 8

(iii) [6 marks] Compute the Fourier series of the periodic function ($T = 2$) that you obtained in (i).

/ 6

(c) [6 marks] Find the general solution of the system

$$y'' + \omega^2 y = g(x),$$

where $g(x)$ is the even function with period 2 constructed in (b.i) above, and ω is not an integer multiple of 2π.

/ 6

Student ID: ....

Question 6. [20 marks]

(a) [4 marks] For each of the equations below, state whether the resulting sytem is (a) autonomous or not; (b) linear or not and (c) homogeneous or not.

(i) [2 marks]

$$\frac{d^2y}{dx^2} = 4y$$

(ii) [2 marks]

$$\frac{d^2x}{dt^2} = 4tx + 4t$$

(b) [4 marks] Write the system of differential equations

$$\begin{aligned} x' + x &= y \\ y' - x &= 4 \end{aligned}$$

in matrix form.

/ 8

(c) [12 marks]

(i) [6 marks] Find the general solution of the system of equations

$$\mathbf{x}' = \begin{pmatrix} -1 & 5 \\ 5 & -1 \end{pmatrix} \mathbf{x}$$

(ii) [4 marks] Define the Wronskian of a system of equations and compute the Wronskian in the example of part (i).

(iii) [2 marks] Write down the solution of the system satisfying

$$\mathbf{x}' = \begin{pmatrix} -1 & 5 \\ 5 & -1 \end{pmatrix} \mathbf{x} \quad \text{with initial conditions} \quad \mathbf{x}(1) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

End of Questions

/ 12

Use this spare page and the other side for rough working if needed.

Student ID: ....

Use this spare page and the other side for rough working if needed.

* * * * * * * * * * * * * * *

Differential Equations Formula Sheet

Trigonometric identities

- $\sin^2 x + \cos^2 x = 1$
- $\sin 2x = 2 \sin x \cos x$
- $\cos 2x = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$
- $\sec^2 x = 1 + \tan^2 x$
- $\csc^2 x = 1 + \cot^2 x$
- $\sin x \cos y = \dfrac{1}{2}\left[\sin(x-y) + \sin(x+y)\right]$
- $\sin x \sin y = \dfrac{1}{2}\left[\cos(x-y) - \cos(x+y)\right]$
- $\cos x \cos y = \dfrac{1}{2}\left[\cos(x-y) + \cos(x+y)\right]$
- if $t = \tan \dfrac{x}{2}$ then $\sin x = \dfrac{2t}{1+t^2}$, $\cos x = \dfrac{1-t^2}{1+t^2}$, and $\tan x = \dfrac{2t}{1-t^2}$

Hyperbolic identities

- $\cosh^2 x - \sinh^2 x = 1$
- $\sinh 2x = 2 \sinh x \cosh x$
- $\cosh 2x = \cosh^2 x + \sinh^2 x = 2\cosh^2 x - 1 = 2\sinh^2 x + 1$
- $\operatorname{sech}^2 x = 1 - \tanh^2 x$
- $\operatorname{csch}^2 x = \coth^2 x - 1$
- $\sinh x \cosh y = \dfrac{1}{2}\left[\sinh(x+y) + \sinh(x-y)\right]$
- $\sinh x \sinh y = \dfrac{1}{2}\left[\cosh(x+y) - \cosh(x-y)\right]$
- $\cosh x \cosh y = \dfrac{1}{2}\left[\cosh(x+y) + \cosh(x-y)\right]$

Derivatives of trigonometric and hyperbolic functions

$$\frac{d}{dx}[\sin x] = \cos x$$

$$\frac{d}{dx}[\cos x] = -\sin x$$

$$\frac{d}{dx}[\tan x] = \sec^2 x$$

$$\frac{d}{dx}[\cot x] = -\csc^2 x$$

$$\frac{d}{dx}[\sec x] = \sec x \ \tan x$$

$$\frac{d}{dx}[\csc x] = -\csc x \ \cot x$$

$$\frac{d}{dx}[\tan^{-1} x] = \frac{1}{1+x^2}$$

$$\frac{d}{dx}[\sinh x] = \cosh x$$

$$\frac{d}{dx}[\cosh x] = \sinh x$$

$$\frac{d}{dx}[\tanh x] = \mathrm{sech}^2 x$$

$$\frac{d}{dx}[\coth x] = -\mathrm{csch}^2 x$$

$$\frac{d}{dx}[\mathrm{sech} x] = -\mathrm{sech} x \ \tanh x$$

$$\frac{d}{dx}[\mathrm{csch} x] = -\mathrm{csch} x \ \coth x$$

Exponents and logarithms

- $y = a^x \quad \Leftrightarrow \quad x = \log_a y$ and $y = e^x \quad \Leftrightarrow \quad x = \ln y$
- $\dfrac{d}{dx}[a^x] = a^x \cdot \ln a$ and $\frac{d}{dx}[\log_a x] = \frac{1}{x \ln a}$

Matrices

Let $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$.

- $A^{-1} = \dfrac{1}{\det A} \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}$.
- Cramer's Rule: the solution of $A\mathbf{x} = \mathbf{b}$ is given by $x_1 = \dfrac{\det A_1}{\det A}$, $x_2 = \dfrac{\det A_2}{\det A}$

  where $A_1 = \begin{pmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{pmatrix}$, $A_2 = \begin{pmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{pmatrix}$.

Sums and series

- $$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$
- $$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$
- $$\sum_{k=1}^{n} k^3 = \left[\frac{n(n+1)}{2}\right]^2$$
- Geometric progression: $\displaystyle\sum_{k=0}^{n-1} r^k = 1 + r + r^2 + \cdots + r^{n-1} = \frac{1-r^n}{1-r}, \quad r \neq 1$

  Geometric series: $\displaystyle\sum_{k=0}^{\infty} r^k = \frac{1}{1-r}$ when $\quad |r| < 1$
- Maclaurin series: $\displaystyle\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k$
- Taylor series: $\displaystyle\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!} (x-a)^k$
- Fourier series for $f(x)$ on $[-\pi, \pi]$: $\displaystyle\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$

  where $\displaystyle a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx, \quad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx.$

Transforms

- Laplace transform: $\mathcal{L}\{f(x)\} = \int_0^\infty f(x)e^{-sx}\,dt = F(s)$

  $\mathcal{L}\{1\} = \frac{1}{s}$

  $\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}, \quad n$ a positive integer

  $\mathcal{L}\{e^{ax}\} = \frac{1}{s-a}$

  $\mathcal{L}\{\cos ax\} = \frac{s}{s^2+a^2}$

  $\mathcal{L}\{\sin ax\} = \frac{a}{s^2+a^2}$

  $\mathcal{L}\{f'(x)\} = sF(s) - f(0)$

  $\mathcal{L}\{f^{(n)}(x\} = s^nF(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - sf^{(n-2)}(0) - f^{(n-1)}(0)$

- Fourier transform: $\mathcal{F}\{f(x)\} = \int_{-\infty}^\infty f(x)e^{i\alpha x}\,dt = F(\alpha)$

  $\mathcal{F}\{f'(x)\} = -i\alpha F(\alpha)$

  Inverse Fourier transform: $\mathcal{F}^{-1}\{F(\alpha)\} = \frac{1}{2\pi}\int_{-\infty}^\infty F(\alpha)e^{-i\alpha x}\,d\alpha = f(x).$