# ENGR 222
## Assignment 3 Submission

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1. **Multiple Integrals**

   (a) $f(x,y,z) = xyz$
   $G = \{(x,y,z) : xy \le z \le 1,\ 0 \le x \le y,\ 0 \le y \le 1\}$

$$\begin{aligned}\iiint_G f(x,y,z)\,dV &= \int_0^1 \int_0^y \int_{xy}^1 xyz\,dzdxdy \\ &= \int_0^1 \int_0^y xy\Big|\frac{z^2}{2}\Big|_{z=xy}^{z=1} dxdy \\ &= \int_0^1 \int_0^y xy\left(\frac{1}{2} - \frac{x^2y^2}{2}\right) dxdy \\ &= \int_0^1 \int_0^y \frac{1}{2}(xy - x^3y^3)dxdy \\ &= \int_0^1 \frac{1}{2}\Big|\frac{x^2y}{2} - \frac{x^4y^3}{4}\Big|_{x=0}^{x=y} dy\end{aligned}$$

$$= \int_0^1 \frac{1}{2}\left(\frac{y^3}{2} - \frac{y^7}{4}\right) dy = \int_0^1 \frac{y^3}{4} - \frac{y^7}{8} dy$$

$$= \Big|\frac{y^4}{16} - \frac{y^8}{64}\Big|_{y=0}^{y=1} = \frac{1}{16} - \frac{1}{64} = \frac{3}{64}$$

   (b) Spherical Coordinates: $f(x,y,z) = x$
   $G = \{x, y, z \ge 0,\ x^2 + y^2 + z^2 \le 1\}$
   To find the spherical region bounds, picture the region as an eight of the unit sphere in the all positive octant.

$$f(r,\theta,\phi) = rcos(\theta)sin(\phi)$$

$$G = \{(r,\theta,\phi) : r \in [0,1], \theta \in [0,\pi/2], \phi \in [0,\pi/2]\}$$

$$\begin{aligned}\iiint_G f(r,\theta,\phi)\,dV &= \int_0^1 \int_0^{\pi/2} \int_0^{\pi/2} rcos(\theta)sin(\phi)\,d\phi d\theta dr \\ &= \int_0^1 r\,dr \int_0^{\pi/2} cos(\theta)\,d\theta \int_0^{\pi/2} sin(\phi)\,d\phi \\ &= \frac{r^2}{2}\Big|_{r=1}^{r=0} \times sin(\theta)\Big|_{\theta=0}^{\theta=\pi/2} \times -cos(\phi)\Big|_{\phi=0}^{\phi=\pi/2}\end{aligned}$$

$$= \frac{1}{2} \times 1 \times 1 = \frac{1}{2}$$

   (c) $f(x,y) = y^{-2}e^{-x}$
   $R = \{(x,y) : x \in [0:\infty], y \in [2,\infty]\}$

$$\begin{aligned}\iint_R f(x,y)\,dA &= \int_2^\infty \int_0^\infty y^{-2}e^{-x}\,dxdy \\ &= \int_2^\infty y^{-2}dy \int_0^\infty e^{-x}\,dx \\ &= \left[-\frac{1}{y}\right]_2^\infty \times \left[-e^{-x}\right]_0^\infty\end{aligned}$$

$$= \left(0 - -\frac{1}{2}\right) \times (0 - -1) = \frac{1}{2}$$

   (d) Centroid: $R = \{(r,\theta) : 0 \le r \le \theta, \theta \in [0, 2\pi]\}$

2. **Vector Fields**

   (a) Divergence: $\mathbf{F} = x^2y^3z^4\mathbf{i} - xyz\mathbf{j} + (x+y+z)\mathbf{k}$

$$\begin{aligned}\text{div}\,\mathbf{F} &= f_x + g_y + h_z \\ &= 2xy^3z^4 - yz + 1\end{aligned}$$

   (b) Curl: $\mathbf{F} = x^2y^3z^4\mathbf{i} - xyz\mathbf{j} + (x+y+z)\mathbf{k}$

$$\begin{aligned}\text{curl}\,\mathbf{F} &= (h_y - g_z)\mathbf{i} + (f_z - h_x)\mathbf{j} + (g_x - f_y)\mathbf{k} \\ &= (1 + xy)\mathbf{i} + (4x^2y^3z^3 - 1)\mathbf{j} + (-yz - 3x^2y^2z^4)\mathbf{k}\end{aligned}$$

   (c) Gradient field: $\phi(x,y,z) = xz^2 + sin(y)e^x$

$$\begin{aligned}\nabla_\phi &= \phi_x\mathbf{i} + \phi_y\mathbf{j} + \phi_z\mathbf{k} \\ &= \left(z^2 + sin(y)e^x\right)\mathbf{i} + (cos(y)e^x)\,\mathbf{j} + (2xz)\,\mathbf{k}\end{aligned}$$

   (d) Laplacian: $\phi(x,y,z) = xz^2 + sin(y)e^x$ (i.e $\nabla \cdot \nabla \phi$)

$$\begin{aligned}\nabla_\phi^2 &= \phi_{xx} + \phi_{yy} + \phi_{zz} \\ &= sin(y)e^x - sin(y)e^x + 2x \\ &= 2x\end{aligned}$$

3. **Line Integrals**

   (a) Calculate $\int_C f\ ds$

$$f(x,y,z) = \frac{y}{x}e^z$$

$$C : (x,y,z) = (2t,\ t^2,\ ln(t)) \text{ for } t \in [1,4]$$

$$\begin{aligned}\int_C \frac{y}{x}e^z\,ds &= \int_1^4 \frac{t^2}{2t}e^{ln(t)}\sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}\,dt \\ &= \int_1^4 \frac{t^3}{2t}\sqrt{(2)^2 + (2t)^2 + \left(\frac{1}{t}\right)^2}\,dt \\ &= \int_1^4 \frac{t^2}{2}\sqrt{4 + 4t^2 + \frac{1}{t^2}}\,dt\end{aligned}$$

$$= \int_1^4 \frac{t^2}{2}\sqrt{\frac{4t^4 + 4t^2 + 1}{t^2}}\,dt = \int_1^4 \frac{t^2}{2}\sqrt{\frac{(2t^2+1)^2}{t^2}}\,dt$$

$$= \int_1^4 \frac{t^2}{2}\frac{2t^2+1}{t} = \int_1^4 \frac{2t^3 + t}{2}\,dt$$

$$\begin{aligned}= \int_1^4 t^3 + \frac{t}{2}\,dt &= \frac{t^4 + t^2}{4}\Big|_1^4 \\ &= (4^4 + 4^2)/4 - (1^4 + 1^2)/4 = 67.5\end{aligned}$$

   (b) Calculate $\int_C \mathbf{F} \cdot\ d\mathbf{r}$

$$\mathbf{F}(x,y,z) = x\mathbf{i} - e^z\mathbf{j} + y\mathbf{k}$$

$$C : \mathbf{r}(t) = 2t\mathbf{i} + t^2\mathbf{j} + ln(t)\mathbf{k} \text{ for } t \in [1,4]$$

$$\mathbf{r}'(t) = 2\mathbf{i} + 2t\mathbf{j} + \frac{1}{t}\mathbf{k}$$

$$\begin{aligned}\int_C \mathbf{F} \cdot\ d\mathbf{r} &= \int_1^4 \left(2t\mathbf{i} - e^{ln(t)}\mathbf{j} + t^2\mathbf{k}\right) \cdot \left(2\mathbf{i} + 2t\mathbf{j} + \frac{1}{t}\mathbf{k}\right)\,dt \\ &= \int_1^4 2(2t) - t(2t) + \frac{1}{t}(t^2)\,dt = \int_1^4 5t - 2t^2\,dt \\ &= \frac{5t^2}{2} - \frac{2t^3}{3}\Big|_1^4 = (5(4^2)/2 - 2(4^3)/3) - (5(1^2)/2 - 2(1^3)/3) = -4.5\end{aligned}$$

   (c)

   (d)

4. **Lab Questions**

   (a) i.

   ii.

   (b) i.

   ii.

   (c) i.

   ii.

```python
import numpy as np
import matplotlib.pyplot as plt
from scipy.integrate import quad

def q4ai():
    pass

def q4aii():
    pass

def q4bi():
    pass

def q4bii():
    pass

def q4ci():
    pass

def q4cii():
    pass
```