ENGR 222
Assignment 3 Submission

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1. **Multiple Integrals**

(a) $f(x,y,z) = xyz$
$G = \{(x,y,z) : xy \le z \le 1,\ 0 \le x \le y,\ 0 \le y \le 1\}$

$$\begin{aligned}\iiint_G f(x,y,z)\,dV &= \int_0^1\int_0^y\int_{xy}^1 xyz\,dzdxdy \\ &= \int_0^1\int_0^y xy\Big|\frac{z^2}{2}\Big|_{z=xy}^{z=1} dxdy \\ &= \int_0^1\int_0^y xy\left(\frac{1}{2}-\frac{x^2y^2}{2}\right)dxdy \\ &= \int_0^1\int_0^y \frac{1}{2}(xy - x^3y^3)dxdy \\ &= \int_0^1 \frac{1}{2}\Big|\frac{x^2y}{2}-\frac{x^4y^3}{4}\Big|_{x=0}^{x=y} dy\end{aligned}$$
$$= \int_0^1 \frac{1}{2}\left(\frac{y^3}{2}-\frac{y^7}{4}\right)dy = \int_0^1 \frac{y^3}{4}-\frac{y^7}{8}dy$$
$$= \Big|\frac{y^4}{16}-\frac{y^8}{64}\Big|_{y=0}^{y=1} = \frac{1}{16}-\frac{1}{64} = \frac{3}{64}$$

(b) Spherical Coordinates: $f(x,y,z) = x$
$G = \{x,y,z \ge 0,\ x^2+y^2+z^2 \le 1\}$
To find the spherical region bounds, picture the region as an eight of the unit sphere in the all positive octant.

$$f(r,\theta,\phi) = rcos(\theta)sin(\phi)$$
$$G = \{(r,\theta,\phi) : r \in [0,1], \theta \in [0,\pi/2], \phi \in [0,\pi/2]\}$$

$$\begin{aligned}\iiint_G f(r,\theta,\phi)\,dV &= \int_0^1\int_0^{\pi/2}\int_0^{\pi/2} rcos(\theta)sin(\phi)\,d\phi d\theta dr \\ &= \int_0^1 r\,dr\int_0^{\pi/2} cos(\theta)\,d\theta\int_0^{\pi/2} sin(\phi)\,d\phi \\ &= \frac{r^2}{2}\Big|_{r=1}^{r=0} \times sin(\theta)\Big|_{\theta=0}^{\theta=\pi/2} \times -cos(\phi)\Big|_{\phi=0}^{\phi=\pi/2}\end{aligned}$$
$$= \frac{1}{2}\times 1 \times 1 = \frac{1}{2}$$

(c) $f(x,y) = y^{-2}e^{-x}$
$R = \{(x,y) : x \in [0:\infty], y \in [2,\infty]\}$

$$\begin{aligned}\iint_R f(x,y)\,dA &= \int_2^\infty\int_0^\infty y^{-2}e^{-x}\,dxdy \\ &= \int_2^\infty y^{-2}dy\int_0^\infty e^{-x}\,dx \\ &= \left[-\frac{1}{y}\right]_2^\infty \times \left[-e^{-x}\right]_0^\infty\end{aligned}$$
$$= \left(0 - -\frac{1}{2}\right)\times(0 - -1) = \frac{1}{2}$$

(d) Centroid: $R = \{(r,\theta) : 0 \le r \le \theta, \theta \in [0,2\pi]\}$

$$x_c = \frac{1}{|R|}\iint_R r^2cos(\theta)\,drd\theta$$
$$y_c = \frac{1}{|R|}\iint_R r^2sin(\theta)\,drd\theta$$

$$\begin{aligned}|R| &= \frac{1}{2}\int_0^{2\pi}\theta^2\,d\theta \\ &= \left[\frac{\theta^3}{3}\right]_0^{2\pi} = \frac{4\pi^3}{3}\end{aligned}$$

$$\begin{aligned}x_c &= \frac{3}{4\pi^3}\int_0^{2\pi}\int_0^\theta r^2cos(\theta)\,drd\theta \\ &= \frac{3}{4\pi^3}\int_0^{2\pi}\left[\frac{r^3}{3}cos(\theta)\right]_0^\theta d\theta \\ &= \frac{1}{4\pi^3}\int_0^{2\pi}\theta^3cos(\theta)d\theta \\ \text{from given} : &= \frac{12\pi^2}{4\pi^3} = \frac{3}{\pi} \\ y_c &= \frac{3}{4\pi^3}\int_0^{2\pi}\int_0^\theta r^2sin(\theta)\,drd\theta \\ &\cdots \\ &= \frac{1}{4\pi^3}\int_0^{2\pi}\theta^3sin(\theta)d\theta \\ \text{from given} : &= \frac{12\pi - 8\pi^3}{4\pi^3} = \frac{3}{\pi^2} - 2 \\ \text{centroid} &= \left(\frac{3}{\pi}, \frac{3}{\pi^2} - 2\right)\end{aligned}$$

2. **Vector Fields**

(a) Divergence: $\mathbf{F} = x^2y^3z^4\mathbf{i} - xyz\mathbf{j} + (x+y+z)\mathbf{k}$

$$\begin{aligned}\text{div }\mathbf{F} &= f_x + g_y + h_z \\ &= 2xy^3z^4 - yz + 1\end{aligned}$$

(b) Curl: $\mathbf{F} = x^2y^3z^4\mathbf{i} - xyz\mathbf{j} + (x+y+z)\mathbf{k}$

$$\begin{aligned}\text{curl }\mathbf{F} &= (h_y - g_z)\mathbf{i} - (f_z - h_x)\mathbf{j} + (g_x - f_y)\mathbf{k} \\ &= (1+xy)\mathbf{i} - (4x^2y^3z^3 - 1)\mathbf{j} + (-yz - 3x^2y^2z^4)\mathbf{k}\end{aligned}$$

(c) Gradient field: $\phi(x,y,z) = xz^2 + sin(y)e^x$

$$\begin{aligned}\nabla_\phi &= \phi_x\mathbf{i} + \phi_y\mathbf{j} + \phi_z\mathbf{k} \\ &= \left(z^2 + sin(y)e^x\right)\mathbf{i} + \left(cos(y)e^x\right)\mathbf{j} + (2xz)\,\mathbf{k}\end{aligned}$$

(d) Laplacian: $\phi(x,y,z) = xz^2 + sin(y)e^x$ (i.e $\nabla\cdot\nabla\phi$)

$$\begin{aligned}\nabla^2_\phi &= \phi_{xx} + \phi_{yy} + \phi_{zz} \\ &= sin(y)e^x - sin(y)e^x + 2x \\ &= 2x\end{aligned}$$

3. **Line Integrals**

(a) Calculate $\int_C f\,ds$

$$f(x,y,z) = \frac{y}{x}e^z$$
$$C : (x,y,z) = (2t,\ t^2,\ ln(t)) \text{ for } t \in [1,4]$$

$$\begin{aligned}\int_C \frac{y}{x}e^z\,ds &= \int_1^4 \frac{t^2}{2t}e^{ln(t)}\sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}\,dt \\ &= \int_1^4 \frac{t^3}{2t}\sqrt{(2)^2 + (2t)^2 + \left(\frac{1}{t}\right)^2}\,dt \\ &= \int_1^4 \frac{t^2}{2}\sqrt{4 + 4t^2 + \frac{1}{t^2}}\,dt\end{aligned}$$
$$= \int_1^4 \frac{t^2}{2}\sqrt{\frac{4t^4+4t^2+1}{t^2}}\,dt = \int_1^4 \frac{t^2}{2}\sqrt{\frac{(2t^2+1)^2}{t^2}}\,dt$$
$$= \int_1^4 \frac{t^2}{2}\frac{2t^2+1}{t} = \int_1^4 \frac{2t^3+t}{2}\,dt$$

$$\begin{aligned}= \int_1^4 t^3 + \frac{t}{2}\,dt &= \frac{t^4+t^2}{4}\Big|_1^4 \\ &= (4^4+4^2)/4 - (1^4+1^2)/4 = 67.5\end{aligned}$$

(b) Calculate $\int_C \mathbf{F}\cdot d\mathbf{r}$

$$\mathbf{F}(x,y,z) = x\mathbf{i} - e^z\mathbf{j} + y\mathbf{k}$$
$$C : \mathbf{r}(t) = 2t\mathbf{i} + t^2\mathbf{j} + ln(t)\mathbf{k} \text{ for } t \in [1,4]$$

$$\mathbf{r}'(t) = 2\mathbf{i} + 2t\mathbf{j} + \frac{1}{t}\mathbf{k}$$

$$\begin{aligned}\int_C \mathbf{F}\cdot\ d\mathbf{r} &= \int_1^4\left(2t\mathbf{i} - e^{ln(t)}\mathbf{j} + t^2\mathbf{k}\right)\cdot\left(2\mathbf{i} + 2t\mathbf{j} + \frac{1}{t}\mathbf{k}\right)dt \\ &= \int_1^4 2(2t) - t(2t) + \frac{1}{t}(t^2)\,dt = \int_1^4 5t - 2t^2\,dt \\ &= \frac{5t^2}{2} - \frac{2t^3}{3}\Big|_1^4 = (5(4^2)/2 - 2(4^3)/3) - (5(1^2)/2 - 2(1^3)/3) = -4.5\end{aligned}$$

(c) Calculate $\int_C \mathbf{F}\cdot d\mathbf{r}$

$$\phi = cos(xsin(ye^z))$$
$$\mathbf{F} = \nabla_\phi$$
$$\int_C \nabla_\phi\cdot\ d\mathbf{r} = \phi(x_1,y_1,z_1) - \phi(x_0,y_0,z_0)$$

$$\begin{aligned}\mathbf{r}(t) &= \pi cos(\pi t/2)\mathbf{i} + \left(\frac{\pi}{2} + sin(8\pi t)\right)\mathbf{j} + t(1-t)\mathbf{k} \text{ for } t \in [0,1] \\ \mathbf{r}(0) &= \pi cos(0)\mathbf{i} + \left(\frac{\pi}{2} + sin(0)\right)\mathbf{j} + 0\mathbf{k} \\ &= \pi\mathbf{i} + \frac{\pi}{2}\mathbf{j} + 0\mathbf{k} \\ \mathbf{r}(1) &= \pi cos(\pi/2)\mathbf{i} + \left(\frac{\pi}{2} + sin(8\pi)\right)\mathbf{j} + 0\mathbf{k} \\ &= 0\mathbf{i} + \frac{\pi}{2}\mathbf{j} + 0\mathbf{k}\end{aligned}$$

$$\begin{aligned}\int_C \nabla_\phi\cdot\ d\mathbf{r} &= \phi(0,\pi/2,0) - \phi(\pi,\pi/2,0) \\ &= cos(0\times sin((\pi/2)e^0)) - cos(\pi\times sin((\pi/2)e^0)) \\ &= 1 - -1 = 2\end{aligned}$$

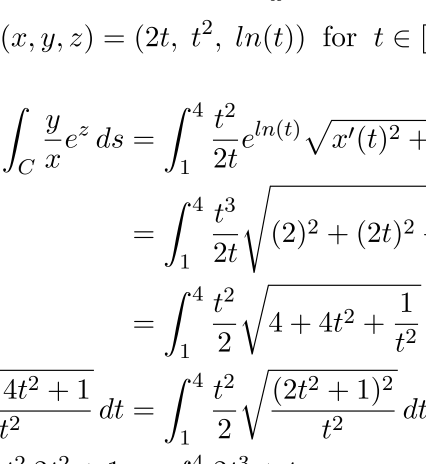
(d)

$$\begin{aligned}\mathbf{F}(x,y) &= -2xe^{-x^2}sin(y)\mathbf{i} + (1 + e^{-x^2}cos(y))\mathbf{j} \\ f_y &= -2xe^{-x^2}cos(y) \\ g_x &= -2xe^{-x^2}cos(y)\end{aligned}$$
$\therefore$ conservative
$$\phi_x = f \therefore \phi = \int -2xe^{-x^2}sin(y)dx = e^{-x^2}\sin(y) + k(y)$$

For any closed curve C, the $\oint \mathbf{F}\cdot d\mathbf{r} = \iint_R (g_x - f_y)dA$, so as $\mathbf{F}$ is conservative, thus the integrand is 0.

4. **Lab Questions**

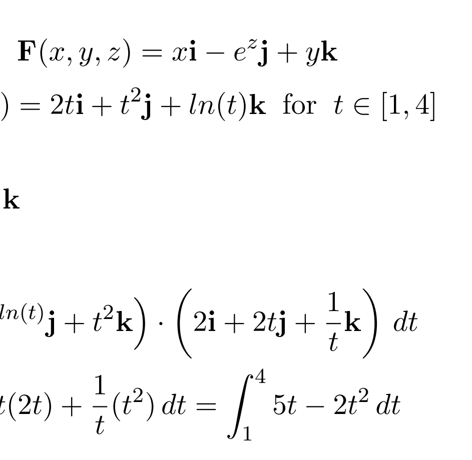
(a) i. `(x(10), y(10)) = (-2.8179467555071627, -0.31125999010827476)`



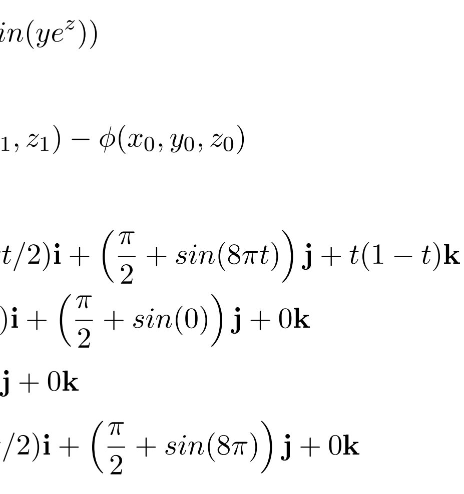
ii.
```
h=0.0001: K(s=5) = 1.2001138830544655
h=0.001: K(s=5) = 1.200113871589581
h=0.003: K(s=5) = 1.2001137789245757
h=0.005: K(s=5) = 1.2001135935941987
h=0.01: K(s=5) = 1.2001127248505328
```

$K(s=5) \approx 1.20011$

(b) i. `t = 2: (x,y,z) = (-0.5580567444088435, -0.2720113135900402, 0.11995195426460033)`



ii. unit tangent, principal unit normal and binormal vectors at $t = 2$.



(c) i. `result: 8906.117634354589, error: 6.960997495135904e-05`
ii. `result: 10.787064853079258, error: 1.1976047768013373e-13`

```python
import numpy as np
from numpy.core.function_base import linspace
from scipy.integrate import cumulative_trapezoid, dblquad, tplquad, quad
import matplotlib.pyplot as plt
from scipy.interpolate import UnivariateSpline
from mpl_toolkits.mplot3d import Axes3D


def q4ai():
    s = linspace(0.0, 10.0, 1000)

    def f(u): return np.cos(np.pi*np.sin(np.log(1.0+u**2)))
    def g(u): return np.sin(np.pi*np.sin(np.log(1.0+u**2)))
    x = cumulative_trapezoid(f(s), s, initial=0)
    y = cumulative_trapezoid(g(s), s, initial=0)

    print(f"(x(10), y(10)) = ({x[-1]}, {y[-1]})")

    plt.plot(x, y)
    plt.grid()
    plt.gca().set_aspect(1.0)
    plt.show()


def q4aii():
    s = linspace(0.0, 10.0, 1000)
    def theta(s): return np.pi*np.sin(np.log(1.0+s**2))
    s_0 = 5

    for h in [1e-4, 1e-3, 3e-3, 5e-3, 10e-3]:
        theta_prime = (theta(s_0+h)-theta(s_0-h))/(2*h)  # 2nd estimation

        print(f"h={h}: K(s=5) = {np.abs(theta_prime)}")


def q4bi():
    ti = [0.0, 0.6, 1.1, 1.5, 1.8, 2.1, 2.3, 2.5, 2.8, 3.2]
    xi = [0.0, -0.44, -0.69, -0.63, -0.62, -0.55, -0.63, -0.67, -0.44, -0.10]
    yi = [0.0, -0.15, -0.41, -0.15, -0.11, -0.31, -0.20, -0.15, -0.23, -0.21]
    zi = [0.0, 0.36, 0.11, 0.09, 0.16, 0.06, -0.04, 0.12, 0.01, 0.05]

    f = UnivariateSpline(ti, xi, s=0)
    g = UnivariateSpline(ti, yi, s=0)
    h = UnivariateSpline(ti, zi, s=0)
    t = np.linspace(ti[0], ti[-1], 200)

    print(f"t = 2: (x,y,z) = ({f(2)}, {g(2)}, {h(2)})")

    fig = plt.figure()
    ax = fig.add_subplot(111, projection='3d')
    ax.plot(f(t), g(t), h(t))
    ax.plot(f(2), g(2), h(2), 'o')
    ax.plot(xi, yi, zi, 'o')
    plt.show()


def q4bii():
    ti = [0.0, 0.6, 1.1, 1.5, 1.8, 2.1, 2.3, 2.5, 2.8, 3.2]
    xi = [0.0, -0.44, -0.69, -0.63, -0.62, -0.55, -0.63, -0.67, -0.44, -0.10]
    yi = [0.0, -0.15, -0.41, -0.15, -0.11, -0.31, -0.20, -0.15, -0.23, -0.21]
    zi = [0.0, 0.36, 0.11, 0.09, 0.16, 0.06, -0.04, 0.12, 0.01, 0.05]

    f = UnivariateSpline(ti, xi, s=0)
    g = UnivariateSpline(ti, yi, s=0)
    h = UnivariateSpline(ti, zi, s=0)

    t = np.linspace(ti[0], ti[-1], 200)

    def r(t): return np.array([f(t), g(t), h(t)]).T
    dfdt = f.derivative()
    dgdt = g.derivative()
    dhdt = h.derivative()
    def v(t): return np.array([dfdt(t), dgdt(t), dhdt(t)]).T
    d2fdt2 = dfdt.derivative()
    d2gdt2 = dgdt.derivative()
    d2hdt2 = dhdt.derivative()
    def a(t): return np.array([d2fdt2(t), d2gdt2(t), d2hdt2(t)]).T

    fig = plt.figure()
    ax = fig.add_subplot(111, projection='3d')
    rk = r(2)
    vk = v(2)
    ak = a(2)
    Tk = vk/np.linalg.norm(vk)
    Nk = ak*np.dot(vk, vk)-vk*np.dot(ak, vk)
    Nk /= np.linalg.norm(Nk)
    Bk = np.cross(Tk, Nk)
    ax.plot([rk[0], rk[0]+Tk[0]], [rk[1], rk[1]+Tk[1]],
            [rk[2], rk[2]+Tk[2]], 'k-')  # plot a unit tangent
    ax.plot([rk[0], rk[0]+Nk[0]], [rk[1], rk[1]+Nk[1]],
            [rk[2], rk[2]+Nk[2]], 'r-')  # plot the principal unit normal
    ax.plot([rk[0], rk[0]+Bk[0]], [rk[1], rk[1]+Bk[1]],
            [rk[2], rk[2]+Bk[2]], 'g-')  # plot the binormal
    ax.plot(f(t), g(t), h(t))  # plot the smooth curve
    plt.show()


def q4ci():
    def f(y, x): return np.cos(x)*np.exp(y)
    def g1(x): return x**2
    def g2(x): return 10+np.sin(x)

    du_int = dblquad(f, -3, 3, g1, g2)
    print(f"result: {du_int[0]}, error: {du_int[1]}")


def q4cii():
    def f(x, y, z): return (4 / (1 + x**2 + y**2 + z**2))

    def F(r, t, p):
        x = r*np.cos(t)*np.sin(p)
        y = r*np.sin(t)*np.sin(p)
        z = r*np.cos(p)
        return f(x, y, z)*r**2*np.sin(p)

    tri_int = tplquad(F, 0, np.pi, 0, 2*np.pi, 0, 1)
    print(f"result: {tri_int[0]}, error: {tri_int[1]}")
```