MATH244: Differential Equations—2019

Part 1: Introduction to DEs and Mathematical Modelling

Contents

1 Introduction

This is a tidier version of the lecture notes for the first half of MATH244. It is a supplement to the hand-written notes that I use to actually lecture from.

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1.1 What is a differential equation?

[Boyce & DiPrima, *Elementary Differential Equations with Boundary Value Problems* (10th edition) Chapter 1.]

Differential equations are one of the most widely used tools for forming a *mathematical model* of phenomena in the observed world: in physics, chemistry, biology, engineering, computer science, ecology, medicine, economics etc. They arise when there are variables that are observed or hypothesised to be related and the relation is between their values and their relative rates of change.

So suppose t is an independent variable and $y = y(t)$ a variable that depends upon it. The independent variable is often denoted t because it represents time, but it could also be in other units.

A **differential equation** (DE) is an equation involving:

$$x, t, \frac{dy}{dt}, \frac{d^2y}{d^2t}, \ldots.$$

and functions of them. A DE is an *equation between functions*—it is true for all values of the independent variable t for which $y(t)$ is defined. More complicated types of differential equations involve several independent or dependent variables and may involve partial derivatives. However in this course we do not study partial differential equations.

The **order** of a differential equation is the highest order of derivative that occurs.

Example 1.1. These are all differential equations:

- $\dfrac{dy}{dt} = y - y^2$
- $\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} - 3y = 0$
- $\dfrac{d^3y}{dt^3} - t\dfrac{dy}{dt} - (1-t^2)y = e^{-t}$
- $\dfrac{d^2y}{du^2} - t\left(\dfrac{dy}{du}\right)^3 - (1-u^2)y = e^{-u}$
- $y'''' - y = 1$

They have order $1, 2, 3, 2, 4$ respectively. Note that in the last equation we used Newton's derivative notation rather than Leibniz's.

Mathematically, the aim is to *solve* the equation—like solving a quadratic or other algebraic equation—to find the function or functions $y = y(t)$ for which it holds true. Any such function is a **solution** of the DE.

1.2 What is a mathematical model?

We have seen some examples of DEs, but why are we interested in them? In many physical situations we are able to identify relationships between measurable quantities that are precisely of this type.

Example 1.2 (Free falling body)**.** Consider a massive object falling through the air.

Figure 1: Falling body

Newton's second law of motion tells us that the acceleration of an object is proportional to the force acting on it. Explicitly

$$F = ma$$

where F is the force (in Newtons), m is the mass of the body (kgs) and a its acceleration (m/s^2). Recall that acceleration is rate of change of velocity so $a = dv/dt$. The forces that may affect a free body are

- gravity (acting downwards) equal to mg where g is a gravitational acceleration for Earth, roughly 9.8 m/s^2;
- drag or air resistance which acts upwards, which is roughly proportional to the velocity so γv, where γ is the *drag coefficient*.

What are the dimensions and units of γ? They have to make ensure that γv has the same dimensions/units as force.

If we take the positive direction for distance from an initial position as pointing downwards then

$$F = mg - \gamma v$$

so finally we have the differential equation

$$m\frac{dv}{dt} = mg - \gamma v. \tag{1.1}$$

Note that if the drag coefficient is dependent on the distance fallen (say, because the atmosphere becomes more dense) then $\gamma = \gamma(y)$ where y measures distance below the body's initial position. Since $v = dy/dt$ and therefore $a = d^2y/dt^2$ we would have a 2nd order DE:

$$m\frac{d^2y}{dt^2} = mg - \gamma(y)\frac{dy}{dt}.$$

Example 1.3 (The Verhulst Equation)**.** In population dynamics, populations that are relatively small in terms of resources are likely to grow proportional to the size of population. If there is a resource constraint then we can expect the rate of growth to diminish as the population starts to consume an increasing portion of its resources. Such a situation can be modelled by an equation of the form:

$$\frac{dy}{dt} = ky(1-y) \tag{1.2}$$

Here, t represents time in appropriate units and y the population (which could be measured in, for example, millions or for populations such as insets or microbes in "biomass", say grams or milligrams). Also k is a growth rate constant, assumed > 0.

The equation was first proposed by Pierre Verhulst, a French mathematician, who studied the demographic work of Thomas Malthus (1766–1834). It was published in his book (written in French) "Mathematical Researches into the Law of Population Growth Increase".

Example 1.4 (LR circuit)**.** Consider an electrical circuit that consists of a voltage source that creates an *electromotive force* $E(t)$ volts (V), together with two components—inductors and resistors—that affect the flow of electric power, the *current*, $i(t)$ amperes (A). These are called LR circuits.

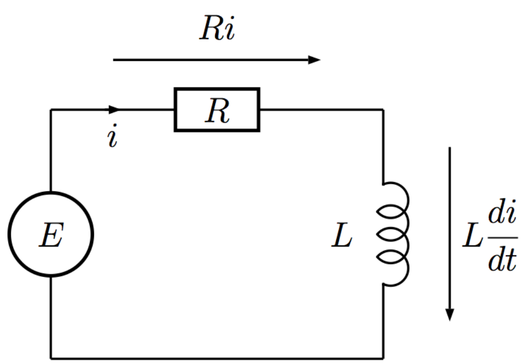


Figure 2: LR circuit

Kirchhoff's Second Law of circuits asserts that the input voltage equals the sum of the voltage drops elsewhere in the circuit. The voltage drops related to the components are as follows:

- inductor: $V = L\dfrac{di}{dt}$, L is the *inductance*, measured in henries (h)—resists change in current
- resistor: $V = Ri$, R is the *resistance*, measured in ohms (Ω)—impedes current

See Figure 2. This gives us the first-order linear DE:

$$L\frac{di}{dt} + Ri = E(t). \tag{1.3}$$

What do we do next? We have created a mathematical surrogate for the real system. We hope we can find solutions—functions that satisfy the differential equation and also some additional conditions that may describe initial or boundary conditions that are fixed.

- Set up the model to capture key features of the observed system, perhaps using some physical analysis or experimentation.
- Try to find solutions to the system. Note that we may need to understand about the mathematical properties of the objects and solution methods to achieve this. Look for underlying mathematical features.
- How well does the solution match observation of the real system being modelled? if it is good then we have a valuable model.
- Are there parameters in the system, e.g. drag coefficient, which for certain values lead to unwanted or desirable behaviour? If so, construct the system to achieve desired outcome.

2 Solutions of a DE

2.1 Solutions and IVPs

Example 2.1. Show that $y = e^{2t}$ is a solution of the first-order DE:

$$\frac{dy}{dt} - y = e^{2t} \tag{2.1}$$

In this case we just have to test that the function works. We get $\frac{dy}{dt} = 2e^{2t}$ so that

$$LHS = 2e^{2t} - e^{2t} = e^{2t} = RHS$$

and this is true for all real numbers t. But it is not the only solution: for any real number C check that $y = e^{2t} + Ce^t$ is a solution:

$$\frac{dy}{dt} - y = (2e^{2t} + Ce^t) - (e^{2t} + Ce^t) = e^{2t}$$

The family of solutions for $C = -3 \ldots 3$ looks like this: with $C = 3$ at the top. The family

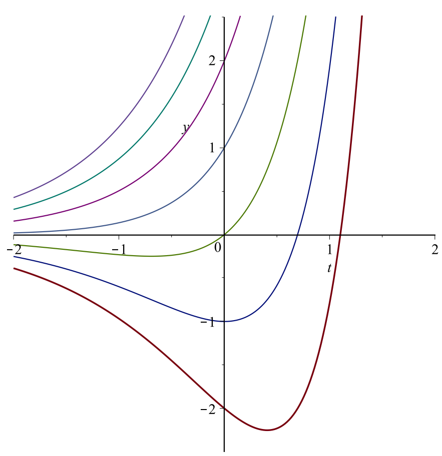


Figure 3: Family of solutions of the DE $\frac{dy}{dt} - y = e^{2t}$

$y = e^{2t} + Ce^t$ is called the **general solution** to the equation—that is, the family of all solutions. We will soon see that having a family of solutions like this is typical. In fact, the number of constants in the general solution is the same as the order of the equation.

A particular value for the $y(t)$ for a fixed value of t is called an **initial condition**. A problem that combines a DE with an initial condition is called an **initial value problem** or IVP. A well-posed IVP usually determines a unique choice of the constant.

Example 2.2. Find the solution of the IVP:

$$\frac{dy}{dt} - y = e^{2t}$$

subject to the initial condition $y(0) = 3$.

We substitute $t = 0$ in the general solution and so:

$$e^0 + Ce^0 = 1 + C = 3 \Rightarrow C = 2$$

so the solution is $y = e^{2t} + 2e^t$.

2.2 Direction fields

How can we relate these solutions to the DE itself? A first order DE of the form

$$\frac{dy}{dt} = f(t, y)$$

defines a **direction field**: it determines the slope of the unknown function $y(t)$ at every point (t, y) where the function f is defined.

Example 2.3. Sketch the **slope field** for the DE $\dfrac{dy}{dt} = e^{2t} + y$ (as in Example 2.1).

The DE tells us the slope of any solution $y = y(x)$ at a point with coordinates (x, y).

This was done in Maple using the command `DEplot` but we can do it by hand by drawing up a table and plotting the resulting grid.

2.3 Finding solutions

How are we to go about solving a DE, as opposed to simply checking a given function *is* a solution? The following simple example gives a clue.

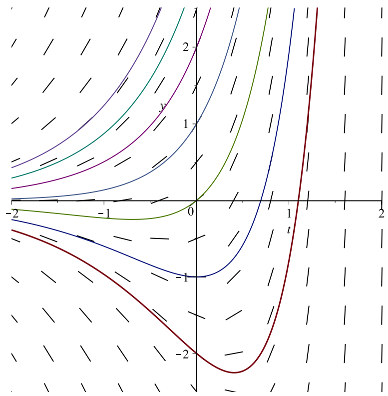


Figure 4: Slope field for the DE $\frac{dy}{dt} = e^{2t} + y$

Example 2.4. Solve the DE:

$$\frac{dy}{dt} = 3t^2$$

This simply says that y must be an anti-derivative of $3t^2$. To put it another way, we can solve the equation by integrating both sides of the equation. The Fundamental Theorem of Calculus tells us that $\int \frac{dy}{dt}\,dt = y$ (plus a constant) so:

$$y = t^3 + C$$

Again, we have a family of solutions and the constant is a constant of integration.

Now let's try something more interesting.

Example 2.5. Solve the DE: $\dfrac{dy}{dt} = 3y$.

Integrating both sides will not help here since we don't know what the integral of y on the right-hand side will be!

First of all note that there is a simple solution $y = 0$, since then $\dfrac{dy}{dt} = 0$ and the equation is trivially true. Now suppose $y \neq 0$, then we can divide through by y and obtain:

$$\frac{1}{y}\frac{dy}{dt} = 3$$

Integrating both sides with respect to t gives

$$\int \frac{y'}{y}\,dt = \int 3\,dt$$

and hence:

$$\ln|y| = 3t + C$$

Note that we could put a constant on both sides, but these could be combined to give a single constant. Taking exponentials:

$$|y| = e^{3t+C} = e^{3t}.e^{C} \Rightarrow y = \pm e^{C}.e^{3t}$$

Since e^C can take any positive value, $\pm e^C$ can be any non-zero number, which we denote by k, i.e. $k = \pm e^C$. Since $y = 0$ is already known to be a solution and corresponds to ke^{3t} with $k = 0$, the general solution is:

$$y = ke^{3t}, \quad k \in \mathbb{R}$$

3 First-Order Differential Equations: Separable equations

A differential equation is called **separable** if it can be written in the form

$$h(y)\frac{dy}{dt} = g(t) \tag{3.1}$$

Sometimes, the equation (3.1) is written in *differential* form as

$$h(y)\,dy = g(t)\,dt.$$

Integrating both sides of (3.1) with respect to t gives:

$$\int h(y)\frac{dy}{dt}\,dt = \int g(t)\,dt$$

and using the substitution $dy = \dfrac{dy}{dt}dt$ means that:

$$\int h(y)\,dy = \int g(t)\,dt$$

So if $H(y)$ is an antiderivative of $h(y)$ and $G(t)$ of $g(t)$ then the equation has solutions of the form

$$H(y) = G(t) + C$$

This is an *implicit* solution for y in terms of t. It may or may not be possible to find $y = f(t)$ explicitly and whether you can may also depend on C or the initial condition (i.c.) if given.

Example 3.1. Solve the DE: $\dfrac{dy}{dt} = \dfrac{t^3}{3y^2}$ with i.c. $y(0) = 2$.

Note that the equation is only defined for $y \neq 0$. Rearranging gives the separable equation (in differential form)

$$3y^2\,dy = t^3\,dt \Rightarrow \int 3y^2\,dy = \int t^3\,dt \Rightarrow y^3 = \frac{1}{4}t^4 + C$$

Substituting the initial condition $t = 0$, $y = 2$ into the general solution gives $C = 8$ so we have the particular solution

$$y^3 = \frac{1}{4}t^4 + 8 \quad \text{or} \quad y = \sqrt[3]{\frac{1}{4}t^4 + 8}$$

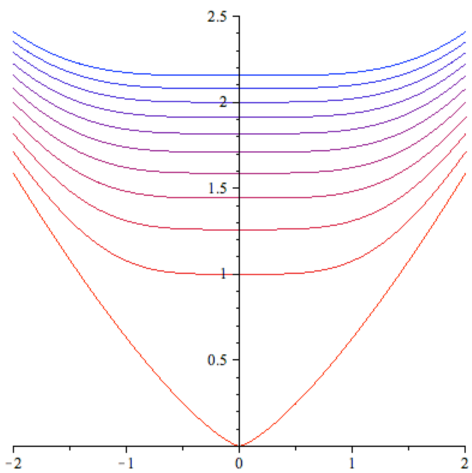


Figure 5: Family of solutions of the DE $\dfrac{dy}{dt} = \dfrac{t^3}{3y^2}$

Example 3.2. Solve the DE

$$(4y - 2\cos y)\frac{dy}{dt} = 3t^2$$

subject to the i.c. $y(1) = 0$.

The equation is separable and can be rearranged to give

$$\int (4y - 2\cos y)\,dy = \int 3t^2\,dt$$
$$\therefore \quad 2y^2 - 2\sin y = t^3 + C$$

Substituting the initial condition gives $0 = 1 + C$, i.e. $C = -1$. In this case we can't solve for y explicitly in terms of t but implicitly

$$2y^2 - 2\sin y = t^3 - 1$$

and this can be plotted using the Maple command `implicitplot`: Strictly, only the lower

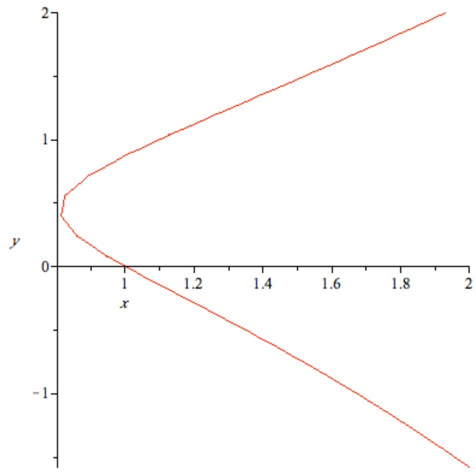


Figure 6: Implicit solution of the DE $(4y - 2\cos y)\dfrac{dy}{dt} = 3t^2$

branch of this curve, passing through the point $(t, y) = (1, 0)$ (the initial condition) is the explicit solution of the equation; the upper branch is a solution for a different initial condition.

4 The Verhulst Equation

Example 4.1. Show the Verhulst equation in Example 1.3 is separable and find its solution.

The DE can be written as

$$\int \frac{dy}{y(1-y)} = \int k\,dt$$

For this we require $y \neq 0, 1$. It is easy to see that in fact $y(t) = 0$ and $y(t) = 1$ are *constant* solutions to the DE. The left-hand side requires partial fractions. Writing

$$\frac{1}{y(1-y)} = \frac{A}{y} + \frac{B}{1-y}$$

gives $1 = A(1-y) + By$. Substituting $t = 0, 1$ gives $A = B = 1$ respectively. Hence:

$$\int \left(\frac{1}{y} + \frac{1}{1-y}\right) dy = \int k\,dt \Rightarrow \ln|y| - \ln|1-y| = kt + C$$

Using properties of logs and exponentials, this can be rewritten as:

$$\ln\left|\frac{y}{1-y}\right| = kt + C \iff \frac{y}{1-y} = de^{kt}$$

where $d = \pm e^C$, as in Example 2.5. This is a good stage to use the initial condition to solve for d:

$$d = \frac{y_0}{1-y_0}$$

and hence

$$\frac{y}{1-y} = \frac{y_0}{1-y_0} e^{kt}. \tag{4.1}$$

Clearing fractions and rearranging (check this for yourself. . .)

$$y(t) = \frac{y_0 e^{kt}}{(1-y_0) + y_0 e^{kt}} = \frac{y_0}{(1-y_0)e^{-kt} + y_0} \tag{4.2}$$

The last form is found by dividing top and bottom by e^{kt}. This is called a **logistic function** and the original DE is often called the **logistic equation**.

We can find out further information about the solution (4.2). First, note that since $k > 0$, $e^{-kt} \to 0$ as $t \to +\infty$ so:

$$\lim_{t\to+\infty} y(t) = \frac{y_0}{0 + y_0} = 1$$

Hence the solutions have a horizontal asymptote $y = 1$. This is independent of $y_0 > 0$. Less important as far as the application to populations is concerned is that, using $e^{kt} \to 0$ as $t \to -\infty$:

$$\lim_{t \to -\infty} y(t) = \frac{0}{(1 - y_0) + 0} = 0$$

Thus $y = 0$ is also a horizontal asymptote in the negative direction. It is clear from the original DE (1.2) that while $0 < y < 1$, $dy/dt > 0$ so that $y(t)$ is an increasing function. Finally we can see that it has an inflection point. This could be derived from the solution (4.2), but differentiating the DE (1.2) with respect to t and using implicit differentiation on the right-hand side $k(y - y^2)$ gives:

$$\frac{d^2y}{dt^2} = k(1 - 2y)\frac{dy}{dt}$$

It follows that $d^2y/dt^2 = 0$ when $y = 1/2$ and the second derivative changes sign so that there is an inflection point there. To find the corresponding value of t, we must solve (4.1) or one of its equivalent forms:

$$\frac{y_0}{1 - y_0}e^{kt} = \frac{\frac{1}{2}}{1 - \frac{1}{2}} = 1 \Rightarrow e^{kt} = \frac{1 - y_0}{y_0} \Rightarrow t = \frac{1}{k}\ln\left(\frac{1 - y_0}{y_0}\right)$$

Here is a plot of solutions for various values of y_0. The line $y = \frac{1}{2}$ is shown along which each solution with $0 < y_0 < 1$ has a point of inflection.

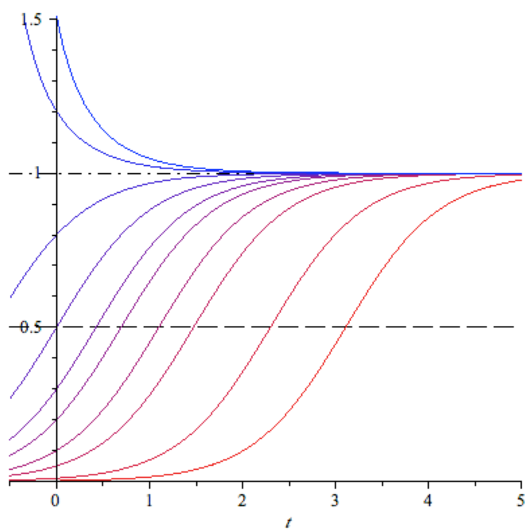


Figure 7: Solutions of the Verhulst equation

5 First-order Differential Equations: Linear equations

A first-order DE is **linear** if it can be written in the form

$$\frac{dy}{dt} + p(t)y = q(t) \tag{5.1}$$

So the dependent variable and its derivative appear on their own. The functions p and q may be anything, but if they have discontinuities then solutions $y(t)$ are only defined on intervals where both functions have no discontinuities.

To solve a linear first-order DE (5.1), we multiply both sides by a function $u(t)$, called an **integrating factor**:

$$u(t)\frac{dy}{dt} + u(t)p(t)y = u(t)q(t) \tag{5.2}$$

The idea is to make the left-hand side equal to

$$\frac{d}{dt}(u(t)y) = u(t)\frac{dy}{dt} + \frac{du}{dt}y \tag{5.3}$$

where we have just applied the Product Rule. Comparing (5.3) with the left-hand side of (5.2) we require

$$\frac{du}{dt} = up(t)$$

This is a differential equation for u and it is separable:

$$\int \frac{du}{u} = \int p(t)\,dt \Rightarrow \ln|u| = \int p(t)\,dt + C \Rightarrow u = ke^{\int p(t)\,dt}$$

We can choose any one of this family of functions, in particular $k = 1$. Then (5.2) can be written

$$\frac{d}{dt}(u(t)y) = u(t)q(t)$$

and this can be solved simply by integrating:

$$u(t)y = \int u(t)q(t)\,dt$$

and then dividing through by u to give y.

Example 5.1. Find an integrating factor for the linear 1st order DE

$$\frac{dy}{dt} + 3t^2y = 9t^2$$

and hence solve the equation.

In this equation $p(t) = 3t^2$ and $q(t) = 9t^2$. The integrating factor is

$$u(t) = e^{\int 3t^2\,dt} = e^{t^3}$$

Multiply the equation by u. We already know the LHS must become $\frac{d}{dt}(uy)$ so can write it like that straight away:

$$\frac{d}{dt}\left(e^{t^3} y\right) = 9t^2 e^{t^3}$$

Important note: The step above is <u>automatic</u> and follows from the way the integrating factor is calculated. You do not need to use the Product Rule!

Integrating, and substituting $u = t^3$, gives

$$e^{t^3} y = \int 9t^2 e^{t^3}\,dt = \int 3e^u\,du = 3e^u + C = 3e^{t^3} + C$$

Finally, divide by e^{t^3} (or multiply by e^{-t^3}) to get

$$y = 3 + Ce^{-t^3}.$$

Example 5.2. Recall the DE modelling an LR circuit:

$$L\frac{di}{dt} + Ri = E(t). \tag{5.4}$$

Find the current if $L = 1, R = 2$ and $E(t) = \sin t$, given $i(0) = 0$.

The equation is:

$$\frac{di}{dt} + 2i = \sin t,$$

so the integrating factor is $u = e^{\int 2\,dt} = e^{2t}$. Multiplying by u, the equation becomes:

$$\frac{d}{dt}\left(e^{2t} i(t)\right) = e^{2t}\sin t \quad \Rightarrow \quad e^{2t} i(t) = \int e^{2t}\sin t\,dt$$

The right-hand side requires integration by parts and we end up with:

$$e^{2t} i(t) = \frac{2}{5}e^{2t}\sin t - \frac{1}{5}e^{2t}\cos t + C$$

Multiply through by e^{-2t} to get:

$$i(t) = \frac{2}{5}\sin t - \frac{1}{5}\cos t + Ce^{-2t}$$

and the initial condition $i(0) = 0$ gives $C = \frac{1}{5}$ and hence:

$$i(t) = \frac{2}{5}\sin t - \frac{1}{5}\cos t + \frac{1}{5}e^{-2t}$$

whose graph is in Figure 8.

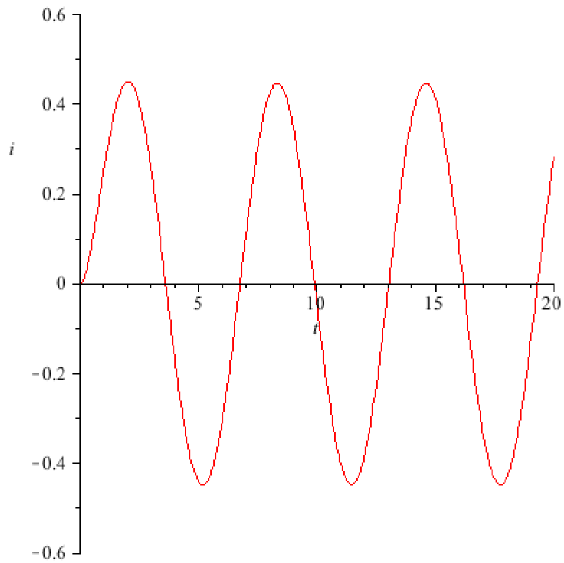


Figure 8: Graph of current in LR circuit

Example 5.3. Find an integrating factor for the linear 1st order DE

$$t^2y' - 2ty = 1, \quad t > 0$$

and hence solve the equation with i.c. $y(1) = 2$.

Note that the equation as written is not in the *standard form* (5.1). We must first divide through by t^2:

$$y' - \frac{2}{t}y = \frac{1}{t^2}$$

Since the coefficient functions now have a discontinuity at $t = 0$ any solution we find will only be valid for either $t > 0$ or $t < 0$. Since the i.c. has $t = 1 > 0$ we look for solutions for $t > 0$.

The integrating factor is

$$u(t) = e^{\int(-2/t)\,dt} = e^{-2\ln t} = e^{\ln(t^{-2})} = \frac{1}{t^2}$$

Don't overlook the minus sign in $p(t) = -2/t$. Multiplying the DE by u gives

$$\frac{d}{dt}\left(\frac{1}{t^2}y\right) = \frac{1}{t^4} \Rightarrow \frac{y}{t^2} = \int \frac{dt}{t^4} = -\frac{1}{3t^3} + C \Rightarrow y = -\frac{1}{3t} + Ct^2.$$

The initial condition gives

$$2 = -\frac{1}{3} + C \Rightarrow C = \frac{7}{3}$$

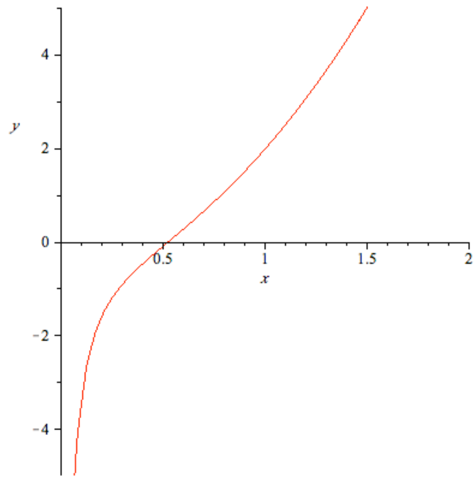


Figure 9: Solution of the linear DE $t^2y' - 2ty = 1$

and the solution is graphed in Figure 9.

Sometimes we may not be able to explicitly calculate one of the integrals.

Example 5.4. Solve the linear equation

$$\frac{dy}{dt} - 2ty = t^2$$

An integrating factor is $u = e^{\int 2t\,dt} = e^{-t^2}$. So after multiplying by u, the equation becomes

$$\frac{d}{dt}(e^{-t^2}y) = t^2e^{-t^2}$$

If you try and integrate the right hand side, nothing seems to work. Maple gives an odd answer:

$$\int t^2e^{-t^2}\,dt = -\frac{1}{2}te^{-t^2} + \frac{\sqrt{\pi}}{4}\mathrm{erf}(t) + C$$

So the best we can do is

$$y = -\frac{1}{2}t + \frac{\sqrt{\pi}}{4}\mathrm{erf}(t)e^{t^2} + Ce^{t^2}$$

In fact, "erf" is really just the name of an integral for which there is not a simple formula.

How can we know if a DE has a solution at all? There are some basic theorems about this.

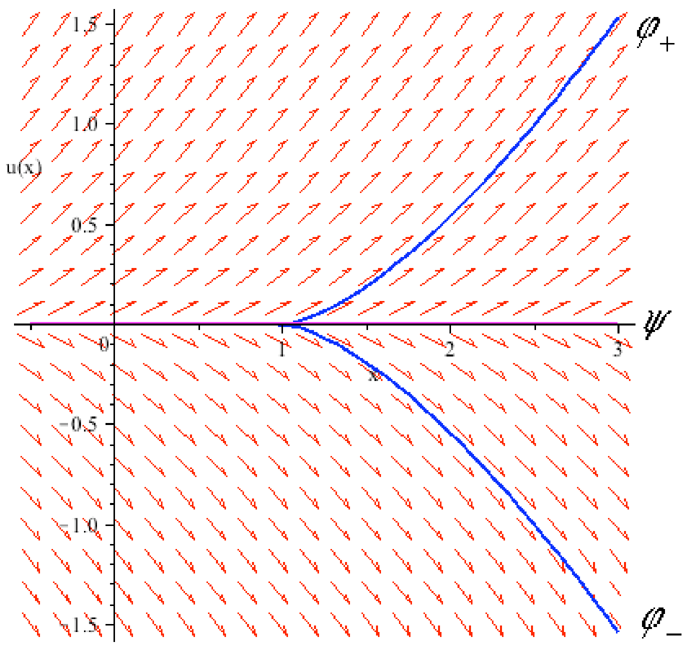
Theorem 5.5 (Existence and Uniqueness of Solutions)**.** *Let $y' + p(t)y = q(t)$ be a 1st order linear equation with p and q continuous functions on some interval $I : \alpha < t < \beta$. If $t_0 \in I$ and $y_0 \in \mathbb{R}$ then there exists a unique solution $y = y(t)$ for $t \in I$ satisfying the IVP $y(t_0) = y_0$.*

Example 5.6. Solve the IVP $y' = y^{1/3}$, $y(1) = 0$.

We get $\frac{3}{2}y^{2/3} = t + c$ while the initial condition gives $c = -1$. Hence

$$y = \pm \left[\frac{2}{3}(t-1)\right]^{3/2}$$

and it is easy to check that both choices of sign give solutions to the IVP, that are only defined for $t \geq 1$. Moreover $y \equiv 0$ is another solution, defined for all $t \in \mathbb{R}$. Why does the fact that there are 3 solutions with the same ic not contradict the theorem?



The general structure of the set of solutions is described in the following theorem. This is a model for higher order equations.

A first-order linear equation (5.1) is called **homogeneous** if $q = 0$.

Theorem 5.7. *If $y' + p(t)y = 0$ is a homogeneous linear 1st order DE and $y = y_1(t)$ is a particular (non-zero) solution (PS), then every solution of the equation has the form $y = ky_1(t)$ for some $k \in \mathbb{R}$. This is called the* **general solution** *(GS) of the equation.*

If $y' + p(t)y = q(t)$ is a non-homogeneous linear 1st order equation with PS $y = y_p(t)$, then its GS is $y = y_p(t) + c_1y_1(t)$ where $cy_1(t)$ is the GS of the associated homogeneous equation (above).

6 Exact Equations

6.1 Exactness

Now let's look at some non-linear equations. We may be able to use separation of variables to solve, but generally not. Here is another method that works sometimes for equations of the form

$$M(x, y) + N(x, y)y' = 0 \quad \text{or equivalently} \quad M\,dx + N\,dy = 0$$

If there is a function of 2 variables $f(x, y)$ which has continuous second-order partial derivatives and $M = f_x$, $N = f_y$, then the equation can be rewritten as

$$\frac{d}{dx}[f(x, y)] = 0$$

thinking of f as implicitly defining y in terms of x. Such an equation is called **exact**. It follows that

$$f(x, y) = c$$

How do we know if an equation is exact? There is a simple test:

Theorem 6.1. *Given functions M, N with continuous first order partial derivatives on a rectangle $a < x < b$, $c < y < d$, the equation $M(x, y) + N(x, y)y' = 0$ is exact if and only if $M_y = N_x$ on the rectangle.*

Idea of proof: If the equation is exact, so there is a function $f(x, y)$ (with continuous 2nd order partial derivatives) for which $M = f_x$ and $N = f_y$ then a theorem of calculus says that $M_y = f_{xy} = f_{yx} = N_x$. On the other hand, suppose $M_y = N_x$. Fix x_0 and do *partial integration* w.r.t. x only (treat y as fixed in the integral) to get:

$$f(x, y) = \int_{x_0}^{x} M(t, y)\,dt + \phi(y)$$

Note the constant of integration may depend on y. Then

$$\begin{aligned} f_y &= \frac{\partial}{\partial y} \int M(x, y)\,dx + \phi_y \\ &= \int M_y(x, y)\,dx + \phi_y \\ &= \int N_x(x, y)\,dx + \phi_y \\ &= N(x, y) - N(x_0, y) + \phi_y \qquad \text{by Fund}^{\ell} \text{ Thm of Calc} \end{aligned}$$

Choose ϕ so that $\phi_y = N(x_0, y)$, that is, for some y_0

$$\phi(y) = \int_{y_0}^{y} N(x_0, u)\, du + C$$

then clearly $f_y = N$ as required. In fact, the proof tells us how to find the correct function f and hence to solve the DE.

Example 6.2. Solve the IVP

$$2xy^3 + y + (3x^2y^2 + x + 1)\frac{dy}{dx} = 0$$

where $y(0) = 1$.

We show the equation is exact. Let $M(x, y) = 2xy^3 + y$ and $N(x, y) = 3x^2y^2 + x + 1$. Then

$$M_y = 6xy^2 = N_x$$

Now, following the idea in the proof, we let

$$f(x, y) = \int 2xy^3 + y\, dx = x^2y^3 + xy + \phi(y)$$

then

$$f_y = 3x^2y^2 + x + \phi_y = N \iff \phi_y = 1 \iff \phi = y + C$$

Hence

$$f(x, y) = x^2y^3 + xy + y + C$$

and since $y = 1$ when $x = 0$, we require $1 + C = 0$, i.e. $C = -1$. Therefore the required solution is

$$x^2y^3 + xy + y = 1.$$

6.2 Integrating factors for exactness

While the equation

$$M(x, y)\, dx + N(x, y)\, dy = 0$$

may not be exact, it may be possible to find a function $u(x, y)$ so that

$$u(x, y)M(x, y)\, dx + u(x, y)N(x, y)\, dy = 0 \tag{6.1}$$

is exact. We would require:

$$(uM)_y = (uN)_x \Rightarrow u_yM - u_xN + u(M_y - N_x) = 0$$

which is a partial differential equation for u. Solving it is likely to be as hard as solving the original DE but becomes simpler if we look only for a solution u that is either a function of x or of y alone. Suppose $u(x, y) = u(x)$ in fact. Then $u_y = 0$ and we obtain:

$$\frac{du}{dx} = \frac{M_y - N_x}{N} u.$$

If $(M_y - N_x)/N$ is a function of x only then this is a linear homogeneous 1st order DE for u which we can really solve.

Example 6.3. Solve the DE

$$(3xy + y^2)\, dx + (x^2 + xy)\, dy = 0$$

using an integrating factor.

The equation is not exact as $M_y = 3x + 2y$ and $N_x = 2x + y$, but

$$\frac{M_y - N_x}{N} = \frac{(3x + 2y) - (2x + y)}{x^2 + xy} = \frac{x + y}{x(x + y)} = \frac{1}{x}$$

is a function of x only. Hence we can find an integrating factor by solving

$$u' = \frac{u}{x} \Rightarrow u = e^{\int (1/x) dx} = e^{\ln x} = x.$$

This gives us the exact equation:

$$(3x^2y + xy^2)\, dx + (x^3 + x^2y)\, dy = 0$$

We therefore look for a function $f(x, y)$ satisfying $f_x = 3x^2y + xy^2$, $f_y = x^3 + x^2y$. As before (but reversing order):

$$f(x, y) = \int (x^3 + x^2y)\, dy = x^3y + \frac{1}{2}x^2y^2 + g(x)$$

and the differentiating w.r.t. x:

$$f_x = 3x^2y + xy^2 + g'(x) = 3x^2y + xy^2 \Rightarrow g'(x) = 0$$

Hence $g(x)$ is a constant and the general solution is:

$$x^3y + \frac{1}{2}x^2y^2 = C.$$

6.3 Homogeneous functions and substitution

Outside the field of DEs, a function of two variables $f(x, y)$ is called **homogeneous of degree** n if for any $\alpha \in \mathbb{R}$, there is a positive integer n so that for all (x, y)

$$f(\alpha x, \alpha y) = \alpha^n f(x, y).$$

Example 6.4. The function $f(x, y) = x^3 - 3xy^2$ is homogeneous of degree 3.

To see this,

$$f(\alpha x, \alpha y) = (\alpha x)^3 - 3(\alpha x)(\alpha y)^2 = \alpha^3(x^3 - 3xy^2) = \alpha^3 f(x, y).$$

Homogeneous functions will be polynomials so that in every term the sum of the degrees of each variable is n.

A DE

$$M(x, y) - N(x, y)\,\frac{dy}{dx} = 0 \quad \text{or equivalently} \quad fracdydx = \frac{M(x, y)}{N(x, y)}$$

that has homogeneous coefficients M, N of the *same* degree can be solved by making a substitution $y = ux$ and converting to a new DE.

Example 6.5. Solve the non-linear DE

$$\frac{dy}{dx} = \frac{x^2 + y^2}{xy - x^2}.$$

We introduce a new dependent variable $u = u(x)$ so that $y = ux$. Differentiating w.r.t. x gives:

$$\frac{dy}{dx} = u + x\frac{du}{dx}.$$

Substituting in the DE gives:

$$u + x\frac{du}{dx} = \frac{x^2 + u^2x^2}{x^2u - x^2} = \frac{1 + u^2}{u - 1}$$

and this equation will always be separable.

7 Phase diagrams and Euler's Method

[Boyce & Di Prima, Sections 2.5, 2.7]

7.1 Autonomous equations and phase diagrams

A first-order DE $y' = f(y)$ is called **autonomous**: the independent variable does not appear explicitly in function f. A zero of the function corresponds to $y' = 0$ and therefore a solution y =constant. Such a value of y is called an **equilibrium point** of the DE. Moreover, the sign of $f(y)$ at other points tells us whether a solution $y(t)$ is increasing ($y' > 0$) or decreasing ($y' < 0$).

Example 7.1 (Logistic equation with threshold)**.** This is a further adaptation of the population dynamics model of growth with constraint (logistic model) that introduces a threshold so that if the population falls below this value it has a negative impact on the growth. The equation has the form

$$\frac{dy}{dt} = r\left(\frac{y}{T} - 1\right)\left(1 - \frac{y}{K}\right)y$$

where r, T, K are positive constants and we further assume that $T < K$.

The DE is clearly autonomous. We can plot a graph of $z = f(y) = r\left(\frac{y}{T} - 1\right)\left(1 - \frac{y}{K}\right)y$.

to complete

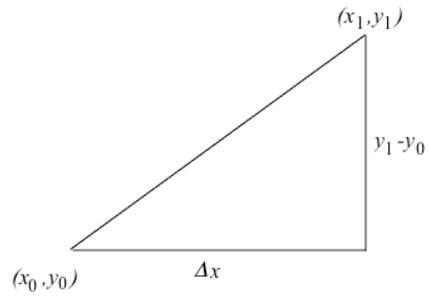
7.2 Euler's Method

Suppose we have a DE $\dfrac{dy}{dt} = f(t, y)$ and an i.c. $y(t_0) = y_0$. We try to approximate a solution $y = y(t)$ by a polygon, that is, by straight line segments. The idea is to choose a *stepsize* for t, say Δt and approximate the solution by the straight line from (t_0, y_0) to (t_1, y_1) where

$$\begin{aligned} t_1 &= t_0 + \Delta t \\ y_1 &= y_0 + f(t_0, y_0)\Delta t \end{aligned}$$

Why do we choose y_1 in this way? Well, the slope of the solution at (t_0, y_0) is given by the DE, so $\dfrac{dy}{dt} = f(t_0, y_0)$. Then the formula just says

$$\frac{y_1 - y_0}{\Delta t} = f(t_0, y_0)$$



Now repeat the procedure starting from (t_1, y_1). In general, let

$$t_{n+1} = t_n + \Delta t$$
$$y_{n+1} = y_n + f(t_n, y_n)\Delta t$$

Proceed for as many steps as you need to approximate the solution for.

Example 7.2. Approximate the solution to the DE $\dfrac{dy}{dt} = t^2 + 2ty$ with initial condition $y(0) = 1$ and stepsize $\Delta t = 0.1$.

We create a table of values (or use a calculator or Maple to find the values):

n	t_n	y_n	$f(t_n, y_n)\Delta t$	y_{n+1}
0	0.0	1.0000	0.0000	1.0000
1	0.1	1.0000	0.0210	1.0210
2	0.2	1.0210	0.0448	1.0658
3	0.3	1.0658	0.0730	1.1388
4	0.4	1.1388	0.1071	1.2459
5	0.5	1.2459	0.1496	1.3955
6	0.6	1.3955	0.2035	1.5989
7	0.7	1.5989	0.2729	1.8718
8	0.8	1.8718	0.3635	2.2353
9	0.9	2.2353	0.4834	2.7186
10	1.0	2.7186	—	—

Note that at each step there is a slight error caused by the approximation and this can build up over time. So using too many steps may result in the approximate solution being a long way from the true solution, but so can using a large stepsize. Here is graph showing the Euler approximation in red and the true solution in blue:

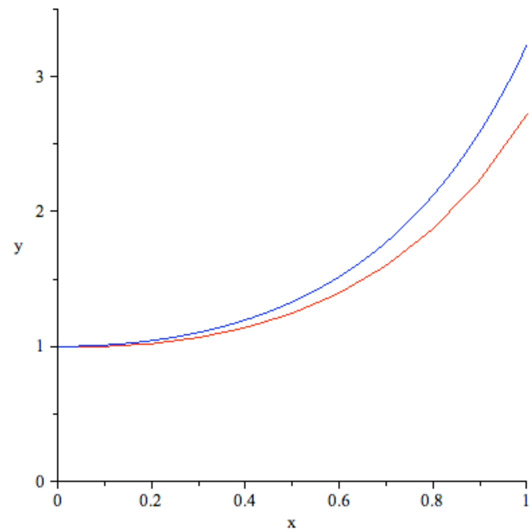


Figure 10: Plot of exact and Euler solutions with same initial condition

8 Linear 2nd Order Differential Equations

8.1 LHCCs

What we have seen for 1st order equations is that we can always solve a linear equation, in theory, by means of an integrating factor. However we may not be able to obtain an actual function as solution since the method requires us to find an integral and there are many functions that do not have a simple functional expression for their integral. Beyond linear equations, there are sometimes special techniques that work: separable or exact equations, but we do not have a general method.

So for 2nd order equations the situation is likely to be more difficult. We therefore start with linear equations.

Applications

(i) *Simple harmonic motion* When a particle oscillates on a straight line between two points, its position x at time t is governed by the equation

$$\frac{d^2x}{dt^2} + \omega^2 x = 0$$

By observation one can spot that the function $x = k \sin \omega t$ is a solution. Here ω represents the frequency of oscillation.

(ii) If the oscillating system is damped then an appropriate equation may be

$$\frac{d^2x}{dt^2} + k\frac{dx}{dt} + \omega^2 x = 0$$

(iii) If the damped oscillating system has some externally imposed oscillating force then the equation becomes

$$\frac{d^2x}{dt^2} + k\frac{dx}{dt} + \omega^2 x = F\cos\eta t$$

(iv) In the study of elastic sheet vibration one finds a model using a partial differential equation. It is possible to reduce this to a linear 2nd order ODE:

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + x^2 y = 0$$

The first 2 equations are particularly simple as they have constant coefficients. The last 2 are still linear (in terms of the unknown function and its derivatives) but involve functions of the independent variable (t in (3) and x in (4)). Linear equations with no constant term are called **homogeneous**: (1), (2) and (4) are all examples.

We start to look for solutions with what looks like the simplest case: *linear homogeneous equations with constant coefficients* (LHCC). They have the general form:

$$\frac{d^2y}{dx^2} + a\frac{dy}{dx} + by = 0 \tag{8.1}$$

The 1st order analogue is just $y' + by = 0$ with general solution $y = ke^{-bx}$. This suggests seeking solutions of the same form, $y = e^{rx}$. Then $y' = re^{rx}$ and $y'' = r^2e^{rx}$ so substituting in (8.1) and dividing through by $e^{rx} \neq 0$:

$$r^2 + ar + b = 0$$

This is called the **characteristic equation** of the DE. It is simply a quadratic in r and we can attempt to solve it by factorising or using the quadratic formula. Obviously the outcome depends on the *discriminant* of the quadratic $\Delta = a^2 - 4b$. If $\Delta > 0$ then there are 2 distinct (real) roots to the quadratic, say $r = r_1, r_2$ and hence we have 2 distinct solutions of the DE, $y = e^{r_1x}$, $y = e^{r_2x}$.

However if $\Delta = 0$ then we have a single (repeated) root, while if $\Delta < 0$ then there are complex conjugate roots. We come back to these cases soon.

It is easy to check that when $\Delta > 0$

$$y = c_1e^{r_1x} + c_2e^{r_2x}$$

is always a solution for any $c_1, c_2 \in \mathbb{R}$. Here c_1, c_2 are in effect constants of integration. We could expect to have two conditions in order to solve for them. See the example in Problem Class 2.

8.2 Complex roots

If the roots are complex numbers then they must be complex conjugates, i.e. $r_1, r_2 = \lambda \pm i\mu$. Now

$$e^{r_1 x} = e^{\lambda x + i\mu x} = e^{\lambda x} e^{i\mu x} = e^{\lambda x}(\cos \mu x + i \sin \mu x)$$

and similarly for r_2. But of course this is a *complex* solution. Nevertheless, if you plug it in and compare real and imaginary parts, it becomes clear that the real $e^{\lambda x} \cos \mu x$ and imaginary $e^{\lambda x} sin \mu x$ parts are solutions independently: hence we can take any solution of the form

$$y = c_1 e^{\lambda x} \cos \mu x + c_2 e^{\lambda x} \sin \mu x$$

Example 8.1. Solve $16y'' - 8y' + 145y = 0$ subject to $y(0) = -2$, $y'(0) = 1$.

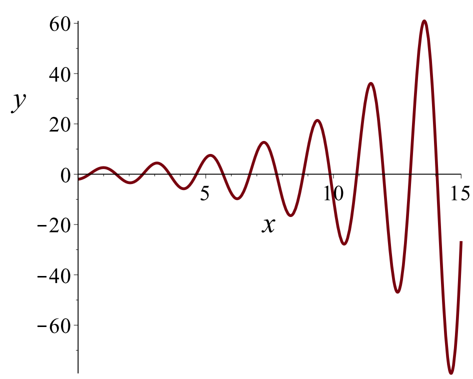
The characteristic equation gives

$$r = \frac{8 \pm \sqrt{64 - 64 \times 145}}{32} = \frac{1}{4} \pm 3i$$

hence a general solution is

$$y = c_1 e^{x/4} \cos 3x + c_2 e^{x/4} \sin 3x$$

and from the initial conditions $c_1 = -2$, $\frac{1}{4}c_1 + 3c_2 = 1 \Rightarrow c_2 = \frac{1}{2}$.



9 Linear 2nd Order Differential Equations II

9.1 LHCCs with Repeated Roots

Recall that we have found 2–parameter families of solutions to 2nd order linear homogeneous equations with constant coefficients in the cases where the characteristic equation has either real distinct or complex conjugate solutions. The remaining case is that of repeated roots. If the characteristic equation $r^2 + ar + b = 0$ has discriminant $a^2 - 4b = 0$ then the unique root is $r = -a/2$. This gives a 1–parameter family of solutions to the equation $y_1(x) = c_1 e^{-ax/2}$. We should expect another independent solution, but how can we find it? D'Alembert's idea was to seek a solution of the form $y = u(x)y_1(x)$.

Example 9.1. Solve the equation $y'' + 4y' + 4y = 0$.

The characteristic equation gives a solution $y_1 = e^{-2x}$. Now look for another solution $y = u(x)e^{-2x}$. We have

$$
\begin{aligned}
y' &= u'e^{-2x} - 2ue^{-2x} \\
y'' &= u''e^{-2x} - 4u'e^{-2x} + 4ue^{-2x}
\end{aligned}
$$

Substituting in the equation gives:

$$[u'' - 4u' + 4u + 4(u' - 2u) + 4u]e^{-2x} = 0$$

Since $e^{-2x} \neq 0$, this simplifies to $u'' = 0$. That can be solved simply:

$$
\begin{aligned}
u' &= c_1 \\
u &= c_1 x + c_2
\end{aligned}
$$

We thus have solutions of the form

$$y = c_1 x e^{-2x} + c_2 e^{-2x}$$

The second term corresponds to the known solution but the first term is new. This looks more like a general solution as there are 2 parameters. In fact, for repeated roots, there will always be 2 solutions of the form $y = e^{rx}$ and $y = xe^{rx}$.

9.2 Inhomogeneous equations: undetermined coefficients

Recall our earlier example of a forced damped oscillator. A typical equation might be:

$$y'' + 5y' + 4y = 2\sin 3x \tag{9.1}$$

Following what we did for the LHCCs, we propose a trial solution: $y = A\sin 3t$. Then $y' = 3A\cos 3x$ and $y'' = -9A\sin 3x$ so substituting in the equation:

$$-9A\sin 3x + 15A\cos 3x + 4A\sin 3x = 2\sin 3x$$

Looking at coefficients of $\sin 3x$ and $\cos 3x$ we get inconsistent equations for A. It would be better to have 2 coefficients, say try

$$y = A\sin 3x + B\cos 3x \Rightarrow y' = 3A\cos 3x - 3B\sin 3x \Rightarrow y'' = -9A\sin 3x - 9B\cos 3x$$

then

$$\begin{aligned}
&(-9A\sin 3x - 9B\cos 3x) + 5(3A\cos 3x - 3B\sin 3x) \\
&\qquad\qquad\qquad\qquad + 4(A\sin 3x + B\cos 3x) = 2\sin 3x \\
\Rightarrow &(-9A - 15B + 4A - 2)\sin 3x + (-9B + 15A + 4B)\cos 3x = 0 \\
\Rightarrow &\begin{cases} 5A + 15B &= -2 \\ 15A - 5B &= 0 \end{cases} \\
\Rightarrow &A = -\frac{1}{25}, B = -\frac{3}{25}
\end{aligned}$$

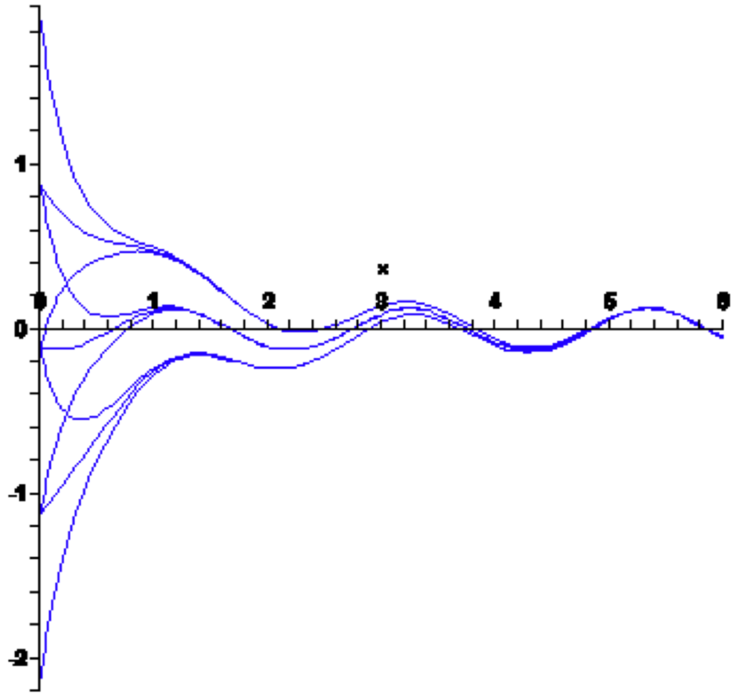
Hence we have a *particular solution*

$$Y = -\frac{1}{25}\sin 3x - \frac{3}{25}\cos 3x$$

Moreover, consider the homogeneous equation $y'' + 5y' + 4y = 0$. If $y = y_h$ is any solution then substituting $y = Y + y_h$ in the LHS of (9.1) gives $0 + 2\sin 3x = 2\sin 3x$ so it is also a solution. We can easily find a general solution to the LHCC: $y = c_1 e^{-x} + c_2 e^{-4x}$ so the *general solution* to (9.1) is

$$y = c_1 e^{-x} + c_2 e^{-4x} - \frac{1}{25}\sin 3x - \frac{3}{25}\cos 3x$$

The diagram shows 9 solutions for $c_1, c_2 = -1, 0, 1$.



9.3 General theory

The pattern we have established for solutions of linear equations is that in the homogeneous case there are 2 'independent' solutions and the general solution is a linear combination of these. In the inhomogeneous case we add a particular solution to the general solution of the corresponding homogeneous equation. We can summarise this in several theorems.

Theorem 9.2. *Given a linear homogeneous equation*

$$y'' + p(x)y' + q(x)y = 0 \tag{9.2}$$

and i.c.s

$$y(x_0) = y_0, \qquad y'(x_0) = y_0' \tag{9.3}$$

where p, q are continuous on an open interval I and $x_0 \in I$, then there is a unique solution $y = y(x)$ defined for all $x \in I$ satisfying (9.2) and (9.3).

Linearity means that if $y = y_1$ and $y = y_2$ are known solutions of the LH (9.2), then so is

$$y = c_1 y_1 + c_2 y_2 \tag{9.4}$$

for any $c_1, c_2 \in \mathbb{R}$. Now if there are i.c.s as in Theorem 9.2, it is clear that we get a pair of simultaneous equations for c_1, c_2:

$$\begin{aligned} c_1 y_1(x_0) + c_2 y_2(x_0) &= y_0 \\ c_1 y_1'(x_0) + c_2 y_2'(x_0) &= y_0' \end{aligned}$$

These will have a unique solution provided the coefficient matrix on the LHS is non-singular (using MATH 114 linear algebra). This determinant of this matrix is called the **Wronskian** and it can be thought of as a function of x_0.

Theorem 9.3. *Suppose $y = y_1$ and $y = y_2$ are solutions of (9.2). If the Wronskian*

$$\begin{vmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{vmatrix} \neq 0$$

then there is a unique solution $y = c_1 y_1 + c_2 y_2$ satisfying the i.c.s (9.3).

In this case the form (9.4) is called the **general solution** of the equation and y_1 and y_2 are called a **fundamental set of solutions**.

Example 9.4. Check that the solutions $y_1 = e^{r_1 x}$ and $y_2 = e^{r_2 x}$ to the LHCC with real roots $r_1 \neq r_2$ of its characteristic equation, are a fundamental set.

The Wronskian is

$$\begin{vmatrix} e^{r_1 x} & e^{r_2 x} \\ r_1 e^{r_1 x} & r_2 e^{r_2 x} \end{vmatrix} = (r_2 - r_1) e^{(r_1 + r_2)x} \neq 0$$

as required.

Theorem 9.5. *Given a linear inhomogeneous equation*

$$y'' + p(x) y' + q(x) y = g(x) \tag{9.5}$$

with p, q, g continuous on an open interval I, suppose y_1, y_2 are a fundamental set of solutions for the corresponding homogeneous equation and $Y(x)$ a particular solution of (10.1), then its general solution has the form

$$y = c_1 y_1(x) + c_2 y_2(x) + Y(x)$$

10 Linear 2nd Order ODEs III

10.1 Undetermined Coefficients

We solved the equation for forced damped oscillation by proposing a test solution and substituting to determine unknown parameters—the*undetermined coefficients*. For a general equation of the form

$$y'' + a y' + b y = g(x) \tag{10.1}$$

we use different test functions depending on $g(x)$. First write $g(x) = g_1(x) + \cdots + g_n(x)$ where each g_i is of a simple form. There is a *principle of superposition* that means a solution to (10.1) can be found by summing solutions to $y'' + ay' + by = g_i(x)$. The following table indicates what test functions $Y(x)$ to use for different types of g_i. Let $p(x) = a_0x^k + a_1x^{k-1} + \cdots + a_{k-1}x + a_k$ denote a general polynomial of degree k.

$g(x)$	$Y(x)$
$p(x)$	$x^m(A_0x^k + A_1x^{k-1} + \cdots + A_k)$
$p(x)e^{\alpha x}$	$x^m(A_0x^k + A_1x^{k-1} + \cdots + A_k)e^{\alpha x}$
$p(x)\sin\beta x$, or $p(x)\cos\beta x$	$x^m[(A_0x^k + A_1x^{k-1} + \cdots + A_k)\cos\beta x + (B_0x^k + B_1x^{k-1} + \cdots + B_k)\sin\beta x]$
$e^{\alpha x}\sin\beta x$, or $e^{\alpha x}\cos\beta x$	$x^m[Ae^{\alpha x}\cos\beta x + Be^{\alpha x}\sin\beta x]$

The extra factor x^m is only required when there is a coincidence between one of the parameters in g and the roots of the characteristic equation of the corresponding homogeneous equation. For $g(x) = p(x)$ use $m = 1$ if $r = 0$ is a root of the characteristic polynomial and $m = 2$ if it is a repeated root. For $g(x)$ involving an exponential function $e^{\alpha x}$, use $m = 1$ if $r = \alpha$ is a root of the c.p. and $m = 2$ if it is a repeated root. For g involving trig functions, you need $m = 1$ if β is the imaginary part of complex conjugate roots of the *c.p.* To find a general solution we simply need to solve the homogeneous equation.

Example 10.1. Find a particular solution to the equation

$$y'' - 3y' - 4y = x^2 + 1 - 4e^x\cos 2x$$

We look for solutions to

$$y'' - 3y' - 4y = x^2 + 1 \tag{10.2}$$
$$y'' - 3y' - 4y = -4e^x\cos 2x \tag{10.3}$$

The characteristic equation is $r^2 - 3r - 4 = (r-4)(r+1) = 0$ which does not have $r = 0$ as a root. So try first $Y(x) = A_0x^2 + A_1x + A_2$ (i.e. $m = 0$). Then $Y' = 2A_0x + A_1$ and $Y'' = 2A_0$ and (10.2) gives

$$2A_0 - 3(2A_0x + A_1) - 4(A_0x^2 + A_1x + A_2) = x^2 + 1$$
$$\Rightarrow \begin{cases} -4A_0 & = 1 \\ -6A_0 - 4A_1 & = 0 \\ 2A_0 - 3A_1 - 4A_2 & = 1 \end{cases}$$

These equations have a unique solution for the A_i: $A_0 = -\frac{1}{4}$, $A_1 = \frac{3}{8}$, $A_2 = -\frac{21}{32}$.

Now try $Y(x) = Ae^x \cos 2x + Be^x \sin 2x$, so

$$\begin{aligned} Y' &= Ae^x \cos 2x - 2Ae^x \sin 2x + Be^x \sin 2x + 2Be^x \cos 2x \\ &= (A + 2B)e^x \cos 2x + (-2A + B)e^x \sin 2x \\ Y'' &= -3Ae^x \cos 2x - 4Ae^x \sin 2x - 3Be^x \sin 2x + 4Be^x \cos 2x \\ &= (-3A + 4B)e^x \cos 2x - (4A + 3B)e^x \sin 2x \end{aligned}$$

then in (10.3):

$$\begin{aligned} &(-3A + 4B)e^x \cos 2x - (4A + 3B)e^x \sin 2x \\ &\qquad - 3[(A + 2B)e^x \cos 2x + (-2A + B)e^x \sin 2x] \\ &\qquad - 4(Ae^x \cos 2x + Be^x \sin 2x) = -4e^x \cos 2x \\ \Rightarrow &\begin{cases} -10A - 2B &= -4 \\ 2A - 10B &= 0 \end{cases} \Rightarrow \left\{ A = \frac{5}{13}, B = \frac{1}{13} \right\} \end{aligned}$$

So a particular solution to the original equation is

$$y(x) = -\frac{1}{32}(8x^2 - 12x + 21) + \frac{1}{13}e^x(5\cos 2x + \sin 2x)$$

10.2 Reduction of Order

In order to solve the LHCC with repeated roots we used a trick of taking a known solution $y = y_1(x)$ and looking for another solution of the form $y = u(x)y_1(x)$. This trick can work in other cases so long as you can spot a solution first.

Example 10.2. Given that $y_1 = 1/x$ is a solution to the equation

$$2x^2 y'' + 3xy' - y = 0$$

find a second independent solution.

In standard form $y'' + \dfrac{3}{2x}y' - \dfrac{1}{2x^2} = 0$, the coefficients have a discontinuity at $x = 0$ so we just look for solutions in the domain $x > 0$. Try $y = uy_1 = u/x$. Then

$$y' = \frac{u'}{x} - \frac{u}{x^2}, \qquad y'' = \frac{u''}{x} - \frac{2u'}{x^2} + \frac{2u}{x^3}$$

and then

$$\begin{aligned} &2x^2\left(\frac{u''}{x} - \frac{2u'}{x^2} + \frac{2u}{x^3}\right) + 3x\left(\frac{u'}{x} - \frac{u}{x^2}\right) - \frac{u}{x} \\ &= 2xu'' + (-4 + 3)u' + (4 - 3 - 1)\frac{u}{x} = \ 2xu'' - u' = 0 \end{aligned}$$

Put $v = u'$ so $2xv' = v \Rightarrow \ln|v| = \frac{1}{2}\ln x + c \Rightarrow v = c_1 x^{1/2}$ $(c_1 = \pm e^c)$ and then

$$u = \int c_1 x^{1/2}\, dx = \frac{2}{3} c_1 x^{3/2} + c_2$$

Then let $y_2 = x^{3/2}/x = x^{1/2}$. It's easy to check that y_1 and y_2 are independent.

10.3 Power series solutions

If all else fails then one can look for a solution to a DE in the form of a power series.

Example 10.3. Find a series solution to **Airy's equation**

$$y'' - xy = 0$$

We propose a series solution $y = \sum_{n=0}^{\infty} a_n x^n$. Assuming this has positive radius of convergence (which we check later) the series can be differentiated term by term to give

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1} \quad \text{and} \quad y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

Substituting into the equation gives

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - x \sum_{n=0}^{\infty} a_n x^n = 0 \Rightarrow \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} a_n x^{n+1}$$

We would like to gather together coefficients from the 2 series but it will be easier if we relabel first so that everything is in terms of say x^k. In the first series let $k = n - 2$ and in the second $k = n + 1$ then we get

$$\sum_{k=0}^{\infty} (k+2)(k+1) a_{k+2} x^k = \sum_{k=1}^{\infty} a_{k-1} x^k$$

Equating coefficients gives $a_2 = 0$ and

$$(k+2)(k+1) a_{k+2} = a_{k-1}, \qquad k = 1, 2, 3, \ldots$$

This is a *recurrence relation* for the coefficients. If we choose a_0, a_1 arbitrarily then the relation tells us successively the values of $a_3, a_4, \ldots$. In particular, $a_{3n+2} = 0$, while

$$a_3 = \frac{a_0}{2 \cdot 3}, \quad a_6 = \frac{a_3}{5 \cdot 6} = \frac{a_0}{2 \cdot 3 \cdot 5 \cdot 6}, \ldots$$
$$a_4 = \frac{a_1}{3 \cdot 4}, \quad a_7 = \frac{a_4}{6 \cdot 7} = \frac{a_1}{3 \cdot 4 \cdot 6 \cdot 7}, \ldots$$

and in general for $n \geq 1$

$$a_{3n} = \frac{a_0}{2 \cdot 3 \cdot 5 \cdot 6 \cdots (3n-4)(3n-3)(3n-1)3n}$$
$$a_{3n+1} = \frac{a_1}{3 \cdot 4 \cdot 6 \cdot 7 \cdots (3n-3)(3n-2)3n(3n+1)}$$

One can show that the resulting series has infinite radius of convergence. Note that $y(0) = a_0$ and $y'(0) = a_1$ so these parameters correspond to initial conditions. It is convenient to split the solution into 2 independent solutions, called Ai and Bi corresponding to the terms in x^{3n} and x^{3n+1} respectively:

$$y = a_0 \left(1 + \sum_{n=1}^{\infty} \frac{x^{3n}}{2 \cdot 3 \cdots (3n-1)3n} \right) + a_1 \left(x + \sum_{n=1}^{\infty} \frac{x^{3n+1}}{3 \cdot 4 \cdots 3n(3n+1)} \right)$$