

# ENGR 222

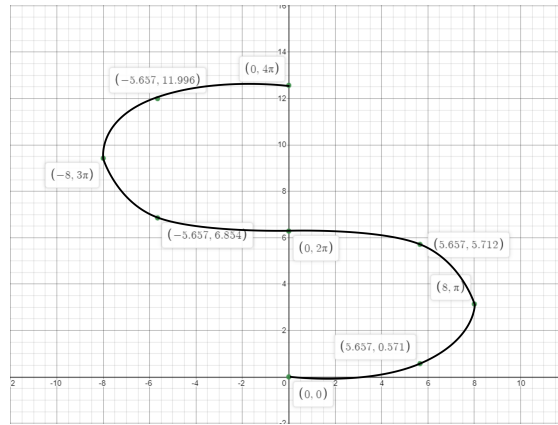
## Assignment 1 Submission

Daniel Eisen : 300447549

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1. Consider the parametric equations:  $(f(t), g(t)) = (8\sin(t), 2t - \sin(2t))$

(a)  $(f(t), g(t)) = [(0, 0), (5.657, 0.571), (8, \pi), (5.657, 5.712), (0, 2\pi), (-5.657, 6.854), (-8, 3\pi), (-5.657, 11.996), (0, 4\pi)]$



(b) Tangent vector:

$$(f'(t), g'(t)) = (8\cos(t), 2 - 2\cos(2t))$$

$$t = \pi/6 : (f'(t), g'(t)) = (6.928203, 1)$$

Unit tangent vector:

$$\frac{(f'(t), g'(t))}{\|(f'(t), g'(t))\|} = \frac{(6.928203, 1)}{\sqrt{6.928203^2 + 1}} = \left( \frac{6.928203}{7}, \frac{1}{7} \right)$$

(c) Tangent Line equation (normalised):

$$= (x, y) + t \cdot \frac{(f'(t), g'(t))}{\|(f'(t), g'(t))\|}$$

$$= (4, 0.181) + t \left( \frac{6.928203}{7}, \frac{1}{7} \right)$$

$$= \left( \frac{6.928203t}{7} + 4, \frac{t}{7} + 0.181 \right)$$

(d) Normal line:

$$= \left( f\left(\frac{\pi}{6}\right) - tg'\left(\frac{\pi}{6}\right), g\left(\frac{\pi}{6}\right) + tf'\left(\frac{\pi}{6}\right) \right)$$

$$= (4 - t, 0.181 + 6.928203t)$$

(e) Arc length:

$$\begin{aligned}
L &= \int_0^{2\pi} \sqrt{f'(t)^2 + g'(t)^2} dt \\
&= \int_0^{2\pi} \sqrt{(8 \cos(t))^2 + (2 - 2 \cos(2t))^2} dt \\
&= \int_0^{2\pi} \sqrt{(32 \cos(2t) + 32) + (4 + 4 \cos(2t)^2 - 8 \cos(2t))} dt \\
&= \int_0^{2\pi} \sqrt{4 \cos(2t)^2 + 24 \cos(2t) + 36} dt \\
&= \int_0^{2\pi} \sqrt{4 (\cos(2t)^2 + 6 \cos(2t) + 9)} dt \\
&= \int_0^{2\pi} \sqrt{4 (\cos(2t) + 3)^2} dt \\
&= 2 \int_0^{2\pi} (\cos(2t) + 3) dt \\
&= \sin(2t) + 6t \Big|_0^{2\pi} \\
&= (0 + 12\pi) - (0 + 0) \\
&= 12\pi \approx 37.6991118431
\end{aligned}$$

2. Consider the curve described by the vector valued function:

$$\mathbf{r}(t) = \frac{e^{2t} - 2t}{4} \mathbf{i} + e^t \mathbf{j}$$

(a)

$$\mathbf{r}(t) \cdot \mathbf{j} = 2$$

$$e^t = 2$$

$$t = \ln(2) \approx 0.693147$$

$$\mathbf{r}(t = \ln(2)) = (0.653, 2)$$

(b)

$$\mathbf{r}'(t) = \frac{2e^{2t} - 2}{4} \mathbf{i} + e^t \mathbf{j}$$

$$\|\mathbf{r}'(t)\| = \sqrt{\left(\frac{2e^{2t} - 2}{4}\right)^2 + (e^t)^2} = \sqrt{\frac{1}{4}(e^{4t} + 2e^{2t} + 1)}$$

$$= \sqrt{\frac{1}{4}(e^{2t} + 1)^2} = \frac{e^{2t} + 1}{2}$$

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|}$$

$$= \frac{\frac{2e^{2t}-2}{4} \mathbf{i} + e^t \mathbf{j}}{\frac{e^{2t}+1}{2}}$$

$$= \left( \frac{\frac{1}{4}(2e^{2t} - 2)}{\frac{1}{2}(e^{2t} + 1)}, \frac{e^t}{\frac{1}{2}(e^{2t} + 1)} \right)$$

$$= \left( \frac{e^{2t} - 1}{e^{2t} + 1}, \frac{2e^t}{e^{2t} + 1} \right)$$

$$\mathbf{T}(t) = \frac{(e^{2t} - 1)\mathbf{i} + 2e^t \mathbf{j}}{e^{2t} + 1}$$

$$\begin{aligned}
 \text{(c)} \quad \mathbf{N}(t) &= \frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|} \\
 \mathbf{T}'(t) &= \left( \frac{2e^{2t}(e^{2t}+1) - 2e^{2t}(e^{2t}-1)}{(e^{2t}+1)^2}, \frac{2e^t(e^{2t}+1) - 2e^t \cdot 2e^{2t}}{(e^{2t}+1)^2} \right) \\
 &= \frac{4e^{2t}}{(e^{2t}+1)^2} \mathbf{i} + \frac{2e^t - 2e^{3t}}{(e^{2t}+1)^2} \mathbf{j} \equiv \operatorname{sech}(t)^2 \mathbf{i} + (-\operatorname{sech}(t) \tanh(t)) \mathbf{j} \\
 \|\mathbf{T}'(t)\| &= \sqrt{\operatorname{sech}(t)^4 + (-\operatorname{sech}(t) \tanh(t))^2} \\
 \mathbf{N}(t) &= \frac{\operatorname{sech}(t)^2 \mathbf{i} + (-\operatorname{sech}(t) \tanh(t)) \mathbf{j}}{\sqrt{\operatorname{sech}(t)^4 + (-\operatorname{sech}(t) \tanh(t))^2}}
 \end{aligned}$$

$$\begin{aligned}
 \text{(d)} \quad \kappa(t) &= \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|} \\
 &= \frac{2\sqrt{\operatorname{sech}(t)^4 + (-\operatorname{sech}(t) \tanh(t))^2}}{e^{2t} + 1}
 \end{aligned}$$

$$\begin{aligned}
 \text{(e)} \quad L &= \int_0^3 \|\mathbf{r}'(t)\| dt \\
 &= \int_0^3 \frac{e^{2t} + 1}{2} dt = \frac{1}{2} \int_0^3 e^{2t} + 1 dt \\
 &= \frac{1}{4} \left| e^{2t} + 2t \right|_0^3 = \frac{1}{4} ((e^6 + 6) - (e^0 + 0)) \\
 &= \frac{e^6 + 5}{4} \approx 102.107198373
 \end{aligned}$$

### 3. Quick questions:

$$\text{(a)} \quad (x, y, z) = (3t, -2 + t, 7 - 5t) : t_0 = 0$$

$$\begin{aligned}
 \mathbf{r}'(t) &= (3, 1, -5) \\
 \|\mathbf{r}'(t)\| &= \sqrt{3^2 + 1 + (-5)^2} = \sqrt{35} = 5.91608 \\
 s &= \int_{t_0}^t \|\mathbf{r}'(t)\| du \\
 &= \int_0^t \sqrt{35} du \\
 s &= \sqrt{35} t \\
 t &= \frac{s}{\sqrt{35}}
 \end{aligned}$$

$$\mathbf{r}(s) = \left( \frac{3s}{\sqrt{35}}, -2 + \frac{s}{\sqrt{35}}, 7 - \frac{5s}{\sqrt{35}} \right)$$

$$\begin{aligned}
 \text{(b)} \quad \mathbf{r}(t) &= (-3 + 5\cos(t))\mathbf{i} + (2 - 5\sin(t))\mathbf{j} : t_0 = 0 \\
 \mathbf{r}'(t) &= (-5\sin(t))\mathbf{i} + (-5\cos(t))\mathbf{j} \\
 \|\mathbf{r}'(t)\| &= \sqrt{(-5\sin(t))^2 + (-5\cos(t))^2} \\
 &= \sqrt{25(\sin(t)^2 + \cos(t)^2)} \\
 &= \sqrt{25}\sqrt{1} = 5
 \end{aligned}$$

$$\begin{aligned}
 s &= \int_0^t 5 \, du = 5t \\
 t &= \frac{s}{5} \\
 \mathbf{r}(s) &= (-3 + 5\cos(\frac{s}{5}))\mathbf{i} + (2 - 5\sin(\frac{s}{5}))\mathbf{j}
 \end{aligned}$$

$$\begin{aligned}
 \text{(c)} \quad \mathbf{r}(t) &= \sqrt{2}\cos(t)\mathbf{i} + \sin(t)\mathbf{j} + \sin(t)\mathbf{k} \\
 \mathbf{T}(t) &= \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} \\
 \mathbf{r}'(t) &= -\sqrt{2}\sin(t)\mathbf{i} + \cos(t)\mathbf{j} + \cos(t)\mathbf{k} \\
 \mathbf{r}'(\pi/3) &= \frac{-\sqrt{6}}{2}\mathbf{i} + 0.5\mathbf{j} + 0.5\mathbf{k} \\
 \|\mathbf{r}'(t)\| &= \sqrt{\left(-\frac{1}{2}\sqrt{6}\right)^2 + 0.5^2 + 0.5^2} = \sqrt{2} \\
 \mathbf{T}(t) &= \frac{-0.5\sqrt{6}\mathbf{i} + 0.5\mathbf{j} + 0.5\mathbf{k}}{\sqrt{2}}
 \end{aligned}$$

$$\begin{aligned}
 \text{(d)} \quad \mathbf{B}(t) &= \frac{\mathbf{r}'(t) \times \mathbf{r}''(t)}{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|} \\
 \mathbf{r}(t) &= t\mathbf{i} - t^3\mathbf{j} + t^2\mathbf{k} \\
 \mathbf{r}'(t) &= \mathbf{i} - 3t^2\mathbf{j} + 2t\mathbf{k} \\
 \mathbf{r}'(1) &= \mathbf{i} - 3\mathbf{j} + 2\mathbf{k} \\
 \mathbf{r}''(t) &= 0\mathbf{i} - 6t\mathbf{j} + 2\mathbf{k} \\
 \mathbf{r}''(1) &= 0\mathbf{i} - 6\mathbf{j} + 2\mathbf{k}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{r}'(t) \times \mathbf{r}''(t) &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -3 & 2 \\ 0 & -6 & 2 \end{vmatrix} \\
 &= \mathbf{i}(2(-3) - 2(-6)) - \mathbf{j}(2(1) - 0(2)) + \mathbf{k}(1(-6) - 0(-3)) \\
 &= 6\mathbf{i} - 2\mathbf{j} - 6\mathbf{k} \\
 \|\mathbf{r}'(t) \times \mathbf{r}''(t)\| &= \sqrt{6^2 + (-2)^2 + (-6)^2} = \sqrt{76}
 \end{aligned}$$

$$\mathbf{B}(t) = \frac{6\mathbf{i} - 2\mathbf{j} - 6\mathbf{k}}{\sqrt{76}}$$

(e)

$$\mathbf{r}(t) = (2 + 3\sin(t))\mathbf{i} + (1 + 2\cos(t))\mathbf{j}$$

$$\kappa(t) = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3}$$

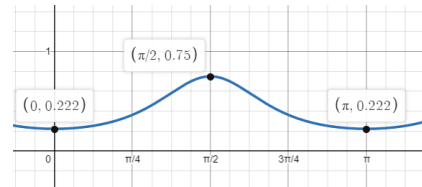
$$\mathbf{r}'(t) = 3\cos(t)\mathbf{i} - 2\sin(t)\mathbf{j}$$

$$\mathbf{r}''(t) = -3\sin(t)\mathbf{i} - 2\cos(t)\mathbf{j}$$

$$\begin{aligned}\|\mathbf{r}'(t)\| &= \sqrt{(3\cos(t))^2 + (-2\sin(t))^2} = \sqrt{9\cos^2(t) + 4\sin^2(t)} \\ &= \sqrt{\left(\frac{9 + 9\cos(2t)}{2}\right) + (2 - 2\cos(2t))} = \sqrt{\frac{13 + 5\cos(2t)}{2}} \\ \|\mathbf{r}'(t)\|^3 &= \left(\sqrt{\frac{13 + 5\cos(2t)}{2}}\right)^3 = \left(\frac{13 + 5\cos(2t)}{2}\right)^{\frac{3}{2}}\end{aligned}$$

$$\begin{aligned}\mathbf{r}'(t) \times \mathbf{r}''(t) &= \begin{vmatrix} \mathbf{i} & \mathbf{j} \\ 3\cos(t) & -2\sin(t) \\ -3\sin(t) & -2\cos(t) \end{vmatrix} \\ &= (-2\cos(t)3\cos(t) - (-2\sin(t)(-3\sin(t))))\mathbf{k} \\ &= -6\cos^2(t) - 6\sin^2(t)\mathbf{k} = -6\mathbf{k}\end{aligned}$$

$$\|\mathbf{r}'(t) \times \mathbf{r}''(t)\| = \sqrt{-6^2} = 6$$



$$\kappa(t) = \frac{6}{\left(\frac{13 + 5\cos(2t)}{2}\right)^{\frac{3}{2}}}$$

$$\cos(2t) \Rightarrow \min @ t = 0, \max @ t = \pi/2$$

$$\min = 0.222 \quad \max = 0.75$$

4. Suppose a roller coaster path described by:

$$\mathbf{r}(t) = \frac{1}{5}t(20 - t)\mathbf{i} + \frac{1}{50}t^2(20 - t)\mathbf{j} + \frac{1}{50}t(10 - t)(20 - t)\mathbf{k} : t \in [0, 20]$$

(a)

$$\mathbf{v}(t) = \mathbf{r}'(t) = \left(\frac{20 - 2t}{5}\right)\mathbf{i} + \left(\frac{40t - 3t^2}{50}\right)\mathbf{j} + \left(\frac{3t^2 - 60t + 200}{50}\right)\mathbf{k}$$

(b)

$$\mathbf{v}(5) = 2\mathbf{i} + 2.5\mathbf{j} - 0.5\mathbf{k}$$

$$v = \|\mathbf{v}(5)\| = \sqrt{2^2 + 2.5^2 + (-0.5)^2} = \sqrt{10.5} = 3.24037$$

(c)

$$\mathbf{a}(t) = \mathbf{v}'(t) = \mathbf{r}''(t) = -\frac{2}{5}\mathbf{i} + \left(\frac{40 - 6t}{50}\right)\mathbf{j} + \left(\frac{6t - 60}{50}\right)\mathbf{k}$$

$$\begin{aligned}
 \text{(d)} \quad \kappa(t) &= \frac{\|\mathbf{v}(t) \times \mathbf{a}(t)\|}{\|\mathbf{v}(t)\|^3} \\
 \mathbf{v}(5) &= 2\mathbf{i} + 2.5\mathbf{j} - 0.5\mathbf{k} \\
 \|\mathbf{v}(5)\|^3 &= \left(\sqrt{10.5}\right)^3 = 34.023889 \\
 \mathbf{a}(5) &= -0.4\mathbf{i} + 0.2\mathbf{j} - 0.6\mathbf{k}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{v}(5) \times \mathbf{a}(5) &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 2.5 & -0.5 \\ -0.4 & 0.2 & -0.6 \end{vmatrix} \\
 &= (2.5(-0.6) - 0.2(-0.5))\mathbf{i} - (2(-0.6) - (-0.4)(-0.5))\mathbf{j} + (2(0.2) - 2.5(-0.4))\mathbf{k} \\
 &= -1.4\mathbf{i} + 1.4\mathbf{j} + 1.4\mathbf{k} \\
 \|\mathbf{v}(5) \times \mathbf{a}(5)\| &= \sqrt{(-1.4)^2 + 1.4^2 + 1.4^2} = \sqrt{5.88} = 2.424871 \\
 \kappa(5) &= \frac{\sqrt{5.88}}{\sqrt{10.5}^3} = 0.07127
 \end{aligned}$$

$$\text{(e)} \quad \mathbf{v}(5) = 2\mathbf{i} + 2.5\mathbf{j} - 0.5\mathbf{k} \quad \mathbf{a}(5) = -0.4\mathbf{i} + 0.2\mathbf{j} - 0.6\mathbf{k}$$

$$\begin{aligned}
 \|\mathbf{v}(5) \times \mathbf{a}(5)\| &= \sqrt{5.88} \\
 \|\mathbf{v}(5)\| &= \sqrt{10.5} \\
 a_N &= \frac{\|\mathbf{v}(5) \times \mathbf{a}(5)\|}{\|\mathbf{v}(5)\|} = \frac{\sqrt{5.88}}{\sqrt{10.5}} = \sqrt{5.88/10.5} = \sqrt{0.56} = 0.748331 \\
 \|\mathbf{a}(5)\| &= \sqrt{(-0.4)^2 + 0.2^2 + (-0.6)^2} = \sqrt{0.56} = 0.748331 \\
 \|\mathbf{a}(5)\| &= a_N
 \end{aligned}$$

From this it can be inferred there is no tangential acceleration.