1.

x	8	5	9	8	4	8
y	18	22	16	9	25	12

a) $\bar{x} = \dfrac{\sum_{i=1}^{N} x_i}{N} = 7$ $\qquad y = \dfrac{\sum_{i=1}^{N} y_i}{N} = 17$

b) $s_x^2 = \dfrac{\sum_{i=1}^{N} (x_i - \bar{x})^2}{N-1} = \dfrac{(8-7)+(5-7)+\cdots}{5} = \dfrac{20}{5} = 4$

$\therefore s_x = 2$

$s_y^2 = \dfrac{\sum_{i=1}^{N} (y_i - \bar{y})^2}{N-1} = \dfrac{(18-17)+\cdots}{5} = \dfrac{180}{5} = 36$

$\therefore s_y = 6$

c) $= \dfrac{\sum_{i=1}^{N} (x_i - \bar{x})(y_i - \bar{y})}{(N-1)\cdot s_x \cdot s_y} = \dfrac{(8-7)(18-17)+(5-7)(22-17)+\cdots}{5\cdot 2\cdot 6}$

$= -\dfrac{48}{60} = \dfrac{-4}{5} = -0.8$

2. Let 'a' denote the 1st inspector finding the flaw
Let 'b' denote the 2nd inspector find the flaw

$\therefore$ $P(a) = 0.9$ $P(b) = 0.7$ $\therefore$ $P(a^c) = 0.1$, $P(b^c) = 0.3$

a) $P(a \cap b) = P(a)P(b) = (0.9)(0.7) = 0.63$

b) $P(a \cup b) = P(a) + P(b) - P(a \cap b)$
$= 0.9 + 0.7 - 0.63 = 0.97$

c) $P(a^c \cap b) = P(a^c)P(b) = [1 - P(a)]P(b)$
$= (1 - 0.9)(0.7) = (0.1)(0.7) = 0.07$

Let f denote an item having a flaw

$P(f) = 0.1$

Assumption: Both inspect every item,
and if there is no flaw (f^c),
they pass it 100% of the time.
i.e. $P(a^c | f^c) = P(b^c | f^c) = 1$

d) $$P(f | a^c) = \frac{P(a^c | f)P(f)}{P(a^c | f)P(f) + P(a^c | f^c)P(f^c)}$$

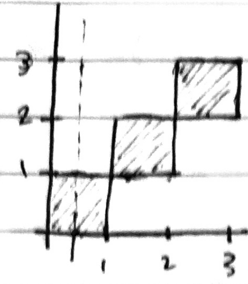
$$= \frac{(0.1)(0.1)}{(0.1)(0.1) + (1)(0.9)} = \frac{0.01}{0.91}$$

$P(f | a^c) = 0.01098...$

e) $$P(f | a^c \cap b^c) = \frac{P(a^c \cap b^c | f)P(f)}{P(a^c \cap b^c | f)P(f) + P(a^c \cap b^c | f^c)P(f^c)}$$

$$= \frac{P(a^c | f)P(b^c | f)P(f)}{P(a^c | f)P(b^c | f)P(f) + P(a^c | f^c)P(b^c | f^c)P(f^c)}$$

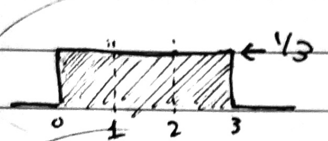
$$= \frac{(0.1)(0.3)(0.1)}{(0.1)(0.3)(0.1) + (1)(1)(0.9)} = \frac{3 \times 10^{-3}}{0.903} = 3.3223 \times 10^{-3} = \frac{1}{301}$$

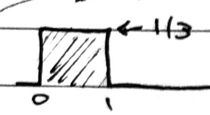
3. $f_{XY}(x,y) \Rightarrow$ 

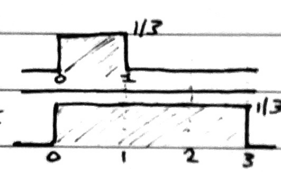
▨ $= C$

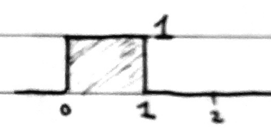
a) 3 zones of equal area and $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dy\,dx = 1$

$\therefore$ $C = \frac{1}{3}$

b) $f_X(x) =$ 

c) $f_{XY}(\frac{1}{2}, y) =$ 

d) $f_{Y|X}(y|\frac{1}{2}) = \frac{f_{XY}(\frac{1}{2}, y)}{f_X(x)} =$ 

$=$ 

e) $E\{X\} = \int x f_X(x)\,dx = \frac{1}{3}\int_0^3 x\,dx = \frac{1}{3} \cdot \frac{x^2}{2}\Big|_0^3$

$= \frac{1}{3}\left(\frac{9}{2} - \emptyset\right) = 3/2$

$E\{X\} = \frac{3}{2}$ (also $= E\{Y\}$)

f) $E\{XY\} = \mu_{XY} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, f_{XY}(x,y)\,dx\,dy$

$= \frac{1}{3}\left[\int_0^1\int_0^1 xy\,dx\,dy + \int_1^2\int_1^2 xy\,dx\,dy + \int_2^3\int_2^3 xy\,dx\,dy\right]$

$= \frac{1}{3}\left(\left[\frac{x^2}{2}\right]_0^1\left[\frac{y^2}{2}\right]_0^1 + \left[\frac{x^2}{2}\right]_1^2\left[\frac{y^2}{2}\right]_1^2 + \left[\frac{x^2}{2}\right]_2^3\left[\frac{y^2}{2}\right]_2^3\right)$

$= \frac{1}{3}\left[\frac{1}{4} + \left(2 - \frac{1}{2}\right)^2 + \left(\frac{9}{2} - 2\right)^2\right]$

$E\{XY\} = \frac{35}{12} \simeq 2.9166$

g) $\mathrm{Cov}(X,Y) = \mathcal{E}\{XY\} - \mathcal{E}\{X\}\mathcal{E}\{Y\}$

$$= \frac{35}{12} - \left(\frac{3}{2}\right)^2$$

$\mathrm{Cov}(X,Y) = 2/3$

4. $f(x,y) = \begin{cases} \frac{1}{6} e^{-\frac{x}{2} - \frac{y}{3}} & , x>0, y>0 \\ 0 & \text{otherwise} \end{cases}$

a) $P(X \leq 2 \text{ e } Y \leq 3) = \frac{1}{6} \int_0^3 \int_0^2 e^{-\frac{x}{2}} e^{-\frac{y}{3}} \, dx\, dy$

$$\int_0^2 e^{-\frac{x}{2}} e^{-\frac{y}{3}} \, dx = e^{-\frac{y}{3}} \int_0^2 e^{-\frac{1}{2}x} \, dx$$

$$= e^{-y/3} \cdot -2\left(e^{-\frac{x}{2}}\right)\Big|_0^2 = -2e^{-\frac{y}{3}}\left(e^{-1} - e^{0}\right)$$

$$= 2e^{-y/3}\left(1 - \frac{1}{e}\right)$$

$$\frac{1}{6} \int_0^3 2e^{-y/3}\left(1 - \frac{1}{e}\right) dy = \frac{1}{6}\left(1 - \frac{1}{e}\right) \int_0^3 2e^{-y/3} \, dy$$

$$\Rightarrow \frac{1}{6}\left(1 - \frac{1}{e}\right) \times -6\left(e^{-\frac{y}{3}}\right)\Big|_0^3 = -6\left(e^{-1} - e^{0}\right) = 6\left(1 - \frac{1}{e}\right)$$

$P(X \leq 2 \text{ e } Y \leq 3) = \left(1 - \frac{1}{e}\right)^2 \simeq 0.3995$

b) $P(X \geq 3 \text{ e } X \geq 3) = \frac{1}{6} \int_3^\infty \int_3^\infty e^{-\frac{x}{2}} e^{-\frac{x}{3}} \, dx\, dy$

$$\int_3^\infty e^{-x/2} e^{-y/3} \, dx = e^{-y/3} \int_3^\infty e^{-x/2} \, dx$$

$$= e^{-y/3}\left[-2\left(e^{-x/2}\right)\Big|_3^\infty\right] = -2e^{-y/3}\left(0 - \frac{1}{e^{3/2}}\right) = \frac{2e^{-y/3}}{e^{3/2}}$$

$$\frac{2}{6e^{3/2}} \int_3^\infty e^{-y/3} \, dy = \frac{2}{6e^{3/2}}\left[-3\left(e^{-y/3}\right)\Big|_3^\infty\right] = \frac{2}{6e^{3/2}} \cdot -3\left(0 - e^{-1}\right)$$

$$\Downarrow \quad \frac{3}{e}$$

$P(X \geq 3 \text{ e } Y \geq 3) = \frac{1}{e^{5/2}} \simeq 0.08208..$

c) $f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy = \frac{1}{6}\int_0^{\infty} e^{-\frac{x}{2}} e^{-\frac{y}{3}}\,dy$

$= \frac{1}{6e^{x/2}} \cdot \int_0^{\infty} e^{-y/3}\,dy = \frac{1}{6e^{x/2}}\left[-3(e^{-y/3})\Big|_0^{\infty}\right]$

$= \text{"} \cdot -3(0-1) \Rightarrow 3$

$f_X(x) = \frac{1}{2e^{x/2}}$

d) $f_Y(y) = \int_0^{\infty} \frac{1}{6} e^{-x/2} e^{-y/3}\,dx = \frac{1}{6e^{y/3}} \int_0^{\infty} e^{-x/2}\,dx$

$= \frac{1}{6e^{y/3}}\left[-2(e^{-x/2})\Big|_0^{\infty}\right]$

$= \text{"} \cdot -2(0-1) \Rightarrow 2$

$f_Y(y) = \frac{1}{3e^{y/3}}$

e) 1st check... $f(1,1) = \frac{1}{6} e^{-\frac{1}{2}-\frac{1}{3}} = 0.0724\ldots$ ✓

$f(x,y) = f_X(x) f_Y(y)$

$f_X(1) \cdot f_Y(1) = \frac{1}{2 \times e^{\frac{1}{2}}} \cdot \frac{1}{3 \times e^{\frac{1}{3}}} = 0.0724\ldots$

2nd check

$f_X(x) > 0$, $f_{Y|X}(y|x) = f_Y(y)$

$f_{Y|X}(y|x) = \frac{f(x,y)}{f_X(x)} \Rightarrow \frac{f(1,1)}{f_X(1)} = \frac{0.072\ldots}{0.303\ldots} = 0.238$

$f_Y(1) = 0.238$ ✓

$\therefore$ Yes, X & Y are independent.