MATH 244 — Modelling with Differential Equations

2018

These notes follow closely Chs 1-3 and Ch 8 of Boyce + DiPrima's "Elementary DE's + BVPs", 10^{th} Ed^n.

1. Introduction
2. First-Order DE's (+ ways to solve them) } Examples
3. Second-Order DE's (constant coeffs, linear) } Examples
4. Numerical Methods (Ch 8)

INTRODUCTION

An equation containing derivatives (rates of change) is a differential equation (DE). Useful for solving in problems involving fluid flow, electrical circuits, transmission lines, heating + cooling, seismic waves, populations / diseases, chemical reactions, big bang theory, gravitational waves, ...

Eg. Newton's 2nd Law – rate of change of momentum equals applied force

or

$$F = ma = m\frac{dv}{dt}$$

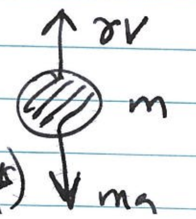
and the force of gravity is mg where in SI units $g = 9.81 \text{ m.s}^{-2}$, so

$$\boxed{mg = m\frac{dv}{dt}},$$ if gravity is the only force acting on an object of mass m

If we assume there is also a drag force due e.g. to air friction, γv with γ a coefficient of drag, that acts in the opposite direction to v:

kg/s

$$\boxed{m\frac{dv}{dt} = mg - \gamma v}$$



~~if object is falling downwards (or going upwards)~~

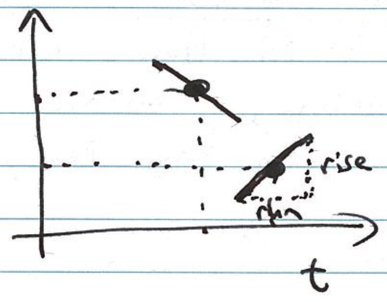
modelling question: What is the equation if the object is thrown upwards, so that it slows down and then begins to fall downwards again?

So we have $\frac{dv}{dt} = g - \left(\frac{\gamma}{m}\right) v$

which is a DE for the unknown function $v(t)$.
How to solve for $v(t)$? Read on.
How to understand solutions without solving?
— use direction field (or slope field):

B+DP p.3

$v(t)$ is a function of t. It's slope is $\frac{dv}{dt}$, and this is given by the DE as $g - \frac{\gamma}{m} v$



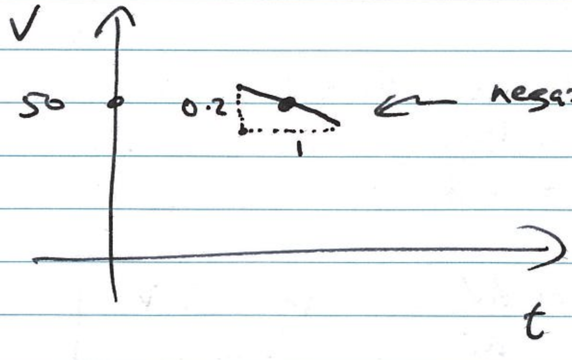
represent slope at a point (t, v) with a short line with slope

$= \frac{\text{rise}}{\text{run}} = g - \frac{\gamma}{m} v$

e.g. if $\gamma = 2$ kg/s and $m = 10$ kg then

$$\frac{dv}{dt} = 9.8 - \frac{v}{5}$$

and at $V = 50$, $\frac{dv}{dt} = 9.8 - 10 = -0.2$ for all t



negative slope decreasing fn.

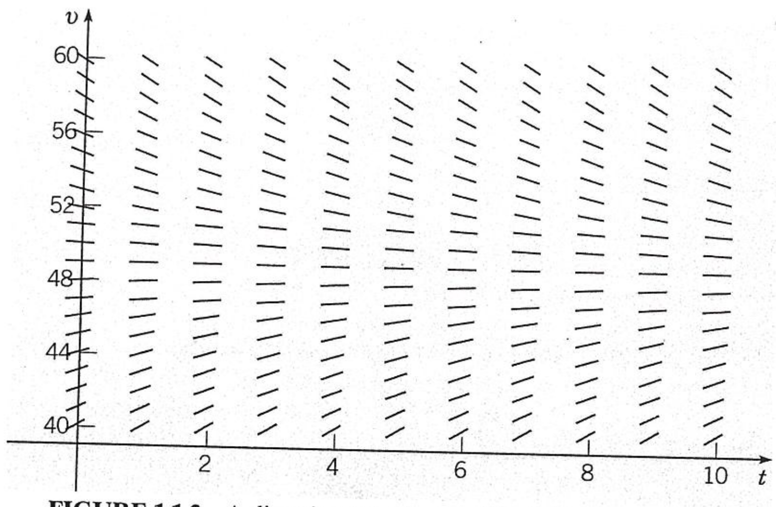


FIGURE 1.1.2 A direction field for Eq. (5): $dv/dt = 9.8 - (v/5)$.

Adding lots of slope marks gives a good picture of solution behaviour

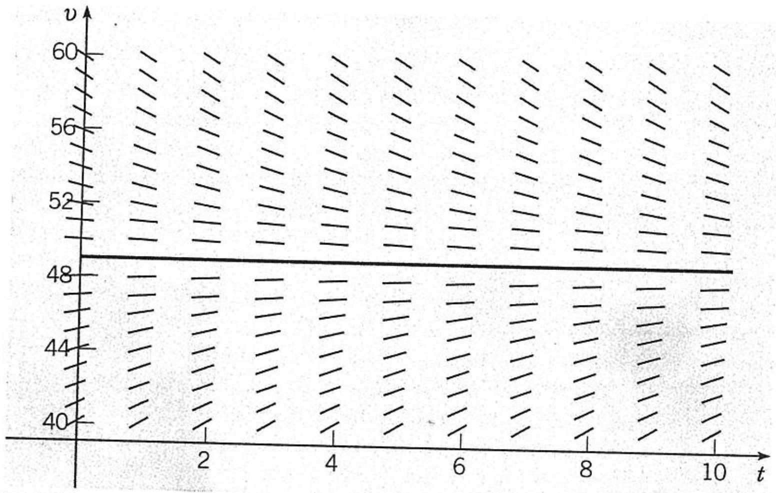


FIGURE 1.1.3 Direction field and equilibrium solution for Eq. (5): $dv/dt = 9.8 - (v/5)$.

$v = 49$ is an equilibrium solution. It is "stable".

Note $\frac{dv}{dt} = 0$ when $g = \left(\frac{\gamma}{m}\right) v$, so that v is not changing with t — an equilibrium solution

[$V = V_e \equiv \frac{mg}{\gamma}$ is an equilibrium solution ; a constant solution.

is when gravity and drag forces exactly balance each other out : "terminal velocity"

skydiver : $V_e \approx$ 200 km/hr $\approx$ 124 mph $\approx$ 56 m/s

[standing up or "balled up" $V_e \approx$ 200 mph] $\approx$ 100 m/s

[Peregrine Falcon $V_e \approx$ 242 mph or 389 kph $\approx$ 108 m/s]

DEs of the form $\dfrac{dy}{dt} = f(t, y)$

where f is some given function of two variables t and y

t ← the independent variable

y ← a dependent variable (depends on t)

are readily analysed using direction fields. See the assignment + tutorial for more examples.

1.2 Solutions of DEs

For the example $\dfrac{dv}{dt} = g - \dfrac{\gamma}{m} v$

we can rearrange it as

$$\frac{dv}{dt} = -\frac{\gamma}{m}\left(v - \frac{mg}{\gamma}\right)$$

$$\Rightarrow \quad \left(\frac{1}{v - \frac{mg}{\gamma}}\right)\frac{dv}{dt} = -\frac{\gamma}{m}$$

and we can get the LHS (left-hand side) by differentiating $\ln\left(v - \frac{mg}{\gamma}\right)$ w.r.t. t:

↖ natural logarithm

$$\frac{d}{dt}\left[\ln\left|v - \frac{mg}{\gamma}\right|\right] =$$

chain rule:

$$= \frac{d}{d\left(v - \frac{mg}{\gamma}\right)}\left[\ln\left|v - \frac{mg}{\gamma}\right|\right]\frac{d\left(v - \frac{mg}{\gamma}\right)}{dt}$$

constant

$$= \left(\frac{1}{v - \frac{mg}{\gamma}}\right)\frac{dv}{dt} \qquad \text{since } \frac{d\,\text{constant}}{dt} = 0$$

since $\dfrac{d}{dx}\left(\ln|x|\right) = \dfrac{1}{x}$ and $x \equiv v - \dfrac{mg}{\gamma}$

$$= \text{LHS} \qquad \left(\text{provided } v \neq \frac{mg}{\gamma}\right)$$

So ~~our~~ DE can be rewritten in the form

$$\frac{d}{dt}\left[\ln\left(v-\frac{mg}{\gamma}\right)\right] = -\frac{\gamma}{m} \quad \text{(a constant)}$$

i.e. the expression in [ ] has constant slope, so <u>we can integrate the DE</u> to get:

$$\ln\left|\left(v-\frac{mg}{\gamma}\right)\right| = -\frac{\gamma}{m}t + C$$

$$\Rightarrow \quad v-\frac{mg}{\gamma} = Ae^{-\frac{\gamma}{m}t} \quad \left(\begin{matrix}\text{where}\\ A=\pm e^{C}\end{matrix}\right)$$

i.e. $\boxed{v = \frac{mg}{\gamma} + Ae^{-\frac{\gamma}{m}t}}$ ⇐ "The General Solution"

<u>Note</u>: since $-\frac{\gamma}{m} < 0$, all solutions decay towards the equilibrium solution $v_e = \frac{mg}{\gamma}$.

Note that $A=\pm e^{C}$ is nonzero. However, if we allow $A=0$, we do still get a solution of the DE, the equilibrium solution. So there is no need to restrict A to be nonzero; A can be any real number. It is truly <u>arbitrary</u>.

Hence we have infinitely many solutions, one for each value of A. "A family of solutions". This agrees with being able to draw so many lines that match the direction field. "Integral Curves", they are sometimes called.

If we choose an initial velocity (say) at $t=0$, say $v(0) = v_0$ (some number) then this initial condition $v(0) = v_0$ leads to a particular value for A:

$$v(0) = v_0 = \frac{mg}{\gamma} + A e^{0} = \frac{mg}{\gamma} + A$$

$$\Rightarrow \quad A = v_0 - \frac{mg}{\gamma}$$

and the solution of this initial value problem is

$$v(t) = \frac{mg}{\gamma} + e^{-\frac{\gamma}{m}t}\left(v_0 - \frac{mg}{\gamma}\right)$$

It corresponds to the integral curve going through the point $(0, v_0)$ when v is graphed against t

i.e. more tidily

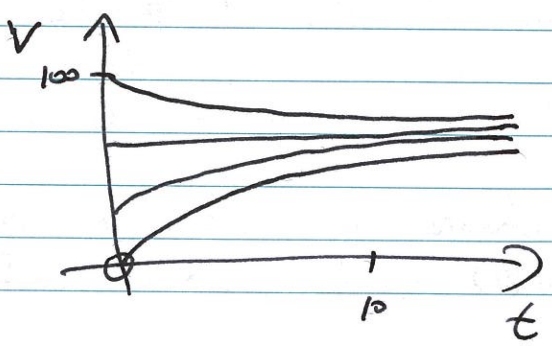
$$\boxed{v(t) = \frac{mg}{\gamma} + \left(v_0 - \frac{mg}{\gamma}\right) e^{-\left(\frac{\gamma}{m}\right)t}} \quad (*)$$

solves the initial-value problem

$$\frac{dv}{dt} = g - \left(\frac{\gamma}{m}\right) v, \qquad v(0) = v_0 .$$

(Read B & DP pp. 10-14 for examples on population dynamics and a falling object with g, γ, m specified.)

(*) Note that graphs of $v(t)$ agree with direction fields.



(*) Note that since $e^{-\frac{\gamma}{m}t}$ decays to zero as $t \to +\infty$,

$V(t) \to \frac{mg}{\gamma}$, $t \to +\infty$, $\forall$ values of v_0.

↑ the equilibrium solution.

(*) Note that if $v_0 = \frac{mg}{\gamma}$, then $V(t) = \frac{mg}{\gamma}$.

(the equilibrium solution)

Classification of DEs (1.3)

Ordinary vs Partial DEs (ODEs vs PDEs)

Ordinary: only ordinary derivatives appear

Partial: only partial derivatives appear

The LRC circuit equation is an ODE:

$$L\frac{d^2Q(t)}{dt^2} + R\frac{dQ(t)}{dt} + \frac{1}{C}Q(t) = E(t)$$

The unknown $Q(t)$ is the charge on a capacitor in a circuit with capacitance C, resistance R, and inductance L and an emf $E(t)$ across it.

The heat conduction equation is a PDE:

$$\frac{\partial T(x,t)}{\partial t} = \alpha^2 \frac{\partial^2 T}{\partial x^2}(x,t)\ , \quad \alpha = \text{constant (perhaps)}$$

note: T is a function of two variables

note: $\frac{\partial T}{\partial t}$ means $\underbrace{\frac{\partial}{\partial t}}_{\text{rate of change with } t}$ holding x constant

In fact, recall $\frac{\partial T(x,t)}{\partial t} = \lim_{h \to 0} \frac{T(x, t+h) - T(x,t)}{h}$

is the definition of a partial derivative.

A System of DEs : If there are two or more unknown functions, a system of DE's is required.

E.g. the Lotka-Volterra eqns

$$\frac{dx}{dt} = ax - \alpha xy \qquad \text{prey } x(t)$$

$$\frac{dy}{dt} = -cy + \gamma xy \qquad \text{predator } y(t)$$

x, y represent populations or total mass or amount of DNA, ...

strictly discrete

$\in \mathbb{R}$

(a, α, c, γ are given constants)

Order of a DE is the highest number of derivatives taken

e.g. $F[\, t, u(t), u'(t), u''(t), \ldots u^{(n)}(t)\,] = 0$

is a general form for an n^{th}-order DE.

We always work with the less general form

$$y^{(n)} = f(t, y, y'', \ldots y^{(n-1)})$$

to avoid multiple roots, e.g. arising in

$$(y')^2 + ty' + 4y = 0 \qquad \text{(First-order)}$$

We would work with, instead, either

$$y' = \sqrt{-ty' - 4y}$$

or $y' = -\sqrt{-ty' - 4y}$

and we would require the $\sqrt{\ }$ to be the mathematical square root function, which for a real $y(t)$ must have non-negative argument, i.e. require $-ty' - 4y \geq 0$.

Linear versus Nonlinear DEs

This is an important distinction, whether the DE is linear in the unknown, or not.

The general linear ODE of order n is

$$a_0(t)\, y^{(n)} + a_1(t)\, y^{(n-1)} + \dots + a_{n-1}(t)\, y' + a_n(t)\, y =$$
$$= g(t)$$

and we will often work with it in standard form, by dividing through by $a_0(t)$

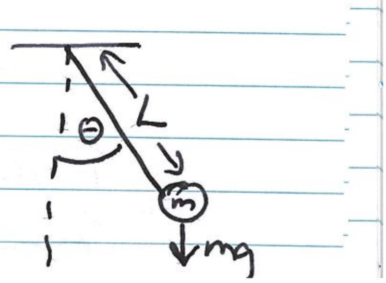
The equation $y''' + 2e^t y'' + yy' = t^4$

is 3rd order, and is nonlinear due to the yy' term.

A classic nonlinear DE is that for a simple pendulum,

nonlinear term

$$\frac{d^2\theta}{dt^2} + \frac{g}{L} \sin\theta = 0$$



to be solved for angle $\theta(t)$.

If we assume θ is small enough,

$$\sin\theta \approx \theta \quad , \text{ and}$$

$$\theta''(t) + \frac{g}{L}\,\theta(t) = 0$$

is the resulting <u>linearised</u> ODE for a simple pendulum.

<u>SOLUTIONS</u> A solution of the ODE

$$y^{(n)} = f(t, y, y', .. y^{(n-1)})$$

on the interval $\alpha < t < \beta$ is a function $\phi(t)$ such that $\phi', \phi'', .. \phi^{(n)}$ exist and satisfy

$$\phi^{(n)}(t) = f\,[\,t, \phi(t), \phi'(t), \dots \phi^{(n-1)}(t)\,]$$

for every t in (α, β). We assume f, y, ϕ are real-valued functions, unless stated otherwise.

It is easy to verify that a given function is a solution! Just substitute it into the DE and see if the DE is satisfied.

Eg. $y = \sin t$ is a solution of

$$y'' + y = 0$$

because $y' = \cos t$, $y'' = -\sin t$, so

$$LHS = y'' + y = -\sin t + \sin t = 0 = RHS,$$

and the DE is <u>satisfied</u>.

There might not <u>be</u> a solution! – existence?
There might be many solutions! – uniqueness?

We are often interested in whether an initial value problem has a unique solution or not.

Computer solutions are always very useful, e.g. using Maple, Matlab or Mathematica. A formula is very powerful for making general statements about how solutions behave, e.g. "all solutions tend towards the equilibrium solution as time increases".

First-Order ODEs (Ch 2, B+DP)

take the standard form $\frac{dy}{dt} = f(t, y)$

where f is a given function of the two variables t, y. We want to find solutions $y = \phi(t)$.

We would like to find ϕ in terms of elementary functions like sin, cos, exp, log, polynomials

2.1 Linear Equations

- use an integrating factor
- always works, although we may find places where the solution does not exist

The standard form for a linear first-order ODE is

$$\frac{dy}{dt} + p(t)\, y = g(t)$$

where p, g are given function of the independent variable t

If the LHS is a perfect derivative, we can simply integrate both sides, e.g.:

$$(4+t^2)\frac{dy}{dt} + 2ty = 4t$$

Since $\frac{d}{dt}\left[(4+t^2)y\right] = (4+t^2)\frac{dy}{dt} + 2ty$

the above DE can be written in the form

$$\frac{d}{dt}\left[(4+t^2)y\right] = 4t$$

Hence, integrating both sides

$$(4+t^2)y = 2t^2 + C$$

arbitrary constant

(Note: to check this, you can now differentiate both sides and get the previous line)

Rearranging we find the general solution explicitly,

$$y = \frac{2t^2 + C}{4 + t^2}.$$

Even if a first-order linear DE is not immediately of the right form (to write the LHS as a derivative) you can always make it so by using an integrating factor:

Consider the standard form of a first-order linear DE,

$$\frac{dy}{dt} + p(t)\,y = g(t)$$

where p, g are given functions. We seek a function $\mu(t)$ which we multiply through by,

$$\mu\frac{dy}{dt} + \mu p\,y = \mu g$$

and we want the LHS to be a total derivative. If we consider

$$\frac{d}{dt}(\mu y) = \mu\frac{dy}{dt} + \frac{d\mu}{dt}y$$

we want $\frac{d}{dt}(\mu y) = \mu \frac{dy}{dt} + \frac{d\mu}{dt} y$

$= LHS \quad = \mu \frac{dy}{dt} + P\mu \, y$

Hence if we set $\boxed{\frac{d\mu}{dt} = p\mu}$, we assure this.

i.e., we choose μ so solve this:

i.e. $\frac{1}{\mu} \frac{d\mu}{dt} = p(t)$

$\Rightarrow \frac{d}{dt}\left(\ln|\mu|\right) = p(t)$

$\Rightarrow \ln \mu = \int p(t)\, dt + \cancel{c}$

and when we work out an antiderivative of $p(t)$, we don't are about any constant of integration — any particular solution will do the job of making the LHS a perfect derivative.

Hence $\mu = e^{\int p(t) dt}$ is our integrating factor; multiplying through by it gives

$$e^{\int p(t) dt} \frac{dy}{dt} + e^{\int p(t) dt} p y = e^{\int p(t) dt} g(t)$$

$$\Rightarrow \frac{d}{dt}\left[e^{\int p(t) dt} y\right] = e^{\int p(t) dt} g(t)$$

$$\Rightarrow e^{\int p(t) dt} y = \int g(t) e^{\int p(t) dt} dt$$

or, written more tidily,

$$\mu(t)y = \int \mu(t)\, g(t)\, dt \quad + c$$

where we write "$+c$" to remind ourselves that when we find an antiderivative of μg, we keep it general by adding an arbitrary constant.

That is, explicitly, our general solution is

$$y(t) = \frac{1}{\mu(t)} \int \mu(t) g(t)\, dt + \frac{c}{\mu(t)}$$

Another way to write this, more carefully, is in the form

$$y(t) = \frac{1}{\mu(t)} \left[\int_{t_0}^{t} \mu(s)\, g(s)\, ds + c \right]$$

and here we do need to include the "c" to allow for the possible initial condition that $y(t_0)$ is nonzero.

[B + DP, p.37]

In general we would not put "+c" until an antiderivative is written down, however.
Note that specifying the value of t_0 is equivalent to specifying "c"

EXAMPLE 4 Solve the initial-value problem

$$t y' + 2y = 4t^2, \qquad y(1) = 2$$

First write the DE in standard form,

$$y' + \underbrace{\frac{2}{t}}_{p} y = \underbrace{4t}_{g}$$

So $\mu = \exp\left(\int \frac{2}{t}\, dt\right) = \exp\left(2 \ln|t|\right)$

i.e. $\mu = \exp\left(\ln|t|^2\right) = t^2$ (choosing the "+" sign, arbitrarily)

is an integrating factor.

Multiply through by μ: <u>it is important to do this to the standard form!</u>

$$\Rightarrow \quad t^2 y' + 2t\,y = 4t^3$$

$$\Rightarrow \quad \frac{d}{dt}\left(t^2 y\right) = 4t^3$$

check: $\frac{d}{dt}$ gets back

$$\Rightarrow \quad t^2 y = t^4 + c$$

integrating both sides

i.e. $y = t^2 + \dfrac{c}{t^2}$ is the general solution to the DE.

The solution to the initial value problem requires us to use $y(1)=2$ to evaluate c:

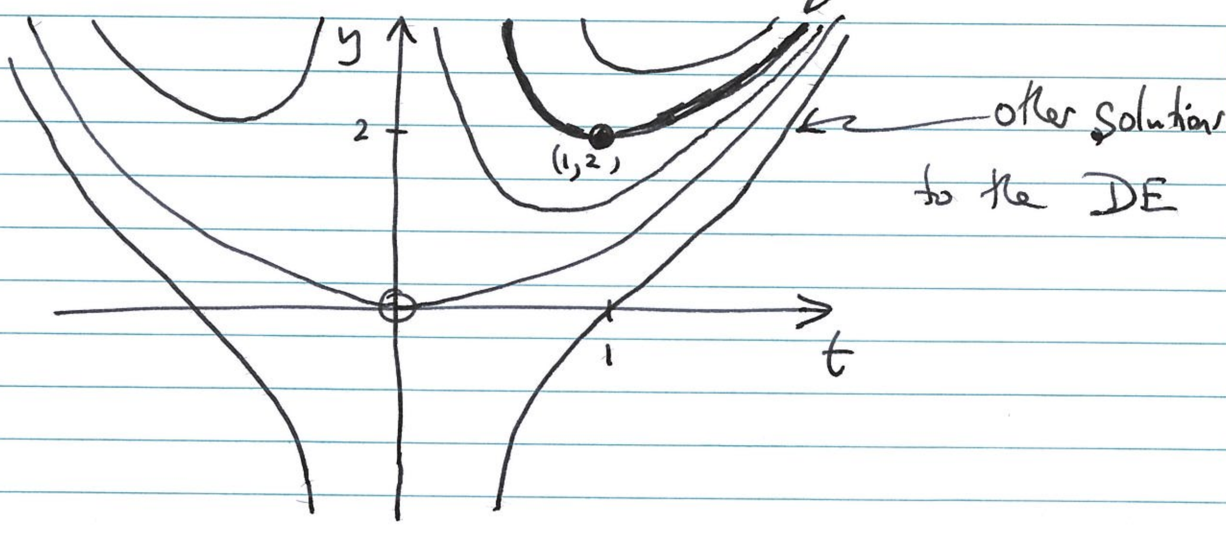
$$y(1) = 2 = 1^2 + \frac{c}{1^2} = 1 + c$$

$$\Rightarrow \quad c = 1$$

$\Rightarrow$ the solution to the IVP is

$$\boxed{y = t^2 + \frac{1}{t^2}}\,.$$

Solution to IVP



Note that $y = t^2 + \frac{1}{t^2}$ has a vertical asymptote at $t=0$. It is undefined there, but is defined everywhere else.

the function $t^2 + \frac{1}{t^2}$

unbounded + not differentiable

The behavior at $t=0$ is due to the standard form of the DE having a $p(t) = \frac{2}{t}$ that is not defined at $t=0$.

However, the solution with $c=0$ is $y=t^2$, and this is defined $\forall t$. It remains bounded, and differentiable, at $t=0$.

SEPARABLE Equations (2.2, B + D.P.)