A one-to-one map (function) remain a function even when x and y are interchanged.

. Many valued function.

Note:

A function can be completely determned from:

1. Its rule or

2. Its graph or

3. The set of ordered pair making it up.

For f a function say that $(x, y) \in f$ to mean x is a first element i.e. the independent for the function f and y is the second element or the dependent variable for the function f.

For the graph displayed immediately behind, we have $(x_1, y_0) \in f$ and $(x_2, y_0) \in f$.

The function f is $1-1$ or one-tone or one-to-one or injetive. If and only if $(x_1, y_0) \in f$ and $(x_2, y_0) \in f$

$\Longrightarrow$ (means) $X_1 = X_2$.

An advantage of injective function is that when the role or X and Y interchanged, we shall have a function

Continuous Function: A function in which the graph is drawn without heling dues pen from the paper.

This is that function whose graph can be drawn without listing the pen from the paper.

POLYNOMIAL FUNCTION

$$P_0(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1 x + a_0, \quad n \in \mathbb{N}$$

$$\mathbb{N} := \{1, 2, 3, \ldots, n, \ldots\} = \text{the set of all natural numbers}$$

$$\mathbb{N}_0 := \{0, 1, 2, 3, \ldots, n, \ldots,\} = \{0\} \cup \mathbb{N}$$

If $n = 3$

$$\begin{aligned} P_3(x) &= a_3x^3 + a_2x^2 + a_1x + a_0 \\ P_5(x) &= a_5x^5 + a_4x^4 + a_3x^3 + a_1x + a_0 \end{aligned}$$

$P_n(x)$ is called a polynomial function of degree n if $a_n \neq 0$

RATIONAL FUNCTIONS

$$\begin{aligned} R(x) &= \frac{P(x)}{Q(x)};\ P(x)\ \text{ and }\ Q(x)\ \text{ are polynomials.} \\ y &= \frac{x^3+4x^2+5}{x-2} \end{aligned}$$

A polynomial function is a rational function with the degrees of.

TRIGONOMETRY FUNCTIONS

If any point P on a circle with centre at the origin and radius one has coordinate $9cx, y)$ and if OP

makes an angle θ with the positive $x-$axis then

$$\begin{aligned} \sin\theta &= \frac{y}{1} = y \\ \cos\theta &= \frac{x}{1} = x \end{aligned}$$

N.B. $-1 \leq \sin\theta \leq 1$.

The graph of $\sin\theta$ of $y = \sin\theta$

Note: $360^0 = 2\pi$ (Radius measure)

$y = \sin\theta$ or $f(\theta) = \sin\theta,\ \ 0 \le \theta \le 2\pi$

or $f : \theta \longmapsto \sin\theta,\ \ 0 \le \theta \le 2\pi$.

Anti-clockwise rotation is taken to be +ve and clockwise rotation is taken the be negative.

Definition: If $f(x+p) = f(x)$ for all values of x, then f is a periodic function with period p. The smallest positive value of p is called the period of f.

$$\begin{aligned}\sin\theta &= \sin(\theta+2\pi) = \sin(\theta+2\pi+2\pi) = \sin(\theta+6\pi) = \cdots = \sin(\theta+2\pi(n-1)) \\ &= \sin(\theta+2\pi(n-1)+2\pi) = \sin(\theta+2\pi n),\ n \in \mathbb{N},\ -\infty < \theta < \infty\end{aligned}$$

RADIANS MEASURE (X^c)

Periodic Extension

$y = \sin\theta$

$f(\theta = \sin\theta \quad 0 \le \theta < 2\pi$

$\sin\theta = \sin(\theta + 2\pi)$ for $\theta \in \mathbb{R},\ \ n \in \mathbb{Z}$

Periodic extension of the sine function

$\mathbb{Z} = \{\cdots, -3, -2, -1, 0, 1, 2, 3, \cdots\}$.

$$
\begin{aligned}
1.\ \sin(750^0) &= \sin(750^0 + 2\pi) \\
&= \sin(370^0 + 360 + 2\pi) \\
&= \sin(390^0 + 4\pi) \\
&= \sin(30^0 + 6\pi) \\
&= \sin 30^0 \\
2.\ \sin(-450^0) &= \sin(270^0 + 4\pi) = \sin 270^0
\end{aligned}
$$

General Rule

$$
\begin{aligned}
\sin\theta &= \sin(\theta + 2\pi) \quad -infty < \theta < +\infty \\
\sin\theta &= \sin(\theta + 2\pi) = \sin(\theta + 2\pi + 2\pi) \\
&= \sin(\theta + 6\pi) = \cdots = \sin[\theta + 2\pi(n-1)] \\
&= \sin(\theta + 2\pi(n-1) + 2\pi) \\
&= \sin(\theta + 2\pi n), \quad n \in \mathbb{N} \qquad (1)
\end{aligned}
$$

Ofcourse,

$$
\therefore \quad \sin\theta = \sin(\theta + x\pi), \quad n \in \mathbb{N} \qquad \sin\theta = \sin(\theta + pi0) \qquad (2)
$$

Graph of $\cos\theta$

$$\begin{aligned} \cos\theta &= \cos(\theta + 2\pi), \quad -\infty < \theta < \infty \\ \cos\theta &= \cos(\theta + 2\pi n), \quad n \in \mathbb{Z} \end{aligned}$$

Tangent Function We define $\tan\theta = \dfrac{\sin\theta}{\cos\theta} = \dfrac{y}{x}$

Graph of $\tan\theta$

θ	0^0	90^0	180^0	270^0	360^0
$f(\theta) = \tan\theta$	0	∞	0	$0 - \infty$	0

$\Longrightarrow \sin(\theta + 2\pi(-n(= \sin(\theta - 2\pi n) = \sin(\theta - 2\pi n) \;\; \forall\, n \in \mathbb{N}$

$\therefore \;\; \sin\theta = \sin(\theta + 2\pi n) = n \in \mathbb{Z}$ from (2) & (3)

$$
\begin{aligned}
\cos\theta &= \cos(\theta + 2\pi) \quad -\infty < \theta < \infty \\
\cos\theta &= \cos(\theta + 2\pi n) \quad n \in \mathbb{Z}
\end{aligned}
$$

$cosec\theta$, secantθ and cotangentθ

$$
\begin{aligned}
cosec\theta &= \frac{1}{\sin\theta} = \frac{1}{y}, \quad y \neq 0 \\
\sec\theta &= \frac{1}{\cos\theta} = \frac{1}{x}, \quad x \neq 0 \\
\cot\theta &= \frac{1}{\tan\theta} = \frac{x}{y}, \quad y \neq 0
\end{aligned}
$$

Graph of $y = cosec\theta$ on the same axis as $y = \sin\theta$

Relationship between the basic curve and others

$$
\begin{aligned}
\cos(\theta - \pi) &= -\cos\theta \\
\sin\left(\theta - \frac{\pi}{\theta}\right) &= -\cos\theta \\
\sin\left(\frac{3\pi}{\theta} - \theta\right) &= -\cos\theta \\
\cos\left(\frac{3\pi}{\theta} + \theta\right) &= \sin\theta
\end{aligned}
$$

$\sin -\theta = -\sin\theta$

$\cos -\theta = \cos\theta$

A function f is odd if $f(-x) = -f(x)$.

$\therefore$ sin is an odd function.

A function f is even if $f(-x) = f(x)$.

$\therefore$ cos is an even function.

Problem 1

1.1 Determine each of the following without using a calculator or table.

$$(a) \quad \sin\frac{\pi}{4} \quad (b) \quad \sec\frac{\pi}{6} \quad (c) \quad cosec\frac{\pi}{2} \quad (d) \quad \tan\tfrac{\pi}{3} \quad (e) \quad \cos\frac{\pi}{3}$$

1.2 Sketch the sine curve

1.3 Use the curve in 1.2 to read off each of the following as circular function of θ

$$(a) \quad \sin\left(\frac{\pi}{2}+\theta\right) \quad (b) \quad \sin\left(\theta-\frac{3\pi}{2}\right) \quad (c) \quad \sin\left(\frac{3\pi}{2}-\theta\right) \quad (d) \quad \sin(\theta-\pi)$$

Problem 2

Give three examples of odd function and two of even functions.

Problem 3

Use the addition formula to prove the following

$$\text{(i)} \;\; \cos x\cos z = \frac{1}{2}(\cos(x+z)+\cos(x-z))$$

$$\text{(ii)} \;\; sinx\sin z = \frac{1}{2}(\cos(x-2)-\cos(x+2))$$

$$\text{(iii)} \;\; \sin x\cos z = \frac{1}{2}(\sin(x+z)+\sin(x-z))$$

Problem 4

With the aid of the substitution

$$x+z=u \;\; \text{and} \;\; x-z=V$$

Prove that the produce formula becomes the following

(a) $\cos U + \cos V = 2\cos\dfrac{U+V}{2} \times \cos\dfrac{U-V}{2}$

(b) $\cos U - \cos V = 2\sin\dfrac{U+V}{2} \times \sin\dfrac{U-V}{2}$

(c) $\sin U + \sin V = 2\sin\dfrac{U+V}{2} \times \cos\dfrac{U-V}{2}$

(d) $\sin U - \sin V = 2\cos\dfrac{U+V}{2}\sin\left(\dfrac{U-V}{2}\right)$

Limits of Functions

The limit of $f(x)$ as x approaches the point $\overset{o}{x}$ is the value that $f(x)$ approaches as $X-\overset{o}{X}$ becomes indefinitely small.

$$\text{Symbolically}\ \lim_{x\to\overset{o}{X}}$$

$f(x) \longmapsto \dfrac{x^2-a^2}{x-a}.$

Symbolically

$$\lim_{x\to\infty} f(x) = \lim_{x\to a}\frac{x^2-a^2}{x-a}.$$

If we are close to a and not at a it is never zero.

$$\begin{aligned} 1.\ f(x) = \lim_{x\to\infty} &= \frac{x^2-a^2}{x-a} \\ &= \frac{(x+a)(x-a)}{(x-a)} \\ &= x+a \ \text{ if } \ x \neq a \end{aligned}$$

$$\therefore\ \lim_{x\to a}\frac{x^2-a^2}{x-a} = \lim_{x\to a}\ x+a = 2a.$$

$$\begin{aligned} 2.\ \lim_{x\to\infty}\frac{x^3-a^3}{x-a} &= \lim_{x\to a}\frac{(x-a)(x^2+ax+\infty)}{(x-a)} \\ &= x^2+x+a \ \text{ if } \ x\neq a \end{aligned}$$

$$\lim_{x\to a}\frac{x^3-a^3}{x-a}=\lim_{x\to a}\ x^2+ax+a^2=3a^2$$

3. $f(x)=\dfrac{Ax^2+Bx+C}{ax^2+bx+C}$

. If ∞ is being considered the

. should be turned around.

$f(x)=?$

$\lim\limits_{x\to\infty}$

$$\frac{Ax^2+Bx+C}{ax^2+bx+c}-\frac{x^2}{x^2}=\frac{A+B\frac{1}{x}+\frac{c}{x^2}}{a+b\frac{1}{x}+\frac{c}{x^2}}\ x\overrightarrow{\to}\infty\ \frac{A}{a}\ \text{ if }\ a\neq 0$$

Arithmetic of Functions

f and g are two functions. Then

$$(f+g)(x)=f(x)+g(x)$$

$y=ax^2+bx$

$$f(x)=ax^2,\ g(x)=bx\quad (f+g)f(x)+g(x)=ax^2+bx$$

$f+g$ is the sum of the two function.

Product

$$\begin{aligned}
(fg)(x) &= f(x)g(x)\\
\text{e.g.}\quad (x^2-a) &= f(x)\\
g(x) &= \frac{1}{x-a}\\
(fg)(x) &= f(x)g(x)\\
&= (x^2-a)\cdot\frac{1}{(x-a)}
\end{aligned}$$

Quotient

$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$

e.g. $f(x) = (x^2 - a^2),\ g(x) = (x - a)$

Then,

$\left(\frac{f}{g}\right)(x) = \frac{x^2 - a^2}{x - a}$

$x \to \infty \cdot \frac{A+0+0}{a+0+0} = \frac{A}{a}.$

Application of Techniques of Limits

$\lim_{\theta \to 0} \frac{\sin\theta}{\theta}$

$$\begin{aligned}
\text{Area of } \triangle OBD &= \text{Area of sector } \triangle OBC \\
&\leq \text{Area of } \triangle OBC \\
\text{Area of} &= \frac{1}{2} OB \times DN \\
&= \frac{1}{2}\sin\theta \\
\text{Area of } \triangle OBC &= \frac{1}{2} OB \times OB\tan\theta \\
&= \frac{1}{2}\tan\theta \\
\text{Area of sector } OBPD &= \pi r^2 \frac{\theta}{2\pi} \\
&= \frac{\theta}{2}
\end{aligned}$$

Area of a sector is the proportion of an angle against the proportion of the whole circle.

$\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = \frac{1}{2}\sin\theta \leq \frac{\theta}{2} = \frac{1}{2}\tan\theta$

$\left(\div \frac{1}{2}\sin\theta\right) = 1 \leq \frac{\theta}{\sin\theta} \leq \frac{1}{\cos\theta}$

$$\lim_{\theta\to\infty} \cos\theta = 1$$
$$= 1 \leq \frac{\theta}{\sin\theta} \leq 1 \lim_{\theta\to 0} \frac{\theta}{\sin\theta} = 1$$
$$\lim_{\theta\to 0} \frac{\theta}{\sin\theta} = 1$$
$$\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1$$

E.G.2

$$\begin{aligned} \lim_{x\to a} \frac{x^3-a^3}{x-a} &= \lim_{x\to a} \frac{(x-a)(x^2+ax+a^2)}{(x-a)} \\ &= \lim_{x\to a} \frac{x-a}{x-a} \lim_{x\to a} x^2+ax+a^2 \end{aligned}$$

E.G. 3

$$\lim_{x\to 1} \frac{\sin 2(x-1)}{x^3-1}$$

$$\frac{\sin 2(x-1)}{x^3-1} = \frac{\sin 2(x-1)}{(x-1)(x^2+x+1)}$$

but $\frac{\sin\theta}{\theta}\ \theta \overrightarrow{\to}\ 0\ 1$ where $2(x-1) = \theta$ $\frac{2\sin 2(x-1)}{2(x-1)(x^2+x+1)}$

$$\begin{aligned} \lim_{x\to 1} \frac{\sin 2(x-1)}{x^3-1} &= 2\lim_{x\to 1} \frac{\sin 2(x-1)}{2(x-1)} \lim_{x\to 1} \frac{1}{x^2+x+1} \\ &= 2\times 1\times \frac{1}{3} \\ &= \frac{2}{3} \end{aligned}$$

E.G. 4

$$\begin{aligned}
\lim_{\theta\to 0}\frac{1-\cos\theta}{\theta} &= \lim_{x\to 0}\frac{2\sin^2\frac{1}{2}\theta-1+1}{\theta}\\
&= \lim_{\theta\to 0}\frac{2\sin^2\frac{1}{2}\theta}{\theta}\\
&= \lim_{\theta\to 0}\frac{\sin^2\frac{1}{2}\theta}{\frac{1}{2}\theta} = \lim_{\theta\to 0}\frac{\sin\frac{1}{2}\theta}{\frac{1}{2}\theta}\lim_{\theta\to 0}\sin\frac{1}{2}\theta\\
&\qquad = 1\times 0\\
&\qquad = 0
\end{aligned}$$

Determine each of the following:

1. $\lim_{x\to a}\dfrac{\lambda^n-a^0}{x-a}$
2. $\lim_{x\to 0}\dfrac{1-\cos\pi x}{x^2}$
3. $\lim_{x\to 0}\dfrac{\sqrt{1+\sin x}-\sqrt{1-\sin x}}{\tan^2 x}$
4. $\lim_{x\to 1}x^3+4x^2+3x+2$
5. $\lim_{t\to 0}\dfrac{1-\cos mt}{1-\cos nt},\quad m,n\in\mathbb{N}$
6. $\lim_{x\to 0}\frac{1-\cos X}{X^2}$

Problems

1. Assume $\lim_{x\to 0}\dfrac{\sin x}{x}=1$
   Compute $\lim_{x\to 0}\dfrac{1-\cos 2x}{x}$
2. Compute each of the following:

(a) $\lim\limits_{x\to 0} \dfrac{\sqrt{(3a-x)}-\sqrt{(x+a)}}{4x-4a}$

(b) $\lim\limits_{x\to 2} \dfrac{\sqrt{(3-x)}-\sqrt{(x-1)}}{6-3x}$

(c) $\lim\limits_{x\to 1} \dfrac{x^2-1}{x-1}$

(d) $\lim\limits_{x\to 1} \dfrac{x-1}{\sqrt{x-1}}$

(e) $\lim\limits_{x\to 1}(2x^2-1)$

(f) $\lim\limits_{x\to 1} \dfrac{1}{\sqrt{x}}$

(g) $\lim\limits_{\theta\to 0} \dfrac{\cos p\theta - \sin q\theta}{\theta^2}$, $p\&q$ are positive integers.

Let P and Q be the points on the continuous curve $y = f(x)$ given by $x = a$ and $x = a$ and $x = a + h$

In the figure a, h is taken to be positive, but it can be either positive or negative. The gradient of the chord PQ is QR/PR, which equals

$$\frac{f(a+h)-f(a)}{h}$$

Keep P fixed and let Q move along the curve towards P. As h tends to zero $f(a+h)-f(a)$ also tends to zero. If the gradient of PQ tendsto a limit, giving the gradient of the curve at P, this limit will be the value at $x = 0$ of a new function. This is known as the derived function of the function f and is denoted by f' or $\dfrac{d}{dx}f(x)$ or $Df(x)$. The value of the derived function f' at any point is called the derivative and is equal to the gradient of the cure $y = f(x)$ at that point. The derivative is defined by

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Differentiation

f is a function with graph

Tangemt os a ;ome tjat touches a point on a curve.

In (1) point $P(x, y)$

In (2) point $P_1(x_1, y_1),\ P_2(x_2, y_2)$

$$\lim_{\substack{x_1 \to x \\ x_2 \to x}} \frac{y_1 - y_2}{x_1 - x_2}$$

$$\lim_{x_1 \to x} \frac{y - y_1}{x - x_1} \quad \text{slope}$$

$$\lim_{x_1 \to x} \frac{f(x) - f(x_1)}{x - x_1}$$

It it exists is the derivative of f at x.

$\dfrac{y - y_1}{x - x_1}$ is a difference quotient.

The limiting derivative of a difference quotient is called differentiation.

Notation

The derivative or f at x is denoted by $(Df)(x)$ or $\dfrac{d}{dx}f(x)$ or $f'(x)$

$$(D.f)(x) = \lim_{x_1 \to x} \frac{f(x) - f(x_1)}{x - x_1}$$

First Principles

1. The constant function $f = C$

$$\begin{aligned} (D.f)(x) &= \lim_{x_1 \to x} \frac{f(x) - f(x_1)}{x - x_1} \\ &= \lim_{x_1 \to x} \frac{C - C}{x - x_1} = 0 \end{aligned}$$

2. The ideality function $I(x) = x$ or $y = x$

$$\begin{aligned} (D.f)(x) &= \lim_{x_1 \to x} \frac{f(x) - f(x_1)}{x - x_1} \\ &= \lim_{x_1 \to x} \frac{x - x_1}{x - x_1} \\ &= 1. \end{aligned}$$

3. $\displaystyle\lim_{x \to a} \frac{x^n - a^n}{x - a} = na^{n-1}$

4. $f := x \to x^n \quad n \in \mathbb{N}$

$$\begin{aligned} (D.f)(x) &= \lim_{x_1 \to x} \frac{f(x) - f(x_1)}{x - x_1} \\ &= \lim_{x_1 \to x} \frac{x^n - x^n}{x - x_1} \\ &= nx^{n-1} \end{aligned}$$

Sum of Two Functions

f and g are differentiable at x with their derivatives.

1. $(Df)(x)$, $(D.g)(x)$ respectively. Then $f+g$ is also differentiable at x with $D(f+g)(x) = (D.f)(x) - (D.g)(x)$.

   Reason:

$$
\begin{aligned}
\frac{(f+g)(x_1+\triangle x)-(f+g)(x)}{\triangle x} &= \frac{f(x+\triangle x)+g(x+\triangle x)-f(x)-g(x)}{\triangle x} \\
&= \frac{f(x+\triangle x)-f(x)}{\triangle x}+\frac{g(x+\triangle x)-g(x)}{\triangle x}
\end{aligned}
$$

$\triangle x \to 0 \quad (D.f)(x)+(D.g)(x)$

$\therefore \quad D(f+g)(x) = (D.f)(x)+(D.g)(x)$

Product of Two Functions

2. Then $-f$ is a differentiable at x with

$$
\begin{aligned}
& D(fg)(x) = f(x)D.g(x)+(Df)(x).g(x) \\
& \frac{(f.g)(x+\triangle x)}{\triangle x} - \frac{(f.g)(x)}{\triangle x} \\
= & \frac{f(x+\triangle x)g(x+\triangle x)-f(x)g(x)}{\triangle x} \\
= & \frac{f(x+\triangle x)g(x+\triangle x)-f(x+\triangle x)g(x)+f(x+\triangle x)g(x)-f(x)g(x)}{\triangle x} \\
= & \frac{f(x+\triangle x)g(x+\triangle x)-f(x+\triangle x)g(x)}{\triangle x}+\frac{f(x+\triangle x)g(x)-f(x)g(x)}{\triangle x} \\
= & \frac{f(x+\triangle x)g(x+\triangle x)-g(x)}{\triangle x}+\frac{f(x+\triangle x)-f(x)g(x)}{\triangle x} \\
\overrightarrow{\triangle x} \to 0 \quad = & f(x)(D.g)(x)+(D.f)(x)g(x).
\end{aligned}
$$

Quotient of Two Functions

$D\left(\frac{1}{g}\right)(x) =?$

$$\begin{aligned}
&= \lim_{\triangle x\to 0} \frac{\frac{1}{g}(x = \triangle x) - \frac{1}{g}(x0}{\triangle x} \\
&= \lim_{\triangle x\to 0} \frac{1}{\triangle x}\left[\frac{1}{g(x = \triangle x)} - \frac{1}{g(x)}\right] \\
&\quad = \frac{1}{\triangle x}\left[\frac{g(x) - g(x+\triangle x)}{g(x+\triangle x)g(x)}\right] \\
&= \lim_{\triangle x\to x} \frac{1}{g(x)}\left[\frac{g(x) - g(x+\triangle x)}{\triangle x}\right] \\
&= -\frac{(D.g)(x)}{[g(x)]^2}.
\end{aligned}$$

If f and g are differentiable

$D\left(\frac{f}{g}\right)$ exists and

$$D\left(\frac{f}{g}\right) = \frac{g\triangle f}{f\triangle g}g^2$$

Reason:

$$\begin{aligned}
D\left(\frac{f}{g}\right) &= D(\left(f \cdot \frac{1}{g}\right) \\
&= f.D\left(\frac{1}{g}\right) + \frac{1}{g}Df \\
&= \frac{f(-Dg)}{g^2} + \frac{1}{g}Df \\
&= \frac{-fDg + gDf}{g^2} \\
&= \frac{g\triangle f - f\triangle g}{g^2}
\end{aligned}$$

$\lim\limits_{\triangle x\to 0} Df(u(x)) \quad Du(x).$

Let $h = f \circ U$.

Then h is differentiable at x. If and only if U is differentiable at x and f is differential at $U(x)$. In this case,

$(Df)u \cdot (Du)x$

$(Dh)(x) = (Df)(u(x))(Du)(x)$

$f := x \longmapsto \sin x$

$(D\sin)(x)$

$$\begin{aligned} \frac{f(x+\triangle x)-f(x)}{\triangle x} &= \frac{\sin(x+\triangle x)-\sin x}{\triangle x} \\ &= \frac{2}{\triangle x}\cos\frac{2x+\triangle x}{2}\sin\frac{\triangle x}{2} \\ &= \frac{\cos(2x+\triangle x)}{2}\frac{\sin\frac{\triangle x}{2}}{\frac{\triangle x}{2}} \\ \lim_{\triangle x\to 0} &= \cos x \\ (D\sin)x &= \cos x \end{aligned}$$

$(D\cos)(x) =?$

$$\frac{\cos(x+\triangle x)-\cos x}{\triangle x} = \frac{-2\sin\left(\frac{\triangle x+2x}{2}\right)\sin\frac{\triangle x}{2}}{\triangle x}$$

$\lim_{\triangle x\to 0} = -\sin x$

E.G. 1

$$\begin{array}{ll} h = (2x-5)(x^5+3x+2) \quad \text{or} & y = x^6\cos x \\ h = fg & \\ f(x) = (2x-5) & Dy = x^6 D(\cos x) + \cos x\ D(x^6) \\ g(x) = (x^5+3x+2) & = x^6(-\sin x) + 6x^5\cos x \\ Dh = f\triangle g + g\triangle f & = 6x^5\cos x - x^6\sin x \end{array}$$

$$
\begin{aligned}
(Dg)(x) &= D(x^3) + D(3x) + D(2) \\
&= 5x^{5-1} + 3 + 0 \\
&= 5x^4 + 3.
\end{aligned}
$$

$$
\begin{aligned}
(Df)(x) &= D(2x) + D(-5) \\
&= 2 + 0 \\
&= 2 \\
Dh &= (2x-5)(5x^4+3) + 2(x^5+3x+2)
\end{aligned}
$$

E.G. 2

$$
\begin{aligned}
h(x) &= \frac{1}{X^5 + 3x + 2} \\
U(x) &= x^5 + 3x + 2
\end{aligned}
$$

$$
\begin{aligned}
f(u(x)) &= \frac{1}{x^5 + 3x + 2} \\
h' &= f'(u) \cdot D(u) \\
D(f) &= -1 - U^{-2} \\
D(u) &= 5x^4 + 3 \\
D(h) &= -U^{-2} + 5x^4 + 3 \\
&= -(x^5 + 3x + 2)^{-2} \cdot 5x^4 + 3
\end{aligned}
$$

E.G. 3

$f := x \longmapsto \sec x$

$\sec x = \dfrac{1}{\cos x} \quad \dfrac{1}{g(x)}$ when $g(x) = \cos x$

$$\begin{aligned}
(D\sec)(x) &= \frac{-Dg(x)}{g^2(x)} \\
(D\sec)x &= \frac{-(-\sin x)}{\cos^2 x} \\
&= \frac{\sin x}{\cos^2 x} \\
&= \frac{\sin x}{\cos x} \cdot \frac{1}{\cos x} \\
&= \tan x \cdot \sec x \\
&= \sec x \tan x
\end{aligned}$$

Problems

1. Determine each of the following:

   $(a)\quad \dfrac{d}{dx}\left(\dfrac{1}{\sqrt{x^2 - 3x - x}}\right) \qquad (b)\quad \dfrac{d}{dx}(\sin x \cos x)$

   $(c)\quad \dfrac{d}{dx}(x^2 + 5)^2 \qquad (d)\quad \dfrac{d}{dx}(\sin x)$

2. Use a geometric approach to show that

$$\lim_{x \to 0} \frac{1 - \cos x}{x} = 0$$

3. Assume $\displaystyle\lim_{x \to 0} \frac{\sin x}{x} = \phi$.

   Show that function f given by

$$f := x \longmapsto \begin{cases} \dfrac{\sin x}{x} & \text{if } x \neq 0 \\[2ex] 1 & \text{if } x = 0 \end{cases}$$

   is differentiable at $x = 0$ and find $(Df)(0)$

Higher Derivative

$$x \longmapsto f(x) \longmapsto (Df)(x) \longmapsto (D(Df))(x)$$

$$\begin{array}{l} D(Df) = (D^2 f)(x) \\ (D(D(Df)))(x) = (D^3 f)(x) \\ \vdots \\ (D \ldots Df)(x) = (D^n f)(x) \\ D(Df) = \lim_{\triangle x \to 0} \frac{(Df)(x+\triangle x) - Df(x)}{\triangle x} \\ \qquad\qquad Df \\ D(Df) = D^2 f \\ (D)(D)(Df) = (D^3 f)(x) \\ DDD(Df) = (D^4 f)(x) \\ \vdots \\ \underbrace{DDD \ldots Df}_{n \text{ tuples}} = (D^n f)(x) \end{array}$$

$$\begin{aligned} D^4(\sin x) &= (D(D(D(D \sin x))) \\ &= DD(D \cos x) \\ &= DD(-\sin x) \\ &= D(-\cos x) \\ &= \sin x \end{aligned}$$

E.G. 1

$$\begin{aligned} (D^3 \sin)(x) &= [D(D(D \sin))](x) \\ &= D(\triangle \cos x) \\ &= D(-\sin x) \\ &= -\cos x \end{aligned}$$

Linearization (Application of Differentiation)

Slope of the straight line $= (Df)x_0 = \dfrac{y - y_0}{x - x_0}$.

$y - y_0 = (Df)(x_0)[x - x_0]$.

This is the linearization of the curve at (x_0, y_0).

Approximation Errors and Small Chances

If f is a function

$$(Df)(x) = \lim_{\triangle x \to 0} \frac{f(x + \triangle x) - f(x)}{\triangle x}$$

For small $\triangle x$

$$\frac{\triangle y}{\triangle x} = \frac{f(x + \triangle x) - f(x)}{\triangle x} = (Df)(x).$$

For Error (change between the value of x and the value near it).

$$f(x + \triangle x) - f(x) \doteqdot (Df)(x)$$

For Error (change between the value of x and the value near it).

$$f(x + \triangle x) - f(x) \doteqdot (Df)(x)\triangle x$$

For Approximation (value of x)

$$f(x + \triangle x) \doteqdot f(x) + (Df)(x)\triangle x.$$

E.G.

Estimate $(0.999)^{10}$

$$(0.999)^{10} = (1 - 0.001)^{10}$$

Now consider

$$f(x) = (x)^{10}$$

then

$$f(x + \triangle x) = [1 + (0001)]^{10}$$

$$x = 1 = \triangle x = -0.001$$

$$f(x + \triangle x) \doteqdot f(x) + (Df)(x)\triangle x$$

$$f[1 + (-00001)]^{10} \doteqdot f(1) + 10(1)(-0.001)$$

$$\begin{aligned} (0.999)^{10} &\doteqdot [1 + -0.001]^{10} \\ &\doteqdot (1)^0 + 10(1)(-0.001) \\ &\doteqdot 1 - 0.01 \\ &= 0.99 \end{aligned}$$

E.G.

A sphere of radius r and volume V

$V = \dfrac{4}{3}\pi r^3$.

Let $\triangle r$ = small change in r

Arising from measuring the radius

$\triangle v$ = the corresponding change in volume.

If the error of measurement arising from $r = 21$ is ± 0.5 find the corresponding error in the volume.

Solution

$$\begin{aligned} V &= \frac{4}{3}\pi r^3 \\ \frac{\triangle V}{\triangle r} &= \frac{d}{dr}(V) \\ \text{Error} &= \frac{dv}{dr}\triangle r \end{aligned}$$

If $r = 21$ and $\triangle r = 0.5$

Then

$$\begin{aligned} \triangle v &= \left(\frac{dv}{dr}\Big/_{r=21}\right)0.5 \\ &\doteqdot \left(4\pi r^2/_{r=21}\right)0.5 \\ &\doteqdot 4\pi(21)^2 \times 0.5 \\ &\doteqdot 2772. \end{aligned}$$

Problem

Calculate $\sqrt[3]{0.5}$ without using a calculator or any other technical aid.

Extrema

X_0 and X_2 = local maximum

V_1 = local minimum

$X_2 =$ Absolute maximum

$e =$ Absolute local minimum.

Given a function R. It is said to have a maximum at the point $X - X - 0$ if $f(x_0) \geq f(x)$ for all x and $f(x_0)$ is the absolute maximum point.

f is said to have a relative maximum at x_0 if $\exists^n \delta > 0] = [X - X_0| < \delta \Longrightarrow f(x_0) \geq f(x)$ and $f(x_0)$ is called a relative maximum.

Given a function f, it is said to have a minimum at the point $X - X_0$ if $f(x_0) \leq f(x)$ for all x and $f(x_0)$ is the absolute minimum point.

f is said to have a relative minimum at X_0 if $\exists\, d > 0$ such that $|X - X_0| < \delta \Longrightarrow f(x_0) \leq f(x)$ and $f(x_0)$ is called a relative minimum.

$$\frac{f(X_0 + \triangle x) - f(x_0)}{\triangle x} \quad x \longmapsto \left| \begin{array}{ll} -5x + 3 & \text{if } x < X_0 \\ \\ +4X - 3 & \text{if } x > X_0 \end{array} \right.$$

When $-\varepsilon = \triangle x$ -ve   When $\varepsilon > 0$

$$\frac{-5(x_0 - \varepsilon) + 3 - (-5x_0) - 3}{-\varepsilon} = \frac{5\varepsilon}{-\varepsilon} = -5$$

When $\varepsilon = \triangle x$ +ve

$$\frac{4(x_0 + \varepsilon)^{-8} - (4x_0) + 8}{\varepsilon} = \frac{4\varepsilon}{\varepsilon} = 4$$

Second Derivative test

For a function f,

$Df(x_0) = 0$ for it to have a max. or min

$(D(Df))(x_0) > 0$ for minimum

$(D(Df))x_0 < 0$ for maximum

Df must exist at and around the point in question i.e. X_0

$(Df)(x_0) = 0$

$(D(Df))x_0 < 0$

$\implies$ Point of inflexion.

E.G. 1

Suppose the value of a car depreciates in such a manner that its value in naira t years after it was purchased is given by:

$$f(t) = 50 + \frac{2000}{t+1}$$

(a) What is the rate of depreciation when the car is three years old?

(b) How does the rate of depreciation at the end of two years compare with the end of three years.

(c) By how much has the car depreciated during the 3rd year?

Solution

$$\begin{aligned}(a) \quad f'(3) &= \left.\frac{-2000}{(t+1)^2}\right|_{t=3} \\ &= \frac{-2000}{16} = -125 \text{ Naira per year} \\ (b) \quad f'(2) &= \left.\frac{-2000}{(t+1)^2}\right|_{t=2} \\ &= \frac{-2000}{9} = -222\frac{2}{9}\end{aligned}$$

$f'(2) - f'(3) - 222\frac{2}{9} + 125 = -97.2$ Naira.

$(c) = (a)$.

E.G. 2

A rocket is fired up. Its distance in cm above the ground at the end of t seconds is given by

$y = 640t - 16t^2$. Describe the motion of the rocket.

Solution

$$\begin{aligned} y' &= 640 - 32t \\ y' &= -32 \end{aligned}$$

When velocity $= 0$

$t = \frac{640}{32} = 20.$

Then

$$\begin{aligned} y &= 640(20) - 16(20)^2 \implies \text{ maximum height reached.} \\ &= 12800 - 6400 \\ &= 6400 \end{aligned}$$

Curve Sketching

To sketch a curve of the graph of a function.

1. Determine when the graph crosses the X-axis i.e. $y = 0$.
2. Determine where the graph crosses the y-axis i.e. $X = 0$.
3. Determine the behaviour of the graph when X is very large.
4. Determine the behaviour of the graph when Y is very large
5. Determine the turning points i.e. the extrema and the point if inflexion.
6. Polynomials are continuous and smooth i.e. they do not have sharp corner.

   Moreover, between any two maxima there is a minimum and vice-versa.
7. Rational functions are in most parts smooth, they have blow-up points (Asymptotes)

$y = [X] \implies y$ is an integer $< x$

It is called the greatest integer function.

If $x = \dfrac{1}{2} \quad y = 0$
If $x = 578999 \quad y = 5.$

Step function

Linear Function

$y = mx + c$

Quadratic Graph

$y = ax^2 + bx + c$

$$
\begin{aligned}
y &= |X| \\
y &= \begin{cases} X & \text{if } X \geq 0 \\ \\ -X & \text{if } X < 0 \end{cases}
\end{aligned}
$$

E.G. 1

$f(x) = 7y = 4x^3 - 5x^2 = 2x - 1.$

When

$X = 0, \;\; y = -1$

$y = 0$ then $4x^3 - 5x^2 + 2x - 1 = 0$

$(x-1)$ is a factor.

$$
\begin{array}{r|l}
 & \underline{4x^2 - x + 1} \\
\underline{x-1} & 4x^3 - 5x^2 + 2x - 1 \\
 & \underline{-(4x^3 - 4x^2)} \\
 & \qquad\quad -x^2 + 2x - 1 \\
 & \qquad\quad -\underline{(-x^2 + x)} \\
 & \qquad\qquad\qquad x - 1 \\
 & \qquad\qquad\qquad \underline{x - 1} \qquad\qquad (1,0)
\end{array}
$$

$x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ complex number

$Dy = 12x^2 - 10x + 2.$

At the T.P. $\quad Dy = 0$

$$
\begin{aligned}
x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\
&= \frac{10 \pm \sqrt{100 - 4(12)(4)}}{24} \\
&= \frac{10 \pm \sqrt{4}}{24} \\
&= \frac{10+2}{24} \text{ or } \frac{10-2}{24} \\
x &= \frac{1}{2} \text{ or } \frac{1}{3} \\
D^3y &= 24x - 10|_{x=\frac{1}{3}} < 0
\end{aligned}
$$

$D^2y|_{x=\frac{1}{3}}$

$\dfrac{1}{3}$ is a max. $\qquad \dfrac{1}{2}$ ia a min.

when $x = \frac{1}{2}$

$$\begin{aligned} y &= 4x^3 - 5x^2 + 2x - 1 \\ &= 4(\frac{1}{3})^2 - 5(\frac{1}{3})^2 + 2(\frac{1}{3}) - 1 \\ &= \frac{4}{27} - \frac{5}{9} + \frac{2}{3} - 1 \\ &= \frac{4 - 15 + 18 - 27}{27} \\ &= \frac{-20}{27}. \end{aligned}$$

When

$$\begin{aligned} x = \frac{1}{2} &= 4(\frac{1}{2})^3 - 5(\frac{1}{2})^2 + 2(\frac{1}{2}) - 1 \\ &= \frac{4}{8} - \frac{5}{4} = 1 \\ &= \frac{4 - 10}{8} = -\frac{6}{8} = \frac{-3}{4} \end{aligned}$$

When x is large and +ve, y is large and +ve

When x is large and -ve, y is large and -ve.

1. $\frac{x^2 - 1}{x^2 + 1}$
2. $y = 4x^3 - 5x^2 - 2x - 1$
3. $y = \frac{1}{x}$

Solution

$f(x) = y = 4x^3 - 5x^2 - 2x - 1$

$(Df)(x) = 12x^2 - 10x - 2$

At A.P. $= 0$

$$\begin{aligned}
&12x^2 + 2x - 12x - 2 = 0\\
&2(6x+1) - 2(6x+1) = 0\\
&(6x+1)(2x-2)\\
&x = 1 \ \text{ or } \ -\frac{1}{6}.
\end{aligned}$$

$(D^f)(x) = 24x - 10 \ 24x - 10|_{x=1} = 14$ min $\qquad 24x - 10|_{x=\frac{-1}{6}} = -14$ max.

$y = 4(1)^3 - 5(1)^2 - 2(1) - 1 = 4 - 5 \cdot 2 - 1 = -1$

$$\begin{aligned}
y = 4\left(-\frac{1}{6}\right)^2 - 5\left(-\frac{1}{6}\right) - 2\left(\frac{1}{6}\right) - 1 = -\frac{4}{216} - \frac{5}{36} + \frac{1}{3} - 1 \quad &= \quad \frac{-4 - 30 + 72 - 216}{216}\\
&= \quad \frac{-175}{216} = \frac{-89}{108}
\end{aligned}$$

$f(x) = y = \dfrac{x^2 - 1}{x^2 + 1}$

$x = 0, \quad y = -1 \quad (0, -1)$

$y = 0, \quad x = \pm 1 \quad \Rightarrow \quad x^2 - 1 = 0$

$(1, 0)$

$; (-1, 0)$

Express X in terms of y

$y = \dfrac{x^2 - 1}{x^2 + 1}$

$x^2 - 1 = y(x^2+1)$
$x^2 - 1 = yx^2 + y$
$x^2 - yx^2 = y + 1$
$x^2(1-y) = y+1$
$x^2 = \dfrac{y+1}{1-y}$
$x = \sqrt{\dfrac{y+1}{1-y}}$
For x to be real $1 - y > 0$ i.e. $y < 1$.

$x \longrightarrow +\infty$

$$f(x) = \frac{x^2+1}{x^2+1} \quad = \quad \frac{1-\frac{1}{x^2}}{1+\frac{1}{x^2}}\ x \overrightarrow{\rightarrow} \infty\ 1$$

$$f(x) \quad x \overrightarrow{\rightarrow -}\infty \quad = \quad 1$$

Determine the T.P.

$$\begin{aligned}(Df)(x) \quad &= \quad \frac{2x(x^2+1) - 2x(x^2-1)}{(x^2+1)^2} \\ &= \quad \frac{2x(2)}{(x^2+1)^2}\end{aligned}$$

At T.P.

$$\frac{2x(2)}{(x^2+1)^2} = 0 \Longrightarrow x = 0$$

$f(x) = y = \dfrac{1}{x}$
$x = 0 \quad y = \infty$
$y = 0 \quad x = \infty$

$$(Df)(x) = \frac{-1}{x^2}$$

(1) $y = \dfrac{1}{|x|}$

(2) $y = \dfrac{2x^2}{x^2 - 1}$

(3) $y = x^3 - 3x^2 - 6$

Integration

$f : \mathbb{R} \longrightarrow \mathbb{R}$

$I = \displaystyle\int f(x)dx$

That function which when differentiated gives $f(x)$.

$I =$ indefinite integral of f is Anti-derivative of f or primitive of f.

i.e. $\dfrac{dI}{dx} = f(x)$.

E.G. 1

$I = \displaystyle\int x^2 dx$

Find I such that $\dfrac{dI}{dx} = X^2$.

$$I = \frac{x^3}{3} + C \qquad \text{i.e. } \frac{d}{dx}\left(\frac{x^3}{3} + C\right) = x^2$$

E.G. 2

$$\begin{aligned} I &= \int \cos x \; dx \\ &= \sin x + C \end{aligned}$$

Reason:

$$\frac{d}{dx}(\sin x + C) = \cos x$$

Integration as Area Under a Curve

Let $A(x)$ be the area under the curve $y = f(x)$ from the limit $X = a_0$ to $X = X$.

Then $A(X + \triangle x)$ the area under the curve from $X = a_0$ to $X = X + \triangle x$.

$$A(X + \triangle x) - A(x) \doteqdot f(x)\triangle x$$

$$\frac{A(X + \triangle x) - Ax}{\triangle x} \doteqdot f(x)$$

$$\frac{dA(x)}{dx} = \lim_{\triangle x \to 0} \frac{A(x + \triangle x) - A(x)}{\triangle x} = f(x)$$

$\therefore$ By the definition of integral

$$A(x) = \int f(x)dx + C.$$

Properties of Indefinite Integrals

1. If f and g are two integrable functions, then

$$\int (f + g)(x)dx = \int f(x)dx + \int g(x)dx$$

Reason.

$$\int (f + g)(x)dx := H(x)$$

$$\int f(x)dx := F(x)$$

$$\int g(x)dx := G(x)$$

$$\begin{aligned} \frac{dH(x)}{dx} &= (f + g)(x) = f(x) + g(x) \\ \frac{dF(x)}{dx} &= f(x) - \frac{dG(x)}{dx} = g(x) \end{aligned}$$

$$
\begin{aligned}
\frac{d}{dx}(F(x)+G(x)) &= \frac{dF(x)}{dx}+\frac{dG(x)}{dx} \\
&= f(x)+g(x) \qquad (**) \\
\frac{dH(x)}{dx} &= \frac{d}{dx}(F(x)+G(x) = \frac{dF(x)}{dx}+\frac{dG(x)}{dx}
\end{aligned}
$$

$H(x) = f(x) + g(x)$.

By definition $\therefore \quad \int (f+g)(x)dx = \int f(x)dx + \int g(x)dx.$

E.G.

$$
\begin{aligned}
\int (x^2+x^5)dx &= \int x^2dx + \int x^5dx \\
&= \frac{x^3}{3}+\frac{x^6}{6}+C.
\end{aligned}
$$

2. Suppose f is an integrable function and a is a constant

$$\int (af)(0)dx = a\int f(x)dx.$$

Reason. Suppose $\int f(x)dx = F(x)$

Then

$$
\begin{aligned}
\frac{d}{dx}(aF(x)+C) &= a\frac{d}{dx}F(x)+0 \\
&= af(x)
\end{aligned}
$$

By definition

$\therefore \quad \int a(f(x0dx = a\int f(x)dx$

f is a functin and $[a,b]$ are interval.

$\therefore \quad \int_a^b f(x)dx$

The definite integral of f w.r.t. x within that limit a and b and it means

$$\int_a^b f(x)dx := [f(x)]_a^b := [f(b) - f(a)]$$

$$\begin{aligned}\int_1^x x^2 dx = \left[\frac{x^3}{3}\right]_1^3 &= \frac{27}{3} - \frac{1}{3} \\ &= \frac{26}{3} = 8\frac{2}{3}\end{aligned}$$

Properties of Definite Integrals

1. $\displaystyle\int_0^b f(x)dx = \int_0^c f(x)dx + \int_c^b f(x)dx$
   Reason: $f(b) - f(a) = f(c) - f(a) + f(b) - f(c)$

   $$\therefore \quad \int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

2. $\displaystyle\int_a^b f(x)dx = -\int_b^a f(x)dx.$
   Reason:

   $$\begin{aligned}f(b) - f(a) &= [f(a) - f(b)] \\ &= f(b) - f(a)\end{aligned}$$

E.G.

$$
\begin{aligned}
\int_1^2 (x+3)dx &= \left[\frac{x^2}{2}+3x\right]_1^2 \\
&= (2+6)\left(\frac{1}{2}+3\right) \\
&= 8-\frac{7}{2}=\frac{9}{2}. \\
\int_2^1 (x+3)dx &= \left[\frac{x^2}{2}+3x\right]_2^1 \\
&= \left(\frac{1}{2}+3\right)-(2+6) \\
&= \frac{7}{2}-5=-\frac{9}{2}.
\end{aligned}
$$

Area under the curve $f(x)$ limit a and b is $\displaystyle\int_a^b f(x)dx$.

Reason:

$$
\begin{aligned}
A(x) &= f(x)+C \\
\int_a^b f(x)dx &= f(b)f(a) \\
&= A(b)-C-(A(a)-C) \\
&= A(b)-A(a)
\end{aligned}
$$

$$\int_c^b X dx = \left[\frac{x^2}{2}\right]_0^3 = \frac{9}{2}$$

Find the Area under the curve $y = x^2$ between $x = 0$ and $x = 2$.

3. Find the area enclosed between the curve $y = x^2$ and $y = x$.

$$\begin{aligned}
\int_0^1 x dx - \int_0^1 x^2 dx &= \int (x - x^2) dx \\
&= \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 \\
&= \frac{1}{2} - \frac{1}{3} = \frac{1}{6}
\end{aligned}$$

Integration by Substitution

Suppose y is a function of X and y is a function of U which is a function of

X. Then $y = y(u)$ and $u = u(x)$.

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} \Longrightarrow y = \int f(x)dx \\ \frac{dy}{du} &= \frac{dy}{dx} \Big/ \frac{du}{dx} \Longrightarrow y = \int f(u)du \end{aligned}$$

E.G. Find

$$\begin{aligned} & \int (2x+3)^2 dx \\ I &= \int (2xrs)^2 dx \\ \frac{dI}{dx} &= (2x+5)^3 \end{aligned}$$

Put $u = 2x + 5$

$$\begin{aligned} \frac{du}{dx} &= 2 \\ \frac{dI}{du} &= \frac{dI}{dx} \Big/ \frac{du}{dx} \\ &= \frac{(2x+s)^2}{2} \\ &= \frac{u^5}{2} \\ I &= \int \frac{U^3}{2} du \\ &= \frac{U^4}{8} + C \\ &= \frac{(2x+5)^4}{8} + C \end{aligned}$$

Find

$$\int \sin(ax+b)dx$$

$$\begin{aligned}
I &= \int \sin(ax+b)dx \\
\frac{dI}{dx} &= \sin(ax+b)
\end{aligned}$$

Let $U = ax + b$

$$\begin{aligned}
\frac{dI}{du} &= \frac{du}{dx} = a \\
\frac{dI}{du} &= \frac{dI}{dx} \Big/ \frac{du}{dx} \\
&= \frac{\sin U}{a} \\
I &= \int \frac{\sin u}{a} \\
&= \frac{1}{a} \int \sin u \\
&= \frac{-\cos u}{a} + C \\
&= \frac{-\cos(ax+b)}{a}
\end{aligned}$$

Solution

1. $\lim_{x\to\infty} \dfrac{1-\cos 2x}{x}$

$$
\begin{aligned}
&= \lim_{x\to 0} \frac{1-(\cos^2 x - \sin^2 x)}{x} \\
&= \lim_{x\to 0} \frac{1-(1-\sin^2 x - \sin^2 x)}{x} \\
&= \lim_{x\to 0} \frac{1-1+2\sin^2 x}{x} \\
&= \lim_{x\to 0} \frac{2\sin^2 x}{x} \\
&= 2\lim_{x\to 0} \frac{\sin x}{x} \quad \lim \sin x \\
&= 2 \times 1 \times 0 \\
&= 0
\end{aligned}
$$

2(a) $\lim_{x\to a} \dfrac{\sqrt{(3a-x)} - \sqrt{(x+a)}}{4(x-a)}$

$$
\begin{aligned}
&= \lim_{x\to a} \frac{(\sqrt{(3a-x)} - \sqrt{(x+a)}) \times (\sqrt{3a-x} + \sqrt{(2x+a)})}{4(x-a)(\sqrt{(3a-x)} + \sqrt{(x+a)})} \\
&= \lim_{x\to a} \frac{3a-x-x-a}{4(x-a)(\sqrt{(3a-x)} + \sqrt{(x+a)})} \\
&= \lim_{x\to a} \frac{2a-2x}{4(x-a)(\sqrt{(3a-x)} + \sqrt{(x+a)})} \\
&= \lim_{x\to a} \frac{-2(x-a)}{4(x-a)(\sqrt{(3a-x)} + \sqrt{(x+a)})} \\
&= \lim_{x\to a} \frac{-1}{2(\sqrt{(3a-x)} + (\sqrt{x+a}))} \\
&= \frac{-1}{2(\sqrt{3a} + \sqrt{2a})}
\end{aligned}
$$

(c) $\lim_{x \to 1} \dfrac{x^2 - 1}{x - 1}$

$$\begin{aligned}
&= \lim_{x \to 1} \frac{(x-1)(x+1)}{(x-1)} \\
&= \lim_{x \to 1} \frac{(x-1)}{(x-1)} \lim_{x \to 1}(x+1) \\
&= 1 \times 2 \\
&= 2
\end{aligned}$$

(d) $\lim_{x \to 1} \dfrac{x - 1}{\sqrt{x - 1}}$

$$\begin{aligned}
&= \lim_{x \to 1} \frac{x - 1(\sqrt{2-1})}{(\sqrt{(x-1)})(\sqrt{(x-1)})} \\
&= \lim_{x \to 1} \frac{(x-1)(\sqrt{(x-1)})}{(x-1)} \\
&= \lim_{x \to 1} \frac{(x-1)}{(x-1)} \lim_{x \to 1} \sqrt{(x-1)} \\
&= 1 \times 0 \\
&= 0
\end{aligned}$$

(e) $\lim_{x \to 1}(2x^2 - 1)$

$$\begin{aligned}
&= 2(1)^2 - 1 \\
&= 2 - 1 \\
&= 1
\end{aligned}$$

(f) $\lim_{x \to 1} \frac{1}{\sqrt{x}}$

$$\begin{aligned} &= \frac{1}{\sqrt{1}} \\ &= \frac{1}{1} \\ &= 1 \end{aligned}$$