2.004 Lab 7 Intro
Flywheel Position Control

Fall, 2021

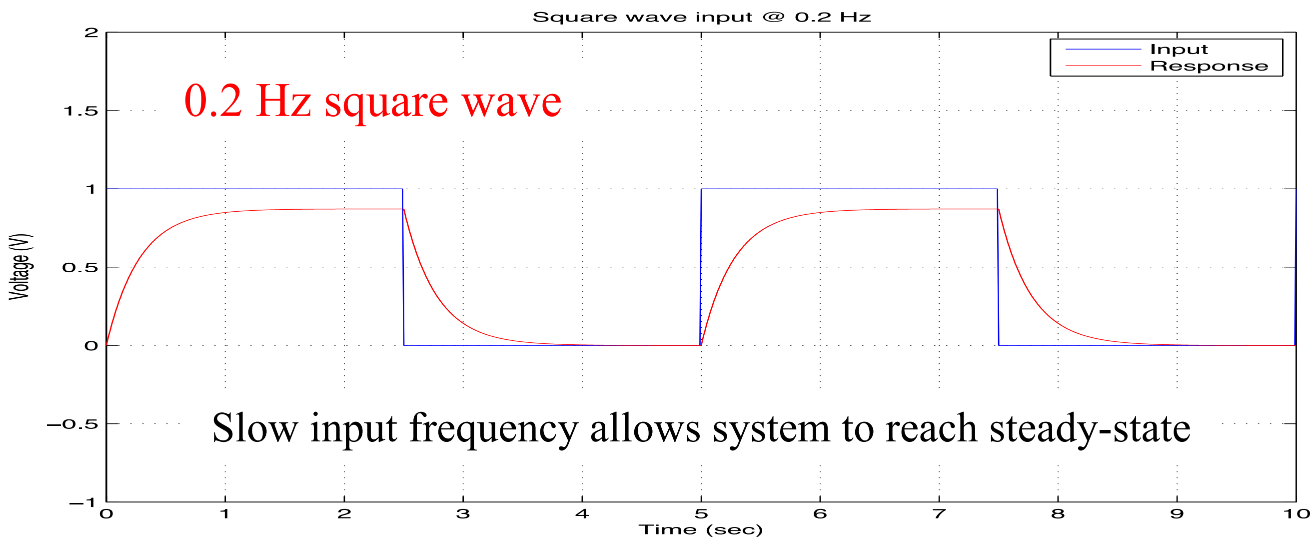
Today's Tasks

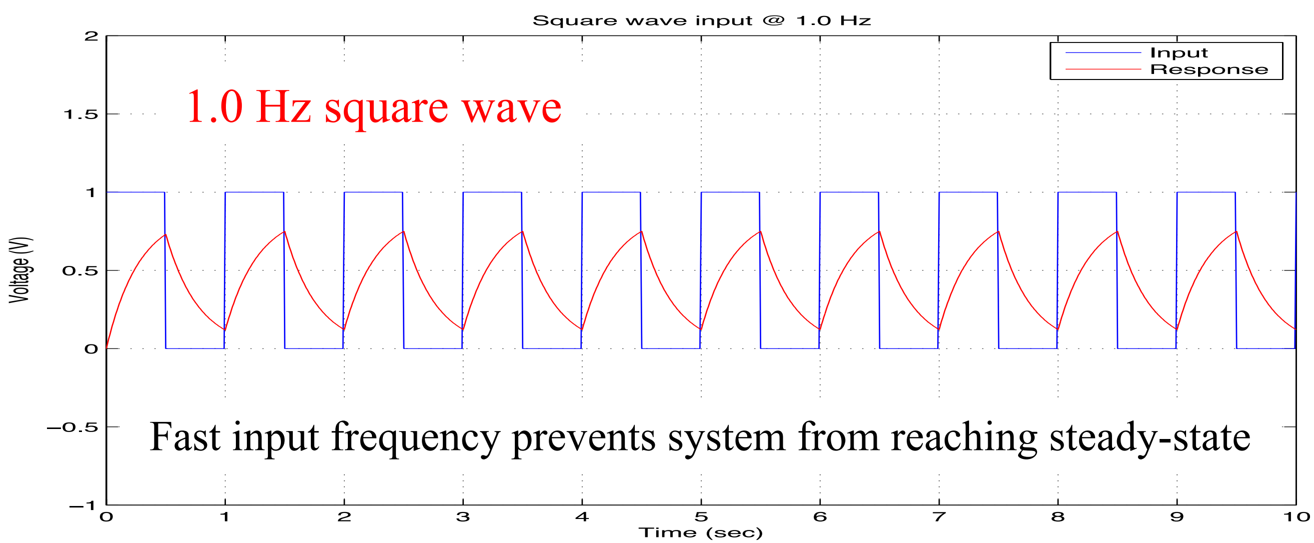
- Feedback control design based on frequency domain method:
  - **Experiment #1:** Acquire experimental Bode plots of the DC motor plant
  - **Experiment #2:** Lead controller (compensator) design
  - **Experiment #3:** Test the controller
- Deliverable:
  - Lab 7 report and prelab (one report per person)

Advantages of Frequency Design Method

- Can infer performance and stability from the same plot
- Can use measured data rather than a transfer function (pole-zero) model
- The design process can be independent of the system order
- Time delays are handled correctly
- Graphical techniques (analysis and synthesis) are relatively simple

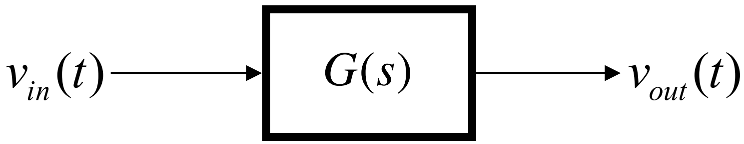
Step Response



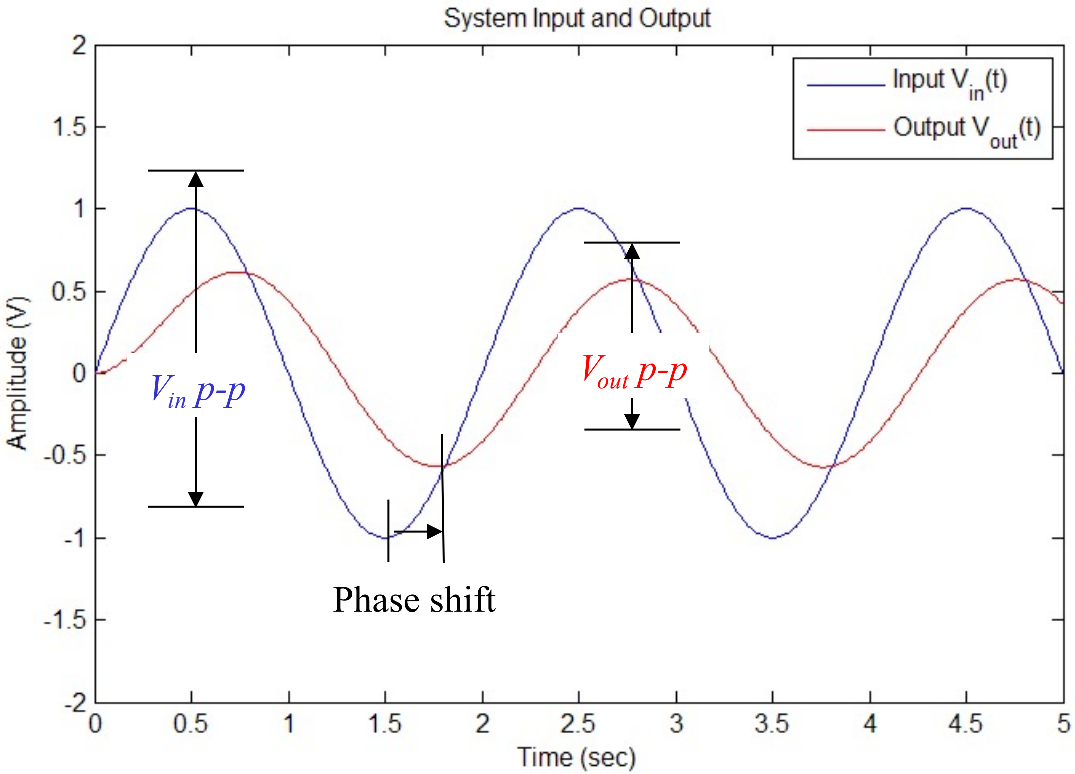


Sinusoidal (Harmonic) Response

Sinusoidal input is typically used to determine a system's frequency response

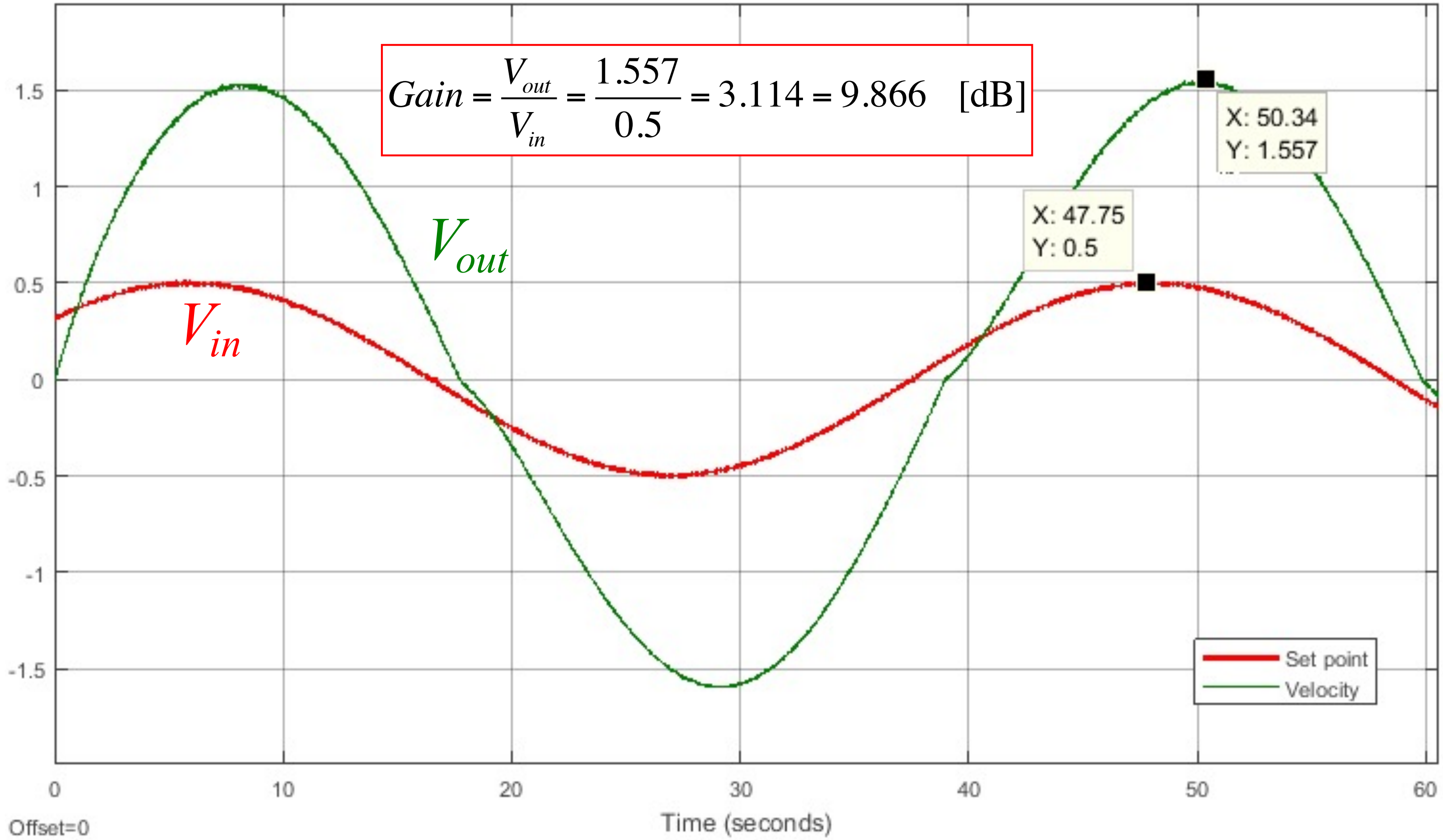


$$s = j\omega$$



Gain and Phase

Input signal: frequency 0.02 Hz, amplitude 1 V_{pp}



$$Gain = \frac{V_{out}}{V_{in}} = \frac{1.557}{0.5} = 3.114 = 9.866 \quad [dB]$$

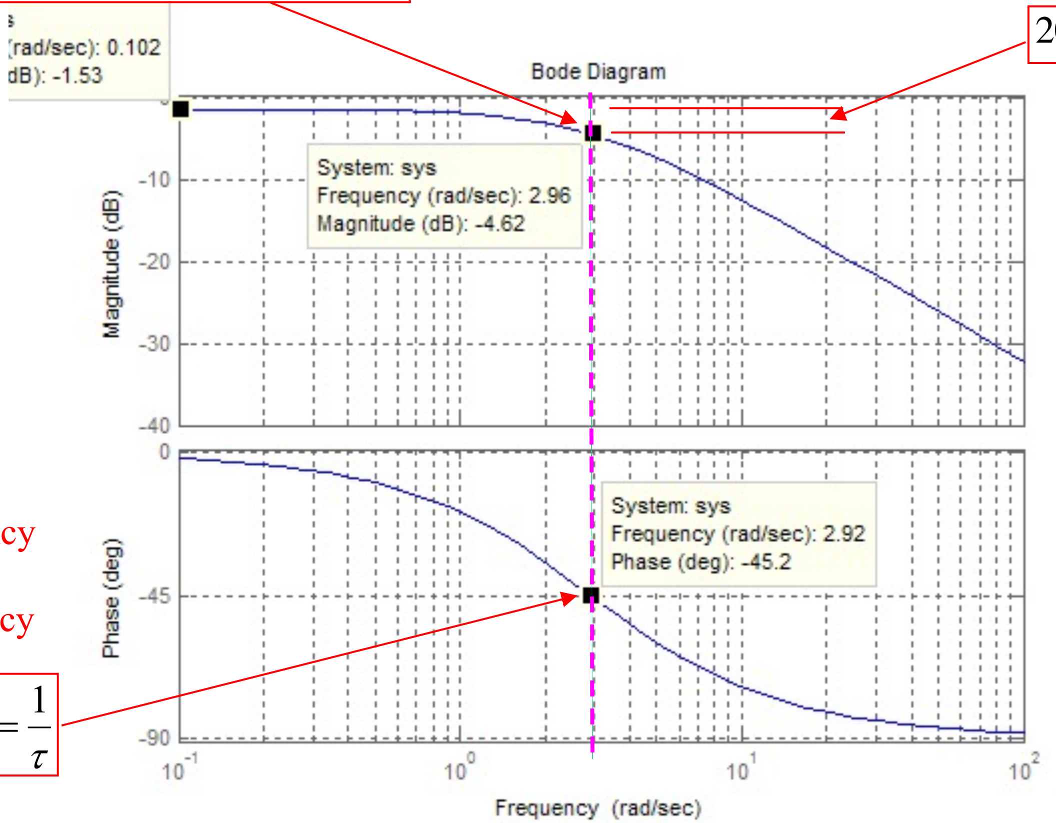
$$Phase = 360^0 \cdot \Delta t \cdot f = 360^0 \cdot (47.75 - 50.34) \cdot 0.02 = -18.65^0$$

Frequency Response (Bode Diagram)

- **The frequency response of a system is typically expressed as Bode diagram.**

$$M_c = M_0 \cdot \sin\left(\frac{\pi}{4}\right) = 0.707 \cdot M_0$$

$$20 \cdot \log_{10}(0.707) = -3 \;\; dB$$



Gain Plot

Phase Plot

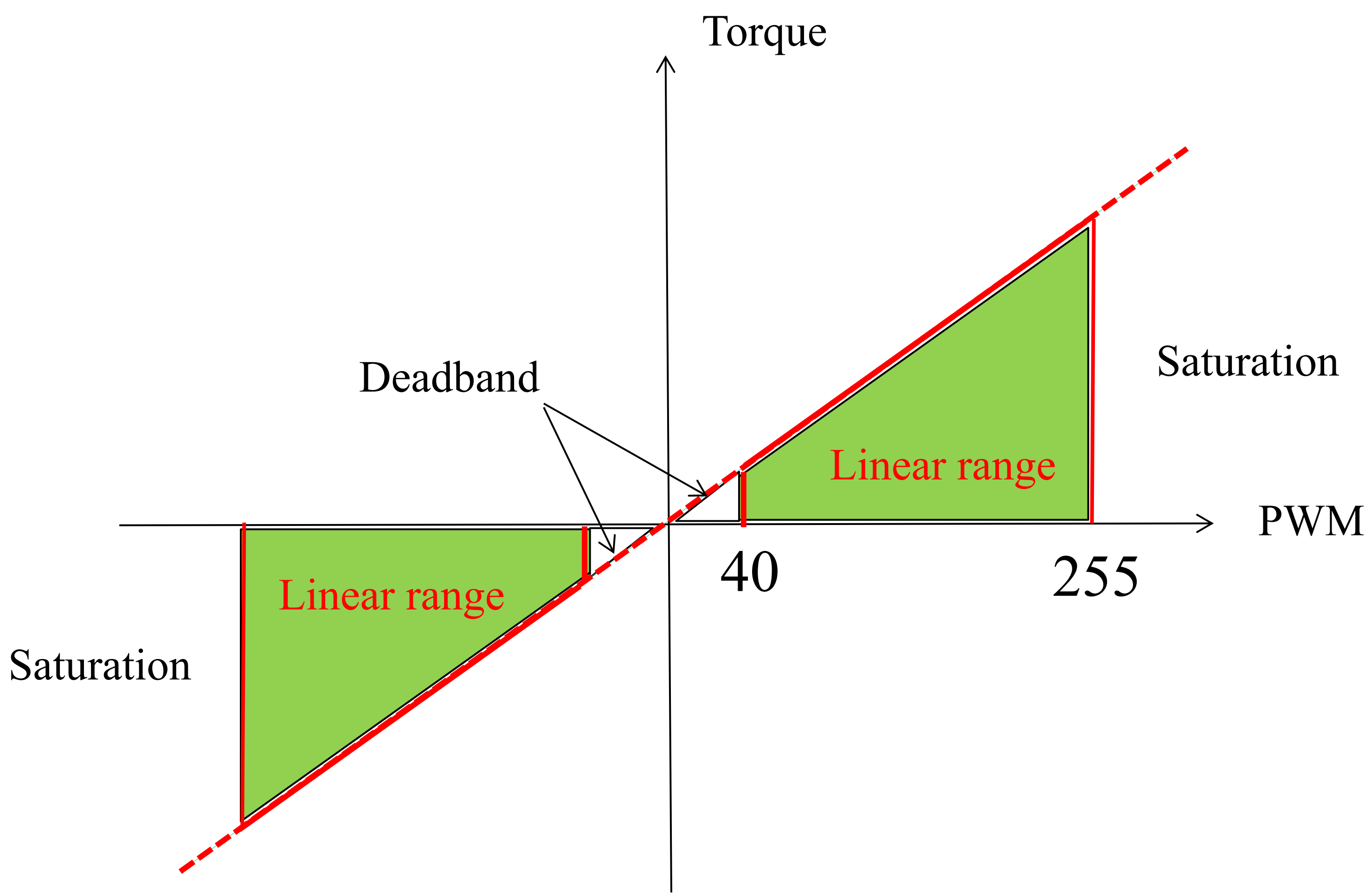
Break frequency
or
Cutoff frequency

$$\omega_c = 2\pi f_c = \frac{1}{\tau}$$

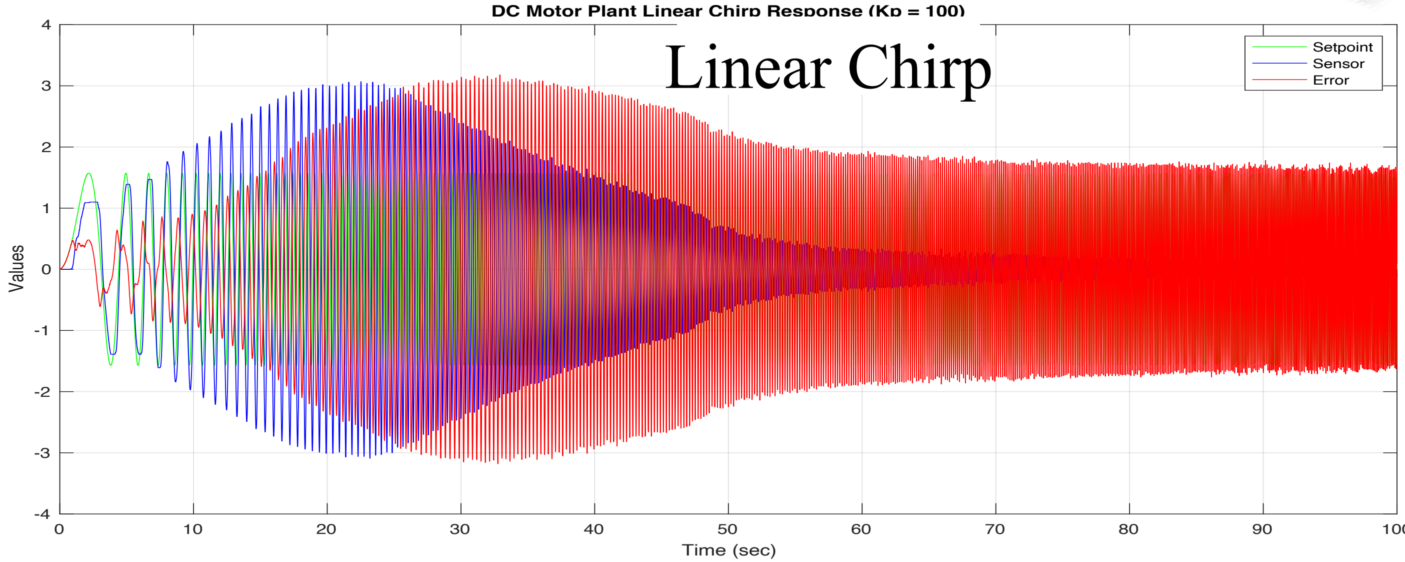
Estimating Frequency Response Function (FRF)

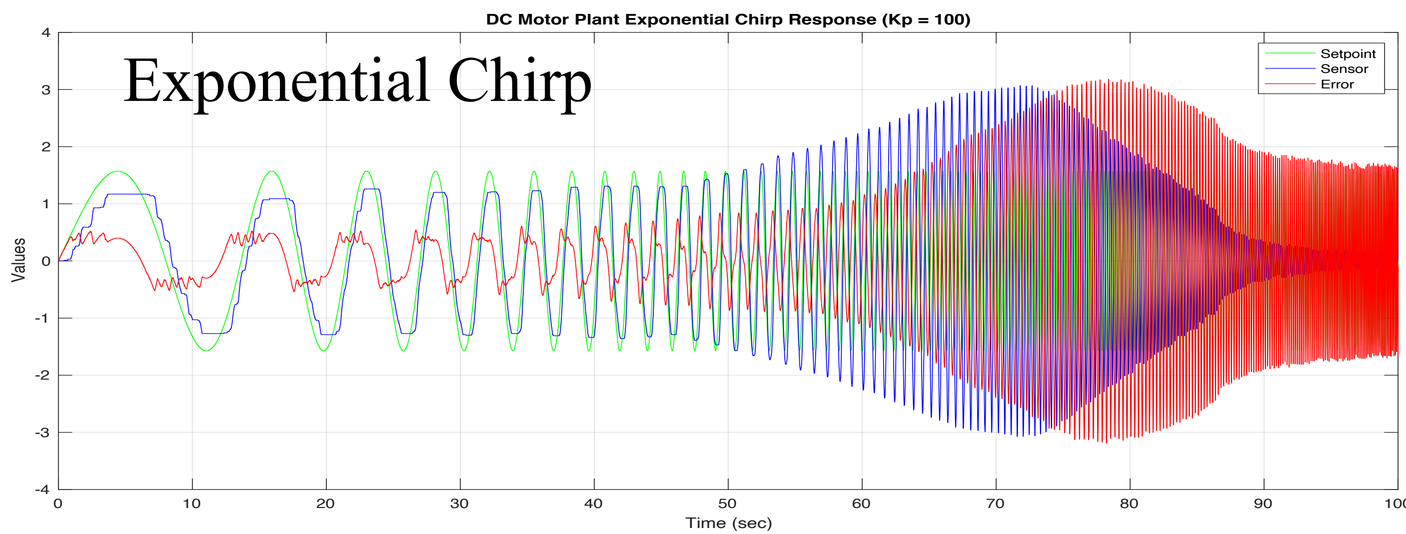
- **Determine the frequency range of interest**
- **Common excitation signals:**
  - Impact; step; burst random signal
  - Stepped sine (sine wave with one discrete frequency at a time); slow sine sweep
  - Band-limited white noise (random signal with constant power spectral density)
  - Chirp signal (fast sine sweep): *to be used in this lab*
- **Dynamic Signal Analyzer, Shaker, Impact Hammer, Sensor**
- **Collect input-output data**
- **Use FFT to compute FRF**

Practical Considerations



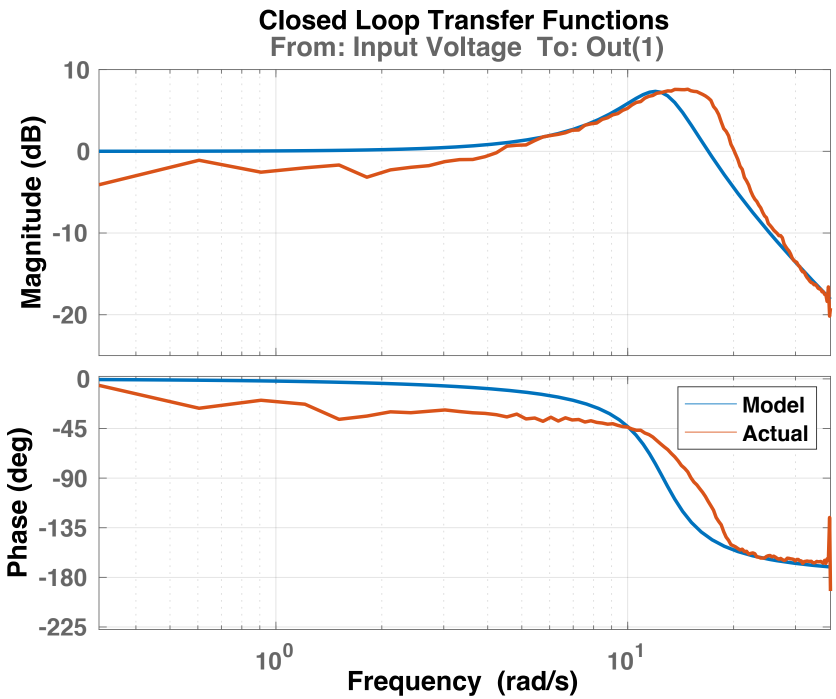
Types of Chirp Signal



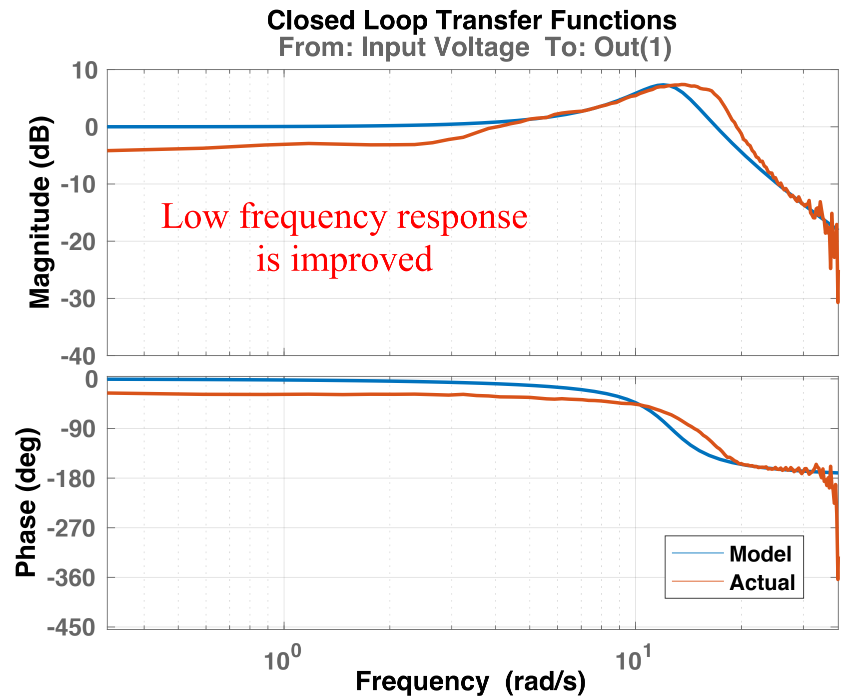


Frequency Responses

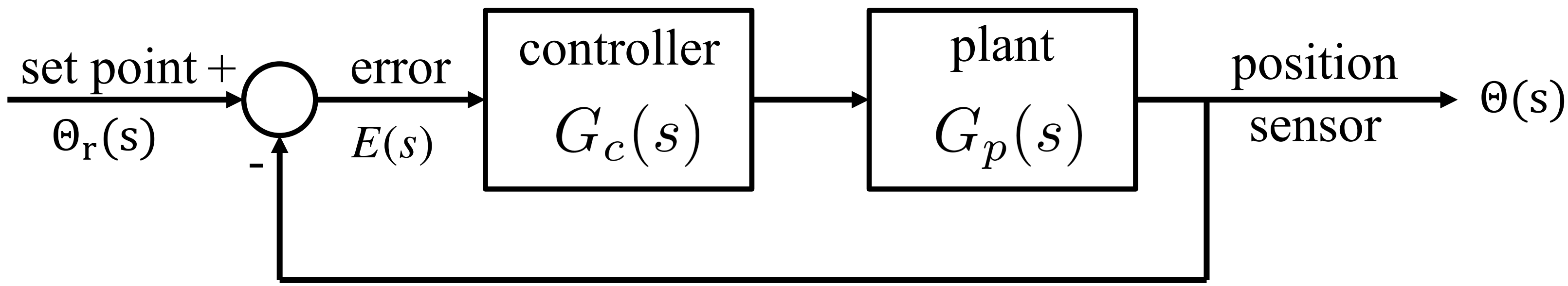
Linear Chirp



Exponential Chirp



Open- and Closed-Loop Transfer Functions



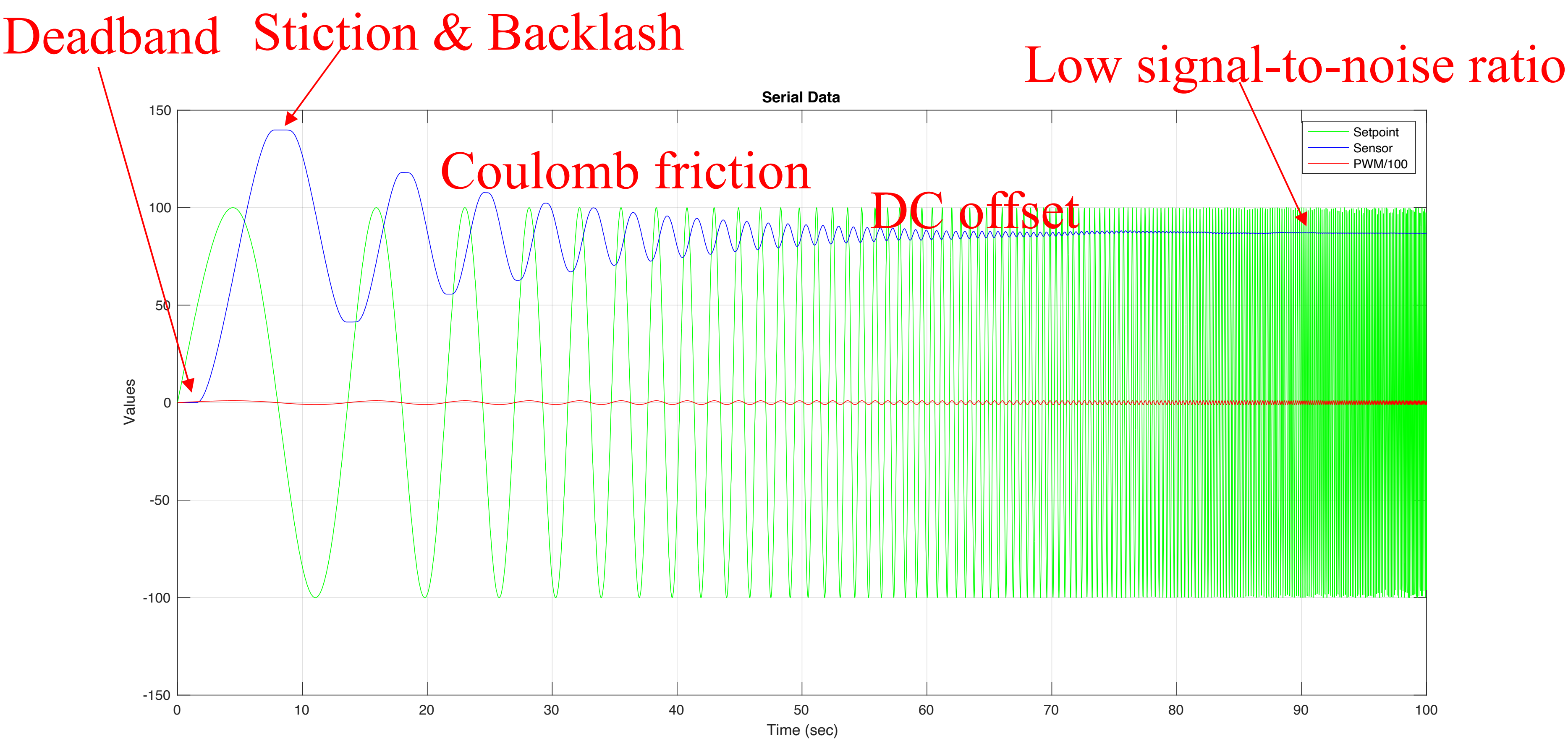
Plant TF: $\mathrm{G_p}(\mathrm{s}) = \dfrac{1}{s}\dfrac{K}{\tau s + 1} = \dfrac{K}{\tau s^2 + s}$

$K = \dfrac{K_{dc}}{vc2pwm}$ $\quad vc2pwm = \dfrac{255\ [PWM]}{5\ [V]} = 51$

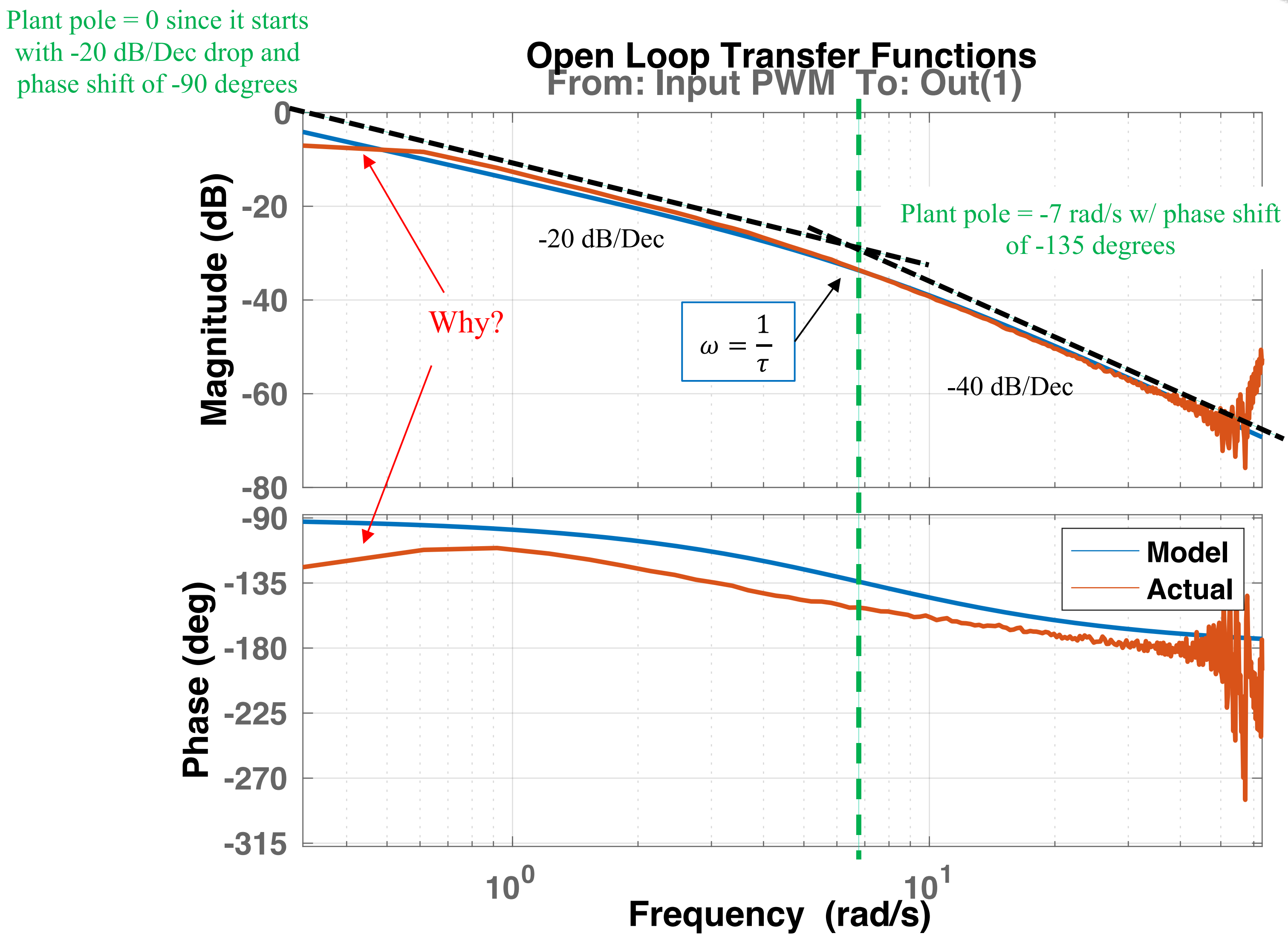
Open-loop transfer function $G_{ol}(s) = \dfrac{\Theta(s)}{E(s)}$

Closed-loop transfer function $G_{cl}(s) = \dfrac{\Theta(s)}{\Theta_r(s)}$

Open-Loop Exponential Chirp (Plant)

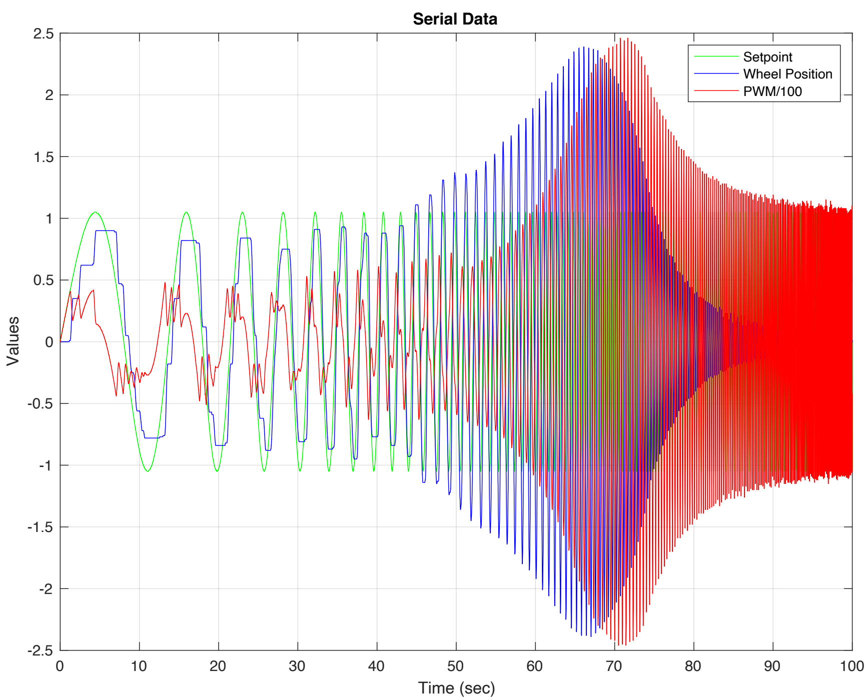


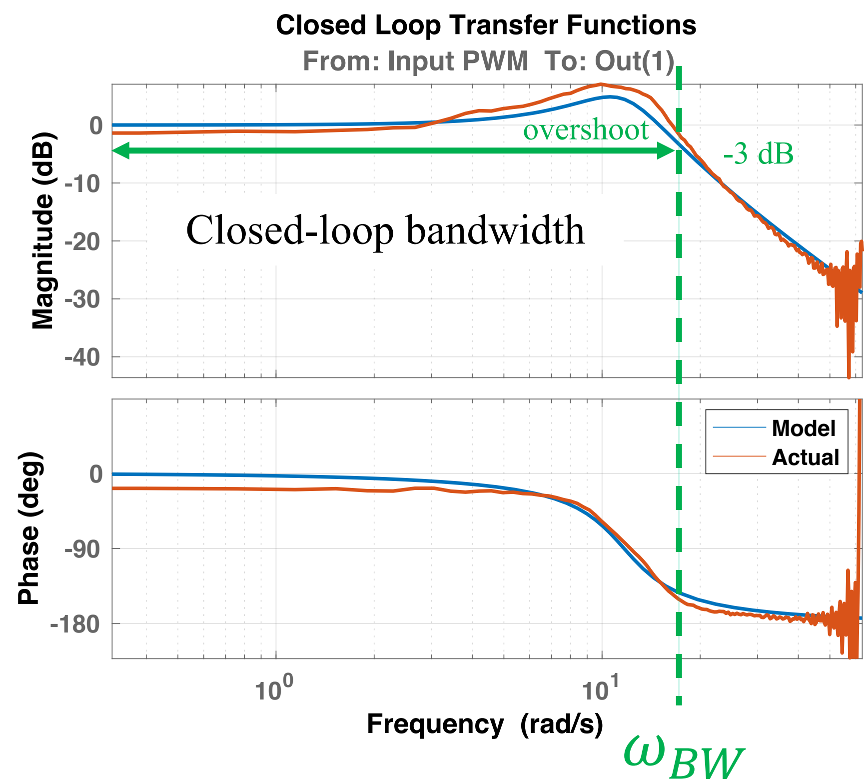
Bode Plots (Plant)



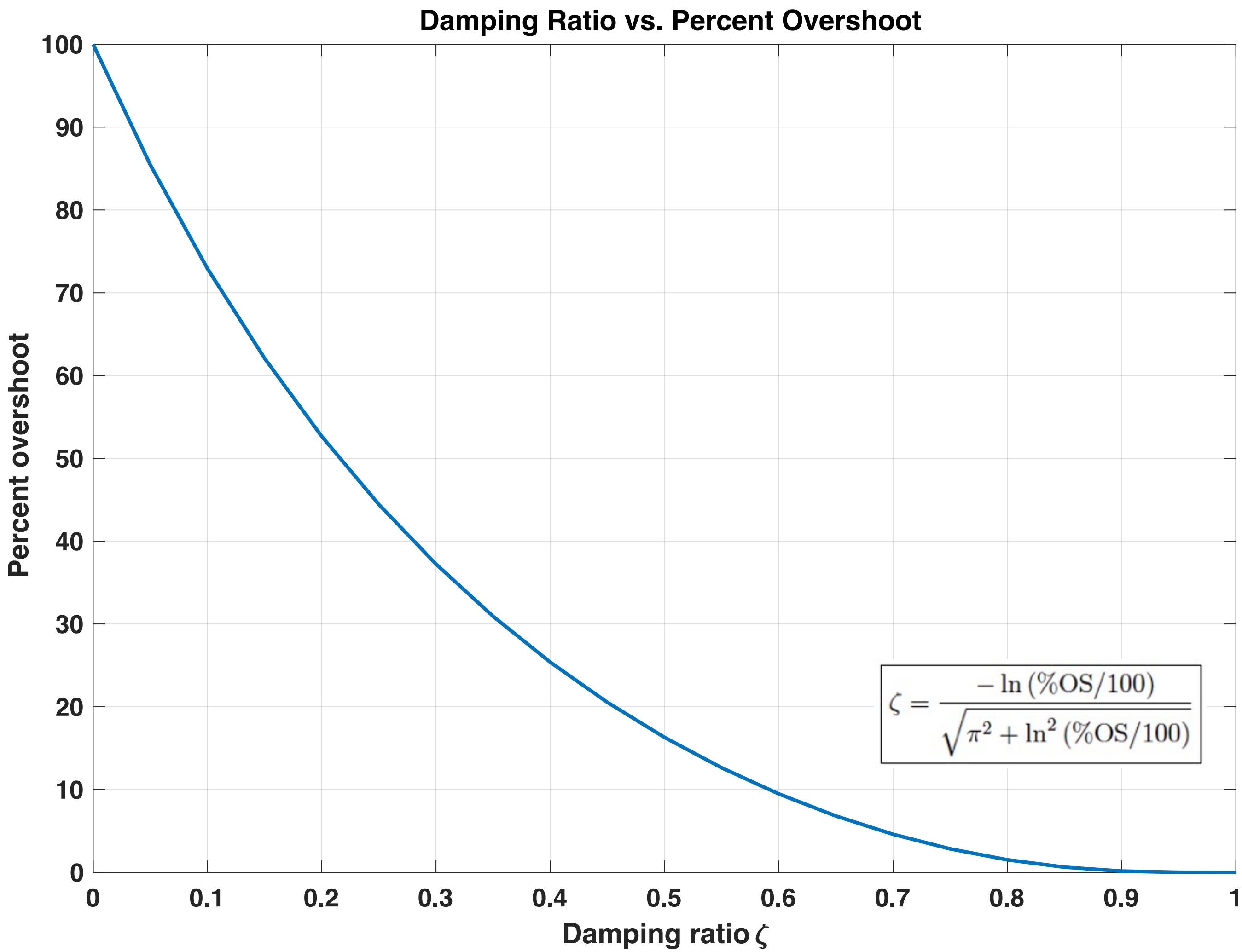
Closed-Loop Response (Kp = 100)

Set chirp amplitude to PI/3 to avoid PWM saturation



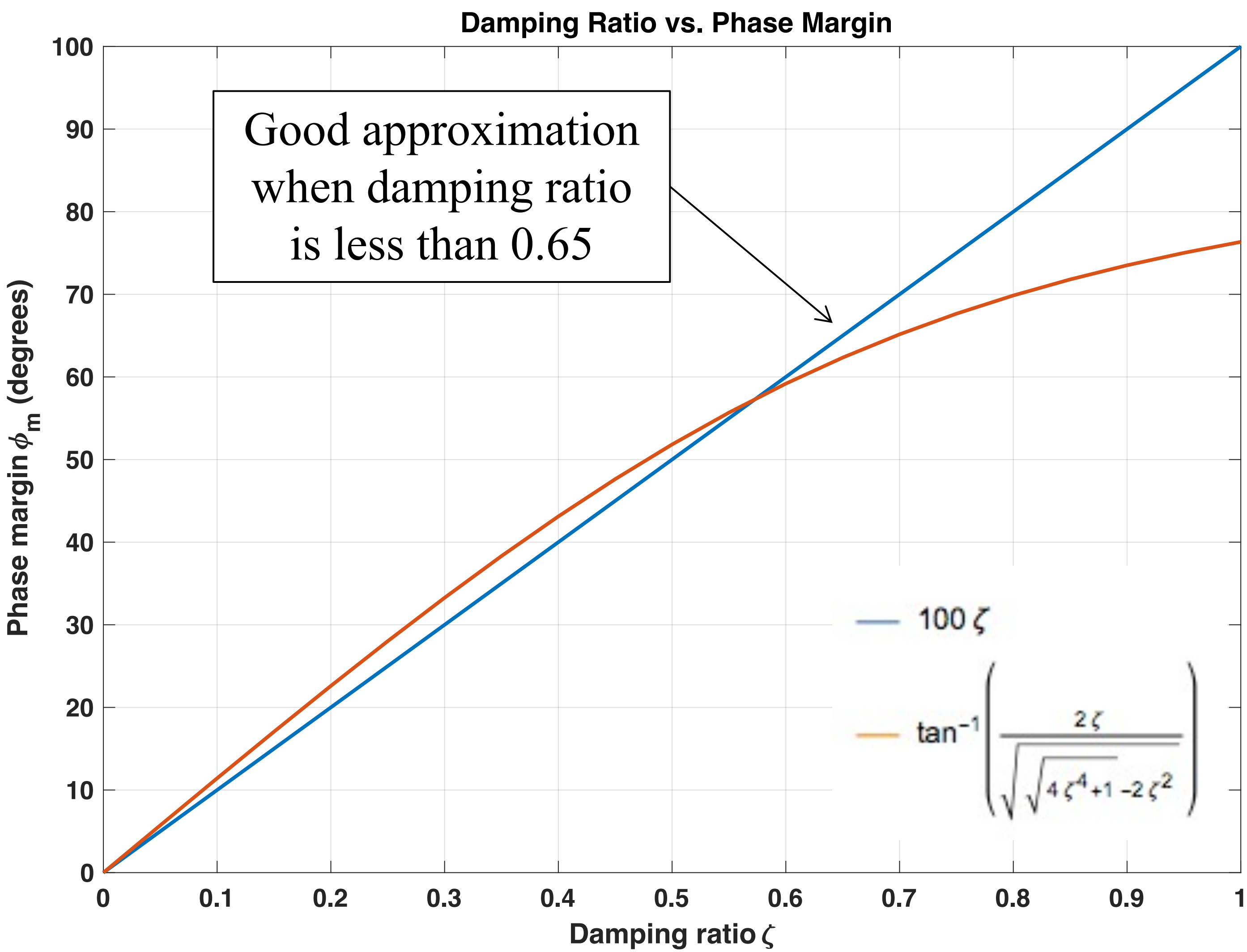


Relationship Between Damping Ratio and Percent Overshoot



$$\zeta = \frac{-\ln(\%OS/100)}{\sqrt{\pi^2 + \ln^2(\%OS/100)}}$$

Relationship Between Damping Ratio and Phase Margin



Lead Controller Design

- Design a lead controller to achieve 7% overshoot and 0.4 sec settling time:
  - $\rightarrow \omega_c$ = ? rad/sec
  - $\rightarrow$ PM = ? ^{o}

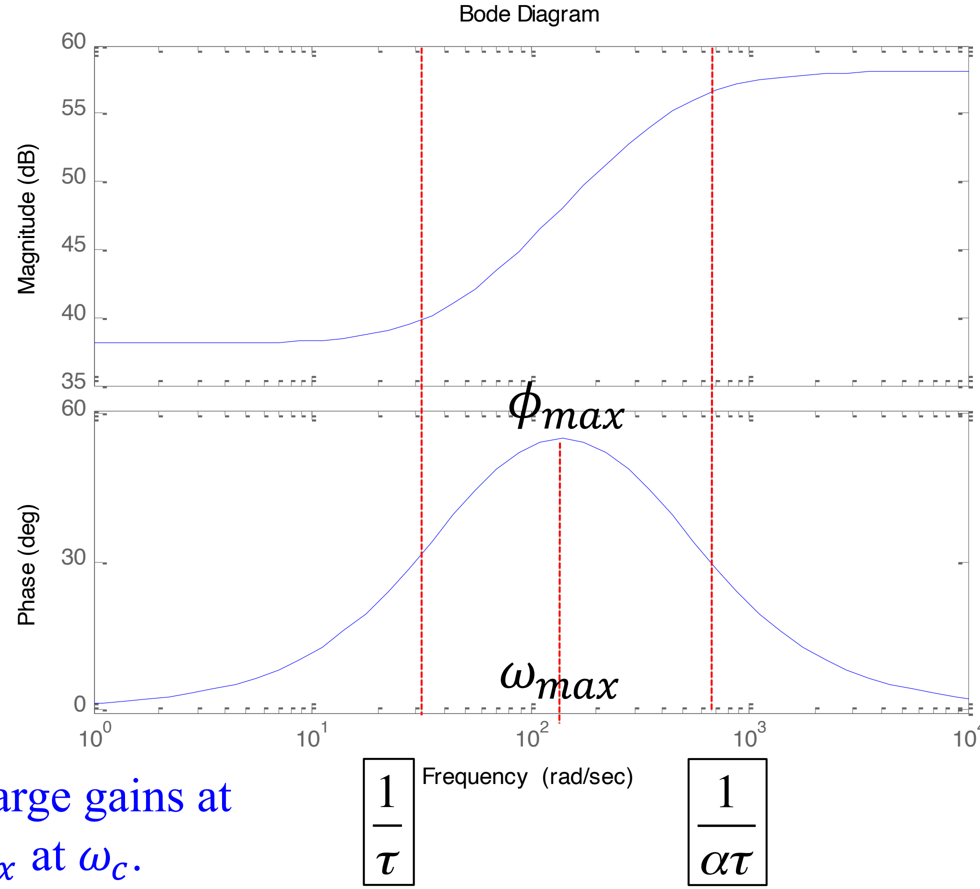
$$G_c(s) = K_c G(s)$$

$$G(s) = \frac{\tau s + 1}{\alpha \tau s + 1}, \qquad 0 \le \alpha < 1$$

$$\phi_{max} = \sin^{-1}\frac{1-\alpha}{1+\alpha}$$

$$\omega_{max} = \frac{1}{\tau\sqrt{\alpha}}$$

$$\left|K_c G(j\omega_c) G_p(j\omega_c)\right| = 1$$

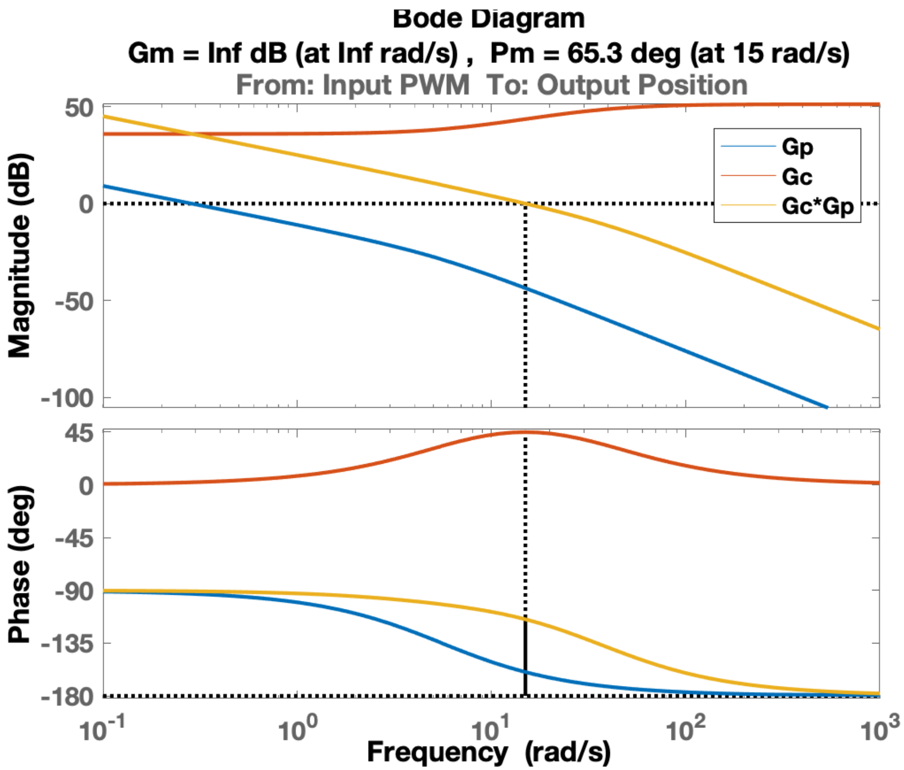


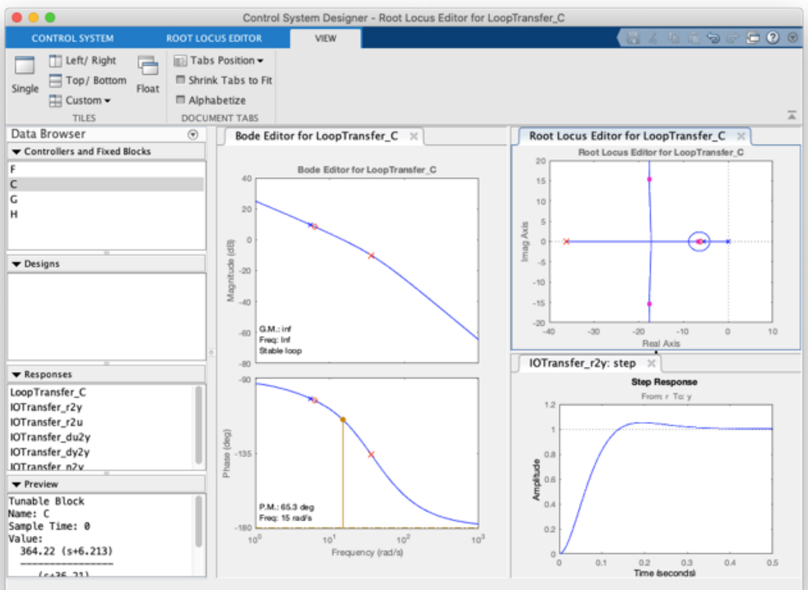
Typically we use an $\alpha \ge 0.1$ to avoid large gains at high frequency, and to place ϕ_{max} at ω_c.

Lead Controller Design

$$\text{Design requirements:} \begin{cases} \%OS \le 7\% \\ T_s \le 0.4s \end{cases}$$

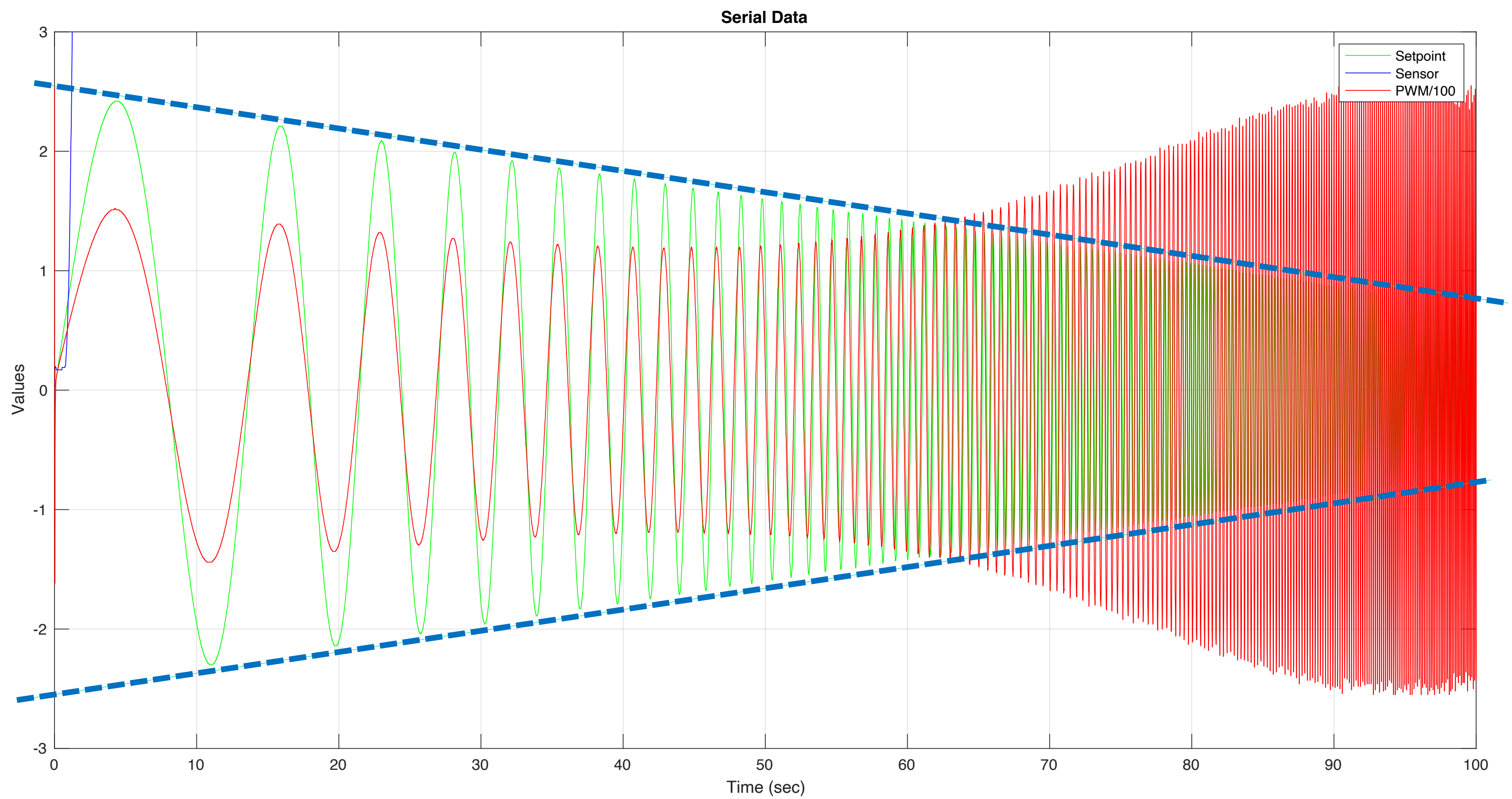
$$\Rightarrow \begin{cases} \phi_m = ? \\ \omega_c = ? \end{cases} \qquad G_c = K_c \frac{T_1 s + 1}{T_2 s + 1}$$





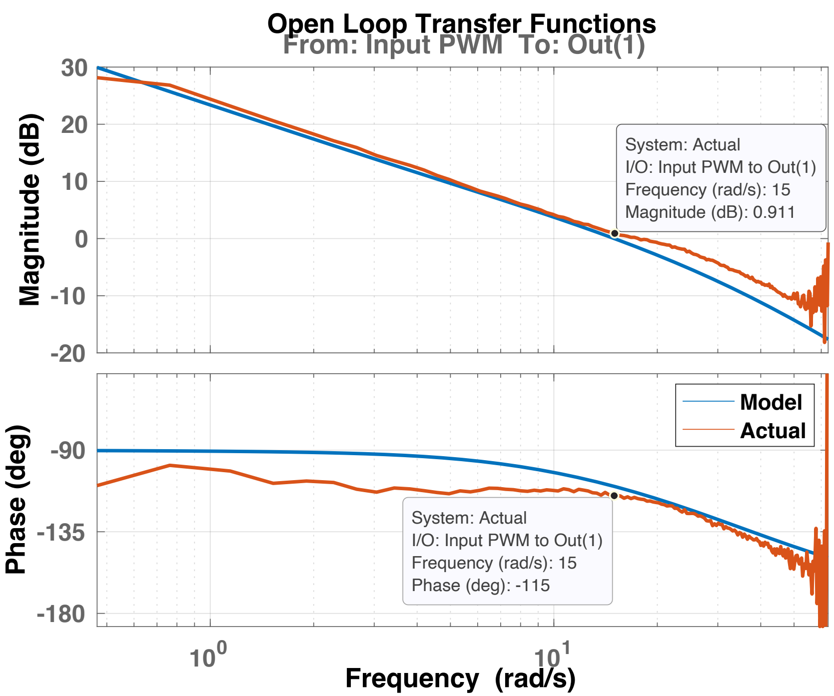
Open-Loop Chirp Response (Lead + Plant)

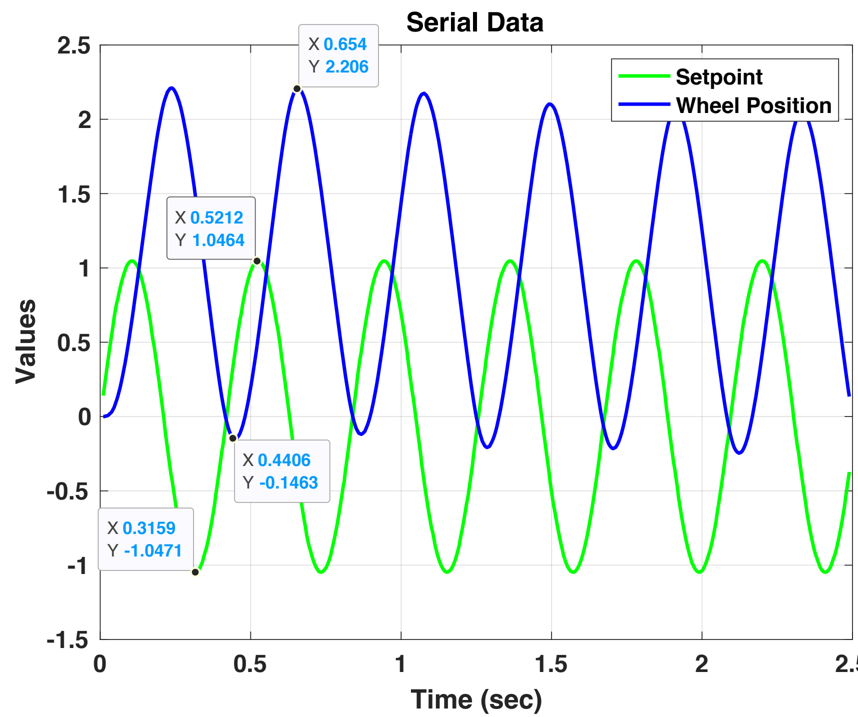
Envelope the Chirp amplitude to avoid saturation



Open Loop Bode Plots (Lead + Plant)

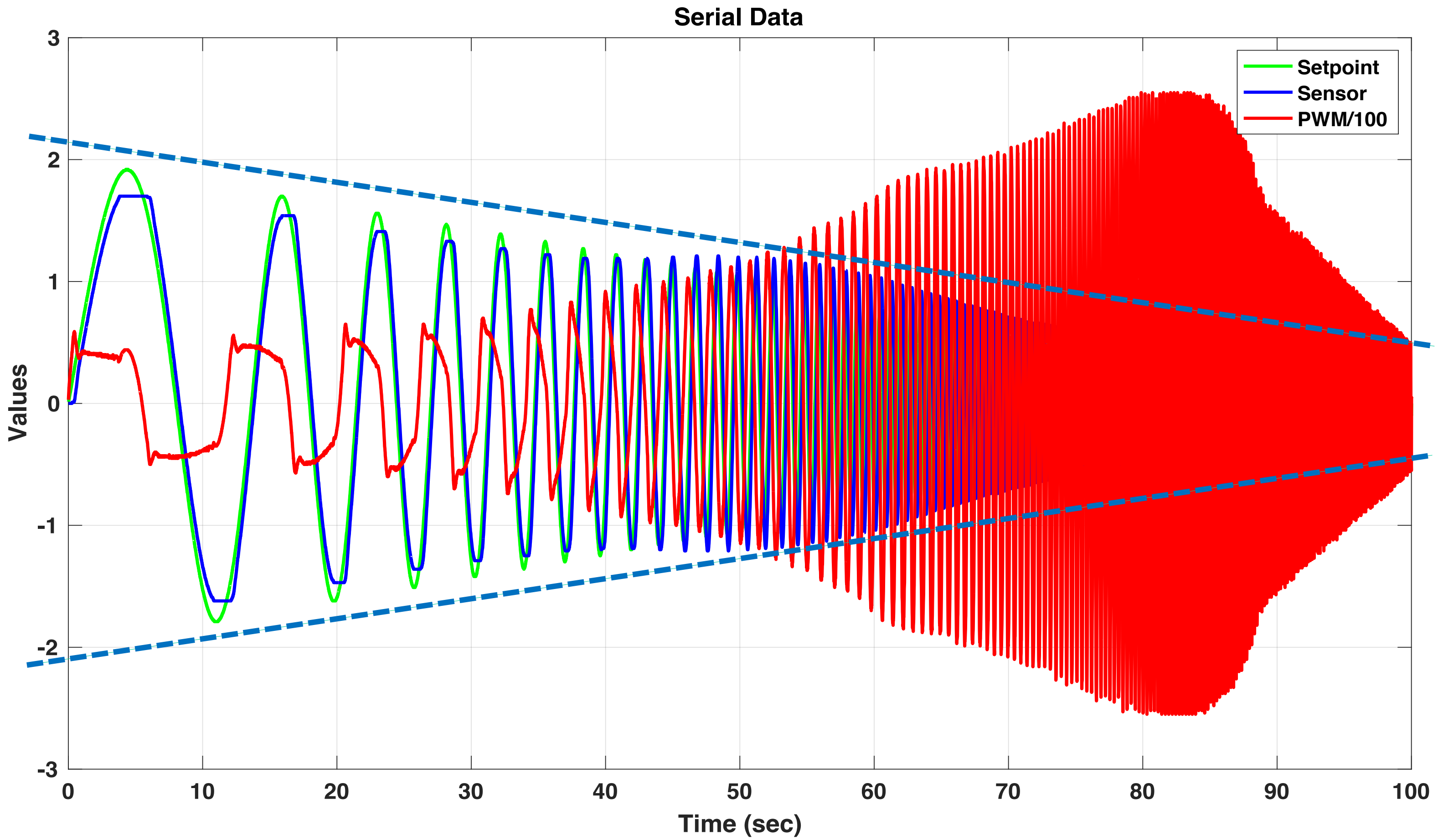
Verify phase margin at the crossover frequency





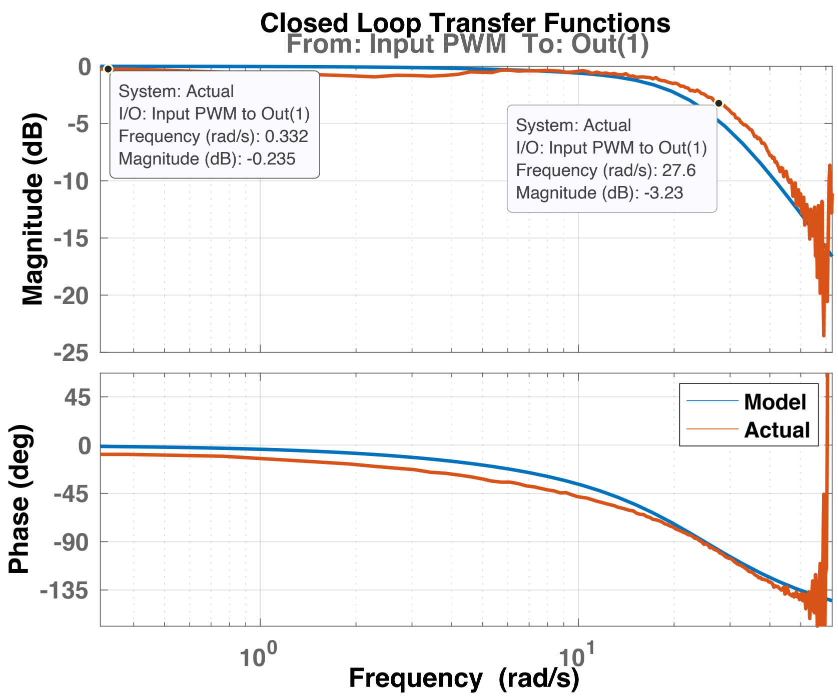
Closed-Loop Chirp Response (Lead)

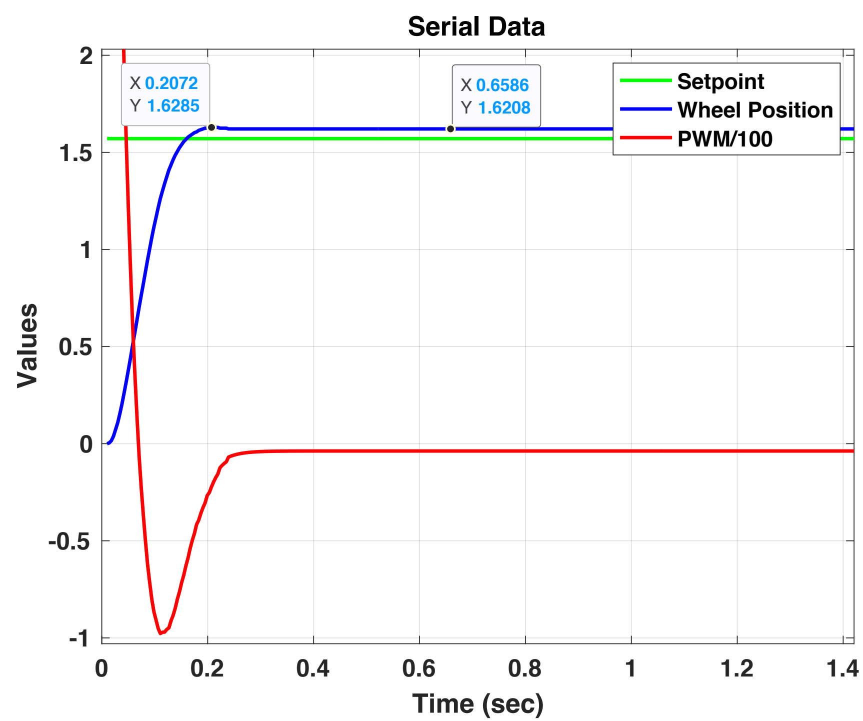
Envelope the Chirp amplitude to avoid saturation



Closed-Loop Bode Plots (Lead)

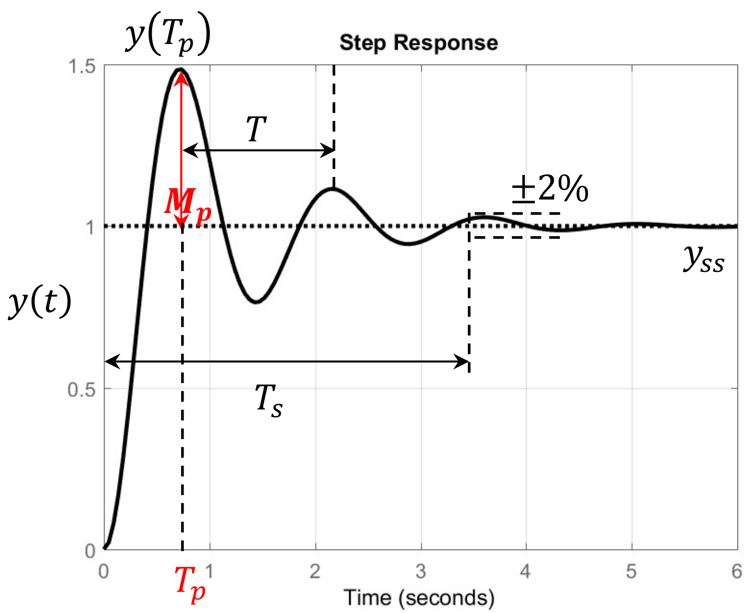
Verify overshoot and check bandwidth





DESIGN PARAMETERS & THEIR RELATIONS

Time Domain Specifications



Example:

$y_{ss} = 1 \qquad y(T_P) = 1.5$

$M_P = 0.5 \qquad \%OS = 50\%$

$$\zeta = -\frac{\ln(0.5)}{\sqrt{\pi^2 + \ln^2(0.5)}} = 0.215$$

$$T_s \approx \frac{4}{\sigma} = \frac{4}{\zeta\omega_n}$$ 2% Settling Time

$$T_s \approx \frac{4.6}{\sigma} = \frac{4.6}{\zeta\omega_n}$$ 1% Settling Time

$$\omega_d = \omega_n\sqrt{1-\zeta^2} = \frac{2\pi}{T}$$ Damped Natural Frequency

$$T_p = \frac{\pi}{\omega_d}$$ Peak Time

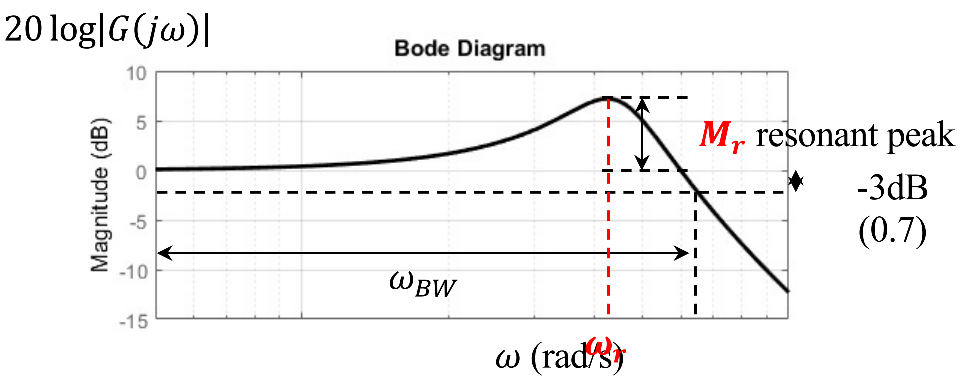
$$M_P \triangleq y(T_p) - y_{ss}$$ Overshoot (Magnitude)

$$\%OS \triangleq \frac{y(T_p) - y_{ss}}{y_{ss}} \times 100$$ Overshoot (Percent)

$$\frac{M_P}{y_{ss}} = \frac{\%OS}{100} = e^{-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}} \quad , 0 \le \zeta < 1$$

$$\ln\left(\frac{M_P}{y_{ss}}\right) = -\frac{\zeta\pi}{\sqrt{1-\zeta^2}} \Rightarrow \qquad \zeta = -\frac{\ln\left(\frac{M_P}{y_{ss}}\right)}{\sqrt{\pi^2 + \ln^2\left(\frac{M_P}{y_{ss}}\right)}}$$ Damping Ratio

Frequency Domain Specifications

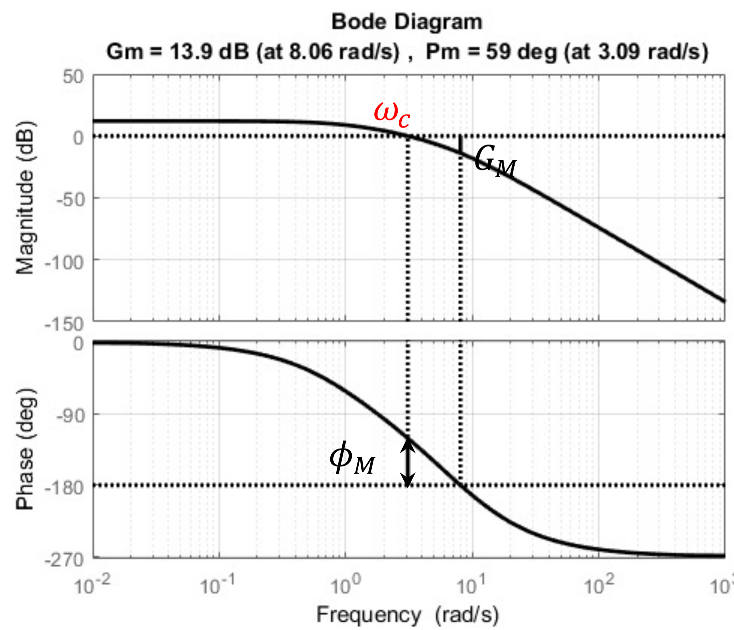


$M_r \triangleq$ resonant peak value

$$M_r = \frac{1}{2\zeta\sqrt{1-\zeta^2}} \qquad 0 \le \zeta \le 0.707$$

$$M_r = 1 \qquad \zeta > 0.707$$

$$\omega_r = \omega_n\sqrt{1-2\zeta^2} \qquad 0 \le \zeta \le 0.707$$

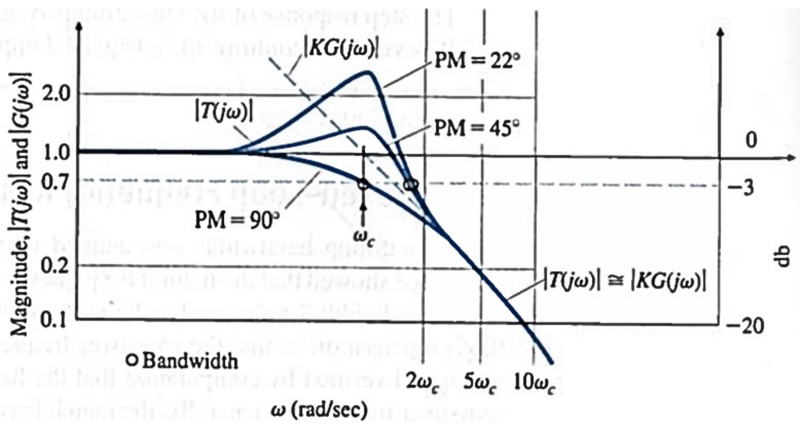


$$\phi_M = \tan^{-1}\left[\frac{2\zeta}{\sqrt{-2\zeta^2+\sqrt{1+4\zeta^4}}}\right]$$

If $0 \le \phi_M \le 70^o$, approximate $\zeta \approx \frac{\phi_M}{100}$

Relationship between Frequency Response and pole-zero locations

Closed-Loop Frequency Response



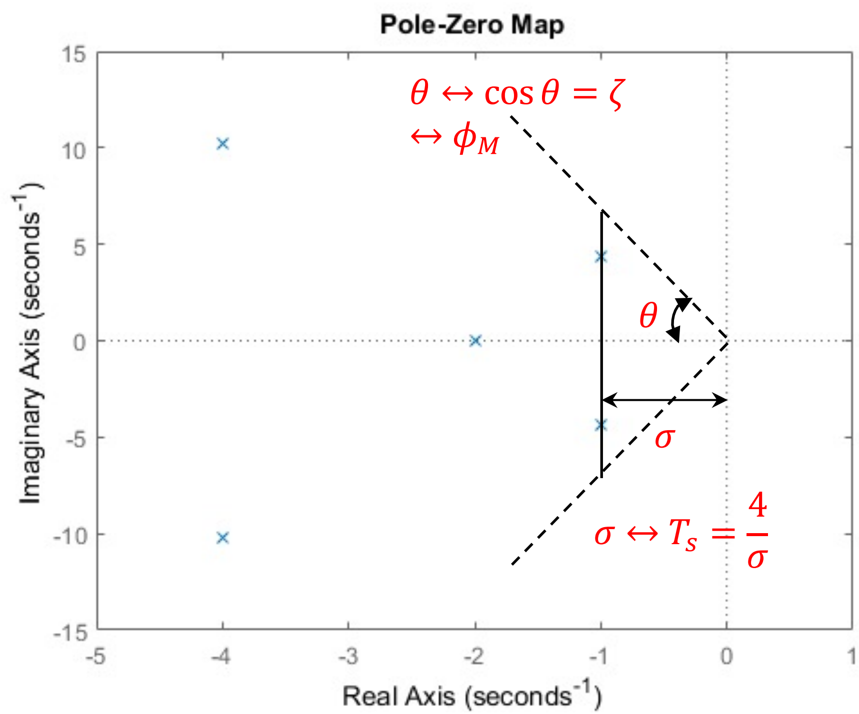
Closed-Loop Bandwidth:

$$\omega_{BW} = \omega_n \sqrt{1 - 2\zeta^2 + \sqrt{4\zeta^4 - 4\zeta^2 + 2}}$$

$$\omega_c \;\leq\; \omega_{BW} \;\leq\; 2\omega_c$$

$$\phi_M \approx 90^o \rightarrow \omega_{BW} \approx \omega_c$$

$$\phi_M \approx 45^o \rightarrow \omega_{BW} \approx 2\omega_c$$

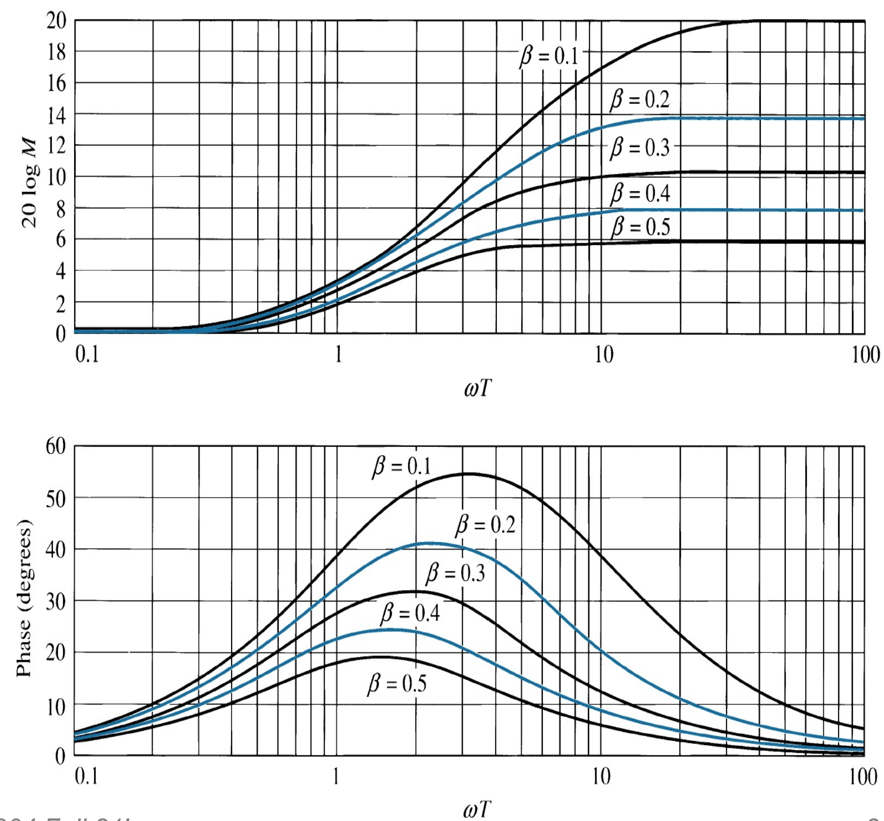


LEAD & LAG COMPENSATORS

Lead Compensator

$$G_c(s) = K_c \frac{(\tau s + 1)}{(\alpha \tau s + 1)} = K_c \frac{\left(\frac{s}{z} + 1\right)}{\left(\frac{s}{p} + 1\right)}, \qquad |z| < |p|$$

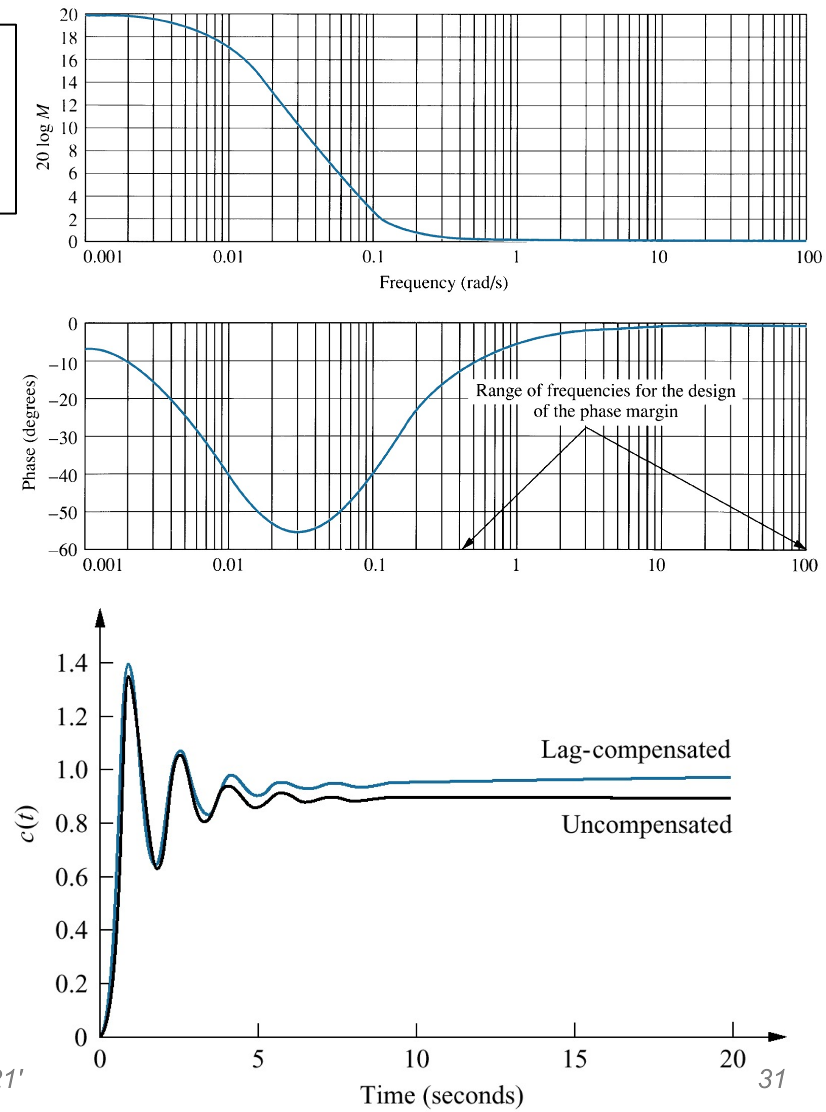
- **Improves stability by adding phase lead.**
- **Dose not change the relative degree of $G_p(s)G_c(s)$.**



Lag Compensator

$$G_c(s) = K_c \frac{(\tau s + 1)}{(\alpha \tau s + 1)} = K_c \frac{\left(\frac{s}{z} + 1\right)}{\left(\frac{s}{p} + 1\right)}, \qquad |z| > |p|$$

- Improves steady state error by increasing gain at low frequency.
- Deteriorates stability by adding phase lag over limited frequency range.
- Typically introduce a slow dominant pole.



Lead Compensator Design Equations

- **Max phase lead**

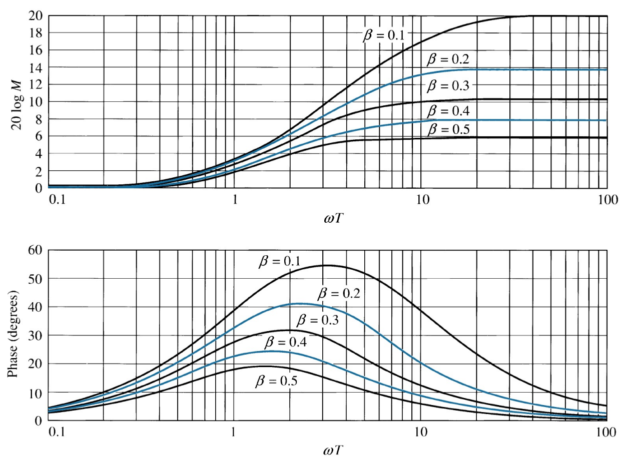
$$\boxed{\phi_{max} = \tan^{-1}\left(\frac{1-\alpha}{2\sqrt{\alpha}}\right) = \sin^{-1}\left(\frac{1-\alpha}{1+\alpha}\right) \quad at \quad \omega_{max} = \frac{1}{\tau\sqrt{\alpha}}}$$

- **Magnitude at max phase lead**

$$\boxed{|G(j\omega_{max})| = \frac{1}{\sqrt{\alpha}}}$$

Lead vs. Lag Compensator

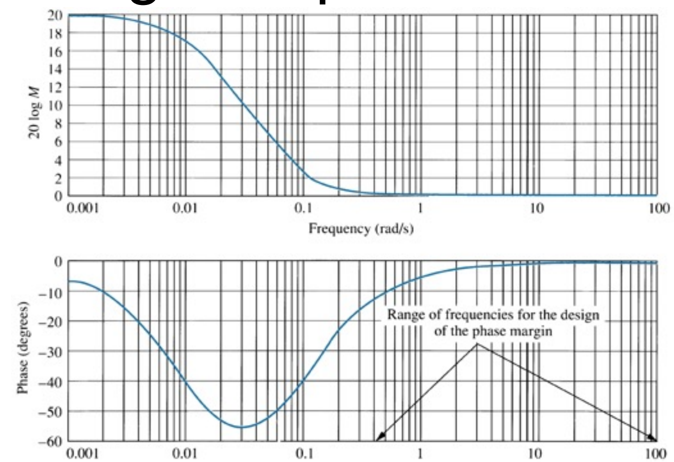
Lead Compensator



Improves stability by adding phase lead.
Improves speed of response.

Use to meet PM, OS%, damping ratio, T_s, T_p, T_r, dominant pole location, etc.

Lag Compensator



Improves steady state error by increasing gain at low frequency.
Deteriorates stability by adding phase lag over limited frequency range.
May introduce a slow dominant pole.

Use to meet steady state error to reference R(s) and disturbance D(s

Lag-Lead Compensator

Lead Lag

$$G_c(s) = K_c\left(\frac{\tau_1 s + 1}{\frac{\tau_1}{\gamma}s + 1}\right)\left(\frac{\tau_2 s + 1}{\gamma\tau_2 s + 1}\right)$$

use $\alpha = \frac{1}{\gamma}$