ME 02.74/740: Bio-inspired Robotics

HW 02: The Double Pendulum

Due Online, Sunday, Sept. 25, 11:59 PM

1 Lagrange's Equations and the Double Pendulum

Using the example given in class as a guide, derive the equations of motion (in MATLAB) for the double pendulum with parameter definitions as in the figure below. A torque τ_1 with vector $\tau_1\hat{k}$ (where $\hat{k} = \hat{\imath} \times \hat{\jmath}$) acts between the base and body 1, and a torque τ_2 with vector $\tau_2\hat{k}$ acts between body 1 and body 2. Assume gravity $g = 9.81\,\mathrm{m/s^2}$ in the $-\hat{\jmath}$ direction. Write a function to simulate the double pendulum, providing animation. Provide a copy of your working code.

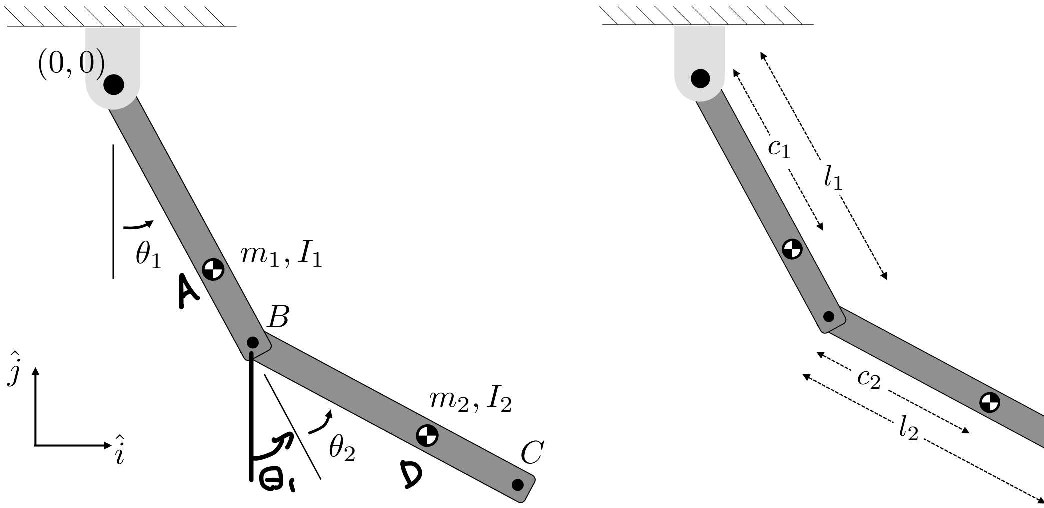


Figure 1: Double pendulum and parameter definitions.

1. With $\tau_1 = \tau_2 = 0$, solve the initial boundary value problem from the initial condition

$$\theta_1 = 3\,\mathrm{rad},\ \theta_2 = 0\,\mathrm{rad},\ \dot{\theta}_1 = \dot{\theta}_2 = 0\,\mathrm{rad/s}$$

on the time interval $t = [0\,\mathrm{s},\ 10\,\mathrm{s}]$. Use parameters

$$m_1 = m_2 = 1\,\mathrm{kg},\ I_1 = I_2 = 0.05\,\mathrm{kg{\cdot}m^2},\ l_1 = 1\,\mathrm{m},\ l_2 = 0.5\,\mathrm{m},\ c_1 = 0.5\,\mathrm{m},\ c_2 = .25\,\mathrm{m}.$$

Plot $\theta_1(t)$ and $\theta_2(t)$. Does the solution display any repetitive and predictable patterns?

2. Plot the total system energy $(T + V)$ over the same interval. Verify energy conservation.

3. (Here to end: Graduate students required, undergraduates optional) Derive the equations again considering the addition of three springs with potential energies

$$V_{e1} = \frac{1}{2}\kappa_1(\theta_1 - \theta_{1,0})^2 \qquad V_{e2} = \frac{1}{2}\kappa_2(\theta_2 - \theta_{2,0})^2$$
$$V_{e3} = \frac{1}{2}k_3(\|\mathbf{r}_C - \mathbf{r}_0\| - l_0)^2$$

The vector $\mathbf{r}_0 = [r_x\ r_y]^T$ represents the attachment point for the last spring.

4. After verifying energy conservation of your equations, simulate the system with

$$\kappa_1 = \kappa_2 = 10\,\mathrm{Nm/rad},\ k_3 = 50\,\mathrm{N/m}$$

$$Q_\tau = M2Q(\tau\hat{k}, \delta\theta\hat{k}) = \text{jacobian}(\delta\theta\hat{k}, dq)' * \tau\hat{k}$$

$$= \left[\frac{\delta\theta}{\delta x}; \frac{\delta\theta}{\delta\theta}\right] * \tau\hat{k}$$

$$Q_F = F2Q(F\hat{\jmath}, rC) = \text{jacobian}(rC, dq)' * F\hat{\jmath}$$

$$= \left[\frac{rc}{\delta x}; \frac{rc}{\delta\theta}\right] * F\hat{\jmath}$$

$$rC = rA + 2\hat{e}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = Q_i$$

jacobian([xyz, y^2, x+z], [x,y,z])

$$\begin{array}{cc} & \begin{array}{ccc} x & y & z \end{array} \\ \begin{array}{c} xyz \\ xyz \\ xyz \end{array} & \begin{bmatrix} yz & xz & xy \\ 0 & 2y & 0 \\ 1 & 0 & 1 \end{bmatrix} \end{array}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{s}}\right) - \frac{\partial L}{\partial s} = F_{ext}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = \tau_{ext}$$

$$g = \overbrace{M(q)}^{\text{Mass Matrix}}\ddot{q} + V(q,\dot{q}) + \overbrace{G(q)}^{\text{Gravitational Terms}} - Q = 0$$

$$\overbrace{M(q)}^{\text{Mass Matrix}}\ddot{q} + \overbrace{C(q,\dot{q})}^{\text{Coriolis Terms}}\dot{q} + \overbrace{G(q)}^{\text{Gravitational Terms}} = \overbrace{\tau_j}^{\text{joint torques}} + J^T \underbrace{F}_{\text{external forces}}$$

$$ma = F_{ext}$$

$$Q_F = J^T F$$

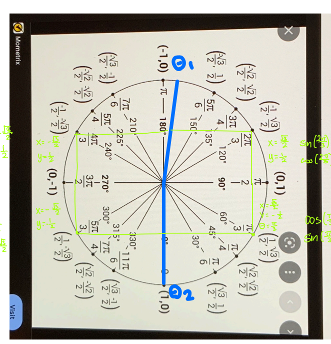
$$g = \frac{d\,\partial L}{d\,\partial\partial q} - \frac{\partial L}{\partial q} - Q$$

$$Q_\tau = \left[\frac{\partial w_\theta}{\partial x} \ \frac{\partial w_\theta}{\partial \theta}\right]^T \tau$$

$$\begin{array}{cc} & \begin{array}{cccc} x_0 & \theta_0 & \dot{x}_0 & \dot{\theta}_0 \end{array} \\ z0 = & [1; 0; 0; 0;] \end{array}$$

dz = dynamics(z_out(:,i), p)

$$\underbrace{z\_out(:,i+1)}_{\text{next}} = \underbrace{z\_out(:,i)}_{\text{prev}} + dz*dt$$



qdd = $\ddot{q}$

dynamics(z,p)

A = A_doublepend(z,p)

u = [0;0;0;]

b = b_doublepend(z,u,p)

qdd = A\b

dz = 0*z

dz(1:2) = z(3:4);

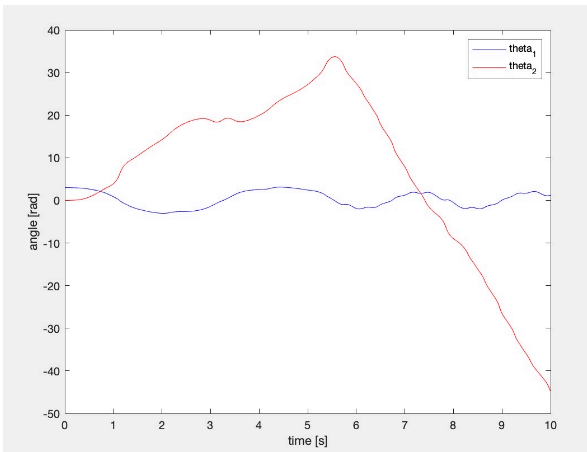
dz(3:4) = qdd

z = [x, θ, δx, δθ]

dz = [δx, δθ, δδx, δδθ]

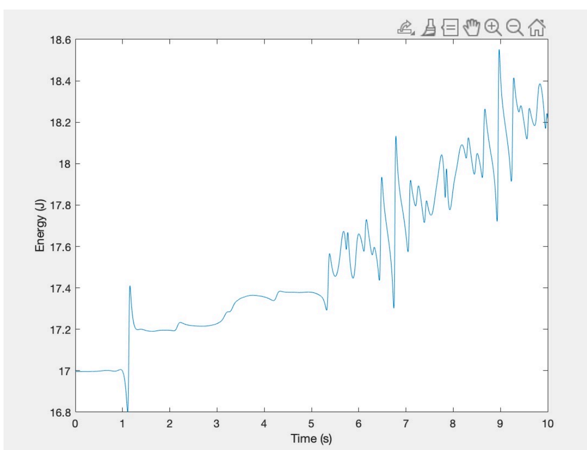
$$z\_out(:,i) = [\theta_{1i}, \theta_{2i}, \delta\theta_{1i}, \delta\theta_{2i}] + \underbrace{[\delta\theta_{1,i-1}, \delta\theta_{2,i-1}, \delta\delta\theta_{1i}, \delta\delta\theta_{2i}] * dt}_{\Delta\theta_1,\ \Delta\theta_2\ \ \Delta\delta\theta_1,\ \Delta\delta\theta_2}$$

1.



theta1 follows a sinusoidal pattern

2.



$$\theta_{1,0} = \theta_{2,0} = 0,\ l_0 = 0,\ r_x = 0\,\mathrm{m},\ r_y = 0.5\,\mathrm{m}$$

Apply $\tau_1 = -\dot{\theta}_1$ and $\tau_2 = -\dot{\theta}_2$ and use the same initial state as in step 2.

5. Create a contour plot of the total potential energy (θ_1 and θ_2 on the x and y axes). Overlay the solution $\theta_1(t)$, $\theta_2(t)$ from step 6 on this contour.
6. Experiment with initial conditions. From these experiments, characterize all possible outcomes of the simulation as $t \to \infty$. (Specifically, what can you say about $\theta_1(t)$, $\theta_2(t)$, $\dot{\theta}_1(t)$, and $\dot{\theta}_2(t)$ as $t \to \infty$?) Provide a physical explanation for your findings.