Time Series Analysis & Forecasting

Class 7

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State Space Models

- Common mathematical formulation that includes regression and ARIMA models
- Based on Markov property that process future is dependent on current process state and independent of process past
- Assume system is represented by state vector X_t at time t
- Has 2 equations:
  - <u>Observation or measurement equation</u> – describes how TS observations are done from the state vector
    $$Y_t = h_t' X_t + \varepsilon_t$$
    h_t' - known vector of constants

    ε_t - observation error
  - <u>State or system equation</u> – describes how the state vector evolves through time
    $$X_t = AX_{t-1} + Ga_t$$
    $A \; and \; G$ – known matrices

    a_t - process noise

VAR Model

- Vector AutoRegressive order 1 VAR(1) represented by

$$\begin{bmatrix} z_{1t} \\ z_{2t} \end{bmatrix} = \begin{bmatrix} \phi_{1,11} & \phi_{1,12} \\ \phi_{1,21} & \phi_{1,22} \end{bmatrix} \begin{bmatrix} z_{1t-1} \\ z_{2t-1} \end{bmatrix} + \begin{bmatrix} a_{1t} \\ a_{2t} \end{bmatrix}$$

- If $\phi_{1,12} = \phi_{1,21} = 0$, then we z_{1t} and z_{2t} are not dynamically correlated

- z_{1t} and z_{2t} are said to have a transfer function relationship => z_{1t} can be adjusted to influence z_{2t} and vice-versa

- Transfer function models are a special case of VARMA models

VARMA Identifiability

- Unlike VAR and VME models, VARMA models encounter problem of identifiability – i.e. non-uniqueness in model specification (get more than 1 VARMA model for the same set of AR and MA polynomials).

- There are cases for which a VMA(1) model can also be written as a VAR(1) model.

- This is harmless since either model can be used in real life application.

Cross-correlation Function (CCF)

- Cross-covariance

$$\gamma_{xy}(k) = E[(x_t - \mu_x)(y_{t+k} - \mu_y)]$$

- CCF

$$\rho_{xy}(k) = \frac{\gamma_{xy}(k)}{\sigma_x \sigma_y}$$

- Note:

$$\gamma_{xy}(k) = \gamma_{yx}(-k)$$

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R code

```
library("MTS", lib.loc="~/R/win-library/3.2")
xt <- matrix(rnorm(1500), 500, 3)
MTSplot(xt)
p1<- matrix(c(0.2,-0.6,0.3,1),2,2)
sig <- matrix(c(4,0.8,0.8,1),2,2)
th1 <- matrix(c(-0.5,0,0,-0.6),2,2)
m1 <- VARMAsim(300, arlags=c(1), malags = c(1),phi=p1,theta=th1,sigma=sig)
zt <- m1$series
MTSplot(zt)
```

R code

- *library("astsa", lib.loc="~/R/win-library/3.2")*
- *data(cmort)*
- *data(tempr)*
- *data(part)*
- *x <- cbind(cmort, tempr, part)*
- *mod1 <- MTS::VARMA(x)*
- *summary(mod1)*
- *acf(resid(mod1)*

R code

- *library("astsa", lib.loc="~/R/win-library/3.2")*
- *data("mts-examples",package="MTS")*
- *ibmspko.mts <- ts(ibmspko[,-1],start=c(1961,1),frequency=12)*
- *mod2 <- VARMA(ibmspko.mts)*
- *summary(mod2)*
- *acf(resid(mod2)*

ARIMA Transfer Function Model

- ARIMA Process

$$\phi(B)Y_t = \theta(B)e_t$$

$$Y_t = \phi^{-1}(B)\theta(B)e_t = \frac{1 - \theta_1 B - \cdots - \theta_q B^q}{1 - \emptyset_1 B - \cdots - \emptyset_p B^p} e_t$$

- TS is represented as output of a dynamic system where the input is white noise
- Transfer function is parsimoniously represented as a ratio of 2 polynomials in *B*

SISO Transfer Function Model

- Assume X_t and Y_t are stationary TS – in a single input single output system they are related through a linear filter:

$$Y_t = \vartheta(B)X_t + \eta_t$$

where η_t is the noise

$\vartheta(B)$ is the transfer function and

$$\vartheta(B) = \sum_{j=0}^{\propto} v_j B^j, \sum_{j=0}^{\propto} |v_j| <\propto$$

- The TF $\vartheta(B)$ may contain an infinite number of coefficients, so represent in the rational form:

$$\vartheta(B) = \frac{\omega(B)B^b}{\delta(B)}$$

SISO Transfer Function Model

- The TF $\vartheta(B)$ may contain an infinite number of coefficients, so represent in the rational form:

$$\vartheta(B) = \frac{\omega(B)B^b}{\delta(B)}$$

Where $\omega(B) = \omega_0 - \omega_1 B - \cdots - \omega_s B^s$

$\delta(B) = 1 - \delta_1 B - \cdots - \delta_r B^r$

b is a delay parameter representing actual time lag before impulse of the input variable produces effect on the output variable

CCF and TF

- If X_t in

$$Y_t = \vartheta(B)X_t + \eta_t$$

Can be whitened to white noise then autocorrelation disappears and TF

$$\vartheta(B) = \frac{\sigma_y}{\sigma_x}\rho_{xy}(k)$$

- So if X_t follows an ARMA process

$$\phi(B)X_t = \theta(B)e_t \Rightarrow e_t = \frac{\phi(B)}{\theta(B)}X_t$$

Then

$$\frac{\phi(B)}{\theta(B)}Y_t = \vartheta(B)\frac{\phi(B)}{\theta(B)}X_t + \frac{\phi(B)}{\theta(B)}\eta_t$$

$$U_t = \vartheta(B)e_t + \eta_t^*$$

Where

$$\vartheta(B) = \frac{\sigma_U}{\sigma_e}\rho_{eU}(k)$$

Textbook Chapters

- Materials covered available in book:
  - FPP: Chapter 9, MJK: Chapter 6