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Knots and Quantum Computation

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ABSTRACT

In this paper, we review the main principles of topological quantum computing, particularly the mathematical tools developed in the theory of knots and used in this context. We start off by giving a brief overview of quantum computing. We then discuss anyons and their use in topological quantum computing. Next, we turn to defining the Jones polynomial and elaborate on how its absolute value can be measured in quantum experiments. We conclude the paper with a summary.

1. INTRODUCTION

The development of quantum theory has inspired a new way of storing, manipulating and transmitting information, called quantum computation. Quantum computation is a type of computation based on a theoretical ability to manufacture, manipulate and measure quantum states. Quantum computation is believed to be particularly suitable for execution of such algorithms as searching a data base (Grover 1997), abelian groups (factoring and discrete logarithm) (Kitaev 1997; Shor 1994), and simulating physical systems (Feynman 1982; Lloyd 1996; Freedman et al. 2002). Additionally, and somewhat surprisingly, quantum computation may also be used to execute algorithms which produce approximate evaluations of many quantum invariants of three dimensional manifolds, e.g., the absolute value of the Jones polynomial of a link L at certain roots of unity. The surprising connection between quantum theory and the Jones polynomial was first discovered by Witten (1989). Since then, however, many more new links between quantum computation and low dimensional topology have been identified (Kauffman 1991). In particular, the theory of anyons, a type of quasiparticle, can be formulated using the braid group (the motion of anyons in a two dimensional system defines a braid in 2 + 1 dimensions). In this paper, we will discuss how this useful connection can serve as a foundation in a model of quantum computation based on topological degrees of freedom.

Why do so many companies and institutions invest in quantum computing research nowadays? First, quantum computing seems inevitable, according to Moore's law. Second, quantum computing can enable fast execution of algorithms that are difficult or otherwise impossible to run with classical computers. Lastly, based on our current understanding of quantum mechanics, quantum computation appears to be feasible (Freedman et al. 2001)

In this paper, we present a model of quantum computation based on anyons which is known as *topological quantum computation* and was first proposed by a Russian-American physicist Alexei Kitaev (Kitaev 2003). Almost by definition, a topological quantum computer is protected from local errors, which is the main challenge in other suggested types of quantum computing that rely on storing information in systems that are either localized in space (e.g., an electron or a nuclear spin) or localized in momentum (e.g., photon polarization) (Freedman et al. 2001). We will discuss the challenge of quantum error correction and how topological quantum computers can overcome it later in the paper.

But first, let us discuss the standard quantum circuit model and its physical implementation. To specify a quantum circuit Γ, we begin with a tensor product $\mathbb{C}^2_1 \otimes ... \otimes \mathbb{C}^2_n$ of n copies of $\mathbb{C}^2$, called qubits. Physically, this models a system of n non-interacting spin$= 1/2$ particles. The circuit then consists of a sequence of K "gates" $U_k, 1 \leq k \leq K$, applied to individual or paired tensor factors. A gate is a unitary transformation on either $\mathbb{C}^2_i$ or $\mathbb{C}^2_i \otimes \mathbb{C}^2_j, 1 \leq i, j \leq n$, and is the identity on all remaining factors. The gates are taken from a fixed set of unitary 2×2 and 4×4 matrices (with respect to a fixed basis $\{|0\rangle, |1\rangle\}$ for each $\mathbb{C}^2$ factor) and must obey the "universality" condition, i.e. the gates must generate the unitary group $\mathbb{U}(2^n)$ densely (up to a physically irrelevant overall phase). Beyond the density requirement the particular choice of gates is not important (Freedman et al. 2001).

Let $W_\Gamma = \prod_{i=1}^m U_i$ denote the operator effected by the circuit Γ.

Formally, information is extracted from the output by measuring the first qubit. The probability of observing $|1\rangle$ is given according to the axioms of quantum mechanics as:

$$p(\Gamma) = \langle 0|W_\Gamma^\dagger \Pi_1 W_\Gamma|0\rangle, \tag{1}$$

where Π_1 is the projection to $|1\rangle$,

$$\begin{pmatrix} 0\,0 \\ 0\,1 \end{pmatrix}, \tag{2}$$

applied to the first qubit.

Now, to build a quantum computer, we may follow three steps: (1) build physical qubits and gates; (2) minimize error level; (3) implement decoherence-protected logical qubits and logical gates.

2. QUANTUM ERROR CORRECTION

Having understood the basics of quantum computing, let us now discuss some of the implementation challenges. Quantum computation consists of two parts: encoding of quantum information and its manipulation with quantum gates. Today, there are two main lines of research in quantum computing: First, to discover new algorithms that go beyond existing algorithms of searching and factorizing. Second, to perform quantum computation in a manner that is resilient to errors.

Let us focus on the latter point. It is not possible to realize a unitary gate precisely. In addition, the environment will always interact with the qubits causing decoherence. Both imprecision and decoherence can be considered as "errors" which can be quantified (Jozsa 1994; Kitaev 1997). Quantum error correction is an algorithmic method that aims to protect encoded information with a sufficiently low error rate. Significant efforts have been made towards the discovery of error-correcting quantum codes (Shor 1995) and fault-tolerant quantum computation (Preskill 1997; Shor 1996).

Recently, a research team at Google has released a paper on the development of quantum processors based on superconducting qubits, which can now perform computations in a Hilbert space of dimension 2^{53}, beyond the reach of the fastest classical supercomputers available today. This experiment marks the first computation that can be performed only on a quantum processor. However, the challenge of finding ways for effective error correction schemes remains unsolved: implementing quantum error correction requires an overhead in qubits and quantum gates (Arute et al. 2019). Topological quantum computing based on nonabelian anyons offers a fresh perspective on quantum error-correction: the idea emerged as an attempt to address error problems at the hardware level. By design, this approach is fault-tolerant to errors and is robust against environmental perturbations. To understand this perspective, we must first discuss anyons and the general principles of topological quantum computing.

3. ANYONS

In two spatial dimensions quantum mechanical particles are not limited to being bosons or fermions as they are in three dimensions, but can be particles known as anyons which obey braiding statistics, i.e. the quantum mechanical state of a system is altered in a particular way when one anyon is moved (braided) around another, independent of the specific path chosen up to a path homotopy (Leinaas & Myrheim 1977).

Let us motivate why anyons can exist in Nature. A simple but physically meaningful observation is that physics should remain unchanged when two identical particles are exchanged in a system. In 3-dimensional space, this implies that only bosons and fermions can exist as point-like particles. A wave function describing the system of either types of particles acquires a +1 or -1 phase, respectively, whenever they are exchanged. However, in 2-dimensional systems, in addition to bosonic and fermionic exchange statistics, arbitrary phase factors, or even non-trivial unitary evolutions, can be obtained when two particles are exchanged. The exchange operator R is no longer constrained to square to identity. Instead, it can be representaed by a complex phase, or even a unitary matrix. In the first case the anyons are called abelian due to their exchange operators commuting, while in the latter case the anyons are called non-abelian. Since we no longer require $R = R^{-1}$ (as we would in the case of bosons and fermions), the order and orientation of the exchanges are physical and the only constraints on the exchange operator R are given by consistency conditions for distinct evolutions. These consistency relations are derived from the braid group. *It is this description of the 2D statistics by the braid group, instead of the permutation group, that allows anyons to exist.*

While this observation motivates the possibility of the existence of anyons in 2-dimensional systems, non-abelian anyons, which are ideal for topological quantum computing, have not been confidently detected yet and lie at the center of the modern condensed matter experimental research (Zhang et al. 2015; Xu et al. 2018). Anyons have also been extensively studied in the gauge field theory context (Dijkgraaf et al. 1991; Alexander Bais et al. 1992; Lo & Preskill 1993).

Non-abelian anyons are believed to exist in the $\nu = 5/2$ fractional quantum Hall, although more experimental work is required to detect them in such physical systems (Clarke et al. 2013; An et al. 2011; Moore & Read 1991; Halperin 1984; Arovas et al. 1984).

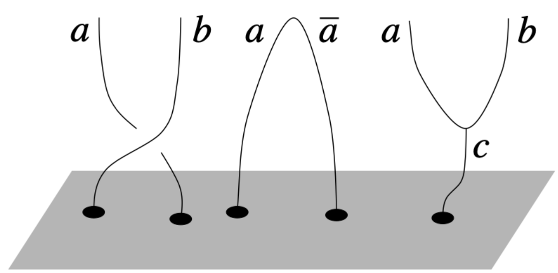

Figure 1. An example of anyon worldlines. Source: Pachos (2012)

4. ANYON MODELS

From now on we adopt the perspective that anyons exist and focus on their defining properties. In addition, we assume that the microscopic details of the physical system that supports anyons can be neglected in the following discussion. Under these assumptions the possible evolutions are limited to three simple scenarios (Figure 1):

- Anyons can be created or annihilated in pairwise fashion.
- Anyons can be fused to form other types of anyons.
- Anyons can be exchanged adiabatically.

Topological quantum field theory is a key framework for studying anyons (Witten 1989). Properties of the anyons corresponding to a particular topological quantum field theory are described by an *anyon model*. The anyon model is specified by: (i) the fusion coefficients N_{ab}^c that describe how many distinct anyons there are and how the anyons fuse, (ii) the F-matrices that describe the structure of the fusion space, and (iii) the R-matrices that describe the mutual statistics of the anyons. Regardless of the microscopics that give rise to anyons in a given system, within a given anyon model all allowed states of the fusion space for arbitrary number of anyons and all the allowed evolutions can be constructed (Lahtinen & Pachos 2017). We discuss each of the (i)-(iii) components of an anyon model below. After that, we provide an example illustrating these concepts: Fibonacci anyons (universal for quantum computation, but so far only a theoretical construction).

4.1. *Fusion channels*

First thing we should note is that there are many different anyonic models. Each model is determined by the statistical properties of its particles. Let us consider such a particular model. To describe it we introduce finitely many different species of particles. In a topological system, they correspond to quasiparticle excitations that can be distinguished according to their properties - physical observables (e.g. topological or anyonic charge). In the following discussion, we use the words “particle”, “anyon” and “quasiparticle” interchangeably.

Let us first specify the number of distinct anyons in a system. Such a list of distinct anyons must always contain a trivial label, 1, corresponding to the vacuum with no anyons. The anyon model is then spanned by a number of particles

$$M = \{1, a, b, c, ...\} \tag{3}$$

where the labels a, b, c are the ”topological charges” carried by each anyon. The fusion corresponds to bringing two anyons together and determines how they behave collectively. These charges must obey conservation rules known as *fusion rules*:

$$a \times b = \sum_{c \in M} N_{ab}^{c} c \tag{4}$$

where the fusion coefficients $N_{ab}^c \in \mathbb{Z}^+$ (i.e. non-negative integers) describing the possible topological charges (fusion channels) a particle fused out of particles a and b can carry. For most physical systems, $N_{ab}^c = 0, 1$. If $N_{ab}^c = 0$, then the fusion of a and b can not yield c. If for all $a, b \in M$ there is only one N_{ab}^c that is different from zero, then the fusion outcome of each pair of particles is unique and the model is called Abelian. Otherwise, the model is called non-Abelian. In a non-Abelian model, the fusion of a and b particles can result in several different anyons. To conserve total topological charge, every particle $a \in M$ must have an anti-particle $b \in M$, in the sense that $N_{ab}^1 = 1$. For instance, a fusion rule $a \times a = 1 + a$ means a is its own anti-particle and fusing a with a has two possible outcomes.

The ordering of a and b is not important, so that

$$a \times b = b \times a \tag{5}$$

In addition, the fusion process can be time reversed. Consider the case where the fusion of a and b gives a specific fusion outcome c. When time is inverted the same process describes the splitting of anyon c into its constituent particles a and b.

Next, if a and b can fuse to several $c \in M$, we can define orthonormal states $|ab; c\rangle$ that satisfy

$$\langle ab; c|ab; d\rangle = \delta_{cd} \tag{6}$$

If there are N distinct fusion channels in the presence of a pair of particles, the system exhibits N-fold degeneracy spanned by these states. We refer to this non-local space shared by the non-Abelian anyons, regardless of where they are located, as the fusion space. Assuming that all microscopics of the system giving rise to the anyons are decoupled from the low-energy physics, the states in the fusion space are perfectly degenerate. Since it is a collective non-local property of the anyons, no local perturbation can lift the degeneracy and hence it is a decoherence-free subspace. Thus, it is ideal for non-local encoding of quantum information.

The fusion space of a pair of non-Abelian anyons can not be used to directly encode a qubit though, since two states that belong to different global topological charge sectors can not be superposed. Instead, we need more than two anyons in the system. The basis in such fusion space is given by a fusion diagram of a fixed fusion order spanned by all possible fusion outcomes. Choosing a different fusion order is equivalent to a change of basis, and the two bases can be related by so called F-matrices, which are obtained as solutions to a set of consistency conditions known as the pentagon equations (Pachos 2012).

To illustrate how the F-matrices give structure to the fusion space, consider a case where three anyons a, b and c are constrained always to fuse to d and assume that several intermediate fusion outcomes are compatible with this constraint. Then there are several fusion states belonging to the same topological charge sector d that can be superposed. For three anyons there are two possible fusion diagrams corresponding to distinct bases. Either one fuses first a and b to give e, in which case the basis states are labeled by e and denoted by $|(ab)c; ec; d\rangle$, or one fuses first b and c to give f , in which case the corresponding basis states are $|a(bc); af; d\rangle$.

These two choices of basis must be related by a unitary matrix F^d_{abc} as

$$|(ab)c; ec; d\rangle = \sum_f (F^d_{abc})_{ef} |a(bc); af; d\rangle \tag{7}$$

where $(F^d_{abc})_{ef}$ are the matrix elements of F^d_{abc}, and f is summed over all the anyons that b and c can fuse to, i.e. for which $N^f_{ab} \neq 0$.

The states and the action of the F-matrices are usually expressed in terms of fusion diagrams, as shown in Figure 2. Such diagrams reflect the topological nature of such processes – two diagrams that can be continuously deformed into each other (i.e. no cutting or crossing of the world lines is allowed) correspond to the same state of the system.

4.2. *Fibonacci anyons*

To better understand the abstract ideas introduced in the previous Section, let us consider now a particular type of anyons, called *Fibonacci anyons*, that satisfy the

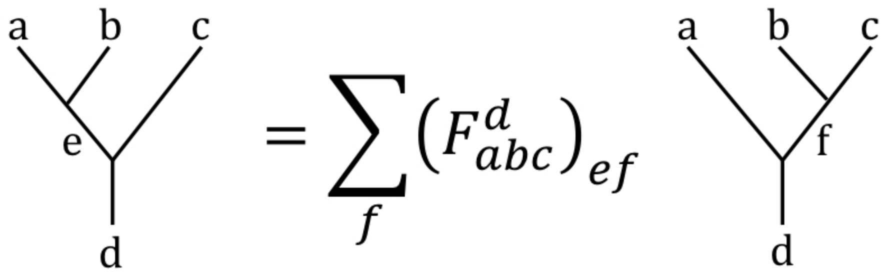

Figure 2. Fusion diagrams and F-matrices. A basis in the fusion space is given by choosing an order in which the anyons are to be fused without exchanging their positions (this results in a unitary evolution in the fusion space as discussed in the next subsection). In the case of three anyons a, b and c that are constrained to fuse to d, there are only two options: Either they are fused pairwise from left to right ($|(ab)c; ec; d\rangle$) or from right to left ($|a(bc); af; d\rangle$). These two bases are related by the unitary matrix F d according to (12). The state in one basis is in general a superposition of the basis states in the other basis.

following fusion rule:

$$\tau \times \tau = 1 + \tau \tag{8}$$

Applying this fusion rule associatively, we observe the following pattern:

$$\tau \times \tau \times \tau = 1 + 2 \cdot \tau \tag{9}$$

$$\tau \times \tau \times \tau \times \tau = 2 \cdot 1 + 3 \cdot \tau \tag{10}$$

$$\tau \times \tau \times \tau \times \tau \times \tau = 3 \cdot 1 + 5 \cdot \tau \tag{11}$$

and so on. In other words, the dimensionality of the fusion space in both topological charge sectors grows as Fibonacci series (i.e. the next number is always the sum of the two preceding numbers). To encode a qubit in the fusion space of Fibonacci anyons, we need three τ anyons that are constrained to fuse to a single τ particle. A basis in this two-dimensional fusion subspace is given by the states $|(\tau\tau)\tau; 1\tau; \tau\rangle$ and $|(\tau\tau)\tau; \tau\tau; \tau\rangle$, with the fusion diagrams shown in Figure 2 (with the corresponding label substituted). For the Fibonacci fusion rules the F-matrix giving the basis transformation and the R-matrix describing the braiding are given by

$$F = \begin{pmatrix} \phi^{-1} & \phi^{-1/2} \\ \phi^{-1/2} & -\phi^{-1} \end{pmatrix}, \tag{12}$$

$$R = \begin{pmatrix} e^{4\pi i/5} & 0 \\ 0 & e^{-3\pi i/5} \end{pmatrix}, \tag{13}$$

where $\phi = \frac{1+\sqrt{5}}{2}$ is the Golden ratio. For a detailed discussion on Fibonacci anyons, we refer to Trebst et al. (2008). We will highlight the main useful property of anyons in the context of topological quantum computing in Section 5.2.

5. TOPOLOGICAL QUANTUM COMPUTER

Now, as in the quantum circuit model mentioned in Section 1, in topological quantum computing, an algorithm is executed by building a sequence of "gates". However, the gates are now the braid group generators $\sigma_i, 1 \leq i \leq 2n-1$, and an approximation theorem (Kitaev 2003) is used to select the braid sequence which approximates the more usual quantum circuits. We will return to this important point in Section 5.2.

Thus, the topological model may be described as:
(1) Initialization of a topological quantum computer.
(2) Classical computation of braid b effecting a desired unitary transformation X of the computational subspace of the Hilbert space of the system.
(3) Adiabatic implementation of the braid by (somehow) moving the anyons around to draw b in 2+1 dimensional space-time.
(4) Application of a projection operator Π to measure the outcome.

In the ideal conditions of zero temperature and infinite anyon separation, the states in the fusion space have three very attractive properties:
(i) All the states are perfectly degenerate.
(ii) They are indistinguishable by local operations.
(iii) They can be coherently evolved by braiding anyons.

If this space of states is used as the computational space of a quantum computer, property (i) implies that the encoded information is free of dynamical dephasing, while property (ii) means that it is also protected against any local perturbations. Property (iii) means that errors could only occur under non-local perturbations to the Hamiltonian. Furthermore, property (iii) implies that all the quantum gates are, in principle, free from errors since they depend only on the topological characteristics of the braiding evolutions given by the F- and R-matrices.

The above discussion shows that in topological quantum computing, errors can be suppressed at the hardware level. Of course, in the real world these ideal conditions are never met and some decoherence of the encoded information always takes place. Still, comparing to other non-topological schemes, the main advantage of topological encoding and processing of quantum information is, in principle, the protection from errors.

In the following sub-sections, we describe initialization of a topological quantum computer and implementation of quantum gates using the braid group language.

5.1. *Initialization of a topological quantum computer: creating and arranging anyons*

To initialize a quantum computer, one needs first to identify the computational space of n qubits. In topological computer this means creating some number of anyons from the vacuum and fixing their positions. A possible configuration and manipulation of anyons that can result in quantum computation is shown in Figure 3. We start with a set of anyons that are prepared in a well-defined fusion state. For example, we may create pairs of non-Abelian anyons a and $\overline{a}$ from the vacuum.

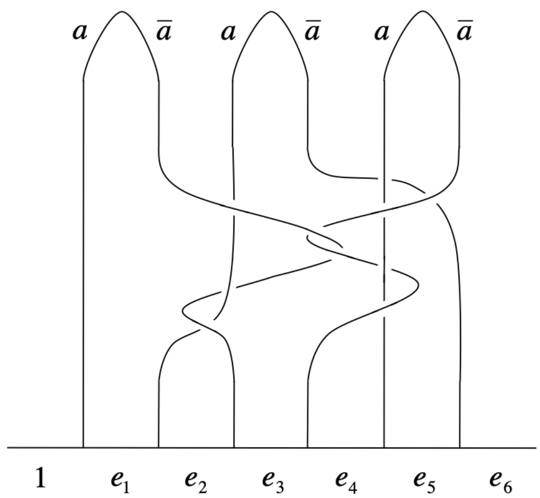

Figure 3. An example configuration of topological quantum computation. Source: Pachos (2012)

5.2. *Quantum gates: braiding anyons*

Performing a computation in the fusion space is equivalent to specifying a braid – a sequence of exchanges of the anyons that corresponds to the desired sequence of logical gates.

Let us provide some background on quantum gates. Quantum gates leverage two properties from quantum mechanics: superposition and entanglement. The processing of the encoded quantum information is usually performed by quantum gates. These are reversible quantum evolutions that operate on one, two or more qubits simultaneously. As quantum evolutions are described by unitary matrices, quantum gates between n qubits are elements of the unitary group $U(2n)$. For example, one qubit gates include the Pauli matrices

$$\sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{14}$$

$$\sigma^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \tag{15}$$

$$\sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{16}$$

Here σ^x is the NOT gate that changes the input 0 to 1 or 1 to 0, respectively. Snother famous gate is the Hadamard gate given by

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \tag{17}$$

that transforms $|0\rangle$ into the quantum superposition $\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ and $|1\rangle$ into $\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$

Quantum evolution of the topological qubit is implemented by successive applications of the elementary exchanges (σ_1 and σ_2 in the simplest case of a single-qubit braiding gate) on its anyons pairs. As an example, Figure 4 shows the braid representing the braiding gate operator G defined as

$$G = \sigma_1^{-2} \sigma_2^2 \sigma_1^2 \tag{18}$$

We now turn to the key property of non-abelian anyons that make them ideal candidates for quantum computation. The assertion rests on the Solovay-Kitaev theorem (Nielsen & Chuang 2010):

Let G be a finite set of elements in $SU(d)$ containing its own inverses, such that the image of G is dense in $SU(d)$, and let a desired accuracy ϵ be given. There exists a constant c such that for any $U \in SU(d)$ there exists a finite sequence S of gates in G of length $O(logc(1/\epsilon))$ such that $d(U, S) < \epsilon$.

This theorem states that it is always possible to approximate any unitary quantum gate U to a desired accuracy ϵ using a long enough braid word S. It turns out the Fibonacci model is universal (Pachos 2012): the braid group generated by R and $F^{-1}RF$ is dense in $SU(2)$ in the sense that any unitary can be approximated to arbitrary accuracy by only braiding the Fibonacci anyons. However, approximating even the simplest gates is not straightforward. Even the NOT-gate requires thousands of braiding operations (Bonesteel et al. 2005; Baraban et al. 2010). Several techniques exist to construct the required braids more efficiently (Xu & Wan 2008, 2009; Xu & Taylor 2011; Burrello et al. 2010, 2011; Rouabah 2020), but the task remains challenging.

Moreover, Fibonacci anyons are difficult to support in physical systems. While there have been proposals involving coupled domain wall arrays of Abelian FQH states (Mong et al. 2014) and the Read-Rezayi states (Read & Rezayi 1999), these systems are fragile and challenging to implement. Nevertheless, the theoretical properties of non-abelian anyons make them attractive for further research and topological quantum computing remains promising.

5.3. *Measurement - fusing anyons and detecting anyonic charge*

The final step of a computation is the read-out. The measurements are performed by bringing the anyons pairwise physically together and detecting the fusion outcomes. The practical implementation of this process depends on the type of anyons

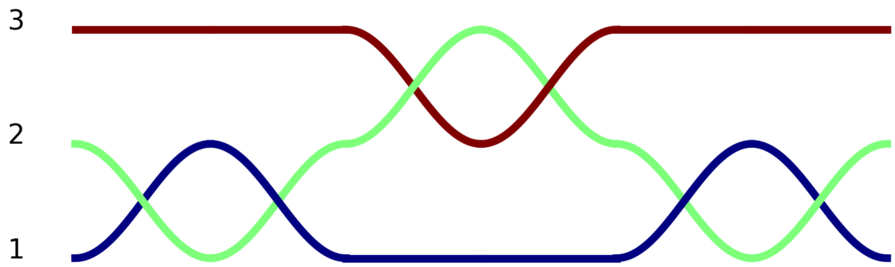

Figure 4. The braid representing the quantum gate $G = \sigma_1^{-2}\sigma_2^2\sigma_1^2$. Time flows to the right.

in question and on the physical system that supports them. For a particular type of anyons which are known as the Ising anyons and which are physically realizable in the laboratory, it is possible to perform measurements by detecting changes in the energy of the system (Pachos 2012).

This concludes our review of topological quantum computing. We now want to discuss the Jones polynomial, which, as we mentioned in Section 1, was critical to finding new and unexpected connections between the theory of knots and quantum computing.

6. THE JONES POLYNOMIAL

Let us now define the Jones polynomial, provide an example calculation, and then return to our discussion of the relation between quantum theory and knots.

We start by introducing a polynomial known as the state sum or Kauffman bracket. The state sum is defined to be invariant under isotopy moves that transform links in a continuous way. The simplest kinds of link configurations comprises of disentangled loops. In this construction, we therefore aim to define a process that relates the state sum of a link to the state sum of disentangled loops. To accomplish this, we use the *Skein relations* that split crossings as shown in Figure 5 ($A, B \in \mathbb{C}$):

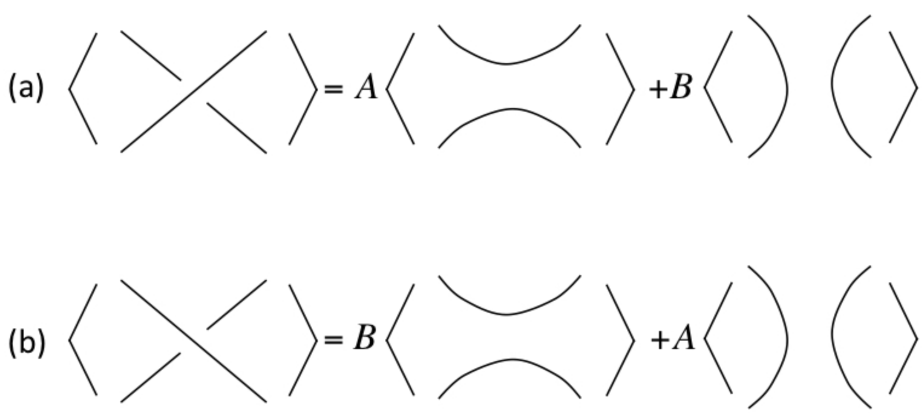

Figure 5. Skein relations. Source: Pachos (2012)

Applying the Skein relation to N crossings of a diagram, we obtain the sum of 2^N different elementary diagrams. The final configuration is a sum of disjoint, unentangled loops denoted by S and weighted by products of A's and B's. Denote the number of such loops at each configuration S by $|S|$. Additionally, we require the following two properties be satisfied by the state sums:

$$\langle K \cup \bigcirc \rangle = d\langle K \rangle$$

$$\langle \bigcirc \rangle = 1$$

The Skein relations in Figure 5 combined with the above two equations uniquely define the state sum of a link L:

$$\langle L \rangle = \sum_{\{S\}} d^{|S|-1} A^i B^j$$

The summation runs over all 2^N possible configurations S that result from the splitting of the diagram. i and $j = N - i$ are the number of times a horizontal or a vertical splitting, respectively, was employed, in order to obtain the S configuration from the link L. Enforcing the state sum to be invariant under the Redemeister moves II and III leads to the following relations being imposed on A, B, and d:

$$B = A^{-1}, d = -A^2 - A^{-2}$$

So the state sum $\langle L \rangle$ is a Laurent polynomial in A, given by

$$\langle L \rangle = \sum_{\{S\}} d^{|S|-1} A^{i-j}$$

with

$$d = -A^2 - A^{-2}$$

An example calculation of the state sum is shown in Figure 6.

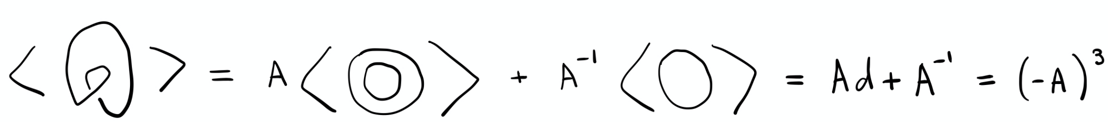

Figure 6. An example calculation of the state sum.

Imposing invariance of $\langle L \rangle$ with respect to the Reidemeister move I leads us to defining the Jones polynomial. The Jones polynomial, which is invariant under all three Redeimeister moves, is defined by

$$V_L(A) = (-A)^{3w(L)} \langle L \rangle$$

where $w(L)$ is the writhe of the link L. To determine the writhe, we give an orientation to all link components. Then $w(L)$ is the sum of signs of all crossings:

$$w(L) = \sum_i w_i$$

While Kauffman invariants are difficult to calculate, quantum computing can provide a way to estimate them (Brennen & Pachos 2007).

Let's return to our discussion of anyons. It turns out that the probability that the particles will annihilate at the end of an experiment is proportional to the Jones polynomial of the knot squared. By conducting physical experiments, we can therefore estimate the probability of anyon annihilation and use this result as a measurement of the Jones polynomial. More specifically, the quantum properties of non-Abelian anyons can be given by the expectation value of their spacetime evolutions. Suppose we create anyons from the vacuum, braid them, and then ask what is the probability of them fusing back to the vacuum. To determine this probability, we need to calculate the expectation value of the anyonic evolution with worldline trajectories that form closed paths (i.e. links). These expectation values are invariant under continuous deformations of trajectories. Hence, they are topological link invariants. Jones polynomials appear as the expectation values of anyonic evolutions described by non-Abelian Chern-Simons theories.

Let us now describe the key ideas of how we may use quantum computation with anyons to estimate Jones polynomials. Consider an anyonic model with statistics corresponding to the following unitary representation ρ_A of the braid group B_n

$$\rho_A(\sigma_i) = AE_i + A^{-1}1$$

$$\rho_A(\sigma_i^{-1}) = A^{-1}E_i + A1$$

where 1 is the identity, A is a newly introduced parameter, $d = -A^2 - A^{-2}$, σ_i are the generators of the braid group, and E_i, $i = 1, ..., n-1$ are the generators of the Temperley-Lieb algebra $TL_n(d)$ that satisfy:

$$E_i E_j = E_j E_i, |i-j| \geq 2$$

$$E_i E_{i\pm1} E_i = E_i$$

$$E_i^2 = dE_i$$

Parameter A provides a particular statistical behavior of the anyons. We can then relate the expectation values of their braiding evolutions to the Jones polynomials in the following way. Consider an anyonic evolution (see Figure 7 for an example) where n anyons are created in pairs from the vacuum. The initial state $|\psi\rangle$ indicates that each anyonic pair is in the vacuum fusion channel. Suppose we perform an arbitraty braiding among half of the anyons (see Figure 7 for an example), described by the

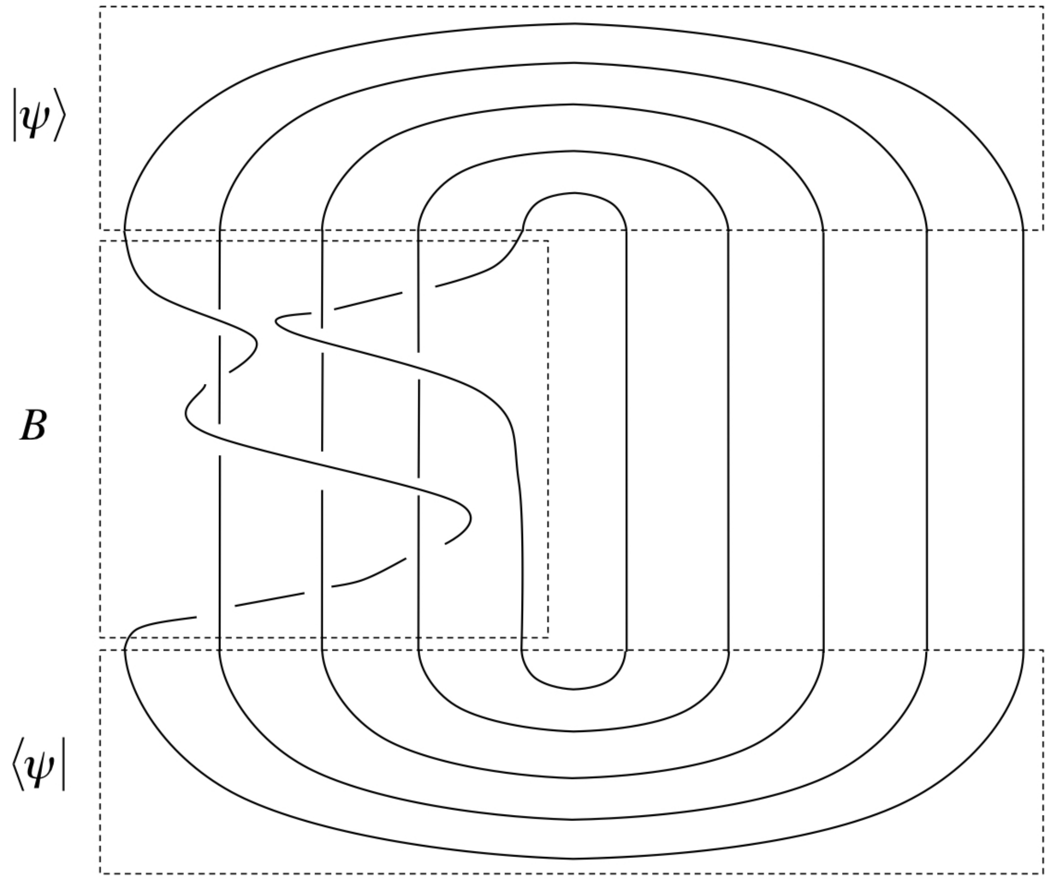

Figure 7. An example of anyon quantum evolution (time flows downward). Five pairs of anyons are created from the vacuum. Source: Pachos (2012)

unitary evolution $B(A)$. The probability of obtaining the vacuum state at the end is then given by the overlap

$$\langle\psi|B(A)|\psi\rangle = (\rho_A(B))$$

Using the braid group, one may show that the Jones polynomial can be expressed in the following way

$$V_L(A) = (-A)^{3w(L)}d^{n-1}(\rho_A(B))$$

Combining the two equations above, we can express the Jones polynomials in terms of outcome amplitudes

$$V_L(A) = (-A)^{3w(L)}d^{n-1}\langle\psi|B(A)|\psi\rangle$$

The amplitudes $\langle\psi|B(A)|\psi\rangle$ can be estimated to high accuracy from repeating the same physical experiments with anyons multiple times.

Freedman and others have laid out the foundation for using the braiding of anyons to efficiently evaluate the Jones polynomial (Freedman et al. 2000; Freedman et al. 2002). Aharonov et al. (2005) later proposed a polynomial-time quantum algorithm to solve this problem. For details of the other proposed quantum algorithms for estimating the amplitude of the Jones polynomial, the reader is referred to Kauffman & Lomonaco (2007) and Marx et al. (2010). To keep this paper manageable in scope, we do not include the details of such quantum algorithms in this manuscript.

7. CONCLUSION

Topological quantum computing is a promising approach to computing that, in theory, addresses the main challenge of the modern quantum computing: error problems that inevitably arise from the environment. Proposed by Kitaev, topological quantum computers employ anyonic statistics to encode and manipulate information. In this paper, we have observed how knot theory provides effective tools for designing quantum algorithms and how quantum computing may in turn assist in solving computationally intensive mathematical problems, such as estimating the amplitude of the Jones polynomial.

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