# The Euler Characteristic Transform

Erik Amézquita
Michigan State University

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# Disclaimer

- My best informed guess of what Turner et al. tried to convey in:
  - K. Turner, S. Mukherjee, D. Boyer "Persistent homology transform for modeling shapes and surfaces". *Information and Inference: A Journal of the IMA* Vol.3 No.4 pp.310–344, 2014
- All the figures were made in either `TikZ` (LaTeX) or `rgl` (`R`).

# Filtrations and Diagrams

- $M \subset \mathbb{R}^d$ a finite simplicial complex, $v \in S^{d-1}$
- Define a *height* filtration

$$M(v)_r = \{x \in M : \langle x, v \rangle \leq r\} \simeq \{\Delta \in M : \langle x, v \rangle \leq r \,\forall\, x \in \Delta\}$$

- Consider the $k$-th persistence diagram $X_k(M, v)$.
- Say $\mathcal{D}$ is the space of all persistence diagrams.
- Due to bottleneck distance stability on $\mathcal{D}$, the following function is Lipschitz

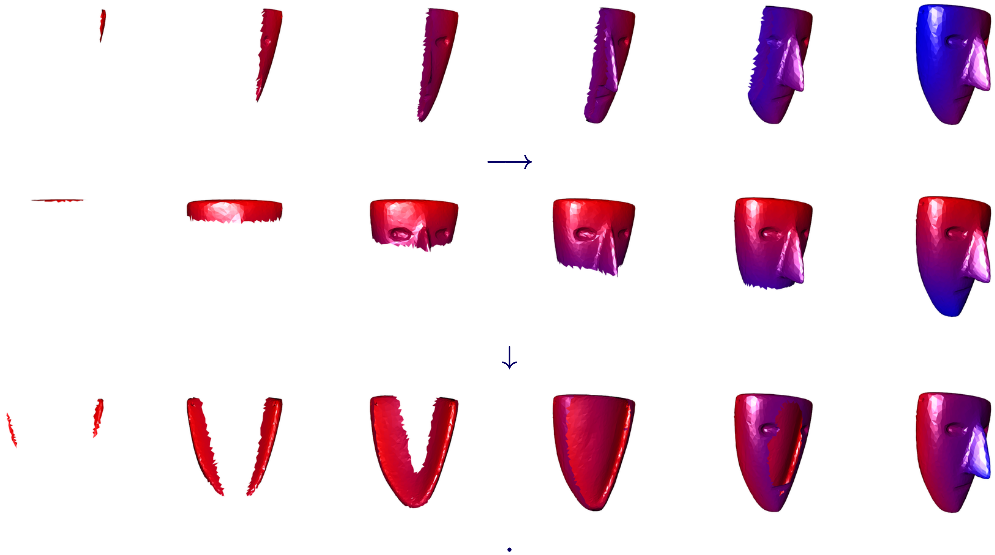
$$v \mapsto X_k(M, v)$$

# Persistence Homology Transform

- Let $\mathcal{M}_d$ be the space of all subsets of $\mathbb{R}^d$ that can be written as finite simplicial complexes.
- Define the Persistence Homology Transform

$$\begin{aligned} PHT : \mathcal{M}_d &\to \mathcal{C}(S^{d-1}, \mathcal{D}^d) \\ PHT(M) : S^{d-1} &\to \mathcal{D}^d \\ v &\mapsto (X_0(M,v), X_1(M,v), \ldots, X_{d-1}(M,v)) \end{aligned}$$

# PHT in Pictures



# Outline of the theorem

**Theorem**

*The PHT is injective when the domain is $\mathcal{M}_3$.*

- The proof is constructive
- Given $PHT(M) : S^2 \to \mathcal{D}^3$ we can find all the vertices in one of the simplest representation of the simplicial complex.
- Then determine the link of each vertex.
    - From the persistence diagrams, we can deduce changes in the Euler characteristic of $M$
- Since $M$ is piecewise linear, vertices and links are enough for reconstruction.

# Injectivity of the PHT

- Given a collection of directions and persistence diagrams $PHT(M)$, there is a procedure to "reconstruct" $M$ in one of the simplest (fewest possible number of vertices) representation of the simplicial complex.
- Fix $x \in M$, $v \in S^2$, $r = \langle x, v \rangle$.

$$M(v)_r = \{z \in M : h_v(z) \leq r\}$$
$$M(v)_r^- = \{z \in M : h_v(z) \leq r - \delta\}$$

- $\delta > 0$ small enough so that no critical values of $h_v$ occur in $(r - \delta, r)$.
- Such $\delta$ exists since $M$ is finite.

# The Long Exact Exact Sequence

$$\ldots \to H_i(M(v)_r^-) \xrightarrow{\iota_*} H_i(M(v)_r) \xrightarrow{\pi_*} H_i(M(v)_r, M(v)_r^-) \xrightarrow{\partial_*} H_{i-1}(M(v)_r^-) \xrightarrow{\iota_*} H_{i-1}(M(v)_r) \to \ldots$$

- We conclude $H_i(M(v)_r, M(v)_r^-) \cong \ker \partial_* \oplus \mathrm{im}\, \partial_*$.
- Namely:

$$\begin{aligned}
H_0(M(v)_r, M(v)_r^-) &\cong \mathrm{coker}\,\{H_0(M(v)_r^-) \to H_0(M(v)_r)\} \\
H_1(M(v)_r, M(v)_r^-) &\cong \mathrm{coker}\,\{H_1(M(v)_r^-) \to H_1(M(v)_r)\} \\
&\quad \oplus \ker\{H_0(M(v)_r^-) \to H_0(M(v)_r)\} \\
H_2(M(v)_r, M(v)_r^-) &\cong \mathrm{coker}\,\{H_2(M(v)_r^-) \to H_2(M(v)_r)\} \\
&\quad \oplus \ker\{H_1(M(v)_r^-) \to H_1(M(v)_r)\} \\
H_i(M(v)_r, M(v)_r^-) &= 0, \quad i \geq 3.
\end{aligned}$$

# Betti numbers and the Euler Characteristic

- Define $\tilde{\beta}_i(x, v) := \mathrm{rank}\,(H_i(M_r, M_r^-))$
- Record the change in $\beta_i$

$$
\begin{aligned}
\tilde{\beta}_0(x, v) &= \#\{\text{classes in } X_0(M_v) \text{ born at height } r\} \\
\tilde{\beta}_1(x, v) &= \#\{\text{classes in } X_1(M_v) \text{ born at height } r\} \\
&\quad + \#\{\text{classes in } X_0(M_v) \text{ that die at height } r\} \\
\tilde{\beta}_2(x, v) &= \#\{\text{classes in } X_2(M_v) \text{ born at height } r\} \\
&\quad + \#\{\text{classes in } X_1(M_v) \text{ that die at height } r\} \\
\tilde{\beta}_i(x, v) &= 0, \quad i \geq 3.
\end{aligned}
$$

- Summarize the homological changes in $(r - \delta, r)$ in persistence diagrams via the Euler Characteristic

$$\tilde{\chi}(x, v) := \tilde{\beta}_0(x, v) - \tilde{\beta}_1(x, v) + \tilde{\beta}_2(x, v).$$

# First Claim

## Proposition

*Changes in homology of sublevel sets of height functions in any direction can only occur at the heights of vertices of $M$.*

## Proof.

- If $x$ is not a vertex, due to finiteness there is a $\delta > 0$ such that

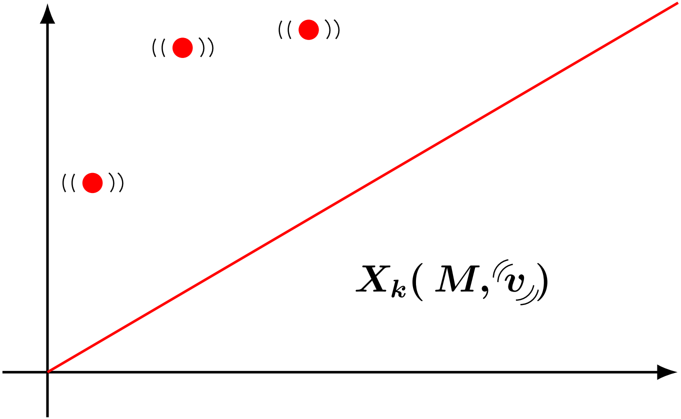
$$H_k(M(v)_r, M(v)_r^-) = 0 \quad \forall\, k \in \mathbb{Z},\ v \in S^2.$$

- There is a corresponding lack of points in the persistence diagrams

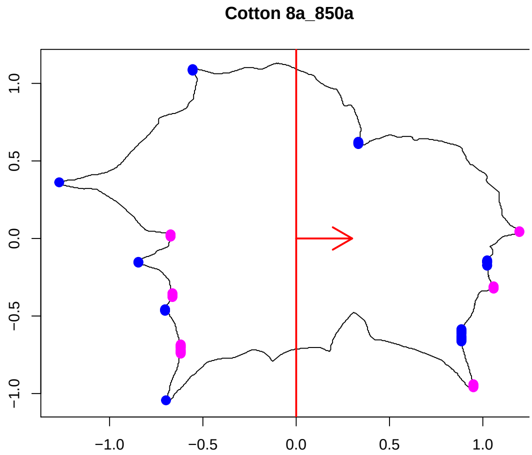
□

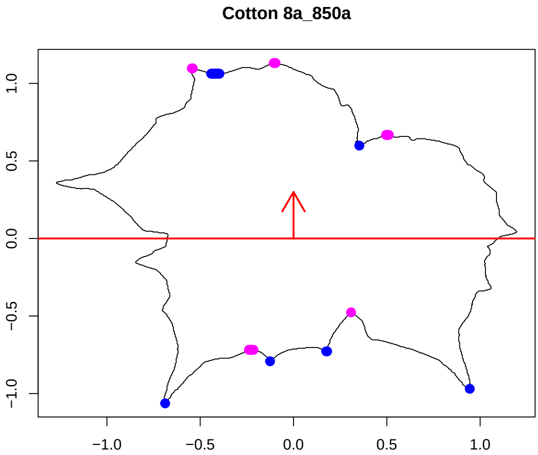
# Find Vertices

1. Choose direction $v \in S^2$, dimension $k$, and point $(b_v, d_v) \in X_k(M, v)$.
2. Since $v \mapsto X_k(M, v)$ is continuous, there is a radius $r > 0$ such that there is a well defined and continuous set of points $(b_u, d_u)$ for each $u \in B(v, r)$.



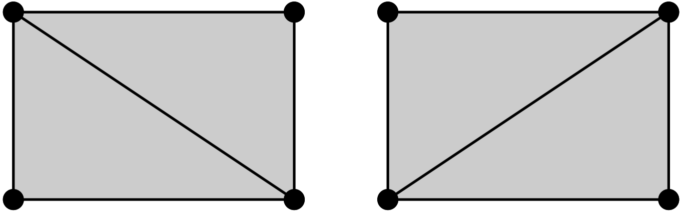
3. Consider $0 < r' < r$. If there exists a point $x \in \mathbb{R}^3$ s.t. $b_u = \langle x, u \rangle$ for every $u \in B(v, r')$, then $x$ must be a vertex of $M$.
4. Consider $0 < r' < r$. If there exists a point $x \in \mathbb{R}^3$ s.t. $d_u = \langle x, u \rangle$ for every $u \in B(v, r')$, then $x$ must be a vertex of $M$.



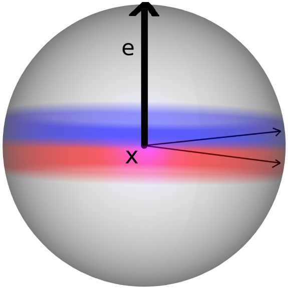
# Finding links

- Given the vertices, we want to find the link structure of each.
- If $x$ is isolated, $\operatorname{Lk} x = \varnothing$ iff an $H_0$ class is born at height $\langle x, v \rangle$ for every direction $v \in S^2$.
- Assume $x$ is not isolated
- Consider only essential edges: every simplicial representation of $M$ with vertices $\{x_i\}_{i=1}^n$ must contain that edge.
- The diagonal of a rectangle is not essential.

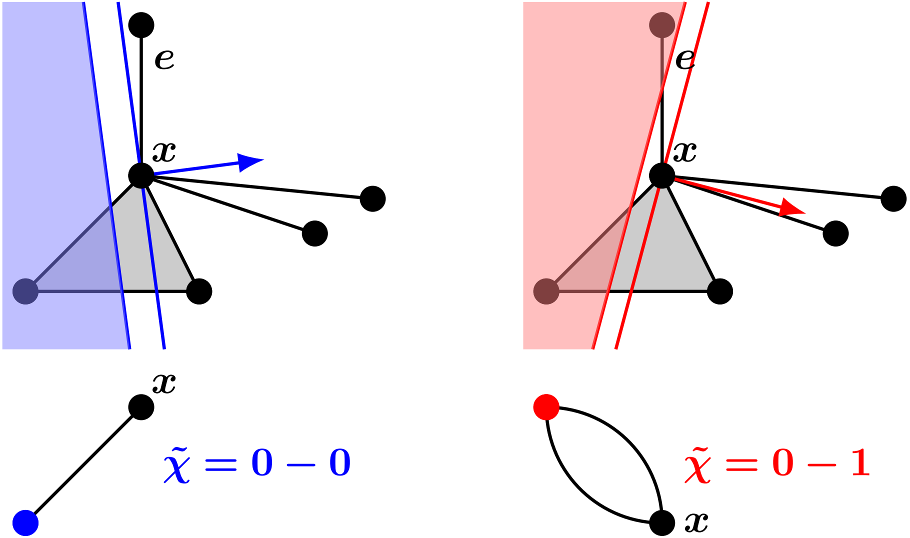


# Keep track of the Euler Characteristic

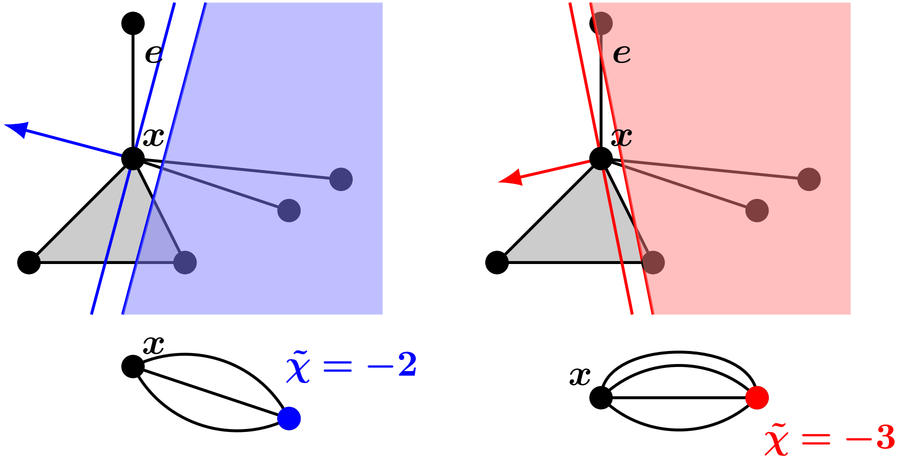
- WLOG Say $x$ is a non-isolated vertex with an edge $e$ pointing north.
- How does $\tilde{\chi}(x, v)$ change whenever $v$ passes from north to south?
- Say first $e$ is isolated, $\mathrm{Lk}\, e = \varnothing$
- $e$ is an extra edge whenever we move southwards
- Thus $\tilde{\chi}$ is reduced by 1 as we gain one cycle or lose one component.



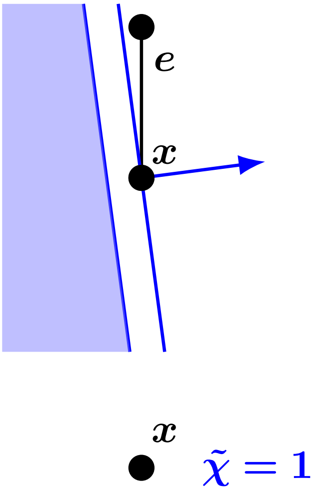
# $\tilde{\chi}$ decreases: one cycle is gained

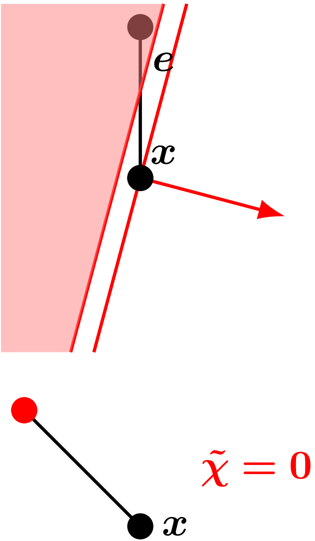


# $\tilde{\chi}$ decreases: one cycle is gained



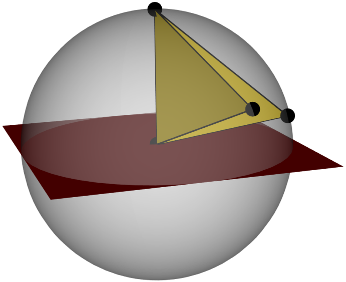
# $\tilde{\chi}$ decreases: one connected component is lost



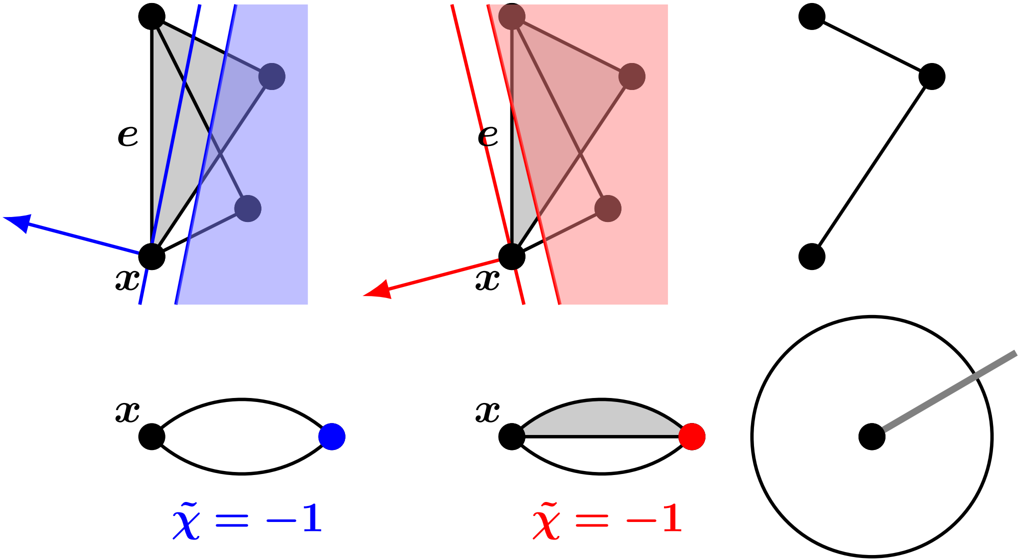



# What if $e$ is not isolated? (temporary reinterpretation)

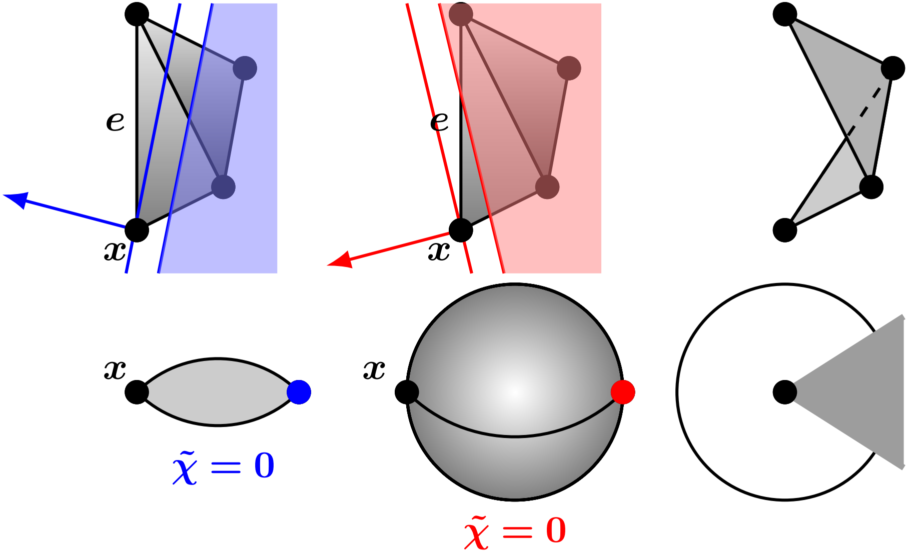
1. Consider the great circle perpendicular to $e$, the equator WLOG
2. Take a bird point of view of the equator
3. Project $\mathrm{Lk}\, e$
4. Split the equator into regions
5. The number of regions tell us how does $\tilde{\chi}$ changes as $v$ passes southwards.



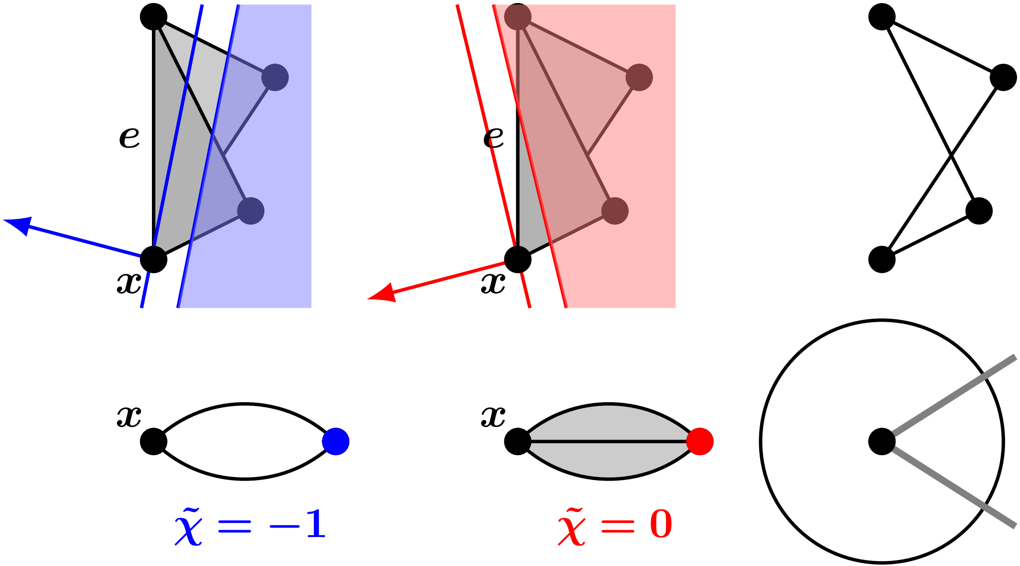
## One component: $\tilde{\chi}$ unchanged



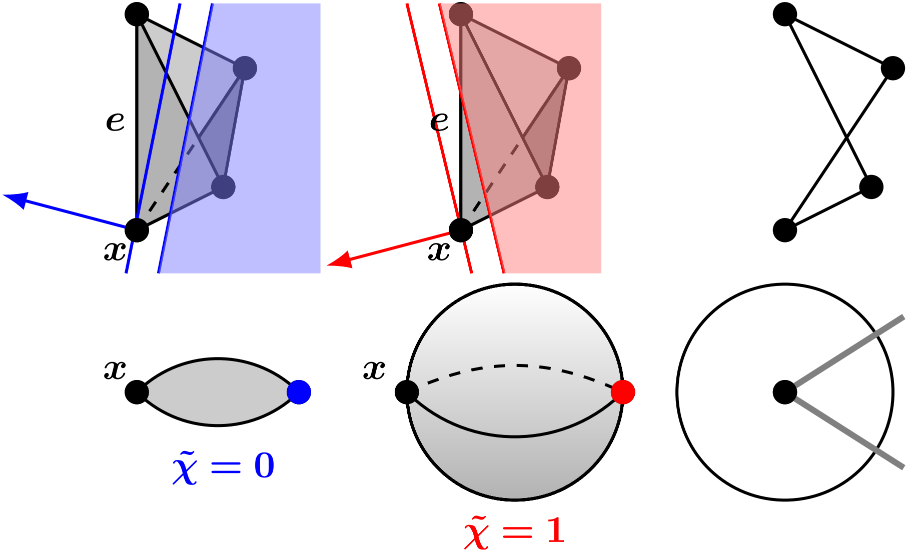
# One component: $\tilde{\chi}$ unchanged



# Two components: Hole filled: $\tilde{\chi}$ increases by 1

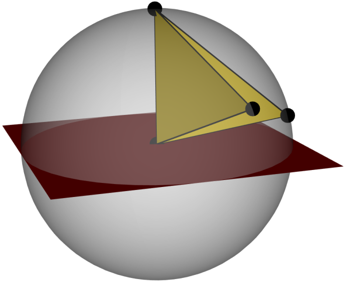


# Two components: Void created: $\tilde{\chi}$ increases by 1

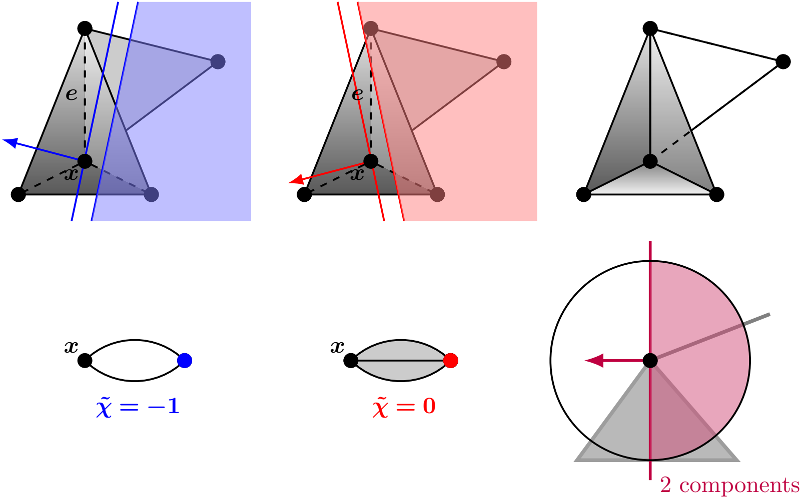


# What if $e$ is not isolated? (Actual interpretation)

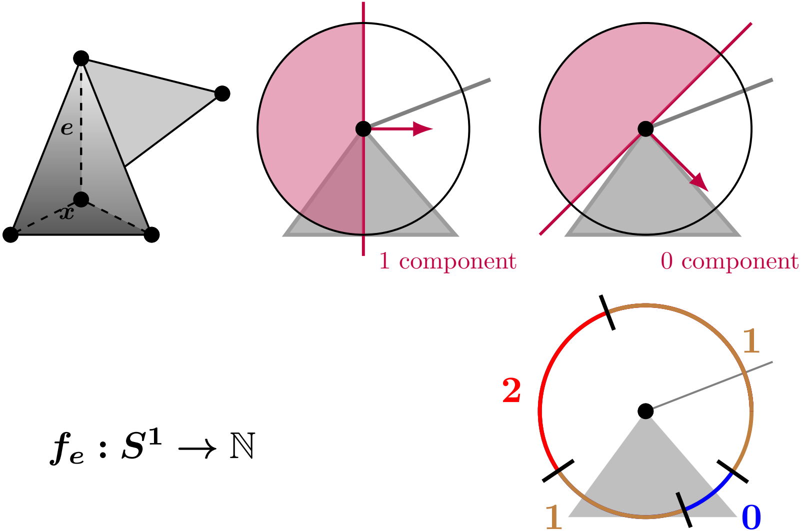
1. Consider the great circle perpendicular to $e$, the equator WLOG
2. Project onto the ball the directions that emanate perpendicularly from $e$ within $M$.
3. Take a bird point of view of the equator
4. Split the equator into regions depending on how many components are in this projection of the link of $e$ intersected with the other half of this equator
5. The number of components tell us how does $\tilde{\chi}$ changes as $v$ passes the equator traveling south.



# Pointing left, we observe two components while moving north to south. Thus $\bar{\chi}$ increases by one.



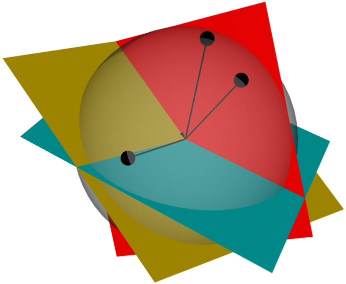
# Define $f_e$ on the greater circle. $f_e$ determined by changes in $\bar{\chi}$.




$$f_e : S^1 \to \mathbb{N}$$

# Link of $e$ = changes in $\tilde{\chi}$

- For each edge $e$, say $f_e : S^2 \to \mathbb{Z}$ tracks the changes in $\tilde{\chi}(x, v)$ as $v$ moves southwards
- $f_e$ equivalent to bird view equivalent to $\mathrm{Lk}\, e$.
- We cannot comment about what happens on the equator, but equator has measure zero.
- The sphere of directions centered at $x$ can be partitioned into regions bounded by finitely many great circles

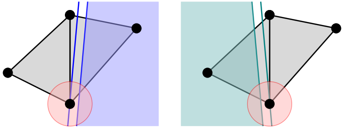


# Second Claim

**Proposition**

- *Every vertex $x$ determines a critical point for an open ball in the set of all directions*
- *Its inclusion causes a birth or death of a homology class*
- *The homology class remains unchanged within the ball*

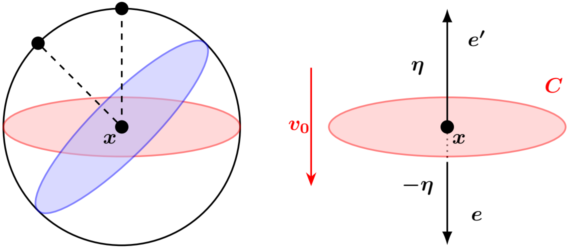
- Within the same region, $\tilde{\chi}$ remains constant
- Since $e$ is essential, the number of components is not 1 for some open interval along the great circle.
- At least one region bounded by great circles has non-zero $\tilde{\chi}$
- $x$ determines a critical point for directions in that region



# Start the reconstruction

1. Scan all directions $v \in S^2$ to find all the vertices of $M$.
    1. Select $v_0$ for which no vertices have the same height in that direction
    2. Order vertices according to their $v_0$-height
    3. Say $x$ is the first vertex
    4. $M(x, v_0) = \{x\}$.
2. Consider the sphere of directions centered at $x$.
3. Based on the change of $\tilde{\chi}$, which can be deduced by the collection of persistence diagrams, we can partition the directions' sphere into homology-constant regions.
4. Just be careful whenever $x$ has diametrically opposite edges.

5. For each great circle $C$ in the partition, we can deduce $\mathrm{Lk}\, e$
6. For each vertex, in the order outlined in (1), find the appropriate great circle $C$ in (3), then do (4) through all great circles.
7. At last the simplicial complex is revealed.



# We got $\mathcal{M}_2$ as well

**Corollary**

*The persistence homology transform is injective when the domain is $\mathcal{M}_2$.*

- Let's consider $\mathbb{R}^2 \hookrightarrow \mathbb{R}^3$ with $(x, y) \mapsto (x, y, 0)$.
- Thus $\tilde{M} \in \mathcal{M}_2$ can be regarded as $M \in \mathcal{M}_3$
- Construct $X_k(M, (v_1, v_2, v_3))$ from $X_k(\tilde{M}, (\tilde{v}_1, \tilde{v}_2))$.

# The Euler Characteristic Transform

- Given the previous height function

$$M(v)_r = \{\Delta \in M : \langle x, v \rangle \leq r \;\; \forall\, x \in \Delta\}.$$

- The Euler Characteristic Curve

$$\chi(M, v)(r) = \chi(M(v)_r) = V - E + F.$$

- Define the ECT

$$\begin{aligned} ECT(M) : S^{d-1} &\to \mathbb{Z}^{\mathbb{R}} \\ v &\mapsto (\chi(M, v)) \end{aligned}$$

# ECT vs PHT

- The injectivity proof of the PHT hinges on keeping track of the Euler Characteristic changes as $v$ moves from north to south.
- The ECT does keep track of the EC
- All the goodies from PHT apply to ECT
  - Injectivity in both $\mathcal{M}_3$ and $\mathcal{M}_2$
  - Sufficient statistic