$\forall\Gamma, \forall e, \forall \tau, \forall x, \forall \tau'\ (((\Gamma \vdash e : \tau) \wedge (x \text{ is not mentioned in } e)) \Rightarrow (\Gamma, x:\tau' \vdash e:\tau)$.

For all contexts Γ, expression e, variable x, and types τ, if e has type τ in context Γ, then adding a new, unused variable x (of type τ') to the context does not change the type of e.

Proof. By structural induction on e.

case $e = x$: can't do this because x should have not been mentioned in e.

case $e = x'$ (where $x \neq x'$) : we must prove $\Gamma, x:\tau' \vdash x' : \tau$; we have assumed that $\Gamma \vdash x' : \tau \wedge x$ is not mentioned in x'. Since $x \neq x'$, and from the assumption that $\Gamma \vdash x' : \tau$ and x is not mentioned in x', we can conclude that $\Gamma, x:\tau' \vdash x' : \tau$ as desired.

case $e = e_1 e_2$: our induction hypothesis tells us that, if $\Gamma \vdash e_1 : \tau_1$ (for some τ_1) and x is not mentioned in e_1 then $\Gamma, x:\tau' \vdash e_1 : \tau_1$ (and similarly for e_2). We have assumed $\Gamma \vdash e_1 e_2 : \tau$ and x is not mentioned in $e_1 e_2$. This can be true only by APP. Thus the premises of APP must be true; we can conclude that $\Gamma \vdash e_1 : \tau_2 \rightarrow \tau$ and $\Gamma \vdash e_2 : \tau_2$ (for some τ_2). We thus use the induction hypothesis on both e_1 and e_2 to conclude that $(\Gamma, x:\tau' \vdash e_1 : \tau_2 \rightarrow \tau)$ and $(\Gamma, x:\tau' \vdash e_2 : \tau_2)$. Thus, we can use APP to conclude $\Gamma, x:\tau' \vdash e_1 e_2 : \tau$ as desired.

ase $e = e_1 + e_2$: This is similar to the previous case, using PLUS instead of APP.

ase $e = n$: we must prove that $\Gamma, x:\tau' \vdash n : \text{Int}$; that is that $\Gamma \vdash n : \text{Int} \wedge x$ is not mentioned as n. We have this by INT and we are done.

case $e = \lambda x' : \tau_1 . e_1$: The induction hypothesis tells us that if $\Gamma \vdash e_1 : \tau_2 \wedge x$ is not mentioned in $\lambda x' : \tau_1 . e_1$ (for some τ_2), then $\Gamma, x : \tau' \vdash e_1 : \tau_2$. We have assumed that $\Gamma \vdash \lambda x' : \tau_1 e_1 : \tau$. This can be only by ABS. Thus, we can assume that the premise of ABS: $\Gamma, x' : \tau_1 \vdash e_1 : \tau_2$ (and it must be that $\tau = \tau_1 \rightarrow \tau_2$). That satisfies the premise of the induction hypothesis and so we conclude $\Gamma, x : \tau', x' : \tau_1 \vdash e_1 : \tau_2$. By, ABS, we can thus conclude $\Gamma, x : \tau' \vdash (\lambda x' : \tau_1 . e_1) : \tau_1 \rightarrow \tau_2$ as desired (remembering that $\tau = \tau_1 \rightarrow \tau_2$).