Problem Set 8, **Math 54-Lec 3, Linear Algebra, Fall 2017** Solutions

September 20th, 2017

This problem was originally assigned on the homework but I felt it deserved extra emphasis.

Problem 1. Let $M_{2\times2}$ be the vector space of all two by two matrices, and define $T : M_{2\times2} \to M_{2\times2}$ by $T(A) = A + A^T$.

Part a. Show that T is a linear transformation.

Let $A = \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix}$, $B = \begin{bmatrix} y_1 & y_2 \\ y_3 & y_4 \end{bmatrix}$, then

$$T(A+B) = (A+B) + (A+B)^T = \begin{bmatrix} x_1+y_1 & x_2+y_2 \\ x_3+y_3 & x_4+y_4 \end{bmatrix} + \begin{bmatrix} x_1+y_1 & x_3+y_3 \\ x_2+y_2 & x_4+y_4 \end{bmatrix}$$

$$= \begin{bmatrix} 2x_1+2y_1 & x_2+x_3+y_2+y_3 \\ x_2+x_3+y_2+y_3 & 2x_4+2y_4 \end{bmatrix} = T(A) + T(B)$$

so $T(A+B) = T(A) + T(B)$, to show $T(cA) = cT(A)$ is similar.

Part b. Let B be any two by two matrix such that $B = B^T$. Find an $A \in M_{2\times2}$ such that $T(A) = B$.

Observe $T(B) = B + B^T = B + B = 2B$. Thus, since

T is linear, $T(\frac{1}{2}B) = \frac{1}{2}T(B) = \frac{1}{2}(2B) = B$. So

Part c. Describe the kernel of T.

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $T(A) = \begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} 2a & b+c \\ c+b & 2d \end{bmatrix}$.

The kernel of T is all matrices $A \in M_{2\times2}$ such that $T(A) = \vec{0}$.

Therefore ~~when~~ if $A \in \ker(T)$, we require that

$a = d = 0$

and $b + c = 0 \Rightarrow c = -b$, so A is of the form $\begin{bmatrix} 0 & b \\ -b & 0 \end{bmatrix}$ for

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Part d. Show that the range of T is the set of B in $M_{2\times 2}$ with the property that $B = B^T$.

We want to show Range(T) = { all 2x2 matrices B such that $B = B^T$ }.

Call this set W. To show the sets are equal we must show

Range(T) $\subseteq$ W and Range(T) $\supseteq$ W because 2 sets are

subsets of each other iff they are equal.

1) Let $\vec{x} \in$ Range(T), then $\vec{x} = T(A)$ for some $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

Therefore $\vec{x} = A + A^T = \begin{bmatrix} 2a & b+c \\ c+b & 2d \end{bmatrix} \in W$. Thus

Range(T) $\subseteq$ W.

2) Let $\vec{B} \in W$. We want to show $B \in$ Range(T) which is

equivalent to showing there exists an $A \in M_{2\times 2}$ such that

$T(A) = B$. We know $B = B^T$, so from part b, we have

$T(\frac{1}{2}B) = B$ so $B \in$ Range(T). Thus $W \subseteq$ Range T.

Thus, Range(T) = W.