**Problem Set 6,** **Math 54-Lec 3, Linear Algebra, Fall 2017**

SEPTEMBER 15TH, 2017

Solutions

**Problem 1.** For vector spaces $V, W$, let $T : V \to W$ be a linear transformation. Additionally let $U$ be a subspace of $V$. Recall $T(U)$ is the set of all $\vec{w} \in W$ such that $\vec{w} = T(\vec{u})$ for some $\vec{u} \in U$. Prove that $T(U)$ is a subspace of $W$.

1.) Since $U$ is a subspace, $\vec{0}_v \in U$. Since $T$ is linear, $T(\vec{0}_v) = \vec{0}_w$ so $\vec{0}_w \in T(U)$ ✓

2.) Let $\vec{w}_1, \vec{w}_2 \in T(U)$. Then $\vec{w}_1 = T(\vec{u}_1)$, $\vec{w}_2 = T(\vec{u}_2)$ for some $\vec{u}_1, \vec{u}_2 \in U$.

Then $\vec{w}_1 + \vec{w}_2 = T(\vec{u}_1) + T(\vec{u}_2) = T(\vec{u}_1 + \vec{u}_2)$. But $(\vec{u}_1 + \vec{u}_2) \in U$ since $U$ is

since $T$ is linear

a subspace so $\vec{w}_1 + \vec{w}_2 \in T(U)$ ✓

3) Let $c$ be a constant, then $c\vec{w}_1 = cT(\vec{u}_1) = T(c\vec{u}_1)$ and $c\vec{u}_1 \in U$ since $U$ is a subspace so $c\vec{w}_1 \in T(U)$ ✓ Hence, $T(U)$ is a subspace of $W$.

**Problem 2.** Let $V$ be a vector space and let $\vec{v_1}, \ldots, \vec{v_n}$ be vectors in $V$. Prove from the definition that $Span\{\vec{v_1}, \ldots, \vec{v_n}\}$ is a subspace of $V$.

1) Observe $0\vec{v}_1 + 0\vec{v}_2 + \ldots + 0\vec{v}_n = \vec{0}$ so $\vec{0} \in Span\{\vec{v}_1, \ldots, \vec{v}_n\}$ ✓

2.) Let $\vec{x}, \vec{y} \in Span\{\vec{v}_1, \ldots, \vec{v}_n\}$, ~~the~~ then $\vec{x} = a_1\vec{v}_1 + \ldots + a_n\vec{v}_n$ and $\vec{y} = b_1\vec{v}_1 + \ldots + b_n\vec{v}_n$ for $a$'s and $b$'s constants. Therefore,

$$\vec{x} + \vec{y} = a_1\vec{v}_1 + \ldots + a_n\vec{v}_n + b_1\vec{v}_1 + \ldots + b_n\vec{v}_n$$

$$= (a_1 + b_1)\vec{v}_1 + \ldots + (a_n + b_n)\vec{v}_n$$, so $\vec{x} + \vec{y} \in Span\{\vec{v}_1, \ldots, \vec{v}_n\}$. ✓

3.) Let $c$ be a constant, observe

$c\vec{x} = ca_1\vec{v}_1 + ca_2\vec{v}_2 + \ldots + ca_n\vec{v}_n$, so $c\vec{x} \in Span\{\vec{v}_1, \ldots, \vec{v}_n\}$ ✓

Hence $Span\{\vec{v}_1, \ldots, \vec{v}_n\}$ is a subspace for any vectors $\vec{v}_1, \ldots, \vec{v}_n$