Solutions

Problem Set 3, Math 54-Lec 3, Linear Algebra, Fall 2017

SEPTEMBER 8TH, 2017

Problem 1. Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Prove that T is one-to-one iff $T(\vec{x}) = \vec{0}$ implies $\vec{x} = \vec{0}$, that is to say the **only** vector in $\mathbb{R}^n$ who's image under T is the zero vector is the zero vector itself.

First Direction ($\Rightarrow$). If T is linear $T(\vec{0}) = \vec{0}$. $\therefore$ If T is one-to-one and $T(\vec{x}) = \vec{0}$, then $\vec{x} = \vec{0}$.

Second Direction ($\Leftarrow$). Suppose T is linear and $T(\vec{x}) = \vec{0}$ only when $\vec{x} = \vec{0}$. Need to prove T is injective. To do so, need to show $T(\vec{u}) = T(\vec{v}) \Rightarrow \vec{u} = \vec{v}$.

So let $\vec{u}, \vec{v} \in \mathbb{R}^n$ s/t $T(\vec{u}) = T(\vec{v}) \Rightarrow T(\vec{u}) - T(\vec{v}) = \vec{0} \Rightarrow T(\vec{u} - \vec{v}) = \vec{0}$. But $T(\vec{x}) = \vec{0} \Rightarrow \vec{x} = \vec{0}$ so this means $\vec{u} - \vec{v} = \vec{0} \Rightarrow \vec{u} = \vec{v}$. $\therefore$ T is one-to-one.

Problem 2. Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Let $\{\vec{v_1}, \vec{v_2}, \ldots, \vec{v_k}\}$ be a linearly independent set in $\mathbb{R}^n$. Prove that if T one-to-one then $\{T(\vec{v_1}), T(\vec{v_2}), \ldots, T(\vec{v_k})\}$ is also linearly independent. [*Hint:* use the result from Problem 1, above.]

Proof by contradiction

Suppose $\{T(\vec{v}_1), \ldots, T(\vec{v}_n)\}$ is not linearly independent.

Then by definition there exist constants: $c_1, c_2, \ldots, c_n$ such that: $c_1 T(\vec{v}_1) + \ldots + c_n T(\vec{v}_n) = \vec{0}$, with not all c_i's $= 0$.

Observe: $c_1 T(\vec{v}_1) + \ldots + c_n T(\vec{v}_n) = T(c_1\vec{v}_1 + \ldots + c_n\vec{v}_n) = \vec{0}$.

Since T is one-to-one, $T(\vec{x}) = \vec{0} \Rightarrow \vec{x} = \vec{0}$ from Problem 1.

$\therefore$ $c_1\vec{v}_1 + \ldots + c_n\vec{v}_n = \vec{0}$, a contradiction since $\{\vec{v}_1, \ldots, \vec{v}_n\}$ is linearly independent and c_i's are not all zero. So it must be that $\{T(\vec{v}_1), \ldots, T(\vec{v}_n)\}$ is linearly independent.