$\boxed{\text{Solutions}}$

Quiz 5 **Math 54-Lec 3, Linear Algebra, Fall 2017**

SECTION: NAME:

You have 30 minutes to complete this quiz. To receive full credit, you must justify your answers.

Problem 1.(12 points)

(a.)(8 points) Find the eigenvalues of the matrix A below and find a basis for each eigen-space. Hence, is A diagonalizable?

$$A = \begin{bmatrix} 3 & 0 & 0 \\ -3 & 4 & 9 \\ 0 & 0 & 3 \end{bmatrix}$$

Take char. eqn: $\det\begin{bmatrix} 3-\lambda & 0 & 0 \\ -3 & 4-\lambda & 9 \\ 0 & 0 & 3-\lambda \end{bmatrix} = (3-\lambda)\det\begin{bmatrix} 4-\lambda & 9 \\ 0 & 3-\lambda \end{bmatrix}$

$$= (3-\lambda)\big[(3-\lambda)(4-\lambda)\big)$$

So e-values are 3 and 4. first find basis for $\lambda = 3$, which is $\mathrm{Null}(A-3I)$:

$$\left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ -3 & 1 & 9 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{array}{l} c_3, c_2 \text{ free and} \\ c_1 = \frac{1}{3}c_2 + 3c_3 \end{array} \quad \text{so gen. sol. } \begin{bmatrix} \frac{1}{3}c_2 + 3c_3 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix} c_2 + \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} c_3$$

Likewise for $\lambda = 4$, find $\mathrm{Null}(A-4I)$:

$$\left[\begin{array}{ccc|c} -1 & 0 & 0 & 0 \\ -3 & 0 & 9 & 0 \\ 0 & 0 & -1 & 0 \end{array}\right] \Rightarrow \begin{array}{l} c_2 \text{ free} \\ c_3 = 0 \\ c_1 = 0 \end{array} \quad \text{so gen. sol. } \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} c_2, \text{ so}$$

$\left\{\begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}\right\}$ is a basis for $\lambda = 3$ and $\left\{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right\}$ is a basis for $\lambda = 4$.

(b.)(4 points) Compute $A^{10}\vec{x}$ for $\vec{x} = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$.

observe $\vec{x} = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, which are e-vectors with e-values $\lambda = 3$ and $\lambda = 4$ respectively.

$$\therefore A^{10}\vec{x} = A^{10}\left(\begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) = A^{10}\begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} + A^{10}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = 3^{10}\begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} + 4^{10}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

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Problem 2. (1 point each) Label the following statements true or false. If the statement is true, explain why. If it is false, explain why or provide a counterexample. Correct answers without justification will receive no credit.

(a.) If A is an $n \times n$ diagonalizable matrix, then A is invertible.

(b.) If λ is eigenvalue of an $n \times n$ matrix A, then the linear transformation defined by the matrix $(A - \lambda I)$ is not injective.

(c.) If λ_0 is a eigenvalue of a matrix A, then the multiplicity of λ_0 as the root of the characteristic polynomial of A is the equal to the dimension of the eigenspace corresponding to λ_0.

a. False, consider the zero-matrix $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ which has e-value 0 and e-basis $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$. So it is diagonalizable but clearly not invertible.

b. True, if λ is an e-value of A then there exists a non-zero vector $\vec{v}$ such that $A\vec{v} = \lambda \vec{v}$

$\Rightarrow (A - \lambda I)\vec{v} = \vec{0}$, so $\text{Null}(A - \lambda I) \neq \{\vec{0}\}$ so $(A - \lambda I)$ is not injective.

c. False, consider $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, which has char. eqn $(1-\lambda)^2$.

However if we try to find e-basis for the sole e-value 1 we see $\text{Null}(A - I) = \text{Nul}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \text{Span}\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$.

So, multiplicity of 1 as a root $>$ ~~multiplicity~~ dimension of e-space corresponding to 1