Quiz 2 Math 54-Lec 3, Linear Algebra, Fall 2017

SECTION: NAME: Solutions

You have 30 minutes to complete this quiz. To receive full credit, you must justify your answers.

Problem 1.(5 points.) Compute the matrix product AB for A, B below.

$$A = \begin{bmatrix} 4 & -3 \\ -3 & 5 \\ 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & 4 \\ 3 & -2 \end{bmatrix},$$

$$AB = \begin{bmatrix} 4 & -3 \\ -3 & 5 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 4 \\ 3 & -2 \end{bmatrix} = \begin{bmatrix} 4-9 , & 16+6 \\ -3+15 , & -12-10 \\ 0+3 , & 0-2 \end{bmatrix} = \begin{bmatrix} -5 & 22 \\ 12 & -22 \\ 3 & -2 \end{bmatrix}$$

Problem 2.(5 points.) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation such that

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix},\right) = \begin{bmatrix} x_1 \\ x_2 \\ -x_3 \end{bmatrix}.$$

Verify that T is linear.

We require, $T(c\vec{x}) = cT(\vec{x})$ and $T(\vec{x}+\vec{y}) = T(\vec{x}) + T(\vec{y})$ for all $\vec{x}, \vec{y} \in \mathbb{R}^3$, and all constants c. Observe for $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ and $\vec{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$, we have

$$T(c\vec{x}) = T\left(c\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = T\left(\begin{bmatrix} cx_1 \\ cx_2 \\ cx_3 \end{bmatrix}\right) = \begin{bmatrix} cx_1 \\ cx_2 \\ -cx_3 \end{bmatrix} = c\begin{bmatrix} x_1 \\ x_2 \\ -x_3 \end{bmatrix} = cT(\vec{x})$$

$$T(\vec{x}+\vec{y}) = T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}\right) = T\left(\begin{bmatrix} x_1+y_1 \\ x_2+y_2 \\ x_3+y_3 \end{bmatrix}\right) = \begin{bmatrix} x_1+y_1 \\ x_2+y_2 \\ -x_3-y_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ -x_3 \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \\ -y_3 \end{bmatrix} = T(\vec{x}) + T(\vec{y})$$

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Problem 3.(5 points.) Let $S : \mathbb{R}^p \to \mathbb{R}^n$ and $T : \mathbb{R}^n \to \mathbb{R}^m$ be one-to-one, linear transformations. Prove that the composition of T and S, $T(S(\vec{x}))$, sometimes written $(T \circ S)$, is also one-to-one.

Recall a lin. transformation $T: \mathbb{R}^n \to \mathbb{R}^m$ is one-to-one iff $T(\vec{x}) = T(\vec{y})$ implies $\vec{x} = \vec{y}$ for all $\vec{x}, \vec{y} \in \mathbb{R}^n$.

Thus, let $\vec{x}, \vec{y} \in \mathbb{R}^p$, ~~and consider~~ such that $(T \circ S)(\vec{x}) = (T \circ S)(\vec{y})$. Note

$(T \circ S)(\vec{x}) = T(S(\vec{x})) = T(S(\vec{y})) = (T \circ S)(\vec{y})$. However;

$S(\vec{x}) \in \mathbb{R}^n$, $S(\vec{y}) \in \mathbb{R}^n$ and T is one-to-one so it must be that $S(\vec{x}) = S(\vec{y})$. But S is also one-to-one so it must be that $\vec{x} = \vec{y}$. Hence $(T \circ S)(\vec{x}) = (T \circ S)(\vec{y})$ implies $\vec{x} = \vec{y}$ so $(T \circ S)$ is one-to-one by definition.