Problem Set 1, **Math 54-Lec 3, Linear Algebra, Fall 2017** Solutions

SEPTEMBER 1ST, 2017

Problem 1. Let $A \in M_{3\times 3}$, that is a 3×3 matrix, such that $A\vec{x} = \vec{b}$ is consistent for all $\vec{b} \in \mathbb{R}^3$. Show that $A\vec{x} = \vec{0}$ has only the trivial solution.

Let $\vec{v}_1, \vec{v}_2, \vec{v}_3$ be the columns of A. Since $A\vec{x} = \vec{b}$ is consistent for all $\vec{b} \in \mathbb{R}^3$, we have $\text{Span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\} = \mathbb{R}^3$. It follows that $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is linearly independent. Thus $c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \vec{0}$ if and only if $c_1 = c_2 = c_3 = 0$. Therefore, $A\vec{x} = \vec{0}$ has only the trivial solution.

Problem 2. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a function with

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix},\right) = \begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix}.$$

Determine if T a linear transformation. Let $\vec{x}, \vec{y} \in \mathbb{R}^3$ with $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ and $y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$. Also let c be a constant. Then,

$$T(\vec{x}+\vec{y}) = T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}\right) = T\left(\begin{bmatrix} x_1+y_1 \\ x_2+y_2 \\ x_3+y_3 \end{bmatrix}\right) = \begin{bmatrix} x_1+y_1 \\ x_2+y_2 \\ 0 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \\ 0 \end{bmatrix} = T(\vec{x}) + T(\vec{y})$$

$$T(c\vec{x}) = T\left(c\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = T\left(\begin{bmatrix} cx_1 \\ cx_2 \\ cx_3 \end{bmatrix}\right) = \begin{bmatrix} cx_1 \\ cx_2 \\ 0 \end{bmatrix} = c\begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix} = cT(\vec{x}).$$

Therefore, T is linear by def.

Problem 3. Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation, and let $\{\vec{v_1}, \vec{v_2}, \vec{v_3}\}$ be linearly dependent in $\mathbb{R}^n$. Determine whether $\{T(\vec{v_1}), T(\vec{v_2}), T(\vec{v_3})\}$ is linearly dependent or independent in $\mathbb{R}^m$.

Since $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ are linearly dependent there exist constants c_1, c_2, c_3 not all zero such that $c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \vec{0}$. Now using the fact that T is linear observe:

$$\vec{0} = T(\vec{0}) = T(c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3) = c_1T(\vec{v}_1) + c_2T(\vec{v}_2) + c_3T(\vec{v}_3).$$

Hence $T(\vec{v}_1), T(\vec{v}_2), T(\vec{v}_3)$ are linearly dependent since c_1, c_2, c_3 are not all zero.