Problem Set 7, Math 54-Lec 3, Linear Algebra, Fall 2017

Solutions

SEPTEMBER 18TH, 2017

Problem 1. The set of solutions to the system of linear equations:

$$x_1 - 2x_2 + x_3 = 0$$
$$2x_1 - 3x_2 + x_3 = 0$$

form a subspace of $\mathbb{R}^3$. Find a basis for this subspace.

First find the solutions via row-reduction:

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 2 & -3 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -1 & 0 \end{bmatrix} \quad \therefore \; x_3 \text{ free}$$

$$x_2 = x_3$$

$$x_1 = 2x_2 - x_3 = x_3.$$

So the solution set is of the form $\begin{bmatrix} x_3 \\ x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ for x_3 a constant. Therefore Span$\left\{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right\}$ = the solution set and $\left\{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right\}$ is linearly independent so $\left\{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right\}$ is a basis.

Problem 2. Let V be a vector space and let $\{\vec{u}, \vec{v}\}$ be a basis for V. Prove that $\{(\vec{u}+\vec{v}), c\vec{u}\}$ is also a basis for V when c is a non-zero constant.

It is easy to see that $\{(\vec{u}+\vec{v}), c\vec{u}\}$ is a linearly independent set since the vectors are not scalar multiples of each other. Have to show their span is all of V. $\therefore$ Let $\vec{x} \in V$, so

$\vec{x} = c_1\vec{u} + c_2\vec{v}$, since $\{\vec{u}, \vec{v}\}$ is a basis for V. Now, let

$c_1 = c_2 + c_3$, for ~~some constant~~ c_3 an appropriate constant.

Then $\vec{x} = c_2\vec{u} + c_3\vec{u} + c_2\vec{v} = c_2(\vec{u}+\vec{v}) + c_3\vec{u}$

$$= c_2(\vec{u}+\vec{v}) + \frac{c_3}{c}(c\vec{u})$$

$\in \{(\vec{u}+\vec{v}), c\vec{u}\}$ so the set is a basis.

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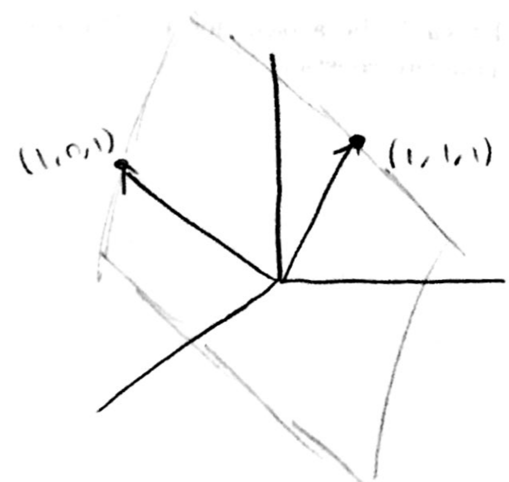
Problem 3. Determine if the following vectors form a basis for a vector space. If so, what is the vector space?

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Observe the vectors are not scalar multiples of each other so they are linearly independent. Hence $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$ is a basis for Span$\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$, which we know is a vector space (see Problem Set 6 #2). This vector space is a plane through the origin in $\mathbb{R}^3$, so it is a subspace of $\mathbb{R}^3$



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