Solutions

Quiz 8 **Math 54-Lec 3, Linear Algebra, Fall 2017**

SECTION: NAME:

You have 30 minutes to complete this quiz. To receive full credit, justify your answers.

Problem 1.(5 points) Let $A = \begin{bmatrix} 2 & 3 \\ 0 & 2 \end{bmatrix}$. Find the singular values of A. Hence, what is the maximum value of the quadratic form $Q(\vec{x}) = \vec{x}^T A^T A \vec{x}$ subject to the constraint that $\vec{x}$ is a unit vector?

$$A^T A = \begin{bmatrix} 2 & 0 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 4 & 6 \\ 6 & 13 \end{bmatrix}$$

$$\det(A^TA - \lambda I) = \det\begin{pmatrix} 4-\lambda & 6 \\ 6 & 13-\lambda \end{pmatrix} = \lambda^2 - 17\lambda + 16 = (\lambda-16)(\lambda-1)$$

$\therefore$ A^TA has e-values $16, 1$ $\Rightarrow$ singular values of A are $\boxed{4, 1}$.

Recall max of $\|A\vec{x}\| = \sigma_1$, the largest singular value of A.

$\therefore$ max of $\|A\vec{x}\| = 4$

$\therefore$ max of $\|A\vec{x}\|^2 = (A\vec{x})^T(A\vec{x}) = \vec{x}^T A^T A \vec{x} = 4^2 = \boxed{16}$.

Note: all max are under constraint $\|\vec{x}\| = 1$.

Problem 2.(6 points) Consider the quadratic form: $Q(\vec{x}) = 2x_1^2 + 6x_1x_2 - 6x_2^2$. Find the matrix form of Q. That is, write $Q(\vec{x}) = \vec{x}^T A\vec{x}$ for some symmetric matrix A. Is Q positive definite, negative definite, or indefinite?

$Q(\vec{x}) = \vec{x}^T A \vec{x}$ for $A = \begin{bmatrix} 2 & 3 \\ 3 & -6 \end{bmatrix}$

Observe, $Q\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = 2$ & $Q\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = -6$,

so Q is indefinite b/c it attains positive and negative values.

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Problem 3.(2 points each) Label the following statements true or false. If the statement is true, explain why. If it is false, explain why or provide a counterexample. Correct answers without justification will receive no credit.

(a.) Let A be an $n \times n$ orthogonally diagonalizable matrix. If A is invertible, then $A^{-1} = A^T$.

(b.) The expression $||x||^2$ is quadratic form.

(a) a matrix, A, is orthogonally diagonalizable iff it is symmetric. $\therefore$ It suffices to find an invertible, symmetric matrix s/t $A^{-1} \neq A^T$. Consider:

$A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$, it is clearly symmetric. It is also invertible b/c its columns are linearly independent.

However $A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} 1 & -2 \\ -2 & 1 \end{bmatrix} = \frac{-1}{3} \begin{bmatrix} 1 & -2 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} -1/3 & 2/3 \\ 2/3 & -1/3 \end{bmatrix}$

so $A^{-1} \neq A^T$. Hence, the statement is $\boxed{\text{false.}}$

(b) True, $||\vec{x}||^2 = \vec{x}^T \vec{x} = \vec{x}^T I \vec{x}$ where I is the $n \times n$ identity matrix.