Problem Set 2, **Math 54-Lec 3, Linear Algebra, Fall 2017** Solutions

September 6th, 2017

Problem 1. Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation, such that $T(\vec{x}) = B\vec{x}$ for some $m \times n$ matrix B. Show that if A is the standard matrix for T, then $A = B$. [*Hint:* Show that A and B have the same columns.]

Let $\vec{a}_i$ be the vector of the i^{th} column of A for $1 \leq i \leq n$. By definition, $\vec{a}_i = T(\vec{e}_i)$ for $\vec{e}_i$ the i^{th} elementary vector (the i^{th} column of the identity matrix I_n).

But $T(\vec{e}_i) = B\vec{e}_i$ = the i^{th} column of B, $\vec{b}_i$. Hence $\vec{a}_i = \vec{b}_i$ for all $1 \leq i \leq n$. Therefore $A = B$.

Problem 2. Let $S : \mathbb{R}^p \to \mathbb{R}^n$ and $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformations. Show that the mapping $(T \circ S)(\vec{x}) = T(S(\vec{x}))$ is a linear transformation.

Let $\vec{x}, \vec{y} \in \mathbb{R}^p$ and let c be a constant.

$T(S(c\vec{x})) = T(c\,S(\vec{x})) = c\,T(S(\vec{x}))$, since S and T linear.

$T(S(\vec{x}+\vec{y})) = T(S(\vec{x})+S(\vec{y})) = T(S(\vec{x})) + T(S(\vec{y}))$, since S & T linear.

Therefore, $(T \circ S)$ linear.

Problem 3. Show that if the columns of B are linearly dependent, then so are the columns of AB.

Let $\vec{b}_1, \vec{b}_2, \ldots, \vec{b}_n$ be the columns of B. Since they are dependent there exist constants $c_1, c_2, \ldots, c_n$ not all zero such that $c_1\vec{b}_1 + \ldots + c_n\vec{b}_n = \vec{0}$. That is, for

$$\vec{c} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}, \quad B\vec{c} = \vec{0} \Rightarrow AB\vec{c} = A\vec{0} = \vec{0}.$$

Hence $c_1 A\vec{b}_1 + c_2 A\vec{b}_2 + \ldots + c_n A\vec{b}_n = \vec{0}$ with $A\vec{b}_i$ being the i^{th} column of AB. Thus, since $c_1, \ldots, c_n$ not all zero, the columns of AB are dependent.