Solutions

Quiz 10 **Math 54-Lec 3, Differential Equations, Fall 2017**

SECTION: NAME:

You have 40 minutes to complete this quiz. To receive full credit, justify your answers.

Problem 1.(4 points) Find the general solution to the inhomogeneous equation $y''(t) - 2y'(t) + y(t) = e^t/t$ by using variation of parameters.

① Find gen. sol. to homogen.

Aux eqn: $r^2 - 2r + 1 = 0$

$\Leftrightarrow (r-1)^2 = 0$

$\Leftrightarrow r = 1$

$\therefore$ gen. soln. to homogeneous eqn is $y_h(t) = c_1 e^t + c_2 t e^t$

② Find particular sol., $y_p(t)$, via variation of parameters.

$y_p(t) = v_1(t) e^t + v_2(t) t e^t$

where $v_1(t) = \int \frac{(-e^t/t) t e^t}{(e^t)(e^t + te^t) - (e^t)(te^t)} dt$

$= \int \frac{-e^{2t}}{e^{2t}} dt = \int -1 = -t$

$v_2(t) = \int \frac{(e^t/t) e^t}{e^t(e^t + te^t) - (e^t)(te^t)} dt$

$= \int \frac{1}{t} dt = \ln(t)$

$\therefore \boxed{y(t) = c_1 e^t + c_2 t e^t - t e^t + t e^t \ln(t)}$

Problem 2.(11 points total) Sometimes we want to find solutions to an inhomogeneous equation of the form: $ay'' + by' + cy = g(t)$ where $g(t)$ is only piecewise continuous (it is discontinuous at finitely many points). In this case we can still come up with a relatively good answer even if no solution exists. As an example consider the differential equation:

$$y'' + 2y' + 5y = g(t); y(0) = 0, \cancel{y(1) = 0} \quad y'(0) = 0$$

where

$$g(t) = \begin{cases} 10 & 0 \le t \le 3\pi/2 \\ 0 & t > 3\pi/2 \end{cases}$$

(a.) (4 points) Find a solution to the initial value problem when $0 \le t \le 3\pi/2$.

(b.) (4 points)Find ~~a solution to the initial value problem~~ a general solution when $t > 3\pi/2$.

(c.) (3 points)Hence, find a function $f(t)$ that is everywhere continuous and differentiable and also satisfies the initial value problem except at $t = 3\pi/2$. [*Hint*: find appropriate constants for the solutions obtained in part (a) and (b).]

Problem 2

a.) Want a solution to $y'' + 2y' + 5y = 10$, w/ $y(0) = 0$, $y'(0) = 0$

Aux eqn: $r^2 + 2r + 5 \Rightarrow r = \frac{-2 \pm \sqrt{4-20}}{2} = \frac{-2 \pm 4i}{2} = -1 \pm 2i$.

So gen. soln to the homogeneous eqn. is

$y_1(t) = e^{-t}(c_1\cos(2t) + c_2\sin(2t))$. By observation, $y = 2$ is a particular solution to the non homogeneous eqn so the general solution is $y(t) = c_1 e^{-t}\cos(2t) + c_2 e^{-t}\sin(2t) + 2$

$$\Rightarrow y'(t) = -c_1 e^{-t}\cos(2t) - 2c_1 e^{-t}\sin(2t) - c_2 e^{-t}\sin(2t) + 2c_2 e^{-t}\cos(2t)$$

Now solve w/ initial values: $y(0) = 0$, $y'(0) = 0$

$y(0) = c_1(1) + c_2(0) + 2 \Rightarrow c_1 = -2$

$y'(0) = -c_1(1) - 2c_1(0) - c_2(0) + 2c_2(1) \Rightarrow 2c_2 = c_1 \Rightarrow c_2 = -1$

$\therefore \boxed{y(t) = -e^{-t}(2\cos(2t) + \sin(2t)) + 2}$

b.) This is just the general sol. to the homogeneous eqn. found in part a, so $\boxed{y(t) = e^{-t}(c_1\cos(2t) + c_2\sin(2t))}$

c.) Let y_a and y_b be the answers to parts a and b. If we define

$f(t) = \begin{cases} y_a(t), & 0 \le t \le 3\pi/2 \\ y_b(t), & t > \frac{3\pi}{2} \end{cases}$, it will satisfy diff. & initial values except at $t = \frac{3\pi}{2}$.

Problem 2 (continued)

To make f continuous and differentiable, we only need to require that $y_a\left(\frac{3\pi}{2}\right) = y_b\left(\frac{3\pi}{2}\right)$ and
$$y_a'\left(\frac{3\pi}{2}\right) = y_b'\left(\frac{3\pi}{2}\right)$$

since y_a and y_b are already continuous & differentiable elsewhere. To do this, just need to find appropriate values for c_1, c_2 in y_b

$$y_a\left(\frac{3\pi}{2}\right) = -e^{-3\pi/2}\left(2\cos(3\pi) + \sin(3\pi)\right) + 2$$
$$= \left(-e^{-3\pi/2}\right)(2)(-1) + 2 = 2e^{-3\pi/2} + 2$$

$$y_b\left(\frac{3\pi}{2}\right) = e^{-3\pi/2}\left(c_1\cos(3\pi) + c_2\sin(3\pi)\right)$$
$$= \left(-e^{-3\pi/2}\right)c_1$$

$$\therefore\ c_1 = \frac{2e^{-3\pi/2} + 2}{-e^{-3\pi/2}} = -2 - 2e^{3\pi/2} = \boxed{-2\left(e^{3\pi/2} + 1\right)}$$

$$y_a'\left(\frac{3\pi}{2}\right) = 2e^{-3\pi/2}\cos(3\pi) - 4e^{-3\pi/2}\sin(3\pi)$$
$$+ e^{-3\pi/2}\sin(3\pi) - 2e^{-3\pi/2}\cos(3\pi) = -2e^{-3\pi/2} + 2e^{-3\pi/2} = 0$$

$$y_b'\left(\frac{3\pi}{2}\right) = -c_1e^{-3\pi/2}\cos(3\pi) - 2c_1e^{-3\pi/2}\sin(3\pi)$$
$$- c_2e^{-3\pi/2}\sin(3\pi) + 2c_2e^{-3\pi/2}\cos(3\pi)$$

$$= e^{-3\pi/2}(c_1 - 2c_2)$$

$$\therefore\ c_1 - 2c_2 = 0 \Rightarrow c_2 = \tfrac{1}{2}c_1 \Rightarrow c_2 = \boxed{-\left(e^{3\pi/2} + 1\right)}$$

Hence $f(t) = \begin{cases} y_a(t), & 0 \le t \le 3\pi/2 \\ y_b(t), & t > \frac{3\pi}{2} \end{cases}$, w/ c_1, c_2 as above.