Problem Set 9, **Math 54-Lec 3, Linear Algebra, Fall 2017**

Solutions

SEPTEMBER 22TH, 2017

This problem was taken from Professor Nadler's Fall 2015 Math 54 midterm.

Problem 1

(a) State the rank theorem for a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$.

$$\dim(\text{Range}(T)) + \dim(\text{Ker}(T)) = n$$

(b) Compute the rank of

$$A = \begin{bmatrix} 2 & 0 & 1 & 1 \\ 3 & -1 & 1 & 2 \\ -1 & -1 & -1 & 1 \end{bmatrix}$$

Row reduction yields:

$$\begin{bmatrix} 2 & 0 & 1 & 1 \\ 3 & -1 & 1 & 2 \\ -1 & -1 & -1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 & -1 \\ 2 & 0 & 1 & 1 \\ 3 & -1 & 1 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 & -1 \\ 0 & -2 & -1 & 3 \\ 0 & -4 & -2 & 5 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 & -1 \\ 0 & -2 & -1 & 3 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

3 pivot columns so $\text{rank}(A) = 3$.

(c) Is the linear transformation defined by A injective? Justify your answer. [*Hint:* It may be helpful to use the previous parts]

Let $T : V \to W$ be the linear transformation defined by A.

Then $\dim(V) = 4$, so by rank thm: $\text{rank}(A) + \dim(\text{ker}(T)) = 4$

$\Rightarrow \dim(\text{ker}(T)) = 1 \neq 0$ so T is not injective.

(d) Use the rank theorem to show that any linear map from $\mathbb{R}^n$ to $\mathbb{R}^m$ cannot be one-to-one (injective) whenever $n > m$.

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ with $n > m$. Then $\dim(\text{range}(T)) \leq m$

so $n - \dim(\text{range}(T)) \geq 1$. But by rank thm.,

$\dim(\text{ker}(T)) = n - \dim(\text{range}(T)) \geq 1$. Thus T is not

injective since $\dim(\text{Ker}(T)) \neq 0$.