$\boxed{\text{Solutions}}$

Problem Set 4, Math 54-Lec 3, Linear Algebra, Fall 2017

SEPTEMBER 11TH, 2017

Problem 1. A matrix $A \in M_{n \times n}$, is called upper-triangular if every entry below the diagonal is 0. Entries on and above the diagonal can be any real number. Let A be an upper-triangular $n \times n$ matrix with diagonal entries $c_1, c_2, \ldots, c_n$. Compute the determinate of A. Justify your answer.

We have $A = \begin{bmatrix} c_1 & & & * \\ & c_2 & & \\ & & c_3 & \\ 0 & & & \ddots & \\ & & & & c_n \end{bmatrix}$ for $*$ any real number.

By expanding over first col, the second, etc. we see

$$\det A = c_1 \det \begin{bmatrix} c_2 & & * \\ & \ddots & \\ 0 & & c_n \end{bmatrix} = c_1 c_2 \det \begin{bmatrix} c_3 & & * \\ & \ddots & \\ 0 & & c_n \end{bmatrix} = \cdots = c_1 c_2 \cdots c_n.$$

Problem 2. Let $T : \mathbb{R}^4 \to \mathbb{R}$ be a function such that:

$$T\left(\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}\right) = \det \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Determine if T is a linear transformation.

Let $\vec{x} = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \in \mathbb{R}^4$, then for k a constant, we have:

$$kT(\vec{x}) = k \det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = k(ad - bc), \text{ while}$$

$$T(k\vec{x}) = \det \begin{bmatrix} ka & kb \\ kc & kd \end{bmatrix} = k^2 ad - k^2 bc = k^2(ad - bc)$$

So $T(k\vec{x}) \neq kT(\vec{x})$ for $k = 2$ for example.

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Problem 3. A matrix $Q \in M_{n \times n}$ is called orthogonal if $QQ^T = I_n$, where I_n is the $n \times n$ identity matrix. If Q is orthogonal, what are the possible values of $\det(Q)$? [*Hint:* take the determinate of both sides of the equality]

$$QQ^T = I_n$$

$$\Rightarrow \det(QQ^T) = \det(I_n)$$

$$\Rightarrow \det(Q)\det(Q^T) = 1$$

$$\Rightarrow \det(Q)^2 = 1$$

$$\Rightarrow \det(Q) = \pm 1$$

Problem 4. Let $A \in M_{n \times n}$. Show that $\det(kA) = k^n \det(A)$.

kA is the equivalent of rescaling the rows of A by k. We know rescaling any 1 row by a factor of k changes determinant by factor of k. Applying this rule n times (for each row ~~column~~ of A) yields

$$\det(kA) = k^n \det(A).$$