Math 54 Study Guide

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I. Linear Algebra

A. Basic Definitions and Properties

Definition. *Give a definition for each of the following: Linear Transformation, Linearly Independence/ Dependence, Injective, Surjective, Bijective, Invertible.*

Exercises

1. Label each of the following true or false. If true, prove it. If false, prove why or provide a counter-example.

   - Let A, B be $n \times n$ matrices. If the columns of B are linearly dependent then so are the columns of AB.
   - Let $S : \mathbb{R}^p \to \mathbb{R}^n$ and $T : \mathbb{R}^n \to \mathbb{R}^m$ be one-to-one, linear transformations. Then the composition of T and S, $T(S(\vec{x}))$, sometimes written $(T \circ S)$, is also one-to-one.

2. Compute the inverse of the following matrix:

$$\begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 4 & -3 & 8 \end{bmatrix}.$$

3. Let A be an $m \times n$ matrix. If the linear transformation defined by A is injective, what can we say about the sizes of n and m? What if A is instead surjective? Bijective? Justify your answers using a pivot argument (Rank theorem argument in section I,C).

B. Determinants

Exercises

1. Label each of the following true or false. If true, prove it. If false, prove why or provide a counter-example.

   - If two rows of a square matrix A are identical, then $\det(A) = 0$.
   - If A is an $n \times n$ matrix then $\det(A^T) = (-1)^n \det(A)$.

2. A matrix $Q \in M_{n\times n}$ is called orthogonal if $QQ^T = I_n$, where I_n is the $n \times n$ identity matrix. If Q is orthogonal, what are the possible values of $\det(Q)$? [*Hint:* take the determinate of both sides of the equality]

3. Let $A \in M_{n\times n}$. Show that $\det(kA) = k^n \det(A)$.

4. List as many conditions as you can think of that show a square matrix is invertible (Ex: "its columns are linearly independent").

5. What is the volume of the parallel-piped determined by the vectors $\begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ in $\mathbb{R}^3$?

C. Vector Spaces and Subspaces

Definition. *Give a definition for each of the following: vector space, subspace, span, basis, null-space/kernel, image/range, rank/nullity. State the Rank-Nullity Theorem.*

Exercises

1. Label each of the following true or false. If true, prove it. If false, prove why or provide a counter-example.

    - Let $W \subseteq \mathbb{R}^n$ be the subset of $\mathbb{R}^n$ containing all vectors whose entries sum to zero. W is a subspace of R^n.
    - Let $\mathbb{P}^n$ be the vector space of polynomials with real coefficients of degree less than or equal to n. Let U be the set of all functions f in $\mathbb{P}^n$ such that the coefficient of x^n in f is non-zero. That is, $f(x) = c_0 + c_1x + c_2x^2 + \ldots + c_nx^n$ is in U if and only if $c_n \neq 0$. Then U is a subspace of $\mathbb{P}^n$.

2. Let A be an $m \times n$ matrix. If the linear transformation defined by A is injective, what can we say about the sizes of n and m? What if A is instead surjective? Bijective? Justify your answers by appealing to the Rank Theorem.

3. Let $M_{n\times n}$ be the vector space of all $n \times n$ matrices, and define $T : M_{n\times n} \to M_{n\times n}$ by $T(A) = A + A^T$.

    - *Part a.* Show that T is a linear transformation.
    - *Part b.* Let B be a symmetric $n \times n$ matrix, so that $B = B^T$. Find an $A \in M_{n\times n}$ such that $T(A) = B$.
    - *Part c.* Find the dimension of the kernel of T.
    - *Part d.* Show that the range of T is the set of symmetric matrices, that is all B in $M_{n\times n}$ with the property that $B = B^T$.

4. Let $\mathbb{P}^n$ be the vector space of polynomials of degree less than or equal to n. Let $T : \mathbb{P}^2 \to \mathbb{P}^3$ be defined by $T(f) = (x + x^2)\frac{df}{dx}$. Find the matrix of T with respect to the basis $\{1, x, x^2\}$.

II. Linear Algebra Continued

A. Eigenvectors and Eigenvalues

Definition. *Give a definition for each of the following: eigenvector, eigenvalue, eigenspace, characteristic equation, similar, diagonalizable.*

Exercises

1. Label each of the following true or false. If true, prove it. If false, prove why or provide a counter-example.

    - Let $\vec{x}, \vec{y}$ be eigenvectors of a matrix A such that $\vec{x}$ and $\vec{y}$ correspond to the distinct (diffferent) eigenvalues λ_1 and λ_2, respectively. Then $\vec{x}, \vec{y}$ are linearly independent.
    - If A is an $n \times n$ diagonalizable matrix, then A is invertible.
    - If A is an $n \times n$ invertible matrix, then A is diagonalizable.

- If λ is eigenvalue of an $n \times n$ matrix A, then the linear transformation defined by the matrix $(A - \lambda I)$ is not injective.
- If λ_0 is a eigenvalue of a matrix A, then the multiplicity of λ_0 as the root of the characteristic polynomial of A is the equal to the dimension of the eigenspace corresponding to λ_0.

2. Compute $A^{10}\vec{x}$ for $A = \begin{bmatrix} 3 & 0 & 0 \\ -3 & 4 & 9 \\ 0 & 0 & 3 \end{bmatrix}$ and $\vec{x} = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$.

B. Inner Product Spaces

Definition. *Give a definition for each of the following: inner-product space (and list the axioms), orthogonal, orthogonal projection, least squares solution.*

Exercises

1. Label each of the following true or false. If true, prove it. If false, prove why or provide a counter-example.
   - Let $\cdot$ denote the dot-product. Then for A an $n \times n$ matrix and $\vec{x}, \vec{y} \in \mathbb{R}^n$, $A\vec{x} \cdot \vec{y} = \vec{x} \cdot A\vec{y}$.
   - Let $\mathbb{P}^2$ be the vectors space of polynomials of degree less than or equal to 2. Then the formula $\langle f, g \rangle = f(1)g(1)$, is an inner-product on $\mathbb{P}^2$.
   - Suppose $\vec{y}$ is orthogonal to the vectors $\vec{u}$ and $\vec{v}$. Then $\vec{y}$ is also orthogonal to every vector in $Span\{\vec{u}, \vec{v}\}$.
   - Let W be a subspace of the inner product space R^n. Then the mapping, $T : \mathbb{R}^n \to \mathbb{R}^n$ defined by $T(\vec{y}) = proj_W(\vec{y})$ is a linear transformation.
2. Let $W = \{x \in \mathbb{R}^4 | x_1 + 2x_2 + 3x_3 + 4x_4 = 0\}$. Find a basis for $W^\perp$.
3. Let $\vec{v_1} = \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}$ and $\vec{v_2} = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$ be vectors in $\mathbb{R}^3$.
   - *Part a.* Find an **orthonormal** basis $\beta = \{\vec{b_1}, \vec{b_2}\}$ for the plane spanned by $\vec{v_1}$ and $\vec{v_2}$.
   - *Part b.* Let $\vec{y} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$. Using whichever method you prefer, find the least square solution(s) to the inconsistent system $A\vec{x} = \vec{y}$, where A is the matrix with columns $\vec{v_1}, \vec{v_2}$.
   - *Part c.* Finally, find a third vector $\vec{b_3}$ such that the matrix B with columns $\vec{b_1}, \vec{b_2}, \vec{b_3}$ is orthogonal.

C. Symmetric Matrices, Quadratic forms, and SVD

Definition. *Give a definition for each of the following: symmetric matrix, quadratic form, singular value.*

Exercises

1. Label each of the following true or false. If true, prove it. If false, prove why or provide a counter-example.

- Let A be an $n \times n$ orthogonally diagonalizable matrix. If A is invertible, then $A^{-1} = A^T$.
- If a matrix is symmetric, then it is orthogonally diagonalizable.
- The quadratic form on R^2 given by $Q(\vec{x}) = x_1^2 + 2x_1x_2 + 2018x_2^2$ is positive definite.

2. Let $A = \begin{bmatrix} 2 & 3 \\ 0 & 2 \end{bmatrix}$.
   - *Part a.* Find the singular values of A.
   - *Part b.* Hence, what is the maximum value of the quadratic form $Q(\vec{x}) = \vec{x}^T A^T A \vec{x}$ subject to the constraint that $\vec{x}$ is a unit vector?
   - *Part c.* By which vector(s) is this maximum achieved?
   - *Part d.* Find the SVD of A.

III. Differential Equations

A. Second Order ODE's

Exercises

1. Find solutions to the following homogeneous differential equations satisfying the given conditions.
   - $y'' - 10y' + 25y = 0$, with initial conditions $y(0) = 1, y'(0) = 6$.
   - $y'' + 4y' + 5y = 0$, with boundary values $y(0) = 2, y(\pi/2) = 0$.
2. Find general solutions to following inhomogeneous differential equations. You may use the method of variation of parameters or the method of undetermined coefficients as you deem appropriate.
   - $y'' - 2y' + y = e^t/t$
   - $y'' - 3y' + 2y = e^t sin(t)$
   - $y'' + 2y' + 5y = t + cos(2t)$

B. Systems of First Order ODE's

Exercises

1. Find general solutions to the following homogeneous systems of first order equations.
   - $x'\vec{(t)} = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} x\vec{(t)}$.
   - $x'\vec{(t)} = \begin{bmatrix} -1 & 2 \\ -1 & -3 \end{bmatrix} x\vec{(t)}$.
2. Consider the differential equation: $y''' + 2y'' - y' - 2y = 0$. Find a general solution by setting $x_1 = y, x_2 = y', x_3 = y''$ and solving the associated system of first order differential equations.
3. Find a general solution of $x'\vec{(t)} = Ax\vec{(t)} + t\vec{g}$ where

$$A = \begin{bmatrix} 1 & -2 & 2 \\ -2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}, \vec{g} = \begin{bmatrix} -9 \\ 0 \\ -18 \end{bmatrix}$$

C. Fourier Series

Exercises

1. State sufficient conditions on a function f defined on a symmetric interval so that the Fourier series of f converges to f at "almost" every point. At which point(s) might the Fourier series not converge to $f(x)$? What does the series converge to in these cases?

2. Let $f(x) = |x|$ for $-\pi \leq x \leq \pi$. Compute the Fourier series of f on $[-\pi, \pi]$.

3. Let $f(x) = sin(x)$ where $0 \leq x \leq \pi$. Show that

$$f(x) = \frac{2}{\pi} + \sum_{n=1}^{\infty} a_n cos(nx)$$

where $a_n = 0$ when n is odd and $a_n = \frac{-4}{\pi(n^2-1)}$ when n is even. *Hint: It will be helpful to recall that* $cos(A)sin(B) = \frac{1}{2}[sin(A+B) - sin(A-B)]$.

D. The Heat Equation

Exercises

1. Consider the heat equation problem:

$$\frac{\partial u}{\partial t} = 5\frac{\partial^2 u}{\partial x^2}$$

with boundary conditions $u(0,t) = u(\pi,t) = 0$ and initial conditions $u(x,0) = -5sin(2x) + sin(3x)$. Solve for $u(x,t)$.

2. Consider the heat equation problem:

$$\frac{\partial u}{\partial t} = 3\frac{\partial^2 u}{\partial x^2}$$

with boundary conditions $\frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(\pi,t) = 0$ and initial conditions $u(x,0) = x^2$ for $0 < x < \pi$. Solve for $u(x,t)$.