Barcode Posets

Combinatorial Properties and Connections

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Introduction: Barcodes

Definition (Barcode, [Ghr08])

A **barcode** is a finite multiset of intervals on the real line, $B = \{(b_i, d_i)\}_{i=1}^n$. Each interval is called a **bar**. The left endpoints, b_i, are often called **birth times** and the right endpoints, d_i, are called **death times**.

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Barcodes arise in persistent homology as summaries of the creation and destruction of homology classes in a filtration [ZC05, EH08, Ghr08, Car09].

Barcodes are also important in the study of interval orders and interval graphs [LB62, Fis85, Gol04].

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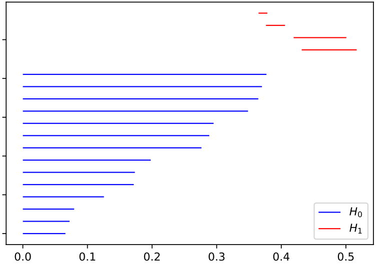

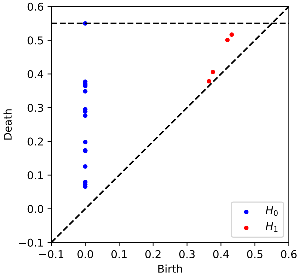

Figure: Barcode (left) and corresponding persistence diagram (right).

From trees to barcodes and back again

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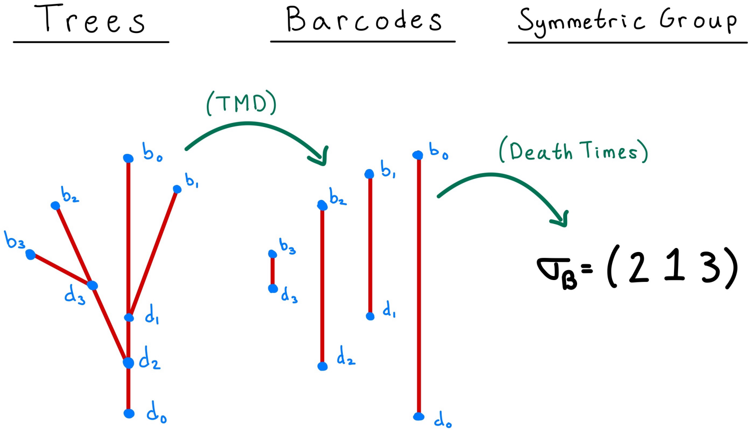

From trees to barcodes and back again

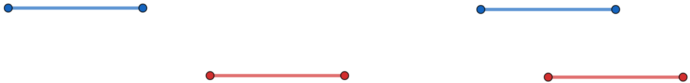

Figure: Disjoint bars (left) and stepped bars (right) produce the same permutation, (2 1).

From trees to barcodes and back again

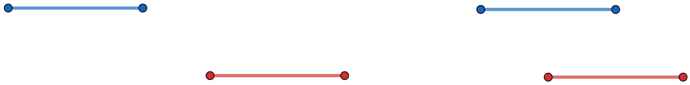

Figure: Disjoint bars (left) and stepped bars (right) produce the same permutation, (2 1).

Definition (Interval graph, [LB62])

Let $B = \{(b_i, d_i)\}_{i=1}^{n}$ be a barcode. The *interval graph* of B is the simple graph $G_B(V, E)$ where $V = [n]$ and an edge (i, j) is in E if and only if $(b_i, d_i) \cap (b_j, d_j) \neq \emptyset$.

Background: Permutahedron

Let $\pi \in \mathfrak{S}_n$. Recall an **inversion** in π is a pair (π_i, π_j) such that $i < j$ and $\pi_i > \pi_j$, i.e., it is a pair of elements that appear out of order.

The **inversion set** of π, $\mathrm{inv}(\pi)$, is the set of all inversions in π.

The **inversion number** of π is the cardinality of its inversion set $\#\,\mathrm{inv}(\pi)$.

Example:

$$\pi = (1\ 2\ 5\ 4\ 3\ 6) \in \mathfrak{S}_6 \implies \mathrm{inv}(\pi) = \{(5,4),(5,3),(4,3)\}.$$

Background: Permutahedron

Definition (Weak Bruhat Order, [Ber71, CSW16])

The **weak Bruhat order** is the relation $\leq_W$ on $\mathfrak{S}_n$ defined by $\pi \leq_W \sigma$ if and only if $\mathrm{inv}(\pi) \subseteq \mathrm{inv}(\sigma)$. The pair $(S_n, \leq_W)$ form a graded lattice known as the **permutahedron**.

One can show that the $\pi \lessdot_W \sigma$ if and only if

$$\#\,\mathrm{inv}(\pi) + 1 = \#\,\mathrm{inv}(\sigma) \text{ and } \sigma = (i \;\; i+1)\pi,$$

for some $i \in [n-1]$, i.e., if σ equals π after transposing a pair of its *adjacent* entries.

Background: Permutahedron

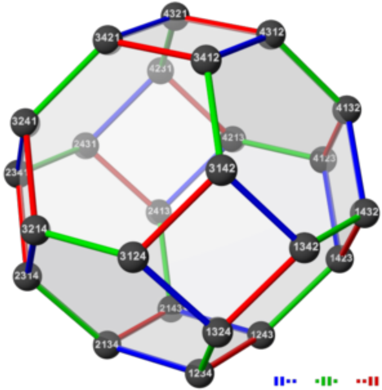

Figure: The permutahedron, $\mathfrak{S}_4$.

Background: Newman Lattices

For $\mathbf{m} = (m_1, \ldots, m_n) \in \mathbb{N}^n$, let $L(\mathbf{m})$ denote the set of permutations of the multiset $M = \{1^{m_1}, \ldots, n^{m_n}\}$

Definition (Multinomial Newman Lattice, [BB94, CSW16])

The **multinomial Newman lattice** $(L(\mathbf{m}), \leq)$ is the poset on $L(\mathbf{m})$ where $s \lessdot t$ if and only if s and t differ only in swapping an adjacent pair of entries, which is in numerical order in s but are reversed in t.

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Theorem ([SW16])

The multinomial Newman lattice $L(\mathbf{m})$ is order isomorphic to a principal ideal of $(\mathfrak{S}_{|\mathbf{m}|}, \leq_W)$. Consequently, $L(\mathbf{m})$ is a graded lattice.

Background: Newman Lattices

Let $S = \{1_1, \ldots, 1_{m_1}, \ldots, n_1, \ldots, n_{m_n}\}$, with the lexicographic order $1_1 \lessdot 1_2 \lessdot \ldots \lessdot 1_{m_1} \lessdot \ldots \lessdot n_{m_n}$.

We can also define multinomial Newman lattice via the map $\iota : L(\mathbf{m}) \to \mathfrak{S}_S \cong \mathfrak{S}_{|\mathbf{m}|}$, where ι sends a multipermutation to the unique, matching permutation in $\mathfrak{S}_S$ such that copies of the same elements appear in lexicographic order.

Example,

$$\iota(1\ 2\ 1\ 3\ 2) = (1_1\ 2_1\ 1_2\ 3_1\ 2_2).$$

Then $s \leq t$ in $L(\mathbf{m}) \iff \iota(s) \leq \iota(t)$ in $\mathfrak{S}_S$ [SW16].

Barcodes as Multipermutations

Definition (Strict barcode)

A barcode $B = \{(b_i, d_i)\}_{i=1}^n$ is called **strict** if $b_i \neq d_j$ for all $(i,j) \in [n]^2$ and $b_i \neq b_j$, $d_i \neq d_j$ for all $i \neq j$, i.e., if no birth/death times are repeated among the bars. Denote the set of strict barcodes with n bars by $\mathcal{B}^n_{st}$.

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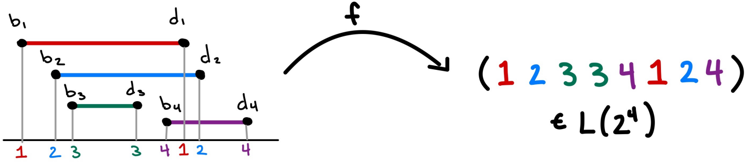
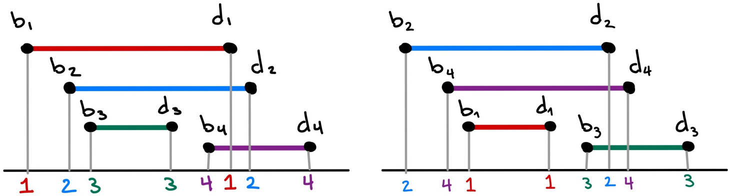
Let $L(2^n)$ denote the multinomial Newman lattice $L(\mathbf{m})$ for $\mathbf{m} = (2, 2, \ldots, 2) \in \mathbb{N}^n$. Define the map $f : \mathcal{B}^n_{st} \to L(2^n)$ such that,



Barcodes as Multipermutations

Note, f relies heavily on the initial labeling of the bars.



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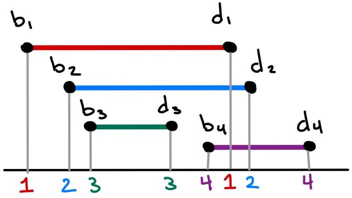

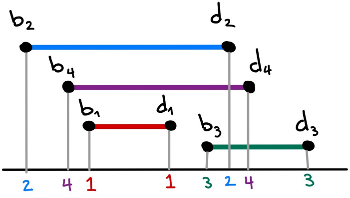

Definition (Combinatorial barcodes)

Let $\mathfrak{S}_n$ act element-wise on $L(2^n)$. For $s \in L(2^n)$, let $[s]$ denote the orbit of s. We call the set of all orbits the space of **combinatorial barcodes** with n bars and denote it $L(2^n)/\mathfrak{S}_n$.

Barcodes as Multipermutations

Define the map $g : \mathcal{B}^n_{st} \to L(2^n)/\mathfrak{S}_n$ where $g(B) = [f(B)]$.

Barcodes as Multipermutations

Define the map $g : \mathcal{B}^n_{st} \to L(2^n)/\mathfrak{S}_n$ where $g(B) = [f(B)]$.

The **canonical representative** of a combinatorial barcode $[s]$ is the multipermutation in $[s]$ where the "bars" are labeled according to increasing birth time.

Let $\psi : L(2^n)/\mathfrak{S}_n \to L(2^n)$ be the map which sends each combinatorial barcode to its canonical representative.

Barcodes as Multipermutations

Note: the following diagram is *not* commutative as $\psi \circ g \neq f$, although it is the case that $q \circ f = g$.

$$\begin{tikzcd}
& \mathcal{B}^n_{st} \arrow[dl, bend right, "g"'] \arrow[d, "f"] & \\
L(2^n)/\mathfrak{S}_n \arrow[r, hook, bend right, "\psi"'] & L(2^n) \arrow[l, bend right, "q"'] \arrow[r, hook, "\iota"] & \mathfrak{S}_s \cong \mathfrak{S}_{2n}
\end{tikzcd}$$

$g(B)$ (equivalently $\psi(g(B))$) provides a new discrete invariant on the space of strict barcodes with n bars.

Combinatorial Barcode Poset

Definition (Combinatorial barcode poset)

For $[s], [t] \in L(2^n)/\mathfrak{S}_n$, let $\leq_c$ be denote relation given by $[s] \leq_c [t]$ if and only if $\psi([s]) \leq \psi([t])$, where $\leq$ denotes the multinomial Newman order. We call the pair $(L(2^n)/\mathfrak{S}_n, \leq_c)$ the **combinatorial barcode poset**.

Combinatorial Barcode Poset

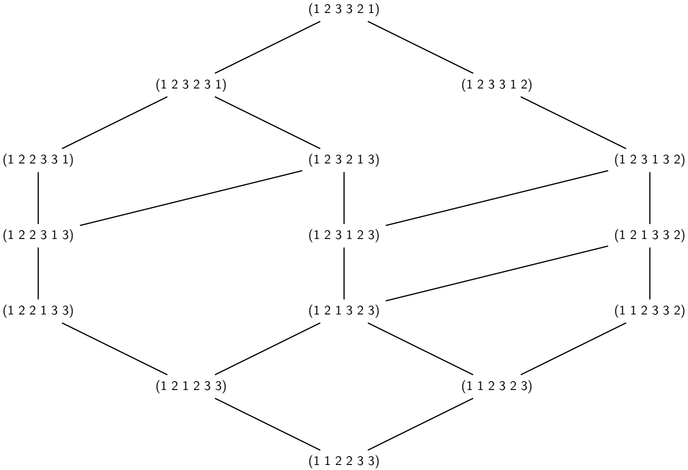

Figure: Hasse Diagram of $L(2^3)/\mathfrak{S}_3$

Combinatorial Barcode Poset

Theorem (J.R.)

The barcode poset $(L(2^n)/\mathfrak{S}_n, \leq_u)$ is order-isomorphic to the principal ideal of $L(2^n)$ generated by the "fully nested" permutation: $(1\ 2\ \ldots\ (n-1)\ n\ n\ (n-1)\ \ldots\ 2\ 1)$. Consequently, $(L(2^n)/\mathfrak{S}_n, \leq_u)$ is a graded lattice.

Inversion Multisets

Definition

Let $[s]$ be a combinatorial barcode and let s denote its canonical representative. The **inversion multiset** of $[s]$ is the multiset of pairs $\{(j,i)^{a_{ij}} : 1 \leq i < j \leq n\}$ where a_{ij} is equal to the number of pairs of i's and j's that appear out of order.

Example,

$$\mathrm{invm}((1\ 2\ 3\ 2\ 4\ 4\ 1\ 3)) = \{(2,1)^2, (3,1)^1, (4,1)^2, (3,2)^1, (4,3)^2\}$$

.

Inversion Multisets

Proposition (J.R.)

For $[s], [t], \in L(2^n)/\mathfrak{S}_n$, *write* $[s] \prec [t]$, *if* $\mathrm{invm}(s) \subseteq \mathrm{invm}(t)$. *Then,* $[s] \prec [t] \iff [s] \leq_c [t]$.

Inversion Multisets

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Proof Sketch:
Let $[s]$ be a combinatorial barcode with canonical representative s. Recall, ι is the embedding of $L(2^n)$ into $\mathfrak{S}_{2n}$.

Suppose $(i, j) \in \mathrm{invm}(s)$ with cardinality k. Let $A_{ij} = \{(j_1, i_1), (j_2, i_1), (j_1, i_2), (j_2, i_2)\}$ be the set of all possible inversions involving i and j in $\iota(s)$. We show that,

$$\mathrm{inv}(\iota(s)) \cap A_{ij} = \begin{cases} \emptyset & , k = 0 \\ \{(j_1, i_2)\} & , k = 1 \\ \{(j_1, i_2), (j_2, i_2)\} & , k = 2 \end{cases}.$$

Crossing Number

Let $B = \{(b_i, d_i)\}_{i=1}^n$ be a barcode. For each pair $b_i < b_j$, define the *crossing number* of bars i and j to be:

$$\text{cross}\#(i,j) = \begin{cases} 0, & d_i < b_j & \text{(disjoint)} \\ 1, & b_j < d_i < d_j & \text{(stepped)} \\ 2, & d_j < d_i & \text{(nested)} \end{cases} .$$

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Corollary (J.R.)

Let $B = \{(b_i, d_i)\}_{i=1}^n$ be a barcode. Then,

$$\rho(g(B)) = \sum_{b_i < b_j} \text{cross\#}(i,j).$$

Crossing Number

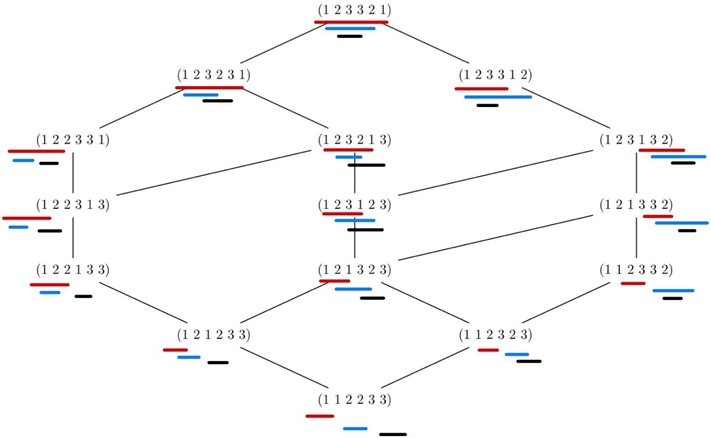
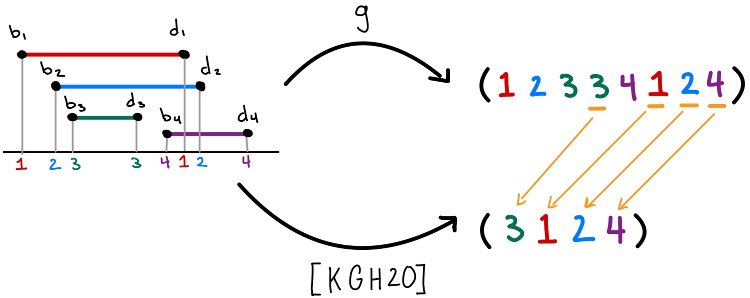


Figure: Hasse diagram for $L(2^3)/\mathfrak{S}_3$.

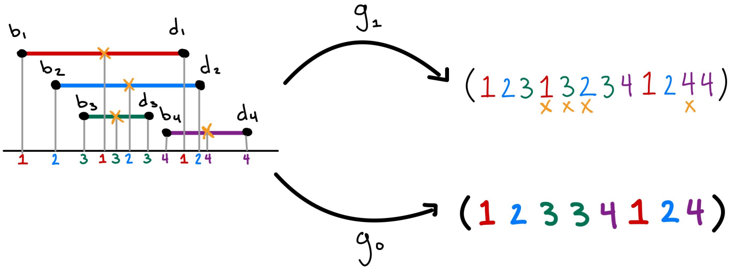
Invariant Inclusions



Power-k Barcode Posets

Let $k \in \mathbb{Z}_{\geq 0}$ and let $B \in B^n_{st}$. From each bar (b_i, d_i) consider the $2^k + 1$ points: $\left\{ b_i + \ell \frac{d_i - b_i}{2^k} : \ell = 0, \ldots, 2^k \right\}$.

Taking the collection of these points for all $i \in [n]$, we can construct maps f_k, g_k as we did for combinatorial barcodes (the k=0 case). We denote the resulting poset the *power-k barcode poset* with n bars and denote it $L((2^k+1)^n)/\mathfrak{S}_n$.



Power-k Barcode Poset Results

Theorem (J.R.)

The power k barcode poset $(L((2^k+1)^n)/\mathfrak{S}_n, \leq_k)$ *is isomorphic to a principal ideal of the multinomial Newman lattice,* $L((2^k+1)^n)$*. Consequently,* $(L((2^k+1)^n)/\mathfrak{S}_n, \leq_k)$ *is a lattice.*

Power-k Barcode Poset Results

Theorem (J.R.)

Let B, B' be fundamentally strict barcodes with n bars, where $B = \{(b_i, d_i)\}_{i=1}^n$ and $B' = \{(b_i', d_i')\}_{i=1}^n$. If $g_k(B) = g_k(B')$ for all $k \in \mathbb{N}$ and the interval graph G_B is connected, then there exist constants $\alpha > 0$ and $\delta \in \mathbb{R}$ such that $B = \alpha B' + \delta$, where $\alpha B' + \delta := \{(\alpha b_i' + \delta, \alpha d_i' + \delta) : i \in [n]\}$.

Power-k Barcode Poset Results

Theorem (J.R.)

Let B, B' be k-strict barcodes with n bars such that $g_k(B) = g_k(B')$. Suppose there exists a bar $(b_*, d_*) \in B$ (or equivalently in B') which contains all others, that is to say $b_* \leq b_i$ and $d_* \geq d_i$ for all $i \in [n]$. Then there exist constants $\alpha > 0$ and $\delta \in \mathbb{R}$ such that

$$d_\infty(B, \alpha B' + \delta) \leq \frac{|d_* - b_*|}{2^k}, \text{ and}$$

$$d_q(B, \alpha B' + \delta) \leq (n-1)^{\frac{1}{q}} \frac{|d_* - b_*|}{2^k}.$$

Thank you!

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