MATH 392 Problem Set 5

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6.39

$\{\hat{\beta}_n\}$ is consistent if and only if $lim_{n\to\infty}P(|\hat{\beta}_n - \beta| < \varepsilon) = 1 \; \forall \; \varepsilon > 0$.

It has been shown before that $\forall$ n ϵ N, $E[\hat{\beta}_n] = \frac{n}{n+1}\beta$. Thus,

$$\begin{aligned} lim_{n\to\infty}P(|\hat{\beta}_n - \beta| < \varepsilon) = 1 &\Leftrightarrow lim_{n\to\infty}P(|E[\hat{\beta}_n] - \beta| < \varepsilon) = 1 \\ &\Leftrightarrow lim_{n\to\infty}|E[\hat{\beta}_n] - \beta| < \varepsilon \\ &\Leftrightarrow lim_{n\to\infty}|\frac{n}{n+1}\beta - \beta| < \varepsilon \\ &\Leftrightarrow lim_{n\to\infty}|(\frac{n}{n+1} - 1)\beta| < \varepsilon \end{aligned}$$

It suffices to show that $lim_{n\to\infty}|\frac{n}{n+1} - 1| < \varepsilon$:

Let $\varepsilon > 0$. Then $\exists m \epsilon \mathbb{N}$ st $\forall n \geq m$,

$$\frac{1}{n} < \varepsilon \Rightarrow \frac{1}{n+1} < \varepsilon \Rightarrow |\frac{-1}{n+1}| < \varepsilon \Rightarrow |\frac{n-n-1}{n+1}| < \varepsilon \Rightarrow |\frac{n}{n+1} - \frac{n+1}{n+1}| < \varepsilon \Rightarrow |\frac{n}{n+1} - 1| < \varepsilon$$

Thus $lim_{n\to\infty}|\frac{n}{n+1} - 1| < \varepsilon$, so $lim_{n\to\infty}P(|\hat{\beta}_n - \beta| < \varepsilon) = 1$. Thus $\{\hat{\beta}_n\}$ is consistent.

6.40

$\{\hat{\sigma}_n^2\}$ is consistent if and only if $lim_{n\to\infty}P(|\hat{\sigma}_n^2 - \sigma^2| < \varepsilon) = 1 \; \forall \; \varepsilon > 0$.

Again, it suffices to show that $lim_{n\to\infty}P(|E[\hat{\sigma}_n^2] - \sigma^2| < \varepsilon) = 1$. From Problem Set 4,

$$E[\hat{\sigma}_n^2] - \sigma^2 = \frac{n-1}{n}\sigma^2 - \sigma^2 = \frac{-1}{n}\sigma^2$$

Let $\varepsilon > 0$. Then $\exists m \epsilon \mathbb{N}$ st $\forall n \geq m$, , $\frac{1}{n} < \varepsilon$. Thus, $\forall n \geq m$,

$$\left|\frac{1}{n}\right| < \varepsilon \Rightarrow \left|\frac{1}{n}\sigma^2\right| < \varepsilon \Rightarrow \left|(\frac{n-1}{n} - \frac{n}{n})\sigma^2\right| < \varepsilon \Rightarrow \left|\frac{n-1}{n}\sigma^2 - \sigma^2\right| < \varepsilon \Rightarrow \left|E[\hat{\sigma}_n^2] - \sigma^2\right| < \varepsilon.$$

Thus, $lim_{n\to\infty}P(|\hat{\sigma}_n^2 - \sigma^2| < \varepsilon) = 1 \forall \varepsilon > 0$.