MATH 392 Problem Set 5 (Corrected)

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6.39

$\{\hat{\beta}_n\}$ is consistent if and only if $\lim_{n\to\infty} P(|\hat{\beta}_n - \beta| < \varepsilon) = 1 \ \forall \ \varepsilon > 0$.

We know that for any n, the distribution of X_{max} follows the distribution

$$F_{X_{max}}(x) = \begin{cases} 1, x > \beta \\ (\frac{x}{\beta})^n, 0 < x < \beta \\ 0, x < 0 \end{cases}$$

Thus

$$\begin{aligned}
\lim_{n\to\infty} P(|\hat{\beta}_n - \beta| < \varepsilon) &= \lim_{n\to\infty} P(\beta - \varepsilon < \hat{\beta}_n < \beta + \varepsilon) \\
&= \lim_{n\to\infty} P(\beta - \varepsilon < X_{max} < \beta) \\
&= \lim_{n\to\infty} (\frac{x}{\beta})^n|_{\beta-\varepsilon}^{\beta} \\
&= 1 - \lim_{n\to\infty} (\frac{\beta - \varepsilon}{\beta})^n \\
&= 1 \forall \varepsilon > 0
\end{aligned}$$

Thus $\{\hat{\beta}_n\}$ is consistent.

6.40

$\{\hat{\sigma}_n^2\}$ is consistent iff

$$\lim_{n\to\infty} P(|\hat{\sigma}_n^2 - \sigma^2| < \varepsilon) = 1 \Leftrightarrow \begin{cases} \lim_{n\to\infty} bias[\hat{\sigma}_n^2] = 0 \\ \lim_{n\to\infty} V[\hat{\sigma}_n^2] = 0 \end{cases}$$

First show that $\lim_{n\to\infty} bias[\hat{\sigma}_n^2] = 0$:

From Problem Set 4,

$$E[\hat{\sigma}_n^2] - \sigma^2 = \frac{n-1}{n}\sigma^2 - \sigma^2 = \frac{-1}{n}\sigma^2$$

Thus $\lim_{n\to\infty} E[\hat{\sigma}_n^2] - \sigma^2 = 0 \Rightarrow bias[\hat{\sigma}_n^2] \to 0$.

Now show that $\lim_{n\to\infty} V[\hat{\sigma}_n^2] = 0$:

From B.16, $\frac{\hat{\sigma}_n^2 n}{\sigma^2} \sim \chi_n^2$, so $V[\frac{\hat{\sigma}_n^2 n}{\sigma^2}] = 2n \ \forall \ n\epsilon\mathbb{N}$. Thus

$$\lim_{n\to\infty}(\frac{n}{\sigma^2})^2 V[\hat{\sigma}_n^2] = \lim_{n\to\infty} V[\frac{\hat{\sigma}_n^2 n}{\sigma^2}] = \lim_{n\to\infty} 2n$$

$$\begin{aligned}\lim_{n\to\infty} V[\hat{\sigma}_n^2] &= \lim_{n\to\infty} \frac{2n}{(\frac{n}{\sigma^2})^2} \\ &= \lim_{n\to\infty} \frac{2\sigma^4}{n} \\ &= 0\end{aligned}$$

Thus $\{\hat{\sigma}_n^2\}$ is consistent.

7.3

a

$$X \sim N(185, 50^2), n = 100, \bar{X} = 210$$

$$P(q_{.05} < \frac{\bar{X}-\mu}{\sigma/\sqrt{n}} < q_{.95}) = 1 - \alpha = .9$$

Solve for μ:

$$P(\bar{X} - q_{.95}\frac{\sigma}{\sqrt{n}} < \mu < \bar{X} - q_{.05}\frac{\sigma}{\sqrt{n}}) = .9$$

$$(210 - 1.645\frac{50}{\sqrt{100}} < \mu < 210 + .1.645\frac{50}{\sqrt{100}}) = (201.8 < \mu < 218.8)$$

Thus the 90% confidence interval is (201.8, 218.8).

b

Solve for n:

$$\bar{X} - LB_{.05} = \bar{X} - (\bar{X} - 1.96(\frac{50}{\sqrt{n}})) \leq 10$$

$$1.96\frac{50}{10} \leq \sqrt{n}$$

$$n \geq 96.04$$

A sample size of at least 97 is needed.

c

Like b,

$$2.576\frac{50}{10} \leq \sqrt{n}$$

$$n \geq 165.9$$

A sample size of at least 166 is needed.

7.8

For n = 30, the code generating the simulation is below, along with the proportion of times the confidence interval missed μ.

```
# Run simulation
low <- 0
high <- 0
n <- 30
q <- qt(.975, n-1)
N <- 10^5
for(i in 1:N){
  x <- rgamma(n,shape = 5, rate = 2)
  xbar <- mean(x)
  s <- sd(x)
  L <- xbar - q*s/sqrt(n)
  U <- xbar + q*s/sqrt(n)
  if(U<5/2){
    low <- low + 1
  }
  if(5/2<L){
    high <- high + 1
  }
}

(high+low)/N
```

```
## [1] 0.05396
```

```
## [1] 0.05194
```

```
## [1] 0.05169
```

```
## [1] 0.05009
```

Compared to the next three numbers for n=60, n=100, and n=250, the frequency of missing μ converges to the thoretical frequency of .05. As n increases, the number of estimates that are too high increases, while the number of estimates that are too low decreases.

7.34

We are given

$$X \sim Gamma(2, \lambda)$$

$$2\lambda x \sim \chi^2_{df=4}, f_{2\lambda X}(2\lambda x)$$

Thus, the quantiles will be calculated in terms of $2\lambda x$, using the $\chi^2_{df=4}$ distribution.

$$\begin{aligned} .95 = P(q_{1,x} < \lambda < q_{2,x}) &= P(q_{1,2\lambda x} < 2\lambda X < q_{2,2\lambda x}) \\ &= P(\frac{q_{1,2\lambda x}}{2X} < \lambda < \frac{q_{2,2\lambda x}}{2X}) \\ &= P(\frac{.2422}{X} < \lambda < \frac{5.572}{X}) \end{aligned}$$

Thus the 95% confidence interval for λ is $(\frac{.2422}{X}, \frac{5.572}{X})$.