MATH 392 Problem Set 4

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6.2

Analytical Solution

Let $x_1, ..., x_n \sim$ Poisson(λ). Show the maximum likelihood estimate of λ is $\hat{\lambda} = \bar{x}$:

$$L(\lambda | x_1, ..., x_n) = P(X_1 = x_1, ..., X_n = x_n)$$

$$L(\lambda | x_1, ..., x_n) = P(X = x_1)...P(X = x_n)$$

$$L(\lambda | x_1, ..., x_n) = \prod_i^n \frac{\lambda^{x_i} e^{-\lambda}}{x_i!}$$

$$L(\lambda | x_1, ..., x_n) = \frac{e^{-\lambda n} \lambda^{\sum_i^n x_i}}{\prod_i^n x_i!}$$

Take the natural log of both sides to make the derivation easier:

$$ln(L(\lambda | x_1, ..., x_n)) = ln(e^{-n\lambda}) + ln(\lambda^{\sum_i^n x_i}) - ln(\prod_i^n x_i!)$$

$$ln(L(\lambda | x_1, ..., x_n)) = -n\lambda + ln(\lambda) \sum_i^n x_i - ln(\prod_i^n x_i!)$$

Derive with respect to λ and set equal to 0:

$$\frac{\partial ln(L(\lambda | x_1, ..., x_n))}{\partial \lambda} = -n + \frac{\sum_i^n x_i}{\lambda} = 0$$

$$\frac{\partial ln(L(\lambda | x_1, ..., x_n))}{\partial \lambda} = \lambda = \frac{\sum_i^n x_i}{n} = \bar{x}$$

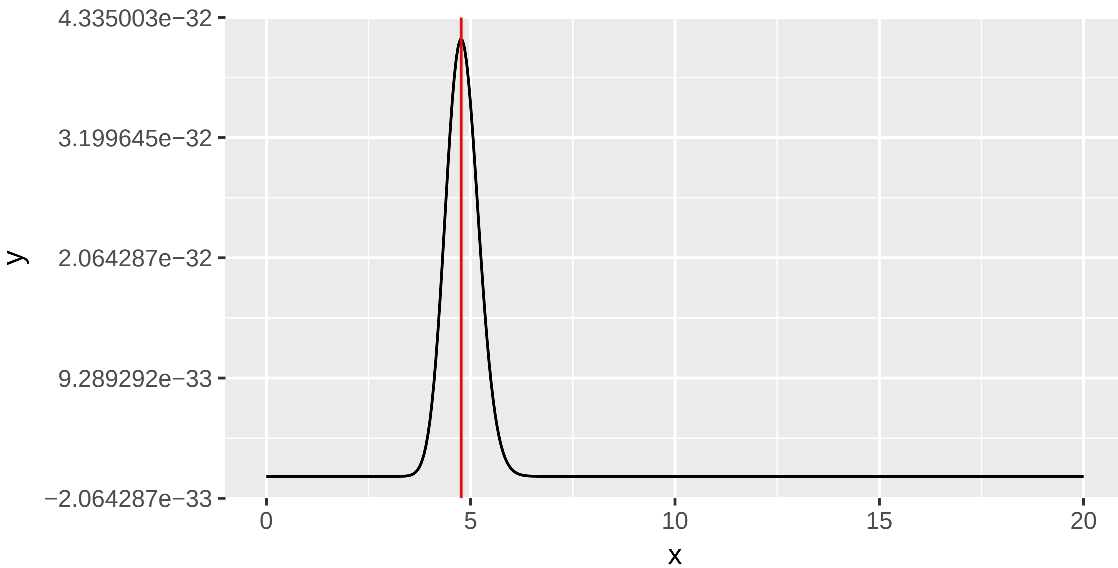
Thus $\lambda_{MLE} = \bar{x}$

Empirical Solution

```
# Build function
e <- exp(1)
pois <- function(lambda, x){
  n <- length(x)
  sum.x <- sum(x)
  prod.fact <- prod(factorial(x))
  e^(-lambda*n)*(lambda^(sum.x))/(prod.fact)
}

# Create sequence of lambdas for MLE plot
lambdas <- seq(from = 0, to = 20, by = .05)
# Draw random sample of x's
x <- sample(1:10,30,replace=T)
# Apply pois function to find likelihoods
L_pois <- pois(lambdas,x)

# Plot
df <- data.frame(x = lambdas,
                 y = L_pois)
ggplot(df, aes(x=x,y=y)) +
  geom_line() +
  geom_vline(xintercept = mean(x), col = "red")
```



A red line above indicates the mean of the random sample. As expected, the highest likelihood for λ is at the sample mean.

6.8

Analytical Solution

$$f(x;\theta) = \frac{\sqrt{2/\pi}x^2 e^{-x^2/2\theta^2}}{\theta^3}$$

$$L(\theta|x_1, ..., x_n) = \prod_i^n f(x_i; \theta)$$

$$L(\theta|x_1, ..., x_n) = \prod_i^n \frac{\sqrt{2/\pi} x_i^2 e^{-x_i^2/2\theta^2}}{\theta^3}$$

$$L(\theta|x_1, ..., x_n) = (\frac{\sqrt{2/\pi}}{\theta^3})^n \prod_i^n x_i^2 e^{-x_i^2/2\theta^2}$$

$$ln(L(\theta|x_1, ..., x_n)) = ln((\frac{\sqrt{2/\pi}}{\theta^3})^n) + \sum_i^n ln(x_i^2) + \sum_i^n ln(e^{-x_i^2/2\theta^2})$$

$$ln(L(\theta|x_1, ..., x_n)) = n(ln\sqrt{2/\pi} - ln\theta^3) + \sum_i^n ln(x_i^2) - \frac{1}{2\theta^2}\sum_i^n x_i^2$$

Derive with respect to θ and set to 0:

$$\frac{\partial(lnL(\theta|x_1, ..., x_n))}{\partial\theta} = -\frac{n}{\theta^3}3\theta^2 + \frac{1}{\theta^3}\sum_i^n x_i^2 = 0$$

$$\frac{3n}{\theta} = \frac{\sum_i^n x_i^2}{\theta^3}$$

$$\theta = \sqrt{\frac{\sum_i^n x_i^2}{3n}}$$

Thus $\theta_{MLE} = \frac{\bar{x}\sqrt{n}}{\sqrt{3}}$.

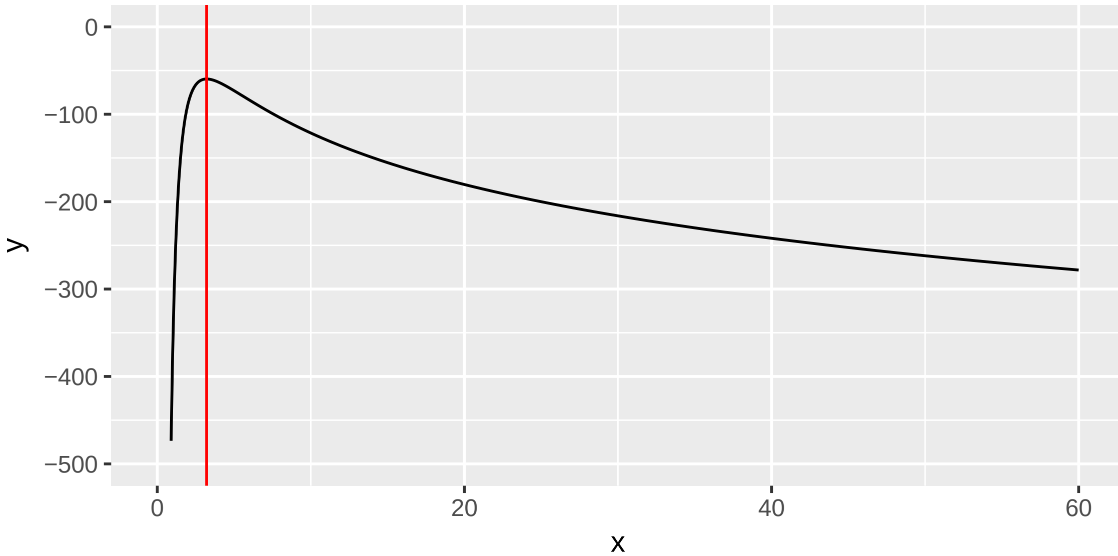
Empirical Solution

```r
# Build log function
f <- function(theta, x){
  n <- length(x)
  n*(log(sqrt(2/pi)) - log(theta^3)) +
    sum(log(x^2)) -
    sum((x^2))/(2*(theta^2))
}

# Create sequence of thetas for MLE plot
thetas <- seq(from = .1, to = 60, by = .1)
# Reset seed and draw random sample of x's
set.seed(23)
x <- rnorm(30,5,2)
# Apply function to find likelihoods
L_theta <- f(thetas,x)
# Find x-intercept from analytical solution
n <- length(x)
analytical.sol <- sqrt(sum(x^2)/(3*n))
```

```r
# Plot
df <- data.frame(x = thetas,
                 y = L_theta)
ggplot(df, aes(x=x,y=y)) +
  geom_line() +
  geom_vline(xintercept = analytical.sol, col = "red") +
  xlim(0,60) +
  ylim(-500,0)
```

```
## Warning: Removed 8 rows containing missing values (geom_path).
```



As expected, the maximum likelihood estimate of our simulation is at about the analytical solution.

6.11

Analytical Solution

$$L(\lambda) = \prod_i^n f(x_i) \prod_j^m f(y_j)$$

$$L(\lambda) = \prod_i^n \lambda e^{-\lambda x_i} \prod_j^m 2\lambda e^{-2\lambda y_j}$$

$$L(\lambda) = 2^m \lambda^{n+m} \lambda e^{-\lambda(\sum_i^n x_i + 2\sum_j^m y_j)}$$

$$ln(L(\lambda)) = (n+m)ln(\lambda) - \lambda(\sum_i^n x_i + 2\sum_j^m y_j)$$

$$\frac{\partial ln(L(\lambda))}{\partial \lambda} = (n+m)/\lambda - (\sum_i^n x_i + 2\sum_j^m y_j) = 0$$

$$\lambda = \frac{n+m}{\sum_i^n x_i + 2\sum_j^m y_j}$$

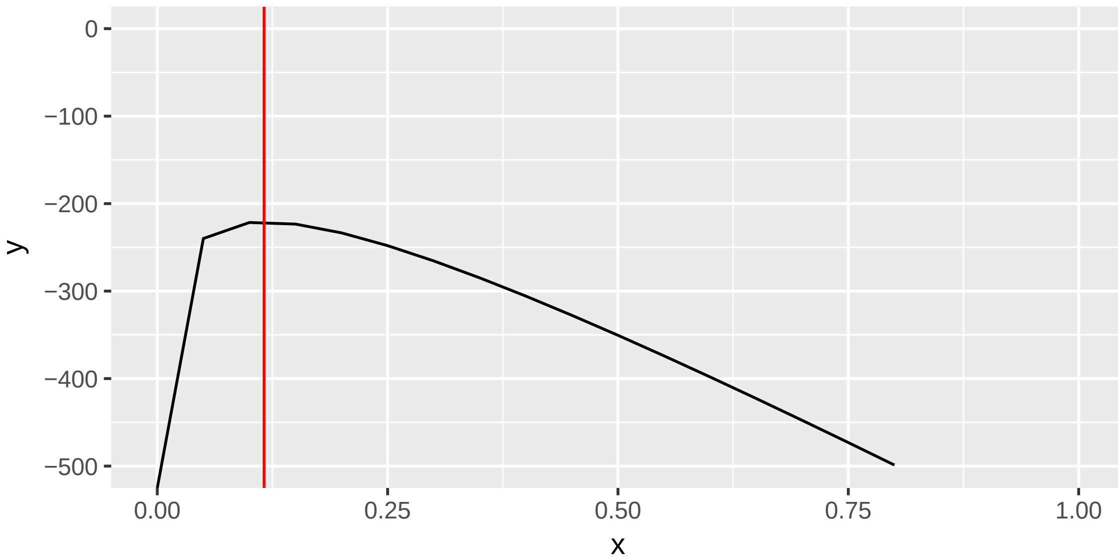
Thus, $\lambda_{MLE} = \frac{n+m}{\sum_i^n x_i + 2\sum_j^m y_j}$.

Empirical Solution

```
# Build log function
pois <- function(lambda,x,y){
  n <- length(x)
  m <- length(y)
  (n+m)*log(lambda) - lambda*(sum(x) + 2*sum(y))
}

# Create sequence of lambdas for MLE plot
lambdas <- seq(from = 0, to = 20, by = .05)
# Draw random sample of x's
set.seed(11)
x <- sample(1:10,30,replace=T)
set.seed(35)
y <- sample(1:10,40,replace=T)
# Apply pois function to find likelihoods
L_pois <- pois(lambdas,x,y)
# Calculate analytical solution
obs <- (length(x)+length(y))/(sum(x) + 2*sum(y))
# Plot
df <- data.frame(x = lambdas,
                 y = L_pois)
ggplot(df, aes(x=x,y=y)) +
  geom_line() +
  geom_vline(xintercept = obs, col = "red") +
  xlim(0,1) +
  ylim(-500,0)
```

```
## Warning: Removed 384 rows containing missing values (geom_path).
```



The analytical solution (the red line) is at around the maximum of the log likelihood function.

6.13

a

$$L(\alpha|X_1,...,X_n;\beta) = \prod_i^n \alpha\beta X_i^{\beta-1}e^{-\alpha X_i^\beta} = (\alpha\beta)^n \prod_i^n X_i^{\beta-1}e^{-\alpha X_i^\beta}$$

$$ln(L(\alpha)) = nln\alpha + nln\beta + (\beta-1)\sum_i^n lnX_i + \sum_i^n ln(e^{-\alpha X_i^\beta})$$

$$ln(L(\alpha|X;\beta)) = nln\alpha + nln\beta + (\beta-1)\sum_i^n lnX_i - \alpha\sum_i^n X_i^\beta$$

$$\frac{\partial ln(L(\alpha|X;\beta))}{\partial\alpha} = \frac{n}{\alpha} - \sum_i^n X_i^\beta = 0$$

Thus $\alpha_{MLE} = \frac{1}{n}\sum_i^n X_i^\beta$.

b

$$L(\alpha;\beta|X_1,...,X_n) = (\alpha\beta)^n \prod_i^n X_i^{\beta-1}e^{-\alpha X_i^\beta}$$

Take the natural log of both sides:

$$ln(L(\alpha;\beta|X)) = nln\alpha + nln\beta + (\beta-1)\sum_i^n lnX_i - \alpha\sum_i^n X_i^\beta$$

Differentiate with respect to α and β and set to 0. Solve simultaneously for a and b:

$$\frac{\partial ln(L(\alpha;\beta|X))}{\partial\alpha} = \frac{n}{\alpha} - \sum_i^n X_i^\beta = 0$$

and

$$\frac{\partial ln(L(alpha;\beta|X))}{\partial\beta} = \frac{n}{\beta} + ln(\sum_i^n X_i) - \alpha\sum_i^n X_i^\beta = 0$$

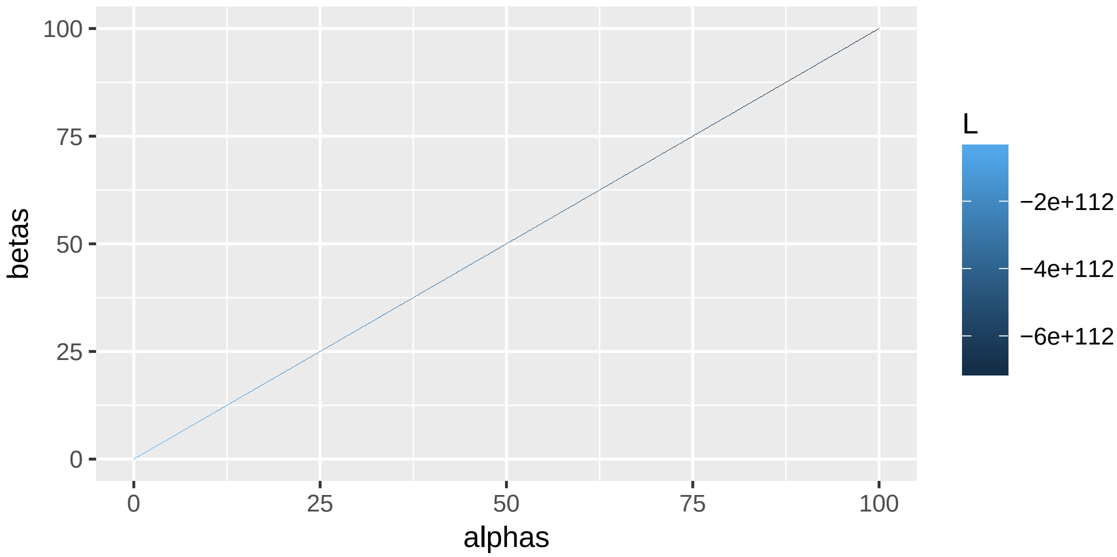
Empirical Solution

```
# Build log function
f <- function(alpha,beta,x){
  n <- length(x)
  n*log(alpha) +
    n*log(beta) +
    (beta-1)*sum(log(x)) -
    alpha*sum(x^beta)
}

# Create sequence of alphas and betas for plot
alphas <- seq(0,100,by=.1)
betas <- seq(0,100,by=.1)
# Draw random sample of x's
set.seed(11)
x <-rnorm(30,9,3)
# Apply function
L_ab <- f(alphas,betas,x)
```

```
## Warning in x^beta: longer object length is not a multiple of shorter object
## length
```

```
df <- data.frame(alphas = alphas,
                 betas = betas,
                 L = L_ab)
ggplot(df, aes(alphas,betas)) +
  geom_tile(aes(fill = L))
```



6.14

Let X = 2,3,5,9,10 ~ Unif$[\alpha, \beta]$. Find $\hat{\alpha}_{MOM}$ and $\hat{\beta}_{MOM}$:

1. Write first and second moments in terms of parameters α and β

$$\mu_1 = E[X] = \int_{\alpha}^{\beta} \frac{1}{\beta - \alpha} x dx = \frac{1}{2(\beta - \alpha)} x^2|_{\alpha}^{\beta} = \frac{\alpha + \beta}{2} = 5.8 = \bar{X}$$

Thus

$$\alpha + \beta = 11.6$$

Now consider the second moment:

$$\mu_2 = E[X^2] = \int_{\alpha}^{\beta} \frac{1}{\beta - \alpha} x^2 dx =$$

Solving the integral and setting μ_2 equal to its theoretical value gives

$$\mu_2 = \frac{\alpha^2 + \alpha\beta + \beta^2}{3} = \frac{\sum_i^5 X_i^2}{5} = 43.8$$

$$\mu_2 = 131.4 = \alpha^2 + \alpha\beta + \beta^2$$

Since $\alpha + \beta = 11.6$, solve for β and plug into the equation for μ_2:

$$\beta = 11.6 - \alpha$$

$$131.4 = \alpha^2 + \alpha(11.6 - \alpha) + (11.6 - \alpha)^2$$

$$0 = 11.6\alpha + 134.6 - 23.2\alpha + \alpha^2 - 131.4$$

$$0 = \alpha^2 - 11.6\alpha + 4.2$$

Using the quadratic formula, solving for α gives

$$\alpha = .27 or 11.32$$

Given the dataset, $\alpha = .27$ makes more sense, since $X_{max} = 10$. Now plug α in to solve for β:

$$.27 + \beta = 11.6$$

$$\beta = 11.33$$

Thus our method of moment estimates are $\hat{\alpha}_{MOM} = .27$, $\hat{\beta}_{MOM} = 11.33$.

6.20

a

$$L(\theta) = \prod_{i=1}^{5} f(x_i|\theta) = \prod_{i=1}^{5} \theta x_i^{\theta-1}$$

$$L(\theta) = \theta(\prod_{i=1}^{5} x_i)^{\theta-1}$$

Differentiate, set equal to 0, and solve for θ:

$$\frac{\partial L(\theta)}{\partial \theta} = 5\theta^4(\prod_{i=1}^{5} x_i)^{\theta-1} + \theta^5(\prod_{i=1}^{5} x_i)^{\theta-1} ln(\prod_{i=1}^{5} x_i) = 0$$

$$\theta^4(\prod_{i=1}^{5} x_i)^{\theta-1}(5 + \theta ln(\prod_{i=1}^{5} x_i)) = 0$$

$$\theta = \frac{-5}{ln(\prod_{i=1}^{5} x_i)} \approx 1.57$$

Thus $\theta_{MLE} \approx 1.57$.

b

$$\mu_1 = E[X] = \int_0^1 \theta x^{\theta-1} x dx = \int_0^1 \theta x^\theta dx$$

$$\mu_1 = \frac{\theta}{\theta+1} x^{\theta+1}|_0^1 = \frac{\theta}{\theta+1}$$

$$\hat{\mu}_1 = \frac{\theta}{\theta+1} = \bar{X}$$

Rearranging gives:

$$\theta = \frac{\bar{X}}{1-\bar{X}} = \frac{.594}{1-.594} = 1.463$$

Thus $\theta_{MOM} \approx 1.463$.

6.25

$\forall i \epsilon$ {1,..,n}, $\sum_i^n a_i = n$

6.27

a

bias $= E[\hat{\sigma}^2] - \sigma^2$

$$E[\hat{\sigma}^2] = E[\frac{1}{n}\sum_{i=1}^n (X_i - \bar{X})^2] = \frac{1}{n} E[\sum_{i=1}^n X_i^2 - n\bar{X}^2]$$

$$E[\hat{\sigma}^2] = \frac{1}{n}(\sum_{i=1}^n E[X_i^2] - E[n\bar{X}^2])$$

$$E[\hat{\sigma}^2] = \frac{n-1}{n}\sigma^2$$

Thus bias $= \frac{n-1}{n}\sigma^2 - \sigma^2 = \frac{-\sigma^2}{n}$

b

From theorem B.16, $\frac{(n-1)S^2}{\sigma^2} \sim \chi^2_{n-1}$, where $S^2 = \frac{1}{n-1}\sum_{i=1}^n (X_i - \bar{X})^2$. Also, from theorem B.12, V[$\frac{(n-1)S^2}{\sigma^2}$] = 2(n-1). Thus

$$\frac{\sum_{i=1}^n (X_i - \bar{X})^2}{\sigma^2} \tilde{\chi}^2_{n-1}$$

Replace with $\hat{\sigma}^2$ to get

$$\frac{\hat{\sigma}^2 n}{\sigma^2} \tilde{} \chi^2_{n-1}$$

Thus

$$V[\frac{\hat{\sigma}^2 n}{\sigma^2}] = 2(n-1)$$

$$V[\hat{\sigma}^2] = \frac{2(n-1)\sigma^4}{n^2}$$

c

$$MSE[\hat{\sigma}^2] = bias[\hat{\sigma}^2]^2 + V[\hat{\sigma}^2]$$

$$MSE[\hat{\sigma}^2] = bias[\hat{\sigma}^2]^2 + V[\hat{\sigma}^2]$$

$$MSE[\hat{\sigma}^2] = \frac{\sigma^4}{n^2} + \frac{2(n-1)\sigma^4}{n^2}$$

$$MSE[\hat{\sigma}^2] = \frac{2n - \sigma^4}{n^2}$$

6.30

a

$$MSE_{\hat{\theta}_1} = bias^2_{\hat{\theta}_1} + Var[\hat{\theta}_1] = 0 + 25 = 25$$

$$MSE_{\hat{\theta}_2} = bias^2_{\hat{\theta}_2} + Var[\hat{\theta}_2] = 3^2 + 4 = 13$$

Thus $MSE_{\hat{\theta}_2} \leq MSE_{\hat{\theta}_1}$.

b

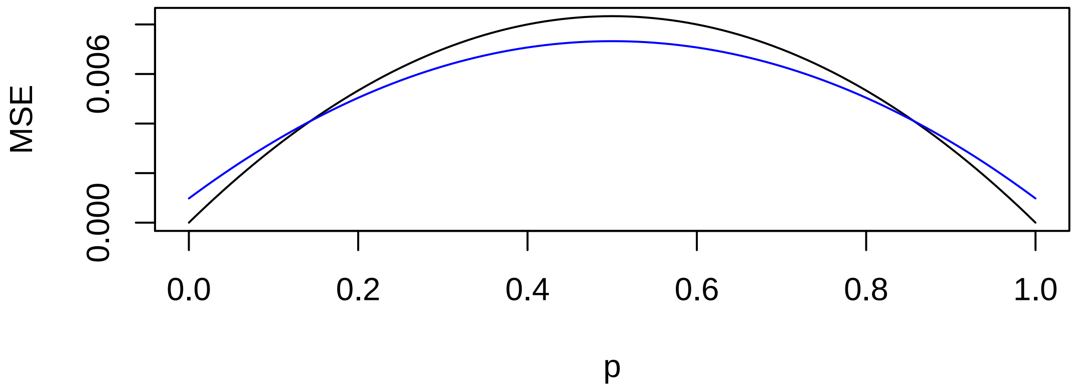
$$MSE_{\hat{\theta}_2} \leq MSE_{\hat{\theta}_1}$$

$$b^2 + 4 \leq 25$$

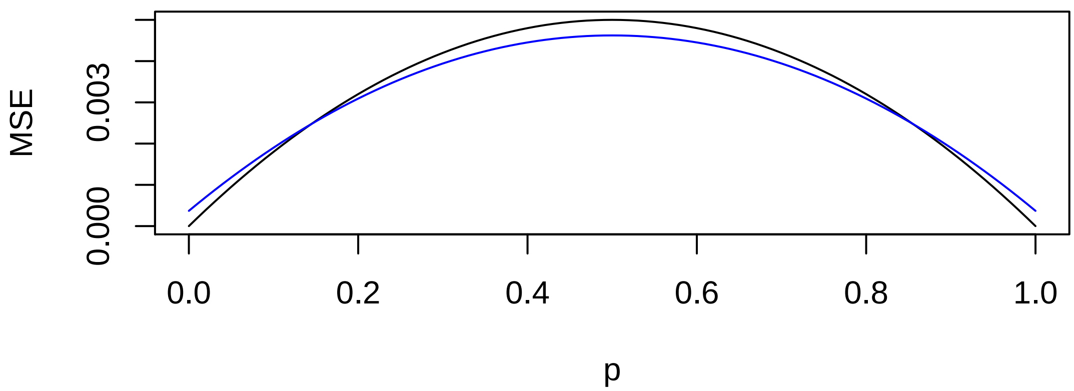
$$b \leq 4.583$$

6.31

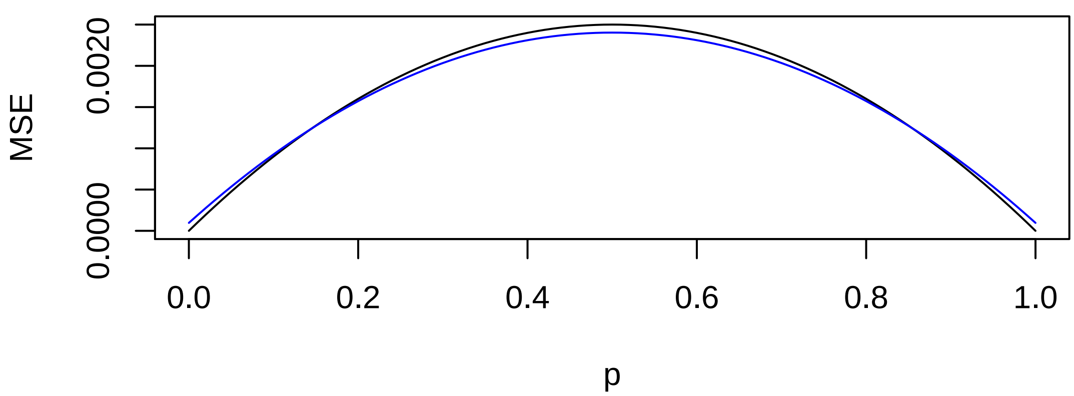
```r
# Recreate plots using code from sec 6.3.3, changing the sample size
n <- 30
curve(x*(1-x)/n,from=0,to=1,xlab="p",ylab="MSE")
curve(n*(1-x)*x/(n+2)^2+(1-2*x)^2/(n+2)^2, add=TRUE, col="blue",lty=1)
```



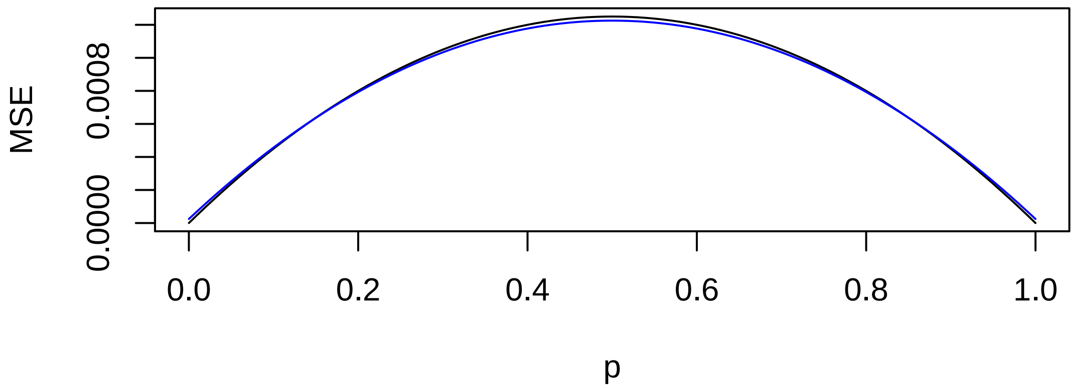

```r
n <- 50
curve(x*(1-x)/n,from=0,to=1,xlab="p",ylab="MSE")
curve(n*(1-x)*x/(n+2)^2+(1-2*x)^2/(n+2)^2, add=TRUE, col="blue",lty=1)
```




```r
n <- 100
curve(x*(1-x)/n,from=0,to=1,xlab="p",ylab="MSE")
curve(n*(1-x)*x/(n+2)^2+(1-2*x)^2/(n+2)^2, add=TRUE, col="blue",lty=1)
```




```r
n <- 200
curve(x*(1-x)/n,from=0,to=1,xlab="p",ylab="MSE")
curve(n*(1-x)*x/(n+2)^2+(1-2*x)^2/(n+2)^2, add=TRUE, col="blue",lty=1)
```



As n increases, the MSE decreases. In each case, the second estimator (the blue line) has a lower MSE except for low and high estimates of p.

6.37

a

$$E[\bar{X}] = E[\frac{X_1 + X_2}{2}] = E[X_1]/2 + E[X_2]/2 = E[X]/2 + E[X]/2 = E[X] = 1/\lambda$$

b

$$Var[\bar{X}] = Var[\frac{X_1 + X_2}{2}] = \frac{1}{4}Var[X_1 + X_2] = \frac{1}{4}(Var[X] + Var[X]) = \frac{1}{4}(\frac{1}{\lambda^2} + \frac{1}{\lambda^2}) = \frac{2}{\lambda^2}$$

c

$$E[\sqrt{X_1 X_2}] = E[\sqrt{X_1}\sqrt{X_2}] = E[\sqrt{X_1}]E[\sqrt{X_2}]$$

By fact,

$$E[\sqrt{X_1 X_2}] = \sqrt{\pi}/2\sqrt{\lambda}(\sqrt{\pi}/2\sqrt{\lambda})) = \pi/4\lambda$$

d

$$bias = |E[\sqrt{X_1 X_2}] - 1/\lambda| = |\pi/4\lambda - 1/\lambda| = |\frac{\pi - 4}{4\lambda}|$$

Thus bias = $0.2146/\lambda$.