MATH 392 Problem Set 4

EJ Arce

17 February 2018

6.2

Let $x_1, ..., x_n \sim$ Poisson(λ). Show the maximum likelihood estimate of λ is $\hat{\lambda} = \bar{x}$:

$$L(\lambda|x_1, ..., x_n) = P(X_1 = x_1, ..., X_n = x_n)$$

$$L(\lambda|x_1, ..., x_n) = P(X = x_1)...P(X = x_n)$$

$$L(\lambda|x_1, ..., x_n) = \prod_i^n \frac{\lambda^{x_i} e^{-\lambda}}{x_i!}$$

$$L(\lambda|x_1, ..., x_n) = \frac{e^{-\lambda n} \lambda^{\sum_i^n x_i}}{\prod_i^n x_i!}$$

Take the natural log of both sides to make the derivation easier:

$$ln(L(\lambda|x_1, ..., x_n)) = ln(e^{-n\lambda}) + ln(\lambda^{\sum_i^n x_i}) - ln(\prod_i^n x_i!)$$

$$ln(L(\lambda|x_1, ..., x_n)) = -n\lambda + ln(\lambda) \sum_i^n x_i - ln(\prod_i^n x_i!)$$

Derive with respect to λ and set equal to 0:

$$\frac{\partial ln(L(\lambda|x_1, ..., x_n))}{\partial \lambda} = -n + \frac{\sum_i^n x_i}{\lambda} = 0$$

$$\frac{\partial ln(L(\lambda|x_1, ..., x_n))}{\partial \lambda} = \lambda = \frac{\sum_i^n x_i}{n} = \bar{x}$$

Thus $\lambda_{MLE} = \bar{x}$

6.8

$$f(x; \theta) = \frac{\sqrt{2/\pi} x^2 e^{-x^2/2\theta^2}}{\theta^3}$$

$$L(\theta|x_1, ..., x_n) = \prod_i^n f(x_i; \theta)$$

$$L(\theta|x_1,...,x_n) = \prod_i^n \frac{\sqrt{2/\pi}x_i^2 e^{-x_i^2/2\theta^2}}{\theta^3}$$

$$L(\theta|x_1,...,x_n) = (\frac{\sqrt{2/\pi}}{\theta^3})^n \prod_i^n x_i^2 e^{-x_i^2/2\theta^2}$$

$$ln(L(\theta|x_1,...,x_n)) = ln(\frac{\sqrt{2/\pi}}{\theta^3})^n + \sum_i^n ln(x_i^2) + \sum_i^n ln(e^{-x_i^2/2\theta^2})$$

$$ln(L(\theta|x_1,...,x_n)) = ln\sqrt{2/\pi} - ln\theta^3 + \sum_i^n ln(x_i^2) + \sum_i^n -x_i^2/2\theta^2$$

Derive with respect to θ and set to 0:

$$\frac{\partial(L(\theta|x_1,...,x_n))}{\partial\theta} = -\frac{n}{\theta^3}3\theta^2 + \sum_i^n x_i^2/\theta^3 = 0$$

$$\frac{3n}{\theta} = \frac{\sum_i^n x_i^2}{\theta^3}$$

$$\theta = \frac{\bar{x}\sqrt{n}}{\sqrt{3}}$$

Thus $\theta_{MLE} = \frac{\bar{x}\sqrt{n}}{\sqrt{3}}$.

6.11

As before,

$$L(\lambda) = \prod_i^n f(x_i) \prod_j^m f(y_i)$$

$$L(\lambda) = \prod_i^n \lambda e^{-\lambda x_i} \prod_j^m \lambda e^{-\lambda y_j}$$

$$L(\lambda) = \lambda^{n+m}\lambda e^{-\lambda(\sum_i^n x_i + \sum_j^m y_j)}$$

$$ln(L(\lambda)) = (n+m)ln(\lambda) - \lambda(\sum_i^n x_i + \sum_j^m y_j)$$

$$\frac{\partial ln(L(\lambda))}{\partial\lambda} = (n+m)/\lambda - (\sum_i^n x_i + \sum_j^m y_j) = 0$$

$$\lambda = \frac{n+m}{\sum_i^n x_i + \sum_j^m y_j}$$

Thus, $\lambda_{MLE} = \frac{n+m}{\sum_i^n x_i + \sum_j^m y_j}$.

6.13

a

$$L(\alpha|X;\beta) = \alpha\beta x^{\beta-1}e^{-\alpha x^{\beta}}$$

$$ln(L(\alpha|X;\beta)) = ln\alpha + ln\beta + (\beta-1)lnx + ln(e^{-\alpha x^{\beta}})$$

$$ln(L(\alpha|X;\beta)) = ln\alpha + ln\beta + (\beta-1)lnx - \alpha x^{\beta}$$

$$\frac{\partial ln(L(\alpha|X;\beta))}{\partial\alpha} = \frac{1}{\alpha} - x^{\beta} = 0$$

$$\alpha = \frac{1}{x^{\beta}}$$

Thus $\alpha_{MLE} = \frac{1}{x^{\beta}}$.

b

$$L(\alpha;\beta|X) = \alpha\beta x^{\beta-1}e^{-\alpha x^{\beta}}$$

Take the natural log of both sides:

$$ln(L(\alpha;\beta|X)) = ln\alpha + ln\beta + (\beta-1)lnx - \alpha x^{\beta}$$

Differentiate with respect to α and β and set to 0. Solve simultaneously for a and b:

$$\frac{\partial ln(L(\alpha|X))}{\partial\alpha} = 1/\alpha - x^{\beta} = 0$$

and

$$\frac{\partial ln(L(\beta|X))}{\partial\beta} = 1/\beta - +lnx - ax^{\beta} = 0$$