

MATH 392 Problem Set 5

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6.39

$\{\hat{\beta}_n\}$ is consistent if and only if $\lim_{n \rightarrow \infty} P(|\hat{\beta}_n - \beta| < \varepsilon) = 1 \forall \varepsilon > 0$.

It has been shown before that $\forall n \in \mathbb{N}, E[\hat{\beta}_n] = \frac{n}{n+1}\beta$. Thus,

$$\begin{aligned} \lim_{n \rightarrow \infty} P(|\hat{\beta}_n - \beta| < \varepsilon) &= 1 \Leftrightarrow \lim_{n \rightarrow \infty} P(|E[\hat{\beta}_n] - \beta| < \varepsilon) = 1 \\ &\Leftrightarrow \lim_{n \rightarrow \infty} |E[\hat{\beta}_n] - \beta| < \varepsilon \\ &\Leftrightarrow \lim_{n \rightarrow \infty} \left| \frac{n}{n+1}\beta - \beta \right| < \varepsilon \\ &\Leftrightarrow \lim_{n \rightarrow \infty} \left| \left(\frac{n}{n+1} - 1 \right) \beta \right| < \varepsilon \end{aligned}$$

It suffices to show that $\lim_{n \rightarrow \infty} \left| \frac{n}{n+1} - 1 \right| < \varepsilon$:

Let $\varepsilon > 0$. Then $\exists m \in \mathbb{N}$ st $\forall n \geq m$,

$$\frac{1}{n} < \varepsilon \Rightarrow \frac{1}{n+1} < \varepsilon \Rightarrow \left| \frac{-1}{n+1} \right| < \varepsilon \Rightarrow \left| \frac{n-n-1}{n+1} \right| < \varepsilon \Rightarrow \left| \frac{n}{n+1} - \frac{n+1}{n+1} \right| < \varepsilon \Rightarrow \left| \frac{n}{n+1} - 1 \right| < \varepsilon$$

Thus $\lim_{n \rightarrow \infty} \left| \frac{n}{n+1} - 1 \right| < \varepsilon$, so $\lim_{n \rightarrow \infty} P(|\hat{\beta}_n - \beta| < \varepsilon) = 1$. Thus $\{\hat{\beta}_n\}$ is consistent.

6.40

$\{\hat{\sigma}_n^2\}$ is consistent if and only if $\lim_{n \rightarrow \infty} P(|\hat{\sigma}_n^2 - \sigma^2| < \varepsilon) = 1 \forall \varepsilon > 0$.

Again, it suffices to show that $\lim_{n \rightarrow \infty} P(|E[\hat{\sigma}_n^2] - \sigma^2| < \varepsilon) = 1$. From Problem Set 4,

$$E[\hat{\sigma}_n^2] - \sigma^2 = \frac{n-1}{n}\sigma^2 - \sigma^2 = \frac{-1}{n}\sigma^2$$

Let $\varepsilon > 0$. Then $\exists m \in \mathbb{N}$ st $\forall n \geq m, \frac{1}{n} < \varepsilon$. Thus, $\forall n \geq m$,

$$\left| \frac{1}{n} \right| < \varepsilon \Rightarrow \left| \frac{1}{n}\sigma^2 \right| < \varepsilon \Rightarrow \left| \left(\frac{n-1}{n} - \frac{n}{n} \right) \sigma^2 \right| < \varepsilon \Rightarrow \left| \frac{n-1}{n}\sigma^2 - \sigma^2 \right| < \varepsilon \Rightarrow |E[\hat{\sigma}_n^2] - \sigma^2| < \varepsilon.$$

Thus, $\lim_{n \rightarrow \infty} P(|\hat{\sigma}_n^2 - \sigma^2| < \varepsilon) = 1 \forall \varepsilon > 0$.

7.3

a

$$X \sim N(185, 50^2), n = 100, \bar{X} = 210$$

$$P(q_{.05} < \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} < q_{.95}) = 1 - \alpha = .9$$

Solve for μ :

$$P(\bar{X} - q_{.95} \frac{\sigma}{\sqrt{n}} < \mu < \bar{X} - q_{.05} \frac{\sigma}{\sqrt{n}}) = .9$$

$$(210 - 1.645 \frac{50}{\sqrt{100}} < \mu < 210 + 1.645 \frac{50}{\sqrt{100}}) = (205.2 < \mu < 209.8)$$

Thus the 90% confidence interval is (201.8, 218.8)

b

Solve for n:

$$\bar{X} - LB_{.05} = \bar{X} - (\bar{X} - 1.96(\frac{50}{\sqrt{n}})) \leq 10$$

$$1.96 \frac{50}{10} \leq \sqrt{n}$$

$$n \geq 96.04$$

A sample size of at least 97 is needed.

c

Like b,

$$2.576 \frac{50}{10} \leq \sqrt{n}$$

$$n \geq 15.9$$

A sample size of at least 166 is needed

7.8

For $n = 30$, the code generating the simulation is below, along with the proportion of times the confidence interval missed μ . Compared to the next three numbers for $n=60$, $n=100$, and $n=250$, the frequency of missing μ converges to the theoretical frequency of .05.

```
# Run simulation
low <- 0
high <- 0
n <- 30
q <- qt(.975, n-1)
N <- 10^5
for(i in 1:N){
  x <- rgamma(n, shape = 5, rate = 2)
  xbar <- mean(x)
  s <- sd(x)
```

```
L <- xbar - q*s/sqrt(n)
U <- xbar + q*s/sqrt(n)
if(U<5/2){
  low <- low + 1
}
if(5/2<L){
  high <- high + 1
}
}

(high+low)/N
```

```
## [1] 0.05396
```

```
## [1] 0.05194
```

```
## [1] 0.05169
```

```
## [1] 0.05009
```

7.34