MATH 392 Problem Set 3

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4.8

$$n = 20, \mu = 6, \sigma^2 = 10$$

$$P(\bar{X} \leq 4.6) = P(\frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \leq \frac{4.6 - \mu}{\sigma/\sqrt{n}})$$

$$P(\bar{X} \leq 4.6) = P(\frac{\bar{X} - 6}{\sqrt{10}/\sqrt{20}} \leq \frac{4.6 - 6}{\sqrt{10}/\sqrt{20}})$$

```
# Calculate Z score of interested statistic
z.obs <- (4.6-6)/(sqrt(10)/sqrt(20))
z.obs
```

```
## [1] -1.979899
```

$$P(\bar{X} \leq 4.6) = P(Z \leq -1.98)$$

```
# Calculate probability using the cdf of N(0,1)
pnorm(z.obs, 0, 1)
```

```
## [1] 0.02385744
```

$$P(\bar{X} \leq 4.6) = .02385$$

4.9

$$f_X(x) = \frac{3}{16}(x-4)^2 for 2 \leq 6$$

Find E[X]:

$$E[X] = \int_2^6 x\frac{3}{16}(x-4)^2 dx$$

$$E[X] = \int_2^6 \frac{3}{16}x(x^2 - 8x + 16)dx$$

$$E[X] = \int_2^6 \frac{3}{16}x^3 - \frac{3}{2}x^2 + 3x dx$$

$$E[X] = \frac{3}{64}x^4 - \frac{1}{2}x^3 + \frac{3}{2}x^2|_2^6$$

$$E[X] = 4$$

Find V[X]:

$$V[X] = E[X^2] - E[X]^2$$

We already calculated that $E[X] = 4$, so $E[X]^2 = 16$. Now solve for $E[X^2]$:

$$E[X^2] = \int_2^6 x^2 f(x)dx$$

$$E[X^2] = \int_2^6 x^2 \frac{3}{16}(x-4)^2 dx$$

$$E[X^2] = \int_2^6 \frac{3}{16}x^4 - \frac{3}{2}x^3 + 3x^2 dx$$

$$E[X^2] = \frac{3}{80}x^5 - \frac{3}{8}x^4 + x^3|_2^6$$

```
# Calculate
e.xsq <- (3*(6^5)/80 - 3*(6^4)/8 +6^3) -
  (3*(2^5)/80 - 3*(2^4)/8 +(2^3))
e.xsq
```

```
## [1] 18.4
```

```
sq.ex <- 4^2
var.x <- e.xsq-sq.ex
sd.x <- sqrt(var.x)
sd.x
```

```
## [1] 1.549193
```

Thus $V[X] = 2.4$, so SD[X] $= \sqrt{2.4} = 1.549$. Now, $n = 244, \mu = 4, \sigma^2 = 2.4$.

$$P(\bar{X} \geq 4.2) = P(\frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \geq \frac{4.2 - \mu}{\sigma/\sqrt{n}})$$

$$P(\bar{X} \geq 4.2) = P(\frac{\bar{X} - 4}{\sqrt{2.4}/\sqrt{244}} \geq \frac{4.2 - 4}{\sqrt{2.4}/\sqrt{244}})$$

$$P(\bar{X} \geq 4.2) = P(Z \geq \frac{4.2 - 4}{\sqrt{2.4}/\sqrt{244}})$$

```
# Calculate Z score of interested statistic
z.obs <- (4.2-4)/(sqrt(2.4)/sqrt(244))
z.obs
```

```
## [1] 2.016598
```

```
# Calculate probability using cdf from N(0,1)
1 - pnorm(z.obs,0,1)
```

```
## [1] 0.02186875
```

4.12

a

Let X be a random sample of size 30 from the exponential distribution with rate $\lambda = .1$. The expected value of the sample mean is the same as the expected value of the population, by linearity of expectation. Thus, E[X] $= \frac{1}{\lambda} = 10$

b

```
# Run simulation
nsim <- 1000
n <- 30
rate <- 1/10
means <- rep(NA,nsim)
for(i in 1:nsim){
  sample <- rexp(n,rate)
  means[i] <- mean(sample)
}
sum(means >= 12)/nsim
```

```
## [1] 0.121
```

c

Since 12.1% of the samples had means of 12 or greater, this observation is not that unusual.

4.13

a

Since X $\sim N(20, 8^2)$ and Y $N(16, 7^2)$ are independent variables, and W $= \bar{X} + \bar{Y}$, then W $\sim N(36, \frac{8^2}{10} + \frac{7^2}{15})$

b

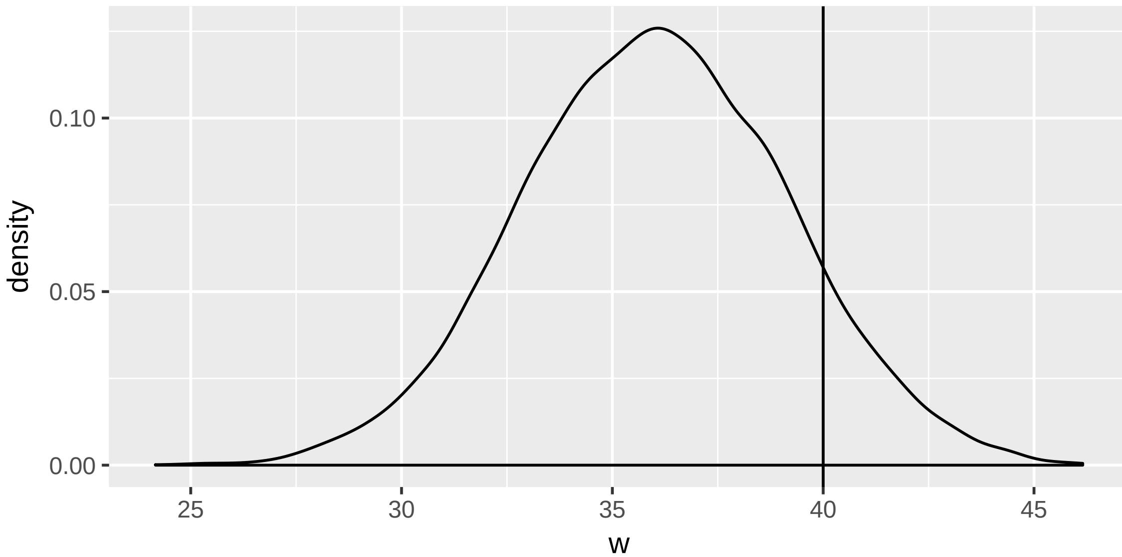
```
nsim <- 10000
w <- rep(NA,nsim)
for(i in 1:nsim){
  x <- rnorm(10,20,8)
  y <- rnorm(15,16,7)
  w[i] <- mean(x) + mean(y)
}
# Compute mean and standard error
mean(w)
```

```
## [1] 36.01261
```

```
sd(w)
```

```
## [1] 3.132671
```

```r
# Plot sampling distribution
w <- data.frame(w)
ggplot(w,aes(x=w)) +
  geom_density() +
  geom_vline(xintercept=40)
```



c

```r
(sum(w<40) + 1) /( nsim + 1)
```

```
## [1] 0.8975102
```

4.18

a

```r
# Simulate sampling distribution
nsim <- 10000
n <- 30
rate <- 1/3
x.bars <- rep(NA,nsim)
for(i in 1:nsim){
  sample <- rexp(n,rate)
  x.bars[i] <- mean(sample)
}
```

b

```r
# Compute and compare simulated mean and standard error with theoretical results
mean.sim <- mean(x.bars)
se.sim <- sd(x.bars)
mean.theory <- 1/rate
```

```
se.theory <- (1/rate)/sqrt(n)
mean.sim
```

```
## [1] 2.996693
```

```
mean.theory
```

```
## [1] 3
```

```
se.sim
```

```
## [1] 0.5416903
```

```
se.theory
```

```
## [1] 0.5477226
```

c

```
# Calculate simulated probability
d <- data.frame(x.bars)
(sum(x.bars <= 3.5) + 1) / (nsim + 1)
```

```
## [1] 0.8244176
```

d

$$n = 30, X\ exp(1/3), \mu = 1/3, \sigma^2 = 9$$

$$P(\bar{X} \leq 3.5) = P(\frac{\bar{X} - \mu}{\sqrt{\sigma^2}/\sqrt{n}} \leq \frac{3.5 - \mu}{\sqrt{\sigma^2}/\sqrt{n}}) = P(\frac{\bar{X} - 3}{\sqrt{9}/\sqrt{30}} \leq \frac{3.5 - 3}{\sqrt{9}/\sqrt{30}})$$

```
# Calculate z score we are testing
z.test <- (3.5-3)/(sqrt(9)/sqrt(30))
z.test
```

```
## [1] 0.9128709
```

$$P(\bar{X} \leq 3.5) = P(Z \leq .9129)$$

```
# Calculate approximated probability
pnorm(z.obs,0,1)
```

```
## [1] 0.9781312
```

$P(\bar{X} \leq 3.5) = .8193$. The approximated result is similar to the simulated probability of .8252.

4.20

$Let X_{1},...,X_{n} be continuouus and i.i.d. random variables with pdf f and cdf F. Show that the pdf's for X_{min} and X_{max} are

$$f_{min}(x) = n(1 - F(x))^{n-1} f(x)$$

$$f_{max}(x) = nF(x)^{n-1}(x)f(x)$$

First, show the pdf of X_{max}:

$$F_{max}(x) = P(maxX_1, ..., X_n \leq x)$$

$$F_{max}(x) = P(X_1 \leq x, ..., X_n \leq x)$$

Because the variables are i.i.d.,

$$F_{max}(x) = P(X_1 \leq x)...P(X_n \leq x)$$

$$F_{max}(x) = F(x)...F(x) = F(x)^n$$

Now, differentiate to find $f_{max}(x)$:

$$f_{max}(x) = \frac{\partial}{\partial x}F_{max}(x) = nF(x)^{n-1}f(x)$$

Now show the pdf$f_{min}(x)$:

$$F_{min}(x) = P(minX_1, ..., X_n \leq x)$$

At least one of the variables $X_i \leq x$, so we can use the probability that none of the random variables will be less than x, and subtract that from 1:

$$F_{min}(x) = (1 - P(X_1 \leq x))...(1 - P(X_n \leq x))$$

$$F_{min}(x) = (1 - P(X \leq x))^n = (1 - F(x))^n$$

Now differentiate to find $f_{min}(x)$:

$$f_{min}(x) = \frac{\partial}{\partial x}F_{min}(x) = n(1 - F(x))^{n-1}f(x)$$

4.21

a

By theorem 4.1, $f_{max}(x) = nF(x)^{n-1}f(x)$. In this case, n = 2, and $f(x) = 2/x^2 for 1 \leq x \leq 2$. So

$$f_{max}(x) = 2(2/x^2)\int_1^x 2/x^{-2}dx$$

$$f_{max}(x) = 2(2/x^2)(-2x^{-1}|_1^x)$$

$$f_{max}(x) = 2(2/x^2)(2(1-1/x))$$

$$f_{max}(x) = \frac{8-8/x}{x^2}$$

b

Solve for E[X]:

$$E[X] = \int_1^2 x f_{max}(x) dx$$

$$E[X] = \int_1^2 x \frac{8-8/x}{x^2} dx$$

$$E[X] = \int_1^2 8x^{-1} - 8x^{-2} dx$$

$$E[X] = 8lnx + 8x^{-1}|_1^2 = (8ln2 + 4) - (8ln1 + 8)$$

$$E[X] = (8ln2 + 4) - (8ln1 + 8) = 8ln2 + 4 - (0 + 8)$$

$$E[X] \approx 1.545.$$