MATH 392 Problem Set 4

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17 February 2018

6.2

Analytical Solution

Let $x_1, ..., x_n \sim$ Poisson(λ). Show the maximum likelihood estimate of λ is $\hat{\lambda} = \bar{x}$:

$$L(\lambda|x_1, ..., x_n) = P(X_1 = x_1, ..., X_n = x_n)$$

$$L(\lambda|x_1, ..., x_n) = P(X = x_1)...P(X = x_n)$$

$$L(\lambda|x_1, ..., x_n) = \prod_i^n \frac{\lambda^{x_i} e^{-\lambda}}{x_i!}$$

$$L(\lambda|x_1, ..., x_n) = \frac{e^{-\lambda n} \lambda^{\sum_i^n x_i}}{\prod_i^n x_i!}$$

Take the natural log of both sides to make the derivation easier:

$$ln(L(\lambda|x_1, ..., x_n)) = ln(e^{-n\lambda}) + ln(\lambda^{\sum_i^n x_i}) - ln(\prod_i^n x_i!)$$

$$ln(L(\lambda|x_1, ..., x_n)) = -n\lambda + ln(\lambda) \sum_i^n x_i - ln(\prod_i^n x_i!)$$

Derive with respect to λ and set equal to 0:

$$\frac{\partial ln(L(\lambda|x_1, ..., x_n))}{\partial \lambda} = -n + \frac{\sum_i^n x_i}{\lambda} = 0$$

$$\frac{\partial ln(L(\lambda|x_1, ..., x_n))}{\partial \lambda} = \lambda = \frac{\sum_i^n x_i}{n} = \bar{x}$$

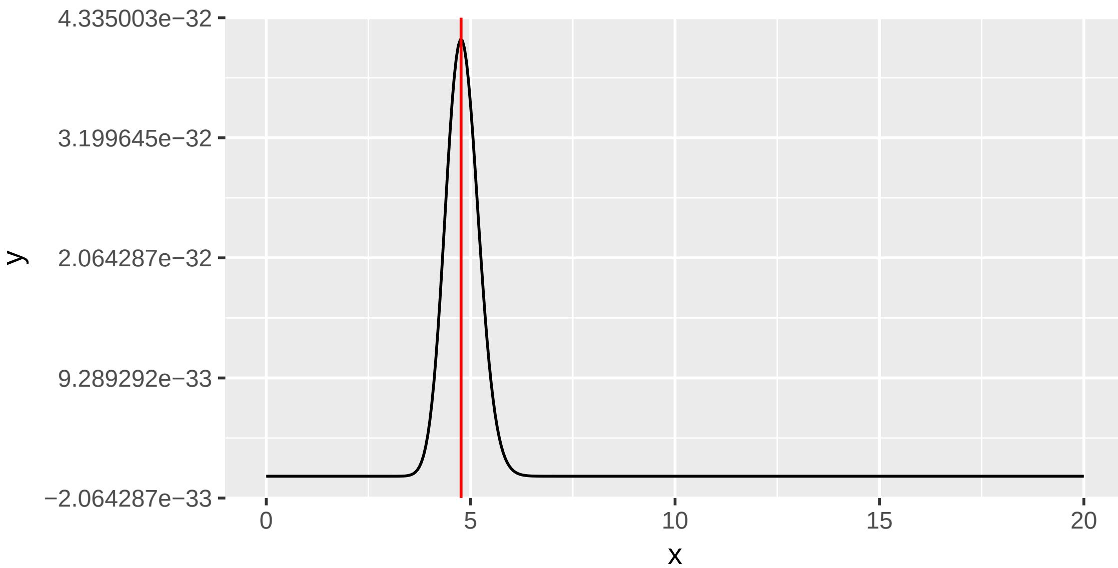
Thus $\lambda_{MLE} = \bar{x}$

Empirical Solution

```r
# Build function
e <- exp(1)
pois <- function(lambda, x){
  n <- length(x)
  sum.x <- sum(x)
  prod.fact <- prod(factorial(x))
  e^(-lambda*n)*(lambda^(sum.x))/(prod.fact)
}

# Create sequence of lambdas for MLE plot
lambdas <- seq(from = 0, to = 20, by = .05)
# Draw random sample of x's
x <- sample(1:10,30,replace=T)
# Apply pois function to find likelihoods
L_pois <- pois(lambdas,x)

# Plot
df <- data.frame(x = lambdas,
                 y = L_pois)
ggplot(df, aes(x=x,y=y)) +
  geom_line() +
  geom_vline(xintercept = mean(x), col = "red")
```



A red line above indicates the mean of the random sample. As expected, the highest likelihood for λ is at the sample mean.

6.8

Analytical Solution

$$f(x;\theta) = \frac{\sqrt{2/\pi}x^2 e^{-x^2/2\theta^2}}{\theta^3}$$

$$L(\theta|x_1, ..., x_n) = \prod_i^n f(x_i;\theta)$$

$$L(\theta|x_1, ..., x_n) = \prod_i^n \frac{\sqrt{2/\pi}x_i^2 e^{-x_i^2/2\theta^2}}{\theta^3}$$

$$L(\theta|x_1, ..., x_n) = (\frac{\sqrt{2/\pi}}{\theta^3})^n \prod_i^n x_i^2 e^{-x_i^2/2\theta^2}$$

$$ln(L(\theta|x_1, ..., x_n)) = ln(\frac{\sqrt{2/\pi}}{\theta^3})^n + \sum_i^n ln(x_i^2) + \sum_i^n ln(e^{-x_i^2/2\theta^2})$$

$$ln(L(\theta|x_1, ..., x_n)) = ln\sqrt{2/\pi} - ln\theta^3 + \sum_i^n ln(x_i^2) + \sum_i^n -x_i^2/2\theta^2$$

Derive with respect to θ and set to 0:

$$\frac{\partial(L(\theta|x_1, ..., x_n))}{\partial\theta} = -\frac{n}{\theta^3}3\theta^2 + \sum_i^n x_i^2/\theta^3 = 0$$

$$\frac{3n}{\theta} = \frac{\sum_i^n x_i^2}{\theta^3}$$

$$\theta = \frac{\bar{x}\sqrt{n}}{\sqrt{3}}$$

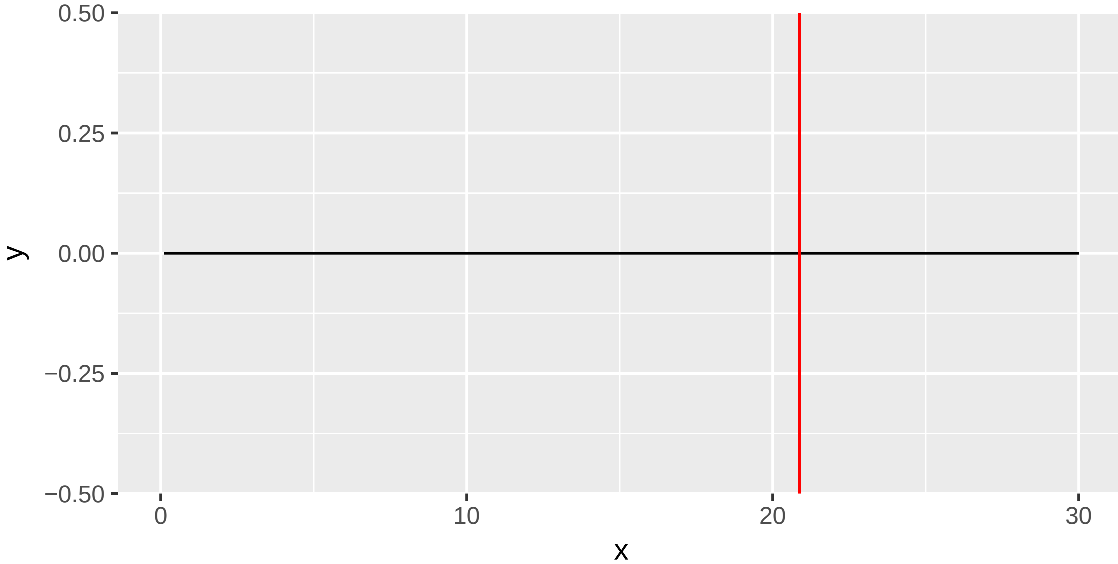
Thus $\theta_{MLE} = \frac{\bar{x}\sqrt{n}}{\sqrt{3}}$.

Empirical Solution

```r
# Build function
f <- function(theta, x){
  e <- exp(1)
  n <- length(x)
  sec.term <- (x^2)*e^((-x^2)/(2*(theta)^2))
  prod.x <- prod(sec.term)
  ((sqrt(2/pi)/theta)^n)*prod.x
}

# Create sequence of thetas for MLE plot
thetas <- seq(from = .1, to = 30, by = .1)
# Reset seed and draw random sample of x's
set.seed(23)
x <- sample(1:10,30,replace=T)
# Apply function to find likelihoods
L_theta <- f(thetas,x)
# Find x-intercept from analytical solution
n <- length(x)
```

```r
analytical.sol <- mean(x)*sqrt(n)/sqrt(3)
# Plot
df <- data.frame(x = thetas,
                 y = L_theta)
ggplot(df, aes(x=x,y=y)) +
  geom_line() +
  geom_vline(xintercept = analytical.sol, col = "red")
```



6.11

As before,

$$L(\lambda) = \prod_i^n f(x_i) \prod_j^m f(y_i)$$

$$L(\lambda) = \prod_i^n \lambda e^{-\lambda x_i} \prod_j^m \lambda e^{-\lambda y_j}$$

$$L(\lambda) = \lambda^{n+m} \lambda e^{-\lambda(\sum_i^n x_i + \sum_j^m y_j)}$$

$$ln(L(\lambda)) = (n+m) ln(\lambda) - \lambda(\sum_i^n x_i + \sum_j^m y_j)$$

$$\frac{\partial ln(L(\lambda))}{\partial \lambda} = (n+m)/\lambda - (\sum_i^n x_i + \sum_j^m y_j) = 0$$

$$\lambda = \frac{n+m}{\sum_i^n x_i + \sum_j^m y_j}$$

Thus, $\lambda_{MLE} = \frac{n+m}{\sum_i^n x_i + \sum_j^m y_j}$.

6.13

a

$$L(\alpha|X;\beta) = \alpha\beta x^{\beta-1}e^{-\alpha x^{\beta}}$$

$$ln(L(\alpha|X;\beta)) = ln\alpha + ln\beta + (\beta-1)lnx + ln(e^{-\alpha x^{\beta}})$$

$$ln(L(\alpha|X;\beta)) = ln\alpha + ln\beta + (\beta-1)lnx - \alpha x^{\beta}$$

$$\frac{\partial ln(L(\alpha|X;\beta))}{\partial\alpha} = \frac{1}{\alpha} - x^{\beta} = 0$$

$$\alpha = \frac{1}{x^{\beta}}$$

Thus $\alpha_{MLE} = \frac{1}{x^{\beta}}$.

b

$$L(\alpha;\beta|X) = \alpha\beta x^{\beta-1}e^{-\alpha x^{\beta}}$$

Take the natural log of both sides:

$$ln(L(\alpha;\beta|X)) = ln\alpha + ln\beta + (\beta-1)lnx - \alpha x^{\beta}$$

Differentiate with respect to α and β and set to 0. Solve simultaneously for a and b:

$$\frac{\partial ln(L(\alpha|X))}{\partial\alpha} = 1/\alpha - x^{\beta} = 0$$

and

$$\frac{\partial ln(L(\beta|X))}{\partial\beta} = 1/\beta - +lnx - ax^{\beta} = 0$$