MATH 392 Problem Set 4

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6.2

Analytical Solution

Let $x_1, ..., x_n \sim$ Poisson(λ). Show the maximum likelihood estimate of λ is $\hat{\lambda} = \bar{x}$:

$$L(\lambda|x_1, ..., x_n) = P(X_1 = x_1, ..., X_n = x_n)$$

$$L(\lambda|x_1, ..., x_n) = P(X = x_1)...P(X = x_n)$$

$$L(\lambda|x_1, ..., x_n) = \prod_i^n \frac{\lambda^{x_i} e^{-\lambda}}{x_i!}$$

$$L(\lambda|x_1, ..., x_n) = \frac{e^{-\lambda n} \lambda^{\sum_i^n x_i}}{\prod_i^n x_i!}$$

Take the natural log of both sides to make the derivation easier:

$$ln(L(\lambda|x_1, ..., x_n)) = ln(e^{-n\lambda}) + ln(\lambda^{\sum_i^n x_i}) - ln(\prod_i^n x_i!)$$

$$ln(L(\lambda|x_1, ..., x_n)) = -n\lambda + ln(\lambda) \sum_i^n x_i - ln(\prod_i^n x_i!)$$

Derive with respect to λ and set equal to 0:

$$\frac{\partial ln(L(\lambda|x_1, ..., x_n))}{\partial \lambda} = -n + \frac{\sum_i^n x_i}{\lambda} = 0$$

$$\frac{\partial ln(L(\lambda|x_1, ..., x_n))}{\partial \lambda} = \lambda = \frac{\sum_i^n x_i}{n} = \bar{x}$$

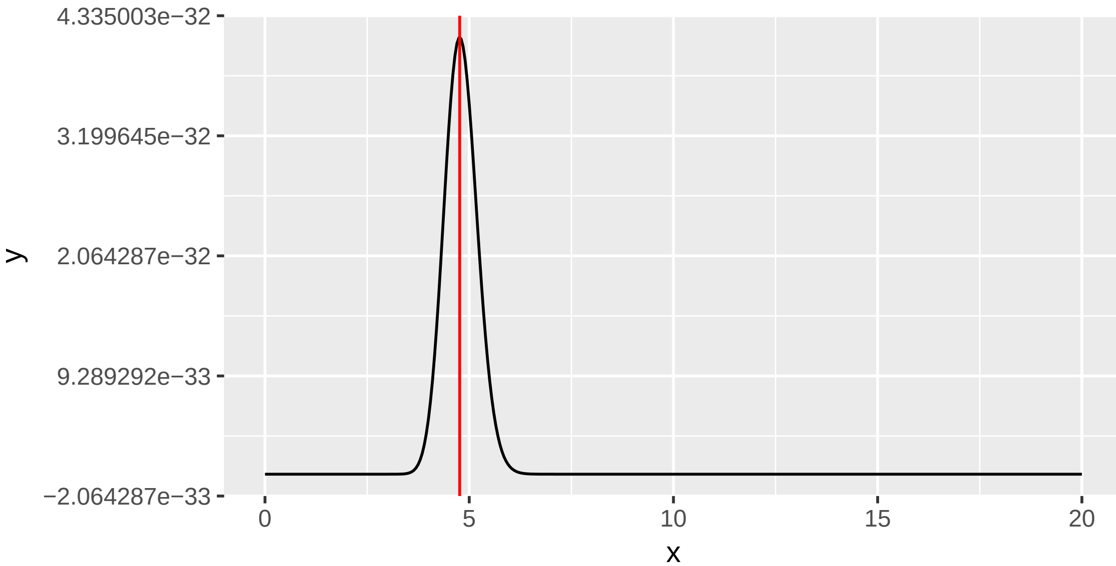
Thus $\lambda_{MLE} = \bar{x}$

Empirical Solution

```r
# Build function
e <- exp(1)
pois <- function(lambda, x){
  n <- length(x)
  sum.x <- sum(x)
  prod.fact <- prod(factorial(x))
  e^(-lambda*n)*(lambda^(sum.x))/(prod.fact)
}

# Create sequence of lambdas for MLE plot
lambdas <- seq(from = 0, to = 20, by = .05)
# Draw random sample of x's
x <- sample(1:10,30,replace=T)
# Apply pois function to find likelihoods
L_pois <- pois(lambdas,x)

# Plot
df <- data.frame(x = lambdas,
                 y = L_pois)
ggplot(df, aes(x=x,y=y)) +
  geom_line() +
  geom_vline(xintercept = mean(x), col = "red")
```



A red line above indicates the mean of the random sample. As expected, the highest likelihood for λ is at the sample mean.

6.8

Analytical Solution

$$f(x;\theta) = \frac{\sqrt{2/\pi}x^2e^{-x^2/2\theta^2}}{\theta^3}$$

$$L(\theta|x_1,...,x_n) = \prod_i^n f(x_i;\theta)$$

$$L(\theta|x_1,...,x_n) = \prod_i^n \frac{\sqrt{2/\pi}x_i^2 e^{-x_i^2/2\theta^2}}{\theta^3}$$

$$L(\theta|x_1,...,x_n) = (\frac{\sqrt{2/\pi}}{\theta^3})^n \prod_i^n x_i^2 e^{-x_i^2/2\theta^2}$$

$$ln(L(\theta|x_1,...,x_n)) = ln(\frac{\sqrt{2/\pi}}{\theta^3})^n + \sum_i^n ln(x_i^2) + \sum_i^n ln(e^{-x_i^2/2\theta^2})$$

$$ln(L(\theta|x_1,...,x_n)) = ln\sqrt{2/\pi} - ln\theta^3 + \sum_i^n ln(x_i^2) + \sum_i^n -x_i^2/2\theta^2$$

Derive with respect to θ and set to 0:

$$\frac{\partial(L(\theta|x_1,...,x_n))}{\partial\theta} = -\frac{n}{\theta^3}3\theta^2 + \sum_i^n x_i^2/\theta^3 = 0$$

$$\frac{3n}{\theta} = \frac{\sum_i^n x_i^2}{\theta^3}$$

$$\theta = \frac{\bar{x}\sqrt{n}}{\sqrt{3}}$$

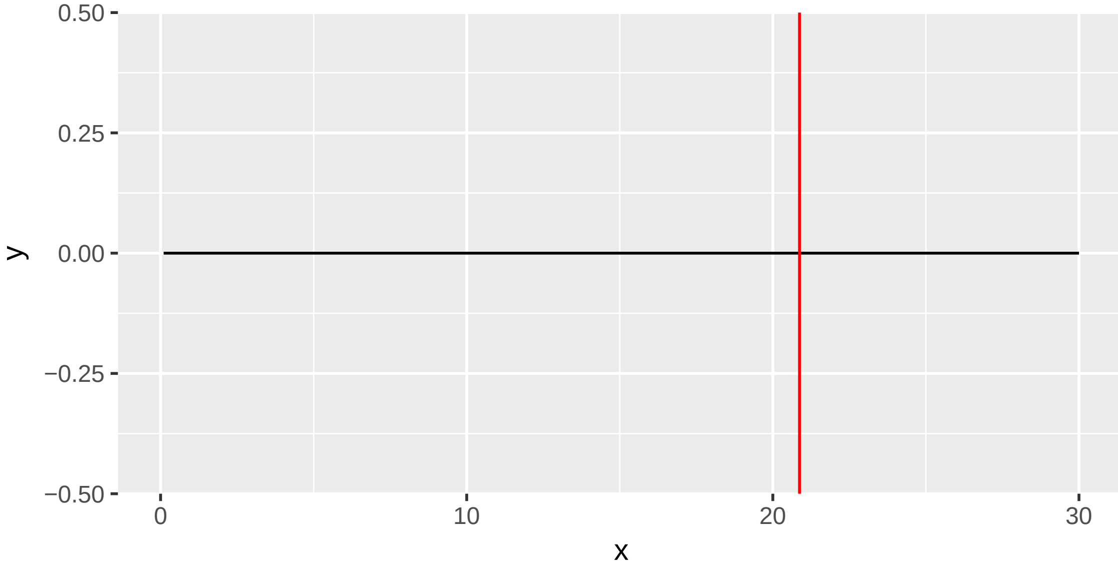
Thus $\theta_{MLE} = \frac{\bar{x}\sqrt{n}}{\sqrt{3}}$.

Empirical Solution

```
# Build function
f <- function(theta, x){
  e <- exp(1)
  n <- length(x)
  sec.term <- (x^2)*e^((-x^2)/(2*(theta)^2))
  prod.x <- prod(sec.term)
  ((sqrt(2/pi)/theta)^n)*prod.x
}

# Create sequence of thetas for MLE plot
thetas <- seq(from = .1, to = 30, by = .1)
# Reset seed and draw random sample of x's
set.seed(23)
x <- sample(1:10,30,replace=T)
# Apply function to find likelihoods
L_theta <- f(thetas,x)
# Find x-intercept from analytical solution
n <- length(x)
```

```r
analytical.sol <- mean(x)*sqrt(n)/sqrt(3)
# Plot
df <- data.frame(x = thetas,
                 y = L_theta)
ggplot(df, aes(x=x,y=y)) +
  geom_line() +
  geom_vline(xintercept = analytical.sol, col = "red")
```



6.11

Analytical Solution

As before,

$$L(\lambda) = \prod_i^n f(x_i) \prod_j^m f(y_j)$$

$$L(\lambda) = \prod_i^n \lambda e^{-\lambda x_i} \prod_j^m \lambda e^{-\lambda y_j}$$

$$L(\lambda) = \lambda^{n+m} \lambda e^{-\lambda(\sum_i^n x_i + \sum_j^m y_j)}$$

$$ln(L(\lambda)) = (n+m) ln(\lambda) - \lambda(\sum_i^n x_i + \sum_j^m y_j)$$

$$\frac{\partial ln(L(\lambda))}{\partial \lambda} = (n+m)/\lambda - (\sum_i^n x_i + \sum_j^m y_j) = 0$$

$$\lambda = \frac{n+m}{\sum_i^n x_i + \sum_j^m y_j}$$

Thus, $\lambda_{MLE} = \frac{n+m}{\sum_i^n x_i + \sum_j^m y_j}$.

Empirical Solution

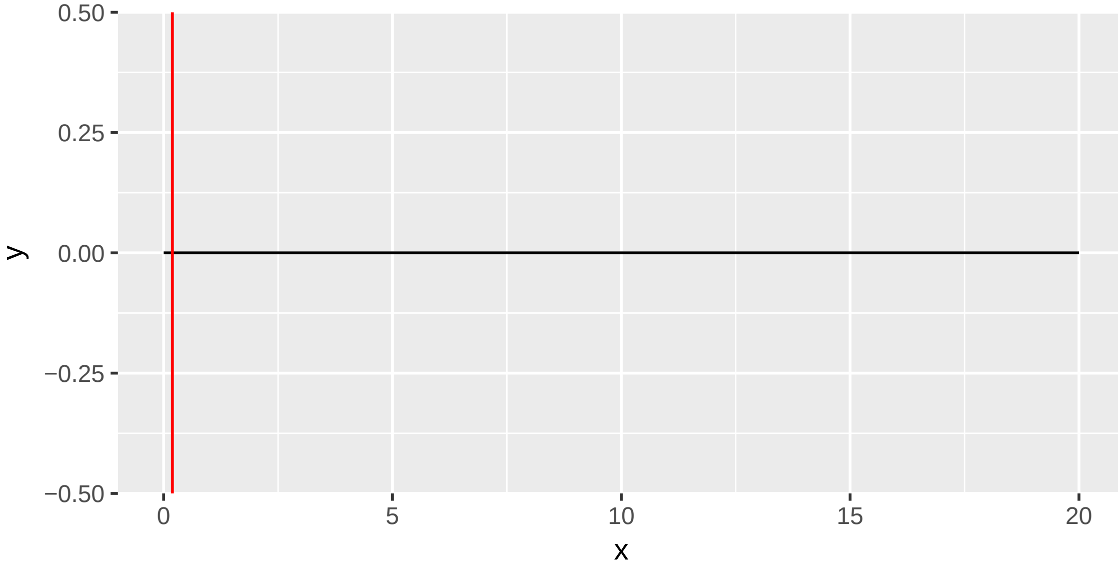
```
# Build function
e <- exp(1)
pois <- function(lambda,x,y){
  n <- length(x)
  m <- length(y)
  p.x <- (lambda^n)*prod((e^(-lambda*x)))
  p.y <- (lambda^m)*prod((e^(-lambda*y)))
  p.x*p.y
}

# Create sequence of lambdas for MLE plot
lambdas <- seq(from = 0, to = 20, by = .05)
# Draw random sample of x's
set.seed(11)
x <- sample(1:10,30,replace=T)
set.seed(35)
y <- sample(1:10,40,replace=T)
# Apply pois function to find likelihoods
L_pois <- pois(lambdas,x,y)
```

```
## Warning in -lambda * x: longer object length is not a multiple of shorter
## object length
```

```
## Warning in -lambda * y: longer object length is not a multiple of shorter
## object length
```

```
# Calculate analytical solution
obs <- (length(x)+length(y))/(sum(x) + sum(y))
# Plot
df <- data.frame(x = lambdas,
                 y = L_pois)
ggplot(df, aes(x=x,y=y)) +
  geom_line() +
  geom_vline(xintercept = obs, col = "red")
```



6.13

a

$$L(\alpha|X;\beta) = \alpha\beta x^{\beta-1}e^{-\alpha x^{\beta}}$$

$$ln(L(\alpha|X;\beta)) = ln\alpha + ln\beta + (\beta-1)lnx + ln(e^{-\alpha x^{\beta}})$$

$$ln(L(\alpha|X;\beta)) = ln\alpha + ln\beta + (\beta-1)lnx - \alpha x^{\beta}$$

$$\frac{\partial ln(L(\alpha|X;\beta))}{\partial\alpha} = \frac{1}{\alpha} - x^{\beta} = 0$$

$$\alpha = \frac{1}{x^{\beta}}$$

Thus $\alpha_{MLE} = \frac{1}{x^{\beta}}$.

b

$$L(\alpha;\beta|X) = \alpha\beta x^{\beta-1}e^{-\alpha x^{\beta}}$$

Take the natural log of both sides:

$$ln(L(\alpha;\beta|X)) = ln\alpha + ln\beta + (\beta-1)lnx - \alpha x^{\beta}$$

Differentiate with respect to α and β and set to 0. Solve simultaneously for a and b:

$$\frac{\partial ln(L(\alpha|X))}{\partial\alpha} = 1/\alpha - x^{\beta} = 0$$

and

$$\frac{\partial ln(L(\beta|X))}{\partial\beta} = 1/\beta - +lnx - ax^{\beta} = 0$$

6.14

Let X = 2,3,5,9,10 ~ Unif$[\alpha, \beta]$. Find $\hat{\alpha}_{MOM}$ and $\hat{\beta}_{MOM}$:

1. Write first and second moments in terms of parameters α and β

$$\mu_1 = E[X] = \int_{\alpha}^{\beta} \frac{1}{\beta-\alpha} x dx = \frac{1}{2(\beta-\alpha)} x^2|_{\alpha}^{\beta} = \frac{\alpha+\beta}{2} = 5.8 = \bar{X}$$

Thus

$$\alpha + \beta = 11.6$$

Now consider the second moment:

$$\mu_2 = E[X^2] = \int_{\alpha}^{\beta} \frac{1}{\beta-\alpha} x^2 dx =$$

Solving the integral and setting μ_2 equal to its theoretical value gives

$$\mu_2 = \frac{\alpha^2 + \alpha\beta + \beta^2}{3} = \frac{\sum_i^5 X_i^2}{5} = 43.8$$

$$\mu_2 = 131.4 = \alpha^2 + \alpha\beta + \beta^2$$

Since $\alpha + \beta = 11.6$, solve for β and plug into the equation for μ_2:

$$\beta = 11.6 - \alpha$$

$$131.4 = \alpha^2 + \alpha(11.6 - \alpha) + (11.6 - \alpha)^2$$

$$0 = 11.6\alpha + 134.6 - 23.2\alpha + \alpha^2 - 131.4$$

$$0 = \alpha^2 - 11.6\alpha + 4.2$$

Using the quadratic formula, solving for α gives

$$\alpha = .27 or 11.32$$

Given the dataset, $\alpha = .27$ makes more sense, since $X_{max} = 10$. Now plug α in to solve for β:

$$.27 + \beta = 11.6 \beta = 11.33$$

Thus our method of moment estimates are $\hat{\alpha}_{MOM} = .27$, $\hat{\beta}_{MOM} = 11.33$.

6.20

a

$$L(\theta) = \prod_{i=1}^{5} f(x_i|\theta) = \prod_{i=1}^{5} \theta x_i^{\theta-1}$$

$$L(\theta) = \theta(\prod_{i=1}^{5} x_i)^{\theta-1}$$

Differentiate, set equal to 0, and solve for θ:

$$\frac{\partial L(\theta)}{\partial \theta} = 5\theta^4(\prod_{i=1}^{5} x_i)^{\theta-1} + \theta^5(\prod_{i=1}^{5} x_i)^{\theta-1} ln(\prod_{i=1}^{5} x_i) = 0$$

$$\theta^4(\prod_{i=1}^{5} x_i)^{\theta-1}(5 + \theta ln(\prod_{i=1}^{5} x_i)) = 0$$

$$\theta = \frac{-5}{ln(\prod_{i=1}^{5} x_i)} \approx 1.57$$

Thus $\theta_{MLE} \approx 1.57$.

b

$$\mu_1 = E[X] = \int_0^1 \theta x^{\theta-1} x dx = \int_0^1 \theta x^{\theta} dx$$

$$\mu_1 = \frac{\theta}{\theta+1} x^{\theta+1}|_0^1 = \frac{\theta}{\theta+1}$$

$$\hat{\mu}_1 = \frac{\theta}{\theta+1} = \bar{X}$$

Rearranging gives:

$$\theta = \frac{\bar{X}}{1-\bar{X}} = \frac{.594}{1-.594} = 1.463$$

Thus $\theta_{MOM} \approx 1.463$.

6.25

6.27