MATH 392 Problem Set 5

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6.39

$\{\hat{\beta}_n\}$ is consistent if and only if $lim_{n\to\infty}P(|\hat{\beta}_n - \beta| < \varepsilon) = 1 \; \forall \; \varepsilon > 0$.

It has been shown before that $\forall$ n ϵ N, $E[\hat{\beta}_n] = \frac{n}{n+1}\beta$. Thus,

$$\begin{aligned} lim_{n\to\infty}P(|\hat{\beta}_n - \beta| < \varepsilon) = 1 &\Leftrightarrow lim_{n\to\infty}P(|E[\hat{\beta}_n] - \beta| < \varepsilon) = 1 \\ &\Leftrightarrow lim_{n\to\infty}|E[\hat{\beta}_n] - \beta| < \varepsilon \\ &\Leftrightarrow lim_{n\to\infty}|\frac{n}{n+1}\beta - \beta| < \varepsilon \\ &\Leftrightarrow lim_{n\to\infty}|(\frac{n}{n+1} - 1)\beta| < \varepsilon \end{aligned}$$

It suffices to show that $lim_{n\to\infty}|\frac{n}{n+1} - 1| < \varepsilon$:

Let $\varepsilon > 0$. Then $\exists m \epsilon \mathbb{N}$ st $\forall n \geq m$,

$$\frac{1}{n} < \varepsilon \Rightarrow \frac{1}{n+1} < \varepsilon \Rightarrow |\frac{-1}{n+1}| < \varepsilon \Rightarrow |\frac{n-n-1}{n+1}| < \varepsilon \Rightarrow |\frac{n}{n+1} - \frac{n+1}{n+1}| < \varepsilon \Rightarrow |\frac{n}{n+1} - 1| < \varepsilon$$

Thus $lim_{n\to\infty}|\frac{n}{n+1} - 1| < \varepsilon$, so $lim_{n\to\infty}P(|\hat{\beta}_n - \beta| < \varepsilon) = 1$. Thus $\{\hat{\beta}_n\}$ is consistent.

6.40

$\{\hat{\sigma}^2_n\}$ is consistent if and only if $lim_{n\to\infty}P(|\hat{\sigma}^2_n - \sigma^2| < \varepsilon) = 1 \; \forall \; \varepsilon > 0$.

From Problem Set 4,

$$E[\hat{\sigma}^2_n] - \sigma^2 = \frac{n-1}{n}\sigma^2 - \sigma^2 = \frac{-1}{n}\sigma^2$$

It suffices to show that $lim_{n\to\infty}|\frac{n-1}{n} - 1| < \varepsilon \; \forall \varepsilon > 0$.

Let $\varepsilon > 0$. Then $\exists m \epsilon \mathbb{N}$ st $\forall n \geq m$, , $\frac{1}{n} < \varepsilon$. Thus, $\forall n \geq m$,

$$\left|\frac{1}{n}\right| < \varepsilon \Rightarrow \left|\frac{1}{n}\right| < \varepsilon \Rightarrow \left|(\frac{n-1}{n} - \frac{n}{n})\right| < \varepsilon \Rightarrow \left|\frac{n-1}{n} - 1\right| < \varepsilon \Rightarrow \left|E[\hat{\sigma}^2_n] - \sigma^2\right| < \varepsilon.$$

Thus, $lim_{n\to\infty}P(|\hat{\sigma}^2_n - \sigma^2| < \varepsilon) = 1 \forall \varepsilon > 0$.

7.3

a

$$X \sim N(185, 50^2), n = 100, \bar{X} = 210$$

$$P(q_{.05} < \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} < q_{.95}) = 1 - \alpha = .9$$

Solve for μ:

$$P(\ \bar{X} - q_{.95}\frac{\sigma}{\sqrt{n}} < \mu < \bar{X} - q_{.05}\frac{\sigma}{\sqrt{n}})\ = .9$$

$$(210 - 1.645\frac{50}{\sqrt{100}} < \mu < 210 + .1.645\frac{50}{\sqrt{100}}) = (201.8 < \mu < 218.8)$$

Thus the 90% confidence interval is (201.8, 218.8).

b

Solve for n:

$$\bar{X} - LB_{.05} = \bar{X} - (\bar{X} - 1.96(\frac{50}{\sqrt{n}})) \leq 10$$

$$1.96\frac{50}{10} \leq \sqrt{n}$$

$$n \geq 96.04$$

A sample size of at least 97 is needed.

c

Like b,

$$2.576\frac{50}{10} \leq \sqrt{n}$$

$$n \geq 165.9$$

A sample size of at least 166 is needed.

7.8

For n = 30, the code generating the simulation is below, along with the proportion of times the confidence interval missed μ. Compared to the next three numbers for n=60, n=100, and n=250, the frequency of missing μ converges to the thoretical frequency of .05.

```r
# Run simulation
low <- 0
high <- 0
n <- 30
q <- qt(.975, n-1)
N <- 10^5
```

```
for(i in 1:N){
  x <- rgamma(n,shape = 5, rate = 2)
  xbar <- mean(x)
  s <- sd(x)
  L <- xbar - q*s/sqrt(n)
  U <- xbar + q*s/sqrt(n)
  if(U<5/2){
    low <- low + 1
  }
  if(5/2<L){
    high <- high + 1
  }
}

(high+low)/N
```

```
## [1] 0.05396
## [1] 0.05194
## [1] 0.05169
## [1] 0.05009
```

7.34

We are given

$$X \sim Gamma(2, \lambda), f_X(x) = \frac{xe^{-x/\lambda}}{\Gamma(2)\lambda^2}$$

$$2\lambda x \sim \chi^2_{df=4}, f_{2\lambda X}(2\lambda x) = \frac{(2\lambda x)^{4/2-1}e^{(-2\lambda x)^2/2}}{2^{4/2}\Gamma(4/2)}$$

Thus, the quantiles will be calculated in terms of $2\lambda x$, using the $\chi^2_{df=4}$ distribution.

$$\begin{aligned} .95 = P(q_{1,x} < \lambda < q_{2,x}) &= P(q_{1,2\lambda x} < 2\lambda X < q_{2,2\lambda x}) \\ &= P(\frac{q_{1,2\lambda x}}{2X} < \lambda < \frac{q_{2,2\lambda x}}{2X}) \\ &= P(\frac{.2422}{X} < \lambda < \frac{5.572}{X}) \end{aligned}$$

Thus the 95% confidence interval for λ is $(\frac{.2422}{X}, \frac{5.572}{X})$.