# NOTES ON THE NUMERICAL SOLUTION OF DIFFERENTIAL EQUATIONS

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## 1 Some Basics

Suppose one wishes to solve, by a numerical method, the simple first-order ordinary differential equation

$$\frac{dY}{dt} = -\lambda\, Y \tag{1}$$

with the initial condition $Y(0) = Y_0$. Equation (1) is the well known equation governing the decay of a radioactive isotope where $\lambda$ is the decay constant. The analytic solution for this equation happens to be $Y(t) = Y_0 \exp(-\lambda t)$ but our aim is to obtain this result numerically rather than by analytical methods. We start by assuming that the time variable $t$ varies discretely, rather than continuously, so that $t = [t_1, t_2, \ldots, t_{i-1}, t_i, t_{i+1}, \ldots, t_{N-1}, t_N]$ where the sample interval is constant at $\Delta t$, e.g. $t_{i+1} - t_i = \Delta t$. The values of $Y(t)$ at times $t_i$ are denoted $Y_i = Y(t_i)$. Referring back to Equation (1), at a particular time $t_i$ the equation could be written

$$\left(\frac{dY}{dt}\right)_i = -\lambda\, Y_i \tag{2}$$

where $(dY/dt)_i$ denotes the time derivative $dY/dt$ evaluated at time $t_i$. From the definition of a derivative we might write this time-derivative as

$$\left(\frac{dY}{dt}\right)_i \approx \frac{Y_{i+1} - Y_i}{\Delta t}, \tag{3}$$

which allows (2) to be approximated by the first-order finite-difference equation

$$\frac{Y_{i+1} - Y_i}{\Delta t} = -\lambda\, Y_i. \tag{4}$$

This procedure leads immediately to the recursion relation

$$Y_{i+1} = Y_i - \lambda \Delta t\, Y_i. \tag{5}$$

(Note that this exercise is equivelant to approximating the derivative with a truncated Taylor series: $y(t + \Delta t) = y(t) + \Delta t y\prime(t) + ((\Delta t)^2/2) y\prime\prime(t) + \ldots \approx y(t) + \Delta t y\prime(t)$) To start the integration, use the initial condition $Y_1 = Y_0$ and march the solution forward in time, by repeated application of the recursion relation (5):

$$\begin{array}{llll} Y_1 & = Y_0 & & (6a) \\ Y_2 & = Y_1 - \lambda \Delta t Y_1 & & (6b) \\ Y_3 & = Y_2 - \lambda \Delta t Y_2 & & (6c) \\ \ldots & = \ldots & & \end{array}$$

A MATLAB script to perform this "step-by-step" integration and plot the result can be downloaded from the course website. It is called "ODEexample1main.m".

A more interesting application of similar thinking is to consider the case of two coupled ordinary equations, for example,

$$\begin{aligned} \frac{dY_1}{dt} &= \alpha Y_2 \qquad (7a) \\ \frac{dY_2}{dt} &= -\alpha Y_1. \quad (7b) \end{aligned}$$

A distinctive feature of the above system is that the solution of (7a) depends on that of (7b) and *vice versa*. Following the steps that lead to Equation (5), I approximate the time derivatives $dY_1/dt$ and $dY_2/dt$ by finite difference approximations and arrive at the recursion relations

$$\begin{aligned} Y_{i+1}^{[1]} &= Y_i^{[1]} + \alpha \Delta t Y_i^{[2]} \qquad (8a) \\ Y_{i+1}^{[2]} &= Y_i^{[2]} - \alpha \Delta t Y_i^{[1]}. \quad (8b) \end{aligned}$$

Assuming initial conditions $Y_1(0) = Y_0^{[1]}$ and $Y_2(0) = Y_0^{[2]}$, one can march the solution forward as for the previous example. A MATLAB script to perform this integration and plot the result can be downloaded from the course website. It is called “ODEexample2main.m”.

## 2 Runge-Kutta Integration

The foregoing examples are chosen for their simplicity and, in fact, are not representative of standard approaches to the numerical solution of initial value problems. The problem with the simple approach is that it is not necessarily accurate—one can confirm this by experimenting with the value of $\Delta t$ in solving Example 2 (see also homework 0). In terms of a Taylor series the method is accurate only to order $\Delta t$ and thus is appropriate only where the time steps are sufficiently small that the crude approximation in Equation 3 holds.

This raises the important issues of the *quality* and *suitability* of the integrator. The recursion schemes of Examples 1 and 2 illustrate the use of a low-quality integrator that is rarely suitable for the accurate solution of differential systems. One way to improve the accuracy of the first order finite difference equation in Equation 3 is to combine higher order Taylor series approximations that lead, for example, to ”forward” ($\Delta t > 0$) and ”backward” ($\Delta t < 0$) difference approximations to the derivative to obtain a centered difference approximation that is accurate to order $\Delta t^2$ rather than $\Delta t$:

$$\left(\frac{dY}{dt}\right)_i \approx \frac{Y_{i+1} - Y_{i-1}}{2\Delta t}.$$

Happily, the development of quality integrators is rarely a concern for the end-user in most programming languages. MATLAB, for example, offers a broad range of integrators of varying quality. One might wonder what the value of low- and intermediate-quality integrators might be: A potential disadvantage of high-quality integrators is that they execute slowly—sometimes prohibitively slowly if the problem is large. So, there is always a tradeoff between accuracy and computational cost.

Improving the accuracy of an integrator in a practical sense is not as simple as adding more terms to a Taylor expansion (as the order of the derivative increases the time steps become vanishingly small such that it takes forever to get a solution). A popular method of great practical importance and that is fourth order $\Delta t)^4$ accurate (i.e., the same as a 5 term Taylor expansion) is the classical Runge-Kutta formula in the interval $t_n \leq t \leq t_{n+1}$:

$$y_{(n+1)} = y_n + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

where

$$\begin{aligned} k_1 &= f(t_n, y_n) \\ k_2 &= f(t_n + 0.5\Delta t, y_n + 0.5\Delta t k_1) \\ k_3 &= f(t_n + 0.5\Delta t, y_n + 0.5\Delta t k_2) \\ k_4 &= f(t_n + \Delta t, y_n + \Delta t k_3). \end{aligned}$$

Here, the sum $\frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4)$ is essentially an average slope (just a better description of the derivative, really). That is, $k_1$ is the slope in the left-hand end of the interval, $k_2$ and $k_3$ are the slopes at the midpoint found with Equation 3 and $k_4$ is the slope at $t_n + \Delta t$ using Equation 3 and the slope $k_3$ to go from $t_n$ to $t_n + \Delta t$.

In MATLAB the choice and implementation of an integrator is the same regardless of the quality of the integrator. In this class, we will mostly rely on Runge-Kutta integration and use the built-in functions **ode45** and (see below) **ode15s** (download and run “odeRKexamplemain” from the course website). The general Runge-Kutta problem is to integrate a coupled system of ODEs such as

$$\begin{aligned} \frac{dY_1}{dt} &= F_1(t, Y_1, Y_2, \ldots, Y_{M-1}, Y_M, F_2, F_3, \ldots, F_M) \\ \frac{dY_2}{dt} &= F_2(t, Y_1, Y_2, \ldots, Y_{M-1}, Y_M, F_1, F_3, \ldots, F_M) \\ \frac{dY_3}{dt} &= F_3(t, Y_1, Y_2, \ldots, Y_{M-1}, Y_M, F_1, F_2, \ldots, F_M) \quad (9) \\ \ldots &= \ldots. \end{aligned}$$

Note in (9) that the form of the left-hand side terms is completely predictable and problem-specific details are all on the right-hand side. In setting up to apply the Runge-Kutta method, all that is required is to specify the exact form of the right-hand side terms. Thus, for the previously-analyzed coupled system,

$$\begin{aligned} \frac{dY_1}{dt} &= \alpha Y_2 \quad (10a) \\ \frac{dY_2}{dt} &= -\alpha Y_1 \quad (10b) \end{aligned}$$

and it is only necessary to define the function

$$\begin{aligned} F_1 &= \alpha Y_2 \quad (11a) \\ F_2 &= -\alpha Y_1 \quad (11b) \end{aligned}$$

and pass this information to a standard Runge-Kutta solver.
A MATLAB script to perform this integration and plot the result can be downloaded from the course website. It is called `ODEexample2main.m`. For this example the standard MATLAB solver `ode45` has been employed. On the website, the function `funcrk4.m` is an explicit definition of the fourth-order Runge-Kutta formula. Solve the equations in `ODEexample2main.m` using this function instead of `ode45` and compare your results for accuracy and speed.

### 2.1 *Solution of higher-order differential equations*

The Runge-Kutta approach is far more powerful than might first be imagined. It is not entirely obvious that the method leads immediately to an approach to solving higher-order differential equations. As an example, consider the nonlinear fourth-order differential equation

$$Y\,\frac{d^4Y}{dt} + \left(\frac{d^2Y}{dt^2}\right)^2 + \exp(-Y) = G(t). \tag{12}$$

It is readily confirmed that, taking $Y_1(t) = Y(t)$, the following system of first-order differential equations is equivalent to (12)

$$\begin{aligned}
\frac{dY_1}{dt} &= \tfrac{dY}{dt} = Y_2 && (13a)\\
\frac{dY_2}{dt} &= \tfrac{d^2Y}{dt^2} = Y_3 && (13b)\\
\frac{dY_3}{dt} &= \tfrac{d^3Y}{dt^3} = Y_4 && (13c)\\
\frac{dY_4}{dt} &= \tfrac{d^4Y}{dt^4} = \tfrac{1}{Y_1}\left(G(t) - Y_3^2 - \exp(-Y_1)\right). && (13d)
\end{aligned}$$

Thus the Runge-Kutta setup for this problem simply entails the definition of the vector function $F_k$

$$\begin{aligned}
F_1 &= Y_2 && (14a)\\
F_2 &= Y_3 && (14b)\\
F_3 &= Y_4 && (14c)\\
F_4 &= \tfrac{1}{Y_1}\left(G - Y_3^2 - \exp(-Y_1)\right) && (14d)
\end{aligned}$$

and integration subject to appropriate initial conditions on $[Y_1, Y_2, Y_3, Y_4]$.

### 2.2 *Shooting Methods*

Runge-Kutta integration is best tailored to solve initial value problems, i.e. problems where $[Y_1, Y_2, Y_3, \ldots, Y_M]$ are known at time $t = 0$. For many interesting problems, for example those involving integration over a space coordinate $x$ rather than a time coordinate $t$, the boundary conditions are given as *endpoint* conditions rather than initial conditions. As an example, the equation for steady fluid flow between two rigid plates separated by a distance $H$ has the mathematical form

$$\frac{d^2Y}{dx^2} = \alpha \tag{15}$$

subject to boundary conditions $Y(0) = 0$ and $Y(H) = 0$ (no flow at the boundaries). Equation (15) can be expressed as two coupled first-order ODEs

$$\begin{aligned}
\frac{dY_1}{dt} &= Y_2 && (16a)\\
\frac{dY_2}{dt} &= \alpha && (16b)
\end{aligned}$$

subject to the endpoint boundary conditions $Y_1(0) = 0$ and $Y_1(H) = 0$. The Runge-Kutta approach requires initial conditions, yet the problem provides only one of these ($Y_1(0) = 0$). To work around this difficulty one adopts the following strategy: (i) a trial value of $Y_2(0)$ is guessed at, say $Y_2(0) = Y_{\text{try}}$; (ii) integration from $x = 0$ to $x = H$ is performed; (iii) the value of $Y_1(H)$ is scrutinized and the error $\varepsilon$ between the actual result $Y_1 = Y_1(H)$ and the target results $Y_1 = 0$ is evaluated; (iv) the starting value of $Y_{\text{try}}$ is adjusted and a new integration is performed; (v) by successive adjustments of $Y_{\text{try}}$, the shooting error is reduced to some predefined value and in this fashion the end-point boundary condition is eventually satisfied.

### 2.3 *Stiffness*

Not all problems are readily solved using the standard Runge-Kutta method. One common problem is that of "stiffness" in the differential equations. A differential system is said to be stiff if two vastly different time scales are simultaneously relevant to the controlling physics. The situation is not uncommon and one can foresee the challenge to a numerical integration scheme. For computational efficiency it is desirable to avoid unnecessarily small time steps, for example, solving a radioactive decay equation for $^{238}$U (half life of 4.468 Gyr) using 1 $\mu$s time steps would be ridiculously wasteful of computing resources, as would integrating the decay equation for $^{3}$H (half life of 12.26 yr) for 12 Gyr. Problems that are numerically stiff combine such extremes in a single system of equations. Fortunately, this need not introduce an extra headache for the MATLAB user because MATLAB includes "stiff solvers" among its optional Runge-Kutta integrators. For our purposes the stiff solver `ode15s` will prove satisfactory.

## 3 Numerical Method of Lines

A remarkably powerful approach, known as the numerical method lines, allows the simplicity of the Runge-Kutta method to be directed to the solution of partial differential equations. It is this application that is most relevant to problems of continuum mechanics because the governing equations for continuous media are commonly partial differential equations rather than ordinary differential equations. Many books have been written on the analytical solution of partial differential equations, in particular those that are viewed as the standard equations of mathematical physics, for example,

$$\begin{array}{lll}\nabla^2\phi & = 0 & (17a)\\ \nabla^2\phi & = \frac{1}{\kappa}\frac{\partial\phi}{\partial t} & (17b)\\ \nabla^2\phi & = \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} & (17c)\end{array}$$

referred to, respectively, as Laplace's equation, the diffusion equation and the wave equation. These equations are examples of linear partial differential equations and even for the simplest of cases the methods of solution can be challenging. By exploiting the method of lines, one can obtain solutions for linear problems as well as for nonlinear and coupled problems for which no analytical solutions exist.

### 3.1 *Finite Difference Approximations of Space Derivatives*

A function $\phi$ uniformly sampled at an interval $\Delta x$ can be represented by the series of values $\phi(x_i) = \phi_i$. The space derivatives of $\phi$ can be approximated by finite-difference expressions. The simplest and least accurate expressions for the first derivative are

$$\begin{aligned}
\left(\frac{\partial \phi}{\partial x}\right)_i &= \tfrac{\phi_{i+1}-\phi_{i-1}}{2\Delta x} \qquad (18a)\\
\left(\frac{\partial \phi}{\partial x}\right)_i &= \tfrac{\phi_{i+1}-\phi_{i}}{\Delta x} \qquad (18b)\\
\left(\frac{\partial \phi}{\partial x}\right)_i &= \tfrac{\phi_{i}-\phi_{i-1}}{\Delta x}. \qquad (18c)
\end{aligned}$$

Note that in (18a) the derivative is centred with respect to the point $x_i$ and for (18b) and (18c) the derivatives are one-sided, Equation (18b) being a forward-ifferencing scheme and (18b) a backward-differencing scheme. Ordinarily, one prefers centred schemes to non-centred ones but at boundaries this preference must scrapped. Note also that the derivative is evaluated on the same $x_i$ grid that is used to define the function $\phi(x_i)$. Such schemes are said to be non-staggered.

An alternative system that can prove useful is to define the derivatives on a staggered grid, i.e. occupying intermediate points between those defined by the $x_i$ spatial positions. To illustrate this idea, consider the following finite difference approximations to the first derivative at points on the staggered grid

$$\begin{aligned}
\left(\frac{\partial \phi}{\partial x}\right)_{i+\frac{1}{2}} &= \tfrac{\phi_{i+1}-\phi_{i}}{\Delta x} \qquad (19a)\\
\left(\frac{\partial \phi}{\partial x}\right)_{i-\frac{i}{2}} &= \tfrac{\phi_{i}-\phi_{i-1}}{\Delta x}. \qquad (19b)
\end{aligned}$$

You will see an example and learn about the utility of staggered grids below as well as in assignment 2.

Finite-difference approximations for the second derivative can be written

$$\begin{aligned}
\left(\frac{\partial^2 \phi}{\partial x^2}\right)_i &= \tfrac{\phi_{i+1}-2\phi_i+\phi_{i-1}}{(\Delta x)^2} \qquad (20a)\\
\left(\frac{\partial^2 \phi}{\partial x^2}\right)_i &= \tfrac{\phi_{i+2}-2\phi_{i+1}+\phi_{i}}{(\Delta x)^2} \qquad (20b)\\
\left(\frac{\partial^2 \phi}{\partial x^2}\right)_i &= \tfrac{\phi_{i-2}-2\phi_{i-1}+\phi_{i}}{(\Delta x)^2} \qquad (20c)
\end{aligned}$$

and respectively correspond to centred, forward and backward differencing schemes.

### 3.2 *Integration of Partial Differential Equations*

To illustrate the numerical method of lines, consider the problem of solving the one-dimensional diffusion equation (see Equation (17b) for the full three-dimensional equation)

$$\frac{\partial \phi}{\partial t} = \kappa \frac{\partial^2 \phi}{\partial x^2}. \tag{21}$$

I have rearranged Equation (17b) to place the time derivative on the left-hand side, as it also appears in Runge-Kutta equations. Imagine $\phi(x,t)$ to be spatially-sampled at a fixed interval $\Delta x$ and define $\phi(x_i,t) := Y_i(t)$. By employing finite-difference approximations to represent $\partial^2\phi/\partial x^2$ one can express (21) as

$$\frac{dY_i}{dt} = \frac{\kappa}{(\Delta x)^2}\,[Y_{i-1} - 2Y_i + Y_{i+1}]. \tag{22}$$

Letting $i$ vary over the range $1 \le i \le M$ generates a system of $M$ first-order ordinary differential equations that approximate the partial differential (21). A problem arises at the boundary points $x_1$ and $x_M$; for example at $i = 1$

$$\frac{dY_1}{dt} = \frac{\kappa}{(\Delta x)^2}\,[Y_0 - 2Y_1 + Y_2] \tag{23}$$

and the point $Y_0$ lies outside the defined grid. The remedy to this difficulty depends on the boundary conditions applying to the given problem. If $\phi(x_1,t) = \phi_0$ (a Dirichlet boundary condition) then $(\partial\phi/\partial t)_1 = 0$ at the boundary and in (22) one would take $dY_1/dt = 0$ as the first equation. Similarly if a Dirichlet condition applied at $x_M$ then $dY_M/dt = 0$ would be used as the $M$-th differential equation for the system of Equation (22). If the derivative $\partial\phi/\partial x$ vanished at $x_1$ then it can be demonstrated that for the "image point" at $x_0$, $\phi_0 = \phi_2$ and the first equation of (22) could thus be written

$$\frac{dY_1}{dt} = \frac{\kappa}{(\Delta x)^2}\,[-2Y_1 + 2Y_2]. \tag{24}$$

In a similar fashion it can be shown that if $(\partial\phi/\partial x)_M = 0$ then the final equation of (22) would become

$$\frac{dY_M}{dt} = \frac{\kappa}{(\Delta x)^2}\,[2Y_{M-1} - 2Y_M]. \tag{25}$$

These examples by no means exhaust the possible boundary conditions that can arise but they give a taste of approaches used to applying them.

A more complex, but also more fascinating, example is afforded by considering the nonlinear thermal diffusion equation

$$\frac{\partial T}{\partial t} = \frac{1}{\rho c}\frac{\partial}{\partial x}\left(K[T, \partial T/\partial x]\,\frac{\partial T}{\partial x}\right) \tag{26}$$

where $T(x,t)$ is temperature, $\rho$ is density, $c$ is specific heat capacity and $K$ the thermal conductivity. For the special case of constant $K$, $\rho$ and $c$ and $\kappa := K/\rho c$, Equation (26) has identical form to Equation (22). Taking $T_i(t) = T(x_i,t)$ and approximating space derivatives in (26) by finite difference approximations one arrives the scheme

$$\frac{dT_i}{dt} = \frac{1}{\rho_i c_i \Delta x}\,[q_{i+\frac{1}{2}} - q_{i-\frac{1}{2}}] \tag{27}$$

with

$$
\begin{aligned}
T_{i-\frac{1}{2}} &= \tfrac{1}{2}[T_{i-1} + T_i] && (28a)\\
T_{i+\frac{1}{2}} &= \tfrac{1}{2}[T_{i+1} + T_i] && (28b)\\
\left(\frac{dT}{dx}\right)_{i-\frac{1}{2}} &= \tfrac{T_i - T_{i-1}}{\Delta x} && (28c)\\
\left(\frac{dT}{dx}\right)_{i+\frac{1}{2}} &= \tfrac{T_{i+1} - T_i}{\Delta x} && (28d)\\
q_{i-\frac{1}{2}} &= K[T_{i-\frac{1}{2}}, (dT/dx)_{i-\frac{1}{2}}] \left(\tfrac{dT}{dx}\right)_{i-\frac{1}{2}} && (28e)\\
q_{i+\frac{1}{2}} &= K[T_{i+\frac{1}{2}}, (dT/dx)_{i+\frac{1}{2}}] \left(\tfrac{dT}{dx}\right)_{i+\frac{1}{2}}. && (28f)
\end{aligned}
$$

Boundary conditions must also be applied but I shall not dwell on this aspect. A MATLAB script illustrating this scheme and plotting the results is included as Example 4 in the Appendix. Approaches to applying the boundary conditions are illustrated.

## 4 Dimensional Analysis and Its Role

A problem that arises repeatedly in the numerical solution of differential equations is to assess the time and space scales that determine such important matters as what space sampling interval to select, how frequently to sample the output functions and how long to perform time integrations. By the use of dimensional analysis it is possible to obtain this information and apply this knowledge to the problem of obtaining numerical solutions. As a first example, I have already introduced the coupled system

$$
\begin{aligned}
\frac{dY_1}{dt} &= \alpha Y_2 && (29a)\\
\frac{dY_2}{dt} &= -\alpha Y_1. && (29b)
\end{aligned}
$$

By dimensional analysis of (29) it is clear that $t$ has dimensions of time and $\alpha$ the dimensions of inverse time. Let us define $[Y_1]$ and $[Y_2]$ as representative magnitudes of the functions $Y_1(t)$ and $Y_2(t)$. We could introduce dimensionless functions $Y_1^*(t)$ and $Y_2^*(t)$ to capture the shape of the dimensioned functions $Y_1(t)$ and $Y_2(t)$ and it follows trivially that

$$
\begin{aligned}
Y_1(t) &= [Y_1]\, Y_1^*(t) && (30a)\\
Y_2(t) &= [Y_2]\, Y_2^*(t). && (30b)
\end{aligned}
$$

Note that in the above separation, $[Y_1]$ and $[Y_2]$ carry the magnitude and dimensions of $Y_1(t)$ and $Y_2(t)$ and the dimensionless functions $Y_1^*(t)$ and $Y_2^*(t)$ carry the information about function shape. In the same spirit, suppose that $[t]$ represents a characteristic time scale for the problem. One can write

$$t = [t]\, t^* \qquad (31)$$

where $t^*$ is a dimensionless time variable. Substituting (30) and (31) into (29) gives

$$\begin{aligned} \frac{[Y_1]}{[t]}\frac{dY_1^*}{dt^*} &= \alpha[Y_2]\,Y_2^* \qquad (32a) \\ \frac{[Y_2]}{[t]}\frac{dY_2^*}{dt^*} &= -\alpha[Y_1]\,Y_1^*. \quad (32b) \end{aligned}$$

Noting that $[Y_1]$ and $[Y_2]$ must have the same dimensions (or equations (32a) and (32b) would each by dimensionally inconsistent), one can define the dimensionless constant $\beta := [Y_2]/[Y_1]$ and rewrite (32) as

$$\begin{aligned} \frac{dY_1^*}{dt^*} &= \alpha\beta[t]\,Y_2^* \qquad (33a) \\ \frac{dY_2^*}{dt^*} &= -\tfrac{\alpha}{\beta}\,[t]\,Y_1^*. \quad (33b) \end{aligned}$$

Examination of (33a) and (33b) suggests that

$$[t] = 1/\alpha \tag{34}$$

defines a natural time constant for the problem and that, with this definition the equations become

$$\begin{aligned} \frac{dY_1^*}{dt^*} &= \beta\,Y_2^* \qquad (34a) \\ \frac{dY_2^*}{dt^*} &= -\tfrac{1}{\beta}\,Y_1^*. \quad (34b) \end{aligned}$$

Equation (34) provides important information about the relevant time scale for the evolution equation. If, for example, $\alpha = 1\,\mathrm{s}^{-1}$ then the characteristic time scale is $[t] = 1\,\mathrm{s}$; if $\alpha = 0.2\,\mathrm{yr}^{-1}$ then $[t] = 5\,\mathrm{yr}$. In the former case the information about time scales would guide the analyst to select an upper integration limit for the time integration: $t_{\mathrm{MAX}} = 1\,\mu\mathrm{s}$ would clearly be much to small and $t_{\mathrm{MAX}} = 1\,\mathrm{h} = 3600\,\mathrm{s}$ way too large. If the analyst chooses to take a further step to nondimensionalize the evolution equation using the characteristic time as the scale for time, integration limits could be set between 0 and some number (usually just a few) of these time scales. For example, if Equation 29 is describing the radiactive decay of U, with a half life of order a few billion years, one choice might be to nondimensionalize time with this half life. Then, carry out the integration using fractions of this half life as the time step (the problem then evolves with the predominant physics in the problem). Integrating between 0 and, say, 4 is easier than figuring out how best to integrate over an exponential that is changing rapidly and then slowly over 0 to 4 billion years.

As a second example of dimensional analysis, consider the nonlinear thermal diffusion equation

$$\frac{\partial T}{\partial t} = \frac{1}{\rho c}\frac{\partial}{\partial x}\left(K[T, \partial T/\partial x]\,\frac{\partial T}{\partial x}\right) \tag{35}$$

and introduce the following scalings

$$\begin{aligned} T &= [T]\,T^* \quad (36a) \\ K &= [K]K^* \quad (36b) \end{aligned}$$

$$\begin{aligned} c &= [c]c^* && (36c) \\ \rho &= [\rho]\rho^* && (36d) \\ x &= [x]x^* && (36e) \\ t &= [t]t^*. && (36f) \end{aligned}$$

Applying these scalings to (35) gives

$$\frac{[T]}{[t]}\frac{\partial T^*}{\partial t^*} = \frac{[K][T]}{[\rho][c][x]^2}\frac{\partial}{\partial x^*}\left(K^*\frac{\partial T^*}{\partial x^*}\right) \tag{37}$$

which simplifies to

$$\frac{\partial T^*}{\partial t^*} = \frac{[K][t]}{[\rho][c][x]^2}\frac{\partial}{\partial x^*}\left(K^*\frac{\partial T^*}{\partial x^*}\right). \tag{38}$$

Examination of (38) reveals that the natural time constant for this problem is given by

$$[t] = \frac{[\rho][c][x]^2}{[K]}. \tag{39}$$

By assigning appropriate magnitudes to $[\rho]$, $[c]$, $[x]$ and $[K]$ the characteristic time scale for diffusion can be determined. As with the example above, this time scale can be used with an appropriate scale length in the X-direction to nondimensionalize the PDE and simplify its solution, while enhancing physical insight.