Problem 1

1) 6^n sequences

2) 5^n sequences do not contain 1
4^n sequences do not contain either [redacted] nor [redacted]

3) $|E \cup F| = |E| + |F| - |E \cap F|$ if E and F are finite.

$\Omega = 6^n, \quad |E| = 6^n - 5^n, \quad |F| = 6^n - 5^n, \quad |E \cap F| = 6^n - 4^n$

so $|E \cup F| = 6^n - 5^n + 6^n - 5^n - 6^n + 4^n$
$= 6^n - 2 \cdot 5^n + 4^n$.

Problem 2

1) $\frac{|E|!}{(|E|-p)!} = \frac{52!}{(52-13)!} = \frac{52!}{39!}$

2) $|E| = 39, \quad p = 26 \Rightarrow \frac{39!}{(39-26)!} = \frac{39!}{13!}$

$\cancel{E = \{a, b, c, d, e\}}$

$\cancel{\Rightarrow |E| = 5 = n}$

$\cancel{k_1 + \dots k_r = n = 5}$ ~~for~~ $k_i > 0$

r

k_i ~~elements~~ receive i

Problem 3

1. There are only 4 ways to receive a royal flush; one per suit
2. 40 ways; 10 per suit.
3. $\binom{13}{5}$ ways to get a flush per 4 suits $\Rightarrow$ 5148 ways.
4. 10 ways in ranks and each card having 4 possible suits $\Rightarrow (10)(4^5) = 10240$ ways
5. $\binom{13}{2} = 78$ is the number of ways to choose two ranks
$\binom{4}{2} = 6$ is the number of ways to choose two suits
$\binom{11}{1} = 11$ is number of ways to choose a rank from remaining options (cannot be of a pair)
$\Rightarrow \binom{13}{2}\binom{4}{2} \quad \binom{11}{1}\binom{4}{2}$ Ahh!

Problem 4

1. $\left.\begin{array}{l} P(1^{st}) = 1 \\ P(2^{nd}) = \frac{5}{6} \\ P(3^{rd}) = \frac{4}{6} \end{array}\right\} P(1^{st}) \cdot P(2^{nd}) \cdot P(3^{rd}) = \frac{5}{9}$

2. $\underset{\text{obj. 1}}{\binom{n}{1}}, \underset{\text{obj. 2}}{\binom{n}{1}}, \cdots \underset{\text{obj. 6}}{\binom{n}{1}}$ each object has n choices for colour

the probability two objects do not have the same colour is $P(2\text{ different}) = \frac{n-1}{n}$

$P(3\text{ different}) = \left(\frac{n-1}{n}\right)\left(\frac{n-2}{n}\right) \quad \cdots \quad P(6\text{ different}) = \frac{(n-1)!}{(n-6)!\,n^6}$

thus probability that $\exists$ a pair of objects with same colour is $\boxed{1 - \frac{(n-1)!}{(n-6)!\,n^6}}$

Problem 5

1. $\{\{a\}, \{b\}, \{c\}\}$,
$\{\{a, b\}, \{c\}\}$,
$\{\{a, c\}, \{b\}\}$,
$\{\{a, b, c\}\}$

2. Proof by induction: $\quad * \binom{n}{k} = \frac{n!}{k!(n-k)!}$

Base case: $B_1 = 1$ since a set of cardinal $n = 1$ has only one possible partition.
let $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ so $B_1 = \sum_{k=0}^{0} \binom{n}{k} B_0 = \binom{0}{0} 1 = 1$ ✓

Inductive step: let $n = m+1 \implies B_{m+2} = \sum_{k=0}^{m+1} \binom{m+1}{k} B_k$

$$= \binom{m+1}{m+1} B_{m+1} + \sum_{k=0}^{m} \binom{m+1}{k} = B_{m+1} + \underbrace{\sum_{k=0}^{m} \binom{m}{k} B_k}_{= B_{m+1}\text{ from base case}} + \sum_{k=0}^{m} \binom{m}{k-1} B_k$$

(pascal's triangle)

$$= 2B_{m+1} + \sum_{k=0}^{m} \binom{m}{k-1} B_k$$

Problem 6

Proof by induction

Base case: let $(i, j) = (1, 1)$ so $\binom{p+q}{2} = \sum_{k=0}^{2} \binom{p}{i}\binom{q}{i} = 2pq$?