MATH 302 Assignment 2

Instructions

- Submit a pdf file of written work on Canvas. Be careful of the size of your file. If it exceeds 5Mb, use a compression tool to reduce it (like this one).
- Each homework assignment is worth 0.5% of your final course mark. They are not graded by the TA; instead they are (randomly) checked for appropriate content. Students who submit significant attempts at solving at least half of the problems in each assignment will receive full mark (*do not submit work otherwise*).
- We implement a "we trust you" policy and assume that all students will try hard to solve the problems in the homework assignments, and will receive full credit for trying hard. However, *students who submit garbage files, work that is not their own or that contains attempted solutions for less than half of the problems will receive a penalty of - 10 points on their final course mark.*

Problem 1

In the article "Croix ou Pile" (Heads or Tails) of the Encyclopédie[1], d'Alembert studied the probability of having at least one time heads with two successive flips of a fair coin. His reasoning is the following: One can either get heads in the first flip, heads in the second flip, or not heads at all. Among these three results, two are favorable, so the probability should be $\frac{2}{3}$.

1. What sample space should we consider to apply such a reasoning?
2. Deduce the actual probability to get heads at least one time.

Problem 2

Let A, B, C be three events of the saple space Ω. Using the set operations seen in class, express the following events

1. No event among A, B, C is realized
2. Only one of the three events A, B, C is realized
3. At least two of the three events A, B, C are realized

Problem 3

We have r balls distributed in n urns with $r \leq n$ and each urn can contain more than one balls. Assume all the way to distribute these balls are equally likely.

1. How many ways are there to distribute these balls?
2. Find the probability of $A =$ "each urn contains at most one ball"
3. Find the probability of $B =$ "there exists an urn containing at least two balls"

[1] Encyclopédie (1754), IV, pp. 512-513, link

Problem 4

In a company, 1% of the articles produced have a defect. An automated quality control process allows to remove 95% of articles that have a defect, but also remove 2% of articles that don't have one.

1. What is the probability that an error of control occurs?
2. What is the probability that an accepted article actually has a defect?

Problem 5 (general inclusion-exclusion formula and matching problem)

1. Show that $P(A_1 \cup A_2 \cup A_3) = S_1 - S_2 + S_3$, where

$$\begin{aligned} S_1 &= P(A_1) + P(A_2) + P(A_3) \\ S_2 &= P(A_1 \cap A_2) + P(A_1 \cap A_3) + P(A_2 \cap A_3) \\ S_3 &= P(A_1 \cap A_2 \cap A_3) \end{aligned}$$

   More generally, we will now admit that $P\left(\cup_{i=1}^n A_i\right) = \sum_{r=1}^n (-1)^{r-1} S_r$, where

$$S_r = \sum_{1 \leq i_1 < \ldots < i_r < n} P(A_{i_1} \cap \ldots \cap A_{i_r}).$$

   (the subscript $1 \leq i_1 < \ldots < i_r < n$ in the sum means that we consider all the positive subset of r integers $i_1, \ldots, i_r$ such that this is true).

2. Suppose one randomly distributes n balls into n urns that are all labeled from 1 to n, so that each urn receives exactly one ball. We also assume that all outcomes are equally likely. Consider r balls $i_1 < \ldots < i_r$. Show that the probability that each ball i_k got assigned to the urn i_k is $\dfrac{n - r!}{n!}$.

3. Show that the probability that the labels of exactly r balls match with their urn is $\dfrac{1}{r!}$.

4. Use the general inclusion-exclusion formula to find the probability that at least one ball got assigned to an urn with the same label.

   Hint: Consider the events A_i = "the ball i got assigned to urn i" ; you can also see example 1.27 from the textbook

Recommended practice exercises (not to be handed in)

Textbook exercices 1.13-1.15, 1.20-1.22 and B.1-B.7