MATH 302 Assignment 9

Instructions

- Submit a pdf file of written work on Canvas. Be careful of the size of your file. If it exceeds 5Mb, use a compression tool to reduce it (like this one).
- Each homework assignment is worth 0.5% of your final course mark. They are not graded by the TA; instead they are (randomly) checked for appropriate content. Students who submit significant attempts at solving at least half of the problems in each assignment will receive full mark (*do not submit work otherwise*).
- We implement a "we trust you" policy and assume that all students will try hard to solve the problems in the homework assignments, and will receive full credit for trying hard. However, *students who submit garbage files, work that is not their own or that contains attempted solutions for less than half of the problems will receive a penalty of - 10 points on their final course mark.*

Problem 1

Recall that for $\alpha > 0$ $\Gamma(\alpha) = \int_0^{+\infty} y^{\alpha-1}e^{-y}dy$ is the normalization constant of the Gamma density with parameter α seen in class.

1. Show that $\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$
2. Show that if n is a positive integer $\Gamma(n) = (n-1)!$
3. Show that if n is an off positive integer, then $\Gamma(n/2) = \frac{\sqrt{\pi}(n-1)!}{2^{n-1}\left(\frac{(n-1)}{2}\right)!}$

Problem 2

The random variables X, Y have joint probability density function

$$f(x,y) = \begin{cases} C\frac{e^{-x}-e^{-x-2y}}{e^y-1} & \text{if } x > 0 \text{ and } y > 0, \\ 0 & \text{otherwise.} \end{cases}$$

1. What is the value of C?
2. Are X and Y independent?
3. Find $\mathbb{P}(X < Y)$.

Problem 3

Find the conditional density functions $f_{X|Y}$ and $f_{Y|X}$ associated with the uniform distribution on the unit disk (use the marginal densities seen in class and be careful of the domain of definition).

Problem 4

Let X, Y be two continuous r.v.'s with joint density function f

$$f(x, y) = ce^{-(x^2 - xy + y^2)/2},$$

where c is a normalization constant to be determined.

1. Show that the marginal density $f_X(x) = c\sqrt{2\pi}e^{-3x^2/8}$ (hint: rewrite $x^2 - xy + y^2$ to "complete the square" so $x^2 - xy + y^2 = P(x, y)^2 + Q(x)^2$ where P and Q are two polynomials of degree 1).
2. Deduce the value of c
3. Are X and Y independent?
4. Find the conditional density of Y given $X = x$.

Problem 5 (t-Distribution)

Let Y, Z be independent r.v's with $Z \sim \mathcal{N}(\prime, \infty)$ and $Y \sim \chi^2(n)$ (so $f_Y(y) = \frac{e^{-y/2}y^{n/2-1}}{2^{n/2}\Gamma(n/2)}$), and let us define $T = \frac{Z}{Y/n}$.

1. Find the conditional density function of T given $Y = y$, i.e. $f_T|Y(t|y)$.
2. Show that $f_T(t) = \frac{\Gamma(\frac{n+1}{2})}{\sqrt{\pi n}\Gamma(\frac{n}{2})}\left(1 + t^2\right)^{-(n+1)/2}$

Recommended practice exercises (not to be handed in)

Textbook exercices 6.5-7, 6.11, 6.13, 6.17, 10.1, 10.2, 10.5-7