University of Dayton

AEE 553 — Compressible Flow

Department of Mechanical and Aerospace Engineering

Homework 3

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24 October 2022

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Contents

Problem 1

(a) Derive the Hugoniot Eqution.

Assumptions:

1-D, steady flow. Inviscid. Calorically perfect gas (constant specific heats). Uniform pressure distribution around the CV. No heat addition (adiabatic). No work done on or by the CV.

Solution:

Beginning with the 1-D continuity equation and expressing both velocities in terms of the densities and the other velocity:

$$\rho_1 u_1 = \rho_2 u_2$$

$$u_1 = u_2 \left(\frac{\rho_2}{\rho_1}\right)$$

$$u_2 = u_1 \left(\frac{\rho_1}{\rho_2}\right)$$

The 1-D momentum equation:

$$p_1 + \rho_1 u_1^2 = p_2 + \rho_2 u_2^2$$

Plugging in velocities in terms of 1-D continuity and rearranging:

$$p_1 + \rho_1 u_1^2 = p_2 + \rho_2 \left[u_1 \left(\frac{\rho_1}{\rho_2}\right)\right]^2$$

$$(p_1 - p_2) = u_1^2 \left[\rho_2 \left(\frac{\rho_1}{\rho_2}\right)^2 - \rho_1\right]$$

$$(p_1 - p_2) = u_1^2 \left[\left(\frac{\rho_1^2}{\rho_2}\right) - \rho_1\right]$$

$$(p_1 - p_2) = u_1^2 \left(\frac{\rho_1}{\rho_2}\right)(\rho_1 - \rho_2)$$

We now have relationships for u_1 and u_2 in terms of pressures and densities.

$$u_1^2 = \left(\frac{p_1 - p_2}{\rho_1 - \rho_2}\right)\left(\frac{\rho_2}{\rho_1}\right)$$

$$u_2^2 = \left(\frac{p_2 - p_1}{\rho_2 - \rho_1}\right)\left(\frac{\rho_1}{\rho_2}\right)$$

Next, the 1-D energy equation (adiabatic, no work):

$$h_1 + \frac{u_1^2}{2} = h_2 + \frac{u_2^2}{2}$$

The definition of enthalpy, h.

$$h = e + \frac{p}{\rho}$$

Recasting the 1-D energy equation with the definition of enthalpy and rearranging:

$$e_1 + \frac{p_1}{\rho_1} + \frac{u_1^2}{2} = e_2 + \frac{p_2}{\rho_2} + \frac{u_2^2}{2}$$

$$e_1 + \frac{p_1}{\rho_1} + \frac{1}{2}\left[\left(\frac{p_1 - p_2}{\rho_1 - \rho_2}\right)\left(\frac{\rho_2}{\rho_1}\right)\right] = \; e_2 + \frac{p_2}{\rho_2} + \frac{1}{2}\left[\left(\frac{p_2 - p_1}{\rho_2 - \rho_1}\right)\left(\frac{\rho_1}{\rho_2}\right)\right]$$

$$0 = (e_2 - e_1) + \left(\frac{p_2}{\rho_2} - \frac{p_1}{\rho_1}\right) + \frac{1}{2}\left[\left(\frac{p_2 - p_1}{\rho_2 - \rho_1}\right)\left(\frac{\rho_1}{\rho_2} - \frac{\rho_2}{\rho_1}\right)\right]$$

$$0 = (e_2 - e_1) + \left(\frac{p_2\rho_1 - p_1\rho_2}{\rho_1\rho_2}\right) + \frac{1}{2}\left[\left(\frac{p_2 - p_1}{\rho_2 - \rho_1}\right)\left(\frac{\rho_1^2 - \rho_2^2}{\rho_1\rho_2}\right)\right]$$

$$0 = (e_2 - e_1) + \left(\frac{1}{\rho_1\rho_2}\right)\left[(p_2\rho_1 - p_1\rho_2) + \frac{1}{2}\left(\frac{p_2 - p_1}{\rho_2 - \rho_1}\right)\left(\rho_1^2 - \rho_2^2\right)\right]$$

$$0 = (e_2 - e_1) + \left(\frac{1}{\rho_1\rho_2}\right)\left[(p_2\rho_1 - p_1\rho_2) + \frac{1}{2}\left(\frac{p_1 - p_2}{\rho_1 - \rho_2}\right)(\rho_1 - \rho_2)(\rho_1 + \rho_2)\right]$$

$$0 = (e_2 - e_1) + \left(\frac{1}{\rho_1\rho_2}\right)\left[(p_2\rho_1 - p_1\rho_2) + \frac{1}{2}(p_1 - p_2)(\rho_1 + \rho_2)\right]$$

$$0 = (e_2 - e_1) + \left(\frac{1}{\rho_1\rho_2}\right)\left[(p_2\rho_1 - p_1\rho_2) + \frac{1}{2}(p_1\rho_1 + p_1\rho_2 - p_2\rho_1 - p_2\rho_2)\right]$$

$$0 = (e_2 - e_1) + \left(\frac{1}{2}\right)\left(\frac{1}{\rho_1\rho_2}\right)(p_1\rho_1 - p_1\rho_2 + p_2\rho_1 - p_2\rho_2)$$

$$(e_2 - e_1) = \left(\frac{1}{2}\right)\left(\frac{p_1}{\rho_2} - \frac{p_1}{\rho_1} + \frac{p_2}{\rho_2} - \frac{p_2}{\rho_1}\right)$$

$$(e_2 - e_1) = \left(\frac{p_1 + p_2}{2}\right)\left(\frac{1}{\rho_1} - \frac{1}{\rho_2}\right)$$

We now have one of the common forms of the Hugoniot Equation:

$$(e_2 - e_1) = \left(\frac{p_1 + p_2}{2}\right)(\nu_1 - \nu_2)$$

For a CPG, internal energy, e:

$$e = c_\nu T$$

Substituting the above relationship into our initial Hugoniot Equation:

$$c_\nu (T_2 - T_1) = \left(\frac{p_1 + p_2}{2}\right)(\nu_1 - \nu_2)$$

For a CPG, c_ν:

$$c_\nu = \frac{R}{\gamma - 1}$$

Substituting and rearranging:

$$\frac{R}{\gamma - 1}(T_2 - T_1) = \left(\frac{p_1 + p_2}{2}\right)(\nu_1 - \nu_2)$$

The ideal gas equation of state:

$$T = \frac{p\nu}{R}$$

Substituting and rearranging to solve for p_2/p_1:

$$\frac{R}{\gamma - 1}\left(\frac{p_2\nu_2}{R} - \frac{p_1\nu_1}{R}\right) = \left(\frac{p_1 + p_2}{2}\right)(\nu_1 - \nu_2)$$

$$\frac{2}{\gamma - 1}(p_2\nu_2 - p_1\nu_1) = (p_1 + p_2)(\nu_1 - \nu_2)$$

$$\frac{2}{\gamma - 1}(p_2\nu_2 - p_1\nu_1) = p_1\nu_1 - p_1\nu_2 + p_2\nu_1 - p_2\nu_2$$

$$\left(\frac{2}{\gamma - 1} + 1\right)(p_2\nu_2 - p_1\nu_1) = (p_2\nu_1 - p_1\nu_2)$$

$$\left(\frac{\gamma+1}{\gamma-1}\right) = \frac{(p_2\nu_1 - p_1\nu_2)}{(p_2\nu_2 - p_1\nu_1)}$$

$$\left(\frac{\gamma+1}{\gamma-1}\right) = \frac{\left(\frac{p_2}{p_1}\nu_1 - \nu_2\right)}{\left(\frac{p_2}{p_1}\nu_2 - \nu_1\right)}$$

$$\left(\frac{\gamma+1}{\gamma-1}\right)\left(\frac{p_2}{p_1}\nu_2 - \nu_1\right) = \left(\frac{p_2}{p_1}\nu_1 - \nu_2\right)$$

$$\left(\frac{\gamma+1}{\gamma-1}\right)\left(\frac{p_2}{p_1}\nu_2 - \nu_1\right) = \left(\frac{p_2}{p_1}\nu_1 - \nu_2\right)$$

$$\frac{p_2}{p_1}\left[\left(\frac{\gamma+1}{\gamma-1}\right)\nu_2 - \nu_1\right] = \left(\frac{\gamma+1}{\gamma-1}\right)\nu_1 - \nu_2$$

$$\frac{p_2}{p_1} = \left[\frac{\left(\frac{\gamma+1}{\gamma-1}\right)\nu_1 - \nu_2}{\left(\frac{\gamma+1}{\gamma-1}\right)\nu_2 - \nu_1}\right]$$

$$\frac{p_2}{p_1} = \left[\frac{\left(\frac{\gamma+1}{\gamma-1}\right)\frac{\nu_1}{\nu_2} - 1}{\left(\frac{\gamma+1}{\gamma-1}\right) - \frac{\nu_1}{\nu_2}}\right]$$

The relation between density, ρ, and specific volume, ν:

$$\rho = \frac{1}{\nu}$$

Substituting the above relation yields the final form of the Hugoniot Equation for pressure ratio across a normal shock in terms of density ratio and γ:

$$\boxed{\frac{p_2}{p_1} = \left[\frac{\left(\frac{\gamma+1}{\gamma-1}\right)\frac{\rho_2}{\rho_1} - 1}{\left(\frac{\gamma+1}{\gamma-1}\right) - \frac{\rho_2}{\rho_1}}\right]}$$

(b) Assume that a turbojet compressor fan isentropically compresses air for a range $1 < \frac{\rho_2}{\rho_1} < 5$. Plot the compression (i.e., $\frac{p_2}{p_1}$) for this range of $\frac{\rho_2}{\rho_1}$ for both traditional isentropic compression (e.g., a turbojet compression fan) and normal-shock-wave compression. These curves should be on the same plot with a clearly labeled legend.

Givens:

Isentropic and normal shock compression of air. $1 < \frac{\rho_2}{\rho_1} < 5$.

Assumptions:

Inviscid, adiabatic, no external work. Air is a CPG with $\gamma = 1.4$.

Solution:

Note: All calculations performed in Python, see Appendix A. The static pressure ratio across a normal shock given by the Hugoniot equation:

$$\frac{p_2}{p_1} = \left[\frac{\left(\frac{\gamma+1}{\gamma-1}\right)\frac{\rho_2}{\rho_1} - 1}{\left(\frac{\gamma+1}{\gamma-1}\right) - \frac{\rho_2}{\rho_1}}\right]$$

The static pressure ratio given as a function of density ratio for isentropic compression:

$$\frac{p_2}{p_1} = \left(\frac{\rho_2}{\rho_1}\right)^{\gamma}$$

Figure 1 shows a comparison of the compression (pressure ratio) as a function of density ratio for both isentropic compression and normal shock compression.

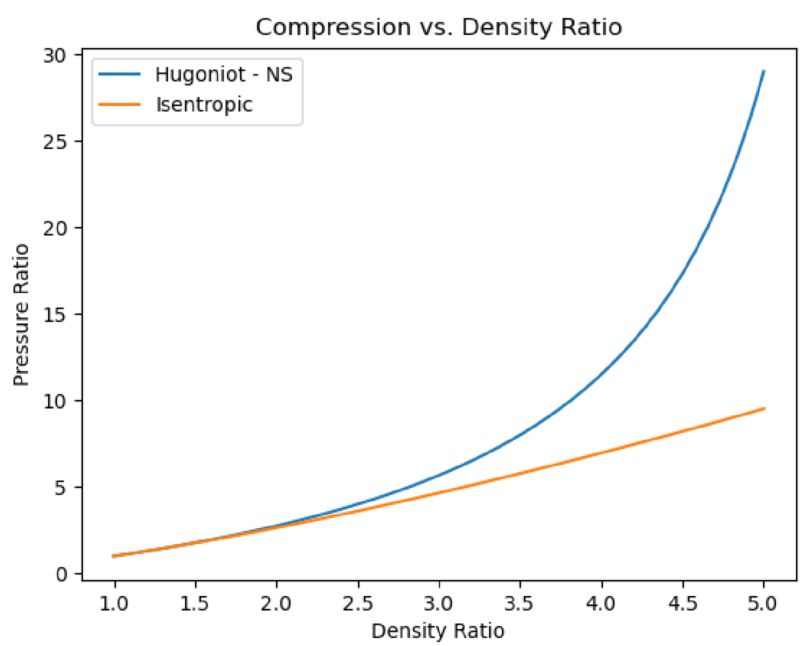

Figure 1: Compression vs. density ratio – normal shock and isentropic compression

(c) If you were tasked with deciding which mechanism to use to compress air, which would you choose? For relatively large $\frac{\rho_2}{\rho_1}$, what other real-world considerations are there for choosing between isentropic versus normal-shock-wave compression? Think of "efficiency".

Discussion:

The normal shock wave delivers much higher static pressure ratios for relatively large density ratios. Absent other context, the normal shock appears to be the ideal solution. However, normal shocks generate large total pressure losses, which are a measure of efficiency. Isentropic compression does not deliver pressure ratios as large as normal shocks, but they are not lossy by definition. Figure 2 shows the total pressure ratio across a normal shock as a function of incoming Mach number. Total pressure ratio plummets as incoming Mach number increases. By approximately Mach 2.5, the flow has lost half of its total pressure because of the presence of a normal shock. For hypersonic Mach numbers (5+), the total pressure recovery is around 10% or less! When the density ratio associated with a compression process is approximately 2.5 or less, both methods of compression deliver approximately the same pressure ratio. Isentropic compression methods require turbomachinery and moving parts which contribute substantial amounts of weight to a vehicle. If the losses are acceptable, normal shock compression at an inlet could be a potentially desirable design choice for minimizing weight.

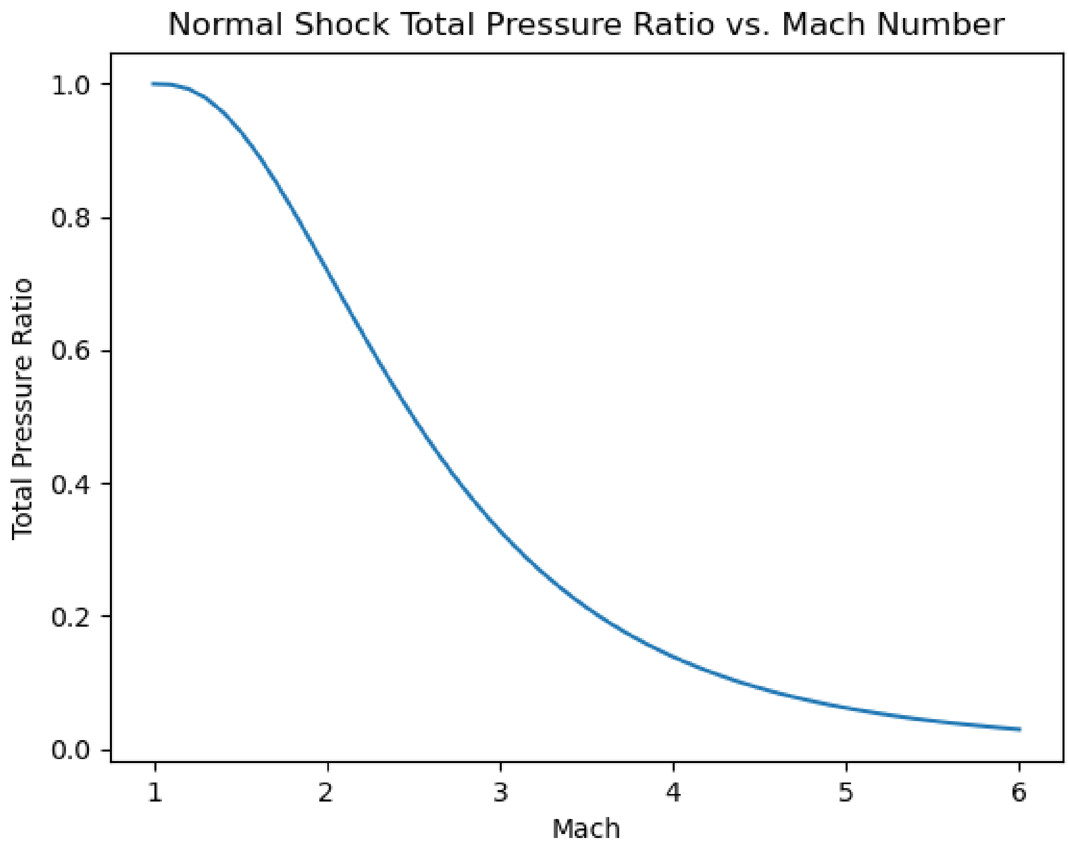

Figure 2: Compression vs. Mach number across normal shock

(d) Calculate the change in entropy $(s_2 - s_1)$ from the given range of $\frac{\rho_2}{\rho_1}$ for both the isentropic and normal-shock compression. Plot $(s_2 - s_1)$ versus $\frac{\rho_2}{\rho_1}$, and comment on what you find. Does this affect your thoughts on part (c)? Research, and then briefly explain, the physical explanation for the $(s_2 - s_1)$ behavior across the normal shock.

Givens:

Isentropic and normal-shock compression.

Assumptions:

Air is a CPG with $\gamma = 1.4$.

Solution:

By definition, $(s_2 - s_1)$ for isentropic flow should be equal to 0. For a CPG, Gibb's equation:

$$s_2 - s_1 = c_p \ln\left(\frac{T_2}{T_1}\right) - R \ln\left(\frac{p_2}{p_1}\right)$$

Recasting the temperature ratio in terms of the density ratio using isentropic relations:

$$\frac{T_2}{T_1} = \left(\frac{\rho_2}{\rho_1}\right)^{\gamma-1}$$

$$s_2 - s_1 = c_p \ln\left(\left(\frac{\rho_2}{\rho_1}\right)^{\gamma-1}\right) - R\ln\left(\frac{p_2}{p_1}\right)$$

The change in density across a normal shock:

$$s_2 - s_1 = -R\ln\frac{p_{t,2}}{p_{t,1}}$$

Solving for $\frac{p_{t,2}}{p_{t,1}}$ across a normal shock is a multi-step process. Beginning with the pressure ratios calculated in part (b) using the Hugoniot Equation, we can solve for upstream Mach number.

$$\frac{p_2}{p_1} = 1 + \frac{2\gamma}{\gamma+1}\left(M_1^2 - 1\right)$$

$$M_1 = \sqrt{\left(\frac{p_2}{p_1} - 1\right)\left(\frac{\gamma+1}{2\gamma}\right) + 1}$$

Next, the total pressure ratio across a normal shock is given by the following equation:

$$\frac{p_{t,2}}{p_{t,1}} = \frac{p_{t,2}}{p_2}\frac{p_2}{p_1}\frac{p_1}{p_{t,1}} = \left[\frac{\frac{\gamma+1}{2}M_1^2}{1 + \frac{\gamma-1}{2}M_1^2}\right]^{\frac{\gamma}{\gamma-1}}\left[\frac{2\gamma}{\gamma+1}M_1^2 - \frac{\gamma-1}{\gamma+1}\right]^{\frac{1}{1-\gamma}}$$

Figure 3 shows the entropy change for both compression methods. As initially presumed, the isentropic compression by definition has no entropy change. The normal shock compression has non-zero entropy change, increasing with density ratio. Shocks are highly lossy in a very thin region, although flow up and downstream can generally be considered isentropic. The near-instantaneous change in flow properties across the thin shock-region cannot be considered "reversible", therefore any work-related assumptions are rendered invalid. The entropy change increases with density ratio because the shock strength required to generate that density ratio is also increasing, and stronger shocks (associated with higher Mach flows) are lossier and less efficient.

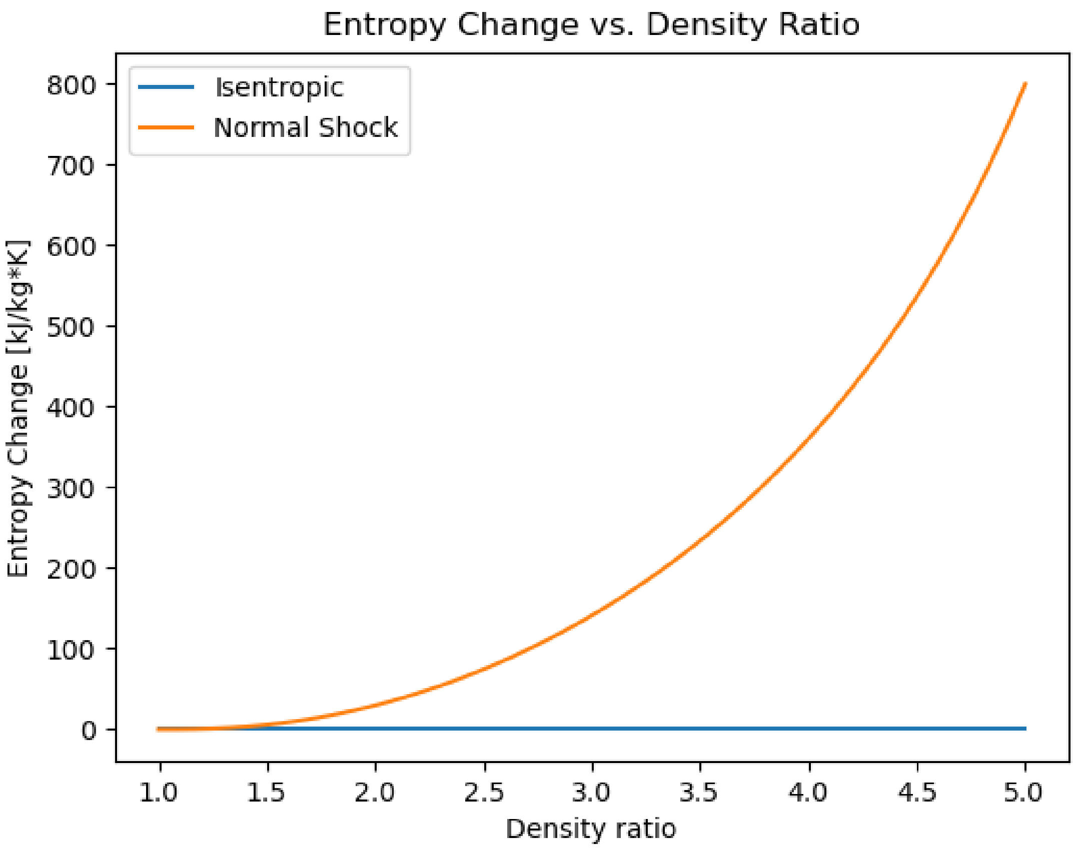

Figure 3: Entropy change for normal-shock and isentropic compression

(e) Using your figure from part (b), for what range of $\frac{\rho_2}{\rho_1}$ are the two processes comparable? Explain why this makes sense. For what Mach-number (i.e., M1) range would this correspond to regarding the normal-shock-wave case? Does this help you explain?

Discussion:

Referencing 1, the two compression processes are comparable up to approximately $\frac{\rho_2}{\rho_1} = 2.5$. From a static pressure ratio standpoint, the processes are nearly identical in output. Examining the entropy change for this density ratio, the normal shock generates entropic losses but the curve of entropy change has not exploded as it does for larger values of $\frac{\rho_2}{\rho_1}$. The range of incoming Mach number, M_1, that this corresponds to is $1 < M_1 < 1.89$. Although the flow is supersonic and experiences a normal shock, it is not extremely supersonic and the associated entropy change and total pressure loss for this relatively low value of M_1 are sufficiently small that the two compression methods are comparable.

Problem 2

Givens:

$M_\infty = 3.0$
$T_\infty = 217\,\mathrm{K}$
$p_\infty = 20\,\mathrm{kPa}$
$\gamma = 1.4$
$R = 287\,\mathrm{J/kg\,K}$
$c_p = 1000\,\mathrm{J/kg\,K}$
$q_{combustor} = 500\,\mathrm{kJ/kg}$

Assumptions:

(i) Steady

(ii) Inviscid

(iii) Uniform velocity, pressure, temperature, density, enthalpy, and energy at each x location.

(iv) The oblique shock from the spike is an attached weak shock from a two-dimensional wedge.

(v) We will neglect the angularity of the streamline through the inlet. For example, we will assume that the fluid is travelling horizontally between the oblique and normal shocks and between the normal shock and fuel injection.

(vi) Our general isentropic, oblique-shock, and normal-shock equations apply (where appropriate).

(vii) We will assume that the static Mach number at the inlet of the combustor is the same value throughout the entire combustor.

(viii) You may use an isentropic relation to calculate the static temperature at the exit of the combustor.

(ix) We will utilize one-dimensional flow with heat transfer to obtain the difference in stagnation temperature between the inlet and exit of the combustor from the given constant value of q.

(x) We will neglect the fuel mixture in the air in the combustor.

(a) Briefly describe how a ramjet works.

Discussion:

A ramjet is a form of high-speed propulsion system that does not involve any turbomachinery to compress the flow. Only capable of starting at higher Mach numbers (3+), a ramjet utilizes a series of oblique and normal shocks (depending on the inlet design) to compress and slow the freestream flow that is being captured for use. The flow must be slowed to subsonic velocities for ramjet operation, as by definition a ramjet utilizes subsonic combustion. As in a turbofan or turbojet engine, fuel is injected into the flow, combusted to generate a large pressure and temperature rise, and then accelerated out of a nozzle to convert pressure and thermal energy into kinetic energy that propels a vehicle forward. Traditional turbomachinery runs into operational issues at high-Mach conditions due to shockwaves generated in the compression section of the flowpath.

(b) A simplified ramjet cycle efficiency is given by:

$$\eta = 1 - \left(\left(\frac{p_{1,\infty}}{p_{3,\infty}}\right)^{\frac{\gamma-1}{\gamma}} \frac{\left(T_{4,\infty} - \left(\frac{p_{3,0}}{p_{1,0}}\right)^{\frac{\gamma-1}{\gamma}} \cdot T_{3,\infty}\right)}{(T_{4,\infty} - T_{3,\infty})} \right)$$

Use this equation to come up with the optimal spike half angle (to the nearest degree) for the given cruise conditions. You must write out your general methodology for the grader. Include a plot of the efficiency versus spike half angle.

Solution:

The solution methodology to calculate the ideal inlet spike half angle, θ, is described below. Assumptions (i)-(iii) allow us to utilize the simplified 1-D forms of continuity, momentum, and energy, which directly lead to the isentropic, normal shock, and oblique shock relations that will be used in this problem, per assumption (vi). For brevity, not all calculations are shown.

Note: All calculations performed in Python, see Appendix B.

1. For a given freestream Mach number, M_∞, and a given inlet half angles, θ, solve for the corresponding shock angle, β, per assumption (v).
2. Using β, determine the component of the incoming Mach normal to the oblique shock, M_{1n}.
3. Use normal shock relationships to calculate the normal component of post-oblique-shock Mach number, M_{2n},
4. Use normal shock relationships to calculate the static pressure ratio across the oblique shock, $\frac{p_2}{p_1}$, as well as the static temperature ratio, $\frac{T_2}{T_1}$..

5. With M_{2n}, β, and θ, determine the magnitude of the post-oblique shock Mach number, M_2.
6. Using assumption (v), treat M_2 as normal to the standing normal shock and use normal shock relationships to determine M_3, the Mach number at the entrance to the combustor
7. Use normal shock relationships to calculate $\frac{p_3}{p_2}$, the post-normal-shock static pressure ratio, as well as the static temperature ratio $\frac{T_3}{T_2}$.
8. Solve for T_3 using $T_3 = \frac{T_3}{T_2}\frac{T_2}{T_1}T_1$ where T_1 is the freestream temperature, T_∞.
9. Use isentropic relationships for M_3 and T_3 to calculate the total temperature at the entrance to the combustor, $T_{t,3}$.
10. Use the energy equation for 1-D flow with heat addition (a.k.a. Rayleigh Flow, assumption (ix)) to calculate the total temperature at the exit of the combustor, $T_{t,4}$, recognizing that total enthalpy can be expressed in terms of specific heats and total temperature.

$$h_{t,3} + q = h_{t,4}$$
$$h = c_p T$$
$$c_p T_{t,3} + q = c_p T_{t,4}$$

11. Assuming that the Mach number is constant throughout the combustor (per assumption (vii)), use isentropic relationships to solve for T_4 with $M_4 = M_3$ (assumption (viii)), neglecting the increased mass from fuel injection (assumption (x)).
12. Calculate the ramjet cycle efficiency, η, using the given equation.

$$\eta = 1 - \left(\left(\frac{p_{1,\infty}}{p_{3,\infty}}\right)^{\frac{\gamma-1}{\gamma}} \frac{\left(T_{4,\infty} - \left(\frac{p_{3,0}}{p_{1,0}}\right)^{\frac{\gamma-1}{\gamma}} \cdot T_{3,\infty}\right)}{(T_{4,\infty} - T_{3,\infty})} \right)$$

For a freestream Mach number $M_\infty = 3$, the maximum turning angle, θ_{max} of the flow is approximately 35 degrees. A range of inlet spike half angles between $1 < \theta < 34$ are iteratively used to calculate ramjet cycle efficiencies to determine the ideal inlet half angle. Figure 4 shows the relationship between ramjet cycle efficiency and inlet half angle for a Mach 3 cruise condition. The ideal inlet half angle is approximately 21 degrees, with a corresponding cycle efficiency of 47.53%, highlighted by a red star.

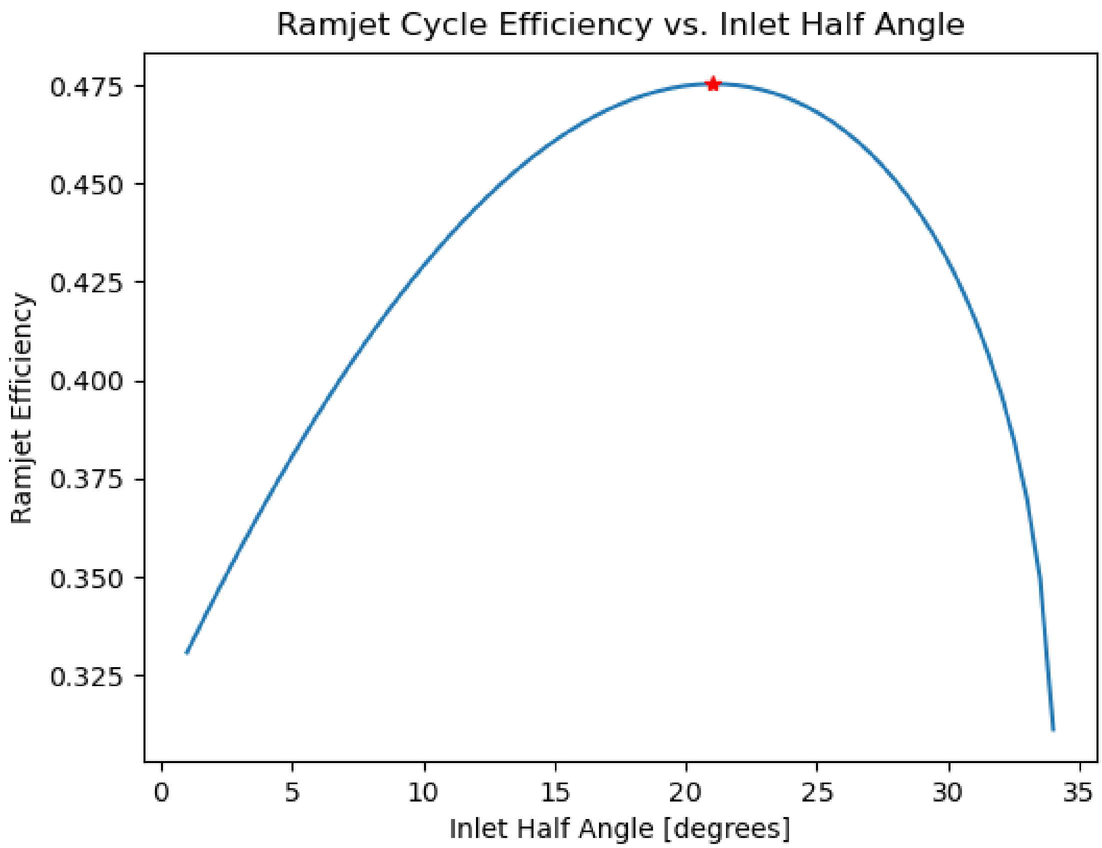

Figure 4: Ramjet cycle efficiency vs. inlet half-angle for Mach 3 cruise

(c) What is the efficiency of the ramjet if we get rid of the spike altogether?

Assumptions:

Remove the inlet spike and corresponding oblique shock while retaining normal shock and all components downstream.

Solution:

Repeat the solution procedure outlined in part (b), removing the oblique-shock relations and setting $M_2 = M_1 = M_\infty$. The ramjet cycle efficiency calculated with a normal shock inlet is 31.7%. For the given flight condition there is only a single solution for cycle efficiency as the variable of inlet geometry has been removed.

(d) Write up a description of your observations of the ramjet with and without the spike. Be sure to include relevant compressible-flow theories/jargon. Be sure to include a conversation of the effect of inlet freestream pressure ratio, inlet total pressure ratio, and combustor freestream temperature difference.

Discussion:

The ramjet with the spike inlet shows a range of efficiencies that initially increases with inlet half angle and decreases beyond a turn angle of approximately 21 degrees. For very small turn angles the oblique shock generated is very nearly a mach wave and does little to compress the flow which then goes through a stronger normal shock, similar to the case without the inlet spike. At the upper end of the turn angle range, efficiences begin to fall off rapidly until the maximum turn angle is reached and the flow becomes separated from the inlet. While the exact details of detached flow are not considered in this analysis, efficiences will tend towards a pure normal shock and dramatically less efficient performance. The spike inlet geometry maintains a much higher inlet total pressure ratio than the normal shock case, which indicates that it is much more efficient. For the ideal case, the efficiency is nearly 2x that of the normal shock case.

The normal shock creates a generally larger inlet static pressure ratio, indicative of more compression. Although this result is desirable, taken in conjunction with the total pressure recovery, the pure normal shock is less desirable. The normal shock case has a smaller difference in freestream temperature across the combustion chamber compared to the inlet case. In general, use of oblique shocks to compress and turn the flow is more efficient than a standing normal shock as entropy changes and total pressure loss are reduce for the oblique shock inlet.

(e) For the optimal spike half angle solved for in part (b), let's now explore the effect of cruise Mach number on efficiency. Plot efficiency versus cruise Mach number.

Solution:

Using the methodology outlined in part (b), we vary Mach number with a fixed inlet half angle of 22 degrees. The corresponding shock angle, β, is be recalculated for each cruise Mach. A range of Mach number $2 < M_\infty < 6$ is analyzed, covering and exceeding the common ramjet operational range. Figure 5 shows the relationship between ramjet cycle efficiency and cruise Mach number. The maximum efficiency is 48.32% at a cruise Mach number of $M_\infty = 3.4$.

(f) Write up a description of your observations of the Mach-number effect on ramjets. Be sure to include the phenomena that limit the ramjet's performance at higher Mach numbers. Be sure to include relevant compressible-flow theories/jargon. Be sure to include a conversation of the effect of inlet freestream pressure ratio, inlet total pressure ratio, and combustor freestream temperature difference.

Discussion:

For a ramjet with a 21 degree inlet half angle the maximum efficiency is 48.32% at a cruise Mach number of $M_\infty = 3.4$. On either side of this ideal Mach number there is a reduced efficiency. For smaller Mach numbers there is less compression generated by the shocks

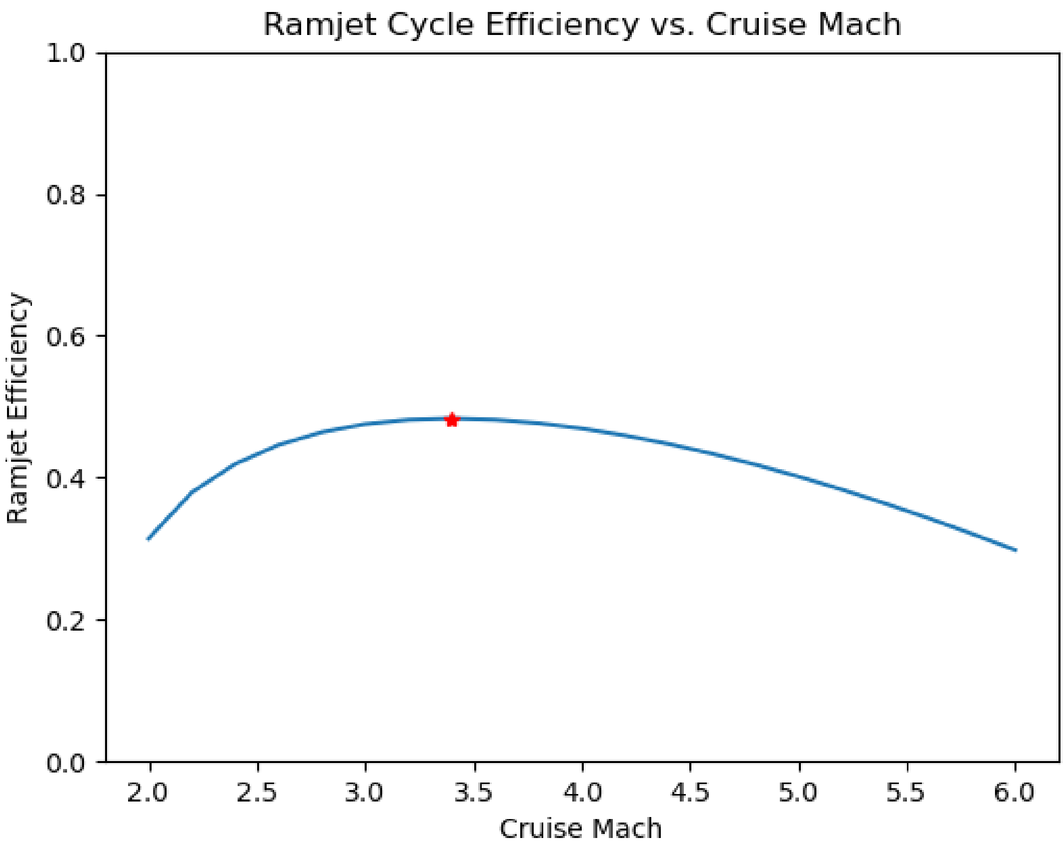

Figure 5: Ramjet cycle efficiency vs. Cruise Mach

indicating that the ramjet inlet is less efficient at compression when operating below its design point. Larger Mach numbers cause the total pressure ratio $\frac{p_{t,3}}{p_{t,1}}$ to become increasingly small as the shock strength grows larger. The combustor static temperature difference increases with increasing cruise Mach, although the sensitivity of the ramjet's efficiency to this parameter is less obvious than the previous two. As the temperature difference increases, the efficiency parameter goes down with it. Ramjets are limited by the subsonic combustion requirement — as M_∞ increases, the losses associated with reducing the flow to a subsonic velocity become large. Scramjets are the solution to the getting around this efficiency issue as they are designed to operate with supersonic combustion.

(g) What do you expect to happen if we instead treat the spike as an axisymmetric cone? Briefly justify your answer with words.

Discussion:

If the inlet spike were treated as an axisymmetric cone instead of a 2-D wedge, the oblique shock would be weaker than the wedge case due to the 3-D relieving effects associated with the geometry. Downstream static temperature and pressure would be smaller than the 2-D case, with a higher total pressure ratio across the shock. Although the weaker shock seems ideal from an efficiency standpoint, we may not be generating the ideal compression required and the downstream flow that passes through the standing normal shock would be at a higher Mach number and therefore experience a stronger shock than the 2-D case. A solution would be to use a series of oblique shocks along a curved ramp instead of a linear cone to gradually induce compression and slow the flow more efficiently than a single oblique shock and a standing normal shock.

Problem 3

(a) Briefly describe why scramjets (theoretically) solve some of the specific issues that ramjets encounter at higher Mach numbers.

(b) How do the inlet freestream pressure ratio, inlet total pressure ratio, and combustor inlet freestream temperature compare for this scramjet design compared to the ramjet (with spike) design? You must actually calculate the numbers. Does this support what you said in part (a)?

(c) Under these conditions, what is the Mach number of the flow at the inlet of the combustor? What combustion challenges do we face in efficiently burning fuel at this Mach number?

(d) Let's now investigate the qualitative effect of viscosity on shock reflections. We will focus our attention to the last reflected shock before the fuel-injection site. An effect of viscosity is to decelerate the flow in the vicinity of the wall such at $u_{wall} = 0$. We will assume that the scramjet is a height of H from the bottom wall to the top wall. We will also assume that u is equal to its local freestream value a distance δ away from the walls. Sketch y versus β for $0 < y < H$. In addition, sketch the more realistic shape of the shock.

Appendix A Problem 1 Python Code

```
# Compressible Flow
# AEE 553
# Homework 4 - Problem 1
# Evan Burke

import numpy as np
from matplotlib import pyplot as plt
import shocks as ns
import isentropic as isen

# b
gamma = 1.4

def hugoniot(gamma=1.4,rho_ratio=None):
    p2_p1 = ( (gamma+1)/(gamma-1) * rho_ratio - 1) / ((gamma+1)/(gamma-1)
    - rho_ratio)
    return p2_p1

rho_ratio = np.linspace(1,5,endpoint=True)

p2_p1_h = [hugoniot(rho_ratio=r) for r in rho_ratio] # normal shock
p2_p1_i = [r**gamma for r in rho_ratio]

fig,ax = plt.subplots()
ax.plot(rho_ratio,p2_p1_h,label='Hugoniot - NS')
ax.plot(rho_ratio,p2_p1_i,label='Isentropic')
ax.legend()
ax.set_xlabel('Density Ratio')
ax.set_ylabel('Pressure Ratio')
ax.set_title('Compression vs. Density Ratio')
plt.savefig('../images/problem_1/hugoniot_vs_isentropic_compression.png')
plt.close()

machs = np.linspace(1,6,endpoint=True)
print(machs)
pt2_pt1 = [ns.get_total_pressure_ratio_normal_shock(M1=m) for m in machs]
    #get_total_pressure worked without a static pressure? need error
    handling
print(pt2_pt1)
fig,ax = plt.subplots()
ax.plot(machs,pt2_pt1)
ax.set_xlabel('Mach')
ax.set_ylabel('Total Pressure Ratio')
ax.set_title('Normal Shock Total Pressure Ratio vs. Mach Number')
plt.savefig('../images/problem_1/compression_efficiency_NS.png')
plt.close()
```

```python
# d
cp = 1004.5
R = 287
ds_isen = [(cp * np.log(r**(gamma-1)) - R * np.log(pr)) for r,pr in zip(
    rho_ratio,p2_p1_i)]

m1 = [ns.get_upstream_mach_normal_shock(p2_p1=pr) for pr in p2_p1_h]
pt2_pt1 = [ns.get_total_pressure_ratio_normal_shock(M1=m) for m in m1]
ds_ns = [-R*np.log(ptr) for ptr in pt2_pt1]
print(ds_isen)
print(ds_ns)

fig,ax = plt.subplots()

ax.plot(rho_ratio,ds_isen,label='Isentropic')
ax.plot(rho_ratio,ds_ns,label='Normal Shock')
ax.set_title('Entropy Change vs. Density Ratio')
ax.set_xlabel('Density ratio')
ax.set_ylabel('Entropy Change [kJ/kg*K]')
ax.legend()

plt.savefig('../images/problem_1/entropy_change.png')

pr_crit = hugoniot(rho_ratio=2.5)
m1_crit = ns.get_upstream_mach_normal_shock(p2_p1=pr_crit)
print(f'Critical Mach: {m1_crit}')
```

Appendix B Problem 2 Python Code

```python
# Compressible Flow
# AEE 553
# Homework 4 - Problem 2
# Evan Burke

from gettext import find
import numpy as np
from matplotlib import pyplot as plt
import shocks as ns
import oblique as os
import isentropic as isen
from scipy.optimize import fsolve

M = 3
T = 217 # K
p = 20000 # Pa
gamma = 1.4
R = 287 # J/kg K
cp = 1000 # J/kg K
q = 500 # kJ/kg

thetas = np.linspace(1,34,num=67,endpoint=True)
print(thetas)

delta = 1 # weak shock solution

def find_beta(M=None,gamma=1.4,theta=None):
    theta = np.deg2rad(theta)
    lamb = ((M**2-1)**2 - 3*(1 + (gamma-1)/2*M**2) * (1 + (gamma+1)/2*M
    **2) * np.tan(theta)**2)**0.5
    chi = ((M**2-1)**3 - 9 * (1 + (gamma-1)/2 * M**2) * (1 + (gamma-1)/2 *
     M**2 + (gamma+1)/4*M**4)*np.tan(theta)**2)/lamb**3
    tan_beta = (M**2 - 1 + 2*lamb*np.cos((4*np.pi*delta+np.arccos(chi))/3)
    ) / (3 * (1 + (gamma-1)/2*M**2)*np.tan(theta))
    beta = np.arctan(tan_beta)
    beta = np.rad2deg(beta)
    print(f'Shock angle = {beta}')
    return beta

betas = [find_beta(M=M,theta=th) for th in thetas if not np.isnan(
    find_beta(M=M,theta=th))]

def rayleigh(M2,M1,Tt2_Tt1,gamma=1.4):
    eq = (M2/M1)**2 * ((1 + gamma*M1**2)/(1 + gamma*M2**2))**2 * ((1 + (
    gamma-1)/2*M2**2)/(1 + (gamma-1)/2 * M1**2)) - Tt2_Tt1
    return eq

```

```
efficiency = []

for beta,theta in zip(betas,thetas): # loop to solve for ideal half angle based on efficiency
     print(f'Theta = {theta}, Beta = {beta}')
     M1n = os.get_m1_normal(M1=M,beta=beta)
     M2n = os.get_m2_normal(M1n=M1n)
     p2_p1 = ns.get_static_pressure_ratio_normal_shock(M1=M1n)
     pt2_pt1 = ns.get_total_pressure_ratio_normal_shock(M1=M1n)
     M2 = os.get_m2(M2n=M2n,beta=beta,theta=theta)
     M3 = ns.get_mach_normal_shock(M1=M2)
     p3_p2 = ns.get_static_pressure_ratio_normal_shock(M1=M2)
     pt3_pt2 = ns.get_total_pressure_ratio_normal_shock(M1=M2)
     p1_p3 = 1/p2_p1 * 1/p3_p2
     pt3_pt1 = pt2_pt1 * pt3_pt2
     T2_T1 = ns.get_static_temperature_ratio_normal_shock(M1=M1n)
     T3_T2 = ns.get_static_temperature_ratio_normal_shock(M1=M2)
     T3_T1 = T3_T2 * T2_T1
     print(f'T3_T1 = {T3_T1}')
     T3 = T3_T2 * T2_T1 * T
     print(f'T3 = {T3}')
     Tt3 = isen.get_total_temperature(M=M3,T=T3)
     Tt4 = Tt3 + (1000*q)/cp
     print(f'Tt4 = {Tt4}')
     Tt4_Tt3 = Tt4/Tt3
     print(f'Tt4/Tt3 = {Tt4_Tt3}')
     # The commmented lines are for solving with Rayleigh
     #M4 = float(fsolve(rayleigh,.5,args=(M3,Tt4/Tt3)))
     #print(f'M4 = {M4}')
     T4 = float(isen.get_static_temperature(M=M3,T_t=Tt4))
     eta = 1 - ((p1_p3)**((gamma-1)/gamma) * (T4 - (pt3_pt1)**((gamma-1)/gamma)*T3) / (T4-T3))
     print(f'Scramjet efficiency = {eta}')
     efficiency.append(float(eta))
     print('\n\n')

max_eta = max(efficiency)
print(f'Max efficiency = {max_eta}')
idx_max = efficiency.index(max_eta)
print(f'idx max = {idx_max}')
theta_ideal = thetas[idx_max]
print(f'Ideal half angle = {theta_ideal}')
print(thetas)

fig,ax = plt.subplots()
ax.set_title("Ramjet Cycle Efficiency vs. Inlet Half Angle")
ax.set_xlabel('Inlet Half Angle [degrees]')
ax.set_ylabel('Ramjet Efficiency')
plt.plot(thetas,efficiency,'-')
plt.plot(theta_ideal,max_eta,'r*')
```

```
plt.savefig('../images/problem_2/idealtheta.png')



# c

M2 = M
M3 = ns.get_mach_normal_shock(M1=M2)
p3_p2 = ns.get_static_pressure_ratio_normal_shock(M1=M2)
pt3_pt2 = ns.get_total_pressure_ratio_normal_shock(M1=M2)
p2_p3 = 1/p3_p2
pt3_pt1 = pt3_pt2
T3_T2 = ns.get_static_temperature_ratio_normal_shock(M1=M2)
T3_T1 = T3_T2
print(f'T3_T1 = {T3_T1}')
T3 = T3_T2 * T
print(f'T3 = {T3}')
Tt3 = isen.get_total_temperature(M=M3,T=T3)
Tt4 = Tt3 + (1000*q)/cp
print(f'Tt4 = {Tt4}')
Tt4_Tt3 = Tt4/Tt3
print(f'Tt4/Tt3 = {Tt4_Tt3}')
# The commented portions are used for solving with Rayleigh flow
#M4 = float(fsolve(rayleigh,.8,args=(M3,Tt4/Tt3)))
#print(f'M4 = {M4}')
T4 = float(isen.get_static_temperature(M=M3,T_t=Tt4))
eta_NS = 1 - ((p2_p3)**((gamma-1)/gamma) * (T4 - (pt3_pt2)**((gamma-1)/
    gamma)*T3) / (T4-T3))
print(f'Scramjet efficiency -- no spike = {eta_NS}')

print('\n\n')

# e

machs = np.linspace(2,6,num=21,endpoint=True)
print(machs)
betas = [find_beta(M=M,theta=theta_ideal) for M in machs]
efficiency = []

for mach,beta in zip(machs,betas): # loop to solve for ideal half angle
    based on efficiency
    print(f'Cruise Mach = {mach}')
    beta = find_beta(M=mach,theta=theta_ideal)

    M1n = os.get_m1_normal(M1=mach,beta=beta)
    M2n = os.get_m2_normal(M1n=M1n)
    p2_p1 = ns.get_static_pressure_ratio_normal_shock(M1=M1n)
    pt2_pt1 = ns.get_total_pressure_ratio_normal_shock(M1=M1n)
    M2 = os.get_m2(M2n=M2n,beta=beta,theta=theta_ideal)
    M3 = ns.get_mach_normal_shock(M1=M2)
```

```
    p3_p2 = ns.get_static_pressure_ratio_normal_shock(M1=M2)
    pt3_pt2 = ns.get_total_pressure_ratio_normal_shock(M1=M2)
    p1_p3 = 1/p2_p1 * 1/p3_p2
    pt3_pt1 = pt2_pt1 * pt3_pt2
    T2_T1 = ns.get_static_temperature_ratio_normal_shock(M1=M1n)
    T3_T2 = ns.get_static_temperature_ratio_normal_shock(M1=M2)
    T3_T1 = T3_T2 * T2_T1
    print(f'T3_T1 = {T3_T1}')
    T3 = T3_T2 * T2_T1 * T
    print(f'T3 = {T3}')
    Tt3 = isen.get_total_temperature(M=M3,T=T3)
    Tt4 = Tt3 + (1000*q)/cp
    print(f'Tt4 = {Tt4}')
    Tt4_Tt3 = Tt4/Tt3
    print(f'Tt4/Tt3 = {Tt4_Tt3}')
    #M4 = float(fsolve(rayleigh,.5,args=(M3,Tt4/Tt3)))
    #print(f'M4 = {M4}')
    T4 = float(isen.get_static_temperature(M=M3,T_t=Tt4))
    print(f'T4 = {T4}')
    print(f'T4-T3 = {T4-T3}')
    print(f'p1/p3 = {p1_p3}')
    print(f'pt3/p13 = {pt3_pt1}')
    eta = 1 - ((p1_p3)**((gamma-1)/gamma) * (T4 - (pt3_pt1)**((gamma-1)/
   gamma)*T3) / (T4-T3))
    print(f'Scramjet efficiency = {eta}')
    efficiency.append(float(eta))
    print('\n\n')

eta_max = max(efficiency)
idx_ideal = efficiency.index(eta_max)
ideal_mach = machs[idx_ideal]

print(f'Ideal Mach = {ideal_mach}, ideal efficiency = {eta_max}')

fig,ax = plt.subplots()
ax.set_title("Ramjet Cycle Efficiency vs. Cruise Mach")
ax.set_xlabel('Cruise Mach')
ax.set_ylabel('Ramjet Efficiency')
ax.set_ylim(bottom=0,top=1)
plt.plot(machs,efficiency,'-')
plt.plot(ideal_mach,eta_max,'r*')
plt.savefig('../images/problem_2/eta_vs_mach.png')
print(f'Max efficiency = {max_eta}')
```