Problem 1 (30 pts)

The final Reynolds Transport Theorem we derived in class looked like:

$$\frac{\mathrm{dB_{sys}}}{\mathrm{dt}} = \frac{\mathrm{d(mb)_{sys}}}{\mathrm{dt}} = \frac{\partial}{\partial t}\int_{\mathrm{CV}} \rho b \mathrm{dV} + \int_{\mathrm{CS,out}} \mathrm{b}\rho|\overrightarrow{V_{\mathrm{n}}}|\mathrm{dA} - \int_{\mathrm{CS,in}} \mathrm{b}\rho|\overrightarrow{V_{\mathrm{n}}}|\mathrm{dA} \qquad .$$

(a) In your own words, describe what each of the three terms on the right-hand-side of the equation mean related to an arbitrary fluid extensive property, B.

(b) If our problem was in $x-y-z$ space, how would you represent the integrals $\int_{\mathrm{CV}} \mathrm{dV}$ and $\int_{\mathrm{CS}} \mathrm{dA}$ in terms of triple and double integrals, respectively?

(c) Why are the last two terms integral terms?

(d) What does the subscript "n" mean for the last two terms? Why do we need that there?

(e) Why do we need the absolute magnitude signs around the $\overrightarrow{V_{\mathrm{n}}}$ terms?

(f) Why is the derivative with-respect-to t a partial derivative?

(g) Explain to a classmate how our

$$\int_{\mathrm{CS,out}} b\rho|\overrightarrow{V_{\mathrm{n}}}|\mathrm{dA} - \int_{\mathrm{CS,in}} \mathrm{b}\rho|\overrightarrow{V_{\mathrm{n}}}|\mathrm{dA}$$

term is equivalent to

$$\int_{\mathrm{CS}} b\rho \boldsymbol{V}\cdot\hat{\boldsymbol{n}}\mathrm{dA} \qquad ,$$

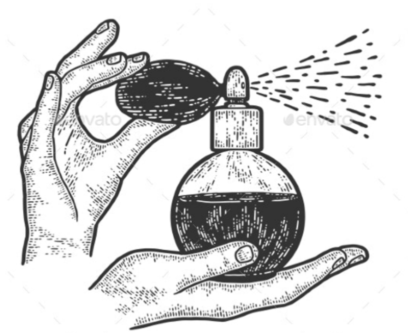
which is equivalent to

$$\int_{\mathrm{CS}} b\rho\overrightarrow{V}\cdot \mathrm{d}\overrightarrow{A} \qquad .$$

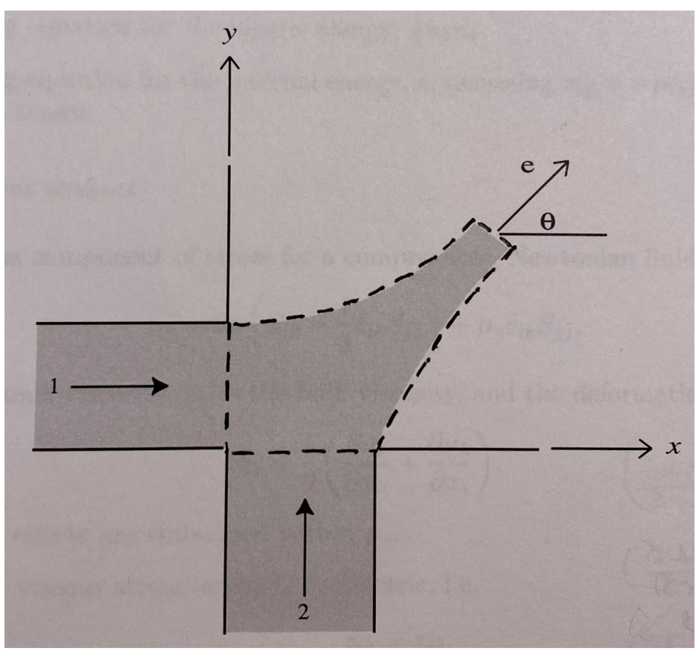
Be sure to explain the different math concepts. You may find it easier to "explain" by using a simple control-volume problem as an illustration.

Problem 2 (12 pts)

Some of you may be familiar with vintage perfume bottles (i.e., atomizers). See the images below. Generally speaking, these work by mixing/combining two fluids (often air and the perfume) initially travelling in different directions. After being mixed, they travel together in a third direction. The problem below will model this phenomena in a general sense.

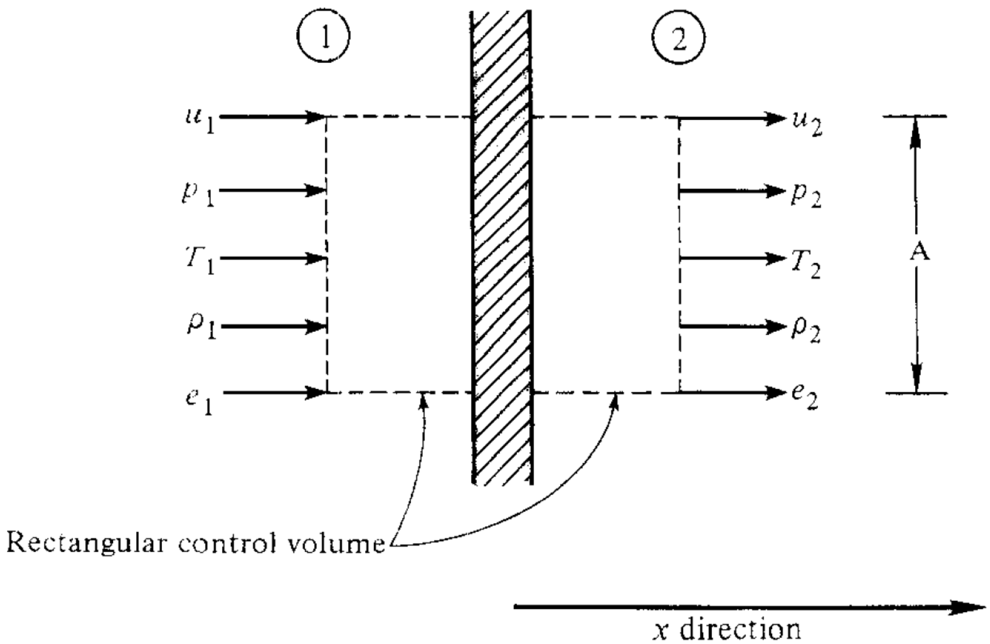


Consider the problem of two steady, uniform, and incompressible fluid jets colliding at right angles as shown to form a common jet at an angle θ. The pressure everywhere is p_{atm}, and gravity/shear stress can be safely neglected. The control volume is given by the dashed lines. Write out, simplify (stating your assumptions), and solve the continuity equation and relevant momentum equations. Find the angle θ in terms of the flow properties u_1, $\dot{m}_1$, v_2, and $\dot{m}_2$ of the two jets (where $\dot{m}$ is the mass flowrate). State your assumptions.



Problem 3 (31 pts)

One-dimensional, steady, compressible flow is used for a number of real-world applications, including: normal shock waves, bow shock waves, etc. Look up some images or videos of normal shock waves and bow shock waves in front of bullets, re-entry vehicles, etc. A schematic illustrating such flow is given below, where the flow entering the dashed control volume is given as state 1 and the flow exiting as state 2. We will learn later in the semester that these properties indeed do change across shock waves. For now, we will focus on simplifying our governing equations for these assumptions.



In our one-dimensional, steady analyses, we will make the following assumptions about our flow:

i One-dimensional in the x direction

ii Steady

iii Uniform velocity, pressure, temperature, density, enthalpy, and energy at each of the two control surfaces

iv Flow is perpendicular to control surfaces 1 and 2

v $A_1 = A_2$

vi No body forces present

vii No friction/shear (i.e., there are no solid boundaries around)

viii No work is done

ix The pressures acting on the control volume in the y and z directions apply no net force

(a) Under these assumptions for one-dimensional, steady flow, show that the integral form of the continuity equation simplifies to,

$$\rho_1 u_1 = \rho_2 u_2 \qquad .$$

You must start with the full integral form and indicate which assumption(s) allowed you to make each simplification.

(b) Can the schematic above, and assumption iii really be valid for a compressible flow? Explain your reasoning.

(c) What would the result be if we assumed "quasi-one-dimensional flow"? Note, the only difference between one-dimensional flow and quasi-one-dimensional flow is that assumption (v) is no longer valid for quasi-one-dimensional flow.

(d) Under the assumptions for one-dimensional, steady flow, show that the integral form of the x-momentum equation simplifies to,

$$p_1 + \rho_1 u_1^2 = p_2 + \rho_2 u_2^2 \qquad .$$

You must start with the full integral form and indicate which assumption(s) allowed you to make each simplification.

(e) What would the y- and z-momentum equations simplify to?

(f) Is assumption (vii) ever a good assumption for compressible flows? If you think it is, give a realistic application/example of when it is. If you don't think it is, explain why not.

(g) Under the assumptions for one-dimensional, steady flow, show that the integral form of the energy equation simplifies to,

$$h_1 + \frac{u_1^2}{2} + q = h_2 + \frac{u_2^2}{2} \qquad .$$

You must start with the full integral form and indicate which assumption(s) allowed you to make each simplification. q is the mass-specific heat.

Problem 4 (60 pts)

You will use this problem to derive the **differential** form of conservation of mass and momentum for **inviscid** flowfields. You will do this by applying the the integral form of these governing equations to a differential-element control volume.

(a) **Conservation of Mass:**

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} = 0$$

Watch the attached lecture from my undergraduate fluid-mechanics class and follow the steps to derive the differential form of the continuity equation. Be explicit about your assumptions and make the derivation your own.

(b) **(Inviscid) Conservation of Momentum:**

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) = \rho g_x - \frac{\partial p}{\partial x} + \cancel{\mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)}$$

$$\rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right) = \rho g_y - \frac{\partial p}{\partial y} + \cancel{\mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right)}$$

$$\rho\left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right) = \rho g_z - \frac{\partial p}{\partial z} + \cancel{\mu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right)}$$

(b1) Draw your differential element with coordinate system. Label each side δx, δy, and δz, respectively.

(b2) Label each of the six sides with the appropriate momentum values. Include the direction of the momentum with arrows. This process will be identical to what we did for Continuity, except now we will have an addition velocity (i.e., $\overrightarrow{V}$) term since we are talking about momentum. For example, the left-hand-side of the differential element should have $\rho u\overrightarrow{V}$ flowing into it instead of just ρu. You can show this analysis on one single differential-element drawing, or three separate ones.

(b3) Now, write out the left-hand-side of the general Integral Momentum Equation, state the relevant assumptions/simplifications based on your differential element control-volume drawing(s), and simplify each term as much as you can.

(b4) Tabulate the influx of momentum and outflux of momentum for each face. Do not forget about the corresponding area terms.

(b5) Populate your simplified Momentum Equation with these influx and outflux terms appropriately.

(b6) Show how three outflux terms cancel with the three influx terms to give:

$$\left[\frac{\partial(\rho\vec{V})}{\partial t}+\frac{\partial(\rho u\vec{V})}{\partial x}+\frac{\partial(\rho v\vec{V})}{\partial y}+\frac{\partial(\rho w\vec{V})}{\partial z}\right]\delta x\delta y\delta z \tag{1}$$

(b7) Each individual term in Eq. 1 can be broken up via the Chain Rule for derivatives, which states:

$$\frac{\partial(ab)}{\partial c}=a\frac{\partial(b)}{\partial c}+b\frac{\partial(a)}{\partial c} \tag{2}$$

Use this rule to break up each of the four partial derivatives in Eq. 1. For terms 2–4, treat the products ρu, ρv, and ρw as a from Eq. 2, and $\vec{V}$ as b. That is, term 2 from Eq. 1 can be split up as:

$$\frac{\partial(\rho u\vec{V})}{\partial x}=\rho u\frac{\partial(\vec{V})}{\partial x}+\vec{V}\frac{\partial(\rho u)}{\partial x} \tag{3}$$

Your updated Eq. 1 should now have eight terms inside the brackets. Four of these terms are the Differential Continuity Equation and their combination can therefore be set to zero. Show this, and show that the new equation is:

$$\left[\rho\frac{\partial(\vec{V})}{\partial t}+\rho u\frac{\partial(\vec{V})}{\partial x}+\rho v\frac{\partial(\vec{V})}{\partial y}+\rho w\frac{\partial(\vec{V})}{\partial z}\right]\delta x\delta y\delta z \tag{4}$$

Woohoo, this is the final form of the left-hand-side of the Differential Momentum Equation!!

(b8) Now calculate the forces due to gravity and pressure on the differential element and add them to the right hand side of your differential momentum equation. Note: for pressure, you may allow for pressure to change in each direction across your differential element via a Taylor-Series-Expansion first-order approximation like you've done in other parts of the problem.

(b9) Put everything together and write the final form appropriately for the x, y, and z directions to give the x, y, and z differential momentum equations.