AEE 553 — Compressible Flow

Department of Mechanical and Aerospace Engineering

Homework 2

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Problem 1

The final Reynolds Transport Theorem we derived in class looked like:

$$\frac{\mathrm{d}B_{sys}}{\mathrm{d}t} = \frac{\mathrm{d}(mb)_{sys}}{\mathrm{d}t} = \frac{\partial}{\partial t}\int_{CV} \rho b \mathrm{d}\forall + \int_{CS,out} b\rho|\vec{V_n}|\mathrm{d}A - \int_{CS,out} b\rho|\vec{V_n}|\mathrm{d}A$$

(a) In your own words, describe what each of the three terms on the right-hand-side of the equation mean related to an arbitrary fluid extensive property, B.

- The first term on the RHS of the equation deals with the time rate change of some extensive property, B, expressed in terms of its intensive version, $b = B/m$. This is specifically the rate of change of the property within some control volume (CV) that we have arbitrarily defined for the purposes of analyzing a problem. In a steady-state problem, this term is 0, as there will be no dependency on time in that case.
- The second term on the RHS of the equation treats the flux of the intensive property b *out* of the CV across every control surface (CS). The velocity term is specifically the velocity normal to each CS, in order to capture the convective effect of a fluid's motion through a CV.
- The third term on the RHS of the equation is identical to the second term except instead of treating flux of b *out* of the CV, it treats flux of b *into* the CV. As before, the velocity term present in this term is specifically the velocity normal to each CS. The sign convention for the fluxes is simplified in this casting of the Reynolds Transport Theorem (RTT).

(b) If our problem was in the $x-y-z$ space, how would you represent the integrals $\int_{CV} \mathrm{d}V$ and $\int_{CS} \mathrm{d}A$ in terms of triple and double integrals, respectively?

In $x-y-z$ space, the integrals can be expressed as follows:

$$\int_{CV} \mathrm{d}\forall \rightarrow \int_x \int_y \int_z \mathrm{d}x\mathrm{d}y\mathrm{d}z$$

$$\int_{CS} \mathrm{d}A \rightarrow \int_x \int_y \mathrm{d}x\mathrm{d}y$$

Note: The two specific dimensions in the second integral will vary depending on the orientation of the control surfaces relative to the major axes.

(c) Why are the last two terms integral terms?

The last two terms are integral terms because the fluid density and velocity can vary across a CS. By integrating across the CS, the full behavior at the boundaries can be captured. If there is no variation in any of the properties in either spatial dimension making up dA, there is no need for an integral.

(d) What does the subscript "n" mean for the last two terms? Why do we need that there?

The subscript "n" indicates that the velocities are **normal** to the CS. In this casting of the RTT we sidestep the need for vector calculus and confusing sign conventions by simply calling for the normal velocity magnitude. Polarity of the terms are simply defined by whether a flux is "in" or "out" of the CS.

(e) Why do we need the absolute magnitude signs around the $\vec{V}_n$ terms?

The absolute magnitude signs around the velocity terms are required for a similar reason as the "n" subscript: to simplify the handling of sign conventions. Taking the absolute magnitude of velocity removes the complexity of juggling multiple conflicting signs between velocity vectors, magnitudes, and CS normal vectors.

(f) Why is the derivative with-respect-to t a partial derivative?

The derivative w.r.t t is a partial derivative because the quantities inside the partial derivative do not only vary in time — they can also vary spatially. Isolating the time-variant component of the intensive property b inside the CV helps capture the generation term required for proper bookkeeping of the extensive property B.

(g) Explain to a classmate how our

$$\int_{CS,out} b\rho|\vec{V}_n|\mathrm{d}A - \int_{CS,out} b\rho|\vec{V}_n|\mathrm{d}A$$

term is equivalent to

$$\int_{CS} b\rho\boldsymbol{V}\cdot\hat{\boldsymbol{n}}\mathrm{d}A\,,$$

which is equivalent to

$$\int_{CS} b\rho\vec{V}\cdot\mathrm{d}\vec{A}\,.$$

Be sure to explain the different math concepts. You may find it easier to "explain" by using a simple control-volume problem as an illustration.

The second and third terms of the RHS of RTT are a decomposition of a term defined using vector notation. Beginning with the third integral form, we isolate the term $\vec{V}\cdot\mathrm{d}\vec{A}$. The mathematical meaning of $\vec{V}$ is the fluid's velocity vector in component form (i.e., $\hat{\imath}, \hat{\jmath}, \hat{k}$ form). Similarly, $\mathrm{d}\vec{A}$ is the vector notation for a differential area *including its normal direction*. As with any other vector, $\mathrm{d}\vec{A}$ can be expressed in terms of a magnitude and direction. For the differential area, the magnitude is simply $\mathrm{d}A$, and the normal direction can be generically expressed as $\hat{\mathbf{n}}$. We can rewrite $\mathrm{d}\vec{A}$ as $\hat{\mathbf{n}}\mathrm{d}A$, as seen in the second integral form. The vector dot product in the third integral now becomes the dot product of the velocity vector and the differential area's normal vector, i.e., $\mathbf{V} \cdot \hat{\mathbf{n}}$.

Isolating the term $\mathbf{V} \cdot \hat{\mathbf{n}}$, we recall the definition of a dot product and scalar projection. The dot product is defined in euclidean space as $\vec{a} \cdot \vec{b} = \|a\|\|b\| \cos\theta$. The scalar projection is defined in euclidean space as $\mathbf{a} \cdot \hat{\mathbf{b}} = \|a\| \cos\theta$. A scalar projection is a unique case of a dot product where one of the vector terms is a unit vector, denoted by $\hat{\mathbf{v}}$. The result of the scalar projection is the magnitude of the vector $\mathbf{a}$ in the direction of the unit vector $\hat{\mathbf{b}}$. Replacing $\hat{\mathbf{b}}$ with $\hat{\mathbf{n}}$, we see that the output of the scalar product $\mathbf{V} \cdot \hat{\mathbf{n}}$ is the magnitude of the velocity vector projected onto the normal vector of the CS, which can be expressed as $\pm|\vec{V}_n|$, depending on the vector orientation.

To understand the polarity associated with the scalar projection, we recall the behavior of $\cos\theta$ on $[0, 2\pi]$ to obtain the following "boundary conditions" for the function:

$$\begin{aligned}
\cos(0) &= 1 \\
\cos(\pi/2) &= 0 \\
\cos(\pi) &= -1 \\
\cos(3\pi/2) &= 0
\end{aligned}$$

Examining these four conditions inform us about the importance of understanding the convective vector's direction relative to a CS. If the velocity vector is flowing *in* to the CS it will be oriented opposite the normal vector defining the CS (outward by convention), making the angle between velocity and normal vector $\theta = \pi$. With this and the behavior of the cos function in mind, we see that $\mathbf{V} \cdot \hat{\mathbf{n}}$ for an influx will have a negative polarity. Following the same logic for an outflux, where the velocity vector and CS normal vector make an angle of $\theta = 0$, we see that an outflux will have a positive polarity.

Note: another interesting observation from the behavior of $\cos()$ *is that perpendicular vectors, with* $\theta = \pi n/2$*, result in 0 flux across a CS.*

We now have all of the building blocks required to connect the three integral forms. We have shown that

$$\vec{V} \cdot \mathrm{d}\vec{A}$$

is equivalent to

$$\mathbf{V} \cdot \hat{\mathbf{n}}\mathrm{d}A\,,$$

and that

$$\mathbf{V} \cdot \hat{\mathbf{n}} = \pm|\vec{V}_n|$$

depending on the vector orientation.

Next, we have proven that velocity vectors flowing *into* a CS will have a negative polarity, and velocity vectors flowing *out of* a CS will have a positive polarity.

$$cos(0) = 1$$
$$cos(\pi) = -1$$

Therefore,

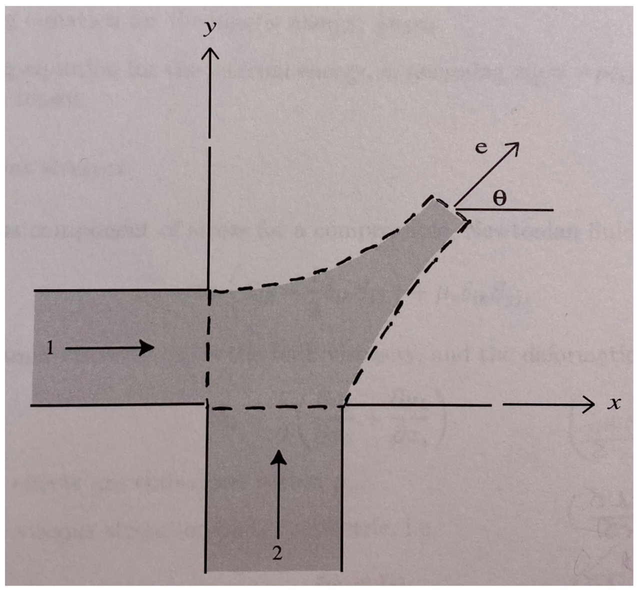
$$\mathbf{V}_{inflow} \cdot \hat{\mathbf{n}} = -|\vec{V}_n|$$
$$\mathbf{V}_{outflow} \cdot \hat{\mathbf{n}} = +|\vec{V}_n|$$

Thus,

$$\int_{CS} b\rho\vec{V} \cdot \mathrm{d}\vec{A} = \int_{CS,out} b\rho|\vec{V}_n|\mathrm{d}A - \int_{CS,out} b\rho|\vec{V}_n|\mathrm{d}A$$

Problem 2

Consider the problem of two steady, uniform, and incompressible fluid jets colliding at right angles as shown to form a common jet at an angle θ. The pressure everywhere is p_{atm}, and gravity/shear stress can be safely neglected. The control volume is given by the dashed lines. Write out, simplify (stating your assumptions), and solve the continuity equation and relevant momentum equations. Find the angle θ in terms of the flow properties $u_1, \dot{m}_1, v_2,$ and $\dot{m}_2$ of the two jets (where $\dot{m}$ is the mass flowrate). State your assumptions.



Givens:

$u_1, \dot{m}_1, v_2, \dot{m}_2, \theta, p_{atm}$

Assumptions:

Steady, uniform, incompressible flow. Gravity and shear stress can be ignored. Pressure everywhere is P_{atm}. Jets collide at a right angle. All flow velocities are normal to inlets/outlets.

Solution:

The full integral form of the conservation of mass equation is given by

$$\frac{\mathrm{d}(m)_{sys}}{\mathrm{d}t} = \frac{\partial}{\partial t}\int_{CV} \rho \mathrm{d}V\!\!\!/ + \int_{CS} \rho \left(\vec{V} \cdot \hat{\mathbf{n}}\right) \mathrm{d}A = 0$$

The assumption of steady flow removes the time-dependency of the continuity equation:

$$\cancelto{0}{\frac{\partial}{\partial t}\int_{CV} \rho \mathrm{d}V\!\!\!/} + \int_{CS} \rho \left(\vec{V} \cdot \hat{\mathbf{n}}\right) \mathrm{d}A = 0$$

$$\int_{CS} \rho \left(\vec{V} \cdot \hat{\mathbf{n}}\right) \mathrm{d}A = 0$$

The assumption of uniform flow removes the spatial dependence of the integrand:

$$\rho \left(\vec{V} \cdot \hat{\mathbf{n}}\right) \int_{CS} \mathrm{d}A = 0$$

This results in the following expression evaluated at every CS:

$$\rho \left(\vec{V} \cdot \hat{\mathbf{n}}\right) A\Big|_{CS} = 0$$

The assumption of incompressibility implies that ρ is constant throughout the flow and the same at every CS, and can therefore be divided out.

$$\left(\vec{V} \cdot \hat{\mathbf{n}}\right) A\Big|_{CS} = 0$$

Recalling that $\left(\vec{V} \cdot \hat{\mathbf{n}}\right)$ can be expressed as $-|\vec{V}_n|$ for influx and $+|\vec{V}_n|$ for outflux, we rewrite the simplified form of the continuity equation in terms of influx and outflux:

$$\sum_{outflux} |\vec{V}_n|A - \sum_{influx} |\vec{V}_n|A = 0$$

$$\sum_{outflux} |\vec{V}_n|A = \sum_{influx} |\vec{V}_n|A$$

The sum of all mass influx terms must be equal to the sum of all outflux terms to satisfy continuity.

Examining our problem, with inlet control surfaces 1 and 2 and outlet control surface e, we apply the simplified form of continuity to yield:

$$V_1 A_1 + V_2 A_2 = V_e A_e$$

where V_e represents the velocity in the exit direction.

Multiplying through by density yields an expression in terms of mass flow rate:

$$\rho V_1 A_1 + \rho V_2 A_2 = \rho V_e A_e$$

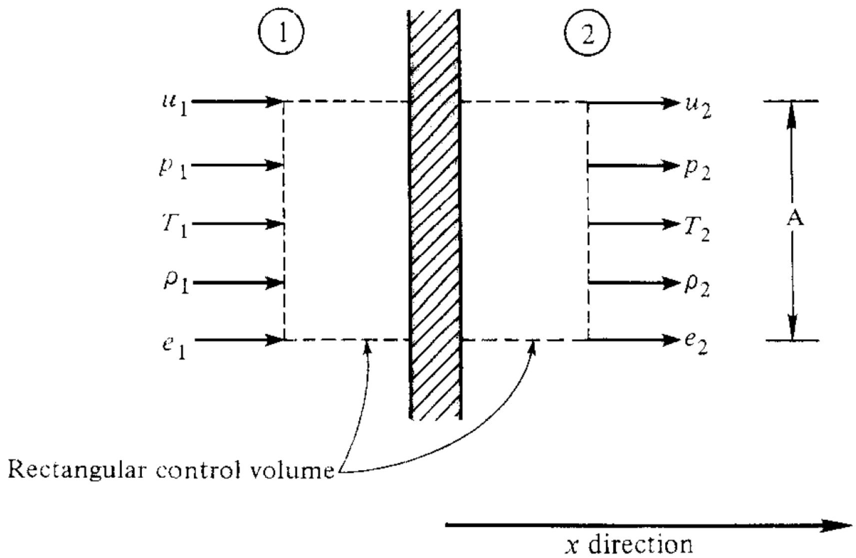
$$\dot{m}_1 + \dot{m}_2 = \dot{m}_e$$

The full integral form of the conservation of momentum equation is given by

$$\frac{\mathrm{d}\left(m\vec{V}\right)_{sys}}{\mathrm{d}t} = \frac{\partial}{\partial t}\int_{CV} \rho\vec{V}\mathrm{d}\forall + \int_{CS} \rho\vec{V}\left(\vec{V}\cdot\hat{\mathbf{n}}\right)\mathrm{d}A = \sum \vec{F}_{sys}$$

Problem 3

One-dimensional, steady, compressible flow is used for a number of real-world applications, including: normal shock waves, bow shock waves, etc. Look up some images or videos of normal shock waves and bow shock waves in front of bullets, re-rentry vehicles, etc. A schematic illustrating such flow is given below, where the flow entering the dashed control volume is given as state 1 and the flow exiting as state 2. We will learn later in the semester that these properties do indeed change across shock waves. For now, we will focus on simplifying our governing equations for these assumptions.



In our one-dimensional, steady analyses, we will make the following assumptions about our flow:

(i) One-dimensional in the x direction

(ii) Steady

(iii) Uniform velocity, pressure, temperature, density, enthalpy, and energy at each of the two control surfaces

(iv) Flow is perpendicular to control surfaces 1 and 2

(v) $A_1 = A_2$

(vi) No body forces present

(vii) No friction/shear (i.e., there are no solid boundaries around)

(viii) No work is done

(ix) The pressures acting on the control volume in the y and z directions apply no net force

(a) Under these assumptions for one-dimensional, steady flow, show that the integral form of the continuity equation simplifies to

$$\rho_1 u_1 = \rho_2 u_2 \,.$$

You must start with the full integral form and indicate which assumption(s) allowed you to make each simplification.

(b) Can the schematic above and assumption (iii) really be valid for compressible flow? Explain your reasoning.

(c) What would the result be if we assumed "quasi-one-dimensional flow"? Note, the only difference between one-dimensional flow and quasi-one-dimensional flow is that assumption (v) is no longer valid for quasi-one-dimensional flow.

(d) Under the assumptions for one-dimensional, steadu flow, show that the integral form of the x-momentum equation simplifies to

$$p_1 + \rho_1 u_1^2 = p_1 + \rho_2 u_2^2 \,.$$

You must start with the full integral form and indicate which assumption(s) allowed you to make each simplification.

(e) What would the y and z-momentum equations simplify to?

(f) Is assumption (vii) ever a good assumption for compressible flows? If you think it is, give a realistic application/example of when it is. If you don't think it is, explain why not.

(g) Under the assumptions for one-dimensional, steadu flow, show that the integral form of the energy equation simplifies to

$$h_1 + \frac{u_1^2}{2} + q = h_2 + \frac{u_2^2}{2}$$

You must start with the full integral form and indicate which assumption(s) allowed you to make each simplification. q is the mass-specific heat.

Problem 4

You will use this problem to derive the **differential** form of conservation of mass and momentum for **inviscid** flowfields. You will do this by applying the integral form of these governing equations to a differential-element control volume.

(a) **Conservation of Mass:**

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} = 0$$

Watch the attached lecture from my undergraduate fluid-mechanics class and follow the steps to derive the differential form of the continuity equation. Be explicity about your assumptions and make the derivation your own.

(b) **(Inviscid) Conservation of Momentum:**

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) = \rho g_x - \frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)$$
$$\rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right) = \rho g_y - \frac{\partial p}{\partial y} + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right)$$
$$\rho\left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right) = \rho g_z - \frac{\partial p}{\partial z} + \mu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right)$$

(b1) Draw your differential element with coordinate system. Label each side $\delta x, \delta y,$ and δz, respectively.

(b2) Label each of the six sides with the appropriate momentum values. Include the direction of the momentum with arrows. This process will be identical to what we did for Continuity, except now we will have an additional velocity (i.e., $\vec{V}$) term since we are talking about momentum. FOr example, the left-hand-side of the differential element should hae $\rho u \vec{V}$ flowing into it instead of just ρu. You can show this analysis on one single differential-element drawing, or three separate ones.

(b3) Now, write out the left-hand-side of the general Integral Momentum Equation, state the relevant assumptions/simplifications based on your differential element control-volume drawing(s), and simplify each term as much as you can.

(b4) Tabulate the influx and outflux of momentum for each face. Do not forget about the corresponding area terms.

(b5) Populate your simplified Momentum Equation with these influx and outflux terms appropriately.

(b6) Show how the three outflux terms cancel with the three influx terms to give:

$$\left[\frac{\partial(\rho\vec{V})}{\partial t} + \frac{\partial(\rho u\vec{V})}{\partial x} + \frac{\partial(\rho v\vec{V})}{\partial y} + \frac{\partial(\rho w\vec{V})}{\partial z}\right]\delta x\delta y\delta z$$

(b7) Each individual term in Eq. can be broken up via the Chain Rule for derivatives, which states:

$$\frac{\partial(ab)}{\partial c} = a\frac{\partial(b)}{\partial c} + b\frac{\partial(a)}{\partial c}$$

Use this rule to break up each of the four partial derivatives in Eq. . For terms 2-4, treat the products ρu, ρv, and ρw as a from Eq. , and $\vec{V}$ as b. That is, term 2 from Eq. can be split up as:

$$\frac{\partial(\rho u\vec{V})}{\partial x} = \rho u\frac{\partial(\vec{V})}{\partial x} + \vec{V}\frac{\partial(\rho u)}{\partial x}$$

Your updated Eq. should now have eight terms inside the brackets. Four of these terms are the Differential Continuity Equation and their combination can therefore be set to zero. Show this, and show that the neq equation is:

$$\left[\rho\frac{\partial(\vec{V})}{\partial t} + \rho u\frac{\partial(\vec{V})}{\partial x} + \rho v\frac{\partial(\vec{V})}{\partial y} + \rho w\frac{\partial(\vec{V})}{\partial z}\right]\delta x\delta y\delta z$$

Woohoo, this is the final form of the left-hand-side of the Differential Momentum Equation!

(b8) Now calculate the forces due to gravity and pressure on the differential element and add them to the right hand side of your differential momentum equation. Note: for pressure, you may allow for pressure to change in each direction across your differential element via a Taylor-Series-Expansion first-order approximation like you've done in other parts of the problem.

(b9) Put everything together and write the final form appropriately for the $x, y, \text{and} z$ directions to give the $x, y, \text{and} z$ differential momentum equations.