AEE 553 — Compressible Flow

Department of Mechanical and Aerospace Engineering

Homework 2

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Problem 1

The final Reynolds Transport Theorem we derived in class looked like:

$$\frac{\mathrm{d}B_{sys}}{\mathrm{d}t} = \frac{\mathrm{d}(mb)_{sys}}{\mathrm{d}t} = \frac{\partial}{\partial t}\int_{CV} \rho b \mathrm{d}\mathcal{V} + \int_{CS,out} b\rho|\vec{V_n}|\mathrm{d}A - \int_{CS,in} b\rho|\vec{V_n}|\mathrm{d}A$$

(a) In your own words, describe what each of the three terms on the right-hand-side of the equation mean related to an arbitrary fluid extensive property, B.

- The first term on the RHS of the equation deals with the time rate change of some extensive property, B, expressed in terms of its intensive version, $b = B/m$. This is specifically the rate of change of the property within some control volume (CV) that we have arbitrarily defined for the purposes of analyzing a problem. In a steady-state problem, this term is 0, as there will be no dependency on time in that case.
- The second term on the RHS of the equation treats the flux of the intensive property b *out* of the CV across every control surface (CS). The velocity term is specifically the velocity normal to each CS, in order to capture the convective effect of a fluid's motion through a CV.
- The third term on the RHS of the equation is identical to the second term except instead of treating flux of b *out* of the CV, it treats flux of b *into* the CV. As before, the velocity term present in this term is specifically the velocity normal to each CS. The sign convention for the fluxes is simplified in this casting of the Reynolds Transport Theorem (RTT).

(b) If our problem was in the $x-y-z$ space, how would you represent the integrals $\int_{CV} \mathrm{d}V$ and $\int_{CS} \mathrm{d}A$ in terms of triple and double integrals, respectively?

In $x-y-z$ space, the integrals can be expressed as follows:

$$\int_{CV} \mathrm{d}\mathcal{V} \rightarrow \int_x \int_y \int_z \mathrm{d}x\mathrm{d}y\mathrm{d}z$$

$$\int_{CS} \mathrm{d}A \rightarrow \int_x \int_y \mathrm{d}x\mathrm{d}y$$

Note: The two specific dimensions in the second integral will vary depending on the orientation of the control surfaces relative to the major axes.

(c) Why are the last two terms integral terms?

The last two terms are integral terms because the fluid density and velocity can vary across a CS. By integrating across the CS, the full behavior at the boundaries can be captured. If there is no variation in any of the properties in either spatial dimension making up dA, there is no need for an integral.

(d) What does the subscript "n" mean for the last two terms? Why do we need that there?

The subscript "n" indicates that the velocities are **normal** to the CS. In this casting of the RTT we sidestep the need for vector calculus and confusing sign conventions by simply calling for the normal velocity magnitude. Polarity of the terms are simply defined by whether a flux is "in" or "out" of the CS.

(e) Why do we need the absolute magnitude signs around the $\vec{V}_n$ terms?

The absolute magnitude signs around the velocity terms are required for a similar reason as the "n" subscript: to simplify the handling of sign conventions. Taking the absolute magnitude of velocity removes the complexity of juggling multiple conflicting signs between velocity vectors, magnitudes, and CS normal vectors.

(f) Why is the derivative with-respect-to t a partial derivative?

The derivative w.r.t t is a partial derivative because the quantities inside the partial derivative do not only vary in time — they can also vary spatially. Isolating the time-variant component of the intensive property b inside the CV helps capture the generation term required for proper bookkeeping of the extensive property B.

(g) Explain to a classmate how our

$$\int_{CS,out} b\rho|\vec{V}_n|\mathrm{d}A - \int_{CS,out} b\rho|\vec{V}_n|\mathrm{d}A$$

term is equivalent to

$$\int_{CS} b\rho\boldsymbol{V}\cdot\hat{\boldsymbol{n}}\mathrm{d}A\,,$$

which is equivalent to

$$\int_{CS} b\rho\vec{V}\cdot\mathrm{d}\vec{A}\,.$$

Be sure to explain the different math concepts. You may find it easier to "explain" by using a simple control-volume problem as an illustration.

The second and third terms of the RHS of RTT are a decomposition of a term defined using vector notation. Beginning with the third integral form, we isolate the term $\vec{V}\cdot\mathrm{d}\vec{A}$. The mathematical meaning of $\vec{V}$ is the fluid's velocity vector in component form (i.e., $\hat{\imath}, \hat{\jmath}, \hat{k}$ form). Similarly, $\mathrm{d}\vec{A}$ is the vector notation for a differential area *including its normal direction*. As with any other vector, $\mathrm{d}\vec{A}$ can be expressed in terms of a magnitude and direction. For the differential area, the magnitude is simply $\mathrm{d}A$, and the normal direction can be generically expressed as $\hat{\mathbf{n}}$. We can rewrite $\mathrm{d}\vec{A}$ as $\hat{\mathbf{n}}\mathrm{d}A$, as seen in the second integral form. The vector dot product in the third integral now becomes the dot product of the velocity vector and the differential area's normal vector, i.e., $\mathbf{V} \cdot \hat{\mathbf{n}}$.

Isolating the term $\mathbf{V} \cdot \hat{\mathbf{n}}$, we recall the definition of a dot product and scalar projection. The dot product is defined in euclidean space as $\vec{a} \cdot \vec{b} = \|a\|\|b\| \cos\theta$. The scalar projection is defined in euclidean space as $\mathbf{a} \cdot \hat{\mathbf{b}} = \|a\| \cos\theta$. A scalar projection is a unique case of a dot product where one of the vector terms is a unit vector, denoted by $\hat{\mathbf{v}}$. The result of the scalar projection is the magnitude of the vector $\mathbf{a}$ in the direction of the unit vector $\hat{\mathbf{b}}$. Replacing $\hat{\mathbf{b}}$ with $\hat{\mathbf{n}}$, we see that the output of the scalar product $\mathbf{V} \cdot \hat{\mathbf{n}}$ is the magnitude of the velocity vector projected onto the normal vector of the CS, which can be expressed as $\pm|\vec{V}_n|$, depending on the vector orientation.

To understand the polarity associated with the scalar projection, we recall the behavior of $\cos\theta$ on $[0, 2\pi]$ to obtain the following "boundary conditions" for the function:

$$
\begin{aligned}
\cos(0) &= 1 \\
\cos(\pi/2) &= 0 \\
\cos(\pi) &= -1 \\
\cos(3\pi/2) &= 0
\end{aligned}
$$

Examining these four conditions inform us about the importance of understanding the convective vector's direction relative to a CS. If the velocity vector is flowing *in* to the CS it will be oriented opposite the normal vector defining the CS (outward by convention), making the angle between velocity and normal vector $\theta = \pi$. With this and the behavior of the cos function in mind, we see that $\mathbf{V} \cdot \hat{\mathbf{n}}$ for an influx will have a negative polarity. Following the same logic for an outflux, where the velocity vector and CS normal vector make an angle of $\theta = 0$, we see that an outflux will have a positive polarity.

Note: another interesting observation from the behavior of $\cos()$ *is that perpendicular vectors, with* $\theta = \pi n/2$, *result in 0 flux across a CS.*

We now have all of the building blocks required to connect the three integral forms. We have shown that

$$\vec{V} \cdot \mathrm{d}\vec{A}$$

is equivalent to

$$\mathbf{V} \cdot \hat{\mathbf{n}}\mathrm{d}A\,,$$

and that

$$\mathbf{V} \cdot \hat{\mathbf{n}} = \pm|\vec{V}_n|$$

depending on the vector orientation.

Next, we have proven that velocity vectors flowing *into* a CS will have a negative polarity, and velocity vectors flowing *out of* a CS will have a positive polarity.

$$cos(0) = 1$$
$$cos(\pi) = -1$$

Therefore,

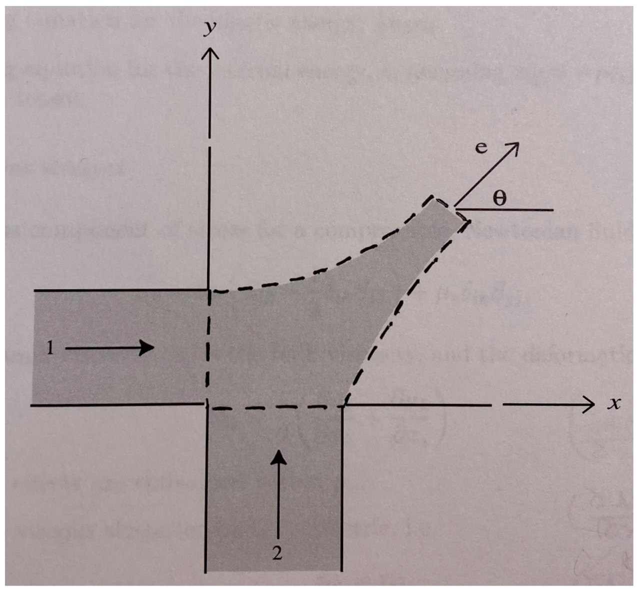
$$\mathbf{V}_{inflow} \cdot \hat{\mathbf{n}} = -|\vec{V}_n|$$
$$\mathbf{V}_{outflow} \cdot \hat{\mathbf{n}} = +|\vec{V}_n|$$

Thus,

$$\int_{CS} b\rho\vec{V} \cdot \mathrm{d}\vec{A} = \int_{CS,out} b\rho|\vec{V}_n|\mathrm{d}A - \int_{CS,out} b\rho|\vec{V}_n|\mathrm{d}A$$

Problem 2

Consider the problem of two steady, uniform, and incompressible fluid jets colliding at right angles as shown to form a common jet at an angle θ. The pressure everywhere is p_{atm}, and gravity/shear stress can be safely neglected. The control volume is given by the dashed lines. Write out, simplify (stating your assumptions), and solve the continuity equation and relevant momentum equations. Find the angle θ in terms of the flow properties $u_1, \dot{m}_1, v_2,$ and $\dot{m}_2$ of the two jets (where $\dot{m}$ is the mass flowrate). State your assumptions.



Givens:

$u_1, \dot{m}_1, v_2, \dot{m}_2, \theta, p_{atm}$

Assumptions:

Steady, uniform, incompressible flow. Gravity and shear stress can be ignored. Pressure everywhere is P_{atm}. Jets collide at a right angle. All flow velocities are normal to inlets/outlets.

Solution:

The full integral form of the conservation of mass equation is given by

$$\frac{\mathrm{d}(m)_{sys}}{\mathrm{d}t} = \frac{\partial}{\partial t}\int_{CV} \rho \mathrm{d}V\!\!\!\!/ + \int_{CS} \rho \left(\vec{V} \cdot \hat{\mathbf{n}}\right) \mathrm{d}A = 0$$

The assumption of steady flow removes the time-dependency of the continuity equation:

$$\cancelto{0}{\frac{\partial}{\partial t}\int_{CV} \rho \mathrm{d}V\!\!\!\!/} + \int_{CS} \rho \left(\vec{V} \cdot \hat{\mathbf{n}}\right) \mathrm{d}A = 0$$

$$\int_{CS} \rho \left(\vec{V} \cdot \hat{\mathbf{n}}\right) \mathrm{d}A = 0$$

The assumption of uniform flow removes the spatial dependence of the integrand:

$$\rho \left(\vec{V} \cdot \hat{\mathbf{n}}\right) \int_{CS} \mathrm{d}A = 0$$

This results in the following expression evaluated at every CS:

$$\rho \left(\vec{V} \cdot \hat{\mathbf{n}}\right) A\Big|_{CS} = 0$$

The assumption of incompressibility implies that ρ is constant throughout the flow and the same at every CS, and can therefore be divided out.

$$\left(\vec{V} \cdot \hat{\mathbf{n}}\right) A\Big|_{CS} = 0$$

Recalling that $\left(\vec{V} \cdot \hat{\mathbf{n}}\right)$ can be expressed as $-|\vec{V}_n|$ for influx and $+|\vec{V}_n|$ for outflux, we rewrite the simplified form of the continuity equation in terms of influx and outflux:

$$\sum_{outflux} |\vec{V}_n| A - \sum_{influx} |\vec{V}_n| A = 0$$

$$\sum_{outflux} |\vec{V}_n| A = \sum_{influx} |\vec{V}_n| A$$

The sum of all mass influx terms must be equal to the sum of all outflux terms to satisfy continuity.

Examining our problem, with inlet control surfaces 1 and 2 and outlet control surface e, we apply the simplified form of continuity to yield:

$$V_1 A_1 + V_2 A_2 = V_e A_e$$

where V_e represents the velocity in the exit direction.

Multiplying through by density yields an expression in terms of mass flow rate:

$$\rho V_1 A_1 + \rho V_2 A_2 = \rho V_e A_e$$

$$\dot{m}_1 + \dot{m}_2 = \dot{m}_e$$

The full integral form of the conservation of momentum equation is given by

$$\frac{\mathrm{d}\left(m\vec{V}\right)_{sys}}{\mathrm{d}t} = \frac{\partial}{\partial t}\int_{CV} \rho\vec{V}\mathrm{d}V\!\!\!/ + \int_{CS} \rho\vec{V}\left(\vec{V}\cdot\hat{\mathbf{n}}\right)\mathrm{d}A = \sum \vec{F}_{CV}$$

This is a vector equation that can be decomposed into its x, y, and z components as shown below:

$$\frac{\partial}{\partial t}\int_{CV} \rho\vec{u}\mathrm{d}V\!\!\!/ + \int_{CS} \rho\vec{u}\left(\vec{V}\cdot\hat{\mathbf{n}}\right)\mathrm{d}A = \sum \vec{F}_{CV_x}$$
$$\frac{\partial}{\partial t}\int_{CV} \rho\vec{v}\mathrm{d}V\!\!\!/ + \int_{CS} \rho\vec{v}\left(\vec{V}\cdot\hat{\mathbf{n}}\right)\mathrm{d}A = \sum \vec{F}_{CV_y}$$
$$\frac{\partial}{\partial t}\int_{CV} \rho\vec{w}\mathrm{d}V\!\!\!/ + \int_{CS} \rho\vec{w}\left(\vec{V}\cdot\hat{\mathbf{n}}\right)\mathrm{d}A = \sum \vec{F}_{CV_z}$$

This problem is 2-dimensional in x and y, so we can safely ignore the z component. Applying our assumptions and observations to the momentum equation greatly simplifies their forms.

By applying the steady flow assumption, the time-dependency of the equations disappears:

$$\cancelto{0}{\frac{\partial}{\partial t}\int_{CV} \rho\vec{u}\mathrm{d}V\!\!\!/} + \int_{CS} \rho\vec{u}\left(\vec{V}\cdot\hat{\mathbf{n}}\right)\mathrm{d}A = \sum \vec{F}_{CV_x}$$
$$\cancelto{0}{\frac{\partial}{\partial t}\int_{CV} \rho\vec{v}\mathrm{d}V\!\!\!/} + \int_{CS} \rho\vec{v}\left(\vec{V}\cdot\hat{\mathbf{n}}\right)\mathrm{d}A = \sum \vec{F}_{CV_y}$$

Next, the uniform flow assumption allows us to remove the integrand from the integral as they are invariant over the control surface, though the terms must still be evaluated at each CS¿

$$\left(\rho\vec{u}\left(\vec{V}\cdot\hat{\mathbf{n}}\right)\int_{CS}\mathrm{d}A\right)\Bigg|_{CS} = \sum \vec{F}_{CV_x}$$
$$\left(\rho\vec{v}\left(\vec{V}\cdot\hat{\mathbf{n}}\right)\int_{CS}\mathrm{d}A\right)\Bigg|_{CS} = \sum \vec{F}_{CV_y}$$

The integral is now simple, and becomes A, the area of each CS that the equation is evaluated upon.

$$\left(\rho\vec{u}\left(\vec{V}\cdot\hat{\mathbf{n}}\right)A\right)\Big|_{CS} = \sum \vec{F}_{CV_x}$$
$$\left(\rho\vec{v}\left(\vec{V}\cdot\hat{\mathbf{n}}\right)A\right)\Big|_{CS} = \sum \vec{F}_{CV_y}$$

The assumption of normal flow velocity at each inlet/outlet allows us to replace $\left(\vec{V}\cdot\hat{\mathbf{n}}\right)$ with $|\vec{V}|$. *Note: the simplification of the dot product removes a polarity on the velocity term. For inlets, the entire term is negative, and for outlets, it will be positive.*

$$\left(\rho\vec{u}|\vec{V}|A\right)\Big|_{CS} = \sum \vec{F}_{CV_x}$$
$$\left(\rho\vec{v}|\vec{V}|A\right)\Big|_{CS} = \sum \vec{F}_{CV_y}$$

The RHS of each equation can be defined by evaluating the forces that act on the control volume. Our assumptions explicitly disregard gravity (body force) and shear stress (surface force). The CV does not have rigid walls or supports, therefore we can ignore reaction forces. The final remaining component is pressure, a surface force. We assume that the pressure on each CS is P_{atm}, therefore there is no net pressure differential acting on the CV. With these assumptions and observations, we conclude that the net force in both x and y is 0.

$$\left(\rho\vec{u}|\vec{V}|A\right)\Big|_{CS} = 0$$
$$\left(\rho\vec{v}|\vec{V}|A\right)\Big|_{CS} = 0$$

We now evaluate the momentum equation in the x and y at each CS, beginning with x, remembering that inlets are negative and outlets are positive.

$$\rho_e u_e|\vec{V}_e|A_e - \rho_1 u_1^2 A_1 = 0$$

We recognize that u_e can be recast as $V_e\cos\theta$.

$$\rho_e\left(V_e\cos\theta\right)|\vec{V}_e|A_e - \rho_1 u_1^2 A_1 = 0$$

The momentum equation in the y direction follows the same logic.

$$\rho_e\left(V_e\sin\theta\right)|\vec{V}_e|A_e - \rho_2 v_v^2 A_2 = 0$$

We now recast the momentum equations in terms of mass flow rates found from the continuity equation:

$$\dot{m}_e V_e\cos\theta - \dot{m}_1 u_1 = 0$$
$$\dot{m}_e V_e\sin\theta - \dot{m}_2 v_2 = 0$$

Rearranging both equations to solve for a common term:

$$\dot{m}_e V_e = \frac{\dot{m}_1 u_1}{\cos\theta}$$
$$\dot{m}_e V_e = \frac{\dot{m}_2 v_2}{\sin\theta}$$

We now set the equations equal and solve for θ.

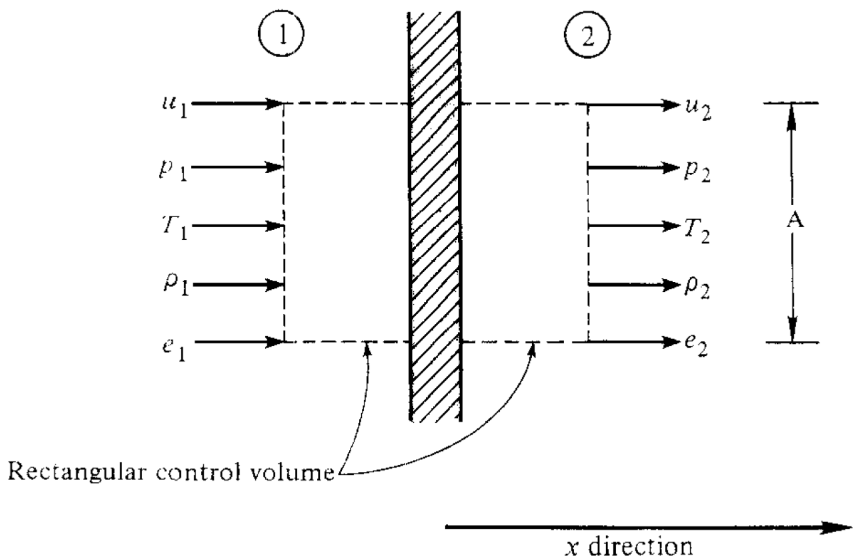
$$\frac{\dot{m}_1 u_1}{\cos\theta} = \frac{\dot{m}_2 v_2}{\sin\theta}$$
$$\frac{\sin\theta}{\cos\theta} = \frac{\dot{m}_2 v_2}{\dot{m}_1 u_1}$$
$$\tan\theta = \frac{\dot{m}_2 v_2}{\dot{m}_1 u_1}$$

Taking the inverse tangent provides our final expression for θ:

$$\boxed{\theta = \tan^{-1}\left(\frac{\dot{m}_2 v_2}{\dot{m}_1 u_1}\right)}$$

Problem 3

One-dimensional, steady, compressible flow is used for a number of real-world applications, including: normal shock waves, bow shock waves, etc. Look up some images or videos of normal shock waves and bow shock waves in front of bullets, re-rentry vehicles, etc. A schematic illustrating such flow is given below, where the flow entering the dashed control volume is given as state 1 and the flow exiting as state 2. We will learn later in the semester that these properties do indeed change across shock waves. For now, we will focus on simplifying our governing equations for these assumptions.



In our one-dimensional, steady analyses, we will make the following assumptions about our flow:

(i) One-dimensional in the x direction

(ii) Steady

(iii) Uniform velocity, pressure, temperature, density, enthalpy, and energy at each of the two control surfaces

(iv) Flow is perpendicular to control surfaces 1 and 2

(v) $A_1 = A_2$

(vi) No body forces present

(vii) No friction/shear (i.e., there are no solid boundaries around)

(viii) No work is done

(ix) The pressures acting on the control volume in the y and z directions apply no net force

(a) Under these assumptions for one-dimensional, steady flow, show that the integral form of the continuity equation simplifies to

$$\rho_1 u_1 = \rho_2 u_2 \,.$$

You must start with the full integral form and indicate which assumption(s) allowed you to make each simplification.

The full form of the continuity equation is given by:

$$\frac{\mathrm{d}(m)_{sys}}{\mathrm{d}t} = \frac{\partial}{\partial t}\int_{CV} \rho \mathrm{d}\forall + \int_{CS} \rho \left(\vec{V} \cdot \hat{\mathbf{n}}\right) \mathrm{d}A = 0$$

The steady flow assumption cancels the time-dependency of the conservation equation:

$$\cancelto{0}{\frac{\partial}{\partial t}\int_{CV} \rho \mathrm{d}\forall} + \int_{CS} \rho \left(\vec{V} \cdot \hat{\mathbf{n}}\right) \mathrm{d}A = 0$$

The uniform flow assumption allows us to remove the integrand terms from the integral because they are not dependent on the CS area:

$$\left(\rho \left(\vec{V} \cdot \hat{\mathbf{n}}\right) \int_{CS} \mathrm{d}A\right)\bigg|_{CS} = 0$$

Now, $\int_{CS} \mathrm{d}A$ simplifies to A:

$$\left(\rho \left(\vec{V} \cdot \hat{\mathbf{n}}\right) A\right)\bigg|_{CS} = 0$$

The normal flow assumption allows us to recast $\left(\vec{V} \cdot \hat{\mathbf{n}}\right)$ as $|\vec{V}|$.

$$\left(\rho |\vec{V}| A\right)\bigg|_{CS} = 0$$

Because the flow is one-dimensional in x, we only apply the continuity equation to the vertical CS on the left and right sides of the CV.

$$\rho_2 u_2 A_2 - \rho_1 u_1 A_1 = 0$$

Because we assume that $A_1 = A_2$, the area terms can be removed from the equation.

$$\boxed{\rho_1 u_1 = \rho_2 u_2}$$

(b) Can the schematic above and assumption (iii) really be valid for compressible flow? Explain your reasoning.

The schematic and assumption (iii) can be compatible for compressible flow because in a given streamtube, there is no requirement that there be a gradient in flow properties at a given CS. Depending on the relevant dimensional scales present in the problem, the uniform flow assumption is sufficiently close to the real-world physics that the answer is not significantly impacted. Even if it is not perfectly accurate, it can be a useful assumption used to solve relevant flow problems.

(c) What would the result be if we assumed "quasi-one-dimensional flow"? Note, the only difference between one-dimensional flow and quasi-one-dimensional flow is that assumption (v) is no longer valid for quasi-one-dimensional flow.

If we assume "quasi-one-dimensional flow", continuity equation still contains area terms and can be represented as

$$\boxed{\rho_1 u_1 A_1 = \rho_2 u_2 A_2}$$

(d) Under the assumptions for one-dimensional, steady flow, show that the integral form of the x-momentum equation simplifies to

$$p_1 + \rho_1 u_1^2 = p_2 + \rho_2 u_2^2\,.$$

You must start with the full integral form and indicate which assumption(s) allowed you to make each simplification.

The full integral form of the momentum equation in the x-direction is given by:

$$\frac{\partial}{\partial t}\int_{CV} \rho\vec{u}\mathrm{d}V\!\!\!\!/ + \int_{CS} \rho\vec{u}\left(\vec{V}\cdot\hat{\mathbf{n}}\right)\mathrm{d}A = \sum \vec{F}_{CV_x}$$

The steady assumption removes the time dependence of the momentum equation:

$$\cancelto{0}{\frac{\partial}{\partial t}\int_{CV} \rho\vec{u}\mathrm{d}V\!\!\!\!/} + \int_{CS} \rho\vec{u}\left(\vec{V}\cdot\hat{\mathbf{n}}\right)\mathrm{d}A = \sum \vec{F}_{CV_x}$$

The perpendicular flow assumption in conjunction with the one-dimensional in x assumption allows us to recast $\left(\vec{V}\cdot\hat{\mathbf{n}}\right)$ as $|\vec{u}|$.

$$\int_{CS} \rho\vec{u}|\vec{u}|\mathrm{d}A = \sum \vec{F}_{CV_x}$$

The uniform flow assumption allows us to pull terms out of the integrand and simplify:

$$\left(\rho u^2 A\right)\Big|_{CS} = \sum \vec{F}_{CV_x}$$

The RHS of the equation can be expressed as the sum of multiple terms: body forces, surface forces, and reaction forces.

$$\sum \vec{F}_{CV_x} = \vec{F}_{body} + \vec{F}_{surface} + \vec{F}_{reaction}$$

We assume that there are no body forces, frictional forces, or solid boundaries around the CV, which implies that there are no reaction forces. The remaining possible forces on the CV are pressure forces. We have assumed that there are no net pressure forces in the y or z-directions. In the x-direction, the pressure forces can be written as:

$$\vec{F}_{pressure} = \int_{CS} p\hat{\mathbf{n}}\mathrm{d}A$$

Because of our assumption that there are no net pressure forces in y, we can evaluate this equation in x alone.

$$\vec{F}_{pressure} = p_2 A_2 - p_1 A_1$$

We now substitute back into our previous momentum equation evaluated at CS *1* and *2*:

$$\rho_2 u_2^2 A_2 - \rho_1 u_1^2 A_1 = p_2 A_2 - p_1 A_1$$

We now rearrange to group terms evaluated at each CS.

$$p_1 A_1 + \rho_1 u_1^2 A_1 = p_2 A_2 + \rho_2 u_2^2 A_2$$

Finally, the assumption that $A_1 = A_2$ results in the final equation:

$$\boxed{p_1 + \rho_1 u_1^2 = p_2 + \rho_2 u_2^2}$$

(e) What would the y and z-momentum equations simplify to?

Because there is no net pressure force in the y or z-directions, the momentum equations would simplify to:

$$\rho_1 v_1^2 = \rho_2 v_2^2$$
$$\rho_1 w_1^2 = \rho_2 w_2^2$$

However, because we have assumed that the flow is one-dimensional in x, we have no flow components in y or z. Therefore, both terms in the equation would cancel, yielding $0 = 0$ for both directions. There is no momentum flux across the top and bottom CS, or in the z-direction through the page, so there is no net force.

(f) Is assumption (vii) ever a good assumption for compressible flows? If you think it is, give a realistic application/example of when it is. If you don't think it is, explain why not.

Assumption (vii) can be a good assumption for compressible flows in a variety of cases. For conceptual design of a high-speed vehicle, initial aerodynamic trends can be evaluated fairly accurately using inviscid methods. Because viscous effects increase the time to perform analysis dramatically, using the inviscid assumption is a desirable and valuable technique in high-speed analyses. Higher-fidelity modeling and ground tests have shown that inviscid modeling does a good job of capturing important flow effects, although more detailed levels of the design process, it is important to take viscous effects into consideration again.

(g) Under the assumptions for one-dimensional, steady flow, show that the integral form of the energy equation simplifies to

$$h_1 + \frac{u_1^2}{2} + q = h_2 + \frac{u_2^2}{2}$$

You must start with the full integral form and indicate which assumption(s) allowed you to make each simplification. q is the mass-specific heat.

The full integral energy equation is given by:

$$\frac{\partial}{\partial t}\int_{CV} \rho \left(e + \frac{1}{2}V^2\right) \mathrm{d}\mathcal{V} + \int_{CS} \rho \left(h + \frac{1}{2}V^2\right) \left(\vec{V} \cdot \hat{\mathbf{n}}\right) \mathrm{d}A = \dot{Q} - \dot{W}_{shaft} - \dot{W}_{shear} - \dot{W}_{other}$$

The steady assumption removes the time-dependency from the equation:

$$\cancelto{0}{\frac{\partial}{\partial t}\int_{CV} \rho \left(e + \frac{1}{2}V^2\right) \mathrm{d}\mathcal{V}} + \int_{CS} \rho \left(h + \frac{1}{2}V^2\right) \left(\vec{V} \cdot \hat{\mathbf{n}}\right) \mathrm{d}A = \dot{Q} - \dot{W}_{shaft} - \dot{W}_{shear} - \dot{W}_{other}$$

We have assumed that no work is done, simplifying the RHS of the equation:

$$\int_{CS} \rho \left(h + \frac{1}{2}V^2\right) \left(\vec{V} \cdot \hat{\mathbf{n}}\right) \mathrm{d}A = \dot{Q}$$

The uniform flow assumption allows us to pull terms out of the integrand and simplify the integral:

$$\left.\left(\rho\left(h+\frac{1}{2}V^2\right)\left(\vec{V}\cdot\hat{\mathbf{n}}\right)A\right)\right|_{CS}=\dot{Q}$$

The perpendicular flow assumption in conjunction with the one-dimensional in x assumption allows us to recast $\left(\vec{V}\cdot\hat{\mathbf{n}}\right)$ as $|\vec{u}|$ and V^2 as u^2.

$$\left.\left(\rho\left(h+\frac{1}{2}u^2\right)|\vec{u}|\ A\right)\right|_{CS}=\dot{Q}$$

We now evaluate the energy equation at each CS, recognizing that there is no flow in the y or z-directions, therefore we need only evaluate at CS *1* and *2*, noting that influx terms are negative and outflux terms are positive:

$$\rho_2\left(h_2+\frac{1}{2}u_2^2\right)|\vec{u_2}|\ A_2-\rho_1\left(h_1+\frac{1}{2}u_1^2\right)|\vec{u_1}|\ A_1=\dot{Q}$$

Recalling the prior definition of mass flow as $\dot{m}=\rho uA$:

$$\dot{m}_2\left(h_2+\frac{1}{2}u_2^2\right)-\dot{m}_1\left(h_1+\frac{1}{2}u_1^2\right)=\dot{Q}$$

Noting that $\dot{m}_1=\dot{m}_2$ and grouping:

$$\dot{m}\left[\left(h_2+\frac{1}{2}u_2^2\right)-\left(h_1+\frac{1}{2}u_1^2\right)\right]=\dot{Q}$$

We can now divide the entire equation by $\dot{m}$, which simplifies $\dot{Q}$ to q:

$$\left(h_2+\frac{1}{2}u_2^2\right)-\left(h_1+\frac{1}{2}u_1^2\right)=q$$

We rearrange to yield the final equation:

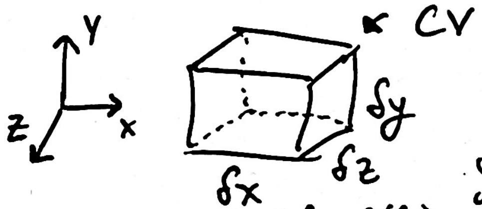
$$\boxed{h_1+\frac{u_1^2}{2}+q=h_2+\frac{u_2^2}{2}}$$

a) Conservation of Mass

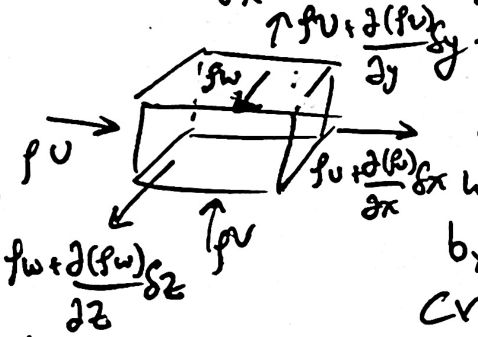
Full integral form of continuity:

$$\frac{\partial}{\partial t}\int_{CV} \rho \, d\forall + \int_{CS} \rho(\vec{V}\cdot\vec{n})\,dA = 0$$

Assume a small differential fluid element:



There is some influx and outflux in each of the x, y, and z-directions.



The fluid velocity in each direction is given as $u, v,$ and w, respectively. The mass carried by the moving fluid into the CV is given by $\vec{V}\rho$.

↳ In x: Influx $= \rho u$, Outflux $= \rho u + \frac{\partial(\rho u)}{\partial x}\delta x$,

where $\frac{\partial(\rho u)}{\partial x}\delta x$ represents a Taylor-series approx. of the density times velocity change through the CV.

↳ Similarly, in y and in z:

$$\text{Influx}_y = \rho v, \quad \text{Outflux}_y = \rho v + \frac{\partial(\rho v)}{\partial y}\delta y$$

$$\text{Influx}_z = \rho w, \quad \text{outflux}_z = \rho w + \frac{\partial(\rho w)}{\partial z}\delta z$$

↳ We now have generic terms for mass flux in each direction across each face of the CV.

↳ Assume that the differential fluid element is small enough that all flow properties are uniform at each CS.

↳ Assume that fluid velocities are normal to each CS.

↳ Using these assumptions, revisit the integral form of continuity:

$$\frac{\partial}{\partial t}\int_{CV} \rho \, d\forall + \int_{CS} \rho (\vec{V}\cdot\hat{n})\, dA = 0$$

$$\frac{\partial \rho}{\partial t}\int_{CV} d\forall + \rho |V_n| \int_{CS} dA = 0$$ (from uniform flow, ρV_n can be removed from the integrand).

$$\frac{\partial \rho}{\partial t}\forall + \rho |V_n| A \Big|_{cs} = 0$$

Note: Influxes are negative, outfluxes are positive.

$$\frac{\partial \rho}{\partial t}\forall + \sum_{out} \rho \vec{V} A - \sum_{in} \rho \vec{V} A = 0$$

Note: $\forall = \delta x \, \delta y \, \delta z$, A can be written as product of differential lengths.

↳ Recognize that $\rho V A = \dot{m}$

↳ Substitute Influx and outflux terms found in x, y, and z-directions:

$$\frac{\partial \rho}{\partial t}\delta x \delta y \delta z + \left[\rho u + \frac{\partial(\rho u)}{\partial x}\delta x\right]\delta y \delta z + \left[\rho v + \frac{\partial(\rho v)}{\partial y}\delta y\right]\delta x \delta z + \left[\rho w + \frac{\partial(\rho w)}{\partial z}\delta z\right]\delta x \delta y - [\rho u]\delta y \delta z - [\rho v]\delta x \delta z - [\rho w]\delta x \delta y = 0$$

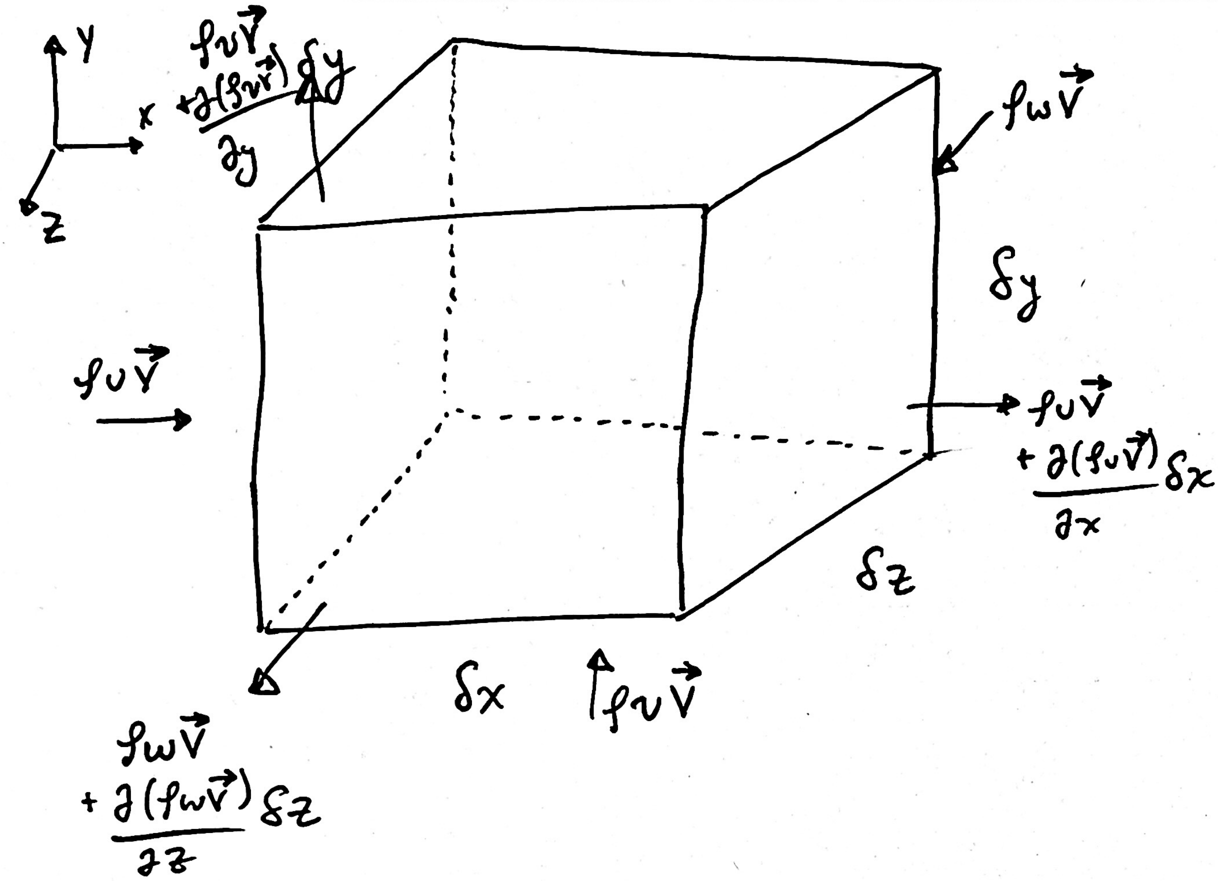
↳ Combine/Simplify like terms:

$$\frac{\partial \rho}{\partial t}\delta x \delta y \delta z + \frac{\partial(\rho u)}{\partial x}\delta x \delta y \delta z + \frac{\partial(\rho v)}{\partial y}\delta x \delta y \delta z + \frac{\partial(\rho w)}{\partial z}\delta x \delta y \delta z = 0$$

↳ Divide by $\delta x \delta y \delta z$:

$$\boxed{\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} = 0}$$

b)



↳ LHS of Integral Momentum Equation

$$\frac{\partial}{\partial t}\int_{cv} \rho \vec{V}\, d\forall + \int_{cs} \rho \vec{V}(\vec{V}\cdot\hat{n})\, dA$$

↳ Assume differential fluid element is small enough flow properties are uniform @ each cs.

↳ Assume influx, outflux, are normal @ each cs.

$$\frac{\partial(\rho\vec{V})}{\partial t}\forall + \rho\vec{V} V_n A\Big|_{cs}$$

↳ Note: Influx terms are negative, outflux are positive.

↳ Influx & Outflux in x, y, z:

	x	y	z
Influx	$[\rho u \vec{V}]\delta y \delta z$	$[\rho v \vec{V}]\delta x \delta y$	$[\rho w \vec{V}]\delta x \delta y$
Outflux	$\left[\rho u \vec{V} + \frac{\partial(\rho u \vec{V})}{\partial x}\delta x\right]\delta y \delta z$	$\left[\rho v \vec{V} + \frac{\partial(\rho v \vec{V})}{\partial y}\delta y\right]\delta x \delta y$	$\left[\rho w \vec{V} + \frac{\partial(\rho w \vec{V})}{\partial z}\delta z\right]\delta x \delta y$

↳ Populate simplified momentum equation:

$$\frac{\partial(\rho\vec{V})}{\partial t}\delta x\delta y\delta z + \left[\rho u\vec{V} + \frac{\partial(\rho u\vec{V})}{\partial x}\delta x\right]\delta y\delta z - \left[\rho u\vec{V}\right]\delta y\delta z +$$

$$\left[\rho v\vec{V} + \frac{\partial(\rho v\vec{V})}{\partial y}\delta y\right]\delta x\delta z - \left[\rho v\vec{V}\right]\delta x\delta z +$$

$$\left[\rho w\vec{V} + \frac{\partial(\rho w\vec{V})}{\partial z}\delta z\right]\delta x\delta y - \left[\rho w\vec{V}\right]\delta x\delta y$$

↳ Cancel out influx terms ($\rho u\vec{V}\delta y\delta z - \rho u\vec{V}\delta y\delta z$)

$$\left[\frac{\partial(\rho\vec{V})}{\partial t} + \frac{\partial(\rho u\vec{V})}{\partial x} + \frac{\partial(\rho v\vec{V})}{\partial y} + \frac{\partial(\rho w\vec{V})}{\partial z}\right]\delta x\delta y\delta z$$

↳ Apply chain rule

$$\left[\rho\frac{\partial(\vec{V})}{\partial t} + \underline{\vec{V}\frac{(\partial\rho)}{\partial t}} + \rho u\frac{\partial(\vec{V})}{\partial x} + \underline{\vec{V}\frac{\partial(\rho u)}{\partial x}} + \rho v\frac{(\partial\vec{V})}{\partial y} + \underline{\vec{V}\frac{\partial(\rho v)}{\partial y}} + \right.$$

$$\left.\rho w\frac{\partial(\vec{V})}{\partial z} + \underline{\vec{V}\frac{\partial(\rho w)}{\partial z}}\right]\delta x\delta y\delta z$$

↳ Recognize the differential form of continuity within the above equation:

$$\vec{V}\left[\frac{\partial\rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z}\right] = 0$$

$$\left[\rho\frac{\partial(\vec{V})}{\partial t} + \rho u\frac{\partial(\vec{V})}{\partial x} + \rho v\frac{\partial(\vec{V})}{\partial y} + \rho w\frac{\partial(\vec{V})}{\partial z}\right]\delta x\delta y\delta z$$

↳ Now, address the RHS of the momentum eqn:

$$\Sigma F = \vec{F}_{Body} + \vec{F}_{Surface} + \vec{R}$$

↳ Assume no reaction forces on differential element

$$\Sigma F = \vec{F}_{Body} + \vec{F}_{Surface}$$

↳ Assume inviscid fluid, no shear forces on CS.

$\vec{F}_{body}$ = gravity force = $m_{cv} g$

$m_{cv} = \rho\, \delta x \delta y \delta z$

$\vec{F}_{surface}$ = pressure force = $p \times A_{cs}$

* Note: outflow pressure in negative direction.

↳ In x: Inflow pressure force: $+p\, \delta y \delta z$

Outflow pressure force: $-(p + \frac{\partial p}{\partial x} \delta x)\, \delta y \delta z$

↳ Same holds for y, z.

↳ RHS in x: $\rho\, \delta x \delta y \delta z\, g_x + [(p\, \delta y \delta z - p\, \delta y \delta z) - \frac{\partial p}{\partial x} \delta x \delta y \delta z]$

↳ $[\rho g_x + \frac{\partial p}{\partial x}]\, \delta x \delta y \delta z$

↳ Same holds for y, z.

↳ Combine RHS w/ LHS in x, y, z:

↳ Note: $\delta x \delta y \delta z$ terms cancel out.

$$x: \rho \frac{\partial u}{\partial t} + \rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} + \rho w \frac{\partial u}{\partial z} = \rho g_x - \frac{\partial p}{\partial x}$$

$$y: \rho \frac{\partial v}{\partial t} + \rho u \frac{\partial v}{\partial x} + \rho v \frac{\partial v}{\partial y} + \rho w \frac{\partial v}{\partial z} = \rho g_y - \frac{\partial p}{\partial y}$$

$$z: \rho \frac{\partial w}{\partial t} + \rho u \frac{\partial w}{\partial x} + \rho v \frac{\partial w}{\partial y} + \rho w \frac{\partial w}{\partial z} = \rho g_z - \frac{\partial p}{\partial z}$$

↳ Factor ρ out of LHS to complete differential form of momentum equations.

↳ X

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) = \rho g_x - \frac{\partial p}{\partial x}$$

↳ Y

$$\rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right) = \rho g_y - \frac{\partial p}{\partial y}$$

↳ Z

$$\rho\left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right) = \rho g_z - \frac{\partial p}{\partial z}$$