Homework 5

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Problem 3

Calculate the freestream pressure in regions 4 and 4' and the flow direction Φ behind the refracted shocks for $M_1 = 3$, $p_1 = 1\,\text{atm}$, $\theta_2 = 20°$, and $\theta_3 = 15°$.

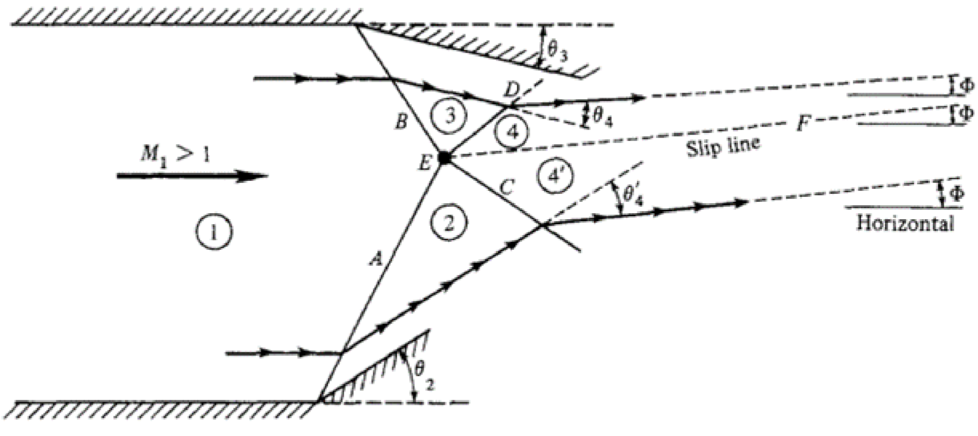

Figure 4.23 | Intersection of shocks of opposite families.

Figure 1: Shock interaction problem setup

Givens:

$M_1 = 3$
$p_1 = 1\,\text{atm}$
$\theta_2 = 20°$
$\theta_3 = 15°$

Assumptions:

Flow in the duct will be considered inviscid, steady, and isentropic outside of the shocks. In each region (1, 2, 3, 4, 4'), flow properties are constant and uniform, only changing across the shocks (A, B, C, D). Changes in area are neglected. There is no heat or work entering/exiting the system. Flow in regions 4 and 4' are oriented in the same direction at an angle Φ from the horizontal, with $p_4 = p_{4'}$. $\Phi = \theta_3 + \theta_4 = \theta_2 + \theta_{4'}$.

Solution:

Note: All calulations performed in MATLAB, see appendix **??**.

Given M_1 and the two ramp angles, θ_2 and θ_3, shock angles β_2 and β_3 are found via the following relation using a numerical solver:

$$\tan\theta_2 = 2\cot\beta_2\left[\frac{M_1^2\sin^2\beta_2 - 1}{M_1^2\left(\gamma + \cos 2\beta_2\right) + 2}\right]$$

$$\tan\theta_3 = 2\cot\beta_3\left[\frac{M_1^2\sin^2\beta_3 - 1}{M_1^2\left(\gamma + \cos 2\beta_3\right) + 2}\right]$$

$$\boxed{\beta_2 = 37.7636^\circ \qquad \beta_3 = 32.2404^\circ}$$

Post-oblique shock Mach numbers are calculated using the component of M_1 normal to shock A and shock B, denoted by $M_{1n,2}$ and $M_{1n,3}$, respectively:

$$M_{1n,2} = M_1 \sin\beta_2$$

$$M_{1n,3} = M_1 \sin\beta_3$$

$$M_{2n}^2 = \frac{M_{1n,2}^2 + \frac{2}{\gamma-1}}{\frac{2\gamma}{\gamma-1}M_{1n,2}^2 - 1}$$

$$M_{3n}^2 = \frac{M_{1n,3}^2 + \frac{2}{\gamma-1}}{\frac{2\gamma}{\gamma-1}M_{1n,3}^2 - 1}$$

$$M_2 = \frac{M_{2n}}{\beta_2 - \theta_2}$$

$$M_3 = \frac{M_{3n}}{\beta_3 - \theta_3}$$

$$\boxed{M_2 = 1.9941 \qquad M_3 = 2.2549}$$

Static pressure ratios across shocks A and B are found using oblique shock relations with M_1 and the shock angles:

$$\frac{p_2}{p_1} = 1 + \frac{2}{\gamma+1}\left(M_1^2\sin^2\beta_2 - 1\right)$$

$$\frac{p_3}{p_1} = 1 + \frac{2}{\gamma+1}\left(M_1^2\sin^2\beta_3 - 1\right)$$

$$\boxed{p_2 = 3.7713\,\text{atm} \qquad p_3 = 2.8216\,\text{atm}}$$

Solving for the conditions in regions 2 and 3 is a relatively trivial procedure. In order to solve for the conditions in regions 4 and 4', an iterative approach must be used. There are 5 unknowns needed to fully solve for the downstream conditions: θ_4, $\theta_{4'}$, β_4, $\beta_{4'}$ and p_4. The corresponding equations used to solve for state 4 and 4':

- Static pressure ratio across an oblique shock given that $p_4 = p_{4'}$:

$$\frac{p_4}{p_3} = 1 + \frac{2}{\gamma + 1}\left(M_3^2 \sin^2 \beta_4 - 1\right)$$

$$\frac{p_4}{p_2} = 1 + \frac{2}{\gamma + 1}\left(M_2^2 \sin^2 \beta_{4'} - 1\right)$$

- $\theta - \beta -$ Mach relations:

$$\tan \theta_4 = 2 \cot \beta_4 \left[\frac{M_3^2 \sin^2 \beta_4 - 1}{M_3^2 \left(\gamma + \cos 2\beta_4\right) + 2}\right]$$

$$\tan \theta_{4'} = 2 \cot \beta_{4'} \left[\frac{M_2^2 \sin^2 \beta_{4'} - 1}{M_2^2 \left(\gamma + \cos 2\beta_{4'}\right) + 2}\right]$$

- The objective function that will be used as a constraint is the relation between turn angles and Φ:

$$\Phi_4 = \theta_3 + \theta_4$$

$$\Phi_{4'} = \theta_2 + \theta_{4'}$$

$$\boxed{\theta_4 - \theta_{4'} + \theta_3 - \theta_2 = 0}$$

The solution technique utilized to determine the correct downstream conditions is know as the secant method and is outlined below:

- Given two points, (x_a, y_a) and (x_b, y_b), the equation for a line connecting these points is given by point slope formula:

$$m = \frac{y_b - y_a}{x_b - x_a}$$

$$y - y_0 = m(x - x_0)$$

- Let $(x_0, y_0) = (x_b, y_b)$:

$$y - y_b = \frac{y_b - y_a}{x_b - x_a}\left(x - x_b\right)$$

- Plug in $y = 0$ to solve for the x-intercept of the line:

$$-y_b = \frac{y_b - y_a}{x_b - x_a}(x - x_b)$$

$$x = x_b - y_b \frac{y_b - y_a}{x_b - x_a}$$

- For an iterative solver this scheme becomes the following, known as secant method:

$$x_{i+1} = x_i - y_i \frac{y_i - y_{i-1}}{x_i - x_{i-1}}$$

This method has the advantage of not needing to bound the true value of zero or determine if the sign of y_i changes relative to y_{i-1}. The solution method will involve iterating across values of downstream pressure, p_4, calculating the value of the objective function, and iterating until the value converges to 0 within a chosen tolerance. To better set up the initial conditions, values of the objective function are calculated until a sign change is observed, indicating that the zero lies between the previous two calculated points. Figure 2 shows the objective function versus p_4 plotted to the point of the sign change. The final two values of p_4 will be used as the points x_1 and x_2 to initialize the secant method algorithm.

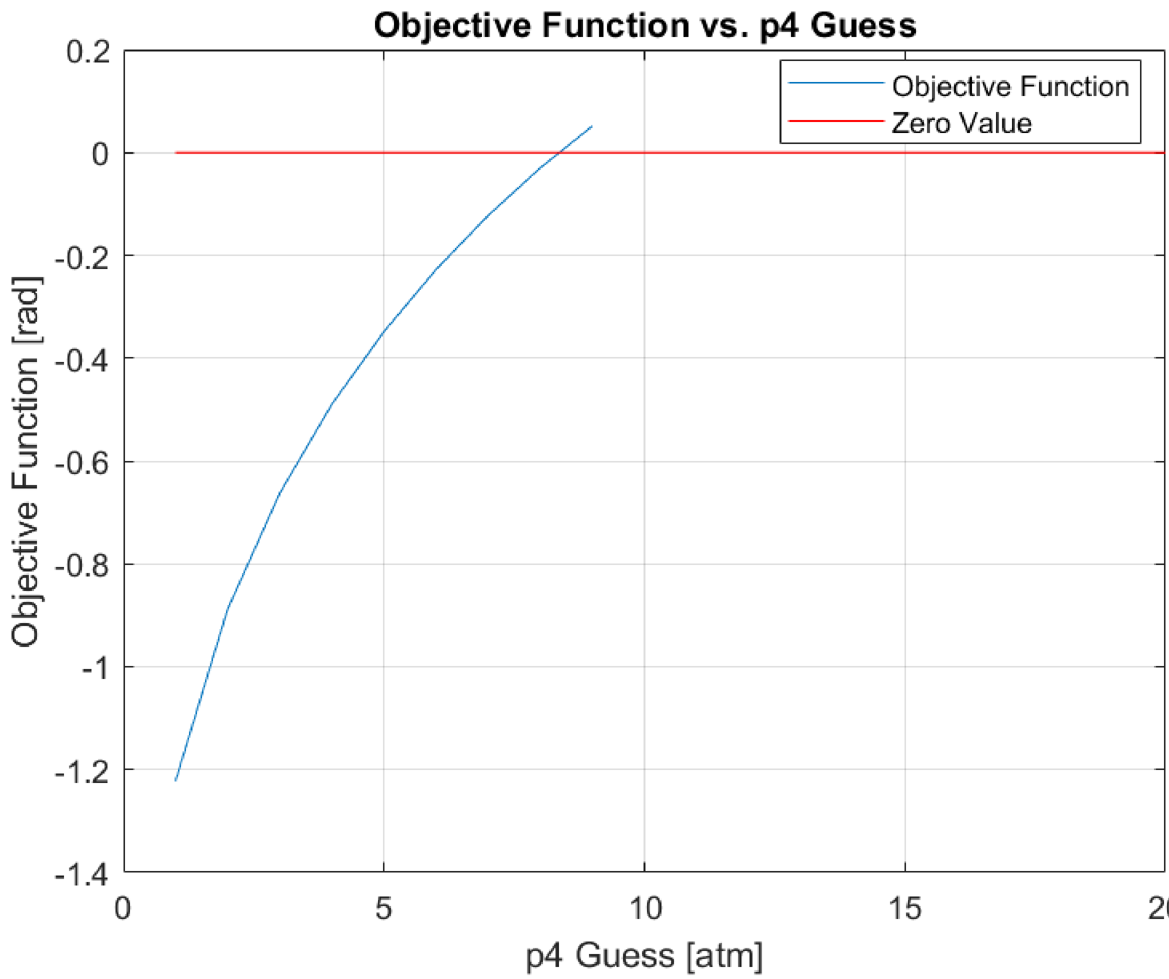

Figure 2: Objective function value plotted against p_4 guess to find sign change

The secant method converges to an objective function value of 0 (tolerance $= 1 \times 10^{-10}$) in 6 iterations, yielding the following values for the unkown variables:

$$\boxed{\theta_4 - \theta_{4'} + \theta_3 - \theta_2 = 0}$$

$$\boxed{p_4 = 8.3526\,\text{atm}}$$

$$\boxed{\theta_4 = 19.80^\circ}$$

$$\boxed{\theta_{4'} = -15.20^\circ}$$

$$\boxed{\Phi = 4.80^\circ}$$

$$\boxed{\beta_4 = 46.55^\circ}$$

$$\boxed{\beta_{4'} = -45.76^\circ}$$