Homework 6

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Problem 1

Starting with $\dot{m} = \rho u A$, prove that the mass flowrate through an isentropic choked nozzle can be written in the form:

$$\dot{m} = \frac{p_0 A^*}{\sqrt{T_0}} \sqrt{\frac{\gamma}{R}\left(\frac{2}{\gamma+1}\right)^{\frac{(\gamma+1)}{(\gamma-1)}}}$$

Assumptions:
Isentropic flow through a nozzle with a choked (sonic) throat.

Solution:
The mass flow at a given cross section in a quasi 1-D flow is given by:

$$\dot{m} = \rho u A$$

Choosing the throat of a choked nozzle as the point of interest, we replace the conditions with sonic conditions, denoted by $*$ and indicating the flow property at the location where $M = 1$.

$$\dot{m} = \rho^* u^* A^*$$

In order to cast this in terms of properties that are more easily known ahead of time, we identify relationships involving the total conditions of the flow, beginning with ρ^*. Using the ideal gas law, we can cast the sonic density in terms of pressure and temperature:

$$p^* = \rho^* R T^*$$

$$\rho^* = \frac{p^*}{RT^*}$$

Now, we find relationships for p^* and T^*.
Beginning with the isentropic relationship between total and static pressure:

$$\frac{p_0}{p} = \left(1 + \frac{\gamma-1}{2}M^2\right)^{\frac{\gamma}{\gamma-1}}$$

Setting $M = 1$:

$$\frac{p_0}{p^*} = \left(1 + \frac{\gamma-1}{2}\right)^{\frac{\gamma}{\gamma-1}}$$

$$p^* = \frac{p_0}{\left(\frac{\gamma+1}{2}\right)^{\frac{\gamma}{\gamma-1}}}$$

For temperature:

$$\frac{T_0}{T} = \left(1 + \frac{\gamma - 1}{2} M^2\right)$$

Setting $M = 1$:

$$\frac{T_0}{T^*} = \left(1 + \frac{\gamma - 1}{2}\right)$$

$$T^* = \frac{T_0}{\left(\frac{\gamma+1}{2}\right)}$$

Substituting into the ideal gas equation yields an expression for ρ^* in terms of p_0, T_0, R, and γ:

$$\rho^* = \frac{p_0}{\left(\frac{\gamma+1}{2}\right)^{\frac{\gamma}{\gamma-1}}} \frac{\left(\frac{\gamma+1}{2}\right)}{T_0} \frac{1}{R}$$

Next, we examine the sonic velocity term, u^*. Noting that for choked flow $M = 1$, we observe that the flow velocity must be equal to the speed of sound, a.

$$M = \frac{u^*}{a^*} = 1 \rightarrow u^* = a^*$$

$$a^* = \sqrt{\gamma R T^*}$$

Subsituting our known equation for T^*:

$$a^* = \sqrt{\frac{\gamma R T_0}{\left(\frac{\gamma+1}{2}\right)}}$$

Substituting everything back into the original mass flow equation:

$$\dot{m} = \frac{p_0}{\left(\frac{\gamma+1}{2}\right)^{\frac{\gamma}{\gamma-1}}} \frac{\left(\frac{\gamma+1}{2}\right)}{T_0} \frac{1}{R} \sqrt{\frac{\gamma R T_0}{\left(\frac{\gamma+1}{2}\right)}} A^*$$

Rearranging:

$$\dot{m} = \frac{p_0 A^*}{RT_0} \frac{\frac{\gamma+1}{2}}{\left(\frac{\gamma+1}{2}\right)^{\frac{\gamma}{\gamma-1}}} \sqrt{\frac{\gamma R T_0}{\left(\frac{\gamma+1}{2}\right)}}$$

$$\dot{m} = p_0 A^* \frac{1}{\left(\frac{\gamma+1}{2}\right)^{\frac{\gamma}{\gamma-1}}} \sqrt{\frac{\gamma R T_0}{R^2 T_0^2} \frac{\left(\frac{\gamma+1}{2}\right)^2}{\left(\frac{\gamma+1}{2}\right)}}$$

$$\dot{m} = \frac{p_0 A^*}{\sqrt{T_0}} \frac{1}{\left(\frac{\gamma+1}{2}\right)^{\frac{\gamma}{\gamma-1}}} \sqrt{\frac{\gamma}{R} \left(\frac{\gamma+1}{2}\right)}$$

$$\dot{m} = \frac{p_0 A^*}{\sqrt{T_0}} \sqrt{\frac{\gamma}{R} \left(\frac{\gamma+1}{2}\right)^{1-\frac{2\gamma}{\gamma-1}}}$$

$$\dot{m} = \frac{p_0 A^*}{\sqrt{T_0}} \sqrt{\frac{\gamma}{R} \left(\frac{\gamma+1}{2}\right)^{\frac{-\gamma-1}{\gamma-1}}}$$

$$\boxed{\dot{m} = \frac{p_0 A^*}{\sqrt{T_0}} \sqrt{\frac{\gamma}{R} \left(\frac{2}{\gamma+1}\right)^{\frac{(\gamma+1)}{(\gamma-1)}}}}$$