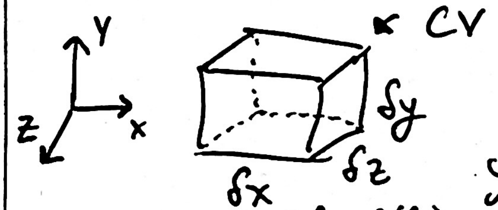
Compressible Flow | HW #2 Problem 4 | 20 September 2022

a) Conservation of Mass

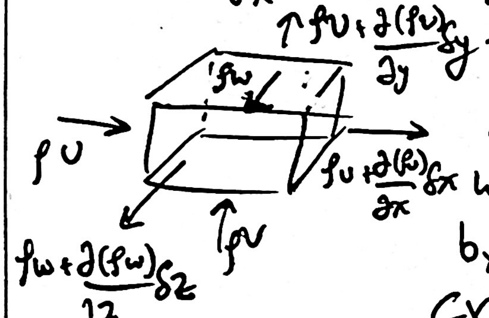
Full integral form of Continuity:

$$\frac{\partial}{\partial t}\int_{cv} \rho \, d\forall + \int_{cs} \rho(\vec{V}\cdot\vec{n})\,dA = 0$$

Assume a small differential fluid element:



There is some influx and outflux in each of the x, y, and z-directions.



The fluid velocity in each direction is given as u, v, and w, respectively. The mass carried by the moving fluid into the CV is given by $\vec{V}\rho$.

↳ In x: Influx $= \rho u$, outflux $= \rho u + \frac{\partial(\rho u)}{\partial x}\delta x$, where $\frac{\partial(\rho u)}{\partial x}\delta x$ represents a Taylor-series approx. of the density times velocity change through the CV.

↳ Similarly, in y and in z:

$$\text{Influx}_y = \rho v, \quad \text{outflux}_y = \rho v + \frac{\partial(\rho v)}{\partial y}\delta y$$

$$\text{Influx}_z = \rho w, \quad \text{outflux}_z = \rho w + \frac{\partial(\rho w)}{\partial z}\delta z$$

↳ We now have generic terms for mass flux in each direction across each face of the CV.

↳ Assume that the differential fluid element is small enough that all flow properties are uniform at each CS.

- Assume that fluid velocities are normal to each CS.

- Using these assumptions, revisit the integral form of continuity:

$$\frac{\partial}{\partial t}\int_{CV} \rho \, d\forall + \int_{CS} \rho(\vec{V}\cdot\hat{n})\, dA = 0$$

$$\frac{\partial \rho}{\partial t}\int_{CV} d\forall + \rho|V_n|\int_{CS} dA = 0$$

(from uniform flow, ρV_n can be removed from the integrand).

$$\frac{\partial \rho}{\partial t}\forall + \rho|V_n|A\Big|_{cs} = 0$$

Note: Influxes are negative, outfluxes are positive.

$$\frac{\partial \rho}{\partial t}\forall + \sum_{out}\rho\vec{V}A - \sum_{in}\rho\vec{V}A = 0$$

Note: $\forall = \delta x\,\delta y\,\delta z$, A can be written as product of differential lengths.

- Recognize that $\rho VA = \dot{m}$
- Substitute Influx and outflux terms found in x, y, and z-directions:

$$\frac{\partial \rho}{\partial t}\delta x\delta y\delta z + \left[\rho u + \frac{\partial(\rho u)}{\partial x}\delta x\right]\delta y\delta z + \left[\rho v + \frac{\partial(\rho v)}{\partial y}\delta y\right]\delta x\delta z + \left[\rho w + \frac{\partial(\rho w)}{\partial z}\delta z\right]\delta x\delta y - [\rho u]\delta y\delta z - [\rho v]\delta x\delta z - [\rho w]\delta x\delta y = 0$$

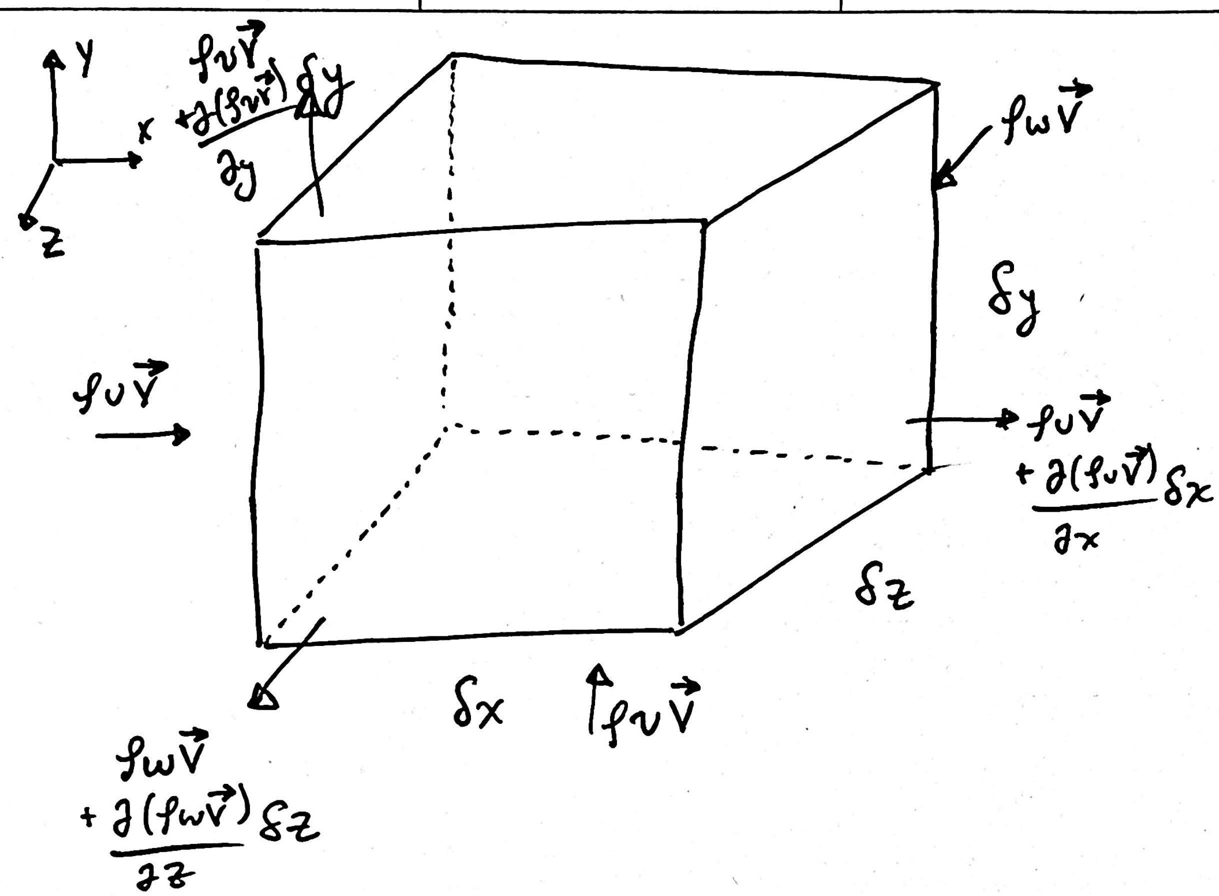
- Combine/Simplify like terms:

$$\frac{\partial \rho}{\partial t}\delta x\delta y\delta z + \frac{\partial(\rho u)}{\partial x}\delta x\delta y\delta z + \frac{\partial(\rho v)}{\partial y}\delta x\delta y\delta z + \frac{\partial(\rho w)}{\partial z}\delta x\delta y\delta z = 0$$

- Divide by $\delta x\delta y\delta z$:

$$\boxed{\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} = 0}$$

b)



↳ LHS of Integral Momentum Equation

$$\frac{\partial}{\partial t}\int_{cv} \rho \vec{V}\, d\forall + \int_{cs} \rho \vec{V}(\vec{V}\cdot\hat{n})\, dA$$

↳ Assume differential fluid element is small enough flow properties are uniform @ each CS.

↳ Assume influx, outflux, are normal @ each CS.

$$\frac{\partial(\rho \vec{V})}{\partial t}\forall + \rho \vec{V} V_n A \Big|_{cs}$$

↳ Note: Influx terms are negative, outflux are positive.

↳ Influx & Outflux in x, y, z:

	x	y	z
Influx	$[\rho u \vec{V}]\delta y \delta z$	$[\rho v \vec{V}]\delta x \delta y$	$[\rho w \vec{V}]\delta x \delta y$
Outflux	$\left[\rho u \vec{V} + \frac{\partial(\rho u \vec{V})}{\partial x}\delta x\right]\delta y \delta z$	$\left[\rho v \vec{V} + \frac{\partial(\rho v \vec{V})}{\partial y}\delta y\right]\delta x \delta y$	$\left[\rho w \vec{V} + \frac{\partial(\rho w \vec{V})}{\partial z}\delta z\right]\delta x \delta y$

↳ Populate simplified momentum equation:

$$\frac{\partial(\rho\vec{V})}{\partial t}\delta x\delta y\delta z + \left[\rho u\vec{V} + \frac{\partial(\rho u\vec{V})}{\partial x}\delta x\right]\delta y\delta z - \left[\rho u\vec{V}\right]\delta y\delta z +$$

$$\left[\rho v\vec{V} + \frac{\partial(\rho v\vec{V})}{\partial y}\delta y\right]\delta x\delta z - \left[\rho v\vec{V}\right]\delta x\delta z +$$

$$\left[\rho w\vec{V} + \frac{\partial(\rho w\vec{V})}{\partial z}\delta z\right]\delta x\delta y - \left[\rho w\vec{V}\right]\delta x\delta y$$

↳ Cancel out influx terms ($\rho u\vec{V}\,\delta y\delta z - \rho u\vec{V}\,\delta y\delta z$)

$$\left[\frac{\partial(\rho\vec{V})}{\partial t} + \frac{\partial(\rho u\vec{V})}{\partial x} + \frac{\partial(\rho v\vec{V})}{\partial y} + \frac{\partial(\rho w\vec{V})}{\partial z}\right]\delta x\delta y\delta z$$

↳ Apply chain rule

$$\left[\rho\frac{\partial(\vec{V})}{\partial t} + \underline{\vec{V}\frac{(\partial\rho)}{\partial t}} + \rho u\frac{\partial(\vec{V})}{\partial x} + \underline{\vec{V}\frac{\partial(\rho u)}{\partial x}} + \rho v\frac{(\partial\vec{V})}{\partial y} + \underline{\vec{V}\frac{\partial(\rho v)}{\partial y}} +\right.$$

$$\left.\rho w\frac{\partial(\vec{V})}{\partial z} + \underline{\vec{V}\frac{\partial(\rho w)}{\partial z}}\right]\delta x\delta y\delta z$$

↳ Recognize the differential form of continuity within the above equation:

$$\vec{V}\left[\frac{\partial\rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z}\right] = 0$$

$$\left[\rho\frac{\partial(\vec{V})}{\partial t} + \rho u\frac{\partial(\vec{V})}{\partial x} + \rho v\frac{\partial(\vec{V})}{\partial y} + \rho w\frac{\partial(\vec{V})}{\partial z}\right]\delta x\delta y\delta z$$

↳ Now, address the RHS of the momentum eqn:

$$\Sigma F = \vec{F}_{Body} + \vec{F}_{Surface} + \vec{R}$$

↳ Assume no reaction forces on differential element

$$\Sigma F = \vec{F}_{Body} + \vec{F}_{Surface}$$

↳ Assume inviscid fluid, no shear forces on CS.

$\vec{F}_{body}$ = gravity force = $m_{cv} g$

$m_{cv} = \rho\, \delta x \delta y \delta z$

$\vec{F}_{surface}$ = pressure force = $p \times A_{cs}$

* Note: outflow pressure in negative direction.

↳ In x: Inflow pressure force: $+p\,\delta y \delta z$

Outflow pressure force: $-(p + \frac{\partial p}{\partial x}\delta x)\,\delta y \delta z$

↳ Same holds for y, z.

↳ RHS in x: $\rho\,\delta x \delta y \delta z\, g_x + [(p\,\delta y \delta z - p\,\delta y \delta z) - \frac{\partial p}{\partial x}\delta x \delta y \delta z]$

↳ $[\rho g_x + \frac{\partial p}{\partial x}]\,\delta x \delta y \delta z$

↳ Same holds for y, z.

↳ Combine RHS w/ LHS in x, y, z:

↳ Note: $\delta x \delta y \delta z$ terms cancel out.

$$x: \rho\frac{\partial u}{\partial t} + \rho u\frac{\partial u}{\partial x} + \rho v\frac{\partial u}{\partial y} + \rho w\frac{\partial u}{\partial z} = \rho g_x - \frac{\partial p}{\partial x}$$

$$y: \rho\frac{\partial v}{\partial t} + \rho u\frac{\partial v}{\partial x} + \rho v\frac{\partial v}{\partial y} + \rho w\frac{\partial v}{\partial z} = \rho g_y - \frac{\partial p}{\partial y}$$

$$z: \rho\frac{\partial w}{\partial t} + \rho u\frac{\partial w}{\partial x} + \rho v\frac{\partial w}{\partial y} + \rho w\frac{\partial w}{\partial z} = \rho g_z - \frac{\partial p}{\partial z}$$

↳ Factor ρ out of LHS to complete differential form of momentum equations.

↳ X

$$\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\right)=\rho g_x-\frac{\partial p}{\partial x}$$

↳ Y

$$\rho\left(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}\right)=\rho g_y-\frac{\partial p}{\partial y}$$

↳ Z

$$\rho\left(\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}\right)=\rho g_z-\frac{\partial p}{\partial z}$$