University of Dayton

AEE 553 — Compressible Flow

Department of Mechanical and Aerospace Engineering

Homework 4

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Contents

Problem 1

(a) Derive the Hugoniot Eqution.

Assumptions:

1-D, steady flow. Inviscid. Calorically perfect gas (constant specific heats). Uniform pressure distribution around the CV. No heat addition (adiabatic). No work done on or by the CV.

Solution:

Beginning with the 1-D continuity equation and expressing both velocities in terms of the densities and the other velocity:

$$\rho_1 u_1 = \rho_2 u_2$$

$$u_1 = u_2 \left(\frac{\rho_2}{\rho_1}\right)$$

$$u_2 = u_1 \left(\frac{\rho_1}{\rho_2}\right)$$

The 1-D momentum equation:

$$p_1 + \rho_1 u_1^2 = p_2 + \rho_2 u_2^2$$

Plugging in velocities in terms of 1-D continuity and rearranging:

$$p_1 + \rho_1 u_1^2 = p_2 + \rho_2 \left[u_1 \left(\frac{\rho_1}{\rho_2}\right)\right]^2$$

$$(p_1 - p_2) = u_1^2 \left[\rho_2 \left(\frac{\rho_1}{\rho_2}\right)^2 - \rho_1\right]$$

$$(p_1 - p_2) = u_1^2 \left[\left(\frac{\rho_1^2}{\rho_2}\right) - \rho_1\right]$$

$$(p_1 - p_2) = u_1^2 \left(\frac{\rho_1}{\rho_2}\right)(\rho_1 - \rho_2)$$

We now have relationships for u_1 and u_2 in terms of pressures and densities.

$$u_1^2 = \left(\frac{p_1 - p_2}{\rho_1 - \rho_2}\right)\left(\frac{\rho_2}{\rho_1}\right)$$

$$u_2^2 = \left(\frac{p_2 - p_1}{\rho_2 - \rho_1}\right)\left(\frac{\rho_1}{\rho_2}\right)$$

Next, the 1-D energy equation (adiabatic, no work):

$$h_1 + \frac{u_1^2}{2} = h_2 + \frac{u_2^2}{2}$$

The definition of enthalpy, h.

$$h = e + \frac{p}{\rho}$$

Recasting the 1-D energy equation with the definition of enthalpy and rearranging:

$$e_1 + \frac{p_1}{\rho_1} + \frac{u_1^2}{2} = e_2 + \frac{p_2}{\rho_2} + \frac{u_2^2}{2}$$

$$e_1 + \frac{p_1}{\rho_1} + \frac{1}{2}\left[\left(\frac{p_1 - p_2}{\rho_1 - \rho_2}\right)\left(\frac{\rho_2}{\rho_1}\right)\right] = e_2 + \frac{p_2}{\rho_2} + \frac{1}{2}\left[\left(\frac{p_2 - p_1}{\rho_2 - \rho_1}\right)\left(\frac{\rho_1}{\rho_2}\right)\right]$$

$$0 = (e_2 - e_1) + \left(\frac{p_2}{\rho_2} - \frac{p_1}{\rho_1}\right) + \frac{1}{2}\left[\left(\frac{p_2 - p_1}{\rho_2 - \rho_1}\right)\left(\frac{\rho_1}{\rho_2} - \frac{\rho_2}{\rho_1}\right)\right]$$

$$0 = (e_2 - e_1) + \left(\frac{p_2\rho_1 - p_1\rho_2}{\rho_1\rho_2}\right) + \frac{1}{2}\left[\left(\frac{p_2 - p_1}{\rho_2 - \rho_1}\right)\left(\frac{\rho_1^2 - \rho_2^2}{\rho_1\rho_2}\right)\right]$$

$$0 = (e_2 - e_1) + \left(\frac{1}{\rho_1\rho_2}\right)\left[(p_2\rho_1 - p_1\rho_2) + \frac{1}{2}\left(\frac{p_2 - p_1}{\rho_2 - \rho_1}\right)\left(\rho_1^2 - \rho_2^2\right)\right]$$

$$0 = (e_2 - e_1) + \left(\frac{1}{\rho_1\rho_2}\right)\left[(p_2\rho_1 - p_1\rho_2) + \frac{1}{2}\left(\frac{p_1 - p_2}{\rho_1 - \rho_2}\right)(\rho_1 - \rho_2)(\rho_1 + \rho_2)\right]$$

$$0 = (e_2 - e_1) + \left(\frac{1}{\rho_1\rho_2}\right)\left[(p_2\rho_1 - p_1\rho_2) + \frac{1}{2}(p_1 - p_2)(\rho_1 + \rho_2)\right]$$

$$0 = (e_2 - e_1) + \left(\frac{1}{\rho_1\rho_2}\right)\left[(p_2\rho_1 - p_1\rho_2) + \frac{1}{2}(p_1\rho_1 + p_1\rho_2 - p_2\rho_1 - p_2\rho_2)\right]$$

$$0 = (e_2 - e_1) + \left(\frac{1}{2}\right)\left(\frac{1}{\rho_1\rho_2}\right)(p_1\rho_1 - p_1\rho_2 + p_2\rho_1 - p_2\rho_2)$$

$$(e_2 - e_1) = \left(\frac{1}{2}\right)\left(\frac{p_1}{\rho_2} - \frac{p_1}{\rho_1} + \frac{p_2}{\rho_2} - \frac{p_2}{\rho_1}\right)$$

$$(e_2 - e_1) = \left(\frac{p_1 + p_2}{2}\right)\left(\frac{1}{\rho_1} - \frac{1}{\rho_2}\right)$$

We now have one of the common forms of the Hugoniot Equation:

$$(e_2 - e_1) = \left(\frac{p_1 + p_2}{2}\right)(\nu_1 - \nu_2)$$

For a CPG, internal energy, e:

$$e = c_\nu T$$

Substituting the above relationship into our initial Hugoniot Equation:

$$c_\nu (T_2 - T_1) = \left(\frac{p_1 + p_2}{2}\right)(\nu_1 - \nu_2)$$

For a CPG, c_ν:

$$c_\nu = \frac{R}{\gamma - 1}$$

Substituting and rearranging:

$$\frac{R}{\gamma - 1}(T_2 - T_1) = \left(\frac{p_1 + p_2}{2}\right)(\nu_1 - \nu_2)$$

The ideal gas equation of state:

$$T = \frac{p\nu}{R}$$

Substituting and rearranging to solve for p_2/p_1:

$$\frac{R}{\gamma - 1}\left(\frac{p_2\nu_2}{R} - \frac{p_1\nu_1}{R}\right) = \left(\frac{p_1 + p_2}{2}\right)(\nu_1 - \nu_2)$$

$$\frac{2}{\gamma - 1}(p_2\nu_2 - p_1\nu_1) = (p_1 + p_2)(\nu_1 - \nu_2)$$

$$\frac{2}{\gamma - 1}(p_2\nu_2 - p_1\nu_1) = p_1\nu_1 - p_1\nu_2 + p_2\nu_1 - p_2\nu_2$$

$$\left(\frac{2}{\gamma - 1} + 1\right)(p_2\nu_2 - p_1\nu_1) = (p_2\nu_1 - p_1\nu_2)$$

$$\left(\frac{\gamma+1}{\gamma-1}\right) = \frac{(p_2\nu_1 - p_1\nu_2)}{(p_2\nu_2 - p_1\nu_1)}$$

$$\left(\frac{\gamma+1}{\gamma-1}\right) = \frac{\left(\frac{p_2}{p_1}\nu_1 - \nu_2\right)}{\left(\frac{p_2}{p_1}\nu_2 - \nu_1\right)}$$

$$\left(\frac{\gamma+1}{\gamma-1}\right)\left(\frac{p_2}{p_1}\nu_2 - \nu_1\right) = \left(\frac{p_2}{p_1}\nu_1 - \nu_2\right)$$

$$\left(\frac{\gamma+1}{\gamma-1}\right)\left(\frac{p_2}{p_1}\nu_2 - \nu_1\right) = \left(\frac{p_2}{p_1}\nu_1 - \nu_2\right)$$

$$\frac{p_2}{p_1}\left[\left(\frac{\gamma+1}{\gamma-1}\right)\nu_2 - \nu_1\right] = \left(\frac{\gamma+1}{\gamma-1}\right)\nu_1 - \nu_2$$

$$\frac{p_2}{p_1} = \left[\frac{\left(\frac{\gamma+1}{\gamma-1}\right)\nu_1 - \nu_2}{\left(\frac{\gamma+1}{\gamma-1}\right)\nu_2 - \nu_1}\right]$$

$$\frac{p_2}{p_1} = \left[\frac{\left(\frac{\gamma+1}{\gamma-1}\right)\frac{\nu_1}{\nu_2} - 1}{\left(\frac{\gamma+1}{\gamma-1}\right) - \frac{\nu_1}{\nu_2}}\right]$$

The relation between density, ρ, and specific volume, ν:

$$\rho = \frac{1}{\nu}$$

Substituting the above relation yields the final form of the Hugoniot Equation for pressure ratio across a normal shock in terms of density ratio and γ:

$$\boxed{\frac{p_2}{p_1} = \left[\frac{\left(\frac{\gamma+1}{\gamma-1}\right)\frac{\rho_2}{\rho_1} - 1}{\left(\frac{\gamma+1}{\gamma-1}\right) - \frac{\rho_2}{\rho_1}}\right]}$$

(b) Assume that a turbojet compressor fan isentropically compresses air for a range $1 < \frac{\rho_2}{\rho_1} < 5$. Plot the compression (i.e., $\frac{p_2}{p_1}$) for this range of $\frac{\rho_2}{\rho_1}$ for both traditional isentropic compression (e.g., a turbojet compression fan) and normal-shock-wave compression. These curves should be on the same plot with a clearly labeled legend.

Givens:

Isentropic and normal shock compression of air. $1 < \frac{\rho_2}{\rho_1} < 5$.

Assumptions:

Inviscid, adiabatic, no external work. Air is a CPG with $\gamma = 1.4$.

Solution:

Note: All calculations performed in Python, see Appendix A. The static pressure ratio across a normal shock given by the Hugoniot equation:

$$\frac{p_2}{p_1} = \left[\frac{\left(\frac{\gamma+1}{\gamma-1}\right)\frac{\rho_2}{\rho_1} - 1}{\left(\frac{\gamma+1}{\gamma-1}\right) - \frac{\rho_2}{\rho_1}}\right]$$

The static pressure ratio given as a function of density ratio for isentropic compression:

$$\frac{p_2}{p_1} = \left(\frac{\rho_2}{\rho_1}\right)^{\gamma}$$

Figure 1 shows a comparison of the compression (pressure ratio) as a function of density ratio for both isentropic compression and normal shock compression.

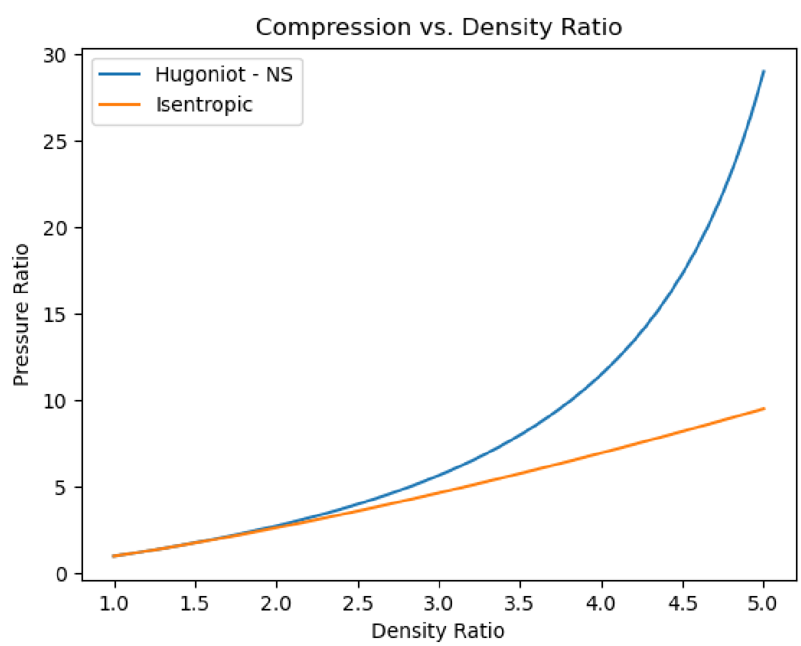

Figure 1: Compression vs. density ratio – normal shock and isentropic compression

(c) If you were tasked with deciding which mechanism to use to compress air, which would you choose? For relatively large $\frac{\rho_2}{\rho_1}$, what other real-world considerations are there for choosing between isentropic versus normal-shock-wave compression? Think of "efficiency".

Discussion:

The normal shock wave delivers much higher static pressure ratios for relatively large density ratios. Absent other context, the normal shock appears to be the ideal solution. However, normal shocks generate large total pressure losses, which are a measure of efficiency. Isentropic compression does not deliver pressure ratios as large as normal shocks, but they are not lossy by definition. Figure 2 shows the total pressure ratio across a normal shock as a function of incoming Mach number. Total pressure ratio plummets as incoming Mach number increases. By approximately Mach 2.5, the flow has lost half of its total pressure because of the presence of a normal shock. For hypersonic Mach numbers (5+), the total pressure recovery is around 10% or less! When the density ratio associated with a compression process is approximately 2.5 or less, both methods of compression deliver approximately the same pressure ratio. Isentropic compression methods require turbomachinery and moving parts which contribute substantial amounts of weight to a vehicle. If the losses are acceptable, normal shock compression at an inlet could be a potentially desirable design choice for minimizing weight.

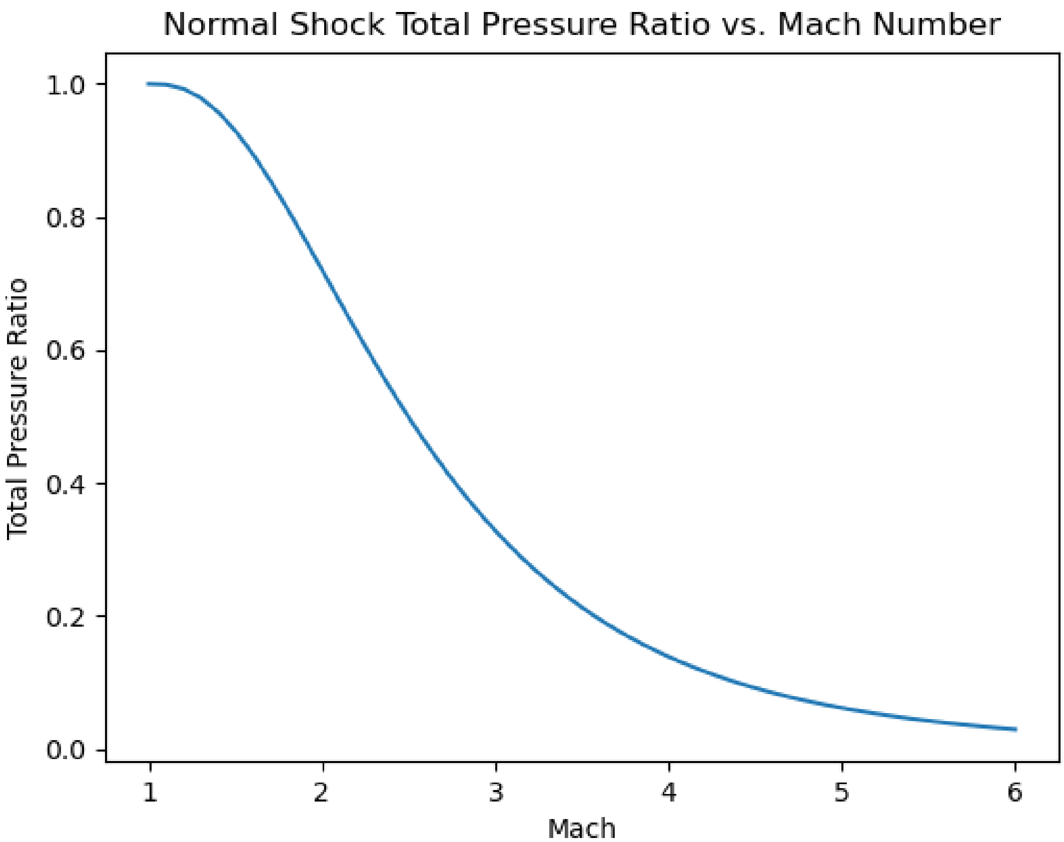

Figure 2: Compression vs. Mach number across normal shock

(d) Calculate the change in entropy $(s_2 - s_1)$ from the given range of $\frac{\rho_2}{\rho_1}$ for both the isentropic and normal-shock compression. Plot $(s_2 - s_1)$ versus $\frac{\rho_2}{\rho_1}$, and comment on what you find. Does this affect your thoughts on part (c)? Research, and then briefly explain, the physical explanation for the $(s_2 - s_1)$ behavior across the normal shock.

Givens:

Isentropic and normal-shock compression.

Assumptions:

Air is a CPG with $\gamma = 1.4$.

Solution:

By definition, $(s_2 - s_1)$ for isentropic flow should be equal to 0. For a CPG, Gibb's equation:

$$s_2 - s_1 = c_p \ln\left(\frac{T_2}{T_1}\right) - R\ln\left(\frac{p_2}{p_1}\right)$$

Recasting the temperature ratio in terms of the density ratio using isentropic relations:

$$\frac{T_2}{T_1} = \left(\frac{\rho_2}{\rho_1}\right)^{\gamma-1}$$

$$s_2 - s_1 = c_p \ln\left(\left(\frac{\rho_2}{\rho_1}\right)^{\gamma-1}\right) - R\ln\left(\frac{p_2}{p_1}\right)$$

The change in density across a normal shock:

$$s_2 - s_1 = -R\ln\frac{p_{t,2}}{p_{t,1}}$$

Solving for $\frac{p_{t,2}}{p_{t,1}}$ across a normal shock is a multi-step process. Beginning with the pressure ratios calculated in part (b) using the Hugoniot Equation, we can solve for upstream Mach number.

$$\frac{p_2}{p_1} = 1 + \frac{2\gamma}{\gamma+1}\left(M_1^2 - 1\right)$$

$$M_1 = \sqrt{\left(\frac{p_2}{p_1} - 1\right)\left(\frac{\gamma+1}{2\gamma}\right) + 1}$$

Next, the total pressure ratio across a normal shock is given by the following equation:

$$\frac{p_{t,2}}{p_{t,1}} = \frac{p_{t,2}}{p_2}\frac{p_2}{p_1}\frac{p_1}{p_{t,1}} = \left[\frac{\frac{\gamma+1}{2}M_1^2}{1+\frac{\gamma-1}{2}M_1^2}\right]^{\frac{\gamma}{\gamma-1}}\left[\frac{2\gamma}{\gamma+1}M_1^2 - \frac{\gamma-1}{\gamma+1}\right]^{\frac{1}{1-\gamma}}$$

Figure 3 shows the entropy change for both compression methods. As initially presumed, the isentropic compression by definition has no entropy change. The normal shock compression has non-zero entropy change, increasing with density ratio. Shocks are highly lossy in a very thin region, although flow up and downstream can generally be considered isentropic. The near-instantaneous change in flow properties across the thin shock-region cannot be considered "reversible", therefore any work-related assumptions are rendered invalid. The entropy change increases with density ratio because the shock strength required to generate that density ratio is also increasing, and stronger shocks (associated with higher Mach flows) are lossier and less efficient. The results shown here support the discussion in part (c), indicating that losses are a major factor in normal shock compression.

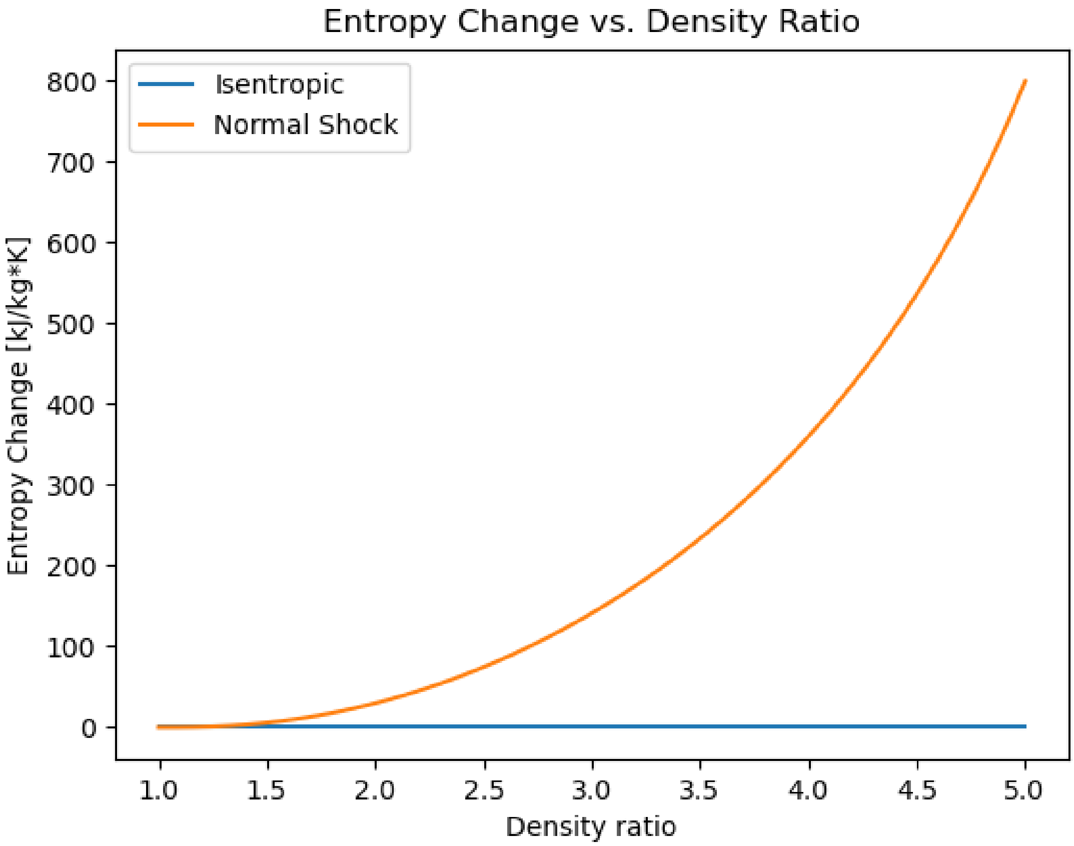

Figure 3: Entropy change for normal-shock and isentropic compression

(e) Using your figure from part (b), for what range of $\frac{\rho_2}{\rho_1}$ are the two processes comparable? Explain why this makes sense. For what Mach-number (i.e., M1) range would this correspond to regarding the normal-shock-wave case? Does this help you explain?

Discussion:

Referencing 1, the two compression processes are comparable up to approximately $\frac{\rho_2}{\rho_1} = 2.5$. From a static pressure ratio standpoint, the processes are nearly identical in output. Examining the entropy change for this density ratio, the normal shock generates entropic losses but the curve of entropy change has not exploded as it does for larger values of $\frac{\rho_2}{\rho_1}$. The range of incoming Mach number, M_1, that this corresponds to is $1 < M_1 < 1.89$. Although the flow is supersonic and experiences a normal shock, it is not extremely supersonic and the associated entropy change and total pressure loss for this relatively low value of M_1 are sufficiently small that the two compression methods are comparable.

Problem 2

Givens:

$M_\infty = 3.0$
$T_\infty = 217\,\mathrm{K}$
$p_\infty = 20\,\mathrm{kPa}$
$\gamma = 1.4$
$R = 287\,\mathrm{J/kg\,K}$
$c_p = 1000\,\mathrm{J/kg\,K}$
$q_{combustor} = 500\,\mathrm{kJ/kg}$

Assumptions:

(i) Steady

(ii) Inviscid

(iii) Uniform velocity, pressure, temperature, density, enthalpy, and energy at each x location.

(iv) The oblique shock from the spike is an attached weak shock from a two-dimensional wedge.

(v) We will neglect the angularity of the streamline through the inlet. For example, we will assume that the fluid is travelling horizontally between the oblique and normal shocks and between the normal shock and fuel injection.

(vi) Our general isentropic, oblique-shock, and normal-shock equations apply (where appropriate).

(vii) We will assume that the static Mach number at the inlet of the combustor is the same value throughout the entire combustor.

(viii) You may use an isentropic relation to calculate the static temperature at the exit of the combustor.

(ix) We will utilize one-dimensional flow with heat transfer to obtain the difference in stagnation temperature between the inlet and exit of the combustor from the given constant value of q.

(x) We will neglect the fuel mixture in the air in the combustor.

(a) Briefly describe how a ramjet works.

Discussion:

A ramjet is a form of high-speed propulsion system that does not involve any turbomachinery to compress the flow. Only capable of starting at higher Mach numbers (3+), a ramjet utilizes a series of oblique and normal shocks (depending on the inlet design) to compress and slow the freestream flow that is being captured for use. The flow must be slowed to subsonic velocities for ramjet operation, as by definition a ramjet utilizes subsonic combustion. As in a turbofan or turbojet engine, fuel is injected into the flow, combusted to generate a large pressure and temperature rise, and then accelerated out of a nozzle to convert pressure and thermal energy into kinetic energy that propels a vehicle forward. Traditional turbomachinery runs into operational issues at high-Mach conditions due to shockwaves generated in the compression section of the flowpath.

(b) A simplified ramjet cycle efficiency is given by:

$$\eta = 1 - \left(\left(\frac{p_{1,\infty}}{p_{3,\infty}}\right)^{\frac{\gamma-1}{\gamma}} \frac{\left(T_{4,\infty} - \left(\frac{p_{3,0}}{p_{1,0}}\right)^{\frac{\gamma-1}{\gamma}} \cdot T_{3,\infty}\right)}{(T_{4,\infty} - T_{3,\infty})} \right)$$

Use this equation to come up with the optimal spike half angle (to the nearest degree) for the given cruise conditions. You must write out your general methodology for the grader. Include a plot of the efficiency versus spike half angle.

Solution:

The solution methodology to calculate the ideal inlet spike half angle, θ, is described below. Assumptions (i)-(iii) allow us to utilize the simplified 1-D forms of continuity, momentum, and energy, which directly lead to the isentropic, normal shock, and oblique shock relations that will be used in this problem, per assumption (vi). For brevity, not all calculations are shown.

Note: All calculations performed in Python, see Appendix B.

1. For a given freestream Mach number, M_∞, and a given inlet half angles, θ, solve for the corresponding shock angle, β, per assumption (v).

2. Using β, determine the component of the incoming Mach normal to the oblique shock, M_{1n}.

3. Use normal shock relationships to calculate the normal component of post-oblique-shock Mach number, M_{2n}.

4. Use normal shock relationships to calculate the static pressure ratio across the oblique shock, $\frac{p_2}{p_1}$, as well as the static temperature ratio, $\frac{T_2}{T_1}$..

5. With M_{2n}, β, and θ, determine the magnitude of the post-oblique shock Mach number, M_2.

6. Using assumption (v), treat M_2 as normal to the standing normal shock and use normal shock relationships to determine M_3, the Mach number at the entrance to the combustor

7. Use normal shock relationships to calculate $\frac{p_3}{p_2}$, the post-normal-shock static pressure ratio, as well as the static temperature ratio $\frac{T_3}{T_2}$.

8. Solve for T_3 using $T_3 = \frac{T_3}{T_2}\frac{T_2}{T_1}T_1$ where T_1 is the freestream temperature, T_∞.

9. Use isentropic relationships for M_3 and T_3 to calculate the total temperature at the entrance to the combustor, $T_{t,3}$.

10. Use the energy equation for 1-D flow with heat addition (a.k.a. Rayleigh Flow, assumption (ix)) to calculate the total temperature at the exit of the combustor, $T_{t,4}$, recognizing that total enthalpy can be expressed in terms of specific heats and total temperature.

$$h_{t,3} + q = h_{t,4}$$
$$h = c_p T$$
$$c_p T_{t,3} + q = c_p T_{t,4}$$

11. Assuming that the Mach number is constant throughout the combustor (per assumption (vii)), use isentropic relationships to solve for T_4 with $M_4 = M_3$ (assumption (viii)), neglecting the increased mass from fuel injection (assumption (x)).

12. Calculate the ramjet cycle efficiency, η, using the given equation.

$$\eta = 1 - \left(\left(\frac{p_{1,\infty}}{p_{3,\infty}}\right)^{\frac{\gamma-1}{\gamma}} \frac{\left(T_{4,\infty} - \left(\frac{p_{3,0}}{p_{1,0}}\right)^{\frac{\gamma-1}{\gamma}} \cdot T_{3,\infty}\right)}{(T_{4,\infty} - T_{3,\infty})} \right)$$

For a freestream Mach number $M_\infty = 3$, the maximum turning angle, θ_{max} of the flow is approximately 35 degrees. A range of inlet spike half angles between $1 < \theta < 34$ are iteratively used to calculate ramjet cycle efficiencies to determine the ideal inlet half angle. Figure 4 shows the relationship between ramjet cycle efficiency and inlet half angle for a Mach 3 cruise condition. The ideal inlet half angle is approximately 21 degrees, with a corresponding cycle efficiency of 47.53%, highlighted by a red star.

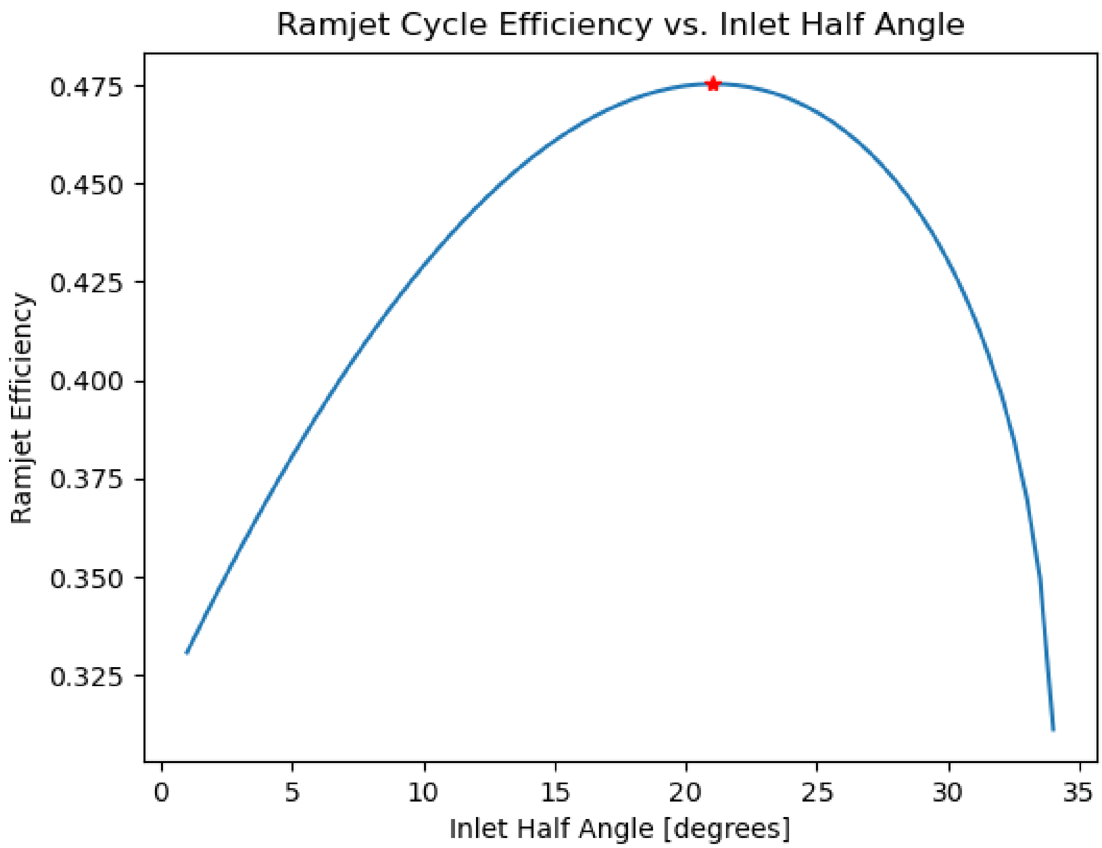

Figure 4: Ramjet cycle efficiency vs. inlet half-angle for Mach 3 cruise

(c) What is the efficiency of the ramjet if we get rid of the spike altogether?

Assumptions:

Remove the inlet spike and corresponding oblique shock while retaining normal shock and all components downstream.

Solution:

Repeat the solution procedure outlined in part (b), removing the oblique-shock relations and setting $M_2 = M_1 = M_\infty$. The ramjet cycle efficiency calculated with a normal shock inlet is 31.7%. For the given flight condition there is only a single solution for cycle efficiency as the variable of inlet geometry has been removed.

$$\eta_{NS} = 31.7\%$$

(d) Write up a description of your observations of the ramjet with and without the spike. Be sure to include relevant compressible-flow theories/jargon. Be sure to include a conversation of the effect of inlet freestream pressure ratio, inlet total pressure ratio, and combustor freestream temperature difference.

Discussion:

The ramjet with the spike inlet shows a range of efficiencies that initially increases with inlet half angle and decreases beyond a turn angle of approximately 21 degrees. For very small turn angles the oblique shock generated is very nearly a mach wave and does little to compress the flow which then goes through a stronger normal shock, similar to the case without the inlet spike. At the upper end of the turn angle range, efficiences begin to fall off rapidly until the maximum turn angle is reached and the flow becomes separated from the inlet. While the exact details of detached flow are not considered in this analysis, efficiences will tend towards a pure normal shock and dramatically less efficient performance. The spike inlet geometry maintains a much higher inlet total pressure ratio than the normal shock case, which indicates that it is much more efficient. For the ideal case, the efficiency is nearly 1.5x that of the normal shock case.

The normal shock creates a generally larger inlet static pressure ratio, indicative of more compression. Although this result is desirable, taken in conjunction with the total pressure recovery, the pure normal shock is less desirable. The normal shock case has a smaller difference in freestream temperature across the combustion chamber compared to the inlet case. The increased temperature at the combustor inlet face effectively reduces the amount of useful thrust that can be extracted from the working fluid. In general, use of oblique shocks to compress and turn the flow is more efficient than a standing normal shock as entropy changes and total pressure loss are reduce for the oblique shock inlet.

(e) For the optimal spike half angle solved for in part (b), let's now explore the effect of cruise Mach number on efficiency. Plot efficiency versus cruise Mach number.

Solution:

Using the methodology outlined in part (b), we vary Mach number with a fixed inlet half angle of 22 degrees. The corresponding shock angle, β, is be recalculated for each cruise Mach. A range of Mach number $2 < M_\infty < 6$ is analyzed, covering and exceeding the common ramjet operational range. Figure 5 shows the relationship between ramjet cycle efficiency and cruise Mach number. The maximum efficiency is 48.32% at a cruise Mach number of $M_\infty = 3.4$, highlighted in figure 5 by a red star.

(f) Write up a description of your observations of the Mach-number effect on ramjets. Be sure to include the phenomena that limit the ramjet's performance at higher Mach numbers. Be sure to include relevant compressible-flow theories/jargon. Be sure to include a conversation of the effect of inlet freestream pressure ratio, inlet total pressure ratio, and combustor freestream temperature difference.

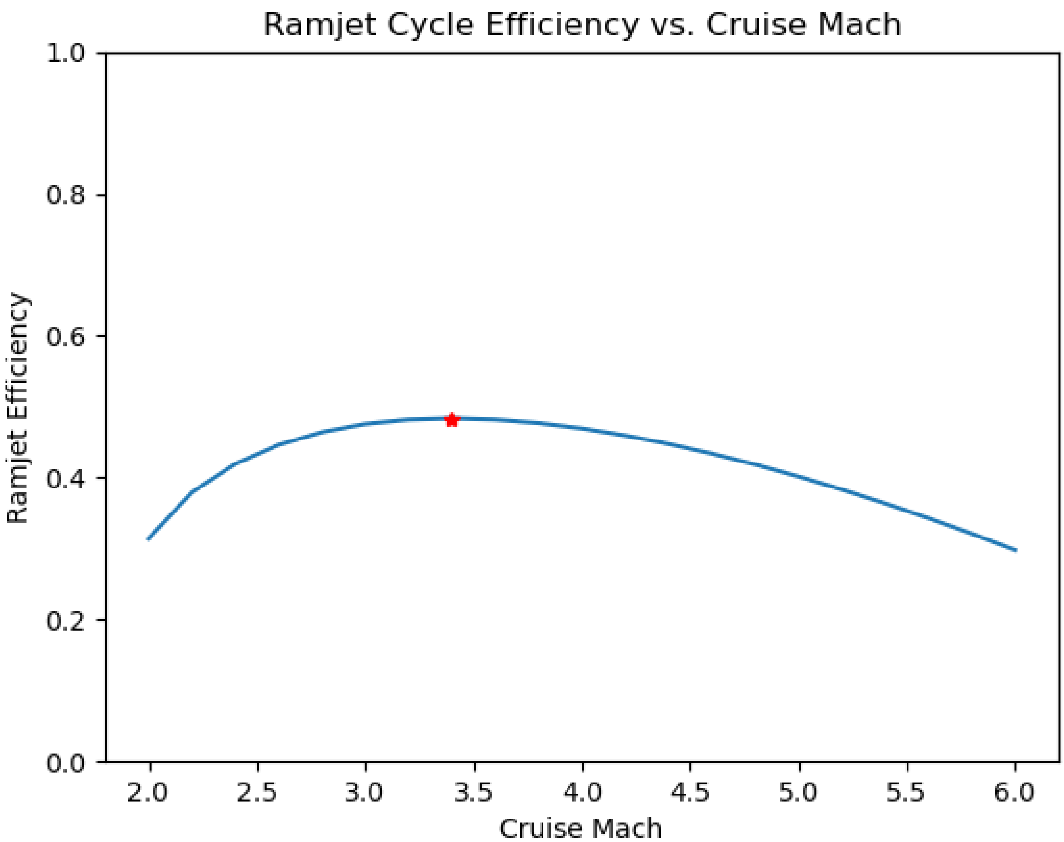

Figure 5: Ramjet cycle efficiency vs. Cruise Mach

Discussion:

For a ramjet with a 21 degree inlet half angle the maximum efficiency is 48.32% at a cruise Mach number of $M_\infty = 3.4$. On either side of this ideal Mach number there is a reduced efficiency. For smaller Mach numbers there is less compression generated by the shocks indicating that the ramjet inlet is less efficient at compression when operating below its design point. Larger Mach numbers cause the total pressure ratio $\frac{p_{t,3}}{p_{t,1}}$ to become increasingly small as the shock strength grows larger. The combustor static temperature difference increases with increasing cruise Mach, although the sensitivity of the ramjet's efficiency to this parameter is less obvious than the previous two. As the temperature difference increases, the efficiency parameter goes down with it, because the propulsion system is less capable of generating useful thrust with a smaller delta in temperature at the beginning and end of the combustion section. Ramjets are limited by the subsonic combustion requirement — as M_∞ increases, the losses associated with reducing the flow to a subsonic velocity become large. Scramjets are the solution to the getting around this efficiency issue as they are designed to operate with supersonic combustion.

Figures 6-8 show the static pressure ratio, total pressure ratio, and combustor inlet static temperature vs. Mach, respectively.

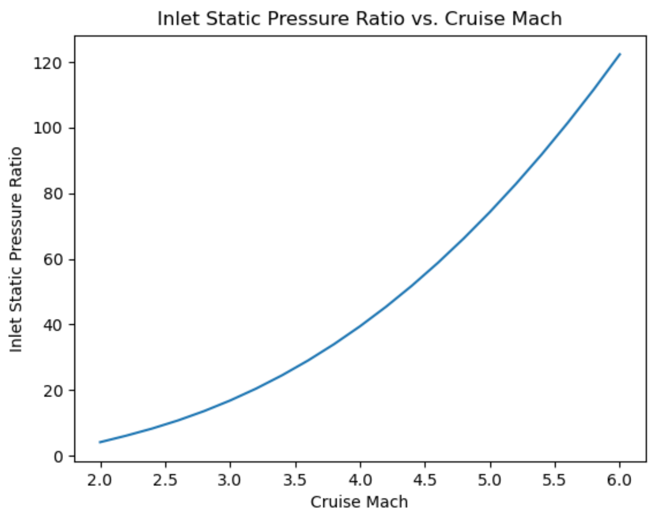

Figure 6: Static pressure ratio vs. Cruise Mach

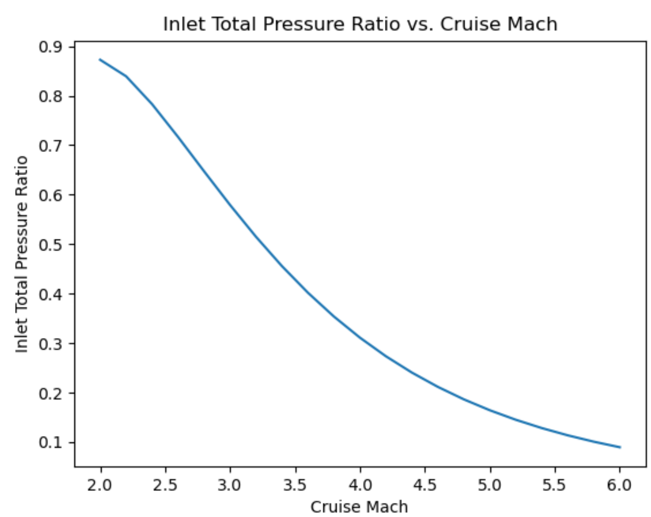

Figure 7: Total pressure ratio vs. Cruise Mach

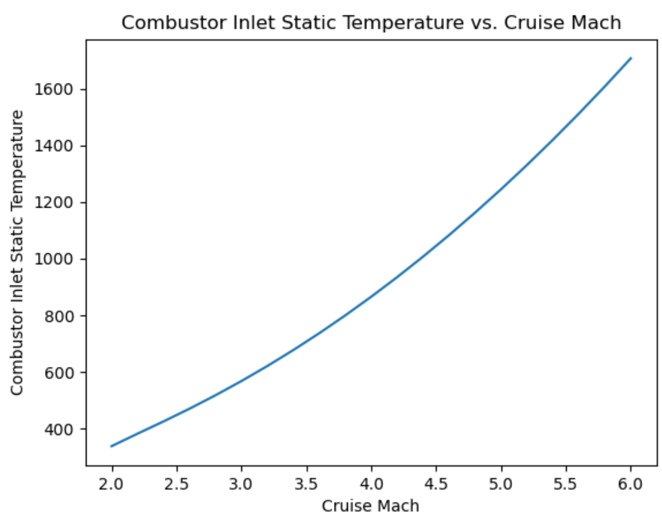

Figure 8: T_3 vs. Cruise Mach

(g) What do you expect to happen if we instead treat the spike as an axisymmetric cone? Briefly justify your answer with words.

<u>Discussion:</u>

If the inlet spike were treated as an axisymmetric cone instead of a 2-D wedge, the oblique shock would be weaker than the wedge case due to the 3-D relieving effects associated with the geometry. Downstream static temperature and pressure would be smaller than the 2-D case, with a higher total pressure ratio across the shock. Although the weaker shock seems ideal from an efficiency standpoint, we may not be generating the ideal compression required and the downstream flow that passes through the standing normal shock would be at a higher Mach number and therefore experience a stronger shock than the 2-D case. A solution would be to use a series of oblique shocks along a curved ramp instead of a linear cone to gradually induce compression and slow the flow more efficiently than a single oblique shock and a standing normal shock.

Problem 3

(a) Briefly describe why scramjets (theoretically) solve some of the specific issues that ramjets encounter at higher Mach numbers.

Discussion:

Scramjets follow the same foundational principles as ramjets with the added caveat that they are capable of supersonic combustion. The need to slow the flow through the ramjet to subsonic velocities adds significant losses at higher Mach numbers. By utilizing supersonic combustion, the compression and velocity reduction in the flow can be performed solely through oblique shocks which are more efficient than normal shocks. This enables air-breathing propulsion at higher Mach numbers than ramjets are capable of supporting, and therefore allows a vehicle to expand its flight envelope and Mach regime. Another issue seen in ramjets at high Mach is the static temperature experienced in the combustion region. As cruise Mach and shock strength increase, the static temperature of the flow grows dramatically. Adding heat to such high-temperature flows reduces the effective amount of thrust that can be extracted from the propulsion system. A scramjet system does not generate static temperatures as high as a ramjet for the given Mach number because there is no need to introduce a normal shock to the flow. In addition to the thrust aspect, limiting the temperature experienced in the flowpath alleviates stress on the vehicle's material requirements and thermal management systems. Despite the lack of moving parts, both ramjets and scramjets are incredibly complex propulsion systems due to the highly complicated flow physics and thermal environments encountered in high supersonic or hypersonic flows.

(b) How do the inlet freestream pressure ratio, inlet total pressure ratio, and combustor inlet freestream temperature compare for this scramjet design compared to the ramjet (with spike) design? You must actually calculate the numbers. Does this support what you said in part (a)?

Assumptions:

Both the ramjet and scramjet are experiencing identical freestream conditions with $M_\infty = 5$, $p_\infty = 20\,\text{kPa}$, and $T_\infty = 217\,\text{K}$. The ramjet has a 21 degree inlet spike half angle, per Problem 2. The assumptions used for Problem 2 are maintained here for use in isentropic relations and normal/oblique shock calculations.

Solution:

The solution methodology for the scramjet case is outlined below. For brevity, not all calculations are shown. *Note: All calculations performed in Python, see Appendix C.*

1. Numerically solve for the shock angle β with $M_\infty = 5$ and $\theta = 7°$. (Polarity of the shock angle does not impact the solution process so for simplicity we take the positive value.)

2. With β, solve for the normal component of the freestream Mach number and use normal

shock relationships to determine the normal component of the post-oblique shock flow.

3. Solve for M_2 using β and θ.

4. Recognize that the second oblique shock is generated by a turn angle of $\theta = 7°$, solve for the second β using M_2 and θ.

5. As before, solve for the component of M_2 normal to the second oblique shock, $M_{2n'}$, then find M_3.

6. Given $M_1, M_{1n}, M_{2n}, M_2, M_{2n'}$, and M_3, isentropic and normal shock relationships can be used to solve for the inlet freestream pressure ratio, the inlet total pressure ratio, and the combustor inlet static temperature. The procedure used in Problem 2 is repeated for many of the calculations shown here.

We now solve for the scramjet inlet freestream pressure ratio, the inlet total pressure ratio, and the combustor inlet static temperature using the above procedure. Results from Problem 2 are used to compare the ramjet performance at the same $M_\infty = 5$ condition. Table 1 compares the two propulsion systems.

Inlet Static Pressure Ratio		**Inlet Total Pressure Ratio**		**Combustor Inlet T_∞ [K]**	
Ramjet	Scramjet	Ramjet	Scramjet	Ramjet	Scramjet
74.30	4.55	0.16	0.92	1244.75	343.04

Table 1: Ramjet vs. Scramjet Performance at $M_\infty = 5$.

The results in table 1 support the discussion in part (a). The ramjet inlet has a static compression ratio over 16x that of the scramjet, with a total pressure ratio approximately 1/6$^{\text{th}}$ that of the scramjet. Despite the scramjet's reduced static compression, the overall efficiency of the propulsion system as measured by total pressure ratio is significantly larger. The combustor inlet temperature is also significantly smaller, approximately 25% of the ramjet value. The reduced combustor static temperature enables more thrust to be pulled out of the working fluid in the scramjet.

(c) Under these conditions, what is the Mach number of the flow at the inlet of the combustor? What combustion challenges do we face in efficiently burning fuel at this Mach number?

Solution:

With $p_{t,3}$ and p_3, the Mach number at the inlet of the combustor, M_3, can be calculated using isentropic relations.

$$M_3 = \sqrt{\left[\left(\frac{p_{t,3}}{p_3}\right)^{\frac{\gamma}{\gamma-1}} - 1\right]\frac{2}{\gamma - 1}}$$

$$M_{combustor} = 3.74$$

Burning fuel at high Mach numbers is difficult for several reasons, some of which inclue flame dynamics, fuel-air mixing, and residence time. Keeping a flameholder operating effectively in such extreme conditions is difficult, a problem exacerbated by the difficulty in thoroughly mixing the fuel and the air going through the combustor to ensure uniform combustion and flow properties. A packet of air moving through a combustor at Mach 3.74 is only exposed to the flameholder and fuel injection for a scant amount of time. Getting good combustion in such a small window of time is a highly complex problem that is still an active research area.

(d) Let's now investigate the qualitative effect of viscosity on shock reflections. We will focus our attention to the last reflected shock before the fuel-injection site. An effect of viscosity is to decelerate the flow in the vicinity of the wall such at $u_{wall} = 0$. We will assume that the scramjet is a height of H from the bottom wall to the top wall. We will also assume that u is equal to its local freestream value a distance δ away from the walls. Sketch y versus β for $0 < y < H$. In addition, sketch the more realistic shape of the shock.

Discussion:

Inclusion of viscous effects changes the way that shocks propogate throughout a flowpath. Because of the non-slip condition with $u_{wall} = 0$, the shock strength required to turn the flow near the wall is smaller and therefore the shock angle β near the wall is also smaller. Figure 9 shows a qualitative sketch of the y vs. β distribution from the bottom surface of the inlet to the top along the reflected oblique shock. Similar to the velocity profile, β begins at 0 and increases until the freestream value of β is reached at a height off of the wall of δ. The shock angle is constant with freestream velocity until a height of $H - \delta$ where the upper surface boundary layer begins to influence the flow. The shock angle then decreses back to a value of 0 at the upper wall.

Figure 10 shows a sketch of the scramjet inlet region with both reflected shocks and viscous effects included. Using knowledge from figure 9 allows us to sketch what happens to the shock in the region near the wall in the viscous boundary layer. The shock is concave on the bottom surface until a height of δ and convex on the top surface from $H - \delta$ to H. The finer details of the shockwave-boundary layer interactions are neglected here but the high-level qualities of the boundary layer's impact on the shockwave are represented qualitatively.

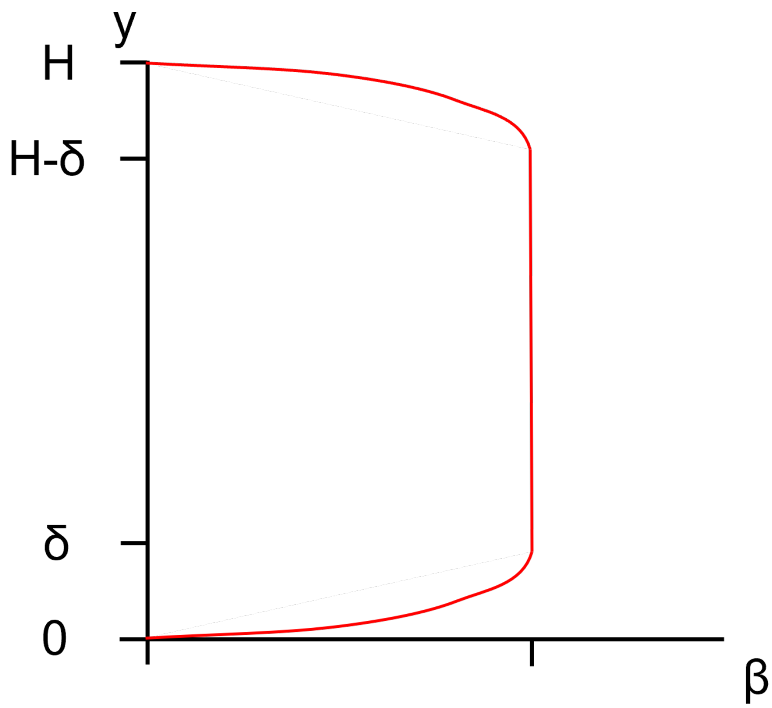

Figure 9: y vs. β

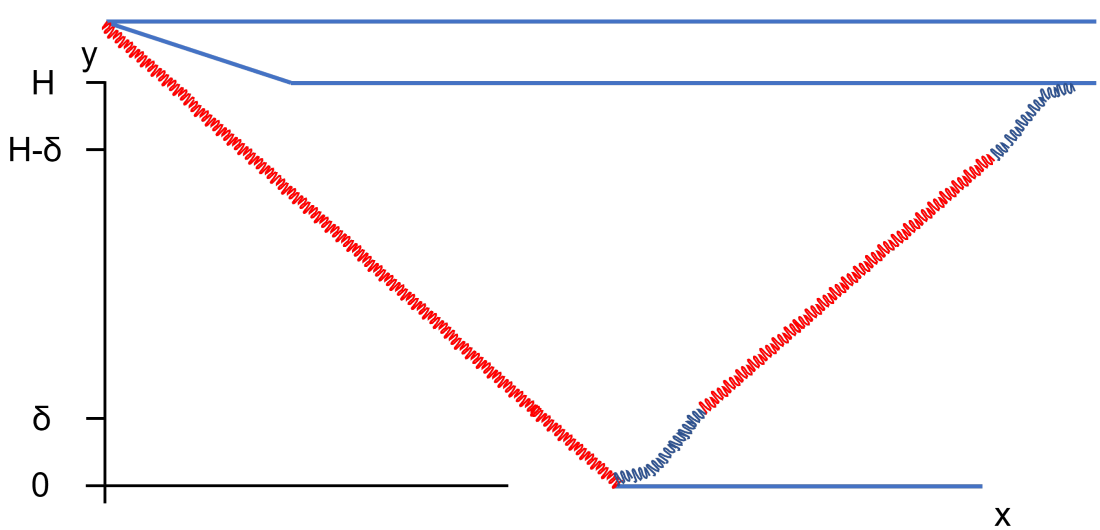

Figure 10: Inlet with viscous effects

Appendix A Problem 1 Python Code

```python
# Compressible Flow
# AEE 553
# Homework 4 - Problem 1
# Evan Burke

import numpy as np
from matplotlib import pyplot as plt
import shocks as ns
import isentropic as isen

# b
gamma = 1.4

def hugoniot(gamma=1.4,rho_ratio=None):
    p2_p1 = ( (gamma+1)/(gamma-1) * rho_ratio - 1) / ((gamma+1)/(gamma-1)
    - rho_ratio)
    return p2_p1

rho_ratio = np.linspace(1,5,endpoint=True)

p2_p1_h = [hugoniot(rho_ratio=r) for r in rho_ratio] # normal shock
p2_p1_i = [r**gamma for r in rho_ratio]

fig,ax = plt.subplots()
ax.plot(rho_ratio,p2_p1_h,label='Hugoniot - NS')
ax.plot(rho_ratio,p2_p1_i,label='Isentropic')
ax.legend()
ax.set_xlabel('Density Ratio')
ax.set_ylabel('Pressure Ratio')
ax.set_title('Compression vs. Density Ratio')
plt.savefig('../images/problem_1/hugoniot_vs_isentropic_compression.png')
plt.close()

machs = np.linspace(1,6,endpoint=True)
print(machs)
pt2_pt1 = [ns.get_total_pressure_ratio_normal_shock(M1=m) for m in machs]
    #get_total_pressure worked without a static pressure? need error
    handling
print(pt2_pt1)
fig,ax = plt.subplots()
ax.plot(machs,pt2_pt1)
ax.set_xlabel('Mach')
ax.set_ylabel('Total Pressure Ratio')
ax.set_title('Normal Shock Total Pressure Ratio vs. Mach Number')
plt.savefig('../images/problem_1/compression_efficiency_NS.png')
plt.close()

```

```
# d
cp = 1004.5
R = 287
ds_isen = [(cp * np.log(r**(gamma-1)) - R * np.log(pr)) for r,pr in zip(
    rho_ratio,p2_p1_i)]

m1 = [ns.get_upstream_mach_normal_shock(p2_p1=pr) for pr in p2_p1_h]
pt2_pt1 = [ns.get_total_pressure_ratio_normal_shock(M1=m) for m in m1]
ds_ns = [-R*np.log(ptr) for ptr in pt2_pt1]
print(ds_isen)
print(ds_ns)

fig,ax = plt.subplots()

ax.plot(rho_ratio,ds_isen,label='Isentropic')
ax.plot(rho_ratio,ds_ns,label='Normal Shock')
ax.set_title('Entropy Change vs. Density Ratio')
ax.set_xlabel('Density ratio')
ax.set_ylabel('Entropy Change [kJ/kg*K]')
ax.legend()

plt.savefig('../images/problem_1/entropy_change.png')

pr_crit = hugoniot(rho_ratio=2.5)
m1_crit = ns.get_upstream_mach_normal_shock(p2_p1=pr_crit)
print(f'Critical Mach: {m1_crit}')
```

Appendix B Problem 2 Python Code

```
# Compressible Flow
# AEE 553
# Homework 4 - Problem 2
# Evan Burke

from gettext import find
import numpy as np
from matplotlib import pyplot as plt
import shocks as ns
import oblique as os
import isentropic as isen
from scipy.optimize import fsolve

class SimpleRamjet:

    def __init__(self,M1,theta,q,cp,gamma,T,delta):
        self.M1 = M1
        self.theta = theta
        self.T=T
        self.delta=delta
        self.beta = SimpleRamjet.find_beta(M=self.M1,theta=self.theta)
        self.M1n = os.get_m1_normal(M1=self.M1,beta=self.beta)
        self.M2n = os.get_m2_normal(M1n=self.M1n)
        self.p2_p1 = ns.get_static_pressure_ratio_normal_shock(M1=self.M1n)
        self.pt2_pt1 = ns.get_total_pressure_ratio_normal_shock(M1=self.M1n)
        self.M2 = os.get_m2(M2n=self.M2n,beta=self.beta,theta=self.theta)
        self.M3 = ns.get_mach_normal_shock(M1=self.M2)
        self.p3_p2 = ns.get_static_pressure_ratio_normal_shock(M1=self.M2)
        self.pt3_pt2 = ns.get_total_pressure_ratio_normal_shock(M1=self.M2)
        self.p1_p3 = 1/self.p2_p1 * 1/self.p3_p2
        self.pt3_pt1 = self.pt2_pt1 * self.pt3_pt2
        self.T2_T1 = ns.get_static_temperature_ratio_normal_shock(M1=self.M1n)
        self.T3_T2 = ns.get_static_temperature_ratio_normal_shock(M1=self.M2)
        self.T3_T1 = self.T3_T2 * self.T2_T1
        self.T3 = self.T3_T2 * self.T2_T1 * self.T
        self.Tt3 = isen.get_total_temperature(M=self.M3,T=self.T3)
        self.q = q
        self.cp = cp
        self.Tt4 = self.Tt3 + (1000*self.q)/self.cp
        self.Tt4_Tt3 = self.Tt4/self.Tt3
        self.T4 = float(isen.get_static_temperature(M=self.M3,T_t=self.Tt4))
```

```python
        self.gamma=gamma
        self.eta = 1 - ((self.p1_p3)**((self.gamma-1)/self.gamma) * (self.T4 - (self.pt3_pt1)**((self.gamma-1)/self.gamma)*self.T3) / (self.T4-self.T3))
        print(f'Ramjet efficiency = {self.eta}')

    def find_beta(M=None,gamma=1.4,theta=None,delta=1):
        theta = np.deg2rad(theta)
        lamb = ((M**2-1)**2 - 3*(1 + (gamma-1)/2*M**2) * (1 + (gamma+1)/2*M**2) * np.tan(theta)**2)**0.5
        chi = ((M**2-1)**3 - 9 * (1 + (gamma-1)/2 * M**2) * (1 + (gamma-1)/2 * M**2 + (gamma+1)/4*M**4)*np.tan(theta)**2)/lamb**3
        tan_beta = (M**2 - 1 + 2*lamb*np.cos((4*np.pi*delta+np.arccos(chi))/3)) / (3 * (1 + (gamma-1)/2*M**2)*np.tan(theta))
        beta = np.arctan(tan_beta)
        beta = np.rad2deg(beta)
        print(f'Shock angle = {beta}')
        return beta

if __name__=='__main__':

    M = 3
    T = 217 # K
    p = 20000 # Pa
    gamma = 1.4
    R = 287 # J/kg K
    cp = 1000 # J/kg K
    q = 500 # kJ/kg

    thetas = np.linspace(1,34,num=67,endpoint=True)

    delta = 1 # weak shock solution
    betas = [SimpleRamjet.find_beta(M=M,theta=th) for th in thetas if not np.isnan(SimpleRamjet.find_beta(M=M,theta=th))]
    ramjets = [SimpleRamjet(M1=M,theta=th,q=q,cp=cp,gamma=1.4,T=T,delta=delta) for th in thetas]
    efficiencies = [ramjet.eta for ramjet in ramjets]

    max_eta = max(efficiencies)
    print(f'Max efficiency = {max_eta}')
    idx_max = efficiencies.index(max_eta)
    print(f'idx max = {idx_max}')
    theta_ideal = thetas[idx_max]
    print(f'Ideal half angle = {theta_ideal}')

    fig,ax = plt.subplots()
    ax.set_title("Ramjet Cycle Efficiency vs. Inlet Half Angle")
    ax.set_xlabel('Inlet Half Angle [degrees]')
    ax.set_ylabel('Ramjet Efficiency')
    plt.plot(thetas,efficiencies,'-')
```

```
    plt.plot(theta_ideal,max_eta,'r*')
    plt.savefig('../images/problem_2/idealtheta.png')

    # c

    M2 = M
    M3 = ns.get_mach_normal_shock(M1=M2)
    p3_p2 = ns.get_static_pressure_ratio_normal_shock(M1=M2)
    pt3_pt2 = ns.get_total_pressure_ratio_normal_shock(M1=M2)
    p2_p3 = 1/p3_p2
    pt3_pt1 = pt3_pt2
    T3_T2 = ns.get_static_temperature_ratio_normal_shock(M1=M2)
    T3_T1 = T3_T2
    print(f'T3_T1 = {T3_T1}')
    T3 = T3_T2 * T
    print(f'T3 = {T3}')
    Tt3 = isen.get_total_temperature(M=M3,T=T3)
    Tt4 = Tt3 + (1000*q)/cp
    print(f'Tt4 = {Tt4}')
    Tt4_Tt3 = Tt4/Tt3
    print(f'Tt4/Tt3 = {Tt4_Tt3}')
    T4 = float(isen.get_static_temperature(M=M3,T_t=Tt4))
    eta_NS = 1 - ((p2_p3)**((gamma-1)/gamma) * (T4 - (pt3_pt2)**((gamma-1)
   /gamma)*T3) / (T4-T3))
    print(f'Scramjet efficiency -- no spike = {eta_NS}')


    # e

    machs = np.linspace(2,6,num=21,endpoint=True)
    ramjets_mach = [SimpleRamjet(M1=Mi,theta=theta_ideal,q=q,cp=cp,gamma
   =1.4,T=T,delta=delta) for Mi in machs]
    efficiencies = [ramjets.eta for ramjets in ramjets_mach]

    max_eta = max(efficiencies)
    idx_ideal = efficiencies.index(max_eta)
    ideal_mach = machs[idx_ideal]

    print(f'Ideal Mach = {ideal_mach}, ideal efficiency = {max_eta}')

    fig,ax = plt.subplots()
    ax.set_title("Ramjet Cycle Efficiency vs. Cruise Mach")
    ax.set_xlabel('Cruise Mach')
    ax.set_ylabel('Ramjet Efficiency')
    ax.set_ylim(bottom=0,top=1)
    plt.plot(machs,efficiencies,'-')
    plt.plot(ideal_mach,max_eta,'r*')
    plt.savefig('../images/problem_2/eta_vs_mach.png')
    plt.close()
    print(f'Max efficiency = {max_eta}')
```

```python

    pt3_pt1s = [ramjets.pt3_pt1 for ramjets in ramjets_mach]

    fig,ax = plt.subplots()
    ax.set_title("Inlet Total Pressure Ratio vs. Cruise Mach")
    ax.set_xlabel('Cruise Mach')
    ax.set_ylabel('Inlet Total Pressure Ratio')
    plt.plot(machs,pt3_pt1s,'-')
    plt.savefig('../images/problem_2/tpr_vs_mach.png')

    p3_p1s = [1/ramjets.p1_p3 for ramjets in ramjets_mach]

    fig,ax = plt.subplots()
    ax.set_title("Inlet Static Pressure Ratio vs. Cruise Mach")
    ax.set_xlabel('Cruise Mach')
    ax.set_ylabel('Inlet Static Pressure Ratio')
    plt.plot(machs,p3_p1s,'-')
    plt.savefig('../images/problem_2/pr_vs_mach.png')

    T3s = [ramjets.T3 for ramjets in ramjets_mach]

    fig,ax = plt.subplots()
    ax.set_title("Combustor Inlet Static Temperature vs. Cruise Mach")
    ax.set_xlabel('Cruise Mach')
    ax.set_ylabel('Combustor Inlet Static Temperature')
    plt.plot(machs,T3s,'-')
    plt.savefig('../images/problem_2/t3_vs_mach.png')
```

Appendix C Problem 3 Python Code

```
# Compressible Flow
# AEE 553
# Homework 4 - Problem 3
# Evan Burke

import numpy as np
from matplotlib import pyplot as plt
import shocks as ns
import isentropic as isen
import oblique as os
from homework_4_problem_2 import SimpleRamjet

M = 5
T = 217
p = 20000

theta1 = 7 #treat this as positive
delta = 1 # weak shock solution

def find_beta(M=None,gamma=1.4,theta=None):
    theta = np.deg2rad(theta)
    lamb = ((M**2-1)**2 - 3*(1 + (gamma-1)/2*M**2) * (1 + (gamma+1)/2*M**2) * np.tan(theta)**2)**0.5
    chi = ((M**2-1)**3 - 9 * (1 + (gamma-1)/2 * M**2) * (1 + (gamma-1)/2 * M**2 + (gamma+1)/4*M**4)*np.tan(theta)**2)/lamb**3
    tan_beta = (M**2 - 1 + 2*lamb*np.cos((4*np.pi*delta+np.arccos(chi))/3)) / (3 * (1 + (gamma-1)/2*M**2)*np.tan(theta))
    beta = np.arctan(tan_beta)
    beta = np.rad2deg(beta)
    print(f'Shock angle = {beta}')
    return beta

beta1 = find_beta(M=M,theta=theta1)
M1n = os.get_m1_normal(M1=M,beta=beta1)
M2n = os.get_m2_normal(M1n=M1n)
M2 = os.get_m2(M2n=M2n,beta=beta1,theta=theta1)

theta2 = 7
beta2 = find_beta(M=M2,theta=theta2)
M2np = os.get_m1_normal(M1=M2,beta=beta2)
M3n = os.get_m2_normal(M1n=M2np)
M3 = os.get_m2(M2n=M3n,beta=beta2,theta=theta2)

pt = isen.get_total_pressure(M=M,p=p)
p2_p1 = ns.get_static_pressure_ratio_normal_shock(M1=M1n)
pt2_pt1 = ns.get_total_pressure_ratio_normal_shock(M1=M1n)
p3_p2 = ns.get_static_pressure_ratio_normal_shock(M1=M2np)
```

```
pt3_pt2 = ns.get_total_pressure_ratio_normal_shock(M1=M2np)
T2_T1 = ns.get_static_temperature_ratio_normal_shock(M1=M1n)
T3_T2 = ns.get_static_temperature_ratio_normal_shock(M1=M2np)

p3_p1 = p2_p1*p3_p2
pt3_pt1 = pt3_pt2*pt2_pt1
T3 = T3_T2 * T2_T1 * T
print(f'T3 = {T3}')
p3 = p3_p1 * p
pt3 = pt3_pt2*pt2_pt1*pt
ramjet_m5 = SimpleRamjet(M1=M,theta=21,q=500,cp=1000,gamma=1.4,T=T,delta=delta)
print('\n\n')
print(f'Inlet Static Pressure Ratio:\nRamjet = {1/ramjet_m5.p1_p3}\nScramjet = {p3_p1}\n')
print(f'Inlet Total Pressure Ratio:\nRamjet = {ramjet_m5.pt3_pt1}\nScramjet = {pt3_pt1}\n')
print(f'Combustor Inlet Static Temperature:\nRamjet = {ramjet_m5.T3}\nScramjet = {T3}')

M3 = isen.get_mach_number(p_t_ratio=pt3/p3)
print(f'Combustor Mach = {M3}')
```