Problem 1

(a) Derive the Hugoniot Eqution.

Assumptions:

1-D, steady flow. Inviscid. Calorically perfect gas (constant specific heats). Uniform pressure distribution around the CV. No heat addition (adiabatic). No work done on or by the CV.

Solution:

Beginning with the 1-D continuity equation and expressing both velocities in terms of the densities and the other velocity:

$$\rho_1 u_1 = \rho_2 u_2$$

$$u_1 = u_2 \left(\frac{\rho_2}{\rho_1}\right)$$

$$u_2 = u_1 \left(\frac{\rho_1}{\rho_2}\right)$$

The 1-D momentum equation:

$$p_1 + \rho_1 u_1^2 = p_2 + \rho_2 u_2^2$$

Plugging in velocities in terms of 1-D continuity and rearranging:

$$p_1 + \rho_1 u_1^2 = p_2 + \rho_2 \left[u_1 \left(\frac{\rho_1}{\rho_2}\right)\right]^2$$

$$(p_1 - p_2) = u_1^2 \left[\rho_2 \left(\frac{\rho_1}{\rho_2}\right)^2 - \rho_1\right]$$

$$(p_1 - p_2) = u_1^2 \left[\left(\frac{\rho_1^2}{\rho_2}\right) - \rho_1\right]$$

$$(p_1 - p_2) = u_1^2 \left(\frac{\rho_1}{\rho_2}\right)(\rho_1 - \rho_2)$$

We now have relationships for u_1 and u_2 in terms of pressures and densities.

$$u_1^2 = \left(\frac{p_1 - p_2}{\rho_1 - \rho_2}\right)\left(\frac{\rho_2}{\rho_1}\right)$$

$$u_2^2 = \left(\frac{p_2 - p_1}{\rho_2 - \rho_1}\right)\left(\frac{\rho_1}{\rho_2}\right)$$

Next, the 1-D energy equation (adiabatic, no work):

$$h_1 + \frac{u_1^2}{2} = h_2 + \frac{u_2^2}{2}$$

The definition of enthalpy, h.

$$h = e + \frac{p}{\rho}$$

Recasting the 1-D energy equation with the definition of enthalpy and rearranging:

$$e_1 + \frac{p_1}{\rho_1} + \frac{u_1^2}{2} = e_2 + \frac{p_2}{\rho_2} + \frac{u_2^2}{2}$$

$$e_1 + \frac{p_1}{\rho_1} + \frac{1}{2}\left[\left(\frac{p_1 - p_2}{\rho_1 - \rho_2}\right)\left(\frac{\rho_2}{\rho_1}\right)\right] = e_2 + \frac{p_2}{\rho_2} + \frac{1}{2}\left[\left(\frac{p_2 - p_1}{\rho_2 - \rho_1}\right)\left(\frac{\rho_1}{\rho_2}\right)\right]$$

$$0 = (e_2 - e_1) + \left(\frac{p_2}{\rho_2} - \frac{p_1}{\rho_1}\right) + \frac{1}{2}\left[\left(\frac{p_2 - p_1}{\rho_2 - \rho_1}\right)\left(\frac{\rho_1}{\rho_2} - \frac{\rho_2}{\rho_1}\right)\right]$$

$$0 = (e_2 - e_1) + \left(\frac{p_2\rho_1 - p_1\rho_2}{\rho_1\rho_2}\right) + \frac{1}{2}\left[\left(\frac{p_2 - p_1}{\rho_2 - \rho_1}\right)\left(\frac{\rho_1^2 - \rho_2^2}{\rho_1\rho_2}\right)\right]$$

$$0 = (e_2 - e_1) + \left(\frac{1}{\rho_1\rho_2}\right)\left[(p_2\rho_1 - p_1\rho_2) + \frac{1}{2}\left(\frac{p_2 - p_1}{\rho_2 - \rho_1}\right)\left(\rho_1^2 - \rho_2^2\right)\right]$$

$$0 = (e_2 - e_1) + \left(\frac{1}{\rho_1\rho_2}\right)\left[(p_2\rho_1 - p_1\rho_2) + \frac{1}{2}\left(\frac{p_1 - p_2}{\rho_1 - \rho_2}\right)(\rho_1 - \rho_2)(\rho_1 + \rho_2)\right]$$

$$0 = (e_2 - e_1) + \left(\frac{1}{\rho_1\rho_2}\right)\left[(p_2\rho_1 - p_1\rho_2) + \frac{1}{2}(p_1 - p_2)(\rho_1 + \rho_2)\right]$$

$$0 = (e_2 - e_1) + \left(\frac{1}{\rho_1\rho_2}\right)\left[(p_2\rho_1 - p_1\rho_2) + \frac{1}{2}(p_1\rho_1 + p_1\rho_2 - p_2\rho_1 - p_2\rho_2)\right]$$

$$0 = (e_2 - e_1) + \left(\frac{1}{2}\right)\left(\frac{1}{\rho_1\rho_2}\right)(p_1\rho_1 - p_1\rho_2 + p_2\rho_1 - p_2\rho_2)$$

$$(e_2 - e_1) = \left(\frac{1}{2}\right)\left(\frac{p_1}{\rho_2} - \frac{p_1}{\rho_1} + \frac{p_2}{\rho_2} - \frac{p_2}{\rho_1}\right)$$

$$(e_2 - e_1) = \left(\frac{p_1 + p_2}{2}\right)\left(\frac{1}{\rho_1} - \frac{1}{\rho_2}\right)$$

We now have one of the common forms of the Hugoniot Equation:

$$(e_2 - e_1) = \left(\frac{p_1 + p_2}{2}\right)(\nu_1 - \nu_2)$$

For a CPG, internal energy, e:

$$e = c_\nu T$$

Substituting the above relationship into our initial Hugoniot Equation:

$$c_\nu (T_2 - T_1) = \left(\frac{p_1 + p_2}{2}\right)(\nu_1 - \nu_2)$$

For a CPG, c_ν:

$$c_\nu = \frac{R}{\gamma - 1}$$

Substituting and rearranging:

$$\frac{R}{\gamma - 1}(T_2 - T_1) = \left(\frac{p_1 + p_2}{2}\right)(\nu_1 - \nu_2)$$

The ideal gas equation of state:

$$T = \frac{p\nu}{R}$$

Substituting and rearranging to solve for p_2/p_1:

$$\frac{R}{\gamma - 1}\left(\frac{p_2\nu_2}{R} - \frac{p_1\nu_1}{R}\right) = \left(\frac{p_1 + p_2}{2}\right)(\nu_1 - \nu_2)$$

$$\frac{2}{\gamma - 1}(p_2\nu_2 - p_1\nu_1) = (p_1 + p_2)(\nu_1 - \nu_2)$$

$$\frac{2}{\gamma - 1}(p_2\nu_2 - p_1\nu_1) = p_1\nu_1 - p_1\nu_2 + p_2\nu_1 - p_2\nu_2$$

$$\left(\frac{2}{\gamma - 1} + 1\right)(p_2\nu_2 - p_1\nu_1) = (p_2\nu_1 - p_1\nu_2)$$

$$\left(\frac{\gamma+1}{\gamma-1}\right) = \frac{(p_2\nu_1 - p_1\nu_2)}{(p_2\nu_2 - p_1\nu_1)}$$

$$\left(\frac{\gamma+1}{\gamma-1}\right) = \frac{\left(\frac{p_2}{p_1}\nu_1 - \nu_2\right)}{\left(\frac{p_2}{p_1}\nu_2 - \nu_1\right)}$$

$$\left(\frac{\gamma+1}{\gamma-1}\right)\left(\frac{p_2}{p_1}\nu_2 - \nu_1\right) = \left(\frac{p_2}{p_1}\nu_1 - \nu_2\right)$$

$$\left(\frac{\gamma+1}{\gamma-1}\right)\left(\frac{p_2}{p_1}\nu_2 - \nu_1\right) = \left(\frac{p_2}{p_1}\nu_1 - \nu_2\right)$$

$$\frac{p_2}{p_1}\left[\left(\frac{\gamma+1}{\gamma-1}\right)\nu_2 - \nu_1\right] = \left(\frac{\gamma+1}{\gamma-1}\right)\nu_1 - \nu_2$$

$$\frac{p_2}{p_1} = \left[\frac{\left(\frac{\gamma+1}{\gamma-1}\right)\nu_1 - \nu_2}{\left(\frac{\gamma+1}{\gamma-1}\right)\nu_2 - \nu_1}\right]$$

$$\frac{p_2}{p_1} = \left[\frac{\left(\frac{\gamma+1}{\gamma-1}\right)\frac{\nu_1}{\nu_2} - 1}{\left(\frac{\gamma+1}{\gamma-1}\right) - \frac{\nu_1}{\nu_2}}\right]$$

The relation between density, ρ, and specific volume, ν:

$$\rho = \frac{1}{\nu}$$

Substituting the above relation yields the final form of the Hugoniot Equation:

$$\boxed{\frac{p_2}{p_1} = \left[\frac{\left(\frac{\gamma+1}{\gamma-1}\right)\frac{\rho_2}{\rho_1} - 1}{\left(\frac{\gamma+1}{\gamma-1}\right) - \frac{\rho_2}{\rho_1}}\right]}$$

<u>Discussion:</u>

(b) <u>Solution:</u>

$$\frac{p_2}{p_1} = \left[\frac{\left(\frac{\gamma+1}{\gamma-1}\right)\frac{\rho_2}{\rho_1} - 1}{\left(\frac{\gamma+1}{\gamma-1}\right) - \frac{\rho_2}{\rho_1}}\right]$$

$$\frac{p_2}{p_1} = \left(\frac{\rho_2}{\rho_1}\right)^{\gamma}$$

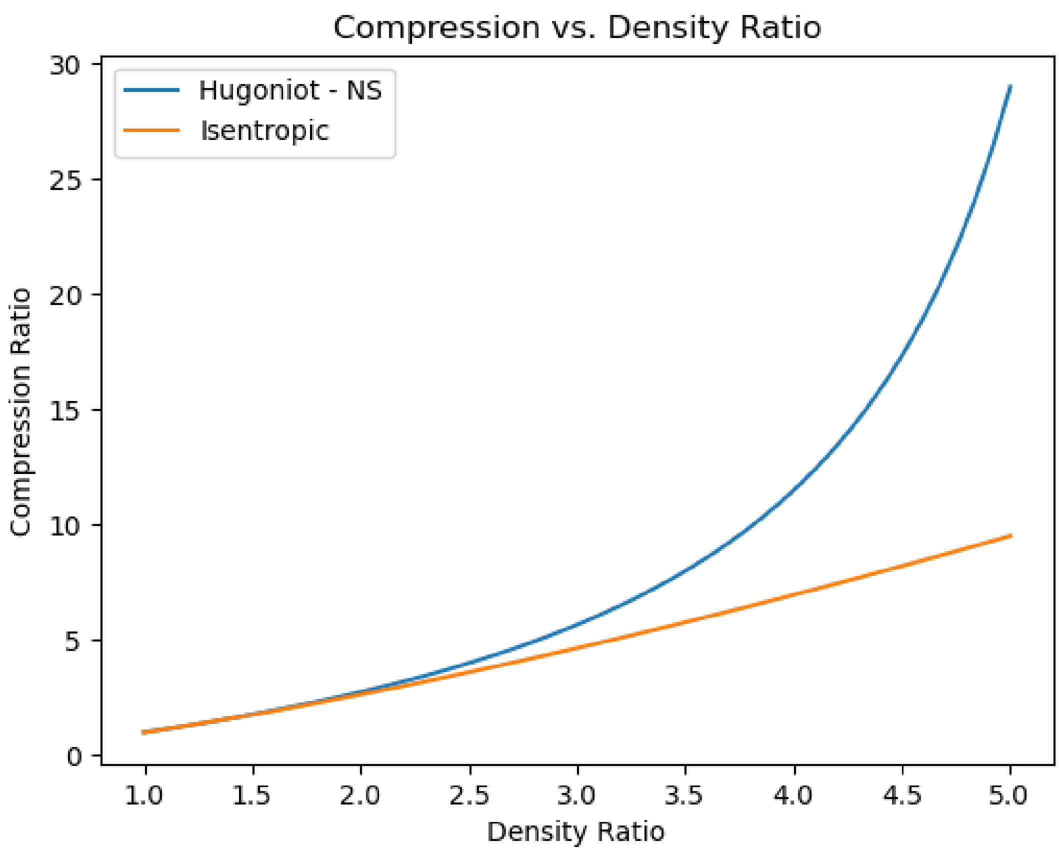

Figure 1: Compression vs. density ratio – normal shock and isentropic compression

(c) Discussion:

(d) Solution:

Isentropic should be equal to 0.

$$s_2 - s_1 = cp \ln\left(\frac{T_2}{T_1}\right) - R \ln\left(\frac{p_2}{p_1}\right)$$

$$\frac{T_2}{T_1} = \left(\frac{\rho_2}{\rho_1}\right)^{\gamma-1}$$

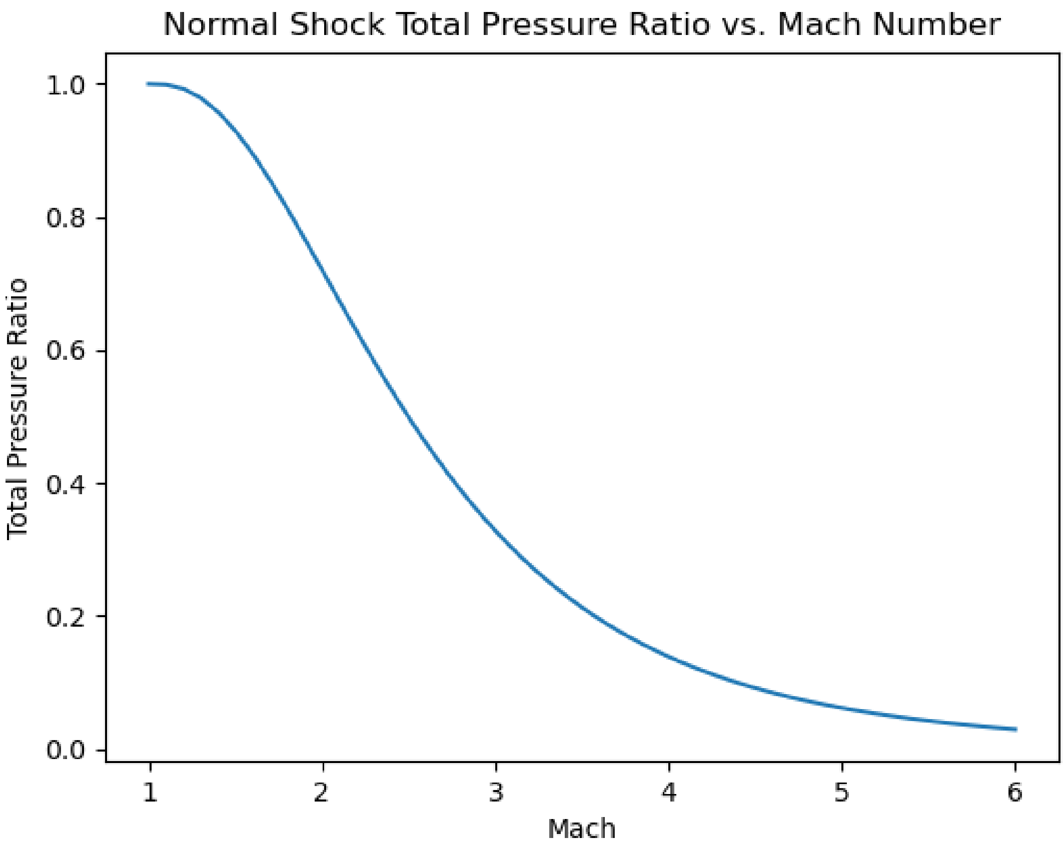

Figure 2: Compression vs. Mach number across normal shock

Across a normal shock:

$$s_2 - s_1 = -R \ln \frac{p_{t,2}}{p_{t,1}}$$

$$\frac{p_2}{p_1} = 1 + \frac{2\gamma}{\gamma + 1} \left(M_1^2 - 1 \right)$$

$$M_1 = \sqrt{\left(\frac{p_2}{p_1} - 1 \right) \left(\frac{\gamma + 1}{2\gamma} \right) + 1}$$

$$\frac{p_{t,2}}{p_{t,1}} = biggrossequation$$