# Problem 3

The potential energy for a simply supported beam under uniform distributed load is

$$\Pi = \int_0^H \left[\frac{EI}{2}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 + \left(\frac{Wx(H-x)}{2}y\right)\right]\mathrm{d}x$$

in which $y$ is the transverse deflection of the beam, $W$ is the transverse distributed load, $E$, $I$, and $H$ are constants independent of $x$ and the boundary conditions are $y(0) = 0$ and $y(H) = 0$.

a) Use the Euler equation to solve for the deflection equation $y(x)$ of the beam.

b) If we were to set $\delta\Pi = 0$, the result would be the Euler equation plus the boundary term

$$y'\delta y\big|_0^H = 0$$

What does this boundary term tell us about the boundary conditions that must exist at the ends of the beam?

## (a) <u>Solution:</u>

Euler's Equation for a functional of the form $I = \int_{x_1}^{x_2} f(x, y, y')\mathrm{d}x = \text{minimum}$:

$$\frac{\partial f}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial f}{\partial y'}\right) = 0$$

Expressing the given potential energy function in terms of $x, y$, and $y'$:

$$\Pi = \int_0^H \left[\frac{EI}{2}(y')^2 + \left(\frac{Wx(H-x)}{2}y\right)\right]\mathrm{d}x$$

$$f(x, y, y') = \left[\frac{EI}{2}(y')^2 + \left(\frac{Wx(H-x)}{2}y\right)\right]$$

$$\frac{\partial f}{\partial y} = \left(\frac{Wx(H-x)}{2}\right)$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial f}{\partial y'}\right) = \left[\frac{\partial^2 f}{\partial y'\partial x} + \frac{\partial^2 f}{\partial y'\partial y}y' + \frac{\partial^2 f}{\partial y'^2}y''\right]$$

$$\frac{\partial f}{\partial y'} = EIy'$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial f}{\partial y'}\right) = EIy''$$

$$\frac{\partial f}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial f}{\partial y'}\right) = 0$$

$$\left(\frac{Wx(H-x)}{2}\right) - EIy'' = 0$$

$$y'' = \left(\frac{Wx(H-x)}{2EI}\right)$$

$$y'' = \frac{W}{2EI}\left(Hx - x^2\right)$$

$$y' = \frac{W}{2EI}\left(\frac{Hx^2}{2} - \frac{x^3}{3}\right) + C$$

$$y = \frac{W}{2EI}\left(\frac{Hx^3}{6} - \frac{x^4}{12}\right) + Cx + D$$

$$y(0) = 0 \rightarrow 0 = D$$

$$y(H) = 0 \rightarrow 0 = \frac{W}{2EI}\left(\frac{H(H)^3}{6} - \frac{H^4}{12}\right) + CH$$

$$0 = \frac{W}{24EI}\left(2H^4 - H^4\right) + CH$$

$$C = -\frac{WH^3}{24EI}$$

$$y = \frac{W}{2EI}\left(\frac{Hx^3}{6} - \frac{x^4}{12}\right) - \frac{WH^3}{24EI}x$$

$$\boxed{y(x) = \frac{W}{24EI}\left(2Hx^3 - x^4 - H^3x\right)}$$

Check boundary conditions:

$$y(0) = \frac{W}{24EI}\left(2Hx^3 - x^4 - H^3x\right) = 0\checkmark$$

$$y(H) = \frac{W}{24EI}\left(2H^4 - H^4 - H^4\right) = 0\checkmark$$

**(b)** Setting $\delta\Pi = 0$:

$$y'\delta y|_0^H = 0$$

$$(y'(H) - y'(0))\delta y = 0$$

Either:

$$y'(H) - y'(0) = 0$$

Or:

$$\delta y(H) - \delta y(0) = 0$$

Examining the first case at $x = 0$ and $x = H$.

$$y(x) = \frac{W}{24EI}\left(2Hx^3 - x^4 - H^3x\right)$$

$$y'(x) = \frac{W}{24EI}\left(6Hx^2 - 4x^3 - H^3\right)$$

$$y'(0) = y'(H)$$

$$\left(6H(0)^2 - 4(0)^3 - H^3\right) = \left(6H(H)^2 - 4(H)^3 - H^3\right)$$

$$-H^3 = H^3 \quad \times$$

This cannot be true, therefore the alternative case must be true:

$$\delta y(H) - \delta y(0) = 0$$

$$\delta y(H) = \delta y(0)$$

At both ends of the beam $y$ must equal 0, which agrees with our initially given boundary conditions.