AEE 546 — Finite Element Analysis I

Department of Mechanical and Aerospace Engineering

Mid Term Exam

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Contents

Problem 1

Instead of the linear shape functions for a 1D bar element, the following shape function shave been proposed for an element with two nodes:

$$N_1 = \frac{-x(1-x)}{2} \qquad N_2 = \frac{x(1+x)}{2}$$

The resulting displacement field is $u = N_1 d_1 + N_2 d_2$.

a) Develop the relation: $\varepsilon = [B]\{d\}$. That is, find the $[B]$ matrix in terms of x.

b) Develop the stiffness matrix, $[K]$.

c) Are these valid shape functions? Why or why not?

(a) <u>**Solution:**</u>

$$\{u\} = [N]\{d\}$$

$$\{\varepsilon\} = [\partial]\{u\}$$

$$\{\varepsilon\} = [\partial][N]\{d\}$$

$$[B] = [\partial][N]$$

$$\{\varepsilon\} = [B]\{d\}$$

$$[N] = [N_1\, N_2] = \begin{bmatrix} \frac{-x(1-x)}{2} & \frac{x(1+x)}{2} \end{bmatrix}$$

$$[B] = [\frac{\partial}{\partial x}]\begin{bmatrix} \frac{-x(1-x)}{2} & \frac{x(1+x)}{2} \end{bmatrix}$$

$$\boxed{[B] = \begin{bmatrix} x - \frac{1}{2} & x + \frac{1}{2} \end{bmatrix}}$$

$$\{\varepsilon\} = \begin{bmatrix} x - \frac{1}{2} & x + \frac{1}{2} \end{bmatrix} \begin{Bmatrix} d_1 \\ d_2 \end{Bmatrix}$$

$$\varepsilon(x) = \left(x - \frac{1}{2}\right) d_1 + \left(x + \frac{1}{2}\right) d_2$$

(b)

$$[K] = \int_{\forall} [B]^T [E][B] \mathrm{d}\forall$$

$$A = \mathrm{d}z\mathrm{d}y$$

$$[K] = \int [B]^T [E][B] A \mathrm{d}x$$

$$[K] = AE \int [B]^T [B] \mathrm{d}x$$

$$[B]^T [B] = \begin{bmatrix} \left(x^2 - x + \frac{1}{4}\right) & \left(x^2 - \frac{1}{4}\right) \\ \left(x^2 - \frac{1}{4}\right) & \left(x^2 + x + \frac{1}{4}\right) \end{bmatrix}$$

$$\int [B]^T [B] \mathrm{d}x = \begin{bmatrix} \left(\frac{x^3}{3} - \frac{x^2}{2} + \frac{x}{4}\right) & \left(\frac{x^3}{3} - \frac{x}{4}\right) \\ \left(\frac{x^3}{3} - \frac{x}{4}\right) & \left(\frac{x^3}{3} + \frac{x^2}{2} + \frac{x}{4}\right) \end{bmatrix}$$

$$[K] = AE \begin{bmatrix} \left(\frac{x^3}{3} - \frac{x^2}{2} + \frac{x}{4}\right) & \left(\frac{x^3}{3} - \frac{x}{4}\right) \\ \left(\frac{x^3}{3} - \frac{x}{4}\right) & \left(\frac{x^3}{3} + \frac{x^2}{2} + \frac{x}{4}\right) \end{bmatrix}$$

$$\boxed{[K] = \frac{AE}{12} \begin{bmatrix} (4x^3 - 6x^2 + 3x) & (4x^3 - 3x) \\ (4x^3 - 3x) & (4x^3 + 6x^2 + 3x) \end{bmatrix}}$$

(c) If we say that node 1 is at an x location of $x = -1$ and node 2 is at an x location of $x = 1$, we see the following behavior of the shape functions:

$$N_1(x_1 = -1) = \frac{-(-1)(1 - (-1))}{2} \qquad N_2(x_1 = -1) = \frac{(-1)(1 + (-1))}{2}$$
$$N_1(x_1 = -1) = 1 \qquad N_2(x_1 = -1) = 0$$

$$N_1(x_2 = 1) = \frac{-(1)(1 - (1))}{2} \qquad N_2(x_2 = 1) = \frac{(1)(1 + (1))}{2}$$
$$N_1(x_2 = 1) = 0 \qquad N_2(x_2 = 1) = 1$$

The shape functions are equal to 1 at the node they are associated with and 0 at the other node. This is a valid shape function for these coordinates. However, if the node coordinates were different then the shape functions would need to be reformulated to ensure this behavior.

Problem 2

A one-dimensional, *second order* element is shown below:

The physical node locations and nodal displacement values are shown in table 1:

Node 1		**Node 2**		**Node 3**	
x_1	d_1	x_2	d_2	x_3	d_3
2 in.	0.15 in.	4 in.	0.05 in.	6 in.	-0.10 in.

Table 1: 1D element coordinates and displacements.

Find the physical location ($x =$) on the element where the displacement is zero.
Solution:

$$u = a_1 + a_2 x + a_3 x^2$$

$$\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = [A] \begin{Bmatrix} 1 \\ x \\ x^2 \end{Bmatrix}$$

$$A = \begin{bmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 4 \\ 1 & 4 & 16 \\ 1 & 6 & 36 \end{bmatrix}$$

$$[N] = \begin{bmatrix} 1 & x & x^2 \end{bmatrix} [A]^{-1}$$

$$[A]^{-1} = \begin{bmatrix} 3 & -3 & 1 \\ -\frac{5}{4} & 2 & \frac{3}{4} \\ \frac{1}{8} & -\frac{1}{4} & \frac{1}{8} \end{bmatrix}$$

$$\begin{bmatrix} N_1 & N_2 & N_3 \end{bmatrix} = \begin{bmatrix} 1 & x & x^2 \end{bmatrix} \begin{bmatrix} 3 & -3 & 1 \\ -\frac{5}{4} & 2 & \frac{3}{4} \\ \frac{1}{8} & -\frac{1}{4} & \frac{1}{8} \end{bmatrix}$$

$$\begin{bmatrix} N_1 & N_2 & N_3 \end{bmatrix} = \begin{bmatrix} \left(\frac{x^2}{8} - \frac{5x}{4} + 3\right) & \left(-\frac{x^2}{4} + 2x - 3\right) & \left(\frac{x^2}{8} - \frac{3x}{4} + 1\right) \end{bmatrix}$$

$$\{u\} = [N]\{d\}$$

$$u = \left[\left(\frac{x^2}{8} - \frac{5x}{4} + 3 \right) \quad \left(-\frac{x^2}{4} + 2x - 3 \right) \quad \left(\frac{x^2}{8} - \frac{3x}{4} + 1 \right) \right] \begin{Bmatrix} 0.15 \\ 0.05 \\ -0.10 \end{Bmatrix}$$

$$u = -\frac{x^2}{160} - \frac{x}{80} + \frac{1}{5}$$

$$\boxed{u(x = 4.7445) = 0}$$

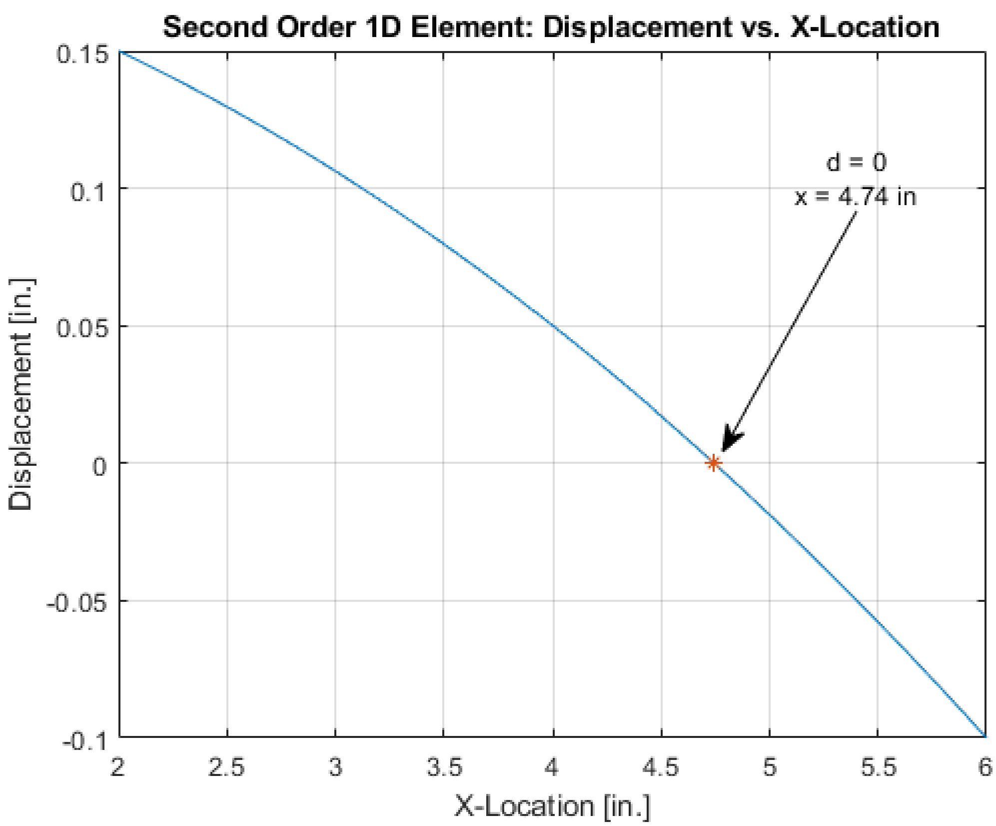

Figure 1: Deflection vs. X-location for second order 1D element. Zero deflection point starred in red.

Problem 3

The potential energy for a simply supported beam under uniform distributed load is

$$\Pi = \int_0^H \left[\frac{EI}{2} \left(\frac{\mathrm{d}y}{\mathrm{d}x} \right)^2 + \left(\frac{Wx(H-x)}{2} y \right) \right] \mathrm{d}x$$

in which y is the transverse deflection of the beam, W is the transverse distributed load, E, I, and H are constants independent of x and the boundary conditions are $y(0) = 0$ and $y(H) = 0$.

a) Use the Euler equation to solve for the deflection equation $y(x)$ of the beam.

b) If we were to set $\delta\Pi = 0$, the result would be the Euler equation plus the boundary term

$$y'\delta y|_0^H = 0$$

What does this boundary term tell us about the boundary conditions that must exist at the ends of the beam?

(a) <u>**Solution:**</u>

Euler's Equation for a functional of the form $I = \int_{x_1}^{x_2} f(x, y, y')\mathrm{d}x =$ minimum:

$$\frac{\partial f}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x} \left(\frac{\partial f}{\partial y'} \right) = 0$$

Expressing the given potential energy function in terms of $x, y,$ and y':

$$\Pi = \int_0^H \left[\frac{EI}{2} (y')^2 + \left(\frac{Wx(H-x)}{2} y \right) \right] \mathrm{d}x$$

$$f(x, y, y') = \left[\frac{EI}{2} (y')^2 + \left(\frac{Wx(H-x)}{2} y \right) \right]$$

$$\frac{\partial f}{\partial y} = \left(\frac{Wx(H-x)}{2} \right)$$

$$\frac{\mathrm{d}}{\mathrm{d}x} \left(\frac{\partial f}{\partial y'} \right) = \left[\frac{\partial^2 f}{\partial y' \partial x} + \frac{\partial^2 f}{\partial y' \partial y} y' + \frac{\partial^2 f}{\partial y'^2} y'' \right]$$

$$\frac{\partial f}{\partial y'} = EIy'$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial f}{\partial y'}\right) = EIy''$$

$$\frac{\partial f}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial f}{\partial y'}\right) = 0$$

$$\left(\frac{Wx(H-x)}{2}\right) - EIy'' = 0$$

$$y'' = \left(\frac{Wx(H-x)}{2EI}\right)$$

$$y'' = \frac{W}{2EI}\left(Hx - x^2\right)$$

$$y' = \frac{W}{2EI}\left(\frac{Hx^2}{2} - \frac{x^3}{3}\right) + C$$

$$y = \frac{W}{2EI}\left(\frac{Hx^3}{6} - \frac{x^4}{12}\right) + Cx + D$$

$$y(0) = 0 \rightarrow 0 = D$$

$$y(H) = 0 \rightarrow 0 = \frac{W}{2EI}\left(\frac{H(H)^3}{6} - \frac{H^4}{12}\right) + CH$$

$$0 = \frac{W}{24EI}\left(2H^4 - H^4\right) + CH$$

$$C = -\frac{WH^3}{24EI}$$

$$y = \frac{W}{2EI}\left(\frac{Hx^3}{6} - \frac{x^4}{12}\right) - \frac{WH^3}{24EI}x$$

$$\boxed{y(x) = \frac{W}{24EI}\left(2Hx^3 - x^4 - H^3x\right)}$$

Check boundary conditions:

$$y(0) = \frac{W}{24EI}\left(2Hx^3 - x^4 - H^3x\right) = 0\checkmark$$

$$y(H) = \frac{W}{24EI}\left(2H^4 - H^4 - H^4\right) = 0\checkmark$$

(b) Setting $\delta\Pi = 0$:

$$y'\delta y\big|_0^H = 0$$

$$(y'(H) - y'(0))\delta y = 0$$

Either:

$$y'(H) - y'(0) = 0$$

Or:

$$\delta y(H) - \delta y(0) = 0$$

Examining the first case at $x = 0$ and $x = H$.

$$y(x) = \frac{W}{24EI}\left(2Hx^3 - x^4 - H^3x\right)$$

$$y'(x) = \frac{W}{24EI}\left(6Hx^2 - 4x^3 - H^3\right)$$

$$y'(0) = y'(H)$$

$$\left(6H(0)^2 - 4(0)^3 - H^3\right) = \left(6H(H)^2 - 4(H)^3 - H^3\right)$$

$$-H^3 = H^3 \quad \times$$

This cannot be true, therefore the alternative case must be true:

$$\delta y(H) - \delta y(0) = 0$$

$$\delta y(H) = \delta y(0)$$

At both ends of the beam y must equal 0, which agrees with our initially given boundary conditions.

Problem 4

Concerning FEA models wih 3-node triangular and 4-node rectangular elements:

a) Using words and/or equations, explain why these elements perform poorly in bending.

b) Does "refining the mesh" (adding more of these elements) improve the performance of models with these elements? Explain.

c) What other approach can be taken to improve the performance of these models? Explain.

(a) ADD MORE ADD MORE ADD MORE 3-node triangular elements are known as linear strain triangles. These elements cause a model to appear overly stiff in bending, requiring a larger-than-realistic moment to generate appropriate bending deflections.

(b) ADD MORE ADD MORE ADD MORE Refining a mesh made up of 3-node triangles and/or 4-node rectangles can improve the accuracy of the model with respect to stiffness and internal strain. However, there is a computational cost that goes hand-in-hand with refining the mesh in this way.

(c) ADD MORE ADD MORE ADD MORE Adding nodes in between existing nodes on the triangular and rectangular elements transforms the elements from linear strain elements to **SOMETHING ELSE**. For the same number of elements, the accuracy with these new elements is significantly greater. However, there is still an added computational cost from the extra nodes. Generally, this cost is small compared to the cost of refinement using solely 3 and 4-node elements.

Problem 5

Consider the bar loaded as shown in figure ??.

Assume $E = 200\,\text{GPa}$ and that the bar is fixed at both ends.

a) Construct a 1D linear bar finite element model of the bar. *Use two elements in each section of the bar (4 elements in total).* Label all nodes and elements.

b) Write the global system of equations $[K]\{d\} = \{R\}$.

c) Apply the boundary conditions to this global system of equations and solve for $\{d\}$.

d) Plot the displacements $u(x)$ vs. x for the entire bar.

e) What are the reaction forces at the two ends?

(a) Figure 2 shows the 1D linear bar finite element model of the bar given in the problem statement. The model has 4 elements and 5 nodes, with the known tractions, forces, and boundary conditions shown in the figure.

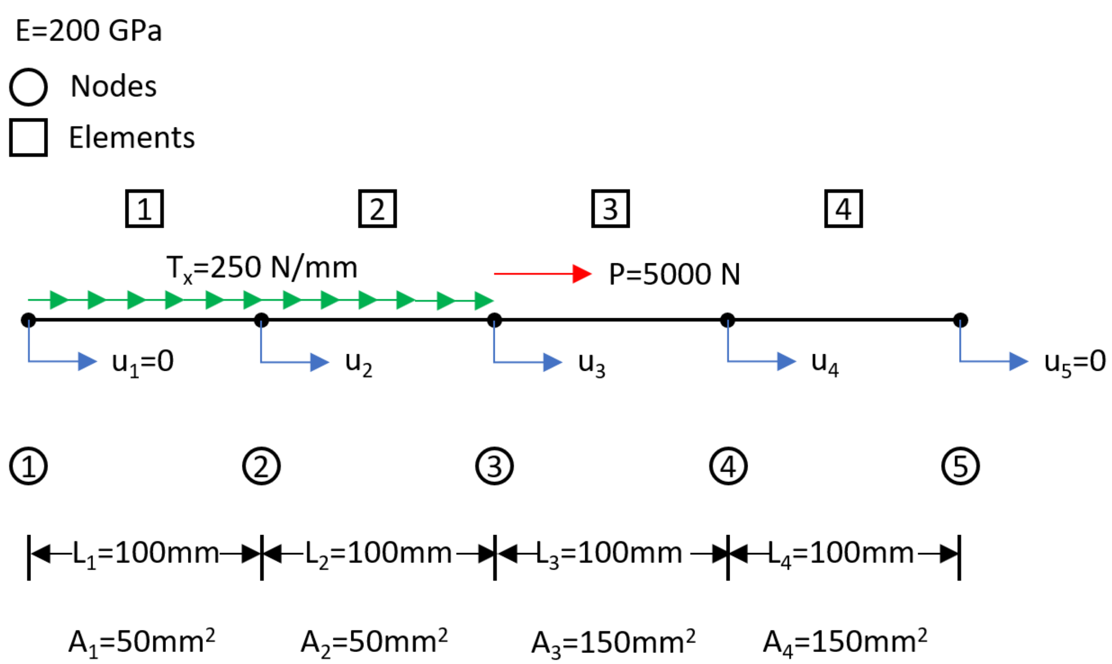

Figure 2: 1D linear bar finite element model of bar.

(b) Let $k_i = \frac{EA_i}{L_i}$. We assemble a local stiffness matrix $[K_i]$ for each element i as shown below:

$$[K_1] = \begin{bmatrix} k_1 & -k_1 \\ -k_1 & k_1 \end{bmatrix}$$

$$[K_2] = \begin{bmatrix} k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix}$$

$$[K_3] = \begin{bmatrix} k_3 & -k_3 \\ -k_3 & k_3 \end{bmatrix}$$

$$[K_4] = \begin{bmatrix} k_4 & -k_4 \\ -k_4 & k_4 \end{bmatrix}$$

Through superposition we combine the local stiffness matrices into a global stiffness matrix, $[K]$:

$$[K] = \begin{bmatrix} k_1 & -k_1 & 0 & 0 & 0 \\ -k_1 & k_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & k_2 & -k_2 & 0 & 0 \\ 0 & -k_2 & k_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & k_3 & -k_3 & 0 \\ 0 & 0 & -k_3 & k_3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & k_4 & -k_4 \\ 0 & 0 & 0 & -k_4 & k_4 \end{bmatrix}$$

$$[K] = \begin{bmatrix} k_1 & -k_1 & 0 & 0 & 0 \\ -k_1 & k_1 + k_2 & -k_2 & 0 & 0 \\ 0 & -k_2 & k_2 + k_3 & -k_3 & 0 \\ 0 & 0 & -k_3 & k_3 + k_4 & -k_4 \\ 0 & 0 & 0 & -k_4 & k_4 \end{bmatrix}$$

Next, we assemble our displacement vector, noting that $d_1 = d_5 = 0$:

$$\{d\} = \begin{Bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \\ d_5 \end{Bmatrix} = \begin{Bmatrix} 0 \\ d_2 \\ d_3 \\ d_4 \\ 0 \end{Bmatrix}$$

Then, the vector of applied forces:

$$\{r\} = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \end{Bmatrix}$$

F_1 and F_5 are unknown because we know that the displacements at both boundaries are 0. The applied forces at nodes 2-4 are outlined below:

$$F_2 = T_x * L_1$$

$$F_3 = T_x * (L_1 + L_2) + P$$

$$F_4 = P$$

$$\{r\} = \begin{Bmatrix} F_1 \\ T_x * L_1 \\ T_x * (L_1 + L_2) + P \\ P \\ F_5 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ 30000 \\ 55000 \\ 5000 \\ F_5 \end{Bmatrix}$$

The global system of equations $[K]\{d\} = \{r\}$:

$$\boxed{\begin{bmatrix} k_1 & -k_1 & 0 & 0 & 0 \\ -k_1 & k_1 + k_2 & -k_2 & 0 & 0 \\ 0 & -k_2 & k_2 + k_3 & -k_3 & 0 \\ 0 & 0 & -k_3 & k_3 + k_4 & -k_4 \\ 0 & 0 & 0 & -k_4 & k_4 \end{bmatrix} \begin{Bmatrix} 0 \\ d_2 \\ d_3 \\ d_4 \\ 0 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ 30000 \\ 55000 \\ 5000 \\ F_5 \end{Bmatrix}}$$

(c) Substituting in given values and solving in MATLAB yields the following for $\{d\}$:

$$\boxed{\{d\} = \begin{Bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \\ d_5 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0.3313 \\ 0.3625 \\ 0.1896 \\ 0 \end{Bmatrix} \text{ mm}}$$

(d) some ldkjfhsdlkfjhdslkjfhdskljfhdskl **MORE HERE MORE HERE MORE HERE** dfgdf dfgdf dfgdf

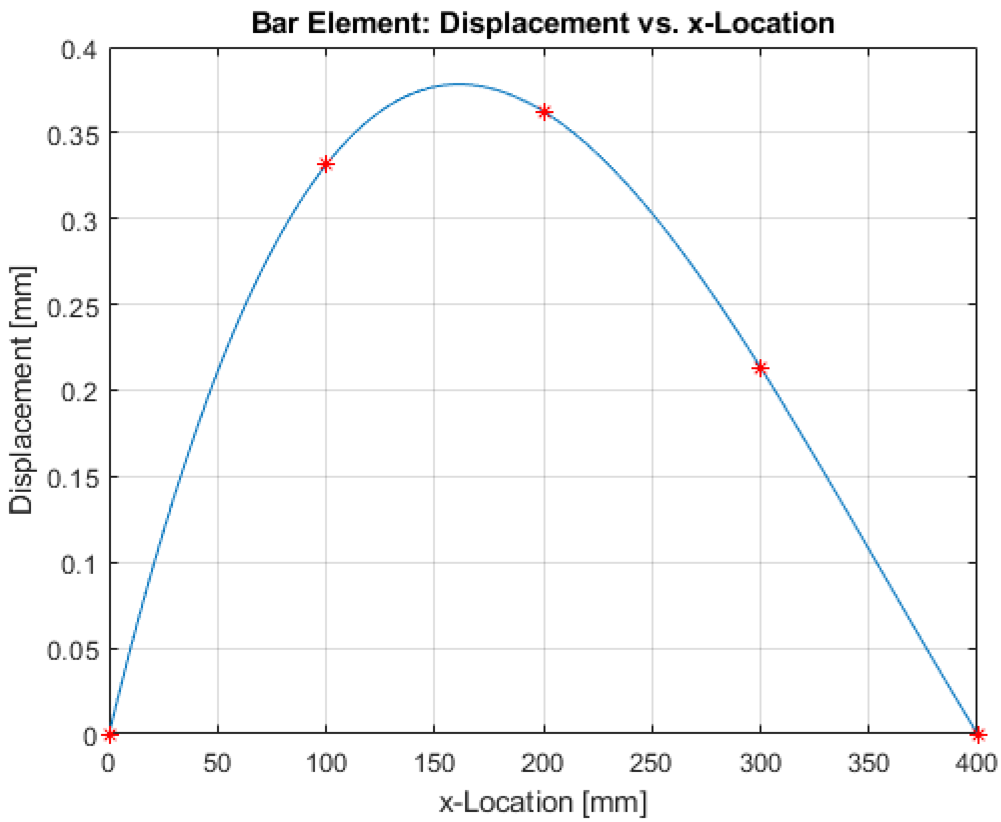

Figure 3: Displacement u vs. x location

(e) Solving the global $[K]\{d\} = \{r\}$ equation in MATLAB yields the following for $\{r\}$:

$$\{r\} = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \end{Bmatrix} = \begin{Bmatrix} -33125 \\ 30000 \\ 55000 \\ 5000 \\ -56875 \end{Bmatrix} \text{ N}$$

The reaction forces at the ends are given by F_1 and F_5:

$$\boxed{F_1 = -33125\,\text{N} \qquad F_5 = -56875\,\text{N}}$$

Appendix A Problem 1 MATLAB Code

```
% FEA Midterm
% Evan Burke

%% Problem 1
clear; close; clc;
syms x d1 d2 E A

N1 = -x*(1-x)/2;
N2 = x*(1+x)/2;
N = [N1 N2];
B = diff(N,x);

d = [d1;d2];
eps = B*d;
pretty(expand(B.'*B))
btb = expand(B.'*B)

K = int(btb)
pretty(expand(K))
```

Appendix B Problem 2 MATLAB Code

```
%% Problem 2
clear; close; clc;
x1 = 2; d1 = 0.15;
x2 = 4; d2 = 0.05;
x3 = 6; d3 = -0.10;

A = [1 x1 x1^2; 1 x2 x2^2; 1 x3 x3^2];

syms x
N = [1 x x^2]/(A);
u = N*[d1;d2;d3];
zero = vpasolve(u,x);
xs = 2:0.05:6;
ds = subs(u,xs);
plot(xs,ds,zero(2),0,'*')
xlabel('X-Location [in.]')
ylabel('Displacement [in.]')
title('Second Order 1D Element: Displacement vs. X-Location')
grid on
fprintf('Zero deflection at x = %f',zero(2))
```

Appendix C Problem 3 MATLAB Code

```
%% Problem 3
clear; close; clc;

syms x y yp ypp H E I W C D

f = E*I/2*yp^2 + (W*x*(H-x)/2)*y
dfdy = diff(f,y)
dfdyp = diff(f,yp)
ddxdfdyp = diff(dfdyp,x) + diff(dfdyp,y)*yp + diff(dfdyp,yp)*ypp

euler = dfdy - ddxdfdyp == 0
RHS = solve(euler,ypp)
yp = int(RHS)
yp = yp + C
y = int(yp)

coeff = solve(y,C)
coeff = subs(coeff,x=H)
y = y- C*x + coeff*x
simplify(y)
```