AEE 546 — Finite Element Analysis I

Department of Mechanical and Aerospace Engineering

Mid Term Exam

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Contents

Problem 1

Instead of the linear shape functions for a 1D bar element, the following shape function shave been proposed for an element with two nodes:

$$N_1 = \frac{-x(1-x)}{2} \qquad N_2 = \frac{x(1+x)}{2}$$

The resulting displacement field is $u = N_1 d_1 + N_2 d_2$.

a) Develop the relation: $\varepsilon = [B]\{d\}$. That is, find the $[B]$ matrix in terms of x.

b) Develop the stiffness matrix, $[K]$.

c) Are these valid shape functions? Why or why not?

(a) <u>**Solution:**</u>

Note: Matrix inversions performed in MATLAB, see appendix A.

The displacement vector for an element is given by the shape functions times the nodal displacement degrees of freedom of the element:

$$\{u\} = [N]\{d\}$$

The strain vector for an element is defined as:

$$\{\varepsilon\} = [\partial]\{u\}$$

Substituting in the definition of the displacement vector:

$$\{\varepsilon\} = [\partial][N]\{d\}$$

The $[B]$ matrix is defined as the derivative of the shape functions, $[N]$:

$$[B] = [\partial][N]$$

Substituting again:

$$\{\varepsilon\} = [B]\{d\}$$

The shape functions defined in the problem statement can be expressed as:

$$[N] = [N_1\; N_2] = \begin{bmatrix} \frac{-x(1-x)}{2} & \frac{x(1+x)}{2} \end{bmatrix}$$

Calculating the $[B]$ matrix:

$$[B] = [\frac{\partial}{\partial x}]\begin{bmatrix} \frac{-x(1-x)}{2} & \frac{x(1+x)}{2} \end{bmatrix}$$

$$\boxed{[B] = \begin{bmatrix} x - \frac{1}{2} & x + \frac{1}{2} \end{bmatrix}}$$

Substituting in the known $[B]$ matrix into the ε equation:

$$\{\varepsilon\} = \begin{bmatrix} x - \frac{1}{2} & x + \frac{1}{2} \end{bmatrix} \begin{Bmatrix} d_1 \\ d_2 \end{Bmatrix}$$

$$\varepsilon(x) = \left(x - \frac{1}{2}\right) d_1 + \left(x + \frac{1}{2}\right) d_2$$

(b) For a given element, the stiffness matrix $[K]$ is defined by:

$$[K] = \int_{V\!\!\!/} [B]^T [E][B] \mathrm{d}V\!\!\!/$$

Noting that $\mathrm{d}V\!\!\!/ = \mathrm{d}x\mathrm{d}y\mathrm{d}z$:

$$A = \mathrm{d}z\mathrm{d}y$$

$$[K] = \int [B]^T [E][B] A \mathrm{d}x$$

Assuming A and E to be constant across the element and solving:

$$[K] = AE \int [B]^T [B] \mathrm{d}x$$

$$[B]^T[B] = \begin{bmatrix} \left(x^2 - x + \frac{1}{4}\right) & \left(x^2 - \frac{1}{4}\right) \\ \left(x^2 - \frac{1}{4}\right) & \left(x^2 + x + \frac{1}{4}\right) \end{bmatrix}$$

$$\int [B]^T [B] \mathrm{d}x = \begin{bmatrix} \left(\frac{x^3}{3} - \frac{x^2}{2} + \frac{x}{4}\right) & \left(\frac{x^3}{3} - \frac{x}{4}\right) \\ \left(\frac{x^3}{3} - \frac{x}{4}\right) & \left(\frac{x^3}{3} + \frac{x^2}{2} + \frac{x}{4}\right) \end{bmatrix}$$

$$[K] = AE \begin{bmatrix} \left(\frac{x^3}{3} - \frac{x^2}{2} + \frac{x}{4}\right) & \left(\frac{x^3}{3} - \frac{x}{4}\right) \\ \left(\frac{x^3}{3} - \frac{x}{4}\right) & \left(\frac{x^3}{3} + \frac{x^2}{2} + \frac{x}{4}\right) \end{bmatrix}$$

$$\boxed{[K] = \frac{AE}{12}\begin{bmatrix} (4x^3 - 6x^2 + 3x) & (4x^3 - 3x) \\ (4x^3 - 3x) & (4x^3 + 6x^2 + 3x) \end{bmatrix}}$$

(c) No information is given regarding the size, position, or orientation of the element. If we make the assumption that node 1 is at an x location of $x = -1$ and node 2 is at an x location of $x = 1$, we see the following behavior of the shape functions:

$$N_1(x_1 = -1) = \frac{-(-1)(1-(-1))}{2} \qquad N_2(x_1 = -1) = \frac{(-1)(1+(-1))}{2}$$
$$N_1(x_1 = -1) = 1 \qquad N_2(x_1 = -1) = 0$$

$$N_1(x_2 = 1) = \frac{-(1)(1-(1))}{2} \qquad N_2(x_2 = 1) = \frac{(1)(1+(1))}{2}$$
$$N_1(x_2 = 1) = 0 \qquad N_2(x_2 = 1) = 1$$

The shape functions are equal to 1 at the node they are associated with and 0 at the other node. This is a valid shape function for these coordinates. However, if the node coordinates were different then the shape functions would need to be reformulated to ensure this behavior. The validity of shape functions is highly dependent on the assumptions that are foundational to the element definition.

Problem 2

A one-dimensional, *second order* element is shown below:

The physical node locations and nodal displacement values are shown in table 1:

Node 1		**Node 2**		**Node 3**	
x_1	d_1	x_2	d_2	x_3	d_3
2 in.	0.15 in.	4 in.	0.05 in.	6 in.	-0.10 in.

Table 1: 1D element coordinates and displacements.

Find the physical location ($x =$) on the element where the displacement is zero.

Solution:

Note: Matrix calculations and plotting performed in MATLAB, see appendix B.

The displacement of the element as a function of x is given by the following second-order equation:

$$u = a_1 + a_2 x + a_3 x^2$$

$$\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = [A] \begin{Bmatrix} 1 \\ x \\ x^2 \end{Bmatrix}$$

$$A = \begin{bmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 4 \\ 1 & 4 & 16 \\ 1 & 6 & 36 \end{bmatrix}$$

Shape functions for the element can be found using the relation below:

$$[N] = \begin{bmatrix} 1 & x & x^2 \end{bmatrix} [A]^{-1}$$

$$[A]^{-1} = \begin{bmatrix} 3 & -3 & 1 \\ -\frac{5}{4} & 2 & \frac{3}{4} \\ \frac{1}{8} & -\frac{1}{4} & \frac{1}{8} \end{bmatrix}$$

$$\begin{bmatrix} N_1 & N_2 & N_3 \end{bmatrix} = \begin{bmatrix} 1 & x & x^2 \end{bmatrix} \begin{bmatrix} 3 & -3 & 1 \\ -\frac{5}{4} & 2 & \frac{3}{4} \\ \frac{1}{8} & -\frac{1}{4} & \frac{1}{8} \end{bmatrix}$$

The final shape functions for the element:

$$\begin{bmatrix} N_1 & N_2 & N_3 \end{bmatrix} = \begin{bmatrix} \left(\frac{x^2}{8} - \frac{5x}{4} + 3\right) & \left(-\frac{x^2}{4} + 2x - 3\right) & \left(\frac{x^2}{8} - \frac{3x}{4} + 1\right) \end{bmatrix}$$

The displacement field for the element can be found using the following relation between shape functions and known nodal degrees of freedom:

$$\{u\} = [N]\{d\}$$

$$u = \begin{bmatrix} \left(\frac{x^2}{8} - \frac{5x}{4} + 3\right) & \left(-\frac{x^2}{4} + 2x - 3\right) & \left(\frac{x^2}{8} - \frac{3x}{4} + 1\right) \end{bmatrix} \begin{Bmatrix} 0.15 \\ 0.05 \\ -0.10 \end{Bmatrix}$$

The following equation defines displacement along the element as a function of x location:

$$u = -\frac{x^2}{160} - \frac{x}{80} + \frac{1}{5}$$

The coordinate of the element where there is no displacement is $x = 4.7445\,\text{in}$.

$$\boxed{u(x = 4.7445) = 0}$$

Figure 1 shows the element displacement versus x-location with the point of zero-displacement annotated.

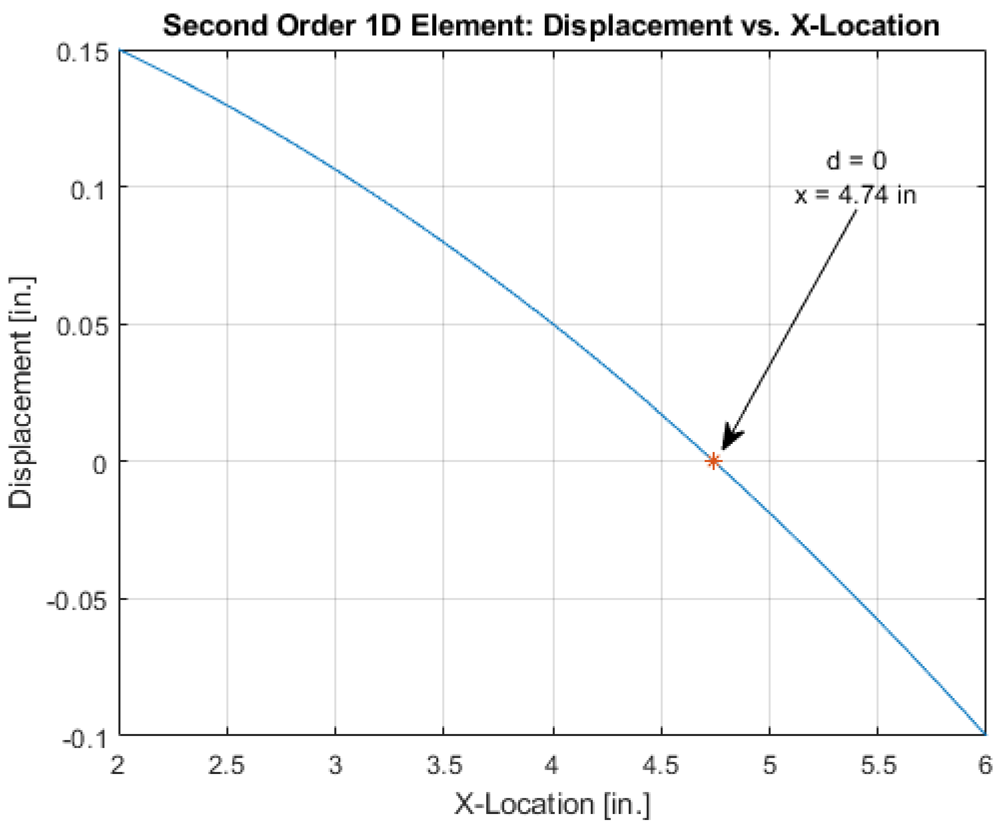


Figure 1: Deflection vs. X-location for second order 1D element. Zero deflection point starred in red.

Problem 3

The potential energy for a simply supported beam under uniform distributed load is

$$\Pi = \int_0^H \left[\frac{EI}{2} \left(\frac{\mathrm{d}y}{\mathrm{d}x} \right)^2 + \left(\frac{Wx(H-x)}{2} y \right) \right] \mathrm{d}x$$

in which y is the transverse deflection of the beam, W is the transverse distributed load, E, I, and H are constants independent of x and the boundary conditions are $y(0) = 0$ and $y(H) = 0$.

a) Use the Euler equation to solve for the deflection equation $y(x)$ of the beam.

b) If we were to set $\delta\Pi = 0$, the result would be the Euler equation plus the boundary term

$$y' \delta y \big|_0^H = 0$$

What does this boundary term tell us about the boundary conditions that must exist at the ends of the beam?

(a) <u>**Solution:**</u>

Note: Integrations and algebra checked in MATLAB, see appendix C.

Euler's Equation for a functional of the form $I = \int_{x_1}^{x_2} f(x, y, y')\mathrm{d}x =$ minimum:

$$\frac{\partial f}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x} \left(\frac{\partial f}{\partial y'} \right) = 0$$

Expressing the given potential energy function in terms of $x, y,$ and y' and solving for the components of Euler's equation:

$$\Pi = \int_0^H \left[\frac{EI}{2} (y')^2 + \left(\frac{Wx(H-x)}{2} y \right) \right] \mathrm{d}x$$

$$f(x, y, y') = \left[\frac{EI}{2} (y')^2 + \left(\frac{Wx(H-x)}{2} y \right) \right]$$

$$\frac{\partial f}{\partial y} = \left(\frac{Wx(H-x)}{2} \right)$$

$$\frac{\mathrm{d}}{\mathrm{d}x} \left(\frac{\partial f}{\partial y'} \right) = \left[\frac{\partial^2 f}{\partial y' \partial x} + \frac{\partial^2 f}{\partial y' \partial y} y' + \frac{\partial^2 f}{\partial y'^2} y'' \right]$$

$$\frac{\partial f}{\partial y'} = EIy'$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial f}{\partial y'}\right) = EIy''$$

$$\frac{\partial f}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial f}{\partial y'}\right) = 0$$

Euler's equation for this functional:

$$\left(\frac{Wx(H-x)}{2}\right) - EIy'' = 0$$

Solving for y via integration and application of the given boundary conditions to determine the value of the constants of integration:

$$y'' = \left(\frac{Wx(H-x)}{2EI}\right)$$

$$y'' = \frac{W}{2EI}\left(Hx - x^2\right)$$

$$y' = \frac{W}{2EI}\left(\frac{Hx^2}{2} - \frac{x^3}{3}\right) + C$$

$$y = \frac{W}{2EI}\left(\frac{Hx^3}{6} - \frac{x^4}{12}\right) + Cx + D$$

$$y(0) = 0 \rightarrow 0 = D$$

$$y(H) = 0 \rightarrow 0 = \frac{W}{2EI}\left(\frac{H(H)^3}{6} - \frac{H^4}{12}\right) + CH$$

$$0 = \frac{W}{24EI}\left(2H^4 - H^4\right) + CH$$

$$C = -\frac{WH^3}{24EI}$$

$$y = \frac{W}{2EI}\left(\frac{Hx^3}{6} - \frac{x^4}{12}\right) - \frac{WH^3}{24EI}x$$

The final equation for y as a function of x, the distributed load W, and the constants H, E, and I:

$$\boxed{y(x) = \frac{W}{24EI}\left(2Hx^3 - x^4 - H^3x\right)}$$

Checking that boundary conditions are satisfied:

$$y(0) = \frac{W}{24EI}\left(2Hx^3 - x^4 - H^3x\right) = 0\checkmark$$

$$y(H) = \frac{W}{24EI}\left(2H^4 - H^4 - H^4\right) = 0\checkmark$$

(b) Setting $\delta\Pi = 0$:

$$y'\delta y\big|_0^H = 0$$

$$(y'(H) - y'(0))\delta y = 0$$

Either:

$$y'(H) - y'(0) = 0$$

Or:

$$\delta y(H) - \delta y(0) = 0$$

Examining the first case at $x = 0$ and $x = H$.

$$y(x) = \frac{W}{24EI}\left(2Hx^3 - x^4 - H^3x\right)$$

$$y'(x) = \frac{W}{24EI}\left(6Hx^2 - 4x^3 - H^3\right)$$

$$y'(0) = y'(H)$$

$$\left(6H(0)^2 - 4(0)^3 - H^3\right) = \left(6H(H)^2 - 4(H)^3 - H^3\right)$$

$$-H^3 = H^3 \quad \times$$

This cannot be true, therefore the alternative case must be true:

$$\delta y(H) - \delta y(0) = 0$$

$$\delta y(H) = \delta y(0)$$

At both ends of the beam y must equal 0, which agrees with our initially given boundary conditions. The variation applied to the potential energy of the beam is equal to zero at both ends and satisfies the fundamental lemma of variational calculus.

Problem 4

Concerning FEA models wih 3-node triangular and 4-node rectangular elements:

a) Using words and/or equations, explain why these elements perform poorly in bending.

b) Does "refining the mesh" (adding more of these elements) improve the performance of models with these elements? Explain.

c) What other approach can be taken to improve the performance of these models? Explain.

(a) 3-node triangular elements are known as constant strain triangles (CSTs). These elements cause a model to appear overly stiff in bending, requiring a larger-than-realistic moment to generate appropriate bending deflections. A mesh made up of CSTs can "lock up" to where the model appears to not bend at all. CSTs are unable to accurately represent stresses and strains and also generate false shear strains which "absorb" internal energy and cause the spurious stiffnesses observed in models using these elements.

A 4-node rectangle, known as a Q4, experiences similar defects to the CST. Q4 elements also fail to accurately represent bending, due to a spurious shear strain. The shear strain that appears even in a plane strain model using Q4s absorbs internal energy so that for a prescribed bending deformation, the moment required to generate said deformation is greater than the correct value.

For Q4 elements, given an applied bending moment of M_b, the following relationship occurs:

$$\frac{\theta_{element}}{\theta_{bending}} = \frac{1-\nu^2}{1+\frac{1-\nu}{2}(\frac{a}{b})^2}$$

As the aspect ratio $(\frac{a}{b})$ increases for a Q4 element, the ratio of rotations tends to 0 and the element locks up.

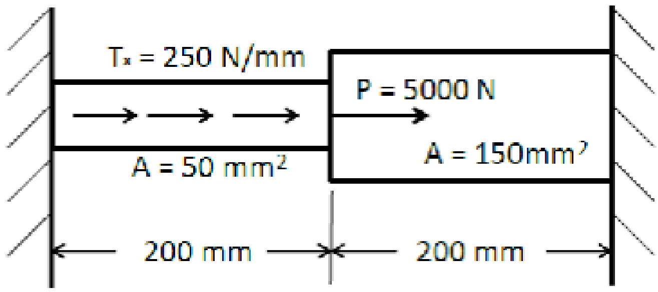
(b) Refining a mesh made up of 3-node triangles and/or 4-node rectangles can improve the accuracy of the model with respect to stiffness and internal strain. However, there is a computational cost that goes hand-in-hand with refining the mesh in this way. In any model, the larger the number of notes, the longer it takes for a solution to be found. For highly-refined models made up of CSTs or Q4s, the elements may "lock up" or experience shear locking, as noted in part (a). The rate of convergence for highly dense models using these elements is slow, computationally expensive, and an inefficient use of resources.

(c) A method of improvement for the CSTs and Q4s is to add nodes in between the existing nodes. This transformation yields linear strain triangles (LSTs) and Q8 elements. LSTs and Q8s are significantly more accurate than CSTs and Q4s; for a given tolerance, the new elements can take advantage of a much coarser mesh. For the same number of elements, the new element types are more computationally expensive due to the increased node count. Generally, this cost is small compared to the cost of refinement using solely 3 and 4-node elements. The tradeoff between reduced node count and element complexity demonstrated here emphasizes the importance of understanding modeling assumptions and limitations for any finite element analysis problem.

Source: *Cook, Robert D., Malkus, David S., Plesha, Michael E. and Witt, Robert J.. Concepts and Applications of Finite Element Analysis, 4th Edition. 4 : Wiley, 2001.*

Problem 5

Consider the bar loaded as shown below:



Assume $E = 200\,\mathrm{GPa}$ and that the bar is fixed at both ends.

a) Construct a 1D linear bar finite element model of the bar. *Use two elements in each section of the bar (4 elements in total).* Label all nodes and elements.

b) Write the global system of equations $[K]\{d\} = \{R\}$.

c) Apply the boundary conditions to this global system of equations and solve for $\{d\}$.

d) Plot the displacements $u(x)$ vs. x for the entire bar.

e) What are the reaction forces at the two ends?

Note: All calculations performed in MATLAB. See appendix D for details.

(a) Figure 2 shows the 1D linear bar finite element model of the bar given in the problem statement. The model has 4 elements and 5 nodes, with the known tractions, forces, and boundary conditions shown in the figure.

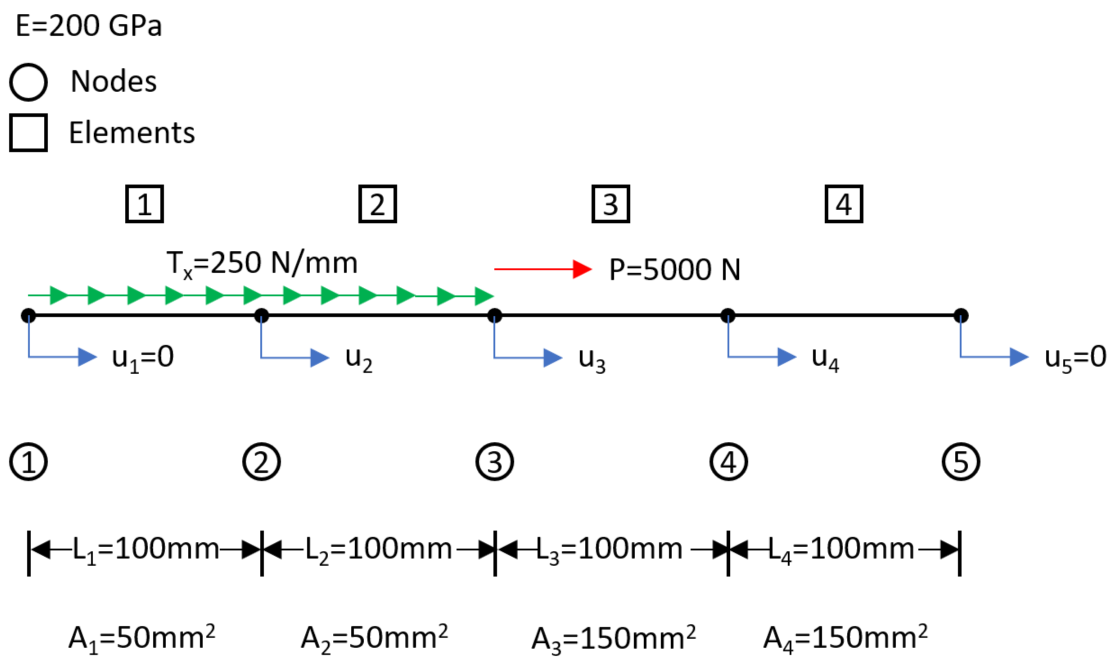


Figure 2: 1D linear bar finite element model of bar.

(b) Let $k_i = \frac{EA_i}{L_i}$. We assemble a local stiffness matrix $[K_i]$ for each element i as shown below:

$$[K_1] = \begin{bmatrix} k_1 & -k_1 \\ -k_1 & k_1 \end{bmatrix}$$

$$[K_2] = \begin{bmatrix} k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix}$$

$$[K_3] = \begin{bmatrix} k_3 & -k_3 \\ -k_3 & k_3 \end{bmatrix}$$

$$[K_4] = \begin{bmatrix} k_4 & -k_4 \\ -k_4 & k_4 \end{bmatrix}$$

Through superposition we combine the local stiffness matrices into a global stiffness matrix, $[K]$:

$$[K] = \begin{bmatrix} k_1 & -k_1 & 0 & 0 & 0 \\ -k_1 & k_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & k_2 & -k_2 & 0 & 0 \\ 0 & -k_2 & k_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} +$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & k_3 & -k_3 & 0 \\ 0 & 0 & -k_3 & k_3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & k_4 & -k_4 \\ 0 & 0 & 0 & -k_4 & k_4 \end{bmatrix}$$

$$[K] = \begin{bmatrix} k_1 & -k_1 & 0 & 0 & 0 \\ -k_1 & k_1 + k_2 & -k_2 & 0 & 0 \\ 0 & -k_2 & k_2 + k_3 & -k_3 & 0 \\ 0 & 0 & -k_3 & k_3 + k_4 & -k_4 \\ 0 & 0 & 0 & -k_4 & k_4 \end{bmatrix}$$

Next, we assemble our displacement vector, noting that $d_1 = d_5 = 0$:

$$\{d\} = \begin{Bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \\ d_5 \end{Bmatrix} = \begin{Bmatrix} 0 \\ d_2 \\ d_3 \\ d_4 \\ 0 \end{Bmatrix}$$

Then, the vector of applied forces:

$$\{r\} = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \end{Bmatrix}$$

F_1 and F_5 are unknown because we know that the displacements at both boundaries are 0. The applied forces at nodes 2-4 are outlined below:

$$F_2 = T_x * L_1 + P$$

$$F_3 = T_x * (L_1 + L_2) + P$$

$$F_4 = P$$

$$\{r\} = \begin{Bmatrix} F_1 \\ T_x * L_1 \\ T_x * (L_1 + L_2) + P \\ P \\ F_5 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ 30000 \\ 55000 \\ 5000 \\ F_5 \end{Bmatrix}$$

The global system of equations $[K]\{d\} = \{r\}$:

$$\boxed{\begin{bmatrix} k_1 & -k_1 & 0 & 0 & 0 \\ -k_1 & k_1 + k_2 & -k_2 & 0 & 0 \\ 0 & -k_2 & k_2 + k_3 & -k_3 & 0 \\ 0 & 0 & -k_3 & k_3 + k_4 & -k_4 \\ 0 & 0 & 0 & -k_4 & k_4 \end{bmatrix} \begin{Bmatrix} 0 \\ d_2 \\ d_3 \\ d_4 \\ 0 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ 30000 \\ 55000 \\ 5000 \\ F_5 \end{Bmatrix}}$$

(c) Substituting in given values and solving in MATLAB yields the following for $\{d\}$:

$$\boxed{\{d\} = \begin{Bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \\ d_5 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0.3313 \\ 0.3625 \\ 0.1896 \\ 0 \end{Bmatrix} \text{ mm}}$$

(d) Following a similar solution procedure as in Problem 2 we solve for displacement, $u(x)$, in MATLAB. Note, values are not shown for legibility:

$$u = a_1 + a_2 x + a_3 x^2 + a_4 x^3 + a_5 x^4$$

$$\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{Bmatrix} = [A] \begin{Bmatrix} 1 \\ x \\ x^2 \\ x^3 \\ x^4 \end{Bmatrix}$$

$$A = \begin{bmatrix} 1 & x_1 & x_1^2 & x_1^3 & x_1^4 \\ 1 & x_2 & x_2^2 & x_2^3 & x_2^4 \\ 1 & x_3 & x_3^2 & x_3^3 & x_3^4 \\ 1 & x_4 & x_4^2 & x_4^3 & x_4^4 \\ 1 & x_5 & x_5^2 & x_5^3 & x_5^4 \end{bmatrix}$$

$$\begin{bmatrix} N_1 & N_2 & N_3 & N_4 & N_5 \end{bmatrix} = \begin{bmatrix} 1 & x & x^2 & x^3 & x^4 \end{bmatrix} [A]^{-1}$$

$$\{u\} = [N]\{d\}$$

Figure 3 shows the displacement u across the finite element bar model. Nodes 1-5 are marked with a red star. The curve obeys the given boundary conditions with $u(x_1) = u(x_5) = 0$.

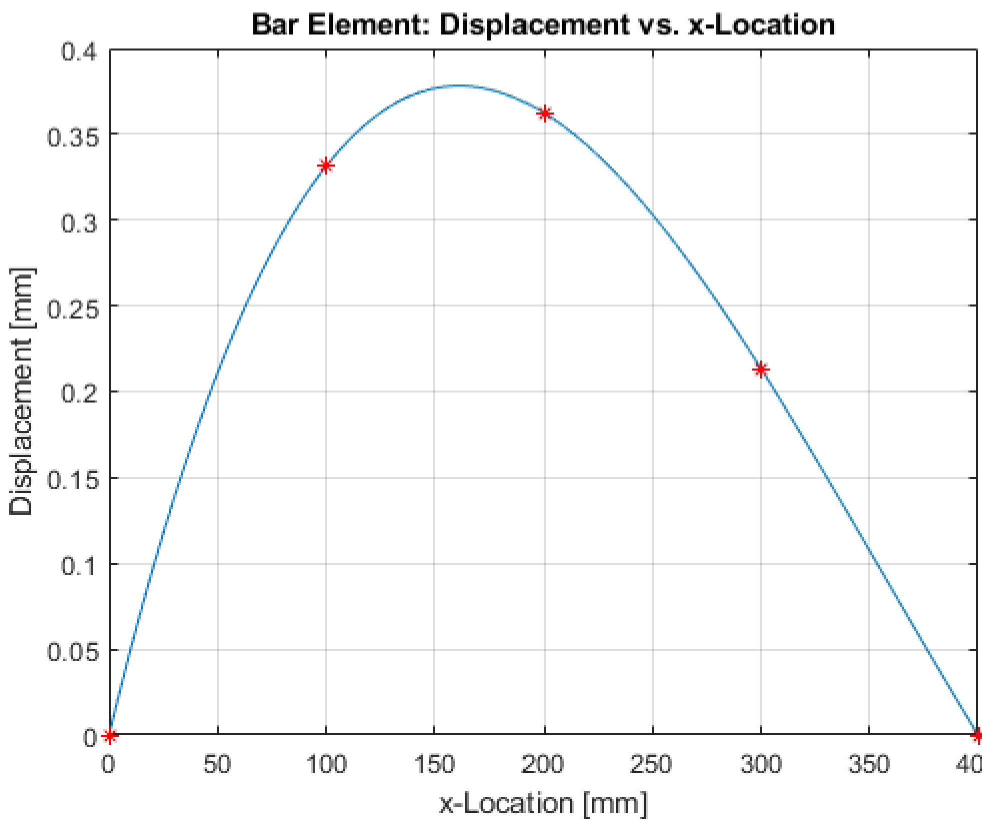


Figure 3: Displacement u vs. x location

(e) Solving the global $[K]\{d\} = \{r\}$ equation in MATLAB yields the following for $\{r\}$:

$$\{r\} = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \end{Bmatrix} = \begin{Bmatrix} -33125 \\ 30000 \\ 55000 \\ 5000 \\ -56875 \end{Bmatrix} \text{ N}$$

The reaction forces at the ends are given by F_1 and F_5:

$$\boxed{F_1 = -33125\,\text{N} \qquad F_5 = -56875\,\text{N}}$$

Appendix A Problem 1 MATLAB Code

```
% FEA Midterm
% Evan Burke

%% Problem 1
clear; close; clc;
syms x d1 d2 E A

N1 = -x*(1-x)/2;
N2 = x*(1+x)/2;
N = [N1 N2];
B = diff(N,x);

d = [d1;d2];
eps = B*d;
pretty(expand(B.'*B))
btb = expand(B.'*B)

K = int(btb)
pretty(expand(K))
```

Appendix B Problem 2 MATLAB Code

```
%% Problem 2
clear; close; clc;
x1 = 2; d1 = 0.15;
x2 = 4; d2 = 0.05;
x3 = 6; d3 = -0.10;

A = [1 x1 x1^2; 1 x2 x2^2; 1 x3 x3^2];

syms x
N = [1 x x^2]/(A);
u = N*[d1;d2;d3];
zero = vpasolve(u,x);
xs = 2:0.05:6;
ds = subs(u,xs);
plot(xs,ds,zero(2),0,'*')
xlabel('X-Location [in.]')
ylabel('Displacement [in.]')
title('Second Order 1D Element: Displacement vs. X-Location')
grid on
fprintf('Zero deflection at x = %f',zero(2))
```

Appendix C Problem 3 MATLAB Code

```
%% Problem 3
clear; close; clc;

syms x y yp ypp H E I W C D

f = E*I/2*yp^2 + (W*x*(H-x)/2)*y
dfdy = diff(f,y)
dfdyp = diff(f,yp)
ddxdfdyp = diff(dfdyp,x) + diff(dfdyp,y)*yp + diff(dfdyp,yp)*ypp

euler = dfdy - ddxdfdyp == 0
RHS = solve(euler,ypp)
yp = int(RHS)
yp = yp + C
y = int(yp)

coeff = solve(y,C)
coeff = subs(coeff,x=H)
y = y- C*x + coeff*x
simplify(y)
```

Appendix D Problem 5 MATLAB Code

```
%% Problem 5
clear; close; clc;

E = 200*1000; % MPa
P = 5000; % N
Tx = 250; % N/mm
A1 = 50; A2 = 50; % mm^2
A3 = 150; A4 = 150; % mm^2
L1 = 100; L2 = 100; % mm
L3 = 100; L4 = 100; % mm

x1 = 0; x2 = x1 + L1; x3 = x2 + L2; x4 = x3 + L3; x5 = x4 + L4;

k1 = A1*E/L1;
k2 = A2*E/L2;
k3 = A3*E/L3;
k4 = A4*E/L4;

K1 = [k1 -k1; -k1 k1];
K2 = [k2 -k2; -k2 k2];
K3 = [k3 -k3; -k3 k3];
K4 = [k4 -k4; -k4 k4];

Ks = {K1, K2, K3, K4};

K = zeros(5,5);

for i=1:4
    K(i:i+1,i:i+1) = K(i:i+1,i:i+1) + Ks{i};
end

syms d2 d3 d4 F1 F5

F2 = Tx*L1 + P;
F3 = Tx*(L1+L2) + P;
F4 = P;

d1 = 0; d5 = 0;

d = [d1; d2; d3; d4; d5];
R = [F1; F2; F3; F4; F5];

Kd = K*d;

eq1 = Kd(1,:) == R(1);
eq2 = Kd(2,:) == R(2);
eq3 = Kd(3,:) == R(3);
```

```
eq4 = Kd(4,:) == R(4);
eq5 = Kd(5,:) == R(5);

eq = K*d == R;

sol = solve(eq);

F1 = double(sol.F1);
F5 = double(sol.F5);
d2 = double(sol.d2);
d3 = double(sol.d3);
d4 = double(sol.d4);

subs(R)
d = double(subs(d));

A = [1 x1 x1^2 x1^3 x1^4;
     1 x2 x2^2 x2^3 x2^4;
     1 x3 x3^2 x3^3 x3^4;
     1 x4 x4^2 x4^3 x5^4;
     1 x5 x5^2 x5^3 x5^4];

syms x

N = [1 x x^2 x^3 x^4]/(A);
u = N*d;

xs = 0:1:x5;
ds = subs(u,xs);
plot(xs,ds,x1,subs(u,x1),'r*',x2,subs(u,x2),'r*',x3,subs(u,x3),'r*',x4,
   subs(u,x4),'r*',x5,subs(u,x5),'r*')
xlabel('x-Location [mm]')
ylabel('Displacement [mm]')
title('Bar Element: Displacement vs. x-Location')
grid on
```