Problem 3

The potential energy for a simply supported beam under uniform distributed load is

$$\Pi = \int_0^H \left[\frac{EI}{2} \left(\frac{\mathrm{d}y}{\mathrm{d}x} \right)^2 + \left(\frac{Wx(H-x)}{2} y \right) \right] \mathrm{d}x$$

in which y is the transverse deflection of the beam, W is the transverse distributed load, E, I, and H are constants independent of x and the boundary conditions are $y(0) = 0$ and $y(H) = 0$.

a) Use the Euler equation to solve for the deflection equation $y(x)$ of the beam.

b) If we were to set $\delta\Pi = 0$, the result would be the Euler equation plus the boundary term

$$y'\delta y\big|_0^H = 0$$

What does this boundary term tell us about the boundary conditions that must exist at the ends of the beam?

(a) <u>**Solution:**</u>

Note: Integrations and algebra checked in MATLAB, see appendix **??**.

Euler's Equation for a functional of the form $I = \int_{x_1}^{x_2} f(x, y, y')\mathrm{d}x = \text{minimum}$:

$$\frac{\partial f}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x} \left(\frac{\partial f}{\partial y'} \right) = 0$$

Expressing the given potential energy function in terms of $x, y,$ and y' and solving for the components of Euler's equation:

$$\Pi = \int_0^H \left[\frac{EI}{2} (y')^2 + \left(\frac{Wx(H-x)}{2} y \right) \right] \mathrm{d}x$$

$$f(x, y, y') = \left[\frac{EI}{2} (y')^2 + \left(\frac{Wx(H-x)}{2} y \right) \right]$$

$$\frac{\partial f}{\partial y} = \left(\frac{Wx(H-x)}{2} \right)$$

$$\frac{\mathrm{d}}{\mathrm{d}x} \left(\frac{\partial f}{\partial y'} \right) = \left[\frac{\partial^2 f}{\partial y' \partial x} + \frac{\partial^2 f}{\partial y' \partial y} y' + \frac{\partial^2 f}{\partial y'^2} y'' \right]$$

$$\frac{\partial f}{\partial y'} = EIy'$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial f}{\partial y'}\right) = EIy''$$

$$\frac{\partial f}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial f}{\partial y'}\right) = 0$$

Euler's equation for this functional:

$$\left(\frac{Wx(H-x)}{2}\right) - EIy'' = 0$$

Solving for y via integration and application of the given boundary conditions to determine the value of the constants of integration:

$$y'' = \left(\frac{Wx(H-x)}{2EI}\right)$$

$$y'' = \frac{W}{2EI}\left(Hx - x^2\right)$$

$$y' = \frac{W}{2EI}\left(\frac{Hx^2}{2} - \frac{x^3}{3}\right) + C$$

$$y = \frac{W}{2EI}\left(\frac{Hx^3}{6} - \frac{x^4}{12}\right) + Cx + D$$

$$y(0) = 0 \rightarrow 0 = D$$

$$y(H) = 0 \rightarrow 0 = \frac{W}{2EI}\left(\frac{H(H)^3}{6} - \frac{H^4}{12}\right) + CH$$

$$0 = \frac{W}{24EI}\left(2H^4 - H^4\right) + CH$$

$$C = -\frac{WH^3}{24EI}$$

$$y = \frac{W}{2EI}\left(\frac{Hx^3}{6} - \frac{x^4}{12}\right) - \frac{WH^3}{24EI}x$$

The final equation for y as a function of x, the distributed load W, and the constants H, E, and I:

$$\boxed{y(x) = \frac{W}{24EI}\left(2Hx^3 - x^4 - H^3x\right)}$$

Checking that boundary conditions are satisfied:

$$y(0) = \frac{W}{24EI}\left(2Hx^3 - x^4 - H^3x\right) = 0\checkmark$$

$$y(H) = \frac{W}{24EI}\left(2H^4 - H^4 - H^4\right) = 0\checkmark$$

(b) Setting $\delta\Pi = 0$:

$$y'\delta y\big|_0^H = 0$$

$$(y'(H) - y'(0))\delta y = 0$$

Either:

$$y'(H) - y'(0) = 0$$

Or:

$$\delta y(H) - \delta y(0) = 0$$

Examining the first case at $x = 0$ and $x = H$.

$$y(x) = \frac{W}{24EI}\left(2Hx^3 - x^4 - H^3x\right)$$

$$y'(x) = \frac{W}{24EI}\left(6Hx^2 - 4x^3 - H^3\right)$$

$$y'(0) = y'(H)$$

$$\left(6H(0)^2 - 4(0)^3 - H^3\right) = \left(6H(H)^2 - 4(H)^3 - H^3\right)$$

$$-H^3 = H^3 \quad \times$$

This cannot be true, therefore the alternative case must be true:

$$\delta y(H) - \delta y(0) = 0$$

$$\delta y(H) = \delta y(0)$$

At both ends of the beam y must equal 0, which agrees with our initially given boundary conditions. The variation applied to the potential energy of the beam is equal to zero at both ends and satisfies the fundamental lemma of variational calculus.