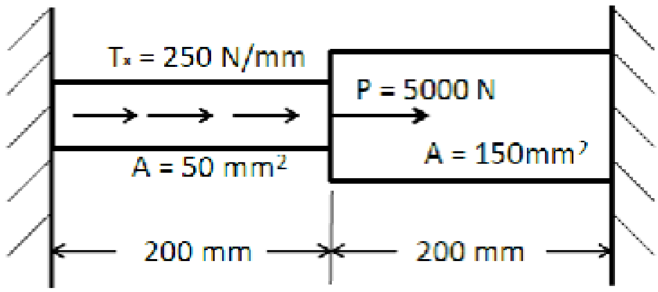
Problem 5

Consider the bar loaded as shown below:



Assume $E = 200\,\text{GPa}$ and that the bar is fixed at both ends.

a) Construct a 1D linear bar finite element model of the bar. *Use two elements in each section of the bar (4 elements in total).* Label all nodes and elements.

b) Write the global system of equations $[K]\{d\} = \{R\}$.

c) Apply the boundary conditions to this global system of equations and solve for $\{d\}$.

d) Plot the displacements $u(x)$ vs. x for the entire bar.

e) What are the reaction forces at the two ends?

Note: All calculations performed in MATLAB. See appendix **??** *for details.*

(a) Figure 1 shows the 1D linear bar finite element model of the bar given in the problem statement. The model has 4 elements and 5 nodes, with the known tractions, forces, and boundary conditions shown in the figure.

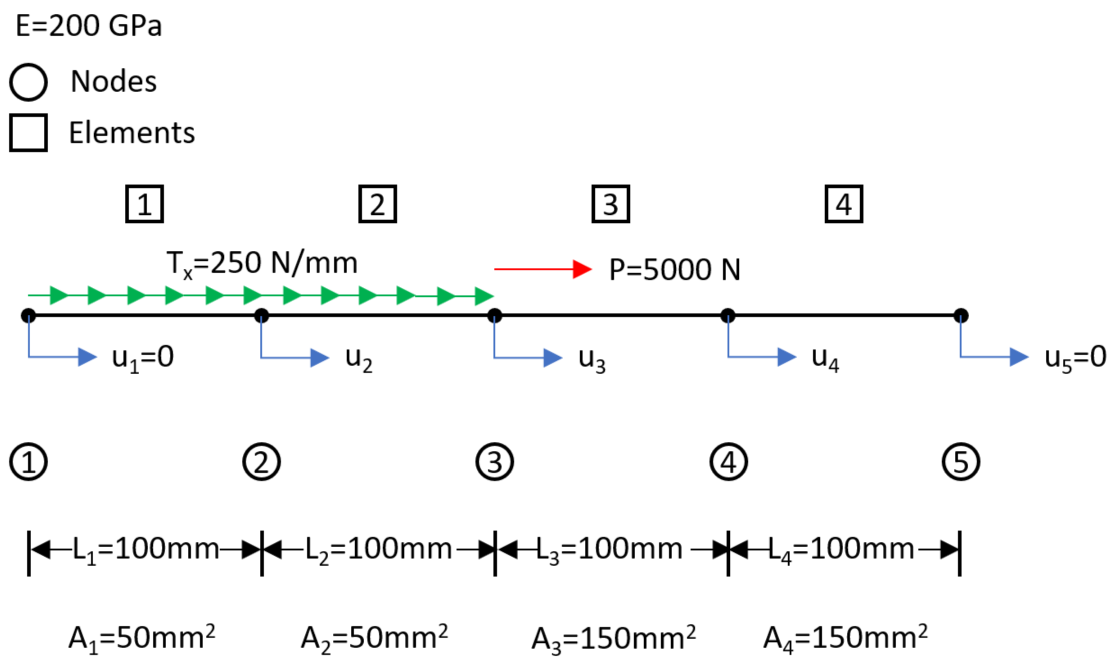


Figure 1: 1D linear bar finite element model of bar.

(b) Let $k_i = \frac{EA_i}{L_i}$. We assemble a local stiffness matrix $[K_i]$ for each element i as shown below:

$$[K_1] = \begin{bmatrix} k_1 & -k_1 \\ -k_1 & k_1 \end{bmatrix}$$

$$[K_2] = \begin{bmatrix} k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix}$$

$$[K_3] = \begin{bmatrix} k_3 & -k_3 \\ -k_3 & k_3 \end{bmatrix}$$

$$[K_4] = \begin{bmatrix} k_4 & -k_4 \\ -k_4 & k_4 \end{bmatrix}$$

Through superposition we combine the local stiffness matrices into a global stiffness matrix, $[K]$:

$$[K] = \begin{bmatrix} k_1 & -k_1 & 0 & 0 & 0 \\ -k_1 & k_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & k_2 & -k_2 & 0 & 0 \\ 0 & -k_2 & k_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} +$$
$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & k_3 & -k_3 & 0 \\ 0 & 0 & -k_3 & k_3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & k_4 & -k_4 \\ 0 & 0 & 0 & -k_4 & k_4 \end{bmatrix}$$

$$[K] = \begin{bmatrix} k_1 & -k_1 & 0 & 0 & 0 \\ -k_1 & k_1 + k_2 & -k_2 & 0 & 0 \\ 0 & -k_2 & k_2 + k_3 & -k_3 & 0 \\ 0 & 0 & -k_3 & k_3 + k_4 & -k_4 \\ 0 & 0 & 0 & -k_4 & k_4 \end{bmatrix}$$

Next, we assemble our displacement vector, noting that $d_1 = d_5 = 0$:

$$\{d\} = \begin{Bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \\ d_5 \end{Bmatrix} = \begin{Bmatrix} 0 \\ d_2 \\ d_3 \\ d_4 \\ 0 \end{Bmatrix}$$

Then, the vector of applied forces:

$$\{r\} = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \end{Bmatrix}$$

F_1 and F_5 are unknown because we know that the displacements at both boundaries are 0. The applied forces at nodes 2-4 are outlined below:

$$F_2 = T_x * L_1$$

$$F_3 = T_x * (L_1 + L_2) + P$$

$$F_4 = P$$

$$\{r\} = \begin{Bmatrix} F_1 \\ T_x * L_1 \\ T_x * (L_1 + L_2) + P \\ P \\ F_5 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ 30000 \\ 55000 \\ 5000 \\ F_5 \end{Bmatrix}$$

The global system of equations $[K]\{d\} = \{r\}$:

$$\boxed{\begin{bmatrix} k_1 & -k_1 & 0 & 0 & 0 \\ -k_1 & k_1 + k_2 & -k_2 & 0 & 0 \\ 0 & -k_2 & k_2 + k_3 & -k_3 & 0 \\ 0 & 0 & -k_3 & k_3 + k_4 & -k_4 \\ 0 & 0 & 0 & -k_4 & k_4 \end{bmatrix} \begin{Bmatrix} 0 \\ d_2 \\ d_3 \\ d_4 \\ 0 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ 30000 \\ 55000 \\ 5000 \\ F_5 \end{Bmatrix}}$$

(c) Substituting in given values and solving in MATLAB yields the following for $\{d\}$:

$$\boxed{\{d\} = \begin{Bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \\ d_5 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0.3313 \\ 0.3625 \\ 0.1896 \\ 0 \end{Bmatrix} \text{ mm}}$$

(d) Following a similar solution procedure as in Problem 2 we solve for displacement, $u(x)$, in MATLAB. Note, values are not shown for legibility:

$$u = a_1 + a_2 x + a_3 x^2 + a_4 x^3 + a_5 x^4$$

$$\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{Bmatrix} = [A] \begin{Bmatrix} 1 \\ x \\ x^2 \\ x^3 \\ x^4 \end{Bmatrix}$$

$$A = \begin{bmatrix} 1 & x_1 & x_1^2 & x_1^3 & x_1^4 \\ 1 & x_2 & x_2^2 & x_2^3 & x_2^4 \\ 1 & x_3 & x_3^2 & x_3^3 & x_3^4 \\ 1 & x_4 & x_4^2 & x_4^3 & x_4^4 \\ 1 & x_5 & x_5^2 & x_5^3 & x_5^4 \end{bmatrix}$$

$$\begin{bmatrix} N_1 & N_2 & N_3 & N_4 & N_5 \end{bmatrix} = \begin{bmatrix} 1 & x & x^2 & x^3 & x^4 \end{bmatrix} [A]^{-1}$$

$$\{u\} = [N]\{d\}$$

Figure 2 shows the displacement u across the finite element bar model. Nodes 1-5 are marked with a red star. The curve obeys the given boundary conditions with $u(x_1) = u(x_5) = 0$.

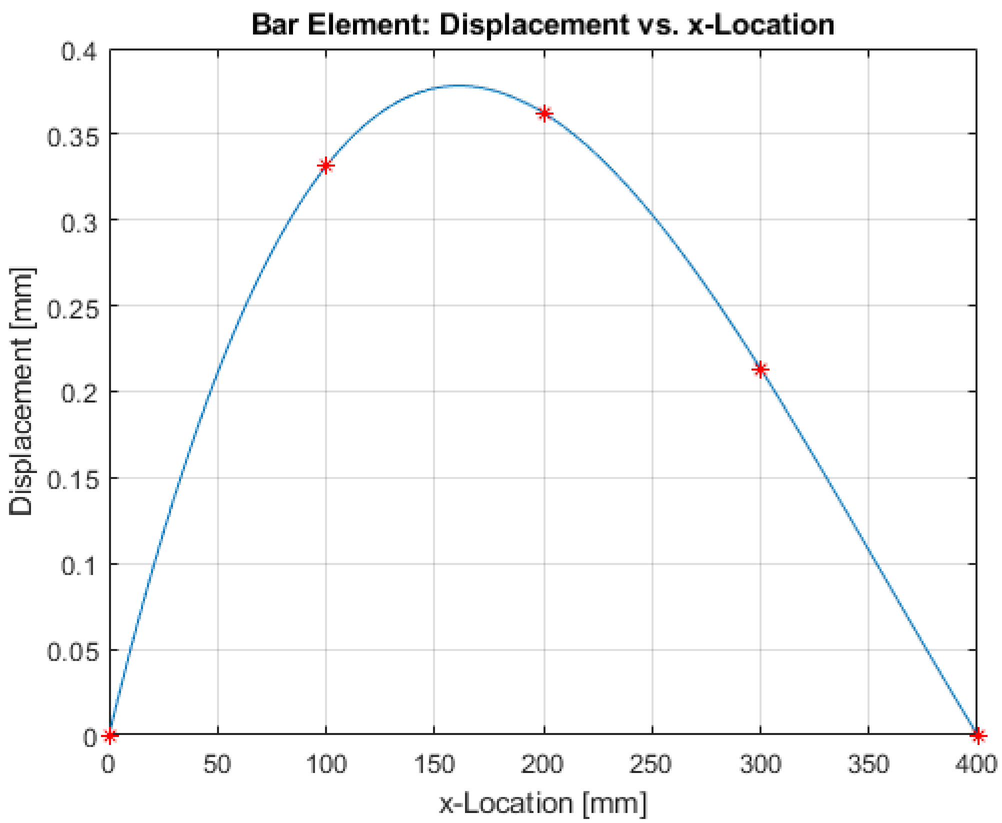


Figure 2: Displacement u vs. x location

(e) Solving the global $[K]\{d\} = \{r\}$ equation in MATLAB yields the following for $\{r\}$:

$$\{r\} = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \end{Bmatrix} = \begin{Bmatrix} -33125 \\ 30000 \\ 55000 \\ 5000 \\ -56875 \end{Bmatrix} \text{N}$$

The reaction forces at the ends are given by F_1 and F_5:

$$\boxed{F_1 = -33125\,\text{N} \qquad F_5 = -56875\,\text{N}}$$