Problem 1

Instead of the linear shape functions for a 1D bar element, the following shape function shave been proposed for an element with two nodes:

$$N_1 = \frac{-x(1-x)}{2} \qquad N_2 = \frac{x(1+x)}{2}$$

The resulting displacement field is $u = N_1 d_1 + N_2 d_2$.

    a) Develop the relation: $\varepsilon = [B]\{d\}$. That is, find the $[B]$ matrix in terms of x.

    b) Develop the stiffness matrix, $[K]$.

    c) Are these valid shape functions? Why or why not?

(a) <u>Solution:</u>

Note: Matrix inversions performed in MATLAB, see appendix **??**.

The displacement vector for an element is given by the shape functions times the nodal displacement degrees of freedom of the element:

$$\{u\} = [N]\{d\}$$

The strain vector for an element is defined as:

$$\{\varepsilon\} = [\partial]\{u\}$$

Substituting in the definition of the displacement vector:

$$\{\varepsilon\} = [\partial][N]\{d\}$$

The $[B]$ matrix is defined as the derivative of the shape functions, $[N]$:

$$[B] = [\partial][N]$$

Substituting again:

$$\{\varepsilon\} = [B]\{d\}$$

The shape functions defined in the problem statement can be expressed as:

$$[N] = [N_1 \ N_2] = \begin{bmatrix} \frac{-x(1-x)}{2} & \frac{x(1+x)}{2} \end{bmatrix}$$

Calculating the $[B]$ matrix:

$$[B] = [\frac{\partial}{\partial x}]\left[\frac{-x(1-x)}{2} \quad \frac{x(1+x)}{2}\right]$$

$$\boxed{[B] = \left[x - \frac{1}{2} \quad x + \frac{1}{2}\right]}$$

Substituting in the known $[B]$ matrix into the ε equation:

$$\{\varepsilon\} = \left[x - \frac{1}{2} \quad x + \frac{1}{2}\right]\begin{Bmatrix} d_1 \\ d_2 \end{Bmatrix}$$

$$\varepsilon(x) = \left(x - \frac{1}{2}\right)d_1 + \left(x + \frac{1}{2}\right)d_2$$

(b) For a given element, the stiffness matrix $[K]$ is defined by:

$$[K] = \int_V [B]^T[E][B]\mathrm{d}V$$

Noting that $\mathrm{d}V = \mathrm{d}x\mathrm{d}y\mathrm{d}z$:

$$A = \mathrm{d}z\mathrm{d}y$$

$$[K] = \int [B]^T[E][B]A\mathrm{d}x$$

Assuming A and E to be constant across the element and solving:

$$[K] = AE\int [B]^T[B]\mathrm{d}x$$

$$[B]^T[B] = \begin{bmatrix} \left(x^2 - x + \frac{1}{4}\right) & \left(x^2 - \frac{1}{4}\right) \\ \left(x^2 - \frac{1}{4}\right) & \left(x^2 + x + \frac{1}{4}\right) \end{bmatrix}$$

$$\int [B]^T[B]\mathrm{d}x = \begin{bmatrix} \left(\frac{x^3}{3} - \frac{x^2}{2} + \frac{x}{4}\right) & \left(\frac{x^3}{3} - \frac{x}{4}\right) \\ \left(\frac{x^3}{3} - \frac{x}{4}\right) & \left(\frac{x^3}{3} + \frac{x^2}{2} + \frac{x}{4}\right) \end{bmatrix}$$

$$[K] = AE\begin{bmatrix} \left(\frac{x^3}{3} - \frac{x^2}{2} + \frac{x}{4}\right) & \left(\frac{x^3}{3} - \frac{x}{4}\right) \\ \left(\frac{x^3}{3} - \frac{x}{4}\right) & \left(\frac{x^3}{3} + \frac{x^2}{2} + \frac{x}{4}\right) \end{bmatrix}$$

$$[K] = \frac{AE}{12} \begin{bmatrix} (4x^3 - 6x^2 + 3x) & (4x^3 - 3x) \\ (4x^3 - 3x) & (4x^3 + 6x^2 + 3x) \end{bmatrix}$$

(c) No information is given regarding the size, position, or orientation of the element. If we make the assumption that node 1 is at an x location of $x = -1$ and node 2 is at an x location of $x = 1$, we see the following behavior of the shape functions:

$$N_1(x_1 = -1) = \frac{-(-1)(1 - (-1))}{2} \qquad N_2(x_1 = -1) = \frac{(-1)(1 + (-1))}{2}$$
$$N_1(x_1 = -1) = 1 \qquad N_2(x_1 = -1) = 0$$

$$N_1(x_2 = 1) = \frac{-(1)(1 - (1))}{2} \qquad N_2(x_2 = 1) = \frac{(1)(1 + (1))}{2}$$
$$N_1(x_2 = 1) = 0 \qquad N_2(x_2 = 1) = 1$$

The shape functions are equal to 1 at the node they are associated with and 0 at the other node. This is a valid shape function for these coordinates. However, if the node coordinates were different then the shape functions would need to be reformulated to ensure this behavior. The validity of shape functions is highly dependent on the assumptions that are foundational to the element definition.