Mid Term Exam

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Problem 1

Instead of the linear shape functions for a 1D bar element, the following shape function shave been proposed for an element with two nodes:

$$N_1 = \frac{-x(1-x)}{2} \qquad N_2 = \frac{x(1+x)}{2}$$

The resulting displacement field is $u = N_1 d_1 + N_2 d_2$.

a) Develop the relation: $\varepsilon = [B]\{d\}$. That is, find the $[B]$ matrix in terms of x.

b) Develop the stiffness matrix, $[K]$.

c) Are these valid shape functions? Why or why not?

(a) <u>**Solution:**</u>

$$\{u\} = [N]\{d\}$$

$$\{\varepsilon\} = [\partial]\{u\}$$

$$\{\varepsilon\} = [\partial][N]\{d\}$$

$$[B] = [\partial][N]$$

$$\{\varepsilon\} = [B]\{d\}$$

$$[N] = [N_1\, N_2] = \begin{bmatrix} \frac{-x(1-x)}{2} & \frac{x(1+x)}{2} \end{bmatrix}$$

$$[B] = [\frac{\partial}{\partial x}]\begin{bmatrix} \frac{-x(1-x)}{2} & \frac{x(1+x)}{2} \end{bmatrix}$$

$$\boxed{[B] = \begin{bmatrix} x - \frac{1}{2} & x + \frac{1}{2} \end{bmatrix}}$$

$$\{\varepsilon\} = \begin{bmatrix} x - \frac{1}{2} & x + \frac{1}{2} \end{bmatrix} \begin{Bmatrix} d_1 \\ d_2 \end{Bmatrix}$$

$$\varepsilon(x) = \left(x - \frac{1}{2}\right) d_1 + \left(x + \frac{1}{2}\right) d_2$$

(b)

$$[K] = \int_{\forall} [B]^T [E][B] \mathrm{d}\forall$$

$$A = \mathrm{d}z\mathrm{d}y$$

$$[K] = \int [B]^T [E][B] A \mathrm{d}x$$

$$[K] = AE \int [B]^T [B] \mathrm{d}x$$

$$[B]^T[B] = \begin{bmatrix} \left(x^2 - x + \frac{1}{4}\right) & \left(x^2 - \frac{1}{4}\right) \\ \left(x^2 - \frac{1}{4}\right) & \left(x^2 + x + \frac{1}{4}\right) \end{bmatrix}$$

$$\int [B]^T[B] \mathrm{d}x = \begin{bmatrix} \left(\frac{x^3}{3} - \frac{x^2}{2} + \frac{x}{4}\right) & \left(\frac{x^3}{3} - \frac{x}{4}\right) \\ \left(\frac{x^3}{3} - \frac{x}{4}\right) & \left(\frac{x^3}{3} + \frac{x^2}{2} + \frac{x}{4}\right) \end{bmatrix}$$

$$[K] = AE \begin{bmatrix} \left(\frac{x^3}{3} - \frac{x^2}{2} + \frac{x}{4}\right) & \left(\frac{x^3}{3} - \frac{x}{4}\right) \\ \left(\frac{x^3}{3} - \frac{x}{4}\right) & \left(\frac{x^3}{3} + \frac{x^2}{2} + \frac{x}{4}\right) \end{bmatrix}$$

$$\boxed{[K] = \frac{AE}{12} \begin{bmatrix} (4x^3 - 6x^2 + 3x) & (4x^3 - 3x) \\ (4x^3 - 3x) & (4x^3 + 6x^2 + 3x) \end{bmatrix}}$$

(c) If we say that node 1 is at an x location of $x = -1$ and node 2 is at an x location of $x = 1$, we see the following behavior of the shape functions:

$$N_1(x_1 = -1) = \frac{-(-1)(1 - (-1))}{2} \qquad N_2(x_1 = -1) = \frac{(-1)(1 + (-1))}{2}$$
$$N_1(x_1 = -1) = 1 \qquad N_2(x_1 = -1) = 0$$

$$N_1(x_2 = 1) = \frac{-(1)(1 - (1))}{2} \qquad N_2(x_2 = 1) = \frac{(1)(1 + (1))}{2}$$
$$N_1(x_2 = 1) = 0 \qquad N_2(x_2 = 1) = 1$$

The shape functions are equal to 1 at the node they are associated with and 0 at the other node. This is a valid shape function for these coordinates. However, if the node coordinates were different then the shape functions would need to be reformulated to ensure this behavior.