Example pset

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Problem 1

Problem 1 (Polynomial time SAT-oracle Turing machine for Hamiltonian path).

Solution: The solution involves three steps:

(1) **Reduce the Hamiltonian path problem to SAT:** We construct the reduction to faithfully simulate testing for whether a given path is a Hamiltonian path from s to t. This will make it easy to recover the Hamiltonian path from a satisfying assignment in step three. A Hamiltonian path travels through the $n = |V|$ nodes of the graph G. Accordingly, we have n^2 variables: for every $1 \leq t, v \leq n$, we have a variable $x_{t,v}$. This variable $x_{t,v}$ is the indicator variable for our path being at vertex v at time t. The boolean formula will then check the following:

- Path is only at one point at any point in time: achieved via
$$\bigwedge_{t=1}^{n} \left(\bigvee_{1 \leq u < v \leq n} (\overline{x_{t,u}} \vee \overline{x_{t,v}}) \right) \wedge (x_{t,1} \vee \cdots \vee x_{t,n}).$$
- Each vertex is visited once: achieved via
$$\bigwedge_{v=1}^{n} \left(\bigvee_{1 \leq i < j \leq n} (\overline{x_{i,v}} \vee \overline{x_{j,v}}) \right) \wedge (x_{1,v} \vee \cdots \vee x_{n,v}).$$
- Directed edge between consecutive vertices: achieved via
$$\bigwedge_{\text{non-edge}(u,v)} \bigwedge_{t=1}^{n-1} (\overline{x_{t,u}} \vee \overline{x_{t+1,v}}).$$
- Path starts at s and ends at t: achieved via $x_{1,s} \wedge x_{n,t}$.

And-ing these requirements together yields a boolean expression ϕ, whose satisfying assignments are in bijection with Hamiltonian paths in G from s to t. Additionally, the construction of ϕ takes only polynomial time (each expression is at most cubic in n).

(2) **Use the SAT oracle to determine a solution to the resulting SAT problem, or conclude that non-exists and report accordingly:** Use the SAT oracle as a sub-routine to determine a satisfying assignment to ϕ by sequentially solving for each variable using the procedure described in problem two, reporting **No Hamiltonian Path** in the case that no satisfying assignment exists.

(3) **Use the solution found in step two to construct a Hamiltonian path:** Check which variables $x_{t,v}$ in the satisfying assignment are set to true. This determines a unique Hamiltonian path from s to t. Return this path.

Finally, M^{SAT} is then just the Turing machine which executes this procedure. □