# The Lax-Wendroff scheme

**Ingress**

A new second order scheme for hyperbolic conversation laws allowing fast and accurate computation of smooth solutions.

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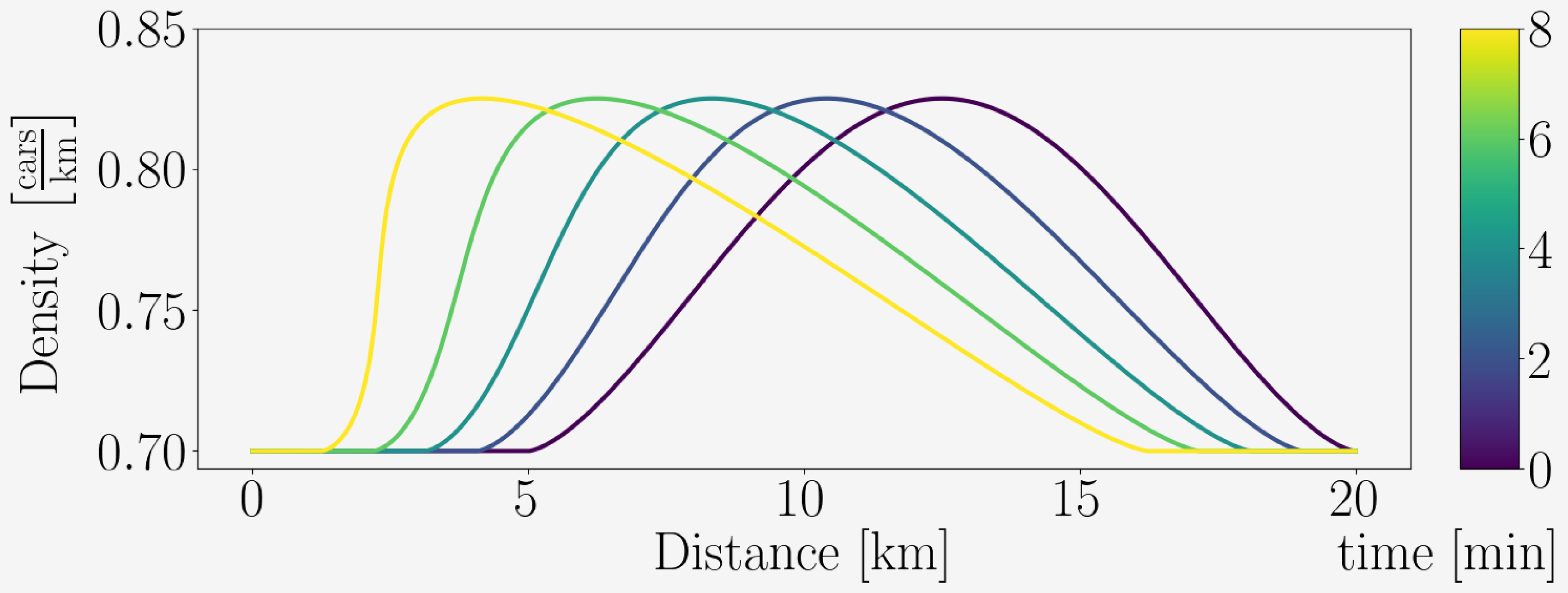


## Abstract

The main topic of this poster is to introduce the Lax-Wendroff scheme for solving the hyperbolic conservations laws. In concrete, it focus on the 1 space dimensional advection equation for different flows to illustrate how the scheme works, and to compare it with other numerical methods. This will lead us to conclude that the Lax-Wendroff scheme works, in general terms, better than the other ones.

## Introduction

Hyperbolic conservation laws are time-dependent systems of partial differential equations that describes the conservation of some quantities. The equation is given as:

$$u_t + f(u)_x = 0$$

Where $u(x,t)$ is a vector of conserved quantities (could be mass, momentum, heat,..) while $f(u)$ is the flux function. This equation must be augmented with initial and boundary conditions. The easiest problem is the Cauchy problem where we need to give initial conditions:

$$u(x,0) = u_0(x)$$

Flux functions are commonly nonlinear functions, leading to nonlinear systems of PDEs that, in general, the exact solution is unknown. Hence numerical methods are used, to compute an approximation.

## The Lax-Wendroff Method

For linear hyperbolic systems ($u_t + au_x = 0$) the Lax-Wendroff methods is given as:

$$U_j^{n+1} = U_j^n - \frac{k}{2h}a(U_{j+1}^n - U_{j-1}^n) + \frac{k^2}{2h^2}a^2(U_{j+1}^n - 2U_j^n + U_{j-1}^n)$$

The idea is to take the first three terms of the Taylor series expansion and use centered difference approximations for the derivatives appearing there. The stepsizes $k$ in time and $h$ in space are chosen such that $|a|\frac{k}{h} \le 1$ to fulfil the CFL condition. For non-linear equations the scheme can be generalized to:

$$U_j^{n+1} = U_j^n - \frac{k}{h}\left(f\left(U_{j+\frac{1}{2}}^{n+\frac{1}{2}}\right) - f\left(U_{j-\frac{1}{2}}^{n+\frac{1}{2}}\right)\right)$$

where

$$U_{j+\frac{1}{2}}^{n+\frac{1}{2}} = \frac{1}{2}(U_j^n + U_{j+1}^n) - \frac{k}{2h}[f(U_{j+1}^n) - f(U_j^n)]$$

while the step sizes are in the following considered to be controlled by $\frac{k}{h} = const$. The computation of the next time step can be written as a function of the solution of the current timestep by $U^{n+1} = \mathcal{H}(U^n)$. This notation can be used to define the local truncation error of a method by

$$L_k(x,t) = \frac{u(x,t+k) - \mathcal{H}(u(\cdot,t);x)}{k}$$

contingent on the used step size in time $k$. This error will now be considered in the linear advection equation with $a > 0$. By Taylor expansion around $u(x,t)$ and some simplifications this yields to:

$$L_k(x,t) = \frac{a}{6}(a^2k^2 - h^2)u_{xxx} + \mathcal{O}(h^3)$$

As $L_k(x,t) = \mathcal{O}(h^2)$ the method is consistent and has order 2. Additional for stability the condition $|\nu| < 1$ where $\nu = a\frac{k}{h}$ is the Courant number must be fulfilled. In the case of an linear advection equation the convergence of a method follows from consistency and stability by the Lax equivalence theorem. In the nonlinear case Lax and Wendroff proved that a consistent numerical method with bounded solutions always converges to a weak solution of the equation. The local truncation error also leads to the modified equation which is given as:

$$u_t + au_x = \frac{a}{6}(a^2k^2 - h^2)u_{xxx} := \mu u_{xxx}$$

This is also a dispersive equation. Regarding the theory of dispersive waves a solution $u(x,t)$ can be represented in Fourier space. By isolating each wavenumber $\xi$ to apply solutions of the form $u(x,t) = e^{i(\xi x - c(\xi)t)}$ to the linear advection equation yields to the dispersion relation:

$$c(\xi) = a\xi + \mu\xi^3$$

Based on this relation its possible to calculate the group velocity $c'(\xi)$ for wavenumber $\xi$ which describes in which speed a wave peak travel. It is given by:

$$c'(\xi) = a + 3\mu\xi^2$$

As $\mu = \frac{a}{6}h^2(\nu^2 - 1)$ and since $a > 0$ and for stability $|\nu| < 1$ the group velocity $c'(\xi)$ is smaller than $a$ for all $\xi$, but tens to $a$ as $h \to 0$. It follows that the scheme leads to an oscillatory wave train lagging behind the discontinuity, which is traveling with speed $a$. This can also later be seen later in the numerical experiments.

## Numerical Analysis

In the following the approximations of the Lax-Wendroff method will be presented in comprehension to the Lax-Friedrich and Godunov method.

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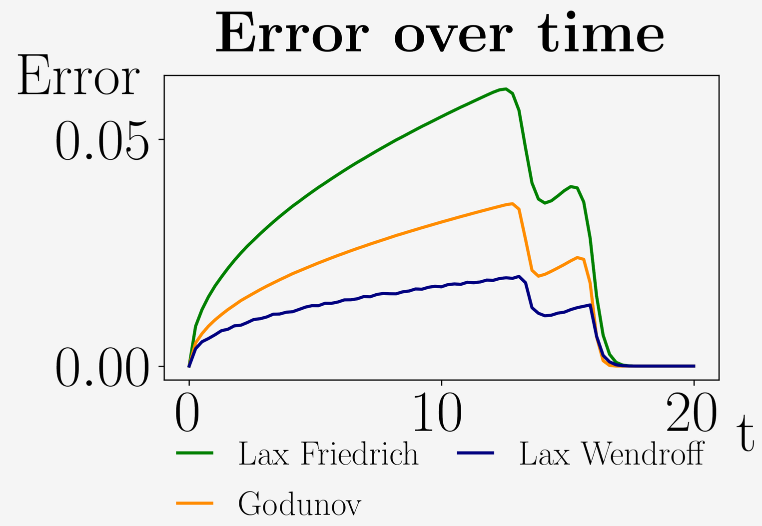


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**REFERENCES**

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