Study of Barotropic Rossby waves, using finite difference methods

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Abstract

This work investigates solutions of the barotropic Rossby wave equation (BRWE) in a closed and periodic domain, using finite difference methods. First we derived the BRWE by linearising the barotropic vorticity equation and then we scaled this linearised equation to obtain dimensionless analytical values for the phase speed in one dimension $c_{\text{periodic}} = -0.0063$, $c_{\text{closed}} = -0.012665$, both with sinusoidal initial states. Next we developed an algorithm for solving the BRWE, which we implemented using both forward time centered space (FTCS) and centered time centered space (CTCS) schemes. For both versions we did an extensive stability analysis and found FTCS to be unconditionally unstable for all combination of $\Delta x, \Delta t$. CTCS was found to be stable for $\Delta x/\Delta t < k^2$. Using the CTCS version we obtained $c_{\text{periodic}} = -0.00625$ and $c_{\text{closed}} = -0.0125$ which is in good agreement with the analytical results.

1 Introduction

Rossby waves was first introduced by the Swedish meteorologist Carl G. Rossby in 1939 (Rossby 1939). The waves are large, with typical scales ranging from hundreds to thousands of kilometres. Rossby waves play a major role in dictating the weather on daily and large time scales and are crucial for the meridional transport of heat, moisture and momentum (Lackmann 2011). Rossby waves are present both in the atmosphere and the ocean and are maintained due to the latitudinal variation in the Coriolis acceleration.

In this study we will examine Rossby waves in a simplified atmosphere/ocean. We will assume that our atmosphere/ocean is barotropic, that is the density depends only on pressure, which via the ideal gas law holds that the temperature is constant on a isobaric surface, i.e. no stratification. We also assume the flow to be frictionless and non-divergent, consequently neglecting vertical velocities. Under these simplifications the quasi geostrophic vorticity equation is reduced to eq. (1), where ζ is the relative vorticity and f is the Coriolis parameter, $f = 2\Omega\sin(\theta)$, where Ω is the rotation rate of the earth $\Omega = 2\pi/\text{day}$.

$$\frac{D(\zeta + f)}{Dt} = 0 \tag{1}$$

Following eq. (1) we have that the change in absolute vorticity $(\zeta + f)$ following the flow is zero, in other words the absolute vorticity is conserved. When you have the vorticity in terms of a conserved quantity it is commonly called potential vorticity (PV). If a fluid parcel is moving poleward towards region of larger f, then there must be a compensating change in ζ in order for the absolute vorticity to remain unchanged. This balancing act between planetary and relative vorticity is the essential mechanism behind Rossby waves. We will derive the barotropic Rossby wave equation (BRWE) from eq. (1) in the subsequent section.

After we have developed our analytical framework, we will discretize the BRWE and solve the BRWE in both one and two dimension, both with periodic and closed boundary conditions. We will examine both the forward difference and central difference scheme and consider their stability.

2 Theory

2.1 The barotropic Rossby wave equation

A problem with the current vorticity eq. (1) is that it is nonlinear. In order to obtain linear wave solutions we need to linearise the equation. We will linearise eq. (1) by supposing that the flow can be represented by a mean flow plus some small perturbation:

$$\begin{aligned} u &= U + u' \quad |U| >> u' \\ v &= V + v' \quad |V| >> v' \end{aligned} \tag{2}$$

Then the vorticity of the mean flow and the perturbation is:

$$Z = \frac{\partial V}{\partial x} - \frac{\partial U}{\partial y}, \quad \zeta' = \frac{\partial v'}{\partial x} - \frac{\partial u'}{\partial y} \tag{3}$$

To linearise the Coriolis parameter, we expand f around a reference latitude θ_0,

$$f \approx f_0 + \beta y \tag{4}$$

where $f_0 = 2\Omega \sin(\theta)$, $\beta = 2\cos(\theta_0)/R_e$ and R_e is the radius of the earth. This linearisation of the Coriolis parameter eq. (4) is commonly referred to as the $\beta -$ plane approximation.

Inserting eq. (4) and eq. (2) in the vorticity equation eq. (1).

$$\frac{\partial \zeta'}{\partial t} + (U + u')\frac{\partial \zeta'}{\partial x} + (V + v')\frac{\partial \zeta'}{\partial y} + \mathbf{U} \cdot \nabla q_s + u'\frac{\partial q_s}{\partial x} + v'\frac{\partial q_s}{\partial y} = 0 \tag{5}$$

$$q_s = Z + f_0 + \beta y \tag{6}$$

The mean flow $\mathbf{U}$ is geostrophic, consequently the flow will be parallel to the contours of stationary potential vorticity q_s, ergo $\mathbf{U} \cdot \nabla q_s = 0$. The perturbations are assumed to be weak so that perturbations advecting perturbations can be neglected. Thus we get the following linearised barotropic vorticity equation:

$$\frac{\partial \zeta'}{\partial t} + U\frac{\partial \zeta'}{\partial x} + V\frac{\partial \zeta'}{\partial y} + \beta v' = 0 \tag{7}$$

We will drop the primes from here on. Next we suppose that the velocity of the mean flow is zero, and express the perturbation velocities in terms of the streamfunction ψ, $u = -\frac{\partial \psi}{\partial y}$, $v = \frac{\partial \psi}{\partial x}$ and $\zeta = \nabla_H^2 \psi$. The equation we end up with is the barotropic Rossby wave equation eq. (8) in the absence of a mean flow.

$$\frac{\partial}{\partial t}\nabla_H^2\psi + \beta\frac{\partial\psi}{\partial x} = 0 \tag{8}$$

2.2 Analytic expression of phase velocities

Proceeding from the barotropic Rossby wave equation eq. (8) we will find analytical expression for the phase velocity in one dimension, first with periodic boundary conditions and then with closed boundaries.

2.2.1 Periodic boundaries

We assume the following periodic solution eq. (9). At $x = 0$ we have $\psi = A\cos(-\omega t)$ and at $x = L$, $\psi = A\cos(2n\pi - \omega)$, and because cosine is periodic with a period of 2π, we see that eq. (9) satisfies the boundary conditions. Next we insert eq. (9) into the barotropic Rossby wave equation eq. (8).

$$\psi = A\cos(2n\pi x/L - \omega t) \tag{9}$$

Using the one dimensional vorticity $\zeta_x = \frac{\partial v}{\partial x} = \frac{\partial^2\psi}{\partial x^2}$

$$\frac{\partial}{\partial t}\left(-A(2n\pi/L)^2\cos(2n\pi x/L - \omega t)\right) - \beta(2n\pi/L)A\sin(2nx\pi/L - \omega t) = 0 \tag{10}$$

$$-(2n\pi/L)\omega A\sin\cancel{(2nx\pi/L - \omega t)} - \beta A\sin\cancel{(2xn\pi/L - \omega t)} = 0 \tag{11}$$

$$(2n\pi/L)\omega = \beta \tag{12}$$

$$\omega = \frac{-\beta L}{2n\pi} \tag{13}$$

The phase speed of the Rossby wave can be derived by considering the phase of the wave eq. (14).

$$\theta = \frac{n\pi x}{L} - \omega t \tag{14}$$

Then the peak of the wave would be at $\theta = 2\pi$, where $\psi = A\cos(2\pi)$. This peak corresponds to a high pressure point in the flow. To get the motion of this point we can solve eq. (14) for x, $x = \theta L/2n\pi + \omega t L/2n\pi$ and then obtaining the phase speed by taking the time derivative.

$$c = \frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\omega L}{2n\pi} \tag{15}$$

Inserting the expression for ω obtained previously eq. (13), we get the following expression for the phase speed.

$$c = \frac{-\beta}{k^2} = \frac{-\beta L^2}{(2n\pi)^2} \tag{16}$$

Then according to eq. (16) the Rossby waves will propagate toward the west.

2.2.2 Closed boundaries

In the ocean the boundaries would be better represented using closed boundary conditions. We will suppose a wavelike solution of eq. (8) where the amplitude of the wave also vary in x. We require there to be no flow at the endpoints, which we can enforce with simple Dirichlet conditions $\psi = 0$ at $x = 0$ and $x = L$.

$$\psi = A(x)\cos(kx - \omega t) \tag{17}$$

Inserting into the BRWE eq. (8), and then differentiating we get the following equation:

$$\begin{gathered} \sin(kx - wt)\Big(\omega(A''(x) - k^2 A(x)) - \beta A(x)\Big) \\ + \cos(kx - \omega t)\left(\omega 2kA'(x) + \beta A'(x)\right) = 0 \end{gathered} \tag{18}$$

The terms in front of the cosine and sine has to be zero, which means that we can split the equation into two equations, one for the sine and one cosine terms respectively (eq. (19), eq. (20)).

$$\omega A''(x) - \omega k^2 A(x) - \beta A(x) = 0 \tag{19}$$

$$\omega 2kA'(x) + \beta A'(x) = 0 \tag{20}$$

Solving the equation for the cosine terms eq. (20) with respect to ω:

$$\omega = -\frac{\beta}{2k} \tag{21}$$

Inserting the expression for omega into eq. (19) and solving the ODE gives us the following expression for $A(x)$

$$A(x) = C_1 \cos(kx) + C_2 \sin(kx) \tag{22}$$

Imposing the boundary condition at $x = 0$ requires C_1 to be zero, ruling out the cosine term in eq. (22). Imposing the boundary condition $\psi = 0$ at $x = L$ yields the following expression for A(x).

$$A(x) = \sin\left(\frac{n\pi}{L}x\right) \tag{23}$$

Then final solution becomes:

$$\psi(x,t) = \sin\left(\frac{n\pi}{L}x\right)\cos(kx - \omega t) \tag{24}$$

The solution contains the same cosine term as in the periodic domain, but is now modulated by a sine term that enforces the boundary conditions and increases the number of throughs and ridges compared to the periodic solution. The phase speed can be calculated in the same manner as in the periodic case, except now $k = \frac{n\pi}{L}$.

$$c = \frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\omega}{k} = -\frac{\beta}{2k^2} = -\frac{\beta L^2}{2\pi^2 n^2} \tag{25}$$

The phase speed for the Rossby waves with solid boundaries is twice compared to the periodic boundaries.

3 Numerical model

In this section we explain the implementation of the numerical model and explore stability . Further details can be found in the python code in the github repository of this project [1].

3.1 Scaling

Before we do the discretisation it is useful to nondimensionalise the BRWE. We scale the BRWE using $x = \tilde{x}L$ and $t = \frac{\tilde{t}}{L\beta}$. The dimensionless phase speed is then $\frac{d\tilde{x}}{d\tilde{t}} = \frac{1}{\beta L^2}c$. For wavenumber two this gives us phase speeds of -0.0063 for

[1] https://github.com/Ovewh/Computilus/tree/master/Project5

the periodic case and $c = -0.012665$ for the closed boundary case. After doing the scaling we end up with the following dimensionless BRWE eq. (26).

$$\frac{\partial \zeta}{\partial \tilde{t}} + \frac{\partial \psi}{\partial \tilde{x}} = 0 \tag{26}$$

3.2 Algorithm and finite differences

The general algorithm for solving the BRWE is outlined in algorithm 1.

```
initialize ζ and ψ using sine or gaussian
for t=0; t<timesteps; t++ do
    advance ζ one timestep
    update ψ using jacobis method
end
```

Algorithm 1: General algorithm for solving the BRWE.

We implemented two methods for advancing ζ using centered (eq. (27)) and forward (eq. (28)) differences. For the planetary vorticity advection term we used centered differences. This lead to the forward time centered space (FTCS) eq. (29) and centered time centered space (CTCS) eq. (30) schemes.

$$\frac{\partial \zeta}{\partial t} \approx \frac{\zeta^{n+1} - \zeta^{n-1}}{2\Delta t} \tag{27}$$

$$\frac{\partial \zeta}{\partial t} \approx \frac{\zeta^{n+1} - \zeta^{n}}{\Delta t} \tag{28}$$

$$\zeta_i^{n+1} = \zeta_i^{n-1} - \frac{\Delta t}{2\Delta x}(\psi_{i+1}^n - \psi_{i-1}^n) \tag{29}$$

$$\zeta_i^{n+1} = \zeta_i^{n} - \frac{\Delta t}{\Delta x}(\psi_{i+1}^n - \psi_{i-1}^n) \tag{30}$$

The truncation error of the centered difference terms are of order $(\Delta x)^2$ and $(\Delta t)^2$, while the forward difference term has a truncation error of order Δt (Langtangen 2014).

3.3 Initalization

We will test different initial condition for ψ. In one dimension we will use the sinusoidal $\psi(x,0) = \sin(4\pi x)$ and Gaussian $\psi(x,0) = \exp\left\{-\left(\frac{x-x_0}{\sigma}\right)^2\right\}$. In two dimensions we will use a two-dimensional sinusoidal $\psi(x, y, 0) = \sin(4\pi x)\sin(4\pi y)$. $\zeta(t = 0)$ was initalized as $\frac{\partial^2\psi}{\partial x^2}$ in one dimension and as $\nabla_H^2\psi$ in two dimensions.

For the CTCS scheme, ζ_j^1 is a function of ζ_j^{-1} which we don't know. We could either set $\zeta_j^{-1} = \zeta_j^0$ or use the FTCS scheme backwards. At first we tried to implement the latter but ran into problems we could not resolve, and opted for the first option.

3.4 Jacobis method

After advancing the vorticity one timestep we adjust the stream function according to $\zeta = \nabla_H^2\psi$. We do this by solving the poisson equation using jacobis method. Jacobis method requires a convergence parameter for deciding when to stop iterating. To determine this parameter we tested our numerical solutions against two analytical solutions of poissons equation, one closed and one periodic.

Figure 1 shows the absolute error of jacobis method compared to the analytical solutions. For small convergence parameters, Jacobis method gives a smaller absolute error on large grids. For our model runs we will be using close to 40 grid points and we found that Jacobis method achieved satisfactory result when setting the convergence parameter to 10^{-8}.

3.5 Stability and comparison of timestepping methods

To assess the stability of the two timestepping methods we can insert solutions of $\psi_j^n = A^n e^{ikj\Delta x}$ into the FTCS and CTCS schemes. For both schemes we need to use that $\frac{\partial\zeta}{\partial t} = -k^2\frac{\partial\psi}{\partial t} = -\frac{1}{c}\frac{\partial\psi}{\partial t}$

3.5.1 FTCS

The FTCS gives,

$$-\frac{1}{c}\frac{A^{n+1}e^{ikj\Delta x} - A^n e^{ikj\Delta x}}{\Delta t} + \frac{A^n e^{ik(j+1)\Delta x} - A^n e^{ik(j-1)\Delta x}}{2\Delta x} \tag{31}$$

$$= A^n e^{ikj}\left[A - 1 - c\frac{\Delta t}{2\Delta x}\left(e^{ik\Delta x} - e^{-ik\Delta x}\right)\right] = 0 \tag{32}$$

Solving for A leads to $A = 1 + \frac{c\Delta t}{\Delta x}sin(k\Delta x)i$. For the numerical solution to stay stable we need $|A|^2 <= 1$, which can not be satisfied. Therefore the

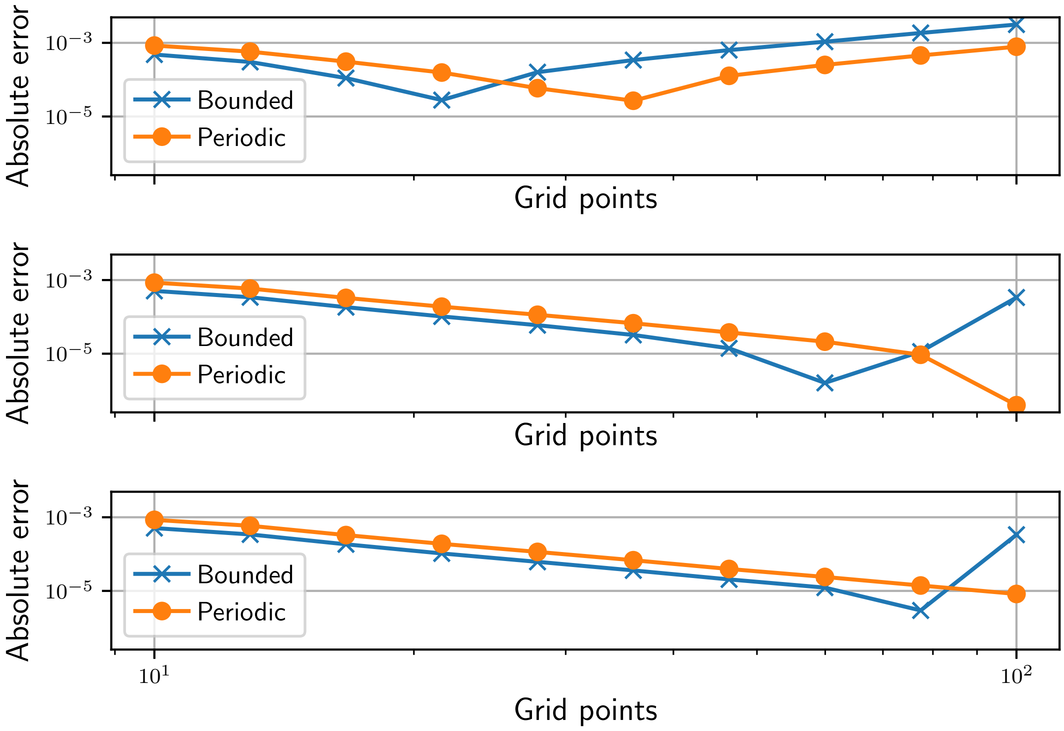

Figure 1: Absolute error of jacobis method compared to analytical solutions of poissons equation. Convergence parameter set to (from top to bottom) 10^{-6}, 10^{-8} and 10^{-10}. The absolute error should have a slope of -2 in loglog space since we use a centered second order scheme with truncation error proportional to $(\Delta x)^2$. The larger values of the convergence parameter has the correct slope for small grids, but performs worse with larger grids. As the parameter is lowered the slope is correct for increasingly larger grids.

FTCS scheme is unconditionally unstable. The instability of the FTCS scheme is shown in fig. 2, where we see the FTCS solutions diverging around t=10. The oscillations might be caused by our lack of initalizing ζ_j^{-1} properly. Another view of the instability is given in section A which shows the whole wave.

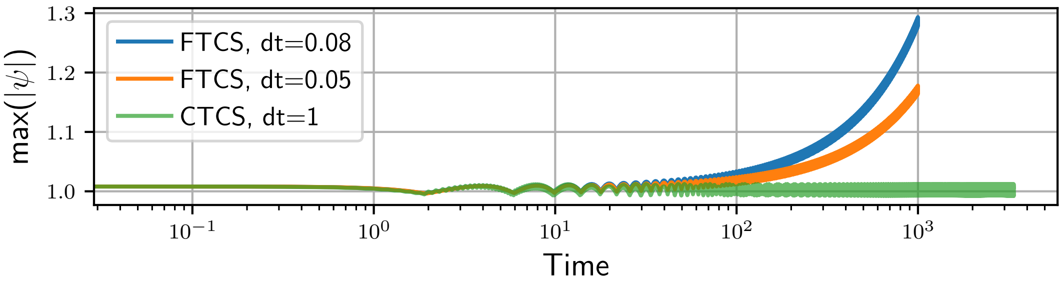

Figure 2: Comparison of maximum value of ψ as a function of time with different timesteps and schemes. The CTCS scheme with a long timestep seems stable, while FTCS with small timesteps both diverges at around t=10.

3.5.2 CTCS

For the CTCS scheme we get,

$$-\frac{1}{c}\frac{A^{n+1}e^{ikj\Delta x}-A^{n-1}e^{ikj\Delta x}}{2\Delta t}+\frac{A^{n}e^{ik(j+1)\Delta x}-A^{n}e^{ik(j-1)\Delta x}}{2\Delta x} \tag{33}$$

$$=A^{n}e^{ikj\Delta x}\left[A-A^{-1}-c\frac{\Delta t}{\Delta x}\left(2sin(k\Delta x)i\right)\right]=0 \tag{34}$$

Setting $c\frac{\Delta t}{\Delta x}=\alpha$ and solving for A gives $A=\alpha sin(k\Delta x)i\pm\sqrt{-\alpha^2 sin(k\Delta x)^2+1}$. This gives us two cases, $|\alpha|>1$ and $|\alpha|\leq 1$ For $|\alpha|<1$ we get $|A|^2=\alpha^2 sin(k\Delta x)+1-\alpha^2 sin(k\Delta x)=1$ which is unconditionally stable. For $|\alpha|>1$ we get $|A|^2=\left(\alpha sin(k\Delta x)\pm\sqrt{\alpha^2 sin(k\Delta x)^2-1}\right)^2$ which will be unstable for some combinations of k and Δx. We therefore find the stability criterion for the CTCS scheme to be $\frac{\Delta t}{\Delta x}<\frac{1}{c}=k^2$.

For the CTCS scheme with a sinusoidal inital state with k = 4π and $\Delta x=\frac{1}{40}$ this corresponds to having $\Delta t\leq\frac{(4\pi)^2}{40}=3.948$. Some runs with periodic boundaries is shown in fig. 3.

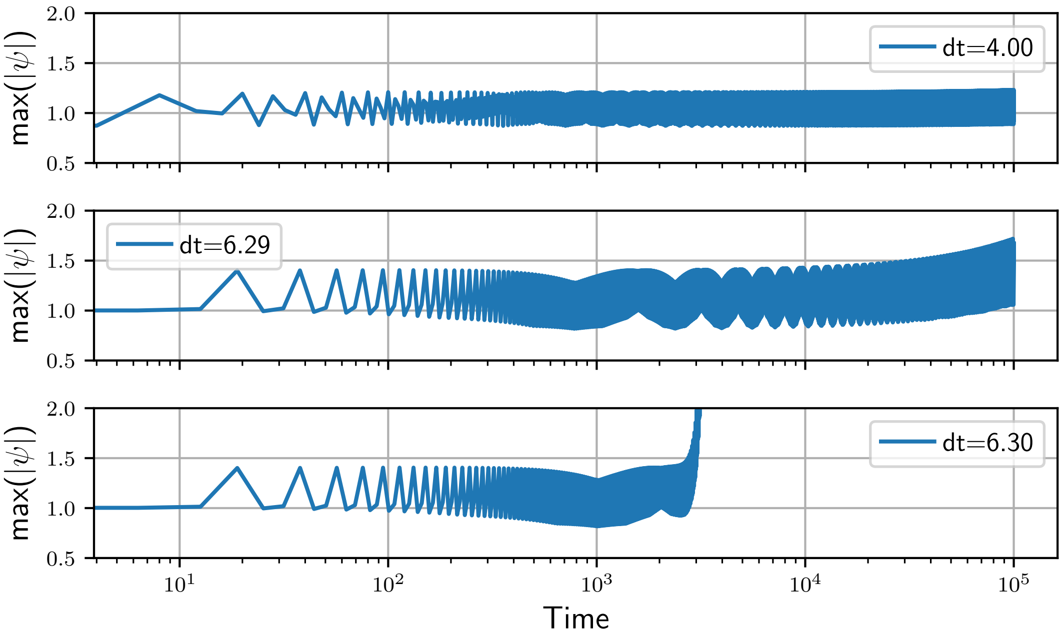

Figure 3: Comparison of maximum value of ψ as a function of time with different timesteps. The top figure is barely outside of the stability criterion and was not run long enough to show any instability. The two bottom figures have very similar timesteps but shows signs off instability at very different times.

4 Results

All figures shown in this section was produced using the CTCS scheme with $\Delta x = \frac{1}{40}$ and $\Delta t = 0.10$.

4.1 Periodic

With periodic boundaries and a sinusoidal inital state (Figure 4) the phase speed is constant and all waves move westward. By following one wave from east to west we can find the phase speed as $\frac{\Delta x}{\Delta t} = \frac{0-1}{160} = -0.00625$.

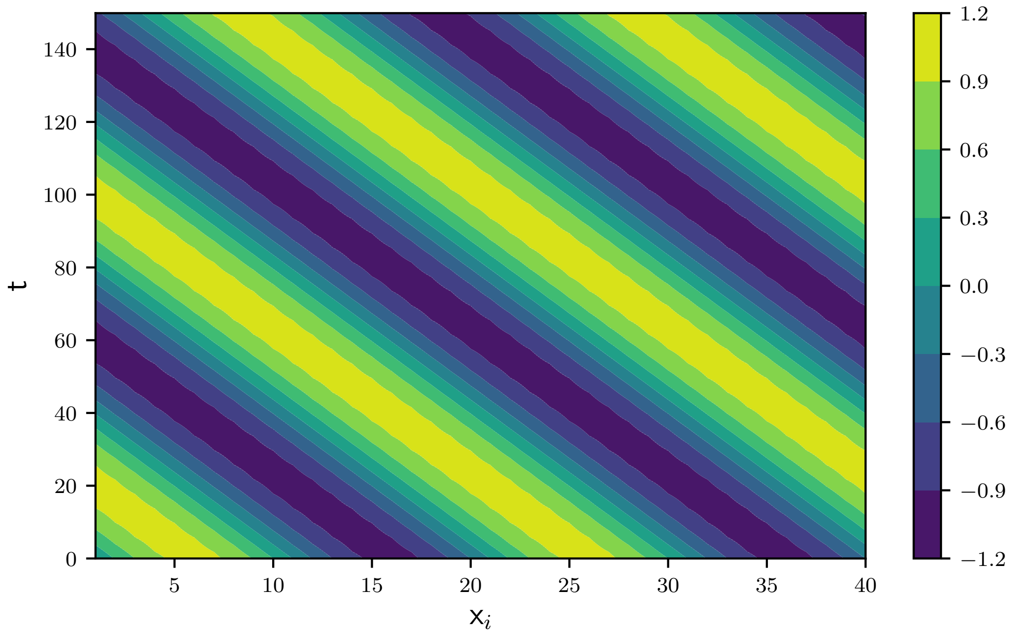

Figure 4: Hovmöller diagram for the periodic domain with a sinusoidal initial state. CTCS, $\Delta x = \frac{1}{40}$, $\Delta t = 0.10$. *Waves propagate westward with phasespeed* -0.00625.

With a Gaussian initial state (fig. 5) the Hovmöller diagram is not as ordered. Due to the less order in the initial state the phase velocity is not constant in time. This effect is most prominent for the cases where σ is small. For large σ the phase velocity seems to be more constant.

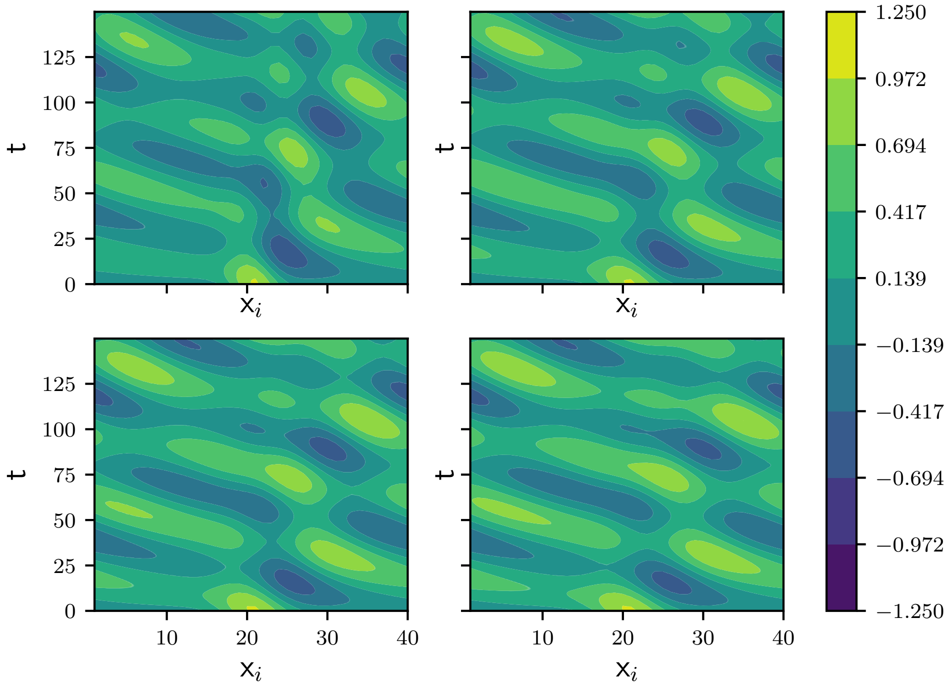

Figure 5: Hovmöller diagram for the periodic domain with a Gaussian initial state.From top left to bottom right: $\sigma = [0.08, 0.10, 0.11, 0.12]$*. CTCS,* $\Delta x = \frac{1}{40}$*,* $\Delta t = 0.10$*.*

4.2 Bounded

With no flow at the boundaries and an initial sinusoidal fig. 6 we can use the same method as for the periodic to find a phase speed of $\frac{\Delta x}{\Delta t} = \frac{0-1}{100-20} = -0.0125$. This is twice the speed as in the periodic case with is in agreement with the analytical phase speeds. The wave pattern also fits well with the theoretical case. Since our initial state has wave number 2 the sine term modulating the solution has one through and one ridge, and is zero at both ends and the middle.

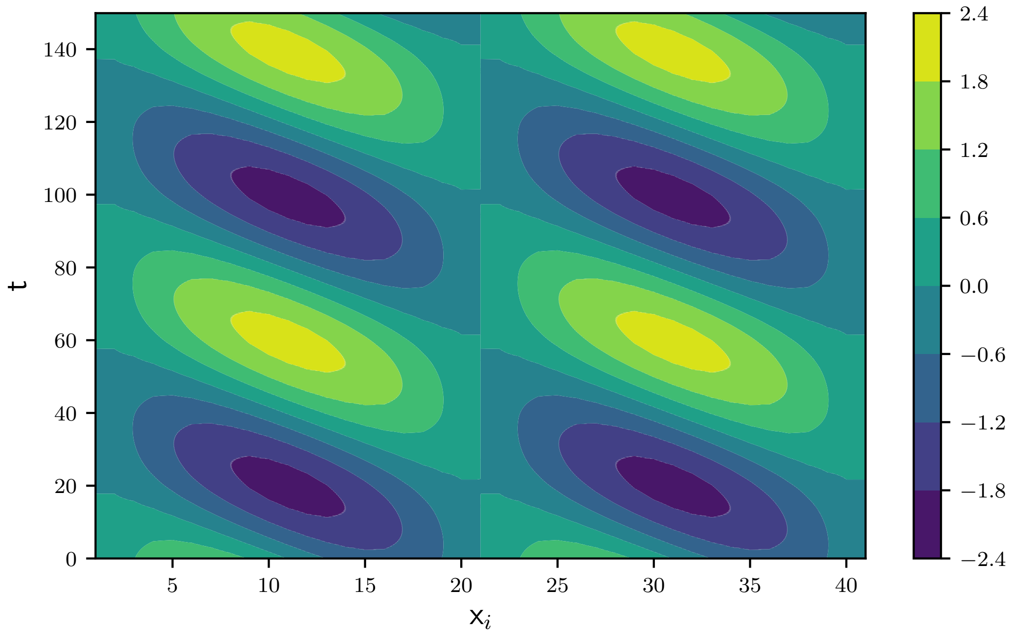

Figure 6: Hovmöller diagram for the bounded domain with a sinusoidal initial state. CTCS, $\Delta x = \frac{1}{40}$*,* $\Delta t = 0.10$*. Waves propagate westward with phasespeed* -0.0125*.*

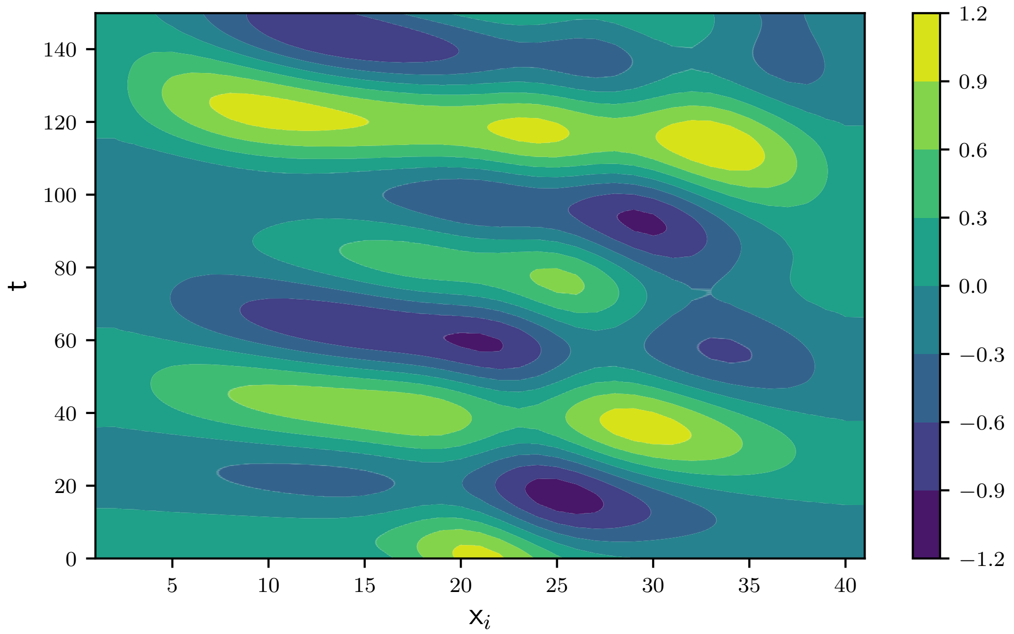

Figure 7: Hovmöller diagram for the bounded domain with a Gaussian initial state. CTCS, $\Delta x = \frac{1}{40}$, $\Delta t = 0.10$.

4.3 2 dimensions

In two dimensions we only looked at the sinusoidal initial state. With both periodic (fig. 8) and closed boundaries (fig. 9) the waves seem to be travelling in very much the same way as in the one dimensional case as the phase speed is not varying in y-direction.

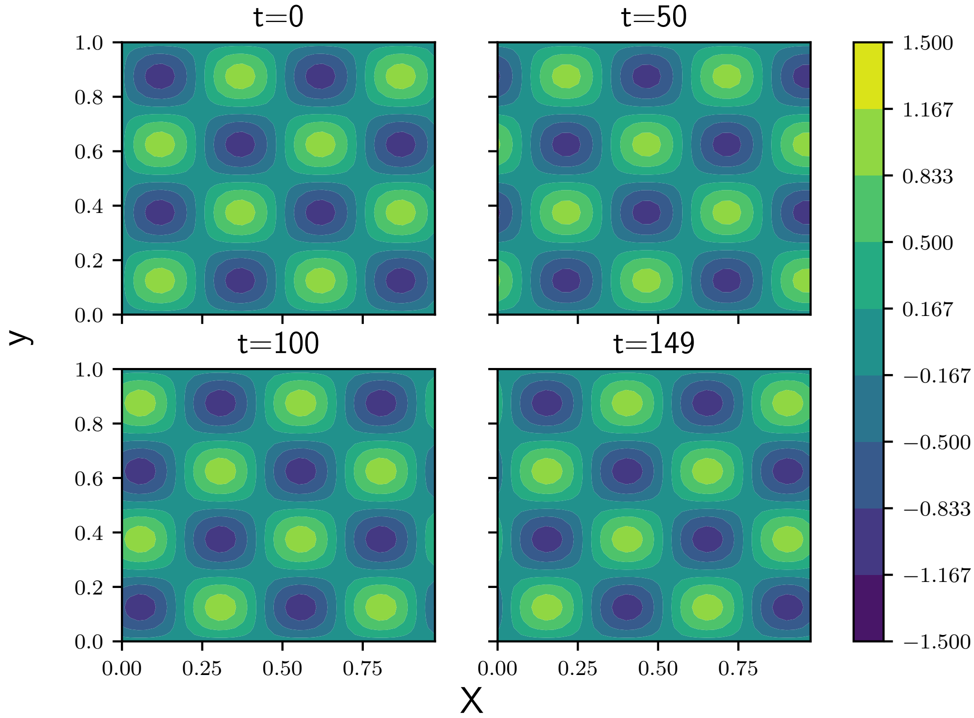

Figure 8: Rossby wave travelling in a 2d periodic domain, plotted is the state of the wave at four different times. CTCS, $\Delta x = \Delta y = \frac{1}{40}$*,* $\Delta t = 0.10$*.*

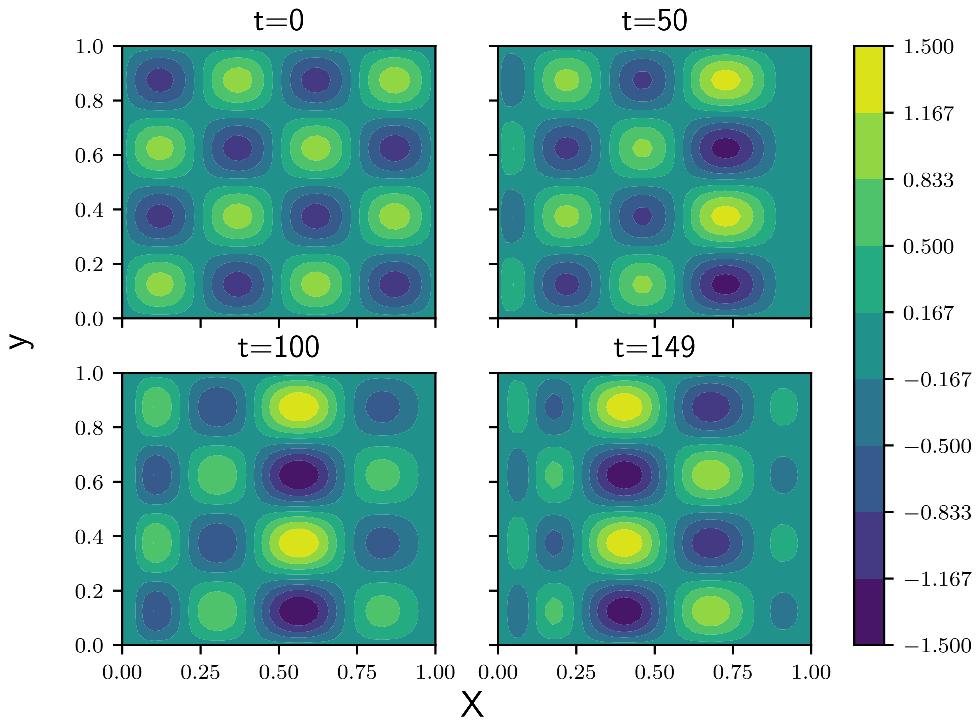

Figure 9: Rossby wave travelling in a 2d basin, plotted is the state of the wave at four different times. CTCS, $\Delta x = \Delta y = \frac{1}{40}$, $\Delta t = 0.10$.

5 Discussion

Our numerical model managed to accurately reproduce the analytical value of the phase speed for the barotropic Rossby waves with sinusoidal initial state. Extending our model to two dimension did not give more insight compared to one dimension, due to the limitation with the β-plane approximation, by which we assume the Coriolis parameter to vary linearly around a reference latitude. Consequently as we move further from the reference latitude the error we are making becomes larger.

Making implications on the real behaviour of observed Rossby waves based on our model is difficult due to our very simplified model. In fact by assuming the flow to be barotropic, we neglect the main driver of motion in the atmosphere/ocean, specifically the poleward temperature gradient. Still, in the end the purpose of using simplified models is not to create an accurate representation of reality, it should rather a be considered a tool for building our physical intuition. This intuition is invaluable when adding complexity to the model, because the simple models can work as a baseline for evaluating of the effects of added complexity. Additions to our model that would be a interesting to follow up on, would be for example to include a mean flow, that would make our model more representative of Rossby waves in the atmosphere, where the westerlies greatly influences the direction of the waves.

6 Conclusion

In this study we examined the barotropic rossby wave equation (BRWE) in one and two dimensions, first we derived the barotropic wave equation and found analytical expressions of the phase speed for a sinusoidal initial state, in a periodic and bounded domain. Next we made a numerical model of the (BRWE), where we made two different version, one using forward differences and a second using central differences. We did an extensive stability analysis of both version, which revealed the forward difference version to be unconditionally unstable, for any combination of step sizes and time steps. For the central difference version we found it to be stable for $\frac{\Delta t}{\Delta x} < k^2$.

The phase velocity of modelled waves $c_{\text{periodic}} = -0.00625$ and $c_{\text{closed}} = -0.0125$ agreed quite well with the analytical phase velocities, $c_{\text{periodic}} = -0.0063$ and $c_{\text{closed}} = -0.012665$ for the sinusoidal initial state, where waves propagated westward with constant phase velocities. We also did model runs with a gaussian initial state, with varying σ. The phase speed of the Rossby waves with gaussian initial states was not constant in time, however the waves with large σ seemed to have a more constant behaviour.

References

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[4] Carl Gustav Rossby. “Relation between variations in the intensity of the zonal circulation of the atmosphere and the displacements of the semipermanent centers of action”. In: *Journal of marine research* (1939). URL: https://peabody.yale.edu/sites/default/files/documents/scientific-publications/jmr02-01-06-CG_ROSSBYetal.pdf.

Appendices

A Stability

Figure 10 shows the blow up of the numerical solution with forward time step with $\Delta t = 1$ and $\Delta x = 0.025$ for the periodic case. With a central difference in time the solution seems stable.

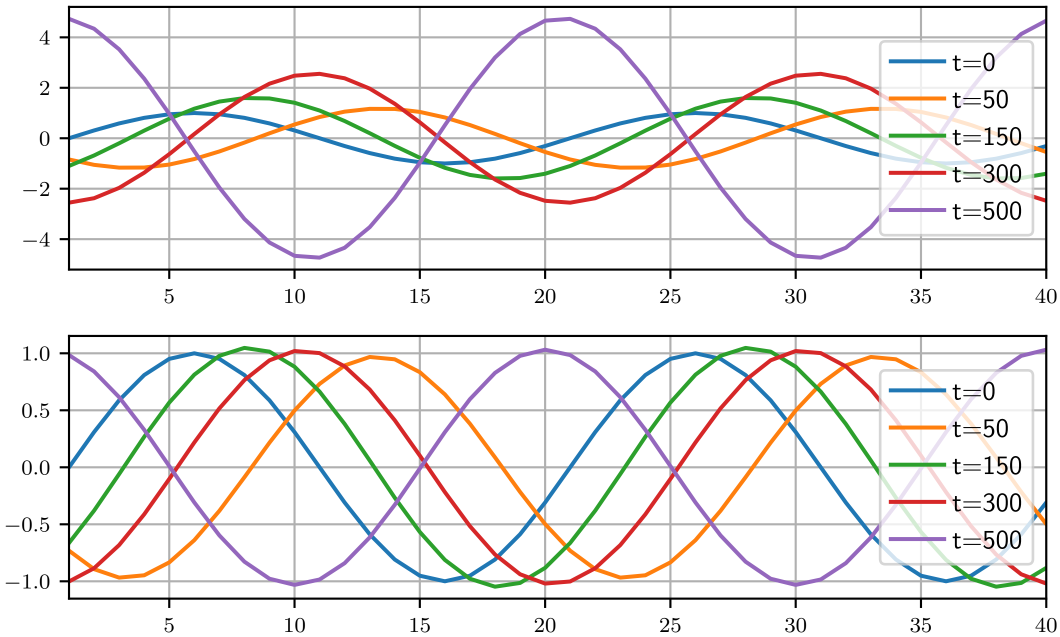

Figure 10: Comparison of solutions with periodic boundaries at different times using forward (top) and centered (bottom) differences in time with $\Delta x = 0.025$ and $\Delta t = 1$. The solution using forward differences blows up quickly while the one using central differences does not.