WEAKLY GROUP SCHEME-THEORETICAL CATEGORIES

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1. Introduction

We begin our proposal with some preliminary definitions. Let $\Bbbk$ be an algebraically closed field.

Definition 1.1. An object in a monoidal category is called rigid if it has both a left and a right dual. A monoidal category $\mathcal{C}$ is called rigid if every object of $\mathcal{C}$ is rigid.

Example 1.1. A vector space has a dual in the above sense if and only if it is finite dimensional.

Definition 1.2. A $\Bbbk$-linear abelian category $\mathcal{C}$ is said to be finite if it is equivalent, as an abelian category, to the category of representations of a finite dimensional algebra.

Definition 1.3. Let $\mathcal{C}$ be a finite $\Bbbk$-linear abelian rigid monoidal category. We will call $\mathcal{C}$ a finite *multitensor category.* If in addition $\mathsf{End}_{\mathcal{C}}(\mathbf{1}) \cong \Bbbk$, then we will call $\mathcal{C}$ a finite *tensor category.*
A *multifusion category* is a finite semisimple multitensor category. A *fusion category* is a multifusion category with $\mathsf{End}_{\mathcal{C}}(\mathbf{1}) \cong \Bbbk$. [EGNO15]

Simply put, tensor categories are categories equipped with a tensor product that endows further "linear-algebraic" properties. The notion of a tensor category was first considered by Deligne and Milne in [DM82], where they considered *symmetric* tensor categories that are not necessarily finite. These types of categories arise very naturally in mathematics; some notable examples that are finite include finite dimensional representations of finite groups, and, more generally, finite group schemes. Removing the symmetric requirement, one can consider representations of an arbitrary finite dimensional Hopf algebra. Restricting further to fusion categories, i.e. finite semisimple tensor categories, there are many exciting examples, with some of the most famous arising from quantum groups and being related to quantum knot invariants. Some of these examples are interesting from the perspective of low-dimensional topology. For example, finite braided tensor categories can be useful in studying low-dimensional manifolds and can also define invariants of 3-manifolds. Fusion categories also have a number of applications in theoretical physics. Namely, highly structured fusion categories can be used to construct topological quantum field theories (TQFTs). Two classical realizations of this idea are the Turaev-Viro model [TV92], and the Reshetikhin-Turaev construction [RT91], both of which take place in a semisimple setting.

The motivations for this project, however, are more algebro-centric and follow the current trend in quantum algebra to move away from semisimplicity, i.e. to study finite tensor categories that are not semisimple. Typically, tensor categories are examined over a field of characteristic zero and under finite semisimplicity conditions [ENO17]. Recently, however, there has been renewed interest and developments in trying to study tensor categories in more generality, especially in positive characteristic, as evidenced in [CEO22] and [Cou20]. This serves as further motivation for the removal of the semisimplicity condition, since it is a poorly behaved condition in positive characteristic. To illustrate, take the prototypical example of $\mathrm{Rep}(G)$, the category of finite dimensional representations of a finite group. By Maschke's theorem, $\mathrm{Rep}(G)$ is finite semisimple when the field has characteristic zero. However, this no longer holds in positive characteristic. It should be emphasized that the resulting category in positive characteristic is still finite, which is why we can keep the finiteness condition. Another motivation for generalizing tensor categories beyond a semisimple setting is that recently, there have been non-semisimple generalizations of the Reshetikhin-Turaev contruction of TQFTs using (braided) finite tensor categories (as seen in [CGHPM23] and [CGPMV23]), but these require some input, i.e. some interesting non-semisimple finite tensor categories.

The outline for this project is as follows: We begin with *group-theoretical* (fusion) categories. These are well-studied, and they are constructed from finite groups and cohomology data [EGNO15]. A step up in generality from these are *weakly group-theoretical* categories [ENO11], which contain the classes of group-theoretical categories and solvable categories [ENO11]. To better define these, consider the notion of upper central series for finite tensor categories (analagous to groups). For every finite tensor category $\mathcal{C}$, there is an associated tensor subcategory $\mathcal{C}'$ called the adjoint category. One can consider the upper central series $\mathcal{C} \supseteq \mathcal{C}' \supseteq ....$ [1] Being weakly group-theoretical is equivalent to the requirement that this upper central series eventually reaches Vec, the category of finite dimensional vector spaces. Group-theoretical categories have trivial upper central series, and are thus inherently weakly group-theoretical. Concurrently, there is another class of finite tensor categories called *group-scheme theoretical* [Gel15, GS23], which are constructed from finite group schemes and cohomological data. Note that it is necessary to study these in positive characteristic, since finite group-scheme theoretical categories are precisely the same as group-theoretical fusion categories in characteristic zero. **This project would be about studying the non-semisimple version of weakly group-theoretical categories, which generalize the class of group-scheme theoretical categories.** These categories have not yet been explored. We will call such categories *weakly group scheme-theoretical categories.*

2. Objectives

In this project, we attempt to prove the following two theorems.

Theorem 2.1. *Any weakly group scheme-theoretical tensor category has the strong Frobenius property.*

Theorem 2.2. *Any weakly group scheme-theoretical category of Frobenius-Perron dimension $p^r q^s$, where p and q are primes, and r, s are nonnegative integers, is solvable.*

The first result is expected to be true very generally, and establishing this theorem would verify this conjecture for a larger class of examples. It can be viewed

[1]In the case of $\mathrm{Rep}(G)$, this series actually recovers the upper central series of G.

as a form of Kaplansky's 6-th conjecture[2] for our type of categories. Adapted to our setting, this result states that the Frobenius-Perron dimension of any simple object divides the Frobenius-Perron dimension of the category. We can think of Frobenius-Perron dimension here as a generalization of the notion of dimension for finite dimensional vector spaces; it assigns positive real numbers to the objects of a finite tensor category in a way that is compatible with the tensor product and direct sums. The second theorem is an analogue of Burnside's theorem, a well-known result for finite groups. In the statement of Theorem 2.2, solvability means being nilpotent with some additional abelianness conditions on the upper central series. Together, these theorems are non-semisimple generalizations of Theorems 1.5 and 1.6 in [ENO11].

3. Approach

While generalizing to the non-semisimple setting is not easy, it is useful to note that [ENO11] would serve as a very good road map for this project and how to approach it. Firstly, during spring term, I will enroll in a reading course on tensor categories and become more familiar with these objects. The book, *Tensor Categories* [EGNO15] is a very good reference, and quite comprehensive on the subject. I will also extensively study the paper on weakly group-theoretical categories [ENO11]. By the time the project starts, I will be familiar with weakly group-theoretical categories and group scheme-theoretical categories, the objects which I am trying to generalize. The first goal of the project is to come up with a good/useful definition of *weakly group scheme-theoretical* tensor categories. After establishing this, I will proceed by studying various properties of these categories, beginning with the Drinfeld center and subcategories. At this point, I will have the machinery to prove the following intermediate result:

Proposition 3.1. *The class of weakly group scheme-theoretical categories is closed under tensor products, taking Drinfeld centers, subcategories, and quotients.*

Then, I will examine (de)equivariantizations and extensions of these categories and their Frobenius-Perron dimensions. This will allow me to prove another intermediate result:

Proposition 3.2. *Let $\mathcal{C}$ be a weakly group scheme-theoretical category having strong Frobenius property, and let G be a finite group. Then*

(1) *A G-equivariantization of $\mathcal{C}$ has the strong Frobenius property, and*
(2) *A G-extension of $\mathcal{C}$ has the strong Frobenius property.*

Finally, I will try to establish some results for weakly group scheme-theoretical categories that are parallel to the character-theoretic lemmas used in the classical proof of Burnside's theorem for groups. For instance, for a simple object X in the category, I will try to find a non-trivial simple object Y such that they projectively centralize each other. All of these properties mentioned above aid in the proofs of the main results from [ENO11] that I am aiming to generalize.

[2] Kaplansky's 6-th conjecture states that the dimension of an irreducible representation of a semisimple Hopf algebra divides the dimension of the Hopf algebra. [ENO11]

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