1 INF5270 Exercise 1 a)

ODE problem: $u'' + w^2 u = f(t)$, $u(0) = I$, $u'(0) = V$, $t \in (0, T]$.

Discretize this according to $[D_t D_t u + w^2 u = f]^n$:

$$\frac{u^{n+1} - 2u^n + u^{n-1}}{\Delta t^2} + w^2 u^n = f(t_n) \qquad (1)$$

Deriving u^1:

Using $\frac{u^1 - u^{-1}}{2\Delta t} = U^0 = V \Rightarrow u^1 - u^{-1} = V 2\Delta t$

$u^{-1} = u^1 - V 2\Delta t$

$n = 0$ in (1):

$$\frac{u^1 - 2u^0 + u^{-1}}{\Delta t^2} + w^2 u^0 = f(t_0)$$

$$u^1 = (f^0 - w^2 I)\Delta t^2 - u^1 + V 2\Delta t + 2I$$

$$u^1 = \frac{(f^0 - w^2 I)\Delta t^2 + V 2\Delta t + 2I}{2}$$

$$\underline{\underline{u^1 = \frac{\Delta t^2}{2}(f(0) - w^2 I) + V\Delta t + I}}$$

b) Using method of manufactured solutions (MMS) with $u_e(x,t) = ct + d$.

$$\left.\begin{aligned} u_e(0) &= c \cdot 0 + d = \underline{\underline{d = I}} \\ u_e'(0) &= \underline{\underline{c = V}} \end{aligned}\right\} \underline{\underline{u_e(t) = Vt + I}}$$

$u'' + w^2 u = f(t) \Rightarrow 0 + w^2(ct + d) = f(t)$

$$\underline{\underline{f(t) = w^2(Vt + I)}}$$

$$[D_t D_t t]^n = \frac{t_{n+1} - 2t_n + t_{n-1}}{\Delta t^2} = \frac{\Delta t - t_n + t_{n-1}}{\Delta t^2} = \frac{\Delta t - \Delta t}{\Delta t^2} = 0$$

Showing that u_e is a perfect solution to the discrete equations:

$$[D_t D_t u_e + w^2 u_e]^n = [D_t D_t (Vt + I) + w^2 (Vt + I)]^n$$

$$= 0 + w^2(Vt_n + I) = \underline{\underline{f(t_n)}}$$

using that $D_t D_t$ is a linear operator