IN5270
MANDATORY EERCISE 3

ERIK JOHANNES B. L. G. HUSOM

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1. Description of the problem

The goal is to compute deflection of a cable with sine functions. We have a hanging cable with tension, and the cable has a deflection $w(x)$ which is governed by:

$$Tw''(x) = l(x), \tag{1}$$

where the variables are:
- L: Length of cable - T: Tension on cable - $w(x)$: Deflection of cable - $l(x)$: Vertical load per unit length

Cable is fixed at $x = 0$ and $x = L$, and the boundary conditions are $w(0) = w(L) = 0$. Deflection is positive upwards and l is positive when it acts downwards.

Assuming $l(x) = \text{const}$, the solution is symmetric around $x = L/2$. For a function $w(x)$ that is symmetric around a point x_0, we have that

$$w(x_0 - h) = w(x_0 + h), \tag{2}$$

which means that

$$(3) lim_{h->0}(w(x_0 + h) - w(x_0 - h))/(2h) = 0. \tag{3}$$

We can therefore halve the domain, since it is symmetric. That limits the problem to find $w(x)$ in $[0, L/2]$, with boundary conditions $w(0) = 0$ and $w'(L/2) = 0$.

Scaling of variables:

$$\overline{x} = x/(L/2) \quad \text{(setting } x = \overline{x} \text{ in code for easier notation)} \tag{4}$$

$$u = w/w_c \quad \text{(where } w_c \text{ is a characteristic size of } w\text{)} \tag{5}$$

By putting this into the original equation we get

$$\frac{4Tw_c}{L^2}u''(\overline{x}) = l = \text{const.} \tag{6}$$

We set $|u''(\overline{x})| = 1$, and we get $w_c = 0.25lL^2/T$, and the scaled problem is

$$u'' = 1, \overline{x} \in (0,1), u(0) = 0, u'(1) = 0. \tag{7}$$

2. Exact solution of u

The exact solution of u is easily found by integrating two times, and using the boudnary conditions to find the unknown constants:

$$u'' = 1, \tag{8}$$
$$u' = x + C, \tag{9}$$
$$u'(1) = 0 \Rightarrow C = -1, \tag{10}$$
$$u = \frac{1}{2}x^2 - x + D, \tag{11}$$
$$u(0) = 0 \Rightarrow D = 0, \tag{12}$$
$$u = \frac{1}{2}x^2 - x. \tag{13}$$

3. Using two P1 elements to approximate the function

In our case, with two elements, we will have three nodes. The basis function of the middle node is easily found by using Lagrange polynomials, which is calculated within each element Ω_i that the basis function is defined in (since we only have two elements in this case, the middle basis function ϕ_1 is defined in both elements. We have a constant distance between each node $h = 0.5$, and our nodes is $x_0 = 0, x_1 = 0.5, x_2 = 1$. For our middle basis function we get:

$$\phi_1 = \begin{cases} \frac{x-x_{i-1}}{x_i-x_{i-1}} = \frac{x-x_0}{h} = 2x & x_0 \leq x < x_1 \\ \frac{x-x_{i+1}}{x_i-x_{i+1}} = \frac{x-x_2}{-h} = -2x+2 & x_1 \leq x < x_2 \end{cases}$$