# IN5270
# WAVE PROJECT

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Contents



## 1. Discretization of equations

In this project we have the following 2D linear wave equation with damping:

$$\frac{\partial^2 u}{\partial t^2} + b\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(q(x,y)\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(q(x,y)\frac{\partial u}{\partial y}\right) + f(x,y,t) \tag{1}$$

The boundary condition is

$$\frac{\partial u}{\partial n} = 0 \tag{2}$$

and the inital conditions are

$$u(x,y,0) = I(x,y) \tag{3}$$

$$u_t(x,y,0) = V(x,y) \tag{4}$$

To use this in our computer calculations, we need a discretized version. Since we have a variable coefficient $q$, we write the inner derivatives (by using a centered derivative) as:

$$\phi_x = q[x,y]\frac{\partial u}{\partial x}, \quad \phi_y = q[x,y]\frac{\partial u}{\partial y} \tag{5}$$

Then we get

$$\left[\frac{\partial \phi_x}{\partial x}\right]_i^n \approx \frac{\phi_{x,i+1/2} - \phi_{x,i-1/2}}{\Delta x} \tag{6}$$

$$\left[\frac{\partial \phi_y}{\partial y}\right]_j^n \approx \frac{\phi_{y,j+1/2} - \phi_{y,j-1/2}}{\Delta y} \tag{7}$$

We then write

$$\phi_{x,i+1/2} = q_{i+1/2,j}\left[\frac{\partial u}{\partial x}\right]^n_{i+1/2} \approx q_{i+1/2,j}\frac{u_{i+1,j} - u^n_{i,j}}{\Delta x} \tag{8}$$

$$\phi_{x,i-1/2} = q_{i-1/2,j}\left[\frac{\partial u}{\partial x}\right]^n_{i-1/2} \approx q_{i-1/2,j}\frac{u_{i,j} - u^n_{i-1,j}}{\Delta x} \tag{9}$$

$$\phi_{y,j+1/2} = q_{i,j++1/2}\left[\frac{\partial u}{\partial y}\right]^n_{j+1/2} \approx q_{i,j+1/2}\frac{u_{i,j+1} - u^n_{i,j}}{\Delta y} \tag{10}$$

$$\phi_{y,j-1/2} = q_{i,j-1/2}\left[\frac{\partial u}{\partial y}\right]^n_{j-1/2} \approx q_{i,j-1/2}\frac{u_{i,j} - u^n_{i,j-1}}{\Delta y} \tag{11}$$

This is then used to discretize equation 1:

$$\begin{aligned}
\frac{u^{n+1}_{i,j}-2u^n_{i,j}+u^{n-1}_{i,j}}{\Delta t^2}+b\frac{u^{n+1}_{i,j}-u^{n-1}_{i,j}}{2\Delta t} &= \frac{1}{\Delta x}\left(q_{i+\frac{1}{2},j}\left(\frac{u^n_{i+1,j}-u^n_{i,j}}{\Delta x}\right)-q_{i-\frac{1}{2},j}\left(\frac{u^n_{i,j}-u^n_{i-1,j}}{\Delta x}\right)\right)\\
&+\frac{1}{\Delta y}\left(q_{i,j+\frac{1}{2}}\left(\frac{u^n_{i,j+1}-u^n_{i,j}}{\Delta y}\right)-q_{i,j-\frac{1}{2}}\left(\frac{u^n_{i,j}-u^n_{i,j-1}}{\Delta y}\right)\right)+f^n_{i,j}\\
\frac{u^{n+1}_{i,j}-2u^n_{i,j}+u^{n-1}_{i,j}}{\Delta t^2} &= -b\frac{u^{n+1}_{i,j}-u^{n-1}_{i,j}}{2\Delta t}\\
&+\frac{1}{\Delta x^2}\left(\frac{1}{2}\left(q_{i+1,j}-q_{i,j}\right)\left(u^n_{i+1,j}-u^n_{i,j}\right)-\frac{1}{2}\left(q_{i,j}-q_{i-1,j}\right)\left(u^n_{i,j}-u^n_{i-1,j}\right)\right)\\
&+\frac{1}{\Delta y^2}\left(\frac{1}{2}\left(q_{i,j+1}-q_{i,j}\right)\left(u^n_{i,j+1}-u^n_{i,j}\right)-\frac{1}{2}\left(q_{i,j}-q_{i,j-1}\right)\left(u^n_{i,j}-u^n_{i,j-1}\right)\right)+f^n_{i,j}\\
u^{n+1}_{i,j}-2u^n_{i,j}+u^{n-1}_{i,j} &= \frac{-b\Delta t}{2}(u^{n+1}_{i,j}-u^{n-1}_{i,j})\\
&+\frac{\Delta t^2}{2\Delta x^2}\left(\left(q_{i+1,j}-q_{i,j}\right)\left(u^n_{i+1,j}-u^n_{i,j}\right)-\left(q_{i,j}-q_{i-1,j}\right)\left(u^n_{i,j}-u^n_{i-1,j}\right)\right)\\
&+\frac{\Delta t^2}{2\Delta y^2}\left(\left(q_{i,j+1}-q_{i,j}\right)\left(u^n_{i,j+1}-u^n_{i,j}\right)-\left(q_{i,j}-q_{i,j-1}\right)\left(u^n_{i,j}-u^n_{i,j-1}\right)\right)+\Delta t^2 f^n_{i,j}\\
u^{n+1}_{i,j}+\frac{b\Delta t}{2}u^{n+1}_{i,j} &= 2u^n_{i,j}-u^{n-1}_{i,j}+\frac{b\Delta t}{2}u^{n-1}_{i,j}\\
&+\frac{\Delta t^2}{2\Delta x^2}\left(\left(q_{i+1,j}-q_{i,j}\right)\left(u^n_{i+1,j}-u^n_{i,j}\right)-\left(q_{i,j}-q_{i-1,j}\right)\left(u^n_{i,j}-u^n_{i-1,j}\right)\right)\\
&+\frac{\Delta t^2}{2\Delta y^2}\left(\left(q_{i,j+1}-q_{i,j}\right)\left(u^n_{i,j+1}-u^n_{i,j}\right)-\left(q_{i,j}-q_{i,j-1}\right)\left(u^n_{i,j}-u^n_{i,j-1}\right)\right)+\Delta t^2 f^n_{i,j}\\
\left(1+\frac{b\Delta t}{2}\right)u^{n+1}_{i,j} &= 2u^n_{i,j}-\left(1-\frac{b\Delta t}{2}\right)u^{n-1}_{i,j}\\
&+\frac{\Delta t^2}{2\Delta x^2}\left(\left(q_{i+1,j}-q_{i,j}\right)\left(u^n_{i+1,j}-u^n_{i,j}\right)-\left(q_{i,j}-q_{i-1,j}\right)\left(u^n_{i,j}-u^n_{i-1,j}\right)\right)\\
&+\frac{\Delta t^2}{2\Delta y^2}\left(\left(q_{i,j+1}-q_{i,j}\right)\left(u^n_{i,j+1}-u^n_{i,j}\right)-\left(q_{i,j}-q_{i,j-1}\right)\left(u^n_{i,j}-u^n_{i,j-1}\right)\right)+\Delta t^2 f^n_{i,j}\\
u^{n+1}_{i,j} &= \frac{1}{1+\frac{b\Delta t}{2}}\Big(\,2u^n_{i,j}-\left(1-\frac{b\Delta t}{2}\right)u^{n-1}_{i,j}\\
&+\frac{\Delta t^2}{2\Delta x^2}\left(\left(q_{i+1,j}-q_{i,j}\right)\left(u^n_{i+1,j}-u^n_{i,j}\right)-\left(q_{i,j}-q_{i-1,j}\right)\left(u^n_{i,j}-u^n_{i-1,j}\right)\right)\\
&+\frac{\Delta t^2}{2\Delta y^2}\left(\left(q_{i,j+1}-q_{i,j}\right)\left(u^n_{i,j+1}-u^n_{i,j}\right)-\left(q_{i,j}-q_{i,j-1}\right)\left(u^n_{i,j}-u^n_{i,j-1}\right)\right)+\Delta t^2 f^n_{i,j}\,\Big)
\end{aligned}$$

The modified scheme for the first step will find by using the initial conditions:

$$\begin{aligned}
u_t(x,y,0) &= V(x,y)\\
\frac{u^0_{i,j}-u^{-1}_{i,j}}{\Delta t} &= V_{i,j}\\
u^{-1}_{i,j} &= u^0_{i,j}-\Delta t V_{i,j}
\end{aligned}$$

The modified scheme at the boundary points is found by using the boundary conditions:

$$\frac{\partial u}{\partial n} = 0 \frac{u^n_{N_x+1,j} - u^n_{N_x-1,j}}{2\Delta x} = 0$$
$$u^n_{N_x+1,j} = u^n_{N_x-1,j}$$