Important exam-relevant topics of IN5270/IN9270

2019

Note:

About 70% of the exam questions will be chosen from the following contents.
No assistance is allowed at the written exam on Dec. 17.

Topic 1

You are asked to approximate the function $f(x) = 1 + 2x - x^2$ in the domain $x \in [0,1]$ by the projection method and using finite element basis functions.

1. Show the details and result of the calculation when a single P2 element is used to cover the domain.
2. Show the details and result of the calculation when two equal-sized P1 elements are used to cover the domain.
3. Extend the projection method to using N equal-sized P1 elements. Show the details of how to set up the corresponding linear system. (There's no need to solve the linear system.)
4. If we want in addition that the approximation result, when using N equal-sized P1 elements, should attain the same value of $f(x)$ at $x = 0$ and $x = 1$, what are the changes needed in the calculation above?

Topic 2

You are asked to solve the 1D Poisson equation

$$-u_{xx} = 1, \quad 0 < x < 1$$

by a finite difference method. On the left boundary point of $x = 0$, the following mixed boundary condition

$$u_x + Cu = 0$$

is valid, where C is a scalar constant. On the right boundary point of $x = 1$, the Dirichlet boundary condition $u = D$ is valid. We assume that a uniform mesh of $N + 1$ points is used by the finite difference method.

1. Discretize the Poisson equation on all the $N-1$ interior points.

2. Discretize the left boundary condition using appropriate finite differencing.

3. Show the details of setting up a linear system $\mathbf{Au} = \mathbf{b}$ which can be used to find the approximations of $u(x)$ on the mesh points. (There's no need to solve the linear system.)

4. How would you validate that the obtained numerical solutions converge towards the exact solution, when the number of mesh points is increased? What is the expected convergence speed?

Topic 3

The following 1D stationary convection diffusion equation

$$u_x = \varepsilon u_{xx}$$

is to be solved by finite differencing in the domain $0 < x < 1$, where $\varepsilon > 0$ is a given constant and the boundary conditions are $u(0) = 0$ and $u(1) = 1$.

1. Show that the above equation has $u(x) = \dfrac{1-e^{x/\varepsilon}}{1-e^{1/\varepsilon}}$ as its exact solution.

2. Assume a uniform mesh that consists of $N+1$ points: $x_0, x_1, \ldots, x_N$, where $x_i = i \cdot h$. Use centered finite differences to discretize the equation, and show the details of how to set up the resulting linear system. (There's no need to solve the linear system.)

3. Prove that the analytical solution of the centered finite difference scheme is of form $u_i = C_1\beta_1^i + C_2\beta_2^i$, where $\beta_1 = 1$ and $\beta_2 = \frac{1+\frac{h}{2\varepsilon}}{1-\frac{h}{2\varepsilon}}$. The values of C_1 and C_2 should be determined using the boundary conditions. What is the stability condition for the numerical solution?

4. Derive another numerical scheme where the convection term u_x is discretized by so-called upwind finite difference. That is, u_x is approximated at $x = x_i$ by

$$\frac{u_i - u_{i-1}}{h}$$

   Any advantage and/or disadvantage of this numerical scheme in comparison with the above scheme?

Topic 4

We consider the following nonlinear diffusion equation (which is applicable for multiple space dimensions):

$$\begin{aligned}
\frac{\partial u}{\partial t} &= \nabla\cdot(\alpha(\mathbf{x},t)\nabla u) + f(u) \quad \mathbf{x}\in\Omega,\ t\in(0,T],\\
u(\mathbf{x},0) &= I(\mathbf{x}) \qquad \mathbf{x}\in\Omega,\\
\frac{\partial u}{\partial n} &= g \qquad \mathbf{x}\in\partial\Omega,\ t\in(0,T].
\end{aligned}$$

Note: $\frac{\partial u}{\partial n}$ denotes the outward normal derivative on the boundary $\partial\Omega$, and g is a constant.

1. Use the Crank-Nicolson scheme in time and show the resulting time discrete problem for each time step.
2. Formulate Picard iterations to linearize the time discrete problem.
3. Use the Galerkin method to discretize the stationary linear PDE per Picard iteration. Show the details of how to derive the corresponding variational form.
4. Restrict now the spatial domain to the 1D case of $x \in (0,1)$, let α be a constant and choose $f(u) = u^2$. (The boundary conditions are now $u_x = -g$ at $x = 0$ and $u_x = g$ at $x = 1$.) Suppose the 1D spatial domain consists of N equal-sized P1 elements. Carry out the calculation in detail for computing the element matrix and vector for the leftmost P1 element.
5. What is the resulting global linear system $\mathbf{Ax} = \mathbf{b}$?

Topic 5

You are asked to solve the 2D Poisson equation:

$$-\nabla \cdot \nabla u = 2$$

in the unit square $(x,y) \in [0,1]^2$. A homogeneous Neumann condition $\frac{\partial u}{\partial n} = 0$ applies on the entire boundary $\partial\Omega$ (which has four sides: $y = 0$, $x = 1$, $y = 1$, $x = 0$).

We will use a 2D uniform mesh consisting of $M \times N$ elements (N elements in the x direction, M elements in the y direction), which all adopt bilinear basis functions.

1. Use the Galerkin method, derive the variational form of the above PDE in detail.
2. What are the degrees of freedom and how many are they in total? How would you number the degrees of freedom, with respect to the rows in a global linear system to be set up?
3. Describe in detail how the bilinear basis functions $\tilde{\varphi}_0(X,Y)$, $\tilde{\varphi}_1(X,Y)$, $\tilde{\varphi}_2(X,Y)$ and $\tilde{\varphi}_3(X,Y)$ are defined in a reference cell $(X,Y) \in [-1,1]^2$.
(Hint: Each basis function is of the form $(aX+b) \cdot (cY+d)$ with suitable choices of the a,b,c,d scalar values.)
4. For element number e, how can the physical coordinates (x,y) be mapped from the local coordinates (X,Y) of the reference cell?
5. Compute the element matrix and vector for element number e, with help of the reference cell.