

IN5270 WAVE PROJECT

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1. DISCRETIZATION OF EQUATIONS

In this project we have the following 2D linear wave equation with damping:

$$(1) \quad \frac{\partial^2 u}{\partial t^2} + b \frac{\partial u}{\partial t} = \frac{\partial}{\partial x} \left(q(x, y) \frac{\partial u}{\partial x} \right) + \frac{\partial}{\partial y} \left(q(x, y) \frac{\partial u}{\partial y} \right) + f(x, y, t)$$

The boundary condition is

$$(2) \quad \frac{\partial u}{\partial n} = 0$$

and the initial conditions are

$$(3) \quad u(x, y, 0) = I(x, y)$$

$$(4) \quad u_t(x, y, 0) = V(x, y)$$

To use this in our computer calculations, we need a discretized version. Since we have a variable coefficient q , we write the inner derivatives (by using a centered derivative) as:

$$(5) \quad \phi_x = q[x, y] \frac{\partial u}{\partial x}, \quad \phi_y = q[x, y] \frac{\partial u}{\partial y}$$

Then we get

$$(6) \quad \left[\frac{\partial \phi_x}{\partial x} \right]_i^n \approx \frac{\phi_{x,i+1/2} - \phi_{x,i-1/2}}{\Delta x}$$

$$(7) \quad \left[\frac{\partial \phi_y}{\partial y} \right]_j^n \approx \frac{\phi_{y,j+1/2} - \phi_{y,j-1/2}}{\Delta y}$$

We then write

$$(8) \quad \phi_{x,i+1/2} = q_{i+1/2,j} \left[\frac{\partial u}{\partial x} \right]_{i+1/2}^n \approx q_{i+1/2,j} \frac{u_{i+1,j} - u_{i,j}^n}{\Delta x}$$

$$(9) \quad \phi_{x,i-1/2} = q_{i-1/2,j} \left[\frac{\partial u}{\partial x} \right]_{i-1/2}^n \approx q_{i-1/2,j} \frac{u_{i,j} - u_{i-1,j}^n}{\Delta x}$$

$$(10) \quad \phi_{y,j+1/2} = q_{i,j+1/2} \left[\frac{\partial u}{\partial y} \right]_{j+1/2}^n \approx q_{i,j+1/2} \frac{u_{i,j+1} - u_{i,j}^n}{\Delta y}$$

$$(11) \quad \phi_{y,j-1/2} = q_{i,j-1/2} \left[\frac{\partial u}{\partial y} \right]_{j-1/2}^n \approx q_{i,j-1/2} \frac{u_{i,j} - u_{i,j-1}^n}{\Delta y}$$

To obtain q at the half steps we use the arithmetic mean $q_{i+1/2} \approx \frac{1}{2}(q_i + q_{i+1})$ and $q_{i-1/2} \approx \frac{1}{2}(q_i + q_{i-1})$.

This is then used to discretize equation 1:

$$\begin{aligned}
\frac{u_{i,j}^{n+1} - 2u_{i,j}^n + u_{i,j}^{n-1}}{\Delta t^2} + b \frac{u_{i,j}^{n+1} - u_{i,j}^{n-1}}{2\Delta t} &= \frac{1}{\Delta x} \left(q_{i+\frac{1}{2},j} \left(\frac{u_{i+1,j}^n - u_{i,j}^n}{\Delta x} \right) - q_{i-\frac{1}{2},j} \left(\frac{u_{i,j}^n - u_{i-1,j}^n}{\Delta x} \right) \right) \\
&\quad + \frac{1}{\Delta y} \left(q_{i,j+\frac{1}{2}} \left(\frac{u_{i,j+1}^n - u_{i,j}^n}{\Delta y} \right) - q_{i,j-\frac{1}{2}} \left(\frac{u_{i,j}^n - u_{i,j-1}^n}{\Delta y} \right) \right) + f_{i,j}^n \\
\frac{u_{i,j}^{n+1} - 2u_{i,j}^n + u_{i,j}^{n-1}}{\Delta t^2} &= -b \frac{u_{i,j}^{n+1} - u_{i,j}^{n-1}}{2\Delta t} \\
&\quad + \frac{1}{\Delta x^2} \left(\frac{1}{2} (q_{i+1,j} - q_{i,j}) (u_{i+1,j}^n - u_{i,j}^n) - \frac{1}{2} (q_{i,j} - q_{i-1,j}) (u_{i,j}^n - u_{i-1,j}^n) \right) \\
&\quad + \frac{1}{\Delta y^2} \left(\frac{1}{2} (q_{i,j+1} - q_{i,j}) (u_{i,j+1}^n - u_{i,j}^n) - \frac{1}{2} (q_{i,j} - q_{i,j-1}) (u_{i,j}^n - u_{i,j-1}^n) \right) + f_{i,j}^n \\
u_{i,j}^{n+1} - 2u_{i,j}^n + u_{i,j}^{n-1} &= \frac{-b\Delta t}{2} (u_{i,j}^{n+1} - u_{i,j}^{n-1}) \\
&\quad + \frac{\Delta t^2}{2\Delta x^2} ((q_{i+1,j} - q_{i,j}) (u_{i+1,j}^n - u_{i,j}^n) - (q_{i,j} - q_{i-1,j}) (u_{i,j}^n - u_{i-1,j}^n)) \\
&\quad + \frac{\Delta t^2}{2\Delta y^2} ((q_{i,j+1} - q_{i,j}) (u_{i,j+1}^n - u_{i,j}^n) - (q_{i,j} - q_{i,j-1}) (u_{i,j}^n - u_{i,j-1}^n)) + \Delta t^2 f_{i,j}^n \\
u_{i,j}^{n+1} + \frac{b\Delta t}{2} u_{i,j}^{n+1} &= 2u_{i,j}^n - u_{i,j}^{n-1} + \frac{b\Delta t}{2} u_{i,j}^{n-1} \\
&\quad + \frac{\Delta t^2}{2\Delta x^2} ((q_{i+1,j} - q_{i,j}) (u_{i+1,j}^n - u_{i,j}^n) - (q_{i,j} - q_{i-1,j}) (u_{i,j}^n - u_{i-1,j}^n)) \\
&\quad + \frac{\Delta t^2}{2\Delta y^2} ((q_{i,j+1} - q_{i,j}) (u_{i,j+1}^n - u_{i,j}^n) - (q_{i,j} - q_{i,j-1}) (u_{i,j}^n - u_{i,j-1}^n)) + \Delta t^2 f_{i,j}^n \\
\left(1 + \frac{b\Delta t}{2}\right) u_{i,j}^{n+1} &= 2u_{i,j}^n - \left(1 - \frac{b\Delta t}{2}\right) u_{i,j}^{n-1} \\
&\quad + \frac{\Delta t^2}{2\Delta x^2} ((q_{i+1,j} - q_{i,j}) (u_{i+1,j}^n - u_{i,j}^n) - (q_{i,j} - q_{i-1,j}) (u_{i,j}^n - u_{i-1,j}^n)) \\
&\quad + \frac{\Delta t^2}{2\Delta y^2} ((q_{i,j+1} - q_{i,j}) (u_{i,j+1}^n - u_{i,j}^n) - (q_{i,j} - q_{i,j-1}) (u_{i,j}^n - u_{i,j-1}^n)) + \Delta t^2 f_{i,j}^n \\
u_{i,j}^{n+1} &= \frac{1}{1 + \frac{b\Delta t}{2}} \left(2u_{i,j}^n - \left(1 - \frac{b\Delta t}{2}\right) u_{i,j}^{n-1} \right. \\
&\quad + \frac{\Delta t^2}{2\Delta x^2} ((q_{i+1,j} - q_{i,j}) (u_{i+1,j}^n - u_{i,j}^n) - (q_{i,j} - q_{i-1,j}) (u_{i,j}^n - u_{i-1,j}^n)) \\
&\quad \left. + \frac{\Delta t^2}{2\Delta y^2} ((q_{i,j+1} - q_{i,j}) (u_{i,j+1}^n - u_{i,j}^n) - (q_{i,j} - q_{i,j-1}) (u_{i,j}^n - u_{i,j-1}^n)) + \Delta t^2 f_{i,j}^n \right)
\end{aligned}$$

The modified scheme for the first step will find by using the initial conditions:

$$\begin{aligned}
u_t(x, y, 0) &= V(x, y) \\
\frac{u_{i,j}^0 - u_{i,j}^{-1}}{\Delta t} &= V_{i,j} \\
u_{i,j}^{-1} &= u_{i,j}^0 - \Delta t V_{i,j}
\end{aligned}$$

The modified scheme at the boundary points is found by using the boundary conditions:

$$\begin{aligned}\frac{\partial u}{\partial n} &= 0 \\ \frac{u_{N_x+1,j}^n - u_{N_x-1,j}^n}{2\Delta x} &= 0 \\ u_{N_x+1,j}^n &= u_{N_x-1,j}^n\end{aligned}$$

2. CONSTANT SOLUTION

The file `test.py` contains test of a constant solution for both the scalar and the vectorized version. With a constant solution $u(x, y, t) = c$, the derivatives become zero, so if we set $f = 0$, our exact solution becomes the same as the initial condition I , which has to be set to $I = c$. In my test case, I have chosen $c = 8$. The code also uses `nose.tools` to assert that the difference between the exact and numerical solution is zero.

3. UNDAMPENED WAVES AND MANUFACTURED SOLUTION

I did not manage to get results that were consistent with the exact solutions. I prioritized to finish the physical problem solving instead of finding the bugs in this part of the project.

4. PHYSICAL PROBLEM

I have implemented to kinds of surfaces, and produced gifs of how the wave motion is.