

# Linear Algebra Review

**Vector Norms**  $\|x\|_2 = \sqrt{\sum_{i=1}^n |x_i|^2}$  (Euclidean norm)

$\|x\|_1 = \sum_{i=1}^n |x_i|$  (absolute value sum/taxi norm)

$\|x\|_\infty = \max_{i=1}^n |x_i|$  (largest component by abs. value)

**Matrix Norms**  $\|A\|_2 = \max_i \sqrt{\text{eig}_i(A^T A)}$

$\|A\|_1 = \max_{1 \leq j \leq n} \sum_{i=1}^m |a_{ij}|$  (max abs. value column sum)

$\|A\|_\infty = \max_{1 \leq i \leq m} \sum_{j=1}^n |a_{ij}|$  (max abs. value row sum)

$\|A\|_F = \sqrt{\sum_{i=1}^m \sum_{j=1}^n |a_{ij}|^2} = \sqrt{\text{tr}(A^* A)}$ . (square root of the sum of the squares of every element in  $A$ ).

## Systems of Linear Equations

**Just Some Basic Stuff**  $XA = B \iff A^T X^T = B^T$

$(UL)^{-1} = L^{-1}U^{-1}$

$\det L = \prod_i L_{ii}$   $\det U = \prod_i U_{ii}$

**Forward Substitution**  $Lx = b$

$L \in \mathbb{R}^{n \times n}$  lower triangular, non-singular

$x_i = \frac{1}{L_{ii}}(b_i - \sum_{j=1}^{i-1} L_{ij} x_j)$  Time cost:  $\mathcal{O}(n^2)$

**Back Substitution**  $Ux = b$

$U \in \mathbb{R}^{n \times n}$  upper triangular, non-singular

$x_i = \frac{1}{U_{ii}}(b_i - \sum_{j=i+1}^n U_{ij} x_j)$  Time cost:  $\mathcal{O}(n^2)$

**LU Decomposition without Pivoting**  $A \in \mathbb{R}^{n \times n}$  diagonally dominant  $\implies \exists A = LU$

*Algorithm:*  $A^{(j)} \equiv A$  on algorithm's  $j$ th iteration.

Start with  $j = 1$  and  $A^{(1)} = A$ . Perform row subtraction on  $A^{(j)}$ 's rows  $j+1$  to  $n$ ; record subtraction coefficient in  $L$ . Increment  $j \rightarrow j+1$ , repeat for all  $n$  rows.

Computational complexity:  $\mathcal{O}(n^3)$

**LU Decomposition with Partial Pivoting**  $A \in \mathbb{R}^{n \times n}$  non-singular  $\implies \exists PA = LU$

*Algorithm:*  $A^{(j)} \equiv A$  on algorithm's  $j$ th iteration.

Start with  $j = 1$  and  $A^{(1)} = A$ . Find largest element  $A_{ij}$  in column  $j$  from rows  $j$  to  $n$ . Switch  $j$ th row with row containing largest element. Record switch in  $P$ . Perform row subtraction on  $A^{(j)}$ 's rows  $j+1$  to  $n$ ; record subtraction coefficient in  $L$ . Increment  $j \rightarrow j+1$ , repeat for all  $n$  rows.

Computational complexity:  $\mathcal{O}(n^3)$

**Applications of LU Decomposition** To find inverse of  $A \in \mathbb{R}^{n \times n}$ : Solve  $AA^{-1} = I$

To find  $\det A$ : Find  $PA = LU$ , use  $\det A = (-1)^s \cdot \det U$  where  $s$  is number of row switches in permutation matrix  $P$

**Cholesky Decomposition**  $A \in \mathbb{R}^{n \times n}$ ,  $A$  symmetric and positive definite.

$A = LL^T$  where  $L$  is lower-diagonal.

*Algorithm:* For  $k = 1$  initialize  $L_1 = \sqrt{A_{11}}$ . For  $k = 2, \dots, n$ , solve  $L_{k-1}l_k = a_k$  for  $l_k$ . Solve  $L_{kk} = \sqrt{A_{kk} - l_k^T l_k}$ . Assemble  $L_k = [L_{k-1} \quad 0_k l_k^T \quad L_{kk}] \in \mathbb{R}^{k \times k}$ ,  $0_k \in \mathbb{R}^k$ . Finish with  $L = L_n$ .

*Notation:* Matrix  $L_k$ :  $k \times k$  principle sub-matrix of  $L$ . Vectors  $a_k, l_k$ : first  $k-1$  entries in column  $k$  of  $A$  and  $L^T$ .

**Solving Common Linear Systems** 1.  $AX = B$  where  $A \in \mathbb{R}^{n \times n}$  and  $B, X \in \mathbb{R}^{n \times m}$

Write  $B = (b_1, \dots, b_m)$ , LU decomposition  $A = LU$ . Solve  $Ly_i = b_i$  with forward substitution, solve  $Ux_i = y_i$  with back substitution, reconstruct  $X = (x_1, \dots, x_m)$ .

2.  $XA = B$  where  $A \in \mathbb{R}^{n \times n}$  and  $B, X \in \mathbb{R}^{m \times n}$

Apply equality  $XA = B \iff A^T X^T = B^T$ . Solve the system  $\tilde{A}\tilde{X} = \tilde{B}$  using (1).

3.  $AXB = C$  where  $A, B, C, X \in \mathbb{R}^{n \times n}$

Let  $Y = XB$  and solve  $LY = C$  for  $Y = (y_1, \dots, y_n)$  using (1). Solve  $XB = Y$  for  $X = (x_1, \dots, x_n)$  using (2).

4.  $AX = B$ ;  $A \in \mathbb{R}^{n \times n}$  symmetric;  $B, X \in \mathbb{R}^{n \times m}$

Let  $B = (b_1, \dots, b_m)$ , Cholesky decomposition  $A = LL^T$ . Solve  $Ly_i = b_i$  with forward substitution, solve  $V^T x_i = y_i$  with back substitution, reconstruct  $X = (x_1, \dots, x_m)$ .

## Non-Linear Equations

**Bisection Method** *Parameters:* Function  $f$ , tolerance  $\epsilon$ , original interval  $[a, b]$ , variable interval  $[\alpha, \beta]$  with midpoint  $c$ .

*Algorithm:* Start with  $\alpha = a, \beta = b$ . Calculate midpoint  $c = \alpha + \frac{\beta - \alpha}{2}$ . If  $\text{sign } f(\alpha) = \text{sign } f(c)$ , let  $\alpha = c$ ; if  $\text{sign } f(\alpha) \neq \text{sign } f(c)$ , let  $\beta = c$ . Repeat until  $|\beta - \alpha| \leq \epsilon$ , then return root  $x_0 = \alpha + \frac{\beta - \alpha}{2}$ .

*Notes:* After  $k$  iterations, interval width is  $\ell = \frac{b-a}{2^k}$ . Tolerance  $\epsilon$  requires  $k \geq \log_2 \frac{b-a}{\epsilon}$  iterations.

**Fixed Point Iteration** *Algorithm:* Solve  $f(x) = 0$  for  $x = g(x)$ . Choose tolerance  $\epsilon$  and initial  $x_0$ , find  $x_{j+1} = g(x_j)$ , repeat until  $|x_{j+1} - x_j| < \epsilon$ .

$\alpha$  fixed point of  $x_{j+1} = g(x_j)$  if  $\alpha = g(\alpha)$

*Convergence:* Fixed point  $\alpha = g(\alpha)$  convergent if  $|g'(\alpha)| \leq 1$ . Fixed point  $\alpha = g(\alpha)$  has order of convergence  $p$  if  $g^{(k)}(\alpha) = 0$  for  $k = 1, \dots, p-1$  and  $g^{(p)}(\alpha) \neq 0$

**Newton's Method** Iteration function:  $g(x) = x - \frac{f(x)}{f'(x)}$

*Algorithm:* Choose tolerance  $\epsilon$  and initial guess  $x_0$ , find  $x_{j+1} = x_j - \frac{f(x_j)}{f'(x_j)}$ , repeat until  $|x_{j+1} - x_j| < \epsilon$ .

*Convergence:* If  $\alpha$  is a simple zero (i.e.  $f'(\alpha) \neq 0$ ), convergence is quadratic if  $f''(\alpha) \neq 0$  and cubic if  $f''(\alpha) = 0$ .

*Secant Method:*  $x_{j+1} = x_j - f(x_j) \frac{x_j - x_{j-1}}{f(x_j) - f(x_{j-1})}$

**Polynomial Roots** Given  $p_n(x) = a_n x^n + \dots + a_1 x + a_0$ , construct  $n \times n$  matrix

$$A_p = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ -\frac{a_0}{a_n} & -\frac{a_1}{a_n} & -\frac{a_2}{a_n} & \dots & -\frac{a_{n-1}}{a_n} \end{bmatrix}$$

The polynomial's  $p_n$ 's roots are  $A_p$ 's eigenvalues  $\text{eig}(A_p)$

## Linear Least Squares

**Problem** Given vector of data points  $b \in \mathbb{R}^m$  and model function  $f = f(t, a_1, \dots, a_m)$ , find vector of parameters  $x = (a_1, \dots, a_n) \in \mathbb{R}^n$  minimizing  $\|Ax - b\|_2$  where  $A \in \mathbb{R}^{m \times n}$ ; rank  $A = n$ .

$A_{ij}$  are coefficients of  $j$ th parameter  $a_j$  at  $i$ th data point.

**Normal System**  $A^T A x = A^T b$ ; solved with Cholesky decomposition.

$x \in \mathbb{R}^n$  is desired vector of parameters

Unstable if  $A$ 's columns are nearly linearly dependent.

**QR Decomposition**  $A \in \mathbb{R}^{m \times n} \implies A = QR$ .  $Q \in \mathbb{R}^{m \times n}$  has orthogonal columns and  $R \in \mathbb{R}^{n \times n}$  is upper triangular.

Solve  $Rx = Q^T b$  for least squares parameter vector  $x$ .

**Householder Reflection**  $\mathbf{P} = \mathbf{I} - \frac{2}{\mathbf{w}^T \cdot \mathbf{w}} \mathbf{w} \mathbf{w}^T$   $\mathbf{P} = \mathbf{P}^T = \mathbf{P}^{-1}$   
For  $\mathbf{A} = [\mathbf{a}_1, \dots, \mathbf{a}_n] \in \mathbb{R}^{m \times n}$  find  $\mathbf{w} \in \mathbb{R}^m$  so  $\mathbf{P} \mathbf{a}_1 = k \mathbf{e}_1$   
 $\mathbf{w} = [a_1 + \text{sgn} \|\mathbf{a}\|_2, a_2, \dots, a_m]^T$   $\|\mathbf{a}\|_2 = k$   
 $\mathbf{Q} = (\mathbf{P}_n \mathbf{P}_{n-1} \dots \mathbf{P}_1)^T = \dots = \mathbf{P}_1 \mathbf{P}_2 \dots \mathbf{P}_n$

**Givens Rotation**  $\mathbf{Q}^T$  Givens rotation such that  $\mathbf{Q}^T \mathbf{A} = \mathbf{R}$   

$$\begin{bmatrix} \cos \phi & -\sin \phi \\ \sin \phi & \cos \phi \end{bmatrix}^T \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} r & R_{12} \\ 0 & R_{22} \end{bmatrix}$$
  
 $r = \sqrt{A_{11}^2 + A_{21}^2}$   $\cos \phi = \frac{A_{11}}{r}$   $\sin \phi = \frac{A_{21}}{r}$

## Eigenvalues and Eigenvectors

**Power Method** Used to find largest eigenvalue of  $\mathbf{A} \in \mathbb{R}^{n \times n}$   
**Algorithm:** Pick initial vector  $\mathbf{z}_0 \in \mathbb{R}^n$  and tolerance  $\epsilon$ . For  $k = 0, 1, \dots$  find  $\mathbf{y}_{k+1} = \mathbf{A} \mathbf{z}_k$  and normalize  $\mathbf{z}_{k+1} = \frac{\mathbf{y}_{k+1}}{\|\mathbf{y}_{k+1}\|}$ .  
Stop when  $\|\mathbf{A} \mathbf{z}_k - \rho_k \mathbf{z}_k\| \leq \epsilon$  where  $\rho(\mathbf{x}, \mathbf{A}) = \frac{\mathbf{x}^H \mathbf{A} \mathbf{x}}{\mathbf{x}^H \mathbf{x}}$ .  
 $\rho_k$  is the approximation for  $\mathbf{A}$ 's largest eigenvalue.

**Eigenvalue Reduction Problem:** For eigenvalue-eigenvector pair  $\lambda_i, \mathbf{x}_i$  of matrix  $\mathbf{A} \in \mathbb{R}^{n \times n}$  where  $\|\mathbf{x}_i\| = 1$ , find  $\mathbf{B} \in \mathbb{R}^{n \times n}$ , so  $\text{eig}(\mathbf{B}) = \text{eig}(\mathbf{A}) \setminus \{\lambda_i\}$  and  $\lambda_i \rightarrow 0$  i.e.  $\lambda_i$  is replaced by 0.

For symmetric matrices:  $\mathbf{B} = \mathbf{A} - \lambda_i \mathbf{x}_i \mathbf{x}_i^T$

For non-symmetric matrices:  $\mathbf{B} = \mathbf{Q} \mathbf{A} \mathbf{Q}^T$  for orthogonal  $\mathbf{Q}$  such that  $\mathbf{Q} \mathbf{x}_i = k \mathbf{e}_i$ ;  $\mathbf{e}_i$  unit vector corresponding to  $\lambda_i$ .

**Inverse Iteration** Used to find smallest eigenvalue of  $\mathbf{A} \in \mathbb{R}^{n \times n}$ .

**Algorithm:** Use power method to find largest eigenvalue  $\psi_{max}$  of  $\mathbf{A}^{-1}$ ;  $\lambda_{min} = \frac{1}{\psi_{max}}$  is smallest eigenvalue of  $\mathbf{A}$ .

**QR Iteration** To find eigenvalues of non-symmetric matrix  $\mathbf{A} \in \mathbb{R}^{n \times n}$ .

**Algorithm:** Start with  $\mathbf{A}_0 = \mathbf{A}$ . For  $k = 0, 1, \dots$  find QR decomposition  $\mathbf{A}_k = \mathbf{Q}_k \mathbf{R}_k$  and calculate  $\mathbf{A}_{k+1} = \mathbf{R}_k \mathbf{Q}_k$ .

If  $\text{eig}(\mathbf{A}) \in \mathbb{R}$ , then for large  $k$ ,  $\mathbf{A}_k$  becomes upper-triangular and  $\mathbf{A}_k$ 's diagonal entries approach  $\mathbf{A}$ 's eigenvalues.

If  $\mathbf{A}$  has  $m < n$  complex eigenvalues, then for large  $k$ ,  $\mathbf{A}_k$  becomes quasi-upper triangular,  $\mathbf{A}_k$ 's diagonal entries approach  $\mathbf{A}$ 's real eigenvalues, and a  $m \times m$  sub-matrix whose eigenvalues are  $\mathbf{A}$ 's complex eigenvalues remains on  $\mathbf{A}_k$ 's diagonal.

**Symmetric and Tridiagonal Matrices** For all symmetric  $\mathbf{A} \in \mathbb{R}^{n \times n}$  there exists permutation matrix  $\mathbf{P} \in \mathbb{R}^{n \times n}$  such that  $\mathbf{T} = \mathbf{P} \mathbf{A} \mathbf{P}^T$  is tridiagonal.

$\mathbf{T}$  irreducible  $\implies \mathbf{T}$  has no zeros on upper tridiagonal.

$$\mathbf{T} = \begin{bmatrix} a_1 & b_1 & & & & \\ b_1 & a_1 & b_2 & & & \\ & \ddots & \ddots & \ddots & & \\ & & b_{n-2} & a_{n-1} & b_{n-1} & \\ & & & b_{n-1} & a_n \end{bmatrix}$$

$\mathbf{T}_i \equiv \mathbf{T}$ 's  $i \times i$  principle sub-matrix.

**Sturm Sequence**  $f_i(\lambda) = \det(\mathbf{T} - \lambda \mathbf{I}) \equiv \mathbf{T}_i$ 's characteristic polynomial.

Sturm sequence:  $f_{i+1}(\lambda) = (a_{i+1} - \lambda) f_i(\lambda) - b_i^2 f_{i-1}(\lambda)$ ,

$$f_0(\lambda) = 1 \text{ and } f_1(\lambda) = a_1 - \lambda \quad i = 0, \dots, n$$

For  $\lambda_0$ ,  $s(\lambda_0)$  is the number of sign agreements between successive terms in the sequence  $f_i(\lambda_0)$ ,  $i = 0, \dots, n$ .

Adjacent terms with same sign and *interior* zeroes of  $f_i(\lambda_0)$  count as sign agreements.

The number of sign agreements  $s(\lambda_0)$  is the number of  $\mathbf{T}$ 's eigenvalues that are strictly larger than  $\lambda_0$ .

## Polynomial Interpolation

Given  $n$  points  $(x_i, y_i)$ ,  $i = 0, 1, \dots, n$ , all  $x_i$  unique, find a polynomial of degree  $\leq n$  such that  $p(x_i) = y_i$  for all  $i$ .

**Classic Form** Interpolation polynomial:  $p_n(x) = a_n x^n + \dots + a_1 x + a_0$   
 $a_n x_1^n + \dots + a_1 x_1 + a_0 = y_1 \implies \mathbf{V} \mathbf{a} = \mathbf{y}$

$$\vdots$$

$$a_n x_n^n + \dots + a_1 x_n + a_0 = y_n$$

Find parameter vector  $\mathbf{a} \in \mathbb{R}^n$  by solving  $\mathbf{V} \mathbf{a} = \mathbf{y}$ .

**Lagrange Polynomial Interpolation** Lagrange polynomials:  $l_i(x) = \prod_{j \neq i} \frac{x - x_j}{x_i - x_j}$ ,  $l_i(x_j) = \delta_{i,j}$

Interpolation polynomial:  $p_n(x) = \sum_{i=0}^n y_i l_i(x)$

If  $\omega(x) = (x - x_0) \dots (x - x_n)$  then  $l_i(x) = \frac{\omega(x)}{(x - x_i) \omega'(x_i)}$

**Newton Polynomial Interpolation** Divided difference for function  $f$  and points  $(x_i, y_i)$  is the leading coefficient of polynomial  $p_n$  interpolating  $f$  at  $(x_i, y_i)$ .

$$[x_0, \dots, x_k] f = \frac{[x_1, \dots, x_k] f - [x_0, \dots, x_{k-1}] f}{x_k - x_0} \quad [x_k] f = f(x_k)$$

$$\lim_{x_1 \rightarrow x_0} [x_0, x_1] f = \lim_{x_1 \rightarrow x_0} \frac{f(x_1) - f(x_0)}{x_1 - x_0} = f'(x_0)$$

If  $x_0 = \dots = x_n$ , then  $[x_0, \dots, x_n] f = \frac{f^{(n)}(x_0)}{n!}$

Interpolation polynomial:  $p_n(x) = [x_0] f + (x - x_0)[x_0, x_1] f + \dots + (x - x_0) \dots (x - x_{n-1})[x_0, \dots, x_n] f$

**Error Estimation for Lagrange and Newton** When interpolating  $f(x), f'(x), \dots$  on the interval  $I = [a, b]$  at  $m$  points  $x_1, \dots, x_m$  with  $n$ th degree polynomial  $p_n$ .

$$f(x) - p_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!} \omega(x), \quad \xi \in [a, b]$$

$$\omega(x) = (x - x_1)^{k_1} (x - x_2)^{k_2} \dots (x - x_n)^{k_n}$$

$$k_i = 1 \text{ for } f(x_i), k_i = 2 \text{ for } f'(x_i) \text{ and } f''(x_i), \text{ etc.}$$

$$\max_{x \in [a, b]} |f(x) - p_n(x)| \leq \frac{1}{(n+1)!} \max_{\xi \in [a, b]} |f^{(n+1)}(\xi)| \max_{x \in [a, b]} |\omega(x)|$$

## Differential Equations

To convert  $a_n y^{(n)} + \dots + a_1 y' + a_0 y = 0$  to a linear system:

$$\text{Let } \tilde{y}_1 = y, \tilde{y}_2 = y', \dots, \tilde{y}_n = y^{(n-1)}$$

$$\mathbf{Y} = [\tilde{y}_1, \dots, \tilde{y}_n] \quad \mathbf{Y}' = \mathbf{F}(x, \mathbf{Y})$$

Explicit Euler:  $y_{n+1} = y_n + h f(x_n, y_n)$

Implicit Euler:  $y_{n+1} = y_n + h f(x_{n+1}, y_{n+1})$

Trapezoid:  $y_{n+1} = y_n + \frac{h}{2} [f(x_n, y_n) + f(x_{n+1}, y_{n+1})]$

**Runge-Kutta RK2**  $k_1 = h f(x_n, y_n)$

$$k_2 = h f(x_n + \frac{1}{2}h, y_n + \frac{1}{2}k_1)$$

$$y_{n+1} = y_n + k_2 + \mathcal{O}(h^3)$$

**Runge-Kutta RK4**  $k_1 = h f(x_n, y_n)$

$$k_2 = h f(x_n + \frac{1}{2}h, y_n + \frac{1}{2}k_1)$$

$$k_3 = h f(x_n + \frac{1}{2}h, y_n + \frac{1}{2}k_2)$$

$$k_4 = h f(x_n + h, y_n + k_3)$$

$$y_{n+1} = y_n + \frac{1}{6}k_1 + \frac{1}{3}k_2 + \frac{1}{3}k_3 + \frac{1}{6}k_4 + \mathcal{O}(h^5)$$