Solved Exercises in Electromagnetism

Notes from Exercises portion of the third-year undergraduate course *Elektromagnetno Polje* (Electromagnetic Field), led by assistant professor Martin Klanjšek at the Faculty of Mathematics and Physics at the University of Ljubljana in the academic year 2020-2021. Credit for the material covered in these notes is due to professor Klanjšek, while the voice, typesetting, and translation to English in this document are my own. The original exercises in Slovene (without solutions) can be found on the course website.

Disclaimer: This document will inevitably contain some mistakes—both simple typos and legitimate errors. Keep in mind that these are the notes of an undergraduate student in the process of learning the material himself, so take what you read with a grain of salt. If you find mistakes and feel like telling me, I will be grateful and happy to hear from you, even for the most trivial of errors. You can reach me by email, in English, Slovene, or Spanish, at `ejmastnak@gmail.com`.

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1 First Exercise Set

Background Theory: Electric Field of a Charge Distribution

The electric field of a spatial charge distribution with volume charge density $\rho(\boldsymbol{r})$ is

$$\boldsymbol{E}(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0} \iiint_V \frac{\rho(\tilde{\boldsymbol{r}})\,\mathrm{d}^3\tilde{\boldsymbol{r}}}{|\boldsymbol{r}-\tilde{\boldsymbol{r}}|^2} \frac{\boldsymbol{r}-\tilde{\boldsymbol{r}}}{|\boldsymbol{r}-\tilde{\boldsymbol{r}}|},$$

where $\tilde{\boldsymbol{r}}$ is a placeholder variable for integration and $\rho(\tilde{\boldsymbol{r}})\,\mathrm{d}^3\tilde{\boldsymbol{r}}$ is an infinitesimal element of charge at the position $\tilde{\boldsymbol{r}}$.

1.1 Electric Field of a Charged Disk

A charged disk has surface charge density σ and radius a. Find the disk's electric field $E(z)$ along an axis through the disk's center and normal to the disk. Analyze the limit behavior of $E(z)$ for small and large z.

We begin by breaking the disk into concentric rings and integrating over the contributions $\mathrm{d}E$ of each ring, where r and $\mathrm{d}r$ represent the radius and thickness of a ring, respectively. We then further divide each ring into infinitesimal segments with area $r\,\mathrm{d}\phi\,\mathrm{d}r$. Along the perpendicular z axis through the disk's center, a single such segment contributes the electric field

$$\mathrm{d}E_1 = \frac{1}{4\pi\epsilon_0} \frac{\sigma r\,\mathrm{d}\phi\,\mathrm{d}r}{z^2+r^2}.$$

Notice the term $r\,\mathrm{d}\phi\,\mathrm{d}r$ must have units of area to produce charge when multiplied by the surface charge density σ. The term z^2+r^2 is simply the squared distance from the charge element to the z axis.

Next, we recognize the circular symmetry of each ring: both the x and y components of the electric field symmetrically cancel along the z axis, so the electric field only has a z component. We then relate the magnitudes of $\mathrm{d}E$ and $\mathrm{d}E_1$ along the z axis with similar triangles to get the contribution $\mathrm{d}E$ of a ring of radius r at the point $(0,0,z)$:

$$\frac{\mathrm{d}E}{\mathrm{d}E_1} = \frac{z}{\sqrt{z^2+r^2}} \implies \mathrm{d}E = \frac{z}{(z^2+r^2)^{3/2}} \frac{\sigma r\,\mathrm{d}\phi\,\mathrm{d}r}{4\pi\epsilon_0}.$$

We find the total electric field $E(z)$ by integrating over the contributions $\mathrm{d}E$:

$$E(z) = \int \mathrm{d}E = \int_0^{2\pi}\int_0^a \mathrm{d}\phi\,\mathrm{d}r \frac{\sigma z r}{4\pi\epsilon_0(z^2+r^2)^{3/2}} = \frac{\sigma z}{2\epsilon_0}\int_0^a \frac{r\,\mathrm{d}r}{(z^2+r^2)^{3/2}}.$$

We solve the integral with the substitution $u = z^2+r^2$, $\mathrm{d}u = 2r\,\mathrm{d}r$:

$$\begin{aligned} E(z) &= \int_{z^2}^{z^2+a^2} \frac{\mathrm{d}u}{u^{3/2}} = -\frac{\sigma z}{2\epsilon_0}\left(\frac{1}{\sqrt{z^2+a^2}} - \frac{1}{\sqrt{z^2}}\right) \\ &= \frac{\sigma}{2\epsilon_0}\left(1 - \frac{z}{\sqrt{z^2+a^2}}\right) = \frac{\sigma}{2\epsilon_0}\left(1 - \frac{1}{1+\frac{a^2}{z^2}}\right). \end{aligned}$$

For $z \ll a$ (very close to the disk), we have $1 + \frac{a^2}{z^2} \to \infty$ and $\frac{1}{1+\frac{a^2}{z^2}} \to 0$, leaving

$$E(z) \to \frac{\sigma}{2\epsilon_0} \qquad (z \ll a),$$

which is the electric field of an infinite charged plane.

For $z \gg a$ (very far from the disk), we have $\frac{a^2}{z^2} \ll 1$ and use the Taylor approximation $(1+x)^p \approx 1 + px$ for $x \ll 1$ to get

$$E(z) = \frac{\sigma}{2\epsilon_0}\left[1 - \left(1 + \frac{a^2}{z^2}\right)^{-1/2}\right] \approx \frac{\sigma}{2\epsilon_0}\left[1 - \left(1 - \frac{a^2}{2z^2}\right)\right] = \frac{\sigma a^2}{4\epsilon_0 z^2}.$$

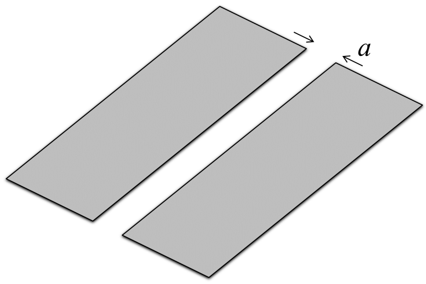
Finally, we multiply above and below by π to match the above result to the expression for a point charge:

$$E(z) = \frac{\pi a^2 \sigma}{4\pi\epsilon_0 z^2} = \frac{\sigma S}{4\pi\epsilon_0 z^2} = \frac{q}{4\pi\epsilon_0 z^2} \qquad (z \gg a),$$

where $q = \sigma S$ is the disk's charge.

1.2 Charged Plate with a Slit

We take a large, rectangular charged plate with surface charged density σ and remove a slit of width a from the plate. Determine the electric field E in the plane perpendicular to the plate and passing through the center of the slit as a function of the vertical distance z from the plate. Analyze the limit behavior of $E(z)$ for small and large z.



We begin by breaking the plate into thin ribbons and integrating over the contributions $\mathrm{d}E$ of each ribbon, where r and $\mathrm{d}r$ will represent the orthogonal distance from the slit and the thickness of each ribbon, respectively.

We find the electric field of a ribbon a distance r from the slit along the z axis using Gauss's law with a cylindrical surface. For a cylinder of radius R and length l, Gauss's law reads

$$\oiint_S \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{S} = 2\pi R l E = \frac{q_{\text{enc}}}{\epsilon_0} \implies E(R) = \frac{q_{\text{enc}}}{2\pi\epsilon_0 l R}.$$

Applied to the ribbon, the enclosed charge q_{enc} is the ribbon's infinitesimal charge $\mathrm{d}q_1 = \sigma\,\mathrm{d}S_1 = \sigma l\,\mathrm{d}r$, while the cylinder's radius R is the distance from the ribbon to the z axis: $R = \sqrt{z^2 + r^2}$, so the contribution $\mathrm{d}E_1$ of one ribbon is

$$\mathrm{d}E_1 = \frac{\mathrm{d}q_1}{2\pi\epsilon_0 l\sqrt{z^2+r^2}} = \frac{\sigma l\,\mathrm{d}r}{2\pi\epsilon_0 l\sqrt{z^2+r^2}} = \frac{\sigma\,\mathrm{d}r}{2\pi\epsilon_0\sqrt{z^2+r^2}}.$$

Because of mirror-image symmetry, both the x and y components of the electric field cancel, leaving only the z component $\mathrm{d}E_z$. We then relate $\mathrm{d}E_z$ and $\mathrm{d}E_1$ using similar triangles

$$\frac{\mathrm{d}E_z}{\mathrm{d}E_1} = \frac{z}{\sqrt{z^2+r^2}} \implies \mathrm{d}E_z = \frac{\sigma z\,\mathrm{d}r}{2\pi\epsilon_0(z^2+r^2)}.$$

We find the total electric field along the z axis by integrating over the contributions $\mathrm{d}E_z$ of all the ribbons. Because of mirror symmetry, we need to calculate only the contribution of e.g. the right plane and multiple the result by two.

$$\begin{aligned} E(z) &= \int \mathrm{d}E_z = 2\int_{a/2}^{\infty} \left(\frac{\sigma z \, \mathrm{d}r}{2\pi\epsilon_0 (z^2 + r^2)} \right) = \frac{\sigma z}{\pi \epsilon_0} \int_{a/2}^{\infty} \frac{\mathrm{d}r}{z^2 + r^2} \\ &= \frac{\sigma z}{\pi\epsilon_0} \left[\frac{1}{z} \arctan \frac{r}{z} \right]_{a/2}^{\infty} = \frac{\sigma}{\pi\epsilon_0} \left[\frac{\pi}{2} - \arctan\left(\frac{a}{2z} \right) \right]. \end{aligned}$$

In the limit $z \gg a$ (very far from the slit), we have $\arctan \frac{a}{2z} \to 0$ and the electric field along the z axis simplifies to

$$E(z) = \frac{\sigma}{2\epsilon_0} \qquad (z \gg a),$$

which is the field of an infinite sheet of charge.

In the limit $z \ll a$ (very close to the slit), we have $\frac{a}{2z} \to \infty$. We use the asymptotic expansion $\arctan x \approx \frac{\pi}{2} - \frac{1}{x}$ for large x to get

$$E(z) \approx \frac{\sigma}{\pi\epsilon_0} \left[\frac{\pi}{2} - \left(\frac{\pi}{2} - \frac{2z}{a} \right) \right] = \frac{2\sigma}{\pi\epsilon_0} \frac{z}{a}.$$

In this case the electric field scales linearly as $E \sim z$.

2 Second Exercise Set

2.1 Theory: The Poisson Equation and the Fourier Transform

We begin with Gauss's law in differential form and the relationship between electric field $\boldsymbol{E}$ and electric potential U

$$\nabla \cdot \boldsymbol{E} = \frac{\rho}{\epsilon_0} \qquad \text{and} \qquad \boldsymbol{E} = -\nabla U$$

We then substitute $\boldsymbol{E} = -\nabla U$ into Gauss's law to get

$$\nabla \cdot \big[-\nabla U\big] = \frac{\rho}{\epsilon_0} \implies \nabla^2 U = -\frac{\rho}{\epsilon_0}.$$

The last equality has the form of a *Poisson equation*,[1] and relates charge density ρ to electric potential U. In other words, if we know a spatial charge distribution ρ, we can find the corresponding electric potential U and thus the electric field $\boldsymbol{E}$ with $\boldsymbol{E} = -\nabla U$.

As a simple example we start with a point charge. The charge distribution is

$$\rho(\boldsymbol{r}) = q\delta(\boldsymbol{r}) \implies \nabla^2 U = -\frac{\rho}{\epsilon_0} = -\frac{q}{\epsilon_0}\delta(\boldsymbol{r}).$$

The above equation is the Poisson equation for a point charge. We will solve the equation with a Fourier transform.

First, we briefly review the Fourier transform, which we can think of as an expansion over a basis of plane waves of the form $e^{i\boldsymbol{k}\cdot\boldsymbol{r}}$. The expression $U(\boldsymbol{k})$ plays the role of a weight function and determines how much each wave contributes to the expansion

$$U(\boldsymbol{r}) = \iiint U(\boldsymbol{k}) e^{i\boldsymbol{k}\cdot\boldsymbol{r}} \,\mathrm{d}^3 k,$$

where $U(\boldsymbol{k})$ is the amplitude of the plane wave with wave vector $\boldsymbol{k}$. To find $U(\boldsymbol{k})$ we take the inner product of both sides of the above equation with the basis function $e^{-i\tilde{\boldsymbol{k}}\cdot\boldsymbol{r}}$, which gives

$$\iiint U(\boldsymbol{r}) e^{-i\tilde{\boldsymbol{k}}\cdot\boldsymbol{r}} \,\mathrm{d}^3 r = \iiint \iiint \mathrm{d}^3 k \,\mathrm{d}^3 r U(\boldsymbol{k}) e^{i(\boldsymbol{k}-\tilde{\boldsymbol{k}})\cdot\boldsymbol{r}}.$$

The integral over $\boldsymbol{r}$ on the right-hand side is in fact a delta function, because the orthogonal plane waves cancel out over all space except at the origin, where they constructively interfere to infinity. Recognizing the delta function simplifies the equation to

$$\iiint U(\boldsymbol{r}) e^{-i\tilde{\boldsymbol{k}}\cdot\boldsymbol{r}} \,\mathrm{d}^3 r = (2\pi)^3 \iiint U(\boldsymbol{k}) \delta(\boldsymbol{k} - \tilde{\boldsymbol{k}}) \,\mathrm{d}^3 k = (2\pi)^3 U(\tilde{\boldsymbol{k}}).$$

The delta function suppresses the integral everywhere except at $(\boldsymbol{k} - \tilde{\boldsymbol{k}})$, which leads to the expression for the amplitude $U(\tilde{\boldsymbol{k}})$ of a wave vector $\tilde{\boldsymbol{k}}$:

$$U(\tilde{\boldsymbol{k}}) = \frac{1}{(2\pi)^3} \iiint U(\boldsymbol{r}) e^{-i\boldsymbol{k}\cdot\boldsymbol{r}} \,\mathrm{d}^3 r.$$

[1] In general any equation of the form $\nabla^2 f(\boldsymbol{r}) = g(\boldsymbol{r})$ is called a Poisson equation.

This expression can also be interpreted as an inverse Fourier transform, used to recover $U(\boldsymbol{k})$ from $U(\boldsymbol{r})$.

As an intermediate step, we analyze the behavior of the gradient operator under the Fourier transform. Since the gradient acts only on $\boldsymbol{r}$, applying the gradient to the earlier expression for $U(\boldsymbol{r})$ gives

$$\nabla U(\boldsymbol{r}) = \iiint U(\boldsymbol{k}) \nabla e^{i\boldsymbol{k}\cdot\boldsymbol{r}} \, \mathrm{d}^3 k.$$

Evaluating the gradient over (x, y, z) components gives

$$\nabla e^{i\boldsymbol{k}\cdot\boldsymbol{r}} = \begin{bmatrix} ikx \\ iky \\ ikz \end{bmatrix} e^{i(k_1 x + k_2 y + k_3 z)} = ike^{i\boldsymbol{k}\cdot\boldsymbol{r}}.$$

The expression for $\nabla U(\boldsymbol{r})$ is thus

$$\nabla U(\boldsymbol{r}) = \iiint U(\boldsymbol{k}) \nabla e^{i\boldsymbol{k}\cdot\boldsymbol{r}} \, \mathrm{d}^3 k = \iiint U(\boldsymbol{k}) i\boldsymbol{k} e^{i\boldsymbol{k}\cdot\boldsymbol{r}} \, \mathrm{d}^3 k,$$

which is a Fourier transform of the function $U(\boldsymbol{k})i\boldsymbol{k}$. In other words, the gradient operator ∇ transforms to multiplication by $i\boldsymbol{k}$ under the Fourier transform. Analogously, the Laplacian ∇^2 transform into multiplication by $(i\boldsymbol{k})^2 = -k^2$.

Finally, we consider the behavior of the delta function under the Fourier transform. Let $\delta(\boldsymbol{k})$ denote the amplitude in the expansion of $\delta(\boldsymbol{r})$, analogous to the relationship between $U(\boldsymbol{k})$ and $U(\boldsymbol{r})$. Using the inverse Fourier transform and the integral properties of the delta function produces

$$\delta(\boldsymbol{k}) = \frac{1}{(2\pi)^3} \iiint \delta(\boldsymbol{r}) e^{-i\boldsymbol{k}\cdot\boldsymbol{r}} \, \mathrm{d}^3 r = \frac{1}{(2\pi)^3} e^{-i\boldsymbol{k}\cdot 0} = \frac{1}{(2\pi)^3}.$$

In other words, the delta function transforms to $1/(2\pi)^3$ in Fourier space.

Recipe: We now state the recipe for solving the Poisson equation in Fourier space. Take the Fourier transform the Poisson equation from $\boldsymbol{r}$ into $\boldsymbol{k}$ space (where the Laplacian operator ∇^2 simplifies to $-ik^2$ under the Fourier transform) and solve for the amplitude $U(\boldsymbol{k})$ of each plane wave $e^{-i\boldsymbol{k}\cdot\boldsymbol{r}}$. Substitute $U(\boldsymbol{k})$ into the Fourier transform,

$$U(\boldsymbol{r}) = \iiint U(\boldsymbol{k}) e^{i\boldsymbol{k}\cdot\boldsymbol{r}} \, \mathrm{d}^3 k,$$

and evaluate the integral—typically in spherical coordinates—to find find $U(\boldsymbol{r})$.

2.2 Poisson Equation for a Point Particle

The Poisson equation for a point particle is

$$\nabla^2 U(\boldsymbol{r}) = -\frac{q}{\epsilon_0} \delta(\boldsymbol{r}).$$

Solve the equation for $U(\boldsymbol{r})$.

The plan is to transform into $\boldsymbol{k}$ space, solve for $U(\boldsymbol{k})$, then transform back to $U(\boldsymbol{r})$.

First, we take the Fourier transform of both sides—use the Fourier transform identities $\nabla^2 \to -k^2$ and $\delta(\boldsymbol{r}) \to \frac{1}{(2\pi)^3}$ from the theory section.

$$-k^2 U(\boldsymbol{k}) = -\frac{q}{\epsilon_0}\frac{1}{(2\pi)^3} \implies U(\boldsymbol{k}) = \frac{q}{(2\pi)^3 \epsilon_0 k^2}.$$

Next, we find $U(\boldsymbol{r})$ using a second Fourier transform

$$U(\boldsymbol{r}) = \iiint \mathrm{d}^3 k \frac{q e^{i\boldsymbol{k}\cdot\boldsymbol{r}}}{k^2 \epsilon_0 (2\pi)^3} = \frac{q}{(2\pi)^3 \epsilon_0} \iiint \frac{e^{i\boldsymbol{k}\cdot\boldsymbol{r}}}{k^2}\,\mathrm{d}^3 k.$$

To solve the integral, we introduce an angle θ between $\boldsymbol{r}$ and $\boldsymbol{k}$ and transform to spherical coordinates to get

$$U(\boldsymbol{r}) = \frac{q}{(2\pi)^3 \epsilon_0} \int_0^{2\pi} \mathrm{d}\phi \int_{-1}^{1} \mathrm{d}[\cos\theta] \int_0^\infty \mathrm{d}k k^2 \frac{e^{ikr\cos\theta}}{k^2}.$$

Integration over the azimuthal angle ϕ is simple, and produces 2π:

$$U(\boldsymbol{r}) = \frac{q}{(2\pi)^2 \epsilon_0} \int_{-1}^{1} \int_0^\infty e^{i\cos\theta kr}\,\mathrm{d}[\cos\theta]\,\mathrm{d}k.$$

Next, we integrate first over θ, (to avoid $e^{i\cos\theta\cdot\infty}$ from the upper k limit) and recognize the sine function in the difference of exponential functions:

$$\begin{aligned} U(\boldsymbol{r}) &= \frac{q}{(2\pi)^2 \epsilon_0} \int_0^\infty \frac{e^{i\cos\theta kr}}{ikr}\bigg|_{\theta=-1}^{1} \mathrm{d}k = \frac{q}{(2\pi)^2 \epsilon_0} \int_0^\infty \frac{e^{ikr} - e^{-ikr}}{ikr}\,\mathrm{d}k \\ &= \frac{q}{(2\pi)^2 \epsilon_0} \int_0^\infty \frac{2\sin(kr)}{kr}\,\mathrm{d}k = \frac{2q}{(2\pi)^2 \epsilon_0} \int_0^\infty \operatorname{sinc}(\kappa r)\,\mathrm{d}k \end{aligned}$$

The integral of the sinc function is

$$\int_0^\infty \frac{\sin x}{x}\,\mathrm{d}x = \frac{\pi}{2},$$

and applying this integral to the expression for $U(\boldsymbol{r})$ gives

$$U(\boldsymbol{r}) = \frac{2q}{(2\pi)^2 \epsilon_0}\frac{\pi}{2r} = \frac{q}{4\pi\epsilon_0 r},$$

which is the electric potential of a point charge. Note that we have derived this expression directly from Maxwell's equations, rather than taking it for granted as in introductory electromagnetism courses.

Finally, we substitute the result for $U(\boldsymbol{r})$ into the Poisson equation for a point charge. The result is

$$\nabla^2 \frac{q}{4\pi\epsilon_0 r} = -\frac{q}{\epsilon_0}\delta(\boldsymbol{r}) \implies \nabla^2 \frac{1}{r} = -4\pi\delta(\boldsymbol{r}),$$

which will be useful in the next problems.

2.3 Theoretical Interlude: Electric Field of a Charge Distribution

We just solved the Poisson equation for the simple case $\rho(\boldsymbol{r}) = \delta(\boldsymbol{r})$. Can we use this result to solve the general case $\rho = \rho(\boldsymbol{r})$? The answer is yes, if we expand $\rho(\boldsymbol{r})$ over a basis of delta functions, as follows:

$$\rho(\boldsymbol{r}) = \iiint \mathrm{d}^3\tilde{r}\rho(\tilde{\boldsymbol{r}})\delta(\boldsymbol{r} - \tilde{\boldsymbol{r}}).$$

In this case, the solution of $U(\boldsymbol{r})$ to the Poisson equation is

$$U(\boldsymbol{r}) = \iiint \mathrm{d}^3\tilde{r}\rho(\tilde{\boldsymbol{r}})\frac{1}{4\pi\epsilon_0|\boldsymbol{r} - \tilde{\boldsymbol{r}}|} = \frac{1}{4\pi\epsilon_0}\iiint \frac{\mathrm{d}^3\tilde{r}\rho(\tilde{\boldsymbol{r}})}{|\boldsymbol{r} - \tilde{\boldsymbol{r}}|}.$$

This result is quite powerful—by solving the Poisson equation for a delta function and then expanding an arbitrary $\rho(\boldsymbol{r})$ in terms of the delta function, we now have the solution to the Poisson equation for any $\rho(\boldsymbol{r})$. We find the corresponding electric field with

$$\boldsymbol{E} = -\nabla U = \frac{1}{4\pi\epsilon_0}\iiint \frac{\mathrm{d}^3\tilde{r}\rho(\tilde{\boldsymbol{r}})}{|\boldsymbol{r} - \tilde{\boldsymbol{r}}|^2}\frac{\boldsymbol{r} - \tilde{\boldsymbol{r}}}{|\boldsymbol{r} - \tilde{\boldsymbol{r}}|},$$

which agrees with the equation quoted in the previous exercise set.

2.4 Electric Field of a Hydrogen Atom

The hydrogen atom has the electric potential

$$U(\boldsymbol{r}) = \frac{q}{4\pi\epsilon_0}\frac{e^{-\alpha r}}{r}\left(1 + \frac{\alpha r}{2}\right), \qquad \alpha = \frac{2}{r_B}.$$

Find the charge density $\rho(\boldsymbol{r})$ that generates this potential.

We use the Poisson equation, which connects U and ρ via

$$\nabla^2 U(\boldsymbol{r}) = -\frac{\rho(\boldsymbol{r})}{\epsilon_0}.$$

We then calculate the Laplacian of our $U(\boldsymbol{r})$ and work in spherical coordinates, since the potential is spherically symmetric (depends only on r). As a review, when acting on a function that depends only on r, ∇^2 in spherical coordinates reads

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right).$$

Applying ∇^2 to $U(r)$, after some straightforward but rather tedious differentiation, leads to

$$\nabla^2 U(\boldsymbol{r}) = \frac{q\alpha^3}{8\pi\epsilon_0}e^{-\alpha r}.$$

Rearranging the Poisson equation then gives

$$\rho(\boldsymbol{r}) = -\frac{q\alpha^3}{8\pi}e^{-\alpha r}.$$

Note that the charge density is negative, which corresponds to the negatively charged electron cloud. Inserting the definition of $\alpha = \frac{2}{r_B}$ gives

$$\rho(\boldsymbol{r}) = -\frac{q}{\pi r_B^3} e^{-\frac{2r}{r_B}}.$$

Another interpretation: $e^{-\frac{2r}{r_B}}$ is equivalent to $\left(e^{-\frac{r}{r_B}}\right)^2$, which is the square of the hydrogen atom's ground state wave function. The square of the wave function is probability, and multiplying the probability by $\frac{q}{r_B^3}$ gives a charge density.

Note that—incorrectly—the proton's charge does not contribute to our expression for $\rho(r)$. This is because the proton occurs at the origin, which corresponds to a charge density singularity at the origin. To avoid dealing with this singularity, we simply ignored it when evaluating the Laplacian ∇^2

We can resolve this problem by separately considering the special case

$$\lim_{r \to 0} U(\boldsymbol{r}) = \frac{q}{4\pi\epsilon_0 r}.$$

which just says the that potential should approach the $\sim 1/r$ proton potential at the origin. We would then have to solve the Poisson equation for this potential. However, we already now the solution—the potential is the potential for a point charge and corresponds to a charge density

$$\rho(\boldsymbol{r}) = q\delta(\boldsymbol{r}).$$

The correct total result for the hydrogen atom is the sum of the electron cloud result and the charge density of the proton nucleus:

$$\rho(\boldsymbol{r}) = q\delta(\boldsymbol{r}) - \frac{q\alpha^3}{8\pi} e^{-\alpha r}.$$

Lesson: Be careful when working with the Poisson equation if $U(r)$ has singularities!

3 Third Exercise Set

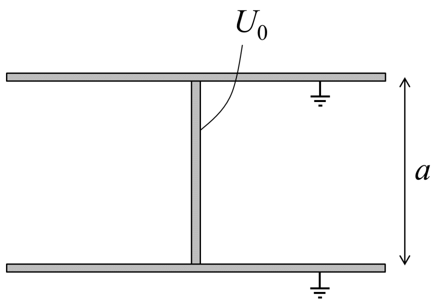
Theory: The Laplace Equation As a quick review from the last exercise set: the Poisson equation used to solve for the electric field potential generated by a charge density ρ is

$$\nabla^2 U(\boldsymbol{r}) = -\frac{\rho(\boldsymbol{r})}{\epsilon_0} \qquad \text{where} \qquad \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}.$$

Often $\rho(\boldsymbol{r}) = 0$ in places we're solving for the electric potential. In this case $\nabla^2 U(\boldsymbol{r}) = 0$. This equation is called a *Laplace equation.*

3.1 Conducting Ribbon in a Parallel-Plate Capacitor

We place a long, thin conducting ribbon between the plates of a large parallel-plate capacitor, perpendicularly to the plates; the ribbon almost touches both plates with a small, insulating layer of air between the ribbon edges and the plates. The ribbon height and distance between the plates is a. We ground both plates and set the ribbon potential to U_0. What is the resulting electric potential inside the capacitor?



First, we decide on a coordinate system—we will use a Cartesian coordinates because of the problem's rectangular symmetry. We choose the y axis to vertically connect the two ribbons (along the same line as a in the figure), the x axis to run from left to right in the plane of the page, and the z axis to point out of the page.

Note that the problem is independent of z (by translation symmetry in the z direction), so we only need $U(x, y)$. More so, the problem has reflection symmetry, so we can find the solution on only one side of the ribbon (one half of the x axis) and reflect the solution about the y axis.

The space between the capacitor plates is empty—there is no charge, and we use the Laplace equation for the space between the plates.

$$\nabla^2 U(x, y) = 0.$$

Charge can occur only along the ribbon or on the capacitor plates.

Next, we determine boundary conditions for $U(x, y)$ so the equation has a unique solution; the problem's boundaries are the ribbon and edges of the capacitor plates.

On the bottom plate, $U(x, 0) = 0$. On the upper plate, $U(x, a) = 0$; both potentials are zero because the plates are grounded, while along the ribbon we have $U(0, y) = U_0$. We need one more boundary—infinity. At infinity, we require only that $U(x \to \infty, y)$ is bounded, i.e. that U does not diverge at ∞.

First, we attempt solving the problem with separation of variables and write $U(x, y) = X(x)Y(y)$—this approach tends to work well with symmetric problems. Substituting this ansatz into the Laplace equation and evaluating ∇^2 gives

$$X^{''}Y + XY^{''} = 0 \implies \frac{X''}{X} = -\frac{Y''}{Y}.$$

The separation of variables is successful—we succeeded in isolating x-dependent and y-dependent terms on different sides of the equation.

We then set the equation equal to a separation constant κ^2 and get two equations,

$$X'' - \kappa^2 X = 0 \qquad \text{and} \qquad Y'' + \kappa^2 Y = 0.$$

Both equations have simple solutions! The equations for X and Y are solved by exponential and sinusoidal functions, respectively.

$$X(x) = Ae^{\kappa x} + Be^{-\kappa x} \qquad \text{and} \qquad Y(y) = C\sin(\kappa y) + D\cos(\kappa y).$$

We then substitute the expressions for X and Y back into the ansatz $U = XY$:

$$U(x, y) = X(x)Y(y) = \left(Ae^{\kappa x} + Be^{-\kappa x}\right)\left(C\sin(\kappa y) + D\cos(\kappa y)\right).$$

We find the coefficients A, B, C and D using the boundary conditions.

- We begin with the most powerful condition, that $U(x \to \infty, y)$ is bounded. This condition implies $A = 0$ to suppress the divergent exponential function e^{κ}.
- Then, we use the next two simplest conditions, the ones requiring $U(x, y) = 0$. Beginning with $U(x, 0) = 0$, we have

  $$0 \equiv U(x, 0) = 1 \cdot (0 + D) \implies D = 0.$$

  With both $A = D = 0$, we're left at this point with only

  $$U(x, y) = Be^{-\kappa x} \cdot C\sin(\kappa y).$$

- Next, we apply the condition $U(x, a) = 0$ to get

  $$0 \equiv U(x, a) = Be^{-\kappa x}C\sin(\kappa a) \equiv Fe^{-\kappa x}\sin(\kappa a),$$

  where we've joined the product of two constants into a single constant $F = BC$.

  We have two options: either $F = 0$ or $\sin(\kappa a) = 0$. The option $F = 0$ gives the trivial solution $U(x, y) = 0$. The non-trivial solution comes from

  $$\sin(\kappa a) = 0 \implies \kappa a = n\pi, \quad n = 1, 2, 3, \ldots.$$

  Note that $n = 0$ leads to a trivial solution $U(x, y) = 0$, which we reject.

  With respect to the quantization of κ and F by the index n, the general solution at this point is the linear superposition

  $$U(x, y) = \sum_{n=1}^{\infty} F_n e^{-\kappa_n x}\sin(\kappa_n y).$$

- To find F_n, we use the final boundary condition $U(0,y) = U_0$.

  $$U_0 = \sum_{n=1}^{\infty} F_n \sin(\kappa_n y) = \sum_{n=1}^{\infty} F_n \sin\Big(\frac{n\pi y}{a}\Big),$$

  where we've used $\kappa_n = \frac{n\pi}{a}$. This is a Fourier expansion of the constant U_0 over sine functions.

  We find the coefficients by taking the inner product of both sides of the equation, which amounts to multiplying both sides by $\sin\frac{m\pi y}{a}$ and integrating both sides over y from 0 to a. The left side of the equation for U_0 becomes

  $$U_0 \int_0^a \sin\Big(\frac{m\pi y}{a}\Big)\,\mathrm{d}y = -\frac{U_0 a}{m\pi}\cos\Big(\frac{m\pi y}{a}\Big)\bigg|_0^a = \frac{U_0 a}{m\pi}\big[1-(-1)^m\big].$$

  On the right hand side, with the sum, we switch the sum and integral to get

  $$\sum_{n=1}^{\infty} F_n \int_0^a \sin\Big(\frac{n\pi y}{a}\Big)\sin\Big(\frac{m\pi y}{a}\Big)\,\mathrm{d}y = \sum_{n=1}^{\infty} F_n \delta_{mn} \int_0^a \sin^2\Big(\frac{m\pi y}{a}\Big)\,\mathrm{d}y = \frac{F_m a}{2}.$$

  Because of the orthogonality of the sine functions, the integral is zero for $m \neq n$, and only the case $m = n$ gives a non-zero result. Equating the two sides gives the desired expression for F_m:

  $$\frac{U_0 a}{m\pi}\big[1-(-1)^m\big] = \frac{F_m a}{2} \implies F_m = \frac{2U_0}{m\pi}\big[1-(-1)^m\big].$$

With F_m known, the final result for $U(x,y)$ is then

$$U(x,y) = \frac{2U_0}{\pi}\sum_{n=1}^{\infty}\frac{1-(-1)^n}{n}\exp\Big(-\frac{n\pi x}{a}\Big)\sin\Big(\frac{n\pi y}{a}\Big).$$

Some limit cases: for $x \gg a$, the exponent terms very small, and we can neglect all terms in the series except the first term $e^{-\frac{\pi x}{a}}$ with $n = 1$. The result is

$$U(x,y) = \frac{4U_0}{\pi}\exp\Big(-\frac{\pi x}{a}\Big)\sin\Big(\frac{\pi y}{a}\Big).$$

A separate case for which we can find a nice analytic solution is at the center of the capacitor at $y = \frac{a}{2}$. The solution reads

$$U(x,\tfrac{a}{2}) = \frac{2U_0}{\pi}\sum_{n=1}^{\infty}\frac{1-(-1)^n}{n}\exp\Big(-\frac{n\pi x}{a}\Big)\sin\Big(\frac{n\pi}{2}\Big).$$

Instead of finding $U(x,\frac{a}{2})$, we'll find the electric field $\boldsymbol{E}(x,\frac{a}{2})$. Because of reflection symmetry across the line $y = \frac{a}{2}$, the electric field cannot have a y component—$\boldsymbol{E}$ only has an x component. We'll find $E_x(x)$ from the potential:

$$E_x(x) = -\frac{\partial}{\partial x}U(x,\tfrac{a}{2}) = -\frac{2U_0}{a}\sum_{n=1}^{\infty}\big[1-(-1)^n\big]\exp\Big(-\frac{n\pi x}{a}\Big)\sin\Big(\frac{n\pi}{2}\Big).$$

Next, note that

$$\left[1-(-1)^n\right]\sin\left(\frac{n\pi}{2}\right)=\begin{cases}0 & n \text{ even}\\ 2 & n=1,5,9,\ldots\\ -2 & n=3,7,11,\ldots.\end{cases}$$

The sum simplifies to

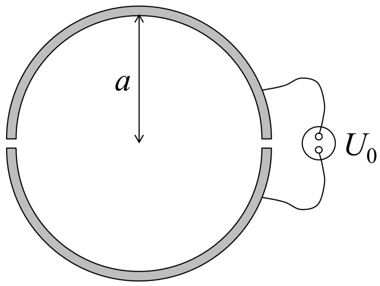
$$E_x(x)=\frac{4U_0}{a}\left[e^{-\frac{\pi x}{a}}-e^{-\frac{3\pi x}{a}}+e^{-\frac{5\pi x}{a}}\mp\ldots\right]=\frac{4U_0}{a}e^{-\frac{\pi x}{a}}\left[1-e^{-\frac{2\pi x}{a}}+\left(e^{-\frac{2\pi x}{a}}\right)^2\mp\ldots\right],$$

which is a geometric series in $e^{-\frac{2\pi x}{a}}$. The result is

$$E_x(x)=\frac{4U_0}{a}\frac{e^{-\frac{\pi x}{a}}}{1+e^{-\frac{2\pi x}{a}}}=\frac{4U_0}{a}\frac{1}{e^{\frac{\pi x}{a}}+e^{-\frac{\pi x}{a}}}=\frac{2U_0}{a\cosh\left(\frac{\pi x}{a}\right)}.$$

3.2 A Halved Conducting Cylinder

Consider a long cylinder of radius a cut in half along a plane running along the cylinder's longitudinal axis. We separate the two cylinder halves by an arbitrarily small amount (so the halves are insulated) and apply a potential difference U_0 between the two halves. The halved cylinder acts as a capacitor. Find the electric potential inside the cylinder.



We must first decide on a coordinate system, and will use cylindrical coordinates because the problem has cylindrical symmetry. Let the z axis run along the cylinder's longitudinal axis. Because of translational symmetry along the z axis, U is independent of z.

There is no charge inside the cylinder, so the electric potential in the cylinder obeys the Laplace equation

$$\nabla^2 U(r,\phi)=0.$$

(I have used U for electric potential and ϕ for the cylindrical coordinate system's angular coordinate.) In cylindrical coordinates (when acting on a function independent of z), the Laplacian operator reads

$$\nabla^2=\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2}{\partial\phi^2}.$$

In our case,

$$\nabla^2 U(r,\phi)=\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial U}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 U}{\partial\phi^2}=0.$$

Following the usual recipe, we separate variables using the ansatz $U(r,\phi)=R(r)\Phi(\phi)$. Substituting this ansatz into the Laplace equation leads to

$$\Phi\frac{1}{r}(rR')'+\frac{R}{r^2}\Phi''=\Phi\left(\frac{R'}{r}+R''\right)+\frac{R}{r^2}\Phi''=0.$$

Note that $r' = 1$. Dividing through by Φ and rearranging gives

$$\frac{rR'}{R} + r^2 \frac{R''}{R} = -\frac{\Phi''}{\Phi}.$$

Following the usual separation procedure, we set both sides equal to the separation constant m^2. The equations for Φ and R read

$$\Phi'' + m^2\Phi = 0 \qquad \text{and} \qquad r^2 R'' + rR' - m^2 R = 0.$$

The solution for Φ is sinusoidal:

$$\Phi(\phi) = A\sin(m\phi) + B\cos(m\phi).$$

Now, our cylindrical problem is periodic in ϕ with period 2π—this just means the cylinder repeats after one revolution. Periodicity in ϕ is possible only if m takes on integer values, so we can immediately index the solutions for Φ with

$$\Phi_m(\phi) = A_m \sin(m\phi) + B_m \cos(m\phi), \quad m = 1, 2, 3, \ldots.$$

We only use positive integers because the odd/even symmetry of sin and cos means negative integers give the same result as positive one—solving for negative m would be redundant. We reject $m = 0$ because this solution leads to $\Phi'' = 0$, meaning Φ is a linear function. But a linear function can't be periodic in ϕ, so we reject $m = 0$.

The second equation for R is solved with powers of r. The result is

$$R_m(r) = C_m r^m + D_m r^{-m}.$$

The general solution is the linear superposition

$$U(r,\phi) = \sum_{m=1}^{\infty} \Phi_m(\phi) R_m(m) = \sum_{m=1}^{\infty} \left(A_m \sin(m\phi) + B_m \cos(m\phi)\right) \left(C_m r^m + D_m r^{-m}\right).$$

Note: We ended the problem at this point (ran out of time) and continued in the fourth exercise set.

4 Fourth Exercise Set

4.1 A Halved Conducting Cylinder (continued)

We left off in the previous exercise set with the general solution

$$\begin{aligned} U(r,\phi) &= \sum_{m=1}^{\infty} \Phi_m(\phi) R_m(m) \\ &= \sum_{m=1}^{\infty} \left(A_m \sin(m\phi) + B_m \cos(m\phi)\right)\left(C_m r^m + D_m r^{-m}\right) \end{aligned}$$

for the potential inside the cylinder. To find a solution specific to our problem, we apply boundary conditions. We already applied the periodic boundary condition $U(r,\phi) = U(r,\phi+2\pi)$, which required m be integer-valued.

A second boundary condition requires the capacitor halves have a potential difference U_0 between them. It is best to write this potential difference in the symmetric form

$$U(a,\phi) = \begin{cases} \frac{U_0}{2} & \phi \in (0,\pi) \\ -\frac{U_0}{2} & \phi \in (\pi, 2\pi). \end{cases}$$

There is another condition—that U does not diverge at $r=0$. This condition implies the D_m coefficients are zero, because the $D_m r^{-m}$ term diverges at $r=0$.

Observation: the second boundary condition is an odd function of ϕ. This implies that only odd (sine) terms can appear in the final solution. This allows us to set the A_m coefficients equal to zero to eliminate the cosine terms. We are left with

$$U(r,\phi) = \sum_{m=1}^{\infty} F_m r^m \sin(m\phi),$$

where we have defined $B_m C_m \equiv F_m$.

Applying the second boundary condition at $r=a$ gives

$$U(a,\phi) = \sum_{m=1}^{\infty} F_m a^m \sin(m\phi).$$

To solve this, we take the scalar product of the equation with $\sin(n\phi)$:

$$\int_0^{2\pi} U(a,\phi)\sin(n\phi)\,\mathrm{d}\phi = \int_0^{2\pi} \sum_{m=1}^{\infty} F_m a^m \sin(m\phi)\sin(n\phi)\,\mathrm{d}\phi.$$

Plugging in the step values of $U(a,\phi)$ gives

$$\frac{U_0}{2}\int_0^{\pi} \sin(n\phi)\,\mathrm{d}\phi - \frac{U_0}{2}\int_{\pi}^{2\pi} \sin(n\phi)\,\mathrm{d}\phi = \int_0^{2\pi} \sum_{m=1}^{\infty} F_m a^m \sin(m\phi)\sin(n\phi)\,\mathrm{d}\phi.$$

First, we solve the left-hand side

$$\begin{aligned} &\frac{U_0}{2}\left[-\frac{1}{n}\cos(n\phi)\Big|_0^{\pi} + \frac{1}{n}\cos(n\phi)\Big|_{\pi}^{2\pi}\right] = \frac{U_0}{2n}\left[-\cos(\pi n) + 1 + 1 - \cos(n\pi)\right] \\ &\quad = \frac{U_0}{n}\left(1 - (-1)^n\right), \end{aligned}$$

where we've used the identity $\cos(n\pi) = (-1)^n$.

Next, we solve the right-hand side. Switching the order of integration and summation gives

$$\sum_{m=1}^{\infty} F_m a^m \int_0^{2\pi} \sin(m\phi)\sin(n\phi)\,\mathrm{d}\phi = \sum_{m=1}^{\infty} F_m a^m \left(\frac{2\pi}{2}\delta_{mn}\right) = F_n a^n \pi.$$

Combining the left and right sides gives

$$F_n = \frac{U_0\big[1-(-1)^n\big]}{\pi n a^n}.$$

So, the solution for $U(r,\phi)$ is

$$U(r,\phi) = \sum_{n=1}^{\infty}\left[\frac{U_0\big[1-(-1)^n\big]}{\pi n a^n}\right] r^n \sin(n\phi) = \frac{U_0}{\pi}\sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n \frac{\big[1-(-1)^n\big]}{n}\sin(n\phi).$$

Next, some limiting cases. It will be easier to work in terms of electric field instead of potential. We will find the electric field in the two planes parallel and perpendicular to the slit between the capacitor halves.

First, in the perpendicular (vertical) plane. The field points from high to low potential, so from the top half of the capacitor to the bottom half. In this plane we can work with just one coordinate r, which represents the vertical distance from the cylinder's center. Note that $\phi = \frac{\pi}{2}$. The component E_r we're after is

$$\begin{aligned} E_r &= -\frac{\partial}{\partial r}U(r,\phi)\bigg|_{\phi=\frac{\pi}{2}} = -\frac{U_0}{\pi}\sum_{n=1}^{\infty} n\left(\frac{r}{a}\right)^{n-1}\frac{1}{a}\frac{\big[1-(-1)^n\big]}{n}\sin(n\phi)\bigg|_{\phi=\frac{\pi}{2}} \\ &= -\frac{U_0}{\pi a}\sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^{n-1}\frac{\big[1-(-1)^n\big]}{n}\sin\Big(\frac{n\pi}{2}\Big). \end{aligned}$$

The sum simplifies considerably when you realize

$$\frac{1-(-1)^n}{n}\sin\Big(\frac{n\pi}{2}\Big) = \begin{cases} 0 & n \text{ even} \\ 2 & n = 1, 5, \ldots \\ -2 & n = 3, 7, \ldots. \end{cases}$$

We can then write the field as a geometric series

$$E_r = \frac{-2U_0}{\pi a}\left[1 - \left(\frac{r}{a}\right)^2 + \left(\frac{r}{a}\right)^4 \mp \cdots\right] = \frac{-2U_0}{\pi a}\frac{1}{1+\left(\frac{r}{a}\right)^2}.$$

Note that E_r is largest at $r = 0$, decreases monotonically with increasing r, and falls to half of its maximum value at $r = a$.

In a field parallel to the slit, we would set $\phi = 0$. This plane is perpendicular to the vertical plane, which used the radial component E_r, so for the parallel plane we work with the ϕ component E_ϕ.

$$E_\phi = -\frac{1}{r}\frac{\partial}{\partial\phi}U(r,\phi)\bigg|_{\phi=0}.$$

As before, the sum simplifies considerably to a geometric series. The result turns out to be

$$E_\phi = -\frac{2U_0}{\pi a}\frac{1}{1-\left(\frac{r}{a}\right)^2}.$$

Note that E_ϕ diverges at $r = a$. This is a consequence of the very small slit spacing between the capacitor halves at $r = a$; schematically have $E = \frac{U_0}{d} \to \infty$ as $d \to 0$.

4.2 Conducting Sphere in a Uniform Electric Field

A conducting sphere of radius a is placed in a uniform electric field E_0 pointing in the z direction. Find the electric potential U inside and outside the sphere.

We will use spherical coordinates to take advantage of spherical symmetry, meaning we aim to solve for electric potential U in the form $U(r, \phi, \theta)$. Because of the problem's rotational symmetry, the solution will be independent of ϕ, and we need only solve for $U(r, \theta)$.

The sphere is at a constant potential because it is a conductor. We'll set $U = 0$ inside the sphere. In the space around the sphere, we solve the Laplace equation

$$\nabla^2 U(r, \theta) = 0.$$

We then separate variables via $U = R(r)\Theta(\theta)$, which leads to the general solution

$$U(r, \theta) = \sum_{l=0}^{\infty}\left[A_l r^l + B_l r^{-(l+1)}\right] P_l(\cos\theta),$$

where P_l are the Legendre polynomials.

On to the boundary conditions. On the surface we'll set $U(a, \theta) = 0$. And at infinity, we use the boundary condition

$$U(r \to \infty, 0) = -E_0 z = -E_0 r\cos\theta.$$

This is the potential of a uniform electric field (at infinity, the potential from the sphere is negligible). This potential is chosen so that

$$-\frac{\partial U}{\partial z} = E_0,$$

i.e. so that the potential at infinity recovers the uniform electric field E_0.

We'll start with the second boundary condition at $r \to \infty$. Applying the condition to the general solution gives

$$\sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta) = -E_0 r\cos\theta.$$

Note that the $r^{-(l+1)}$ terms vanish as $r \to \infty$. The entire series equals a single term proportional to $\cos\theta$ and the equality holds if only the $l = 1$ term in the series is non-zero, which generates a corresponding $\cos\theta$ term from $P_1(\cos\theta) = \cos\theta$ ie. $P_1(x) = x$. The $l = 1$ term is

$$A_1 r\cos\theta = -E_0 r\cos\theta \implies A_1 = -E_0,$$

so we have $A_l = -E_0\delta_{l1}$. The solution for $U(r,\theta)$ simplifies to

$$U(r,\theta) = -E_0 r\cos\theta + \sum_{l=1}^{\infty} B_l r^{-(l+1)} P_l(\cos\theta).$$

Next, the second boundary condition: $U(a,\theta) = 0$. Substituting the condition into the intermediate solution gives

$$E_0 a\cos\theta = \sum_{l=1}^{\infty} B_l a^{-(l+1)} P_l(\cos\theta).$$

Again, the entire series equals only a single term. Again, this will be the $l = 1$ term corresponding to $P_1(\cos\theta) = \cos\theta$. For $l \neq 1$ we have $B_l = 0$. The $l = 1$ term gives

$$E_0 a\cos\theta = B_1 a^{-2}\cos\theta \implies B_1 = E_0 a^3.$$

With A_l and B_l known for all l, the final result is

$$U(r,\theta) = -E_0 r\cos\theta + \frac{E_0 a^3}{r^2}\cos\theta.$$

The first term, $-E_0 r\cos\theta$, is the potential of the uniform external field E_0. The second term comes from the sphere. In fact, this second term has the same form as the potential of an electric dipole!

Limiting cases are discussed in the next exercise set.

5 Fifth Exercise Set

5.1 Conducting Sphere in a Uniform Electric Field (continued)

We left off in the previous exercise set with the electric potential due to the conducting sphere, which as

$$U(r,\theta) = -E_0\cos\theta + \frac{E_0 a^3}{r^2}\cos\theta.$$

More so, we had recognized that the sphere's contribution $\frac{E_0 a^3}{r^2}\cos\theta$ corresponded to the potential of an electric dipole. Our next step is to explore the sphere's dipole behavior.

We consider an infinitesimal element of the sphere's surface at the angle θ carrying charge $\mathrm{d}q$, on the upper hemisphere with positive charge and θ. Recall the electric field, as for any conductor, is perpendicular to the surface.

We write Gauss's law for the small surface element, which is simple because the electric field is perpendicular to the surface

$$E_\perp \,\mathrm{d}S = \frac{\mathrm{d}q}{\epsilon_0} \implies \frac{\mathrm{d}e}{\mathrm{d}S} = \sigma = \epsilon_0 E_\perp.$$

This equality gives us an expression for σ in terms of the electric field $E_\perp$ perpendicular to the surface. We can find $E_\perp$ from the potential:

$$E_\perp = -\frac{\partial U}{\partial r}\bigg|_{r=a} = E_0\cos\theta + 2E_0\cos\theta \implies \sigma = 3\epsilon_0 E_0\cos\theta.$$

The charge density's dependence on θ quantitatively demonstrates the sphere's dipole-like charge distribution.

With charge density σ known, we can find the sphere's electric dipole moment via

$$\boldsymbol{p}_e = \int \tilde{\boldsymbol{r}}\,\mathrm{d}q.$$

We qualitatively expect $\boldsymbol{p}_e$ to point upward (from the negative to the positively charged hemisphere), and confirm this analytically. By spherical symmetry, only the z component of $\boldsymbol{p}_e$ is non-zero; this is

$$p_{e_z} = \int \tilde{z}\,\mathrm{d}q = \iint (a\cos\theta)\cdot(\sigma\,\mathrm{d}S) = \iint (a\cos\theta)\cdot(3\epsilon_0 E_0\cos\theta)\cdot\mathrm{d}S.$$

To find $\mathrm{d}S$, we find the area of a small band of width $\mathrm{d}a$ around the sphere's surface. The band's area is $2\pi r\,\mathrm{d}a = 2\pi(a\sin\theta)(a\,\mathrm{d}\theta)$. The dipole moment p_{e_z} is then

$$\begin{aligned} p_{e_z} &= \int_0^\pi (3\epsilon_0 E_0 a\cos^2\theta)\cdot 2\pi(a\sin\theta)(a\,\mathrm{d}\theta) = 6a^3\pi\epsilon_0 E_0 \int_{-1}^{1}\cos^2\theta\,\mathrm{d}[\cos\theta] \\ &= 6a^3\pi\epsilon_0 E_0\left[\frac{1}{3}\cos^3\theta\right]_{-1}^{1} = 4\pi\epsilon_0 E_0 a^3. \end{aligned}$$

5.2 Electric Dipole in a Conducting Spherical Shell

We place an electric dipole with dipole moment $\boldsymbol{p}_e$ in the center of a conducting spherical shell of radius a. What is the electric potential inside the shell?

We will work in spherical coordinates, which are best suited to the problem's spherical symmetry. By rotational symmetry, the potential depends only on the coordinates r and θ, not ϕ. Besides at the center, the charge density inside the sphere is zero, so we solve the Laplace equation

$$\nabla^2 U(r,\theta) = 0.$$

The general solution is

$$U(r,\theta) = \sum_{l=0}^{\infty} \left(A_l r^l + B_l r^{-(l+1)} \right) P_l(\cos\theta).$$

where P_l are the Legendre polynomials.

We then apply boundary conditions to find a solution specific to our problem. First, the potential on the shell's surface is constant, since the shell is a conductor. For convenience, we'll set $U(a,\theta) = 0$. The second boundary condition concerns the dipole at the sphere's center. Namely, the potential approaches the potential of an electric dipole near the sphere's center. Quantitatively, this condition reads

$$U(r \to 0, \theta) = \frac{p_e \cos\theta}{4\pi\epsilon_0 r^2}.$$

We begin with the simpler second boundary condition, (the boundary $r \to 0$ eliminates the r^l-dependent term). Inserted into the general solution, the second condition reads

$$U(r \to 0, \theta) = \sum_{l=0}^{\infty} B_l r^{-(l+1)} P_l(\cos\theta) = \frac{p_e \cos\theta}{4\pi\epsilon_0} r^{-2}.$$

Note that the entire series sums to only a single term; for this to work, only the $l = 1$ term in the series can be non-zero, leaving

$$B_1 r^{-2} \cos\theta = \frac{p_e \cos\theta}{4\pi\epsilon_0} r^{-2} \implies B_1 = \frac{p_e}{4\pi\epsilon_0} \qquad \text{and} \qquad B_{l\neq 1} = 0.$$

Note the use of $P_1(\cos\theta) = \cos\theta$. The intermediate solution at this stage is

$$U(r,\theta) = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta) + \frac{p_e}{4\pi\epsilon_0} r^{-2} \cos\theta.$$

We apply the second boundary condition $U(a,\theta) = 0$ to get

$$\sum_{l=0}^{\infty} A_l a^l P_l(\cos\theta) = -\frac{p_e}{4\pi\epsilon_0} a^{-2} \cos\theta.$$

As before, only the $l = 1$ term can be non-zero to satisfy the equality. The result is

$$A_1 a \cos\theta = -\frac{p_e}{4\pi\epsilon_0} a^{-2} \cos\theta \implies A_1 = -\frac{p_e}{4\pi\epsilon_0 a^3} \qquad \text{and} \qquad A_{l\neq 1} = 0.$$

With the coefficients A_l and B_l known, the solution for $U(r,\theta)$ is

$$U(r,\theta) = \left[\frac{p_e}{4\pi\epsilon_0 r^2} - \frac{p_e}{4\pi\epsilon_0 a^3} r\right]\cos\theta = \frac{p_e\cos\theta}{4\pi\epsilon_0}\left[\frac{1}{r^2} - \frac{r}{a^3}\right].$$

The $\frac{1}{r^2}$ term is the dipole's contribution. The $\frac{r}{a^3}$ comes from the charge induced on the conducting shell.

The induced term is worth a closer look, noting that $r\cos\theta = z$.

$$U_{\text{induced}} = -\frac{p_e}{4\pi\epsilon_0 a^3} r\cos\theta = -\frac{p_e}{4\pi\epsilon_0 a^3} z.$$

In particular, the associated electric field is

$$E_{\text{induced}} = -\frac{\partial}{\partial z} U_{\text{induced}} = \frac{p_e}{4\pi\epsilon_0 a^3}.$$

In other words, the electric field generated by the induced charge is constant! The uniform field also tells us about the charge distribution on the sphere's surface: to create a uniform field in the z direction, the shell must have a dipole-like charge distribution, with positive charge on the lower hemisphere and negative charge on the upper hemisphere.

Next, we're interested in the analytic expression for the surface charge density σ. We consider a small surface element $\mathrm{d}S$, and consider the total electric field at that surface. The electric field must be perpendicular to the surface, since the shell is a conductor. Gauss's law applied to the surface element reads

$$-E_\perp \,\mathrm{d}S = \frac{\mathrm{d}q}{\epsilon_0} \implies \sigma \equiv \frac{\mathrm{d}q}{\mathrm{d}S} = -\epsilon_0 E_\perp.$$

Note the minus sign, indicating the field's electric flux leaving the surface element from inside the shell. We find an expression for $E_\perp$ from U:

$$E_\perp = -\frac{\partial U}{\partial r}\bigg|_{r=a} = +\frac{p_e\cos\theta}{4\pi\epsilon_0}\left[\frac{2}{r^3} + \frac{1}{a^3}\right]_{r=a} = \frac{3p_e}{4\pi\epsilon_0 a^3}\cos\theta.$$

The associated surface charge density is

$$\sigma = -\epsilon_0 E_\perp = -\frac{3p_e}{4\pi a^3}\cos\theta.$$

5.3 Point Charge Above a Conducting Plane

Consider a positive point charge q a distance d above a large, grounded conducting plane. What is the electric potential in space due to the charge-plane system?

We will use a trick called the *method of images* to solve the problem. Namely, we imagine a negative point charge $-q$ a distance d below the plane—a mirror image of the original positive charge. The resulting charge distribution is an electric dipole.

Note: placing an imaginary negative charge a distance d below the plane does not change the field above the plane due to the positive charge.

Considering both points, the potential at an arbitrary position $\boldsymbol{r}$ from the origin is

$$U(\boldsymbol{r}) = \frac{q}{4\pi\epsilon_0}\frac{1}{|\boldsymbol{r}-\boldsymbol{d}|} - \frac{q}{4\pi\epsilon_0}\frac{1}{|\boldsymbol{r}+\boldsymbol{d}|},$$

where the vector $\boldsymbol{d}$ points perpendicularly up from the plane toward the positive charge. Introducing an angle θ between $\boldsymbol{d}$ and $\boldsymbol{r}$, we have

$$|\boldsymbol{r} \pm \boldsymbol{d}| = \sqrt{r^2 + d^2 \pm 2rd\cos\theta}.$$

The expression for $U(\boldsymbol{r})$ for the two charges is then simply

$$U_2(\boldsymbol{r}) = \frac{q}{4\pi\epsilon_0}\left[\frac{1}{\sqrt{r^2 + d^2 \pm 2rd\cos\theta}} - \frac{1}{\sqrt{r^2 + d^2 \pm 2rd\cos\theta}}\right].$$

For the original configuration of a single positive charge a distance d above the plane, the potential above the plane agrees with U_2, while the potential below the plane, where there is in reality no charge, is zero. The correct expression for the single charge $+q$ is then

$$U(\boldsymbol{r}) = \begin{cases} U_2(\boldsymbol{r}) & \text{above the plane} \\ 0 & \text{below the plane.} \end{cases}$$

Next, we're interested in the surface charge density $\sigma(\rho)$ on the plane where ρ is the radial distance in the plane from the origin. As usual, we start with Gauss's law for a small surface element of the plane:

$$-E_\perp \,\mathrm{d}S = \frac{\mathrm{d}q}{\epsilon_0} \implies \sigma \equiv \frac{\mathrm{d}q}{\mathrm{d}S} = -\sigma E_\perp.$$

To find $E_\perp$, we differentiate U with respect to the vertical coordinate z. First, we introduce z into the expression $|\boldsymbol{r} \pm \boldsymbol{d}|$

$$|\boldsymbol{r} \pm \boldsymbol{d}| = \sqrt{r^2 + d^2 \pm 2rd\cos\theta} = \sqrt{\rho^2 + z^2 + d^2 \pm 2dz}.$$

We then have

$$U(\rho, z) = \frac{q}{4\pi\epsilon_0}\left[\frac{1}{\sqrt{\rho^2 + z^2 + d^2 - 2dz}} - \frac{1}{\sqrt{\rho^2 + z^2 + d^2 + 2dz}}\right].$$

We then find $E_\perp$ and then σ with

$$\begin{aligned}\sigma = -\epsilon_0 E_\perp = -\epsilon_0 \frac{\partial U}{\partial z}\bigg|_{z=0} &= -\frac{q}{4\pi}\left[\frac{d}{(\rho^2 + d^2)^{3/2}} + \frac{d}{(\rho^2 + d^2)^{3/2}}\right] \\ = -\frac{q}{2\pi}\frac{d}{(\rho^2 + d^2)^{3/2}}.\end{aligned}$$

With surface charge density σ known, we then ask what is the total charge on the plane. Integrating the plane over rings with area $\mathrm{d}S = 2\pi\rho\,\mathrm{d}\rho$, we have

$$q_{\text{plane}} = \iint \sigma \,\mathrm{d}S = -\int_0^\infty \frac{qd}{2\pi(\rho^2 + d^2)^{3/2}}(2\pi\rho\,\mathrm{d}\rho).$$

In terms of the new variable $u = \rho^2 + d^2$, the integral evaluates to

$$q_{\text{plane}} = -\frac{qd}{2}\int_{d^2}^{\infty}\frac{\mathrm{d}u}{u^{3/2}} = qd\left[\frac{1}{u^{1/2}}\right]_{d^2}^{\infty} = -q.$$

Summary of what we did: recognize that the field above the plane from the positive charge looks like half the field of an electric dipole. Since we know the solution for a dipole, instead of solving the charge-plane system, we solve the (imaginary) two-charge system, which gives the same field above the plane anyway. We then reuse the upper half of the dipole solution for the single-charge plane system, and set the field below the plane equal to zero. The basic idea is: the field above the plane is the same for both the positive-charge plane system and for a dipole system, so we can use either approach to solve for the field above the plane.

Next, we ask what is the electrostatic force on the point charge above the plane? First, some theory: **Theory: Electrostatic Force**

The total electrostatic force $\boldsymbol{F}$ acting on the charges enclosed in a region of space V permeated with an electric field $\boldsymbol{E}$ is

$$\boldsymbol{F} = \epsilon_0 \oiint_{\partial V}\left[\boldsymbol{E}(\boldsymbol{E}\cdot\hat{\mathbf{n}}) - \frac{1}{2}E^2\,\hat{\mathbf{n}}\right]\mathrm{d}S,$$

where $\hat{\mathbf{n}}$ is the normal to the surface ∂V enclosing the charges. Like with Gauss's law, a good choice of the boundary surface, usually taking advantage of the problem's symmetries, tends to simplify the problem. Alternatively, if the electric field vanishes at infinity, we choose a surface that closes at infinity.

Back to our problem: we choose an infinite surface whose base runs along the plane, then turns upward and closes at infinity to enclose the upper half of space above the plane. The field in this case is the same $E_\perp$ calculated above:

$$E_\perp = \frac{q}{2\pi\epsilon_0}\frac{d}{(\rho^2+d^2)^{3/2}}.$$

For the part of the surface running parallel to the plane, the normal to the surface $\hat{\mathbf{n}}$ points perpendicularly into the plane, parallel to the electric field. The force equation for the bottom half of the surface reads

$$\boldsymbol{F}_e = \epsilon_0 \iint_{\text{bottom}}\left[E^2\,\hat{\mathbf{n}} - \frac{1}{2}E^2\,\hat{\mathbf{n}}\right]\mathrm{d}S = \frac{\epsilon_0}{2}\,\hat{\mathbf{n}}\iint_{\text{bottom}} E_\perp^2\,\mathrm{d}S.$$

In fact, the contribution from the upper half of the surface is zero—the upper half extends to infinity, where the electric field vanishes. We only need to integrate over the bottom of the surface, running parallel to the plane. Writing $\mathrm{d}S = 2\pi\rho\,\mathrm{d}\rho$ and substituting in the expression for $E_\perp$, the force reads

$$\begin{aligned}\boldsymbol{F}_e &= \hat{\mathbf{n}}\frac{\epsilon_0}{2}\int_0^{\infty}\frac{q^2}{4\pi^2\epsilon_0^2}\frac{d^2}{(\rho^2+d^2)^3}2\pi\rho\,\mathrm{d}\rho = \frac{q^2d^2}{4\pi\epsilon_0}\,\hat{\mathbf{n}}\int_0^{\infty}\frac{\rho}{(\rho^2+d^2)^3}\,\mathrm{d}\rho\\ &= \frac{q^2d^2}{8\pi\epsilon_0}\,\hat{\mathbf{n}}\int_{d^2}^{\infty}\frac{\mathrm{d}u}{u^3} = -\frac{q^2d^2}{16\pi\epsilon_0}\,\hat{\mathbf{n}}\left[\frac{1}{u^2}\right]_{d^2}^{\infty} = \frac{q^2}{16\pi\epsilon_0 d^2}\,\hat{\mathbf{n}}.\end{aligned}$$

The force points in the direction of $\hat{\mathbf{n}}$—downward into the plane. A final note: if we write

$$\boldsymbol{F}_e = \frac{q^2}{4\pi\epsilon_0 (2d)^2}\,\hat{\mathbf{n}},$$

the force takes the form of the electric force between a positive and negative charge separated by a distance $2d$—the same situation we used in the method of images above.

6 Sixth Exercise Set

Theory: Electrostatic Force Recall from the previous exercise set that the total electrostatic force $\boldsymbol{F}$ acting on the charges enclosed in a region of space V permeated with an electric field $\boldsymbol{E}$ is

$$\boldsymbol{F}_e = \epsilon_0 \oiint_{\partial V} \left[\boldsymbol{E}(\boldsymbol{E} \cdot \hat{\mathbf{n}}) - \frac{1}{2} E^2 \, \hat{\mathbf{n}} \right] \mathrm{d}S,$$

where $\hat{\mathbf{n}}$ is the normal to the surface ∂V enclosing the charges.

6.1 Force on a Conducting Spherical Shell

We place a conducting sphere of radius a in a homogeneous electric field E_0. Find the electrostatic force on the upper half of the sphere.

Suppose the field points in the z direction. Recall from the previous exercise set that the potential from the sphere and electric field is

$$U(r, \theta) = -E_0 r \cos\theta + \frac{E_0 a^3}{r^2} \cos\theta.$$

Qualitatively, there are two main contributions to the force on the sphere: an upwards contribution in the positive z direction from the external electric field, and a downward contribution in the negative z direction from the negative charge accumulated on the bottom half of the sphere.

We are interested in the force on the upper half of the sphere—the next step is to choose a surface around the sphere's upper half that will simplify the force calculation. Recall the field points perpendicularly out of the conducting sphere's surface at all points.

With this perpendicular field in mind, choose a surface that tightly hugs the sphere's upper half—in this case, the field and normal to the surface $\hat{\mathbf{n}}$ are parallel at all points outside the sphere. In the hemisphere plane inside the sphere, there is no field at all. These to facts simplify the dot product $\boldsymbol{E} \cdot \hat{\mathbf{n}}$ in the force equation.

We then have $\boldsymbol{E} \cdot \hat{\mathbf{n}} = E$ and $\boldsymbol{E}(\boldsymbol{E} \cdot \hat{\mathbf{n}}) = E^2 \, \hat{\mathbf{n}}$. The contribution to the force on along the sphere's outside surface is

$$\boldsymbol{F}_e = \epsilon_0 \iint_{\text{outer}} \frac{1}{2} E^2 \, \hat{\mathbf{n}} \, \mathrm{d}S.$$

The contribution from the hemisphere plane through the sphere zero, since $\boldsymbol{E} = 0$ inside the sphere.

Next, we find the magnitude E on the sphere's surface from the potential $U(r, \theta)$. The field points radially outwards, so we differentiate U with respect to r to get

$$E = \left. \frac{\partial U}{\partial r} \right|_{r=a} = E_0 \cos\theta + 2E_0 \cos\theta = 3E_0 \cos\theta.$$

Inserting E into the force equation gives

$$\boldsymbol{F}_e = \epsilon_0 \iint \frac{1}{2} \big(3E_0 \cos\theta\big)^2 \, \hat{\mathbf{n}} \, \mathrm{d}S.$$

In spherical coordinates, the unit normal $\hat{\mathbf{n}}$ to the sphere's surface is $\hat{\mathbf{n}} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$. The surface element $\mathrm{d}S$ at the surface $r = a$ is (just like in the previous exercise sets) $\mathrm{d}S = a^2\,\mathrm{d}\phi\sin\theta\,\mathrm{d}\theta$. The force on the sphere's upper half is then

$$\boldsymbol{F}_e = \frac{9\epsilon_0 E_0^2}{2}\int_{\theta=0}^{\pi/2}\int_{\phi=0}^{2\pi}\cos^2\theta\begin{bmatrix}\sin\theta\cos\phi\\ \sin\theta\sin\phi\\ \cos\theta\end{bmatrix}(a^2\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi).$$

Both the x and y components will be zero—integrating $\cos\phi$ and $\sin\phi$ over a full period 2π give zero, while the ϕ contribution to the z component is 2π. We make this explicit with

$$\boldsymbol{F}_e = \frac{9\epsilon_0 E_0^2}{2}\int_{\theta=0}^{\pi/2}\cos^2\theta\begin{bmatrix}0\\ 0\\ 2\pi\cos\theta\end{bmatrix}(a^2\sin\theta\,\mathrm{d}\theta).$$

The non-zero z component F_z is

$$\begin{aligned}F_z &= \frac{9\epsilon_0 E_0^2}{2}\int_{\theta=0}^{\pi/2}2\pi\cos^3\theta(a^2\sin\theta\,\mathrm{d}\theta) = 9\pi\epsilon_0 E_0^2 a^2\int_{\theta=0}^{\pi/2}\cos^3\theta(\sin\theta\,\mathrm{d}\theta)\\ &= 9\pi\epsilon_0 E_0^2 a^2\int_0^1\cos^3\theta\,\mathrm{d}[\cos\theta] = \frac{9\pi\epsilon_0 a^2 E_0^2}{4}.\end{aligned}$$

The vector force can be written simply as

$$\boldsymbol{F} = \frac{9\pi\epsilon_0 a^2 E_0^2}{4}\,\hat{\mathbf{z}}.$$

In other words, the force on the upper half points upward in the positive z direction.

6.2 Point Charge Between Two Conducting Plates

We place two large conducting plates at a right angle to each other, so that the plates come close together but just barely do not touch. We then place a point charge q along the line bisecting the right angle between the plates, at a perpendicular distance a from each plate. Both plates are grounded. What is the electric potential in the region bounded by the plates at large distances from the plates' intersection?

Assume $r = 0$ along the line connecting the two plates. For a single plate, we could solve the problem with the method of images—see the previous exercise set. With two plates we proceed analogously, with a mirror image for each plate. Because of the two reflections from the two plates, we end up with three imaginary charges plus the one original one in a quadrupole arrangement. (This is hard to describe in words, it is best to see a picture). For large r, the charge arrangement will have the field of an electric quadrupole. Solving the problem thus reduces to a multipole expansion of the electric potential to the quadrupole term.

6.2.1 Theory: The Multipole Expansion

The multipole expansion of U to quadrupole order, using the Einstein summation convention, is

$$U(r) = \frac{1}{4\pi\epsilon_0}\left[\frac{q}{r} + \frac{p_i r_i}{r^3} + \frac{\mathrm{Q}_{ij} r_i r_j}{r^5}\right],$$

where $\boldsymbol{p}$ is the electric dipole moment and $\mathbf{Q}$ is the quadrupole moment tensor. *Note*: we could think of the charge q is a scalar monopole moment, creating a logical progression from scalar monopole moment to vector dipole moment to tensor quadrupole moment.

We find the dipole moment with

$$\boldsymbol{p} = \iiint \mathrm{d}^3\tilde{r}\rho(\tilde{\boldsymbol{r}})\tilde{\boldsymbol{r}}.$$

We find the quadrupole moment by components:

$$\mathrm{Q}_{ij} = \iiint \mathrm{d}^3\tilde{r}\rho(\tilde{\boldsymbol{r}})\big(3\tilde{r}_i\tilde{r}_j - \delta_{ij}\tilde{r}^2\big).$$

The discrete analog a configuration of N charges reads

$$\mathrm{Q}_{ij} = \sum_{n=1}^{N} q_n\left(3r_{n_i}r_{n_j} - \delta_{ij}r_n^2\right).$$

Note that both definitions produces a symmetric tensor. Also important: the tensor's trace—the sum of the diagonal elements is zero:

$$\operatorname{tr}\mathbf{Q} = \sum_n q_n\left[3x_n^2 - r_n^2 + 3z_n^2 - r_n^2 + 3z_n^2 - r_n^2\right] = \sum_n q_n\left[3r_n^2 - 3r_n^2\right] = 0.$$

6.2.2 Returning to the original problem

For our imaginary quadrupole configuration of four charges, the total charge, and thus the monopole moment, is zero. Analogously, the total dipole moment of the arrangement, which consists of two positive and two negative charges, is zero—the two dipoles cancel each other out.

From the three terms in our multipole expansion of $U(r)$, only the quadrupole term remains. We just have to calculate the quadrupole tensor Q_{ij}. We label the four charges in the imaginary quadrupole configuration as 1, 2, 3, and 4, where 1 is the original positive charge in the upper right corner, 2 is the negative image charge in the upper left corner, 3 is the positive image charge in the lower left corner, and 4 is the negative image charge in the lower right corner.

Using the discrete formula for Q_{ij}, the first component Q_{xx} is

$$\begin{aligned}\mathrm{Q}_{xx} &= \sum_{n=1}^{N} q_n\left(3x_n^2 - \delta_{ij}r_n^2\right) = q\left(3a^2 - 2a^2\right) + (-q)\left(3a^2 - 2a^2\right)\\ &= q\left(3a^2 - 2a^2\right) + (-q)\left(3a^2 - 2a^2\right) = 0.\end{aligned}$$

The other diagonal terms Q_{yy} and Q_{zz} will analogously sum to zero.

All off-diagonal terms with a z component are also zero, since the charges lie in a plane with $z = 0$. We thus have $\mathrm{Q}_{xz} = \mathrm{Q}_{zx} = \mathrm{Q}_{yz} = \mathrm{Q}_{zy} = 0$. We have just two terms left calculate: Q_{xy} and Q_{yx}. By the tensor's symmetry, the two are equal, so we really only have one term:

$$\begin{aligned}\mathrm{Q}_{xy} &= \sum_{n=1}^{N} q_n \left(3x_n y_n - 0 \cdot r_n^2\right) = 3qa^2 + 3(-q)(-a^2) + 3qa^2 + 3(-q)(-a^2)\\ &= 12qa^2.\end{aligned}$$

The quadrupole tensor is

$$\mathbf{Q} = \begin{bmatrix} 0 & 12qa^2 & 0 \\ 12qa^2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

As expected, the tensor is symmetric with trace $\operatorname{tr}\mathbf{Q} = 0$.

Recall the quadrupole expansion of $U(r)$:

$$U(r) = \frac{1}{4\pi\epsilon_0}\left[\frac{q}{r} + \frac{p_i r_i}{r^3} + \frac{\mathrm{Q}_{ij} r_i r_j}{r^5}\right].$$

In our case, with $q = 0$ and $\boldsymbol{p} = 0$, we have

$$\begin{aligned}U(r) &= \frac{1}{4\pi\epsilon_0}\left[\frac{12qa^2 xy}{r^5} + \frac{12qa^2 yx}{r^5} + 0 + \cdots + 0\right] = \frac{6qa^2}{\pi\epsilon_0}\frac{xy}{r^5}\\ &= \frac{6qa^2}{\pi\epsilon_0}\frac{\cos\phi\sin\phi\sin^2\theta}{r^3}.\end{aligned}$$

The second line uses the spherical coordinates $x = r\cos\phi\sin\theta$ and $y = r\sin\phi\sin\theta$. In fact, the expression for $U(r)$ takes the exact same form as the wave function of a d electron orbital (angular momentum quantum number $l = 2$) in a hydrogen atom. The equipotential surfaces of $U(r)$ have the same spatial distribution as the d_{xy} orbitals for a hydrogen wave function.

7 Seventh Exercise Set

7.1 Theory: Magnetic Vector Potential and the Biot-Savart Law

We will need to use two more Maxwell equations for magnetostatics. The first is

$$\nabla \cdot \boldsymbol{B} = 0.$$

This equation rules out the possibility of magnetic monopoles and allows $\boldsymbol{B}$ to be written as the curl of a vector potential $\boldsymbol{A}$ as

$$\boldsymbol{B} = \nabla \times \boldsymbol{A}.$$

The second Maxwell equation is

$$\nabla \times \boldsymbol{B} = \mu_0 \left(\boldsymbol{j} + \epsilon_0 \frac{\partial \boldsymbol{E}}{\partial t} \right).$$

For static situations with a constant electric field this simplifies to

$$\nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{j}.$$

Substituting the expression $\boldsymbol{B} = \nabla \times \boldsymbol{A}$ into the static second Maxwell equation produces

$$\nabla \times \big(\nabla \times \boldsymbol{A}\big) = \nabla \cdot \big(\nabla \cdot \boldsymbol{A}\big) - \nabla^2 \boldsymbol{A} = \mu_0 \boldsymbol{j}.$$

The magnetic vector potential is defined only up to a constant; we usually choose $\boldsymbol{A}$ so that $\nabla \cdot \boldsymbol{A} = 0$. In this convention, we have

$$\nabla^2 \boldsymbol{A} = -\mu_0 \boldsymbol{j},$$

where $\boldsymbol{j}$ is the current density vector. This equation is a vector analog of the Poisson equation $\nabla^2 U = -\frac{\rho}{\epsilon_0}$ from electrostatics. Similarly to how the electrostatic potential U at a point $\boldsymbol{r}$ in region of space with charge density ρ is found with

$$U = \frac{1}{4\pi\epsilon_0} \iiint \frac{\rho(\tilde{\boldsymbol{r}})\,\mathrm{d}^3\tilde{r}}{|\boldsymbol{r} - \tilde{\boldsymbol{r}}|},$$

the magnetic potential at a point $\boldsymbol{r}$ in region of space with current density $\boldsymbol{j}$ is

$$\boldsymbol{A}(\boldsymbol{r}) = \frac{\mu_0}{4\pi} \iiint \frac{\boldsymbol{j}(\tilde{\boldsymbol{r}})\,\mathrm{d}^3\tilde{r}}{|\boldsymbol{r} - \tilde{\boldsymbol{r}}|}.$$

In problems involving one-dimensional conductors, where $\boldsymbol{j}$ is non-zero only along the conductor, the expression for $\boldsymbol{j}(\tilde{\boldsymbol{r}})\,\mathrm{d}^3\tilde{r}$ simplifies to

$$\boldsymbol{j}(\tilde{\boldsymbol{r}})\,\mathrm{d}^3\tilde{r} = \boldsymbol{j}(\tilde{\boldsymbol{r}})\,\mathrm{d}\tilde{S}\,\mathrm{d}\tilde{l} = I\,\hat{\mathbf{t}}\,\mathrm{d}\tilde{l},$$

where $\hat{\mathbf{t}}$ is the unit normal vector tangent to the conductor, I is the current through the conductor at the point $\tilde{\boldsymbol{r}}$ and $\mathrm{d}\tilde{l}$ is a small distance element along the conductor's length. The magnetic vector potential for a one-dimensional conductor carrying a current I then simplifies to

$$\boldsymbol{A}(\boldsymbol{r}) = \frac{\mu_0 I}{4\pi} \int \frac{\hat{\mathbf{t}}\,\mathrm{d}l}{|\boldsymbol{r} - \tilde{\boldsymbol{r}}|}.$$

Recall $\boldsymbol{B} = \nabla \times \boldsymbol{A}$. Taking the curl of the general expression for $\boldsymbol{A}$ in terms of current density $\boldsymbol{j}$ gives general expression for the magnetic field

$$\boldsymbol{B}(\boldsymbol{r}) = \frac{\mu_0}{4\pi} \int \frac{\boldsymbol{j}(\tilde{\boldsymbol{r}}) \times (\boldsymbol{r} - \tilde{\boldsymbol{r}})}{|\boldsymbol{r} - \tilde{\boldsymbol{r}}|^3} \,\mathrm{d}^3\tilde{r}.$$

This is a general form of the Biot-Savart law for the magnetic field $\boldsymbol{B}$ of a current distribution.

7.2 Magnetic Field of a Circular Current Loop

A closed conducting loop of radius a carries current I. What is the magnetic vector potential far from the conducting loop?

Our starting point is the vector potential of a one-dimensional conductor from the theory section, i.e.

$$\boldsymbol{A}(\boldsymbol{r}) = \frac{\mu_0 I}{4\pi} \int \frac{\hat{\mathbf{t}}\,\mathrm{d}l}{|\boldsymbol{r} - \tilde{\boldsymbol{r}}|}.$$

We need expressions for $\boldsymbol{r}$, $\tilde{\boldsymbol{r}}$ and $\hat{\mathbf{t}}$.

Assume the loop lies in the x, y plane with the z axis normal to the loop. For mathematical convenience, we rotate the x, y plane so $\boldsymbol{r}$ lies in the x, z plane (i.e. $\phi = 0$); this just gives us one less non-zero component to work with, since the $\sin\phi$ term in the y component of $\boldsymbol{r}$ is zero. In polar coordinates $\boldsymbol{r}$ reads

$$\boldsymbol{r} = \big(r\sin\theta, 0, r\cos\theta\big)$$

, where θ is the angle between $\boldsymbol{r}$ and the z axis.

The integration variable $\tilde{\boldsymbol{r}}$, which runs over the current loop in the x, y plane, reads

$$\tilde{\boldsymbol{r}} = \big(a\cos\tilde{\phi}, a\sin\tilde{\phi}, 0\big),$$

where $\tilde{\phi}$ is the azimuthal angle between $\tilde{\boldsymbol{r}}$ and the x axis.

Finally, the expression for $\hat{\mathbf{t}}$, the tangent to the current loop, is

$$\hat{\mathbf{t}} = \big(-\sin\tilde{\phi}, \cos\tilde{\phi}, 0\big).$$

The small distance element is $\mathrm{d}\tilde{l} = a\tilde{\phi}$.

We can now put the pieces together in the equation for $\boldsymbol{A}$. First,

$$\begin{aligned}
|\boldsymbol{r} - \tilde{\boldsymbol{r}}| &= \sqrt{(r\sin\theta - a\cos\tilde{\phi})^2 + a^2\sin^2\tilde{\phi} + r^2\cos^2\theta} \\
&= \sqrt{r^2 + a^2 - 2ra\sin\theta\cos\tilde{\phi}} \\
&\approx \sqrt{r^2 - 2ra\sin\theta\cos\tilde{\phi}} = r\sqrt{1 - \frac{2a}{r}\sin\theta\cos\tilde{\phi}},
\end{aligned}$$

where the last lines uses $r \gg a$ (recall we're interested in the solution far from the conducting loop). We now have, again using $a \ll r \implies \frac{a}{r} \ll 1$,

$$\frac{1}{|\boldsymbol{r} - \tilde{\boldsymbol{r}}|} = \frac{1}{r}\left(1 - \frac{2a}{r}\sin\theta\cos\tilde{\phi}\right)^{-1/2} \approx \frac{1}{r}\left(1 + \frac{1}{2}\frac{2a}{r}\sin\theta\cos\tilde{\phi}\right).$$

Substituting the expressions for $\frac{1}{|\boldsymbol{r}-\tilde{\boldsymbol{r}}|}$, $\hat{\mathbf{t}}$ and $\mathrm{d}\tilde{l}$ into expression for $\boldsymbol{A}$ gives

$$\boldsymbol{A}(\boldsymbol{r}) = \frac{\mu_0 I}{4\pi}\int \frac{\hat{\mathbf{t}}\,\mathrm{d}l}{|\boldsymbol{r} - \tilde{\boldsymbol{r}}|} = \frac{\mu_0 I}{4\pi}\frac{a}{r}\int_0^{2\pi} \mathrm{d}\tilde{\phi}\begin{bmatrix} -\sin\tilde{\phi} \\ \cos\tilde{\phi} \\ 0 \end{bmatrix}\left(1 + \frac{1}{2}\frac{2a}{r}\sin\theta\cos\tilde{\phi}\right).$$

The integrals conveniently simplify, since we are integrating sinusoidal terms over an entire period. Only the integral of $\cos^2\tilde{\phi}$ in the y component is nonzero. We end up with $A_x = A_z = 0$ and

$$A_y(r) = \frac{\mu_0 I}{4\pi}\frac{a}{r}\int_0^{2\pi} \frac{a\sin\theta}{r}\cos^2\tilde{\phi}\,\mathrm{d}\tilde{\phi} = \frac{\mu_0 I}{4}\frac{a^2}{r^2}\sin\theta,$$

or, in vector form,

$$\boldsymbol{A}(\boldsymbol{r}) = \frac{\mu_0 I}{4}\frac{a^2}{r^2}\sin\theta\,\hat{\mathbf{y}} = \frac{\mu_0 I}{4}\frac{a^2}{r^2}\hat{\mathbf{z}} \times \hat{\mathbf{r}}.$$

The second expression is preferable: the first, in terms of $\hat{\mathbf{y}}$, holds only with x, y plane rotated so $\phi = 0$ and $\boldsymbol{r}$ lies in the x, z plane. The second, which uses $\sin\theta\,\hat{\mathbf{y}} = \hat{\mathbf{z}} \times \hat{\mathbf{r}}$, holds for any orientation of the x, y plane.

Finally, using $I\pi a^2 = |\boldsymbol{m}|$ (where $\boldsymbol{m}$ is the loop's magnetic dipole moment, which points in the direction of the loop's normal), the expression for $\boldsymbol{A}$ simplifies to

$$\boldsymbol{A}(\boldsymbol{r}) = \frac{\mu_0 \boldsymbol{m} \times \hat{\mathbf{r}}}{4\pi r^2} = \frac{\mu_0 \boldsymbol{m} \times \boldsymbol{r}}{4\pi r^3}.$$

This is the same form of magnetic vector potential as for a magnetic dipole. In other words, a circular current loop behaves as magnetic dipole at long distances.

7.3 Magnetic Field of a Rotating Charged Disk

Consider a charged, insulating disk with uniform surface charge density σ and radius a. The disk rotates uniformly about an axis through its center with constant angular speed ω. Find the magnetic field along the axis of rotation.

We choose a coordinate system so the disk lies in the x, y plane and the rotation axis coincides with the z axis, so $\boldsymbol{\omega} = (0, 0, \omega)$. Start with the general Biot-Savart law

$$\boldsymbol{B}(\boldsymbol{r}) = \frac{\mu_0}{4\pi}\iiint \frac{\boldsymbol{j}(\tilde{\boldsymbol{r}}) \times (\boldsymbol{r} - \tilde{\boldsymbol{r}})}{|\boldsymbol{r} - \tilde{\boldsymbol{r}}|^3}\,\mathrm{d}^3\tilde{r},$$

and, like in the previous problem, find expressions for each vector quantity in the equation. The expression for $\boldsymbol{r}$ along the z axis is simply $\boldsymbol{r} = (0, 0, z)$, while the expression for $\tilde{\boldsymbol{r}}$, which lies in the disk in the x, y plane, is

$$\tilde{\boldsymbol{r}} = \left(\tilde{r}\cos\tilde{\phi}, \tilde{r}\sin\tilde{\phi}, 0\right).$$

The difference of $\boldsymbol{r}$ and $\tilde{\boldsymbol{r}}$ and its magnitude is

$$\boldsymbol{r} - \tilde{\boldsymbol{r}} = \big(-\tilde{r}\cos\tilde{\phi}, -\tilde{r}\sin\tilde{\phi}, z\big) \qquad \text{and} \qquad |\boldsymbol{r} - \tilde{\boldsymbol{r}}| = \sqrt{\tilde{r}^2 + z^2}.$$

Finally, the current density $\boldsymbol{j}$, which points tangent to the disk's rotation, is

$$\boldsymbol{j} = j(-\sin\tilde{\phi}, \cos\tilde{\phi}, 0).$$

Using the expressions for our vector quantities, the cross product $\boldsymbol{j}(\tilde{\boldsymbol{r}}) \times (\boldsymbol{r} - \tilde{\boldsymbol{r}})$ is

$$\boldsymbol{j}(\tilde{\boldsymbol{r}}) \times (\boldsymbol{r} - \tilde{\boldsymbol{r}}) = j \begin{bmatrix} -\sin\tilde{\phi} \\ \cos\tilde{\phi} \\ 0 \end{bmatrix} \begin{bmatrix} -\tilde{r}\cos\tilde{\phi} \\ -\tilde{r}\sin\tilde{\phi} \\ z \end{bmatrix} = j \begin{bmatrix} z\cos\tilde{\phi} \\ z\sin\tilde{\phi} \\ \tilde{r} \end{bmatrix}.$$

Substituting $\boldsymbol{j}(\tilde{\boldsymbol{r}}) \times (\boldsymbol{r} - \tilde{\boldsymbol{r}})$ and $|\boldsymbol{r} - \tilde{\boldsymbol{r}}|$ into the Biot-Savart law gives

$$\boldsymbol{B}(z) = \frac{\mu_0}{4\pi} \iiint \frac{j \, \mathrm{d}^3\tilde{r}}{\big(\tilde{r}^2 + z^2\big)^{3/2}} \begin{bmatrix} z\cos\tilde{\phi} \\ z\sin\tilde{\phi} \\ \tilde{r} \end{bmatrix}.$$

Next, we write $j \, \mathrm{d}^3\tilde{r} = j \, \mathrm{d}\tilde{S} \, \mathrm{d}\tilde{l} = j \, \mathrm{d}\tilde{S}(\tilde{r} \, \mathrm{d}\tilde{\phi}) = \mathrm{d}I \tilde{r} \, \mathrm{d}\tilde{\phi}$. Note that the product $j \, \mathrm{d}\tilde{S}$ is the current element $\mathrm{d}I$ in the surface element $\mathrm{d}\tilde{S}$. The current $\mathrm{d}I$ at the radius $\tilde{r}$ on the disk rotating with period $t_0 = \frac{2\pi}{\omega}$ is

$$\mathrm{d}I = \frac{\mathrm{d}q}{t_0} = \frac{\mathrm{d}q}{2\pi}\omega = \frac{(\sigma 2\pi\tilde{r} \, \mathrm{d}\tilde{r})}{2\pi}\omega = \sigma\omega\tilde{r} \, \mathrm{d}\tilde{r}.$$

Substituting $j \, \mathrm{d}^3\tilde{r} = \mathrm{d}I \tilde{r} \, \mathrm{d}\tilde{\phi} = \sigma\omega\tilde{r}^2 \, \mathrm{d}\tilde{r} \, \mathrm{d}\tilde{\phi}$ into the Biot-Savart law gives

$$\boldsymbol{B}(z) = \frac{\mu_0}{4\pi} \int_0^a \mathrm{d}\tilde{r} \int_0^{2\pi} \mathrm{d}\tilde{\phi} \frac{\sigma\omega\tilde{r}^2}{\big(\tilde{r}^2 + z^2\big)^{3/2}} \begin{bmatrix} z\cos\tilde{\phi} \\ z\sin\tilde{\phi} \\ \tilde{r} \end{bmatrix}.$$

The first and second components of $\boldsymbol{B}$ contain integrals cos and sin terms over a full period—the result is zero. After integrating over $\tilde{\phi}$, the magnetic field simplifies to

$$\boldsymbol{B}(z) = \frac{\mu_0}{4\pi} \int_0^a \mathrm{d}\tilde{r} \frac{\sigma\omega\tilde{r}^2}{\big(\tilde{r}^2 + z^2\big)^{3/2}} \begin{bmatrix} 0 \\ 0 \\ 2\pi\tilde{r} \end{bmatrix}.$$

Only the z component of $\boldsymbol{B}$ is non-zero; it is

$$B_z(z) = \frac{\mu_0}{2} \int_0^a \mathrm{d}\tilde{r} \frac{\sigma\omega\tilde{r}^3}{\big(\tilde{r}^2 + z^2\big)^{3/2}}.$$

In terms of the new variable $u = \tilde{r}^2 + z^2$, the integral evaluates to

$$\begin{aligned} B_z(z) &= \frac{\mu_0}{2}\frac{\sigma\omega}{2} \int_{z^2}^{z^2+a^2} \frac{u - z^2}{u^{3/2}} \, \mathrm{d}u = \frac{\mu_0\sigma\omega}{4} \Big[2u^{1/2} + 2z^2u^{-1/2} \Big]_{a^2}^{z^2+a^2} \\ &= \frac{\mu_0\sigma\omega}{2} \left(\sqrt{z^2 + a^2} - z + \frac{z^2}{\sqrt{z^2 + a^2}} - z \right) \\ &= \frac{\mu_0\sigma\omega}{2} \left(\frac{2z^2 + a^2}{\sqrt{z^2 + a^2}} - 2z \right). \end{aligned}$$

The magnetic field along the z axis is thus $\boldsymbol{B} = (0, 0, B_z)$ with B_z as above.

Next, we consider the limit case $z \gg a$, far from the rotating disk. Expanding the square root to fourth order in the small quantity $\frac{a}{z}$, multiplying out and simplifying like terms gives

$$\begin{aligned}
B_z &= \frac{\mu_0 \sigma \omega}{2} \left(\frac{2z^2 + a^2}{z\sqrt{1 + \frac{a^2}{z^2}}} - 2z \right) \approx \frac{\mu_0 \sigma \omega}{2} \left[\left(2z + \frac{a^2}{z} \right) \left(1 - \frac{a^2}{2z^2} + \frac{3}{8}\frac{a^4}{z^4} \right) - 2z \right] \\
&= \frac{\mu_0 \sigma \omega}{2} \left[2z + \frac{a^2}{z} - \frac{a^2}{z} - \frac{1}{2}\frac{a^4}{z^3} + \frac{3}{4}\frac{a^4}{z^3} + \frac{3}{8}\frac{a^6}{z^5} - 2z \right] = \frac{\mu_0 \sigma \omega}{2} \left[\frac{1}{4}\frac{a^4}{z^3} + \frac{3}{8}\frac{a^6}{z^5} \right].
\end{aligned}$$

Neglecting the highest-order $\frac{a^6}{z^5}$ term gives the simple result

$$B_z \approx \frac{\mu_0 \sigma \omega}{8} \frac{a^4}{z^3}, \qquad z \gg a.$$

In other words, far from the disk, the magnetic field falls off as z^{-3}, just like the field of a magnetic dipole.

Next, we will try to write the magnetic field in the form $B_z \propto \frac{|\boldsymbol{m}|}{z^3}$ where $\boldsymbol{m}$ is the disk's magnetic dipole moment. Integrating over concentric rings with area S carrying current $\mathrm{d}I$, the disk's magnetic dipole moment is

$$|\boldsymbol{m}| = \int S \,\mathrm{d}I = \int_0^a (\pi \tilde{r}^2) \cdot (\sigma \omega \tilde{r} \,\mathrm{d}\tilde{r}) = \pi \sigma \omega \int_0^a \tilde{r}^3 \,\mathrm{d}\tilde{r} = \frac{\pi}{4} \sigma \omega a^4.$$

Comparing this expression for $|\boldsymbol{m}|$ to the similar expression for B_z leads to

$$B_z \approx \frac{\mu_0 \sigma \omega}{8} \frac{a^4}{z^3} = \frac{\mu_0 |\boldsymbol{m}|}{2\pi z^3},$$

which is in the desired form $B_z \propto \frac{|\boldsymbol{m}|}{z^3}$. The general form for the magnetic field of a magnetic dipole is

$$\boldsymbol{B}(\boldsymbol{r}) = \frac{\mu_0}{4\pi} \frac{3(\boldsymbol{m} \cdot \boldsymbol{r})\boldsymbol{r} - \boldsymbol{m} r^2}{r^5}.$$

In fact, this expression is equivalent to our result $B_z(z) = \frac{\mu_0 |\boldsymbol{m}|}{2\pi z^3}$. Since $\boldsymbol{m}$ and $\boldsymbol{r}$ both point along the z axis, their dot product is $\boldsymbol{m} \cdot \boldsymbol{r} = |m| r$. Along the z axis, $\boldsymbol{r} = (0, 0, z)$ and the general expression for the dipole magnetic field simplifies to

$$\boldsymbol{B}(\boldsymbol{r}) = \frac{\mu_0}{4\pi} \frac{3|m|z^2 - |m|z^2}{z^5} \,\hat{\mathbf{z}} = \frac{\mu_0 |\boldsymbol{m}|}{2\pi z^3} \,\hat{\mathbf{z}},$$

in agreement with our expression for B_z for $z \gg a$.

8 Eighth Exercise Set

Theory: Magnetic Force

The magnetic force $\boldsymbol{F}$ on the matter contained in the region of space enclosed in the region V and permeated by the magnetic field $\boldsymbol{B}$ is

$$\boldsymbol{F} = \frac{1}{\mu_0} \oiint_{\partial V} \left[\boldsymbol{B}(\boldsymbol{B} \cdot \hat{\mathbf{n}}) - \frac{1}{2}\boldsymbol{B}^2 \, \hat{\mathbf{n}} \right] \mathrm{d}S,$$

where $\hat{\mathbf{n}}$ is the normal vector to the region's boundary ∂V.

8.1 Magnetic Force in a Coaxial Cable

A long coaxial cable consists of a thin inner wire and outer sheath with radius a. The inner wire carries a current I, and we connect the sheath to the inner wire at the cable's ends so that the current I returns along the outer sheath in the opposite direction as the current along the inner wire. Find the magnetic force per unit length on the outer sheath.

There are two contributions to the magnetic force on the sheath: the repulsive, radially outward force between the sheath and the inner wire and an attractive "surface tension" force distributed across the sheath's surface, which carries uniformly distributed current I (think of the sheath as a collection of parallel conducting wires, which attract each other).

Consider one-half of the sheath, which forms a semicircular cross section. There are downward forces F_1 at end of the semicircle and an upward magnetic force F_m acting on the top of the sheath. In equilibrium, the forces are related by $F_m = 2F_1$.

First, we find the magnetic field in the conductor. Inside the sheath, Ampere's law with a circular path encircling the inner wire reads

$$\oint \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{l} = \mu_0 I \implies B(2\pi r) = \mu_0 I \implies B = \frac{\mu_0 I}{2\pi r}.$$

Outside the conductor, the net current enclosed by a circular path encircling both the outer sheath and inner wire is zero, and the Ampere's law reads

$$\oint \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{l} = \mu_0(+I - I) = 0 \implies B = 0.$$

In other words, there is no magnetic field outside the conductor.

With the coaxial cable's magnetic field known, we now find the magnetic force $\boldsymbol{F}_m$ on a semi-circular half of the outer sheath.

$$\boldsymbol{F}_m = \frac{1}{\mu_0} \oiint_{\partial V} \left[\boldsymbol{B}(\boldsymbol{B} \cdot \hat{\mathbf{n}}) - \frac{1}{2}B^2 \, \hat{\mathbf{n}} \right] \mathrm{d}S.$$

We choose an integration surface tightly hugging the half-sheath and work in cylindrical coordinates r, ϕ. Outside the sheath, the magnetic field is zero and there is no contribution to $\boldsymbol{F}_m$. Inside the sheath, the magnetic field is tangent to the semicircle, so $\boldsymbol{B} \cdot \hat{\mathbf{n}} = 0$, and the force integral reads

$$\boldsymbol{F}_m = \frac{1}{\mu_0} \iint \left[0 - \frac{1}{2} B^2 \, \hat{\mathbf{n}}\right] \mathrm{d}S.$$

The magnetic field at along the sheath (where $r = a$) is constant and equal to $B = \frac{\mu_0 I}{2\pi a}$ and can be moved outside the integral. The normal vector in cylindrical coordinates reads

$$\hat{\mathbf{n}} = (-\cos\phi, -\sin\phi, 0),$$

while the surface element is $\mathrm{d}S = al \,\mathrm{d}\phi$. The force integral reads

$$\boldsymbol{F}_m = -\frac{1}{2} \frac{\mu_0 I^2}{4\pi^2 a^2} \int_0^\pi \begin{bmatrix} -\cos\phi \\ -\sin\phi \\ 0 \end{bmatrix} la \,\mathrm{d}\phi.$$

The x component with $\cos\phi$ integrates to zero over $\phi \in [0, \pi]$. Only the y component is nonzero, and the vector force reads

$$\boldsymbol{F}_m = +\frac{1}{2} \frac{\mu_0 I^2}{4\pi^2 a^2} (2al) \, \hat{\mathbf{y}} = \frac{\mu_0 I^2 l}{4\pi^2 a} \, \hat{\mathbf{y}}.$$

The forces F_1 at the two ends of the semicircle are then

$$F_1 = \frac{F_m}{2} = \frac{\mu_0 I^2 l}{8\pi^2 a} \implies \frac{F}{l} = \frac{\mu_0 I^2}{8\pi^2 a}.$$

8.2 Tension in a Toroidal Inductor

A toroidal inductor with N coils radius r_1 and cross-sectional radius r_2 carries a current I. Find the tension force on a single coil if $r_1 \gg r_2$.

First, we find the magnetic field inside the inductor with Ampere's law using a closed circular path of radius $r \in (r_1, r_1 + r_2)$ in the inductor's equatorial plane. Ampere's law reads

$$B_{\text{in}}(2\pi r) = \mu_0 (NI) \implies B_{\text{in}} = \frac{\mu_0 NI}{2\pi r}.$$

Outside the inductor, with a circular path of radius $r > r_1 + r_2$ in the inductor's equatorial plane, Ampere's law reads

$$B_{\text{out}}(2\pi r) = \mu_0 (NI + N(-I)) = 0 \implies B_{\text{out}} = 0.$$

Consider a semicircular half of a single inductor coil. As in the previous problem, the magnetic force with magnitude F_m acts upwards on the top of the semicircle, while two "surface tension" forces $F_1 = \frac{F_m}{2}$ act downward at the semicircle's two ends. As usual, we find the magnetic force $\boldsymbol{F}_m$ using

$$\boldsymbol{F}_m = \frac{1}{\mu_0} \oiint_{\partial V} \left[\boldsymbol{B}(\boldsymbol{B} \cdot \hat{\mathbf{n}}) - \frac{1}{2} B^2 \, \hat{\mathbf{n}}\right] \mathrm{d}S.$$

For the integration surface (awkward to describe, best to see a picture) we choose a surface that looks like a coin cut in half, basically a thin three-dimensional extension of a semicircle enclosing the coil's semicircular upper half. We split the surface into four parts: the left and right semicircular faces, the circular ribbon along the surface's outer radius, and the thin rectangular plane along the surface's bottom.

The circular upper ribbon occurs just outside the inductor coil (where $B = 0$) and thus does not contribute the magnetic force. The left and right semicircular faces have equal magnitude and opposite-sign contributions, since the normal to the surface changes sign for each face.

Only the rectangular plane has a nonzero contribution to $\boldsymbol{F}_m$. Under the assumption $r_1 \gg r_2$, the magnetic field along the plane simplifies to

$$B = \int_{r_1}^{r_1+r_2} \frac{\mu_0 NI}{2\pi\tilde{r}} \,\mathrm{d}\tilde{r} \approx \frac{\mu_0 NI}{2\pi r_1}.$$

The magnetic field $\boldsymbol{B}$ points along the toroid's longitudinal axis (into the plane of a cross-sectional coil) and is perpendicular to the normal $\hat{\mathbf{n}}$ to the planar integration surface, so $\hat{\mathbf{B}} \cdot \hat{\mathbf{n}} = 0$. The magnetic force simplifies to

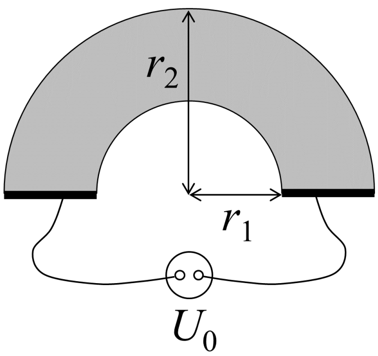
$$\begin{aligned} \boldsymbol{F}_m &= \frac{1}{\mu_0} \left[0 - \frac{1}{2} B^2 \,\hat{\mathbf{n}}\right] \mathrm{d}S = -\frac{1}{2\mu_0} \left(\frac{\mu_0 NI}{2\pi r_1}\right)^2 \hat{\mathbf{n}} \iint \mathrm{d}S \\ &= -\frac{1}{2\mu_0} \left(\frac{\mu_0 NI}{2\pi r_1}\right)^2 \hat{\mathbf{n}} \left(r_2 \frac{2\pi r_1}{N}\right) = -\frac{\mu_0 NI^2}{2\pi} \frac{r_2}{r_1} \,\hat{\mathbf{n}}. \end{aligned}$$

The unit vector $\hat{\mathbf{n}}$ points downward, so $-\hat{\mathbf{n}}$ and thus $\boldsymbol{F}_m$ point upward and pull the coil apart. The magnitude of the tension on the coil is then

$$F_1 = \frac{F_m}{2} = \frac{\mu_0 NI^2}{4\pi} \frac{r_2}{r_1}.$$

8.3 Resistance of a Thin Conducting Plate

Consider a conducting plate consisting of half of a thin annulus with inner radius r_1, outer radius r_2, thickness h and conductivity σ, with electrodes placed at each end. Find the plate's electric resistance R when we establish a potential difference U_0 between the electrodes at the plate's ends.



We first find the current density in the conducting plate using Ohm's law

$$\boldsymbol{j} = \sigma \boldsymbol{E},$$

where $\boldsymbol{j}$ and $\boldsymbol{E}$ are the current density and electric field in the conductor. We write $\boldsymbol{j}$ in terms of the continuity equation

$$\nabla \cdot \boldsymbol{j} + \frac{\partial \rho}{\partial t} = 0.$$

For a stationary charge distribution, we have $\nabla \cdot \boldsymbol{j} = 0$. We then take the divergence of both sides of the equation $\boldsymbol{j} = \sigma \boldsymbol{E} = \rho$ and apply $\boldsymbol{E} = -\nabla U$ to get

$$\nabla \cdot \boldsymbol{j} = \sigma \nabla \cdot \boldsymbol{E} = -\sigma \nabla^2 U = \nabla \cdot \boldsymbol{j} = 0 \implies \nabla^2 U = 0.$$

We end up with a Laplace equation for U in the conductor. The plan is to find $U(\boldsymbol{r})$, then $\boldsymbol{E}(r)$, then $\boldsymbol{j}$ via $\boldsymbol{j} = \sigma \boldsymbol{E}$, then I and finally resistance with $R = \frac{U}{I}$.

The general solution of the Laplace equation in cylindrical coordinates (including the $m = 0$ term) is

$$U(r, \phi) = \sum_{m=1}^{\infty} \big[A_m \cos(m\phi) + B_m \sin(m\phi)\big] \cdot \big[C_m r^m + D_m r^{-m}\big] \\ + (a\phi + b)(c \ln r + d).$$

To find a unique solution, we need boundary conditions for our particular problem. At the first electrode at $\phi = 0$, the electric potential is constant; we'll set $U(r, 0) = 0$ for convenience. At the second electrode, the potential is $U(r, \pi) = -U_0$ to make a potential difference U_0 between electrodes (we choose $-U_0$ so the current runs in the direction of increasing ϕ).

Along the annulus's semicircular boundaries the electric field must be tangent to the surface to satisfy $\nabla \cdot \boldsymbol{j} = 0$. Along these surfaces the radial component of both $\boldsymbol{j}$ and $\boldsymbol{E}$ is zero; $E_r = 0$ gives the boundary condition

$$E_r = -\left.\frac{\partial U}{\partial r}\right|_{r_1, r_2} = 0.$$

The general solution for $U(r, \phi)$ satisfies the boundary condition $\frac{\partial U}{\partial r}\big|_{r_1, r_2} = 0$ for all ϕ only if $C_m = D_m = c = 0$ (try finding $\frac{\partial U}{\partial r}\big|_{r_1, r_2}$ to see for yourself). With $C_m = D_m = c = 0$, the expression for $U(r, \phi)$ simplifies to

$$U(r, \phi) = (a\phi + b)d \equiv \tilde{a}\phi + \tilde{b}.$$

The boundary condition $U(r, 0) = 0$ gives

$$U(r, 0) \equiv= 0 + \tilde{b} \implies \tilde{b} = 0.$$

The final boundary condition $U(r, \pi) = -U_0$ gives

$$U(r, \pi) \equiv= -U_0 \implies \tilde{a} = -\frac{U_0}{\pi} \implies U(r, \phi) = -\frac{U_0}{\pi}\phi.$$

In other words, the electric potential is a linear function of ϕ.

With $U(r, \phi)$ known, we find the tangential electric field E_t with

$$E_t = -\frac{1}{r}\frac{\partial U}{\partial \phi} = \frac{U_0}{\pi r},$$

from which we find the tangential current density j_t with

$$j_t = \sigma E_t = \frac{\sigma U_0}{\pi r}.$$

We find the total current I from the current density via

$$I = \iint j_t \, \mathrm{d}S = \int_{r_1}^{r_2} \left(\frac{\sigma U_0}{\pi r} \right) \cdot h \, \mathrm{d}r = \frac{\sigma U_0 h}{\pi} \ln \frac{r_2}{r_1},$$

where h is the plate's thickness. The electrical resistance is then

$$R = \frac{U_0}{I} = \frac{\pi}{\sigma h \ln \frac{r_2}{r_1}}.$$

9 Ninth Exercise Set

Theory: Inductance

Inductance is the proportionality constant between magnetic flux Φ and current I. For a single object, e.g. a current loop, the relationship between Φ and I reads

$$\Phi_1 = L_{11} I_1,$$

where the quantity L_{11} is called self-inductance.

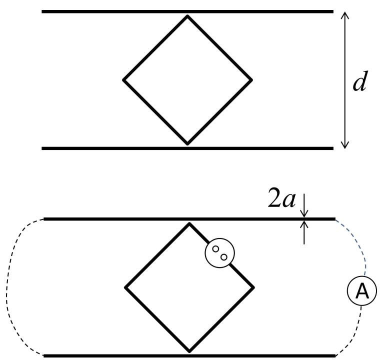
For a system of two current-carrying loops, where the current through one loop generates a magnetic field and thus magnetic flux through the second loop, and vice versa, the relationships between Φ and I read

$$\Phi_1 = L_{12} I_2 \qquad \text{and} \qquad \Phi_2 = L_{21} I_1,$$

where the quantities L_{12} and L_{21} are the loops' mutual inductances. Without derivation, we state that $L_{12} = L_{21}$ for reasons of symmetry.

9.1 Mutual Inductance

Consider two long, parallel wires of length l separated by a distance $d \ll l$ and connected at their endpoints to form a long conducting loop. We place a square frame with side length $\sqrt{2}d$ between the wires and supply the frame with an alternating current source $I_1 = I_{1_0} \sin \omega t$. Find the system's mutual inductivity and the induced current in the wires. Neglect Ohmic losses, and assume $a \ll d$ where a is the wires' radius.



Finding Mutual Inductance

We label the frame as object 1 and the wires as object 2 and introduce a coordinate y separating the parallel wires so that the bottom wire occurs at $y = 0$ and the top wire at $y = d$. To find the system's mutual inductance L_{12}, we consider the hypothetical magnetic flux Φ_1 through the frame due to a current I_2 in the parallel wires. We then find L_{12} with

$$L_{12} = \frac{\Phi_1}{I_2}.$$

If the parallel wires carry a current I_2 (in opposite directions, since they form a closed loop), the corresponding magnetic field is

$$B = \frac{\mu_0 I_2}{2\pi} \left(\frac{1}{y} + \frac{1}{d - y} \right) = \frac{\mu_0 I_2}{2\pi} \frac{d}{y(d - y)}.$$

We find magnetic flux with $\Phi = \int \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{S}$; the dot product drops because $\boldsymbol{B}$ is parallel to the frame's cross section $\mathrm{d}\boldsymbol{S}$. The square frame's surface element is $\mathrm{d}S = 2y\,\mathrm{d}y$

(the frame's width is 2y), and the magnetic flux through the square frame is

$$\begin{aligned}\Phi_1 &= \iint B\,\mathrm{d}S = 2\int_0^{d/2} \frac{\mu_0 I_2 d}{2\pi}\frac{2y\,\mathrm{d}y}{y(d-y)} = -\frac{2\mu_0 I_2 d}{\pi}\ln(d-y)\Big|_0^{d/2} \\ &= \frac{2\mu_0 I_2 d}{\pi}\left[\ln d - \ln\frac{d}{2}\right] = \frac{2\mu_0 d\ln 2}{\pi} I_2.\end{aligned}$$

Note the use of symmetry—we integrate only from 0 to $d/2$ and multiply by two.

The mutual inductance—the proportionality between Φ and I_2 is

$$L_{12} = \frac{\Phi_1}{I_2} = \frac{2\ln 2}{\pi}\mu_0 d.$$

Note that L_{12} depends only on system's geometry.

Induced Current in the Parallel Wires

Recall the square frame carries an alternating current

$$I_1(t) = I_{1_0}\sin\omega t,$$

so we expect the induced current $I_2(t)$ in the parallel wires to alternate with the same frequency ω and a general form

$$I_2(t) = I_{2_0}\sin(\omega t).$$

We will find the ratio of current amplitudes I_{2_0}/I_{1_0}.

We'll solve the problem as follows: use the frame current I_1 to find the magnetic flux Φ_2 through the parallel wires, then use Φ_2 to find induced voltage U_2 in the wires, and finally use U_2 to find the induced current I_2.

We find Φ_2 from $\Phi_2 = L_{21}I_1$ and the symmetry relation $L_{12} = L_{21}$, i.e.

$$\Phi_2 = L_{21}I_1 = L_{12}I_1,$$

where L_{12} was found the first part of the problem. We then find U_2 with

$$U_{12} = -\dot{\Phi}_2 = -L_{12}\dot{I}_1.$$

Next, we find I_2 from U_2 from the general circuit equation

$$U = RI + L_\mathrm{s}\dot{I},$$

which relates the voltage in a loop of resistance R and self-inductance L_s to the current I through the loop. In our case, neglecting resistance, we have

$$U_{12} \approx L_{22}\dot{I}_2,$$

where L_{22} is the parallel wire's self-inductance. Substituting U_{12} into the earlier expression $U_{12} = -L_{12}\dot{I}_1$ gives

$$-L_{12}\dot{I}_1 = L_{22}\dot{I}_2.$$

If we assume both I_1 and I_2 are sinusoidal with amplitudes I_{1_0} and I_{2_0}, the above reduces to

$$\frac{I_{2_0}}{I_{1_0}} = \frac{L_{12}}{L_{22}}.$$

What about the minus sign?

To find the current amplitude ratio, we just need to find the parallel wires self-inductance L_{22} using $\Phi_2 = L_{22} I_2$

If we send a hypothetical current current I_2 through the wires, the corresponding magnetic field is, as before,

$$B = \frac{\mu_0 I_2}{2\pi} \left(\frac{1}{y} + \frac{1}{d-y} \right).$$

The field and surface are parallel, so $\boldsymbol{B} \cdot \mathrm{d}\boldsymbol{S} = B \, \mathrm{d}S$, and the magnetic flux is

$$\Phi_2 = \iint B \, \mathrm{d}S = \frac{\mu_0 I_2 l d}{2\pi} \int_a^{d-a} \left(\frac{1}{y} + \frac{1}{d-y} \right) \mathrm{d}y,$$

where a is the wire's radius. The integral evaluates to

$$\Phi_2 = \frac{\mu_0 I_2 l}{2\pi} \left[\ln \frac{d-a}{a} - \ln \frac{a}{d-a} \right] = \frac{\mu_0 I_2 l}{\pi} \ln \frac{d-a}{a}.$$

Applying $d \gg a$ we have

$$\Phi_2 = \frac{\mu_0 I_2 l}{\pi} \ln \frac{d}{a}.$$

The self-inductance—the proportionality constant between Φ_2 and I_2—is thus

$$L_{22} = \frac{\mu_0 l}{\pi} \ln \frac{d}{a}.$$

With L_{22} known, we return to the current amplitude ratio to get

$$\frac{I_{2_0}}{I_{1_0}} = \frac{\frac{2 \ln 2}{\pi} \mu_0 d}{\frac{\mu_0 l}{\pi} \ln \frac{d}{a}} = \frac{2 \ln 2}{\frac{l}{d} \ln \frac{d}{a}}.$$

To get a better feel for the numbers involved, if we assume $l/d = d/a = 10$, we have

$$\frac{I_{2_0}}{I_{1_0}} \approx 0.06.$$

9.2 The Cabrera Experiment and Magnetic Monopoles

Consider a superconducting current loop with non-zero inductance L, radius a and a built-in ammeter. Assume a magnetic monopole passes through the loop along the axis of symmetry. Find the resulting current pulse in the loop.

First, the magnetic field of a hypothetical monopole is

$$\boldsymbol{B}(\boldsymbol{r}) = \frac{\mu_0 g}{4\pi} \frac{\boldsymbol{r}}{r^3},$$

where g is the "magnetic charge", with units A m. Our plan is to find the magnetic flux through the loop, use this to find voltage induced in the loop, use the induced voltage to find current.

Let $d(t)$ be the monopole's perpendicular distance from the loop's center, and let ρ denote radial distance from the current loop's center (in the plane of the loop).

The magnetic field magnitude a distance r from the monopole is

$$B = \frac{\mu_0}{4\pi}\frac{g}{r^2} = \frac{\mu_0 g}{4\pi}\frac{1}{d^2+\rho^2}.$$

We find the magnetic field component $B_\perp$ perpendicular to the current loop with similar triangles:

$$B_\perp = \frac{d}{\sqrt{d^2+\rho^2}}B = \frac{\mu_0 g d}{4\pi}\frac{1}{(\rho^2+d^2)^{3/2}}.$$

The magnetic flux Φ through the loop, using $\mathrm{d}S = 2\pi\rho\,\mathrm{d}\rho$, is

$$\begin{aligned}\Phi &= \iint B_\perp\,\mathrm{d}S = \frac{\mu_0 g d}{2}\int_0^a \frac{\rho\,\mathrm{d}\rho}{(\rho^2+d^2)^{3/2}} = \frac{\mu_0 g d}{4}(-2)\frac{1}{\sqrt{\rho^2+d^2}}\bigg|_0^a \\ &= \frac{\mu_0 g}{2}\left(1-\frac{d}{\sqrt{a^2+d^2}}\right).\end{aligned}$$

We assume the monopole moves with constant speed v and passes through the loop's center at $t=0$. On the left of the loop (and thus for negative time), the monopole's perpendicular distance from the loop is

$$d(t) = -vt.$$

The time-dependent flux on the left of the loop is thus

$$\Phi(t) = \frac{\mu_0 g}{2}\left(1+\frac{vt}{\sqrt{a^2+v^2t^2}}\right).$$

On the right of loop (and for positive time), the distance from the loop is $d = vt$, and thus the magnetic flux through the loop is

$$\Phi(t) = -\frac{\mu_0 g}{2}\left(1-\frac{vt}{\sqrt{a^2+v^2t^2}}\right).$$

Note the additional minus sign, since on the right of the loop, once the particle passes through, the magnetic flux points in the opposite direction.

Note that $\Phi(t)$ increases from 0 at $t\to-\infty$ to a maximum value of $\frac{\mu_0 g}{2}$ as $t\to 0$ from the left. As the particle passes through the loop at $t=0$, Φ jumps discontinuously to $-\frac{\mu_0 g}{2}$, and then decreases back to 0 as $t\to\infty$.

Next, we find the induced voltage in the loop. We begin with a modified Maxwell equation:

$$\nabla\times\boldsymbol{E} = -\frac{\partial\boldsymbol{B}}{\partial t} - \mu_0\boldsymbol{j}_{\mathrm{m}},$$

where $\boldsymbol{j}_{\mathrm{m}}$ is magnetic current density and accounts for the possible existence of magnetic monopoles. We then use this modified Maxwell equation to re-derive the law of induction in the presence of magnetic monopoles. We integrate the equation over the loop's surface to get

$$\iint_S (\nabla \times \boldsymbol{E})\,\mathrm{d}\boldsymbol{S} = -\frac{\partial}{\partial t}\iint_S \boldsymbol{B}\cdot\mathrm{d}\boldsymbol{S} - \mu_0 \iint_S \boldsymbol{j}_{\mathrm{m}}\,\mathrm{d}S.$$

We then apply Stokes' theorem, which leads to

$$\oint \boldsymbol{E}\cdot\mathrm{d}\boldsymbol{l} = U_i = -\dot{\Phi} - \mu_0 I_{\mathrm{m}},$$

where I_{m} is magnetic current through the loop's cross section and U_i is the induced voltage in the loop. The magnetic current through the loop is nonzero only at the singular instant when the magnetic monopole passes through, which we model with the delta function:

$$I_{\mathrm{m}} = g\delta(t),$$

where $\delta(t)$ has units s^{-1}.

If we neglect resistive and capacitive effects, the loop's circuit equation reads $U_i = L\dot{I}$. Substituting in $U_i = -\dot{\Phi} - \mu_0 I_{\mathrm{m}}$ gives

$$U_i = U_i = -\dot{\Phi} - \mu_0 I_{\mathrm{m}} = -\dot{\Phi} - \mu_0 g\delta(t) = L\dot{I}.$$

We then integrate this equation over time to get

$$\begin{aligned} L\int_{-\infty}^{t} \dot{I}\,\mathrm{d}\tilde{t} &= -\int_{-\infty}^{t} \dot{\Phi}\,\mathrm{d}\tilde{t} - \mu_0 g \int_{-\infty}^{t} \delta(\tilde{t})\,\mathrm{d}\tilde{t} \\ LI(\tilde{t})\Big|_{-\infty}^{t} &= -\Phi(\tilde{t})\Big|_{-\infty}^{t} - \mu_0 g H(t), \end{aligned}$$

where $H(t)$ is the Heaviside step function. Since $I(t \to -\infty) = \Phi(t \to -\infty) = 0$ (when the monopole is infinitely far from the loop), the equation reduces to

$$LI(t) = -\Phi(t) - \mu_0 g H(t).$$

Remember that Φ changes from $\frac{\mu_0 g}{2}$ to $-\frac{\mu_0 g}{2}$ as the monopole passes through the loop at $t = 0$. The discontinuity in Φ is exactly balanced by the Heaviside step function activating with magnitude $\mu_0 g$ at $t = 0$, and the effect is that $LI(t)$ and thus the current through the loop is as a continuous, measurable quantity.

If we use the Dirac quantization of magnetic charge, which reads

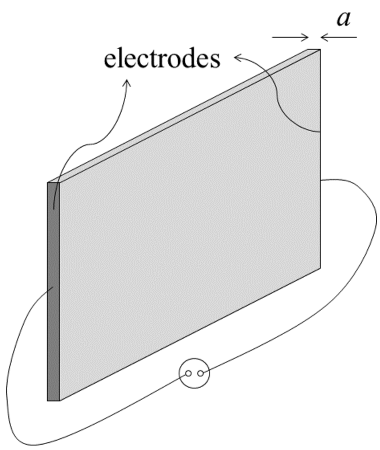
$$\frac{\mu_0 g e_0}{2} = h \implies \mu_0 g = \frac{2h}{e_0},$$

we get a numerical result for the quantity $\mu_0 g$, and a theoretically expected value value of $LI(t)$.

10 Tenth Exercise Set

10.1 Skin Effect in a Ribbon-Like Conductor

Consider a long conducting ribbon of conductivity σ, width a, height b and length l where $l \gg b \gg a$. We place ideal electrodes at each end of the ribbon, and connect the electrodes to an alternating voltage source with frequency ω. Find the dependence of the conductor's impedance on frequency and investigate the high- and low-frequency limits.



Our plan is to solve for the electric field $\boldsymbol{E}(t)$ in the ribbon, use $\boldsymbol{E}$ to find the potential difference U, and finally use U to find the ribbon's impedance. We begin with the relevant Maxwell equations, which are

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t} \qquad \nabla \cdot \boldsymbol{E} = 0$$

$$\nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{j} + \epsilon_0\mu_0 \frac{\partial \boldsymbol{E}}{\partial t} \qquad \nabla \cdot \boldsymbol{B} = 0.$$

Note that $\nabla \cdot \boldsymbol{E} = 0$ because the conductor is neutral—no electric field escapes.

We make a quasi-static approximation to simplify the equations, neglecting the displacement current $\epsilon_0\mu_0 \frac{\partial \boldsymbol{E}}{\partial t}$ to get

$$\nabla \times \boldsymbol{B} \approx \mu_0 \boldsymbol{j}.$$

We take the curl of the first Maxwell equation to get

$$\nabla \times (\nabla \times \boldsymbol{E}) = \nabla(\nabla \cdot \boldsymbol{E}) - \nabla^2 \boldsymbol{E} = -\frac{\partial}{\partial t} \nabla \times \boldsymbol{B}.$$

We then apply $\nabla \cdot \boldsymbol{E} = 0$ and $\nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{j} = \mu_0(\sigma \boldsymbol{E})$ to get

$$-\nabla^2 \boldsymbol{E} = -\frac{\partial}{\partial t} \mu_0 \sigma \boldsymbol{E}.$$

Since the ribbon is attached to an alternating voltage source, the electric field $\boldsymbol{E} = \boldsymbol{E}(t)$ reads

$$\boldsymbol{E}(t) = \boldsymbol{E}_0 e^{i\omega t} \implies \frac{\partial}{\partial t} \boldsymbol{E}(t) = i\omega \boldsymbol{E}(t).$$

Finally, we substitute the $\frac{\partial E}{\partial t}$ into the earlier equation $-\nabla^2 \boldsymbol{E} = -\frac{\partial}{\partial t} \mu_0 \sigma \boldsymbol{E}$ to get

$$\nabla^2 \boldsymbol{E} - i\omega\mu_0\sigma \boldsymbol{E} \equiv \nabla^2 \boldsymbol{E} - k^2 \boldsymbol{E} = 0,$$

where we have defined the amplitude

$$k^2 \equiv i\omega\mu_0\sigma \implies k = \sqrt{i\omega\mu_0\sigma} = \frac{1+i}{\sqrt{2}} \sqrt{\omega\mu_0\sigma}.$$

Next, we define coordinate system whose x axis aligns with the ribbon's width a and whose y axis aligns with the ribbon's height. The conductor's length corresponds to the z axis. We choose the origin so that $x \in [-a/2, a/2]$, meaning the ribbon's center occurs at $x = 0$.

With respect to this coordinate system, we can then simplify the Laplacian $\nabla^2 \boldsymbol{E}$. Since the ribbon's width a is much smaller than the height and length, the Laplacian's derivatives with respect to y and z are negligible, i.e.

$$\nabla^2 \boldsymbol{E} = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) \boldsymbol{E} \approx \frac{\partial^2 \boldsymbol{E}}{\partial x^2}.$$

The alternating voltage is applied along the conductor's length—along the z axis—so the electric field reads $\boldsymbol{E} = E_z(x)\,\hat{\mathbf{z}}$. Plugging all of these simplifications into the amplitude equation gives

$$\nabla^2 \boldsymbol{E} - k^2 \boldsymbol{E} \approx \frac{\partial^2}{\partial x^2} \big[E_z(x)\,\hat{\mathbf{z}} \big] - k^2 E_z(x)\,\hat{\mathbf{z}} = E_z''(x) - k^2 E_z(x) = 0.$$

The solutions to the equation

$$E_z'' - k^2 E_z = 0$$

can be written either as exponents or hyperbolic functions; we will use hyperbolic functions, which are best suited to the problem's reflection symmetry about the y axis. The general solution is

$$E_z(x) = A \cosh kx + B \sinh kx.$$

However, the problem's reflection symmetry means E_z will have only the even component cosh, and the solution simplifies to

$$E_z(x) = A \cosh kx.$$

We find A from the problem's boundary conditions. Assuming $E_z(\pm a/2) = E_0$, we have

$$A \cosh \left(\pm \frac{ka}{2} \right) = A \cosh \frac{ka}{2} = E_0 \implies A = \frac{E_0}{\cosh \frac{ka}{2}}.$$

With the constant A known, the solution for $E_z(x)$ is then

$$E_z(x) = A \cosh kx = E_0 \frac{\cosh kx}{\cosh \frac{ka}{2}}.$$

Next, we draw some qualitative sketches of $E(x)$ with k as a parameter. The resulting curves show that as k (and thus frequency ω) increases, $E(x)$ becomes concentrated near the ribbon's outer surfaces $x = \pm a/2$. This is a qualitative demonstration of the skin effect, where electric field and current become concentrated along a conductor's surface at high frequencies.

Potential Difference, Current and Impedance

Next, with $E_z(x)$ known, the potential difference across the ribbon's is simply

$$U_z(x) = E_z(x) l,$$

where l is the conductor length.

With potential difference known, we find the current through the conductor with

$$I = \iint j \, \mathrm{d}S = \iint (\sigma E_z) \, \mathrm{d}S = \frac{\sigma E_0}{\cosh \frac{ka}{2}} \int_{-a/2}^{a/2} \cosh(kx)(b \, \mathrm{d}x),$$

where we have written the surface element $\mathrm{d}S = b \, \mathrm{d}x$ in terms of the conductor's height b. The integral evaluates to

$$\begin{aligned} I &= \frac{\sigma E_0 b}{\cosh \frac{ka}{2}} \int_{-a/2}^{a/2} \cosh(kx) \, \mathrm{d}x = \frac{\sigma E_0 b}{\cosh \frac{ka}{2}} \cdot \frac{2}{k} \cdot \sinh kx \Big|_{x=0}^{a/2} \\ &= \frac{2\sigma E_0 b}{k} \tanh \frac{ka}{2}. \end{aligned}$$

Next, we introduce the dimensionless quantity $\kappa = \frac{ka}{2}$ and substitute in $U_0 = E_0 l$ to get

$$I = \frac{\sigma U_0 ba}{l} \frac{2}{ka} \tanh \frac{ka}{2} = \frac{\sigma U_0 ba}{l} \frac{\tanh \kappa}{\kappa} = \frac{U_0}{R_0} \frac{\tanh \kappa}{\kappa},$$

where, in the last equality, we have substituted in the conductor's static resistance

$$R_0 = \frac{l}{\sigma S}.$$

In terms of R_0, the conductor's impedance is then

$$Z \equiv \frac{U_0}{I} = R_0 \frac{\kappa}{\tanh \kappa}.$$

High and Low Frequency Limits

In the low frequency limit, k and thus κ are small ($\kappa \ll 1$) and we expand the $\tanh x \approx$ function to get

$$Z \approx R_0 \frac{\kappa}{\kappa} = R_0.$$

In other words, the ribbon's impedance approaches the static resistance R_0.

Finally, for large frequencies and thus $\kappa \gg 1$, we use the asymptotic expansion $\tanh x \approx 1$ to get

$$Z \approx R_0 \kappa = \frac{R_0 a}{2} k = \frac{R_0 a}{2} \sqrt{\frac{\sigma \mu_0 \omega}{2}} (1 + i).$$

The real component $\operatorname{Re} Z$ corresponds to resistance $R(\omega)$, while the imaginary component $\operatorname{Im} Z$ corresponds to reactance. Reactance is out of phase with resistance by $\pi/2$. At high frequencies, the resistance is

$$R(\omega) = \frac{r_0 a}{2} \sqrt{\frac{\sigma \mu_0 \omega}{2}}.$$

Note the relationship $R \propto \sqrt{\omega}$, meaning resistance increases with frequency, since the electric field is more concentrated at the conductor edges (the "skin") and the current has a smaller effective cross section and thus larger resistance.

10.2 Theory: Conservation of Electromagnetic Energy

Conservation of electromagnetic energy is written as the energy balance

$$\frac{\partial w}{\partial t} + \nabla \cdot \mathbf{S} + \boldsymbol{j} \cdot \boldsymbol{E} = 0,$$

where w is electromagnetic energy density

$$w = \frac{1}{2}\left(\epsilon_0 E^2 + \frac{1}{\mu_0} B^2\right)$$

and $\mathbf{S}$ is the Poynting vector, defined as

$$\mathbf{S} = \frac{1}{\mu_0}\boldsymbol{E} \times \boldsymbol{B} = \boldsymbol{E} \times \boldsymbol{H}.$$

In integral form for a region of space V, the energy equation reads

$$\frac{\partial}{\partial t} \iiint_V w \, \mathrm{d}V + \oiint_{\partial V} \mathbf{S} \cdot \mathrm{d}\boldsymbol{S} + \iiint_V \boldsymbol{j} \cdot \boldsymbol{E} \, \mathrm{d}V = 0,$$

where we have used the divergence theorem to convert the Poynting vector term to a surface integral.

- The w term corresponds to the changing electromagnetic field energy within the region V
- The $\mathbf{S}$ term encodes energy flow (power) through the surface
- The $\boldsymbol{j} \cdot \boldsymbol{E}$ term corresponds to Ohmic energy losses within the region.

10.3 Power in a Coaxial and Cylindrical Conductor

Find the electromagnetic energy flow (power) through:

1. *The cross section of a coaxial cable with inner and outer radii* a *and* b*, respectively, and carrying current* I*, of length* l*, with potential difference* U *between the inner and outer conductor. Neglect resistive losses.*
2. *The lateral surface of a long, straight cylindrical conductor of radius* a*, length* l*, conductivity* σ *and carrying a current* I *with a potential difference* U *between the cylinder ends.*

Lossless Coaxial Conductor

In a coaxial cable, the magnetic field $\boldsymbol{B}$ points tangent to the circular cross section (perpendicular to the radial direction), while the electric field $\boldsymbol{E}$ points radially outward. We assume positive charge accumulates on the inner conductor and negative charge on the outer conductor.

We find the magnetic field with Ampere's law using a loop around the inner conductor. The result is

$$\mu_0 I = B \cdot 2\pi r \implies B = \frac{\mu_0 I}{2\pi r}.$$

Meanwhile, we find electric field with Gauss's law, using a cylinder enclosing the inner conductor:

$$Q = \epsilon_0 E \cdot 2\pi r l \implies E = \frac{Q}{2\pi\epsilon_0 l r}.$$

The induced charge Q and potential difference between the inner and outer conductor are related by

$$U = \int_a^b E \,\mathrm{d}r = \frac{Q}{2\pi\epsilon_0 l} \ln\frac{b}{a}.$$

In terms of U, the electric field is thus

$$E = \frac{Q}{2\pi\epsilon_0 l r} = \frac{U}{r \ln\frac{b}{a}}.$$

The Poynting vector points in the direction $\boldsymbol{E} \times \boldsymbol{B}$, along the conductor's longitudinal axis. In our case we have $\boldsymbol{E} \perp \boldsymbol{B}$, so the Poynting vector magnitude S is

$$\mathrm{S} = \frac{1}{\mu_0} E B = \frac{1}{\mu_0}\left(\frac{\mu_0 I}{2\pi r}\right)\left(\frac{U}{r\ln\frac{b}{a}}\right) = \frac{UI}{2\pi \ln\frac{b}{a} r^2}.$$

The electromagnetic power through the cross section is thus

$$\begin{aligned} P &= \iint \mathbf{S} \cdot \mathrm{d}\boldsymbol{S} = \iint \mathrm{S}\,\mathrm{d}S = \int_a^b \left(\frac{UI}{2\pi\ln\frac{b}{a} r^2}\right)(2\pi r \,\mathrm{d}r) \\ &= \frac{UI}{\ln\frac{b}{a}} \int_a^b \frac{\mathrm{d}r}{r} = \frac{UI}{\ln\frac{b}{a}} \ln\frac{b}{a} = UI, \end{aligned}$$

where $\mathbf{S} \cdot \mathrm{d}\boldsymbol{S} = \mathrm{S}\,\mathrm{d}S$ because $\mathbf{S}$ is parallel to coaxial cable's cross section.

Cylindrical Conductor

As for the coaxial cable, the cylindrical conductor's magnetic field is tangent to the conductor's circular cross section. Meanwhile, the electric field points in the direction of the potential difference—along the conductor's longitudinal axis. The Poynting vector $\mathbf{S} = \boldsymbol{E} \times \boldsymbol{B}$ thus along the radial direction. We choose I to flow so that $\mathbf{S}$ points radially inward towards the conductor's center (as opposed to radially outward for an opposite current).

The electric field in the conductor, which arises from the potential difference U between the conductor's ends, is simply

$$E = \frac{U}{l},$$

where l is the conductor's length.

We find the conductor's magnetic field with Ampere's law, using a circular loop centered along the conductor's longitudinal axis. Assuming current is uniformly distributed across the cross section, the magnetic field is

$$B \cdot (2\pi r) = \mu_0 I \left(\frac{r}{a}\right)^2 \implies B = \frac{\mu_0 I}{2\pi a^2} r.$$

Since $\boldsymbol{E}$ and $\boldsymbol{B}$ is perpendicular, the Poynting vector is

$$\mathrm{S} = \frac{1}{\mu_0} EB = \frac{1}{\mu_0}\left(\frac{U}{L}\right)\left(\frac{\mu_0 I}{2\pi a^2} r\right) = \frac{UI}{2\pi a^2 l} r.$$

Note that S increases linearly with r.

We find electromagnetic power by integrating S over the cylinder's lateral surface (not the cross section!). The power through a lateral surface at radius r is

$$P(r) = \iint_{S_{\mathrm{lat}}} \mathbf{S} \cdot \mathrm{d}\boldsymbol{S} = \iint_{S_{\mathrm{lat}}} \mathrm{S}\,\mathrm{d}S = \left(\frac{UI}{2\pi a^2 l} r\right)(2\pi r l) = \frac{UIr^2}{a^2}.$$

The power through the entire conductor's lateral surface occurs at $r = a$, which produces the familiar result

$$P = \frac{UIa^2}{a^2} = UI.$$

Next, we consider the conductor's energy balance

$$-\frac{\partial}{\partial t} \iiint_V w\,\mathrm{d}V = \oiint_{\partial V} \mathbf{S} \cdot \mathrm{d}\boldsymbol{S} + \iiint_V \boldsymbol{j} \cdot \boldsymbol{E}\,\mathrm{d}V = P + \iiint_V \boldsymbol{j} \cdot \boldsymbol{E}\,\mathrm{d}V.$$

Because the situation is stationary we have $\frac{\partial w}{\partial t} = 0$, and thus

$$P + \iiint_V \boldsymbol{j} \cdot \boldsymbol{E}\,\mathrm{d}V \equiv P + P_{\mathrm{loss}} = 0 \implies P_{\mathrm{loss}} = -P = -UI.$$

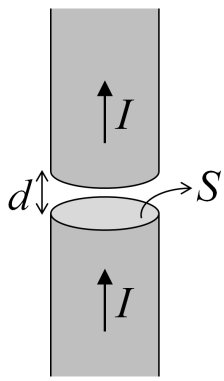
Note that Ohmic losses amount to $P_{\mathrm{loss}} = -UI$, which is the familiar expression from e.g. high school physics. The energy balance reads

$$P + \iiint \boldsymbol{j} \cdot \boldsymbol{E}\,\mathrm{d}V = P + P_{\mathrm{loss}} = UI - UI = 0,$$

and energy is conserved, at it must be.

10.4 Cylindrical Conductor with a Slit

Consider a long cylindrical conductor of radius a and cross-sectional area S carrying a current I. We cut a narrow slit of width $d \ll a$ through the conductor in a plane parallel to the circular cross section. Determine the magnitude and direction of the electric and magnetic fields in the slit, then calculate the energy flow through the slit's lateral surface and confirm the validity of Poynting's theorem for conservation of electromagnetic energy.



Because of the current I, positive charge accumulates on one surface of the slit and negative charge on the other. Because $d \ll a$, we can thus treat the two surfaces of the slit as parallel-plate capacitor.

The charge accumulating on the slit is $q(t) = It$, and, using the model of a parallel-plate capacitor, the corresponding electric field is

$$E(t) = \frac{\sigma(t)}{\epsilon_0} = \frac{1}{\epsilon_0}\frac{q(t)}{S} = \frac{I}{\epsilon_0 S}t.$$

Next, we find the magnetic field in the slit. The time-varying changing electric field $E(t)$ creates displacement current, and the Maxwell equation for $\nabla \times \boldsymbol{B}$ reads

$$\nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{j} + \epsilon_0\mu_0\frac{\partial \boldsymbol{E}}{\partial t} = \epsilon_0\mu_0\frac{\partial \boldsymbol{E}}{\partial t}.$$

Note that $\mu_0 \boldsymbol{j} = 0$ inside the slit, since there's no free current. We integrate the equation over the conductor's cross-sectional surface S to get

$$\iint_S \nabla \times \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{S} = \oint \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{l} = \epsilon_0\mu_0\frac{\partial}{\partial t}\iint_S \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{S} = \epsilon_0\mu_0\frac{\partial \boldsymbol{E}}{\partial t} \cdot (\pi r^2).$$

We evaluate the line integral over the surface's boundary and substitute in electric field to get

$$\oint \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{l} = B \cdot (2\pi r) = \epsilon_0\mu_0\frac{\partial E(t)}{\partial t} \implies B(r) = \frac{\mu_0 I}{2S}r = \frac{\mu_0 I}{2\pi r},$$

where $S = \pi r^2$ is the cross-sectional surface area. The magnetic field points tangent to the cylindrical lateral, just like the magnetic field of a conductor, except that free current is replaced by displacement current.

Since $\boldsymbol{E}$ points along the conductor's longitudinal axis and $\boldsymbol{B}$ is tangent to the lateral surface, $\mathbf{S} = \boldsymbol{E} \times \boldsymbol{B}$ points along the radial direction. Because $\boldsymbol{B} \perp \mathbf{S}$, the power through the lateral surface is simply

$$P = \iint_{S_{\mathrm{lat}}} \mathrm{S}\,\mathrm{d}S = \frac{1}{\mu_0}\iint_{S_{\mathrm{lat}}} EB\,\mathrm{d}S = \frac{1}{\mu_0}\iint_{S_{\mathrm{lat}}} \left(\frac{I}{\epsilon_0 S}t\right)\left(\frac{\mu_0 I}{2S}r\right)\mathrm{d}S.$$

We set $r = a$ at the lateral surface to get

$$P = \frac{I^2 a}{2\epsilon_0 S^2}t(2\pi a d) = \frac{I^2 d}{\epsilon_0 S}t,$$

where $S = \pi a^2$ is cross sectional area at $r = a$.

Finally we consider the conductor's energy balance. If we neglect Ohmic losses, the energy balance reads

$$\frac{\partial}{\partial t}\iiint_V w\,\mathrm{d}V + \oiint_{\partial V} \mathbf{S} \cdot \mathrm{d}\boldsymbol{S} = \frac{\partial W_{\mathrm{EM}}}{\partial t} + P = 0,$$

where $W_{\mathrm{EM}} = W_{\mathrm{E}} + W_{\mathrm{B}}$ is the sum of magnetic and electric field energy. Note that $B = B(r)$ and thus magnetic energy W_{B} is independent of time. It follows that

$$\frac{\partial W_{\mathrm{EM}}}{\partial t} = \frac{\partial W_{\mathrm{E}}}{\partial t} + 0.$$

Using $W_\mathrm{E} = w_\mathrm{E}V$, the electric field energy in the slit is

$$W_\mathrm{E} = w_\mathrm{E}V = \left(\frac{1}{2}\epsilon_0 E^2\right) \cdot (Sd) = \frac{\epsilon_0}{2}\frac{I^2}{\epsilon_0^2 S^2} S d t^2 = \frac{I^2 d}{2\epsilon_0 S} t^2.$$

In terms of W_E, the time derivative of total magnetic field energy is thus

$$\frac{\partial W_\mathrm{EM}}{\partial t} = \frac{\partial W_\mathrm{E}}{\partial t} = \frac{\partial}{\partial t}\frac{I^2 d}{2\epsilon_0 S} t^2 = \frac{I^2 d}{\epsilon_0 S} t = P.$$

Up to a negative sign (depending on the direction of current), $\frac{\partial W_\mathrm{EM}}{\partial t}$ and P are equal, indicating that energy is conserved, as it must be.

11 Eleventh Exercise Set

11.1 A Radially Polarized Sphere

Consider a radially polarized sphere of radius a in which electric polarization points in the radial direction and grows with increasing radius as $\boldsymbol{P}(\boldsymbol{r}) = k\boldsymbol{r}$ where k is a constant. Find the volume density of bound charges, the surface density of bound charges, and the total bound charge. Finally, find the electric field due to the polarized sphere.

Polarization and bound charge density ρ_{b} are related by

$$\rho_{\mathrm{b}} = -\nabla \cdot \boldsymbol{P},$$

while electric field and total charge density ρ are related by Gauss's law

$$\rho = \epsilon_0 \nabla \cdot \boldsymbol{E}.$$

We find volume charge density with $\rho_{\mathrm{b}} = -\nabla \cdot \boldsymbol{P}$ and the known expression for $\boldsymbol{P}$:

$$\rho_{\mathrm{b}} = -\nabla \cdot \boldsymbol{P} = -\nabla \cdot (k\boldsymbol{r}) = -k\nabla \cdot \boldsymbol{r} = -3k,$$

where we've used $\nabla \cdot \boldsymbol{r} = 3$. Note the bound volume charge density is negative, since the electric dipoles in the sphere have their positive poles radially outward and negative pole radially inward, so negative charge is concentrated toward the sphere's center.

To relate surface and volume charge density, we integrate $\rho_{\mathrm{b}} = -\nabla \cdot \boldsymbol{P}$ over the sphere's volume and apply the divergence theorem

$$\iiint_V \rho_{\mathrm{b}} \,\mathrm{d}V = q_{\mathrm{b}} = -\iiint_V \nabla \cdot \boldsymbol{P} \,\mathrm{d}V = -\iint_S \boldsymbol{P} \cdot \mathrm{d}\boldsymbol{S} = -\boldsymbol{P} \cdot \boldsymbol{S}\Big|_{\mathrm{in}}^{\mathrm{out}},$$

where the last step writes $\boldsymbol{S} = \hat{\mathbf{n}} S$ and notes that polarization is zero outside the sphere. The bound surface charge density is then

$$q_{\mathrm{b}} = \boldsymbol{P} \cdot \hat{\mathbf{n}} S \implies \sigma_{\mathrm{B}} = \frac{q_{\mathrm{b}}}{S} = \boldsymbol{P} \cdot \hat{\mathbf{n}}.$$

At the surface $r = a$, charge density σ_{b} evaluates to

$$\sigma_{\mathrm{b}} = \boldsymbol{P} \cdot \hat{\mathbf{n}} = (k\boldsymbol{r}) \cdot \hat{\mathbf{n}} = kr\big|_{r=a} = ka.$$

Note that surface charge density is positive, since the electric dipoles have their positive pole oriented radially outward.

We find total bound charge from ρ_{b} and σ_{b} by integrating both volume and surface charge densities:

$$q_{\mathrm{b}} = \iiint_V \rho_{\mathrm{b}} \,\mathrm{d}V + \iint_S \sigma_{\mathrm{b}} \,\mathrm{d}S = -3k\frac{4}{3}\pi a^3 + ka4\pi a^2 = 0.$$

We should expect total charge to be zero, since all dipoles within the sphere should cancel out.

Finding the Electric Field

By Gauss's law, the electric field outside the sphere is zero, since the sphere's total charge is zero.

Inside the sphere, using the problem's spherical symmetry, we use Gauss's law with a spherical surface to get

$$\epsilon_0 E 4\pi r^2 = \rho_b V = \rho_b \frac{4}{3}\pi r^3 \implies E(r) = \frac{\rho_b}{3\epsilon_0} r.$$

The electric field points radially outwards:

$$\boldsymbol{E}(r) = \frac{\rho_b}{3\epsilon_0}\boldsymbol{r}.$$

We then substitute in the charge density $\rho_b = -3k$ to get

$$\boldsymbol{E} = \frac{-k}{\epsilon_0}\boldsymbol{r} \implies \boldsymbol{E} = -\frac{\boldsymbol{P}}{\epsilon_0}.$$

Note the linear relationship between electric field and polarization.

The linear relationship arises because the sphere contains only bound charge, meaning total charge and bound charge are equal, i.e. $\rho = \rho_b$. We could then combine $\rho_b = \rho = -\nabla \cdot \boldsymbol{P}$ and Gauss's law $\rho = \nabla \cdot (\epsilon_0 \boldsymbol{E})$ to get $\boldsymbol{E} = -\frac{\boldsymbol{P}}{\epsilon_0}$.

11.2 A Halved Polarized Sphere

Consider a sphere of radius a with homogeneous (uniform) polarization $\boldsymbol{P}$ pointing in the z direction. Find the electric field inside and outside the sphere. We then cut the sphere in half in a plane perpendicular to the direction of polarization and slightly separate the halves to form a capacitor. Find the electric field in the slit between the halves.

First, we find volume density of bound charges, which is

$$\rho_b = -\nabla \cdot \boldsymbol{P} = 0,$$

since the divergence of $\boldsymbol{P}$, which is homogeneous, is zero by definition. In other words, there are no bound charges in the sphere's volume.

Next, we find bound surface charge density $\sigma_b = \boldsymbol{P} \cdot \hat{\mathbf{n}}$ where $\hat{\mathbf{n}}$ is the normal to the surface. In terms of an θ between the normal $\hat{\mathbf{n}}$ and polarization $\boldsymbol{P}$, surface charge density is

$$\sigma_b = \boldsymbol{P} \cdot \hat{\mathbf{n}} = P\cos\theta.$$

We proceed with the Poisson equation, which simplifies to the Laplace equation within the sphere were $\rho_b = 0$.

$$\nabla^2 U(\boldsymbol{r}) = -\frac{\rho_b}{\epsilon_0} = 0.$$

We account for the non-zero surface charge density with boundary conditions.

The general solution for $U(\boldsymbol{r})$ is

$$U(\boldsymbol{r}) = U(\boldsymbol{r}, \theta) = \sum_{l=0}^{\infty} (A_l r^l + B_l r^{-(l+1)}) P_l(\cos\theta).$$

Since the surface charge distribution σ_{b} depends only on $\cos\theta$, our solution can include only $\cos\theta$-dependent terms, which occurs for $l = 1$. The general solution simplifies to

$$U(\boldsymbol{r}) = (A_1 r + B_1 r^{-2}) \cos\theta.$$

To avoid divergence at $r \to 0$ and $r \to \infty$, we separate $U(\boldsymbol{r})$ according to

$$U(\boldsymbol{r}) \equiv \begin{cases} U_{\text{in}} = A_1 r \cos\theta & r < a \\ U_{\text{out}} = B_1 r^{-2} \cos\theta & r > a. \end{cases}$$

We find A_1 and B_1 with boundary conditions.

First boundary condition: the electric potential must be continuous at the boundary (a discontinuous potential would imply infinite charge, which is nonphysical). The continuity condition at $r = a$ requires

$$A_1 a = \frac{B_1}{a^2} \implies B_1 = A_1 a^3.$$

For the second boundary condition we require electric field is perpendicular to the surface. We then apply Gauss's law near the sphere's surface to get

$$q_{\mathrm{b}} = \epsilon_0 S (E_{\text{out}}^{\perp} - E_{\text{in}}^{\perp}) \implies \sigma_{\mathrm{b}} = \epsilon_0 (E_{\text{out}}^{\perp} - E_{\text{in}}^{\perp}).$$

We know $\sigma_{\mathrm{b}} = P\cos\theta$ and use $E_{\text{in/out}}^{\perp} = -\frac{\partial U_{\text{in/out}}}{\partial r}\big|_{r=a}$ to get

$$\sigma_{\mathrm{b}} = P\cos\theta = \epsilon_0 (E_{\text{out}}^{\perp} - E_{\text{in}}^{\perp}) = \epsilon_0 \left[\frac{2B_1}{a^3} + A_1\right] \cos\theta.$$

We substitute the first boundary condition $B_1 = A_1 a^3$ into the second to get

$$P\cos\theta = \epsilon_0 (2A_1 + A_1) \cos\theta \implies P = 3\epsilon_0 A_1$$

and thus

$$A_1 = \frac{P}{3\epsilon_0} \qquad \text{and} \qquad B_1 = \frac{Pa^3}{3\epsilon_0}.$$

The electric potential is then

$$U(\boldsymbol{r}) \equiv \begin{cases} U_{\text{in}} = \dfrac{P}{3\epsilon_0} r \cos\theta & r < a \\ U_{\text{out}} = \dfrac{Pa^3}{3\epsilon_0} \dfrac{1}{r^2} \cos\theta & r > a. \end{cases}$$

We now discuss the solution. The 3 in the denominators is called the depolarization factor. Next, we note that $r\cos\theta = z$—since U_{in} depends only on z, the field inside the sphere is homogeneous:

$$E_z = -\frac{\partial U_{\text{in}}}{\partial z} = -\frac{P}{3\epsilon_0} \implies \boldsymbol{E} = -\frac{\boldsymbol{P}}{3\epsilon_0},$$

where we note that $\boldsymbol{P}$ points in the z direction. The field is negative because with our choice of polarization, positive charges are in the positive z direction and negative charges in the negative z direction.

The field outside the sphere is field of an electric dipole; the cosine term is analogous to the dipole dot product term $\boldsymbol{p}_{\mathrm{e}} \cdot \boldsymbol{r}$. A dipole field is expected, since the sphere is polarized like an electric dipole.

Halved Sphere; Electric Field in the Slit

We now consider the electric field in the slit when the sphere is cut in half in a plane perpendicular to the polarization.

Because of the slit through the middle, bound charges accumulates on the cut surfaces as well as the outer surface. Since polarization points "upwards" in the positive z direction, the bound charges on cut surface of the upper hemisphere are negative, while the bound charges on the cut surface of the lower sphere are positive. The relationship $\sigma_{\mathrm{b}} = \boldsymbol{P} \cdot \hat{\mathbf{n}}$ is preserved.

To find electric field because of the slit, we model the slit as a parallel-plate capacitor, for which the electric field reads

$$\tilde{\boldsymbol{E}} = \frac{\tilde{\sigma}_{\mathrm{b}}}{\epsilon_0} \hat{\mathbf{n}} = \frac{(\boldsymbol{P} \cdot \hat{\mathbf{n}})}{\epsilon_0} \hat{\mathbf{n}} = \frac{\boldsymbol{P}}{\epsilon_0},$$

where tilde corresponds to the flat surface and $\hat{\mathbf{n}}$ is the normal to the flat surface.

We must also consider the electric field contribution from the bound charge on the hemispherical surfaces. Reusing the already derived electric field from the bound charges on the spherical surfaces, which read

$$\boldsymbol{E}_{\mathrm{sphere}} = -\frac{\boldsymbol{P}}{3\epsilon_0},$$

the total field in the slit is

$$\boldsymbol{E} = \tilde{\boldsymbol{E}} + \boldsymbol{E}_{\mathrm{sphere}} = \frac{\boldsymbol{P}}{\epsilon_0} - \frac{\boldsymbol{P}}{3\epsilon_0} = \frac{2\boldsymbol{P}}{3\epsilon_0}.$$

11.3 Theory: Dielectric and Displacement Field

The previous two problems involved ferro-electric materials in which the polarization was built in to the material, even in the absence of an external electric field. We now consider dielectric materials, in which polarization occurs only in the presence of an external field.

Recall the equations $\rho_{\mathrm{b}} = -\nabla \cdot \boldsymbol{P}$ and Gauss's law $\rho = \epsilon_0 \nabla \cdot \boldsymbol{E}$. We subtract the equations to get

$$\rho_{\mathrm{f}} = \rho - \rho_{\mathrm{b}} = \nabla \cdot (\epsilon_0 \boldsymbol{E} + \boldsymbol{P}) \equiv \nabla \cdot \boldsymbol{D},$$

where the free charge density $\rho_{\mathrm{f}} = \rho - \rho_{\mathrm{b}}$ is difference between the total and bound charge densities. The electric displacement field is defined as

$$\boldsymbol{D} = \epsilon_0 \boldsymbol{E} + \boldsymbol{P}.$$

The $\boldsymbol{D}$ field arises from free charge and is useful when analyzing dielectrics.

For small fields, we make the approximation $\boldsymbol{D} \propto \boldsymbol{E}$ where $\boldsymbol{D}$ and $\boldsymbol{E}$ are linearly dependent, which results in the approximate linear relationship

$$\boldsymbol{D} = \epsilon_0 \boldsymbol{\chi} \boldsymbol{E},$$

where $\boldsymbol{\chi}$ is the permittivity tensor.

11.4 Parallel-Plate Capacitor with an Anisotropic Dielectric

Consider a parallel-plate capacitor with plate separation d, plate surface area S, and the intra-plate space filled with a dielectric insulator whose dielectric tensor has components ϵ_1, ϵ_2 and ϵ_2. The principle axes corresponding to ϵ_1 and ϵ_3 are parallel to the capacitor plates, but the principle axis corresponding to ϵ_2 makes an angle ϕ with the normal to the plates. Find the capacitor's capacitance C.

First, recall that capacitance is in general defined as

$$C = \frac{q}{U},$$

where q is the charge on the capacitor plates and U is the potential difference between the plates.

Because $\boldsymbol{\chi}$ is angled (equivalently, because $\boldsymbol{\chi}$'s second principle axis does not align with the normal to the capacitor plates), $\boldsymbol{E}$ and $\boldsymbol{D}$ are not parallel.

From the boundary conditions for Maxwell's equations, we know $\boldsymbol{E}$ must be perpendicular to the capacitor plates. To show this, for review, we consider the Maxwell equation

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}.$$

In our static case we have $\frac{\partial \boldsymbol{B}}{\partial t} = 0$. We integrate the equation over a surface hugging the boundary and apply Stokes' theorem to get

$$\iint_S \nabla \times \boldsymbol{E} \cdot \nabla \cdot \boldsymbol{S} = \oint \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{l} = 0,$$

where the line integral runs over a rectangular path thinly hugging a capacitor plate. Because the path thinly hugs the plate, the integral registers only the component of $\boldsymbol{E}$ parallel to the plates:

$$E_{\text{out}}^{(\parallel)} \cdot l - E_{\text{in}}^{(\parallel)} \cdot l = 0 \implies E_{\text{out}}^{(\parallel)} = E_{\text{in}}^{(\parallel)}.$$

Because the electric field outside the capacitor is zero, we have

$$E_{\text{out}}^{(\parallel)} = 0 \implies E_{\text{in}}^{(\parallel)} = 0.$$

The result $E_{\text{in}}^{(\parallel)} = 0$ implies the electric field has only a component perpendicular to the capacitor plates.

Working in the two-dimensional x, y plane (since the problem is invariant to translation in the z direction along the plates) the dielectric tensor in the principle axis system, which we denote $\tilde{\boldsymbol{\chi}}$, reads

$$\tilde{\boldsymbol{\chi}} = \begin{pmatrix} \epsilon_1 & 0 \\ 0 & \epsilon_2 \end{pmatrix}.$$

The capacitor's coordinate system is rotated by ϕ relative to the principle axis system. To transform from $\tilde{\boldsymbol{\chi}}$ to $\boldsymbol{\chi}$, we rotate the principle axes tensor by rotation matrices:

$$\begin{aligned} \boldsymbol{\chi} = \mathbf{R}\tilde{\boldsymbol{\chi}}\mathbf{R} &= \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} \epsilon_1 & 0 \\ 0 & \epsilon_2 \end{pmatrix} \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} \\ &= \begin{bmatrix} \epsilon_1\cos^2\phi + \epsilon_2\sin^2\phi & (\epsilon_2 - \epsilon_1)\sin\phi\cos\phi \\ (\epsilon_2 - \epsilon_1)\sin\phi\cos\phi & \epsilon_1\sin^2\phi + \epsilon_2\cos^2\phi \end{bmatrix}. \end{aligned}$$

Note that $\boldsymbol{\chi}$ is symmetric, as is expected for the dielectric tensor.

Because of the boundary condition, $\boldsymbol{E}_\parallel = 0$, if follows that $\boldsymbol{E}$ has only a y component (where the y axis is normal to the plates). The electric field then reads $\boldsymbol{E} = (E_x, E_y) \equiv (0, E)$, which we combine with $\boldsymbol{D} = \epsilon_0\boldsymbol{\chi}\boldsymbol{E}$ to get

$$\begin{pmatrix} D_x \\ D_y \end{pmatrix} = \epsilon_0 \begin{bmatrix} \epsilon_1\cos^2\phi + \epsilon_2\sin^2\phi & (\epsilon_2 - \epsilon_1)\sin\phi\cos\phi \\ (\epsilon_2 - \epsilon_1)\sin\phi\cos\phi & \epsilon_1\sin^2\phi + \epsilon_2\cos^2\phi \end{bmatrix} \begin{pmatrix} 0 \\ E \end{pmatrix}.$$

In terms of E, the electric field between the capacitor plates is simply

$$U = Ed,$$

where d is the distance between the plates.

Next, we use the boundary condition on the $\boldsymbol{D}$ field to find the charge q on the plates. We derive the relevant boundary condition, we apply Gauss's law to a thin region tightly enclosing a capacitor plate. For the $\boldsymbol{D}$ field, which obeys $\nabla \cdot \boldsymbol{D} = \rho_{\mathrm{f}}$, Gauss's law reads

$$\iiint_V \rho_{\mathrm{f}}\,\mathrm{d}V = \iint_S \boldsymbol{D} \cdot \mathrm{d}\boldsymbol{S} \implies q_{\mathrm{f}} = \left[D_{\text{out}}^{(\perp)} - D_{\text{in}}^{(\perp)}\right] S.$$

In our case, where $D_{\text{out}}^{(\perp)} = 0$ and $D_{\text{out}}^{(\perp)} = D_y$ the charge on the capacitor plates (up to a minus sign depending on the definition of the surface normal) is

$$q_{\mathrm{f}} = D_y S.$$

We find D_y from the earlier matrix equation:

$$D_y = \epsilon_0\left(\epsilon_1\sin^2\phi + \epsilon_2\cos^2\phi\right)E = \epsilon_0\left(\epsilon_1\sin^2\phi + \epsilon_2\cos^2\phi\right)\frac{U}{d}.$$

The charge on the capacitor plates is then

$$q_{\mathrm{f}} = D_y S = \frac{\epsilon_0 SU}{d}\left(\epsilon_1\sin^2\phi + \epsilon_2\cos^2\phi\right).$$

In terms of q_{f}, the capacitor's capacitance C is

$$C = \frac{q_{\mathrm{f}}}{U} = \frac{\epsilon_0 S}{d}\left(\epsilon_1\sin^2\phi + \epsilon_2\cos^2\phi\right) = C_0\left(\epsilon_1\sin^2\phi + \epsilon_2\cos^2\phi\right),$$

where $C_0 = \frac{\epsilon_0 S}{d}$ is the capacitance of an empty capacitor.

Note that for $\phi = 0$, corresponding to an isotropic dielectric in which the dielectric tensor's second principle axes *does* align with normal to the capacitor plates, the capacitor's capacitance reduces to $C = C_0 \epsilon_2$.

12 Twelfth Exercise Set

12.1 Point Dipole in a Spherical Dielectric Cavity

Consider an empty spherical cavity of radius a within a vast dielectric material with permittivity ϵ. We place an electric dipole with dipole momentum $\boldsymbol{p}_e$ in the cavity. What is the resulting electric potential inside and outside the cavity? Find the resulting effective dipole moment $\boldsymbol{p}_e'$ in the dielectric.

We choose our coordinate system so that the dipole $\boldsymbol{p}_\mathrm{e}$ within the cavity points in the z direction.

We find the electric potential due to the dipole by solving the Laplace equation

$$\nabla^2 U(\boldsymbol{r}) = 0.$$

In spherical coordinates, which are best suited to the problem's spherical geometry, the general solution is

$$U(r, \theta) = \sum_{l=0}^{\infty} (A_l r^l + B_l r^{-(l+1)}) P_l(\cos\theta),$$

where P_l are the Legendre polynomials.

We then consider boundary conditions along the spherical cavity's surface and at the center of the cavity, where the dipole $\boldsymbol{p}_\mathrm{e}$ is placed. The potential at the sphere's center should approach the potential of an electric dipole:

$$U(r \to 0, \theta) = U_\mathrm{dipole} = \frac{p_e \cos\theta}{4\pi\epsilon_0 r^2}.$$

Because this solution contains only $\cos\theta$ to the first power, only the $l = 1$ term containing $\cos\theta$ can occur in the general solution for U, simplifying the solution to

$$U(r, \theta) = (A_1 r + B_1 r^{-2}) \cos\theta.$$

For the potential to converge at $r \to \infty$, the $A_1 r$ term must vanish at large r (outside the cavity). The general solution further simplifies to

$$\begin{aligned} U(r) &= \begin{cases} U_\mathrm{dipole} + A_1 r \cos\theta & r < a \\ B_1 r^{-2} \cos\theta & r > a \end{cases} \\ &= \begin{cases} \dfrac{p_e \cos\theta}{4\pi\epsilon_0 r^2} + A_1 r \cos\theta & r < a \\ \dfrac{B_1}{r^2} \cos\theta & r > a. \end{cases} \end{aligned}$$

Next, we require U is continuous at the cavity's boundary $r = a$, which produces

$$U(r \to a^+) = U(r \to a^-) \implies \frac{p_e}{4\pi\epsilon_0 a^2} + A_1 a = \frac{B_1}{a^2}.$$

Note: We could derive the same result using the boundary conditions for Maxwell's equations in materials, among which is required that the component of electric field

$E_\parallel$ tangent to the boundary surface must be equal on both sides of the boundary. The tangential component of electric field is found with

$$E_\parallel = -\frac{1}{r}\frac{\partial U}{\partial \theta}\bigg|_{r=a},$$

and the boundary condition $E_{\text{in}}^\parallel = E_{\text{out}}^\parallel$ then implies

$$-\frac{p_e}{4\pi\epsilon_0 a^2}\sin\theta - A_1 a\sin\theta = \frac{B_1}{a^2}\sin\theta,$$

which is the same result we arrived at from requiring continuity of electric potential U at the boundary.

Finally, we apply the boundary condition on the $\boldsymbol{D}$ field, which applies to the components $D_\perp$ perpendicular to the boundary surface and reads

$$D_{\text{in}}^\perp - D_{\text{out}}^\perp = \sigma_{\text{f}},$$

where σ_{f} is the surface density of free charges along the boundary. We find $D_\perp$ with

$$D_\perp = -\epsilon\epsilon_0 \frac{\partial U}{\partial r}\bigg|_{r=a},.$$

In our case, which involves a dielectric with only bound charges, we have σ_{f}, and the boundary condition on D reduces to

$$D_{\text{in}}^\perp = D_{\text{out}}^\perp \iff \epsilon_0\epsilon_{\text{in}}\frac{\partial U_{\text{in}}}{\partial r}\bigg|_{r=a} = \epsilon_0\epsilon_{\text{out}}\frac{\partial U_{\text{out}}}{\partial r}\bigg|_{r=a}.$$

The dielectric constant inside the empty cavity is $\epsilon_{\text{in}} = 1$, and we denote $\epsilon_{\text{out}} \equiv \epsilon$ in the dielectric. We then substitute U_{in} and U_{out} into the boundary condition and simplify to get

$$2\frac{p_e}{4\pi\epsilon_0 a^3} - A_1 = 2\epsilon\frac{B_1}{a^3}.$$

We then add the earlier boundary condition requiring continuity of U, which read

$$\frac{p_e}{4\pi\epsilon_0 a^3} + A_1 = \frac{B_1}{a^3},$$

to the just-derived boundary condition on $D_\perp$ to get

$$3\frac{p_e}{4\pi\epsilon_0 a^3} = \frac{B_1}{a^3}(1+2\epsilon) \implies B_1 = \frac{3}{1+2\epsilon}\frac{p_e}{4\pi\epsilon_0}.$$

With B_1 known, we can find A_1 according to

$$A_1 = \frac{B_1}{a^3} - \frac{p_e}{4\pi\epsilon_0 a^3} = \frac{p_e}{4\pi\epsilon_0 a^3}\left(\frac{3}{1+2\epsilon} - 1\right) = \frac{2(1-\epsilon)}{1+2\epsilon}\frac{p_e}{4\pi\epsilon_0 a^3}.$$

With the coefficients A_1 and B_1 known, the solution for $U(r,\theta)$ is then

$$U(r,\theta) = \frac{p_e}{4\pi\epsilon_0}\cos\theta\begin{cases}\dfrac{1}{r^2} - \dfrac{2(\epsilon-1)}{1+2\epsilon}\dfrac{r}{a^3} & r < a\\ \dfrac{3}{1+2\epsilon}\dfrac{1}{r^2} & r > a.\end{cases}$$

Inside the cavity where $r < a$, the first $1/r^2$ term is the potential of the electric dipole $\boldsymbol{p}_\mathrm{e}$, while the term containing $r\cos\theta \equiv z$ corresponds to a homogeneous field inside the cavity.

Outside the cavity where $r > a$, we have another $1/r^2$ term—again like a dipole, but with an additional factor $\frac{3}{1+2\epsilon}$, which would reduce to the usual 1 for a dielectric constant $\epsilon = 1$. This is the so-called effective dipole term in the problem instructions. Outside the cavity we thus have the effective dipole moment

$$p_e' = \frac{3}{1+2\epsilon} p_e.$$

To find the surface density of bound charges, we have two options. We could could use Gauss's law in the form

$$\sigma_\mathrm{b} = \epsilon_0 \left(E_\mathrm{out}^\perp - E_\mathrm{in}^\perp \right), \qquad E_\perp = - \frac{\partial U}{\partial r}\bigg|_{r=a}.$$

Alternatively, we could use the relationship between surface charge density and polarization, which reads

$$\sigma_\mathrm{b} = \boldsymbol{P} \cdot \hat{\mathbf{n}}.$$

Note that the surface's normal vector $\hat{\mathbf{n}}$ points from the dielectric material's surface into the cavity, while the polarization $\boldsymbol{P}$ points out of the cavity. The product $\boldsymbol{P} \cdot \hat{\mathbf{n}}$ thus evaluates to

$$\boldsymbol{P} \cdot \hat{\mathbf{n}} \equiv -P_\perp,$$

where $P_\perp$ is the component of polarization normal to the cavity's surface. Assuming a linear relationship between $\boldsymbol{D}$ and $\boldsymbol{E}$, we then find polarization with

$$\epsilon_0 \epsilon \boldsymbol{E} \approx \boldsymbol{D} = \epsilon_0 \boldsymbol{E} + \boldsymbol{P} \implies \boldsymbol{P} = \epsilon_0(\epsilon - 1)\boldsymbol{E}.$$

We then use $\boldsymbol{P} = \epsilon_0(\epsilon-1)\boldsymbol{E}$ to find surface charge density via

$$\sigma_\mathrm{b} = \boldsymbol{P} \cdot \hat{\mathbf{n}} = -P_\perp = -\epsilon_0(\epsilon - 1)\boldsymbol{E}_\perp\big|_{r=a}.$$

12.2 Dielectric Constant of Cold Plasma

Find the relative permittivity ϵ of a cold plasma, then find the dispersion relation for electromagnetic waves in the plasma.

Plasma is a gas of ions, i.e. a gas of free electrons that have been ionized from their positive nuclear cores. We assume the plasma is cold to neglect thermal motion. Since the nuclei are much more massive than electrons, we assume the atoms are at rest as a first approximation.

We then ask how electromagnetic waves propagates through the plasma. We write the electromagnetic waves in terms of the electric field, which we model with the oscillatory ansatz

$$\boldsymbol{E}(\boldsymbol{r}, t) = \boldsymbol{E}_0 e^{i(kz - \omega t)}.$$

Assuming the positive ion cores have charge q and electrons have charge $-q$, the force on a representative free electron of mass m is

$$\boldsymbol{F} = m\ddot{\boldsymbol{r}} = -q\boldsymbol{E} = -q\boldsymbol{E}_0 e^{i(kz-\omega t)},$$

where we have introduced the coordinate $\boldsymbol{r}$ to measure the electron's displacement. Since force is oscillatory, the resulting motion is modeled by

$$\boldsymbol{r} = \boldsymbol{r}_0 e^{i(kz-\omega t)}.$$

Substituting the expressions for $\boldsymbol{r}$ and $\boldsymbol{E}$ into Newton's law $m\ddot{\boldsymbol{r}} = -q\boldsymbol{E}$ gives

$$-m\omega^2 \boldsymbol{r}_0 e^{i(kz-\omega t)} = -q\boldsymbol{E}_0 e^{i(kz-\omega t)},$$

from which we can find the position's oscillation amplitude $\boldsymbol{r}_0$ via

$$m\omega^2 \boldsymbol{r}_0 = q\boldsymbol{E}_0 \implies \boldsymbol{r}_0 = \frac{q}{m\omega^2}\boldsymbol{E}_0.$$

The displacement $\boldsymbol{r}_0$ of the negative electron from the positive ion core creates a dipole moment with amplitude

$$\boldsymbol{p}_{e_0} = -q\boldsymbol{r}_0 = -\frac{q^2}{m\omega^2}\boldsymbol{E}_0.$$

In terms of $\boldsymbol{p}_{e_0}$, the polarization of the plasma is then

$$\boldsymbol{P}_0 = n\boldsymbol{p}_{e_0} = -\frac{nq^2}{m\omega^2}\boldsymbol{E}_0,$$

where n is the number density of electric dipoles in the material.

We then use the relationship $\boldsymbol{P}$, $\boldsymbol{E}$ and permittivity, which reads

$$\boldsymbol{P}_0 = \epsilon_0(\epsilon - 1)\boldsymbol{E}_0,$$

to find the plasma's dielectric constant (relative permittivity)

$$\epsilon = 1 + \frac{\boldsymbol{P}_0}{\epsilon_0 \boldsymbol{E}_0} = 1 - \frac{nq^2}{m\epsilon_0\omega^2}.$$

Next, we note that $\frac{nq^2}{m\epsilon_0}$ has units of frequency, and define the plasma frequency

$$\omega_{\mathrm{p}}^2 = \frac{nq^2}{m\epsilon_0},$$

which describes the frequency at which displaced electrons oscillated about the positive ion cores. In terms of plasma frequency, the plasma's dielectric constant is

$$\epsilon = 1 - \frac{\omega_{\mathrm{p}}^2}{\omega^2}.$$

Note that for electromagnetic wave frequencies $\omega > \omega_{\mathrm{p}}$ we have $\epsilon > 0$ and for $\omega < \omega_{\mathrm{p}}$ we have $\epsilon < 0$. However, electromagnetic waves cannot propagate through a material with dielectric constant $\epsilon < 0$, which we will discuss more shortly. **Dispersion Relation for EM Waves in Plasma**

Next, we will find the dispersion relation for the electromagnetic waves in the plasma. In material with zero electric current density (i.e. $\boldsymbol{j} = 0$) the Maxwell equations read

$$\begin{aligned} \nabla \cdot \boldsymbol{E} &= 0 \qquad & \nabla \times \boldsymbol{E} &= -\mu_0 \frac{\partial \boldsymbol{H}}{\partial t} \\ \nabla \cdot \boldsymbol{H} &= 0 \qquad & \nabla \times \boldsymbol{H} &= \epsilon_0 \epsilon \frac{\partial \boldsymbol{E}}{\partial t} \end{aligned}.$$

We take the curl of the equations for $\nabla \cdot \boldsymbol{E}$ and $\nabla \times \boldsymbol{E}$ to get

$$\begin{aligned} \nabla \times (\nabla \times \boldsymbol{E}) &= -\mu_0 \frac{\partial}{\partial t} \left(\epsilon_0 \epsilon \frac{\partial \boldsymbol{E}}{\partial t} \right) \\ \nabla(\nabla \cdot \boldsymbol{E}) - \nabla^2 \boldsymbol{E} &= -\mu_0 \epsilon \epsilon_0 \frac{\partial^2 \boldsymbol{E}}{\partial t^2}. \end{aligned}$$

The relationship $\nabla \cdot \boldsymbol{E} = 0$ simplifies the second equation to the wave equation

$$\nabla^2 \boldsymbol{E} - \mu_0 \epsilon_0 \epsilon \frac{\partial^2 \boldsymbol{E}}{\partial t^2} = 0.$$

We then substitute in the oscillatory electric field ansatz $\boldsymbol{E} = \boldsymbol{E}_0 e^{i(kz - \omega t)}$ to get

$$-k^2 \boldsymbol{E} - \mu_0 \epsilon_0 \epsilon (-\omega^2) \boldsymbol{E} = 0 \implies \boldsymbol{E}(\mu_0 \epsilon_0 \epsilon \omega^2 - k^2) = 0.$$

The equality $\boldsymbol{E}(\mu_0 \epsilon_0 \epsilon \omega^2 - k^2) = 0$ holds for all $\boldsymbol{E}$ only if

$$\mu_0 \epsilon_0 \epsilon \omega^2 - k^2 = 0 \implies \frac{\omega^2}{k^2} = \frac{1}{\epsilon_0 \mu_0} \frac{1}{\epsilon} = \frac{c_0^2}{\epsilon}.$$

Next, we substitute in the plasma's dielectric constant $\epsilon = 1 - \frac{\omega_{\mathrm{p}}^2}{\omega^2}$ to get

$$c_0^2 = \epsilon \frac{\omega_2}{k^2} = \frac{\omega^2}{k^2} \left(1 - \frac{\omega_{\mathrm{p}}^2}{\omega^2} \right) = \frac{\omega^2}{k^2} - \frac{\omega_{\mathrm{p}}^2}{k^2}.$$

Finally, we rearrange the relationship between c_0, k and ω to get the dispersion relation

$$k^2 c_0^2 = \omega^2 - \omega_{\mathrm{p}}^2 \implies \omega = \sqrt{\omega_{\mathrm{p}}^2 + c_0^2 k^2}.$$

This is a dispersion relation for electromagnetic waves in the plasma—it relates the waves' frequency ω and wave vector k. Note that the dispersion relation prohibits $\omega < \omega_{\mathrm{p}}$ since at the minimum possible k, i.e. $k = 0$, frequency is $\omega = \omega_{\mathrm{p}}$. The prohibited region of $\omega < \omega_{\mathrm{p}}$ corresponds to $\epsilon < 0$—this relationship quantifies the previous claim that EM waves cannot propagate through materials with $\epsilon < 0$.

Note that in a vacuum, with $\epsilon = 1$, EM waves obey the linear dispersion relation

$$\frac{\omega^2}{k^2} = \frac{c_0^2}{\epsilon} = c_0^2 \implies \omega = c_0 k.$$

We now consider the limiting cases for the dispersion relation in plasma, i.e.

$$\omega = \sqrt{\omega_{\mathrm{p}}^2 + c_0^2 k^2}.$$

For large k, such that $c_0 k \gg \omega_{\mathrm{p}}$, we approach the linear relation $\omega = c_0 k$. Meanwhile, very small k, such that $c_0 k \ll \omega_{\mathrm{p}}$, result in the constant dispersion relation $\omega \to \omega_{\mathrm{p}}$. We get a more accurate approximation for small k with the first-order Taylor approximation

$$\omega = \omega_{\mathrm{p}} \left(1 + \frac{c_0^2}{\omega_{\mathrm{p}}^2} k^2\right)^{1/2} \approx \omega_{\mathrm{p}} \left[1 + \frac{c_0^2}{2\omega_{\mathrm{p}}^2} k^2\right], \qquad \left(k \ll \frac{w_{\mathrm{p}}}{c_0}\right).$$

As a side note, we remark that in this limit of small k, where the dispersion relation obeys $\omega \sim k^2$. A quadratic dispersion relation means the photons making up the electric field behave like particles with mass, which leads to the concept of effective mass, like in solid state physics.

Further Discussion

Next, we more thoroughly consider the regime of $\epsilon < 0$. Assuming $\epsilon < 0$, we rearranging the equation

$$\frac{\omega^2}{k^2} = \frac{c_0^2}{\epsilon} \qquad \text{to get} \qquad k = \frac{\omega}{c_0}\sqrt{\epsilon} = \frac{\omega}{c_0} i \sqrt{\epsilon_R},$$

where k is imaginary because $\epsilon < 0$. Substituting this expression for k into the oscillator ansatz for the electric field $\boldsymbol{E}$ gives

$$\boldsymbol{E} = \boldsymbol{E}_0 e^{i(kz - \omega t)} = \boldsymbol{E}_0 e^{-\frac{\omega}{c_0}\sqrt{\epsilon_R} z} e^{-i\omega t}.$$

The position-dependent term is proportional to e^{-z}, which means that an electromagnetic wave will exponentially decay with position in a material with $\epsilon < 0$.

Next, we use the dispersion relation to find the group and phase velocity of the electromagnetic waves in the plasma. The phase velocity is

$$v_{\mathrm{phase}} = \frac{\omega_{\mathrm{p}}}{k} = \frac{\sqrt{\omega_{\mathrm{p}}^2 + c_0^2 k^2}}{k} = \sqrt{c_0^2 + \frac{\omega_{\mathrm{p}}^2}{k^2}}.$$

Note that $v_{phase} > c_0$—the phase velocity is greater than the speed of light in vacuum.

The group velocity—the quantity relevant to the universal speed limit of c_0—is

$$v_{\mathrm{group}} = \frac{\partial \omega}{\partial k} = \frac{2 c_0^2 k}{2\sqrt{\omega_{\mathrm{p}}^2 + c_0^2 k^2}} = \frac{c_0}{\sqrt{1 + \frac{\omega_{\mathrm{p}}^2}{c_0^2 k^2}}}.$$

As expected, $v_{\mathrm{group}} < c_0$, in agreement with special relativity. As a final comment, note that $v_{\mathrm{phase}} v_{\mathrm{group}} = c_0^2$.

13 Thirteenth Exercise Set

13.1 Theory: Electromagnetic Wave Propagation in Waveguides

Consider an empty waveguide (i.e. with relative permittivity $\epsilon = 1$). The wave equation for EM waves reads

$$\nabla^2 \boldsymbol{E} - \frac{1}{c_0^2}\frac{\partial^2 \boldsymbol{E}}{\partial t^2} = 0,$$

where c_0 is the speed of light in a vacuum. As a simplification, we assume the waveguide has a bounded, pipe-like waveguide, with the longitudinal axis coinciding with the $\hat{\mathbf{z}}$ axis, so that EM waves are restricted to propagation in the $\hat{\mathbf{z}}$ direction. In this case, the electric field takes the form

$$\boldsymbol{E}(\boldsymbol{r}, t) = \boldsymbol{E}(\boldsymbol{\rho})e^{i(kz-\omega t)},$$

where $\boldsymbol{\rho}$ is the radial position within the waveguide's planar cross section, i.e. the (x, y) plane perpendicular to the direction of EM propagation along $\hat{\mathbf{z}}$, k is the wave vector corresponding to wave propagation in the $\hat{\mathbf{z}}$ direction, and ω is the wave's frequency. With respect to this ansatz, the z and t derivatives produce

$$\frac{\partial}{\partial z} = ik \implies \frac{\partial^2}{\partial z^2} = -k^2 \qquad \text{and} \qquad \frac{\partial}{\partial t} = -i\omega \implies \frac{\partial^2}{\partial t^2} = -\omega^2.$$

Using the expression for $\frac{\partial}{\partial z}$, we then write the Laplacian in the separated form

$$\nabla^2 = \frac{\partial^2}{\partial z^2} + \nabla_\perp^2 = -k^2 + \nabla_\perp^2,$$

where $\nabla_\perp^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ corresponds to differentiation in the (x, y) plane, i.e. the waveguide's cross section. The wave equation in a waveguide then simplifies to

$$\left[\nabla_\perp^2 + \left(\frac{\omega^2}{c_0^2} - k^2\right)\right]\boldsymbol{E}(\boldsymbol{\rho}, t) = 0.$$

Although the electric field above is $\boldsymbol{E} = \boldsymbol{E}(\rho)$, we can generalize this to

$$\left[\nabla_\perp^2 + \left(\frac{\omega^2}{c_0^2} - k^2\right)\right]\boldsymbol{E}(\boldsymbol{r}, t) = 0.$$

We can include the $e^{i(kz-\omega t)}$, whose only variables are z and t, because the above wave equation doesn't contain $\frac{\partial}{\partial z}$ or $\frac{\partial}{\partial t}$ derivatives (but only $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ derivatives because of $\nabla_\perp^2$).

Since $\boldsymbol{E} \in \mathbb{R}^3$ has three components, the above wave equation in vector form really has 3 equations for each of $\boldsymbol{E}$'s three components.

Similarly, we could derive an analogous vector wave equation for magnetic field $\boldsymbol{H}$:

$$\left[\nabla_\perp^2 + \left(\frac{\omega^2}{c_0^2} - k^2\right)\right]\boldsymbol{H}(\boldsymbol{r}, t) = 0.$$

This vector equation also contain three equations for each of $\boldsymbol{H}$'s three components.

As a result, the electromagnetic waves' propagation in the vacuum is nominally described by $3+3=6$ equations. However, it turns out that we can express multiple $\boldsymbol{H}$ in terms of $\boldsymbol{E}$, which simplifies the equations involved. Our goal in the coming page will be to express E_x, E_y, H_x and H_y in terms of only E_z and H_z.

Relationship Between $\boldsymbol{E}$ and $\boldsymbol{H}$

We begin with the Maxwell equations

$$\nabla \times \boldsymbol{E} = -\mu_0 \frac{\partial \boldsymbol{H}}{\partial t} \qquad \text{and} \qquad \nabla \times \boldsymbol{H} = \epsilon_0 \frac{\partial \boldsymbol{E}}{\partial t}.$$

In component form, the equation for $\nabla \times \boldsymbol{E}$ reads

$$\begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix} \times \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix} = \begin{bmatrix} \frac{\partial E_z}{\partial y} - ikE_y \\ ikE_x - \frac{\partial E_z}{\partial x} \\ \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} \end{bmatrix} = i\mu_0\omega \begin{bmatrix} H_x \\ H_y \\ H_z \end{bmatrix},$$

where we have used $\frac{\partial}{\partial z} = ik$ and $\frac{\partial}{\partial t} = -i\omega$. The second equation for $\nabla \times \boldsymbol{H}$ reads

$$\begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix} \times \begin{bmatrix} H_x \\ H_y \\ H_z \end{bmatrix} = \begin{bmatrix} \frac{\partial H_z}{\partial y} - ikH_y \\ ikH_x - \frac{\partial H_z}{\partial x} \\ \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} \end{bmatrix} = -i\epsilon_0\omega \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix}.$$

The first and fifth equations both contain H_x and E_y. They read

$$\frac{\partial E_z}{\partial y} - ikE_y = i\mu_0\omega H_x \qquad \text{and} \qquad ikH_x - \frac{\partial H_z}{\partial x} = -i\epsilon_0\omega E_y.$$

Our next step is to eliminate H_x: we multiply the first equation by k, the second by $\mu_0\omega$, and subtract the two equations to get

$$k\frac{\partial E_z}{\partial y} - ik^2 E_y = \mu_0\omega\left(\frac{\partial H_z}{\partial x} - i\epsilon_0\omega E_y\right).$$

We then rearrange to get

$$k\frac{\partial E_z}{\partial y} - \mu_0\omega\frac{\partial H_z}{\partial x} = iE_y\left(k^2 - \omega^2\mu_0\epsilon_0\right) = iE_y\left(k^2 - \frac{\omega^2}{c^2}\right),$$

and finally solve for E_y in terms of the z componentsE_z and H_z:

$$E_y = i\frac{k\frac{\partial E_z}{\partial y} - \mu_0\omega\frac{\partial H_z}{\partial x}}{\frac{\omega^2}{c^2} - k^2}.$$

We could perform an analogous procedure the remaining equations to solve for the remaining components E_x, H_x and H_y in terms of E_z and H_z. Without derivation, the results are

$$H_x = i\frac{k\frac{\partial H_z}{\partial x} - \omega\epsilon_0\frac{\partial E_z}{\partial y}}{\frac{\omega^2}{c^2} - k^2}$$

$$E_x = i\frac{k\frac{\partial E_z}{\partial x} + \omega\mu_0\frac{\partial H_z}{\partial y}}{\frac{\omega^2}{c^2} - k^2}$$

$$H_y = i\frac{k\frac{\partial H_z}{\partial y} + \omega\epsilon_0\frac{\partial E_z}{\partial x}}{\frac{\omega^2}{c^2} - k^2}.$$

Note that these equations apply only in Cartesian coordinates for a pipe-like waveguide where waves propagate along the $\hat{\mathbf{z}}$ direction.

Next, we consider boundary conditions at the waveguide's walls. We begin with the Maxwell equations

$$\nabla \times \boldsymbol{E} = -\mu_0\frac{\partial \boldsymbol{H}}{\partial t} \qquad \text{and} \qquad \nabla \cdot \boldsymbol{H} = 0,$$

and encircle a small portion of the waveguide boundary wall with a hypothetical closed loop. We then send to loop to zero thickness while applying Stokes' law, which results in

$$\nabla \times \boldsymbol{E} \to 0 \implies E_{\parallel} = 0 \qquad \text{and} \qquad H_{\perp} \to 0.$$

In other words, the component of electric field parallel to the waveguide boundary and the component of magnetic field normal to the waveguide boundary are zero.

Finally, we find E_z and H_z from the waveguide wave equations

$$\left[\nabla_\perp^2 + \left(\frac{w^2}{c^2} - k^2\right)\right]\begin{Bmatrix} E_z \\ H_z \end{Bmatrix} = 0.$$

By convention, since the wave equation is a second-order order equation, we consider two linear independent solutions with orthogonal components. These are:

- Transverse electric (TE) wave propagation with $E_z = 0$ and $H_z \neq 0$. This is propogation mode is called transverse electric because, from $E_z = 0$, the electric field is perpendicular (transverse) to the direction of EM wave propagation along the $\hat{\mathbf{z}}$ axis.
- Transverse magnetic (TM) wave propagation with $H_z = 0$ and $E_z \neq 0$. The first propogation mode is called transverse magnetic (TM) because the magnetic field is perpendicular (transverse) to the direction of EM wave propagation (since $H_z = 0$).

The general solution for EM waves in the waveguide is then a linear combination of the two solutions.

13.2 A Parallel-Plate Waveguide

Consider a waveguide consisting to two large parallel conducting plates separated by the distance a, in which waves propagate along the z axis. For both transverse magnetic and transverse electric wave propagation, find:

1. *The solution for the electric field in the waveguide*
2. *The dispersion relation relating wave vector and wave frequency*
3. *The wave impedance Z*
4. *The ratio of electric field amplitudes E_x to E_z*

Transverse Magnetic Solution

First, we define our coordinate system, which is easiest to visualize in terms of two horizontal, parallel plates drawn on a piece of paper.

We choose the direction of wave propagation z to run from left to right along the paper; the x axis, which runs from bottom to top in the paper, separates the two plates, and the y axis points out of the page. Because of translational symmetry along the y axis (i.e. because the plates are "large"), their is no change in physical quantities with respect to y, so $\frac{\partial}{\partial y} = 0$.

In transverse magnetic (TM) wave propagation, we have $H_z = 0$ and $E_z \neq 0$. From the introductory theory section, recall the equations

$$E_x = i\frac{k\frac{\partial E_z}{\partial x} + \omega\mu_0\frac{\partial H_z}{\partial y}}{\frac{\omega^2}{c^2} - k^2} \qquad E_y = i\frac{k\frac{\partial E_z}{\partial y} - \mu_0\omega\frac{\partial H_z}{\partial x}}{\frac{\omega^2}{c^2} - k^2}$$

$$H_x = i\frac{k\frac{\partial H_z}{\partial x} - \omega\epsilon_0\frac{\partial E_z}{\partial y}}{\frac{\omega^2}{c^2} - k^2} \qquad H_y = i\frac{k\frac{\partial H_z}{\partial y} + \omega\epsilon_0\frac{\partial E_z}{\partial x}}{\frac{\omega^2}{c^2} - k^2},$$

which express E_x, E_y, H_x and H_y in terms of E_z and H_z. Using $H_z = 0$, which holds in general for TM waves, and $\frac{\partial}{\partial y} = 0$, which holds for our translationally-invariant waveguide geometry, the field components simplify to $E_y = H_x = 0$ and

$$E_x = \frac{ik}{\kappa^2}\frac{\partial E_z}{\partial x} \qquad \text{and} \qquad H_y = \frac{i\omega\epsilon_0}{\kappa^2}\frac{\partial E_z}{\partial x}, \qquad \kappa^2 = \frac{w^2}{c^2} + k^2.$$

Note that $\boldsymbol{H} = (0, H_y, 0)$ has only an y component, while $\boldsymbol{E} = (E_x, 0, E_z)$ has both x and z components.

Next, we find E_z using the wave equation

$$\left[\nabla_\perp^2 + \kappa^2\right] E_z = \left(\frac{\partial^2}{\partial x^2} + \kappa^2\right) E_z = 0 \implies \frac{\partial^2 E_z}{\partial x^2} = -\kappa^2 E_z,$$

where we have again used $\kappa^2 = \frac{w^2}{c^2} + k^2$ for conciseness. The last equality holds because $\frac{\partial}{\partial y} = 0$. The equation is solved with the oscillatory ansatz

$$E_z(x) = A\sin\kappa x + B\cos\kappa x.$$

Next, we consider boundary conditions along the waveguide plates. The general condition $E_\parallel = 0$ applies to the parallel components to waveguide surface, which in our geometry are E_z and E_y. The result is $E_z = E_y = 0$—note that we knew the result E_y from earlier. Evaluated at the boundaries surfaces $x = 0$ and $x = a$, the condition reads

$$E_\parallel = E_z(0) = E_z(a) = 0.$$

We substitute the boundary condition $E_z(0) = 0$ into the solution ansatz to get

$$E_z(0) \equiv 0 = A \sin \kappa \cdot 0 + B \cos \kappa \cdot 0 \implies B = 0.$$

The solution then simplifies to $E_z(x) = A \sin \kappa x$.

Next, we apply the condition $E_z(a) = 0$, which gives

$$E_z(a) \equiv 0 = A \sin \kappa a \implies \kappa = \frac{n\pi}{a}, \qquad n = 1, 2, 3, \ldots.$$

Note that $n = 0$ gives a trivial solution. The solution for E_z in TM mode is then

$$E_z(x) = A \sin \kappa x = A \sin \frac{n\pi x}{a}.$$

Dispersion Relation

Next, we aim to find the dispersion relation between frequency ω and wave vector k. Combining the definition $\kappa^2 = \frac{w^2}{c^2} + k^2$ with the just-derived result $\kappa = \frac{n\pi}{a}$ shows the dispersion relation is

$$\kappa^2 \equiv \frac{\omega^2}{c^2} - k^2 = \left(\frac{n\pi}{a}\right)^2 \implies \omega = c_0 \sqrt{k^2 + \left(\frac{n\pi}{a}\right)^2}.$$

At large k, this approaches the free space relation $\omega = c_0 k$

Note that frequencies below $c_0 \frac{n\pi}{a}$ are unattainable, since k^2 cannot decrease below zero. In other words, there is a minimum possible frequency at which EM waves can propagate through the waveguide. When $k = 0$, the frequencies are quantized according to

$$\omega_n = c_0 \frac{n\pi}{a}.$$

We call the frequency band $\Delta\omega \equiv \omega_2 - \omega_1$ the waveguide bandwidth. In our case, the bandwidth is

$$\Delta\omega = \frac{c_0 \pi}{a}.$$

Wave Impedance

Wave impedance (not to be confused with electrical impedance), is the ratio of the transverse components of the electric and magnetic fields with respect to the direction of wave propagation.

For TM waves, wave impedance is defined in terms of the $\boldsymbol{E}$ and $\boldsymbol{H}$ fields as

$$Z \equiv \frac{E_\perp}{H_\parallel}.$$

For our parallel-plate waveguide with $E_\perp = E_x$ and $H_x = 0 \implies H_\parallel = H_y$, the waveguide impedance is

$$Z = \frac{E_x}{H_y}.$$

Using the earlier equations for the components E_x and H_y, we have

$$Z = \frac{E_x}{H_y} = \frac{ik\frac{\partial E_z}{\partial x}}{i\omega\epsilon_0\frac{\partial E_z}{\partial x}} = \frac{k}{\omega\epsilon_0}.$$

We then use the dispersion relation between ω and k to get

$$Z = \frac{\sqrt{\frac{\omega^2}{c_0^2} - \left(\frac{n\pi}{a}\right)^2}}{\omega\epsilon_0} = \frac{1}{c_0\epsilon_0}\sqrt{1 - \frac{1}{\omega^2}\left(\frac{n\pi c_0}{a}\right)^2}.$$

The coefficient $\frac{1}{c_0\epsilon_0}$ has a special meaning—it is the impedance of free space Z_0, which is more visible in the form

$$\frac{1}{c_0\epsilon_0} = \frac{\sqrt{\epsilon_0\mu_0}}{\epsilon_0} = \sqrt{\frac{\mu_0}{\epsilon_0}} \equiv Z_0.$$

In terms of Z_0 and the frequency $\omega_n = \frac{n\pi c_0}{a}$, the dispersion relation is written concisely as

$$Z(\omega) = Z_0\sqrt{1 - \left(\frac{\omega_n}{\omega}\right)^2}.$$

For large ω, the impedance approaches Z_0.

Ratio of Electric Field Amplitudes

Next, we find the ratio of electric field amplitudes $\frac{E_z}{E_x}$. Substituting in the equation for E_x gives

$$\frac{E_z}{E_x} = \frac{E_z}{ik\frac{\partial E_z}{\partial x}}\kappa^2.$$

Next, we substitute in the earlier result $E_z = A\sin\kappa x$ to get

$$\frac{E_z}{E_x} = \frac{E_z}{ik\frac{\partial E_z}{\partial x}}\kappa^2 = \frac{A\sin\kappa x}{ik\kappa A\sin\kappa x}\kappa^2 = \frac{\kappa}{k}.$$

We can interpret κ as a wave vector in the x direction, while k is the wave vector along the direction of propagation z.

TE Propagation Mode

In TE mode, we have $E_z = 0$ and we find $H_z \neq 0$. Once again, we start with the field component equations

$$E_x = \frac{i}{\kappa}\left(k\frac{\partial E_z}{\partial x} + \omega\mu_0\frac{\partial H_z}{\partial y}\right) \qquad E_y = \frac{i}{\kappa}\left(k\frac{\partial E_z}{\partial y} - \mu_0\omega\frac{\partial H_z}{\partial x}\right)$$
$$H_x = \frac{i}{\kappa}\left(k\frac{\partial H_z}{\partial x} - \omega\epsilon_0\frac{\partial E_z}{\partial y}\right) \qquad H_y = \frac{i}{\kappa}\left(k\frac{\partial H_z}{\partial y} + \omega\epsilon_0\frac{\partial E_z}{\partial x}\right),$$

which express E_x, E_y, H_x and H_y in terms of E_z and H_z. Using $E_z = 0$ and $\frac{\partial}{\partial y} = 0$, the field components simplify to $E_x = H_y = 0$ and

$$E_y = -\frac{i\mu_0\omega}{\kappa}\frac{\partial H_z}{\partial x} \qquad \text{and} \qquad H_x = \frac{ik}{\kappa}\frac{\partial H_z}{\partial x}.$$

Note that $\boldsymbol{E} = (0, E_y, 0)$ has only an y component, while $\boldsymbol{H} = (H_x, 0, H_z)$ has both x and z components.

Next, as for E_z, we find H_z using the wave equation

$$\left[\nabla_\perp^2 + \kappa^2\right] H_z = \left(\frac{\partial^2}{\partial x^2} + \kappa^2\right) H_z = 0 \implies \frac{\partial^2 H_z}{\partial x^2} = -\kappa^2 H_z.$$

The equation is solved with

$$H_z = A \sin \kappa x + B \cos \kappa x.$$

The boundary conditions in TE mode are different than in TM mode. For TE mode, we use the boundary condition $H_\perp = 0$. For our coordinate system and parallel-plate waveguide, the component of $\boldsymbol{H}$ perpendicular to the surface is H_x, which is

$$H_x = \frac{ik}{\kappa} \frac{\partial H_z}{\partial x}.$$

Using the above expression for H_x in terms of $\frac{\partial H_z}{\partial x}$, the condition $H_\perp = H_x = 0$ at the boundary surfaces reads

$$H_x(x = 0, a) \propto \left.\frac{\partial H_z}{\partial x}\right|_{x=0,a} = 0.$$

Thus, in TE mode, the boundary condition is written in terms of a derivative—note that $\frac{\partial H_z}{\partial x}$ is a "normal" derivative; since $\frac{\partial H_z}{\partial x}$ points in the direction x, which is normal to the waveguide's boundary.

In general, the boundary condition in TE mode involves the normal derivative of H_z with respect to the waveguide boundary surface, although the relevant coordinate will naturally not always be x for different geometries and coordinate systems.

Taking the derivative of H_z and evaluating the boundary condition gives

$$H_z(x) = B \cos \frac{n\pi x}{a}, \qquad n = 1, 2, 3, \ldots.$$

Note that $n = 0$ produces the constant solution $H_z = B$ which is not a trivial solution in itself, but the other field components, which involve derivatives of the constant quantity H_z, are zero, so we again reject $n = 0$.

Since the quantity κ is the same in TE mode as in TM mode, the dispersion relation in TE is the same, and reads

$$\omega = c_0 \sqrt{k^2 + \left(\frac{n\pi}{a}\right)^2}.$$

For TE waves, wave impedance is defined in terms of the $\boldsymbol{E}$ and $\boldsymbol{H}$ fields as

$$Z \equiv \frac{E_\parallel}{H_\perp}.$$

For our parallel-plate waveguide with $E_\parallel = E_y$ and $H_\perp = H_x$, the waveguide impedance is

$$Z = \frac{E_y}{H_x}.$$

Using the earlier equations for the components E_y and H_x (although I've ignored the negative sign in E_y), we have

$$Z = \frac{E_y}{H_x} = \frac{i\mu_0\omega\frac{\partial H_z}{\partial x}}{ik\frac{\partial H_z}{\partial x}} = \frac{\mu_0\omega}{k}.$$

We then substitute in the dispersion relation between w and k to get

$$Z = \frac{\mu_0\omega}{k} = \frac{\mu_0\omega}{\sqrt{\frac{\omega^2}{c_0^2} - \left(\frac{n\pi}{a}\right)^2}} = \frac{\mu_0 c_0}{\sqrt{1 - \frac{\omega_n^2}{\omega^2}}},$$

where we've used $\omega_n = \frac{n\pi c_0}{a}$. Finally, we use the relationship $c_0 = \frac{1}{\sqrt{\epsilon_0\mu_0}}$ to get

$$\mu_0 c_0 = \frac{\mu_0}{\sqrt{\mu_0\epsilon_0}} = \sqrt{\frac{\mu_0}{\epsilon_0}} = Z_0 \implies Z(\omega) = \frac{Z_0}{\sqrt{1 - \left(\frac{\omega_n}{\omega}\right)^2}}.$$

Generalization: Waveguide With a Rectangular Cross Section

Finally, we consider a waveguide with a rectangular cross section with width and height a and b, corresponding to the x and y directions, respectively. Waves propagate along the z axis. As before, the "cross-sectional" Laplacian reads

$$\nabla_\perp^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.$$

Note both $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ are non-zero, since the waveguide is not finite-dimensional in the y direction. The wave equation reads

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \kappa^2\right) E_z(x, y) = 0.$$

We solve the equation with separation of variables: $E_z(x, y) = X(x)Y(y)$. We then substitute the ansatz for E_z into the wave equation to get

$$X''Y + XY'' + \kappa^2 XY = 0 \implies \frac{X''}{X} + \frac{Y''}{Y} = -\kappa^2.$$

Since the X terms depend only on x and the Y terms only on y, but $\frac{X''}{X}$ and $\frac{Y''}{Y}$ must be constant for the equation to hold. We thus define

$$\frac{X''}{X} = -\kappa_x^2 \implies X'' = -\kappa_x^2 X \qquad \text{and} \qquad \frac{Y''}{Y} = -\kappa_y^2 \implies Y'' = -\kappa_y^2 Y.$$

Both equations are solved with sinusoidal solutions of the form

$$X = A\sin\kappa_x x + B\cos\kappa_x x \qquad \text{and} \qquad Y = C\sin\kappa_y y + D\cos\kappa_y y.$$

The boundary conditions now involve both a and b; the results are

$$\kappa_x = \frac{n\pi}{a} \qquad \text{and} \qquad \kappa_y = \frac{m\pi}{b}, \qquad n, m = 1, 2, 3, \ldots,$$

where $\kappa^2 = \kappa_x^2 + \kappa_y^2$. The dispersion relation now reads

$$\omega = c_0 \sqrt{k^2 + \left(\frac{n\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2}.$$

In TM mode, both indices run over $n, m = 1, 2, 3, \ldots$.

Meanwhile, in TE mode, which has cosine solutions, because of the presence of two indices n and m one of either n or m can be zero.

14 Fourteenth Exercise Set

14.1 Cylindrical Waveguide

Consider a waveguide with a circular cross section of radius a. Find the time-dependent solution for the electric and magnetic field in the waveguide, along with the dispersion relation, for both TM and TE propagation modes.

We begin with the wave equation for either E_z or H_z in a pipe-like waveguide:

$$\left[\nabla_\perp^2 + \kappa^2\right] \begin{Bmatrix} E_z(x,y) \\ H_z(x,y) \end{Bmatrix} = 0.$$

Next, we write the Laplacian in cylindrical coordinates—in cylindrical geometry, the coordinates perpendicular to the propagation direction z are r and ϕ. The Laplacian reads:

$$\nabla_\perp^2 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \phi^2}.$$

We begin with TM mode, in which we have $H_z = 0$ and solve for E_z. We find E_z with separation of variables, using the ansatz

$$E_z(r,\phi) = R(r)\Phi(\phi).$$

We substitute this ansatz into wave equation and get

$$\frac{1}{r}\left(rR'\right)'\Phi + \frac{1}{r^2}R\Phi'' + \kappa^2 R\Phi = 0.$$

We then rearrange and get

$$\frac{rR' + r^2R''}{R} + \frac{\Phi''}{\Phi} + \kappa^2r^2 = 0 \implies \frac{r^2R'' + rR' + \kappa^2r^2R}{R} = -\frac{\Phi''}{\Phi} = m^2,$$

where, following the usual separation of variables recipe, we have set the r and ϕ-dependent terms equal to the separation constant m^2. The two equations are

$$\Phi'' + m^2\Phi = 0$$

and

$$r^2R'' + rR' + (\kappa^2r^2 - m^2)R = 0.$$

The equation for Φ has a sinusoidal solution, which we write in the form

$$\Phi(\phi) \propto \sin(m\phi + \phi_m).$$

Note that we've written the solution with a phase shift ϕ_m instead of as a linear combination of sine and cosine terms.

The radial equation is a Bessel equation and is solved by the Bessel functions J_m and the Neumann functions N_m. The general solution is a linear combination of the form

$$R(r) \propto J_m(\kappa r) + N_m(\kappa r).$$

The Neumann functions apply in situations with divergence near the origin (e.g. a coaxial cable), and the Bessel functions for convergence near the origin. Our problem is free of divergence near the origin, so we will use only the Bessel functions. In this case, the general solution for $E_z = \Phi R$ is

$$E_z(r,\phi) = \sum_{m=0}^{\infty} A_m J_m(\kappa_m r) \sin(m\phi + \phi_m),$$

where m indexes the Bessel functions.

We now consider boundary conditions. The electric field in waveguides must satisfy

$$E_{\|}\big|_{\partial} = 0,$$

where $E_{\|}$ is the component of electric field tangent (parallel) to the waveguide's surface and ∂ denotes the boundary surface. For our cylindrical geometry both E_z and E_ϕ are tangent, and the boundary corresponds to $r = a$. We apply the condition to E_z to get

$$E_z(a,\phi) = \sum_{m=0}^{\infty} A_m J_m(\kappa a) \sin(m\phi + \phi_m) \equiv 0.$$

This implies κ must satisfy

$$J_m(\kappa a) = 0 \implies \kappa a = \xi_{m,n},$$

where $m = 0, 1, 2, \ldots$ indexes the Bessel functions and $n = 1, 2, 3, \ldots$ indexes the zero of the m-th Bessel function. The solution for κ is then

$$\kappa = \frac{\xi_{mn}}{a},$$

where the zeros ξ_{mn} are tabulated in the table below. As such, the general solution for E_z is more appropriately written with two indices in the form

$$E_z(r,\phi) = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} A_{mn} J_m\left(\xi_{mn}\frac{r}{a}\right) \sin(m\phi + \phi_{mn}),$$

where m indexes the Bessel functions and n indexes the zeros of a given Bessel function.

n	$J_0(x)$	$J_1(x)$	$J_2(x)$	$J_3(x)$	$J_4(x)$	$J_5(x)$
1	2.4048	3.8317	5.1356	6.3802	7.5883	8.7715
2	5.5201	7.0156	8.4172	9.7610	11.0647	12.3386
3	8.6537	10.1735	11.6198	13.0152	14.3725	15.7002
4	11.7915	13.3237	14.7960	16.2235	17.6160	18.9801
5	14.9309	16.4706	17.9598	19.4094	20.8269	22.2178

Table 1: The first 5 zeros ξ_{mn} of the first 6 Bessel functions $J_m(x)$.

With κ known, we find the dispersion relation via

$$\kappa^2 = \frac{\omega^2}{c_0^2} - k^2 \implies \omega = c_0\sqrt{k^2 + \frac{\xi_{mn}^2}{a^2}}.$$

We then find the waveguide's bandwidth from the difference between the lowest two frequency modes, which occur for the smallest two zeros ξ_{mn} when $k = 0$, where the dispersion relation simplifies to

$$\omega(k = 0) = \frac{c_0}{a}\xi_{mn}.$$

The bandwidth is then the difference between the two smallest possible frequencies ω, which occur for the two smallest values of ξ_{mn}. These are:

$$\Delta\omega = \frac{c_0}{a}(\xi_{11} - \xi_{10}) \approx \frac{c_0}{a}(3.8317 - 2.4048) = 1.47 \cdot \frac{c_0}{a}.$$

TE Mode

In TE mode we have $E_z = 0$ and solve for H_z. As in TM, we use separation of variables with the ansatz

$$H_z(r, \phi) = R(r)\Phi(\phi).$$

Following an analogous separation of variables procedure as in TM mode produces the general solution

$$H_z(r, \phi) = \sum_{m=0}^{\infty} A_m J_m(\kappa r) \sin(m\phi + \phi_m).$$

In TE mode, we boundary condition applies to $H_\perp$ and reads

$$H_\perp\big|_\partial = 0,$$

meaning the component of H perpendicular to the surface ∂ must be zero. In cylindrical geometry the radial coordinate is perpendicular to the surface, meaning $H_\perp = H_r$. We find H_r with

$$H_r \propto \frac{\partial H_z}{\partial r} \qquad \text{and} \qquad H_r\big|_\partial = 0 \implies \frac{\partial H_z}{\partial r} = 0.$$

Applying this boundary condition to the general solution for H_z gives

$$J_m'(\kappa a) = 0 \implies \kappa = \frac{\xi_{mn}'}{a},$$

where ξ_{mn}' are the zeros of the Bessel function derivatives, tabulated below.

n	$J_0'(x)$	$J_1'(x)$	$J_2'(x)$	$J_3'(x)$	$J_4'(x)$	$J_5'(x)$
1	3.8317	1.8412	3.0542	4.2012	5.3175	6.4156
2	7.0156	5.3314	6.7061	8.0152	9.2824	10.5199
3	10.1735	8.5363	9.9695	11.3459	12.6819	13.9872
4	13.3237	11.7060	13.1704	14.5858	15.9641	17.3128
5	16.4706	14.8636	16.3475	17.7887	19.1960	20.5755

Table 2: The first 5 zeros ξ_{mn} of the first 6 Bessel function derivatives $J_m'(x)$.

The correct general solution for H_z is written with two indices in the form

$$H_z(r, \phi) = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} A_{mn} J_m \left(\xi'_{mn} \frac{r}{a} \right) \sin(m\phi + \phi_{mn}),$$

where m indexes the Bessel functions and n indexes the zeros of the Bessel function derivatives ξ'_{mn}.

With κ known, the dispersion relation in TE mode is

$$\kappa^2 = \frac{\omega^2}{c_0^2} - k^2 \implies \omega = c_0 \sqrt{k^2 + \frac{(\xi'_{mn})^2}{a^2}}.$$

We find the bandwidth analogously to TM mode, we just using the new zeros ξ'_{mn} instead. The difference between the lowest two frequencies is

$$\Delta\omega = \frac{c_0}{a}(\xi'_{21} - \xi_{11}) \approx \frac{c_0}{a}(3.0542 - 1.8412) \approx 1.21 \cdot \frac{c_0}{a}.$$

14.2 Overview: Waveguide with a Quarter-Circle Cross Section

We briefly sketch the typical problem solving procedure for a waveguide with a quarter-circle cross section. As for a circular cross section, we work in cylindrical coordinates and use separation of variables with the ansatzes

$$E_z(r, \phi) = R(r)\Phi(\phi) \qquad \text{or, in TE mode,} \qquad H_z(r, \phi) = R(r)\Phi(\phi).$$

Substituting these ansatzes into the waveguide wave equation leads to the familiar general solutions

$$\begin{Bmatrix} E_z(r, \phi) \\ H_z(r, \phi) \end{Bmatrix} = \sum_{m=0}^{\infty} A_m J_m(\kappa r) \sin(m\phi + \phi_m).$$

Boundary Conditions in TM Mode

A waveguide with quarter-circular cross section has two additional flat boundaries, in addition to its circular boundary. In TM mode, we require $E_z(\phi = 0, r) = 0$ along the bottom flat surface and $E_z(\phi = \pi/2, r) = 0$ along the vertical flat surface.

The boundary conditions along the flat surfaces, which are conditions involving the coordinate ϕ, apply to the angular component of the general solution, i.e. $\sin(m\phi + \phi_m)$. The first condition, $E_z(0, r) = 0$ requires $\phi_m = 0$.

The second condition, $E_z(\pi/2, r) = 0$, leads to

$$\sin\left(m\frac{\pi}{2}\right) = 0,$$

which means $m\frac{\pi}{2} = n\pi$ where $n \in \mathbb{N}$. The result is

$$m = 2, 4, 6, \cdots.$$

In other words, only even values of m are allowed.

Finding the radial component of the general solution is analogous to the procedure for a circular cross section and involves finding zeros of the Bessel functions.

Boundary Conditions in TE Mode

In TE mode, the relevant boundary condition involves the perpendicular magnetic field $H_\perp$. Along the circular surface, the perpendicular magnetic field is $H_r \propto \frac{\partial H_z}{\partial r}$. Requiring $H_\perp = 0$ leads to the condition $m = 0, 2, 4, 6, \ldots$, as in TE mode.

Along the bottom flat surface at $\phi = 0$, the perpendicular component of magnetic field is

$$H_\perp = \frac{\partial H_z}{\partial \phi}.$$

Applying the boundary condition $H_\perp = 0$ leads to $\cos(\phi_m) = 0$, which means the phase term is $\phi_m = \frac{\pi}{2}$. In other words, the angular solutions are cosine terms, since

$$\sin\left(m\phi + \frac{\pi}{2}\right) = \cos(m\phi).$$

Along the vertical flat surface at $\phi = \pi/2$, the boundary condition reads

$$H_\perp = \frac{\partial H_z}{\partial \phi}\left(\frac{\pi}{2}, r\right) = 0.$$

This condition leads to the requirement $m = 0, 2, 4, \ldots$. In TE mode, the case $m = 0$ is valid because the angular cosine solutions are constant and thus non-trivial, at $m = 0$. Similarly, the radial Bessel functions are nontrivial for $m = 0$, and we get a valid solution. In TE mode for a quarter-circle cross section, the valid values of m are thus $m = 0, 2, 4, \ldots$.

14.3 Theory: TEM Waves in Waveguides

In TEM (transverse electric and transverse magnetic) mode we have both $E_z = 0$ and $H_z = 0$, which is a more strict condition than either TE or TM mode individually. In TEM mode both $\boldsymbol{E}$ and $\boldsymbol{H}$ are perpendicular to the direction of wave propagation, i.e. both $\boldsymbol{E}$ and $\boldsymbol{B}$ are perpendicular to the wave vector $\boldsymbol{k}$. We now $\boldsymbol{E} \perp \boldsymbol{k}$ and $\boldsymbol{B} \perp \boldsymbol{k}$ is satisfied for EM waves in empty space, and it turns out this condition is also possible is specialized waveguides, which we consider in this section.

We begin the analysis of TEM waves in waveguides with the familiar ansatz

$$\boldsymbol{E}(\boldsymbol{r}) = \boldsymbol{E}(\boldsymbol{\rho})e^{i(kz-\omega t)},$$

which assumes $\boldsymbol{E}$ propagates through the waveguide in the z direction.

First Step

Our first step is to show $\nabla \times \boldsymbol{E}$ and $\nabla \times \boldsymbol{H}$ obey the relationships

$$\nabla \times \boldsymbol{E} = i\boldsymbol{k} \times \boldsymbol{E} \qquad \text{and} \qquad \nabla \times \boldsymbol{H} = i\boldsymbol{k} \times \boldsymbol{H}.$$

We begin the proof with the intermediate calculation

$$\nabla \times \boldsymbol{E}(\boldsymbol{r}) = \nabla \times \left[\boldsymbol{E}(\boldsymbol{\rho})e^{i(kz-\omega t)}\right] = \nabla \times \boldsymbol{E}(\boldsymbol{\rho})e^{i(kz-\omega t)} + \nabla e^{i(kz-\omega t)} \times \boldsymbol{E}(\boldsymbol{\rho}),$$

where the second equation holds because the exponent term is a scalar quantity. Next, we make the calculation

$$\nabla e^{i(kz-\omega t)} = i\boldsymbol{k}e^{i(kz-\omega t)} \implies \nabla e^{i(kz-\omega t)} \times \boldsymbol{E}(\boldsymbol{\rho}) = i\boldsymbol{k} \times \boldsymbol{E}(\boldsymbol{r}).$$

To show $\nabla \times \boldsymbol{E}(\boldsymbol{r}) = 0$, we just need to show that the term $\nabla \times \boldsymbol{E}(\boldsymbol{\rho})e^{i(kz-\omega t)}$ equals zero. Noting that $\boldsymbol{\rho} = (x, y, 0)$ i.e. the $\rho_z = 0$, and also that $E_z \equiv 0$ in TEM mode, we have

$$\begin{aligned} \nabla \times \boldsymbol{E}(\boldsymbol{\rho})e^{i(kz-\omega t)} &= \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix} \times \begin{bmatrix} E_x(\boldsymbol{\rho}) \\ E_y(\boldsymbol{\rho}) \\ E_z(\boldsymbol{\rho}) \end{bmatrix} e^{i(kz-\omega t)} = \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ 0 \end{bmatrix} \times \begin{bmatrix} E_x(\boldsymbol{\rho}) \\ E_y(\boldsymbol{\rho}) \\ 0 \end{bmatrix} e^{i(kz-\omega t)} \\ &= \left(\frac{\partial E_y(\boldsymbol{\rho})}{\partial x} - \frac{\partial E_x(\boldsymbol{\rho})}{\partial y} \right) e^{i(kz-\omega t)}\, \hat{\mathbf{z}} = \big[\nabla \times \boldsymbol{E}(\boldsymbol{r})\big]_z \hat{\mathbf{z}}. \end{aligned}$$

Applying Maxwell's equations shows that

$$\big[\nabla \times \boldsymbol{E}(\boldsymbol{r})\big]_z = \mu_0 \frac{\partial H_z}{\partial t} \equiv 0,$$

since $H_z = 0$ in TEM mode. The result is the desired equality

$$\nabla \times \boldsymbol{E} = i\boldsymbol{k} \times \boldsymbol{E},$$

where $\boldsymbol{k} = (0, 0, k)$.

Proving $\nabla \times \boldsymbol{H} = i\boldsymbol{k} \times \boldsymbol{H}$ analogous; left as an exercise to the reader i.e. I'm too lazy to type it out right now.

Second Step

Next, we aim to prove that TEM mode has the dispersion relation $\omega = c_0 k$. We begin with the just-derived equality $\nabla \times \boldsymbol{E} = i\boldsymbol{k} \times \boldsymbol{E}$ and proceed with

$$i\boldsymbol{k} \times \boldsymbol{E} = \nabla \times \boldsymbol{E} = -\mu_0 \frac{\partial \boldsymbol{H}}{\partial t} = +i\mu_0\omega\boldsymbol{H} \implies \boldsymbol{k} \times \boldsymbol{E} = \mu_0\omega\boldsymbol{H}.$$

Analogously, using $\nabla \times \boldsymbol{H} = i\boldsymbol{k} \times \boldsymbol{H}$, we get

$$i\boldsymbol{k} \times \boldsymbol{H} = \nabla \times \boldsymbol{H} = \epsilon_0 \frac{\partial \boldsymbol{E}}{\partial t} = -i\epsilon_0\omega\boldsymbol{E} \implies \boldsymbol{k} \times \boldsymbol{H} = -\epsilon_0\omega\boldsymbol{E}.$$

Next, we take the cross product of the above equations with $\boldsymbol{k}$ to get

$$\boldsymbol{k} \times (\boldsymbol{k} \times \boldsymbol{E}) = \mu_0\omega\boldsymbol{k} \times \boldsymbol{H} = -\epsilon_0\mu_0\omega^2\boldsymbol{E}.$$

The left hand side $\boldsymbol{k} \times (\boldsymbol{k} \times \boldsymbol{E})$ evaluates to

$$\boldsymbol{k} \times (\boldsymbol{k} \times \boldsymbol{E}) = \boldsymbol{k}(\boldsymbol{k} \cdot \boldsymbol{E}) - k^2\boldsymbol{E} = 0 - k^2\boldsymbol{E},$$

since $\boldsymbol{k} \cdot \boldsymbol{E} = 0$ in TEM mode. Using $\boldsymbol{k} \times (\boldsymbol{k} \times \boldsymbol{E}) = -\epsilon_0\mu_0\omega^2\boldsymbol{E}$ leads to

$$-k^2\boldsymbol{E} = -\epsilon_0\mu_0\omega^2\boldsymbol{E} \implies \boldsymbol{E}\left(k^2 - \frac{\omega^2}{c_0^2}\right) = 0 \implies \omega = c_0 k.$$

In other words, we have proven that TEM mode has the dispersion relation $\omega = c_0 k$

Last Step

We begin with the waveguide wave equation, which reads

$$\left[\nabla_\perp^2 + \left(\frac{\omega^2}{c_0^2} - k^2\right)\right] \begin{Bmatrix} \boldsymbol{E} \\ \boldsymbol{H} \end{Bmatrix} = 0.$$

Because of the dispersion relation $\omega = ck$, the equation simplifies to

$$\nabla_\perp^2 \begin{Bmatrix} \boldsymbol{E} \\ \boldsymbol{H} \end{Bmatrix} = 0.$$

This equation corresponds to a solution with $\omega = 0$ and $k = 0$, which corresponds to a static electromagnetic wave. Mathematically, this means TEM wave propagation reduces to solving a static (time-independent) Laplace equation.

As an example, we consider the electric field $\boldsymbol{E}$. Since $\nabla_\perp^2 \boldsymbol{E} = 0$, (which applies to the entire electric field), we can also consider only the component $E_\parallel$ and write

$$\nabla_\perp^2 E_\parallel = 0.$$

Meanwhile, the boundary condition for waveguides reads $E_\parallel\big|_\partial = 0$. The Laplace equation for $E_\parallel$ requires that the Laplacian of $E_\parallel$ is zero everywhere along the waveguide cross section, while the boundary condition requires that $E_\parallel$ itself is zero along the boundary. The only possible solution for simple[2] surfaces is $E_\parallel = 0$.

However, for e.g. a coaxial cable with an annular (not simple) cross section, $E_\parallel \neq 0$ (and thus TEM mode) is a valid solution. Another possible waveguide that allows for TEM mode is a parallel-plate waveguide in which the plates extend to infinity, meaning the "infinite" cross section is not a simple surface. In general, only waveguides with non-simple cross sections can carry TEM waves.

14.4 TEM Waves in a Coaxial Waveguide

Consider a coaxial waveguide with an inner conductor of radius a and an outer sheath of radius b. To make things more interesting, we fill the interior with a material with the same dielectric constant as the cold plasma in Subsection 12.2. *Find the dispersion relation and impedance for TEM waves in the coaxial waveguide.*

Because the waveguide is filled with a dielectric material, the wave speed is $c < c_0$, and the dispersion relation is

$$\omega = ck.$$

The wave speeds c and c_0 are related by

$$c = \frac{c_0}{\sqrt{\epsilon}} \implies \omega = \frac{c_0 k}{\sqrt{\epsilon}}.$$

[2]By simple surface, we refer to a surface enclosed by a single boundary. For example, a circle would be a simple surface, but an annulus, which has two boundaries, is not.

We then substitute in the dielectric constant of cold plasma to get

$$\omega = \frac{c_0 k}{\sqrt{1 - \frac{\omega_{\mathrm{p}}^2}{\omega^2}}} \implies \omega = \sqrt{c_0^2 k^2 + \omega_{\mathrm{p}}^2}.$$

Note that this is the same dispersion relation as for cold plasma, even though the plasma problem was solved in unbounded space, while the coaxial waveguide has bounded geometry.

Impedance

We find the waveguide's impedance with the relationship $Z = \frac{U}{I}$ where U and I are the potential difference between the inner and outer sheath and the current through the conductor, respectively, corresponding to TEM waves in the waveguide. In TEM mode, the EM field is static, which simplifies finding U and I.

We assume the coaxial cable carries a hypothetical current I. Using Ampere's law, the current through the inner conductor leads to a tangential magnetic field H_ϕ of the form

$$I = H_\phi \cdot 2\pi r \implies H_\phi = \frac{I}{2\pi r},$$

where H_ϕ is the ϕ component of H (tangential to the waveguide's circular cross section).

Next, assume a potential difference U between the inner and outer conductors. We find the associated radial electric field E_r using Gauss's law:

$$Q = \epsilon_0 \epsilon \cdot 2\pi r l \cdot E_r \implies E_r = \frac{Q}{2\pi\epsilon\epsilon_0 l r},$$

where q is the charge on the inner conductor. Note that because of the coaxial conductor's concentric geometry, $\boldsymbol{E}$ has only a radial component and $\boldsymbol{H}$ has only a tangential component. With E_r known, we find the associated potential difference with

$$U = \int_a^b E_r \, \mathrm{d}r = \frac{q}{w\pi\epsilon\epsilon_0 l} \int_a^b \frac{\mathrm{d}r}{r} \implies U = \frac{q}{2\pi\epsilon\epsilon_0 l} \ln \frac{b}{a}.$$

We can relate U and E_r with

$$U = E_r r \ln \frac{b}{a}.$$

Next, we substitute U and I into the impedance equation to get

$$Z = \frac{U}{I} = \frac{1}{2\pi} \frac{E_r}{H_\phi} \ln \frac{b}{a}.$$

We can relate E_r and H_ϕ with the earlier theoretical TEM result

$$\boldsymbol{H} = \frac{\boldsymbol{k}}{\omega\mu_0} \times \boldsymbol{E}.$$

Since $\boldsymbol{E} = (E_r, 0, 0)$ and $\boldsymbol{H} = (0, H_\phi, 0)$ are perpendicular, the above vector equation simplifies to

$$H_\phi = \frac{k}{\omega\mu_0} E_r = \frac{1}{c\mu_0} E_r = \sqrt{\frac{\epsilon\epsilon_0\mu_0}{\mu_0^2}} E_r = \frac{\sqrt{\epsilon}}{Z_0} E_r,$$

where $Z_0 = \frac{1}{\sqrt{\epsilon_0 \mu_0}}$ is the impedance of free space. We substitute this result for H_ϕ in terms of E_r into the impedance equation to get

$$Z = \frac{Z_0}{2\pi\sqrt{\epsilon}} \ln\frac{b}{a} = \frac{1}{2\pi}\frac{Z_0}{\sqrt{1 - \frac{\omega_\mathrm{p}^2}{\omega^2}}} \ln\frac{b}{a}.$$