Solved Exercises in Continuum Mechanics

Notes from the Exercises component of the third-year undergraduate course *Mehanika kontinuov* (Continuum Mechanics), led by prof. dr. Daniel Svenšek at the Faculty of Mathematics and Physics at the University of Ljubljana in the academic year 2020-21. Credit for the material covered in these notes is due to professor Svenšek, while the voice, typesetting, and translation to English in this document are my own. At the time of writing, a Slovene manuscript of the original exercises is available on prof. Svenšek's website.

Disclaimer: Mistakes—both simple typos and legitimate errors—are likely. Keep in mind that these are the notes of an undergraduate student in the process of learning the material himself—take what you read with a grain of salt. If you find mistakes and feel like telling me, for example by email, I'll be happy to hear from you, even for the most trivial of errors.

For more notes like this, visit `ejmastnak.github.io/fmf`

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1 Mathematics

1.1 A Few Words on Vectors and Tensors

- We define physical vectors, i.e. vectors representing physically observable quantities, as those n-tuples with the same transformation properties as the position vector $\boldsymbol{r}$. Such "proper" vectors are called polar vectors, to distinguish from axial vectors, which we discuss shortly.

- Second-rank tensors, which we will refer to simply as *tensors* when the context is clear, are those (m, n)-tuples whose elements transform as the products of the corresponding components of polar vectors—only tensors with the transformation property correspond to physically observable quantities.

- In a given basis, we can represent a vector as a column and a tensor as a matrix. Note that every tensor can be represented by a matrix, but not every matrix represents a tensor, since not every matrix will satisfy the tensor transformation conditions.

- An axial vector $\boldsymbol{\omega}$ is a vector product of two polar vectors $\boldsymbol{a}$ and $\boldsymbol{b}$. In both vector and component form, an axial vector is given by

  $$\boldsymbol{\omega} = \boldsymbol{a} \times \boldsymbol{b} \qquad \text{or, in component form,} \qquad \omega_i = \epsilon_{ijk} a_j b_k.$$

  An axial vector, unlike a polar vector, is unchanged under space inversion.

  Formally—and I found this very helpful the first time I read this—*an axial vector is not a vector*, because it does not obey a vector's transformation properties. An axial vector is an *antisymmetric second-rank tensor with three independent components.* Because an axial vector/tensor has only three components, it is possible to pack these three components into a single-index, three-dimensional object similar to a vector. The "vector" and tensor forms of an axial vector are related by

  $$\omega_k = \frac{1}{2}\epsilon_{ijk} W_{ij} \qquad \text{and} \qquad W_{ij} = \epsilon_{ijk}\omega_k,$$

  where W_{ij} is an antisymmetric second-rank tensor. We prove the two forms are equivalent by substituting one into the other. For example, substituting ω_k into W_{ij} and applying the general identity $\epsilon_{ijk}\epsilon_{lmk} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$ gives

  $$\begin{aligned} W_{ij} = \epsilon_{ijk}\omega_k &= \epsilon_{ijk}\left(\tfrac{1}{2}\epsilon_{lmk} W_{lm}\right) = \tfrac{1}{2}\left(\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}\right) W_{lm} \\ &= \tfrac{1}{2}(W_{ij} - W_{ji}) = \tfrac{1}{2}(W_{ij} + W_{ij}) = W_{ij}, \end{aligned}$$

  where the last equality hold because W_{ij} is antisymmetric, i.e. $W_{ij} = -W_{ji}$.

  To summarize: we can represent a cross product $\boldsymbol{a} \times \boldsymbol{b}$ as an axial vector $\boldsymbol{\omega}$ where

  $$\omega_k = \epsilon_{ijk} a_i b_j,$$

  or as a second-rank antisymmetric tensor W where

  $$W_{ij} = a_i b_j - a_j b_i.$$

  We show these forms are equivalent with

  $$\omega_k = \epsilon_{ijk} a_i b_j = \tfrac{1}{2}(\epsilon_{ijk} a_i b_j + \epsilon_{jik} a_j b_i) = \tfrac{1}{2}(\epsilon_{ijk} a_i b_j - \epsilon_{ijk} a_j b_i) = \tfrac{1}{2}\epsilon_{ijk} W_{ij}.$$

1.1.1 Exercise: Tensor Transformation

Show that the components of a tensor transform as the products of vector components.

- We begin with the vector

$$\boldsymbol{a} = a_i \, \hat{\mathbf{e}}_i, \quad \text{where} \;\; a_i = \boldsymbol{a} \cdot \hat{\mathbf{e}}_i. \tag{1.1}$$

  We then consider an orthogonal transformation $\mathcal{M}$ between the orthonormal bases

$$\hat{\mathbf{e}}_i = M_{ij} \, \hat{\mathbf{e}}'_j \tag{1.2}$$

  In the $\{\hat{\mathbf{e}}'_i\}$ basis, a generic vector $\boldsymbol{a} = a_i \, \hat{\mathbf{e}}_i$ is written

$$\boldsymbol{a} = \underbrace{a_i M_{ij}}_{\equiv a'_j} \hat{\mathbf{e}}'_j = a'_j \, \hat{\mathbf{e}}'_j \implies a'_j = M_{ij} a_i.$$

- We now consider a tensor T, which we write in the form

$$\mathsf{T} = T_{ij} \, \hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j, \quad \text{where} \;\; T_{ij} = \hat{\mathbf{e}}_i \cdot \mathsf{T} \cdot \hat{\mathbf{e}}_j.$$

  This is just a generalization of Equation 1.1 to a tensor. The expression $T_{ij} = \hat{\mathbf{e}}_i \cdot \mathsf{T} \cdot \hat{\mathbf{e}}_j$ allows to determine the tensor components T_{ij} as long as we know how the tensor transforms vectors.

  Under the same change of basis $\mathcal{M}$ from Equation 1.2, the tensor transforms as

$$\mathsf{T} = T_{ij} \, \hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j = T_{ij}(M_{ik} \, \hat{\mathbf{e}}'_k) \otimes (M_{jl} \, \hat{\mathbf{e}}'_l) = \underbrace{T_{ij} M_{ik} M_{jl}}_{\equiv T'_{kl}} \hat{\mathbf{e}}'_k \otimes \hat{\mathbf{e}}'_l = T'_{kl} \, \hat{\mathbf{e}}'_k \otimes \hat{\mathbf{e}}'_l.$$

  The tensor components after the transformation $\mathcal{M}$ are thus

$$T'_{kl} = T_{ij} M_{ik} M_{jl}.$$

  We then compare above tensor component transformation to the vector component transformation $a'_j = M_{ij} a_i$, which shows that the components of a tensor transform as the product of vector components. This follows from

$$a'_k a'_l = a_i a_j M_{ik} M_{jl} \iff T'_{kl} = T_{ij} M_{ik} M_{jl}.$$

- As a side note, assuming $\mathcal{M}$ is symmetric, we can write the new components in the form

$$T'_{kl} = T_{ij} M_{ki}^{\top} M_{jl} = M_{ki}^{\top} T_{ij} M_{jl},$$

  which we can interpret as matrix multiplication.

1.1.2 Exercise: Torque

Find the matrix $\mathbf{R}$ *mapping force* $\boldsymbol{F}$ *to torque* $\boldsymbol{M}$.

- In other words, we are looking for the matrix transformation $\mathbf{R}$ satsifying the equation $\boldsymbol{M} = \mathbf{R}\boldsymbol{F}$. We begin by writing torque as a vector product, which gives

$$\boldsymbol{M} = \boldsymbol{r} \times \boldsymbol{F} = \begin{vmatrix} \hat{\mathbf{e}}_i & \hat{\mathbf{e}}_i & \hat{\mathbf{e}}_i \\ x & y & z \\ F_x & F_y & F_z \end{vmatrix} = \begin{bmatrix} yF_z - zF_y \\ zF_x - xF_z \\ xF_y - yF_x \end{bmatrix}.$$

  We then compare this result for vector torque to the matrix transformation

$$\boldsymbol{M} = \mathbf{R}\boldsymbol{F},$$

  which motivates the definition of the matrix $\mathbf{R}$ as

$$\mathbf{R} = \begin{bmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{bmatrix}. \tag{1.3}$$

- Alternatively, we could write torque in component form using the Levi-Civita tensor, which reads

$$M_i = \epsilon_{ijk} r_j F_k$$

  We then compare this to the component-form matrix equation $M_i = R_{ik}F_k$, which implies the desired transformation matrix has components

$$R_{ik} = \epsilon_{ijk} r_j,$$

  which agrees with the result in Equation 1.3.

- Finally, we note that torque is not a (polar) vector, in that it does not satisfy a vector's transformation properties. Formally, torque is an antisymmetric second-rank tensor with components

$$M_{ij} = r_i F_j - r_j F_i,$$

  which may be written as a single-index axial vector in the form

$$M_k = \frac{1}{2}\epsilon_{kij} M_{ij}.$$

1.2 Vector Operators in Orthogonal Curvilinear Coordinates

Excellent introductory paragraph from Svenšek: I like the spirit it captures and have translated it directly. Let's take a look at how the differentiation operator ∇ is written in orthogonal, curvilinear coordinate systems. Instead of simply quoting formulas, we are interested in understanding. Specifically, we are interested in a general procedure for expressing vector operators in curvilinear coordinates, and how this is used in practice in the case of cylindrical and spherical coordinates. Once we understand the general procedure, we will then happily use the various formulas, with better insight into their origin and meaning.

1.2.1 Coordinates and Basis Vectors

- We begin by considering a set of orthogonal curvilinear coordinates q_i, where $i = 1, 2, 3$. Each coordinate corresponds to a basis vector $\hat{\mathbf{e}}_i$, which points in the direction of increasing q_i; orthogonal means the three basis vectors $\hat{\mathbf{e}}_i$ are mutually perpendicular at all points in space.

  We now give the procedure for finding the basis vectors $\hat{\mathbf{e}}_i$ for an arbitrary set of orthogonal curvilinear coordinates q_i.

1.2.2 Finding Orthogonal Curvilinear Basis Vectors

This is written for three-dimensional Euclidean space, but generalizes directly to higher- or lower-dimensional Euclidean spaces. Assuming you have decided on a set of orthogonal coordinates (q_1, q_2, q_3)...

1. Write the Cartesian coordinates (x, y, z) in terms of the coordinates q_i. That is, determine the functions

$$x = x(q_1, q_2, q_3) \qquad y = y(q_1, q_2, q_3) \qquad z = z(q_1, q_2, q_3).$$

   Use the above expressions for (x, y, z) in terms of (q_1, q_2, q_3) to write the position vector $\boldsymbol{r}$ in the form

$$\boldsymbol{r} = x(q_1, q_2, q_3)\,\hat{\mathbf{e}}_x + y(q_1, q_2, q_3)\,\hat{\mathbf{e}}_y + z(q_1, q_2, q_3)\,\hat{\mathbf{e}}_z.$$

2. Use the above expression for $\boldsymbol{r}(q_1, q_2, q_3)$ to compute the unnormalized basis vectors

$$\mathbf{e}_i = \frac{\partial \boldsymbol{r}}{\partial q_i}.$$

3. Use the unnormalized basis vectors $\mathbf{e}_i$ to compute the *scale factors* according to

$$h_i = |\mathbf{e}_i| = \sqrt{\mathbf{e}_i \cdot \mathbf{e}_i}.$$

   The normalized basis vectors $\hat{\mathbf{e}}_i$ are then given by

$$\hat{\mathbf{e}}_i = \frac{1}{h_i}\mathbf{e}_i.$$

   Often, the last two steps are combined, and the normalized basis vectors are found directly from

$$\hat{\mathbf{e}}_i = \frac{1}{h_i}\frac{\partial \boldsymbol{r}}{\partial q_i}, \quad \text{where } h_i = \left|\frac{\partial \boldsymbol{r}}{\partial q_i}\right|.$$

A Few Comments

- Interpretted physically, the scale factors h_i encode the distance associated with an infinitesimal change in the coordinate q_i. To be precise, the distance $\mathrm{d}l$ associated with an infinitesimal change $\mathrm{d}q_i$ is

$$\mathrm{d}l = h_i\,\mathrm{d}q_i,$$

  with no summation implied over i. For sure?

- **TODO:** interpret

$$\mathrm{d}s^2 = \sum_i h_i^2\,\mathrm{d}q_i^2.$$

  and $h_i\,\mathrm{d}q_i$ is a length during a displacement of the coordinate.

$$|\mathrm{d}\boldsymbol{r}| = \mathrm{d}l_i = h_i\,\mathrm{d}q_i.$$

Later

- Nabla is the operator of differentiation with respect to position. Note the intentional use of the word position (and not coordinates) since the concept of position is more general. The general formula for nabla is

$$\nabla = \mathbf{e}_i \frac{\partial}{\partial q_i} = \sum_i \hat{\mathbf{e}}_i \frac{1}{h_i} \frac{\partial}{\partial q_i}.$$

- Side note: for general, non-orthogonal coordinates, the scale factors h_i are replaced by a second-rank *metric tensor* g with components

$$g_{ij} = \frac{\partial \boldsymbol{r}}{\partial q_i} \cdot \frac{\partial \boldsymbol{r}}{\partial q_j}.$$

For orthogonal coordinate systems, the metric tensor is diagonal and obeys

$$g_{ij}^{\text{(orthogonal)}} = \frac{\partial \boldsymbol{r}}{\partial q_i} \cdot \frac{\partial \boldsymbol{r}}{\partial q_j} \delta_{ij},$$

and the corresponding scale factors can be written in terms of a single index as

$$h_i = \sqrt{g_{ii}}.$$

1.2.3 The Vector Operators in Orthogonal Coordinates

- The above is a foundation of sorts. We will now construct the vector operators, starting with the gradient of a scalar field, which, for generic, non-normalized basis vectors $\mathbf{e}_i$, reads

$$\nabla f = \mathbf{e}_i \frac{\partial}{\partial q_i} f,$$

In terms of normalized unit vectors $\hat{\mathbf{e}}_i$, the gradient of a scalar field reads

$$\nabla f = \sum_i \hat{\mathbf{e}}_i \frac{1}{h_i} \frac{\partial}{\partial q_i} f \tag{1.4}$$

The summation is written explicitly only for clarity, since the index i appears thrice (instead of twice, which is the convention for implicit summation).

- Next, we consider the divergence of a vector field. First we write the vector field as

$$\boldsymbol{v} = v_i \, \hat{\mathbf{e}}_i.$$

To find divergence, we simply take the dot product of the known expressions for nabla and $\boldsymbol{v}$. The result is

$$\nabla \cdot \boldsymbol{v} = \left(\sum_i \hat{\mathbf{e}}_i \frac{1}{h_i} \frac{\partial}{\partial q_i} \right) \cdot \left(\sum_j v_j \, \hat{\mathbf{e}}_j \right).$$

Note the need to use different indices (e.g. i and j) in each term when multiplying. We leave the expression in general form, and would expand it out for a concrete case once we have decided on a set of basis vectors and coordinates.

- We find the curl of a vector field analogously, from the cross product of the basis vector expressions for nabla and a generic vector field $\boldsymbol{v}$. The result is

$$\nabla \times \boldsymbol{v} = \left(\sum_i \hat{\mathbf{e}}_i \frac{1}{h_i} \frac{\partial}{\partial q_i} \right) \times \left(\sum_j v_j \, \hat{\mathbf{e}}_j \right).$$

- And the Laplacian of a scalar function. Again very much by definition. The Laplacian is the divergence of a gradient of a scalar function, so we just combined gradient and divergence, which we already know.

  We assemble, from *right to left*, the scalar function f, the gradient operator with index j, the dot product, and the gradient operator with a new index, e.g. i.

$$\nabla^2 f = \nabla \cdot \nabla f = \left(\sum_i \hat{\mathbf{e}}_i \frac{1}{h_i} \frac{\partial}{\partial q_i} \right) \cdot \left(\sum_j \hat{\mathbf{e}}_j \frac{1}{h_j} \frac{\partial}{\partial q_j} \right) f$$

- And the Laplacian of a vector field. We take the Laplacian of a scalar field and replace the scalar function f with a basis vector expression for a vector field.

$$\nabla^2 \boldsymbol{v} = \nabla \cdot \nabla \boldsymbol{v} = \left(\sum_i \hat{\mathbf{e}}_i \frac{1}{h_i} \frac{\partial}{\partial q_i} \right) \cdot \left(\sum_j \hat{\mathbf{e}}_j \frac{1}{h_j} \frac{\partial}{\partial q_j} \right) \otimes \left(\sum_k v_k \, \hat{\mathbf{e}}_k \right)$$

  The gradient of a vector is a tensor. The tensor product just helps us to write that. The end result is a vector: $\nabla \boldsymbol{v}$ produces a rank-two tensor, and $\nabla \cdot (\nabla \boldsymbol{v})$ reduces the tensor to a vector.

  Read the derivatives from left to right. So the derivative $\frac{\partial}{\partial q_i}$ applies to everything. Also note that in general we must differentiate both the coordinates and basis vectors.

- Just another note that these expressions aren't immediately practical. We have to first choose a set of coordinates q_i and basis vectors $\hat{\mathbf{e}}_i$ and evaluate the relevant vector operator formula to get a practically useful result.

1.2.4 Spherical Coordinates

- Let's take a look at spherical coordinates (r, θ, ϕ). We begin with the expressions for Cartesian coordinates in terms of the spherical coordiantes.

$$x = r \sin\theta \cos\phi \qquad y = r \sin\theta \sin\phi \qquad z = r \cos\theta.$$

  Using **TODO:** , the corresponding (un-normalized) basis vectors are

$$\begin{aligned}
\mathbf{e}_r &= \frac{\partial \boldsymbol{r}}{\partial r} = \sin\theta \cos\phi \, \hat{\mathbf{e}}_x + \sin\theta \sin\phi \, \hat{\mathbf{e}}_y + \cos\theta \, \hat{\mathbf{e}}_z \\
\mathbf{e}_\theta &= \frac{\partial \boldsymbol{r}}{\partial \theta} = r \cos\theta \cos\phi \, \hat{\mathbf{e}}_x + r \cos\theta \sin\phi \, \hat{\mathbf{e}}_y - r \sin\theta \, \hat{\mathbf{e}}_z \\
\mathbf{e}_\phi &= \frac{\partial \boldsymbol{r}}{\partial \phi} = -r \sin\theta \sin\phi \, \hat{\mathbf{e}}_x - r \sin\theta \cos\phi \, \hat{\mathbf{e}}_y + 0 \cdot \hat{\mathbf{e}}_z.
\end{aligned}$$

  The scale factors, using the identity $\cos^2 x + \sin^2 y = 1$, come out to

$$h_r = |\mathbf{e}_r| = 1 \qquad h_\theta = |\mathbf{e}_\theta| = r \qquad h_\phi = |\mathbf{e}_\phi| = r \sin\theta.$$

The normalized basis vectors are then

$$\begin{aligned}
\hat{\mathbf{e}}_r &= \frac{\mathbf{e}_r}{h_r} = \sin\theta\cos\phi\,\hat{\mathbf{e}}_x + \sin\theta\sin\phi\,\hat{\mathbf{e}}_y + \cos\theta\,\hat{\mathbf{e}}_z \\
\hat{\mathbf{e}}_\theta &= \frac{\mathbf{e}_\theta}{h_\theta} = \cos\theta\cos\phi\,\hat{\mathbf{e}}_x + \cos\theta\sin\phi\,\hat{\mathbf{e}}_y - \sin\theta\,\hat{\mathbf{e}}_z \\
\hat{\mathbf{e}}_\phi &= \frac{\mathbf{e}_\phi}{h_\phi} = -\sin\phi\,\hat{\mathbf{e}}_x - \cos\phi\,\hat{\mathbf{e}}_y.
\end{aligned}$$

Note that $|\hat{\mathbf{e}}_i| = 1$ for $i = r, \theta, \phi$, as required normalized basis vectors.

1.2.5 Example: Vector Operators in Cylindrical Coordinates

Find the explicit expressions for gradient, divergence and curl in cylindrical coordinates (r, ϕ, z) and the corresponding cylindrical basis.

- Our first step is to compute the cylindrical basis vectors $\hat{\mathbf{e}}_r$, $\hat{\mathbf{e}}_\phi$ and $\hat{\mathbf{e}}_z$. To do this, we first write the Cartesian coordinates (x, y, z) in terms of the cylindrical coordiantes (r, ϕ, z):

  $$x = r\cos\phi \qquad y = r\sin\phi \qquad z = z.$$

  In terms of spherical coordinates, the position vector then reads

  $$\boldsymbol{r} = r\cos\phi\,\hat{\mathbf{e}}_x + r\sin\phi\,\hat{\mathbf{e}}_y + z\,\hat{\mathbf{e}}_z.$$

- Using Equation **TODO** with the above expression for $\boldsymbol{r}$, the cylindrical basis vectors are

  $$\begin{aligned}
  \mathbf{e}_r &= \frac{\partial \boldsymbol{r}}{\partial r} = \cos\phi\,\hat{\mathbf{e}}_x + \sin\phi\,\hat{\mathbf{e}}_y \\
  \mathbf{e}_\phi &= \frac{\partial \boldsymbol{r}}{\partial \phi} = -r\sin\phi\,\hat{\mathbf{e}}_x + r\cos\phi\,\hat{\mathbf{e}}_y \\
  \mathbf{e}_z &= \frac{\partial \boldsymbol{r}}{\partial z} = \hat{\mathbf{e}}_z.
  \end{aligned}$$

  The associated scale factors, using $\sin^2\phi + \cos^2\phi = 1$ for h_r and h_ϕ, are

  $$h_r = |\hat{\mathbf{e}}_r| = 1 \qquad h_\phi = |\hat{\mathbf{e}}_\phi| = r \qquad h_z = |\hat{\mathbf{e}}_z| = 1.$$

  Using $\hat{\mathbf{e}}_i = \mathbf{e}_i/h_i$, the cylindrical unit basis vectors are thus

  $$\hat{\mathbf{e}}_r = \cos\phi\,\hat{\mathbf{e}}_x + \sin\phi\,\hat{\mathbf{e}}_y \qquad \hat{\mathbf{e}}_\phi = -\sin\phi\,\hat{\mathbf{e}}_x + \cos\phi\,\hat{\mathbf{e}}_y \qquad \hat{\mathbf{e}}_z = \hat{\mathbf{e}}_z.$$

- Using the general definition in Equation 1.4 and the just-derived unit vectors, the cylindrical gradient is

  $$\nabla = \sum_i \frac{\hat{\mathbf{e}}_i}{h_i}\frac{\partial}{\partial q_i} = \hat{\mathbf{e}}_r\frac{\partial}{\partial r} + \frac{1}{r}\,\hat{\mathbf{e}}_\phi\frac{\partial}{\partial \phi} + \hat{\mathbf{e}}_z\frac{\partial}{\partial z}.$$

  Note that basis vector is written to the left of derivatives—this is important, since we are not taking a derivative of the basis vectors!

- Next, using the just-derived expression for gradient, the divergence of an arbitrary vector $\boldsymbol{v}$ in cylindrical coordinates is

$$\nabla \cdot \boldsymbol{v} = \left(\hat{\mathbf{e}}_r \frac{\partial}{\partial r} + \frac{1}{r}\,\hat{\mathbf{e}}_\phi \frac{\partial}{\partial \phi} + \hat{\mathbf{e}}_z \frac{\partial}{\partial z} \right) \cdot \left(v_r\,\hat{\mathbf{e}}_r + v_\phi\,\hat{\mathbf{e}}_\phi + v_z\,\hat{\mathbf{e}}_z \right)$$

Before evaluating the expression, we first make the auxiliary calculations

$$\frac{\partial\,\hat{\mathbf{e}}_r}{\partial \phi} = -\sin\phi\,\hat{\mathbf{e}}_x + \cos\phi\,\hat{\mathbf{e}}_y = \hat{\mathbf{e}}_\phi \qquad \text{and} \qquad \frac{\partial\,\hat{\mathbf{e}}_\phi}{\partial \phi} = -\cos\phi\,\hat{\mathbf{e}}_x - \sin\phi\,\hat{\mathbf{e}}_y = -\,\hat{\mathbf{e}}_r.$$

Second, although it is trivial, we note that because the Cartesian basis vectors are constant, we have

$$\frac{\partial\,\hat{\mathbf{e}}_x}{\partial q_i} = \frac{\partial\,\hat{\mathbf{e}}_y}{\partial q_i} = \frac{\partial\,\hat{\mathbf{e}}_z}{\partial q_i} = 0$$

for $q_i \in \{\hat{\mathbf{e}}_r, \hat{\mathbf{e}}_\phi, \hat{\mathbf{e}}_z\}$. This identity saves us a good deal of work when evaluating the dot product. Note also that we must differentiate both the coordinate and basis vectors. An example calculation for a diagonal term gives

$$\begin{aligned} \hat{\mathbf{e}}_r \frac{\partial}{\partial r} \cdot \big(v_r\,\hat{\mathbf{e}}_r \big) &= \hat{\mathbf{e}}_r \cdot \hat{\mathbf{e}}_r \frac{\partial v_r}{\partial r} + v_r\,\hat{\mathbf{e}}_r \cdot \frac{\partial\,\hat{\mathbf{e}}_r}{\partial r} = \frac{\partial v_r}{\partial r} + v\,\hat{\mathbf{e}}_r \cdot \left[\frac{\partial}{\partial r} (\cos\phi\,\hat{\mathbf{e}}_x + \sin\phi\,\hat{\mathbf{e}}_y) \right] \\ &= \frac{\partial v_r}{\partial r} + v\,\hat{\mathbf{e}}_r \cdot (0 + 0) = \frac{\partial v_r}{\partial r}. \end{aligned}$$

In addition the non-zero contributions from the diagonal terms $\frac{\hat{\mathbf{e}}_i}{h_i} \frac{\partial}{\partial q_i} \cdot (v_i\,\hat{\mathbf{e}}_i)$, an additional off-diagonal term is nonzero; this is

$$\frac{v_r}{r}\,\hat{\mathbf{e}}_\phi \cdot \frac{\partial\,\hat{\mathbf{e}}_r}{\partial \phi} = \frac{\partial v_r}{\partial r}\,\hat{\mathbf{e}}_\phi \cdot \hat{\mathbf{e}}_\phi = \frac{v_r}{r}.$$

Without showing the derivation that all other terms are zero, the divergence comes out to

$$\nabla \cdot \boldsymbol{v} = \frac{\partial v_r}{\partial r} + \frac{v_r}{r} + \frac{1}{r}\frac{\partial v_\phi}{\partial \phi} + \frac{\partial v_z}{\partial z}.$$

- Next, again using the cylindrical gradient operator, the curl of a vector in cylindrical coordinates is

$$\nabla \times \boldsymbol{v} = \left(\hat{\mathbf{e}}_r \frac{\partial}{\partial r} + \frac{1}{r}\,\hat{\mathbf{e}}_\phi \frac{\partial}{\partial \phi} + \hat{\mathbf{e}}_z \frac{\partial}{\partial z} \right) \times \left(v_r\,\hat{\mathbf{e}}_r + v_\phi\,\hat{\mathbf{e}}_\phi + v_z\,\hat{\mathbf{e}}_z \right)$$

We first quote, without proof, the useful identities

$$\hat{\mathbf{e}}_\phi \times \hat{\mathbf{e}}_z = \hat{\mathbf{e}}_r \qquad \hat{\mathbf{e}}_z \times \hat{\mathbf{e}}_r = \hat{\mathbf{e}}_\phi \qquad \hat{\mathbf{e}}_r \times \hat{\mathbf{e}}_\phi = \hat{\mathbf{e}}_z.$$

Note how to determine signs of the cylindrical basis vector cross products. In general we have $\hat{\mathbf{e}}_\alpha \times \hat{\mathbf{e}}_\beta = \pm\,\hat{\mathbf{e}}_\gamma$. The sign is positive if (α, β, γ) is a positive permutation of (r, ϕ, z). Some examples:

$$\hat{\mathbf{e}}_z \times \hat{\mathbf{e}}_r = +\,\hat{\mathbf{e}}_\phi,$$

since (z, r, ϕ) is a positive permutation of (r, ϕ, z). Similarly, we have

$$\hat{\mathbf{e}}_\phi \times \hat{\mathbf{e}}_r = \hat{\mathbf{e}}_z,$$

since (ϕ, r, z) is a negative permutation of (r, ϕ, z).

- Anyway, aside from the diagonal terms, we have a non-zero off-diagonal contribution from
$$\frac{v_\phi}{r}\,\hat{\mathbf{e}}_\phi \times \frac{\partial\,\hat{\mathbf{e}}_\phi}{\partial \phi} = -\frac{v_\phi}{r}\,\hat{\mathbf{e}}_\phi \times \hat{\mathbf{e}}_r = \frac{v_\phi}{r}\,\hat{\mathbf{e}}_z$$
Without full derivation, the curl comes out to
$$\nabla \times \boldsymbol{v} = \left(\frac{1}{r}\frac{\partial v_z}{\partial \phi} - \frac{\partial v_\phi}{\partial z}\right)\hat{\mathbf{e}}_r + \left(\frac{\partial v_r}{\partial z} - \frac{\partial v_z}{\partial r}\right)\hat{\mathbf{e}}_\phi + \left(\frac{\partial v_\phi}{\partial r} + \frac{v_\phi}{r} - \frac{1}{r}\frac{\partial v_r}{\partial \phi}\right)\hat{\mathbf{e}}_z.$$

2 Elastomechanics

2.1 The Strain Tensor

- Consider a deformation in which a reference point $\boldsymbol{x}$ in a body shifts to the new position $\boldsymbol{x}'$. We first define the corresponding *displacement vector* $\boldsymbol{u}$ as

$$\boldsymbol{u} \equiv \boldsymbol{x}' - \boldsymbol{x}.$$

  The displacement vector is just the vector displacement between the deformed and reference position.

- The displacement vector is used to define a fundamental quantity in continuum mechanics: the *strain tensor* u, whose components are given by

$$u_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} + \frac{\partial u_k}{\partial x_i}\frac{\partial u_k}{\partial u_j}\right)$$

  Since the indices i and j aren't repeated in the definition, u_{ij} is a rank-two tensor. The strain tensor is generally separated into linear and quadratic terms; these are

$$u_{ij} = \frac{1}{2}\underbrace{\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)}_{\text{linear}} + \frac{1}{2}\underbrace{\frac{\partial u_k}{\partial x_i}\frac{\partial u_k}{\partial x_j}}_{\text{quadratic}}.$$

  For the majority of this course, we will work only in the linear regime, which is adequately models many physical deformations.

- The strain tensor is useful for describing the change in distance between neighboring points after a deformation. More specifically, consider two nearby reference points in a body initially separated by a vector distance $\mathrm{d}\boldsymbol{x}$ and scalar distance $\mathrm{d}l = \mathrm{d}\boldsymbol{x} \cdot \mathrm{d}\boldsymbol{x}$. Then a deformation occurs, after which the two points are separated by the new vector distance $\mathrm{d}\boldsymbol{x}'$. The new distance between the two points after the deformation, i.e, $\mathrm{d}l'$, is given in terms of the strain tensor as

$$(\mathrm{d}l')^2 = (\mathrm{d}l)^2 + 2u_{ij}\,\mathrm{d}x_i\,\mathrm{d}x_j,$$

  where u_{ij} are the components of the strain tensor. In other words, the strain tensor u is a mechanism for encoding how distances between points in a body change after a deformation.

  Why change in distance is significant in the first place: change in distance corresponds to change in elastic energy!

- We now focus on the strain tensor's linear term. Here is a cool interpretation: the linear term is the symmetrized gradient of the displacement vector $\boldsymbol{u}$, i.e.

$$u_{ij}^{\text{lin}} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) = \frac{1}{2}\left[(\nabla\boldsymbol{u})_{ij} + (\nabla\boldsymbol{u})_{ji}\right]$$

  The symmetrization (instead of just defining $\mathsf{u}_{\text{lin}} = \nabla\boldsymbol{u}$ and thus $u_{ij}^{\text{lin}} = \frac{\partial u_i}{\partial x_j}$) ensures the strain tensor associated with a rigid rotation is zero. We will discuss this more below, but the strain tensor being zero for a rigid rotation just encodes the physical fact that a rigid body doesn't deform under rotations.

2.1.1 Strain Tensor for a Rigid Rotation

Find the strain tensor corresponding to the rotation of a rigid body by an angle ϕ about the z axis.

- The matrix for a rotation by an angle ϕ about the z axis is

$$\mathbf{R} = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

  This rotation matrix transforms the position vector $\boldsymbol{x}$ as

$$\boldsymbol{x}' = \mathbf{R}\boldsymbol{x}.$$

  The corresponding displacement vector $\boldsymbol{u}$ after the rotation is

$$\boldsymbol{u} = \boldsymbol{x}' - \boldsymbol{x} = \mathbf{R}\boldsymbol{x} - \boldsymbol{x} = (\mathbf{R} - \mathbf{I})\boldsymbol{x}.$$

  After substituting in the rotation matrix, the displacement vector comes out to

$$\boldsymbol{u} = \begin{bmatrix} \cos\phi - 1 & -\sin\phi & 0 \\ \sin\phi & \cos\phi - 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x(\cos\phi - 1) - y\sin\phi \\ x\sin\phi + y(\cos\phi - 1) \\ 0 \end{bmatrix}.$$

- Next, recall the strain tensor is defined as

$$u_{ij} = \frac{1}{2}\underbrace{\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)}_{\text{linear}} + \frac{1}{2}\underbrace{\frac{\partial u_k}{\partial x_i}\frac{\partial u_k}{\partial x_j}}_{\text{quadratic}}$$

  When accounting only for the linear term, and using the just-derived expression for $\boldsymbol{u}$, the linear strain tensor comes out to

$$\mathsf{u}_{\text{lin}} = \begin{bmatrix} \cos\phi - 1 & 0 & 0 \\ 0 & \cos\phi - 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

  This is not physical behavior, because the strain tensor is zero for a rigid rotation. We expect $\mathsf{u} = 0$.

  Note: To compute the strain tensor u, corresponding to some transformation, all you need to know is the displacement vector u corresponding to transformation, and the coordinates x_i. End note.

- We recover the desired result $\mathsf{u} = 0$ by including the non-linear term. We only need to focus on u_{xx} and u_{yy}—the non-zero terms in the linear formulation. In the quadratic expression, these terms are

$$u_{xx}^{\text{quad}} = \frac{1}{2}\frac{\partial u_k}{\partial x}\frac{\partial u_k}{\partial x} = \frac{1}{2} = \frac{1}{2}\left[(\cos\phi - 1)^2 + \sin^2\phi\right] = 1 - \cos\phi$$

  Note that

$$u_{xx}^{\text{lin}} + u_{xx}^{\text{quad}} = (\cos\phi - 1) + (1 - \cos\phi) = 0$$

  Similarly, we have

$$u_{yy}^{\text{quad}} = 1 - \cos\phi \qquad \text{and} \qquad u_{yy}^{\text{lin}} + u_{yy}^{\text{quad}} = 0.$$

- We could ignore the off-diagonal quadratic terms because (without proof) they all come out to zero, and thus won't affect the linear result. The possible concern here is that although the linear off-diagonal terms are not zero, the quadratic terms might be nonzero, and then the quadratic strain tensor is nonzero. Anyway, this doesn't happen. We won't derive this for all components, but instead show a representative calculation for u_{xy}.

$$u_{xy}^{\text{quad}} = \frac{1}{2}\left[\frac{\partial u_x}{\partial x}\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\frac{\partial u_y}{\partial y}\right] = \frac{1}{2}\left[(\cos\phi - 1)(-\sin\phi) + \sin\phi(\cos\phi - 1)\right] = 0$$

  The lesson is that the off-diagonal terms are zero, and won't change the linear result on the off-diagonal.

- The main lesson here is that the strain tensor is zero for all rigid rotations when we include the linear term.

2.1.2 TODO: Strain Tensor in General (Orthogonal) Curvilinear Coordinates

- We will consider only linear strain tensor

$$u_{ij}^{\text{lin}} = \frac{1}{2}\left(\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j}\right) = \frac{1}{2}\left[(\nabla \boldsymbol{u})_{ij} + (\nabla \boldsymbol{u})_{ji}\right]$$

  Something about non symmetrization to have fewer terms, and then we can symmetrize later.

  Recall

$$\nabla = \sum_l \frac{1}{h_l}\,\hat{\mathbf{e}}_l \frac{\partial}{\partial q_l}$$

- We first write an expression for the gradient of the displacement vector $\nabla \boldsymbol{u}$, which is a tensor. If we write $\boldsymbol{u}$ as $\boldsymbol{u} = u_k\,\hat{\mathbf{e}}_k$, the corresponding expression for $\nabla \boldsymbol{u}$ is

$$\nabla \boldsymbol{u} = \sum_l \frac{1}{h_l}\,\hat{\mathbf{e}}_l \frac{\partial}{\partial q_l} \otimes \sum_k u_k\,\hat{\mathbf{e}}_k.$$

  We then differentiate using the product rule—note that we must differentiate both the coordinates u_k and the basis vectors $\hat{\mathbf{e}}_k$ with respect to q_l. This reads

$$\nabla \boldsymbol{u} = \sum_{l,k}\left[\frac{1}{h_l}\frac{\partial u_k}{\partial q_l}\,\hat{\mathbf{e}}_l \otimes \hat{\mathbf{e}}_k + u_k\,\hat{\mathbf{e}}_l \otimes \frac{1}{h_l}\frac{\partial\,\hat{\mathbf{e}}_k}{\partial q_l}\right]$$

  In the first term we differentiate the coordinates u_k; in the second term we differentiate the basis vectors $\hat{\mathbf{e}}_k$.

- For shorter notation, we then introduce the quantity

$$\frac{1}{h_i}\frac{\partial\,\hat{\mathbf{e}}_j}{\partial q_i} \equiv \sum_k \Gamma_{ij}^k\,\hat{\mathbf{e}}_k,$$

  where Γ_{ij}^k is called a *Christoffel symbol*. It is just a way to write the derivatives of the basis vectors. This notation then gives

$$\nabla \boldsymbol{u} = \sum_{k,l}\left(\frac{1}{h_l}\frac{\partial u_k}{\partial q_l}\,\hat{\mathbf{e}}_l \otimes \hat{\mathbf{e}}_k + \sum_m u_k\,\hat{\mathbf{e}}_l \otimes \Gamma_{lk}^m\,\hat{\mathbf{e}}_m\right).$$

In component form, the gradient is

$$(\nabla \boldsymbol{u})_{ij} = \frac{1}{h_i}\frac{\partial u_j}{\partial q_i} + \sum_k u_k \Gamma^j_{ik},$$

where we have used the identity

$$(\nabla \boldsymbol{u})_{ij} = \hat{\mathbf{e}}_i \cdot (\nabla \boldsymbol{u}) \cdot \hat{\mathbf{e}}_j.$$

The idea is: to find components, just multiply from left and right by corresponding unit vectors and apply the orthonormality of the unit vectors.

- In elastomechanics, we must symmetrize the expression $\nabla \boldsymbol{u}$. This comes from the requirement that we are interested only in changes in distance between parts of a body. And because the antisymmetric component of a gradient of a deformation corresponds to a rotation, and a rotation preserves distances! So we need a symmetrized $\nabla \boldsymbol{u}$. We then have

$$u^{\text{lin}}_{ij} = \frac{1}{2}\left(\frac{1}{h_j}\frac{\partial u_i}{\partial q_j} + \frac{1}{h_i}\frac{\partial u_j}{\partial q_i}\right) + \frac{1}{2}\sum_k u_k \left(\Gamma^i_{jk} + \Gamma^j_{ik}\right).$$

Then we note that for Cartesian coordinates we recover the familiar expression

$$u^{\text{lin}}_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial q_j} + \frac{\partial u_j}{\partial q_i}\right)$$

This is because the $h_i = 1$ for Cartesian coordinates, while the basis vectors are constant, so the Christoffel symbols, which just represent derivatives of the basis vectors, are all zero.

2.1.3 Exercise: Strain Tensor in Spherical Coordinates

On page 16. Write the strain tensor in spherical coordinates (r, ϕ, θ).

- First, let's recall

$$x = r\sin\theta\cos\phi \qquad y = r\sin\theta\sin\phi \qquad z = r\cos\theta$$

And then sum stuff lol

$$h_r\,\hat{\mathbf{e}}_r = \frac{\partial \boldsymbol{r}}{\partial r} = \cos\theta\cos\phi\,\hat{\mathbf{e}}_x + \sin\theta\sin\phi\,\hat{\mathbf{e}}_y + \cos\theta\,\hat{\mathbf{e}}_z$$

and $h_r = 1$

And then

$$h_\theta\,\hat{\mathbf{e}}_\theta = \frac{\partial \boldsymbol{r}}{\partial \theta} = r\big[\cos\theta\cos\phi\,\hat{\mathbf{e}}_x + \cos\theta\sin\phi\,\hat{\mathbf{e}}_y - \sin\theta\,\hat{\mathbf{e}}_z\big]$$

and so $h_\theta = r$

and finally

$$h_\phi\,\hat{\mathbf{e}}_\phi = \frac{\partial \boldsymbol{r}}{\partial \phi} = r\sin\theta\big(-\sin\phi\,\hat{\mathbf{e}}_x + \cos\phi\,\hat{\mathbf{e}}_y\big)$$

and $h_\phi = r\sin\theta$

- Okay, we have our basis vectors. Derivtives of the basis vectors are
- Then just apply formulae. Well, go to it and get

$$\nabla \boldsymbol{u} = \cdots + \frac{u_r}{r}\,\hat{\mathbf{e}}_\theta \otimes \hat{\mathbf{e}}_\theta - \frac{u_\theta}{r}\,\hat{\mathbf{e}}_\theta \otimes \hat{\mathbf{e}}_r + \\ + \frac{u_r}{r}\,\hat{\mathbf{e}}_\phi \otimes \hat{\mathbf{e}}_\phi + \frac{u_\theta}{r\tan\theta}\,\hat{\mathbf{e}}_\phi \otimes \hat{\mathbf{e}}_\phi - \frac{u_\phi}{r}\,\hat{\mathbf{e}}_\phi \otimes \hat{\mathbf{e}}_r - \frac{u_\phi}{r\tan\theta}\,\hat{\mathbf{e}}_\phi \otimes \hat{\mathbf{e}}_\theta.$$

  Then: account for terms in "other" sum, write the equation in component form, and symmetrize:

$$u_{ij} = \frac{1}{2}\big[(\nabla\boldsymbol{u})_{ij} + (\nabla\boldsymbol{u})_{ji}\big]$$

- We have

$$\begin{aligned}
u_{rr} &= \frac{\partial u_r}{\partial r} \\
u_{\theta\theta} &= \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r} \\
u_{\phi\phi} &= \frac{1}{r\sin\theta}\frac{\partial u_\phi}{\partial \phi} + \frac{u_r}{r} + \frac{u_\theta}{r\tan\theta} \\
u_{r\theta} &= \frac{1}{2}\left(\frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} + \frac{1}{r}\frac{\partial u_r}{\partial \theta}\right) \\
u_{r\phi} &= \frac{1}{2}\left(\frac{\partial u_\phi}{\partial r} - \frac{u_\phi}{r} + \frac{1}{r\sin\theta}\frac{\partial u_r}{\partial \phi}\right) \\
u_{\theta\phi} &= \frac{1}{2}\left(\frac{1}{r}\frac{\partial u_\phi}{\partial \theta} - \frac{u_\phi}{r\tan\theta} + \frac{1}{r\sin\theta}\frac{\partial u_\theta}{\partial \phi}\right)
\end{aligned}$$

  Note that the off-diagonal terms are equal on a change of index, because the strain tensor is symmetric.

2.1.4 TODO: Strain Tensor in Cylindrical Coordinates

On page 16. Write the strain tensor in the cylindrical coordinates (r, ϕ, z).

- Finally, the result for cylindrical coordinates is

$$\begin{aligned}
u_{rr} &= \frac{\partial u_r}{\partial r} \\
u_{\phi\phi} &= \frac{1}{r}\frac{\partial u_\phi}{\partial \phi} + \frac{u_r}{r} \\
u_{zz} &= \frac{\partial u_z}{\partial z} \\
u_{r\phi} &= \frac{1}{2}\left(\frac{\partial u_\phi}{\partial r} - \frac{u_\phi}{r} + \frac{1}{r}\frac{\partial u_r}{\partial \phi}\right) \\
u_{rz} &= \frac{1}{2}\left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}\right) \\
u_{\phi z} &= \frac{1}{2}\left(\frac{1}{r}\frac{\partial u_z}{\partial \phi} = \frac{u_\phi}{z}\right).
\end{aligned} \tag{2.1}$$

2.2 Free Energy in Deformations

2.2.1 Some Theory

- Free energy is a scalar quantity, and is thus invariant under inversions, rotations, and time inversion.

- We are interested in the dependence of free energy on the defomration tensor for an isotropic body.

- Because free energy is a scalar invariant, we can write free energy purely in terms of the strain tensor's scalar invariants.

  So what are these scalar invariants? First we review the component expressions of the dot product and cross product:

  $$\boldsymbol{a} \cdot \boldsymbol{b} = a_i b_j \delta_{ij} \qquad \text{and} \qquad (\boldsymbol{a} \times \boldsymbol{b})_i = \epsilon_{ijk} a_j b_k$$

  Interpretation: the dot product decreases the number of free indices by two, and cross product by one. The decrease in the number of indices is called contraction.

- Example: here are the nonzero scalars that we can form from a vector $\boldsymbol{a}$ and rank-two tensor A.

  $$A_{ij}\delta_{ij} = A_{ii} \qquad a_i a_j \delta_{ij} = a_i^2 \qquad A_{ij} A_{kl} \delta_{ij} \delta_{jl} = A_{ii}^2$$

  And another three are

  $$A_{ij} A_{kl} \delta_{ik} \delta_{jl} = A_{ij}^2 \qquad A_{ij} a_k a_l \delta_{ij} \delta_{kl} = A_{ii} a_k^2 \qquad A_{ij} a_k a_l \delta_{ik} \delta_{jl} = A_{ij} a_i a_j$$

- Next, we note that for an antisymmetric tensor $W_{ij} = -W_{ji}$, we have two additional non-zero terms. These are

  $$W_{ij} a_k \epsilon_{ijk} \qquad \text{and} \qquad W_{ij} W_{kl} a_m \epsilon_{ijk} \delta_{lm} = W_{ij} W_{kl} a_l \epsilon_{ijk}$$

  However, if $\boldsymbol{a}$ is a true (polar) vector, these terms are not scalars but "pseudo-scalars", which change sign under sign inversion. This conflicts with the behavior of standard scalars under space inversion, which preserve their sign.

Free Energy Near Equilbrium

- For a body near equilibrium, the free energy must be positive definite.

  TODO: mathematical formulation.

  Interpretted physically, this requirement means that for any non-zero deformation u_{ij} in the neighborhood of equilbrium at $u_{ij} = 0$, the free energy must increase. This just means that the free energy should have a local minimum at the equilibrium point $u_{ij} = 0$.

- Near equilibrium (by the way, this is just in the harmonic regime, like small oscillations in classical mechanics) we can thus work only with a symmetric tensor u_{ij}.

  We can immediately reject the diagonal terms (we also called them first-order scalars) u_{ii}. First-order as in first order in the strain tensor u. Why can we reject them? Probably because they are not positive-definite. They are not harmonic!

The second-order terms are good though. These are u_{ii}^2, the square of the trace, and $u_{ij}u_{ji}$, which is the trace of the square (of u). After rejecting u_{ii}, the free energy reads

$$f(\mathsf{u}) = \frac{1}{2}\lambda u_{ii}^2 + \mu u_{ij}^2,$$

where we have defined the Lamé constants μ and λ.

- The Lamé constants are elastic constants. To see this, we write the free energy into an isotropic (diagonal) and shear (trace-less) term:

$$f(\mathsf{u}) = \tfrac{1}{2}K u_{ii}^2 + \mu\left(u_{ij} - \tfrac{1}{3}u_{kk}\delta_{ij}\right)^2$$

where $K = \lambda + \frac{2}{3}\mu$ is the bulk modulus. For physical behavior, the bulk and shear moduli K and μ must be positive.

2.2.2 Free Energy of Crystals

From page 20. Write deformation free energy for a crystal with tetragonal and cubic symmetry

- First consider tetragonal. We classify this structure by its symmetry properties. We have an n-fold rotation axis about the longitudinal (height or z) axis. We write the symmetry group as C_{4v} where the v refers to two planes of reflection; the normals to these planes lie along the x and y axes in our coordinate system.

- We begin by writing free energy in the most general form (1.83):

$$f(\mathsf{u}) = \frac{1}{2}C_{ijkl}u_{ij}u_{kl}$$

Note that the energy must be quadratic in u for harmonic theory to hold, to have a minimum and thus a point of equilibrium.

We then simplify the equation by noting various symmetry requirements for our chosen crystal.

- For tetragonal and consider the four-fold axis and two symmetry reflection planes. The two reflections give

$$x \to -x, \quad y \to y, \quad z \to z \qquad \text{and} \qquad x \to x, \quad y \to -y, \quad z \to z.$$

Recall that tensor components (from introduction of this text!) transform under the same properties as the components of vectors. We thus get, from the first transformation about the x-facing plane, for example $C_{xyyy} \to -C_{xyyy}$ or $C_{xxyy} \to C_{xxyy}$.

- Start note: We know from the cute arrow things how the components of the vectors transform. Example: C_{xyyy} (using the first plane) maps into product of vector components $-C_{xyyy}$. That's above but why? Well y is unchanged, and x changes sign. Hugh? What's new i.e. where does the product of tensor stuff apply here? Symmetry then requires $C_{xyyy} = 0$.

End note.

- All components C_{ijkl} that transform into eachother must be equal! It also follows, then, that all C_{ijkl} with an odd number of x or y indexes must be nonzero.
- Next consider rotations by $\pi/2$ about the z axis. This maps

$$x \to y, \quad y \to -x \quad z \to z,$$

which implies $C_{xxxx} = C_{yyyy}$, $C_{xxzz} = C_{yyzz}$, and $C_{xzxz} = C_{yzyz}$. The tetragonal free energy is thus

$$f(\mathsf{u}) = \frac{1}{2}C_{xxxx}(u_{xx}^2 + u_{yy}^2) + \frac{1}{2}C_{zzzz}u_{zz}^2 + C_{xxzz}(u_{xx}u_{zz} + u_{yy}u_{zz}) + C_{xxyy}u_{xx}u_{yy} + 2C_{xyxy}u_{xy}^2 + 2C_{xzxz}(u_{xz}^2 + u_{yz}^2).$$

Cubic

- A cubic crystal system has all of a tetragonal crystal's symmetry. In addition, we may rotate by $\pi/2$ about both the x and y axes, which produce, respectively

$$x \to x, \quad y \to z, \quad z \to -y \qquad \text{and} \qquad x \to -z, \quad y \to y, \quad z \to x.$$

This implies $C_{xxxx} = C_{zzzz}$, $C_{xxzz} = C_{xxyy}$ and $C_{xyxy} = C_{xzxz}$. The free energy for a cubic system is thus

$$f(\mathsf{u}) = \frac{1}{2}C_{xxxx}(u_{xx}^2 + u_{yy}^2 + u_{zz}^2) + C_{xxyy}(u_{xx}u_{yy} + u_{xx}u_{zz} + u_{yy}u_{zz}) = 2C_{xyxy}(u_{xy}^2 + u_{xz}^2 + u_{yz}^2).$$

2.3 General Deformations

2.3.1 Theory: Quantities Used to Describe General Deformations

We quickly review the quantities we will use to describe general deformations.

- We will use the linear strain tensor

$$u_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right).$$

- Relationship between Lamé constants and the Young's modulus and Poisson ratio

$$\mu = \frac{E}{2(1+\sigma)} \qquad \text{and} \qquad \lambda = \frac{E\sigma}{(1-2\sigma)(1+\sigma)}$$

Expressions for E and σ in terms of μ and λ:

$$E = \frac{\mu(2\mu + 3\lambda)}{\mu + \lambda} \qquad \text{and} \qquad \sigma = \frac{\lambda}{2(\lambda + \mu)}.$$

- In terms of λ and μ, strain and stress are related by Hooke's law

$$p_{ij} = 2\mu u_{ij} = \lambda u_{kk}\delta_{ij}.$$

The reverse relationship for strain in terms of stress is

$$u_{ij} = \frac{1}{2\mu}\left(p_{ij} - \frac{\lambda}{2\mu_3\lambda}p_{kk}\delta_{ij}\right).$$

Hooke's law in terms of E and σ reads

$$p_{ij} = \frac{E}{1+\sigma}\left(u_{ij} + \frac{\sigma}{1-2\sigma}u_{kk}\delta_{ij}\right) \qquad \text{and} \qquad u_{ij} = \frac{1}{E}\Big[(1+\sigma)p_{ij} - \sigma p_{kk}\delta_{ij}\Big]$$

- Volume equilibrium condition:

$$\frac{\partial p_{ij}}{\partial x_j} + f_i^{\text{ext}} = 0$$

  More generally, the Cauchy equation reads

$$\rho \ddot{u}_i = f_i^{\text{ext}} + \frac{\partial p_{ij}}{\partial x_j}.$$

- Surface equilibrium condition:

$$p_{ij} n_j = F_i^{\text{ext}}$$

  $\hat{\mathbf{n}}$ is the normal vector to the surface (TODO in or out?).

  And F_i^{ext} is surface density of external forces applied to the body's surface.

- Describe statics and dynamics of a loaded body with the Navier equation. In component and vector form, this reads

$$\rho \ddot{u}_i = f_i^{\text{ext}} + \mu \nabla^2 u_i + (\mu + \lambda) \frac{\partial}{\partial x_i} \left(\frac{\partial u_k}{\partial x_k} \right)$$
$$\rho \ddot{\boldsymbol{u}} = \boldsymbol{f}^{\text{ext}} + \mu \nabla^2 \boldsymbol{u} + (\mu + \lambda) \nabla (\nabla \cdot \boldsymbol{u}).$$

  Navier equation in terms of E and σ

$$\rho \ddot{u}_i = f_i^{\text{ext}} + \frac{E}{2(1+\sigma)} \left[\nabla^2 u_i + \frac{1}{1-2\sigma} \frac{\partial}{\partial x_i} \left(\frac{\partial u_k}{\partial x_k} \right) \right]$$
$$\rho \ddot{\boldsymbol{u}} = \boldsymbol{f}^{\text{ext}} + \frac{E}{2(1+\sigma)} \left(\nabla^2 \boldsymbol{u} + \frac{1}{1-2\sigma} \nabla (\nabla \cdot \boldsymbol{u}) \right)$$

2.3.2 Theory: Relative Change in Volume and the Strain Tensor's Trace

- Consider an infinitesimal cube element with volume

$$\mathrm{d}V = \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z.$$

  Let cube's principle axes align with x, y and z Cartesian axes.

- Deformation: expand cube equally along each principle axis. Begin with relative change in volume after deformation

$$(\mathrm{d}l')^2 = (\mathrm{d}l)^2 + 2u_{ij} \, \mathrm{d}x_i \, \mathrm{d}x_j$$

  Rearrange, with the goal of an expansion in $(\mathrm{d}x_i)/(\mathrm{d}l)$.

$$(\mathrm{d}l')^2 = (\mathrm{d}l)^2 \left[1 + \frac{2u_{ij} \, \mathrm{d}x_i \, \mathrm{d}x_j}{(\mathrm{d}l)^2} \right]$$

  Solve for $\mathrm{d}l'$

$$\mathrm{d}l' = \mathrm{d}l \sqrt{1 + \frac{2u_{ij} \, \mathrm{d}x_i \, \mathrm{d}x_j}{(\mathrm{d}l)^2}}$$

  Example to first order in $(\mathrm{d}x_i)/(\mathrm{d}l)$ and get

$$\mathrm{d}l' \approx \mathrm{d}l \left(1 + u_{ij} \frac{\mathrm{d}x_i}{\mathrm{d}l} \frac{\mathrm{d}x_j}{\mathrm{d}l} \right) \equiv \mathrm{d}l (1 + u_{ij} l_i l_j),$$

  where l_i is the direction cosine between an generic displacement $\mathrm{d}\boldsymbol{x}$ and the $\hat{\mathbf{e}}_i$ coordinate axis. Note the implicit sum over the indices i and j.

- Consider change in length for a vector $\mathrm{d}\boldsymbol{x}$ along the x axis (a principle axis). In this case we have $l_x = 1$ and $l_y = l_z = 0$. Change in length is then

$$\mathrm{d}x' = \mathrm{d}x(1 + u_{xx})$$

Similarly, for displacements along the y and z axes we have

$$\mathrm{d}y' = \mathrm{d}y(1 + u_{yy}) \qquad \text{and} \qquad \mathrm{d}z' = \mathrm{d}z(1 + u_{zz}).$$

- In this problem we consider loads only on the principle (coordinate) axes. The corresponding strain tensor is diagonal, and the cube deforms into a cuboid. The orientation of the cube's mutually perpendicular surfaces is preserved.

The post-deformation volume is

$$\begin{aligned}\mathrm{d}V' &= \mathrm{d}x' \,\mathrm{d}y' \,\mathrm{d}z' = \mathrm{d}x \,\mathrm{d}y \,\mathrm{d}z(1 + u_{xx}(1 + u_{yy}))(1 + u_{zz})\\ &\mathrm{d}V(1 + u_{xx}(1 + u_{yy}))(1 + u_{zz}).\end{aligned}$$

Keep only linear terms in u_{ij} and get

$$\mathrm{d}V' = \mathrm{d}V(1 + u_{xx} + u_{yy}u_{zz}) + \mathrm{d}V(1 + \operatorname{Tr}\mathsf{u}).$$

Relative change in the cube's volume is

$$\frac{\mathrm{d}V' - \mathrm{d}V}{\mathrm{d}V} = \operatorname{Tr}\mathsf{u} = \nabla \cdot \boldsymbol{u}. \tag{2.2}$$

Lesson: to lowest order in $(\mathrm{d}x_i)/(\mathrm{d}l)$, the relative change in volume (of a cuboid for loads along the principle axes?) equals the trace of the strain tensor. (Or: to lowest order relative volume change is $\operatorname{Tr}\mathsf{u}$ for any body?)

Important: the quantity $\operatorname{Tr}\mathsf{u}$ is invariant to coordinate transformations. The above result thus holds in any coordinate system; the coordinate axes need not align with the cube's principle axes. We just performed the calculation in the principle axes for convenience.

TODO: Because u is position-independent, relative change in total volume equals local relative change in volume, which is $\operatorname{Tr}\mathsf{u}$.

2.3.3 Example: Change in Volume for a Shear Deformation

Compute the change in volume for cube which undergoes a shear deformation along the x axis.

- Assume displacement vector

$$\boldsymbol{u} = (u_x, 0, 0), \text{ where } u_x = \frac{b}{a} \cdot y \equiv \alpha y,$$

where we have defined $\alpha = b/a$ for shorthand. The corresponding strain tensor is

$$u_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \implies \mathsf{u} = \frac{1}{2}\begin{pmatrix} 0 & \alpha & 0 \\ \alpha & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

This strain tensor's trace is zero, so from Equation 2.2, the relative change in the cube's volume for a shear deformation is also zero, i.e.

$$\frac{\mathrm{d}V' - \mathrm{d}V}{\mathrm{d}V} = \operatorname{Tr}\mathsf{u} = 0.$$

Note

- Locally, every deformation may be written as pure compression or extension in mutually perpendicular directions in the form

$$\mathsf{u}' = \begin{pmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{pmatrix}.$$

  This applies *locally*. Writing u locally in a diagonal form is allowed because u is always symmetric, and thus may be diagonalized in an orthogonal basis.

- Goal: diagonalize the above shear strain tensor u. Do this by finding u's eigenvalues and eigenvectors by solving the characteristic polynomial

$$\det(\mathsf{u} - \lambda \mathsf{I}) = \det \begin{pmatrix} -\lambda & \frac{\alpha}{2} & 0 \\ \frac{\alpha}{2} & -\lambda & 0 \\ 0 & 0 & -\lambda \end{pmatrix} = -\lambda \left[\lambda^2 - \left(\frac{\alpha}{2} \right)^2 \right] = 0.$$

  The three solutions are $\lambda_1 = \frac{\alpha}{2}$, $\lambda_2 = -\frac{\alpha}{2}$ and $\lambda_3 = 0$. Without derivation, the corresponding eigenvectors are

$$\boldsymbol{a}_1 = (1, 1, 0) \qquad \boldsymbol{a}_2 = (1, -1, 0) \qquad \boldsymbol{a}_3 = (0, 0, 1),$$

  and the corresponding diagonalized strain tensor is

$$\mathsf{u}' = \begin{pmatrix} \frac{\alpha}{2} & 0 & 0 \\ 0 & -\frac{\alpha}{2} & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

  Interpretation: we have resolved the strain deformation into a expansion in the direction of $\boldsymbol{a}_1$ and a compression in the direction of $\boldsymbol{a}_2$, while there is no deformation in the direction of $\boldsymbol{a}_3$ because $\lambda_3 = 0$. More so, the increase in volume from the extension along $\boldsymbol{a}_1$ equals the decrease in volume from the compression along $\boldsymbol{a}_2$.

2.3.4 Exercise: Surface with Maximum Applied Force

Given the stress tensor

$$\mathsf{p} = \begin{pmatrix} p_1 & 0 & 0 \\ 0 & p_2 & 0 \\ 0 & 0 & p_3 \end{pmatrix} = \begin{pmatrix} 50 & 0 & 0 \\ 0 & -50 & 0 \\ 0 & 0 & 75 \end{pmatrix} \mathrm{N\,cm^{-2}},$$

find the cross sectional surface on which the shear stress is maximal.

- Describe desired surface using the to-be-determined normal vector $\hat{\mathbf{n}}$.

  Surface force density on this surface is

$$f_i = p_{ik} n_k.$$

  Decompose force density on the desired surface into two components: a normal component $\boldsymbol{N}$ parallel to the surface normal and a shear component $\boldsymbol{S}$ in the plane of the surface.

- The normal component of surface force density is

$$N_i = f_j n_j n_i = (p_{ik} n_k) n_j n_i.$$

The shear component of surface force density is found by subtracting the normal comonent $\boldsymbol{N}$ from the total force density $\boldsymbol{f}$; this reads

$$S_i = p_{ik} n_k - p_{ik} n_k n_j n_i = p_{jk} n_k (\delta_{ij} - n_i n_j) = f_j (\delta_{ij} - n_i n_j).$$

We then compute the squared shear force density. Somehow this comes out to

$$S^2 = (p_{ik} n_k)^2 - (p_{ik} n_i n_k).$$

- Stess tensor is diagonal, so

$$p_{ik} n_k = (p_1 n_1, p_2 n_2, p_3 n_3).$$

From this we get

$$(p_{ik} n_k)^2 = p_1^2 n_1^2 + p_2^2 n_2^2 + p_3^2 n_3^2 \qquad \text{and} \qquad p_{ik} n_i n_k = p_1 n_1^2 + p_2 n_2^2 + p_3 n_3^2.$$

The dependence of S^2 on p_{ik} is then

$$\begin{aligned} S^2 &= (p_{ik} n_k)^2 - (p_{ik} n_i n_k) \\ &= p_1^2 n_1^2 + p_2^2 n_2^2 + p_3^2 n_3^2 - (p_1 n_1^2 + p_2 n_2^2 + p_3 n_3^2)^2 \end{aligned}$$

Because $\hat{\mathbf{n}}$ is unit vector and thus $n_i n_i = 1$, we can eliminate one component of $\hat{\mathbf{n}}$ above. We choose to eliminate n_3 (supposedly write n_3 in terms of n_1 and n_2 because $n_1^2 + n_2^2 + n_3^2 = 1$), and multiply out to get

$$\begin{aligned} S^2 = (p_1^2 - p_3^2) n_1^2 + (p_2^3 - p_3^2) n_2^2 + p_3^2 \\ - \Big[(p_1 - p_3) n_1^2 + (p_2 - p_3) n_2^2 + p_3 \Big]^2. \end{aligned}$$

- The extrema of the shear force occurs where

$$\frac{\mathrm{d}(S^2)}{\mathrm{d}n_1} = \frac{\mathrm{d}(S^2)}{\mathrm{d}n_1} = 0$$

This would lead to the conditions

$$\Big\{ (p_1 - p_3) - 2\big[(p_1 - p_3) n_1^2 + (p_2 - p_3) n_2^2\big] \Big\} n_1 = 0 \Big\{ (p_2 - p_3) - 2\big[(p_1 - p_3) n_1^2 + (p_2 - p_3) n_2^2\big] \Big\} n_1 \quad = 0$$

Note that we reject the solution $n_1 = n_2 = 0$ and thus $n_3 = 1$, because, from the expression for S^2, this corresponds to a surface with no shear forces.

- Possible solutions are:

  1. $n_1 \neq 0$ and $n_2 = 0$, which gives

$$(p_1 - p_3)(1 - 2n_1^2) = 0 \implies \hat{\mathbf{n}} = \left(\pm \frac{1}{\sqrt{2}}, 0, \pm \frac{1}{\sqrt{2}} \right).$$

2. $n_1 = 0$ and $n_2 \neq 0$, which gives
$$(p_2 - p_3)(1 - 2n_2^2) = 0 \implies \hat{\mathbf{n}} = \left(0, \pm\frac{1}{\sqrt{2}}, \pm\frac{1}{\sqrt{2}}\right).$$
3. $n_1 \neq 0$ and $n_2 \neq 0$. This appears to correspond to $p_1 = p_2 = p_3$.
However: if we had eliminated either n_1 or n_2 instead of n_3 above, choosing $n_1 \neq 0$ and $n_2 = 0$ would lead to
$$\hat{\mathbf{n}} = \left(\pm\frac{1}{\sqrt{2}}, \pm\frac{1}{\sqrt{2}}, 0\right).$$

- **TODO:** why did we solve for S^2?

2.3.5 Exercise: Relative Volume Change for a Uniaxially-Loaded Rod

We compress a rectangular rod with cross section S by applying a force of magnitude F unformly across the rod's two cross sections. Calculate the resulting change in the rod's volume. Neglect the rod's weight.

- We first define a coordinate system such that the rod's longitudinal axis aligns with the z axis and the rod's cross section is parallel to the xy plane.

  Since the applied force is independent of the position (x, y) in the rod's cross section, the deformation is homogeneous; thus, we don't need the Navier equation, which considerably simplifies the problem.

- We begin with the Cauchy condition for equilibrium, which reads
$$\frac{\partial p_{ij}}{\partial x_j} + f_i^{\text{ext}} = 0,$$
  where p_{ik} is the stress tensor and f_i^{ext} is the volume density of external forces acting throughout the body's volume (e.g. weight). The Cauchy condition must hold across the entire volume of any body in equilibrium.

  - *Important*: p_{ik} corresponds only to internal, volume, surface-contact force and not directly the external force F. However, the internal and external forces are related by the Cauchy equilibrium condition.
  - *Important*: to be clear, the term f_i^{ext} is *internal* volume-density forces inside the rod and has nothing to do with external forces like F applied *only to the surface.*
  - The surface-applied external forces (in our problem, the force applied to the rod's z faces) are accounted for in the boundary conditions on p_{ij}.

- The Cauchy condition applies to the body's volume; we must also satisfy the problem's boundary conditions, which apply to the body's surface.

  General process: determine which stress tensor components p_{ij} are relevenat along each of the body's boundaries.

  Interpreting the component p_{ij}: the first index i corresponds to the i-th component of an applied load, while the index j corresponds to the normal to the cross-sectional surface along which we view this load.

  In the case of the rod, we have six faces as boundaries.

- Our applied force reads $\boldsymbol{F} = (0, 0, F)$, so only stress tensor components p_{zj} will be nonzero.
- The component p_{zz} (where the first index corresponds to the z component of the applied force and the the second index z corresponds to the rod's cross-section with normal in the z direction) is

$$p_{zz} = -\frac{F}{S}$$

  Minus sign because force points into rod, normal points out of rod.
- The components p_{zx} and p_{zy} because the applied forces does not act on the rod's faces with normals along the x and y axes.
- The componets p_{xj} and p_{yj} are zero, since $F_x = F_y = 0$, regardless of which face we consider.

These are our problem's boundary conditions. They are conditions for the stress tensor *on the faces*—they apply only on the surface. In other words, boundary conditions just means determine the value of p_{ij} along each of the body's boundaries.

- We neglect weight, meaning $f_i^{\text{ext}} = 0$, so the Cauchy condition simplifies to

$$\frac{\partial p_{ij}}{\partial x_j} = 0.$$

In principle, this equation has many possible solutions. We first test the simplest option: $p_{ij} \neq p_{ij}(\boldsymbol{r})$, i.e. a constant stress tensor.

Well, the solution $p_i = \text{constant}$ automatically satisfies the volume problem (the Cauchy equation), and also satisfies the boundary conditions (the surface problem) if we choose

$$p_{ij} = \begin{cases} -\frac{F}{S} & \text{if } i = j = z \\ 0 & \text{otherwise.} \end{cases}$$

Calculating Change in Volume

- From Section 2.3.2, we will find change in the rod's volume ΔV under the uniaxial load from the equation

$$\frac{\Delta V}{V} \approx \frac{\mathrm{d}V - \mathrm{d}V'}{\mathrm{d}V} = \operatorname{tr} \mathsf{u}.$$

Note: If $\nabla \cdot \boldsymbol{u}$ were position-dependent (which it is not in our case), we would integrate $\operatorname{tr} \mathsf{u}$ over the body's volume.

- To compute volume change, we must find the problem's strain tensor u. We begin with the just-derived stress tensor, which reads

$$\mathsf{p} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -F/S \end{pmatrix},$$

We find the strain tensor from Hooke's law, which (for isotropic materials) reads

$$p_{ij} = 2\mu u_{ij} + \lambda u_{kk}\delta_{ij}.$$

We now solve Hooke's law for u_{ij} in terms of p_{ij}.[1]

We first solve Hooke's law for the diagonal components u_{kk}, which reads

$$p_{kk} = 2\mu u_{kk} + \lambda u_{ll}\delta_{kk} = 2\mu u_{kk} + 3\lambda u_{kk} = u_{kk}(2\mu + 3\lambda) \implies u_{kk} = \frac{p_{kk}}{2\mu + 3\lambda}.$$

We then substitute u_{kk} into the Hooke's law expression for p_{ij} to get

$$p_{ij} = 2\mu u_{ij} + \lambda\delta_{ij}\frac{p_{kk}}{2\mu + 3\lambda} \implies u_{ij} = \frac{1}{2\mu}p_{ij} - \frac{\lambda}{2\mu(2\mu + 3\lambda)}p_{kk}\delta_{ij}. \tag{2.3}$$

- This strain tensor's diagonal terms are

$$u_{kk} = \frac{p_{kk}}{2\mu + 3\lambda},$$

and the change in the rod's volume is

$$\frac{\Delta V}{V} = \operatorname{tr}\mathsf{u} = u_{kk} = \frac{1}{2\mu + 3\lambda}p_{kk} = -\frac{F}{S}\frac{1}{2\mu + 3\lambda}.$$

2.3.6 Exercise: A Uniaxially-Loaded Rectangular Rod, Revisited

As in the previous exercise, a rectangular rod is compressed with a force of magnitude F applied uniformly along the longitudinal axis. Find the pressure p which we should apply to the rod's lateral surface such that the total change in rod's volume is zero.

- The problem requires zero change in volume, which we write as the condition

$$\frac{\Delta V}{V} = \operatorname{tr}\mathsf{u}.$$

Plan: find the load's stress tensor p, use this to find the strain tensor u, then require that $\operatorname{tr}\mathsf{u} = 0$.

- As in the previous problem, align z axis with the rod's longitudinal axis, so that the external force resolves to

$$\boldsymbol{F} = (0, 0, F).$$

On the z face, which is loaded only by the force $\boldsymbol{F}$ (and not pressure, which we apply only to the lateral surface), the stress tensor thus reads $p_{zz} = -F/S$ and $p_{xz} = p_{yz} = 0$.

Meanwhile, the pressure applied to the lateral surface (we assume pressure is uniformly distributed across the lateral surface) contributes the boundary conditions $p_{xx} = p_{yy} = -p$ and $p_{xy} = 0$. The boundary conditions on the stress tensor are then

$$\mathsf{p}^{\text{surface}} = \begin{pmatrix} -p & 0 & 0 \\ 0 & -p & 0 \\ 0 & 0 & -F/S \end{pmatrix}.$$

[1] Technically, since we are only interested in volume change, we only need the strain tensor's diagonal components, but we will find the entire strain tensor for the sake of completeness.

- Within the rod's volume, we have an equilibrium situation as in the previous exercse. Neglecting weight or other external volume forces, the Cauchy condition reads
$$\frac{\partial p_{ik}}{\partial x_k} = 0.$$
This is solved by a constant stress tensor
$$p_{ik} = \text{constant},$$
which satisfies the boundary conditions if we choose
$$\mathsf{p} = \begin{pmatrix} -p & 0 & 0 \\ 0 & -p & 0 \\ 0 & 0 & -F/S \end{pmatrix}.$$

- From Equation 2.3 in the previous exercise, we find the corresponding strain tensor componets from
$$u_{ij} = \frac{1}{2\mu} p_{ij} - \frac{\lambda}{2\mu(2\mu + 3\lambda)} p_{kk}\delta_{ij}. \tag{2.4}$$
We are interested only in the diagonal components (which encode change in volume). These are
$$\begin{aligned} u_{xx} &= \frac{1}{2\mu}\left(p_{xx} - \frac{\lambda}{2\mu + 3\lambda} \cdot p_{kk}\right) \\ u_{yy} &= \frac{1}{2\mu}\left(p_{yy} - \frac{\lambda}{2\mu + 3\lambda} \cdot p_{kk}\right) = u_{xx} \\ u_{zz} &= \frac{1}{2\mu}\left(p_{zz} - \frac{\lambda}{2\mu + 3\lambda} \cdot p_{kk}\right). \end{aligned}$$
This strain tensor's trace is
$$\operatorname{tr} \mathsf{u} = u_{xx} + u_{yy} + u_{zz} = \frac{1}{2\mu}\left(p_{kk} - \frac{3\lambda}{2\mu + 3\lambda} \cdot p_{kk}\right) = \frac{p_{kk}}{2\mu + 3\lambda}$$
The requirement $\operatorname{tr} \mathsf{u} = 0$ for zero volume change then gives
$$\operatorname{tr} \mathsf{u} = \frac{p_{kk}}{2\mu + 3\lambda} = 0 \implies p_{kk} = p_{xx} + p_{yy} + p_{zz} = -2p - \frac{F}{S} = 0 \implies p = -\frac{F}{2S}.$$

2.3.7 Exercise: Radial Deformation of a Rotating Cylinder

A homogeneous cylinder of radius R rotates about its longitudinal axis with angular speed ω. Find the components of the displacement vector corresponding to this rotation. You may assume the cylinder does not deform along its longitudinal axis, but be sure to give a physical interpretation of this assumption.

- We will work in cylindrical coordinates, in which the z axis aligns with the cylinder's longitudinal axis. In this case the angular velocity reads $\boldsymbol{\omega} = (0, 0, \omega)$.

  The cylinder deforms because of the centrifugal force, which acts radially outward. The is no deformation in the ϕ direction because of the homogeneous cylinder's rotational symmetry, and no deformation along the z direction under the assumption in the problem's instructions. The displacement is thus purely radial, and we write the displacement vector in the form
$$\boldsymbol{u} = u(r)\,\hat{\mathbf{e}}_r.$$

- Since a rotating cylinder presents a dynamics problem, we will use the Navier equation, which we write in terms of the Young's modulus and Poisson ratio in the form
$$\rho \ddot{\boldsymbol{u}} = \boldsymbol{f}^{\text{ext}} + \frac{E}{2(1+\sigma)}\left[\nabla^2 \boldsymbol{u} + \frac{1}{1-2\sigma}\nabla(\nabla\cdot\boldsymbol{u})\right],$$
where ρ is the cylinder's density, $\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{r})$ (in our case simply $\boldsymbol{u} = u(r)\,\hat{\mathbf{e}}_r$) is the displacement vector, and $\boldsymbol{f}^{\text{ext}}$ represents the volume density of external forces.

- We will solve the problem in the (non-inertial) coordinate system attached to the rotating cylinder. In this case, neglecting weight, the only force acting on the cylinder is the centrifugal force, which reads
$$\boldsymbol{f}^{\text{cent}}(r) = \rho\omega^2 r\,\hat{\mathbf{e}}_r.$$
Derivation: begin with $F = (mv^2)/r$, divide through by volume to get $f = (\rho v^2)/r$, use $v = \omega r$ to get $F = \rho\omega^2 r$. The Navier equation then reads
$$\mathbf{0} = \rho\omega^2 r\,\hat{\mathbf{e}}_r + \frac{E}{2(1+\sigma)}\left[\nabla^2 \boldsymbol{u} + \frac{1}{1-2\sigma}\nabla(\nabla\cdot\boldsymbol{u})\right].$$
Note that we have dropped $\rho\ddot{\boldsymbol{u}}$. I suppose for a steady state solution in which the rod is not deforming?

- Next, we use the general vector calculus identity
$$\nabla(\nabla\cdot\boldsymbol{u}) = \nabla^2\boldsymbol{u} + \nabla\times(\nabla\times\boldsymbol{u}) \implies \nabla^2\boldsymbol{u} = \nabla(\nabla\cdot\boldsymbol{u}) - \nabla\times(\nabla\times\boldsymbol{u})$$
to rewrite the Navier equation in the form
$$\rho\omega^2 r\,\hat{\mathbf{e}}_r + \frac{E}{2(1+\sigma)}\left[\nabla(\nabla\cdot\boldsymbol{u}) - \nabla\times(\nabla\times\boldsymbol{u}) + \frac{1}{1-2\sigma}\nabla(\nabla\cdot\boldsymbol{u})\right] = \mathbf{0}.$$
It turns out that the curl of $\nabla\times\boldsymbol{u}$ comes out to zero. Qualitatively, $\nabla\times\boldsymbol{u} = \mathbf{0}$ because $\boldsymbol{u}$ depends only on the radial coordinate r and is thus irrotational. More formally, the curl operator in cylindrical coordinates reads
$$\nabla\times\boldsymbol{u} = \left(\frac{1}{r}\frac{\partial u_z}{\partial\phi} - \frac{\partial u_\phi}{\partial z}\right)\hat{\mathbf{e}}_r + \left(\frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r}\right)\hat{\mathbf{e}}_\phi + \left(\frac{1}{r}\frac{\partial(ru_\phi)}{\partial r} - \frac{1}{r}\frac{\partial u_r}{\partial\phi}\right)\hat{\mathbf{e}}_z.$$
Since we have assumed $\boldsymbol{u} = u(r)\,\hat{\mathbf{e}}_r = (u(r),0,0)$, i.e. that $\boldsymbol{u}$ has only an r component and depends only on r, all of the terms in $\nabla\times\boldsymbol{u}$ evaluate to zero. Since $\nabla\times\boldsymbol{u} = \mathbf{0}$, the Navier equation, after combining like terms in $\nabla(\nabla\cdot\boldsymbol{u})$, simplifies to
$$\rho\omega^2 r\,\hat{\mathbf{e}}_r + \frac{E(1-\sigma)}{(1+\sigma)(1-2\sigma)}\nabla(\nabla\cdot\boldsymbol{u}) = \mathbf{0}.$$

- Next, we recall the general expressions for gradient and divergence in cylindrical coordinates, which read
$$\nabla f = \frac{\partial f}{\partial r}\,\hat{\mathbf{e}}_r + \frac{1}{r}\frac{\partial f}{\partial\phi}\,\hat{\mathbf{e}}_\phi + \frac{\partial f}{\partial z}\,\hat{\mathbf{e}}_z \qquad \text{and} \qquad \nabla\cdot\boldsymbol{u} = \frac{1}{r}\frac{\partial(ru_r)}{\partial r} + \frac{1}{r}\frac{\partial u_\phi}{\partial\phi} + \frac{\partial u_z}{\partial z}. \tag{2.5}$$

We will note evaluate the expressions explicitly quite yet. However, we note that because $\boldsymbol{u} = (u(r), 0, 0)$, our problem simplifies to

$$\nabla \cdot \boldsymbol{u} = \frac{1}{r}\frac{\partial(ru_r)}{\partial r} \implies \nabla(\nabla \cdot \boldsymbol{u}) = \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial(ru_r)}{\partial r}\right)\hat{\mathbf{e}}_r,$$

in terms of which the Navier equation becomes to

$$\rho\omega^2 r\,\hat{\mathbf{e}}_r + \frac{E(1-\sigma)}{(1+\sigma)(1-2\sigma)} \cdot \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial(ru_r)}{\partial r}\right)\hat{\mathbf{e}}_r = \mathbf{0}.$$

In other words, we have simplified our problem to the scalar equation

$$\frac{E(1-\sigma)}{(1+\sigma)(1-2\sigma)}\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial(ru_r)}{\partial r}\right) = -\rho\omega^2 r.$$

- For shorthand, we then define the constant

  $$\alpha \equiv \frac{(1+\sigma)(1-2\sigma)}{E(1-\sigma)}\rho\omega^2,$$

  in terms of which the Navier equation takes the more compact form

  $$\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial r u_r}{\partial r}\right) = -\alpha r,$$

  which we then integrate over r to get

  $$\frac{1}{r}\frac{\partial(ru_r)}{\partial r} = -\alpha\frac{r^2}{2} + A.$$

  We then multiply through by r, integrate once more and solve for u_r, producing

  $$u_r = -\alpha\frac{r^3}{8} + \frac{A}{2}r + \frac{B}{r}.$$

  We will determine the integration constants A and B with boundary conditions.

- We immediately recognize that the constant B must equal zero, since the term B/r would cause the radial deformation u_r to diverge along the cylinder's axis; this is clearly non-physical.

  We find A from the requirement that the stress on the cylinder's lateral surface must be zero (since no external forces act on the cylinder's surface). Mathematically, the zero surface stress condition reads

  $$p_{rr}\big|_{r=R} = 0,$$

  where we choose the component p_{rr} because p_{rr} corresponds to radial component of force acting in the radial direction.

  Note that the centrifugal force is a "volume force", so the presence of the centrifugal force does not contradict the assumption of zero stress on the surface. Here is a nice interpretation: the vanishingly thin surface has zero *volume* mass density ρ, and so the centrifugal force $f^{\text{cent}} = \rho\omega^2 r$ vanishes on the surface.

- Plan: use the just-derived displacement vector $\boldsymbol{u} = (u_r, 0, 0)$ to find the strain tensor u and use u to find the stress tensor p; hence apply the boundary condition $p_{rr}\big|_{r=R} = 0$ to find the constant A.

- For reference, we first write Hooke's law in the form

$$p_{ij} = \frac{E}{1+\sigma}\left[u_{ij} + \frac{\sigma}{1-2\sigma}(\nabla\cdot\boldsymbol{u})\delta_{ij}\right],$$

where we have written the strain tensor's trace u_{kk} in the equivalent form $u_{kk} = \nabla\cdot\boldsymbol{u}$. Lesson: use the coordinate-independent representation $\nabla\cdot\boldsymbol{u}$ to find $\operatorname{tr}\mathsf{u}$ when unsure about coordinates.

We are interested only in the component p_{rr}, which reads

$$p_{rr} = \frac{E}{1+\sigma}\left[u_{rr} + \frac{\sigma}{1-2\sigma}(\nabla\cdot\boldsymbol{u})\right]. \tag{2.6}$$

- We have already written down $\nabla\cdot\boldsymbol{u}$ in Equation 2.5; for $\boldsymbol{u} = (u_r, 0, 0)$ this is

$$\nabla\cdot\boldsymbol{u} = \frac{1}{r}\frac{\partial(ru_r)}{\partial r} = \frac{1}{r}\frac{\partial}{\partial r}\left(-\frac{\alpha r^4}{8} + \frac{Ar^2}{2}\right) = -\frac{\alpha r^2}{2} + A.$$

Meanwhile, from Equation 2.1 the component u_{rr} is given by

$$u_{rr} = \frac{\partial u_r}{\partial r} = \frac{\partial}{\partial r}\left(-\frac{\alpha r^3}{8} + \frac{Ar}{2}\right) = -\frac{3}{8}\alpha r^2 + \frac{A}{2}.$$

Substituting u_{rr} and $\nabla\cdot\boldsymbol{u}$ into Equation 2.6 and a few steps of algebra leads to

$$p_{rr} = \frac{E}{2(1+\sigma)(1-2\sigma)}\left(-\frac{3-2\sigma}{4}\sigma r^2 + A\right).$$

Applying the condition $p_{rr}\big|_{r=R} = 0$ and solving for A gives

$$A = \frac{3-2\sigma}{4}\alpha R^2.$$

With A known, the complete solution for u_r is thus

$$u_r(r) = \frac{(1+\sigma)(1-2\sigma)}{8E(1-\sigma)}\rho\omega^2\Big[-r^3 + (3-2\sigma)R^2 r\Big].$$

This solves our problem.

- Cool note: Hooke's law (i.e. getting p_{ij} from u_{ij}) holds in all orthogonal coordinates (once we know u_{ij}) because it does not contain derivatives of coordinates or basis vectors (which are what get ugly during change of coordinate systems).

Next: Other Stress Tensor Components

- The other nonzero components of the stress tensor in our calculations are

$$p_{\phi\phi} = \frac{E}{1+\sigma}\left[\frac{u_r}{r} + \frac{\sigma}{1-2\sigma}\frac{1}{r}\frac{\partial(ru_r)}{\partial r}\right]$$

Calculation from vaje using

$$p_{\phi\phi} = \frac{E}{1+\sigma}\left[u_{\phi\phi} + \frac{\sigma}{1-2\sigma}\frac{1}{r}\frac{\partial(ru_r)}{\partial r}\right]$$

where $u_{\phi\phi} = u/r$

In general we would use Hooke's law

$$p_{ij} = \frac{E}{1+\sigma}\left[u_{ij} + \frac{\sigma}{1-2\sigma}u_{kk}\delta_{ij}\right]$$

- And the component p_{zz} is found from

$$p_{zz} = \frac{E}{1+\sigma}\frac{\sigma}{1-2\sigma}\frac{1}{r}\frac{\partial ru_r}{\partial r}.$$

- Interpretation: p_{zz} is not only nonzero, but also dependent on r. This behavior corresponds to a cylinder whose bases are fixed in such a way that the cylinder cannot stretch or compress along its longitudinal axis (think of a cylinder with constant height).

  Physical interpretation: normally if the cylinder moves radially outwards it would compress a little in the z direction to compensate.

  In this case our solution satisfies the boundary condition $u_z = 0$ at $z = 0$ and $z = h$, i.e. the ends of the cylinder, while the aforementioned fixing of the end faces accounts for the nonzero p_{zz}.

- Meanwhile: if the cylinder is free, then $p_{zi} = 0$ on the end faces. Our solution (see p_{zz}) does not satisfy this condition. The solution for a free cylinder should be written with the ansatz

$$\boldsymbol{u} = u_r(r,z)\,\hat{\mathbf{e}}_r + u_z(r,z)\,\hat{\mathbf{e}}_z,$$

  and is thus a two-dimensional problem.

2.3.8 Exercise: A "Pressure-Loaded" Pipe

Consider a cylindrical pipe with inner and outer radii R_1 and R_2 respectively. The pressure outside the pipe is p_2. Analyze the pipe's deformation if the pressure inside the pipe si increased from $p \approx 0$ to $p = p_1$. Neglect deformation along the pipe's longitudinal axis.

- We will work in cylindrical coordinates, with the z axis aligned with the rod's longitudinal axis.
  - Because of the pipe's rotational symmetry about the azimuthal angle ϕ, the displacement vector $\boldsymbol{u}$ is independent of ϕ.
  - Under the assumption of no deformation along the pipe's longitudinal axis, $\boldsymbol{u}$ is also independent of the coordinate z.
  - The pressure acts radially outward, so $\boldsymbol{u}$ will have only a r component.

We conclude the deformation vector for this problem takes the form

$$\boldsymbol{u} = u_r(r)\,\hat{\mathbf{e}}_r = (u_r(r), 0, 0).$$

- We will solve the problem using the Navier equation in terms of E and σ, i.e.

$$\rho\ddot{\boldsymbol{u}} = \boldsymbol{f}^{\text{ext}} + \frac{E}{2(1+\sigma)}\left(\nabla^2\boldsymbol{u} + \frac{1}{1-2\sigma}\nabla(\nabla\cdot\boldsymbol{u})\right).$$

Neglecting weight, no external forces act on the rod's volume, so $\boldsymbol{f}^{\text{ext}}$ is zero. Next, we use the vector calculus identity

$$\nabla^2\boldsymbol{u} = \nabla(\nabla\cdot\boldsymbol{u}) - \nabla\times(\nabla\times\boldsymbol{u}),$$

together with $\boldsymbol{f}^{\text{ext}} = \mathbf{0}$, to rewrite the Navier equation in the form

$$\frac{E}{1+\sigma}\left(\frac{1-\sigma}{1-2\sigma}\cdot\nabla(\nabla\cdot\boldsymbol{u}) - \nabla\times(\nabla\times\boldsymbol{u})\right) = \mathbf{0}.$$

Things simplify further: namely, $\nabla\times\boldsymbol{u} = \mathbf{0}$. To verify this, we note that our displacement vector $\boldsymbol{u} = u_r(r)\,\hat{\mathbf{e}}_r$ has only an r component, which itself depends only on r. From the definition of the curl operator in cylindrical coordinates, i.e.

$$\nabla\times\boldsymbol{u} = \left(\frac{1}{r}\frac{\partial u_z}{\partial\phi} - \frac{\partial u_\phi}{\partial z}\right)\hat{\mathbf{e}}_r + \left(\frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r}\right)\hat{\mathbf{e}}_\phi + \left(\frac{1}{r}\frac{\partial(ru_\phi)}{\partial r} - \frac{1}{r}\frac{\partial u_r}{\partial\phi}\right)\hat{\mathbf{e}}_z,$$

it follows that $\nabla\times\boldsymbol{u} = \mathbf{0}$, since every single term vanishes. The Navier equation then reduces to

$$\nabla(\nabla\cdot\boldsymbol{u}) = \mathbf{0}. \tag{2.7}$$

We then recall the definitions of gradient and divergence in cylindrical coordinates, (see also the analogous discussion in the context of Equation 2.5), which for review read

$$\nabla f = \frac{\partial f}{\partial r}\,\hat{\mathbf{e}}_r + \frac{1}{r}\frac{\partial f}{\partial\phi}\,\hat{\mathbf{e}}_\phi + \frac{\partial f}{\partial z}\,\hat{\mathbf{e}}_z \qquad \text{and} \qquad \nabla\cdot\boldsymbol{u} = \frac{1}{r}\frac{\partial(ru_r)}{\partial r} + \frac{1}{r}\frac{\partial u_\phi}{\partial\phi} + \frac{\partial u_z}{\partial z}.$$

After substituting in $\boldsymbol{u} = u_r(r)\,\hat{\mathbf{e}}_r$, Equation 2.7 then becomes

$$\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial(ru_r)}{\partial r}\right] = 0.$$

We then integrate twice over r and solve for u_r to get the general solution

$$u_r = \frac{1}{2}Ar + \frac{B}{r}.$$

- We determine the constants A and B from boundary conditions, which arise from the pressures p_1 and p_2 on the cylinder's inner and outer surfaces and read

$$p_{rr}\big|_{R_1} = -p_1 \qquad \text{and} \qquad p_{rr}\big|_{R_2} = -p_2$$

The minus signs occur because the normal points out of the surface and the pressure inward, producing a minus sign. We find the strain tensor from Hooke's law

$$p_{ij} = \frac{E}{1+\sigma}\left(u_{ij} + \frac{\sigma}{1-2\sigma}u_{kk}\delta_{ij}\right).$$

Writing the trace as $u_{kk} = \nabla \cdot \boldsymbol{u}$ and applying

$$u_{rr} = \frac{\partial u_r}{\partial r} \qquad \text{and} \qquad \nabla \cdot \boldsymbol{u} = \frac{1}{r}\frac{\partial (r u_r)}{\partial r}$$

leads to

$$p_{rr} = \frac{E}{1+\sigma}\left[\frac{\partial u_r}{\partial r} + \frac{\sigma}{1-2\sigma}\frac{1}{r}\frac{\partial (r u_r)}{\partial r}\right].$$

Evaluating the derivatives with the known form of u_r leads to

$$p_{rr} = \frac{E}{1+\sigma}\left[\frac{A}{2(1-2\sigma)} - \frac{B}{r^2}\right],$$

Without derivation, solving this equation for A and B using the two boundary conditions $p_{rr}\big|_{R_1} = -p_1$ and $p_{rr}\big|_{R_2} = -p_2$ leads to

$$\begin{aligned} A &= \frac{2(1+\sigma)(1-2\sigma)}{E}\left[\frac{p_2 - p_1}{1-(R_2/R_1)^2} - p_2\right] \\ B &= \frac{1+\sigma}{E}\frac{(p_1 - p_2)R_1^2 R_2^2}{R_2^2 - R_1^2}. \end{aligned}$$

With A and B known, the solution for u_r and thus the complete displacement vector $\boldsymbol{u} = u_r(r)\,\hat{\mathbf{e}}_r$ is fully determined.

2.3.9 Exercise: Radial Eigenmodes of an Elastic Sphere

Find the eigenfrequencies for radial oscillations of an elastic ball of radius R.

- We will work in a spherical coordinate system whose origin coincides with the ball's center. Since we are interested in radial oscillations, we will search for solutions with a displacement vector of the form

  $$\boldsymbol{u}(\boldsymbol{r}) = u_r(r)\,\hat{\mathbf{e}}_r.$$

  In other words, the sphere's displacement from equilibrium depends only on the distance from the center of the sphere.

- We will solve for the sphere's dynamics with the Navier equation in terms of E and σ, i.e.

  $$\rho\ddot{\boldsymbol{u}} = \boldsymbol{f}^{\text{ext}} + \frac{E}{2(1+\sigma)}\left(\nabla^2\boldsymbol{u} + \frac{1}{1-2\sigma}\nabla(\nabla\cdot\boldsymbol{u})\right).$$

  Neglecting weight, no external forces act on the rod's volume, so $\boldsymbol{f}^{\text{ext}}$ is zero. Following the same pattern as in previous problems, we use the vector calculus identity

  $$\nabla^2\boldsymbol{u} = \nabla(\nabla\cdot\boldsymbol{u}) - \nabla\times(\nabla\times\boldsymbol{u}),$$

  together with $\boldsymbol{f}^{\text{ext}} = \boldsymbol{0}$, to rewrite the Navier equation in the form

  $$\frac{E}{1+\sigma}\left(\frac{1-\sigma}{1-2\sigma}\cdot\nabla(\nabla\cdot\boldsymbol{u}) - \nabla\times(\nabla\times\boldsymbol{u})\right) = \boldsymbol{0}.$$

Again as in previous problems, $\nabla \times \boldsymbol{u} = \mathbf{0}$, which, using the expression for curl in spherical coordinates, we confirm with the calculation

$$\begin{aligned}\nabla \times \boldsymbol{u} =& \frac{1}{r\sin\theta}\left(\frac{\partial(\sin\theta u_\phi)}{\partial\theta} - \frac{\partial u_\theta}{\partial\phi}\right)\hat{\mathbf{e}}_r + \frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial u_r}{\partial\phi} - \frac{\partial(ru_\phi)}{\partial r}\right)\hat{\mathbf{e}}_\theta \\ &+ \frac{1}{r}\left(\frac{\partial(ru_\theta)}{\partial r} - \frac{\partial u_r}{\partial\theta}\right)\hat{\mathbf{e}}_\phi \\ &= \mathbf{0},\end{aligned}$$

where all terms vanish because of the simplified ansatz $\boldsymbol{u} = u_r(r)\,\hat{\mathbf{e}}_r$. The Navier equation thus simplifies to

$$\frac{E(1-\sigma)}{(1+\sigma)(1-2\sigma)}\nabla(\nabla\cdot\boldsymbol{u}) = \rho\ddot{\boldsymbol{u}}. \tag{2.8}$$

- Since we are searching for oscillating solutions, we use the sinusoidal ansatz $\boldsymbol{u} = \boldsymbol{u}_0 e^{i\omega t}$. We then substitute this ansatz into Equation 2.8 and cancel like terms, which leads to the amplitude equation

$$\nabla(\nabla\cdot\boldsymbol{u}_0) + k^2\boldsymbol{u}_0 = 0, \quad \text{where} \;\; k^2 \equiv \frac{(1+\sigma)(1-2\sigma)}{E(1-\sigma)}\rho\omega^2 \tag{2.9}$$

Our next step is to write out the nabla operator. For review, the general expressions for gradient and divergence in spherical coordinates are

$$\begin{aligned}\nabla\cdot\boldsymbol{u} &= \frac{1}{r^2}\frac{\partial(r^2u_r)}{\partial r} + \frac{1}{r\sin\theta}\left(\frac{\partial(\sin\theta u_\theta)}{\partial\theta} + \frac{\partial u_\phi}{\partial\phi}\right) \\ \nabla f &= \frac{\partial f}{\partial r}\,\hat{\mathbf{e}}_r + \frac{1}{r}\frac{\partial f}{\partial\theta}\,\hat{\mathbf{e}}_\theta + \frac{1}{r\sin\theta}\frac{\partial f}{\partial\phi}\,\hat{\mathbf{e}}_\phi\end{aligned}$$

Using these expression and recalling the purely radial ansatz $\boldsymbol{u} = u_r(r)\,\hat{\mathbf{e}}_r$, Equation 2.9 reduces to the purely radial equation

$$\frac{\partial}{\partial r}\left[\frac{1}{r^2}\frac{\partial(r^2u_0)}{\partial r}\right] + k^2u_0 = \frac{\partial}{\partial r}\left[\frac{1}{r^2}\left(2ru_0 + r^2\frac{\partial u_0}{\partial r}\right)\right] + k^2u_0 = 0.$$

Evaluating the final derivative with respect to r produces

$$\frac{\partial^2u_0}{\partial r^2} + \frac{2}{r}\frac{\partial u_0}{\partial r} + \left(k^2 - \frac{2}{r^2}\right)u_0 = 0.$$

Finally, we divide the equation through by k^2 to produce

$$\frac{\partial^2u_0}{\partial(kr)^2} + \frac{2}{kr}\frac{\partial u_0}{\partial(kr)} + \left(1 - \frac{2}{(kr)^2}\right)u_0 = 0.$$

Note that we have reduced the problem of finding the ball's eigenmodes to solving a scalar equation.

- We solve Equation 2.3.9 by recalling the spherical Bessel differential equation, i.e.

$$\frac{\partial^2 j_l}{\partial r^2} + \frac{2}{r}\frac{\partial j_l}{\partial r} + \left(1 - \frac{l(l+1)}{r^2}\right)j_l = 0.$$

This matches Equation 2.3.9 if we choose $l = 1$ and $r \to kr$, which implies our amplitude equation for u_0 is solved by $j_1(kr)$.[2] The solution for the sphere's displacement u_0 thus takes the form

$$u_0(r) = Aj_1(kr),$$

where A is the (unknown) amplitude of the radial eigenmode. Note that A being is not a problem—amplitude is generally unknown for eigenmode problems, which are determined only up to a constant coefficient.

- Boundary condition: there is no force on the surface of the sphere at $r = R$. In terms of the stress tensor p_{ij}, this condition reads

$$p_{rr}\big|_{r=R} = 0.$$

The general expression for the stress tensor reads

$$p_{ij} = \frac{E}{1+\sigma}\left(u_{ij} + \frac{\sigma}{1-2\sigma}u_{kk}\delta_{ij}\right),$$

and the specific expression for the component p_{rr} reads evaluated at $r = R$ is

$$p_{rr} = \frac{E}{1+\sigma}\left(u_{rr} + \frac{\sigma}{1-2\sigma}\nabla\cdot\boldsymbol{u}\right)_{r=R} = 0,$$

where we have used the identity $u_{kk} = \nabla\cdot\boldsymbol{u}$. Using spherical coordinate identity $u_{rr} = \frac{\partial u_r}{\partial r}$ and writing out the divergence spherical coordinates gives

$$\left(\frac{\partial u_r}{\partial r} + \frac{\sigma}{1-2\sigma}\frac{1}{r}\frac{\partial(ru_r)}{\partial r}\right)_{r=R} = 0.$$

We then substitute in the solution $u_0 = Aj_1(kr)$, and evaluate the derivative over kr (instead of over r), which produces

$$\left[k\frac{\partial j_1(kr)}{\partial(kr)} + \frac{\sigma}{1-2\sigma}\frac{1}{kr^2}\frac{\partial\big[(kr)^2 j_1(kr)\big]}{\partial(kr)}\right]_{r=R} = 0.$$

- We will use the following relationships between the spherical Bessel functions:

$$-x^{-l}j_{l+1} = (x^{-l}z_l)' \qquad \text{and} \qquad x^{l+1}z_{l-1} = (x^{l+1}z_l)'.$$

We use the first identity with $l = 0$, combined with $j_0(x) = \frac{\sin x}{x}$, to write j_1 in the form

$$1\cdot j_1 = -(1\cdot j_0)' = \frac{\mathrm{d}}{\mathrm{d}x}\frac{\sin x}{x} = -\frac{\cos x}{x} + \frac{\sin x}{x^2}.$$

We match the second identity to the second term in Equation 2.3.9, which reduces to

$$\frac{\sin(kR)}{kR} + 2\frac{\cos kR}{(kR)^2} - 2\frac{\sin kR}{(kR)^3} + \frac{\sigma}{1-2\sigma}\frac{\sin(kR)}{kR} = 0.$$

We then rearrange this to get

$$\tan(kR) = \frac{kR}{1 - \frac{1-\sigma}{2(1-2\sigma)(kR)^2}}.$$

TODO: define $\xi = kR$.

[2]Formally, the corresponding Neumann function $n_1(kr)$ is also a mathematical solution, but the Neumann function diverges at the origin, so we reject it on physical grounds.

- For large values of kR, the solutions $\xi_i \equiv k_i R$ appraoch $n\pi$ from below (i.e. the solutions approach integer multiples of π for large values of kR). (Confirm this with a graphical solution to Equation 2.3.9).

  We assume a value $\sigma = 0.25$ (a typical Poisson ratio for steel), and numerically find the solutions

$$\xi_i = k_i R = 2.563, 6.059, 9.280, 12.459, \ldots$$

  From the expression for k in Equation 2.9, we find the eigenfrequencies are

$$\omega_i = \frac{\xi_i}{R}\sqrt{\frac{E(1-\sigma)}{\rho(1+\sigma)(1-2\sigma)}}$$

  For a steel sphere with radius $5\,\mathrm{cm}$, which has tabulated density and Young's modulus $\rho = 7800\,\mathrm{kg\,m^{-3}}$ and $E = 2.0 \cdot 10^{11}\,\mathrm{N\,m^{-2}}$, the frequencies (not angular, but actual frequencies) are, for $i = 1, 2, 3, 4, \ldots$

$$\nu_i \approx \{90.5, 214, 328, 440, \ldots\}\,\mathrm{kHz}$$

  Note that these frequencies are well above the audible range.

2.4 Plates

2.4.1 Theory: Deformation of Plates

- Formally, a plate is an object whose thickness is much smaller than its other two dimensions. We will work in a coordinate system in which the xy plane aligns aligns with the plate's equilibrium state, while the z axis is normal to the plate.

- We will denote the displacement of the plate in the z direction with $u(x, y)$; this displacement u is governed by the equation

$$D\Delta^2 u(x, y) - P(x, y) = 0, \tag{2.10}$$

  where $P(x, y)$ is surface force density of external forces, D is the plate's flexural rigidity (an elastic constant encoding the plate's resistance to flexure), and $\Delta \equiv \nabla^2$ is the Laplacian operator. Note that $\Delta^2 \equiv \nabla^2\nabla^2$ is the Lapplacian applied twice.

  For the entirety of the chapter on plates, we assume acts only on the x and y coordinates.

- An isotropic plate's flexural rigidity D is given by

$$D = \frac{Eh^3}{12(1-\sigma^2)}, \tag{2.11}$$

  where h is the plate's thickness and σ is the plate material's Poisson ratio.

Boundary Conditions

- The plate's displacement $u(x, y)$ depends on how its edges and surface are fixed.

- If the plate is rigidly fixed at its edges, the boundary conditions on u read

$$u(x,y)\big|_{\text{edge}} = 0 \qquad \text{and} \qquad \frac{\partial u}{\partial \hat{\mathbf{n}}}\bigg|_{\text{edge}} = 0. \tag{2.12}$$

The unit vector $\hat{\mathbf{n}}$ is the normal to the plate's edge (*not* the normal to the plate itself!) and lies in the xy plane. The operation $\frac{\partial}{\partial \hat{\mathbf{n}}}$ thus denotes differentiation in the direction perpendicular ot the plate's edge and lying in the plane of the sphere.

- If the plate is supported at its edge but not rigidly fixed (e.g. the plate leans on a block), the boundary conditions read

$$u(x,y)\big|_{\text{edge}} = 0 \qquad \text{and} \qquad \left[\frac{\partial^2 u}{\partial \hat{\mathbf{n}}^2} + \sigma \frac{\partial \theta}{\partial \hat{\mathbf{l}}}\frac{\partial u}{\partial \hat{\mathbf{n}}}\right]_{\text{edge}} = 0, \tag{2.13}$$

where θ is the angle between the tangent to the edge and the x axis, while $\frac{\partial}{\partial \hat{\mathbf{l}}}$ denotes differentiation along the tangent to the edge. The derivative $\frac{\partial \theta}{\partial \hat{\mathbf{l}}}$ gives the shape of the rod's edge.

By convention, we assume the normal to the edge $\hat{\mathbf{n}}$ points out of the plate and the tangent $\hat{\mathbf{l}}$ points along the plate's edge such that the plate's interior is on the left.

2.4.2 Exercise: A Disk's Deformation Under its Own Weight

An isotropic disk of radius R, density ρ and thickness h is rigidly fixed around its circumference but is otherwise unsupported. Determine the displacement $u = u(\boldsymbol{r})$ resulting from the disk sagging under its own weight.

- Begin with the equilibrium equation

$$D\Delta^2 u - P = 0,$$

where P is the surface density of external forces acting on the plate and D is the disk's flexural rigidity. In principle we could compute D from Equation 2.11, but we assume it is known.

In our case the only external force is the disk's weight, and P reads

$$P = \frac{F}{S} = \frac{mg}{S} = \frac{\rho g V}{S} = \frac{\rho g(hS)}{S} = \rho g h.$$

The disk's equilibrium equation then reads

$$D\Delta^2 u = \rho g h \implies \Delta^2 u = \frac{\rho g h}{D} \equiv \alpha. \tag{2.14}$$

- We will work in a plane polar coordinate system; because of the disk's symmetry with respect to the azimuthal angle ϕ, the displacement depends only the radial coordinate r, i.e. $u = u(r)$. Assuming $u = u(r)$, the Laplacian operation in plane polar coordinates simplifies to

$$\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right),$$

and Equation 2.14 then reads

$$\Delta^2 u = \nabla^2(\nabla^2 u) = \frac{1}{r}\frac{\partial}{\partial r}\left\{r\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)\right]\right\} = \alpha. \tag{2.15}$$

The remainder of the problem is just integrating over r and applying appropriate boundary conditions. A first integration over r produces

$$r\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)\right] = \int \alpha r \,\mathrm{d}r = \frac{\alpha r^2}{2} + C_1. \tag{2.16}$$

We can then set $r = 0$ to obtain $C_1 = 0$. More formally, we can be assured the term in the square brackets in the LHS of Equation 2.16 does not diverge as $r \to 0$ since we know from Equation 2.15 ensures this term must be finite as $r \to 0$ (since α is finite, the LHS must also remain finite).

- Setting $C_1 = 0$ and integrating Equation 2.16 over r produces

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) = \frac{\alpha}{2}\int r\,\mathrm{d}r = \frac{\alpha r^2}{4} + C_2 \tag{2.17}$$

This time around, we have no gaurantee that the LHS Equation 2.17 is finite as $r \to 0$, so we cannot simply set $r = 0$ and eliminate C_2. Instead we press onwards, integrating Equation 2.17 over r to get

$$r\frac{\partial u}{\partial r} = \frac{\alpha}{4}\int r^3\,\mathrm{d}r + C_2\int r\,\mathrm{d}r = \frac{\alpha r^4}{16} + \frac{C_2 r^2}{2} + C_3. \tag{2.18}$$

In this case we set $r = 0$ (the LHS of Equation 2.18 does not contain $1/r$ terms, so there is no risk of divergence) to conclude $C_3 = 0$.

- Setting $C_3 = 0$ in Equation 2.18 and dividing through by r results in

$$\frac{\partial u}{\partial r} = \frac{\alpha r^3}{16} + \frac{C_2 r}{2}, \tag{2.19}$$

and integrating once more over r gives

$$u(r) = \frac{\alpha r^4}{64} + \frac{C_2 r^2}{4} + C_4. \tag{2.20}$$

- Next, we turn to the boundary conditions for a rigidly fixed plate in Equation 2.12, which in our case read

$$u(r)\big|_{r=R} = 0 \qquad \text{and} \qquad \frac{\partial u}{\partial r}\bigg|_{r=R} = 0.$$

Applying these conditions to Equations 2.19 and 2.20 produces

$$C_2 = -\frac{\alpha}{8}R^2 \qquad \text{and} \qquad C_4 = \frac{\alpha}{64}R^4,$$

which we then substitute into Equation 2.20 to get the final solution

$$u(r) = \frac{\alpha}{64}\left(r^4 - r^2R^2 + R^4\right) = \frac{\alpha}{64}\left(r^2 - R^2\right)^2.$$

2.4.3 Exercise: A Disk's Deformation Under a Point Force

An isotropic disk of radius R is supported, but not rigidly fixed, around its circuference, and a point force of magnitude F_0 is applied at the disk's center, normal to the disk. Find the corresponding deformation $u = u(\boldsymbol{r})$ throughout the plate.

- As in the previous problem, we will work in a plane polar coordinate system. Because of the disk's symmetry with respect to the azimuthal angle ϕ, the displacement depends only the radial coordinate r, i.e. $u = u(r)$.

- We will find the plate's deformation from the equilibrium equation

$$D\Delta^2 u - P = 0,$$

where P is the surface density of external forces acting on the plate and D is the disk's flexural rigidity. Our first step is to determine the surface density P associated with the point force.

We will write this surface density in the form

$$P(r) = f(r)\delta(r),$$

where $\delta(r)$ is the Dirac delta function, while the to-be-determined function $f(r)$ is found from the condition

$$F_0 = \iint P \,\mathrm{d}S,$$

which simply states that the integral of the external surface force density over the entire disk must equal the total surface force. Substituting in $P(r)$, we find

$$F_0 = \iint f(r)\delta(r)\,\mathrm{d}S = 2\pi \int_0^\rho f(r)\delta(r) r\,\mathrm{d}r, \tag{2.21}$$

where ρ is any finite radius in the range $(0, R]$. (We don't need to integrate all the way to $r = R$, because the external force is completely concentrated at the origin $r = 0$). We see that Equation 2.21 is satisfied if we choose

$$f(r) = \frac{F_0}{2\pi}\frac{1}{r},$$

which produces

$$F_0 = 2\pi \int_0^\rho \frac{F_0}{2\pi}\frac{1}{r} \cdot \delta(r) r\,\mathrm{d}r = F_0 \int_0^\rho \delta(r)\,\mathrm{d}r = F_0,$$

where the last equality follows from the integral definition of the delta function. Using the just-deduced expression for $P(r)$, the disk's equilibrium equation reads

$$D\Delta^2 u - P = 0 \implies \Delta^2 u = \frac{P}{D} = \frac{1}{D} f(r)\delta(r) = \frac{F_0}{2\pi D}\frac{\delta(r)}{r} \equiv \alpha \frac{\delta(r)}{r}.$$

We're left with the equation

$$\Delta^2 u \equiv \nabla^2(\nabla^2 u) = \alpha \frac{\delta(r)}{r}, \tag{2.22}$$

which we will integrate to find the disk's deformation u.

- Assuming $u = u(r)$, the Laplacian operation in plane polar coordinates simplifies to
$$\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right),$$
and Equation 2.22 then reads
$$\frac{1}{r}\frac{\partial}{\partial r}\left\{r\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)\right]\right\} = \alpha\frac{\delta(r)}{r}. \tag{2.23}$$
Our first step involves a definite integral from zero to a radius $r \in (0, R]$, which reads
$$\rho\frac{\partial}{\partial \rho}\left[\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial u}{\partial \rho}\right)\right]_{\rho=0}^{r} = \alpha\int_0^r \delta(\rho)\,\mathrm{d}\rho.$$
Again leveraging the integral definition of the delta function, this integration produces
$$r\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)\right] = \alpha.$$
The next integral over r leads to
$$\begin{aligned}\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) &= \alpha\int\frac{1}{r}\,\mathrm{d}r = \alpha\ln r + B_1 \equiv \alpha\ln r - \alpha\ln R + C_1\\ &= \alpha\ln\frac{r}{R} + C_1, \end{aligned} \tag{2.24}$$
where we have manipulated the original integration constant via $B_1 \equiv C_1 - \alpha\ln R$ to ensure the argument of the logarithm remains dimensionless.

- We then integrate Equation 2.25 again to get
$$r\frac{\partial u}{\partial r} = \alpha\int r\ln\frac{r}{R}\,\mathrm{d}r + C_1 r + C_2. \tag{2.25}$$
We can immediately set $r = 0$ to conclude $C_2 = 0$ (since $\frac{\partial u}{\partial r}$ must remain finite on physical grounds, while $\lim_{x\to 0} x\ln x = 0$) to conclude $C_2 = 0$. The remaining integral is solved by defining the new variable $x = r/R$ and using the integral
$$R^2\int x\ln x\,\mathrm{d}x = \frac{R^2 x^2}{2}\left(\ln x - \frac{1}{2}\right) + C. \tag{2.26}$$
This is found with integration by parts and the identity $\int \ln x = x(\ln x - 1) + C$; I've left out the full derivation for conciseness. The end result for Equation 2.25, after dividing through by r, is
$$\frac{\partial u}{\partial r} = \frac{\alpha r}{4}\left(2\ln\frac{r}{R} - 1\right) + \frac{C_1 r}{2}. \tag{2.27}$$
One final integration, involving a further application of Equation 2.26, leads to
$$u(r) = \frac{\alpha r^2}{4}\left(\ln\frac{r}{R} - 1\right) + \frac{C_1 r^2}{4} + C_3. \tag{2.28}$$

- We determine the constants C_1 and C_3 in Equation 2.28 from the boundary equations for a supported plate (Eq. 2.13), which in our case, for a disk, read

$$u(r)\big|_{r=R} = 0 \qquad \text{and} \qquad \left(\frac{\partial^2 u}{\partial r^2} + \sigma \frac{1}{R}\frac{\partial u}{\partial r}\right)_{r=R} = 0.$$

The first boundary condition on $u(r)$ produces the relationship

$$\frac{\alpha R^2}{4} = \frac{C_1 R^2}{4} + C_3. \tag{2.29}$$

- To make use of the second boundary condition, we must first compute $\frac{\partial^2 u}{\partial r^2}$. By differentiating Equaiton 2.27, this comes out to

$$\frac{\partial^2 u}{\partial r^2} = \frac{\alpha}{2}\ln\frac{r}{R} + \frac{\alpha}{4} + \frac{C_1}{2}.$$

There is an important subtelty here—since we artificially introduced the expression $\ln(r/R)$ instead of just $\ln r$ way back in Equation 2.25 to ensure consistent units, we must first decompose $\ln(r/R)$ back into $\ln r - \ln R$ before differentiating in Equation 2.27, to ensure units match up.

In any case, with $\frac{\partial^2 u}{\partial r^2}$ known, the second boundary condition produces

$$\left(\frac{\alpha}{4} + \frac{C_1}{2}\right) + \frac{\sigma}{R}\left(-\frac{\alpha R}{4} + \frac{C_1 R}{2}\right) = 0 \implies C_1 = \frac{\alpha}{2}\frac{\sigma - 1}{\sigma + 1}.$$

Substituting C_1 into Equation 2.29 and solving for C_3 then gives

$$C_3 = \frac{R^2\alpha}{8}\frac{3+\sigma}{1+\sigma}.$$

Substituting C_1 and C_3 into Equation 2.28 produces the final solution

$$u(r) = \frac{\alpha r^2}{4}\ln\frac{r}{R} + \frac{\alpha}{8}\frac{3+\sigma}{1+\sigma}(R^2 - r^2).$$

2.4.4 Exercise: A Disk's Deformation Under a Ring-Like Force

A disk of radius R is supported at its edges, and a force of magnitude F_0 is then distribued around a thin ring of radius $r_0 < R$, normal to the disk. Find the disk's resulting displacement? Assume the disk's weight is negligible relative to the external force.

- As usual for this section, we will find the plate's deformation from the general plate equilibrium equation

$$D\Delta^2 u - P = 0.$$

Working in a plane polar coordinate system and following the same initial steps as in Exercise 2.4.3, we deduce that the appropriate surface density of external forces P corresponding to the ring-like load is

$$P = \frac{F_0}{2\pi}\frac{\delta(r - r_0)}{r}.$$

With P known, the disk's corresponding equilibrium equation reads

$$\Delta^2 u = \frac{P}{D} = \frac{F_0}{2\pi D}\frac{\delta(r - r_0)}{r} \equiv \alpha\frac{\delta(r - r_0)}{r}.$$

- Again following the process of the previous problem (see e.g. Equation 2.23), we note that $u = u(r)$ (i.e. the displacement has only radial dependency because of the disk's symmetry about the azimuthal angle ϕ), in which case the double Laplace operator Δ^2 evaluates to
$$\frac{1}{r}\frac{\partial}{\partial r}\left\{r\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)\right]\right\} = \alpha\frac{\delta(r-r_0)}{r}.$$
Next, we multiply through by r and then integrate over r to any radius in the range (r_0, R). We use the Heaviside step function to elegantly account for the integral over delta function, and the result, after dividing through by r, is
$$\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)\right] = \frac{\alpha}{r}\cdot \mathrm{H}(r-r_0). \tag{2.30}$$
Next, we first note the general Heaviside function identity
$$\int \phi(x)\mathrm{H}(x-x_0)\,\mathrm{d}x = \big[\Phi(x)-\Phi(x_0)\big]\cdot \mathrm{H}(x-x_0) + C,$$
where $\Phi(x)$ is the indefinite integral of $\phi(x)$. We then integrate Equation 2.30 over r to get
$$\begin{aligned}\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) &= \alpha\int\frac{1}{r}\mathrm{H}(r-r_0)\,\mathrm{d}r = \alpha\ln r\cdot\mathrm{H}(r-r_0) + B_1\\ &\equiv (\alpha\ln r - \alpha\ln r_0)\mathrm{H}(r-r_0) + C_1\\ &= \alpha\ln\frac{r}{r_0}\mathrm{H}(r-r_0) + C_1,\end{aligned}$$
where, just like in Equation 2.25 in the previous exercise, we have reshifted the integration constant B_1 via $B_1 \equiv C_1 - \alpha\ln r_0$, which ensures the argument of the logarithm $\ln(r/r_0)$ remains dimensionless. After multiplying through by r, the above equation reads
$$\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) = \alpha r\ln\frac{r}{r_0}\mathrm{H}(r-r_0) + C_1 r. \tag{2.31}$$

- Next, we integrate Equation 2.31 over r, which gives
$$\begin{aligned} r\frac{\partial u}{\partial r} &= \alpha\int r\ln\frac{r}{r_0}\mathrm{H}(r-r_0)\,\mathrm{d}r + C_1\int r\,\mathrm{d}r\\ &= \frac{\alpha}{4}\left[2r^2\ln\frac{r}{r_0} - (r^2-r_0^2)\right]\mathrm{H}(r-r_0) + \frac{C_1}{2}r^2 + C_2,\end{aligned} \tag{2.32}$$
where we have evaluated the first integral with the identity given in Equation 2.26, which for review reads
$$r_0^2\int x\ln x\,\mathrm{d}x = \frac{r_0^2x^2}{2}\left(\ln x - \frac{1}{2}\right) + C. \tag{2.33}$$
We then set $r = 0$ in Equation 2.32, to conclude $C_2 = 0$, before solving for $\frac{\partial u}{\partial r}$ to get
$$\frac{\partial u}{\partial r} = \frac{\alpha}{4}\left[2r\ln\frac{r}{r_0} - \frac{(r^2-r_0^2)}{r}\right]\mathrm{H}(r-r_0) + \frac{C_1 r}{2}.$$
We integrate over r once more, again using the integral in Equation 2.33, to get
$$u = \frac{\alpha}{4}\left[(r^2+r_0^2)\ln\frac{r}{r_0} - (r^2-r_0^2)\right]\mathrm{H}(r-r_0) + \frac{C_1 r^2}{4} + C_3. \tag{2.34}$$

- We determine the constants C_1 and C_3 in Equation 2.34 from the boundary equations for a supported plate (Eq. 2.13), which in our case, for a disk, read

$$u(r)\big|_{r=R} = 0 \qquad \text{and} \qquad \left(\frac{\partial^2 u}{\partial r^2} + \sigma \frac{1}{R}\frac{\partial u}{\partial r}\right)_{r=R} = 0. \tag{2.35}$$

For use with the boundary conditions, we first make the auxiliary calculations

$$\left.\frac{\partial u}{\partial r}\right|_{r=R} = \frac{\alpha}{4}\left(2R\ln\frac{R}{r_0} - \frac{R^2 - r_0^2}{R}\right) + \frac{C_1 R}{2} \equiv 0,$$

and

$$\begin{aligned}\frac{\partial^2 u}{\partial r^2} &= \frac{\alpha}{4}\left(2\ln\frac{r}{r_0} - \frac{r_0^2}{r^2} + 1\right)\mathrm{H}(r - r_0) + \frac{C_1}{2},\\ \left.\frac{\partial^2 u}{\partial r^2}\right|_{r=R} &= \frac{\alpha}{4}\left(2\ln\frac{R}{r_0} - \frac{r_0^2}{R^2} + 1\right) + \frac{C_1}{2} \equiv 0.\end{aligned}$$

The boundary conditions in Equation 2.35, after combining like terms, produce

$$\begin{aligned}0 &= \frac{\alpha}{4}\left[(R^2 + r_0^2)\ln\frac{R}{r_0} - (R^2 - r_0^2)\right] + \frac{C_1 R^2}{4} + C_3\\ 0 &= \frac{1+\sigma}{2}\left(C_1 + \alpha\ln\frac{R}{r_0}\right) + \frac{\alpha(1-\sigma)}{4}\left(1 - \frac{r_0^2}{R^2}\right).\end{aligned}$$

Without derivation, the solutions for the integrations constants are

$$\begin{aligned}C_1 &= \alpha\left(\frac{\sigma - 1}{\sigma + 1}\frac{R^2 - r_0^2}{2R^2} - \ln\frac{R}{r_0}\right)\\ C_3 &= \frac{\alpha}{4}\left[\frac{3+\sigma}{2(1+\sigma)}(R^2 - r_0^2) - r_0^2\ln\frac{R}{r_0}\right].\end{aligned}$$

With C_1 and C_3 known, we have fully determine the disk's displacement $u(r)$ in Equation 2.34, thus completing this exercise.

2.4.5 Exercise: A Rectangular Plate Under a Pile of Sand

A very long steel plate of width l and thickness d is supported at its edges. A pile of sand is then placed on the plate, with the height h of the pile varying along the plate's width according to

$$h(x) = a\sin^2\frac{\pi x}{l},$$

while the pile height is constant along the plate's length. Find the plate's deformation under the total load from both the sand and the plate's own weight.

- Sketched. Begin with the equilibrium equation

$$D\Delta^2 u = P$$

The surface force density on the plate, accounting for both plate and sand's weight, is

$$P = \rho_{\mathrm{s}} g \cdot a\sin^2\frac{\pi x}{l} + \rho_{\mathrm{p}} g d.$$

The plate's equilibrium condition is then

$$\Delta^2 u = \frac{1}{D}\left(\rho_{\mathrm{s}} g \cdot a \sin^2 \frac{\pi x}{l} + \rho_{\mathrm{p}} g d\right) \equiv A \sin^2 \frac{\pi x}{l} + B. \tag{2.36}$$

Since the plate is very long (i.e. the length dimension is much longer than the width) we can approximate the Laplacian as

$$\nabla^2 u \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \approx \frac{\partial^2}{\partial x^2}.$$

Equation 2.36 then becomes

$$\frac{\partial^4 u}{\partial x^4} = A \sin^2 \frac{\pi x}{l} + B. \tag{2.37}$$

We first note the integral identity

$$\int \sin^2 ax \, \mathrm{d}x = \frac{x}{2} - \frac{\sin(2ax)}{4a}.$$

We then integrate Equation 2.37 for times over x to get

$$u(x) = A\left(\frac{x^4}{48} - \frac{l^4}{32\pi^4}\cos\frac{2\pi x}{l}\right) + \frac{Bx^4}{24} + \frac{C_1 x^3}{6} + \frac{C_2 x^2}{2} + C_3 x + C_4.$$

- We find the integration constants C_i from the problem's boundary conditions. The plate is supported at its edges, so we use the boundary conditions in Equation 2.13, which for review read

$$u(x,y)\big|_{\text{edge}} = 0 \qquad \text{and} \qquad \left[\frac{\partial^2 u}{\partial \hat{\mathbf{n}}^2} + \sigma \frac{\partial \theta}{\partial \hat{\mathbf{l}}}\frac{\partial u}{\partial \hat{\mathbf{n}}}\right]_{\text{edge}} = 0,$$

where θ is the angle between the tangent to the edge and the x axis, while $\frac{\partial}{\partial \hat{\mathbf{l}}}$ denotes differentiation along the tangent to the edge. Since this angle is constant and equal to $\pi/2$ for a rectangular plate, the term $\frac{\partial \theta}{\partial \hat{\mathbf{l}}}$ conveniently vanishes.

The normal derivative to our rectangular plate's edge is $\frac{\partial}{\partial x}$, while the boundaries occur at $x = 0$ and $x = l$, so the boundary conditions read

$$u(0) = u(l) = 0 \qquad \text{and} \qquad \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial x^2} = 0.$$

The conditions at $x = 0$ lead to

$$C_4 = \frac{Al^4}{32\pi^4} \qquad \text{and} \qquad C_2 = -\frac{Al^2}{8\pi^2}.$$

The boundary condition on u's second derivatve at $x = l$ gives

$$(A + 2B)\frac{l^2}{4} + \frac{Al^2}{8\pi^2} + C_1 l - \frac{Al^2}{8\pi^2} = 0 \implies C_1 = -\frac{l}{4}(A + 2B).$$

The final boundary condition on u's second derivatve at $x = 0$ gives

$$C_3 = \frac{l^3}{48}\left(\frac{3A}{\pi^2} + A + 2B\right).$$

The complete solution for the plates's displacement is thus

$$u = \frac{l^4}{48}\Bigg[(A + 2B)\left(\frac{x}{l}\right)^4 - 2(A + 2B)\left(\frac{x}{l}\right)^3 - \frac{3A}{\pi^2}\left(\frac{x}{l}\right)^2 + \left(\frac{3A}{\pi^2} + A + 2B\right)\frac{x}{l} + \frac{3A}{2\pi^4}\left(1 - \cos\frac{2\pi x}{l}\right)\Bigg].$$

2.5 Rod Flexure

2.5.1 Theory: Equilibrium of Forces and Torques

- Consider a small rod segment of length $\mathrm{d}l$. The equilibrium condition for this rod element reads
$$\frac{\mathrm{d}\boldsymbol{F}}{\mathrm{d}l} + \boldsymbol{K} = \boldsymbol{0}, \tag{2.38}$$
where $\boldsymbol{F} + \mathrm{d}\boldsymbol{F}$ is the elastic force of a neighboring rod element on the front cross-sectional surface of a the given rod element of length $\mathrm{d}l$, where "front" corresponds to the direction of increasing length l. The force on the given rod elements opposite cross-sectional surface is $-\boldsymbol{F}$. The quantity $\boldsymbol{K}$ is the linear density of external force on the rod (at the position of the given rod element).

  Alternative approach: in equilibrium, the difference between the forces $\boldsymbol{F} + \mathrm{d}\boldsymbol{F}$ and $-\boldsymbol{F}$ on a rod element's two faces balances the contribution $\boldsymbol{K}\,\mathrm{d}l$ of external forces. This interpretation gives
$$\boldsymbol{F} + \mathrm{d}\boldsymbol{F} - \boldsymbol{F} + \boldsymbol{K}\,\mathrm{d}l = \boldsymbol{0} \implies \frac{\mathrm{d}\boldsymbol{F}}{\mathrm{d}l} + \boldsymbol{K} = \boldsymbol{0}.$$
  Note that in the absence of external forces, i.e. if $\boldsymbol{K} = \boldsymbol{0}$, then the internal elastic force $\boldsymbol{F}$ on the rod elements' cross sectional faces is constant throughout the rod.

On Torque

- Neighboring rod elements can act on each other, via their mutually contacting surfaces, with both force and torque.
- Torque arises from force pairs. Note that the net force on a cross section can be zero with a non-zero net torque, for example for two equal-magnitude forces acting on a cross-sectional surface in opposite directions.
- The equilibrium condition for torque reads
$$\frac{\mathrm{d}\boldsymbol{M}}{\mathrm{d}l} + \boldsymbol{t} \times \boldsymbol{F} = \boldsymbol{0},$$
where $\boldsymbol{M} + \mathrm{d}\boldsymbol{M}$ is the elastic torque of the "next" rod element on a given rod element with respect to the centroid of the chosen rod element's "front" cross section, with "next" and "front" defined from the direction of increasing l.

  The torque of the "previous" rod element on the given element, again with respect to the centroid of the given element's cross section, is $-\boldsymbol{M}$.

  The vector $\boldsymbol{t}$ is the unit vector tangent to the rod, defined via
$$\boldsymbol{t} = \frac{\mathrm{d}\boldsymbol{l}}{\mathrm{d}l} \equiv \boldsymbol{l}'.$$
- Another take on torque equilibrium. Begin with condition
$$\boldsymbol{M} + \mathrm{d}\boldsymbol{M} - \boldsymbol{M} - \mathrm{d}\boldsymbol{l} \times (-\boldsymbol{F}) = \boldsymbol{0}$$
  Divde by $\mathrm{d}l$ (scalar length element) and define $\frac{\mathrm{d}\boldsymbol{l}}{\mathrm{d}l} = \boldsymbol{t}$ and get
$$\frac{\mathrm{d}\boldsymbol{M}}{\mathrm{d}l} + \boldsymbol{t} \times \boldsymbol{F} = \boldsymbol{0}. \tag{2.39}$$

- Note that we neglect the torque of external forces $\boldsymbol{K}\,\mathrm{d}l$, which is $\boldsymbol{M}_{\mathrm{ext}} = \mathrm{d}\boldsymbol{l} \times (\boldsymbol{K}\,\mathrm{d}l)$. We ignore $\boldsymbol{M}_{\mathrm{ext}}$ because this quantity is second order in the infinitesimal length element $\mathrm{d}l$.

- A rod's curvature $\boldsymbol{t}'$ is related to the torque $\boldsymbol{M}$ with which the rod is loaded by

$$\boldsymbol{M} = G\tau\boldsymbol{t} + EI\boldsymbol{t} \times \boldsymbol{t}',$$

  where G is the rod's torsion constant, τ is the torsion angle per unit rod length, and I is the rod's cross sectional geometric moment of inertia.

- We will generally restrict ourselves to torsion-free situations with $\tau = 0$, which gives

$$\boldsymbol{M} = EI\boldsymbol{t} \times \boldsymbol{t}'.$$

  We then make the auxiliary calculation

$$\frac{\mathrm{d}\boldsymbol{M}}{\mathrm{d}l} = EI(\boldsymbol{t}' \times \boldsymbol{t}' + \boldsymbol{t} \times \boldsymbol{t}'') = EI\boldsymbol{t} \times \boldsymbol{t}'', \tag{2.40}$$

  since $\boldsymbol{a} \times \boldsymbol{a} = \boldsymbol{0}$ for any vector $\boldsymbol{a}$. We then combine Equations 2.40 and 2.39 to get

$$EI\boldsymbol{t} \times \boldsymbol{t}'' + \boldsymbol{t} \times \boldsymbol{F} = \boldsymbol{0}.$$

2.5.2 Theory: Linear Equations for Small Flexure

Torque

- Begin with the torsion-free rod flexure equation

$$EI\boldsymbol{t} \times \boldsymbol{t}'' + \boldsymbol{t} \times \boldsymbol{F} = \frac{\mathrm{d}\boldsymbol{M}}{\mathrm{d}l} + \boldsymbol{t} \times \boldsymbol{F} = \boldsymbol{0}$$

  and differentiate with respect to l to get

$$\frac{\mathrm{d}^2\boldsymbol{M}}{\mathrm{d}l^2} + \boldsymbol{t} \times \frac{\mathrm{d}\boldsymbol{F}}{\mathrm{d}l} + \frac{\mathrm{d}\boldsymbol{t}}{\mathrm{d}l} \times \boldsymbol{F} = \boldsymbol{0}.$$

- Define a coordinate system in which the rod, in its equilibrium reference state, aligns with the z axis, while the rod's cross section is parallel to the xy plane. We will then parameterize the position along the deformed rod as $\boldsymbol{r}(l) = \big(x(l), y(l), z(l)\big)$, where l is the arc length along the rod. The tangent vector to the rod then reads

$$\begin{aligned}\boldsymbol{t} &\equiv \frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}l} = \frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}z}\frac{\mathrm{d}z}{\mathrm{d}l} = \left(\frac{\mathrm{d}x}{\mathrm{d}z}, \frac{\mathrm{d}y}{\mathrm{d}z}, 1\right) \cdot \frac{\mathrm{d}z}{\big[(\mathrm{d}x)^2 + (\mathrm{d}y)^2 + (\mathrm{d}z)^2\big]} \equiv \frac{(\dot{x}, \dot{y}, 1)}{\sqrt{1 + \dot{x}^2 + \dot{y}^2}} \\ &\approx (\dot{x}, \dot{y}, 1),\end{aligned}$$

  where the dot denotes differentiation with respect to z, and the last approximation assumes the $\frac{\mathrm{d}x}{\mathrm{d}z}$ and $\frac{\mathrm{d}y}{\mathrm{d}z}$ are much less than one (essentially the limit of small flexure, saying the cross-sectional dimensions remain roughly constant along the rod's longitudinal axis).

- Using the approximation $\boldsymbol{t} \approx (\dot{x}, \dot{y}, 1)$, the torque on the rod's cross section (and replacing $\boldsymbol{t}' = \frac{\mathrm{d}\boldsymbol{t}}{\mathrm{d}l}$ with $\dot{\boldsymbol{t}} = \frac{\mathrm{d}\boldsymbol{t}}{\mathrm{d}z}$, which holds for small flexure) becomes

$$\begin{aligned}
\boldsymbol{M} = EI\boldsymbol{t} \times \boldsymbol{t}' &\approx EI\left(\frac{\mathrm{d}x}{\mathrm{d}z}, \frac{\mathrm{d}y}{\mathrm{d}z}, 1\right) \times \left(\frac{\mathrm{d}^2 x}{\mathrm{d}z^2}, \frac{\mathrm{d}^2 y}{\mathrm{d}z^2}, 0\right) \\
&= EI\left(-\frac{\mathrm{d}^2 y}{\mathrm{d}z^2}, \frac{\mathrm{d}^2 x}{\mathrm{d}z^2}, \frac{\mathrm{d}x}{\mathrm{d}z} \cdot \frac{\mathrm{d}^2 x}{\mathrm{d}z^2} - \frac{\mathrm{d}y}{\mathrm{d}z} \cdot \frac{\mathrm{d}^2 y}{\mathrm{d}z^2}\right) \\
&\approx EI\left(-\frac{\mathrm{d}^2 y}{\mathrm{d}z^2}, \frac{\mathrm{d}^2 x}{\mathrm{d}z^2}, 0\right)
\end{aligned}$$

  Written more in more concise dot notation, the torque on the rod's cross section in the regime of small flexure is

$$\boldsymbol{M} = EI(-\ddot{y}, \ddot{x}, 0). \tag{2.41}$$

Force

- First, using the small-flexure approximation $\boldsymbol{t} \approx (\dot{x}, \dot{y}, 1)$, we make the auxiliary calculation

$$\begin{aligned}
\boldsymbol{t} \times \boldsymbol{F} = (\dot{x}, \dot{y}, 1) \times (F_x, F_y, F_z) &= (\dot{y}F_z - F_y, F_x - \dot{x}F_z, \dot{x}F_y - \dot{y}F_x) \\
&\approx (\dot{y}F_z - F_y, F_x - \dot{x}F_z, 0).
\end{aligned}$$

  We neglect the z component of $\boldsymbol{t} \times \boldsymbol{F}$, which contains both $\dot{x}$ and $\dot{y}$; these are both much smaller than unity for small flexure.

- Next, return to the equation

$$\frac{\mathrm{d}\boldsymbol{M}}{\mathrm{d}l} + \boldsymbol{t} \times \boldsymbol{F} = \boldsymbol{0}.$$

  Use the small-flexure approximations $\frac{\mathrm{d}}{\mathrm{d}z} \approx \frac{\mathrm{d}}{\mathrm{d}l}$ and $\boldsymbol{M} \approx EI(-\ddot{y}, \ddot{x}, 0)$ to get

$$\frac{\mathrm{d}\boldsymbol{M}}{\mathrm{d}l} + \boldsymbol{t} \times \boldsymbol{F} \approx \frac{\mathrm{d}}{\mathrm{d}l}\Big[EI(-\ddot{y}, \ddot{x}, 0)\Big] + (\dot{y}F_z - F_y, F_x - \dot{x}F_z, 0) = \boldsymbol{0}.$$

  After rearranging, the resulting equation is

$$EI(-\dddot{y}, \dddot{x}, 0) = (F_y - \dot{y}F_z, \dot{x}F_z - F_x, 0).$$

  We have two component equations for x and y. These are

$$EIy^{(3)} = \dot{y}F_z - F_y \qquad \text{and} \qquad EIx^{(3)} = \dot{x}F_z - F_x. \tag{2.42}$$

  Finally, we note that one more differentiation of these equations with respect to l leads to

$$EIy^{(4)} = \ddot{y}F_z + \dot{y}\dot{F}_z + K_y \qquad \text{and} \qquad EIx^{(4)} = \ddot{x}F_z + \dot{x}\dot{F}_z + K_x, \tag{2.43}$$

  where we have used the approximations $\frac{\mathrm{d}F_x}{\mathrm{d}z} \approx -K_x$ and $\frac{\mathrm{d}F_y}{\mathrm{d}z} \approx -K_y$ (adapted from the equilibrium condition $\frac{\mathrm{d}\boldsymbol{F}}{\mathrm{d}l} = -\boldsymbol{K}$ in Equation 2.38.

  Note that $\dot{F}_z$ is typically non-zero for a rod with non-negligible weight.

2.5.3 Exercise: A Rod's Deformation Under its Own Weight

A horizontally-lying rod of length L, density ρ and cross-sectional area S is rigidly fixed at one end, but is otherwise unsupported. Find the rod's deformation under its own weight, and the resulting force and torque at the fixation point.

- We will work in a Cartesian coordinate system with the y axis pointing in the direction of gravitational acceleration and the z axis pointing along the (undeformed) rod's longitudinal axis, away from the fixed end.

- We denote the rod's displacement in the y direction by $u = u(z)$. We will find this displacement from Equation 2.43, which states

$$EIu^{(4)} - F\ddot{u} - \dot{F}\dot{u} - K = 0,$$

where F is the elastic force on the rod's cross section, while K is the linear density of external forces—in our case the rod's weight. Neglecting internal forces, we have simply

$$EIu^{(4)} = K = \frac{mg}{L} = \frac{\rho g S L}{L} = \rho g S.$$

The displacement is then given by

$$u^{(4)} = \frac{\rho g S}{EI} \equiv \alpha.$$

Integrating four times over z leads to the general solution

$$u = \frac{\alpha}{24} z^4 + \frac{Az^3}{6} + \frac{Bz^2}{2} + Cz + D. \tag{2.44}$$

- We determine the integration constants A, B, C and D from the problem's boundary conditions, which are:

1. $u(z)\big|_{z=0} = 0$, since the rod is rigidly fixed
2. $u'(z)\big|_{z=0} = 0$, again because the rod is rigidly fixed
3. $\boldsymbol{M}(z)\big|_{z=L} = \boldsymbol{0}$, because there are no internal torques on the cross section of the rod's free end. We then combine this condition with Equation 2.41 to produce
$$\boldsymbol{M} = EI(-\ddot{y}, \ddot{x}, 0) \implies u''(z)\big|_{z=L} = 0.$$
4. $\boldsymbol{F}(z)\big|_{z=L} = \boldsymbol{0}$, because there are no internal forces on the cross section of the rod's free end. Combined with Equation 2.42, this produces
$$EIy^{(3)} = \dot{y}F_z - F_y \implies u^{(3)}(z)\big|_{z=L} = 0.$$

Substituting the first boundary condition into Equation 2.44 immediately produces $A = 0$. Computing u' and applying $u'(z)\big|_{z=0} = 0$ gives $B = 0$, while the third and fourth conditions lead to

$$A = -\alpha L \qquad \text{and} \qquad B = \frac{\alpha L^2}{2}.$$

The rod's displacement is then

$$u(z) = \alpha\left(\frac{z^4}{24} - \frac{Lz^3}{6} + \frac{L^2z^2}{4}\right).$$

This equation is enough to find force and torque on any segment of the rod.

- We find the force exerted by the rod on the wall from

$$F_y = -EIu^{(3)}\big|_{z=0} = -EI\alpha(z-L)_{z=0} = -EI\frac{mg}{EI}\cdot(-L) = mg;$$

Note that the sign is consistent because the y axis points in the direction of gravitational acceleration.

Finally, the torque on the rod is

$$\boldsymbol{M} = EI(-y'', x'', 0).$$

The rod would rotate in the yz plane, so the relavant torque component is M_x, which reads

$$M_y = -EI\alpha\left(\frac{z^2}{2} - \frac{Lz}{2} + \frac{L^2}{2}\right) \implies M_y(0) = -\frac{EI\alpha L^2}{2}.$$

After substituting in the original value of α, we have

$$M_y = -\frac{EIL^2}{2}\cdot\left(\frac{\rho g S}{EI}\right) = -\frac{(\rho SL)\cdot gL}{2} = -\frac{mgL}{2},$$

where $\rho SL = \rho V = m$ is the rod's mass.

2.5.4 Exericse: Euler Instability I

A rod of length L is held in place but free to rotate at both of its ends. We then compress the rod with a uniaxial force. Determine the critical magnitude F_{crit} at which the rod buckles. You may neglect the rod's weight.

- Define coordinate system: z along rod's longitudinal axis, x and y normal to rod.

Begin with general equilibrium condition

$$\frac{\mathrm{d}\boldsymbol{F}}{\mathrm{d}l} + \boldsymbol{K} = \boldsymbol{0}.$$

The compressive force applied to the rod's end faces corresponds to the $\boldsymbol{F}$ term; for unaxial compression along the rod's longitudinal axis this reads $\boldsymbol{F} = (0, 0, F)$.

Neglecting the rod's weight, no other external forces act on the rod, so $\boldsymbol{K} = \boldsymbol{0}$, from which we conclude

$$\frac{\mathrm{d}\boldsymbol{F}}{\mathrm{d}l} = \boldsymbol{0} \implies \boldsymbol{F}(l) = \text{constant}.$$

In other words, the force on the rod's cross-sectional faces is constant throughout the rod's length.

- We find the rod's displacement from either of the expressions in Equation 2.43, which for review read

$$EIy^{(4)} = \ddot{y}F_z + \dot{y}\dot{F}_z + K_y \qquad \text{and} \qquad EIx^{(4)} = \ddot{x}F_z + \dot{x}\dot{F}_z + K_x.$$

In the absence of weight, there is no preferential transverse direction, so the x and y expressions are equivalent. From $\boldsymbol{K} = 0$, $F_z = F$ and $\boldsymbol{F} = \text{constant} \implies \dot{\boldsymbol{F}} = \boldsymbol{0}$ we have

$$EIu^{(4)} = \ddot{u}F \implies u^{(4)} = \frac{F}{EI}\ddot{u} \equiv k^2\ddot{u}.$$

We solve the equation by first defining the new variable $v \equiv \ddot{u}$, which gives

$$\ddot{v} = k^2 v \implies v(z) = \ddot{u}(z) = A \sin kz + B \cos kz. \tag{2.45}$$

We then integrate $\ddot{u}(z)$ twice more over z to get

$$u(z) = -\frac{A}{k^2}\sin(kz) - \frac{B}{k^2}\cos(kz) + Cz + D. \tag{2.46}$$

- As usual, we determine the integration constants in Equation 2.46 from the problem's boundary conditions. Because the rod is held in place at its two ends we have

  $$u(z)\big|_{z=0} = u(z)\big|_{z=L} = 0.$$

  Meanwhile, because the rod is free to rotate at its ends, the torque on the rod's cross-sectional end faces must be zero. Combined with $\boldsymbol{M} = EI(-\ddot{y}, \ddot{x}, 0)$ (i.e. torque components are proportional to the second derivative of displacement) we conclude

  $$\ddot{u}(z)\big|_{z=0} = \ddot{u}(z)\big|_{z=L} = 0.$$

  Applying the condition $\ddot{u}(z)\big|_{z=0} = 0$ to Equation 2.45 gives $B = 0$, and then applying $u(z)\big|_{z=0} = 0$ to Equation 2.46 gives $D = 0$. We're left with

  $$u(z) = -\frac{A}{k^2}\sin(kz) + Cz.$$

  The boundary condition $u(z)\big|_{z=L} = 0$ can hold only if $C = 0$; what remains is

  $$u(L) = -\frac{A}{k^2}\sin(kL) = 0.$$

  Physical interpretation: the trivial solution $A = 0$ corresponds to buckling (basically linear displacement theory fails and predicts $u = 0$). The alternate (non-trivial) solution for $u(z)$ if $kL = \pi$ corresponds to the limit of linear theory, just before the rod begins to buckle. For $kL = \pi$ we have

  $$kL = \pi \implies k = \frac{\pi}{L}.$$

  Combined with the general expression $k^2 \equiv F/(EI)$ we have, at the limit point before buckling,

  $$k^2 = \frac{\pi^2}{L^2} = \frac{F_{\text{crit}}}{EI} \implies F_{\text{crit}} = \frac{\pi^2 EI}{L^2}.$$

2.5.5 Exericse: Euler Instability II

A rod of length L is rigidly fixed at one end and held in place but free to rotate the other end. The rod is then compressed with a uniaxial force. Determine the critical magnitude F_{crit} at which the rod buckles. You may neglect the rod's weight.

- This is a nearly identical problem to the previous exercise; only the boundary conditions at one end have changed. Re-using the results of the previous exercise, we begin at the general solution for the rod's displacement:

  $$u(z) = -\frac{A}{k^2}\sin(kz) - \frac{B}{k^2}\cos(kz) + Cz + D. \tag{2.47}$$

For later use with the boundary conditions, the displacement's derivatives are

$$\dot{u}(z) = -\frac{A}{k}\cos(kz) + \frac{B}{k}\sin(kz) + C \qquad \text{and} \qquad \ddot{u} = A\sin(kz) + B\cos(kz).$$

- Assuming the rod's rigidly fixed end occurs at $z = 0$ and the end which is free to rotate at $z = L$, the boundary conditions read

$$\ddot{u}(z)\big|_{z=L} = 0 \qquad \text{and} \qquad u(z)\big|_{z=L} = 0.$$

and

$$\dot{u}(z)\big|_{z=0} = 0 \qquad \text{and} \qquad u(z)\big|_{z=0} = 0.$$

Use $u(0) = 0$ to get

$$0 = -\frac{B}{k^2} + D \implies B = k^2 D.$$

Use $\dot{u}(0) = 0$ to get

$$0 = -\frac{A}{k} + C \implies A = kC.$$

Use $\ddot{u}(L) = 0$ to get

$$0 = A\sin(kL) + B\cos(kL). \tag{2.48}$$

Use $u(L) = 0$ with $B = k^2 D$ and $A = kC$ to get

$$0 = \left(\frac{L}{k} - \frac{\sin(kL)}{k^2}\right)A + \left(\frac{1}{k^2} - \frac{\cos(kL)}{k^2}\right)B. \tag{2.49}$$

Equation 2.48 and 2.49 are a homogeneous system of a equatoins with a non-trivial solution for A and B under the condition

$$\det\begin{pmatrix} \sin(kL) & \cos(kL) \\ \frac{L}{k} - \frac{\sin(kL)}{k^2} & \frac{1}{k^2} - \frac{\cos(kL)}{k^2} \end{pmatrix} = \frac{\sin(kL)}{k^2} - \frac{L\cos(kL)}{k} = 0 \implies \tan(kL) = kL.$$

The result is a transcendental equation, which we could solve graphically or numerically. We note that $kL = 0$ gives an immediate trivial solution, while the first nontrivial solution occurs for

$$kL \approx 4.49.$$

The corresponding critical force, using $k^2 \equiv F(EI)$, is then

$$k^2 = \frac{F_{\text{crit}}}{EI} \approx \frac{(4.49)^2}{L^2} \implies F_{\text{crit}} \approx \frac{EI}{L^2}(4.49)^2.$$

2.5.6 Exercise: Normal Modes of a Rod Fixed at One End

Page 76. Find the normal modes of a light homogeneous rod of length L which is rigidly fixed at one end and free to oscillate at the other end.

- We will work in a coordinate system in which the z axis aligns with the rod's longitundinal axis. The rod is light, so we neglect weight; this means the transverse dimensions x and y are equivalent with respect to the rod's dynamics.

- We will solve the rod's dynamics using a dynamic generalization of Equation 2.43, which reads

$$EIy^{(4)} + \mu\frac{\partial^2 y}{\partial t^2} = \ddot{y}F_z + \dot{y}\dot{F}_z + K_y$$
$$EIx^{(4)} + \mu\frac{\partial^2 x}{\partial t^2} = \ddot{x}F_z + \dot{x}\dot{F}_z + K_x,$$

where μ is the rod's linear mass density. In our case there are no external forces, so $\boldsymbol{K} = \mathbf{0}$). More so, for small oscillations, the elastic force F_z along the rod's longitudinal axis is vanishingly small compared to transverse elastic forces. (TODO reference theory, see e.g. Svensek eq. 5.11). Finally, we levarage the equivalence of the x and y directions reach the simplified dyamics equation

$$EIu^{(4)} + \mu\frac{\partial^2 u}{\partial t^2} = 0, \tag{2.50}$$

where u denotes a generic displacement transverse to the rod's longitudinal axis.

- We will find the rod's normal modes with an oscillatory ansatz of the form

$$u(z,t) = v(z)e^{-i\omega t},$$

which we substitute into Equation 2.50 to get the amplitude equation

$$v^{(4)} - \alpha^4 v = 0, \quad \alpha^4 \equiv \frac{\mu\omega^2}{EI}.$$

This fourth-order equation has the general solution

$$v(z) = A\sinh\alpha z + B\cosh\alpha z + C\sin\alpha z + D\cos\alpha z, \tag{2.51}$$

where A, B, C and D are to-be-determined constants, found from the problem's boundary conditions.

- Boundary conditions: the rod's rigidly fixed end gives the conditions

$$v(z)\big|_{z=0} = 0 \qquad \text{and} \qquad \dot{v}(z)\big|_{z=0} = 0,$$

while the free end, which by definition has neither elastic force nor torque on the cross section, gives the conditions

$$F(z)\big|_{z=L} = 0 \implies v^{(3)}\big|_{z=L} = 0,$$
$$M(z)\big|_{z=L} = 0 \implies v^{(2)}\big|_{z=L} = 0.$$

The first two conditions at $z = 0$ produce

$$B = -D \qquad \text{and} \qquad A = -C, \tag{2.52}$$

while the boundary conditions at $z = L$, using $B = -D$ and $A = -C$, give

$$A(\sinh\alpha L + \sin\alpha L) + B(\cosh\alpha L + \cos\alpha L) = 0, \tag{2.53}$$
$$A(\cosh\alpha L + \cos\alpha L) + B(\sinh\alpha L - \sin\alpha L) = 0. \tag{2.54}$$

The result is a system of homogeneous equations for the coefficients A and B, which has a nontrivial solution only if the determinant of the coefficient matrix is nonzero. We thus require

$$\begin{aligned} 0 &\equiv \det \begin{pmatrix} \sinh \alpha L + \sin \alpha L & \cosh \alpha L + \cos \alpha L \\ \cosh \alpha L + \cos \alpha L & \sinh \alpha L - \sin \alpha L \end{pmatrix} \\ &= (\sinh \alpha L + \sin \alpha L)(\sinh \alpha L - \sin \alpha L) - (\cosh \alpha L + \cos \alpha L)^2 \\ &= \sinh^2 \alpha L - \cos^2 \alpha L - 1 - 2 \cosh \alpha L \cos \alpha L \\ &= -2(1 + \cosh \alpha L \cos \alpha L), \end{aligned}$$

where we have used the identity $\cosh^2 x - \sinh^2 x = 1$. We thus require

$$\cosh \alpha L \cos \alpha L = -1 \implies \cos \alpha L = -\frac{1}{\cosh \alpha L}. \tag{2.55}$$

- Equation 2.55 is a transcendental equation, which we would, in principle, have to solve numerically. However, there is a nice analytic approximation: we first define the dimensionless variable $\xi \equiv \alpha L$, which gives

$$\cos \xi = -\frac{1}{\cosh \xi}.$$

We then note that for large values of ξ, the function $\cosh \xi$ tends to infinity, leaving

$$\cos \xi \approx 0 \quad (\text{for } \xi \gg 1).$$

This equation, meanwhile, has the simple solutions

$$\xi_n \equiv \alpha_n L = \frac{(2n+1)}{2}\pi \quad (\text{for } \xi \gg 1).$$

The corresponding eigenfrequencies ω_n, recalling the original definition of α, are

$$\frac{\xi_n^4}{L^4} = \alpha_n^4 \equiv \frac{\mu \omega_n^2}{EI} \implies \omega_n = \frac{\xi_n^2}{L^2}\sqrt{\frac{EI}{\mu}}.$$

Meanwhile, solved numerically, the first few solutions for small ξ come out to

$$\xi_0 \approx 1.875 \quad \xi_1 \approx 4.694 \quad \xi_2 \approx 7.855.$$

Note that the eigenfrequencies are not integer multiples of the fundamental frequency as for harmonic systems (such as a light string), because $\omega_n \propto \xi_n^2$ and not $\omega_n \propto \xi_n$.

- Finally, in passing, we note that we could compute the position dependence of $v(x)$ by, say, solving for B in terms of A in Equations 2.53 and 2.54, apply $D = -B$ and $C = -A$ from Equation 2.52, and substitute the result into Equation 2.51 for the amplitude $v(z)$. Without derivation, the result is

$$v(z) = A\left[\sinh \alpha z - \sin \alpha z - \frac{\sinh \alpha L + \sin \alpha L}{\cosh \alpha L + \cos \alpha L}(\cosh \alpha z - \cos \alpha z)\right].$$

2.5.7 Exercise: Anharmonic Corrections to a Guitar String

Consider a steel E *string guitar of radius* $2R = 0.25\,\mathrm{mm}$ *and length* $L = 0.8\,\mathrm{m}$ *tuned to a fundamental frequency* $\nu_0 = 660\,\mathrm{Hz}$. *Compute the deviations of the rod's eigenfrequencies from the harmonic idealization* $\nu_n = n\nu_0$, $n \in \mathbb{N}$. *Assume the density and Young's modulus of steel are* $\rho = 7800\,\mathrm{kg\,m^{-3}}$ *and* $E = 2 \cdot 10^{11}\,\mathrm{Pa}$.
Although it is a rough approximation at best, you may simplify the boundary conditions by assuming the string's ends are held in place but free to rotate at each end.

- Align z axis with the string's longitudinal axis in equilibrium. Neglect weight, in which case x and y are dynamically equivalent coordinates.

  Begin with the general dynamic equation

$$EIu^{(4)} + \mu\frac{\partial^2 u}{\partial t^2} = F_z\ddot{u} + K.$$

  In the absence of weight, the external linear force density K vanishes. Writing the uniaxial force F_z as simply F, the string's dynamics read

$$EIu^{(4)} - F\ddot{u} = -\mu\frac{\partial^2 u}{\partial t^2}.$$

  We will find the string's normal modes with the oscillatory ansatz

$$u(z,t) = u_0\sin(kz)e^{-i\omega t}. \tag{2.56}$$

  The key bit of physics here is recognizing the string is fixed at its endpoints $x = 0, L$, which implies $u(0,t) = u(L,t) = 0$. This requirement, together with the sinuidal ansatz, produces to wave vectors of the form

$$k_n = \frac{(n+1)\pi}{L}.$$

  Substituting the amplitude ansatz in Equation 2.56 into the string's dynamics equation and differentiating produces

$$EIu_0k_n^4\sin(k_nz) + Fu_0k_n^2\sin(k_nz) = \mu\omega^2u_0\sin(k_nz).$$

  We then cancel like terms and divide through by EI, to get the dispersion relation.

$$k_n^4 + \frac{F}{EI}k_n^2 = \frac{\rho_l}{EI}\omega^2,$$

  which we can immediately sovlve for the string eigenfrequencies:

$$\omega_n = ck_n\sqrt{1 + \frac{EI}{F}k_n^2} = \omega_n^{(0)}\sqrt{1 + \frac{EI}{F}k_n^2}, \tag{2.57}$$

  where we have defined the string's ideal, harmonic eigenfrequencies by $\omega_n^{(0)} = ck_n$ and introduced the wave speed

$$c = \sqrt{\frac{F}{\rho_l}}.$$

- Perform Taylor expansion of Equation 2.57 to get
$$\omega_n \approx \omega_n^{(0)} \left(1 + \frac{1}{2}\frac{EI}{F}k_n^2\right).$$
The relative deviation of the string's eigenfrequencies ω_n from the harmonic eigenfrequencies $\omega_n^{(0)}$ is thus
$$\frac{\omega_n - \omega_n^{(0)}}{\omega_n^{(0)}} = \frac{\Delta\omega_n}{\omega_n^{(0)}} = \frac{1}{2}\frac{EI}{F}k_n^2 = \frac{1}{2}\frac{EI}{F}\frac{(n+1)^2\pi^2}{l^2}.$$
Lesson: the deviation increases quadratically with harmonic index n!

Clean This Up: Computing Numerical Values

- We need to compute F also! We do this using the given fundamental frequency is $\nu_0 = 660\,\mathrm{Hz}$.

  Use a harmonic approximation:
$$\omega_0 = k_0 c = 2\pi\nu_0$$
  Use
$$k_n = \frac{(n+1)\pi}{l} \implies k_0 = \frac{\pi}{l}$$
  And so for the funamental
$$\frac{\pi}{l}\sqrt{\frac{F}{\rho_l}} = 2\pi \cdot 660\,\mathrm{Hz}$$
  And then we get supposedly
$$F = \frac{\rho_l(2\pi\nu_0)^2}{k_0^2} = 4\pi\rho R^2 l^2 \nu_0^2 \approx 107\,\mathrm{N}$$
  Convert linear mass density ρ_l to density ρ which is given.

- And then meanwhile, we find a numerical value of I. To get a numerical value for $\Delta\omega_n/\omega_n$.

  This comes out to
$$\frac{\Delta\omega_n}{\omega_n} = 2.76 \cdot 10^{-6}\,(n+1)^2$$
  Ah. So the deviation is small because the factor in front is so large.

- Next we compute the string's area moment of inertia I.

  Begin with (supposedly approximate a square root)
$$\omega \approx \omega_n\left(1 + \frac{1}{2}\frac{EI}{F}k_n^2\right), \qquad k_n = \frac{(n+1)\pi}{l}$$
  To find I we have, assuming rod's radius is R,
$$I = \int_0^R \int_0^{2\pi} r\,\mathrm{d}r\,\mathrm{d}\phi r^2 \sin_2\phi = \frac{2\pi}{2}\int_0^R r^3\,\mathrm{d}r = \frac{\pi R^4}{4}$$
  Further steps are then quoted up to
$$\frac{\Delta\omega_n}{\omega_n} = 2.76 \cdot 10^{-6}(n+1)^2$$

2.6 Elastic Waves

2.6.1 Specific Impedance of Elastic Waves in Unbounded Matter

Page 82. Specific impedance.

- Specific impedance (at a given frequency) is defined as

  $$z = \frac{p}{v}$$

  where p is the pressure amplitude at a position in the matter and v is the oscillation speed at that position.

  Classically, $Z = F/v$ is how much force to move stuff?

- Consider a plane wave moving in one direction

  $$\boldsymbol{u} = \boldsymbol{u}_0 e^{i(kx-\omega t)}$$

- We find pressure from stress tensor.

- For longitudinal we have $\boldsymbol{u} = u_x\,\hat{\mathbf{e}}_x$, since wave moves in $\hat{\mathbf{x}}$ direction. Impedance is

  $$z_{\mathrm{L}} = -\frac{p_{xx}}{v_x}$$

  where $v_x = \dot{u}_x$. Negative sign because pressure is negative stress.

- Use Hooke's law in terms of speeds:

  $$p_{ij} = 2\rho c_{\mathrm{T}}^2 u_{ij} + \rho(c_{\mathrm{L}}^2 - 2c_{\mathrm{T}}^2)u_{kk}\delta_{ij}$$

  This comes out to

  $$p_{xx} = 2\rho c_{\mathrm{T}}^2 u_{xx} + \rho(c_{\mathrm{L}}^2 - 2c_{\mathrm{T}}^2)\nabla\cdot\boldsymbol{u} = 2\rho c_{\mathrm{T}}^2 iku_x + \rho(c_{\mathrm{L}}^2 - 2c_{\mathrm{T}}^2)iku_x = ik\rho c_{\mathrm{L}}^2 u_x.$$

  Note u_{kk} is like diverence of $\boldsymbol{u}$.

- Wave speed is $v_x = \dot{u}_x = -i\omega u_x$. Longitudinal impedance for a plane wave is then

  $$z_{\mathrm{L}} = \frac{ik\rho c_{\mathrm{L}}^2}{i\omega} = \frac{\rho c_{\mathrm{L}}^2}{c_{\mathrm{L}}} = \rho c_{\mathrm{L}}.$$

Transverse

- For transverse waves we choose either $\boldsymbol{u} = u_y\,\hat{\mathbf{e}}_y$ or $\boldsymbol{u} = u_z\,\hat{\mathbf{e}}_z$; both $\hat{\mathbf{y}}$ and $\hat{\mathbf{z}}$ are transverse to the wave direction of propagation in the $\hat{\mathbf{x}}$ direction.

  We then define impedance as

  $$z_{\mathrm{T}} = -\frac{p_{yx}}{v_y}$$

  for force applied to surface normal to $\hat{\mathbf{y}}$ for wave in the x direction. **TODO:** formulation of directions.

  "Force in the y direction" for wave moving the x direction.

- Hooke's then gives

$$p_{yx} = 2\rho c_{\mathrm{T}}^2 u_{yx} = 2\rho c_{\mathrm{T}}^2 \frac{1}{2}\left(\underbrace{\frac{\partial u_x}{\partial y}}_{=0} \frac{\partial u_y}{\partial x}\right) = ik\rho c_{\mathrm{T}}^2 u_y.$$

  Note no trace term for the shear off-diagonal term p_{yx}.

- Wave speed is $v_y - iwu_y$.

- Impedance is then

$$z_{\mathrm{T}} = \frac{ik\rho c_{\mathrm{T}}^2}{i\omega} = \frac{\rho c_{\mathrm{T}}^2}{c_{\mathrm{T}}} = \rho c_{\mathrm{T}}.$$

- In passing, note that in liquid we have

$$z = \frac{p}{v} = \frac{\rho}{c},$$

  where c is the speed of acoustic waves.

2.6.2 Reflection and Refraction of Elastic Waves

Page 83, longitudinal waves, angle α_0...

- z axis out of paper, x axis from left to right, y axis from bottom to top.

  Wave travels in the positive $\hat{\mathbf{x}}$ direction.

  The plane $z = 0$ (i.e. the xy plane) is the plane of incidence.

  Note that z componet of incident wave is unchanged because z is transverse to the plane of incidence.

- Define angle of incident α_0.

  Angles of reflection for longitudinal and transverse are α_{L} and α_{L}.

  Define unit vector $\boldsymbol{a}_{\mathrm{L}}$ in the direction of reflected longitudinal.

  Define unit vector $\boldsymbol{a}_{\mathrm{T}}$ in the direction of reflected transverse.

- In first material, incident longitudinal waves and reflected longitudinal and transverse waves:

$$\boldsymbol{u}_1 = \left(A_0 \boldsymbol{a}_0 e^{i\boldsymbol{k}_0\cdot\boldsymbol{r}} + A_{\mathrm{L}} \boldsymbol{a}_{\mathrm{L}} e^{i\boldsymbol{k}_{\mathrm{L}}\cdot\boldsymbol{r}} + A_{\mathrm{T}} \boldsymbol{a}_{\mathrm{T}} e^{i\boldsymbol{k}_{\mathrm{T}}\cdot\boldsymbol{r}}\right) e^{-i\omega t}$$

- In the second material we have transmitted longitudinal and and transverse waves

$$\boldsymbol{u}_2 = \left(B_{\mathrm{L}} \boldsymbol{b}_{\mathrm{L}} e^{i\boldsymbol{q}_{\mathrm{L}}\cdot\boldsymbol{r}} + B_{\mathrm{T}} \boldsymbol{b}_{\mathrm{T}} e^{i\boldsymbol{q}_{\mathrm{T}}\cdot\boldsymbol{r}}\right) e^{i\omega t}.$$

- The solutions $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ must be equal along the entire boundary at every point in time!

  This requires wave frequencies are all equal (this is already assumed above).

- Requiring agreement of $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ along $x = 0$ for all y produces

$$A_0\boldsymbol{a}_0 e^{ik_{0_y}y} + A_\mathrm{L}\boldsymbol{a}_\mathrm{L} e^{ik_{\mathrm{L}_y}y} + A_\mathrm{T}\boldsymbol{a}_\mathrm{T} e^{ik_{\mathrm{T}_y}y} = B_\mathrm{L}\boldsymbol{b}_\mathrm{L} e^{iq_{\mathrm{L}_y}y} + B_\mathrm{T}\boldsymbol{b}_\mathrm{T} e^{iq_{\mathrm{T}_y}y} +$$

  This condition can hold for all y if the y components of all wave vectors are equal. So like

$$k_{0_y} = k_{\mathrm{L}_y} = k_{\mathrm{T}_y} = q_{\mathrm{L}_y} = q_{\mathrm{T}_y}$$

  And then get, from $k_y = k\sin\alpha$ where $k = \omega/c$.

$$\frac{\sin\alpha_0}{c_{\mathrm{L}_1}} = \frac{\sin\alpha_{\mathrm{L}_1}}{c_{\mathrm{L}_1}} = \frac{\sin\alpha_\mathrm{T}}{c_{\mathrm{T}_1}} = \frac{\sin\beta_\mathrm{L}}{c_{\mathrm{L}_2}} = \frac{\sin\beta_\mathrm{T}}{c_{\mathrm{T}_2}}$$

  Here we recognize the law of reflection:

$$\frac{\sin\alpha_0}{c_{\mathrm{L}_1}} = \frac{\sin\alpha_{\mathrm{L}_1}}{c_{\mathrm{L}_1}} = \frac{\sin\alpha_\mathrm{T}}{c_{\mathrm{T}_1}}$$

  and the law of refraction:

$$\frac{\sin\beta_\mathrm{L}}{c_{\mathrm{L}_2}} = \frac{\sin\beta_\mathrm{T}}{c_{\mathrm{T}_2}} = \frac{\sin\alpha_\mathrm{T}}{c_{\mathrm{T}_1}}$$

  So here for example c_{L_1} is the speed of longitudinal waves in material one.

3 Hydrodynamics

3.1 Ideal Fluids

3.1.1 Review of Theory

- The Euler equation for ideal fluids is

$$\frac{\mathrm{D}\boldsymbol{v}}{\mathrm{D}t} \equiv \frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{v} = -\frac{\nabla p}{\rho} + \frac{\boldsymbol{f}}{\rho}. \tag{3.1}$$

- Deriving a continuity equation: quantity that is conserved (mass). Determine its volume density and take partial time derivative. Determine the density of its current ($æ_\mathrm{m} = \rho \boldsymbol{v}$) and take divergence. For fluid flow this reads

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{v}) = 0.$$

- Continuity equation for momentum. We write fluid momentum density as $\boldsymbol{\pi}$ to avoid conflict with pressure p. The continuity equation for fluid momentum is

$$\frac{\partial \pi_i}{\partial t} + \frac{\partial}{\partial x_j}\Pi_{ij} = 0 \qquad \text{or} \qquad \frac{\partial \pi_i}{\partial t} + \frac{\partial}{\partial x_j}\Pi_{ij} = f_i,$$

where the latter form holds in the presence of an external force with force density $\boldsymbol{f}$. The tensor quantity Π is momentum current density, and $\frac{\partial}{\partial x_j}$ is just the divergence operator in component form.

- For isentropic flow, in which a fluid's specific entropy s is independent of time and position, we can write the Euler equation in terms of the fluid's specific enthalpy h via $\nabla p/\rho = \nabla h$. Assuming no external force acts on the fluid (i.e. that $\boldsymbol{f} = \boldsymbol{0}$), the Euler equation for an isentropic fluid is

$$\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{v} = -\nabla h.$$

- Next, we'll write above equation in terms of only curl. First use the vector identity

$$\frac{1}{2}\nabla v^2 = (\boldsymbol{v} \cdot \nabla)\boldsymbol{v} + \boldsymbol{v} \times (\nabla \times \boldsymbol{v}) \implies (\boldsymbol{v} \cdot \nabla)\boldsymbol{v} = \frac{1}{2}\nabla v^2 - \boldsymbol{v} \times (\nabla \times \boldsymbol{v}).$$

Substitute into Euler equation and rearrange to get

$$\frac{\partial \boldsymbol{v}}{\partial t} = \boldsymbol{v} \times (\nabla \times \boldsymbol{v}) - \nabla \left(h + \frac{v^2}{2} \right).$$

We then take the curl of this equation, switch the order of curl and time differentiation, and apply $\nabla \times (\nabla \phi) = \boldsymbol{0}$ for any scalar function ϕ to get

$$\frac{\partial}{\partial t}(\nabla \times \boldsymbol{v}) = \nabla \times \big[\boldsymbol{v} \times (\nabla \times \boldsymbol{v})\big]. \tag{3.2}$$

This equation contains only velocity $\boldsymbol{v}$.

- For **incompressible flow**: Next, use the vector identity

$$\nabla \times (\boldsymbol{a} \times \boldsymbol{b}) = (\boldsymbol{b} \cdot \nabla)\boldsymbol{a} - (\boldsymbol{a} \cdot \nabla)\boldsymbol{b} + \boldsymbol{a}(\nabla \cdot \boldsymbol{b}) - \boldsymbol{b}(\nabla \cdot \boldsymbol{a})$$

with $\boldsymbol{a} = \boldsymbol{v}$ and $\boldsymbol{b} = \nabla \times \boldsymbol{v}$. Apply $\nabla \cdot (\nabla \times \boldsymbol{v}) = \mathbf{0}$ for any vector field. Assume incompressible flow $\nabla \cdot \boldsymbol{v} = 0$. Define vorticity $\boldsymbol{\omega} \equiv \nabla \times \boldsymbol{v}$. The result is the *Helmholtz equation*

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{\omega} = (\boldsymbol{\omega} \cdot \nabla)\boldsymbol{v}.$$

Irrotational Flow

- Irrotational flow obeys $\nabla \times \boldsymbol{v} = \mathbf{0}$. By Equation 3.2, irrotational flow always remains irrotational. Since $\nabla \times \boldsymbol{v} = \mathbf{0}$ initially, and remains so for all time, because the rate of change $\frac{\partial}{\partial t}(\nabla \times \boldsymbol{v})$ is also zero.
- Potential flow has no internal dynamics and is determined only by boundary conditions. Potential flow can change only if the boundary changes.

Bernoulli

- For irrotational (potential) flow, Bernoulli holds for all space.

$$\nabla \left(\frac{v^2}{2} + h + \frac{\partial \phi}{\partial t} + gz \right) = 0$$

Assuming $\boldsymbol{g} = -g\,\hat{\mathbf{e}}_z$. Stress that Bernoulli includes a $\frac{\partial \phi}{\partial t}$ term. Only if the flow is in addition stationary, the $\frac{\partial \phi}{\partial t}$ vanishes.

- For stationary but in general rotational flow, the Bernoulli equation applies only to a single streamline or line vortex. Begin with Euler equation

$$\frac{\partial \boldsymbol{v}}{\partial t} = \boldsymbol{v} \times (\nabla \times \boldsymbol{v}) - \nabla \left(\frac{v^2}{2} + h + gz \right)$$

Scalar-multiply through by either $\boldsymbol{v}$ or $\nabla \times \boldsymbol{v}$ to get

$$\boldsymbol{v} \cdot \nabla \left(\frac{v^2}{2} + h + gz \right) = 0$$
$$(\nabla \times \boldsymbol{v}) \cdot \nabla \left(\frac{v^2}{2} + h + gz \right) = 0$$

Interpretation: the terms in parentheses are constant only along curves in the direction of $\boldsymbol{v}$ (a streamline) or curves in the direction $\nabla \times \boldsymbol{v}$ (vortex lines).

The term h is free enthalpy density. In principle h encodes the effects of all relevant potentials (including gravitational) and pressure. However, it is standard to add a separate gravitational pressure term gz.

Kelvin Theorem

- The material derivative of circulation is zero.

$$\frac{\mathrm{D}\Gamma}{\mathrm{D}t} = \frac{\mathrm{D}}{\mathrm{D}t} \oint \boldsymbol{v} \cdot \mathrm{d}\boldsymbol{l} = 0.$$

The circulation along any given curve of fluid elements is constant as the fluid element curve travels through the flow.

Biot-Savart Analogy

- Suppose we know a velocity field's vorticity $\boldsymbol{\omega} = \nabla \times \boldsymbol{v}$, we can find the corresponding velocity field from the fluid analog of the Biot-Savart law. This reads

$$\boldsymbol{v}(\boldsymbol{r}) = \frac{1}{4\pi} \iiint \frac{\boldsymbol{\omega}(\boldsymbol{r}')\nabla \times (\boldsymbol{r} - \boldsymbol{r}')}{|\boldsymbol{r} - \boldsymbol{r}'|^3} \,\mathrm{d}^3\boldsymbol{r}'.$$

  Analogy is $\boldsymbol{v} \longleftrightarrow \boldsymbol{H}$ and $\boldsymbol{\omega} = \nabla\times\boldsymbol{v} \longleftrightarrow \boldsymbol{j} = \nabla\times\boldsymbol{H}$. If all vorticity is concentrated along a single space curve (like all current along a single wire) around which the circulation is

$$\Gamma = \oint \boldsymbol{v} \cdot \mathrm{d}\boldsymbol{l} = \iint \nabla \times \boldsymbol{v} \cdot \mathrm{d}\boldsymbol{S} = \iint \boldsymbol{\omega} \cdot \mathrm{d}\boldsymbol{S},$$

  (here Γ is analogous to current because $I = \iint \boldsymbol{j} \cdot \mathrm{d}\boldsymbol{S}$) the Biot-Savart law simplifies to a line integral along the vortex curve:

$$\boldsymbol{v}(\boldsymbol{r}) = \frac{\Gamma}{4\pi} \int \frac{\mathrm{d}\boldsymbol{l}' \times (\boldsymbol{r} - \boldsymbol{r}')}{|\boldsymbol{r} - \boldsymbol{r}'|^3},$$

  where $\mathrm{d}\boldsymbol{l}'$ is the tangent to the curve.

3.1.2 Exercise: Force of a Water Jet on a Wall

Jet of cross section S and speed $\boldsymbol{v}_0$ is normally incident on a wall and, on hitting the wall, spreads out symmetrically. Assume there is no gravity, and that the jet density and velocity is uniform across its cross section.

- Align z axis with jet velocity $\boldsymbol{v}_0$. Begin with momentum continuity equation in the absence of external forces

$$\frac{\partial \pi_i}{\partial t} + \frac{\partial}{\partial x_j}\Pi_{ij} = 0, \quad \text{where } \Pi_{ij} = \rho v_i v_j + p\delta_{ij}$$

  Choose a closed cylindrical integration volume around the jet, starting where the jet contacts the wall and extending an arbitrary distance backward along the jet.

  Convert momentum continuity equation to integral form and apply divergence theorem

$$\iiint \frac{\partial \pi_i}{\partial t} \,\mathrm{d}V = -\oiint \Pi_{ij} \,\mathrm{d}S_j$$

  The LHS is the change in the fluid's momentum with time and is zero in a stationary state, leaving

$$\oiint \Pi_{ij} \,\mathrm{d}S_j = 0.$$

  Consider z component (force on wall will have only a z component from the problem's geometry) and get

$$\oiint \Pi_{zj} \,\mathrm{d}S_j = 0.$$

  Split the integration volume's surface into three components, S_0 for the jet's cross section far from the wall, S_{lat} for the lateral surface, and S_{wall} for the jet's cross section along the wall.

  Set pressure $p = 0$ along S_0 and S_{lat} (e.g. "in air") and $p = p_{\text{wall}}$ along the wall.

Momentum current density tensor at each surface is

$$\Pi^{(0)}_{zj} = \rho v_z v_j \qquad \text{cross section} \tag{3.3}$$

$$\Pi^{(\text{lat})}_{zj} = 0 \qquad \text{lateral surface} \tag{3.4}$$

$$\Pi^{(\text{wall})}_{zj} = p_{\text{wall}} \cdot \delta_{zj} \qquad \text{cross section on wall} \tag{3.5}$$

Thus

$$\begin{aligned}\oiint \Pi_{zj}\,\mathrm{d}S_j &= \iint_{S_0} \Pi^{(0)}_{zj}\,\mathrm{d}S_j + \iint_{S_{\text{lat}}} \Pi^{(\text{lat})}_{zj}\,\mathrm{d}S_j + \iint_{S_{\text{wall}}} \Pi^{(\text{wall})}_{zj}\,\mathrm{d}S_j \\ &= \rho \iint_{S_0} v_z v_j\,\mathrm{d}S_j + 0 + \iint_{S_{\text{wall}}} p_{\text{wall}}\,\mathrm{d}S_z\end{aligned}$$

For our geometry $\mathrm{d}S = \mathrm{d}S_z$ because $\hat{\mathbf{n}} = \hat{\mathbf{z}}$. We conclude

$$F_{\text{wall}} = \iint_{S_{\text{wall}}} \Pi^{(\text{wall})}_{zj}\,\mathrm{d}S_j = -\rho \iint_{S_0} v_z v_j\,\mathrm{d}S_j = -\rho \iint_{S_0} v_z \boldsymbol{v} \cdot \mathrm{d}\boldsymbol{S}.$$

The LHS is the force on the wall. The RHS, assuming $\boldsymbol{v}_0$ is constant across S_0, evaluates to

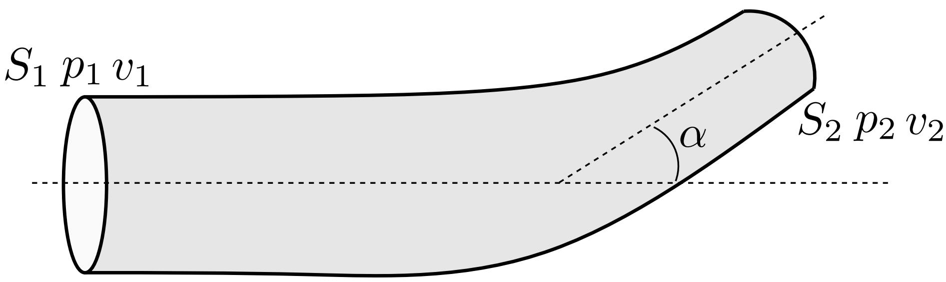
$$F_{\text{wall}} = -\rho v_z \boldsymbol{v}_0 \cdot \boldsymbol{S}_0 = +\rho v_0^2 S_0.$$

Change of sign because surface normal to S_0 points outward of integration volume and jet direction $\boldsymbol{v}_0$ points inward.

Note that $v_z = |\boldsymbol{v}_0|$.

3.1.3 Exercise: A Bent Hose with a Changing Cross Section

Page 24. Direction of flow changes by angle α. Given input and output cross sections, pressures and volumes. Find force on pipe.
Assume ideal fluid and homogeneous flow through both cross sections.



- Define coordinate system in xy plane so flow before bend aligns with the x axis.
- Assume no external force. For review equation **TODO:** reference reads

$$\frac{\partial \pi_i}{\partial t} + \frac{\partial}{\partial x_j}\Pi_{ij} = 0$$

Integrate over fluid between the two cross sections and apply divergence theorem to get

$$\iiint \frac{\partial \pi_i}{\partial t}\,\mathrm{d}V + \oiint \Pi_{ij}\,\mathrm{d}S_j = 0.$$

Assume a steady state in which momentum of fluid in pipe is constant so that

$$\frac{\partial \boldsymbol{\pi}}{\partial t} = \mathbf{0} \implies \oiint \Pi_{ij} \, \mathrm{d}S_j = 0.$$

Split integration surface into S_1, lateral surface, and S_2. Work in 2D xy plane and observe x and y components separately.

$$\oiint \Pi_{xj} \, \mathrm{d}S_j = 0 \qquad \text{and} \qquad \oiint \Pi_{yj} \, \mathrm{d}S_j = 0.$$

Conclude that force on pipe comes from lateral surface integral

$$F_x = \iint_{S_{\text{lat}}} \Pi_{xj} \, \mathrm{d}S_j \qquad \text{and} \qquad F_y = \iint_{S_{\text{lat}}} \Pi_{yj} \, \mathrm{d}S_j$$

Then write out first the x component, noting $\boldsymbol{v}_1 = v_1 \, \hat{\mathbf{e}}_x$ etc...

$$\begin{aligned} 0 &= \iint_{S_1} (\rho v_{1_x} v_{1_j} + p_1 \delta_{xj}) \, \mathrm{d}S_j + \iint_{S_2} (\rho v_{2_x} v_{2_j} + p_2 \delta_{xj}) \, \mathrm{d}S_j + F_x \\ &= \Big(\rho v_1 \boldsymbol{v}_1 \cdot \boldsymbol{S}_1 + p_1 S_1\Big) + \Big(\rho (v_2 \cos\alpha) \boldsymbol{v}_2 \cdot \boldsymbol{S}_2 + p_2 S_2 \cos\alpha\Big) + F_x \end{aligned}$$

Then write out first the y component, noting $\boldsymbol{v}_{1_y} = 0$ etc...

$$\begin{aligned} 0 &= \iint_{S_1} (\rho v_{1_y} v_{1_j} + p_1 \delta_{yj}) \, \mathrm{d}S_j + \iint_{S_2} (\rho v_{2_y} v_{2_j} + p_2 \delta_{yj}) \, \mathrm{d}S_j + F_y \\ &= 0 + \Big(\rho (v_2 \sin\alpha) \boldsymbol{v}_2 \cdot \boldsymbol{S}_2 + p_2 S_2 \sin\alpha\Big) + F_y \end{aligned}$$

Simplify dot products (use $\boldsymbol{v}_1 \cdot \boldsymbol{S}_1 = -v_1 S_1$ (anti-parallel) and $\boldsymbol{v}_2 \parallel \boldsymbol{S}_2$) and solve for forces and get

$$\begin{aligned} F_x &= +\big(\rho v_1^2 + p_1\big) S_1 - \big(\rho v_2^2 + p_2\big) S_2 \cos\alpha \\ F_y &= -\big(\rho v_2^2 + p_2\big) S_2 \sin\alpha \end{aligned}$$

- Interpretation: F_y (the y force on the *hose*) points in the negative y direction, which should make sense because the hose is angled in the positive y direction. More formally, pointing the pipe in the positive y direction increases the y momentum component of the water passing through the bend. So the corresponding force on the pipe should be $\parallel -\hat{\mathbf{y}}$.

TODO: interpretations.

3.1.4 Exercise: Two Orbiting Irrotational Line Vortices

Compute velocity field of a long straight "vortex string" with circulation Γ.

- Vortex line is an idealized distribution of vorticity along a straight line. Define so that

$$\iint \boldsymbol{\omega} \cdot \mathrm{d}\boldsymbol{S} = \Gamma$$

for any planar surface which intersects the vortex line.

Think of vortex line as a delta function distribution of vorticity along a line.

Fluid rotates around the line in a circle—the circulation of the velocity field for a closed curve around the vortex line is Γ, but the flow's vorticity $\boldsymbol{\omega} = \nabla \times \boldsymbol{v}$ is zero everywhere except along the vortex line, where it is undefined (or "infinity").

A vortex curve may be closed but otherwise cannot just stop.

Analogy to magnetostatics: $\Gamma \leftrightarrow I$; $\boldsymbol{\omega} \leftrightarrow \boldsymbol{j}$; $\boldsymbol{v} \leftrightarrow \boldsymbol{B}$.

We would find $\boldsymbol{v}$ associated with $\boldsymbol{\omega}$ from an analog of the Biot-Savart law:

$$\boldsymbol{v}(\boldsymbol{r}) = \frac{1}{4\pi} \iiint \frac{\boldsymbol{\omega}(\boldsymbol{r}') \times (\boldsymbol{r} - \boldsymbol{r}')}{|\boldsymbol{r} - \boldsymbol{r}'|^3} \, \mathrm{d}^3\boldsymbol{r}'$$

For a single vortex line (again compare to EMP) we can integrate over the vortex line get

$$\boldsymbol{v}(\boldsymbol{r}) = \frac{\Gamma}{4\pi} \int \frac{\mathrm{d}\boldsymbol{l}' \times (\boldsymbol{r} - \boldsymbol{r}')}{|\boldsymbol{r} - \boldsymbol{r}'|^3}$$

where $\mathrm{d}\boldsymbol{l}'$ is the tangent to the line at the point $\boldsymbol{r}'$.

One Vortex Line

- The vortex line creates a velocity field of the general form

  $$\boldsymbol{v}(\boldsymbol{r}) = v_\varphi(r) \, \hat{\mathbf{e}}_\varphi$$

  Compute velocity field from

  $$\Gamma \equiv \iint \boldsymbol{\omega} \, \mathrm{d}\boldsymbol{S} = \iint \nabla \times \boldsymbol{v} \, \mathrm{d}\boldsymbol{S} = \oint \boldsymbol{v} \cdot \mathrm{d}\boldsymbol{s} = 2\pi r v$$

  Choose a circle of radius r normal to the vortex line

  $$\Gamma \equiv \iint \boldsymbol{\omega} \, \mathrm{d}\boldsymbol{S} = \iint \nabla \times \boldsymbol{v} \, \mathrm{d}\boldsymbol{S} = \oint \boldsymbol{v} \cdot \mathrm{d}\boldsymbol{s} = 2\pi r v_\varphi(r)$$

  The vortex line's velocity field is thus

  $$\boldsymbol{v}(\boldsymbol{r}) = \frac{\Gamma}{2\pi r} \, \hat{\mathbf{e}}_\varphi.$$

Two Parallel Vortex Lines

- Velocity field of both lines is sum of individual velocity fields from linearity of the Biot-Savart equation in $\boldsymbol{\omega}$. The result is

  $$\boldsymbol{v}(\boldsymbol{r}) = \frac{\Gamma_1}{2\pi|\boldsymbol{r} - \boldsymbol{r}_1|} \, \hat{\mathbf{e}}_\varphi + \frac{\Gamma_2}{2\pi|\boldsymbol{r} - \boldsymbol{r}_2|} \, \hat{\mathbf{e}}_\varphi.$$

  Assuming the two lines occur at $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$.

- Assume vortex lines are separated by distance d, so that, for an origin e.g. halfway between the lines,

  $$\boldsymbol{v}(\boldsymbol{r}) = \frac{\Gamma_1}{2\pi(r - d/2)} \, \hat{\mathbf{e}}_\varphi + \frac{\Gamma_2}{2\pi(r + d/2)} \, \hat{\mathbf{e}}_\varphi.$$

Assume first line occurs at $r = +d/2$ and second at $r = -d/2$. Then velocities at first and second line are (contribution of a vortex line to velocity along its own axis is zero)

$$v_1 = \frac{\Gamma_2}{2\pi d}\hat{\mathbf{e}}_\varphi \qquad \text{and} \qquad v_2 = \frac{\Gamma_1}{2\pi d}\hat{\mathbf{e}}_\varphi$$

The two vortex lines thus orbit a common center of rotation (assuming Γ_1 and Γ_2 have the same sign).

- Consider angular velocity of the two-vortex line system (e.g. the angular velocity of the line joining the two vortex lines) about a point shifted a distance x from midway point between lines towards the second line. Then, equating orbital velocity of lines and angular velocity basically using $v = \Omega r$ get

$$\Omega \cdot \left(\frac{d}{2} - x\right) = v_2 = \frac{\Gamma_1}{2\pi d} \qquad \text{and} \qquad \Omega \cdot \left(\frac{d}{2} + x\right) = v_1 = \frac{\Gamma_2}{2\pi d}$$

Solve system of equations for x and Ω to get

$$x = \frac{\Gamma_2 - \Gamma_1}{\Gamma_1 + \Gamma_2}\frac{d}{2} \qquad \text{and} \qquad \Omega = \frac{\Gamma_1 + \Gamma_2}{2\pi d^2}$$

If Γ are equal (equal circulations along each string) then $x = 0$, i.e. the lines orbit their halfway point.

3.1.5 Exercise: Rankine Vortex

Idea: replace singularity along the axis of a line vortex with a cylindrical core of radius a with non-zero vorticity. The field outside $r = a$ is identical to a line vortex's velocity field. Assume the line vortex and core hypothetically end, e.g. in the plane $z = 0$. Find the fluid's height profile.

- The velocity field in the core for $r < a$ must match the $\Gamma(2\pi r)$ field for $r > a$ computed above. Without proof, this field is

$$v = \begin{cases} \Omega r & r < a \\ \frac{\Gamma}{2\pi r} & r > a \end{cases}$$

Interpret $v = \Omega r$ as orbital velocity of fluid around the z axis. We first find Ω such that solutions are continuous at a. This is

$$\Omega a = \frac{\Gamma}{2\pi a} \implies \Omega = \frac{\Gamma}{2\pi a^2}.$$

- Recall Bernoulli equation applies everywhere (not just one streamline) for irrotational flow Bernoulli. In the irrotational region, we'll find height profile with Bernoulli equation.

The Irrotational Region $r > a$

- The region $r > a$ is irrotational, since $\nabla \times \boldsymbol{v} = \mathbf{0}$ in this region. **TODO:** write out. Assume flow is incompressible and use the classic Bernoulli equation

$$p + \frac{1}{2}\rho v^2 + \rho g z = \text{constant}.$$

Decide coordinate system such that $z = 0$ at top of vortex. Ignore pressure (air pressure) along the surface of the vortex and substitute in $v(r)$ for $r > a$ to get

$$\frac{\rho}{2}\frac{\Gamma^2}{4\pi^2 r^2} + \rho g z(r) = 0 \implies z(r) = -\frac{\Gamma^2}{8\pi^2 r^2 g}.$$

Interpretation: the water surface is "flat" far from the vortex. That makes sense.

Second Region, for $r < a$

- The field $\boldsymbol{v}(r) = \omega r\,\hat{\mathbf{e}}_\varphi$ is rotational. In this case Bernoulli applies only along a single streamline, which is not helpful.
- Instead use definition of vorticity $\boldsymbol{\omega}$ via

$$\boldsymbol{\omega} = \nabla \times \boldsymbol{v}.$$

Work with Equation 3.2, which reads

$$\frac{\partial(\nabla \times \boldsymbol{v})}{\partial t} = \nabla \times \big[\boldsymbol{v} \times (\nabla \times \boldsymbol{v})\big]$$

- Interpret $\boldsymbol{v} = \boldsymbol{\Omega} \times \boldsymbol{r}$ for $r < a$, so non-zero vorticity means

$$\boldsymbol{\omega} = \nabla \times \boldsymbol{v} = \nabla \times (\boldsymbol{\Omega} \times \boldsymbol{r}) \neq \mathbf{0}.$$

We aim to compute $\nabla \times (\boldsymbol{\Omega} \times \boldsymbol{r})$. We'll do this by components.

$$\big[\nabla \times (\boldsymbol{\Omega} \times \boldsymbol{r})\big]_i = \epsilon_{ijk}\partial_j \epsilon_{klm}\Omega_l r_m,$$

Use the double Levi-Civita symbol identity and $\partial_j r_m = \delta_{jm}$ to get

$$\epsilon_{ijk}\partial_j \epsilon_{klm}\Omega_l r_m = (\delta_{il}\delta_{jm} - \delta_{im}\delta_{lj})\Omega_l \delta_{jm} = 3\Omega_i - \Omega_i = 2\Omega_i.$$

Note cyclic permutation $\epsilon_{klm} = \epsilon_{lkm}$. Using $\frac{\partial}{\partial j} r_m = \delta_{jm}$ holds because Ω is constant and not differentiated. Note that double deltas sum to three—the trace of the δ tensor. Result is

$$\big[\nabla \times (\boldsymbol{\Omega} \times \boldsymbol{r})\big]_i = 2\Omega_i.$$

This might have just been for practice throwing around indices and whatnot.

Using Euler Equation

- We proceed with the Euler equation, which is more general than the Bernoulli equation. Include an external gravitational term $\boldsymbol{g} \parallel -\hat{\mathbf{z}}$.

$$\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{v} = -\frac{\nabla p}{\rho} + \boldsymbol{g},$$

Consider stationary flow, so $\frac{\partial \boldsymbol{v}}{\partial t} = \mathbf{0}$. For the velocity field $\boldsymbol{v}(\boldsymbol{r}) = \Omega r\,\hat{\mathbf{e}}_\varphi$, the φ derivative of $\boldsymbol{v}$ components is zero, but the derivative of the $\hat{\mathbf{e}}_\varphi$ basis vector is nonzero.

First compute directional derivative $(\boldsymbol{v} \cdot \nabla)\boldsymbol{v}$ (in cylindrical coordinates). For review, the nabla operator in cylindrical coordinates reads

$$\nabla = \hat{\mathbf{e}}_r \frac{\partial}{\partial r} + \frac{\hat{\mathbf{e}}_\varphi}{r} \frac{\partial}{\partial \varphi} + \hat{\mathbf{e}}_z \frac{\partial}{\partial z}$$

The directional derivative of $\boldsymbol{v}$ (writing every step out—feel free to skip) is

$$\begin{aligned}(\boldsymbol{v} \cdot \nabla)\boldsymbol{v} &= (\Omega r\, \hat{\mathbf{e}}_\varphi) \cdot \left[\hat{\mathbf{e}}_r \frac{\partial}{\partial r} + \frac{\hat{\mathbf{e}}_\varphi}{r} \frac{\partial}{\partial \varphi}\right] \Omega r\, \hat{\mathbf{e}}_\varphi \\ &= \Omega^2 r (\hat{\mathbf{e}}_\varphi \cdot \hat{\mathbf{e}}_r)\, \hat{\mathbf{e}}_\varphi + \Omega^2 r (\hat{\mathbf{e}}_\varphi \cdot \hat{\mathbf{e}}_\varphi) \frac{\partial\, \hat{\mathbf{e}}_\varphi}{\partial \varphi} \\ &= \Omega^2 r (-\,\hat{\mathbf{e}}_r)\end{aligned}$$

Interpretation: $\Omega^2 r(-\,\hat{\mathbf{e}}_r)$ corresponds to cetripetal force in the $-\,\hat{\mathbf{e}}_r$ direction.

- Next compute ∇p in the Euler equation. This is just

$$\nabla p = \hat{\mathbf{e}}_r \frac{\partial p}{\partial r} + \hat{\mathbf{e}}_z \frac{\partial p}{\partial z}.$$

The r component of ∇p is nonzero—pressure changes within the core with radial distance from the central axis. The z component is nonzero "because of buoyancy". The φ component of p is zero because of rotational symmetry about the z axis.

- Substitute results into Euler equation and and use $\boldsymbol{g} = -g\, \hat{\mathbf{e}}_z$ toget

$$-\Omega^2 r\, \hat{\mathbf{e}}_r = -\frac{1}{\rho}\left(\hat{\mathbf{e}}_r \frac{\partial p}{\partial r} + \hat{\mathbf{e}}_z \frac{\partial p}{\partial z}\right) - g\, \hat{\mathbf{e}}_z.$$

Note this results in two independent equations for the componets $\hat{\mathbf{e}}_r$ and $\hat{\mathbf{e}}_z$. Pressure comes from the z component of the above equation.

$$\frac{\partial p}{\partial z} = -\rho g \implies p(z) = -\rho g z + C(r)$$

To find the integration constant, we note that on the fluid surface exposed to air, pressure is zero. This gives **TODO:** maybe better to go back to h. To differentiate general z coordinate and fluid height h.

$$0 = -\rho g h(r) + C(r) \implies C(r) = \rho g h(r)$$

Pressure $p(z)$ is then

$$p(z) = -\rho g\big[z - h(r)\big]$$

- Next consider the z component of the **TODO:** reference equation. This is

$$\rho \Omega^2 r = \frac{\partial p}{\partial r}$$

Solve this using the earlier expression for p from the z component

$$p(z) = -\rho g\big[z - h(r)\big] \implies \frac{\partial p}{\partial r} = \rho g h'(r)$$

Then substitute into $\frac{\partial p}{\partial r}$ above and integrate over r to get

$$\rho\Omega^2 r = \rho g h'(r) \implies \frac{1}{2}\Omega^2 r^2 = gh(r) - D$$

Solve for $h(r)$ and get.

$$h(r) = \frac{1}{2}\frac{\Omega^2 r^2}{g} + D$$

- Next step is to determine integration constant D. We equate the two expressions for $h(r)$ at $r = a$ and require continuity; this gives

$$\frac{1}{2}\frac{\Omega^2 a^2}{g} + D = -\frac{\Gamma^2}{8\pi^2 r^2 g}$$

From this we find

$$D = -\frac{1}{2}\frac{\Omega^2 a^2}{g} - \frac{\Gamma^2}{8\pi^2 r^2 g}$$

The expression for $h(r)$ (for $r < a$) is then

$$h(r) = \frac{1}{2}\frac{\Omega^2}{g}(r^2 - a^2) - \frac{\Gamma^2}{8\pi^2 a^2 g}$$

After substituting in $\Gamma = 2\pi\Omega a^2$, we have

$$h(r) = -\frac{\Gamma^2}{8\pi^2 a^2 g}\left(2 - \frac{r^2}{a^2}\right)$$

Interpretation: the height profile is a parabola, which joins continuously to the $h(r) \sim 1/r$ profile for $r > a$.

3.1.6 Exercise: A Radially Collapsing Air Bubble

Air bubble of radius R_0 inside an incompressible, irrotational fluid with density ρ and pressure p_0. Hypothetically, all air vanishes from inside bubble, leaving only vacuum. Compute dynamics of how surrounding water fills the buble.

- Center origin at center of bubble. Assume an ansatz of the form

$$\boldsymbol{v}(\boldsymbol{r}, t) = v(r, t)\,\hat{\mathbf{e}}_r.$$

Write continuity equation $\nabla \cdot \boldsymbol{v} = 0$ in spherical coordinates:

$$\nabla \cdot \boldsymbol{v} = \frac{1}{r^2}\frac{\partial}{\partial r}\big(r^2 v\big) = 0.$$

Integrate over r and get

$$r^2 v(r, t) = f(t),$$

where the integration constant $f(t)$ may also be a function of time.

- Incompressible potential flow depends on time only indirectly, through a problem's boundary conditions—for this problem, the decrease in bubble size as water fills the bubble. This idea is encoded in $r^2 v(r,t) = f(t)$.

  Write radius of bubble as $R(t)$, and let $v(R,t) \equiv V(t)$ denote the speed at which the bubble boundary moves radially inward. On the bubble boundary where $r = R(t)$, the solution $r^2 v(r,t) = f(t)$ gives

$$f(t) = V(t)R^2(t).$$

  TODO: just formulate this as finding f from boundary conditions.

  The general expression for velocity field is then

$$v(r,t) = \frac{f(t)}{r^2} = V(t)\frac{R^2(t)}{r^2}.$$

  INT: we have written entire fluid velocity field in terms of speed at which bubble changes.

- Bubble contraction speed depends on fluid pressure on bubble surface.

  Write out Euler equation

$$\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{v} = -\frac{\nabla p}{\rho}.$$

  In cylindrical coordinates, assuming $\boldsymbol{v} = v(r)\,\hat{\mathbf{e}}_r$ and $p = p(r)$, this simplifies considerably to

$$\frac{\partial v}{\partial t}\,\hat{\mathbf{e}}_r + v\frac{\partial v}{\partial r}\,\hat{\mathbf{e}}_r = -\frac{1}{\rho}\frac{\partial p}{\partial r}\,\hat{\mathbf{e}}_r.$$

  Substitute in velocity from **TODO:** ref into $\frac{\partial v}{\partial t}$ and cancel $\hat{\mathbf{e}}_r$ to get

$$\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}t}\Big[V(t)R^2(t)\Big] + v\frac{\partial v}{\partial r} = -\frac{1}{\rho}\frac{\partial p}{\partial r}.$$

  Integrate over r from $r = R(t)$ to $r \to \infty$ to get

$$\left\{-\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}t}\Big[V(t)R^2(t)\Big]\right\}_{R(t)}^{\infty} + \left[\frac{v(r)^2}{2}\right]_{R(t)}^{\infty} = -\left[\frac{p(r)}{\rho}\right]_{R(t)}^{\infty}$$

  Use $v\big(r = R(t)\big) = V(t)$, $v(r \to \infty) = 0$, $p(r \to \infty) = p_0$ and $p\big(r = R(t)\big) = 0$ to get

$$\frac{1}{R}\frac{\mathrm{d}}{\mathrm{d}t}\Big[V(t)R^2(t)\Big] - \frac{V(t)^2}{2} = -\frac{p_0}{\rho}.$$

  We then compute:

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big[V(t)R^2(t)\Big] = R^2\frac{\mathrm{d}V}{\mathrm{d}t} + 2VR\frac{\mathrm{d}R}{\mathrm{d}t} = R^2\frac{\mathrm{d}V}{\mathrm{d}t} + 2RV^2,$$

  where we have used $\frac{\mathrm{d}R}{\mathrm{d}t} = V(t)$ in the last equality. Substitute in above and get

$$R(t)\frac{\mathrm{d}V}{\mathrm{d}t} + 2V(t)^2 - \frac{V(t)^2}{2} = -\frac{p_0}{\rho}.$$

Then rewrite $\frac{\mathrm{d}V}{\mathrm{d}t}$ with the chain rule according to

$$\frac{\mathrm{d}V}{\mathrm{d}t} = \frac{\mathrm{d}V}{\mathrm{d}R}\frac{\mathrm{d}R}{\mathrm{d}t} = V\frac{\mathrm{d}V}{\mathrm{d}R}.$$

Substitute in above and get

$$RV\frac{\mathrm{d}V}{\mathrm{d}R} + \frac{3}{2}V^2 = -\frac{p_0}{\rho}.$$

This is a separable equation. Divide through by V and form a common denominator for V terms.

$$\frac{V\,\mathrm{d}V}{(p_0/\rho) + (3/2)V^2} = -\frac{\mathrm{d}R}{R}$$

Integrate over variables from $t = 0$ when $R = R_0$ and $V = 0$ to arbitrary t when $V = V(t)$ and $R = R(t)$. Velocity integral solved with $u = (p_0/\rho) + (3/2)V^2$. Result is

$$\ln\left(1 + \frac{3}{2}\frac{\rho V^2}{p_0}\right) = \ln\frac{R_0}{R}.$$

Solve for $V(R)$ and get

$$V(R) = \sqrt{\frac{2}{3}\frac{p_0}{\rho}\left(\frac{R_0^3}{R^3} - 1\right)}$$

Importantly, V rapidly increases as $R \to 0$—this behavior is the basis of a phenomenon called sonoluminescence. See https://en.wikipedia.org/wiki/Rayleigh-Plesset_equation for a similar analysis, but accounting for viscosity and surface tension.

3.2 Potential Flow

3.2.1 Theory

- Consider flow in two dimensions, so that, in Cartesian coordinates,

$$\boldsymbol{v} = v_x(x,y)\,\hat{\mathbf{e}}_x + v_y(x,y)\,\hat{\mathbf{e}}_y.$$

We will restrict ourselves to incompressible and irrotational (potential) two-dimensional flow. Cool things happen in this case.

- Incompressible flow gives

$$\nabla\cdot\boldsymbol{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0$$

Potential flow gives

$$\boldsymbol{v} = v_x\,\hat{\mathbf{e}}_x + v_y\,\hat{\mathbf{e}}_y = \nabla\phi = \frac{\partial\phi}{\partial x}\,\hat{\mathbf{e}}_x + \frac{\partial\phi}{\partial y}\,\hat{\mathbf{e}}_y.$$

- We then define a *stream function* ψ according to

$$v_x = \frac{\partial\psi}{\partial y} \qquad \text{and} \qquad v_y = -\frac{\partial\psi}{\partial x}.$$

This definition is intentionally desiged to satisfy the continuity equation $\nabla\cdot\boldsymbol{v} = 0$:

$$0 = \frac{\partial^2\psi}{\partial x\partial y} - \frac{\partial^2\psi}{\partial y\partial x} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = \nabla\cdot\boldsymbol{v} = 0$$

- Combine $\boldsymbol{v} = \nabla\phi$ together with $v_x = \frac{\partial\psi}{\partial y}$ and $v_y = -\frac{\partial\psi}{\partial x}$ to get

$$v_x = \frac{\partial\phi}{\partial x} = \frac{\partial\psi}{\partial y} \qquad \text{and} \qquad v_y = \frac{\partial\phi}{\partial y} = -\frac{\partial\psi}{\partial x}.$$

This is a set of Cauchy-Riemann equations for ϕ and ψ, and motivates the definition of a complex potential

$$w(z) \equiv \phi(x, y) + i\psi(x, y), \quad \text{where } z = x + iy.$$

- Next step is to show the derivative of w with respect to z is related to velocity components

$$\frac{\mathrm{d}w}{\mathrm{d}z} = \frac{\partial\phi}{\partial x} + i\frac{\partial\psi}{\partial x} = \frac{\partial\phi}{\partial(iy)} + i\frac{\partial\psi}{\partial(iy)} = v_x - iv_y.$$

Idea is we differentiate w with respect to both the x and y directions, which are orthogonal, and show the result is the same. The result should be the same if w is indeed complex-differentiable.

Calculations read

$$\frac{\partial\phi}{\partial x} + i\frac{\partial\psi}{\partial x} = v_x - iv_y \qquad \text{and} \qquad \frac{\partial\phi}{\partial(iy)} + i\frac{\partial\psi}{\partial(iy)} = \frac{1}{i}v_y + i\frac{1}{i}v_x = v_x - iv_y.$$

We conclude

$$\frac{\mathrm{d}w}{\mathrm{d}z} = v_x - iv_y. \tag{3.6}$$

Stream Function

- Note $\nabla^2\phi = \nabla^2\psi = 0$. We then compute the directional derivative

$$\boldsymbol{v} \cdot \nabla\psi = (v_x, v_y) \cdot (-v_y, v_x) = 0$$

Lesson: ψ is constant along the direction of $\boldsymbol{v}$, i.e. is ψ is constant along streamlines.

- Equation of streamline in 2D is

$$\frac{\mathrm{d}x}{v_x} = \frac{\mathrm{d}y}{v_y} \implies v_y \,\mathrm{d}x = v_x \,\mathrm{d}y$$

Substitute in v_x and v_y in terms of stream function to get

$$-\frac{\partial\psi}{\partial x}\,\mathrm{d}x = \frac{\partial\psi}{\partial y}\,\mathrm{d}y \implies \frac{\partial\psi}{\partial x}\,\mathrm{d}x + \frac{\partial\psi}{\partial y}\,\mathrm{d}y = \mathrm{d}\psi = 0.$$

Lesson: another way to see that ψ is constant along a streamline.

Flow Rate

- In 2D we speak of area flow across a curve. This is

$$Q_{\mathrm{A}} = \int_{\boldsymbol{r}_1}^{\boldsymbol{r}_2} \boldsymbol{v} \cdot \hat{\mathbf{n}} \,\mathrm{d}l = \int_{\boldsymbol{r}_1}^{\boldsymbol{r}_2} v_x \,\mathrm{d}y - v_y \,\mathrm{d}x = \psi(\boldsymbol{r}_2) - \psi(\boldsymbol{r}_1),$$

where $\hat{\mathbf{n}} = \left(\frac{\mathrm{d}y}{\mathrm{d}l}, -\frac{\mathrm{d}x}{\mathrm{d}l}\right)$ is the normal to the curve. The tangent to the curve is just $\mathrm{d}\boldsymbol{l} = (\mathrm{d}x, \mathrm{d}y)$. A fuller derivation reads

$$Q_{\mathrm{A}} = \int_{\boldsymbol{r}_1}^{\boldsymbol{r}_2} \boldsymbol{v} \cdot \hat{\mathbf{n}} \,\mathrm{d}l = \int_{\boldsymbol{r}_1}^{\boldsymbol{r}_2} \left(\frac{\partial\psi}{\partial y}\,\mathrm{d}y + \frac{\partial\psi}{\partial x}\,\mathrm{d}x\right)\frac{\mathrm{d}l}{\mathrm{d}l} = \int_{\boldsymbol{r}_1}^{\boldsymbol{r}_2} \mathrm{d}\psi = \psi(\boldsymbol{r}_2) - \psi(\boldsymbol{r}_1).$$

- Mass flow rate across a curve, assuming incompressible fluid and so $\psi =$ constant, is
$$Q_{\mathrm{m}} = \rho \int_{\boldsymbol{r}_1}^{\boldsymbol{r}_2} \boldsymbol{v} \cdot \hat{\mathbf{n}} \, \mathrm{d}l = \rho \big[\psi(\boldsymbol{r}_2) - \psi(\boldsymbol{r}_1) \big].$$

3.2.2 Exercise: Velocity Potentials of a Point Source and Dipole in 3D

Derive the velocity potentials corresponding to a point flow source and a point flow dipole in three dimensions.

Point Source

- Assume irrotational flow $\nabla \times \boldsymbol{v} = \mathbf{0}$ so that $\boldsymbol{v} = \nabla \phi$, where ϕ is the velocity potential. Also assume incompressible flow, in which case $\nabla \cdot \boldsymbol{v} = 0$ and thus
$$\nabla \cdot \boldsymbol{v} = \nabla^2 \phi = 0.$$
Begin by quoting the general solution of the Laplace equation $\nabla^2 \phi = 0$ in three dimensions in spherical coordinates:
$$\phi(\rho, \theta, \varphi) = -\frac{1}{4\pi} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \Big[A_{lm} r^l + B_{lm} r^{-(l+1)} \Big] \sqrt{\frac{4\pi}{2l+1}} Y_{lm}(\theta, \varphi) \tag{3.7}$$
The minus sign occurs because $\boldsymbol{v} = +\nabla \phi$.

- We then write, by analogy with electrostatics, the point source potential
$$\phi(r) = -\frac{a}{4\pi r}$$
This potential corresponds to a general solution of Equation 3.7 with $l = 0$. Our goal is to find the volume flow rate from this source. We begin by computing the corresponding velocity field from
$$\boldsymbol{v} = \nabla \phi = -\frac{a}{4\pi} \nabla \frac{1}{r} = +\frac{a}{4\pi} \frac{\boldsymbol{r}}{r^3}.$$
Note that the velocity field points radially outward from the source, which is the expected behavior for a source. The volume flow rate through a closed spherical surface containing the source is
$$Q_{\mathrm{V}} = \oiint \boldsymbol{v} \cdot \mathrm{d}\boldsymbol{S} = \frac{a}{4\pi} \oiint \frac{\boldsymbol{r} \cdot \hat{\mathbf{n}}}{r^3} \, \mathrm{d}S = \frac{a}{4\pi} \oiint \frac{\mathrm{d}S}{r^2} = \frac{a}{4\pi} \frac{4\pi r^2}{r^2} = a.$$

Point Dipole

- Skipping the derivation in equation (2.66), the result is
$$\phi(\boldsymbol{r}) = -\frac{1}{4\pi} \frac{\boldsymbol{p} \cdot \boldsymbol{r}}{r^3},$$
where the dipole moment $\boldsymbol{p}$ is given by
$$\boldsymbol{p} = 2a\Delta z \, \hat{\mathbf{e}}_z$$

- Derivation actually

$$\phi_{\text{dipole}}(r) = \phi_a(r) + \phi_{-a}(r) = -\frac{a}{4\pi r_1} + \frac{a}{4\pi r_2}$$

Assume dipoles are separated by a distance $2d$ along the z axis. The source with output $+a$ at $z = d$ and the sink with "output" $-a$ at $z = -d$. Then (writing $\phi_{\text{dipole}} \to \phi_{\text{d}}$ for conciseness) we have

$$\phi = \phi_a + \phi_{-a} = -\frac{a}{4\pi}\left[\frac{1}{\sqrt{x^2+y^2+(z-d)^2}} - \frac{1}{\sqrt{x^2+y^2+(z+d)^2}}\right]$$

We then expand the expressions in the point dipole limit $d/r \to 0$. This gives

$$\begin{aligned}\phi &= -\frac{a}{4\pi}\left[\frac{1}{\sqrt{x^2+y^2+(z-d)^2}} - \frac{1}{\sqrt{x^2+y^2+(z+d)^2}}\right]\\ &\approx -\frac{a}{4\pi}\left(\frac{1}{\sqrt{r^2-2zd}} - \frac{1}{\sqrt{r^2+2zd}}\right) = \frac{-a}{4\pi r}\left(\frac{1}{\sqrt{1-\frac{2zd}{r^2}}} - \frac{1}{\sqrt{1+\frac{2zd}{r^2}}}\right)\end{aligned}$$

Expand square roots and get

$$\begin{aligned}\phi_{\text{d}} &= \frac{-a}{4\pi r}\left(\frac{1}{\sqrt{1-\frac{2zd}{r^2}}} - \frac{1}{\sqrt{1+\frac{2zd}{r^2}}}\right)\\ &\approx -\frac{a}{4\pi r}\left(1+\frac{zd}{r^2} - 1 + \frac{zd}{r^2}\right) = -\frac{2ad}{r\pi}\frac{z}{r^3}.\end{aligned}$$

Define dipole momentum $p_z \equiv 2ad$ and get

$$\phi_{\text{d}} = \frac{p_z}{4\pi}\frac{z}{r^3}.$$

TODO: change $\phi(r)$ to $\phi(\boldsymbol{r})$.

In vector form

$$\phi_{\text{d}}(\boldsymbol{r}) = -\frac{1}{4\pi}\frac{\boldsymbol{p}\cdot\boldsymbol{r}}{r^3}, \quad \text{where } \boldsymbol{p} \equiv 2ad\,\hat{\mathbf{e}}_z.$$

Multipole Construction

- We now consider the construction of an arbitrary dipole. Begin with a monopole

$$\phi_0 = \frac{c}{4\pi r}$$

This potential solves the Laplace equation (Eq. 3.7) $\nabla^2\phi = 0$. The derivatives $\partial_i\phi_0$ also solve Equation 3.7, since, switching order of differentiation and using $\nabla^2\phi_0 = 0$, we find

$$\nabla^2(\partial_i\phi_0) = \partial_i(\nabla^2\phi_0) = \partial_i 0 = 0$$

- To get a scalar potential, we scalar-multiply the solution $\partial_i\phi_0$ by a constant vector A_i to get $A_i\partial_i\phi_0$. This field still satisfies the Laplace equation:

$$\nabla^2(A_i\partial_i\phi_0) = \partial_i\left(A_i\nabla^2\phi_0\right) = 0$$

(Recall $\boldsymbol{A}$ is constant). We define the resulting potential as

$$\phi_1 \equiv A_i \partial_i \phi_0.$$

This potential turns out to be the potential of a point dipole with dipole moment $p_i = -CA_i$. Where for review $\phi_0 = C/(4\pi r)$.

- In our case above $C \to -a$. We write $\phi_0 = -\frac{a}{4\pi r}$ and from this construct

$$\phi_1 = \frac{p_i}{C}\partial_i\left(-\frac{a}{4\pi r}\right) = \frac{p_i}{C}\left(\frac{a}{4\pi r^2}\frac{r_i}{r}\right) = -\frac{p_i}{a}\frac{a}{4\pi r^3}r_i = -\frac{\boldsymbol{p}\cdot\boldsymbol{r}}{4\pi r^3},$$

where we have used $C = -a$ in the last equality. Lesson: the result agrees with the earlier contruction.

3.2.3 Exercise: Velocity Potentials of a Point Source and Vortex in 2D

Compute the velocity potential and velocity field of a point source and point vortex in two dimensions.

- Fundamental goal is to solve the Laplace equation

$$\nabla^2\phi = 0.$$

We will work in in cylindrical polar coordinates, for which the Laplace equation reads

$$\nabla^2\phi(r,\varphi) = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\varphi^2} = 0.$$

Boundary conditions are time independent and flow is potential and incompressible, so the eventual solutions for $\boldsymbol{v}$ will be stationary.

Point Source

- The source points radially outward is symmetric with respect to φ.

  Possible solutions are $\phi = \text{constant}$ and $\phi \propto \ln r$.

  The first solution is trivial and leads to $\boldsymbol{v} = \nabla\phi = \mathbf{0}$.

  The second corresponds to a velocity field of a point source with output Q_0

$$\phi = \frac{Q_0}{2\pi}\ln r \implies \boldsymbol{v} = \nabla\phi = \frac{Q_0}{2\pi r}\,\hat{\mathbf{e}}_r.$$

We now compute the source's area flow rate through a circle containing the source. This is

$$Q_\mathrm{A} = \oint \boldsymbol{v}\cdot\mathrm{d}\hat{\mathbf{n}} = \oint\left(\frac{Q_0}{2\pi r}\,\hat{\mathbf{e}}_r\right)\cdot\hat{\mathbf{e}}_r\,\mathrm{d}s = \frac{Q_0}{2\pi}\oint\frac{\mathrm{d}s}{r} = \frac{Q_0}{2\pi}\frac{2\pi r}{r} = Q_0.$$

Lesson: we were correct to label Q_0 is the the point source's output because $Q_\mathrm{A} = Q_0$.

For a Vortex

- A point vortex is a two-dimension analog of the line vortex in **TODO:** reference. The corresponding velocity field is irrotational everywhere except at the origin, where it is undefined (or "infinite").

- We proceed from first principles as follows: a vortex's velocity field is purely tangential (i.e. purely in the $\hat{\mathbf{e}}_\varphi$ direction), so the corresponding velocity potential must depend only on φ. The possible solutions to $\nabla^2\phi = 0$ are $\phi = \text{constant}$ and $\phi \propto \varphi$.

  $\phi = \text{constant}$ is trivial and we reject it.

- Instead choose $\phi \propto \varphi$. For a vortex with circulation Γ_0 this potential is

$$\phi = \frac{\Gamma_0}{2\pi}\varphi \implies \boldsymbol{v} = \nabla\phi = \frac{\Gamma_0}{2\pi r}\,\hat{\mathbf{e}}_\varphi.$$

  In detail, the computation of $\nabla\phi$ reads

$$\nabla\phi = \left(\hat{\mathbf{e}}_r\frac{\partial}{\partial r} + \frac{\hat{\mathbf{e}}_\varphi}{r}\frac{\partial}{\partial\varphi}\right)\frac{\Gamma_0}{2\pi}\varphi = \frac{\Gamma_0}{2\pi r}\,\hat{\mathbf{e}}_\varphi.$$

- We then compute that the Γ_0 is indeed the vortex's circulation

$$\Gamma = \oint \boldsymbol{v}\cdot \mathrm{d}\boldsymbol{l} = \oint\left(\frac{\Gamma_0}{2\pi r}\,\hat{\mathbf{e}}_\varphi\right)\cdot(\hat{\mathbf{e}}_\varphi\,\mathrm{d}l) = \frac{\Gamma_0}{2\pi}\frac{\mathrm{d}l}{r} = \frac{\Gamma_0}{2\pi}\frac{2\pi r}{r} = \Gamma_0$$

  Lesson: we were correct to label Γ_0 is the the point vortex's circulation because $\Gamma = \Gamma_0$.

Point Source: Complex Formalism

- This is done heuristically. The analytic complex function correponding to a point source with output Q_0 is

$$w(z) = \frac{Q_0}{2\pi}\ln z.$$

  Write in the polar form $z = re^{i\varphi}$ which gives

$$w(z) = \frac{Q_0}{2\pi}\ln\left(re^{i\varphi}\right) = \frac{Q_0}{2\pi}\left(\ln r + \ln e^{i\varphi}\right) = \frac{Q_0}{2\pi}\ln r + i\frac{Q_0}{2\pi}\varphi$$

  Comparing to $w(z) = \phi + i\psi$ we read read off ϕ and ψ are

$$\phi = \frac{Q_0}{2\pi}\ln r \qquad \text{and} \qquad \psi = \frac{Q_0}{2\pi}\varphi,$$

  Note that ϕ is consistent with **TODO:** reference. Interpretation: recall a streamline equation is $\psi = \text{constant}$. For the above $\psi \propto \varphi$, $\psi = \text{constant}$ reduces to $\varphi = \text{constant}$, and lines of constant φ (i.e. lines pointing radially outward from the origin) are precisely the stream lines of a point source.

- The area flow rate through a closed circle enclosing the source (i.e. a circle begining at $\varphi_1 = 0$ and ending at $\varphi_2 = 2\pi$) is, via Equation **TODO:** reference

$$Q_{\mathrm{A}} = \psi(\varphi)\big|_{\varphi=2\pi} - \psi(\varphi)\big|_{\varphi=0} = \frac{Q_0}{2\pi}(2\pi - 0) = Q_0.$$

Point Vortex: Complex Formalism

- For a point vortex, the roles of ϕ and ψ are switched relative to those for a point source. Equipotential lines are now radially outward and streamlines are in the direction of $\hat{\mathbf{e}}_\varphi$.
- We guess the correct complex potential for a point vortex with circulation Γ_0 is

$$w(z) = -i\frac{\Gamma_0}{2\pi}\ln z = \frac{\Gamma_0}{2\pi i}\ln\left(re^{i\varphi}\right) = \frac{\Gamma_0}{2\pi i}\ln r + \frac{\Gamma_0}{2\pi}\varphi = \frac{\Gamma_0}{2\pi}\varphi - i\frac{\Gamma_0}{2\pi}\ln r. \tag{3.8}$$

From here we read off

$$\phi = \frac{\Gamma_0}{2\pi}\varphi \qquad \text{and} \qquad \psi = -\frac{\Gamma_0}{2\pi}\ln r$$

These agree with the results in **TODO:** reference.

Note: as expected, the vortex's circulation is zero around any closed circle of radius R between the points $\boldsymbol{r}_1 = (R, 0)$ and $\boldsymbol{r}_2 = (R, 2\pi)$. This is because ψ is independent of ϕ.

$$Q_\mathrm{A} = \psi(\boldsymbol{r}_2) - \psi(\boldsymbol{r}_1) = -\frac{\Gamma_0}{2\pi}(\ln R - \ln R) = 0.$$

3.2.4 Exercise: Velocity Potential of a Point Dipole 2D

Write velocity potential for a dipole formed by two points sources along the x axis: one with output Q at $x = a$ and one with output Q at $x = -a$.
Use this to find find potential of a point dipole. Find the corresponding velocity field. Work in Cartesian coordinates.

- **TODO:** make sure i wrote separated by $2a$ and not a.
- Begin with potential of a point source at the origin:

$$w_0(z) = \frac{Q}{2\pi}\ln z$$

Linearity (of what!): velocity of sums is sum of velocities. Translate by $\pm a$ and get

$$w(z) = \frac{Q}{2\pi}\ln(z-a) - \frac{Q}{2\pi}\ln(z+a) = \frac{Q}{2\pi}\ln\frac{z-a}{z+a}.$$

- **TODO:** dude! a is in general complex. Drop absval below.
- Point dipole: approximation $a \ll |z|$. Expand to first order in a/z and use $\ln(1+x) \approx x$ to get

$$w(z) = \frac{Q}{2\pi}\left[\ln\left(1 - \frac{a}{z}\right) - \ln\left(1 + \frac{a}{z}\right)\right] \approx \frac{Q}{2\pi}\left[-\frac{a}{z} - \left(+\frac{a}{z}\right)\right] = -\frac{Qa}{\pi z}.$$

Define dipole moment $p \equiv 2Qa$ to get

$$w(z) = -\frac{p}{2\pi z}.$$

- Note: in principle a (which is currently the dipole separation along the x axis) could also be an arbitrary vector in the complex plane.

  Cool: rotating $\boldsymbol{a} = a + 0 \cdot i$ by $-\pi/2$ to get $\boldsymbol{a}' = -ia$ is equivalent to keeping a and defining $Q \to -i\Gamma$. Interpretation: this just replaces the source and sink Q and $-Q$ with $-i\Gamma$ and $+i\Gamma$, which **TODO:** reference above produces the potentials of point vortices along the x axis separated by $2a$.

Velocity Field

- Return to
$$w(z) = -\frac{p}{2\pi}\frac{1}{z} = \frac{p}{2\pi}\frac{1}{x+iy} = -\frac{p}{2\pi}\frac{x-iy}{x^2+y^2} \equiv \phi(x,y) + i\psi(x,y).$$
  Velocity potential is
$$\phi(x,y) = -\frac{p}{2\pi}\frac{x}{x^2+y^2}.$$
  The corresponding velocity field is
$$\boldsymbol{v}(x,y) = \nabla\phi = \frac{p}{\pi(x^2+y^2)}\left(\frac{x^2-y^2}{x^2+y^2}\,\hat{\mathbf{e}}_x + \frac{2xy}{x^2+y^2}\,\hat{\mathbf{e}}_y\right).$$

- The stream function is
$$\psi(x,y) = \frac{p}{\pi}\frac{y}{x^2+y^2}$$
  Streamlines are given by
$$\psi(x,y) = \text{constant} \implies \frac{y}{x^2+y^2} \equiv \text{constant}.$$
  Denote constant by A and rearrange to get
$$y = A(x^2+y^2) \implies x^2+y^2-\frac{y}{A} \equiv x^2+y^2-2B = 0$$
  We then complete the square to get the streamline
$$x^2 + (y-A)^2 = A^2.$$
  INT: Streamlines are circles with center on the y axis at $z = iA$, with radius A.

3.2.5 Dynamics of Two Point Vortices

A pair of point vortices, both with circulation Γ, initially at $x = \pm a$.
Find time dependence of complex potential and velocity field far from the vortices.

- Begin with potential of a point vortex at the origin: **TODO:** origin
$$w_0(z) = \frac{\Gamma}{2\pi i}\ln z$$
  Linearity (of what!): velocity of sums is sum of velocities. Translate each vortex by $\pm a$ and get
$$w(z) = \frac{\Gamma}{2\pi i}\ln(z-a) + \frac{\Gamma}{2\pi i}\ln(z+a) = \frac{\Gamma}{2\pi i}\ln\big(z^2-a^2\big).$$

Expand for $|a| \ll |z|$ (a can in general be complex) using $\ln(1-x) \approx -x$ and get

$$\begin{aligned} w(z) &= \frac{\Gamma}{2\pi i} \ln\left[z^2\left(1-\frac{a^2}{z^2}\right)\right] = \frac{\Gamma}{2\pi i}\left[\ln z^2 + \ln\left(1-\frac{a^2}{z^2}\right)\right] \\ &\approx \frac{2\Gamma}{2\pi i}\ln z - \frac{\Gamma}{2\pi i}\frac{a^2}{z^2} \end{aligned}$$

Interpretation: first term is a gain a point vortex, but with double circulation 2Γ. The second term is a correction.

Dynamics

- **TODO:** recall from section reference...

  Dynamics of two vortices with equal circulations is: the line connecting the two vortices rotates in the xy plane about the common center with angular velocity

  $$\Omega = \frac{\Gamma}{4\pi a^2}.$$

  (Orbital velocity of one vortex is the fluid velocity caused by the other vortex at the point $2a$ from the other vortex.). This is $v = \Gamma/(4\pi a)$ so $v = \Omega a$ leads to angular velocity about center a distance a away.

- The vector a connecting the two vortices rotates in the complex plane as

  $$a(t) = |a|e^{i\Omega t}.$$

  Substitute this into **TODO:** reference leads to

  $$w(z) = \frac{\Gamma}{\pi i}\ln z - \frac{\Gamma}{2\pi i}\frac{|a|^2 e^{2i\Omega t}}{z^2}$$

  First term in polar coordinates is

  $$\frac{\Gamma}{\pi i}\ln z = \frac{\Gamma}{\pi i}\ln re^{i\varphi} = \frac{\Gamma}{\pi i}\big(\ln r + i\varphi\big) = \frac{\Gamma}{\pi}\varphi - i\frac{\Gamma}{\pi}\ln r.$$

  Second term in polar coordinates follows from

  $$\frac{1}{z^2} = \frac{1}{r^2 e^{2i\varphi}} = \frac{e^{-2i\varphi}}{r^2} \implies \frac{\Gamma}{2\pi i}\frac{|a|^2 e^{2i\Omega t}}{z^2} = \frac{\Gamma}{2\pi i}\frac{|a|^2}{r^2}e^{2i(\Omega t - \varphi)}$$

  The velocity potential is then

  $$\begin{aligned} w(r,\varphi,t) &= \frac{\Gamma}{\pi}\varphi - i\frac{\Gamma}{\pi}\ln r + i\frac{\Gamma}{2\pi}\frac{|a|^2}{r^2}e^{2i(\Omega t-\varphi)} \\ &= \frac{\Gamma}{\pi}\varphi - i\frac{\Gamma}{\pi}\ln r + \frac{\Gamma}{2\pi}\frac{|a|^2}{r^2}\Big\{i\cos\big[2(\Omega t-\varphi)\big] - \sin\big[2(\Omega t-\varphi)\big]\Big\} \end{aligned}$$

  And so

  $$\begin{aligned} \phi &= \operatorname{Re} w = \frac{\Gamma}{\pi}\varphi - \frac{\Gamma}{2\pi}\frac{|a|^2}{r^2}\sin\big[2(\Omega t-\varphi)\big] \\ \psi &= \operatorname{Im} w = -\frac{\Gamma}{\pi}\ln r + \frac{\Gamma}{2\pi}\frac{|a|^2}{r^2}\cos\big[2(\Omega t-\varphi)\big] \end{aligned}$$

- The corresponding velocity field components are

  $$\begin{aligned} v_r &= \frac{\partial\phi}{\partial r} = +\frac{2\Gamma}{2\pi}\frac{|a|^2}{r^3}\sin\big[2(\Omega t-\varphi)\big] \\ v_\varphi &= \frac{1}{r}\frac{\partial\phi}{\partial\varphi} = \frac{\Gamma}{\pi r} + \frac{2\Gamma}{2\pi}\frac{|a|^2}{r^3}\cos\big[2(\Omega t-\varphi)\big] \end{aligned}$$

3.2.6 Exercise: A Point Source Between Two Vortices

A pair of point vorticies, both with circulation Γ, initially separated by $2a_0$.
Place a 2D isotropic point source symmetrically between the vortices. Area output is Q.
Find curve $r(\varphi)$ traced out by the vortex lines in the $r\varphi$ plane.

- Place origin at midpoint between vortices. Line connecting vortices will orbit around the origin. Because equal circulations.
- Velocity field of point source is **TODO:** reference

  $$v_r(r) = \frac{Q}{2\pi r}.$$

  Velocity fields contribution of opposite vortices at the other vortex is azimuthal and equal to

  $$v_\varphi(r) = \frac{\Gamma}{2\pi \cdot (2r)}.$$

- Velocity fields of all contributions are additive because linearity blah blah **TODO:** explain.

  The vortices travel with the fluid at the current vortex position. Thus: each vortex has a radial component, from the point source, of $v_R(r)$, and tangential component, from the other vortex $v_\varphi(r)$. Result: spiral motion.

  Angular velocity changes as

  $$\Omega = \frac{\Gamma}{4\pi r^2}.$$

- We now compute the trajectory of a given vortex

  $$\frac{\mathrm{d}r}{\mathrm{d}\varphi} = \frac{\mathrm{d}r}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}\varphi} = v_r \cdot \frac{1}{\Omega} = \frac{Q}{2\pi r}\frac{4\pi r^2}{\Gamma}.$$

  Separate variables to get

  $$\frac{\mathrm{d}r}{r} = \frac{2Q}{\Gamma}\,\mathrm{d}\varphi$$

  Integrate:

  $$\int_{a_0}^{r} \frac{\mathrm{d}r'}{r'} = \frac{2Q}{\Gamma}\int_0^{\varphi} \mathrm{d}\varphi' \implies \ln\frac{r}{a_0} = \frac{2Q}{\Gamma}\varphi$$

  The solution, in terms of either r or φ, is

  $$\varphi(r) = \frac{\Gamma}{2Q}\ln\frac{r}{a_0} \qquad \text{or} \qquad r(\varphi) = a_0 \exp\left(\frac{2Q}{\Gamma}\varphi\right).$$

  The result is a well-known curve called a logarithmic spiral.

3.2.7 Exercise: Potential Flow In a Wedge

Find the incompressible potential velocity field of a fluid in a region bounded by "semi-planes" at an angle α.

By Solving Laplace Equation

- Begin by quoting solutions to $\nabla^2\phi = 0$ in (r, φ) coordinates.

$$\phi(r,\varphi) = \begin{cases} \big(Ar^m + Br^{-m}\big)\cdot\big(C\cos m\varphi + D\sin m\varphi\big) & m \neq 0 \\ F + G\ln r & m = 0 \end{cases}$$

- Which solutions are allowed? First, we note the $m = 0$ solution $\phi \propto F$ results in a trivial velocity field $\boldsymbol{v} = \nabla\varphi = \mathbf{0}$, so we require $m > 1$.

- In our geometry, $\hat{\mathbf{e}}_\varphi$ points normal to the boundaries. At the boundaries at $\varphi = 0$ and $\varphi = \alpha$ for all r we must have

$$v_\varphi = \frac{1}{r}\frac{\partial\phi}{\partial\varphi} = 0$$

Require only cosine solutions to satsify $\frac{\partial\phi}{\partial\varphi} = 0$ at $\varphi = 0$. At $\varphi = \alpha$ we require

$$0 \equiv \frac{\mathrm{d}}{\mathrm{d}\varphi}\big(\cos m\varphi\big)_{\varphi=\alpha} \propto \sin m\alpha \implies m\alpha = \pi n \implies m = \frac{\pi n}{\alpha},$$

where $n \in \mathbb{N}$. (In principle $m \in \mathbb{Z}$, but cosine is even).

- Finally, avoid divergence of ϕ at $r = 0$ by eliminating r^{-m} solutions. We're left with

$$\phi_n(r,\varphi) = Ar^{\frac{n\pi}{\alpha}}\cos\left(\frac{n\pi}{\alpha}\varphi\right)$$

The corresponding velocity field is

$$\boldsymbol{v}_n = \nabla\phi_n \propto \frac{n\pi}{\alpha}r^{\frac{n\pi}{\alpha}-1}\left[\cos\left(\frac{n\pi}{\alpha}\varphi\right)\hat{\mathbf{e}}_r - \sin\left(\frac{n\pi}{\alpha}\varphi\right)\hat{\mathbf{e}}_\varphi\right]$$

We will restrict ourselves to the simple solution $n = 1$. Physical interpretation: fluid flows along one border and then leaves on the other, roughly like a hyperbolic comet orbit.

Note: for $\alpha = \pi$ and $n = 1$, i.e. for no corner at all, something cool happens:

$$\begin{aligned} \boldsymbol{v}_1(r,\varphi) &\propto \cos(\varphi)\,\hat{\mathbf{e}}_r - \sin(\varphi)\,\hat{\mathbf{e}}_\varphi \\ &= \cos\varphi\cdot\big[\cos(\varphi)\,\hat{\mathbf{e}}_x + \sin(\varphi)\,\hat{\mathbf{e}}_y\big] - \sin\varphi\cdot\big[-\sin(\varphi)\,\hat{\mathbf{e}}_x + \cos(\varphi)\,\hat{\mathbf{e}}_y\big] \\ &= (\cos^2\varphi + \sin^2\varphi)\,\hat{\mathbf{e}}_x \\ &= \hat{\mathbf{e}}_x. \end{aligned}$$

The solution $\boldsymbol{v}_1(\boldsymbol{r},\varphi)$ is just homogeneous flow through the xy plane along the x axis.

Complex Solution

- **TODO:** record complex potential for homogeneous velocity field in the x direction is

$$w(z) = v_0 z$$

Confirm by computing $\frac{\mathrm{d}w}{\mathrm{d}x} = v_0$. Match to Equation 3.6, i.e. $\frac{\mathrm{d}w}{\mathrm{d}z} = v_x - iv_y$ to conclude $v_x = v_0$ and $v_y = 0$ as expected for homogeneous flow.

- Summary: we found a complex potential for $\alpha = \pi$; we now aim to find a complex potential for arbitrary α.

  Plan: Find $w'(z')$ for arbitrary α region.

  Boundary condition: normal component of velocity to boundary is zero. But another way of saying this is that velocity is purely tangent to boundary, i.e. that the boundary region is a streamline.

  This links nicely to $w = \phi + i\psi$, where ψ is the stream function.

  For region z' we thus require

  $$\psi'(z') = \operatorname{Im} w'(z') = \text{constant}$$

  for all z' along the boundary.

  When the problem has only a single boundary—i.e. the boundary is a single contour, we can freely set $\psi'(z') = 0$ along the boundary contour.

- Goal: find a conformal transformation mapping the boundary region between the walls at an angle α onto the x axis. Basically transform z' into z. This transformation is

  $$z = \big(z'\big)^{\pi/\alpha} \implies w'\big(z'\big) = w\big(z(z')\big) = v_0\big(z'\big)^{\pi/\alpha}.$$

  First write potential in polar form

  $$w'(z') \propto \big(z'\big)^{\pi/\alpha} = r^{\pi/\alpha} e^{i\frac{\pi}{\alpha}\varphi} = r^{\pi/a}\left(\cos\frac{\pi\varphi}{\alpha} + i\sin\frac{\pi\varphi}{\alpha}\right) \equiv \phi' + i\psi'.$$

  From here we can read off

  $$\phi' = r^{\pi/a}\cos\frac{\pi\varphi}{\alpha} \qquad \text{and} \qquad \psi' = r^{\pi/a}\sin\frac{\pi\varphi}{\alpha}$$

  The corresponding velocity field is

  $$\boldsymbol{v} \propto \nabla\phi'(r,\varphi) = \frac{\pi}{\alpha} r^{\frac{\pi}{\alpha}-1}\left(\cos\frac{\pi\varphi}{\alpha}\,\hat{\mathbf{e}}_r - \sin\frac{\pi\varphi}{\alpha}\,\hat{\mathbf{e}}_\varphi\right).$$

  This agrees with the solution from Equation **TODO:** reference for $n = 1$.

3.2.8 Exercise: Potential Flow in a Wedge with a Rounded Corner

Two walls at an angle α like in previous exercise. But in addition a circle of radius a centered at the corner.

- Let region z be a flat without circle. Like for homogeneous velocity field like in Exercise **TODO:** reference.

  Let region z' flat but with circle of radius a'.

  Let z'' be bent and with circle of radius a, basically what's in the problem

- First map bent and circle into flat and circle with

  $$z' = (z'')^{\pi/\alpha}$$

  In the flat and circle region, the circle radius a from bent and circle region transforms into $a' = a^{\pi/\alpha}$.

Then map flat and circle into flat without circle with

$$z(z') = z' + \frac{(a')^2}{z'} = z' + \frac{(a^{\pi/\alpha})^2}{z'} = z' + \frac{a^{2\pi/\alpha}}{z'}$$

The complete transformation from bend and circle (z'') to flat and no circle (z) is

$$w''(z'') = w'\big(z'(z'')\big) = w\Big(z\big(z'(z'')\big)\Big) \propto (z'')^{\pi/\alpha} + \frac{a^{2\pi/\alpha}}{(z'')^{\pi/\alpha}}$$

- **TODO:** to find velocity. First write $z = re^{i\varphi}$ and find $w(r, \varphi)$. Match to $w = \phi + i\psi$. Find velocity from $\boldsymbol{v} = \nabla\phi$.

Flow Around Cylinder

- **TODO:** explain see vaje page 46.

$$w(z) = v_0 \left(z + \frac{a^2}{z}\right) \tag{3.9}$$

Sum of homongeneous flow and a point dipole with dipole moment $p = -2\pi v_0 a^2$.

Corresponds to fluid flowing around a cylinder of radius a, where homogeneous flow far from cylinder has speed v_0.

Dipole term ensures flow around cylinder surface is a constant streamline.

3.2.9 A Circular Droplet's Deformation in Potential Flow

From **TODO:** reference the complex potential in a 2D wedge with angle α is

$$w(z) = v_0 z^{\pi/\alpha}$$

For a perpendicular wedge, compute the corresponding velocity strain tensor

$$v_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right).$$

In Cartesian coordinates. Determine in which directions the flow expands and contracts. Find time evolution of a circular droplet's length $a(t)$ and width $b(t)$ when placed in the flow.

- For a perpendicular wedge $\alpha = \pi/2$, and the velocity potential reads

$$w(z) \propto z^{\pi/\alpha} = z^2 = (x + iy)^2.$$

The complex derivative is

$$\frac{\mathrm{d}w}{\mathrm{d}z} = v_x - iv_y \propto z.$$

We can then read off

$$v_x = x \qquad \text{and} \qquad v_y = -y$$

The corresponding velocity field is

$$\boldsymbol{v}(x, t) = A(x\,\hat{\mathbf{e}}_x - y\,\hat{\mathbf{e}}_y).$$

The corresponding velocity strain tensor, in Cartesian coordinates, is

$$v_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) = \begin{pmatrix} A & 0 \\ 0 & -A \end{pmatrix}.$$

Since $v_{xx} > 0$ and $v_{yy} < 0$, the flow expands in the x direction and contracts in the y direction.

In general, the tensor field $v_{ij}(\boldsymbol{r})$ encodes the local fluid deformation at the point $\boldsymbol{r}$. In a short time, δt, two neighboring points separated a small distance $\delta x\,\hat{\mathbf{e}}_x$ shift in the x

$$\frac{\mathrm{d}(\delta x)}{\mathrm{d}t} = \frac{\delta v_x}{\delta x}\delta x = v_{xx}\delta x$$

Analogously, two neighboring points separated by $\delta y\,\hat{\mathbf{e}}_y$ shift

$$\frac{\mathrm{d}(\delta y)}{\mathrm{d}t} = \frac{\delta v_y}{\delta y}\delta y = v_{yy}\delta y.$$

- In this problem, the velocity strain tensor is very simple—it is independent of both time and position. As a result, the above considerations also apply to arbitrarily-spaced points over arbitrarily long time periods.

  A circular droplet dropped in this problem's flow will expand along the x axis and contract along the y axis, formin an ellipse, while the droplet's center of mass will travel in the average flow direction (averaged over the droplet's surface).

  For this problem, Equations **TODO:** reference, for macroscopic separations Δx and Δy, read

$$\frac{\mathrm{d}(\Delta x)}{\mathrm{d}t} = v_{xx}\Delta x = A\Delta x \qquad \text{and} \qquad \frac{\mathrm{d}(\Delta y)}{\mathrm{d}t} = v_{yy}\Delta y = -A\Delta y$$

  The corresponding solutions, after integration, are

$$\Delta x(t) = \Delta x_0 e^{At} \qquad \text{and} \qquad \Delta y(t) = \Delta y_0 e^{-At}$$

  For a circular droplet with initial radius R, the major axis a (assumed to be parallel to the x axis) and minor axis b (assumed to be parallel to the y axis) thus evolve as

$$a(t) = Re^{At} \qquad \text{and} \qquad b(t) = Re^{-At}.$$

3.2.10 Exercise: Point Source Near a Large Wall

- Begin with **TODO:** reference point source potential

$$w(z) = \frac{Q_0}{2\pi}\ln z = \frac{Q_0}{2\pi}\ln r + i\frac{Q_0}{2\pi}\varphi$$

  In our problem place the actual point source at $z = a$, then method of images, to satisfy velocity-tangent-to-wall BC, by placing a second point source as z = - a. This gives

$$w(z) = \frac{Q_0}{2\pi}\big[\ln(z-a) + \ln(z+a)\big]$$

  Convert to polar form and get

$$w(r,\varphi) = \frac{Q_0}{2\pi}(\ln r_1 + \ln r_2) + i\frac{Q_0}{2\pi}(\varphi_1 + \varphi_2) \equiv \phi + i\psi.$$

  The r_1 and r_2 are the distance from a point (r,φ) to source 1 and 2, while φ_1 and φ_2 are the angle between either source and an arbitrary (r,φ) point.

- Recognize ϕ and convert to Cartesian coordinates and take $1/2$ power from square root from logarithm to get

$$\phi = \frac{Q_0}{2\pi} \cdot \frac{1}{2}\Big\{ \ln\big[(x-a)^2 + y^2\big] + \big[(x+a)^2 + y^2\big]\Big\}$$

Velocity components of $\boldsymbol{v}(x, y) = \nabla\varphi$ are

$$\begin{aligned} v_x &= \frac{Q_0}{2\pi}\left[\frac{x-a}{(x-a)^2+y^2} + \frac{x+a}{(x+a)^2+y^2}\right] \\ v_y &= \frac{Q_0}{2\pi}\left[\frac{y}{(x-a)^2+y^2} + \frac{y}{(x+a)^2+y^2}\right]. \end{aligned}$$

- The stream function, in Cartesian coordinates, is

$$\psi = \frac{Q_0}{2\pi}\left(\arctan\frac{y}{x-a} + \arctan\frac{y}{x+a}\right).$$

- We find pressure from the Bernoulli equation for stationary irrotational flow in the absence of external forces. This is

$$p + \frac{\rho}{2}v^2 = \text{constant} = 0.$$

Far from the source, $v = 0$ and $p = 0$, thus the constant is zero. **TODO:** write as p_0 and v_0, then set to zero. On the wall at $x = 0$ we have $v_x = 0$, so $v^2 = v_y^2$ and

$$p(0, y) = -\frac{\rho}{2}\frac{Q_0^2}{\pi^2}\frac{y^2}{(a^2+y^2)^2}.$$

- The force on the wall, per unit length L along the z axis I believe, is

$$\frac{F_x}{L} = -\int_{-\infty}^{\infty} p(0, y)\,\mathrm{d}y = \frac{\rho}{2}\frac{Q_0^2}{\pi^2}\int_{-\infty}^{\infty}\frac{y^2}{(a^2+y^2)^2}\,\mathrm{d}y = \frac{\rho}{2}\frac{Q_0^2}{\pi^2}\frac{\pi}{2a} = \frac{\rho Q^2}{4\pi a}.$$

TODO: directions. "Force on the wall is positive, i.e. in the direction of the source. The force on the source is thus towards the wall. The source is thus attracted to the wall".

- Uses the integral

$$\int_{-\infty}^{\infty}\frac{x^2}{(a^2+x^2)^2}\,\mathrm{d}x = -\frac{x}{2(a^2+x^2)} + \frac{1}{2a}\arctan\frac{x}{a}.$$

3.2.11 Exercise: Flow Around a Cylinder with Additional Circulation

Flow around a cylinder, with additional circulation around cylinder from a line vortex.

- Flow around a cylinder (or a 2D cross section in the plane) is given by complex potential in Equation 3.9, which for review reads

$$w(z) = v_0\left(z + \frac{a^2}{z}\right) = v_0\left(re^{i\varphi} + \frac{a^2}{r}e^{-i\varphi}\right).$$

- Take gradient of real component of potential. We find velocity field from

$$v_r = \frac{\mathrm{d}}{\mathrm{d}r}\operatorname{Re} w = v_0 \frac{\mathrm{d}}{\mathrm{d}r}\left(r + \frac{a^2}{r}\right)\cos\varphi = v_0\left(1 - \frac{a^2}{r^2}\right)\cos\varphi \tag{3.10}$$

  Note that $v_r = 0$ for all φ at the cylinder's surface $r = a$; this makes sense, because fluid cannot flow through the cylinder. The velocity's azimuthal component is

$$v_\varphi = \frac{1}{r}\frac{\partial}{\partial\varphi}\operatorname{Re} w = \frac{v_0}{r}\left(r + \frac{a^2}{r}\right)\frac{\partial}{\partial\varphi}\cos\phi = -v_0\left(1 + \frac{a^2}{r^2}\right)\sin\varphi. \tag{3.11}$$

Adding Vortex Line Circulation

- Next, we add the potential of an ideal line vortex with circulation Γ centered on the cylinder's longitudinal. For review from Equation 3.8, this potential is

$$w_\Gamma(z) = \frac{\Gamma_0}{2\pi i}\ln z = \frac{\Gamma_0}{2\pi}\varphi - i\frac{\Gamma_0}{2\pi}\ln r.$$

  The corresponding real potential and stream function are

$$\phi = \frac{\Gamma_0}{2\pi}\varphi \qquad \text{and} \qquad \psi = -\frac{\Gamma_0}{2\pi}\ln r,$$

  and, from $\boldsymbol{v} = \nabla\phi$, the associated velocity field is

$$\boldsymbol{v}_\Gamma = \nabla\phi = \frac{\Gamma_0}{2\pi}\nabla\varphi = \frac{\Gamma}{2\pi r}\hat{\mathbf{e}}_\varphi.$$

  In practice, the extra circulation around the cylinder comes from rotation of the cylinder in a real (and thus viscious) fluid.

- We then add the contributions from the vortex line and the flow around the cylinder in Equations 3.10 and 3.11 to get

$$\boldsymbol{v}_{\mathrm{tot}} = \boldsymbol{v}_{\mathrm{cyl}} + \boldsymbol{v}_\Gamma = v_0\left(1 - \frac{a^2}{r}\right)\cos\varphi\,\hat{\mathbf{e}}_r + \left[\frac{\Gamma}{2\pi r} - v_0\left(1 + \frac{a^2}{r^2}\right)\sin\varphi\right]\hat{\mathbf{e}}_\varphi$$

  We aim to find force on the cylinder, and to do this we need the pressure on the cylinder's surface $r = a$. Note that $v_r = 0$ at $r = a$, so we will be interested in only v_φ, which at $r = a$ reads

$$v_\varphi(r)\big|_{r=a} = \frac{\Gamma}{2\pi a} - 2v_0\sin\varphi. \tag{3.12}$$

- We then recall the Bernoulli equation for incompressible, potential, isentropic flow from Equation **TODO:** reference; for review this reads

$$\nabla\left(\frac{v^2}{2} + \frac{p}{\rho}\right) = 0 \implies \frac{p}{\rho} + \frac{v^2}{2} = \frac{v_0^2}{2},$$

  where v_0 is the homogeneous reference fluid speed far from the cylinder and p and v are the fluid pressure and velocity immediately at the cylinder's surface.

  Recall from **TODO:** reference that the Bernoulli equation applies everywhere for irrotational velocity fields (and not just along given streamlines). This problem's velocity field is irrotational (except along the hypothetical vortex line itself), so the Bernoulli equation applies everywhere.

- The corresponding pressure at $r = a$ is, using $v_\varphi(a)$ from Equation 3.12, is

$$p = \frac{\rho}{2}(v_0^2 - v^2) = \frac{\rho}{2}\left[v_0^2 - \left(\frac{\Gamma}{2\pi a}\right)^2 - 4v_0^2\sin^2\varphi + \frac{2\Gamma v_0}{\pi a}\sin\varphi\right] = 0.$$

Finding Force

- The force on the cylinder per unit cylinder length is

$$\frac{\boldsymbol{F}}{L} = -\oint \hat{\mathbf{n}}\,\mathrm{d}l$$

We first find the force in the direction of the homogeneous flow $\boldsymbol{v} = v_0\,\hat{\mathbf{e}}_x$ far from the cylinder. This is

$$\begin{aligned}\frac{F_x}{L} &= -\int_0^{2\pi} a\cos(\varphi)\cdot p(\varphi)\,\mathrm{d}\varphi\\ &= -\frac{\rho a}{2}\int_0^{2\pi}\left[v_0^2 - \left(\frac{\Gamma}{2\pi a}\right)^2 - 4v_0^2\sin^2\varphi + \frac{2\Gamma v_0}{\pi a}\sin\varphi\right]\cos\varphi\,\mathrm{d}\varphi\end{aligned}$$

Functions of the form $A\cos\varphi$, $A\sin^2\varphi\cos\varphi$ and $\sin\varphi\cos\varphi$ are all zero when integrated over the full period $\varphi \in [0, 2\pi]$, leaving

$$\frac{F_x}{L} = 0.$$

There is no force in the direction of steady flow!

- Meanwhile, the force in the y direction is

$$\begin{aligned}\frac{F_y}{L} &= -\int_0^{2\pi} a\sin(\varphi)\cdot p(\varphi)\,\mathrm{d}\varphi\\ &= -\frac{\rho a}{2}\int_0^{2\pi}\left[v_0^2 - \left(\frac{\Gamma}{2\pi a}\right)^2 - 4v_0^2\sin^2\varphi + \frac{2\Gamma v_0}{\pi a}\sin\varphi\right]\sin\varphi\,\mathrm{d}\varphi\end{aligned}$$

All terms integrate to zero over a full period $\varphi \in [0, 2\pi]$ except for $\frac{2\Gamma v_0}{\pi a}\sin^2\varphi$, which comes out to

$$\frac{F_y}{L} = -\frac{\rho\Gamma v_0}{\pi}\int_0^{2\pi}\sin^2\varphi\,\mathrm{d}\varphi = -\frac{\rho v_0\Gamma}{\pi}\cdot\pi = -\rho v_0\Gamma$$

TODO: interpretation: this is just dynamic buoyancy, or lift. This is the same physics as what makes airplanes fly!

3.3 Viscous Fluids

3.3.1 Theory: Review

- The Euler equation, when including a viscous stress tensor $p^{(\mathrm{v})}$, reads

$$\rho\left[\frac{\partial \boldsymbol{v}}{\partial t}+(\boldsymbol{v}\cdot\nabla)\boldsymbol{v}\right]=-\nabla p+\nabla\cdot p^{(\mathrm{v})}+\boldsymbol{f},$$

The viscous stress tensor is given by

$$p_{ij}^{(\mathrm{v})}=2\eta\left(v_{ij}-\frac{1}{3}v_{kk}\delta_{ij}\right)+\zeta v_{kk}\delta_{ij},$$

where η is dynamic viscosity and ζ is volume viscosity. As in **TODO:** reference the velocity strain tensor reads

$$v_{ij}=\frac{1}{2}\left(\frac{\partial v_j}{\partial x_i}+\frac{\partial v_i}{\partial x_j}\right)$$

- The Navier-Stokes equation is

$$\rho\left[\frac{\partial \boldsymbol{v}}{\partial t}+(\boldsymbol{v}\cdot\nabla)\boldsymbol{v}\right]=-\nabla p+\eta\nabla^2\boldsymbol{v}+\left(\zeta+\frac{1}{3}\eta\right)\nabla(\nabla\cdot\boldsymbol{v})+\boldsymbol{f}.$$

For an incompressible fluid in which $\nabla\cdot\boldsymbol{v}=0$, the Navier-Stokes equation simplifies to

$$\rho\left[\frac{\partial \boldsymbol{v}}{\partial t}+(\boldsymbol{v}\cdot\nabla)\boldsymbol{v}\right]=-\nabla p+\eta\nabla^2\boldsymbol{v}+\boldsymbol{f}. \tag{3.13}$$

We will mostly work with this incompressible form of the Navier-Stokes equation for the remainder of this course.

- Viscous relaxation time τ_{v} (time in which, because of viscous effects, there is established a stationary velocity flow) is estimated from the equality

$$\left|\rho\frac{\partial \boldsymbol{v}}{\partial t}\right|\sim\eta\left|\nabla^2\boldsymbol{v}\right|.$$

The result for viscous relaxation time is

$$\tau_{\mathrm{v}}=\frac{\rho L^2}{\eta},$$

where L is a characteristic length dimension of the velocity field.

- The Reynolds number is the ratio of the magnitude of the advective component (force density, required to accelerate a portion of fluid through a velocity field gradient) and the magnitude of the viscious term (density of viscious forces). The Reynolds number is

$$\mathrm{Re}=|\rho(\boldsymbol{v}\cdot\nabla)\boldsymbol{v}|\cdot\frac{1}{|\eta\nabla^2\boldsymbol{v}|}.$$

For a system with a single spatial scale L and a characteristic velocity v, the above definition reduces to the more familiar expression

$$\mathrm{Re}=\frac{\rho v L}{\eta}.$$

- Next, ratio of characteristic times for (i) changing velocity field and (ii) traversing the velocity gradient is

$$\frac{\tau_{\boldsymbol{v}}}{\tau_{\nabla \boldsymbol{v}}} = |(\boldsymbol{v} \cdot \nabla)\boldsymbol{v}| \cdot \left|\frac{\partial \boldsymbol{v}}{\partial t}\right|^{-1}$$

  This ratio equals the Reynolds number if: the fluid dynamics are not externally driven, but arise only from the internal dynamics of the viscous fluid (with characteristic time τ_{v}).

- If fluid dynamics are externally driven...

  assign a characteristic driving time τ_{ext}, e.g. $\tau_{\mathrm{ext}} = 1/\omega$ for a periodic driving force. In this case we define the *Strouhal number*

$$\mathrm{St} = \left|\frac{\partial \boldsymbol{v}}{\partial t}\right| \cdot \frac{1}{|(\boldsymbol{v} \cdot \nabla)\boldsymbol{v}|} \sim \frac{L}{\tau_{\mathrm{ext}} v}.$$

  This is really

$$\mathrm{St} = \frac{\tau_{\nabla \boldsymbol{v}}}{\tau_{\mathrm{ext}}}$$

  i.e. ratio of characteristic time to traverse velocity gradient and the characteristic time with which the velocity field changes because of the external driving force.

  Note that $\mathrm{St} \gg 1$ corresponds to highly unstationary flow.

- If $\tau_{\mathrm{ext}} = \tau_{\mathrm{v}}$, then $\mathrm{St} = 1/\mathrm{Re}$. See this from

$$\mathrm{St} = \frac{L}{\tau v} = \frac{L}{\tau_{\mathrm{v}} v} = \frac{L}{v} \frac{\eta}{\rho L^2} = \frac{\eta}{v \rho L} = \frac{1}{\mathrm{Re}}.$$

Stokes Approximation

- The Stokes approximation or Stokes flow is an approximation of the Navier-Stokes equation for low Reynolds numbers. It reads

$$\mathbf{0} = -\nabla p + \eta \nabla^2 \boldsymbol{v} + \boldsymbol{f}.$$

- For the Navier-Stokes equation (i) without external forces and (ii) for constant density, we reach the Helmholtz equation for vorticity in a viscous fluid

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{\omega} = (\boldsymbol{\omega} \cdot \nabla)\boldsymbol{v} + \frac{\eta}{\rho} \nabla^2 \boldsymbol{\omega}.$$

  TODO: derive.

3.3.2 Exercise: Power Dissipation of a Rotating Cylinder in a Viscous Fluid

A long cylinder rotates with angular frequency Ω in a large body of viscous oil with dynamic viscosity η and uniform density ρ. Find difference in oil pressure near cylinder's surface and reference state. Find power to rotate the cylinder.

- Assume a tangentail ansatz for fluid velocity around the rotating cylinder.

$$\boldsymbol{v} = v(r)\,\hat{\mathbf{e}}_\varphi$$

Density is uniform, so flow may be treated as incompressible. Begin with general Navier-Stokes equation for incompressible flow (Eq. 3.13), which for review reads

$$\rho\left[\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{v}\right] = -\nabla p + \eta\nabla^2\boldsymbol{v} + \boldsymbol{f}.$$

For our problem: no external force so $\boldsymbol{f} = \mathbf{0}$.
Find stationary state solution, so $\frac{\partial \boldsymbol{v}}{\partial t} = \mathbf{0}$.
Navier-Stokes equation simplifies to

$$\rho\,(\boldsymbol{v}\cdot\nabla)\,\boldsymbol{v} = -\nabla p + \eta\nabla^2\boldsymbol{v} \tag{3.14}$$

- Next, we will take advantage of the fact that $\nabla\cdot\boldsymbol{v} = 0$ in an incompressible fluid. Use the vector identity

$$\nabla(\nabla\cdot\boldsymbol{v}) = \nabla^2\boldsymbol{v} + \nabla\times(\nabla\times\boldsymbol{v}),$$

then apply $\nabla\cdot\boldsymbol{v} = 0$ and rearrange to get

$$\nabla^2\boldsymbol{v} = -\nabla\times(\nabla\times\boldsymbol{v}).$$

Substitute into Equation 3.14 and get the further simplified Navier-Stokes equation

$$\rho(\boldsymbol{v}\cdot\nabla)\boldsymbol{v} = -\nabla p - \eta\nabla\times(\nabla\times\boldsymbol{v}). \tag{3.15}$$

- In cylindrical coordinates curl reads

$$\nabla\times\boldsymbol{v} = \left(\frac{1}{r}\frac{\partial v_z}{\partial\varphi} - \frac{\partial v_\varphi}{\partial z}\right)\hat{\mathbf{e}}_z + \left(\frac{\partial v_r}{\partial z} - \frac{\partial v_z}{\partial r}\right)\hat{\mathbf{e}}_\varphi + \left(\frac{1}{r}\frac{\partial(r v_\varphi)}{\partial r} - \frac{1}{r}\frac{\partial v_r}{\partial\varphi}\right)\hat{\mathbf{e}}_z$$

For this problem's simple ansatz $\boldsymbol{v} = v(r)\,\hat{\mathbf{e}}_\varphi$, only the $\hat{\mathbf{e}}_z$ term is non-zero:

$$\nabla\times\boldsymbol{v} = \frac{1}{r}\frac{\partial(rv)}{\partial r}\,\hat{\mathbf{e}}_z.$$

We then take one more curl to compute $\nabla\times(\nabla\times\boldsymbol{v})$. This time only the $\frac{\partial v_z}{\partial r}\,\hat{\mathbf{e}}_\varphi$ term is non-zero, producing

$$\nabla\times(\nabla\times\boldsymbol{v}) = -\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial(rv)}{\partial r}\right]\hat{\mathbf{e}}_\varphi.$$

The advective derivative, in cylindrical coordinates for the simple ansatz $\boldsymbol{v} = v(r)\,\hat{\mathbf{e}}_\varphi$, comes out to

$$(\boldsymbol{v}\cdot\nabla)\boldsymbol{v} = (v_\varphi\,\hat{\mathbf{e}}_\varphi)\cdot\frac{\hat{\mathbf{e}}_\varphi}{r}v_\varphi\frac{\partial\,\hat{\mathbf{e}}_\varphi}{\partial\varphi} = -\frac{v^2}{r}\,\hat{\mathbf{e}}_r.$$

Finally, we compute ∇p. Assume $p = p(r)$ and get simply

$$\nabla p = \frac{\partial p}{\partial r}\,\hat{\mathbf{e}}_r.$$

- Substitute results into Navier-Stokes equation in Eq. 3.15 to get

$$-\frac{\rho v^2}{r}\,\hat{\mathbf{e}}_r = -\frac{\partial p}{\partial r}\,\hat{\mathbf{e}}_r + \eta\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial(rv)}{\partial r}\right]\hat{\mathbf{e}}_\varphi. \tag{3.16}$$

  Result: two independent equations for the r and φ components.

- First take azimuthal component to get

$$\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial(rv)}{\partial r}\right] = 0$$

  Integrate twice over r to get

$$v(r) = \frac{A}{2}r + \frac{B}{r}.$$

  As a first boundary conditions, we require that velocity converges as $r \to \infty$, requiring $A = 0$. Velocity field reduces to

$$v(r) = \frac{B}{r}.$$

  Second boundary condition: $v(R) = \Omega R$, i.e. that fluid velocity at cylinder surface equals orbital speed at cylinder surface. This condition produces $B = \Omega R^2$. The problem's velocity field is thus

$$\boldsymbol{v}(r) = \frac{\Omega R^2}{r}\,\hat{\mathbf{e}}_\varphi$$

- Next take radial component of Equation 3.16 and substitute in $v(r) = \Omega R^2/r$ to get

$$\frac{\rho v^2}{r} = \frac{\partial p}{\partial r} \implies \frac{\partial p}{\partial r} = \rho(\Omega R^2)^2\frac{1}{r^3} \implies p(r) = C - \frac{\rho}{2}\frac{\Omega^2 R^4}{r^2}$$

  Apply boundary condition $p(r \to \infty) = p_0$ to get $C = p_0$ and thus

$$p(r) = p_0 - \frac{\rho}{2}\frac{\Omega^2 R^4}{r^2}$$

Method One For Power

- We first compute power directly from the torque of the cylinder rotatining in the viscous fluid. In a stationary state, the torque of the cylinder on the fluid is equal in magnitude to the viscous torque of the fluid on the cylinder, which is determined by the viscous stress tensor

$$p_{ij}^{(\mathrm{v})} = 2\eta\left(v_{ij} - \frac{1}{3}v_{kk}\delta_{ij}\right) + \zeta v_{kk}\delta_{ij}.$$

  Ignore ζ term in an incompressible fluid. In this problem, the viscous torque arises from forces in the tangential $\hat{\mathbf{e}}_\varphi$ direction acting on the radial $\hat{\mathbf{e}}_{\boldsymbol{r}}$ surface, so we compute the viscous torque from the stress tensor component

$$p_{\varphi r}^{(\mathrm{v})} = 2\eta v_{\varphi r}$$

For review from **TODO:** reference in cylindrical coordinates the component $v_{\varphi r}$ is

$$v_{\varphi r} = \frac{1}{2}\left(\frac{\partial v_\varphi}{\partial r} - \frac{v_\varphi}{r} + \frac{1}{r}\frac{\partial v_r}{\partial \varphi}\right)$$

Using TODO $v_\varphi(r) = \Omega R^2/r$ and $v_r = 0$, this simplifies to

$$v_{\varphi r} = -\frac{\Omega R^2}{2}\left(\frac{1}{r^2} + \frac{1}{r^2}\right) \implies p_{\varphi r}^{(\mathrm{v})} = 2\eta v_{\varphi r} = -2\eta\Omega R^2 \frac{1}{r^2}$$

- The φ component of force at $r = R$ is

$$\mathrm{d}F_\varphi = p_{\varphi r}^{(\mathrm{v})}\,\mathrm{d}S = p_{\varphi r}^{(\mathrm{v})} RL\,\mathrm{d}\varphi \implies \frac{F_\varphi}{L} = 2\pi R p_{\varphi r}^{(\mathrm{v})}.$$

We then substitute in $p_{\varphi r}^{(\mathrm{v})}$ to get

$$\frac{F_\varphi}{L} = 2\pi R \cdot \left(-2\eta\Omega R^2 \frac{1}{R^2}\right) = -4\pi\eta\Omega R.$$

The corresponding torque per unit length at the cylinder's surface (noting that the radial lever arm is perpendicular to the tangential displacement, so that $M = F_\varphi R$) is simply

$$\frac{|M|}{L} = \frac{|F|}{L}R = 4\pi\eta\Omega R^2.$$

The corresponding power, using the same principles as the simple Newtonian formula $P = \Omega M$, is just

$$\frac{P}{L} = \frac{M}{L}\Omega = 4\pi\eta\Omega^2 R^2.$$

Finding Dissipated Power from Viscous Energy Losses

- We now find the power associated with viscous energy dissipation throughout the entire fluid volume, and check that the result agrees with the mechanically-derived result in **TODO:** ref.

- From first principles, the power dissipated by the work of viscous forces encoded by a stress tensor $p_{ij}^{(\mathrm{v})}$ is

$$P = \iiint p_{ij}^{(\mathrm{v})} v_{ij}\,\mathrm{d}V = 2\eta \iiint v_{ij}v_{ij}\,\mathrm{d}V$$

The corresponding power dissipation per unit length is

$$\frac{P}{L} = 2\eta \iint v_{ij}v_{ij}\,\mathrm{d}S = 2\eta \iint \operatorname{tr}\mathsf{v}^2\,\mathrm{d}S.$$

Dropping z terms, which do not contribute in the cross section $\mathrm{d}S$, the trace $\operatorname{tr}\mathsf{v}^2$ in cylindrical coordinates reads

$$\operatorname{tr}\mathsf{v}^2 = v_{ij}v_{ij} = v_{rr}^2 + v_{\varphi\varphi}^2 + v_{r\varphi}^2 + v_{\varphi r}^2,$$

We first recall TODO

$$v_{\varphi r} = v_{\varphi r} = -\Omega\frac{R^2}{r^2}.$$

Meanwhile for $\boldsymbol{v}(\boldsymbol{r}) = \frac{\Omega R^2}{r}\,\hat{\mathbf{e}}_\varphi$, we have

$$v_{rr} = \frac{\partial v_r}{\partial r} = 0 \qquad \text{and} \qquad v_{\varphi\varphi} = \frac{1}{r}\frac{\partial v_\varphi}{\partial \varphi} + \frac{v_r}{r} = 0.$$

Desired trace is then

$$\operatorname{tr} \mathsf{v}^2 = v_{r\varphi}^2 + v_{\varphi r}^2 = 2\left(\Omega \frac{R^2}{r^2}\right)^2$$

- Then

$$\frac{P}{L} = 2\eta \iint \operatorname{tr} \mathsf{v}^2 \,\mathrm{d}S = 4\pi\eta \int_R^\infty \operatorname{tr} \mathsf{v}^2 r \,\mathrm{d}r$$

 Substitute in $\operatorname{tr} \mathsf{v}^2$ and get

$$\begin{aligned}\frac{P}{L} &= 4\pi\eta \int_R^\infty 2\left(\Omega\frac{R^2}{r^2}\right)^2 r\,\mathrm{d}r = 8\pi\eta\Omega^2 R^4 \int_R^\infty \frac{\mathrm{d}r}{r^3} \\ &= 4\pi\eta\Omega^2 R^2\end{aligned}$$

 This agrees with **TODO:** blah blah.

 Lesson: we were able to find the power dissipated by viscous forces acting on the cylinder from the viscous energy dissipation in the entire fluid volume.

3.3.3 Exercise: An Oscillatory Shear Deformation (Reometer)

A reometer is an instrument used to measure the frequency dependence of fluids' mechanical properties. We model a simple reometer as a two large parallel plates enclosing a fluid, which we wish to measure, with density ρ and viscosity η. On plane oscillates relative to the other with angular frequency $\omega = 2\pi\nu$.
Compute the fluid's time-dependent velocity field.
Compute the resulting shear deformation's skin depth.
How does phase shift (of? oscillation?) change with distance from the oscillating plate.
Assume large plate separation compared to skin depth.

- Decide coordinate system: Cartesian coordinate system. Plates are parallel to the xy plane. Let upper, oscillating plate occur at $z = 0$. Let z axis point "down", i.e. from oscillating top plate to bottom plate. Let the plate oscillate along the x axis.

- Assume velocity field takes the general form

$$\boldsymbol{v}(\boldsymbol{r}) = v(z,t)\,\hat{\mathbf{e}}_x$$

 Velocity varies with distance z from oscillating plate and points only in the direction of oscillation. Assuming $\boldsymbol{v} \parallel \hat{\mathbf{e}}_x$ is valid as long as the shear deformation is weak enough that neighboring velocity planes don't "buckle".

- Begin with Navier-Stokes equation for incompressible fluids REF

$$\rho\left[\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{v}\right] = -\nabla p + \eta\nabla^2\boldsymbol{v}$$

 Note that we don't need to assume incompressible fluid for this simple form $\nabla\cdot\boldsymbol{v} = 0$ holds simply from the ansatz $\boldsymbol{v}(\boldsymbol{r}) = v(z,t)\,\hat{\mathbf{e}}_x$, since $v_x \neq v_x(x)$.

Other terms simplify too. Writing all steps out (feel free to skip):

$$\begin{aligned}(\boldsymbol{v}\cdot\nabla)\boldsymbol{v} &= v(z,t)\,\hat{\mathbf{e}}_x\cdot\left(\hat{\mathbf{e}}_x\frac{\partial}{\partial x}+\hat{\mathbf{e}}_y\frac{\partial}{\partial y}+\hat{\mathbf{e}}_z\frac{\partial}{\partial z}\right)v(z,t)\,\hat{\mathbf{e}}_x\\ &= v(z,t)\,\hat{\mathbf{e}}_x\cdot\hat{\mathbf{e}}_z\frac{\partial v(z,t)}{\partial z}\,\hat{\mathbf{e}}_x\\ &= \mathbf{0}.\end{aligned}$$

Next, because of translational symmetry in the x and y directions (and assuming no weight), we have $\nabla p = \mathbf{0}$. We are left with only a simplified x component version of **TODO:** eq, which reads

$$\rho\frac{\partial v}{\partial t} = \eta\nabla^2 v.$$

Assume an oscillatory ansatz for v of the form

$$v(z,t) = v_0 e^{i(kz-\omega t)}.$$

Substitute in above and simplify to get

$$-i\omega\rho v_0 e^{i(kz-\omega t)} = -\eta k^2 v_0 e^{i(kz-\omega t)} \implies -i\omega\rho = -\eta k^2.$$

In terms of plate oscillation frequency ω, the two possible wave vector solutions are

$$k_\pm = \pm\sqrt{\frac{\rho\omega}{\eta}}\sqrt{i} = \pm\sqrt{\frac{\rho\omega}{\eta}}\cdot\frac{1}{\sqrt{2}}(1+i) \equiv \pm(k_{\mathrm{Re}}+ik_{\mathrm{Im}}),$$

where we have defined $k = k_{\mathrm{Re}} + ik_{\mathrm{Im}}$ where

$$k_{\mathrm{Re}} = k_{\mathrm{Im}} = \pm\sqrt{\frac{\rho\omega}{2\eta}}(1+i).$$

The solution for $v(z,t)$ in Equation **TODO:** reference is then

$$\begin{aligned}v(z,t) &= v_0 e^{i(k_+ z-\omega t)} + v_0' e^{i(k_- z-\omega t)}\\ &= v_0 e^{i(k_{\mathrm{Re}}z-\omega t)}e^{-k_{\mathrm{Im}}z} + v_0' e^{i(-k_{\mathrm{Re}}z-\omega t)}e^{k_{\mathrm{Im}}z}.\end{aligned}$$

The corresponding skin depth (or characteristic attentuation length for the shear oscillations in the fluid) is

$$\xi = \frac{1}{k_{\mathrm{Im}}} = \sqrt{\frac{2\eta}{\rho\omega}}.$$

- If there is no lower plate, or the lower plate is far away relative to the skin depth ξ, then we require $v_0' \to 0$ to avoid non-physical divergence of $v(z,t)$ for large z. Even if the lower plate occurs at finite $z = h$, but $h \gg \xi$, then we must have a solution with $v_0' \ll v_0$, again to avoid non-physical divergence of $v(z,t)$ as $z \to h$. Assume v_0', so solution is

$$v(z,t) = v_0 e^{i(k_{\mathrm{Re}}z-\omega t)}e^{-i\frac{z}{\xi}}.$$

- The phase shift profile is

$$\Delta\phi(z) = k_{\mathrm{Re}}z = \sqrt{\frac{\rho\omega}{2\eta}}z.$$

This is phase shift between oscillation of the oscillating plate and the fluid located at a distance z from the oscillating plate.

3.3.4 Exercise: Vortex Boundary Layer Along a Moving Plate

Viscous liquid in the region $z > 0$, bounded by the xy plane at $z = 0$. The xy plane begins to move at speed v_0 along the x axis. Determine the vorticity of the boundary layer with respect to time.

- When the plate just begins to move at $t = 0$, fluid is at rest everywhere except for an infinitesimally thin layer just above the plate, which moves with the plate speed v_0. The corresponding vorticity $\boldsymbol{\omega} = \nabla \times \boldsymbol{v}$ is zero everywhere except in the xy plane, where it is undefined (or infinite).
- Write velocity at

$$\boldsymbol{v} = v(z,t)\,\hat{\mathbf{e}}_x \implies \boldsymbol{\omega} = \nabla \times \boldsymbol{v} = \frac{\partial v(z,t)}{\partial z}\,\hat{\mathbf{e}}_y \equiv \omega\,\hat{\mathbf{e}}_y.$$

- Begin with Helmholtz equation for vorticity REFERENCE and get

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{\omega} = (\boldsymbol{\omega}\cdot\nabla)\boldsymbol{v} + \frac{\eta}{\rho}\nabla^2\boldsymbol{\omega}.$$

  From $\boldsymbol{\omega} = \omega(z,t)\,\hat{\mathbf{e}}_y$ and $\boldsymbol{v} = v(z,t)\,\hat{\mathbf{e}}_x$, we have

$$(\boldsymbol{v}\cdot\nabla)\boldsymbol{\omega} = \mathbf{0} \qquad \text{and} \qquad (\boldsymbol{\omega}\cdot\nabla)\boldsymbol{v} = \mathbf{0}.$$

  The Helmholtz equation simplifies to

$$\frac{\partial \boldsymbol{\omega}}{\partial t} - \frac{\eta}{\rho}\nabla^2\boldsymbol{\omega} = \frac{\partial w}{\partial t}\,\hat{\mathbf{e}}_y - \frac{\eta}{\rho}\frac{\partial^2 \omega}{\partial z^2}\,\hat{\mathbf{e}}_y = 0$$

  Only the y component remains; this is the one-dimensional diffusion equation

$$\frac{\partial \omega}{\partial t} - \frac{\eta}{\rho}\frac{\partial^2 \omega}{\partial z^2} \equiv \frac{\partial \omega}{\partial t} - D\frac{\partial^2 \omega}{\partial t^2} = 0, \quad \text{where } D \equiv \frac{\eta}{\rho}.$$

- The initial vorticity reads

$$\omega(z, t=0) = \left(\frac{\partial v(z,t)}{\partial z}\right)_{t=0} = -v_0\delta(z).$$

  The delta function $\delta(z)$ has units length^{-1}.

  Add initial conditino into diffusion equation by including $\delta(t)$. This gives

$$\frac{\partial \omega}{\partial t} - D\frac{\partial^2 \omega}{\partial t^2} = -v_0\delta(z)\delta(t).$$

  For $z \in (-\infty, \infty)$, this equation is solved by

$$\omega(z,t) = -\frac{v_0}{\sqrt{4\pi D t}}\exp\left(-\frac{z^2}{4Dt}\right).$$

  We are interested only in the region $z \in [0, \infty)$.

  Goal: modify diffusion equation initial condition so we can use $z \in (-\infty, \infty)$ solution for our $z \in [0, \infty)$ problem. Do this by reflecting the $z > 0$ solution about the xy plane. This produces a jump of $-2v_0$ in velocity across the plane $z = 0$, and Equation **TODO:** reads

$$\frac{\partial \omega}{\partial t} - D\frac{\partial^2 \omega}{\partial z^2} = -2v_0 \delta(z)\delta(t).$$

The coresponding solution for diffusion of vorticity is

$$\omega(z,t) = -\frac{2v_0}{\sqrt{4\pi D t}} \exp\left(-\frac{z^2}{4Dt}\right).$$

- The boundary layer's characteristic depth iz

$$\xi^2 = 4Dt = \frac{4\eta}{\rho}t \implies \xi = \sqrt{\frac{4\eta}{\rho}}.$$

Solution With Navier-Stokes Equation

- Begin with Navier-Stokes equation for an incompressible fluid

$$\rho\left[\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{v}\right] = -\nabla p + \eta\nabla^2\boldsymbol{v}$$

Our problem has no pressure gradient, while $(\boldsymbol{v}\cdot\nabla)\boldsymbol{v} = \mathbf{0}$, leaving

$$\frac{\partial \boldsymbol{v}}{\partial t} - \frac{\eta}{\rho}\nabla^2\boldsymbol{v} \equiv \frac{\partial \boldsymbol{v}}{\partial t} - D\nabla^2\boldsymbol{v} = \mathbf{0}.$$

For the $\hat{\mathbf{e}}_x$ component, this simplifies to

$$\frac{\partial v}{\partial t} - D\frac{\partial^2 v}{\partial z^2} = 0$$

Boundary conditions are $v(z=0,t) = v_0$ and $v(z\to\infty, t) = 0$. Initial condition is $v(z>0, t=0) = 0$.

- Since it is uncertain how to proceed, we are given a hint: use the new variable

$$\zeta \equiv \frac{z}{\sqrt{4Dt}}.$$

We first compute derivatives in terms of the new variable:

$$\begin{aligned}
\frac{\partial}{\partial t} &= \frac{\partial \zeta}{\partial t}\frac{\partial}{\partial \zeta} = -\frac{1}{2}\frac{z}{\sqrt{4D}}\frac{1}{t^{3/2}}\frac{\partial}{\partial \zeta}\\
\frac{\partial}{\partial z} &= \frac{\partial \zeta}{\partial z}\frac{\partial}{\partial \zeta} = \frac{1}{\sqrt{4Dt}}\frac{\partial}{\partial \zeta}\\
\frac{\partial^2}{\partial z^2} &= \frac{\partial}{\partial z}\left(\frac{1}{\sqrt{4Dt}\frac{\partial}{\partial \zeta}}\right) = \frac{1}{\sqrt{4Dt}}\frac{\partial}{\partial z}\frac{\partial}{\partial \zeta} = \frac{1}{4Dt}\frac{\partial^2}{\partial \zeta^2}.
\end{aligned}$$

Equation NS **TODO:** simplifies to

$$-\frac{1}{2}\frac{z}{\sqrt{4D}}\frac{1}{t^{3/2}}\frac{\mathrm{d}v}{\mathrm{d}\zeta} - D\frac{1}{4Dt}\frac{\mathrm{d}^2 v}{\mathrm{d}\zeta^2} = 0 \implies \frac{1}{4Dt}\frac{\partial^2}{\partial \zeta^2}.$$

We then recognize the original definition of ζ and introduce the variable $w \equiv \frac{\mathrm{d}v}{\mathrm{d}\zeta}$ to produce the straightforward, separable ordinary differential equation.

$$\frac{\mathrm{d}w}{\mathrm{d}\zeta} = -2\zeta w \implies \frac{\mathrm{d}w}{w} = -2\zeta \,\mathrm{d}\zeta.$$

We then integrate both sides to get

$$\ln \frac{w}{w_0} = -\zeta^2 \implies w = w_0 e^{-z^2}.$$

Next, we recognize

$$w = \frac{\mathrm{d}v}{\mathrm{d}\zeta} = \frac{\mathrm{d}v}{\mathrm{d}z}\frac{\mathrm{d}z}{\mathrm{d}\zeta} = \sqrt{4Dt}\,\frac{\mathrm{d}v}{\mathrm{d}z}$$

And thus

$$\frac{\mathrm{d}v}{\mathrm{d}z} = \frac{C}{\sqrt{4Dt}} \exp\left(-\frac{z^2}{4Dt}\right) \equiv \frac{C'}{\sqrt{4\pi Dt}} \exp\left(-\frac{z^2}{4Dt}\right)$$

Then integrate over $z \in [0, \infty)$ and apply $v(z = 0, t) = v_0$ and $v(z \to \infty, t) = 0$ to get

$$v(\infty) - v(0) = -v_0 = \int_0^\infty \frac{\mathrm{d}v}{\mathrm{d}z}\,\mathrm{d}z = \int_0^\infty \frac{C'}{\sqrt{4\pi Dt}} \exp\left(-\frac{z^2}{4Dt}\right)\mathrm{d}z = \frac{C'}{2}.$$

We thus conclude $C' = -2v_0$, giving

$$\frac{\mathrm{d}v}{\mathrm{d}z} = \frac{-2v_0}{\sqrt{4\pi Dt}} \exp\left(-\frac{z^2}{4Dt}\right)$$

This agrees with **TODO:** reference.

3.3.5 Exercise: Time Evolution of a Line Vortex

Consider velocity field of an ideal line vortex with circulation Γ at $t = 0$. Determine how the velocity field changes with time in a fluid with viscocity η.

- From **TODO:** reference the ideal vortex line's velocity field is

$$\boldsymbol{v}(\boldsymbol{r}) = \frac{\Gamma}{2\pi r}\,\hat{\mathbf{e}}_\varphi$$

This velocity field's corresponding vorticity $\boldsymbol{\omega} = \nabla \times \boldsymbol{v}$ is

$$\boldsymbol{\omega}(\boldsymbol{r}) = \Gamma\delta^2(\boldsymbol{r})\,\hat{\mathbf{e}}_z = \Gamma\delta(x)\delta(y)\,\hat{\mathbf{e}}_z,$$

where the delta function $\delta^2(\boldsymbol{r})$ is centered at the (two-dimensional) origin $(0, 0, z)$ and is understood to have units length^{-2}. We will solve the problem in terms of vorticity $\boldsymbol{\omega} = \nabla \times \boldsymbol{v}$. We begin with the Helmholtz equation for vorticity from **TODO:** reference

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{\omega} = (\boldsymbol{\omega} \cdot \nabla)\boldsymbol{v} + \frac{\eta}{\rho}\nabla^2\boldsymbol{\omega}. \tag{3.17}$$

It turns out this equation simplifies considerably for this problem's simple velocity field $\boldsymbol{v}(\boldsymbol{r}) = v_\varphi(r)\,\hat{\mathbf{e}}_\varphi$. The velocity field is independent of the z coordinate, so the corresponding vorticity points purely along the z axis, i.e.

$$\boldsymbol{\omega} = \nabla \times \boldsymbol{v} = \omega\,\hat{\mathbf{e}}_z$$

Because $\boldsymbol{\omega} \parallel \hat{\mathbf{e}}_z$ and $\boldsymbol{v} \parallel \hat{\mathbf{e}}_\varphi$ we have both $(\boldsymbol{\omega} \cdot \nabla)\boldsymbol{v} = \mathbf{0}$ and $(\boldsymbol{v} \cdot \nabla)\boldsymbol{\omega} = \mathbf{0}$, and so Equation 3.17 simplifies to

$$\frac{\partial \omega}{\partial t} = \frac{\eta}{\rho}\nabla^2 \omega.$$

Interpretation: this is just a diffusion equation for vorticity ω. We thus define a diffusion constant $D \equiv \eta/\rho$ and have

$$\frac{\partial w}{\partial t} - D\nabla^2 \omega = 0.$$

- Recall from **TODO:** reference that the vorticity's initial condition at $t = 0$, corresponding to an ideal line vortex, reads

$$\boldsymbol{\omega}(\boldsymbol{r}) = \hat{\mathbf{e}}_z \Gamma \delta(x)\delta(y) = \hat{\mathbf{e}}_z \Gamma \delta^2(\boldsymbol{r})\delta(t).$$

Add in $\delta(t)$ for initial condition, which allows easy incorporation of initial condition into the diffusion equation:

$$\frac{\partial \omega}{\partial t} - D\nabla^2 \omega = \Gamma \delta^2(\boldsymbol{r})\delta(t)$$

- The problem's diffusion equation REF takes the general form

$$\frac{\partial u}{\partial t} - D\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = \delta(x)\delta(y)\delta(t).$$

This equation has the general Gaussian solution

$$u(x, y, t) = \frac{1}{4\pi D t} e^{-\frac{x^2}{4Dt}} e^{-\frac{y^2}{4Dt}}$$

We then write this as

$$u(\boldsymbol{r}, t) = u(r, t) = \frac{1}{4\pi D t} e^{-\frac{r^2}{4Dt}}$$

- In our case solution of Equation **TODO:** reference for vorticity is

$$\omega(\boldsymbol{r}, t) = \frac{\Gamma}{4\pi D t} e^{-\frac{r^2}{4Dt}}, \quad \text{where } D \equiv \frac{\eta}{\rho}.$$

This describes how, because of viscocity, vorticity spreads through space with respect to time.

Velocity Field

- We now consider velocity field. Because cylindrical symmetry of $\boldsymbol{\omega}$ about φ, we can quickly find velocity field from the known vorticity $\boldsymbol{\omega}$ using Stokes' theorem

$$\Gamma \equiv \oint \boldsymbol{v} \cdot \mathrm{d}\boldsymbol{l} = \iint \nabla \times \boldsymbol{v} \cdot \mathrm{d}\boldsymbol{S} = \iint \boldsymbol{\omega} \cdot \mathrm{d}\boldsymbol{S}.$$

Cylindrical symmetry of $\boldsymbol{v}$: the LHS evaluates to

$$\Gamma = \oint \boldsymbol{v} \cdot \mathrm{d}\boldsymbol{l} = \int_0^{2\pi} v(r, t) r \, \mathrm{d}\varphi = 2\pi r v(r, t).$$

Cylindrical symmetry of $\boldsymbol{\omega}$: for a circle of radius r:

$$\iint \boldsymbol{\omega} \cdot \mathrm{d}\boldsymbol{S} = 2\pi \int_0^r \omega(r', t) r' \, \mathrm{d}r' = \frac{2\pi \cdot \Gamma}{4\pi D t} \int_0^r r' e^{-\frac{(r')^2}{4Dt}} \, \mathrm{d}r' = \Gamma \cdot \left(1 - e^{-\frac{r^2}{4Dt}}\right).$$

We then equate

$$2\pi r v(r, t) = \Gamma \cdot \left(1 - e^{-\frac{r^2}{4Dt}}\right) \implies v(r, t) = \frac{\Gamma}{2\pi r}\left(1 - e^{-\frac{r^2}{4Dt}}\right).$$

3.3.6 Exercise: Practice with Reynolds and Strouhal Numbers

Two plates of radius a separated by a vertical distance $h \ll a$ contain an incompressible viscous fluid with viscosity ρ. Find the force (between?) the distances if they are pulled apart or pushed together by a speed $u = \frac{\mathrm{d}h}{\mathrm{d}t}$.

- Begin with Navier-Stokes equation for incompressible flow:
$$\rho\left[\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{v}\right] = -\nabla p + \eta\nabla^2\boldsymbol{v}.$$
- Solving the problem for arbitrary speeds u is beyond the scope of this course. We will only consider the case when u is small enough that the entire LHS of the Navier-Stokes equation is negligible.

Condition for Neglecting Navier-Stokes Equation LHS

- When disk spacing changes, the enclosed volume changes, and surrounding fluid flows either in or out to compensate.

  We first consider pulling the disks apart, so surrounding fluid flows into the vacated region.

  Approximation: the radial velocity into the vacated region is constant and equal to v_0 across the vacated gap in lateral surface (not the entire lateral surface, just the small vacated gap).

  Conservation of volume flow rate requires
$$\pi a^2 u + 2\pi a h v_0 = 0 \implies v_0 \sim \frac{a}{h}u.$$
- We estimate the problem's Reynold's number from TODO skripta 3.5.
$$\mathrm{Re} \equiv |\rho(\boldsymbol{v}\cdot\nabla)\boldsymbol{v}| \cdot \frac{1}{|\eta\nabla^2\boldsymbol{v}|}$$

  Radial velocity into the lateral surface changes from the center to the disk edge from 0 to v_0, so
$$\rho v_0 \frac{\partial v_0}{\partial r} \sim \rho\frac{v_0^2}{a}$$

  TODO: change one to v_r and keep the constant as v_0.

  To estimate the viscous term in the Reynold's number definition, we take as characteristic quantities the magnitude of radial speed v_0 and the separation h, and estimate
$$\eta\nabla^2 v_0 \sim \frac{\eta v_0}{h^2}.$$

  The problem's Reynold's number comes out to
$$\mathrm{Re} \sim \left(\frac{\rho v_0^2}{a}\right)\cdot\left(\frac{h^2}{\eta v_0}\right) = \frac{\rho v_0 h^2}{\eta a} = \frac{\rho u h}{\eta}.$$

  TODO: formulate Note that a basic Reynolds number estimate with REFERENCE would fail because the problem has multiple distance scales.

  The condition for "small speed" for a small Reynolds number is thus
$$u \ll \frac{\eta}{\rho h}.$$

Condition for Stationary Solutions

- Characteristic time scale for change in the problem's velocity field is

  $$\tau = \frac{h}{u}.$$

  Meanwhile, the characteristic time for traversing the velocity field gradient is

  $$\tau_2 = \frac{a}{v_0} = a\frac{h}{au} = \frac{h}{u} = \tau$$

  TODO: notation The problem's **TODO:** name number is thus

  $$\mathrm{St} \sim 1.$$

  Conclusion: the partial and advective derivatives $\frac{\partial \boldsymbol{v}}{\partial t}$ and $(\boldsymbol{v} \cdot \nabla)\boldsymbol{v}$ on the LHS of the Navier-Stokes equation are of similar order.

  Conclusion: the NAME number does not play an important role in establishing the problem as stationary; only the Reynolds number does. Condition for stationary problem is $\mathrm{Re} \ll 1$ for this problem.

- Another look: the problem's viscous relaxation time (SKRIPTA 3.4) (taking h as a characteristic distance scale) is

  $$\tau_{\mathrm{v}} = \frac{\rho h^2}{\eta} \cdot \frac{u}{h} = \frac{\rho u h}{\eta}$$

  This is the same result as Re.

  Another look: The travel time from edge to center of the disks is of the same order as the characteristic velocity field changing time τ. However, for $u \ll \frac{\eta}{\rho h}$, the viscous relaxation time is much smaller than both of these other two times.

Solution

- In any case, we restrict ourselves to the stationary regime $u \ll \frac{\eta}{\rho h}$ and work in the Stokes approximation

  $$-\nabla p + \eta \nabla^2 \boldsymbol{v} = \mathbf{0}.$$

  The problem has rotational symmetry about the z axis. Use an velocity field ansatz of the form

  $$\boldsymbol{v}(\boldsymbol{r}, t) = v_r(r, z, t)\,\hat{\mathbf{e}}_r + v_z(r, z, t)\,\hat{\mathbf{e}}_z.$$

  Our problem has $\tau_{\mathrm{v}} \ll \tau$. The problem is thus approximately adiabatic—the velocity field is to a first approximation stationary, and changes only because of (assumed-to-be slow) change in plate separation $h(t)$.

- This problem has (assumes) $h \ll a$. Because the problem's z coordinate (related to h) occurs on a much smaller scale than the radial coordinate (related to a), we have

  $$\frac{\partial}{\partial z} \gg \frac{\partial}{\partial r} \implies \nabla^2 \approx \frac{\partial^2}{\partial z^2}.$$

From $v_0 \sim \frac{a}{h}u$ and the limit $h \ll a$ it follows that $v_z \ll v_r$ (since v_0 is radial speed). From this we also approximate

$$\frac{\partial^2 v_z}{\partial z^2} \ll \frac{\partial^2 v_z}{\partial z^2}$$

We then have **TODO:** go through

$$\nabla \cdot \boldsymbol{v} \approx \frac{1}{r}\frac{\partial}{\partial r}(r v_r) + \frac{\partial v_z}{\partial z} = 0$$

While $-\nabla p + \eta \nabla^2 \boldsymbol{v}$ becomes, in components,

$$-\frac{\partial p}{\partial r} + \eta \frac{\partial^2 v_r}{\partial z^2} = 0 \qquad \text{and} \qquad -\frac{\partial p}{\partial z} \approx 0.$$

We conclude, in this approximation, that $p \neq p(z)$. The then twice integrate the radial component equation to get

$$v_r = \frac{1}{2\eta}\frac{\partial p}{\partial r} z^2 + A(r) z + B(r).$$

Boundary conditions: $v_r(z = 0) = v_r(z = h) = 0$ **TODO:** why not v_0? This produce

$$B = 0 \qquad \text{and} \qquad A = A(r) = \frac{1}{2\eta}\frac{\partial p}{\partial r} h.$$

Thus

$$v_r = \frac{1}{2\eta}\frac{\partial p}{\partial r} z \cdot (z - h).$$

Substitute v_r into continuity equation and integrate over z to get

$$v_z = -\frac{1}{r}\frac{\partial}{\partial r}\int_0^z r v_r \, \mathrm{d}z = -\frac{1}{\eta}\frac{1}{r}\frac{\partial}{\partial r}\left[r \frac{\partial p}{\partial r}\left(\frac{z^3}{3} - \frac{h z^2}{2}\right)\right]$$

This assumes the lower disk is at rest, i.e. $v_z(z = 0) = 0$. We then use $v_z(z = h) = u$ to get

$$u = \frac{h^3}{12\eta}\frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial p}{\partial r}\right).$$

Integrate once and get

$$\frac{\partial p}{\partial r} = \frac{6\eta u}{h^3} + \frac{C}{r}.$$

To avoid divergence at the origin we require $C = 0$. (Symmetry WRT φ requires even $\frac{\partial p}{\partial r} = 0$). We then have

$$\frac{\partial p}{\partial r} = \frac{6\eta u}{h^3}.$$

Integrate once more to get

$$p(r) = \frac{3\eta u}{h^3}(r^2 - a^2).$$

For $u > 0$ (disks moving apart) pressure is zero everywhere in the disk region. That makes sense—disks moving apart would create a vacuum.

- Total force on the upper disk to to pressure is

$$F = \iint p \, \mathrm{d}S = 2\pi \int_0^a p(r) r \, \mathrm{d}r = -\frac{3\pi}{2} \frac{\eta a^4}{h^3} \frac{\mathrm{d}h}{\mathrm{d}t}.$$

  The direction of F depends on the sign of $\frac{\mathrm{d}h}{\mathrm{d}t}$. If $\dot{h} > 0$ (disks moving apart), the force on the upper disk points downwards (pulling the disks closer together—this makes sense, resulting vacuum should pull disks together).

- For review, the use of the Stokes approximation to the Navier-Stokes equation is valid in the regime

$$\left| \frac{\mathrm{d}h}{\mathrm{d}t} \right| \ll \frac{\eta}{\rho h}$$

  Thus, the expression for F in REFERENCE holds for

$$|F| \ll 3\pi \frac{\eta^2 a^4}{2\rho h^4}.$$