Classical Physics

Lecture notes from the first-year undergraduate course *Klasična fizika*, taught by prof. dr. Marko Mikuž at the Faculty of Mathematics and Physics at the University of Ljubljana in the 2020-2021 academic year. The notes were transformed from recorded lectures into textbook form, translated to English, and expanded with additional material in the 2021-2022 academic year by Elijan Mastnak.

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Last update: December 3, 2021

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1 Introduction

Measurements are the foundation of science. For our purposes, measurements are quantified observations of the natural world in which we associate *numerical* values with a physical phenomenon we wish to better understand. The essence of measurement in physics goes roughly as follows:

1. observe a process in the natural world,

2. make numerical measurements of the observed process, and

3. study the measurements for patterns and structure. On the basis of these patterns, formulate physical laws—essentially relationships between measured quantities—in the language of *mathematics*.

When a well-defined, higher-level structure emerges from the jumble of experimentally-determined observations and patterns, the resulting conclusions are called a *physical model*. As a model is fine-tuned and continually stands the test of experiment—meaning the model's theoretical predictions are found to agree with experimental observations—the model becomes a *physical theory*.[1]

Meanwhile, if, when testing the model, we find situations in which experimental results consistently deviate from the model's theoretical predictions, we have two options:

1. reject the theory entirely, or

2. conclude that the theory is only valid within the limits of a specific regime.

Example: A short history of Newtonian gravitation

As a concrete example, we now give a quick history of the theory of Newtonian gravitation that illustrates all of the important steps listed above. Newtonian gravitation's roots trace back to early astronomers observing the sky and recognizing patterns and structure in distribution of stars. These observations eventually took on a quantitative nature (e.g. times of orbit, when certain stars appear in the sky, estimates of distances between the Earth and Sun, etc...) and eventually evolved into the Copernican model of planets orbiting the Sun and the current solar system model. These early astronomical models were later polished and summarized mathematically by Kepler's laws of planetary motion, which eventually evolved into Newton's theory of gravitation.

Later measurements with more advanced 20^{th} century instruments revealed slight but consistent discrepancies between experiment and the predictions of Newtonian gravitation.[2] These discrepancies were later explained by the 20^{th} century theory of general relativity, and Newtonian gravitation was eventually deemed to hold in the limits of small gravitational fields and low speeds.

[1] Note that the definitions of and borders between concepts like "model" and "theory" are not rigidly defined. Don't worry if the distinctions seem vague—they often are, and you will gain intuition with experience. For now it suffices to remember that a theory is more general and better tested than a model.

[2] For example discrepancies involving the perihelion precession of Mercury or the angular deflection of light around massive objects

1.1 The landscape of physical theories

Many theories exist to describe the natural world; the most important of these are:

1. **Classical physics** (For our purposes[3] classical physics is the material covered in this course, i.e. Newtonian mechanics, classical thermodynamics, and classical electrodynamics.)

2. **Special relativity** (covered in the second-year course *Moderna fizika 1*)

3. **Quantum mechanics** (covered initially in *Moderna fizika 1* and in more detail in the aptly-named third-year course *Kvantna mehanika*)

4. **General relativity**, **quantum field theory**, and various theories of **quantum gravity** (all beyond the scope of undergraduate study at FMF)

Each theory is valid (and useful) in a specific physical regime; which theory applies in which regime depends largely on three physical quantities. These are:

1. speed,

2. distance (informally, size), and

3. gravitational field strength.

To say whether a physical system[4] is, say, "big" or "small", we have to compare it to something. Ideally, this reference value for the system's size should be something with fundamental physical meaning, such as a natural constant. Although somewhat ahead of our discussion of physical quantities in Section 1.3, the idea of comparison to meaningful physical constants motivates describing a physical system in terms of *dimensionless parameters* (dimensionless meaning without units). We construct these dimensionless parameters by multiplying or dividing dimensioned quantities characteristic of the system by fundamental constants with deeper physical meaning (these constants set a meaningful scale against which to compare the system's values) and choose a combination such that the net result is dimensionless.

Since that might sound rather abstract, we now show how to describe a system's speed, size, and the surrounding gravitational field strength in dimensionless form. Importantly, this exercise will give us a simple way to estimate which of the theories mentioned earlier applies in which physical regime.

- **Speed**: we first determine a good universal reference speed. As you might guess, a natural choice is the speed of light, *defined*[5] as of May 2019 as

$$c \equiv 2.997\,924\,58 \cdot 10^{8}\,\mathrm{m\,s^{-1}}. \tag{1.1}$$

  We then analyze a system with speed v in terms of the dimensionless ratio

$$\frac{v}{c}. \tag{1.2}$$

[3] Note that some authors also define special (and sometimes general) relativity as "classical physics", and only theories involving quantum physics are deemed "non-classical" or "modern".

[4] For orientation, this "physical system" could be, for example, a mass on a spring, a ball tossed in the air, a planet orbiting a sun, two protons colliding in a particle accelerator, a black hole...

[5] More on the (re-)definition of physical units coming soon in Section 1.3.1

The quantity v/c is less than one for any physical system (since the speed of light c is a universal speed limit), and how much less than one gives a physically meaningful statement of the system's speed. Systems with $v/c \ll 1$ are accurately described by classical physics, while systems with $v/c \lesssim 1$ require special relativity or more advanced theories.[6]

- **Size**: A fundamental distance characteristic of the boundary regime between classical and quantum physics is a quantity called the *electron Compton wavelength*, denoted by λ_C and equal to

$$\lambda_\mathrm{C} \equiv \frac{hc}{m_\mathrm{e}c^2} = 2.426\,310\,238\,67(73) \cdot 10^{-12}\,\mathrm{m}, \tag{1.3}$$

where h is the Planck constant and m_e is the electron mass. The Compton wavelength sets the scale for distances at the level of atoms and fundamental particles—you will hear more about it in *Moderna fizika 1.* We then represent the size of a system with characteristic length L in terms of the dimensionless parameter

$$\frac{\lambda_\mathrm{C}}{L}. \tag{1.4}$$

Systems with $\lambda_\mathrm{C}/L \ll 1$ (i.e. lengths much larger than λ_C) are accurately described by classical physics, while systems with $\lambda_\mathrm{C}/L \gtrsim 1$ require a quantum-mechanical treatment (although quantum mechanics is often relevant even at nanometer scales).

Alternatively, one could also separate classical and quantum mechanics with the Bohr radius

$$a_0 = 5.291\,772\,109\,03(80) \cdot 10^{-11}\,\mathrm{m} \tag{1.5}$$

and the associated dimensionless parameter a_0/L. Physically, the Bohr radius represents the expected distance between the electron and nucleus in the ground state of a hydrogen atom; you'll get plenty of experience with both λ_C and a_0 in *Moderna fizika 1.*

To be clear, the exact values of both λ_C and a_0 to the long strings of decimals given above are not important now; the takeaway here is that both are characteristic of distances at the atomic scale, which occurs at roughly $10^{-12}\,\mathrm{m}$ and is much smaller than anything familiar from everyday life (the width of a human hair, for instance, is of the order $10\,\mathrm{\mu m}$ to $100\,\mathrm{\mu m}$.)

- **Gravitational Field Strength**: Only for the sake of completeness—this predates our treatment of gravitation—a characteristic *dimensioned* quantity representing a system of a mass m a distance r from a massive body of mass M generating a gravitational field is the gravitational potential

$$\phi_\mathrm{g} = \frac{GM}{r}, \tag{1.6}$$

where $G = 6.674\,30(15) \cdot 10^{-11}\,\mathrm{m^3\,kg^{-1}\,s^{-2}}$ is a universal constant called the *gravitational constant*. We then turn this into a *dimensionless* parameter by

[6] In case you haven't seen it before, the symbol $\ll$ means "much less than", while $\lesssim$ means "less than, but of the same order". For example, one might reasonably write $1 \ll 1000$ and $1 \lesssim 1.1$.

dividing through by c^2 and including the mass m of the body exposed to the gravitational field:

$$\frac{m\phi_{\mathrm{g}}}{mc^2} = \frac{GmM}{rmc^2}. \tag{1.7}$$

It might seem unproductive to include the mass m in both the numerator and denominator without simplifying, but this has an instructive physical interpretation: mc^2, as you might have recognized from the famous equation $E = mc^2$, is the rest energy inherently associated with the mass m, while $m\phi_{\mathrm{g}}$ is the gravitational energy associated with immersing the mass m in the gravitational field generated by the mass M. In other words, $m\phi_{\mathrm{g}}/mc^2$ is a ratio of two energies—one associated with the gravitational field and one inherent to the mass m—and energies are easier to interpret than an abstract quantity like the gravitational potential ϕ_{g} alone.

To a first approximation, systems with $m\phi_{\mathrm{g}}/mc^2 \ll 1$ are well-described by classical physics, while systems with $m\phi_{\mathrm{g}}/mc^2 \gtrsim 1$ require general relativity. For orientation, the Earth-Sun system, using $M_{\mathrm{sun}} \approx 2 \cdot 10^{30}\,\mathrm{kg}$ and an Earth-Sun separation $r \approx 1.5 \cdot 10^{11}\,\mathrm{m}$, produces $m\phi_{\mathrm{g}}/mc^2 \sim 10^{-8} \ll 1$. In other words, the Earth-Sun system, as might be expected, is well-described by simple Newtonian gravitation. Don't worry if you don't understand the equation for ϕ_{g}—you are not expected too, since we have not yet covered gravitation.

Figure 1 shows where each of the important theories mentioned earlier falls in the "phase space"[7] of physical theories spanned by the parameters v/c, λ_{C}/L and ϕ_{g}/c^2.

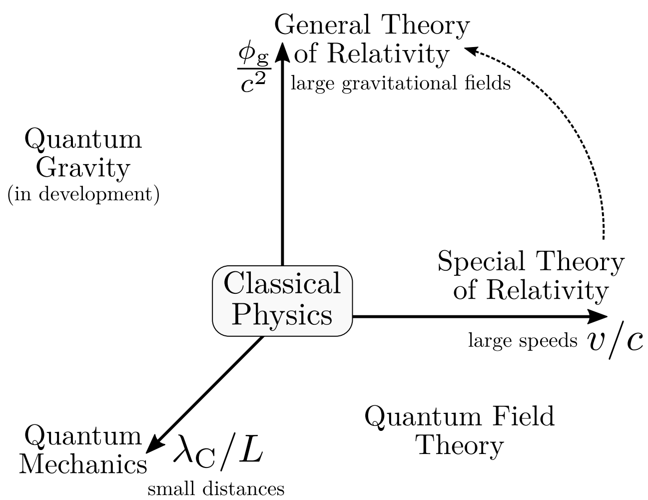

Figure 1: The landscape of physical theories. The appropriate theory to describe a given system depends on the system's speed, size, and the gravitational field to which it is exposed.

Validity of Classical Physics

[7] To not leave you in the dark, a physical system's *phase space* refers to the space of the system's possible states. This concept turns out to be useful in many branches of physics (you'll meet it, for example, in the second-year courses *Statistična termodinamika* and *Klasična mehanika*). But we use the term here only loosely to refer to the space of all existing physical theories.

Classical physics holds at "not too small" distances and "not too large" speeds and gravitational fields. More quantitatively, the theoretical predictions of Newtonian mechanics agree with experiment for physical systems with characteristic speed v, distance L and gravitational potential ϕ_{g} satisfying

$$\frac{v}{c} \ll 1, \qquad \frac{\lambda_{\mathrm{C}}}{L} \ll 1, \quad \text{and} \quad \frac{\phi_{\mathrm{g}}}{c^2} \ll 1. \tag{1.8}$$

The physical world we experience in our everyday lives generally falls in this regime.

Of course, we also wish to describe the physical world in the more exotic regimes shown in Figure 1, where classical physics fails. For the most part, each exotic regime has its own specialized theory; a very active subject of research involves connecting the exotic theories among themselves in a self-consistent manner. Following is a whirlwind tour of the physical regimes and theories shown in Figure 1.

- The first theory to transcend Newtonian mechanics was the **special theory of relativity** (STR), which holds at large speeds ($v/c \lesssim 1$) but only "everyday" distances and gravitational fields, i.e. at $\lambda_{\mathrm{C}}/L \ll 1$ and $\phi_{\mathrm{g}}/c^2 \ll 1$.

- **Quantum mechanics** holds at small distances but, in its basic form, only at everyday speeds and gravitational field strengths.

- The **general theory of relativity** (GTR) accurately describes the physical world at large gravitational field strengths. Happily, the general theory of relativity (as the name might suggest) generalizes the special theory of relativity, i.e. the GTR covers everything described by the STR, in addition to large gravitational fields.

- **Quantum field theory** (QFT) is a generalization of quantum mechanics and special relativity and accurately describes both small distances and large speeds.

- Theories of small distances and large gravitational fields are collectively called **quantum gravity**. This part of the physics landscape is still poorly understood and a topic of active research; there is not a yet a universally-accepted theory of quantum gravity.

  Development of quantum gravity is plagued by both (i) a lack of measurements due to its extreme physical conditions (subatomic particles and black holes are notoriously challenging to measure, let alone simultaneously) and (ii) the lack of a well-established mathematical formalism with which to describe the theory. In other words, we lack both the mathematics to formulate theory and the measurements to test it.

  To illustrate the experimental problem, one current theory of quantum gravity involves eleven dimensions; confirming this theory would require (somehow) a transition from 11 dimensional space to the three dimensional space of our everyday lives. As you can imagine, that poses quite an experimental challenge!

The main lesson here is that classical physics covers only a small regime of the vast and diverse physical phase space, plenty of which, particularly in the more exotic corners of small distances and large gravitational fields, remains poorly understood.

This course will cover only the "center" of the physical phase space shown in Figure 1, in which Newtonian mechanics, classical thermodynamics, and classical electrodynamics accurately describe the physical world. We will, however, (for example when studying electrical conduction) to some extent consider the connection between the microscopic and macroscopic worlds, and aim to describe physical processes in both worlds and find connections between them. We will only qualitatively mention the deviations of special relativity from Newtonian mechanics at large speeds, and leave the other exotic theories for future courses.

1.2 A tour of the mathematics used in this course

Before solving problems, we first summarize the mathematical and physical quantities with which we will operate. We will represent physical quantities mathematically mostly in terms of scalars and vectors. Vectors have both magnitude and direction in space and are well-suited to describing physical processes in the three-dimensional Euclidean space of classical physics. Both the magnitude and direction of a vector can *change* over the course of a physical process. Thus, we need a mathematical formalism describing change of both scalars and vectors.

The change of scalars is governed by basic differential and integral calculus, which forms the backbone of the Newtonian physics covered in this course. As an example, the speed $v(t)$ of an object is related to the total distance $s(t)$ traveled by the object according to

$$v = \frac{\mathrm{d}s}{\mathrm{d}t} \qquad \text{and} \qquad s = \int v \, \mathrm{d}t, \tag{1.9}$$

where the derivative and integral are performed with respect to time.

The change of vector quantities is governed by a branch of mathematics called *vector calculus*, which is the generalization of scalar differential and integral calculus to vector quantities. Vector calculus is covered formally in the second-year course *Matematika 3*, but we will dabble with it in this course as well. For orientation, here are some physical examples of calculus involving vector quantities:

- The momentum $\boldsymbol{p}(t)$ of an object of mass m with position vector $\boldsymbol{r}(t)$ is
$$\boldsymbol{p} = m\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}. \tag{1.10}$$
- The work[8] W done by a force $\boldsymbol{F}$ as the force's point of application moves along a curve in space from position $\boldsymbol{r}_1$ to $\boldsymbol{r}_2$ is given by the line integral
$$W = \int_{\boldsymbol{r}_1}^{\boldsymbol{r}_2} \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{s}. \tag{1.11}$$
- The power P (energy per unit time) through a surface S through which flows heat current density $\boldsymbol{j}$ (energy per unit time per unit area) is given by the surface integral
$$P = \iint_S \boldsymbol{j} \cdot \mathrm{d}\boldsymbol{S}. \tag{1.12}$$

[8] I've denoted work by the symbol W, from the English word "work", because this is most common in the modern scientific literature. In Slovenia, at least in introductory courses, work is commonly denoted by A, from the German word "Arbeit".

Don't worry if you don't understand all of the equations quite yet—you'll gain experience with them in this course, and learn them formally when your mathematics courses catch up with our curriculum over the next few months. The goal here is just to give a taste of how calculus is used to formulate the concepts of physics.

1.3 Physical quantities

After our quick tour through the *mathematical* formalism of vectors and calculus, we now consider how *physical* quantities are described and what they represent. For our purposes, a physical quantity is a quantity that can be quantitatively measured (i.e. measured with numbers).

1.3.1 Base quantities and the 2019 redefinition of units

Physics involves seven *base quantities* from which all other physical quantities are derived. These base quantities are time, distance, mass, electric current, temperature, amount of substance, and luminous intensity. These are shown in Table 1. The final quantity—luminous intensity—occurs rarely outside of photometry, and we will not study it in this course.

Different physical quantities are distinguished by their units. The International System of Units (abbreviated "SI") established by an international organization called the International Bureau of Weights and Measures defines the SI *base units* for each of the base quantities; we will quote these units shortly. But first, a (recent) historical aside: in May 2019, all SI units were completely redefined in terms of only physical constants. Using physical constants is good! Namely, units defined in terms of physical constants are more fundamental and more stable than arbitrarily-defined prototypes of human creation.

Quantity	Unit	Unit symbol
Time	second	s
Distance	meter	m
Mass	kilogram	kg
Electric current	ampere	A
Temperature	kelvin	K
Amount of substance	mole	mol
Luminous intensity	candela	cd

Table 1: The seven base quantities and their SI base units (discussed shortly).

As an example of the problems associated with human-created standards, the International Prototype of the Kilogram (a cylinder made of a platinum-iridium alloy whose mass served as the definition of the kilogram before the 2019 redefinition) was found to have a fluctuating mass (of the order $\Delta m \sim 10\,\mu\mathrm{g}$); we still lack a universally accepted explanation why. As a more amusing example, the yard (an imperial unit roughly equal to the meter) was once defined as the length from the nose to the thumb of the outstretched arm of King Henry I of England. Not exactly a scientifically rigorous definition of distance!

1.3.2 The SI base units

We now summarize the base quantities, their units, and the natural constants on the basis of which the units are defined.

1. The SI base unit of **time** is the **second** and is defined in terms of the unperturbed, ground state hyperfine transition frequency of the cesium-133 atom. Concretely, the second (symbol s) is defined such that this transition frequency takes exactly

$$\Delta\nu_{\mathrm{Cs}} \equiv 9\,192\,631\,770\,\mathrm{s}^{-1}. \tag{1.13}$$

   The relevant physics will be more clear after *Moderna fizika 1*, but the takeaway here is that the second is based on a precise and fundamental natural process—the cesium atom's transition frequency is an extremely consistent timekeeper.

   Previously, the second was defined in terms of the Earth's rotation cycle over the course of a day. Aside from being rather anthropocentric, this definition was plagued by instabilities in the Earth's rotation.

2. The SI base unit of **distance** is the **meter**. The meter (symbol m) is based on the speed of light in vacuum c and the above-defined second. The meter is defined such that speed of light in vacuum is exactly

$$c \equiv 299\,792\,458\,\mathrm{m\,s}^{-1}, \tag{1.14}$$

   assuming the second is defined in terms of $\Delta\nu_{\mathrm{Cs}}$.

3. The SI base unit of **mass** is the **kilogram** (symbol kg) and is based on a universal constant called Planck's constant, denoted by h. The kilogram is defined such that Planck's constant is exactly

$$h \equiv 6.626\,070\,15 \cdot 10^{-34}\,\mathrm{kg\,m^2\,s^{-1}}, \tag{1.15}$$

   assuming the second is defined in terms of $\Delta\nu_{\mathrm{Cs}}$ and the meter in terms of c. You'll hear more the Planck constant in second-year courses.

   Actually implementing the relationship between the kilogram and Planck's constant involves an extremely precise electromechanical scale called a *Kibble balance*. You can read more about this in the BIPM's publication "*Mise en pratique* for the definition of the kilogram" available on the BIPM's page on the practical realization of the definition of the base units.

4. The SI base unit of **electric current** is the **ampere** (symbol A) and is based on the elementary charge (the charge of single proton). The ampere is defined such that the elementary charge is exactly

$$e_0 \equiv 1.602\,176\,634 \cdot 10^{-19}\,\mathrm{A\,s}, \tag{1.16}$$

   assuming the second is defined in terms of $\Delta\nu_{\mathrm{Cs}}$.

5. The SI base unit of **temperature** is the **kelvin** and is based on a constant called the Boltzman constant, denoted by k_{B}. The kelvin (symbol K) is defined such that the Boltzman constant is exactly

$$k_{\mathrm{B}} \equiv 1.380\,649 \cdot 10^{-23}\,\mathrm{kg\,m^2\,s^{-2}\,K^{-1}}, \tag{1.17}$$

assuming the kilogram, meter, and second are defined in terms of natural constants as described above.

6. The SI base unit of **amount of substance** (e.g. the number atoms in a sample of carbon, the number of gas particles in closed chamber of gas, etc...) is the **mole** and is based on the Avogadro constant N_A. The mole (symbol mol) is defined such that Avogadro is exactly

$$N_\mathrm{A} \equiv 6.022\,140\,76 \cdot 10^{23}\ /\text{mole}. \tag{1.18}$$

In other words, one mole of "stuff" contains exactly $6.02214076{\cdot}10^{-23}$ elementary entities.

7. For the sake of completeness, the SI base unit of **luminous intensity** (which no one really uses besides photometrists, and we won't mention further in this course) is the **candela**. If you feel inspired, you can read about its formal definition in the SI Brochure, or just check the Wikipedia article on the candela.

1.3.3 Derived quantities

Derived quantities are constructed by combining base quantities through multiplication and division. We already encountered some derived quantities above. For example, speed is, mathematically, the quotient of distance and time, and the speed of light was quoted above in meters per second, i.e. units of distance per units of time.

Derived quantities deemed important enough are assigned their own *derived unit*; for example, energy is assigned a unit called the *joule*. Derived units are convenient: they save us from writing out long strings of base units. For example, you will probably agree that it is more convenient to quote energy in joules than in the equivalent base unit combination $\mathrm{kg\,m^2\,s^{-2}}$. Some of the first derived quantities that we will meet in this course, together with their units and conventional symbols, are given in Table 2.

Quantity	**Symbol**	**SI Base Unit**	**Derived Unit**
Velocity	v	$\mathrm{m\,s^{-1}}$	-
Momentum	p	$\mathrm{kg\,m\,s^{-1}}$	-
Acceleration	a	$\mathrm{m\,s^{-2}}$	-
Force	F	$\mathrm{kg\,m\,s^{-2}}$	N (newton)
Energy	E	$\mathrm{kg\,m^2\,s^{-2}}$	J (joule)
Power	P	$\mathrm{kg\,m^2\,s^{-3}}$	W (watt)

Table 2: A few of the first derived physical quantities we will encounter in mechanics, together with their SI base units and derived unt, if applicable.

1.3.4 Measuring average and instantaneous quantities

We go directly to an example: measuring a quantity called *volume flow rate*. Physically, volume flow rate represents the volume of "stuff" (typically fluid) moving past a region of space (such as a given point in a hose) per unit time.

Suppose we wish to measure the volume flow rate of water out of a pipe's faucet. If our tools are limited to everyday objects, we could measure the VFR using a bucket of known volume[9] ΔV and a stopwatch. The measurement proceeds in two steps:

(a) Simultaneously start the stopwatch and place the (empty) bucket under the faucet and observe the rising water level in the bucket as the bucket fills.

(b) Stop the stopwatch when the bucket is full and record the time interval Δt between the stopwatch stop and start times.

This measurement gives us the time needed to fill the known bucket volume. From ΔV and Δt, we determine the *average volume flow rate* out of the faucet over the course of the measurement via

$$\overline{Q} = \frac{\Delta V}{\Delta t}. \tag{1.19}$$

We stress that $\overline{Q}$ is a single scalar quantity[10] representing the average volume flow rate over the course of the entire measurement. It contains no information about the volume flow rate at a specific time during the measurement process.

Alternatively, suppose we have an advanced measurement instrument capable of continuously measuring the small volume of water $\mathrm{d}V$ leaving the faucet every small time interval $\mathrm{d}t$ (e.g. of milli- or microsecond order). The continuous stream of $\{\mathrm{d}V, \mathrm{d}t\}$ measurements at each point in time until the bucket is full can be used to approximate the *instantaneous volume flow rate*

$$Q(t) = \frac{\mathrm{d}V}{\mathrm{d}t}. \tag{1.20}$$

Importantly, $Q(t)$ gives the VFR through the faucet as a *function of time* throughout the measurement, while $\overline{Q}$ from Equation 1.19 is a single scalar value giving the average VFR over the course of the entire measurement. To be clear, however, both $\overline{Q}$ and $Q(t)$ represent the same physical phenomena (i.e. flow of fluid volume with respect to time) and have the same units (volume per time). Thus, $\overline{Q}$ and $Q(t)$ still correspond to the same physical quantity, i.e. volume flow rate.

[9] If unknown, we could estimate the bucket's volume with a tape measure and geometrical measurements. Better yet, if we had a scale handy, we could measure the mass of water needed to fill the bucket and convert to volume using water's known (room temperature and pressure) density $\rho \approx 1\,\mathrm{g\,cm^{-3}}$.

[10] I have used the symbol Q to denote volume flow rate, which might seem rather arbitrary at first glance. I agree. But Q is the conventional symbol for VFR and I figure it is better to encounter it sooner than later.

2 Mechanics

Mechanics is the field of physics that describes and predicts the motion of bodies. Mechanics divides into two sub-fields:

1. *kinematics*, which *describes* the motion of bodies, and

2. *dynamics*, which *predicts* the motion of bodies.

Kinematics is observational (i.e. quantifies a body's *current motion*) while dynamics, which is more powerful, can predict a body's *future motion* given its current state.

2.1 Physical models

Real-life physical objects are complicated—they are asymmetric, anisotropic, deformable, have non-homogeneous mass distributions, and so on. Exactly describing and predicting their motion in all its detailed complexity is difficult—in fact, exact analytical solutions are often impossible. Instead, physicists make a compromise: we *approximate* bodies and physical systems with simple models that are relatively easy to analyze, at the expense of a perfectly exact prediction. A simplification of a physical system that...

1. dramatically improves analysis and also

2. preserves a result reasonably in agreement with real-life behavior...

is called a *physical model*. Models aim to preserve only those properties of an object that are essential to predicting its motion and remove secondary details that significantly complicate analysis but produce only small corrections in observed motion.

Our plan in this chapter is to first study the kinematics and dynamics of a few useful models in a theoretical sense, and then apply the developed theory to approximate the motion of real-life objects. Certain models are particularly well-suited to mechanics (and many other branches of physics, too), and you will encounter them in any standard physics course. These are:

1. the point mass,

2. the system of point masses,

3. the rigid body, and

4. the linearly deformable body.

We will define these models shortly. Each is a successively better approximation of real-life objects, but each is also more difficult to work with analytically. In this course we will begin with the mechanics of a point mass, which we will later generalize to a system of point masses. We introduce rigid bodies in our treatment of rotational mechanics, and briefly cover deformable bodies in the lectures on elasticity and deformation. Rigid bodies are covered in much more detail in the second-year course *Klasična mehanika*, while deformable bodies and continuous media are covered in the third-year elective course *Mehanika kontinuov*.

2.1.1 The point mass

The simplest and most fundamental model is the point mass: a hypothetical object with its *entire mass concentrated at a single point in space.* To specify a point mass's state, you need to describe only its mass and its position at a *single* spatial coordinate. Importantly, you *don't* have to worry about a point mass's size, geometry, orientation in space, mass distribution, and so on—a spatial coordinate and the mass are all you need, at least for kinematics applications. In the context of dynamics, if you also specify the point mass's velocity at any given moment in time (in addition to its mass and position), you can predict everything there is to know about its future state within the scope of Newtonian mechanics.

Physical point masses do not occur in nature in the sense that you cannot walk down the street and find a body with infinitesimal size but finite mass. The closest physical objects to point masses, within the scope of current scientific knowledge, are elementary particles with no (currently) known internal structure, such as electrons or quarks. But this might be a limitation of current experimental technology rather than a fundamental truth. For our purposes, it suffices to remember that perfect point masses do not occur in nature, although some elementary particles currently appear to come close.

Validity of the point mass model

A finite-sized object may be treated as a point mass if the object's size is insignificant in the analysis—this is often the case when the ambient physical system or the distance traveled by the object is much larger than object itself. For example, the Earth on the scale of the solar system, an electron on the scale of an atomic nucleus, or an ion on the scale of a particle accelerator are all excellent candidates for a point mass approximation.

A word of caution: The validity of a point mass (or any other physical model) in describing a physical object *depends crucially upon context and scale.* For example, the Earth is very well described as a point mass on the scale of the solar system (for example in the context of predicting planetary orbits using Newtonian gravitation), but the Earth is certainly nothing like a point mass from the perspective of a human being walking on its surface.[11] Note the difference in scales: the Earth's size is insignificant on the scale of the solar system but very large on the scale of a human being. A general statement like "we can model the Earth as a point mass" is not well defined—you must say "we can model the Earth as a point mass at XYZ scale in the context of ABC analysis".

To conclude, here are a few miscellaneous comments about point masses:

- Within the realm of experimental error, a point mass approximation and an exact analysis of a body may be indistinguishable.
- Spherically-symmetrical objects under the influence of inverse square laws (such as the gravitational and electrostatic forces) behave *exactly* as point particles.
- The terms "point particle" and "point mass" are often used interchangeably.

[11] Well, its gravitational field actually comes pretty close—more on this in the lectures on gravitation—but at least the Earth's geometry is not remotely point-like in this context.

2.1.2 Other physical models

In passing, we now briefly define the other common physical models mentioned in this chapter's introduction.

- A **system of point masses** is exactly what it sounds like—a set of multiple point masses. This model can be useful for describing mutually-interacting (often spherical) objects. The Earth-Moon-Sun system, for example, is well-described as a system of point masses (a point-mass approximation is valid because the distances between the planets are much larger than the planets themselves).

  Alternatively, when the number of points is large, systems of point masses can be used to model continuous objects, such as a ball, disk, or rod.[12] This approximation works particularly well if the object is homogeneous, and we will return to this concept in the lectures on center of mass, center of gravity, and moment of inertia.

- A **rigid body** is an object in which the relative distances between all constituent points are constant. In practice, this means a rigid body cannot deform when subject to outside forces—all points in the body retain their original orientation no matter how hard you push it. For orientation, on the scale of everyday stresses, a slab of concrete is very well-described as a rigid body, while human muscle or a piece of gelatin are poor examples of rigid bodies—both deform if you poke or squeeze them. Note that a rigid body and a system of point masses are not mutually exclusive—any system of point masses in which the distances between all points is fixed is also a rigid body.

  Rigid bodies are the canonical model in the elementary theory of rotational mechanics—we will return to them when studying rotation, and you will study them in more detail in the second-year course *Klasična mehanika*.

- A **linearly deformable body** is a generalization of the rigid body in which the relative distances between constituent points are allowed to vary in response to outside stresses, but the resulting deformation must be *linear* in the applied stress. Linearly deformable bodies are the canonical model in the elementary theory of elastomechanics, and are covered in detail, using a tensor formalism, in the third-year course *Mehanika kontinuov*. We will cover them only briefly in the lectures on deformation using scalars and vectors.

2.2 Kinematics

Kinematics is a quantitative description of the motion of bodies. After covering the *description* of motion using kinematics, we will be equipped to *predict* motion using dynamics.

[12] In practice, *numerical* computer simulations treat physical objects as systems of discrete points. Although the number of points can be very large on modern hardware, it is still finite, since no physical computer can store in memory the theoretically infinite number of points needed to describe a continuous body.

2.2.1 Position and trajectory

Position is a vector quantity, conventionally denoted by $\boldsymbol{r}$, that specifies a body's location in space at a given point in time. But we usually aren't satisfied with knowing a body's position at a just *single* point in time—we want to know the position for *all* times (or at least over an interval of time). In the language of kinematics, this desired quantity is called a *trajectory*, which is a body's position in space *as a function of time*.

Position, in the three-dimensional Euclidean space of everyday life and classical physics, is fully specified by a three-dimensional position vector $\boldsymbol{r} \in \mathbb{R}^3$. A trajectory, which describes how a body's position vector changes over time, is specified by a vector-valued function $\boldsymbol{r}(t)$. A trajectory associates with every time t (a scalar quantity $t \in \mathbb{R}$) a corresponding position vector $\boldsymbol{r}$ (a three-dimensional vector $\boldsymbol{r} \in \mathbb{R}^3$).[13]

Coordinate representation of position

After specifying a coordinate system and basis for the space $\mathbb{R}^3$, a position vector $\boldsymbol{r}$ can be represented with three coordinates, for example the (x, y, z) coordinates of the Cartesian coordinate system you are probably familiar with from high school. In a Cartesian coordinate system, a position vector and a trajectory would be represented in coordinate form as

$$\boldsymbol{r} = (x, y, z) \qquad \text{and} \qquad \boldsymbol{r}(t) = \big(x(t), y(t), z(t)\big). \tag{2.1}$$

For the purposes of this course, unless explicitly stated otherwise, we will work in three-dimensional Euclidean space and perform analysis in a Cartesian coordinate system using the standard basis. Translated to everyday language, this means we will continue using the (x, y, z) coordinate system you are familiar with from high school, and, if desired, you can forget about coordinate systems, bases, and basis vectors until *Matematika 2*.[14]

Note: distinguishing $\boldsymbol{r}$ and r

In your study of physics you will often encounter both the symbols $\boldsymbol{r}$ and r. These always represent different physical quantities:

- $\boldsymbol{r}$ is a vector quantity and, nearly universally, is used to represent a position in three-dimensional space.
- r is a scalar quantity usually used to represent a body's distance from a coordinate system's origin. In this usage, r would be the length (also "magnitude" or "norm") of a position vector, i.e. $r = |\boldsymbol{r}|$.[15]

[13] To make a connection to the formal notation of real analysis introduced in *Matematika 1*, you would define a trajectory as a vector-valued, single-variable function $\boldsymbol{r} : \mathbb{R} \to \mathbb{R}^3$ with $t \mapsto \boldsymbol{r}(t)$.

[14] In general, representing a vector in terms of coordinates requires specifying a basis and coordinate system for the ambient vector space, which in this course will always be the 3D Euclidean space familiar from everyday life—this space is called $\mathbb{R}^3$ in the notation of mathematics. You will learn about bases and coordinates formally in *Matematika 2* and get practical knowledge with common coordinate systems in *Proseminar A/B*. The point, for now, is to know that coordinate systems other than the (x, y, z) Cartesian system familiar from high school exist, and $\boldsymbol{r} = (x, y, z)$ is just a special case of writing a vector in coordinate form.

[15] The mathematically correct notation for a vector norm would actually be $\|\boldsymbol{r}\|$ (with $|\cdot|$ reserved for the magnitude of scalars) but physicists are sloppy and use the absolute value sign $|\boldsymbol{r}|$ for vector magnitudes, too. If you ever see the absolute value of a vector in a physics context, you can assume it denotes the vector's magnitude (i.e. the length or norm).

The symbol r can also denote the radial coordinate in a spherical coordinate system—you'll learn more about this in *Proseminar A/B*. Loosely, the net effect is the same: r represents the scalar distance from the origin.

Plotting trajectories

In one spatial dimension, or for a single coordinate, we conventionally show a trajectory by plotting the coordinate (on the ordinate axis) as a function of time (on the abscissa), while two-dimensional trajectories are often plotted in a coordinate plane with time as a parameter. Three-dimensional trajectories can also be plotted parametrically in three dimensions, but the result is often difficult to instructively interpret.

TODO: create example plots. Some ideas (for example):

- $z(t)$ for a particle in free fall
- $(x(t), y(t))$ parametrically for a particle in an EM field
- $\boldsymbol{r}(t)$ parametrically for the moon's orbit around the earth

2.2.2 Displacement

Consider a particle with initial position $\boldsymbol{r}_1$ and final position $\boldsymbol{r}_2$. The difference of these two vectors is called a *displacement* and is denoted by

$$\Delta \boldsymbol{r} = \boldsymbol{r}_2 - \boldsymbol{r}_1; \tag{2.2}$$

it is conventional to formulate displacement as final position minus initial position. Displacements have a very useful property: *a displacement between two positions is the same regardless of the coordinate system origin with respect to which the positions are measured*. This is because any relative offset of a coordinate system's origin is canceled in the difference of $\boldsymbol{r}_2$ and $\boldsymbol{r}_1$.

Differences and differentials

Consider a moving object traveling through space along some trajectory $\boldsymbol{r}(t)$, and imagine repeatedly observing the object's position. Now imagine measuring this position more and more frequently—over smaller and smaller time intervals. Of course, in practice you can't measure over arbitrarily small times—the smallest time interval can be as short, but no shorter, than the available experimental equipment can accurately measure. This experimentally bounded time interval between successive position measurements is a *finite*, measurable quantity, and is mathematically classified as a *difference*—we denote it by Δt. Associated with each interval Δt is the corresponding displacement $\Delta \boldsymbol{r}$ between the body's position at the, say, i-th and $(i+1)$-th measurement, i.e. the change in position between two subsequent measurements. Interpreted physically, $\Delta \boldsymbol{r}$ encodes the net direction and distance the body moved between two measurements. Like Δt, the difference $\Delta \boldsymbol{r}$ is finite and measurable and (assuming the body is not at rest) can only be as small, but no smaller, as the available experimental equipment allows.

But in *theory*—and this concept might be familiar from a differential calculus course from high school—the time interval between successive displacement measurements, and thus the corresponding displacement, can be made arbitrarily small. So small, in fact, that successive measurements are separated by nothing more than a hypothetical instant. This "infinitely small" time interval is mathematically classified as a *differential*, and we denote it by $\mathrm{d}t$. The associated differential change in position during the time $\mathrm{d}t$ is denoted by $\mathrm{d}\boldsymbol{r}$. Differentials are *infinitesimal*—they are so small that they exist only in theory, but are too small to be actually measured. Loosely, it may help to think of $\mathrm{d}t$ as the limit of the observation interval Δt as Δt approaches zero, i.e.

$$\mathrm{d}t = \lim_{\Delta t \to 0} \Delta t. \tag{2.3}$$

Note that if you are not familiar with limits and differentials from high school, it would be completely understandable if these concepts don't yet make sense on the basis of the above explanations alone. We are covering the material much faster than one would in a dedicated differential calculus course, since most students will have already seen the material in high school. If that is not the case for you, don't worry—you will catch up soon in *Matematika 1*, and for now just try to understand the general concepts.

Relating infinitesimal and differential quantities

Our goal in this section is to show how to analytically relate differences—which are finite—and differentials—which are infinitesimal.

- Let t_0 and t_N denote the time at the first and last measurement, and let $\boldsymbol{r}_0 = \boldsymbol{r}(t_0)$ and $\boldsymbol{r}_N = \boldsymbol{r}(t_N)$ denote the positions of the measured body at the initial and final times t_0 and t_N.
- Let $\Delta t = t_N - t_0$ and $\Delta \boldsymbol{r} = \boldsymbol{r}_N - \boldsymbol{r}_0$ denote the differences in initial and final time, and initial and final position, for the entire experiment.
- Let $\Delta t_i = t_i - t_{i-1}$ and $\Delta \boldsymbol{r}_i = \boldsymbol{r}_i - \boldsymbol{r}_{i-1}$ denote the differences in time and position between the i-th and $(i-1)$-th measurements.

We can then relate the net changes in time and position Δt and $\Delta \boldsymbol{r}$ for the entire experiment to the changes over individual measurements according to

$$\Delta t = \sum_{i=1}^{N} \Delta t_i \qquad \text{and} \qquad \Delta \boldsymbol{r} = \sum_{i=1}^{N} \Delta \boldsymbol{r}_i. \tag{2.4}$$

Next, consider the (theoretical) limit of infinitely frequent measurements, meaning that $N \to \infty$ and $\Delta t_i \to 0$ for all i. In this limit case (2.4) generalizes to

$$\Delta t = \lim_{N \to \infty} \sum_{i=1}^{N} \Delta t_i \overset{\text{(a)}}{\equiv} \int_{t_0}^{t_N} \mathrm{d}t \qquad \text{and} \qquad \Delta \boldsymbol{r} = \lim_{N \to \infty} \sum_{i=1}^{N} \Delta \boldsymbol{r}_i \overset{\text{(b)}}{\equiv} \int_{\boldsymbol{r}_0}^{\boldsymbol{r}_N} \mathrm{d}\boldsymbol{r}, \tag{2.5}$$

where in (a) and (b) we have written the infinite sums in a more convenient notation using *integral symbols*. Equation (2.5) is important—it provides a formalism for converting between finite quantities and differentials. Loosely, but instructively, we can interpret the relationship between differentials and differences in (2.5) as follows:

(a) First, for orientation, we stress that the differences Δt and $\Delta \boldsymbol{r}$ are *finite* quantities (i.e. macroscopic and measurable), while the differentials $\mathrm{d}t$ and $\mathrm{d}\boldsymbol{r}$ are *infinitesimal* quantities (i.e. so small they exist only in a theoretical sense).

(b) We then interpret (2.5) as formalizing the intuitive idea that a sum of infinitely many infinitesimally small quantities produces a finite quantity.

Mathematically, the procedure we have just performed is called Riemann integration—you will study it formally in *Matematika 1* in the chapters on integral calculus. Limits and infinite sums like in (2.5) raise questions of convergence, and in *Matematika 1* you will cover the conditions that the functions t and $\boldsymbol{r}$ must satisfy for the sums in (2.5) to be well-defined. Fortunately, physical quantities in Newtonian mechanics are well-behaved, and in this course we won't have to worry about convergence. For our purposes, for the time being, we are satisfied with the following ideas:

- We convert from between differentials and differences (e.g. between $\mathrm{d}\boldsymbol{r}$ and $\Delta\boldsymbol{r}$) through a sum of infinitely many infinitesimal quantities, which is formally called a Riemann integral.
- If a sum of individual differences (e.g. Δt_i or $\Delta \boldsymbol{r}_i$) converges to a finite net quantity (e.g. Δt or $\Delta\boldsymbol{r}$) for ever more frequent measurements (larger N) and continually smaller times and displacements, *within the scope of physical measurement*, we'll assume the sum's limit, as defined in (2.5), exists and is well-defined.

2.2.3 Total distance traveled

Suppose you travel from Ljubljana to Cambridge and then return to Ljubljana. Since the journey begins and ends in Ljubljana, the journey's displacement is zero, but—as anyone who has made the journey can immediately tell you—the total distance traveled is certainly not zero. In fact, the net displacement along any trajectory with the same start and end position (for example any closed loop) is zero, but the trajectory's total arc length could take on any value. Motivated by the wish to better distinguish such trajectories, we associate total distance traveled with a new physical quantity, typically denoted by s. Precisely, the total distance s traveled along a trajectory $\boldsymbol{r}(t)$ beginning at initial position $\boldsymbol{r}_1$ and ending at final position $\boldsymbol{r}_2$ is

$$s = \int_{\boldsymbol{r}_1}^{\boldsymbol{r}_2} |\mathrm{d}\boldsymbol{r}| \overset{\text{(a)}}{=} \int_{\boldsymbol{r}_1}^{\boldsymbol{r}_2} \mathrm{d}s, \tag{2.6}$$

where in (a) we have introduced the shorthand notation $\mathrm{d}s = |\mathrm{d}\boldsymbol{r}|$ to denote the magnitude of the vector differential $\mathrm{d}\boldsymbol{r}$.[16] Equation (2.6) formalizes a simple idea: to get the total distance traveled over an entire journey, divide the journey into many small steps $\Delta\boldsymbol{r}$, record the length $\mathrm{d}s = |\Delta\boldsymbol{r}|$ of each step as you go, and add the lengths together. Intuitively, imagine following a trajectory through space from $\boldsymbol{r}_1$ to $\boldsymbol{r}_2$ and maintaining a running sum of your step lengths as you go. We stress that s is a scalar quantity—this follows (i) intuitively because total distance traveled is just a single number and (ii) mathematically because integrating a scalar magnitude $|\mathrm{d}\boldsymbol{r}|$ produces a scalar result.

Computing total distance traveled

[16]The use of $\mathrm{d}s$ (instead of $\mathrm{d}r$) to represent the magnitude $|\mathrm{d}\boldsymbol{r}|$ is intentional, since $\mathrm{d}r$ can be confused with the differential of the radial coordinate in spherical coordinate systems, while $\mathrm{d}s$ is exclusively used for total distance traveled.

In Cartesian coordinates, we compute $\mathrm{d}s$ with the Pythagorean theorem,

$$\mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2, \tag{2.7}$$

where in this context $\mathrm{d}s^2$ is shorthand for $(\mathrm{d}s)^2$ and not for $\mathrm{d}(s^2)$. In other coordinate systems the expression for $\mathrm{d}s$ can be more complicated, and in two or more spatial dimensions practical computation of (2.6) involves a line integral, which is a subject we leave for *Matematika 3*. You won't be expected to compute line integrals in this course; the point for now is to introduce the *concept* of total distance traveled and show how one formalizes the idea, as in (2.6), in the language of vector calculus.

2.2.4 Velocity

Recall from Section 2.2.2 the idea of observing a body moving through space along some trajectory $\boldsymbol{r}(t)$, and continuously measuring the body's position over smaller and smaller time intervals Δt. Let $\Delta \boldsymbol{r}$ denote the body's change in position over a measurement interval Δt. Then associated with each measurement pair $(\Delta \boldsymbol{r}, \Delta t)$ is an *average velocity*, defined as

$$\overline{\boldsymbol{v}} = \frac{\Delta \boldsymbol{r}}{\Delta t}. \tag{2.8}$$

Interpreted physically, $\overline{\boldsymbol{v}}$ encodes in which direction and how fast a body moved between the two different positions and times associated with the differences $\Delta \boldsymbol{r}$ and Δt. From vector algebra, a vector's direction remains the same after multiplication or division with a scalar, so $\overline{\boldsymbol{v}}$ has the same direction in space as the displacement $\Delta \boldsymbol{r}$ (since Δt is a scalar). And because the time interval Δt carries a physical unit, i.e. time, the quotient $\Delta \boldsymbol{r}/\Delta t$ is a new physical quantity—which we have already defined as velocity—with units of distance per time (meters per second in SI units).

Instantaneous velocity

The question that naturally arises next is:

> If (2.8) gives a body's velocity between *two different* positions and times along a given trajectory, what is a body's velocity at *every* position and time?

The desired quantity is called *instantaneous velocity* and is conventionally denoted by $\boldsymbol{v}(t)$. To determine velocity at every point on a given trajectory, one observes the average velocity $\overline{\boldsymbol{v}} = \Delta \boldsymbol{r}/\Delta t$ over smaller and smaller time intervals—mathematically, in the limit as $\Delta t \to 0$. Intuitively, this just means observing the moving body's position at every possible instant in time. In any case, instantaneous velocity is defined as

$$\boldsymbol{v} = \lim_{\Delta t \to 0} \frac{\Delta \boldsymbol{r}}{\Delta t} = \frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}. \tag{2.9}$$

In Cartesian coordinates, instantaneous velocity is represented component-wise as

$$\boldsymbol{v} = (v_x, v_y, v_z) = \left(\frac{\mathrm{d}x}{\mathrm{d}t}, \frac{\mathrm{d}y}{\mathrm{d}t}, \frac{\mathrm{d}z}{\mathrm{d}t}\right). \tag{2.10}$$

Convergence of the difference quotient

The limit in (2.9) raises the question of convergence. Loosely, the expression $\Delta \boldsymbol{r}/\Delta t$

as $\Delta t \to 0$ is essentially the quotient $\mathbf{0}/0$, which is undefined in general. But in the scope of physics, the limit in (2.9) exists and converges to well-defined velocity (at least on physical grounds this is obvious: any physical object's velocity is always finite—you cannot walk down the street and encounter a body with infinite velocity). On mathematical grounds, the limit defining $\boldsymbol{v}$ exists under the assumption that the trajectory $\boldsymbol{r}(t)$ is a continuously differentiable function of time.

You will study the existence of limits and difference quotients formally using the tools of real analysis developed in *Matematika 1*, but in classical physics there is not much to worry about. Because physical quantities are generally well-behaved, we can safely assume expressions like (2.9) are well-defined. That said, it is important to mention the general issue of convergence at least once.[17]

In the language of differential calculus, which you have probably encountered in high school and will soon cover in *Matematika 1*, the instantaneous velocity from (2.9) is called the *first derivative of position with respect to time.* Mathematicians and physicists have come up with various ways to write derivatives, all of which are equivalent. Here are some examples in the case of velocity:

$$\boldsymbol{v} = \frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t} = \dot{\boldsymbol{r}}(t) = \boldsymbol{r}'(t). \tag{2.11}$$

In physics, a dot over a quantity, as in $\dot{\boldsymbol{r}}$, universally denotes the quantity's derivative with respect to time. You'll see plenty of this dot notation in the second-year course *Klasična mekanika*. The notation $\boldsymbol{r}'(t)$ is more common in mathematics—a primed function denotes the function's derivative with respect to its argument. When a function's argument is time, such as $\boldsymbol{r}(t)$, the notations $\dot{\boldsymbol{r}}(t)$ and $\boldsymbol{r}'(t)$ are equivalent.

Finally, we note that the direction of the instantaneous velocity vector $\boldsymbol{v}$ at time t is tangent to the corresponding trajectory $\boldsymbol{r}(t)$. Loosely, this is just a generalization of average velocity $\overline{\boldsymbol{v}}$ being parallel to the corresponding secant line $\Delta\boldsymbol{r}$; in the limit of instantaneous velocity, the secant line to the trajectory converges to the tangent line. More formally, you will show that velocity is tangent to trajectory in the lectures on space curves in *Matematika 1* and *Matematika 3*.

Velocity and differential equations

We now return to the concept, first introduced in (2.5), of converting between finite and infinitesimal quantities. Specifically, we will show how, given a particle's velocity $\boldsymbol{v}(t)$, it is possible to recover the particle's trajectory by solving a *differential equation* involving the infinitesimals $\mathrm{d}\boldsymbol{r}$ and $\mathrm{d}t$.

We begin with the definition of velocity, $\boldsymbol{v} = \mathrm{d}\boldsymbol{r}/\,\mathrm{d}t$, and, loosely, imagine multiplying through[18] by $\mathrm{d}t$ to get

$$\boldsymbol{v} = \frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t} \quad \longrightarrow \quad \mathrm{d}\boldsymbol{r} = \boldsymbol{v}\,\mathrm{d}t. \tag{2.12}$$

[17] In physics questions like these don't bother us too much, largely because they don't have too—physicists are blessed with quantities described by well-behaved functions. We are more interested in the practical use of given equations instead of the details of their existence. In fact, for better or worse, physicists often take pleasure in a tongue-in-cheek disregard for mathematical rigor. But this is of course tongue-in-cheek, and any serious physicist will have a strong training in mathematics.

[18] Formally, multiplication by differentials is not quite the same as regular scalar multiplication, even though the process appears analogous. But the details are beyond the scope of this course—you will discuss this in *Matematika 1*.

Caution: when manipulating equations involving differentials, the left- and right-hand sides of the resulting equality *must* be of the same multiplicity in the differentials. For example, $\mathrm{d}\boldsymbol{r} = v\,\mathrm{d}t$ is a valid expression (both sides involve first-order differentials) while $\mathrm{d}\boldsymbol{r} = v \cdot t$ is not (the left-hand side contains a first-order differential and the right-hand side does not contain any differentials). Remember that a differential is an infinitesimally small quantity; loosely, a differential on one side of an equation but not the other would be like equating a finite value to zero.

In any case, to recover a finite-valued trajectory $\boldsymbol{r}(t)$ from the differential equation $\mathrm{d}\boldsymbol{r} = \boldsymbol{v}\,\mathrm{d}t$, we must integrate both sides of the equation, written schematically as

$$\int \mathrm{d}\boldsymbol{r} = \int \boldsymbol{v}\,\mathrm{d}t. \tag{2.13}$$

Of course we must also specify the integration limits, and in this context there are two common choices:

(a) One could integrate over a *definite* interval, say from t_1 to t_2 for time and $\boldsymbol{r}_1 \equiv \boldsymbol{r}(t_1)$ to $\boldsymbol{r}_2 \equiv \boldsymbol{r}(t_2)$ for position. In this case both the upper and lower integration limits are fixed values.

(b) Alternatively, one integrates over an *indefinite* interval in which the lower limits, say t_0 and $\boldsymbol{r}_0 \equiv \boldsymbol{r}(t_0)$ are fixed, but the upper limits are allowed to vary arbitrarily and are written as the generic functions t and $\boldsymbol{r}(t)$.

Depending on the notation (we will comment on this shortly), these two choices could be written

$$\int_{\boldsymbol{r}_1}^{\boldsymbol{r}_2} \mathrm{d}\boldsymbol{r} = \int_{t_1}^{t_2} \boldsymbol{v}(t)\,\mathrm{d}t \tag{2.14a}$$

$$\int_{\boldsymbol{r}_0}^{\boldsymbol{r}(t)} \mathrm{d}\boldsymbol{\rho} = \int_{t_0}^{t} \boldsymbol{v}(\tau)\,\mathrm{d}\tau. \tag{2.14b}$$

In either case the limits of integrations over time and position must correspond. For example, in (2.14a) the position limit $\boldsymbol{r}_2 = \boldsymbol{r}(t_2)$ is the position at the time limit t_2, just like $\boldsymbol{r}(t_1)$ is the position at time t_1.

We now comment on the notation in (2.14b). When, as in (2.14b), the symbol for the variable of integration would clash with the symbol for a limit of integration, one should change the integration variable to a different symbol. Since that might sound vague, here is an example of what *not* to do:

$$\int_{\boldsymbol{r}_0}^{\boldsymbol{r}(t)} \mathrm{d}\boldsymbol{r} = \int_{t_0}^{t} \boldsymbol{v}(t)\,\mathrm{d}t. \qquad \text{(example of poor notation)} \tag{2.15}$$

Notice how the upper integration limits—$\boldsymbol{r}(t)$ and t—clash with the variables of integration, which are also written as $\boldsymbol{r}$ and t.[19] Although admittedly somewhat

[19] On the RHS the clash involving time t is immediately obvious; on the LHS, even though the limit $\boldsymbol{r}(t)$ and differential $\mathrm{d}\boldsymbol{r}$ don't match explicitly, we must recall that the differential $\mathrm{d}\boldsymbol{r}$ is a function of time and could really be written $\mathrm{d}\boldsymbol{r}(t)$, in which case the clash with the limit $\boldsymbol{r}(t)$ is clear. We generally don't write differentials' functional dependence explicitly, since equations would get too cluttered, and infer their dependence on other variables from context.

pedantic, this sort of conflict means the integral is not mathematically well-defined. To resolve this clash, one modifies the symbol used for the integration variable—two common choices are adding a tilde, e.g. $\widetilde{\boldsymbol{r}}$ to replace $\boldsymbol{r}$, or replacing the letter with its Greek equivalent, e.g. $\boldsymbol{\rho}$ for $\boldsymbol{r}$ or τ for t. The result would be

$$\int_{\boldsymbol{r}_0}^{\boldsymbol{r}(t)} \mathrm{d}\widetilde{\boldsymbol{r}} = \int_{t_0}^{t} v(\widetilde{t}) \, \mathrm{d}\widetilde{t} \qquad \text{or} \qquad \int_{\boldsymbol{r}_0}^{\boldsymbol{r}(t)} \mathrm{d}\boldsymbol{\rho} = \int_{t_0}^{t} \boldsymbol{v}(\tau) \, \mathrm{d}\tau, \tag{2.16}$$

just like in (2.14b). Of course, since physicists are sloppy with notation, you will see plenty of examples of (2.15) during your course of study, but keep in mind that (2.16) is a better choice, since the integration variable and limit don't conflict. And, if you see strange Greek letters appearing in integral expressions, you now know why.

After commenting on notation, we now return to the initial goal of solving (2.14) for a trajectory $\boldsymbol{r}(t)$; for review, (2.14) reads

$$\int_{\boldsymbol{r}_1}^{\boldsymbol{r}_2} \mathrm{d}\boldsymbol{r} = \int_{t_1}^{t_2} \boldsymbol{v}(t) \, \mathrm{d}t \tag{2.17a}$$

$$\int_{\boldsymbol{r}_0}^{\boldsymbol{r}(t)} \mathrm{d}\boldsymbol{\rho} = \int_{t_0}^{t} \boldsymbol{v}(\tau) \, \mathrm{d}\tau, \tag{2.17b}$$

We treat the two cases separately:

(a) Integrating the left-hand side of (2.17a) gives

$$\boldsymbol{r}_2 - \boldsymbol{r}_1 = \int_{t_1}^{t_2} \boldsymbol{v}(t) \, \mathrm{d}t \implies \boldsymbol{r}_2 = \boldsymbol{r}_1 + \int_{t_1}^{t_2} \boldsymbol{v}(t) \, \mathrm{d}t \tag{2.18}$$

Interpreted physically, (2.18) expresses the final position on the trajectory $\boldsymbol{r}_2$ in terms of the initial position $\boldsymbol{r}_1$ and an integral of velocity over the interval $[t_1, t_2]$. By the way, (2.5), which we met earlier, is an example of this type of definite integral.

(b) Integrating the left-hand side of (2.14b) gives

$$\boldsymbol{r}(t) - \boldsymbol{r}_0 = \int_{t_0}^{t} \boldsymbol{v}(t) \, \mathrm{d}t \implies \boldsymbol{r}(t) = \boldsymbol{r}_0 + \int_{t_0}^{t} \boldsymbol{v}(t) \, \mathrm{d}t. \tag{2.19}$$

Interpreted physically, (2.19) gives a body's trajectory $\boldsymbol{r}(t)$ in terms of an initial position $\boldsymbol{r}_0$ and an integral of velocity from the initial time t_0 to the time t at which the trajectory is evaluated.

In either case, actually solving for $\boldsymbol{r}_2$ or $\boldsymbol{r}(t)$ requires evaluating the integral of velocity; this could be anywhere from trivial to analytically impossible, depending on the functional form of $\boldsymbol{v}(t)$.

2.2.5 Acceleration

Having just introduced velocity—the rate of change of *position* with respect to time—it is natural to ask how a body's *velocity* itself changes with time. The rate of change of velocity with respect to time is called *acceleration*, and is derived in an analogous manner to the derivation of velocity in Section 2.2.4—it might be helpful to revisit Section 2.2.4 now.

Average acceleration

As in Section 2.2.4 and Section 2.2.2 before it, imagine observing a body moving through space along some trajectory $\boldsymbol{r}(t)$. To derive acceleration, imagine repeatedly measuring the body's velocity $\boldsymbol{v}(t)$, and continuously computing the changes in the body's velocity $\Delta \boldsymbol{v}$ between sequential measurements. Once again, let Δt denote the difference in time between subsequent measurements. Then associated with each measurement pair $(\Delta \boldsymbol{v}, \Delta t)$ is an *average acceleration*, defined as

$$\overline{\boldsymbol{a}} = \frac{\Delta \boldsymbol{v}}{\Delta t}. \tag{2.20}$$

Interpreted physically, $\overline{\boldsymbol{a}}$ encodes in which direction, and how quickly, a body's *velocity* changed over the measurement interval Δt. Note that an immediate, intuitive interpretation of acceleration is often difficult for people, so don't worry if it takes some time to come to terms with the concept. This is probably because the majority of daily life—besides brief spurts of speeding up or slowing down—is carried out with constant velocity, or zero acceleration. In any case, you will soon become familiar with acceleration from your study of mechanics.

Instantaneous acceleration

To derive instantaneous acceleration, just like in the derivation of instantaneous velocity, one imagines computing average acceleration $\overline{\boldsymbol{a}} = \Delta \boldsymbol{v}/\Delta t$ over smaller and smaller time intervals—mathematically, in the limit as $\Delta t \to 0$. The resulting instantaneous acceleration is defined as

$$\boldsymbol{a} = \lim_{\Delta t \to 0} \frac{\Delta \boldsymbol{v}}{\Delta t} = \frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t}. \tag{2.21}$$

Acceleration is the first derivative of velocity with respect to time and has SI units of meters per second squared. Acceleration can also be interpreted as the second derivative of position with respect to time, which is clear if one writes

$$\boldsymbol{a} = \frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} \overset{\text{(a)}}{=} \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}\right) \overset{\text{(b)}}{=} \frac{\mathrm{d}^2\boldsymbol{r}}{\mathrm{d}t^2}, \tag{2.22}$$

where in (a) we have used the definition of velocity $\boldsymbol{v} = \mathrm{d}\boldsymbol{r}/\,\mathrm{d}t$ and in (b) we have introduced the standard notation for higher-order derivatives, which you will soon encounter in *Matematika 1*. In Cartesian coordinates, instantaneous acceleration has the following, equivalent, coordinate representations:

$$\boldsymbol{a} = (a_x, a_y, a_z) = \left(\frac{\mathrm{d}v_x}{\mathrm{d}t}, \frac{\mathrm{d}v_y}{\mathrm{d}t}, \frac{\mathrm{d}v_z}{\mathrm{d}t}\right) = \left(\frac{\mathrm{d}^2 x}{\mathrm{d}t^2}, \frac{\mathrm{d}^2 y}{\mathrm{d}t^2}, \frac{\mathrm{d}^2 z}{\mathrm{d}t^2}\right). \tag{2.23}$$

Decomposition of acceleration

Consider a body moving along a trajectory through space with velocity $\boldsymbol{v}(t)$ and acceleration $\boldsymbol{a}(t)$. For the decomposition of acceleration (which we will introduce shortly) to make sense, let $\hat{\mathbf{v}}$ denote the dimensionless unit vector[20] in the direction

[20]The hat notation (e.g. $\hat{\mathbf{v}}$) is conventionally used to denote dimensionless direction vectors of unit norm—for example, you might have seen $\hat{\mathbf{x}}$ or $\hat{\mathbf{\imath}}$ used to denote the direction in space of the x axis.

of the body's instantaneous velocity,

$$\hat{\mathbf{v}}(t) = \frac{\boldsymbol{v}(t)}{|\boldsymbol{v}(t)|} \overset{\text{(a)}}{\equiv} \frac{\boldsymbol{v}(t)}{v(t)}, \tag{2.24}$$

where in (a) we have introduced the common shorthand notation $v(t) = |\boldsymbol{v}(t)|$ for the magnitude of the body's velocity—this is easier to write than $|\boldsymbol{v}(t)|$. In everyday language, $\hat{\mathbf{v}}(t)$ is just the direction in which the body is moving at time t. More precisely, $\hat{\mathbf{v}}$ is the unit tangent vector to the body's trajectory and would usually be written as $\mathbf{T}$ in the mathematical context of space curves.

Having introduced $\hat{\mathbf{v}}$, it is often useful to decompose the body's acceleration $\boldsymbol{a}$ into two independent components:

1. a component $\boldsymbol{a}_{\parallel}$ *parallel* to the body's instantaneous velocity, i.e. parallel to $\hat{\mathbf{v}}$, and

2. a component $\boldsymbol{a}_{\perp}$ *perpendicular* to the body's instantaneous velocity, i.e. perpendicular to $\hat{\mathbf{v}}$.

One could then decompose a body's total acceleration into the vector sum

$$\boldsymbol{a} = \boldsymbol{a}_{\parallel} + \boldsymbol{a}_{\perp}. \tag{2.25}$$

Both components have instructive physical interpretations:

- The component of acceleration parallel to velocity changes the *magnitude* of a body's velocity. In everyday language, $\boldsymbol{a}_{\parallel}$ is responsible for a body speeding up or slowing down.

- The component of acceleration perpendicular to velocity changes the *direction* of velocity. In everyday language, $\boldsymbol{a}_{\perp}$ is responsible for making a body's path twist, turn, or bend as the body moves through space.

Our goal here is just to gain some insight into what acceleration physically does, and we introduced the $\boldsymbol{a} = \boldsymbol{a}_{\parallel} + \boldsymbol{a}_{\perp}$ decomposition in the hope that changing a body's speed and direction of motion are two easily-understood concepts. Note that actually computing the components $\boldsymbol{a}_{\parallel}$ and $\boldsymbol{a}_{\perp}$ for the general case of arbitrary three-dimensional motion is more complicated and requires the study of space curves, a branch of vector analysis. You will analyze this general case in the lectures on the Frenet-Serret formulas in *Matematika 1* and in a dedicated chapter on space curves in *Matematika 3*. In this course we will return to the above decomposition of acceleration in a few lectures, but we will mostly restrict ourselves to the special case of circular motion.

2.2.6 Kinematics of one-dimensional free fall

In this section we analyze the kinematics of a point mass in free fall in a uniform gravitational field. Physically, imagine a point-like body launched straight upward or straight downward with initial speed v_0, then left to fall under the influence of gravity. We will find that the object's height, say $z(t)$, changes quadratically with time according to

$$z(t) = z_0 + v_0 t - \frac{g}{2}t^2, \tag{2.26}$$

where z_0 and v_0 are the initial height and velocity, and g is the magnitude of the gravitational acceleration. For simplicity, we begin with a one-dimensional analysis and neglect air resistance, and return to two-dimensional free fall in the next section.

Note: Choosing a coordinate system

We must first define a coordinate system in which to perform the analysis. Since choosing coordinate systems will come up again and again in your studies, now is a good opportunity to offer some general advice.

- *In theory*, the choice of coordinate system used to analyze any physical problem is arbitrary. Coordinate systems are a human construction used to facilitate analysis—physical objects have no knowledge of them and behave the same way regardless of the choice of coordinate system. A tossed ball, for example, will fall back down to the Earth in whatever coordinate system you choose to analyze it. You'll get the same physical results in any coordinate system, it will just be *much easier in a well-chosen one*. This brings us to the second point.
- *In practice*, a good choice of coordinate system can dramatically simplify the mathematical analysis needed to solve a physical problem. There is often a particular coordinate system well-suited to a particular problem. As your studies progress, you will find that good coordinate systems tend to reflect a problem's symmetries or preferential directions.

The problem of a free-falling body has a clear preferential direction—the direction of the gravitational acceleration, which we represent with the vector quantity $\boldsymbol{g}$. We will solve this problem in a Cartesian coordinate system whose z axis points opposite the direction of $\boldsymbol{g}$ and thus perpendicularly upward from the Earth's surface. In everyday language, this just means the z axis points in the direction we think of as "up", while $\boldsymbol{g}$ points "down". Since we consider only one-dimensional motion along the axis of free fall, there is no need to bother defining either the x or y axes.[21]

Solving for the falling body's velocity

A free-falling object, in the absence of air resistance, accelerates with the acceleration of the ambient gravitational field, which we have called $\boldsymbol{g}$. The current problem has a *uniform* gravitational field, which means that $\boldsymbol{g}$ is constant—this considerably simplifies analysis. Let $g \equiv |\boldsymbol{g}|$ denote the magnitude of the gravitation acceleration; for orientation, the value of g on the Earth's surface is approximately $g \approx 9.8\,\mathrm{m\,s^{-2}}$.

In this problem's Cartesian coordinate system, $\boldsymbol{g}$ can be written in either vector or component form as

$$\boldsymbol{g} = -g\,\hat{\mathbf{e}}_z \qquad \text{or} \qquad \boldsymbol{g} = (0, 0, -g), \tag{2.27}$$

where $\hat{\mathbf{e}}_z$ is the unit vector in the direction of the z axis. The minus sign in corresponds to the fact that $\boldsymbol{g}$ points opposite $\hat{\mathbf{e}}_z$ or, in everyday language, the fact that $\boldsymbol{g}$ points "down" while $\hat{\mathbf{e}}_z$ points "up". Since a free-falling body accelerates with the acceleration of the ambient gravitational field, the body's acceleration $\boldsymbol{a}$ is just

$$\boldsymbol{a} = \boldsymbol{g} \qquad \text{or, in component form,} \qquad (a_x, a_y, a_z) = (0, 0 - g). \tag{2.28}$$

[21] More precisely, one might say that the problem of one-dimensional free fall is *invariant under rotation* about the axis of free fall. This means that as long as the z axis points upward, our analysis will be the same, and produce the same result, regardless of how we rotate the x and y axes with respect to the z axis.

The equation's z component, which corresponds to vertical motion, is

$$a_z = -g. \tag{2.29}$$

In fact, the simple equation $a_z = -g$ encodes everything there is to know about a free-falling particle in one dimension. Our goal is to solve (2.29) for the falling body's position and velocity as a function of time. Like in (2.12) in the discussion of velocity and differential equations, we first turn $a_z = -g$ into a differential equation:

$$a_z \overset{(a)}{=} \frac{\mathrm{d}v_z}{\mathrm{d}t} = -g \implies \mathrm{d}v_z = -g\,\mathrm{d}t, \tag{2.30}$$

where in (a) we have used the definition of acceleration $a_z = \mathrm{d}v_z/\,\mathrm{d}t$ from (2.23). For shorthand, let $v \equiv v_z$ and $a \equiv a_z$, since there is only one dimension involved and the subscripts would only clutter the problem. Dropping the subscripts, (2.30) reads

$$\mathrm{d}v = -g\,\mathrm{d}t. \tag{2.31}$$

Along the same lines as (2.14b) in the section on velocity and differential equations, we first let $v_0 \equiv v(t_0)$ denote the free-falling body's vertical velocity at some initial time t_0, then integrate (2.31) from t_0 to an arbitrary time t to get

$$\int_{v_0}^{v(t)} \mathrm{d}\nu = v(t) - v_0 = \int_{t_0}^{t} (-g)\,\mathrm{d}\tau = -g\int_{t_0}^{t} \mathrm{d}\tau = -g(t - t_0). \tag{2.32}$$

After clearing things up and rearranging we have

$$v(t) - v_0 = -g(t - t_0) \implies v(t) = v_0 - g(t - t_0). \tag{2.33}$$

Equation (2.33) is the general result for the velocity of a free-falling object in one dimension as a function of time. If we choose to begin counting time at t_0, so that in this problem $t_0 = 0$, we recover the familiar kinematics formula

$$v(t) = v_0 - gt. \tag{2.34}$$

Solving for the falling body's position

Next, we will solve (2.34)[22] for the free-falling particle's position $z(t)$. As in (2.30), we first write (2.34) as a differential equation using the definition of velocity to get

$$v(t) = \frac{\mathrm{d}z}{\mathrm{d}t} = v_0 - gt. \tag{2.35}$$

We then rearrange, let $z_0 \equiv z(0)$, and integrate (recalling the choice $t_0 \equiv 0$) to get

$$\int_{z_0}^{z(t)} \mathrm{d}\zeta = \int_0^t v_0\,\mathrm{d}\tau - \int_0^t g\tau\,\mathrm{d}\tau. \tag{2.36}$$

(We have used the Greek letters zeta and tau for integration variables, as discussed in the context of (2.14b).) Solving the integral produces

$$z(t) - z_0 = v_0 t - \frac{g}{2}t^2 \implies z(t) = z_0 + v_0 t - \frac{g}{2}t^2. \tag{2.37}$$

[22] One could also solve the more general (2.33), which leaves t_0 arbitrary, but for our present purposes dragging along the extra t_0 just muddles the message without introducing any new physics.

Equation (2.37) gives the position of a free-falling particle in one dimension as a function of time, assuming time is counted so that the particle began falling at $t_0 = 0$.

At what time is the body at a given height?

We now ask the following question: if a body is launched upward at time $t_0 = 0$ from an initial height z_0, how much time does it take to reach an arbitrary height z? To answer the question, we first rearrange (2.37) into a standard-form quadratic equation in time, i.e.

$$\frac{g}{2}t^2 - v_0 t + z - z_0 = 0. \tag{2.38}$$

Our idea is to solve for t as a function of z. To do this, we apply the quadratic formula with $a = g/2$, $b = -v_0$ and $c = (z - z_0)$ to get

$$t = \frac{v_0 \pm \sqrt{v_0^2 - 2g(z - z_0)}}{g}. \tag{2.39}$$

By the fundamental theorem of algebra, (2.39) is guaranteed to have two (in general complex) solutions. But before diving right into mathematics, it's more instructive to think about the problem physically. Forgetting equations for a moment, imagine an object launched vertically upward at $t_0 = 0$ with initial velocity v_0 from initial height z_0. Everyday intuition holds good here: the object will intially rise, reach a maximum height $z_{\text{max}} > z_0$ at which it stops rising, and then begin falling back down. We can immediately recognize the following: the object...

(a) ...won't ever reach heights above z_{max}, so (2.39) shouldn't have real-valued solutions for $z > z_{\text{max}}$.

(b) ...reaches heights in the range z_0 to z_{max} twice—once coming up and once coming down. We thus expect two solutions to (2.39) for $z \in (z_0, z_{\text{max}})$, both at times after the object was launched.

(c) ...reaches its initial height z_0 twice—once exactly at launch time (at $t = 0$) and once at a later time when falling back down.

(d) ...reaches heights less than the initial height z_0 after passing z_0 on the way down, i.e. for times larger than $2v_0/g$.

After examining the problem physically, we can follow up with a mathematical analysis of each point predicted in the heuristic analysis above

(a) Equation (2.39) has no real solutions when the discriminant is zero, which occurs for height z larger than

$$v_0^2 - 2g(z - z_0) < 0 \implies z > \frac{v_0^2}{2g} + z_0. \tag{2.40}$$

The maximum height mentioned in point (a) above is thus

$$z_{\text{max}} = \frac{v_0^2}{2g} + z_0. \tag{2.41}$$

Jumping ahead somewhat, in the future lectures on energy we will find that the object cannot pass z_{max} if its initial kinetic energy is too small to overcome the gravitational potential energy at z_{max}.

(b) It is a straightforward exercise in grade school algebra to show that (2.39) has two positive solutions only for z in the range $(z_0, z_{\max})$. In this problem positive times are those after the object was launched, so this mathematical result agrees with the physical intuition of point (b) above.

(c) Similarly, (2.39) has two solutions for $z = z_0$, once at $t = 0$ (launch time) and once at $t = 2v_0/g$, in agreement with point (c) above.

(d) Careful here—(2.39) has *two* solutions for any $z < z_0$, one negative and one positive. Suppose you solve (2.39) for a height $z_- < z_0$ get the two solutions t_- and t_+. Physically, the positive time corresponds to the falling-down phase predicted in point (d) above.

What the negative solution tells you is this: if the object had been traveling along the trajectory determined by its initial height and velocity for *all times*, and had passed through the launch point with velocity v_0 at time $t_0 = 0$, it would have passed through the height z_- (on the way up) t_- seconds *before* passing reaching the launch point z_0.

It is tempting to throw this answer out with an argument like "negative times are non-physical", but that's not quite correct. *Complex-valued* times are non-physical—this is why the object never reaches heights above $z_{\max}$. But negative times can be perfectly physical—which times are negative and which are positive only depends on the arbitrary decision of when you begin to start counting time, just like which heights are negative and which are positive depends on where you define a coordinate system's origin.

Solving for final velocity

Finally, we answer the following question: if a body is launched upward at time $t_0 = 0$ from an initial height z_0, what is its velocity at an arbitrary height z? We begin with (2.34), $v(t) = v_0 - gt$, (the body's velocity as a function of time), and substitute in time as a function of height from (2.39). The result is

$$v = v_0 - gt \tag{2.42}$$

$$\stackrel{\text{(a)}}{=} v_0 - g\left(\frac{v_0 \pm \sqrt{v_0^2 - 2g(z - z_0)}}{g}\right) \tag{2.43}$$

$$= \pm\sqrt{v_0^2 - 2g(z - z_0)}, \tag{2.44}$$

where (a) uses (2.39). Like in the discussion in point (d) above, the two solutions correspond to a free-falling body reaching a given height twice—once going up and once going down. Note also that (2.44) has no real solutions for $z > z_{\max}$, which must be the case for consistency with point (a) above. It is also instructive to check that $v = 0$ at $z = z_{\max}$, which corresponds to a free-falling object stopping momentarily at its peak height $z_{\max}$ before falling back down again.

Additionally, to conceal questions of plus or minus, one can square (2.44), which produces the well-known kinematics formula

$$v^2 = v_0^2 - 2g(z - z_0). \tag{2.45}$$

A good lesson from (2.45) is that magnitude of final velocity v increases with an increasing distance $\Delta z = z - z_0$. Interpreted physically, this just means that the farther an object falls, the faster it gets.

In passing, is also possible, and instructive, to derive (2.45) using the chain rule from differential calculus. We begin with the definition of acceleration $a = \mathrm{d}v/\,\mathrm{d}t$ and apply the chain rule to get

$$a = \frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\mathrm{d}v}{\mathrm{d}z}\frac{\mathrm{d}z}{\mathrm{d}t} = v\frac{\mathrm{d}v}{\mathrm{d}z} \implies v\,\mathrm{d}v = a\,\mathrm{d}z. \tag{2.46}$$

We then integrate (2.46), producing

$$\int_{v_0}^{v} \nu\,\mathrm{d}\nu = \int_{z_0}^{z} a\,\mathrm{d}\zeta \implies \frac{1}{2}(v^2 - v_0^2) = a(z - z_0), \tag{2.47}$$

which we rearrange to get

$$v^2 = v_0^2 + 2a(z - z_0). \tag{2.48}$$

Substituting in $a = -g$ for free fall recovers (2.45).