Mathematics 4

Lecture notes from the second-year undergraduate course *Mathematics 4* (Matematika 4) given by prof. dr. Bojan Magajna at the Faculty of Mathematics and Physics at the University of Ljubljana in the academic year 2019-20. The course covers complex analysis, Fourier analysis, and ordinary and partial differential equations. Credit for the material covered in these notes is due to professor Magajna, while the voice, typesetting, and translation to English in this document are my own. Note: proofs are mostly missing, so these notes are incomplete.

Disclaimer: Mistakes—both simple typos and legitimate errors—are likely. Keep in mind that these are the notes of an undergraduate student in the process of learning the material himself—take what you read with a grain of salt. If you find mistakes and feel like telling me, for example by email, I'll be happy to hear from you, even for the most trivial of errors.

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1 Complex Analysis

1.1 Introduction to the Complex Plane

The complex numbers $\mathbb{C}$ are points in the complex plane. I use the terms complex numbers and complex plane synonymously.

1.1.1 Basic Properties of Functions from $\mathbb{R}$ to $\mathbb{C}$

Let $\gamma : [a, b] \subset \mathbb{R} \to \mathbb{C}$ be a function mapping from the real line to the complex plane. The function γ can be written in the form

$$\gamma(t) = \gamma_1(t) + i\gamma_2(t)$$

where $\gamma_1, \gamma_2 : [a, b] \to \mathbb{R}$ are both real functions and i is the imaginary number.

1. γ is continuous if γ_1 and γ_2 are both continuous.
2. γ is (continuously) differentiable if γ_1 and γ_2 are both (continuously) differentiable. In this case,
$$\gamma'(t) = \lim_{h\to 0} \frac{\gamma(t+h) - \gamma(t)}{h} = \gamma_1'(t) + i\gamma_2'(t)$$

1.1.2 Definition: Piecewise Continuously Differentiable

The function $\gamma : [a, b] \subset \mathbb{R} \to \mathbb{C}$ is *piecewise continuously differentiable* if the interval $[a, b]$ can be divided into a finite set of disjoint sub-intervals $I_1, I_2, \ldots, I_n$ where $[a, b] = I_1 \cup I_2 \cup \cdots \cup I_n$ such that γ is continuously differentiable on each sub-interval I_i.

1.1.3 Definitions: Path, Path-Connected Set and Region

1. A piecewise differentiable function $\gamma : [a, b] \subset \mathbb{R} \to \mathbb{C}$ is called a *path* in $\mathbb{C}$.
2. The open complex subset $U \in \mathbb{C}$ is *path-connected* if for all $\alpha, \beta \in U$ there exists a path $\gamma : [a, b] \to U$ such that
$$\gamma(a) = \alpha \quad \text{and} \quad \gamma(b) = \beta$$
3. An open, path-connected subset of $\mathbb{C}$ is called a *region*.

1.1.4 Definition: The Expanded Complex Plane

The expanded complex plane, denoted by $\hat{\mathbb{C}}$, is the union of the complex plane and ∞.

$$\hat{\mathbb{C}} = \mathbb{C} \cup \infty$$

Any distance infinitely far from the origin of the expanded complex plane has the value ∞.

1.2 Differentiability in Complex Analysis

1.2.1 Basics: Complex Differentiability

- Let $U \subset \mathbb{C}$ be a region in the complex plane and let $f : U \to \mathbb{C}$ be a complex function. The function f is *complex-differentiable at the point* $z_0 \in \mathbb{C}$ if the difference quotient
$$\lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h}$$
exists at z_0. In this case, we define the derivative of f at z_0, denoted by $f'(z_0)$, as
$$f'(z_0) = \lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h}$$
- The function f is *complex-differentiable* if it is complex-differentiable for all $z \in U$.
- The function f is called *holomorphic* if it is complex-differentiable for all $z \in U$.

1.2.2 Basic Properties of Complex Differentiation

Differentiation of complex functions behaves similarly to differentiation of real functions. Let $U \subset \mathbb{C}$ be a region in the complex plane and let $f, g : U \to \mathbb{C}$ be differentiable complex functions and let $c \in \mathbb{C}$ be a constant. In this case:

$$\begin{aligned} (cf)' &= cf' & (f \pm g)' &= f' \pm g' \\ (fg)' &= fg' + f'g & \left(\frac{f}{g}\right)' &= \frac{f'g - fg'}{g^2} \\ (c)' &= 0 & (z^n)' &= nz^{n-1} \end{aligned}$$

1.2.3 Proposition: Convergence of Complex Power Series

The complex power series $\displaystyle\sum_{k=0}^{\infty} a_k z^k$ is convergent if it satisfies the Cauchy criterion; that is, if for all $\epsilon > 0$ there exists $M_0 \in \mathbb{N}$ such that
$$\left| \sum_{k=n}^{m} a_k z^k \right| < \epsilon \qquad \text{for all } m \geq n \geq M_0$$

1.2.4 Definition: Radius of Convergence of Complex Power Series

Let $\mathcal{S} = \displaystyle\sum_{k=0}^{\infty} a_k (z - z_0)^k$ be a complex power series centered at $z_0 \in \mathbb{C}$. The radius of convergence R of $\mathcal{S}$ is defined as
$$R = \frac{1}{\limsup \sqrt[n]{|a_n|}}$$

1.2.5 Theorem: Convergence In Terms of Radius of Convergence

Let $\mathcal{S} = \sum_{k=0}^{\infty} a_k(z - z_0)^k$ be a complex power series with radius of convergence R, and let $D(z_0, R) \subset \mathbb{C}$ be an open disk in the complex plane of radius R centered at z_0 given by $D(z_0, R) = \{z \in \mathbb{C};\ |z - z_0| < R\}$. In this case:

1. $\mathcal{S}$ converges absolutely for all $z \in D(z_0, R)$,
2. $\mathcal{S}$ diverges for all $z \notin U$.

Additionally, the function series $f(z) = \sum_{k=0}^{\infty} a_k(z - z_0)^k$ converges uniformly for all $z \in D(z_0, R)$.

1.2.6 Differentiation of Complex Power Series

Let $\mathcal{S} = \sum_{k=0}^{\infty} a_k(z - z_0)^k$ be a complex power series with radius of convergence R and let $f(z)$ be the function series

$$f(z) = \mathcal{S}(z) = \sum_{k=0}^{\infty} a_k(z - z_0)^k$$

In this case, for all $z \in D(z_0, R)$ (because $\mathcal{S}(z)$'s uniform convergence allows us to differentiate term by term) we define $f(z)$'s derivative $f'(z)$ as

$$\begin{aligned}
f'(z) &= \sum_{k=1}^{\infty} k a_k (z - z_0)^{k-1} \\
f''(z) &= \sum_{k=2}^{\infty} k(k-1) a_k (z - z_0)^{k-2} \\
&\vdots \\
f^{(n)}(z) &= \sum_{k=n}^{\infty} k(k-1)\cdots(k-n-1)(z - z_0)^{k-n}
\end{aligned}$$

1.2.7 Coefficients and Differentiability of Complex Function Series

Let $f(z) = \sum_{k=0}^{\infty} a_k(z - z_0)^k$ be a complex function series with radius of convergence R. In this case, for all $z \in D(z_0, R)$, $f(z)$:

1. $f(z)$ is infinitely continuously differentiable in the complex sense
2. The coefficients a_k are defined by the relationship $a_k = \dfrac{f^{(k)}(\alpha)}{k!}$

1.2.8 Identity: Euler's Identity

For all $z \in \mathbb{C}$, the complex exponential, sine, and cosine functions are related by the identity

$$e^{iz} = \cos z + i \sin z$$

1.2.9 Definitions: Complex Exponent, Sine and Cosine Functions

1. The complex exponential function e^z, where $z \in \mathbb{C}$, is defined in terms of complex function series as:

$$e^z = 1 + z + \frac{z^2}{2!} + \cdots + \frac{z^k}{k!} + \cdots = \sum_{k=0}^{\infty} \frac{z^k}{k!}$$

   For all $z, w \in \mathbb{C}$, $e^z e^w = e^{z+w}$.

2. Using Euler's identity and the power series definition of the complex exponential function, we define the complex sine and cosine functions as:

$$\begin{aligned}
\sin z &= \frac{e^{iz} - e^{-iz}}{2i} = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} \pm \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n+1} \\
\cos z &= \frac{e^{iz} + e^{-iz}}{2} = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} \pm \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} z^{2n}
\end{aligned}$$

3. The complex hyperbolic sine and cosine functions are defined in terms of the complex sine and cosine functions as

$$\begin{aligned}
\sinh z &= -i \sin(iz) = z + \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots = \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!} \\
\cosh z &= \cos(iz) = 1 + \frac{z^2}{2!} + \frac{z^4}{4!} + \cdots = \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!}
\end{aligned}$$

1.3 The Cauchy-Riemann Equations

1.3.1 Note: Complex Functions in Terms of Real Components

Let $U \subset \mathbb{C}$ be a region in $\mathbb{C}$, let $z \in \mathbb{C}$ be a complex number and let $f : U \to \mathbb{C}$ be a complex function mapping $z \mapsto f(z)$. We write f in terms of real components as

$$f(z) = u(z) + iv(z)$$

where $u, v : U \to \mathbb{R}$ are real functions mapping from the complex plane to the real numbers. Typically, we go one step further and write z in terms of its real components as $z = x + iy$ where $x, y \in \mathbb{R}$. In this case, we re-define $u(z) = u(x, y)$ and $v(z) = v(x, y)$ as real functions of two variables, $u, v : \mathbb{R}^2 \to \mathbb{R}$. In this case, we write f as

$$f(z) = u(x, y) + iv(x, y) \qquad \text{where} \qquad z = x + iy; \quad x, y \in \mathbb{R}$$

1.3.2 Definition: Cauchy-Riemann Equations

The Cauchy-Riemann equations for two real, continuously differentiable functions $u, v : \mathbb{R}^2 \to \mathbb{R}$ mapping $(x, y) \mapsto u(x, y)$ and $(x, y) \mapsto v(x, y)$ are

$$\frac{\partial}{\partial x}u(x,y) = \frac{\partial}{\partial y}v(x,y) \qquad \frac{\partial}{\partial x}v(x,y) = -\frac{\partial}{\partial x}u(x,y)$$

Note that two arbitrary functions u, v do not in general satisfy the Cauchy-Riemann equations and this definition is not saying they do; it just presents the definition of the equations.

1.3.3 Complex Differentiability and the Cauchy-Riemann Equations

Let $U \subset \mathbb{C}$ be a region in $\mathbb{C}$, let $f : U \to \mathbb{C}$ be a differentiable complex function, and let $u, v : \mathbb{R}^2 \to \mathbb{R}$ be continuously differentiable real functions such that

$$f(z) = u(x,y) + iv(x,y) \qquad \text{where} \qquad z = x + iy; \quad x, y \in \mathbb{R}$$

In this case, f is holomorphic if and only if u and v satisfy the Cauchy-Riemann equations

$$\frac{\partial}{\partial x}u(x,y) = \frac{\partial}{\partial y}v(x,y) \qquad \frac{\partial}{\partial x}v(x,y) = -\frac{\partial}{\partial x}v(x,y)$$

1.3.4 Definition: Laplacian In the Real Plane

Let $u : U \subset \mathbb{R}^2 \to \mathbb{R}$ be a twice differentiable scalar function. The Laplacian of u, denoted by Δf or $\nabla^2 f$, is the scalar function

$$\Delta f = \nabla \cdot (\nabla f) = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$

1.3.5 Definition: Harmonic Functions

Scalar functions whose Laplacian is zero are called harmonic functions. The twice-differentiable function $u : U \subset \mathbb{R}^2 \to \mathbb{R}$ is harmonic on $\mathbb{R}^2$ if

$$\Delta u = 0 \qquad \text{(condition for harmonic functions)}$$

1.3.6 Note: Components of a Holomorphic Function are Harmonic

Let $z \in \mathbb{C}$ be a complex number, let $f : U \subset \mathbb{C} \to \mathbb{C}$ be a holomorphic function with continuously differentiable harmonic components $u, v : \mathbb{R}^2 \to \mathbb{R}^2$, written

$$f(z) = u(x,y) + iv(x,y)$$

where $z = x + iy$. In this case, if f is holomorphic, both u and v are harmonic functions.
Proof: We prove the proposition for the real component u; the proof for v is analogous. By definition, the Laplacian of u is

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial x}\left[\frac{\partial u}{\partial x}\right] + \frac{\partial}{\partial y}\left[\frac{\partial u}{\partial y}\right]$$

Because f is holomorphic, u and v satisfy the Cauchy-Riemann equations. Continuing from before, we use the Cauchy-Riemann equations to write

$$\Delta u = \frac{\partial}{\partial x}\left[\frac{\partial v}{\partial y}\right] - \frac{\partial}{\partial y}\left[\frac{\partial v}{\partial x}\right] = \frac{\partial^2 v}{\partial x \partial y} - \frac{\partial^2 v}{\partial y \partial x} = 0 \implies \Delta u = 0$$

The mixed partial derivatives are equal because we have assumed u and v are continuous.

1.3.7 Proposition: Harmonic and Holomorphic Functions

Let $U \subset \mathbb{C}$ be an arbitrary subset of the complex plane and let $u : U \to \mathbb{R}$ be a twice-continuously differentiable harmonic function.

1. If u is the real component of some holomorphic function, then u must be harmonic. That is, u being harmonic is a necessary condition for u to be the real part of some holomorphic function.

2. More so, if U is a *connected* subset of C, then u being harmonic becomes a sufficient condition for u to be the real part of some holomorphic function. That is, if U is connected and u is harmonic, there exists holomorphic function $f : U \to \mathbb{C}$ of which u is the real part.

1.4 Integration of Complex Functions

1.4.1 Definition: Integral of the Complex Functions of a Real Variable

Let $I = [a, b] \subset \mathbb{R}$ be an interval on the real line and let $f : I \to \mathbb{C}$ be a complex function. Let $\mathcal{P} = \{a = x_0, x_1, \ldots, x_i, \ldots, x_n = b\}$ be a partition of the interval I, let $\Delta_i x = x_i - x_{i-1}$ be the ith sub-interval of I and let $\xi_i \in \Delta x_i$ be an element of the ith sub-interval Δx_i. If the sum

$$\lim_{n\to\infty} \sum_{i}^{n} f(\xi_i)\Delta x_i$$

converges to a complex number, we say that f is *integrable* on I and define the integral of f on I as

$$\int_a^b f(t)\,\mathrm{d}t = \lim_{n\to\infty} \sum_{i}^{n} f(\xi_i)\Delta x_i$$

1.4.2 Integral of the Complex Functions of a Real Variable by Components

Let $I = [a, b] \subset \mathbb{R}$ be an interval on the real line and let $f : I \to \mathbb{C}$ be a complex function with integrable real and imaginary components $u, v : I \to \mathbb{R}$, respectively. In this case, the integral of f is defined as

$$\int_a^b f(t)\,\mathrm{d}t = \int_a^b u(t)\,\mathrm{d}t + i\int_a^b v(t)\,\mathrm{d}t$$

1.4.3 Proposition: Properties of Complex Integrals

Let $I = [a, b] \subset \mathbb{R}$ be an interval on the real line, let $f, g : I \to \mathbb{C}$ be integrable complex functions, and let $\alpha, \beta \in \mathbb{C}$ be constants. In this case

1. $\displaystyle\int_a^b (\alpha f(t) + \beta g(t))\,\mathrm{d}t = \alpha\int_a^b f(t)\,\mathrm{d}t + \beta\int_a^b g(t)\,\mathrm{d}t$

2. $\displaystyle\left|\int_a^b f(t)\,\mathrm{d}t\right| \le \int_a^b |f(t)|\,\mathrm{d}t$

1.4.4 Definition: Complex Line Integral

Let $I = [a, b] \subset \mathbb{R}$ be an interval on the real line, let $\gamma : I \to \mathbb{C}$ be a path, let $\Gamma = \{\gamma(t); t \in [a, b]\}$ be the image of γ and let $f : \Gamma \to \mathbb{C}$ be an integrable complex function. The line integral of f along the curve Γ is defined as

$$\int_\Gamma f(z)\,\mathrm{d}z = \int_a^b f(\gamma(t))\dot{\gamma}(t)\,\mathrm{d}t$$

where $\dot{\gamma}(t) = \frac{\mathrm{d}\gamma(t)}{\mathrm{d}t}$ denotes differentiation with respect to t.

1.4.5 Note: Component Form of a Complex Line Integral

Let $I = [a, b] \subset \mathbb{R}$ be an interval on the real line, let $\gamma : I \to \mathbb{C}$ be a path, let $\Gamma = \{\gamma(t); t \in [a, b]\}$ be the image of γ and let $f : \Gamma \to \mathbb{C}$ be an integrable complex function with the integrable components $u, v : \Gamma \to \mathbb{R}$. If we write f in the form

$$f(z) = u(x, y) + iv(x, y)$$

where $z = x + iy$ is a point on Γ, we can equivalently write the line integral of f along Γ in terms of u and v as

$$\int_\Gamma f(z)\,\mathrm{d}z = \int_\Gamma (u\,\mathrm{d}x - v\,\mathrm{d}y) + i\int_\Gamma (v\,\mathrm{d}x - u\,\mathrm{d}y)$$

where $\int_\Gamma (u\,\mathrm{d}x - v\,\mathrm{d}y)$ and $\int_\Gamma (v\,\mathrm{d}x - u\,\mathrm{d}y)$ are line integrals of the real functions u and v.

1.4.6 Proposition: Line Integral is Independent of Parameterization

The definition of the line integral of a complex function is independent of the parametrization of the curve along which the integral is evaluated.
In symbols, let $\Gamma \subset \mathbb{C}$ be a complex curve and let $f : \Gamma \to \mathbb{C}$ be an integrable complex function. More so, let $I = [a, b] \subset \mathbb{R}$ and $J = [\alpha, \beta] \subset \mathbb{R}$ be intervals on the real line and let $\gamma : I \to \Gamma$ and $\psi : J \to \Gamma$ be two arbitrary parameterizations of the curve Γ. In this case,

$$\int_a^b f(\gamma(t))\dot{\gamma}(t)\,\mathrm{d}t = \int_\alpha^\beta f(\psi(t))\dot{\psi}(t)\,\mathrm{d}t$$

1.4.7 Proposition: Properties of Complex Line Integrals

Let $I = [a, b] \subset \mathbb{R}$ be an interval on the real line, let $\gamma : I \to \mathbb{C}$ be a path, let $\Gamma = \{\gamma(t); t \in [a, b]\}$ be the image of γ, let $f, g : \Gamma \to \mathbb{C}$ be integrable complex functions, and let $\alpha, \beta \in \mathbb{C}$ be constants. In this case

1. $\displaystyle\int_\Gamma (\alpha f(z) + \beta g(z))\,\mathrm{d}z = \alpha\int_\Gamma f(z)\,\mathrm{d}z + \beta\int_\Gamma g(z)\,\mathrm{d}z$
2. $\displaystyle\left|\int_\Gamma f(z)\,\mathrm{d}z\right| \leq \int_\Gamma |f(z)|\,\mathrm{d}s$

where $\mathrm{d}s = |\mathrm{d}z|$ denotes a line element of the curve Γ.

1.4.8 Note: Connected Combination of Two Curves

The purpose of this note is two formalize the intuitive notion of piecing two curves together by placing the beginning of the second curve at the end of the first curve to form a single, connected curve. Along these lines, let $\gamma : [a,b] \to \mathbb{C}$ and $\psi : [c,d] \to \mathbb{C}$ be two paths such that $\gamma(b) = \psi(c)$; i.e. the end of the curve described by γ coincides with the beginning of the curve described by ψ. In this case, we define a joint parametrization of $\gamma \dot{+} \psi$ describing the entire joined curve with the relation

$$(\gamma \dot{+} \psi)(t) = \begin{cases} \gamma(t); & t \in [a,b] \\ \psi(t-b+c); & t \in [b, b+d-c] \end{cases}$$

The rather complicated-looking choice of the time domain for ψ is made to ensure that $\gamma(b) = \psi(c)$.

For any complex function $f : [\gamma \dot{+} \psi] \to \mathbb{C}$

$$\int_{\gamma \dot{+} \psi} f(z)\,\mathrm{d}z = \int_\gamma f(z)\,\mathrm{d}z + \int_\psi f(z)\,\mathrm{d}z$$

1.4.9 Note: Oppositely Parameterized Path

Let $\gamma : [a,b] \to \mathbb{C}$ be a path with image Γ. We denote the oppositely parameterized path by γ_- and define it with the relation

$$\gamma^-(t) = \gamma(-t); \quad t \in [-b, -a]$$

Both paths have the same image (i.e. describe the same curve Γ) but traverse the curve in opposite directions.

For any complex function $f : [\gamma] \to \mathbb{C}$

$$\int_\gamma f(z)\,\mathrm{d}z = -\int_{\gamma^-} f(z)\,\mathrm{d}z$$

1.4.10 Definition: Closed Path and Closed Curve

Let $\gamma : [a,b] \to \mathbb{C}$ be a path with image is the curve $\Gamma \in \mathbb{C}$. If $\gamma(a) = \gamma(b)$, then γ is called a *closed path* and Γ is a *closed curve*. The line integral of a function along a closed curve is denoted by the symbol $\oint$. For instance, the line integral of some function f along the closed curve γ is written:

$$\oint_\gamma f(z)\,\mathrm{d}z \qquad \text{(notation for closed line integral)}$$

1.4.11 Fundamental Theorem of Calculus for Curve Integrals

Let $U \subset \mathbb{C}$ be a region in $\mathbb{C}$, let $\gamma : [a,b] \to U$ be a path mapping to U, and let $F : U \to \mathbb{C}$ be a holomorphic function. In this case

$$\int_\gamma F'(z)\,\mathrm{d}z = F(\gamma(b)) - F(\gamma(a))$$

Proof: Applying the definition of a complex line integral and then the chain rule gives

$$\int_\gamma F'(z)\,\mathrm{d}z = \int_a^b F'(\gamma(t))\dot{\gamma(t)}\,\mathrm{d}t = \int_a^b \frac{\mathrm{d}}{\mathrm{d}t}\big[F(\gamma(t))\big]\,\mathrm{d}t = F(\gamma(b)) - F(\gamma(a))$$

1.4.12 Implication: Integral Along a Closed Curve is Zero

Let $U \subset \mathbb{C}$ be a region in $\mathbb{C}$, let $\gamma : [a, b] \to U$ be a closed path mapping to U and let $F : U \to \mathbb{C}$ be a holomorphic function. In this case

$$\oint_\gamma F'(z)\,\mathrm{d}z = 0 \qquad\qquad (\gamma \text{ closed path})$$

1.4.13 Theorem

Let $U \subset \mathbb{C}$ be a region in $\mathbb{C}$, let $f : U \to \mathbb{C}$ be a continuous function such that

$$\oint f(z)\,\mathrm{d}z = 0$$

for all closed paths $\gamma : [a, b] \to U$. In this case, there exists holomorphic function $F : U \to \mathbb{C}$ such that

$$F'(z) = f(z)$$

1.5 Winding Number and Cauchy's Theorem

1.5.1 Definition: Winding Number

Let $\gamma : [a, b] \to \mathbb{C}$ be closed path with corresponding closed curve $\Gamma \subset \mathbb{C}$. The *winding number* of the point $z_0 \in \mathbb{C} \setminus \Gamma$ with respect to the closed curve Γ, also called the index of z_0 with respect to Γ and denoted by $\mathrm{Ind}_\gamma(z_0)$, is the number of times the curve Γ wraps around z_0 in the counterclockwise direction. The winding number is given by the formula

$$\mathrm{Ind}_\gamma(z_0) = \frac{1}{2\pi i} \oint_\gamma \frac{\mathrm{d}z}{z - z_0} = \frac{1}{2\pi i} \int_a^b \frac{\dot{\gamma}(t)\,\mathrm{d}t}{\gamma(t) - z_0}$$

1.5.2 Proposition: Properties of the Winding Number

Let $\gamma : [a, b] \to \mathbb{C}$ be closed path with corresponding closed curve $\Gamma \subset \mathbb{C}$, let $\Omega = \mathbb{C} \setminus \Gamma$ be the complement of Γ and define the point $\mathrm{Ind}_\gamma(z_0)$. In this case:

1. $\mathrm{Ind}_\gamma(z_0)$ is an integer number for all $z_0 \in \Omega$.
2. $\mathrm{Ind}_\gamma(z_0)$ has the same value for all z_0 in a given maximal connected component of Ω.
3. $\mathrm{Ind}_\gamma(z_0) = 0$ if z_0 is in the unbounded component of Ω.

1.5.3 Cauchy's Theorem

Let $U \subset \mathbb{C}$ be a region in $\mathbb{C}$, let $f : U \to \mathbb{C}$ be a holomorphic function and let $\gamma : [a, b] \to U$ be closed path that does not wind around any points in U's complement, i.e. $\mathrm{Ind}_\gamma(z_0) = 0$ for all $z_0 \in \mathbb{C} \setminus U$. In this case

$$\oint_\gamma f(z)\,\mathrm{d}z = 0$$

Proof: This proof is only for the special case in which f has the continuously differentiable components $u, v : \mathbb{R}^2 \to \mathbb{R}$ and the curve $[\gamma]$ does not cross over itself. We write f in terms of its components as

$$f(z) = u(x, y) + iv(x, y); \qquad z = x + iy, \quad \mathrm{d}z = \mathrm{d}x + i\,\mathrm{d}y$$

Let $\Gamma \subset U$ be the region bounded by the curve $[\gamma]$. We split the complex line integral into real components and apply Green's formula on the region Γ, followed by Cauchy's equations for u and v to get

$$\begin{aligned}\oint_\gamma f(z)\,\mathrm{d}z &= \oint_\gamma (u + iv)(\mathrm{d}x + i\,\mathrm{d}y)\\ &= \oint_\gamma u\,\mathrm{d}x - v\,\mathrm{d}y + i\oint_\gamma v\,\mathrm{d}z + u\,\mathrm{d}y\\ &= \iint_\Gamma (-v_x - u_y)\,\mathrm{d}A + \iint_\Gamma (u_x - v_y)\,\mathrm{d}A\\ &= \iint_\Gamma (0)\,\mathrm{d}A + \iint_\Gamma (0)\,\mathrm{d}A = 0\end{aligned}$$

Because f is holomorphic u and v satisfy the Cauchy equations, so $u_x = v_y$ and $u_y = -v_x$.

1.5.4 Cauchy's Integral Formula

Let $U \subset \mathbb{C}$ be a region in $\mathbb{C}$, let $f : U \to \mathbb{C}$ be a holomorphic function, let $\gamma : [a, b] \to U$ be closed path that does not wind around any points in U's complement, i.e. $\mathrm{Ind}_\gamma(z) = 0$ for all $z \in \mathbb{C} \setminus U$, and let $z_0 \in U \setminus [\gamma]$ be complex number in U that does not lie on the curve $[\gamma]$. In this case

$$\mathrm{Ind}_\gamma(z_0) f(z_0) = \frac{1}{2\pi i}\oint_\gamma \frac{f(z)}{z - z_0}\,\mathrm{d}z$$

1.5.5 Note: Cauchy's Integral Formula on an Annulus

On an annulus with outer radius R and inner radius r, where the inner disk is bounded by γ_r and the outer disk by γ_R, and where z_0 is a point in the annulus, Cauchy's integral formula reads

$$f(z_0) = \frac{1}{2\pi i}\left[\oint_{\gamma_R} \frac{f(z)}{z - z_0}\,\mathrm{d}z - \oint_{\gamma_r} \frac{f(z)}{z - z_0}\,\mathrm{d}z\right]$$

1.5.6 Cauchy's Differentiation Formula

Let $U \subset \mathbb{C}$ be a region in $\mathbb{C}$, let $f : U \to \mathbb{C}$ be a holomorphic function, let $\gamma : [a, b] \to U$ be closed path that does not wind around any points in U's complement, i.e. $\mathrm{Ind}_\gamma(z) = 0$ for all $z \in \mathbb{C} \setminus U$, and let $z_0 \in U \setminus [\gamma]$ be complex number in U that does not lie on the curve $[\gamma]$. In this case

$$\mathrm{Ind}_\gamma(z_0) f^{(n)}(z_0) = \frac{n!}{2\pi i}\oint_\gamma \frac{f(z)}{(z - z_0)^{n+1}}\,\mathrm{d}z$$

1.5.7 Cauchy's Inequality For Derivatives

Let $U \subset \mathbb{C}$ be a region in $\mathbb{C}$, let $z_0 \in U$ be a point in U, and let $D(\omega, R) \subset U$ be an open disk of radius R centered at ω. Finally, let $M \in \mathbb{R}^+$ be a constant and let $f : U \to \mathbb{C}$ be a holomorphic function bounded on D such that $|f(z)| \leq M$ for all $z \in D$. In this case

$$\left| f^{(n)}(z_0) \right| \leq \frac{Mn!}{R^n}$$

1.5.8 Liouville's Theorem

Let $f : \mathbb{C} \to \mathbb{C}$ be a holomorphic function. If there exists constant $M \in \mathbb{R}^+$ such that $|f(z)| \leq M$ for all $z \in \mathbb{C}$, then f is a constant function. Equivalently, all non-constant holomorphic functions $f : \mathbb{C} \to \mathbb{C}$ have unbounded images.

1.5.9 Note: Fundamental Theorem of Algebra

Every non-constant single-variable constant polynomial has at least one complex zero.

1.6 Laurent Series

1.6.1 Definition: Laurent Series

Let $U \subset \mathbb{C}$ be a region, let $z_0 \in \mathbb{C}$ be a complex number, let $r, R \in \mathbb{R}^+$ be positive real numbers such that $R > r$, and let $A \subset U$ be the open annulus centered at z_0 with outer and inner radii R and r, respectively:

$$A = \{z \in \mathbb{C};\ r < |z - z_0| \leq R\}$$

Finally, let the complex function $f : U \to \mathbb{C}$ be holomorphic in a neighborhood of the annulus A. In this case, for all $z \in A$, f's *Laurent series* about the point $z_0 \in A$ is defined as

$$f(z) = \sum_{-\infty}^{\infty} a_n (z - z_0)^n$$

The coefficients a_n are defined in terms of the Cauchy integration formula on an annulus:

$$a_n = \begin{cases} \dfrac{1}{2\pi i} \displaystyle\oint_{\gamma_R} \dfrac{f(z)}{(z - z_0)^{n+1}} \, \mathrm{d}z & n \geq 0 \\ \dfrac{1}{2\pi i} \displaystyle\oint_{\gamma_r} \dfrac{f(z)}{(z - z_0)^{n+1}} \, \mathrm{d}z & n \leq 0 \end{cases}$$

The paths of integration γ_R and γ_r describe the positively oriented circles

$$\begin{aligned} y_R: &\quad \{z \in \mathbb{C};\ |z - z_0| = R\} \\ y_r: &\quad \{z \in \mathbb{C};\ |z - z_0| = r\} \end{aligned}$$

More generally, the coefficients a_n may be written

$$a_n = \frac{1}{2\pi i} \oint_{\gamma} \frac{f(z)}{(z - z_0)^{n+1}} \, \mathrm{d}z$$

where the path of integration γ describes a positively oriented Jordan curve in the annulus A enclosing z_0.

1.6.2 Principle and Regular Part of Laurent Series

Let $U \subset \mathbb{C}$ be a region, let $z_0 \in \mathbb{C}$ be a complex number, let $r, R \in \mathbb{R}^+$ be positive real numbers such that $R > r$, let $A \subset U$ be the open annulus centered at z_0 with outer and inner radii R and r. And let $f : U \to \mathbb{C}$ be a holomorphic function in a neighborhood of the annulus A with Laurent series

$$f(z) = \sum_{-\infty}^{\infty} a_n (z - z_0)^n$$

In this case, the terms of f's Laurent series with positive degree, i.e. $\sum_{n=0}^{\infty} a_n (z - z_0)^n$, are called the *regular part* of the Laurent series. Meanwhile, the terms with negative degree, i.e. $\sum_{n=-\infty}^{-1} a_n (z - z_0)^n$, are called the *principle part* of the Laurent series.

1.6.3 Consequence: Existence of Taylor Series

Let $U \subset \mathbb{C}$ be a region, let $z_0 \in \mathbb{C}$ be a complex number, let $R \in \mathbb{R}^+$ be a positive real number and let $D(z_0, R) \subset U$ be an open disk centered at z_0. Finally, let the function $f : U \to \mathbb{C}$ be a holomorphic in a neighborhood of D. In this case, for all $z \in D$, there exists a Taylor series for f about the point $z_0 \in D$ that converges for all $z \in D$:

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \qquad\qquad a_n = \frac{1}{2\pi i} \oint_{|z|=R} \frac{f(z)}{(z - z_0)^{n+1}} \, \mathrm{d}z$$

Interpretation: The expression for f's Taylor series is derived by applying f's Laurent series expansion to an annulus centered at z_0 with outer radius R and inner radius $r \to 0$. As $r \to 0$, the annulus converges to the disk D, the principle part of f's Laurent series vanishes, only the regular part remains, and we are left with the above expression for the Taylor series.

1.6.4 Proposition: Zeros of Holomorphic Functions are Isolated

Let $U \subset \mathbb{C}$ be a region. The zeros of every non-constant holomorphic function $f : U \to \mathbb{C}$ are isolated. More formally, for every zero $z_0 \in U$ of f, there exists positive rela number $\delta \in \mathbb{R}^+$ such that $f(z) \neq 0$ for all z in the open disk $D(z_0, \delta)$ of radius δ centered at z_0.

1.6.5 Consequences: Cluster Points of Zeros

Let $U \subset \mathbb{C}$ be a region and $f : U \to \mathbb{C}$ be a non-constant holomorphic function. In this case, all of f's zeros do not have cluster points in U.
More so, if any two holomorphic functions $f, g : U \to \mathbb{C}$ coincide on a subset $E \subset U$ that has cluster points in U, then $f \equiv g$ everywhere on U. We show this by applying the previous statement to the function $f - g$.

1.7 Isolated Singularities

1.7.1 Definition: Isolated Singularity

Let $U \subset \mathbb{C}$ be a region, $z_0 \in U$ be a complex number, $R \in \mathbb{R}^+$ be a positive real number, and $f : U \setminus \{z_0\} \to \mathbb{C}$ be a complex function. In this case, z_0 is an *isolated singularity* of the function f if there exists a disk $D(z_0, R)$ centered at z_0 such that f is holomorphic on $D \setminus \{z_0\}$, i.e. everywhere on D except at z_0 itself.

1.7.2 Definition: Removable Singularity

Let $U \subset \mathbb{C}$ be a region and let $z_0 \in U$ be an isolated singularity of the function $f : U \setminus \{z_0\} \to \mathbb{C}$. If all coefficients a_n in the principle part of f's Laurent series about z_0 are equal to zero, then z_0 is a *removable singularity* of f. In this case, we can write

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

Equivalently, z_0 is a removable singularity of f if there exists holomorphic function $g : U \to \mathbb{C}$ that coincides with f on $U \setminus \{z_0\}$ defined by

$$g(z) = \begin{cases} f(z), & z \in U \setminus \{z_0\} \\ c_0, & z = z_0 \end{cases}$$

where c_0 is the 0-index coefficient of f's Laurent series.

1.7.3 Proposition: Bounded Functions and Removable Singularities

Let $U \subset \mathbb{C}$ be a region and let $z_0 \in U$ be an isolated singularity of the function $f : U \setminus \{z_0\} \to \mathbb{C}$. If f is bounded in some neighborhood of z_0, then z_0 is a removable singularity of f.

1.7.4 Definition: Pole

Let $U \subset \mathbb{C}$ be a region and let $z_0 \in U$ be an isolated singularity of the function $f : U \setminus \{z_0\} \to \mathbb{C}$. If only finitely many coefficients a_n in the principle part of f's Laurent series about z_0 are nonzero, then z_0 is a *pole* of f.

1.7.5 Proposition: Pole and Divergence

Let $U \subset \mathbb{C}$ be a region and let $z_0 \in U$ be a pole of the function $f : U \setminus \{z_0\} \to \mathbb{C}$. In this case $\lim\limits_{z \to z_0} |f(z)| = \infty$.

1.7.6 Definition: Essential Singularity

Let $U \subset \mathbb{C}$ be a region and let $z_0 \in U$ be an isolated singularity of the function $f : U \setminus \{z_0\} \to \mathbb{C}$. If all coefficients a_n in the principle part of f's Laurent series about z_0 are nonzero, then z_0 is a *essential singularity* of f.

1.7.7 Casorati Weierstrass Theorem

Let z_0 be an essential singularity of the function f. In this case for all $w \in \mathbb{C}$ and for all real $\epsilon, \delta > 0$ there exists z in $D(z_0, \delta) \setminus \{z_0\}$ such that $|f(z) - w| < \epsilon$.
Interpretation: A function $f(z)$ takes on values arbitrarily close to *every* complex number $w \in \mathbb{C}$ in every neighborhood of an essential singularity.

1.7.8 Infinity as an Isolated Singularity

- The point ∞ is an isolated singularity of the function f if the point 0 is an isolated singularity of the function $\tilde{f}(z) := f\left(\frac{1}{z}\right)$.
- If the point 0 is a removable singularity, pole, or essential singularity of $\tilde{f}(z) := f\left(\frac{1}{z}\right)$, then the point ∞ is correspondingly a removable singularity, pole, or essential singularity of the function $f(z)$.
- If $\tilde{f}(z) := f\left(\frac{1}{z}\right)$ has the Laurent series $\tilde{f}\left(\dfrac{1}{z}\right) = \displaystyle\sum_{-\infty}^{\infty} c_n z^n$ about the point 0, then $f(z)$ has the Laurent series
$$f(z) = \sum_{-\infty}^{\infty} c_n z^{-n}$$
about the point ∞.
- If the point 0 is a zero/pole of degree n of the function $\tilde{f}$, then the point ∞ is correspondingly a zero/pole of degree n of the function f.

1.7.9 Definition: Meromorphic Function

The function f is *meromorphic* on the open subset $U \subset \mathbb{C}$ of the complex plane if there exists subset $S \subset U$ such that

1. S does not have cluster points in D
2. Every point in S is a pole of f
3. f is holomorphic on $D \setminus S$

Interpretation: A meromorphic function f is a function that is holomorphic on the subset U, except for on a set S of isolated points, which must be poles of f.
Note: Every meromorphic function on the open subset U can be expressed as the ratio of two holomorphic functions.

1.8 Complex Powers and Logarithms

1.8.1 Complex Logarithm

Complex Logarithm: A complex logarithm of the complex number $z \in \mathbb{C}$ is any complex number $w = \ln z$ for which $z = e^w$. Because the complex exponent has periodic behavior, a complex number z in general has multiple complex logarithms. If $z = |z|e^{i(\phi+2\pi k)}$, then
$$(\ln z)_k = \ln|z| + i(\phi + 2\pi k)$$
are all complex logarithms of z, and the complex logarithmic function is not well-defined.
Complex Logarithmic Function: We define the complex logarithmic function as $\ln : \mathbb{C} \setminus (-\infty, 0] \to \mathbb{C}$ with the relation
$$\ln z = \ln\Big(|z|e^{i\phi}\Big) = \ln|z| + i\phi$$
for $\phi \in (-\pi, \pi)$. The left half of the real line is removed from the domain so that ln is holomorphic on its domain.

1.8.2 Complex Logarithm: Integration and Differentiation

Let $U = \mathbb{C} \setminus (-\infty, 0]$ and define the point $w_0 \in U$. In this case, for all $w \in U$:

- $\ln w - \ln w_0 = \displaystyle\int_{w_0}^{w} \frac{\mathrm{d}z}{z} = \int_{\gamma} \frac{\mathrm{d}z}{z}$
  where the path $\gamma \in C^1$ is a continuously differentiable path in U with w_0 as its starting point and w as its ending point.
- $\dfrac{\mathrm{d}}{\mathrm{d}w} \ln w = \dfrac{1}{w}$

1.8.3 Proposition: Branches of the Complex Logarithm

Let $U \subset \mathbb{C}$ be a region such that $\mathrm{Ind}_\gamma(z_0) = 0$ for all closed paths γ in U and all points $z_0 \in U^c$ i.e. all closed curves in U don't wind around points outside of U. In this case, for all holomorphic functions $f : U \to \mathbb{C}$ without zeros on U there exists holomorphic function $g = \ln f$, i.e. holomorphic function $g : D \to \mathbb{C}$ such that $e^{g(z)} = f(z)$ for all $z \in U$. More so, any two such holomorphic functions g differ by $2\pi i n$ where $n \in \mathbb{Z}$.
Interpretation: We can define different branches of the complex logarithm on arbitrary complex regions; such branches of the logarithm differ by $2\pi i n$.

1.8.4 Definition: Complex Power Function

For all $\alpha \in \mathbb{C} \setminus (-\infty, 0]$ and all $z \in \mathbb{C}$ we define

$$z^\alpha = e^{\alpha \ln z}$$

Interpretation: We use the definition of the complex logarithm to define the complex power function.

1.9 Residue

1.9.1 Definition: Residue

Let f be a holomorphic function and z_0 be an isolated singularity of f. The coefficient a_{-1} in the Laurent series of the holomorphic function f expanded about the isolated singularity z_0 is called the *residue* of the function of f at the point z_0 and is denoted by $\mathrm{Res}(f; z_0)$.
Interpretation: If the function f has an isolated singularity at z_0, then the residue $\mathrm{Res}(f; z_0)$ is the coefficient a_{-1} in the Laurent series

$$f(z) = \sum_{-\infty}^{\infty} a_n (z - z_0)^n$$

1.9.2 Lemma: Residue and Integration

Let $U \subset \mathbb{C}$ be a region, $z_0 \in U$ be a point in U and let $\gamma : \mathbb{R} \to U$ be a closed path in U for which $\mathrm{Ind}_\gamma(z_0) = 1$. If the function $f : U \to \mathbb{C}$ is holomorphic on U, except possibly at z_0, then

$$\frac{1}{2\pi i} \oint_\gamma f(z)\,\mathrm{d}z = \mathrm{Ind}_\gamma(z_0) a_{-1} = \mathrm{Res}(f; z_0)$$

Interpretation: We write $f(z)$ as in infinite sum in terms of its Laurent series expansion and apply the integral identity

$$\oint_\gamma z^k \,\mathrm{d}z = \begin{cases} 2\pi i, & k = -1 \\ 0, & \text{otherwise} \end{cases}$$

where γ is a simple closed curve enclosing the origin, for instance the unit circle $C(0,1)$. From this identity, it follows that all terms in f's Laurent series integrate to zero except the term with $n = -1$, what remains is the coefficient a_{-1}, which is the residue $\operatorname{Res}(f; z_0)$.

1.9.3 Residue Theorem

Let $U \subset \mathbb{C}$ be a region, let $S \subset U$ be a subset of U such that S does not have cluster points in U (i.e. S is a set of isolated points in U), let the function f be holomorphic on $U \setminus S$ and define the closed path γ in $U \setminus S$ such that $\operatorname{Ind}_\gamma(z_0) = 0$ for all $z_0 \in U^c$. In this case

$$\frac{1}{2\pi i} \oint_\gamma f(z) \,\mathrm{d}z = \sum_{z_0 \in S} \operatorname{Res}(f; z_0) \operatorname{Ind}_\gamma(z_0)$$

Interpretation: In practice, S is the set of f's isolated singularities. The theorem expresses f's line integral via that sum of f's residues at the isolated singularities.

1.9.4 Integrals of Rational Functions with Sinusoidal Arguments

Let R be a rational function of the form $R(\cos\phi, \sin\phi)$ that is continuous on the unit circle $z = e^{i\phi}$. In this case

$$\int_0^{2\pi} R(\cos\phi, \sin\phi) \,\mathrm{d}\phi = -i \oint_{|z|=1} R\left[\frac{1}{2}\left(z + \frac{1}{2}\right), \frac{1}{2i}\left(z - \frac{1}{z}\right)\right] \frac{\mathrm{d}z}{z}$$

Interpretation: We calculate a difficult-to-evaluate integral in terms of the integrand's residue.

1.9.5 Integrals of Single-Variable Rational Functions

Let R be a single-variable holomorphic function with a finite number of isolated singularities z_0 in the upper half of the complex plane, no singularities on the real line, and a zero of degree at least two at ∞. In this case:

$$\int_{-\infty}^{\infty} R(x) \,\mathrm{d}x = 2\pi i \sum_{\operatorname{Im} z_0 > 0} \operatorname{Res}(R; z_0)$$
$$\int_{-\infty}^{\infty} R(x) e^{ix} \,\mathrm{d}x = 2\pi i \sum_{\operatorname{Im} z_0 > 0} \operatorname{Res}\big(R e^{ix}; z_0\big)$$

Interpretation: We calculate a difficult-to-evaluate integral in terms of the integrand's residue.

1.9.6 Mellin Transform in Terms of Residue

The Mellin transform of the function f is the parameter-dependent integral

$$\{\mathcal{M}f\}(t) = \int_0^\infty f(x)x^{t-1}\,\mathrm{d}x \qquad (t > 0)$$

Let f be holomorphic on $\mathbb{C}$ except possibly at a finite number of points $\alpha_i = \alpha_1, \dots, \alpha_n$ that do not lie on the line $[0, \infty)$ and let a, b be constants such that $a < b$ and $\left|f(z)z^b\right|$ is bounded for large z and $|f(z)z^a|$ is bounded in a neighborhood of 0. In this case the Mellin transform of f is

$$\{\mathcal{M}f\}(t) = \int_0^\infty f(x)x^{t-1}\,\mathrm{d}x = -\frac{\pi e^{-\pi i t}}{\sin \pi t}\sum_i \operatorname{Res}\bigl(f(z)z^{t-1}; \alpha_i\bigr) \qquad (t \in (a, b) \setminus \mathbb{Z})$$

Interpretation: If f meets certain conditions, we can calculate f's Mellin transform in terms of its residues at the isolated singularities α_i, which is often easier than by definition.

1.9.7 Residue at a First Degree Zero

Let the point $z_0 \in \mathbb{C}$ be a first degree zero of the function g and let f be holomorphic in a neighborhood of z_0. In this case

$$\operatorname{Res}\left(\frac{f}{g}; z_0\right) = \frac{f(z_0)}{g'(z_0)}$$

Interpretation: Under the above conditions, we can calculate the residue of the quotient f/g at the point z_0 with the above formula, which is easier than by definition with a series expansion.

1.9.8 Residue at Poles

Let the function f have a pole of degree n at the point $z_0 \in \mathbb{C}$ and be holomorphic on a neighborhood of z_0. In this case

$$\operatorname{Res}(f; z_0) = \lim_{z \to z_0} \frac{\bigl((z - z_0)^n f(z)\bigr)^{(n-1)}}{(n-1)!} \equiv \lim_{z \to z_0} \frac{\mathrm{d}^{(n-1)}}{\mathrm{d}z^{(n-1)}}\left[\frac{(z - z_0)^n f(z)}{(n-1)!}\right]$$

Interpretation: We can calculate the residue of f at its poles with the above formula instead of using the definition.

1.10 Open Mapping Theorem and Maximum Modulus Principle

1.10.1 Definition: Open Map

The map $f : X \to \mathbb{C}$ is an *open map* if for every open subset $Y \subset X$, the set $f(Y)$ is also open.
Interpretation: An open map maps all open subsets of its domain to open subsets.

1.10.2 Meromorphic Line Integral On a Disk via Poles and Zeros

Let f be meromorphic on the region $U \subset \mathbb{C}$, let $\overline{D}$ be a closed disk such that f has neither poles nor zeros on the border ∂D. Finally, let $\mathcal{Z}$ be the sum of the multiplicities of f's zeros on D and let $\mathcal{P}$ be the sum of the degrees of the f's poles on D. In this case

$$\oint_{\partial D} \frac{f'(z)}{f(z)} \, \mathrm{d}z = 2\pi i(\mathcal{Z} - \mathcal{P})$$

Interpretation: The line integral of a meromorphic function f along the border of a disk is related directly to the number of f's poles and zeros *inside* the disk.

1.10.3 Lemma: Disks and Zeros

Let f be holomorphic on an open neighborhood U of the point α and let $\beta = f(\alpha)$, meaning the function $z \mapsto f(z) - \beta$ has a zero at α. Let the multiplicity of this zero be n. In this case, there exist disks $D(\alpha, \delta) \subset U$ and $D(\beta, \epsilon)$ such that the equation $f(z) = w$ has exactly n unique solutions in the disk $D(\alpha, \delta)$ for all $w \in D(\beta, \epsilon) \setminus \{\beta\}$.
Interpretation: The function f maps the disk $D(\alpha, \delta)$ surjectively to the disk $D(\beta, \epsilon)$ and exactly n unique points in the disk $D(\alpha, \delta)$ map to the point w in the disk $D(\beta, \epsilon)$.

1.10.4 Consequence: Open Mapping Theorem

Let $U \subset \mathbb{C}$ be a region and let $f : U \to \mathbb{C}$ be a non-constant holomorphic function. In this case, f is an open map.

1.10.5 Consequence: Bijectivity of Holomorphic Maps

Let the function f be non-constant and holomorphic on a neighborhood of the point z_0 (implying f is an open map by the open mapping theorem). In this case there exists an open neighborhood U of z_0 such that:

1. f maps U bijectively to the open subset $f(U)$ and
2. the inverse function $f^{-1} : f(U) \to U$ is also holomorphic.

1.10.6 Maximum Modulus Principle

Let f be a non-constant holomorphic function defined on the region $U \in \mathbb{C}$. In this case the modulus $|f|$ does not attain a maximum value at any point $z \in U$. More so, $|f|$ can reach a minimum value only its zeros.
Reciprocally, let $U \subset C$ be a region and let $g : U \to \mathbb{C}$ be an arbitrary function. In this case, if there exists a point $z_0 \in U$ for which

$$|g(z_0)| \geq |g(z)| \text{ for all } z \in U$$

then g is a constant function.

1.10.7 Consequence: Maximum Modulus Principle on a Compact Set

Let $K \subset \mathbb{C}$ be a compact set and let the function f be holomorphic and non-constant on a neighborhood of K. In this case, the restricted function $|f|\big|_K$ can attain its maximum only at points $z \in \partial K$ on the border ∂K and its minimum only either on the border ∂K or at f's zeros.

1.10.8 Schwarz Lemma

Let $R, r \in \mathbb{R}^+$ be positive real numbers and let $f : D(0,R) \to D(0,r)$ be a holomorphic function such that $f(0) = 0$. In this case, either

$$|f(z)| < \frac{r}{R}|z| \quad \text{for all} \quad z \in D(0,R) \setminus \{0\} \qquad\qquad \text{and} \qquad\qquad \left|f'(0)\right| < \frac{r}{R}$$

or

$$f(z) = \frac{r}{R}wz \quad \text{for all } z \in D(0,R) \text{ and } w \in \mathbb{C} \text{ such that } |w| = 1$$

1.10.9 Definition: Biholomorphism and Automorphism

Let $U, V \in \mathbb{C}$ be two open regions. A bijective holomorphic map $f : U \to V$ between U and V is called a *biholomorphism* from U to V. A bijective holomorphic map $f : U \to U$ mapping U to itself is called an *automorphism* of U.

1.10.10 Automorphisms of the Unit Disk

- The function of the form

  $$f_\alpha(z) = \frac{z - \alpha}{1 - \overline{\alpha}z}; \quad \alpha \in D(0,1)$$

  is an automorphism of the disk D(0, 1).

- Every automorphism $f : D(0,1) \to D(0,1)$ of the unit disk is of the form

  $$f(z) = wf_\alpha(z)$$

  for some $\alpha \in D(0,1)$ and $w \in \mathbb{C}$ such that $|w| = 1$.

2 Harmonic Functions

Definition: Harmonic Functions
Harmonic functions are functions $u : \mathbb{R}^n \to \mathbb{R}$ satisfying the equality

$$\Delta u = \sum \frac{\partial^2 u}{\partial x_i^2} = 0$$

Interpretation: Harmonic functions are functions whose Laplacian is zero.

2.1 Harmonic Functions in the Plane

2.1.1 Poisson Kernel on the Unit Disk

The Poisson kernel on the unit disk is the function $P_r : D(0,1) \to \mathbb{R}$ given by

$$P_r(\theta) = \frac{1 - r^2}{1 - 2r\cos\theta + r^2}; \quad r \in [0,1), \theta \in \mathbb{R}$$

Interpretation: The values of r and θ for which P_r is defined correspond to points on the open unit disk.
Sketched Derivation: Applying Cauchy's formula to curve γ along the border of the unit disk in the complex plane leads to the equality that for all $z \in \overline{D}(0,1)$:

$$f(z) = \frac{1}{2\pi i} \oint_\gamma f(\zeta) \frac{1 - |z|^2}{|1 - \overline{z}\zeta|^2} \frac{\mathrm{d}\zeta}{\zeta}$$

If we write $\zeta = e^{i\theta}$ and $z = re^{i\phi}$ for $r \in [0,1), \theta \in \mathbb{R}$ and parameterize the line integral, the expression becomes

$$f(re^{i\phi}) = \frac{1}{2\pi} \int_0^{2\pi} \frac{1 - r^2}{1 - 2r\cos(\theta - \phi) + r^2} f(e^{i\phi})\,\mathrm{d}\theta$$

The integrand with $\phi = 0$ is the Poisson kernel.

2.1.2 Poisson Formula on the Unit Disk

For every function u that is harmonic on the open unit disc $D(0,1)$ and continuous on the closed unit disk $\overline{D}(0,1)$:

$$u(re^{i\phi}) = \frac{1}{2\pi} \int_0^{2\pi} P_r(\theta - \phi) u\left(e^{i\phi}\right) \mathrm{d}\theta$$

for all $r \in [0,1), \phi \in \mathbb{R}$.
Interpretation: The value of a harmonic function at a point $re^{i\phi}$ on the open unit disk is directly related to an integral of the Poisson kernel and the function's value on the unit circle at the point $e^{i\phi}$.

2.1.3 Properties of the Poisson Kernel

For all $r \in [0,1)$ and $\theta \in \mathbb{R}$, the Poisson kernel P_r is positive, even, and continuous with period 2π. More so

1. For $\theta \in [0, 2\pi]$,
$$\lim_{r \to 1} P_r(\theta) = \begin{cases} 0, & \theta \neq 0 \\ \infty, & \theta = 0 \end{cases}$$

2. For angles ϕ, θ such that $0 \le \phi \le \theta \le \pi$ we have $P_r(\theta) \le P_r(\phi)$. In other words, at a fixed radius r, the value of the Poisson kernel decreases with increasing angle.

3. $\dfrac{1}{2\pi} \displaystyle\int_0^{2\pi} P_r(\theta)\, \mathrm{d}\theta = 1$. This is simply a special case of the Poisson formula for $u = 1$ and $\phi = 0$.

2.1.4 Poisson Formula on an Arbitrary Disk

On an arbitrary disk $D(\alpha, R)$ centered at the point α with radius R, the Poisson formula states that for all $r \in D(\alpha, R)$
$$u(\alpha + re^{i\phi}) = \frac{1}{2\pi} \int_0^{2\pi} \frac{R^2 - r^2}{R^2 - 2Rr\cos(\theta - \phi) + r^2} u(\alpha + re^{i\phi})\, \mathrm{d}\theta$$

2.1.5 Mean Value Property in the Plane

For every function u that is harmonic on the open disc $D(\alpha, R)$ and continuous on the closed disk $\overline{D}(\alpha, R)$:
$$u(\alpha) = \frac{1}{2\pi} \int_0^{2\pi} u(\alpha + Re^{i\theta})\, \mathrm{d}\theta$$
Interpretation: This is a special case of the Poisson formula on a disk with $r = 0$. The property equates the value of the function u at the center of a disk to the average value function along the disk's border.

2.1.6 Maximum Principle in the Plane

1. If u is a non-constant, harmonic function on the *open* region U, then u cannot attain either a maximum or a minimum anywhere on U.

2. If u is a non-constant, harmonic function on the *compact* region K. then u can attain an a maximum or a minimum only on the border ∂K.

2.1.7 Dirichlet Problem on the Unit Disk

Let $f : \partial D(0, 1) \to \mathbb{R}$ be a continuous function defined on the unit circle. In this case, there exists exactly one function u continuous on the closed unit disk $\overline{D}(0, 1)$ and harmonic on the open unit disk for which
$$f = u\big|_{\partial D(0,1)}$$
Such a function u is given in terms of the Poisson formula as
$$u(re^{i\phi}) = \begin{cases} \dfrac{1}{2\pi} \displaystyle\int_0^{2\pi} P_r(\phi - \theta) g(e^{i\phi})\, \mathrm{d}\phi, & r < 1 \\ f(z) & r = 1 \end{cases}$$
Interpretation: Given a function f defined on the unit circle, there exists exactly one harmonic function u defined on the unit disk such that u is equal to f on the unit circle. More so, f and u are closely related by the Poisson formula for a disk.

2.2 Harmonic Functions in Space

Notation: For this section, unless explicitly stated otherwise:

- Let the region $U \subset \mathbb{R}^3$ be an open, connected, and bounded subset of $\mathbb{R}^3$ and let $\overline{U} = U \cup \partial U$ be the closed set U.
- Let U have a smooth, outward-oriented surface $\partial U \in C^1$, meaning that we can locally parameterize the ∂U at every point on the surface with a continuously differentiable vector function $\boldsymbol{r} : \mathbb{R}^2 \to \mathbb{R}^3$, $\boldsymbol{r} = \boldsymbol{r}(t, s)$, such that $\boldsymbol{r}_t \times \boldsymbol{r}_s \neq 0$ for all values of (t, s).
- Let $\boldsymbol{n}(\boldsymbol{r})$ denote the unit normal vector to the surface ∂U at the point $\boldsymbol{r}$.

2.2.1 Fundamental Solution of the Laplace Equation in Space

- The Laplace equation $\Delta u = 0$ for harmonic functions $u : \mathbb{R}^3 \to \mathbb{R}$ reads

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

  The Laplace equation can be written $\Delta u(r) = u''(r) + \frac{2}{r}u'(r) = 0$ for functions $u = u(r)$ whose value depends only on the distance $r = \sqrt{x^2 + y^2 + z^2}$ from the origin.

  The general solution to the equation $u''(r) + \frac{2}{r}u'(r) = 0$ is $u(r) = a + \frac{b}{r}$.

- The *fundamental solution* of the Laplace equation in space is the function $u : \mathbb{R}^+ \to \mathbb{R}$

$$u(r) = -\frac{1}{4\pi r} \qquad (r > 0)$$

2.2.2 Quick Review of Gauss's and Directional Derivatives

We need these three concepts to understand Green's identities, which follow.

1. For the continuously differentiable vector field $\boldsymbol{F} : \overline{U} \to \mathbb{R}^3$

$$\iiint_U (\nabla \cdot \boldsymbol{F})\, \mathrm{d}V = \oint\!\!\oint_{\partial U} (\boldsymbol{F} \cdot \boldsymbol{n})\, \mathrm{d}S$$

   Interpretation: The volume integral of the divergence $\nabla \cdot \boldsymbol{F}$ over the entire region U equals the surface integral of $\boldsymbol{F}$ over the region's boundary ∂U.

2. The *gradient* of the scalar function $u : U \to \mathbb{R}$ is $\nabla u = \left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}\right)$

3. The *directional derivative* of the continuously differentiable scalar function $u : U \to \mathbb{R}$ in the direction of the vector $\hat{\mathbf{n}} \in \mathbb{R}^3$ is $\dfrac{\partial u}{\partial} \equiv u_{\hat{\mathbf{n}}} = \nabla u \cdot \hat{\mathbf{n}}$

2.2.3 Green's Identities in Space

- *Green's identities* for all twice-continuously differentiable scalar functions $u, v : U \to \mathbb{R}$ defined on a neighborhood of U and for the point $\boldsymbol{r}_0 \in U$ are
  - $\iiint_U (v\Delta u + \nabla u \cdot \nabla v)\,\mathrm{d}V = \oiint_{\partial U} v u_{\hat{\mathbf{n}}}\,\mathrm{d}S$
  - $\iiint_U (v\Delta u - u\Delta v)\,\mathrm{d}V = \iint_{\partial U} (v u_{\hat{\mathbf{n}}} - u v_{\hat{\mathbf{n}}})\,\mathrm{d}S$
  - $u(\boldsymbol{r}_0) = \dfrac{1}{4\pi}\iint_{\partial U}\left(\dfrac{u_{\hat{\mathbf{n}}}(\boldsymbol{r})}{\|\boldsymbol{r}-\boldsymbol{r}_0\|} - u(\boldsymbol{r})\dfrac{\partial}{\partial\,\hat{\mathbf{n}}}\dfrac{1}{\|\boldsymbol{r}-\boldsymbol{r}_0\|}\right)\mathrm{d}S - \dfrac{1}{4\pi}\iiint_U \dfrac{\Delta u}{\|\boldsymbol{r}-\boldsymbol{r}_0\|}\,\mathrm{d}V$
- *Implication:* For any harmonic function u on a neighborhood of the closed region $\overline{U}$

  $$\oiint_{\partial U} u_{\hat{\mathbf{n}}}\,\mathrm{d}S \equiv \oiint_{\partial U} (\nabla u \cdot \hat{\mathbf{n}})\,\mathrm{d}S = 0$$

  Interpretation: The closed surface integral of a harmonic function u's directional derivative normal to the surface of integration is zero. Derived directly from the first identity with $v = 1$ and $u : U \to \mathbb{R}$ an arbitrary harmonic function.

2.2.4 Mean Value Property for Harmonic Functions in Space

Consider the closed sphere $\overline{K}(\boldsymbol{r}_0, R) \subset U$ of radius R centered at $\boldsymbol{r}_0$ and contained completely in the region $U \subset \mathbb{R}^3$. For all harmonic functions $u : U \to \mathbb{R}$

$$u(\boldsymbol{r}_0) = \frac{1}{4\pi R^2}\iint_{\partial K} u(\boldsymbol{r})\,\mathrm{d}S$$

Interpretation: The value of the function u at the center of the closed sphere K equals the average value of u on the sphere's surface.

2.2.5 Maximum and Minimum Principle for Harmonic Functions in Space

Consider the *open* subset $U \subset \mathbb{R}^3$ and the *compact* subset $\overline{K} \subset \mathbb{R}^3$.

- Any non-constant function $u : U \to \mathbb{R}$ that is harmonic on U cannot attain either a maximum or a minimum on U.
- Any non-constant function $u : \overline{K} \to \mathbb{R}$ that is harmonic on K and continuous on $\overline{K}$ can attain a maximum or minimum only on the border ∂K.

2.2.6 Dirichlet Problem in Space

For the region $U \subset \mathbb{R}^3$ and given the function $f : \partial U \to \mathbb{R}$, the Dirichlet problem is to find a function u that is continuous on the border $\overline{U}$ and harmonic on the interior U for which $f = u\big|_{\partial U}$; i.e, to find a harmonic function u that has the same values as the given function f on the boundary ∂U.
If it exists, the solution u to the Dirichlet problem is unique.

2.2.7 Green's Function in Space

In three dimensions, a Green's function (formally, a Green's function of the Laplacian operator) on the region $U \subset \mathbb{R}^3$ with smooth border ∂U is a function $G : \overline{U} \times U \to \mathbb{R}$ such that for all $\boldsymbol{r}_0 \in U$:

1. The function $\boldsymbol{r} \mapsto G(\boldsymbol{r}, \boldsymbol{r}_0) + \dfrac{1}{4\pi\|\boldsymbol{r} - \boldsymbol{r}_0\|}$ is continuous on $\overline{U}$ and harmonic on U.

2. For all $\boldsymbol{r} \in \partial U$, $G(\boldsymbol{r}, \boldsymbol{r}_0) = 0$

2.2.8 Poisson Kernel and Formula in Terms of Green's Function

In three-dimensional space, the Poisson kernel and Poisson formula are defined in terms of Green's function.

1. **Poisson Kernel**: In terms of Green's function, the Poisson kernel $P_r : \partial U \times U \to \mathbb{R}$ in space is given by

$$P_r(\boldsymbol{r}, \boldsymbol{r}_0) = \frac{\partial}{\partial\,\hat{\mathbf{n}}} G(\boldsymbol{r}, \boldsymbol{r}_0) = \big(\nabla G(\boldsymbol{r}, \boldsymbol{r}_0)\big) \cdot \hat{\mathbf{n}}$$

2. **Poisson Formula**: Let $u : U \to \mathbb{R}$ be a harmonic function. In this case, for all $\boldsymbol{r}_0 \in U$

$$u(\boldsymbol{r}_0) = \oiint_{\partial U} \frac{\partial}{\partial\,\hat{\mathbf{n}}} G(\boldsymbol{r}, \boldsymbol{r}_0) u(\boldsymbol{r})\,\mathrm{d}S = \oiint_{\partial U} P_r(\boldsymbol{r}, \boldsymbol{r}_0) u(\boldsymbol{r})\,\mathrm{d}S$$

3 Fourier Analysis

3.1 Convolution

Convolutions are used, among other things, to approximate a given non-differentiable function in terms of smooth functions.

3.1.1 Definition: Convolution

The *convolution* of the functions $f, g : \mathbb{R} \to \mathbb{C}$, denoted by $f * g$, is the parameter-dependent integral

$$(f * g)(x) = \int_{-\infty}^{\infty} f(x-t)g(t)\,\mathrm{d}t$$

assuming the integral converges absolutely.
Note: Condition for Absolute Convergence The integral

$$\int_{-\infty}^{\infty} f(x-t)g(t)\,\mathrm{d}t$$

converges absolutely if f is bounded and piecewise continuous on its domain and g is piecewise continuous on its domain and has compact support. Of course, there are other conditions for absolute convergence; this is just one.

3.1.2 Properties of Convolutions

Convolutions are linear, commutative, and associative. In terms of symbols, let $f, g, h : \mathbb{R} \to \mathbb{C}$ be functions. In this case:

1. $(\alpha f + \beta g) * h = \alpha(f * g) + \beta(g * h)$ (linearity)
2. $g * f = f * g$ (commutativity)
3. $f * (g * h) = (f * g) * h$ (associativity)

3.1.3 Differentiating Convolutions

Let $f, g : \mathbb{R} \to \mathbb{C}$ be two functions.

- If f is a continuously differentiable function, and the integral $\int_{-\infty}^{\infty} |f'(x-t)g(t)|\,\mathrm{d}t$ converges uniformly on every finite interval of x then

$$(f * g)' = f' * g$$

- Similarly, f is an arbitrary function, g is continuously differentiable and the integral $\int_{-\infty}^{\infty} |g'(x-t)f(t)|\,\mathrm{d}t$ converges uniformly on every finite interval of x, then

$$(f * g)' = f * g'$$

- If g is a smooth function (is infinitely continuously differentiable) with compact support then $f * g$ is also smooth and

$$(f * g)^{(n)} = f * g^{(n)}; \quad n \in \mathbb{N}$$

3.1.4 Definition: The Space $L^1(\mathbb{R})$

The space $L^1(\mathbb{R})$ is the extension of the space $C_c(\mathbb{R})$ (the set of all continuous complex functions $f : \mathbb{R} \to \mathbb{C}$ with compact support) with respect to the norm

$$\|f\|_1 = \int_{\mathbb{R}} |f(x)| \, \mathrm{d}x$$

In other words, we create the $L^1(\mathbb{R})$ space by adding to the $C_c(\mathbb{R})$ space all functions that are limits of functions sequences in $C_c(\mathbb{R})$ with respect to the $\|\cdot\|_1$ norm.

Note: Interpreting the $L^1(\mathbb{R})$ Space
Elements of the $L^1(\mathbb{R})$ space may be viewed as functions $f : \mathbb{R} \to \mathbb{C}$ with a finite Lebesgue integral $\int_{\mathbb{R}} |f(x)| \, \mathrm{d}x$ (i.e. $\|f\|_1$ norm). In this interpretation, we view two elements $f \in L^1(\mathbb{R})$ as equal if their values agree except possibly on a set with zero measure.

3.1.5 Proposition: Convolutions of Functions in $L^1(\mathbb{R})$

Let $f, g : \mathbb{R} \to \mathbb{C}$ be two functions.

- If f is an element of $L^1(\mathbb{R})$ and g is a bounded, piecewise continuous function, then $f * g$ is a continuous function.
- If f, g are both elements of $L^1(\mathbb{R})$:
  1. The convolution $f * g$ exists and is itself an element of $L^1(\mathbb{R})$
  2. $\|f * g\|_1 \leq \|f\|_1 \|g\|_1$

3.1.6 Definition: The Function $g_{(\delta)}$

Let $g : L^1(\mathbb{R})$ be a function and define $\delta \in \mathbb{R}$ such that $\delta > 0$. In this let $g_{(\delta)}(x)$ denote the function

$$g_{(\delta)}(x) \coloneqq \frac{1}{\delta} g\left(\frac{x}{\delta}\right) \qquad (\delta > 0)$$

3.1.7 Proposition: Properties of $g_{(\delta)}$

Let $g : L^1(\mathbb{R})$ be a function for which $\int_{-\infty}^{\infty} g(x) \, \mathrm{d}x = 1$.

- In this case, for all $\delta > 0$,

$$\int_{-\infty}^{\infty} g_\delta(x) \, \mathrm{d}x = 1$$

- If g has compact support then, for small δ, $g_{(\delta)}$ is non-zero only in a small neighborhood of the point 0. In this case, for all continuous $f : \mathbb{R} \to \mathbb{C}$

$$(f * g_{(\delta)}) = \int_{-\infty}^{\infty} f(x - t) g_{(\delta)}(t) \, \mathrm{d}t \approx f(x) \int_{-\infty}^{\infty} g_{(\delta)}(t) \, \mathrm{d}t = f(x)$$

3.1.8 Theorem: Convergence of Convolutions Involving $g_{(\delta)}$

Let $g \in L^1(\mathbb{R})$ be a function for which $\int_{-\infty}^{\infty} g(x)\,\mathrm{d}x = 1$.

- For all bounded, continuous $f : \mathbb{R} \to \mathbb{C}$, the convolutions $f * g_{(\delta)}$ converge to f uniformly on all finite intervals $[a, b] \subset \mathbb{R}$ as δ approaches 0. In equation form:
$$\lim_{\delta \to 0} (f * g_{(\delta)})(x) = f(x)$$
- For all functions $f \in L^1(\mathbb{R})$ the convolutions $f * g_{(\delta)}$ converge to f with respect to the norm $\|\cdot\|_1$ as δ approaches 0.
$$\lim_{\delta \to 0} \left\| f * g_{(\delta)}) - f \right\| = 0$$

Note: If we choose g to be a smooth function, it follows that we can uniformly approximate a continuous function f with smooth functions on every finite interval $[a, b] \subset \mathbb{R}$. More so, if f has compact support in the interval of approximation $[a, b]$ and g is a smooth function with a with compact support in $[a, b]$, we are led to the following consequence:

3.1.9 Consequence: Approximation By Smooth Functions

Let $[a, b] \subset \mathbb{R}$ be a finite interval and let $f : \mathbb{R} \to \mathbb{C}$ be a continuous function with compact support in $[a, b]$. In this case, for all $\epsilon > 0$, there exists a sequence (f_n) of smooth functions with supports in the interval $[a - \epsilon, b + \epsilon]$ that converges uniformly to f.

3.1.10 Weierstrass Approximation Theorem

Every continuous function $f : \mathbb{R} \to \mathbb{R}$ can be uniformly approximated by polynomials on the compact interval $[a, b] \subset \mathbb{R}$. In other words, for all $\epsilon > 0$ there exists polynomial $p : \mathbb{R} \to \mathbb{C}$ such that
$$\max_{x \in [a,b]} |f(x) - p(x)| < \epsilon$$

3.1.11 Definition: Schwartz Space $\mathcal{S}(\mathbb{R})$

The Schwartz space $\mathcal{S}(\mathbb{R})$ of rapidly decreasing functions is the set of all smooth functions $f : \mathbb{R} \to \mathbb{C}$ for which the functions of the form $x \mapsto f^{(m)}(x)x^n$ are bounded for all $m, n \in \mathbb{N}$.
Note: The Schwartz space is a subset of $L^1(\mathbb{R})$ since the function $x \mapsto (1 + x^2)f(x)$ is bounded and continuous (and thus an element of $L^1(\mathbb{R})$) for all $f \in \mathcal{S}(\mathbb{R})$.

3.1.12 Proposition: Properties of the Schwartz Space

- For every function $f \in \mathcal{S}(\mathbb{R})$,
  - all translations $f_t(x) = f(x - t)$,
  - all functions of the form $x \mapsto f(ax)$ for all $a \in \mathbb{R}$,
  - all derivatives $f^{(n)}$ and
  - all products of the form fp where $p : \mathbb{R} \to \mathbb{C}$ is a polynomial

  are also elements of $\mathcal{S}(\mathbb{R})$.
- If $f, g : \mathbb{R} \to \mathbb{C}$ are elements of $\mathcal{S}(\mathbb{R})$, then the convolution $f * g$ is also in $\mathcal{S}(\mathbb{R})$.

3.2 The Fourier Transform in $\mathbb{R}$

3.2.1 Motivation

The expansion of the function f into a Fourier series in terms of the complex sinusoidal function $e^{ik\frac{2\pi}{w}x}$ where $k \in \mathbb{Z}$ is

$$f(x) = \sum_{-\infty}^{\infty} c_k e^{ik\frac{2\pi}{w}x} = \sum_{-\infty}^{\infty} \left(\frac{1}{w} \int_{-w/2}^{w/2} f(t) e^{-ik\frac{2\pi}{w}t} \, \mathrm{d}t \right) e^{ik\frac{2\pi}{w}x}$$

for $x \in \left(-\frac{w}{2}, \frac{w}{2}\right)$.
For large ω, in the limit where $f(x)$ rapidly approaches zero as $|x|$ approaches ∞, we make the approximation

$$f(x) = \int_{-w/2}^{w/2} f(t) e^{-ik\frac{2\pi}{w}t} \, \mathrm{d}t \approx \int_{-\infty}^{\infty} f(t) e^{-ik\frac{2\pi}{w}t} \, \mathrm{d}t \qquad \text{(large } w\text{)}$$

Next, we introduce the new variables $\Delta\xi = \frac{2\pi}{w}$ and $\xi_k = k\frac{2\pi}{w}$ to obtain

$$\frac{1}{w} \int_{-\infty}^{\infty} f(t) e^{-ik\frac{2\pi}{w}t} \, \mathrm{d}t = \frac{\Delta\xi}{2\pi} \int_{-\infty}^{\infty} f(t) e^{-i\xi_k t} \, \mathrm{d}t$$

Finally, we define the function $\widehat{f}(\xi_k) \coloneqq \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{-\infty}^{\infty} f(t) e^{-i\xi_k t} \, \mathrm{d}t$ and obtain

$$f(x) \approx \frac{1}{\sqrt{2\pi}} \sum_{-\infty}^{\infty} \widehat{f}(\xi_k) e^{i\xi_k x} \Delta\xi \qquad \text{(large } w\text{)}$$

$$\lim_{w\to\infty} f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \widehat{f}(\xi) e^{i\xi x} \, \mathrm{d}\xi$$

3.2.2 Definition of the Fourier Transform

The *Fourier transform* of the function $f \in L^1(\mathbb{R})$, denoted by $\widehat{f}$, is defined as

$$\widehat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\xi x} \, \mathrm{d}x$$

Note that the above integral is absolutely convergent because $\left|e^{-i\xi x}\right| = 1$.

3.2.3 Properties of the Fourier Transform

Define the function $f \in L^1(\mathbb{R})$. In this case:

1. The Fourier transform $\widehat{f}$ is continuous and $\left|\widehat{f}(\xi)\right| \leq \|f\|_1$ for all $\xi \in \mathbb{R}$.
2. Let $e_t(x) \coloneqq e^{itx}$ for all $t \in \mathbb{R}$. In this case $\widehat{fe_t}(\xi) = \widehat{f}(\xi - t)$
3. Let $f_{[a]}(x) \coloneqq f(ax)$ for all $a > 0$. In this case $\widehat{f_{[a]}}(\xi) = \dfrac{1}{a}\widehat{f}\left(\dfrac{\xi}{a}\right)$
4. Let $f_t \coloneqq f(x - t)$ for all $t \in \mathbb{R}$. In this case $\widehat{f_t}(\xi) = e^{-it\xi}\widehat{f}(\xi)$

5. Let $\chi(x) = x$ for all $x \in \mathbb{R}$. If $\xi f \in L^1(\mathbb{R})$, then $\widehat{f}$ is differentiable and

$$\left(\widehat{f}\right)'(\xi) = -i\left(\widehat{\chi f}\right)(\xi)$$

6. If f is continuously differentiable and $f' \in L^1(\mathbb{R})$ then $\widehat{f'}(\xi) = i\xi\widehat{f}(\xi)$.
7. For all $g \in L^1(\mathbb{R})$, $\widehat{f * g} = \sqrt{2\pi}\widehat{f}\widehat{g}$.
8. For all $f \in \mathcal{S}(\mathbb{R})$, $\widehat{f} \in \mathcal{S}(\mathbb{R})$.

3.2.4 Inverse Fourier Transform

Lemma (Special Case): Let $g_0(x) := e^{-\frac{1}{2}x^2}$ and $(g_0)_{[a]}(x) := e^{-\frac{1}{2}(ax)^2}$ for $a > 0$. In this case,

$$\widehat{g_0} = g_0 \quad \text{and} \quad \widehat{(g_0)_{[a]}}(\xi) = \frac{1}{a}e^{-\frac{\xi^2}{2a^2}}$$

General Statement: Let $f \in L^1(\mathbb{R})$ be a function for which $\widehat{f} \in L^1(\mathbb{R})$. In this case

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\widehat{f}(\xi)e^{ix\xi}\,\mathrm{d}\xi$$

for almost all $x \in \mathbb{R}$.

Inverse Fourier Transform: The inverse Fourier transform of the function $f \in L^1(\mathbb{R})$ function for which $\widehat{f} \in L^1(\mathbb{R})$, denoted by $\check{f}$, is defined as

$$\check{f}(x) := \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)e^{ix\xi}\,\mathrm{d}\xi = \widehat{f}(-x)$$

3.2.5 Inverse Fourier Transform and Schwartz Space

The Fourier transform maps the space $\mathcal{S}(\mathbb{R})$ bijectively onto itself.

3.2.6 Riemann-Lebesgue Lemma

For all $f \in L^1(\mathbb{R})$, $\lim\limits_{|\xi|\to\infty}\widehat{f}(\xi) = 0$. In other words, the Fourier transform of a function vanishes the argument ξ approaches ∞ in absolute value.

3.2.7 Fourier Transform and Lipschitz Continuity

Let the continuous function $f \in L^1(\mathbb{R})$ be Lipschitz continuous at the point $x \in \mathbb{R}$. In this case

$$f(x) = \lim_{A\to\infty}\frac{1}{\sqrt{2\pi}}\int_{-A}^{A}\widehat{f}(\xi)e^{ix\xi}\,\mathrm{d}\xi$$

We can define the value of Lipschitz continuous function in $L^1(\mathbb{R})$ in terms of the function's Fourier transform.

3.2.8 Plancherel Theorem: Motivation and Background

Let $f, g \in \mathcal{S}(\mathbb{R})$ be two functions in the Schwartz space. In terms of the inverse Fourier transform, we can manipulate the inner product $\langle f, g\rangle$ as follows:

$$\begin{aligned}\langle f, g\rangle &:= \int_{-\infty}^{\infty} f(x)\overline{g}(x)\,\mathrm{d}x = \int_{-\infty}^{\infty} f(x)\overline{g(x)}\,\mathrm{d}x = \int_{-\infty}^{\infty} f(x)\left(\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\overline{\widehat{g}(\xi)e^{ix\xi}}\,\mathrm{d}\xi\right)\mathrm{d}x \\ &= \int_{-\infty}^{\infty}\overline{\widehat{g}(\xi)}\left(\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-ix\xi}\,\mathrm{d}x\right)\mathrm{d}\xi = \int_{-\infty}^{\infty}\overline{\widehat{g}(\xi)}\widehat{f}(\xi)\,\mathrm{d}\xi = \left\langle \widehat{f}, \widehat{g}\right\rangle\end{aligned}$$

This means that the Fourier transform, i.e. the mapping $f \mapsto \widehat{f}$, preserves the inner product on $\mathcal{S}(\mathbb{R})$ and thus also the derived norm, meaning $\left\|\widehat{f}\right\|_2 = \|f\|_2$ for all $f \in \mathcal{S}(\mathbb{R})$.
Recall that we can define the space $L^2(\mathbb{R})$ as a completion of the set of all continuous functions with compact support. It follows from the section on convolutions that all functions in $L^2(\mathbb{R})$ can be uniformly approximated with smooth functions with support in a fixed interval. From these two points, it follows that every function $f \in L^2(\mathbb{R})$ is the limit of some sequence (f_n) of smooth functions with compact support; i.e.

$$\lim_{n\to\infty}\|f - (f_n)\|_2 = 0.$$

Because all functions $f_n - f_m$ are in $\mathcal{S}(\mathbb{R})$, if follows that

$$\left\|\widehat{f_n} - \widehat{f_m}\right\|_2 = \|f_n - f_m\|_2$$

for all f_n, f_m in such a s sequence (f_n).
Because the sequence (f_n) is convergent by Cauchy's criterion, it follows that the sequence $(\widehat{f}_n)$ is also convergent by Cauchy's criterion.
The limit of the sequence $(\widehat{f}_n)$ may then be defined as the Fourier transform $\mathcal{F}(f)$ of the function f.
In this way, the Fourier transform becomes a linear map of the space $L^2(\mathbb{R})$ onto itself, since the transform preserves the inner product. This forms the basis for the Plancherel theorem.

3.2.9 Plancherel Theorem

The Fourier transform in $\mathcal{S}(\mathbb{R})$ can be uniquely extended to a unitary operator on the space $L^2(\mathbb{R})$.

3.3 Basic Concepts: The Fourier Transform in $\mathbb{R}^n$

3.3.1 The Spaces $L^1(\mathbb{R}^n)$ and $L^2(\mathbb{R}^n)$

- The space $L^1(\mathbb{R}^n)$ is a completion of the space $C_c(\mathbb{R}^n)$ of all continuous functions $f : \mathbb{R}^n \to \mathbb{R}$ with compact support in the one-norm

$$\|f\|_1 = \int_{\mathbb{R}^n} |f(\boldsymbol{x})|\,\mathrm{d}V$$

  where $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)$ is an element of $\mathbb{R}^n$ and $\mathrm{d}V = \mathrm{d}x_1, \mathrm{d}x_2, \ldots, \mathrm{d}x_n$ is a volume element of $\mathbb{R}^n$.

- Analogously, the space $L^2(\mathbb{R}^n)$ is a completion of the space $C_c(\mathbb{R}^n)$ of all continuous functions $f : \mathbb{R}^n \to \mathbb{R}$ with compact support in the two-norm

$$\|f\|_2 = \int_{\mathbb{R}^n} |f(\boldsymbol{x})| \, \mathrm{d}V$$

- Both norms are induced by the inner product

$$\langle f, g \rangle = \int_{\mathbb{R}^n} f(\boldsymbol{x}) \overline{g(\boldsymbol{x})} \, \mathrm{d}V$$

where $f, g : \mathbb{R}^n \to \mathbb{R}$ are integrable functions.

3.3.2 Convolution in $\mathbb{R}^n$

A convolution of the functions $f, g : \mathbb{R}^n \to \mathbb{R}$ in $\mathbb{R}^n$ is defined as

$$f * g = \int_{\mathbb{R}^n} f(\boldsymbol{t}) g(\boldsymbol{x} - \boldsymbol{t}) \, \mathrm{d}V(\boldsymbol{t})$$

assuming the integral is absolutely convergent.
The properties of convolutions in $\mathbb{R}^n$ are analogous to those of convolutions in $\mathbb{R}$.

3.3.3 Note

Let $g \in {}^1(\mathbb{R}^n)$ be a function, let $\delta > 0$ be a constant, and define

$$g_{(\delta)} \coloneqq \delta^{-n} g(\delta^{-1}\boldsymbol{x}) \qquad (\delta > 0)$$

In this case, if $\int_{\mathbb{R}^n} g(\boldsymbol{x}) \, \mathrm{d}V = 1$, then analogously $\int_{\mathbb{R}^n} g_{(\delta)}(\boldsymbol{x}) \, \mathrm{d}V = 1$

3.3.4 Theorem: Approximation by Convolution

Let $g \in {}^1(\mathbb{R}^n)$ be a function such that $\int_{\mathbb{R}^n} g(\boldsymbol{x}) \, \mathrm{d}V = 1$. In this case:

1. For all bounded continuous functions $f : \mathbb{R}^n \to \mathbb{C}$ the sequence of functions $f * g_{(\delta)}$ converges uniformly to f on all compact subsets of $\mathbb{R}^n$ as δ approaches zero.

2. For all functions $f \in L^1(\mathbb{R}^n)$,

$$\lim_{\delta \to 0} \left\| f * g_{(\delta)} - f \right\|_1 = 0$$

In other words, as δ approaches zero, the function $f * g_{(\delta)}$ becomes arbitrarily close to f.

3.3.5 Schwartz Space in $\mathbb{R}^n$

The Schwartz space $\mathcal{S}(\mathbb{R}^n)$ consists of all infinitely continuously differentiable functions $f : \mathbb{R}^n \to \mathbb{C}$ for which all the derivatives

$$x_1^{k_1} \ldots x_n^{k_n} \frac{\partial_{j_1 + \cdots + j_n} f}{\partial x_1^{j_1} \ldots \partial x_n^{j_n}} \qquad (k_1, \ldots, k_n, j_1, \ldots, j_n \in \mathbb{N})$$

are bounded.

3.4 The Fourier Transform in $\mathbb{R}^n$

3.4.1 Definition: The Fourier Transform in $\mathbb{R}^n$

For Fourier transform of the function $f \in L^1(\mathbb{R}^n)$ is

$$\widehat{f}(\boldsymbol{\xi}) = \frac{1}{(\sqrt{2\pi})^n} \int_{\mathbb{R}^n} f(\boldsymbol{x}) e^{-i\langle \boldsymbol{x}, \boldsymbol{\xi} \rangle} \, \mathrm{d}V(\boldsymbol{x})$$

where $\langle \boldsymbol{x}, \boldsymbol{\xi} \rangle$ denotes the standard inner product on $\mathbb{R}^n$, namely $\langle \boldsymbol{x}, \boldsymbol{\xi} \rangle = x_1\xi_1 + \cdots + x_n\xi_n$.

3.4.2 Properties of the Fourier Transform in $\mathbb{R}^n$

The properties of the Fourier transform in $\mathbb{R}^n$ are analogous to those of the Fourier transform in $\mathbb{R}$.
For the function $f \in L^1(\mathbb{R}^n)$

1. The function $\widehat{f}$ is continuous on $\mathbb{R}^n$ and $\left|\widehat{f}(\boldsymbol{\xi})\right| \leq \|f\|_1$ for all $\boldsymbol{\xi} \in \mathbb{R}^n$.
2. For all $\boldsymbol{t} \in \mathbb{R}^n$, let $e_{\boldsymbol{t}}$ denote the function $e_{\boldsymbol{t}}(\boldsymbol{x}) := e^{i\langle \boldsymbol{t}, \boldsymbol{x} \rangle}$. In this case
$$\widehat{fe_{\boldsymbol{t}}}(\boldsymbol{\xi}) = \widehat{f}(\boldsymbol{\xi} - \boldsymbol{t})$$
3. Scaling: for all $a > 0$, let $f_{[a]}(\boldsymbol{x}) := f(a\boldsymbol{x})$. In this case
$$\widehat{f_{[a]}}(\boldsymbol{\xi}) = \frac{1}{a^n} \widehat{f}\left(\frac{\boldsymbol{\xi}}{a}\right)$$
4. Translation: for all $\boldsymbol{t} \in \mathbb{R}^n$, let $f_{\boldsymbol{t}} := f(\boldsymbol{x} - \boldsymbol{t})$. In this case
$$\widehat{f_{\boldsymbol{t}}}(\boldsymbol{\xi}) = e^{-i\langle \boldsymbol{t}, \boldsymbol{\xi} \rangle} \widehat{f}(\boldsymbol{\xi})$$
5. Let $\chi_j(\boldsymbol{x}) = x_j$ be the function mapping $\boldsymbol{x} \in \mathbb{R}^n$ to the jth component of the vector $\boldsymbol{x}$ and let $f : \mathbb{R}^n \to \mathbb{R}$ be a function such that $\chi_j f$ is an element of $L^1(\mathbb{R}^n)$. In this case,
$$\frac{\partial \widehat{f}}{\partial \xi_j} = -i\widehat{(\chi_j f)}(\boldsymbol{\xi})$$
6. If $\frac{\partial f}{\partial x_j}$ is an element of $L^1(\mathbb{R}^n)$, then
$$\widehat{\frac{\partial f}{\partial x_j}}(\boldsymbol{\xi}) = i\xi_j \widehat{f}(\boldsymbol{\xi})$$
7. Convolutions: for all functions $g \in L^1(\mathbb{R}^n)$, $\widehat{f * g} = \left(\sqrt{2\pi}\right)^n \widehat{f}\widehat{g}$.
8. For all functions $f \in \mathcal{S}(\mathbb{R}^n)$, the transform $\widehat{f}$ is also an element of $\mathcal{S}(\mathbb{R}^n)$.
9. The Fourier transform commutes with rotations (orthogonal transformations with determinant $+1$) in $\mathbb{R}^n$. If $\mathcal{R}$ denotes a rotation in $\mathbb{R}^n$ and $\mathcal{R}f$ denotes the function $(\mathcal{R}f)(\boldsymbol{x}) = f\left(\mathcal{R}^{-1}(\boldsymbol{x})\right)$, then for all $\boldsymbol{\xi} \in \mathbb{R}^n$
$$\widehat{(\mathcal{R}f)}(\boldsymbol{\xi}) = \widehat{f}\left(\mathcal{R}^{-1}(\boldsymbol{\xi})\right)$$

3.4.3 Note: Products and the Gaussian Kernel

If the function $f : \mathbb{R}^n \to \mathbb{R}$ is a product of n single-variable functions, i.e.

$$f(\boldsymbol{x}) = f_1(x_1) \cdots f_n(x_n) \qquad (f_i : \mathbb{R} \to \mathbb{R})$$

then the Fourier transform $\widehat{f}$ can also be written as a product of n single-variable functions.
An important example of such a function is the Gaussian kernel

$$\frac{1}{(\sqrt{2\pi})^n} e^{-\frac{1}{2}\|x\|^2} = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x_1^2} \cdots \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x_n^2}$$

3.4.4 Inverse Fourier Transform

If $f \in L^1(\mathbb{R}^n)$ is a function for which $\widehat{f}$ is also an element of $L^1(\mathbb{R}^n)$, then

$$f(\boldsymbol{x}) = \frac{1}{(\sqrt{2\pi})^n} \int_{\mathbb{R}^n} \widehat{f}(\boldsymbol{\xi}) e^{i\langle \boldsymbol{x}, \boldsymbol{\xi} \rangle} \, \mathrm{d}V(\boldsymbol{\xi})$$

3.4.5 Plancherel Theorem in $\mathbb{R}^n$

The Fourier transform can be uniquely extended to an unitary operator on $L^2(\mathbb{R}^n)$ and

$$\left\|\widehat{f}\right\|_2 = \|f\|_2 \quad \text{for all} \quad f \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$$

3.4.6 Riemann-Lebesgue Lemma in $\mathbb{R}^n$

For all functions $f \in L^1(\mathbb{R}^n)$, $\lim\limits_{\|\boldsymbol{\xi}\| \to \infty} \widehat{f}(\boldsymbol{\xi}) = 0$. In other words, the Fourier transform vanishes for large values of the argument $\boldsymbol{\xi}$.

4 Partial Differential Equations

4.1 The Fourier Transform and the Heat Equation

4.1.1 Definition: The Heat Equation

The heat equation is a partial differential equation describing the change of the quantity $u(\boldsymbol{r}, t)$ in space and time of the form

$$\frac{\partial u}{\partial t} = c\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) = c\Delta u$$

where $c > 0$ is a constant and Δu denotes the Laplacian operation.

4.1.2 Background and Motivation

We are interested in solutions of the heat equation for the initial condition $u(\boldsymbol{r}, 0) = f(\boldsymbol{r})$ where f is a given function that approaches zero as $\|r\|$ approaches ∞.

4.1.3 Fourier Transform of Heat Equation

The Fourier transform of the function $u(\boldsymbol{r}, t)$ at a fixed time t is

$$\widehat{u}(\boldsymbol{\rho}, t) = \frac{1}{(\sqrt{2\pi})^2} \int_{\mathbb{R}^3} u(\boldsymbol{r}, t) e^{-i\langle \boldsymbol{r}, \boldsymbol{\rho}\rangle} \, \mathrm{d}V$$

and the Fourier transform of the heat equation at fixed t is

$$\widehat{\left(\frac{\partial u}{\partial t}\right)} = \frac{\partial \widehat{u}}{\partial t} = -c\|\boldsymbol{\rho}\|^2 \widehat{u}$$

This expression can be written in the form

$$\frac{\partial}{\partial t}\left[e^{c\|\boldsymbol{\rho}^2\|t} \widehat{u}\right] = 0$$

which shows that the derivative of the function $e^{c\|\boldsymbol{\rho}^2\|t}\widehat{u}$ with respect to time is zero; i.e. $e^{c\|\boldsymbol{\rho}^2\|t}\widehat{u}$ is constant with respect to time. We then write

$$e^{c\|\boldsymbol{\rho}^2\|t}\widehat{u}(\boldsymbol{\rho}, t) = A(\boldsymbol{\rho}) \tag{4.1}$$

where A is a function of only position $\boldsymbol{\rho}$.
Next, applying the Fourier transform to the initial condition $u(\boldsymbol{r}, 0) = f(\boldsymbol{r})$ gives

$$\widehat{u}(\boldsymbol{\rho}, 0) = \widehat{f}(\boldsymbol{\rho})$$

Evaluating Equation 4.1 at $t = 0$ results in $A(\boldsymbol{\rho}) = \widehat{f}(\boldsymbol{\rho})$, leading to

$$\widehat{u}(\boldsymbol{\rho}, t) = \widehat{f}(\boldsymbol{\rho}) e^{-c\|\boldsymbol{\rho}^2\|t}$$

The plan is to apply the inverse Fourier transform to this equation to map $\widehat{u}(\boldsymbol{\rho}, t)$ back into position-time space, thus solving the heat equation.

Lemma: From the earlier properties of the Fourier transform, we have

$$\mathcal{F}\left(e^{-a^2\|\boldsymbol{r}\|^2}\right) = \mathcal{F}\left(e^{-\frac{1}{2}\left\|\sqrt{2}a\boldsymbol{r}\right\|^2}\right) = \frac{1}{(\sqrt{2}a)^3} e^{-\frac{1}{2}\left\|\frac{1}{\sqrt{2}a}\boldsymbol{\rho}\right\|^2} = \frac{1}{(\sqrt{2}a)^3} e^{-\frac{\|\boldsymbol{\rho}\|^2}{4a^2}}$$

Next we choose the constant $a > 0$ such that

$$\frac{1}{4a^2} = ct \implies a = \frac{1}{2\sqrt{ct}}$$

It follows that for fixed time t,

$$\mathcal{F}\left(e^{-c\|\boldsymbol{\rho}\|^2 t}\right)(\boldsymbol{r}) = \frac{1}{(\sqrt{2ct})^3} e^{-\frac{\|\boldsymbol{r}\|^2}{4ct}} \equiv K_t(\boldsymbol{r})$$

The function $K_t(\boldsymbol{r})$ is called the *heat kernel.*
We now return to the equation

$$\widehat{u}(\boldsymbol{\rho}, t) = \widehat{f}(\boldsymbol{\rho}) e^{-c\|\boldsymbol{\rho}^2\|t}$$

and apply the inverse Fourier transform with our intermediate results to get

$$u(\boldsymbol{r}, t) = \frac{1}{(\sqrt{2\pi})^3}(f * K_t)(\boldsymbol{r}) = \frac{1}{(\sqrt{4\pi ct})^3} \int_{\mathbb{R}^3} f(\boldsymbol{r} - \boldsymbol{s}) e^{-\frac{\|\boldsymbol{s}\|^2}{4ct}} \,\mathrm{d}V(\boldsymbol{s})$$

4.2 Vibration of Strings and the Wave Equation

4.2.1 Finite One-Dimensional String with Fourier Method

- The one-dimensional wave equation is

  $$u_{tt} = c^2 u_{xx}$$

  If the string is fixed at the endpoints, the boundary conditions are $u(0,t) = u(l,t) = 0$. The initial position and velocity distributions $u(x,0)$ and $u_t(x,0)$ are given by $f(x)$ and $g(x)$.

  The goal is to solve the wave equation $u_{tt} = c^2 u_{xx}$ for the boundary conditions $u(0,t) = u(l,t) = 0$ and initial conditions $u(x,0) = f(x,0)$ and $u_t(x,0) = g(x,0)$.

- Separate the solution into the product $u(x,t) = X(x)T(t)$ where X and T are twice-differentiable non-zero functions. The wave equation becomes

  $$\frac{1}{c^2}\frac{T''(t)}{T(t)} = \frac{X''(x)}{X(x)}$$

  Because the left and right sides of the equation are simultaneously equal and dependent on different variables, they must be constant. We write

  $$\frac{1}{c^2}\frac{T''(t)}{T(t)} = \frac{X''(x)}{X(x)} \equiv -\lambda$$

  which leads to the two equations

  $$X'' + \lambda X = 0 \quad \text{and} \quad T'' + c^2\lambda T = 0$$

  Because $T(t)$ is non-zero, the boundary conditions lead to $X(0) = X(l) = 0$.

- The general solution to $X'' + \lambda X = 0$

$$X(x) = A\cos\omega x + B\sin\omega x \qquad (\lambda = \omega^2)$$

The boundary conditions lead to $A = 0$ and n eigenvalues $\omega_n = \frac{n\pi}{l}, n \in \mathbb{N}$ and eigenfunctions $X_n(x) = B_n \sin\frac{n\pi}{l}$.

- The solutions to $T'' + c^2\lambda T = 0$ with $\lambda = \omega^2 = \left(\frac{n\pi}{l}\right)^2$ are

$$T_n(t) = A_n\cos(\omega_n ct) + B_n\sin(\omega_n ct)$$

- The solution to $u(x,t) = X(x)T(t)$ is thus

$$u_n(x,t) = \left[A_n\cos\left(\frac{n\pi c}{l}t\right) + B_n\sin\left(\frac{n\pi c}{l}t\right)\right]\sin\left(\frac{\pi n}{l}x\right)$$

Because the wave equation is linear, the general solution is a linear superposition of the solutions:

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty}\left(A_n\cos\frac{n\pi c}{l}t + B_n\sin\frac{n\pi c}{l}t\right)\sin\frac{\pi n}{l}x$$

- The initial conditions $u(x,0) = f(x)$ and $u_t(x,0) = g(x)$ lead to

$$f(x) = \sum_{n=1}^{\infty} A_n\sin\frac{\pi n}{l}x \quad \text{and} \quad g(x) = \sum_{n=1}^{\infty}\frac{n\pi c}{l}B_n\sin\frac{\pi n}{l}x$$

- The coefficients A_n and B_n are found with a sine expansion of $f(x)$ and $g(x)$. This is possible if $f, g \in L^2(0,l)$. First, f, g are expanded to the interval $[-l, l]$ using $f(x) = -f(x)$ on the interval $[-a, 0)$ (analogously for g), then expanded onto the entire real line with period $2l$.

The functions f, g can then be written as a Fourier sine series with the function $\sin\frac{n\pi}{l}x$. The Fourier sine expansions produce the solutions for the coefficients:

$$A_n = \frac{2}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi}{l}x\,\mathrm{d}x \quad \text{and} \quad B_n = \frac{l}{n\pi c}\frac{2}{l}\int_{-l}^{l} g(x)\sin\frac{n\pi}{l}x\,\mathrm{d}x$$

4.2.2 d'Alembert's Formula

- We again look for solutions to the wave equation $u_{tt} = c^2u_{xx}$ with the initial conditions $u(x,0) = f(x)$ and $u_t(x,0) = 0$.

- Assume solutions of the form

$$u(x,t) = F(x - ct) + G(x + ct)$$

where F, G are twice differentiable functions.

The initial conditions read

$$u(x,0) = F(x) + G(x) = f(x) \qquad u_t(x,0) = c(G'(x) - F(x)) = g(x)$$

- Differentiating the first equation, solving for $F'(x)$, and inserting this into the second equation leads to

$$G'(x) = \frac{f'(x)}{2} + \frac{g'(x)}{2c} \quad \text{and} \quad F'(x) = \frac{f'(x)}{2} - \frac{g'(x)}{2c}$$

- Integrating the expression for $G'(x)$ with respect to x gives

$$G(x) = \frac{f(x)}{2} + \frac{1}{2c}\int_0^x g(\chi)\,\mathrm{d}\chi + C$$

Combining this with $F(x) + G(x) = f(x)$ leads to

$$F(x) = \frac{f(x)}{2} - \frac{1}{2c}\int_0^x g(\chi)\,\mathrm{d}\chi + C$$

- The solution for $u(x,t) = F(x-ct) + G(x+ct)$ is thus

$$\begin{aligned} u(x,t) &= \frac{1}{2}\big[f(x-ct) + f(x+ct)\big] + \frac{1}{2c}\left[\int_0^{x+ct} g(\chi)\,\mathrm{d}\chi - \int_0^{x-ct} g(\chi)\,\mathrm{d}\chi\right] \\ &= \frac{1}{2}\big[f(x-ct) + f(x+ct)\big] + \frac{1}{2c}\left[\int_0^{x+ct} g(\chi)\,\mathrm{d}\chi + \int_{x-ct}^{0} g(\chi)\,\mathrm{d}\chi\right] \\ &= \frac{1}{2}\big[f(x-ct) + f(x+ct)\big] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\chi)\,\mathrm{d}\chi \end{aligned}$$

which is the *d'Alembert formula* for the one-dimensional wave equation. To satisfy the wave equation, f must be twice-differentiable and g once-differentiable.

- If additionally f and g are odd periodic functions with period $2l$ for which $f(l) = g(l) = 0$, then $u(0,t) = u(a,t) = 0$ and the d'Alembert formula satisfies the boundary conditions $u(0,t) = u(a,t)$.

The equality $u(0,t) = 0$ follows from the odd condition $f(-x) = -f(x)$ and identity $\int_{-a} af(x)\,\mathrm{d}x = 0$ for odd functions

$$u(0,t) = \frac{1}{2}\big[f(-ct) + f(ct)\big] + \frac{1}{2c}\int_{-ct}^{+ct} g(\chi)\,\mathrm{d}\chi = \frac{1}{2}\big[-f(ct) + f(ct)\big] + 0 = 0$$

The equality $u(a,t) = 0$ follows from the periodic condition $f(x+a) = f(x-a)$ and the fact that the integral of an odd function over a full period is zero.

$$\begin{aligned} u(a,t) &= \frac{1}{2}\big[f(a-ct) + f(a+ct)\big] + \frac{1}{2c}\int_{a-ct}^{a+ct} g(\chi)\,\mathrm{d}\chi \\ &= \frac{1}{2}\big[-f(ct-a) + f(ct+a)\big] + \frac{1}{2c}\int_{a-ct}^{a+ct} g(\chi)\,\mathrm{d}\chi \\ &= 0 + 0 = 0 \end{aligned}$$

5 Second-Order Homogeneous LDEs

5.1 Zeros of Solutions to Second-Order Homogeneous LDEs

For the entirety of this section, let $I \subset \mathbb{R}$ by an interval on the real line and let $p, q : I \to \mathbb{R}$ be continuous functions on I.

5.1.1 Homogeneous 2nd Order Linear Differential Equation

A homogeneous second-order linear differential equation is a differential equation of the form

$$y'' + p(x)y' + q(x)y = 0$$

where y is a function of x.
The *trivial solution* to such an equation is $y \equiv 0$. All other solutions are non-trivial.

5.1.2 Zeros

- The zeros of nontrivial solutions to 2nd order homogeneous LDEs $y''+py'+qy = 0$ cannot have (finite) cluster points.
- Every zero of a nontrivial solution is a simple zero. In a equation form:

$$y(x_0) = 0 \implies y'(x_0) \neq 0$$

5.1.3 Shared Zeros and Linear Dependence

Let y_1 and y_2 be two solutions of the equation $y'' + py' + qy = 0$. If y_1 and y_2 have a shared zero, then y_1 and y_2 are linearly independent.
Expressed in symbols, if there exists $x_0 \in I$ such that $y_1(x_0) = y_2(x_0) = 0$, then there exists constant $\alpha \in \mathbb{R}$ such that $y_1 = \alpha y_2$.

5.1.4 Zeros and Linear Independence

Let y_1 and y_2 be linear independent solutions of $y'' + py' + qy = 0$, and let $x_1, x_2 \in I$ be two zeros of y_1. In this case, the solution y_2 has exactly one zero in the open interval (x_1, x_2).

5.1.5 Normal Form of a Homogeneous 2nd-Order LDE

The motivation is comparing the solutions to $y'' + py' + qy = 0$ to the solutions of the simpler equation $y'' + \alpha y = 0$ where $\alpha \in \mathbb{R}$ is a constant.
If p is continuously differentiable and p is continuous on I, then the homogeneous second-order linear differential equation $y'' + p(x)y' + q(x)y = 0$ can be written in the *normal form*

$$u'' + Q(x)u = 0$$

with the substitution $y = uv$ where

$$v = \exp\left(-\frac{1}{2}\int p(x)\,\mathrm{d}x\right) \qquad \text{and} \qquad Q(x) = q(x) - \frac{p(x)^2}{4} - \frac{p'(x)}{2}$$

5.1.6 Theorem: Sturm's Comparison Criterion

Let $q, r : I \to \mathbb{R}$ be continuous functions on I such that $q(x) > r(x)$ for all $x \in I$.
In this case, between any two zeroes x_1, x_2 of the nontrivial solution u of the equation

$$u'' + ru = 0$$

there is at least one zero the solution to the equation

$$y'' + qy = 0$$

in the interval (x_1, x_2).

5.1.7 Implication:

If $q(x) < 0$ for all $x \in I$, then any non-trivial solution y of the equation $y'' + qy = 0$ has at most one zero in I.

5.1.8 Example: Zeroes of the Bessel Equation

Let y be a non-trivial solution to the Bessel equation

$$x^2 y'' + xy' + (x^2 - \nu^2)y = 0 \qquad (x > 0, \nu \geq 0)$$

The number of zeros depends on ν as follows

- If $\nu \in \left[0, \frac{1}{2}\right)$, then every interval of width π contains at least one zero of the solution y.
- If $\nu = \frac{1}{2}$, then the distance between subsequent zeroes of y is exactly π.
- If $\nu > \frac{1}{2}$, then every interval of width π contains at most one zero of y.

In any case, y has infinitely many zeros on the positive real line $(0, \infty)$

5.2 Introduction to Sturm-Liouville Theory

For the entirety of this section, let $I = [a, b] \subset \mathbb{R}$ be an interval on the real line.

5.2.1 Definition: The Sturm-Liouville Problem

We are interested in solving problems of the form

$$P(x)y'' + Q(x)y' + R(x)y = -\lambda y$$

on the interval $[a, b]$ with boundary conditions

$$\begin{aligned} &\alpha_1 y(a) + \alpha_2 y'(a) = 0 \qquad && \alpha_1^2 + \alpha_2^2 \neq 0 \\ &\beta_1 y(b) + \beta_2 y'(b) = 0 && \beta_1^2 + \beta_2^2 \end{aligned}$$

where $P, Q, R : [a, b] \to \mathbb{R}$ are real continuous functions and $\alpha_i, \beta_i \in \mathbb{R}$ are real constants.
The constant parameter λ must be determined to fit the boundary conditions.

5.2.2 Space of Continuous and Twice-Differential Functions on $[a,b]$

Let $C[a,b]$ be the space consisting of all continuous complex functions on the interval $[a,b]$ equipped with the inner product

$$\langle f,g\rangle = \int_a^b f(x)\overline{g(x)}\,\mathrm{d}x$$

Meanwhile, $C^2[a,b]$ denotes the space of all twice-differentiable complex-valued functions on the interval $[a,b]$.

5.2.3 The Second-Degree Linear Differential Operator L

For all continuous functions $P,Q,R:[a,b]\to\mathbb{R}$, the mapping L defined by

$$L: C^2[a,b]\to C[a,b], \qquad y\mapsto Py''+Qy'+Ry$$

is linear, and is called a second-degree linear differential operator.
In terms of L, we can write the differential equation in the Sturm Liouville problem as

$$Ly=-\lambda y$$

which is an eigenvalue problem for the operator L; we are interested in eigenvectors $y\in C^2[a,b]$ that solve the eigenvalue problem while also satisfying the border conditions of the Sturm-Liouville problem.

5.2.4 Review of Self-Adjoint Operators

An operator is self-adjoint (or symmetric for real operators) if it can be diagonalized, i.e. has a complete set of eigenvectors.
The adjoint L^* of a linear operator L is defined by the relationship

$$\langle Lu,v\rangle = \langle u, L^*v\rangle$$

for all u in the domain of L and all v in the domain of L^*.
In the case of a differential operator, u,v are twice-differentiable functions on $[a,b]$.
In this case, from the definition of the differential operator L, we have

$$\langle Lu,v\rangle = \int_a^b (Pu''+Qu'+Ru)\overline{v}\,\mathrm{d}x$$

Assuming P and Q are twice and once continuously differentiable, respectively, integration by parts gives:

$$\langle Lu,v\rangle = \cdots = \big[P(u'\overline{v}-u\overline{v}')+(Q-P)u\overline{v}\big]_a^b + \int_a^b u\overline{[(Pv)''-(Qv)'+Rv]}\,\mathrm{d}x$$

5.2.5 Defining the Adjoint of the Differential Operator

From the above equality, we define the adjoint L^* of the differential operator L with the relationship

$$\begin{aligned} L^*v &:= (Pv)''-(Qv)'+Rv \\ &= Pv''+(2P'-Q)v'+(P''-Q'+R)v \end{aligned}$$

where $v \in \mathbb{C}^2[a,b]$ is a twice-continuously differentiable function.
With this definition, we recover the identity

$$\langle Lu, u\rangle = \langle u, L^* v\rangle$$

as long as

$$\left[P(u'\overline{v} - u\overline{v}') + (Q - P')u\overline{v}\right]_a^b = 0$$

5.2.6 Formally Self-Adjoint Differential Operator

In this case, the operator L is called *formally self-adjoint* if $L^* = L$. In this case

$$2P' - Q = Q \qquad \text{and} \qquad P'' - Q' + R = R$$

which is equivalent to the condition $Q = P'$. If $Q = P'$, we the earlier condition for the existence of the adjoint becomes

$$\left[P(u'\overline{v} - u\overline{v}')\right]_a^b = 0$$

and we can write L in the form

$$Ly = (Py')' + Ry$$

When L can be written in this form, integration by parts shows that L is formally self-adjoint, i.e.

$$\langle Lu, v\rangle = \langle u, Lv\rangle + \left[P(u'\overline{v} - u\overline{v}')\right]_a^b$$

even when P is only a once continuously differentiable function.
Condition: L is formally self-adjoint on $[a,b]$ when $Q = P'$ and P is a real, continuously differentiable function and R is a continuous real function.

5.2.7 Self-Adjoint and Identity

If the differential operator L is formally self-adjoint on the interval $[a,b]$, then

$$\langle Lu, v\rangle = \langle u, Lv\rangle + \left[P(u'\overline{v} - u\overline{v}')\right]_a^b$$

5.2.8 Implication

If the functions $u, v \in C^2[a,b]$ satisfy the boundary conditions noted earlier, then $\langle Lu, v\rangle = \langle u, Lv\rangle$ for all formally self-adjoint differential operators L.
This is shown by showing that $u'\overline{v} - u\overline{v}' = 0$ at the points a and b.

5.3 More General Sturm-Liouville Problem

5.3.1 Sturm-Liouville Problem with a Weight

In this case, the earlier equation becomes

$$Ly = -\lambda wy$$

instead of $Ly = -\lambda y$. In this case, $w : [a,b] \to \mathbb{R}$ is a positive, continuous function called the *weight* and L is the earlier-defined differential operator.
As before, λ are the eigenvalues of the operator L and solutions y satisfying the equation and boundary conditions are the operator L's eigenfunctions.

5.3.2 Inner Product, Norm, and Orthogonal Functions with a Weight

The positive continuous function w defines an inner product on the space $C[a,b]$ given by

$$\langle f, g\rangle_w = \int_a^b f(x)\overline{g(x)}w(x)\,\mathrm{d}x$$

The functions f, g are called *orthogonal with respect to the weight* w if $\langle f, g\rangle_w = 0$.
The norm induced by the inner product $\langle \cdot, \cdot\rangle_w$ is denoted by $\|\cdot\|_w$ and

$$\|f\|_w = \sqrt{\langle f, f\rangle_w}$$

Finally we denote the completion of the space $C[a,b]$ in the norm $\|\cdot\|_w$ by $L^2_w(a,b)$.

5.3.3 Proposition: Eigenvalues of Self-Adjoint Differential Operators

The eigenvalues of the formally self-adjoint differential operator L of the form $Ly = (Py')' + Ry$ where P has no zeroes on $[a,b]$ with respect to the standard Sturm-Liouville boundary conditions are real.
More so, eigenfunctions of L corresponding to different eigenvalues are mutually orthogonal with respect to the weight w.
Finally, for any two of L's eigenfunctions corresponding to the same eigenvalue are linearly dependent.
Summary: Eigenvalues of formally self-adjoint differential operators are real. Eigenfunctions of different eigenvalues are orthogonal, and eigenfunctions of the same eigenvalue are linearly dependent.

5.3.4 Motivation

When solving

$$Ly = -\lambda w y$$

with respect to the usual boundary conditions and when L is formally self-adjoint, we are interested in when L has sufficiently many non-zero, mutually orthogonal (with respect to the weight w) eigenfunctions to form an orthogonal basis of the Hilbert space $L^2_w(a,b)$.

5.3.5 Definition: Regular Sturm-Liouville Problem

Let $L : C^2[a,b] \to C[a,b]$ be a self-adjoint differential operator of the form

$$Ly = (Py')' + Ry$$

where $P \in C^1[a,b]$ is a real, continuously differentiable function and $R \in C[a,b]$ is a real, continuous function. A *regular Sturm-Liouville* is the problem of finding all eigenvalues $\lambda \in \mathbb{C}$ of the differential operator L for which the equation

$$Ly = -\lambda w y$$

has a non-trivial solution $y \in C^2[a,b]$ with respect to the boundary conditions

$$\begin{aligned} &\alpha_1 y(a) + \alpha_2 y'(a) = 0 \qquad && \alpha_1^2 + \alpha_2^2 \neq 0 \\ &\beta_1 y(b) + \beta_2 y'(b) = 0 && \beta_1^2 + \beta_2^2 \end{aligned}$$

and finding the solution y.

5.3.6 Sturm-Liouville Theorem

For all regular Sturm-Liouville problems there exists an orthonormal basis of the Hilbert space $L^2_w(a,b)$ consisting of real eigenfunctions $u_n, n \in \mathbb{N}$ of the operator L.
For the eigenvalue λ_n corresponding to the eigenfunction u_n,

$$\lim_{n\to\infty} \lambda_n = \infty$$

For all functions $y \in C^2[a,b]$ satisfying the boundary conditions, the series

$$\sum_{n=1}^{\infty} \langle y, u_n \rangle_w \, u_n$$

converges uniformly to y on the interval $[a,b]$.

5.3.7 Lemma With Green's Function

The idea is that this would be used to prove the Sturm-Liouville theorem.
If y_1 and y_2 are two linearly independent solutions of the differential equation

$$y'' + qy = 0$$

satisfying the boundary conditions $y_1(a) = 0$ and $y_2(b) = 0$ and the Green's function G defined by

$$G(x,t) = \frac{1}{W} \begin{cases} y_1(t)y_2(x), & t \le x \\ y_1(x)y_2(t), & x \le t \end{cases} = \frac{1}{W} y_1\big(\min\{t,x\}\big) y_2\big(\max\{t,x\}\big)$$

then

$$y_p = \int_a^b G(x,t) f(t) \, \mathrm{d}t$$

solves the non-homogeneous equation

$$y'' + qy = f$$

and satisfies the boundary conditions $y(a) = 0$ and $y(b) = 0$.

5.4 Solving Linear Differential Equations with Power Series

For the entirety of this section, let p, q be holomorphic functions in a neighborhood of 0, meaning we can expand them into the power series

$$p(z) = \sum_{n=0}^{\infty} p_N z^n \qquad \text{and} \qquad \sum_{n=0}^{\infty} q_n z^n$$

where $p_k, q_k \in \mathbb{C}$ are complex coefficients.
The goal is to solve the equation

$$y'' + py' + qy = 0$$

with the aid of the power series

$$y = \sum_{n=0}^{\infty} c_n z^n$$

where the coefficients $c_n \in \mathbb{C}$ are found by inserting the series for p and q into the differential equation and finding a recurrence relation.
A similar approach follows for functions holomorphic on a neighborhood of some point z_0 other than zero. In that case, p, q must be holomorphic on a neighborhood of z_0, and the solution is of the form

$$y = \sum_{n=0}^{\infty} c_n (z - z_0)^n$$

The series for y converges on every disk on which the series for p and q also converge.

5.4.1 Theorem: Power Series Solutions

If p, q are holomorphic functions on the disk $D(z_0, R)$ of radius R centered at the point $z_0 \in \mathbb{C}$, then for any complex constants $c_0, c_1 \in \mathbb{C}$ there exists a solution y of the equation

$$y'' + py' + qy = 0$$

that is holomorphic on the on the disk $D(z_0, R)$ and satisfies the boundary conditions $y(z_0) = c_0$ and $y'(z_0) = c_1$.

5.4.2 Legendre Equation

The Legendre equation is a second-order differential equation of the form

$$(z^2 - 1)y'' + 2z' - \nu(\nu + 1)y = 0$$

where

$$p(z) = \frac{2z}{z^2 - 1} \quad \text{and} \quad q(z) = -\frac{\nu(\nu + 1)}{z^2 - 1}$$

are holomorphic functions on the unit disk $D(0, 1)$.
Plugging $y = \sum_{0}^{\infty} c_k z^k$ into the equation leads to

$$\begin{aligned}
c_{2n} &= \frac{(-1)^n}{(2n)!} \nu(\nu - 2) \cdots (\nu - 2n + 2)(\nu + 1)(\nu + 3) \cdots (\nu + 2n - 1)c_0 \\
2_{2n+1} &= \frac{(-1)^n}{(2n + 1)!} (\nu - 1)(\nu - 3) \cdots (\nu - 2n + 1)(\nu + 2)(\nu + 4) \cdots (\nu + 2n)c_1
\end{aligned}$$

with $c_0 = 1$ and $c_1 = 0$, we get the solution

$$y_1 = \sum_{n=0}^{\infty} c_{2n} z^{2n} = 1 - \frac{\nu(\nu + 1)}{2!} z^2 + \frac{\nu(\nu - 2)(\nu + 1)(\nu + 3)}{4!} z^4 - \ldots$$

and with $c_0 = 0$ and $c_1 = 1$, we get the linearly independent solution

$$y_2 = \sum_{n=0}^{\infty} c_{2n+1} z^{2n+1} = z - \frac{(\nu - 1)(\nu + 2)}{3!} z^3 + \frac{(\nu - 1)(\nu - 3)(\nu + 2)(\nu + 4)}{5!} z^5 - \ldots$$

Both solutions converge for $|z| < 1$.

5.4.3 Legendre Polynomials

The polynomials of the form

$$P_n(z) = \sum_{k=0}^{[\frac{n}{2}]} (-1)^k \frac{(2n-2k)!}{2^n k!(n-k)!(n-2k)!} z^{n-2k}$$

are called Legendre polynomials for $n \in \mathbb{N}$. The Legendre polynomials may be generated by the Rodrigues formula

$$P_n(z) = \frac{1}{2^n n!} \frac{\mathrm{d}^n \; {}^n}{\mathrm{d}z^n}$$

The Legendre polynomial P_n generated by the Rodgrigues formula satisfies the Legendre equation for $\nu = n$.

For small $|t|$

$$\frac{1}{\sqrt{1-2zt+t^2}} = \sum_{n=0}^{\infty} P_n(z) t^n$$

Additionally,

$$(n+1)P_{n+1}(z) = (2n+1)zP_n(z) - nP_{n-1}(z)$$

This identity allows us to recursively calculate all Legendre polynomials once $P_0 = 1$ and $P_1 = z$ are known.

5.4.4 Orthogonality of the Legendre Polynomials

$$\int_{-1}^{1} P_m(x)P_n(x)\,\mathrm{d}x = \delta_{m,n} \frac{2}{2n+1}$$

where

$$\delta_{m,n} = \begin{cases} 1, & m = n \\ 0, & \text{otherwise} \end{cases}$$

In other words, the Legendre polynomials are orthogonal when integrated over the interval $[-1, 1]$.

5.4.5 Convergence of Legendre Polynomials in the $L^2(-1,1)$ Space

For all functions $f \in L^2(-1,1)$, the series

$$\sum_{n=0}^{\infty} a_n P_n \qquad a_n = \frac{2n+1}{2} \int_{-1}^{1} f(x)P_n(x)\,\mathrm{d}x$$

converges uniformly to f in the $L^2(-1,1)$ norm. In other words,

$$\lim_{N\to\infty} \left\| \sum_{n=0}^{N} \right\| a_n P_n - f = 0 \qquad \|f\| = \sqrt{\int_{-1}^{1} |f(x)|^2} \quad \text{for all} \quad f \in L^2(-1,1)$$

5.4.6 Zeros of General Orthogonal Polynomials

Let $I = (a, b) \subseteq \mathbb{R}$ be a real interval and let $w : I \to \mathbb{R}^+$ be a positive function such that

$$\int_a^b x^n w(x)\,\mathrm{d}x < \infty \quad \text{for all} \quad n \in \mathbb{N}$$

Note that this condition is automatically fulfilled if I is finite and w is continuous on I. In this case, the Hilbert space

$$L^2_w(a, b) \coloneqq \{f : (a, b) \to \mathbb{C}; \int_a^b |f(x)|^2 w(x)\,\mathrm{d}x < \infty\}$$

contains all polynomials.
The family of polynomials $(p_n), n \in \mathbb{N}$ is called the family of orthogonal polynomials with weight w if $\langle p_n, p_m \rangle_w = 0$ for $n \neq m$ where

$$\langle f, g \rangle_w = \int_a^b f(x)\overline{g}(x)w(x)\,\mathrm{d}x$$

5.4.7 Oscillation of Orthogonal Polynomials

Let (p_n) be a sequence of orthogonal polynomials with weight w on the interval (a, b) such that p_n is of degree n. In this case, p_n has exactly n zeros on the interval (a, b).

5.4.8 Associated Legendre Polynomials

The associated Legendre polynomials associated with the Legendre polynomial P_n, denoted by P_n^m, are

$$P_n^m(z) \coloneqq (1 - z^2)^{\frac{m}{2}} \frac{\mathrm{d}^m}{\mathrm{d}z^m} P_n(z), \qquad m = 0, 1, \ldots, n$$

Note that the associated Legendre polynomials satisfy the differential equation

$$\left[(1 - z^2)y'\right]' + \left[n(n + 1) - \frac{m^2}{1 - z^2}\right] y = 0$$

Additionally, for $m > 0$

$$P_n^m(1) = P_n^m(-1) = 0$$

For a fixed m, the associated polynomials $P_n^m (n \geq m)$ are mutually orthogonal.

5.4.9 Theorem: Associated Legendre Polynomials as a Basis of $L^2(-1, 1)$

For a fixed m, the family of associated Legendre polynomials $\left(P_n^m\right)_{n=m}^{\infty}$ for an orthogonal basis of the Hilbert space $L^2(-1, 1)$ and

$$\|P_n^m\|^2 = \frac{2}{2n + 1} \frac{(n + 1)!}{(n - m)!}$$

5.4.10 Example: Stationary-State Temperature Distribution on a Sphere

Let the center of the sphere be the origin, let $a \in \mathbb{R}^+$ be the radius of the sphere. We use a spherical coordinates system. Let the temperature at the point (r, ϕ, θ) at time t be $u(r, \phi, \theta, t)$.
Temperature satisfies the heat equation $\frac{\partial u}{\partial t} = c\Delta u$ where c is a positive constant. In spherical coordinates, the Laplace operator Δ reads

$$\Delta u = u_{rr} + \frac{2}{r}u_r + \frac{1}{r^2 \sin\theta}(u_\theta \sin\theta)_\theta + \frac{1}{r^2 \sin^2\theta}u_{\phi\phi}$$

Let the temperature at the surface of the sphere be given by $u(a, \theta, \phi) = f(\theta, \phi)$ where f is a given function. For the stationary state solution, $\frac{\partial u}{\partial t} = 0$, and the heat equation becomes the Dirichlet problem

$$\Delta u = 0 \quad \text{for} \quad r < a \qquad \text{and} \qquad u(a, \phi, \theta) = f(\phi, \theta)$$

Using the Fourier method of separation of variables leads to the ansatz

$$u(r, \theta, \phi) = R(r)\Theta(\theta)\Phi(\phi)$$

which, when plugged in to the original differential equation and evaluating the spherical Laplacian, yields

$$r^2 \sin^2\theta \left(\frac{R''}{R} + \frac{2}{r}\frac{R'}{R} \right) + \sin\theta \frac{(\Theta' \sin\theta)'}{\Theta} = -\frac{\Phi''}{\Phi}$$

The eventual solution is in terms of the spherical harmonics, which are based on the associated Legendre polynomials.