A Concise Summary of Wave Optics

Condensed notes of the material covered in the third-year undergraduate course *Optika* (Optics) at the Faculty of Mathematics and Physics at the University of Ljubljana in the academic year 2020-21. The course covers wave optics at an undergraduate level.

Disclaimer: This document will inevitably contain some mistakes—both simple typos and legitimate errors. Keep in mind that these are the notes of an undergraduate student in the process of learning the material himself, so take what you read with a grain of salt. If you find mistakes and feel like telling me, I will be grateful and happy to hear from you, even for the most trivial of errors. You can reach me by email, in English, Slovene, or Spanish, at ejmastnak@gmail.com.

Elijan J. Mastnak
Last update: October 10, 2022
Faculty of Mathematics and Physics, University of Ljubljana

Contents

Review of Geometrical Optics

Assumptions: light consists of rays that..

- propagate in straight-line paths in *homogeneous* media,
- change direction, and may split in two, at the interface between two media with *different optical properties*,
- propagate in curved paths in a media with a *continuously-changing* refractive index, and
- may be absorbed and reflected.

$c_0 \approx 3.0\,\mathrm{m\,s^{-1}}$ (speed of light in vacuum)
$c = c_0/n$ (light speed in medium with refractive index n)
$n = n(\mathbf{r}, \omega)$ is a material-dependent property and may change with light frequency ω.

Fermat's Principle

Light travels between any two points $\mathbf{r}_2$ and $\mathbf{r}_2$ along the path minimizing the travel time between the two points.
$\mathcal{S} \equiv \int_{s_1}^{s_2} n\big(\mathbf{r}(s)\big)\,\mathrm{d}s = c_0 \int_{t_1}^{t_2} \mathrm{d}t$ (optical path length)
$\mathcal{S} = \int_{s_1}^{s_2} \big(\mathbf{r}(s)\big)\,\mathrm{d}s = \min$ (**Fermat's principle**)

The Eikonal Ray Equation

Consider light propagating through a material with position-dependent refractive index $n = n(\mathbf{r})$.
$\nabla n = \frac{\mathrm{d}}{\mathrm{d}s}\left(n \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}s}\right)$ (**ray equation**)

Deriving the Ray Equation

(Source: `https://www.fields.utoronto.ca/programs/scientific/12-13/Marsden/FieldsSS2-FinalSlidesJuly2012.pdf`)
By Fermat's principle, the path taken by a light ray between two points $\mathbf{r}_1$ and $\mathbf{r}_2$ leaves the optical path length $\mathcal{S}$ stationary under variations in the family of paths $\mathbf{r}(s, \varepsilon)$ connecting $\mathbf{r}_1$ and $\mathbf{r}_2$ that vary smoothly with the parameter ε.
$0 \equiv \delta\mathcal{S} = \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\left[\int_{s_1}^{s_2} n\big(\mathbf{r}(s,\varepsilon)\big)\,\mathrm{d}s\right]_{\varepsilon=0}$ (Fermat's principle)
$\mathbf{r}(s, 0) = \mathbf{r}(s)$ (assume vanishing ε recovers path $\mathbf{r}(s)$)
$\mathbf{r}(s_1, \varepsilon) = \mathbf{r}(s_1)$ for all ε (fixed endpoints)
$\mathbf{r}(s_2, \varepsilon) = \mathbf{r}(s_2)$ for all ε (fixed endpoints)
$\delta\mathbf{r}(s) \equiv \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\Big[\mathbf{r}(s,\varepsilon)\Big]_{\varepsilon=0}$ (variation in path)
$\dot{\mathbf{r}} \equiv \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}s}$ (shorthand notation)
$|\dot{\mathbf{r}}| = \sqrt{\dot{\mathbf{r}} \cdot \dot{\mathbf{r}}} = \sqrt{\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}s} \cdot \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}s}}$
$0 \equiv \delta\mathcal{S} = \delta \int_{s_1}^{s_2} n\big(\mathbf{r}(s)\big)|\dot{\mathbf{r}}|\,\mathrm{d}s$ (alternate fomulation of FP)
$= \int_{s_1}^{s_2} \left[|\dot{\mathbf{r}}| \frac{\partial n}{\partial \mathbf{r}} \cdot \delta\mathbf{r} + \left(n\big(\mathbf{r}(s)\big)\frac{\dot{\mathbf{r}}}{|\dot{\mathbf{r}}|}\right) \cdot \delta\dot{\mathbf{r}}\right]\mathrm{d}s$ (product rule)
$= \int_{s_1}^{s_2} \left[\frac{\partial n}{\partial \mathbf{r}} \cdot \delta\mathbf{r} + n\big(\mathbf{r}(s)\big)\dot{\mathbf{r}} \cdot \delta\dot{\mathbf{r}}\right]\mathrm{d}s$ (using $|\dot{\mathbf{r}}| = 1$)
$\mathrm{I} \equiv \int_{s_1}^{s_2} \left[n\big(\mathbf{r}(s)\big)\dot{\mathbf{r}} \cdot \delta\dot{\mathbf{r}}\right]\mathrm{d}s$ (second integral)
Use integration by parts with $\mathrm{d}v = \delta\dot{\mathbf{r}}\,\mathrm{d}s$ to get...
$\mathrm{I} = \left[n\big(\mathbf{r}(s)\big)\dot{\mathbf{r}} \cdot \delta\mathbf{r}\right]_{s_1}^{s_2} - \int_{s_1}^{s_2} \left\{\frac{\mathrm{d}}{\mathrm{d}s}\left[n\big(\mathbf{r}(s)\big)\dot{\mathbf{r}}\right]\right\} \cdot \delta\mathbf{r}\,\mathrm{d}s$
$= 0 - \int_{s_1}^{s_2} \left\{\frac{\mathrm{d}}{\mathrm{d}s}\left[n\big(\mathbf{r}(s)\big)\dot{\mathbf{r}}\right]\right\} \cdot \delta\mathbf{r}\,\mathrm{d}s$ (stationary endpoints)
$\implies \delta\mathcal{S} = \int_{s_1}^{s_2} \left\{\frac{\partial n}{\partial \mathbf{r}} - \frac{\mathrm{d}}{\mathrm{d}s}\left[n\big(\mathbf{r}(s)\big)\dot{\mathbf{r}}\right]\right\} \cdot \delta\mathbf{r}\,\mathrm{d}s \equiv 0$
$\implies \frac{\partial n}{\partial \mathbf{r}} - \frac{\mathrm{d}}{\mathrm{d}s}\left[n\big(\mathbf{r}(s)\big)\dot{\mathbf{r}}\right] = 0$ (ray equation)
$\nabla n = \frac{\mathrm{d}}{\mathrm{d}s}\left[n\big(\mathbf{r}(s)\big)\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}s}\right]$ (in original form)

Paraxial Approximation

Assume light propagates through the xz plane.
The paraxial approximation, with $\hat{\mathbf{z}}$ as the optical axis, holds if
$\frac{\mathrm{d}x}{\mathrm{d}z} \ll 1$ for all z (**paraxial approximation**)
$\theta \equiv \frac{\mathrm{d}x}{\mathrm{d}z}$ (angle between tangent to ray path and optical axis)
$\sin\theta \approx \theta, \quad \tan\theta \approx \theta, \quad \cos\theta \approx 1$
$\mathrm{d}s \equiv \sqrt{(\mathrm{d}x)^2 + (\mathrm{d}z)^2} \approx \mathrm{d}z$ (path length differential)
$\frac{\mathrm{d}^2 x}{\mathrm{d}z^2} = \frac{1}{n(x)}\frac{\mathrm{d}n}{\mathrm{d}x}$ (ray equation for $n = n(x)$ and $\frac{\mathrm{d}x}{\mathrm{d}z} \ll 1$)

Optical Transfer Matrices

Assume light propagates through the xz plane.
Assume paraxial approximation with $\hat{\mathbf{z}}$ as optical axis.
$\theta = \frac{\mathrm{d}x}{\mathrm{d}z}$ (angle between tangent to ray path and optical axis)
Represent rays with the coordinates (x, θ).

Goal: given initial ray position (x_1, θ_1), find position (x_2, θ_2) after the ray passes through an optical medium.
$x_2 = Ax_1 + B\theta_1$
$\theta_2 = Cx_1 + D\theta_1$
$\begin{pmatrix} x_2 \\ \theta_2 \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} x_1 \\ \theta_1 \end{pmatrix} \equiv \mathbf{M}\begin{pmatrix} x_1 \\ \theta_1 \end{pmatrix}$
The determinant of a transfer matrix between e.g. material 1 and 2 equals the ratio of refractive indices: $\det\mathbf{M} = (n_1)/(n_2)$.

Common Transfer Matrices

$\begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix}$ (through homogeneous material of length L)
$\begin{pmatrix} 1 & 0 \\ 0 & \frac{n_1}{n_2} \end{pmatrix}$ (through straight interface)
$\begin{pmatrix} 1 & 0 \\ \frac{n_1 - n_2}{n_2 R} & \frac{n_1}{n_2} \end{pmatrix}$ (through curved interface of radius R)
$\begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix}$ (through thin lens with focus $f = \frac{R}{2(n-1)}$)
$\begin{pmatrix} x_n \\ \theta_n \end{pmatrix} = \mathbf{M}_n\mathbf{M}_{n-1}\cdots\mathbf{M}_2\mathbf{M}_1\begin{pmatrix} x_1 \\ \theta_1 \end{pmatrix}$ (n consecutive interfaces)

Light as Electromagnetic Waves

Notation

$\mathbf{E}$ and $\mathbf{B}$ are electric and magnetic field.
$\mathbf{D}$ and $\mathbf{H}$ are "$\mathbf{D}$" and "$\mathbf{H}$" field.
ρ and ρ_{f} are total and free electric charge density.
$\mathbf{j}$ and $\mathbf{j}_{\mathrm{f}}$ are total and free electric current density.
$\varepsilon_0 = 8.85 \cdot 10^{-12}\,\mathrm{F\,m^{-1}}$ (vacuum permittivity)
$\mu_0 = 4\pi \cdot 10^{-7}\,\mathrm{H\,m^{-1}}$ (vacuum permeability)

Maxwell Equations In Free Space

$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}$
$\nabla \cdot \mathbf{B} = 0$
$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$
$\nabla \times \mathbf{B} = \mu_0 \mathbf{j} + \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}$

Maxwell Equations In Matter

$\nabla \cdot \mathbf{D} = \rho_{\mathrm{f}}$
$\nabla \cdot \mathbf{B} = 0$
$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$
$\nabla \times \mathbf{H} = \mathbf{j}_{\mathrm{f}} + \frac{\partial \mathbf{D}}{\partial t}$

Electric Field in Matter

$\mathbf{P}$ is a material's electric polarization.
ρ_{b} is a material's bound electric charge density.
ε is a material's relative permittivity.
$\rho_{\mathrm{b}} = -\nabla \cdot \mathbf{P}$ (implicit definition for polarization)
$\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P}$ (definition of $\mathbf{D}$ field)
$\mathbf{P} = \mathbf{P}(\mathbf{D})$ (general constitutive relation)
$\mathbf{P}(\mathbf{D}) \approx \chi_{\mathrm{E}} \mathbf{D} + \mathcal{O}(\mathbf{D}^2)$ (linear approximation of CR)
$\chi_{\mathrm{E}} = 1 - \frac{1}{\varepsilon}$ (electric susceptibility)
$\mathbf{D} = \varepsilon \varepsilon_0 \mathbf{E}$ (in linear, isotropic matter)
$\mathbf{P} = \varepsilon_0 (\varepsilon - 1) \mathbf{E}$ (in linear, isotropic matter)

Magnetic Field in Matter

$\mathbf{M}$ is a material's magnetiziation.
$\mathbf{j}_{\mathrm{b}}$ is a material's bound electric current density.
μ is a material's relative permeability.
$\mathbf{j}_{\mathrm{b}} = \nabla \times \mathbf{M} + \frac{\partial \mathbf{P}}{\partial t}$ (implicit definition for magnetization)
$\mathbf{H} = \frac{\mathbf{B}}{\mu_0} - \mathbf{M}$ (definition of $\mathbf{H}$ field)
$\mathbf{M} = \mathbf{M}(\mathbf{H})$ (general constitutive relation)
$\mathbf{M}(\mathbf{H}) \approx \chi_{\mathrm{M}} \mathbf{H} + \mathcal{O}(\mathbf{H}^2)$ (linear approximation of CR)
$\chi_{\mathrm{M}} = \mu - 1$ (magnetic susceptibility)
$\mathbf{B} = \mu \mu_0 \mathbf{H}$ (in linear, isotropic matter)
$\mathbf{M} = \left(1 - \frac{1}{\mu}\right) \frac{\mathbf{B}}{\mu_0}$ (in linear, isotropic matter)

Simplifying Assumptions

We assume the matter in which we will analyze EM waves is...

(i) homogeneous: the material's properties are identical throughout the material (so $\varepsilon \neq \varepsilon(\mathbf{r})$ and $\mu \neq \mu(\mathbf{r})$),

(ii) isotropic: the material's properties are identical for all orientations of the material (so ε and μ are scalars and not rank-two tensors)

(iii) nondispersive: the material's properties are independent of EM wave frequency (so $\varepsilon \neq \varepsilon(\omega)$ and $\mu \neq \mu(\omega)$),

(iv) charge-free: the material is free of net electric charge (so $\rho = 0$),

(v) nonconducting: an electric field in the material does not establish electric currents (so $\mathbf{j} = \mathbf{0}$), and

(vi) linear: $\mathbf{D} = \varepsilon \varepsilon_0 \mathbf{E}$ and $\mathbf{B} = \mu \mu_0 \mathbf{H}$.

Maxwell Equations Under Above Assumptions

$\nabla \cdot \mathbf{D} = 0$
$\nabla \cdot \mathbf{B} = 0$
$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$
$\nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t}$

Electromagnetic Wave Equation

Consider material with relative permittivity ε and relative permeability μ, so that $\varepsilon_0 \to \varepsilon \varepsilon_0$ and $\mu_0 \to \mu \mu_0$.
Begin derivation for $\mathbf{E}$ with $\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$.
$\nabla \times (\nabla \times \mathbf{E}) = -\frac{\partial (\nabla \times \mathbf{B})}{\partial t} = -\mu \mu_0 \varepsilon \varepsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2}$ (assuming $\mathbf{j} = \mathbf{0}$)
$\nabla \times (\nabla \times \mathbf{E}) = \nabla(\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E}$ (general identity)
$= -\nabla^2 \mathbf{E}$ (assuming $\nabla \cdot \mathbf{E} = 0$)
Begin derivation for $\mathbf{B}$ with $\nabla \times \mathbf{B} = \mu \mu_0 \varepsilon \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}$.
$\nabla \times (\nabla \times \mathbf{B}) = \mu \mu_0 \varepsilon \varepsilon_0 \frac{\partial (\nabla \times \mathbf{E})}{\partial t}$ (assuming $\mathbf{j} = \mathbf{0}$)
$= -\mu \mu_0 \varepsilon \varepsilon_0 \frac{\partial^2 \mathbf{B}}{\partial t^2}$
$\nabla \times (\nabla \times \mathbf{B}) = \nabla(\nabla \cdot \mathbf{B}) - \nabla^2 \mathbf{B}$ (general identity)
$= -\nabla^2 \mathbf{B}$ (using $\nabla \cdot \mathbf{B} = 0$)
$\nabla^2 \mathbf{E} = \varepsilon \varepsilon_0 \mu \mu_0 \frac{\partial^2 \mathbf{E}}{\partial t^2}$ (**wave equation for E**)
$\nabla^2 \mathbf{B} = \varepsilon \varepsilon_0 \mu \mu_0 \frac{\partial^2 \mathbf{B}}{\partial t^2}$ (**wave equation for B**)
$c = 1/\sqrt{\varepsilon \varepsilon_0 \mu \mu_0}$ (EM wave speed)
$c_0 = 1/\sqrt{\varepsilon_0 \mu_0} \approx 3.0 \cdot 10^8\,\mathrm{m\,s^{-1}}$ (EM wave speed in vacuum)

Plane Wave Solutions to the Wave Eq.

Mathematical Solutions

Mathematically, the EM wave equation has complex plane wave solutions:
$\mathbf{E}(\mathbf{r}, t) = \mathbf{E}_0 e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)}$ ($\mathbf{E}_0 \in \mathbb{C}^3$)
$\mathbf{B}(\mathbf{r}, t) = \mathbf{B}_0 e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)}$ ($\mathbf{B}_0 \in \mathbb{C}^3$)
$\mathbf{E}_0 = (E_x, E_y, E_z) \in \mathbb{C}^3$
$E_x = |E_x| e^{i\phi_x},\ E_y = |E_y| e^{i\phi_y},\ E_z = |E_z| e^{i\phi_z}$

Physically Observable Solutions

But only the real part of the the plane wave solutions are physically observable:
$\mathbf{E}(\mathbf{r}, t) = \mathrm{Re}\left[\mathbf{E}_0 e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)}\right] = \mathrm{Re}[\mathbf{E}_0] \cos(\mathbf{k}\cdot\mathbf{r} - \omega t)$
$\mathbf{B}(\mathbf{r}, t) = \mathrm{Re}\left[\mathbf{B}_0 e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)}\right] = \mathrm{Re}[\mathbf{B}_0] \cos(\mathbf{k}\cdot\mathbf{r} - \omega t)$

Plane Waves Traveling Along the z Axis

Align coordinate system so that the z axis aligns with the direction of EM wave propagation, i.e. $\mathbf{k} = k\,\hat{\mathbf{e}}_z$.
$\mathbf{E}(z, t) = \mathrm{Re}\left[\mathbf{E}_0 e^{i(kz - \omega t)}\right] = \mathrm{Re}[\mathbf{E}_0] \cos(kz - \omega t)$
$\mathbf{B}(z, t) = \mathrm{Re}\left[\mathbf{B}_0 e^{i(kz - \omega t)}\right] = \mathrm{Re}[\mathbf{B}_0] \cos(kz - \omega t)$
At any *fixed time* t, $\mathbf{E}$ and $\mathbf{B}$ are sinusoidal functions of position z with wavelength $\lambda = 2\pi/k$.
At any *fixed position* z, $\mathbf{E}$ and $\mathbf{B}$ are sinusoidal functions of time t with frequency $\nu = \omega/2\pi$.

Phase and Wave Fronts

Let $\phi \equiv \mathbf{k}\cdot\mathbf{r} - \omega t$.
Constant $\phi \implies$ constant EM wave phase.
$\phi = \text{constant} \implies \mathbf{k}\cdot\mathbf{r} = \phi + \omega t_0 = \text{constant}$ (at $t = t_0$)
$\mathbf{k}\cdot\mathbf{r} = \text{constant}$ defines a plane of constant phase at $t = t_0$
Planes of constant EMW phase are called *wave fronts*.
Wave fronts are normal to $\mathbf{k}$ by construction $\mathbf{k}\cdot\mathbf{r} = \text{constant}$.

Phase Velocity

Phase velocity is the velocity at which wave fronts move through space.
$v_{\mathrm{p}} = \omega/k$ (definition of phase velocity)
Substitute $\mathbf{E} = \mathbf{E}_0 e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)}$ into wave equation and get...
$v_{\mathrm{p}} = \frac{1}{\sqrt{\varepsilon_0 \mu_0}} \frac{1}{\sqrt{\varepsilon \mu}}$ (for plane wave solutions to EM wave eq.)
$c_0 = \frac{1}{\sqrt{\varepsilon_0 \mu_0}}$ (EM wave phase velocity in vacuum)
$v_{\mathrm{p}} \equiv c = \frac{c_0}{\sqrt{\varepsilon \mu}} \equiv \frac{c_0}{n}$ (EM wave phase velocity in matter)
$n = \sqrt{\varepsilon \mu}$ (refractive index)
$\nu_{\mathrm{vacuum}} = \nu_{\mathrm{matter}} \equiv \nu$ (frequency preserved in all matter)

$\lambda = \lambda_0/n = \frac{c_0}{n\nu}$ (wavelength decreases in matter)
$k_0 \equiv \omega/c_0$ (wave vector in vacuum)
$k = nk_0$ (wave vector increases in matter)

Directions of the Vectors E, B and k

Assumptions as in Simplifying Assumptions.
Assume plane wave solutions to EM wave eq. of the form...
$\mathbf{E}(\mathbf{r},t) = \mathbf{E}_0 e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}$
$\mathbf{B}(\mathbf{r},t) = \mathbf{B}_0 e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}$
$\mathbf{E}\cdot\mathbf{k} = 0 \implies \mathbf{k}\perp\mathbf{E}$ (from $\nabla\cdot\mathbf{D} = 0$ and $\mathbf{D} = \varepsilon\varepsilon_0\mathbf{E}$)
$\mathbf{B}\cdot\mathbf{k} = 0 \implies \mathbf{k}\perp\mathbf{B}$ (from $\nabla\cdot\mathbf{B} = 0$)
$\mathbf{k}\times\mathbf{E} = \omega\mathbf{B}$ (from $\nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t}$)
$\mathbf{k}\perp\mathbf{E}$ and $\mathbf{k}\perp\mathbf{B} \implies$ EM waves are transverse waves!
$\mathbf{k}\times\mathbf{E} \propto \mathbf{B}$ and $\mathbf{E}\perp\mathbf{k} \implies$ **E**, **B**, **k** are mutually orthogonal!

Ratio of Field Amplitudes

Assumptions as in Simplifying Assumptions.
$\mathbf{k}\times\mathbf{E} = \omega\mathbf{B}$ and $\mathbf{k}\perp\mathbf{E}\perp\mathbf{B} \implies kE_0 = \omega B_0$
$E_0 = \frac{\omega}{k}B_0 = cB_0$
$Z \equiv \frac{E_0}{H_0} = \frac{\mu\mu_0 E_0}{B_0} = \mu\mu_0 c = \sqrt{\frac{\mu\mu_0}{\varepsilon\varepsilon_0}}$ (impedance)
$Z_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}} \approx 377\,\Omega$ (impedance of free space)

EM Energy and Power

$u_\mathrm{E} = \frac{\varepsilon\varepsilon_0}{2}E^2$ (electric field energy density)
$u_\mathrm{B} = \frac{1}{2\mu\mu_0}B^2$ (magnetic field energy density)
$u_\mathrm{EM} = u_\mathrm{E} + u_\mathrm{B}$ (EM field energy density)
$\mathbf{j}(\mathbf{r},t) = u_\mathrm{EM} c\,\hat{\mathbf{c}}$ (instantaneous energy current density)
c and $\hat{\mathbf{c}}$ are speed and direction of EM energy propagation.
$\hat{\mathbf{c}} \parallel \mathbf{k}$ in isotropic, linear, charge-free materials.

Energy Density for Sinusoidal Solutions

Assumptions as in Simplifying Assumptions.
Assume sinusoidal solutions to the EM wave eq. of the form...
$\mathbf{E}(\mathbf{r},t) = \mathrm{Re}\left[\mathbf{E}_0 e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\right]$
$\mathbf{B}(\mathbf{r},t) = \mathrm{Re}\left[\mathbf{B}_0 e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\right]$
$u_\mathrm{EM} = \varepsilon\varepsilon_0 E_0^2\cos^2(\mathbf{k}\cdot\mathbf{r}-\omega t)$ (from $E_0 = cB_0$; $c = \frac{1}{\sqrt{\varepsilon\varepsilon_0\mu\mu_0}}$)
$\langle u_\mathrm{EM}\rangle = \frac{\varepsilon\varepsilon_0}{2}E_0^2$ (average EM energy density)

Energy Current Density

Assumptions as in "**Energy Density for Sinusoidal Solutions**"
$\langle\mathbf{j}\rangle = \langle u_\mathrm{EM}\rangle c\,\hat{\mathbf{c}} = \frac{\varepsilon\varepsilon_0}{2}cE_0^2\,\hat{\mathbf{c}}$ (average energy current density)
$\langle j\rangle = \langle u_\mathrm{EM}\rangle c = \frac{\varepsilon\varepsilon_0}{2}cE_0^2$ (EM energy current density)
$\mathbf{S} \equiv \frac{1}{\mu_0}\mathbf{E}\times\mathbf{B}$ (Poynting vector)
$\mathbf{S} \equiv \mathbf{E}\times\mathbf{H}$ (Poynting vector, alternate definition)
$\langle|\mathbf{S}|\rangle = \langle|\mathbf{j}|\rangle = \frac{\varepsilon\varepsilon_0}{2}cE_0^2$

Polarization

Polarization refers to the geometrical orientation of the electric and magnetic field's oscillations.

Complex Analysis of Polarization

Assumptions as in Simplifying Assumptions.
Assume plane wave solutions to EM wave equation.
$\mathbf{E}(\mathbf{r},t) = \mathbf{E}_0 e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}$
$\mathbf{B}(\mathbf{r},t) = \mathbf{B}_0 e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}$
Assume non-dispersive material with $\omega = c|\mathbf{k}|$.
$\mathbf{E}_0 \in \mathbb{C}$ and $\mathbf{B}_0 \in \mathbb{C}$ are *polarization vectors*.
B_0 is fully determined by E_0 and c (from $E_0 = cB_0$)
$\mathbf{B}_0/B_0$ is fully determined by **E** and **k** (from $\mathbf{k}\perp\mathbf{E}\perp\mathbf{B}$)
$\implies$ **B** is fully determined by **E**, **k** and c
Define coordinate system so that $\hat{\mathbf{e}}_z \parallel \mathbf{k}$.
Notation: to distinguish complex and real field amplitudes, we will denote complex quantities with an underline.
$\mathbf{E}_0 = (\underline{E}_x, \underline{E}_y, \underline{E}_z) \in \mathbb{C}^3$ (in general)
$\mathbf{E}_0 = (\underline{E}_x, \underline{E}_y, 0) \in \mathbb{C}^3$ (if $\hat{\mathbf{e}}_z \parallel \mathbf{k}$)
$\mathbf{E} = \mathbf{E}(z,t) = \mathbf{E}_0 e^{i(kz-\omega t)}$ (if $\hat{\mathbf{e}}_z \parallel \mathbf{k}$)
$\underline{E}_x = E_x e^{i\phi_x}$ (complex component in polar form)
$\underline{E}_y = E_y e^{i\phi_y}$ (complex component in polar form)
Define phase difference $\phi = \phi_x - \phi_y$ and global phase Φ.
$\underline{E}_x = E_x e^{i\Phi}$ (in terms of global phase)
$\underline{E}_y = E_y e^{i(\Phi+\phi)}$ (in terms of global phase)
$\mathbf{E}(z,t) = \left(E_x e^{i\Phi}\,\hat{\mathbf{e}}_x + E_y e^{i(\Phi+\phi)}\,\hat{\mathbf{e}}_y\right)e^{i(kz-\omega t)}$

Real Analysis of Polarization

Assumptions as in Complex Analysis of Polarization.
Define coordinate system so that $\hat{\mathbf{e}}_z \parallel \mathbf{k}$.
$\mathbf{E}(z,t) = E_x\cos(kz-\omega t+\phi_x)\,\hat{\mathbf{e}}_x$ (if $\hat{\mathbf{e}}_z \parallel \mathbf{k}$)
$\quad + E_y\cos(kz-\omega t+\phi_y)\,\hat{\mathbf{e}}_y$ ($E_x, E_y \in \mathbb{R}$)
Define phase difference $\phi = \phi_x - \phi_y$ and global phase Φ.
$\mathbf{E}(z,t) = E_x\cos(kz-\omega t+\Phi)\,\hat{\mathbf{e}}_x$ (in terms of global phase)
$\quad + E_y\cos(kz-\omega t+\Phi+\phi)\,\hat{\mathbf{e}}_y$

Linear Polarization

$\phi = \phi_x - \phi_y = n\pi,\ n\in\mathbb{Z}$ (definition of linear polarization)
Choose global phase $\Phi = 0$.
$\underline{E}_x = E_x \in \mathbb{R}$ (in general if $\Phi = 0$)
$\underline{E}_y = \pm E_y \in \mathbb{R}$ (for linear polarization $\phi = n\pi$)
$\mathbf{E}_0 = (E_x, \pm E_y, 0) \in \mathbb{R}^3$ ($\mathbf{E}_0$ is real for LP)
$\mathbf{E}(z,t) = \left(E_x\,\hat{\mathbf{e}}_x + E_y\,\hat{\mathbf{e}}_y\right)\cos(kz-\omega t)$ (if n is even)
$\mathbf{E}(z,t) = \left(E_x\,\hat{\mathbf{e}}_x - E_y\,\hat{\mathbf{e}}_y\right)\cos(kz-\omega t)$ (if n is odd)
$E_0 = \sqrt{E_x^2 + E_y^2}$ (field magnitude)
$\hat{\mathbf{e}}_\mathrm{E} = \frac{1}{E_0}(E_x, \pm E_y, 0)$ (field direction)
Find $\hat{\mathbf{e}}_\mathrm{B}$ by rotating $\hat{\mathbf{e}}_\mathrm{E}$ by $+\pi/2$ in the xy plane.
$\mathbf{B} = (E_0/c)\cos(kz-\omega t)\,\hat{\mathbf{e}}_\mathrm{B}$ (magnetic field for LP)

Circular Polarization

$\phi = \phi_x - \phi_y = \frac{\pi}{2} + n\pi,\ n\in\mathbb{Z}$ (definition of CP)
$E_x = E_y \equiv E_0$ (definition of CP)
Choose global phase $\Phi = 0$.
$\underline{E}_x = E_0 \in \mathbb{R}$ (for $\Phi = 0$ and $E_x = E_0$)
$\underline{E}_y = \pm E_0$ (for CP with $\phi = \frac{\pi}{2} + n\pi$)
$\mathbf{E}_0 = E_0(1, \pm i, 0)$
$\mathbf{E} = E_0(1, \pm i, 0)e^{i(kz-\omega t)}$
$\mathrm{Re}[\mathbf{E}] = E_0\left(\cos(kz-\omega t), \mp\sin(kz-\omega t), 0\right)$

Left-Hand Circular Polarization (LHC)

(Definitions vary, the one used in this course appears below.)
For an observer at fixed position z *facing* the source of EM waves, **E** rotates *counterclockwise* with respect to *time* in the plane perpendicular to the direction of EM wave propagation.
$\mathbf{E}_\mathrm{lhc} = E_0\left(e^{i(kz-\omega t)}, +ie^{i(kz-\omega t)}, 0\right) \in \mathbb{C}^3$
$\mathrm{Re}[\mathbf{E}_\mathrm{lhc}] = E_0\left(\cos(kz-\omega t), -\sin(kz-\omega t), 0\right)$
$\mathrm{Re}[\mathbf{E}_\mathrm{lhc}]_x = E_0\cos(\omega t)$ (with respect to time at $z = 0$)
$\mathrm{Re}[\mathbf{E}_\mathrm{lhc}]_y = E_0\sin(\omega t)$ (with respect to time at $z = 0$)
LHC polarization occurs when n is even; $\phi = \pi/2 + 2\pi k,\ k\in\mathbb{Z}$

Right-Hand Circular Polarization (RHC)

For an observer at fixed position z *facing* the source of EM waves, **E** rotates *clockwise* with respect to *time* in the plane perpendicular to the direction of EM wave propagation.
$\mathbf{E}_\mathrm{rhc} = E_0\left(e^{i(kz-\omega t)}, -ie^{i(kz-\omega t)}, 0\right) \in \mathbb{C}^3$
$\mathrm{Re}[\mathbf{E}_\mathrm{rhc}] = E_0\Big(\cos(kz-\omega t), \sin(kz-\omega t), 0\Big)$
$\mathrm{Re}[\mathbf{E}_\mathrm{lhc}]_x = E_0\cos(\omega t)$ (with respect to time at $z = 0$)
$\mathrm{Re}[\mathbf{E}_\mathrm{lhc}]_y = -E_0\sin(\omega t)$ (with respect to time at $z = 0$)
RHC polarization occurs when n is odd; $\phi = -\pi/2 + 2\pi k,\ k\in\mathbb{Z}$

Combining Polarizations

$\mathbf{E}_\mathrm{lhc} + \mathbf{E}_\mathrm{rhc} = 2E_0\big(\cos(kz-\omega t), 0, 0\big) = 2\mathbf{E}_\text{lin-x}$ (real parts)
$\mathbf{E}_\mathrm{rhc} - \mathbf{E}_\mathrm{lhc} = 2E_0\big(0,\ \sin(kz-\omega t),\ 0\big) = 2\mathbf{E}_\text{lin-y}$ (real parts)
Any LP can be constructed from a linear combination of CP!

Elliptical Polarization
$|E_x| \neq |E_y|$ (definition of EP)
$\phi = \phi_x - \phi_y$ is an arbitrary real constant (definition of EP)
Choose global phase $\Phi = 0$
$\underline{E}_x = E_x \in \mathbb{R}$ (for $\Phi = 0$)
$\underline{E}_y = E_y e^{i\phi}$ (E_y and ϕ are arbitrary)
$\mathbf{E}_0 = \big(E_x, E_y e^{i\phi}, 0\big)$
$\mathbf{E} = \big(E_x, E_y e^{i\phi}, 0\big) e^{i(kz-\omega t)}$
$\mathrm{Re}[\mathbf{E}] = E_x \cos(kz - \omega t)\,\hat{\mathbf{e}}_x + E_y \cos(kz - \omega t + \phi)\,\hat{\mathbf{e}}_y$
$\mathbf{E}(z,t)$ traces out an ellipse in the plane perpendicular to the direction of wave propagation; the orientation of the ellipse itself is fixed in the xy plane.

Geometry of Elliptical Polarization
$\hat{\mathbf{e}}_a$ is the direction of the ellipse's semi-major axis.
$\hat{\mathbf{e}}_b$ is the direction of the ellipse's semi-minor axis.
θ is angle of $\hat{\mathbf{e}}_a$ and $\hat{\mathbf{e}}_b$ relative to $\hat{\mathbf{e}}_x$ and $\hat{\mathbf{e}}_y$.
$E_0 \equiv \sqrt{E_x^2 + E_y^2}$
$\tan(2\theta) = \frac{2E_xE_y}{E_0^2}\cos\phi$
If $\phi = \frac{\pi}{2} + n\pi,\ n \in \mathbb{Z}$ then $\hat{\mathbf{e}}_a$ and $\hat{\mathbf{e}}_b$ align with $\hat{\mathbf{e}}_x$ and $\hat{\mathbf{e}}_y$.
$\frac{b}{a} = \frac{E_y \cos\theta \sin\phi}{E_x \cos\theta + E_y \sin\theta \cos\phi}$ (ratio of elliptical axes)

Jones Calculus
Assumptions as in Simplifying Assumptions.
Define coordinate system so that $\hat{\mathbf{e}}_z \parallel \mathbf{k}$.

Jones Vector
Jones vectors encode the polarization state of EM plane waves.
$\mathbf{E}(z,t) = \big(E_x\,\hat{\mathbf{e}}_x + E_y\,\hat{\mathbf{e}}_y e^{i\phi}\big)\, e^{i(kz-\omega t)}$ (general polarization)
$\mathbf{E}(z,t) = e^{i(kz-\omega t)}\begin{pmatrix} E_x \\ E_y e^{i\phi}\end{pmatrix}$ (vector representation in xy plane)
$E_0 \equiv \sqrt{E_x^2 + E_y^2}$ (for shorthand)
$\mathbf{J} \equiv \frac{1}{E_0}\begin{pmatrix} E_x \\ E_y e^{i\phi}\end{pmatrix}$ (definition of Jones vector)

Jones Vectors for Common Polarizations
$\mathbf{J}_{\text{lin-x}} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ (linear polarization along $\hat{\mathbf{e}}_x$)
$\mathbf{J}_{\text{lin-y}} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ (linear polarization along $\hat{\mathbf{e}}_y$)
$\mathbf{J}_{\text{lin-}\theta} = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$ (LP at angle θ relative to $\hat{\mathbf{e}}_x$)
$\mathbf{J}_{\text{lhc}} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ +i \end{pmatrix}$ (left-hand circular polarization)
$\mathbf{J}_{\text{rhc}} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix}$ (right-hand circular polarization)

Jones Matrices for Common Polarizing Elements
$\mathbf{M}_{\text{lin-x}} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ (transmission axis along $\hat{\mathbf{e}}_x$)
$\mathbf{M}_{\text{lin-y}} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ (transmission axis along $\hat{\mathbf{e}}_y$)
$\mathbf{M}_{\text{lin-}\theta} = \begin{pmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{pmatrix}$ (TA at angle θ relative to $\hat{\mathbf{e}}_x$)
$\mathbf{M}_{\text{lhc}} = \frac{1}{2}\begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix}$ (transmits LHC polarized light)
$\mathbf{M}_{\text{rhc}} = \frac{1}{2}\begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}$ (transmits RHC polarized light)

Phase Retarders
Phase retarders are made from uniaxial birefringent materials.
Let $\hat{\mathbf{e}}_x$, $\hat{\mathbf{e}}_y$, and $\hat{\mathbf{e}}_z$ be the principal axes of the uniaxial material's dielectric tensor.
Define dielectric tensor eigenvalues so that $n_x = n_y \neq n_z$.
$n_x = n_y \equiv n_{\text{o}}$ (ordinary refractive index)
$n_z \equiv n_{\text{e}}$ (extraordinary refractive index)
Fast axis is axis with slower n (and faster phase velocity).
Slow axis is axis with larger n (and slower phase velocity).
Negative uniaxial crystals have n_{e} as fast axis and $n_{\text{e}} < n_{\text{o}}$.
Positive uniaxial crystals have n_{e} as slow axis and $n_{\text{e}} > n_{\text{o}}$.

Jones Matrices For Common Phase Retarders
QWPs introduce phase difference $\pm\pi/2$ between E_x and E_y.
QWPs transform linear polarization into elliptical polarization (and linear polarization with $E_x = E_y$ into circular polarization).
$\mathbf{M}_{\text{qw}} = e^{\pm i\frac{\pi}{4}}\begin{pmatrix} 1 & 0 \\ 0 & \mp i \end{pmatrix}$ (quarter waveplate)
$\mathbf{M}_{\text{qw-}\theta} = e^{\pm i\frac{\pi}{4}}\begin{pmatrix} \cos^2\theta + i\sin^2\theta & (1-i)\sin\theta\cos\theta \\ (1-i)\sin\theta\cos\theta & \sin^2\theta \mp i\cos^2\theta \end{pmatrix}$
HWPs introduce phase difference $\pm\pi$ between E_x and E_y.
HWPs transform RHC polarization into LHC polarization and reflect linear polarization about the coordinate axes.
$\mathbf{M}_{\text{hw}} = e^{\pm i\frac{\pi}{2}}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ (half waveplate)
$\mathbf{M}_{\text{hw-}\theta} = e^{\pm i\frac{\pi}{2}}\begin{pmatrix} \cos^2\theta - \sin^2\theta & 2\sin\theta\cos\theta \\ 2\sin\theta\cos\theta & \sin^2\theta - \cos^2\theta \end{pmatrix}$

EM Waves in Conductive Materials
Simplifying Assumptions in Condutors
We assume conducting matter in which we analyze EM waves is...

(i) homogeneous: the material's properties are identical throughout the material (so $\varepsilon \neq \varepsilon(\mathbf{r})$ and $\mu \neq \mu(\mathbf{r})$),

(ii) isotropic: the material's properties are identical for all orientations of the material (so ε and μ are scalars and not rank-two tensors),

(iii) charge-free: the material is free of net electric charge (so $\rho = 0$),

(iv) linear: $\mathbf{D} = \varepsilon\varepsilon_0\mathbf{E}$ and $\mathbf{B} = \mu\mu_0\mathbf{H}$, and

(v) an Ohmic conductor: $\mathbf{j}_{\text{f}} = \sigma_{\text{E}}\mathbf{E}$.

σ_{E} is the conduting material's electrical conductivity.

Maxwell Equations Under Above Assumptions
$\nabla \cdot \mathbf{D} = 0$
$\nabla \cdot \mathbf{B} = 0$
$\nabla \times \mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t}$
$\nabla \times \mathbf{H} = \mathbf{j}_{\text{f}} + \frac{\partial\mathbf{D}}{\partial t} = \sigma_{\text{E}}\mathbf{E} + \frac{\partial\mathbf{D}}{\partial t}$

"Wave Equations" in Conducting Material
$\nabla^2\mathbf{E} = \mu\mu_0\left(\sigma_{\text{E}}\frac{\partial\mathbf{E}}{\partial t} + \varepsilon\varepsilon_0\frac{\partial^2\mathbf{E}}{\partial t^2}\right)$ (**E in conducting media**)
$\nabla^2\mathbf{B} = \mu\mu_0\left(\sigma_{\text{E}}\frac{\partial\mathbf{B}}{\partial t} + \varepsilon\varepsilon_0\frac{\partial^2\mathbf{B}}{\partial t^2}\right)$ (**B in conducting media**)
$\mathbf{E}(\mathbf{r},t) = \mathbf{E}_0 e^{i(\boldsymbol{\mathcal{K}}\cdot\mathbf{r}-\omega t)}$ (ansatz; $\mathbf{E}_0, \boldsymbol{\mathcal{K}} \in \mathbb{C}^3$)
$\mathbf{B}(\mathbf{r},t) = \mathbf{B}_0 e^{i(\boldsymbol{\mathcal{K}}\cdot\mathbf{r}-\omega t)}$ (ansatz; $\mathbf{B}_0, \boldsymbol{\mathcal{K}} \in \mathbb{C}^3$)
$\mathcal{K}^2 = k_0^2\left(\varepsilon\mu + i\frac{\sigma_{\text{E}}\mu}{\varepsilon_0\omega}\right),\ k_0 = \omega/c_0$ (wave vector in conductors)

Refractive Index in Conducting Material
$\mathcal{N}^2 \equiv \varepsilon\mu + i\frac{\sigma_{\text{E}}\mu}{\varepsilon_0\omega}$ (refractive index in conductors; $\mathcal{N} \in \mathbb{C}$)
$\mathcal{N} \equiv n_{\text{Re}} + i n_{\text{Im}}$
$n_{\text{Re}}^2 = \frac{1}{2}\left(\varepsilon\mu + \sqrt{(\varepsilon\mu)^2 + \left(\frac{\sigma_{\text{E}}\mu}{\varepsilon_0\omega}\right)^2}\right)$
$n_{\text{Im}}^2 = \frac{1}{2}\left(-\varepsilon\mu + \sqrt{(\varepsilon\mu)^2 + \left(\frac{\sigma_{\text{E}}\mu}{\varepsilon_0\omega}\right)^2}\right)$

Limit Cases in a Good Conductor
Consider limit case of EM waves in a material with...

(i) $\mu = 1$ (a non-magnetic material)

(ii) $\frac{\sigma_{\mathrm{E}}}{\varepsilon_0 \omega} \gg \varepsilon$ (good conductor; low frequencies)

$$n_{\mathrm{Re}}^2 \approx \frac{1}{2}\left(+\varepsilon + \frac{\sigma_{\mathrm{E}}}{\varepsilon_0 \omega}\right) \approx \frac{\sigma_{\mathrm{E}}}{2\varepsilon_0 \omega}$$
$$n_{\mathrm{Im}}^2 \approx \frac{1}{2}\left(-\varepsilon + \frac{\sigma_{\mathrm{E}}}{\varepsilon_0 \omega}\right) \approx \frac{\sigma_{\mathrm{E}}}{2\varepsilon_0 \omega}$$

Electric Field Solution in Conducting Material

Align coordinate system so $\hat{\mathbf{e}}_z$ aligns with direction of wave front propagation.

$$\mathbf{E} = \mathbf{E}(z,t) = \mathbf{E}_0 e^{i(\mathcal{K}z - \omega t)}$$
$$\mathbf{E}(z,t) = \mathbf{E}_0 e^{i(n_{\mathrm{Re}} k_0 z - \omega t)} e^{-n_{\mathrm{Im}} k_0 z} \qquad \text{(using } \mathcal{K} = \mathcal{N} k_0 \in \mathbb{C}\text{)}$$
$$z_0 \equiv \frac{1}{k_0 n_{\mathrm{Im}}} = \frac{c_0}{\omega n_{\mathrm{Im}}} \qquad \text{(definition of skin depth)}$$
$$z_0 \approx \sqrt{\frac{2}{\sigma_{\mathrm{E}} \mu_0 \omega}} \qquad \text{(limit case in good conductors)}$$

Reflection and Refraction

Maxwell Equations In Matter (for review)

$\nabla \cdot \mathbf{D} = \rho_\mathrm{f}$
$\nabla \cdot \mathbf{B} = 0$
$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$
$\nabla \times \mathbf{H} = \mathbf{j}_\mathrm{f} + \frac{\partial \mathbf{D}}{\partial t}$

Boundary Conditions

Consider a boundary btwn. two materials with different μ and ε.
$\hat{\mathbf{n}}$ is normal vector to boundary from material 2 to material 1.

Boundary Condition for B

$\mathbf{B}_1$ and $\mathbf{B}_2$ are the fields in material 1 and 2, respectively
$\nabla \cdot \mathbf{B} = 0 \implies \iiint_V \nabla \cdot \mathbf{B} \, \mathrm{d}^3\mathbf{r} = \oiint \mathbf{B} \cdot \mathrm{d}\mathbf{S} = 0$
Consider Gaussian pillbox of height $\mathrm{d}\mathbf{h} \to \mathbf{0}$ enclosing boundary.
$\iint_{S_1} \mathbf{B}_1 \cdot \hat{\mathbf{n}} \, \mathrm{d}S - \iint_{S_2} \mathbf{B}_2 \cdot \hat{\mathbf{n}} \, \mathrm{d}S + 0 = 0$
$(\mathbf{B}_1 - \mathbf{B}_2) \cdot \hat{\mathbf{n}} = 0$ (BC on **B** field)
$B_1^\perp = B_2^\perp$ (alternate formulation)
$B_\perp$ is magnitude of **B** normal to boundary.

Boundary Condition for D

$\mathbf{D}_1$ and $\mathbf{D}_2$ are the fields in material 1 and 2, respectively.
$\nabla \cdot \mathbf{D} = \rho_\mathrm{f} \implies \iiint_V \nabla \cdot \mathbf{D} \, \mathrm{d}^3\mathbf{r} = \oiint \mathbf{D} \cdot \mathrm{d}\mathbf{S} = \rho_\mathrm{f}$
Consider Gaussian pillbox of height $\mathrm{d}\mathbf{h} \to \mathbf{0}$ enclosing boundary.
$\iint_{S_1} \mathbf{D}_1 \cdot \hat{\mathbf{n}} \, \mathrm{d}S - \iint_{S_2} \mathbf{D}_2 \cdot \hat{\mathbf{n}} \, \mathrm{d}S + 0 = \iint_S \sigma_\mathrm{f} \, \mathrm{d}S$
$(\mathbf{D}_1 - \mathbf{D}_2) \cdot \hat{\mathbf{n}} = \sigma_\mathrm{f}$ (BC on **D** field)
$D_1^\perp - D_2^\perp = \sigma_\mathrm{f}$ (alternate formulation)
σ_f is free charge density along boundary.

Boundary Condition for E

$\mathbf{E}_1$ and $\mathbf{E}_2$ are the fields in material 1 and 2, respectively.
$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$
$\iint_S \nabla \times \mathbf{E} \cdot \mathrm{d}\mathbf{S} = \oint_{\partial S} \mathbf{E} \cdot \mathrm{d}\mathbf{s} = -\frac{\partial}{\partial t} \iint_S \mathbf{B} \cdot \mathrm{d}\mathbf{S}$
Consider rectangular surface of length l and width $a \to 0$.
$\hat{\mathbf{t}}_1$ and $\hat{\mathbf{t}}_2$ are tangents to perimeter in material 1 and 2
$\int_l (\mathbf{E}_1 \cdot \hat{\mathbf{t}}_1 + \mathbf{E}_2 \cdot \hat{\mathbf{t}}_2) \, \mathrm{d}l + 0 + 0 = -\frac{\partial}{\partial t} \iint_S \mathbf{B} \cdot \mathrm{d}\mathbf{S} \to 0$
$\mathbf{E}_1 \cdot \hat{\mathbf{t}}_1 + \mathbf{E}_2 \cdot \hat{\mathbf{t}}_2 = 0$ (BC on **E** field)
$E_1^\parallel = E_2^\parallel$ (alternate formulation)
$(\mathbf{E}_1 - \mathbf{E}_2) \times \hat{\mathbf{n}} = \mathbf{0}$ (alternate formulation)

Boundary Condition for H

$\mathbf{H}_1$ and $\mathbf{H}_2$ are the fields in material 1 and 2, respectively.
$\nabla \times \mathbf{H} = \mathbf{j}_\mathrm{f} - \frac{\partial \mathbf{D}}{\partial t}$
$\iint_S \nabla \times \mathbf{H} \cdot \mathrm{d}\mathbf{S} = \oint_{\partial S} \mathbf{H} \cdot \mathrm{d}\mathbf{s} = \iint_S \mathbf{j}_\mathrm{f} \cdot \mathrm{d}\mathbf{S} - \frac{\partial}{\partial t} \iint_S \mathbf{D} \cdot \mathrm{d}\mathbf{S}$
Consider rectangular surface of length l and width $a \to 0$.
$\hat{\mathbf{t}}_1$ and $\hat{\mathbf{t}}_2$ are tangents to perimeter in material 1 and 2.
$\int_l (\mathbf{H}_1 \cdot \hat{\mathbf{t}}_1 + \mathbf{H}_2 \cdot \hat{\mathbf{t}}_2) \, \mathrm{d}l + 0 + 0 = 0 + \int_l \mathbf{K} \cdot \mathrm{d}\mathbf{l}$
K is surface current density in boundary (units $\mathrm{A\,m^{-1}}$).
$\mathbf{H}_1 \cdot \hat{\mathbf{t}}_1 + \mathbf{H}_2 \cdot \hat{\mathbf{t}}_2 = K$ (BC on **H** field)
$H_1^\parallel - H_2^\parallel = K$ (alternate formulation)
$(\mathbf{H}_1 - \mathbf{H}_2) \times \hat{\mathbf{n}} = \mathbf{K}$ (alternate formulation)

Boundary Conditions In Dielectrics

Assume both material 1 and 2 are ideal dielectrics.
$\sigma_\mathrm{f} = 0$ (no surface charge density along boundary)
$\mathbf{K} = \mathbf{0}$ (no surface current density along boundary)

$$
\begin{aligned}
(\mathbf{B}_1 - \mathbf{B}_2) \cdot \hat{\mathbf{n}} &= 0 \implies B_1^\perp = B_2^\perp \\
(\mathbf{D}_1 - \mathbf{D}_2) \cdot \hat{\mathbf{n}} &= 0 \implies D_1^\perp = D_2^\perp \\
(\mathbf{E}_1 - \mathbf{E}_2) \times \hat{\mathbf{n}} &= 0 \implies E_1^\parallel = E_2^\parallel \\
(\mathbf{H}_1 - \mathbf{H}_2) \times \hat{\mathbf{n}} &= 0 \implies H_1^\parallel = H_2^\parallel
\end{aligned}
$$

Reflection and Refraction

Consider a plane wave incident on a planar interface from material 1 with refractive indices n_1 into material 2 with refractive index n_2.
Assume both material 1 and 2 are ideal dielectrics.
Let interface lie in xy plane.
Let z axis point from material 1 into material 2.

Notation
The subscript $_\mathrm{i}$ denotes incident quantities.
The subscript $_\mathrm{r}$ denotes reflected quantities.
The subscript $_\mathrm{t}$ denotes transmitted quantities.

$\mathbf{E}_\mathrm{i}(\mathbf{r}, t) = \mathbf{E}_{\mathrm{i}_0} e^{i(\mathbf{k}_\mathrm{i} \cdot \mathbf{r} - \omega_\mathrm{i} t + \phi_\mathrm{i})}$ (incident wave)
$\mathbf{E}_\mathrm{r}(\mathbf{r}, t) = \mathbf{E}_{\mathrm{r}_0} e^{i(\mathbf{k}_\mathrm{r} \cdot \mathbf{r} - \omega_\mathrm{r} t + \phi_\mathrm{r})}$ (reflected wave)
$\mathbf{E}_\mathrm{t}(\mathbf{r}, t) = \mathbf{E}_{\mathrm{t}_0} e^{i(\mathbf{k}_\mathrm{t} \cdot \mathbf{r} - \omega_\mathrm{t} t + \phi_\mathrm{t})}$ (transmitted wave)

Applying Boundary Conditions

$E_\mathrm{i}^\parallel + E_\mathrm{r}^\parallel = E_\mathrm{t}^\parallel$ for all $\mathbf{r} = (x, y, 0)$ in interface and for all t
$E_{\mathrm{i}_0}^\parallel e^{i\phi_\mathrm{i}} + E_{\mathrm{r}_0}^\parallel e^{i\phi_\mathrm{r}} = E_{\mathrm{t}_0}^\parallel e^{i\phi_\mathrm{t}}$ (for $x = y = z = 0$ and $t = 0$)
$\implies \phi_\mathrm{i} = \phi_\mathrm{r} = \phi_\mathrm{t} \equiv \phi$ (phases are equal)
$E_{\mathrm{i}_0}^\parallel e^{-i\omega_\mathrm{i} t} e^{i\phi} + E_{\mathrm{r}_0}^\parallel e^{-i\omega_\mathrm{r} t} e^{i\phi} = E_{\mathrm{t}_0}^\parallel e^{-i\omega_\mathrm{t} t} e^{i\phi}$ (for $\mathbf{r} = \mathbf{0}$)
$\implies \omega_\mathrm{i} = \omega_\mathrm{r} = \omega_\mathrm{t} \equiv \omega$ (frequencies are equal)
$E_{\mathrm{i}_0}^\parallel e^{i\mathbf{k}_\mathrm{i} \cdot \mathbf{r}} e^{i\phi} + E_{\mathrm{r}_0}^\parallel e^{i\mathbf{k}_\mathrm{r} \cdot \mathbf{r}} e^{i\phi} = E_{\mathrm{t}_0}^\parallel e^{i\mathbf{k}_\mathrm{t} \cdot \mathbf{r}} e^{i\phi}$ (for $t = 0$)
$\implies \mathbf{k}_\mathrm{i} \cdot \mathbf{r} = \mathbf{k}_\mathrm{r} \cdot \mathbf{r} = \mathbf{k}_\mathrm{t} \cdot \mathbf{r} = \text{constant}$
Geometrically: $\mathbf{k}_\mathrm{i}$, $\mathbf{k}_\mathrm{r}$ and $\mathbf{k}_\mathrm{t}$ lie in the same *plane of incidence.*
Convention: plane of incidence is xz plane for interface in xy plane.

Geometry of Reflection and Refraction

Let interface lie in xy plane.
Let plane of incidence lie in xz plane.
Let z axis point from material 1 into material 2.
θ_i is angle of incidence.
θ_r is angle of reflection.
θ_t is angle of transmission.
All angles measured with respect to interface normal vector $\hat{\mathbf{n}}$.
$\mathbf{k}_\mathrm{i} = k_0 n_1 (\sin\theta_\mathrm{i}, 0, \cos\theta_\mathrm{i})$ (incident wave vector)
$\mathbf{k}_\mathrm{r} = k_0 n_1 (\sin\theta_\mathrm{r}, 0, -\cos\theta_\mathrm{r})$ (reflected wave vector)
$\mathbf{k}_\mathrm{t} = k_0 n_2 (\sin\theta_\mathrm{t}, 0, \cos\theta_\mathrm{t})$ (transmitted wave vector)

Laws of Reflection and Refraction

Substitute $\mathbf{k}_\mathrm{i}$, $\mathbf{k}_\mathrm{r}$ into $\mathbf{k}_\mathrm{i} \cdot \mathbf{r} = \mathbf{k}_\mathrm{r} \cdot \mathbf{r}$; apply $\mathbf{r} = (x, y, 0)$
$\implies \theta_\mathrm{i} = \theta_\mathrm{r}$ (**law of reflection**)
Substitute $\mathbf{k}_\mathrm{i}$, $\mathbf{k}_\mathrm{t}$ into $\mathbf{k}_\mathrm{i} \cdot \mathbf{r} = \mathbf{k}_\mathrm{t} \cdot \mathbf{r}$; apply $\mathbf{r} = (x, y, 0)$
$\implies n_1 \sin\theta_\mathrm{i} = n_2 \sin\theta_\mathrm{t}$ (**law of refraction**)

Transverse Electric (TE) Polarization

$\mathbf{E}_\mathrm{i}$, $\mathbf{E}_\mathrm{r}$ and $\mathbf{E}_\mathrm{t}$ are normal (*transverse*) to the plane of incidence and tangent to boundary interface.
$\mathbf{B}_\mathrm{i}$, $\mathbf{B}_\mathrm{r}$ and $\mathbf{B}_\mathrm{t}$ lie in the plane of incidence and are perpendicular to $\mathbf{E}_\mathrm{i}, \mathbf{k}_\mathrm{i}$ / $\mathbf{E}_\mathrm{r}, \mathbf{k}_\mathrm{r}$ / $\mathbf{E}_\mathrm{t}, \mathbf{k}_\mathrm{t}$.
TE polarized-quantities are denoted by the subscript $_\mathrm{s}$.

Transverse Magnetic (TM) Polarization

$\mathbf{B}_\mathrm{i}$, $\mathbf{B}_\mathrm{r}$ and $\mathbf{B}_\mathrm{t}$ are normal (*transverse*) to the plane of incidence and tangent to boundary interface.
$\mathbf{E}_\mathrm{i}$, $\mathbf{E}_\mathrm{r}$ and $\mathbf{E}_\mathrm{t}$ lie in the plane of incidence and are perpendicular to $\mathbf{B}_\mathrm{i}, \mathbf{k}_\mathrm{i}$ / $\mathbf{B}_\mathrm{r}, \mathbf{k}_\mathrm{r}$ / $\mathbf{B}_\mathrm{t}, \mathbf{k}_\mathrm{t}$.
TM polarized-quantities are denoted by the subscript $_\mathrm{p}$.

Fresnel Equations

Situation as in "**Reflection and Refraction**"
Additionally assume $\mu_1 = \mu_2 = 1$.
$r \equiv \frac{E_{\mathrm{r}_0}}{E_{\mathrm{i}_0}}$ (definition of reflection coefficient)

$t \equiv \frac{E_{\mathrm{t}_0}}{E_{\mathrm{i}_0}}$ (definition of transmission coefficient)

Fresnel Equations for TE Waves

$E_{\mathrm{i}}^{\|} + E_{\mathrm{r}}^{\|} = E_{\mathrm{t}}^{\|}$ (general boundary condition)

$E_{\mathrm{i}_0} + E_{\mathrm{r}_0} = E_{\mathrm{t}_0}$ (for TE-polarized waves)

$H_{\mathrm{i}}^{\|} + H_{\mathrm{r}}^{\|} = H_{\mathrm{t}}^{\|}$ (general boundary condition)

$B_{\mathrm{i}}^{\|} + B_{\mathrm{r}}^{\|} = B_{\mathrm{t}}^{\|}$ (from $\mathbf{B} = \mu_0 \mathbf{H}$)

$(B_{\mathrm{r}_0} - B_{\mathrm{i}_0}) \cos\theta_{\mathrm{i}} = B_{\mathrm{t}_0} \cos\theta_{\mathrm{t}}$ (after geometry)

$(E_{\mathrm{r}_0} - E_{\mathrm{i}_0}) n_1 \cos\theta_{\mathrm{i}} = E_{\mathrm{t}_0} n_2 \cos\theta_{\mathrm{t}}$ (after $E_0 = cB_0$)

$r_{\mathrm{s}} = \frac{E_{\mathrm{r}_0}}{E_{\mathrm{i}_0}} = \frac{n_1\cos\theta_{\mathrm{i}} - n_2\cos\theta_{\mathrm{t}}}{n_1\cos\theta_{\mathrm{i}} + n_2\cos\theta_{\mathrm{t}}}$ (**Fresnel equation for** r_{s})

$t_{\mathrm{s}} = \frac{E_{\mathrm{t}_0}}{E_{\mathrm{i}_0}} = 1 + r_{\mathrm{s}}$ (**Fresnel equation for** t_{s})

Alternate Formulations

$r_{\mathrm{s}} = -\frac{\sin(\theta_{\mathrm{i}} - \theta_{\mathrm{t}})}{\sin(\theta_{\mathrm{i}} + \theta_{\mathrm{t}})}$ (using Snell's law)

$$r_{\mathrm{s}} = \frac{n_1\cos\theta_{\mathrm{i}} - n_2\sqrt{1-\left(\frac{n_1}{n_2}\right)^2\sin^2\theta_{\mathrm{i}}}}{n_1\cos\theta_{\mathrm{i}} + n_2\sqrt{1-\left(\frac{n_1}{n_2}\right)^2\sin^2\theta_{\mathrm{i}}}}$$ (Snell's law and trig. identities)

Fresnel Equations for TM Waves

$H_{\mathrm{i}}^{\|} + H_{\mathrm{r}}^{\|} = H_{\mathrm{t}}^{\|}$ (general boundary condition)

$B_{\mathrm{i}}^{\|} + B_{\mathrm{r}}^{\|} = B_{\mathrm{t}}^{\|}$ (from $\mathbf{B} = \mu_0 \mathbf{H}$)

$B_{\mathrm{i}_0} + B_{\mathrm{r}_0} = B_{\mathrm{t}_0}$ (for TM-polarized waves)

$E_{\mathrm{i}_0} n_1 + E_{\mathrm{r}_0} n_1 = E_{\mathrm{t}_0} n_2$ (after $E_0 = cB_0$)

$E_{\mathrm{i}}^{\|} + E_{\mathrm{r}}^{\|} = E_{\mathrm{t}}^{\|}$ (general boundary condition)

$(E_{\mathrm{i}_0} - E_{\mathrm{r}_0}) \cos\theta_{\mathrm{i}} = E_{\mathrm{t}_0} \cos\theta_{\mathrm{t}}$ (after geometry)

$r_{\mathrm{p}} = \frac{E_{\mathrm{r}_0}}{E_{\mathrm{i}_0}} = \frac{n_2\cos\theta_{\mathrm{i}} - n_1\cos\theta_{\mathrm{t}}}{n_2\cos\theta_{\mathrm{i}} + n_1\cos\theta_{\mathrm{t}}}$ (**Fresnel equation for** r_{p})

$t_{\mathrm{p}} = \frac{E_{\mathrm{t}_0}}{E_{\mathrm{i}_0}} = \frac{n_1}{n_2}(1 + r_{\mathrm{p}})$ (**Fresnel equation for** t_{p})

Alternate Formulations

$r_{\mathrm{p}} = \frac{\sin\theta_{\mathrm{i}}\cos\theta_{\mathrm{i}} - \sin\theta_{\mathrm{t}}\cos\theta_{\mathrm{t}}}{\sin\theta_{\mathrm{i}}\cos\theta_{\mathrm{i}} + \sin\theta_{\mathrm{t}}\cos\theta_{\mathrm{t}}}$ (using Snell's law)

$r_{\mathrm{p}} = \frac{\tan(\theta_{\mathrm{i}} - \theta_{\mathrm{t}})}{\tan(\theta_{\mathrm{i}} + \theta_{\mathrm{t}})}$ (Snell's law and trig. identities)

$$r_{\mathrm{p}} = \frac{n_2\cos\theta_{\mathrm{i}} - n_1\sqrt{1-\left(\frac{n_1}{n_2}\right)^2\sin^2\theta_{\mathrm{i}}}}{n_2\cos\theta_{\mathrm{i}} + n_1\sqrt{1-\left(\frac{n_1}{n_2}\right)^2\sin^2\theta_{\mathrm{i}}}}$$ (Snell's law and trig. identities)

Power Coefficients

Situation and assumptions as in Fresnel Equations.

Let $\mathbf{j}_{\mathrm{i}}$, $\mathbf{j}_{\mathrm{r}}$, and $\mathbf{j}_{\mathrm{t}}$ denote incident, reflected, and transmitted energy current densities, respectively.

Let $\hat{\mathbf{z}}$ denote normal to boundary (from material 1 into material 2).

$\mathbf{j} \parallel \mathbf{k}$ (in isotropic materials)

$\langle \mathbf{j} \rangle = \frac{1}{2}\varepsilon\varepsilon_0 c E_0^2 \,\hat{\mathbf{k}}$ (in isotropic materials)

$\langle \mathbf{j} \rangle = \frac{1}{2}\varepsilon_0 c_0 n E_0^2 \,\hat{\mathbf{k}}$ (assuming $\mu = 1 \implies \varepsilon = n^2$)

$\langle \mathbf{j}_{\mathrm{i}} \rangle \cdot \hat{\mathbf{z}} = \frac{1}{2}\varepsilon_0 c_0 n_1 E_{\mathrm{i}_0}^2 \cos\theta_{\mathrm{i}}$

$\langle \mathbf{j}_{\mathrm{r}} \rangle \cdot \hat{\mathbf{z}} = -\frac{1}{2}\varepsilon_0 c_0 n_1 E_{\mathrm{r}_0}^2 \cos\theta_{\mathrm{i}}$ (using $\theta_{\mathrm{r}} = \theta_{\mathrm{i}}$)

$\langle \mathbf{j}_{\mathrm{t}} \rangle \cdot \hat{\mathbf{z}} = \frac{1}{2}\varepsilon_0 c_0 n_2 E_{\mathrm{t}_0}^2 \cos\theta_{\mathrm{t}}$

$R \equiv \frac{|\langle \mathbf{j}_{\mathrm{r}} \rangle \cdot \hat{\mathbf{z}}|}{|\langle \mathbf{j}_{\mathrm{i}} \rangle \cdot \hat{\mathbf{z}}|} = \left(\frac{E_{\mathrm{r}_0}}{E_{\mathrm{i}_0}}\right)^2 = |r|^2$ (reflectance)

$T \equiv \frac{|\langle \mathbf{j}_{\mathrm{t}} \rangle \cdot \hat{\mathbf{z}}|}{|\langle \mathbf{j}_{\mathrm{i}} \rangle \cdot \hat{\mathbf{z}}|} = \frac{n_2}{n_1}\frac{\cos\theta_{\mathrm{t}}}{\cos\theta_{\mathrm{i}}}\left(\frac{E_{\mathrm{t}_0}}{E_{\mathrm{i}_0}}\right)^2 = \frac{n_2}{n_1}\frac{\cos\theta_{\mathrm{t}}}{\cos\theta_{\mathrm{i}}}|t|^2$ (transmittance)

$R + T = 1$

$\mathbf{J}_{\mathrm{r}} = \begin{pmatrix} r_{\mathrm{s}} & 0 \\ 0 & r_{\mathrm{p}} \end{pmatrix}\mathbf{J}_{\mathrm{i}}$ (Jones vectors; general polarization)

$\mathbf{J}_{\mathrm{t}} = \begin{pmatrix} t_{\mathrm{s}} & 0 \\ 0 & t_{\mathrm{p}} \end{pmatrix}\mathbf{J}_{\mathrm{i}}$ (Jones vectors; general polarization)

Passage into Optically Denser Material

"Optical density" refers to value of refractive index n.

Optically denser material $\iff$ material larger n.

Optically less dense material $\iff$ material smaller n.

Passage into optically denser material $\implies n_2 > n_1$.

TE Polarization: Reflection Coefficients

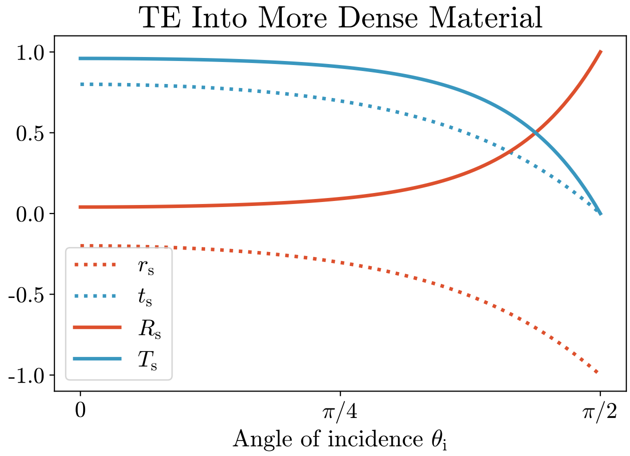
$$r_{\mathrm{s}}(\theta_{\mathrm{i}}) = \frac{n_1\cos\theta_{\mathrm{i}} - n_2\sqrt{1-\left(\frac{n_1}{n_2}\right)^2\sin^2\theta_{\mathrm{i}}}}{n_1\cos\theta_{\mathrm{i}} + n_2\sqrt{1-\left(\frac{n_1}{n_2}\right)^2\sin\theta_{\mathrm{i}}}} \qquad n_2 > n_1 \implies r_{\mathrm{s}} < 0$$

$r_{\mathrm{s}} \in [r_{\max} < 0, -1]$ for $\theta_{\mathrm{i}} \in [0, \pi/2]$

$t_{\mathrm{s}} \in [t_{\max} > 0, 0]$ for $\theta_{\mathrm{i}} \in [0, \pi/2]$; $\quad t_{\mathrm{s}} = 1 + r_{\mathrm{s}}$

$R_{\mathrm{s}} \in [R_{\min}, 1]$ for $\theta_{\mathrm{i}} \in [0, \pi/2]$; $\quad R_{\mathrm{s}} = |r_{\mathrm{s}}|^2$

$T_{\mathrm{s}} \in [T_{\max}, 0]$ for $\theta_{\mathrm{i}} \in [0, \pi/2]$; $\quad T_{\mathrm{s}} = 1 - R_{\mathrm{s}}$



TM Polarization: Brewster's Angle

Brewster's angle θ_{B}: angle of incidence θ_{i} at which $r_{\mathrm{p}} = 0$.

$r_{\mathrm{p}} = \frac{\tan(\theta_{\mathrm{i}} - \theta_{\mathrm{t}})}{\tan(\theta_{\mathrm{i}} + \theta_{\mathrm{t}})}$ (for TM polarization in general)

$\theta_{\mathrm{i}} + \theta_{\mathrm{t}} = \pi/2 \implies \theta_{\mathrm{i}} \equiv \theta_{\mathrm{B}} = \pi/2 - \theta_{\mathrm{t}}$ (for $r_{\mathrm{p}} = 0$)

$\sin\theta_{\mathrm{t}} = \sin\left(\frac{\pi}{2} - \theta_{\mathrm{B}}\right) = \cos\theta_{\mathrm{B}}$

$\theta_{\mathrm{B}} = \tan^{-1}\frac{n_2}{n_1}$ (from Snell's law)

TM Polarization: Reflection Coefficients

$$r_{\mathrm{p}}(\theta_{\mathrm{i}}) = \frac{n_2\cos\theta_{\mathrm{i}} - n_1\sqrt{1-\left(\frac{n_1}{n_2}\right)^2\sin\theta_{\mathrm{i}}}}{n_2\cos\theta_{\mathrm{i}} + n_1\sqrt{1-\left(\frac{n_1}{n_2}\right)^2\sin\theta_{\mathrm{i}}}}$$

$r_{\mathrm{p}} \in [r_{\max} > 0, -1]$ for $\theta_{\mathrm{i}} \in [0, \pi/2]$

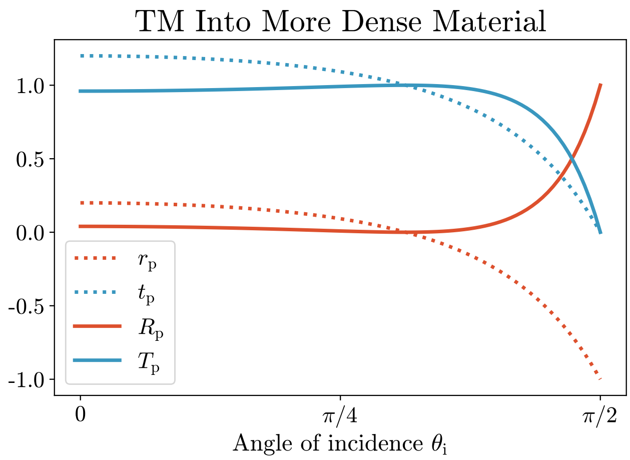
$t_{\mathrm{p}} \in [t_{\max} > 0, 0]$ for $\theta_{\mathrm{i}} \in [0, \pi/2]$

$R_{\mathrm{p}} \in [R_0, 0]$ for $\theta_{\mathrm{i}} \in [0, \theta_{\mathrm{B}}]$

$R_{\mathrm{p}} \in [0, 1]$ for $\theta_{\mathrm{i}} \in [\theta_{\mathrm{B}}, \theta_{\mathrm{c}}]$

$T_{\mathrm{p}} \in [T_0, 1]$ for $\theta_{\mathrm{i}} \in [0, \theta_{\mathrm{B}}]$

$T_{\mathrm{p}} \in [1, 0]$ for $\theta_{\mathrm{i}} \in [\theta_{\mathrm{B}}, \theta_{\mathrm{c}}]$



Passage into Less Dense Material

Passage into optically less dense material $\implies n_1 > n_2$.

Total Internal Reflection

$\theta_{\mathrm{t}} = \sin^{-1}\left(\frac{n_1}{n_2}\sin\theta_{\mathrm{i}}\right)$ (from Snell's law)

Critical angle: angle of incidence θ_i beyond which all incident light is reflected (*total internal reflection*).

$\theta_\mathrm{c} \equiv \sin^{-1} \frac{n_2}{n_1}$ (critical angle)

TE Polarization: Reflection Coefficients

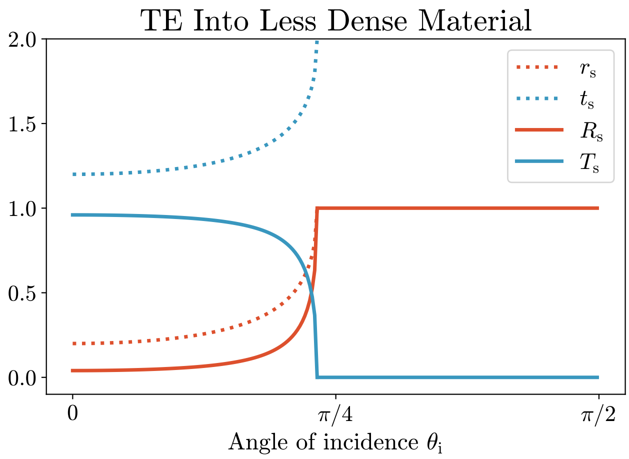
$$r_\mathrm{s}(\theta_\mathrm{i}) = \frac{n_1 \cos\theta_\mathrm{i} - n_2 \sqrt{1-\left(\frac{n_1}{n_2}\right)^2 \sin^2\theta_\mathrm{i}}}{n_1 \cos\theta_\mathrm{i} + n_2 \sqrt{1-\left(\frac{n_1}{n_2}\right)^2 \sin\theta_\mathrm{i}}} \qquad r_\mathrm{s} \in \mathbb{C} \text{ for } \theta_\mathrm{i} > \theta_\mathrm{c}$$

$$r_\mathrm{s}(\theta_\mathrm{i}) \equiv \frac{n_1 \cos\theta_\mathrm{i} - i n_2 \kappa}{n_1 \cos\theta_\mathrm{i} + i n_2 \kappa} \qquad (r_\mathrm{s} \text{ for } \theta_\mathrm{i} > \theta_\mathrm{c})$$

$$\kappa \equiv \sqrt{\left(\frac{n_1}{n_2}\right)^2 \sin^2\theta_\mathrm{i} - 1} = \sqrt{\left(\frac{\sin\theta_\mathrm{i}}{\sin\theta_\mathrm{c}}\right)^2 - 1}$$

$$R_\mathrm{s} = |r_\mathrm{s}|^2 = r_\mathrm{s} r_\mathrm{s}^* = 1 \qquad (\text{for } \theta_\mathrm{i} > \theta_\mathrm{c})$$

$r_\mathrm{s} \in [r_\mathrm{min} > 0, 1]$ for $\theta_\mathrm{i} \in [0, \theta_\mathrm{c}]$
$t_\mathrm{s} \in [t_\mathrm{min} > 1, 2]$ for $\theta_\mathrm{i} \in [0, \theta_\mathrm{c}]$
$R_\mathrm{s} \in [R_\mathrm{min}, 1]$ for $\theta_\mathrm{i} \in [0, \theta_\mathrm{c}]$
$R_\mathrm{s} = 1$ for $\theta_\mathrm{i} \in [\theta_\mathrm{c}, \pi/2]$
$T_\mathrm{s} \in [T_\mathrm{max}, 0]$ for $\theta_\mathrm{i} \in [0, \theta_\mathrm{c}]$
$T_\mathrm{s} = 0$ for $\theta_\mathrm{i} \in [\theta_\mathrm{c}, \pi/2]$



TM Polarization: Reflection Coefficients

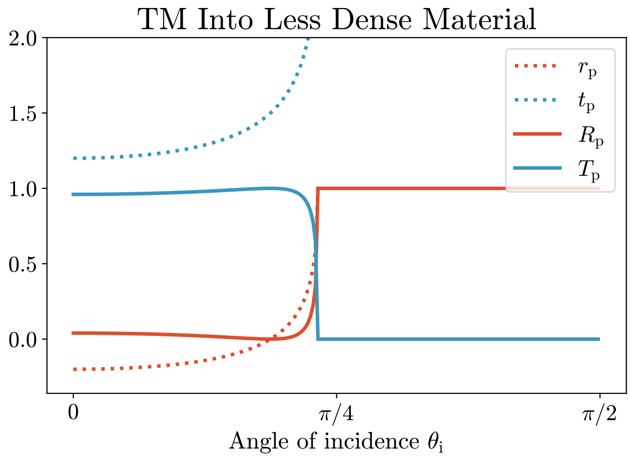
$$r_\mathrm{p}(\theta_\mathrm{i}) = \frac{n_2 \cos\theta_\mathrm{i} - n_1 \sqrt{1-\left(\frac{n_1}{n_2}\right)^2 \sin^2\theta_\mathrm{i}}}{n_2 \cos\theta_\mathrm{i} + n_1 \sqrt{1-\left(\frac{n_1}{n_2}\right)^2 \sin\theta_\mathrm{i}}} \qquad r_\mathrm{p} \in \mathbb{C} \text{ for } \theta_\mathrm{i} > \theta_\mathrm{c}$$

$$r_\mathrm{p}(\theta_\mathrm{i}) \equiv \frac{n_2 \cos\theta_\mathrm{i} - i n_1 \kappa}{n_2 \cos\theta_\mathrm{i} + i n_1 \kappa} \qquad (r_\mathrm{s} \text{ for } \theta_\mathrm{i} > \theta_\mathrm{c})$$

$$\kappa \equiv \sqrt{\left(\frac{n_1}{n_2}\right)^2 \sin^2\theta_\mathrm{i} - 1} = \sqrt{\left(\frac{\sin\theta_\mathrm{i}}{\sin\theta_\mathrm{c}}\right)^2 - 1}$$

$$R_\mathrm{p} = |r_\mathrm{p}|^2 = r_\mathrm{p} r_\mathrm{p}^* = 1 \qquad (\text{for } \theta_\mathrm{i} > \theta_\mathrm{c})$$

$r_\mathrm{p} \in [r_\mathrm{min} < 0, 0]$ for $\theta_\mathrm{i} \in [0, \theta_\mathrm{B}]$
$r_\mathrm{p} \in [0, 1]$ for $\theta_\mathrm{i} \in [\theta_\mathrm{B}, \theta_\mathrm{c}]$
$t_\mathrm{p} \in [t_\mathrm{min} > 1, t_\mathrm{max} > 2]$ for $\theta_\mathrm{i} \in [0, \theta_\mathrm{c}]$
$R_\mathrm{p} \in [R_0, 0]$ for $\theta_\mathrm{i} \in [0, \theta_\mathrm{B}]$
$R_\mathrm{p} \in [0, 1]$ for $\theta_\mathrm{i} \in [\theta_\mathrm{B}, \theta_\mathrm{c}]$
$R_\mathrm{p} = 1$ for $\theta_\mathrm{i} \in [\theta_\mathrm{c}, \pi/2]$
$T_\mathrm{p} \in [T_0, 1]$ for $\theta_\mathrm{i} \in [0, \theta_\mathrm{B}]$
$T_\mathrm{p} \in [1, 0]$ for $\theta_\mathrm{i} \in [\theta_\mathrm{B}, \theta_\mathrm{c}]$
$T_\mathrm{p} = 0$ for $\theta_\mathrm{i} \in [\theta_\mathrm{c}, \pi/2]$



Evanescent Field for TE Polarization

Situation as in Geometry of Reflection and Refraction.
Assume $n_2 < n_1$ (passage into optically less dense material).

$\mathbf{k}_\mathrm{t} = k_0 n_2 (\sin\theta_\mathrm{t}, 0, \cos\theta_\mathrm{t})$ (in general)

$\mathbf{k}_\mathrm{t} = k_0 (n_1 \sin\theta_\mathrm{i}, 0, n_2 \cos\theta_\mathrm{t})$ (after Snell's law)

$\mathbf{k}_\mathrm{t} = k_0 (n_1 \sin\theta_\mathrm{i}, 0, i n_2 \kappa)$ (after $\cos\theta_\mathrm{t} \to i\kappa$)

$$r_\mathrm{s}(\theta_\mathrm{i}) = \frac{n_1 \cos\theta_\mathrm{i} - n_2 \sqrt{1-\left(\frac{n_1}{n_2}\right)^2 \sin^2\theta_\mathrm{i}}}{n_1 \cos\theta_\mathrm{i} + n_2 \sqrt{1-\left(\frac{n_1}{n_2}\right)^2 \sin\theta_\mathrm{i}}}$$

$$r_\mathrm{s}(\theta_\mathrm{i}) \equiv \frac{n_1 \cos\theta_\mathrm{i} - i n_2 \kappa}{n_1 \cos\theta_\mathrm{i} + i n_2 \kappa} \qquad (r_\mathrm{s} \text{ for } \theta_\mathrm{i} > \theta_\mathrm{c})$$

$$\kappa \equiv \sqrt{\left(\frac{n_1}{n_2}\right)^2 \sin^2\theta_\mathrm{i} - 1} = \sqrt{\left(\frac{\sin\theta_\mathrm{i}}{\sin\theta_\mathrm{c}}\right)^2 - 1}$$

$\mathbf{E}_\mathrm{t} = \mathbf{E}_{\mathrm{t}_0} e^{i k_0 (n_1 x \sin\theta_\mathrm{i} + i n_2 \kappa z)} e^{-i\omega t}$ (transmitted $\mathbf{E}$ field)

$\mathbf{E}_\mathrm{t} = \mathbf{E}_{\mathrm{t}_0} e^{i k_0 n_1 x \sin\theta_\mathrm{i}} e^{-z/z_0} e^{-i\omega t}$ (in terms of skin depth)

$z_0 \equiv \frac{1}{n_2 \kappa k_0}$ (skin depth)

$z_0 = \frac{\lambda_0}{2\pi} \frac{1}{\sqrt{n_1^2 \sin^2\theta_\mathrm{i} - n_2^2}}$; $\lambda_0 = 2\pi/k_0$ (alternate expression)

$z_0 \to \frac{\lambda_0}{2\pi} \frac{1}{\sqrt{n_1^2 - n_2^2}}$ (as $\theta_\mathrm{i} \to \pi/2$)

$z_0 \to \infty$ (regular transmission as $\theta_\mathrm{i} \to \theta_\mathrm{c}^+$)

Reflected and Transmitted Field for TE Polarization

Situation, assumptions as in Evanescent Field for TE Pol.

$\hat{\mathbf{E}}_y \parallel \hat{\mathbf{e}}_y$ (for TE polarization and xz plane of incidence)

Assume $E_{\mathrm{r}_0} = E_{\mathrm{i}_0}$ and get define $E_0 \equiv E_{\mathrm{r}_0} = E_{\mathrm{i}_0}$.

$\mathbf{E}_1 = \mathbf{E}_\mathrm{i} + \mathbf{E}_\mathrm{r} = 2E_0 e^{i k_0 n_1 \sin\theta_\mathrm{i} x} \cos\left[k_0 n_1 z \cos\theta_\mathrm{i}\right] e^{-i\omega t} \, \hat{\mathbf{e}}_y$

$\operatorname{Re} \mathbf{E}_1 = 2E_0 \cos\left[k_0 n_1 x \sin\theta_\mathrm{i} - \omega t\right] \cos\left[k_0 n_1 z \cos\theta_\mathrm{i}\right] \hat{\mathbf{e}}_y$

$E_\mathrm{t}^{\parallel} = E_\mathrm{i}^{\parallel} + E_\mathrm{r}^{\parallel}$ (general boundary condition)

$E_{\mathrm{t}_0} = E_{\mathrm{i}_0} + E_{\mathrm{r}_0}$ (for TE-polarized waves)

$\mathbf{E}_\mathrm{t} = 2E_0 \cos(k_0 n_1 x \sin\theta_\mathrm{i} - \omega t) e^{-z/z_0} \, \hat{\mathbf{e}}_y$ ($E_{\mathrm{t}_0} = 2E_0$)

Reflected and Transmitted TE Poynting Vectors

Situation, assumptions as in Evanescent Field for TE Pol.
Assume non-magnetic materials with $\mathbf{B} = \mu_0 \mathbf{H}$.

$\mathbf{E}_\mathrm{t} = E_{\mathrm{t}_0} \cos(k_0 n_1 x \sin\theta_\mathrm{i} - \omega t) e^{-z/z_0} \, \hat{\mathbf{e}}_y$ (transmitted $\mathbf{E}$ field)

$\nabla \times \mathbf{E}_\mathrm{t} = -\frac{\partial \mathbf{B}_\mathrm{t}}{\partial t} = -\mu_0 \frac{\partial \mathbf{H}_\mathrm{t}}{\partial t} \implies \ldots$

$\mathbf{H}_\mathrm{t} = \frac{E_{\mathrm{t}_0}}{\mu_0} \left[\frac{1}{\omega z_0} \sin(k_x x - \omega t) \, \hat{\mathbf{e}}_x + \frac{k_x}{\omega} \cos(k_x x - \omega t) \, \hat{\mathbf{e}}_z\right] e^{-z/z_0}$

$\mathbf{S}_\mathrm{t} = \mathbf{E}_\mathrm{t} \times \mathbf{H}_\mathrm{t}$

$= \frac{E_{\mathrm{t}_0}^2}{\mu_0 \omega} \left[k_x \cos^2(k_x x - \omega t) \, \hat{\mathbf{e}}_x - \frac{1}{z_0} \sin(k_x x - \omega t) \cos(k_x x - \omega t) \, \hat{\mathbf{e}}_z\right] e^{-2z/z_0}$

$\langle \mathbf{S}_\mathrm{t} \rangle = \frac{E_{\mathrm{t}_0}^2}{\mu_0 \omega} \frac{k_x}{2} e^{-2z/z_0} \, \hat{\mathbf{e}}_x$ (note that $\langle \mathbf{S}_\mathrm{t} \rangle \cdot \hat{\mathbf{e}}_z = 0$)

$\langle |\mathbf{S}_\mathrm{t}| \rangle = \frac{E_{\mathrm{t}_0}^2}{\mu_0 \omega} \frac{k_0 n_1}{2} e^{-2z/z_0} \sin\theta_\mathrm{i}$ (using $k_x = k_0 n_1 \sin\theta_\mathrm{i}$)

$\langle |\mathbf{S}_\mathrm{t}| \rangle = \frac{1}{2} \varepsilon_0 c_0 n_1 E_{\mathrm{t}_0}^2 \sin\theta_\mathrm{i} e^{-2z/z_0}$ (using $\varepsilon_0 \mu_0 = 1/c_0^2$)

Phase Shift During Reflection

Situation and assumptions as in Fresnel Equations.

Phase Shift During Regular Reflection

ϕ is phase shift between incident and reflected light.

$\phi_s = 0$ if $r_s > 0$

$\phi_s = \pi$ if $r_s < 0$

$\phi_p = \pi$ if $r_p > 0$

$\phi_p = 0$ if $r_p < 0$

TE Phase Shift During Total Internal Reflection

Assume $\theta_i > \theta_c = \arcsin(n_2/n_1)$.

$r_s(\theta_i) \equiv \frac{n_1 \cos\theta_i - i n_2 \kappa}{n_1 \cos\theta_i + i n_2 \kappa}$

$\kappa \equiv \sqrt{\left(\frac{n_1}{n_2}\right)^2 \sin^2\theta_i - 1} = \sqrt{\left(\frac{\sin\theta_i}{\sin\theta_c}\right)^2 - 1}$

$r_s = e^{-i\phi_s}$ (in complex polar form)

$|r_s| = 1$

$\phi_s = 2\arctan\frac{n_2\kappa}{n_1\cos\theta_i} = 2\arctan\left[\frac{n_2}{n_1\cos\theta_i}\cdot\sqrt{\left(\frac{\sin\theta_i}{\sin\theta_c}\right)^2 - 1}\right]$

$\phi_s = 0$ at $\theta_i = \theta_c$ and increases to $\phi_s \to \pi$ as $\theta_i \to \pi/2$

TM Phase Shift During Total Internal Reflection

Assume $\theta_i > \theta_c = \arcsin(n_2/n_1)$.

$r_p(\theta_i) \equiv \frac{n_2 \cos\theta_i - i n_1 \kappa}{n_2 \cos\theta_i + i n_1 \kappa}$

$r_p = e^{-i\phi_p}$ (in complex polar form)

$|r_p| = 1$

$\phi_p = 2\arctan\frac{n_1\kappa}{n_2\cos\theta_i} = 2\arctan\left[\frac{n_1}{n_2\cos\theta_i}\cdot\sqrt{\left(\frac{\sin\theta_i}{\sin\theta_c}\right)^2 - 1}\right]$

$\phi_p = 0$ at $\theta_i = \theta_c$ and increases to $\phi_p \to \pi$ as $\theta_i \to \pi/2$

$\phi_p > \phi_s$ for $\theta_i \in (\theta_c, \pi/2)$ (because $n_1 > n_2$)

Reflection From Metals

Suppose light is incident from dielectric onto metal.
Material 1 is a non-conducting dielectric with RI n_1.
Material 2 is a conducting metal with conductivity σ_2.

For TE Polarization

$\boldsymbol{\mathcal{K}} = (\mathcal{K}_x, 0, \mathcal{K}_z) \in \mathbb{C}$ (transmitted wave vector in metal)

$\mathcal{K}_x = k_{i_x} = k_0 n_1 \sin\theta_i$ (x component preserved)

$\mathcal{K}_z \equiv k_{t_z} + i\kappa_{t_z}$ (Re and Im components)

$\mathcal{N}^2 = \varepsilon\mu + i\frac{\sigma_E \mu}{\varepsilon_0\omega}$ (refractive index in metal)

$|\boldsymbol{\mathcal{K}}| = \mathcal{K} = \mathcal{N} k_0$

$|\boldsymbol{\mathcal{K}}|^2 = \mathcal{K}_x^2 + (k_z + i\kappa_{t_z})^2 = \mathcal{N}^2 k_0^2$

$k_{t_z}^2 = \frac{1}{2}\left[\sqrt{(k_0^2\varepsilon_2 - \mathcal{K}_x^2)^2 + \left(\frac{\sigma_2 k_0^2}{\varepsilon_0\omega}\right)^2} + k_0^2\varepsilon_2 - \mathcal{K}_x^2\right]$ (if $\mu = 1$)

$\kappa_{t_z}^2 = \frac{1}{2}\left[\sqrt{(k_0^2\varepsilon_2 - \mathcal{K}_x^2)^2 + \left(\frac{\sigma_2 k_0^2}{\varepsilon_0\omega}\right)^2} - k_0^2\varepsilon_2 + \mathcal{K}_x^2\right]$ (if $\mu = 1$)

Coefficients for TE Polarization

We quote the following results without derivation...

$ik_{i_z}E_{i_0} - ik_{i_z}E_{r_0} = (ik_{t_z} + \kappa)E_{0_t}$ (boundary conditions)

$k_{i_z}E_{i_0} - k_{i_z}E_{r_0} = (k_{t_z} - i\kappa)E_{0_t}$ (after rearranging)

$r_s = \frac{E_{r_0}}{E_{i_0}} = \frac{k_{i_z} - k_{t_z} - i\kappa}{k_{i_z} + k_{t_z} + i\kappa}$ (using $E_{t_0} = E_{i_0} + E_{r_0}$)

$r_s = \frac{n_1\cos\theta_i - \mathcal{N}_2\cos\theta_t}{n_1\cos\theta_i + \mathcal{N}_2\cos\theta_t}$ (alternate formulation)

$r_s = \frac{n_1\cos\theta_i - n_2\cos\theta_t}{n_1\cos\theta_i + n_2\cos\theta_t} = \frac{k_{i_z} - k_{t_z}}{k_{i_z} + k_{t_z}}$ (dielectric to dielectric)

Coefficients For TM Polarization

We quote the following result without derivation...

$r_p = \frac{\mathcal{N}_2\cos\theta_i - n_1\cos\theta_t}{\mathcal{N}_2\cos\theta_i + n_1\cos\theta_t}$ (alternate formulation)

Normal Incidence

Let $\mathcal{N} \equiv n_{Re} + i n_{Im}$.

$r_s = \frac{n_1 - \mathcal{N}}{n_1 + \mathcal{N}} = \frac{n_1 - n_{Re} - i n_{Im}}{n_1 + n_{Re} + i n_{Im}}$

$r_p = \frac{\mathcal{N} - n_1}{\mathcal{N} + n_1} = -r_s$

$R = |r_s|^2 = |r_p|^2 = \frac{(n_1 - n_{Re})^2 + n_{Im}^2}{(n_1 + n_{Re})^2 + n_{Im}^2}$

$R \to \frac{n_1^2 + n_{Im}^2}{n_1^2 + n_{Im}^2} = 1$ as $n_{Re} \to 0$ (in good conductors)

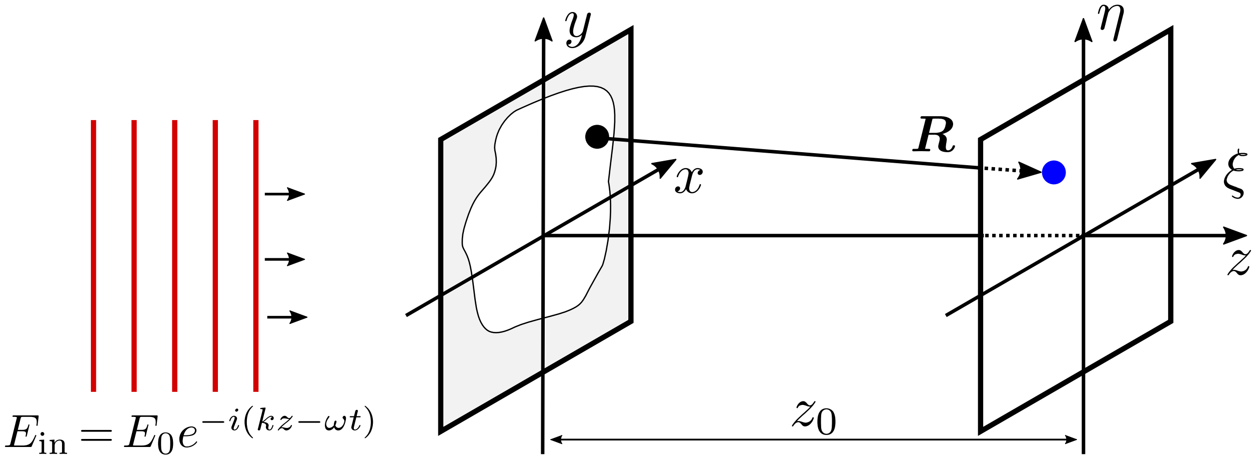


Figure 1: Geometry of Fraunhofer diffraction.

Diffraction

Light undergoes diffraction when incident on, or passing through, obstacles or openings with characteristic linear dimensions comparable to the light's wavelength.

Situation, Geometry, and Coordinate System

Monochromatic plane waves with $\mathbf{k} \parallel \hat{\mathbf{e}}_z$ and amplitude E_0 are normally incident on a diffracting aperture in the xy plane.

Goal: determine spatial distribution of electric field magnitude on a distant observation screen as a function of electric field magnitude $E(x, y)$ in diffracting aperture.

z axis is optical axis.

S_{a} is the planar diffracting aperture.

Origin is the intersection of aperture plane and optical axis.

Observation screen is parallel to xy plane at $z = z_0$.

(x, y) are coordinates in diffracting aperature.

(ξ, η) are coordinates in observation screen.

$\mathbf{R}$ points from arbitrary point in aperture to arbitrary point in observation screen.

$R = |\mathbf{R}| = \sqrt{(\xi - x)^2 + (\eta - y)^2 + z_0^2}$

$f_{\text{a}}(x, y) \equiv \begin{cases} 1 & (x, y) \in S_{\text{a}} \\ 0 & \text{otherwise} \end{cases}$ (aperture function)

Diffraction Integral

Assume R is much larger than aperture's linear dimensions.

$|\mathbf{E}(\mathbf{r})| \equiv E_0$ for all $\mathbf{r}$ in S_{a} (for *plane waves* incident on S_{a})

Huygen's principle: Every point in S_{a} acts as a source of secondary spherical EM waves whose amplitude is determined by the plane waves incident on S_{a}.

$E(r) = \frac{A}{r} e^{i(kr - \omega t)}$ (a general spherical wave)

$E(\xi, \eta) = \frac{1}{i\lambda} \iint_{S_{\text{a}}} E_0 \frac{e^{ikR}}{R} \, \mathrm{d}S$ (field at observation screen)

$\quad = \frac{1}{i\lambda} \iint f_{\text{a}}(x, y) E_0 \frac{e^{ikR}}{R} \, \mathrm{d}S$

Fraunhofer Diffraction

See "**Situation, Geometry, and Coordinate System**"

$R_0 \equiv \xi^2 + \eta^2 + z_0^2$ (distance from origin to obs. screen)

$R = \sqrt{(\xi - x)^2 + (\eta - y)^2 + z_0^2}$ (in general)

Assume $R_0^2 \gg (x^2 + y^2)$

$R \approx R_0 - \frac{\xi x}{R_0} - \frac{\eta y}{R_0}$ (Fraunhofer diffraction approximation)

$\frac{e^{ikR}}{R} \approx \frac{1}{R_0} \exp\left[ik\left(R_0 - \frac{\xi x}{R_0} - \frac{\eta y}{R_0}\right)\right]$

$E(\xi, \eta) = \frac{1}{i\lambda} \frac{E_0 e^{ikR_0}}{R_0} \iint_{S_{\text{a}}} \exp\left[-ik\left(\frac{\xi x}{R_0} + \frac{\eta y}{R_0}\right)\right] \mathrm{d}x \, \mathrm{d}y$ (**FraD**)

$\quad = \frac{1}{i\lambda} \frac{E_0 e^{ikR_0}}{R_0} \iint f_{\text{a}}(x, y) \exp\left[-ik\left(\frac{\xi x}{R_0} + \frac{\eta y}{R_0}\right)\right] \mathrm{d}x \, \mathrm{d}y$

Fraunhofer Diffraction, Alternate Expression

$\sin\theta_\xi \equiv \frac{\xi}{R_0}; \qquad \theta_\xi \approx \frac{\xi}{R_0}$ (for $R_0 \gg \xi$)

$\sin\theta_\eta \equiv \frac{\eta}{R_0}; \qquad \theta_\eta \approx \frac{\eta}{R_0}$ (for $R_0 \gg \eta$)

$\kappa_\xi \equiv k \sin\theta_\xi = \frac{2\pi \sin\theta_\xi}{\lambda}$

$\kappa_\eta \equiv k \sin\theta_\eta = \frac{2\pi \sin\theta_\eta}{\lambda}$

$E(\kappa_\xi, \kappa_\eta) = \frac{1}{i\lambda} \frac{E_0 e^{ikR_0}}{R_0} \iint_{S_{\text{a}}} e^{-i(\kappa_\xi x + \kappa_\eta y)} \, \mathrm{d}x \, \mathrm{d}y$

$\quad = \frac{1}{i\lambda} \frac{E_0 e^{ikR_0}}{R_0} \iint f_{\text{a}}(x, y) e^{-i(\kappa_\xi x + \kappa_\eta y)} \, \mathrm{d}x \, \mathrm{d}y$

Fresnel Number and Validity of Fraunhofer Diffraction

Fraunhofer approximation neglects the $\frac{x^2+y^2}{2R_0}$ term in R.

$\exp\left(ik\frac{x^2+y^2}{2R_0}\right)$ (phase contribution of neglected term)

$k\frac{x^2+y^2}{2R_0} \ll 2\pi$ for all $(x, y) \in S_{\text{a}}$ (condition for Fra. approx.)

$L^2 \equiv \max\left[x^2 + y^2\right]_{(x,y) \in S_{\text{a}}}$ (characteristic aperture size)

$\frac{kL^2}{2R_0} = \frac{2\pi}{\lambda} \frac{L^2}{2R_0} \ll 2\pi$ (condition in terms of L)

$\frac{L^2}{\lambda z_0} \ll 1$ (using $R_0 \sim z_0$ and $2 \sim 1$)

$F \equiv \frac{L^2}{\lambda z_0}$ (definition of Fresnel number)

$F \ll 1$ (condition for Fraunhofer approx)

Fraunhofer Diffraction; Thin Slit

Consider monochromatic plane wave light of wavelength λ normally incident on a thin slit of width a in the xy plane.

Let the slit width span $x \in [-a/2, a/2]$.

Assume translational invariance along the y axis.

$f_{\text{a}}(x) = \begin{cases} 1 & x \in [-a/2, a/2] \\ 0 & \text{otherwise} \end{cases}$ (aperture function)

$E(\kappa_\xi) = \frac{1}{i\lambda} \frac{E_0 e^{ikR_0}}{R_0} \int_{-\infty}^{\infty} f_{\text{a}}(x) e^{-i\kappa_\xi x} \, \mathrm{d}x$

$\quad \equiv A \int_{-\infty}^{\infty} f_{\text{a}}(x) e^{-i\kappa_\xi x} \, \mathrm{d}x$ $\left(A \equiv \frac{1}{i\lambda} \frac{E_0 e^{ikR_0}}{R_0}\right)$

$\quad = Aa \operatorname{sinc} \frac{\kappa_\xi a}{2}$

$E(\theta_\xi) = Aa \operatorname{sinc}\left(\frac{\pi a \sin\theta_\xi}{\lambda}\right)$ (alternate expression)

$\sin\theta_{\text{min}} = \frac{n\lambda}{a}; \; n \in \mathbb{Z}$ (diffraction pattern minima)

Fraunhofer Diffraction; Rectangular Aperture

Consider monochromatic plane wave light of wavelength λ normally incident on a rectangular aperture of width a and height b in the xy plane.

Let the aperture width span $x \in [-a/2, a/2]$.

Let the aperture height span $y \in [-b/2, b/2]$.

$f_{\text{a}}(x, y) = \begin{cases} 1 & x \in [-a/2, a/2] \text{ and } y \in [-b/2, b/2] \\ 0 & \text{otherwise} \end{cases}$

$E(\kappa_\xi, \kappa_\eta) = A \int_{-\infty}^{\infty} \mathrm{d}x \int_{-\infty}^{\infty} \mathrm{d}y f_{\text{a}}(x, y) e^{-i(\kappa_\xi x + \kappa_\eta y)}$

$\quad = Aab \operatorname{sinc} \frac{\kappa_\xi a}{2} \operatorname{sinc} \frac{\kappa_\eta b}{2}$

$E(\theta_\xi, \theta_\eta) = Aab \operatorname{sinc}\left(\frac{\pi a \sin\theta_\xi}{\lambda}\right) \operatorname{sinc}\left(\frac{\pi b \sin\theta_\eta}{\lambda}\right)$

Fraunhofer Diffraction; Diffraction Grating
Consider monochromatic plane wave light of wavelength λ normally incident on a series of N thin slits of width a, uniformly separated by distance D, in the xy plane.
Assume translational invariance along the y axis.
Assume width of central slit spans $x \in [-a/2, a/2]$.

$$E(\kappa_\xi) = A \sum_{n=0}^{N} \int_{nD-a/2}^{nD+a/2} e^{-in\kappa_\xi x}\, \mathrm{d}x$$
$$= Aa \operatorname{sinc} \tfrac{\kappa_\xi a}{2} \sum_{n=0}^{N} \left(e^{-i\kappa_\xi D}\right)^n$$
$$= Aa \operatorname{sinc} \tfrac{\kappa_\xi a}{2} \tfrac{1-e^{-i\kappa_\xi DN}}{1-e^{-i\kappa_\xi D}}$$
$$E(\theta) = Aa \operatorname{sinc} \tfrac{ka\sin\theta}{2} \tfrac{1-e^{-ikDN\sin\theta}}{1-e^{-ikD\sin\theta}} \qquad \text{(using } \theta \equiv \theta_\xi\text{)}$$
$$j(\theta) \propto (Aa)^2 \operatorname{sinc}^2 \tfrac{ka\sin\theta}{2} \cdot \frac{\sin^2\left(\frac{kDN\sin\theta}{2}\right)}{\sin^2\left(\frac{kD\sin\theta}{2}\right)} \qquad \text{(intensity)}$$

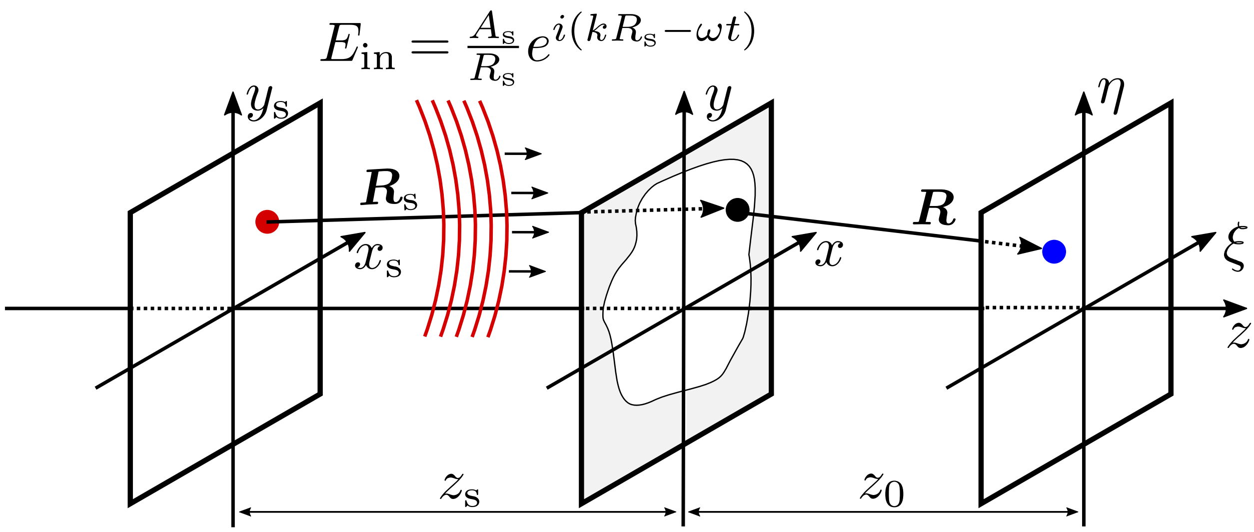


Figure 2: Geometry of Fresnel diffraction.

Fresnel Diffraction

See "**Situation, Geometry, and Coordinate System**"
Assume light originates from a point source of spherical waves.
Source lies in plane parallel to xy plane at $z = -z_{\text{s}}$.
$(x_{\text{s}}, y_{\text{s}})$ are coordinates in source plane.
$\mathbf{R}_{\text{s}}$ points from arbitrary point in source plane to arbitrary point in diffracting aperture.
$R_{\text{s}} = |\mathbf{R}_{\text{s}}| = \sqrt{(x_{\text{s}} - x)^2 + (y_{\text{s}} - y)^2 + z_{\text{s}}^2}$
$\mathbf{R}$ points from arbitrary point in aperture to arbitrary point in observation screen.
$R = |\mathbf{R}| = \sqrt{(\xi - x)^2 + (\eta - y)^2 + z_0^2}$
$E_0 = E_0(x, y)$ (electric field may vary in diffracting aperture)
$E_0(x, y) = \frac{A_{\text{s}}}{R_{\text{s}}} e^{ikR_{\text{s}}}$ (field amplitude in aperture)

Distance Approximations

$$R = \sqrt{(\xi - x)^2 + (\eta - y)^2 + z_0^2} \qquad \text{(in general)}$$
$$R_{\text{s}} = \sqrt{(x_{\text{s}} - x)^2 + (y_{\text{s}} - y)^2 + z_{\text{s}}^2} \qquad \text{(in general)}$$
$$R \approx z_0 \left(1 + \frac{(x-\xi)^2}{2z_0^2} + \frac{(y-\eta)^2}{2z_0^2}\right) \qquad \text{(Fresnel approximation)}$$
$$R_{\text{s}} \approx z_{\text{s}} \left(1 + \frac{(x-x_{\text{s}})^2}{2z_{\text{s}}^2} + \frac{(y-y_{\text{s}})^2}{2z_{\text{s}}^2}\right) \qquad \text{(Fresnel approximation)}$$

Fresnel Diffraction Integral

$$E(\xi, \eta) = \frac{1}{i\lambda} \iint_{S_{\text{a}}} E_0 \frac{e^{ikR}}{R}\, \mathrm{d}S \qquad \text{(general diffraction integral)}$$
$$E(\xi, \eta) = \frac{A_{\text{s}}}{i\lambda} \iint_{S_{\text{a}}} \frac{e^{ikR_{\text{s}}}}{R_{\text{s}}} \frac{e^{ikR}}{R}\, \mathrm{d}S \qquad \text{(for above } E_0(x,y)\text{)}$$
$$E(\xi, \eta, x_{\text{s}}, y_{\text{s}}) = \frac{A_{\text{s}}}{i\lambda} \frac{e^{ik(z_{\text{s}}+z_0)}}{R_{\text{s}} R} \iint_{S_{\text{a}}} e^{\frac{ik}{2z_{\text{s}}}\left[(x-x_{\text{s}})^2 + (y-y_{\text{s}})^2\right]} \times e^{\frac{ik}{2z_0}\left[(x-\xi)^2 + (y-\eta)^2\right]}\, \mathrm{d}x\, \mathrm{d}y$$

Validity of the Fresnel Approximation

$a^2 \equiv (x - \xi)^2 + (y - \eta)^2$
$$R = z_0 \sqrt{1 + a^2/z_0^2} \qquad \text{(origin-observation screen distance)}$$
$$R \approx z_0 + \frac{a^2}{2z_0} - \frac{a^4}{8z_0^3} \qquad \text{(for } a \ll z_0\text{)}$$
Fresnel approximation neglects the $\frac{a^4}{8z_0^3}$ term.
$$\exp\left(ik\frac{a^4}{8z_0^3}\right) \qquad \text{(phase contribution of neglected term)}$$
$$k\frac{a^4}{8z_0^3} \ll 2\pi \qquad \text{(condition for Fresnel approximation)}$$
$$\frac{a^4}{z_0^4} \ll \frac{8\lambda}{z_0} \qquad \text{(alternate expression)}$$

Fresnel Diffraction; Circular Aperture
Consider aperture of radius a centered on z (optical) axis.
Source lies on optical axis a distance z_{s} from aperture.
Observation point centered on OA a distance z_0 from aperture.
Let $1/L \equiv (1/z_0) + (1/z_{\text{s}})$.
$$E = \frac{A_{\text{s}}}{i\lambda} \frac{e^{ik(z_0+z_{\text{s}})}}{RR_{\text{s}}} \iint_{S_{\text{a}}} e^{\frac{ik}{2z_0}(x^2+y^2)} e^{\frac{ik}{2z_{\text{s}}}(x^2+y^2)}\, \mathrm{d}x\, \mathrm{d}y$$
$$= 2\pi \frac{A_{\text{s}}}{i\lambda} \frac{e^{ik(z_0+z_{\text{s}})}}{RR_{\text{s}}} \int_0^a \rho e^{\frac{ik\rho^2}{2L}}\, \mathrm{d}\rho \qquad \text{(polar coordinates)}$$
$$E(z_0) \approx 2\pi \frac{A_{\text{s}}}{i} \frac{e^{ik(z_0+z_s)}}{z_0+z_{\text{s}}} e^{i\frac{ka^2}{4L}} \sin \frac{ka^2}{4L} \qquad (z_0 \approx R,\ z_s \approx R_{\text{s}})$$
$$I(z_0) \propto 4\pi^2 \frac{A_{\text{s}}^2}{(z_0+z_s)^2} \sin^2 \frac{ka^2}{4L} \qquad \text{(intensity)}$$

Fresnel Zones
Reconsider the intensity from the circular aperture...
$I = 4I_0 \sin^2 \frac{ka^2}{4L}; \qquad I_0 = \frac{\varepsilon_0 c}{2} \frac{A_{\text{s}}^2}{(z_0+z_{\text{s}})^2}$
I_0 is intensity at observation point of the same point source *without* a diffracting screen placed between source and OP.
I oscillates with aperture radius a between $0 \cdot I_0$ and $4I_0$.
Values of a for which I attains maxima and minima define the boundaries of *Fresnel zones*—concentric annuli centered on the circular aperture.
$$a_{n-1} = \sqrt{(n-1)\lambda L} \qquad \text{(inner radius of } n\text{-th FZ)}$$
$$a_n = \sqrt{n\lambda L} \qquad \text{(outer radius of } n\text{-th FZ)}$$

Phase
Situation as in Fresnel Diffraction; Circular Aperture.
Assume $z_{\text{s}} \to \infty \implies L = z_0$.
Consider light from (a) aperture center to observation point and (b) radial distance a in aperture to observation point.
$R_{\text{a}} = z_0$
$R_{\text{b}} = \sqrt{z_0^2 + a^2} \approx z_0 + \frac{a^2}{2z_0}$
$\phi_{\text{a}} = kz_0$
$\phi_{\text{b}} \approx kz_0 + \frac{ka^2}{2z_0}$
$\Delta\phi = \phi_{\text{b}} - \phi_{\text{a}} \approx \frac{ka^2}{2z_0}$
$\Delta\phi < \pi \implies$ constructive interference between (a) and (b)
$\Delta\phi > \pi \implies$ destructive interference between (a) and (b)

Covering every other Fresnel zone produces a Fresnel lens.

$\frac{1}{f} = \frac{1}{z_0} \approx \frac{1}{L} \implies f \approx L$ (focus, assuming $z_{\mathrm{s}} \to \infty$)

$f = a_1^2/\lambda$ (from $a_1 = \sqrt{\lambda L}$)

$a_1^2 = a_{n+1}^2 - a_n^2 = (a_{n+1} + a_n)(a_{n+1} - a_n) \approx 2a_n \Delta a_n$

(assuming $a_{n+1} + a_n \approx 2a_n$ for large n)

$f \approx \frac{2a_n \Delta a_n}{\lambda}$ (focus for multi-zone lens)

Interference

Both diffraction and interference are fundamentally the same phenomenon: superposition of electromagnetic waves.
Superposition of vector field $\mathbf{E}$ applies in general.
Superposition of scalar field $E = |\mathbf{E}|$ applies only for EM waves with equal polarizations.
Simplification: we consider only scalar electric field magnitude.
Resulting restriction: all light in this section's analyses must have the same polarization to apply superposition principles.
Assumption: we consider only superposition of *plane* waves.
Assumption: consider EM waves only in nonmagnetic materials with $\mu = 1 \implies n = \sqrt{\varepsilon}$.

Superposition of Plane Waves

Consider the two plane waves with equal frequency ω.

$E_1 = E_{1_0} e^{i(\mathbf{k}_1 \cdot \mathbf{r}_1 - \omega t + \phi_1)}$ (assume $E_{1_0} \in \mathbb{R}$)
$E_2 = E_{2_0} e^{i(\mathbf{k}_2 \cdot \mathbf{r}_2 - \omega t + \phi_2)}$ (assume $E_{2_0} \in \mathbb{R}$)
$E_1 = E_{1_0} e^{i(\Phi_1 - \omega t)}$ (alternate expression; $\Phi_1 \equiv \mathbf{k}_1 \cdot \mathbf{r}_1 - \phi_1$)
$E_2 = E_{2_0} e^{i(\Phi_2 - \omega t)}$ (alternate expression; $\Phi_2 \equiv \mathbf{k}_2 \cdot \mathbf{r}_2 - \phi_2$)
$E = E_{1_0} e^{i(\Phi_1 - \omega t)} + E_{2_0} e^{i(\Phi_2 - \omega t)}$ (superposed wave)

$\langle j \rangle = \frac{1}{2}\varepsilon\varepsilon_0 c |E|^2 = \frac{1}{2}\varepsilon_0 n c_0 |E|^2$ (if $\varepsilon = n^2$)
$|E| = E_{1_0}^2 + E_{2_0}^2 + E_{1_0}E_{2_0}\left(e^{i(\Phi_1 - \Phi_2)} + e^{-i(\Phi_1 - \Phi_2)}\right)$
$\quad = E_{1_0}^2 + E_{2_0}^2 + 2E_{1_0}E_{2_0}\cos\Delta\Phi$ ($\Delta\Phi \equiv \Phi_1 - \Phi_2$)
$\langle j \rangle = \frac{1}{2}\varepsilon_0 n c_0 \left(E_{1_0}^2 + E_{2_0}^2 + 2E_{1_0}E_{2_0}\cos\Delta\Phi\right)$
$\quad = \langle j_1 \rangle + \langle j_2 \rangle + 2\sqrt{\langle j_1 \rangle \langle j_2 \rangle}\cos\Delta\Phi$ (and not $\langle j_1 \rangle + \langle j_2 \rangle$!)
$\nu \equiv \frac{j_{\max} - j_{\min}}{j_{\max} + j_{\min}} \in (0, 1)$ (interferometric visibility)

$2\sqrt{\langle j_1 \rangle \langle j_2 \rangle}\cos\Delta\Phi$ is observed only if $\Delta\Phi$ is *constant*!
$\Delta\Phi = \text{constant} \implies$ light must be *coherent* to observe interference

Superposition of Equal-Amplitude Plane Waves

Situation as in Superposition of Plane Waves.
Additionally assume $E_{1_0} = E_{2_0} \equiv E_0$.

$\langle j_1 \rangle = \langle j_2 \rangle \equiv \langle j_0 \rangle = \frac{1}{2}\varepsilon_0 n c_0 E_0^2$ (if $E_{1_0} = E_{2_0}$)
$\langle j \rangle = 2\langle j_0 \rangle (1 + \cos\Delta\Phi)$ (superposed intensity)
$\quad = 4\langle j_0 \rangle \cos^2\frac{\Delta\Phi}{2}$ (superposed intensity)
$\overline{\langle j \rangle} = 4j_0\overline{\cos^2\frac{\Delta\Phi}{2}} = 2j_0$ (conservation of energy)

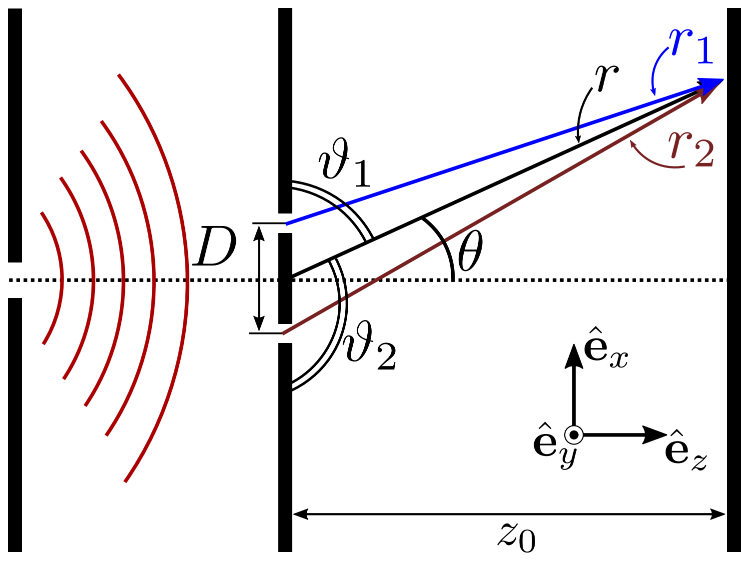


Figure 3: Geometry of Young's double slit experiment.

Young's Double-Slit Experiment

Principle: interference via wavefront splitting
Assume monochromatic point source with well-defined phase.

Young's Double-Slit Experiment

Consider two slits separated by distance D in xy plane.
z (optical) axis points from source to midpoint between slits
Let slit width run along x axis.
Work in xz plane; assume translational invariance along y axis.
Principle: thin slits split point source's spherical wavefront.
Because slits are symmetrically spaced about optical axis, light leaving each slit has equal phase.
Observe interference between light from slits on distance screen.

Geometry

r_1 and r_2 are distances from each slit to observation point.
r is distance from midpoint between slits to observation point.
θ is angle between optical axis and r.
ϑ_1 is angle between $+\hat{\mathbf{e}}_x$ and r.
ϑ_2 is angle between $-\hat{\mathbf{e}}_x$ and r.

$\vartheta_1 + \vartheta_2 = \pi \implies \cos\vartheta_2 = -\cos\vartheta_1$ (by construction)
$\vartheta_1 + \theta = \pi/2 \implies \cos\vartheta_1 = \sin\theta$ (by construction)
$r_1^2 = r^2 + (D/2)^2 - 2r(D/2)\cos\vartheta_2$ (law of cosines)
$r_2^2 = r^2 + (D/2)^2 - 2r(D/2)\cos\vartheta_1$ (law of cosines)
$r_1^2 - r_2^2 = 2rD\cos\vartheta_1$ ($\cos\vartheta_2 = -\cos\vartheta_1$)
$\quad = 2rD\sin\theta$ ($\cos\vartheta_1 = \sin\theta$)
Assume $r_1, r_2 \gg D \implies r_1 + r_2 \approx 2r$.
$2rD\sin\theta = (r_1 + r_2)(r_1 - r_2) \approx 2r\Delta r$
$\Delta r \approx D\sin\theta$ (difference in optical path lengths to OP)

Intensity

$\Delta\Phi = k(r_1 - r_2) \approx kD\sin\theta$ (phase difference at OP)
For shorthand let $j_0 \equiv \langle j_0 \rangle$.
$\langle j \rangle = 4j_0\cos^2\frac{\Delta\Phi}{2}$ (superposed intensity at OP)
$\quad = 4j_0\cos^2\left(\frac{kD\sin\theta}{2}\right)$ (superposed intensity at OP)
$\frac{kD\sin\theta}{2} = \frac{k\Delta r}{2} = m\pi$ (condition for intensity maxima)
$r_1 - r_2 = m\lambda$ (path difference for maxima)
$\frac{kD\sin\theta}{2} = \frac{k\Delta r}{2} = \frac{\pi}{2} + m\pi$ (condition for intensity minima)
$r_1 - r_2 = \lambda(m + 1/2)$ (path difference for minima)
$\Delta\theta \approx \frac{\lambda}{D}$ (approx. angular spacing btwn. extrema)
$\Delta x \approx \frac{\lambda z_0}{D}$ (approx. position spacing btwn. extrema)

Relationship to Two-Slit Fraunhofer Diffraction

$j(\theta) = j_0 \operatorname{sinc}^2\left(\frac{ka\sin\theta}{2}\right)\frac{\sin^2\left(\frac{kDN\sin\theta}{2}\right)}{\sin^2\left(\frac{kD\sin\theta}{2}\right)}$ (N slits of width a)
$j(\theta) \approx j_0\frac{\sin^2\left(\frac{2kD\sin\theta}{2}\right)}{\sin^2\left(\frac{kD\sin\theta}{2}\right)}$ (for $\theta \ll 1$ and two slits)

$= j_0 \frac{4\sin^2 \frac{kD\sin\theta}{2} \cos^2 \frac{kD\sin\theta}{2}}{\sin^2 \frac{kD\sin\theta}{2}}$ (trig. identities) $= 4j_0 \cos^2\left(\frac{kD\sin\theta}{2}\right)$ (same as in Young's experiment!)

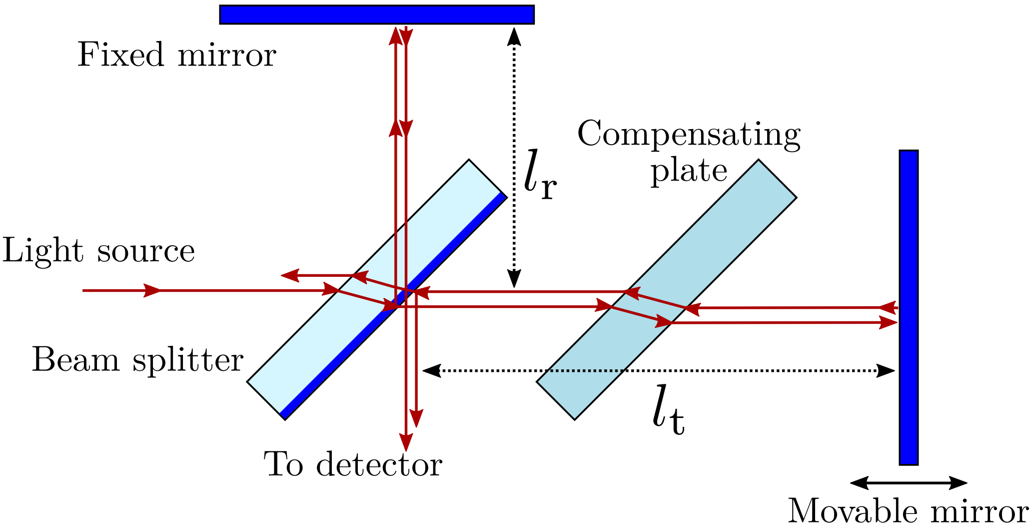


Figure 4: Geometry of a Michelson interferometer.

Interference via Amplitude Splitting

Principle: split a single incident beam into two equal parts with beam splitter. The split beams create an interference pattern.

Michelson Interferometer

Consider plane wave light incident on a beam splitter with $R = T = 0.5$. Transmitted light travels through compensator to movable mirror and back, then reflects to detector. Reflected light travels to fixed mirror and back, then on to detector. Compensator ensures reflected and transmitted beams travel equal optical paths.

l_t is distance btwn. beam splitter and mirror for transmitted beam.

l_r is distance btwn. beam splitter and mirror for reflected beam.

Transmitted and reflected beam interfere.

$\langle j_\mathrm{det}\rangle = 4j_0\cos^2\frac{\Delta\Phi}{2}$ (superposed intensity at detector)

$\Delta\Phi = 2k(l_1 - l_2) \equiv 2k\Delta l$ (phase difference between beams)

$\Delta l = \frac{m\pi}{k} = \frac{\lambda m}{2};\ m \in \mathbb{N}$ (interference maxima condition)

Variation: *Twynman-Green interferometer*: light source is always a point source. Source light is first expanded with diverging lens, then collimated into a parallel beam incident on beam splitter.

Sagnac Interferometer

Mirrors arranged periodically around a circular loop.

Incident beam passes through beam splitter. Reflected and transmitted beams travel in opposite directions around the interferometer into detector, guided by mirrors.

In an inertial frame: transmitted and reflected beams travel equal optical paths. No phase difference and perfect constructive interference at detector.

Rotating frame: beams travels different optical path lengths around interferometer. Beams have difference phase at detector $\implies$ some destructive interference and weaker signal.

Sagnac Interferometer: Analysis

R is interferometer radius.

Ω is angular speed of interferometer rotation relative to inertial reference frame. Typically $\omega R \sim 1\,\mathrm{m\,s^{-1}}$.

ω is angular frequency of light waves.

t_1 is time required for beam traveling *opposite* direction of interferometer rotation to circumvent interferometer.

l_1 is orbital distance traced out by intf. edge in time t_1.

$t_1 = \frac{2\pi R - l_1(\Omega)}{c}$

$l_1(\Omega) = \Omega R t_1$

$t_1 = \frac{2\pi R}{c + \Omega R}$ (using $l_1 = \Omega R t_1$)

t_2 is time required for beam traveling *in* direction of interferometer rotation to circumvent interferometer.

l_2 is orbital distance traced out by intf. edge in time t_2.

$t_2 = \frac{2\pi R + l_2(\Omega)}{c}$

$l_2 = \Omega R t_2$

$t_2 = \frac{2\pi R}{c - \Omega R}$ (using $l_2 = \Omega R t_2$)

$\Delta t = t_2 - t_1$ (time btwn. beams reach detector)

$\Delta\Phi = \omega\Delta t$ (phase difference btwn. beams at detector)

$\Delta\Phi = \frac{4\pi R^2\omega\Omega}{c^2 - \Omega^2R^2} \approx \frac{4\pi S\Omega}{c^2}$ ($S = \pi R^2;\quad c \gg \Omega R$)

Thin Film Interference

Consider plane waves with amplitude E_0 incident at an angle α on a thin film of width a and refractive index n_2 surrounded on either side by a material with refractive index n_1.

The incident plane wave undergoes both reflection and refraction at both film surfaces.

Goal: determine average transmitted intensity $\langle j\rangle$ on an observation screen on the opposite side of the film.

Subscript $_{12}$ denotes transition from n_1 (surroundings) to n_2 (film).

Subscript $_{21}$ denotes transition from n_2 (film) to n_1 (surroundings).

Reflection and Refraction at Boundaries

Assumption: consider only light with TE polarization

$r_{12} = \frac{n_1\cos\alpha - n_2\cos\beta}{n_1\cos\alpha + n_2\cos\beta}$ (surroundings into film)

$r_{21} = \frac{n_2\cos\beta - n_1\cos\alpha}{n_2\cos\beta + n_1\cos\alpha}$ (film into surroundings)

$r_{12} = -r_{21}$

$t_{12} = 1 + r_{12}$

$t_{21} = 1 + r_{21} = 1 - r_{12}$

$E_1 = t_{21}t_{12}E_0$ (after passing directly through film)

$E_2 = t_{21}(r_{21}r_{21}e^{i\Phi})t_{12}E_0$ (after one internal reflection)

$E_{m+1} = t_{21}r_{21}^{2m}e^{im\Phi})t_{12}E_0$ (after $m+1$ internal reflections)

$\Delta L = 2a\cos\beta$ (difference in optical path length between adjacent transmitted waves)

$\Phi = 2n_2k_0a\cos\beta$ (phase shift btwn. adjacent trans. waves)

$\Phi = n_2k_0\Delta L$ (in terms of OPL difference)

$E_\mathrm{t} = \sum_m E_m$ (total transmitted field passing through film)

$= t_{21}t_{12}E_0\left(1 + r_{21}^2e^{i\Phi} + r_{21}^4e^{i2\Phi} + \cdots\right)$

$= \frac{t_{21}t_{12}E_0}{1 - r_{21}^2e^{i\Phi}}$ (assuming $m \to \infty$)

Transmittance of Thin Films

$T = \frac{n_\mathrm{out}}{n_\mathrm{in}}\frac{\cos\theta_\mathrm{out}}{\cos\theta_\mathrm{in}}|t|^2$ (general transmittance)

$\theta_\mathrm{in} = \theta_\mathrm{out} = \alpha$ and $n_\mathrm{in} = n_\mathrm{out} = n_1$ (for a single thin film)

$$T_f = |t| = \left|\frac{E_t}{E_i}\right|^2 = \frac{1}{E_0^2}\cdot\left|\frac{t_{21}t_{12}E_0}{1-r_{21}^2 e^{i\Phi}}\right|^2 \quad \text{(thin film's transmittance)}$$
$$= \frac{(1-R)^2}{1+R^2-2R\cos\Phi} \quad (R \equiv |r_{12}|^2 = |r_{21}|^2)$$
$$= \frac{1}{1+\frac{4R^2}{(1-R)^2}\sin^2(\Phi/2)} \quad \text{(using } \cos\Phi = 1-2\sin^2\frac{\Phi}{2})$$
$$\equiv \frac{1}{1+F\sin^2(\Phi/2)} \quad (F \equiv \frac{4R}{(1-R)^2} \text{ is film's finesse coefficient})$$

Thin Film as a Frequency Filter

$T_f = \frac{1}{1+F\sin^2(\Phi/2)}$ (thin film's transmittance)
$T_f = 1$ when $\Phi = 2\pi m;\ m \in \mathbb{N}$ (maximum transmittance)
$2a\cos\beta = m\lambda_2;\ \lambda_2 = \lambda_0/n_2$ (max. transmittance condition)

Fabry-Perot Interferometer

Principle: thin film with adjustable thickness a. Observe transmittance T of incident light as a function of a.
$\Phi = 2n_2k_0a\cos\beta$ (phase shift btwn. adjacent trans. waves)
$= \frac{2\omega}{c_0}n_2a\cos\beta$ (in terms of wave frequency ω)
Free spectral range $\Delta\omega_{FSR}$ is frequency spacing between adjacent transmittance peaks in $T(\omega)$ plot.
$\Delta\omega_{FSR} = \frac{\pi c_0}{n_2a\cos\beta}$ (found by setting $\Delta\Phi = 2\pi$)
$\Delta\lambda_{FSR} = \frac{\lambda^2}{2n_2a\cos\beta}$
$\Delta\lambda_{FSR}$ may be e.g. $\sim 1\,$nm in practice.

Multiple Thin Films

Consider plane waves with amplitude E_{in} incident on a sequence of films of width a_m and refractive index n_m surrounded by a material with refractive index n_0.
Subscript $_{lm}$ denotes transition from film l to film m.
Restriction: consider only normal incidence ($\alpha = \beta = 0$).
$r_{ml} = \frac{n_m-n_l}{n_m+n_l}$ (TE polarization; normal incidence)
$r_{lm} = \frac{n_l-n_m}{n_l+n_m} = -r_{ml}$ (TE polarization; normal incidence)
$t_{ml} = 1 + r_{ml}$
$t_{lm} = 1 + r_{lm} = 1 - r_{ml}$

Notation

$\rightarrow$ denotes plane waves moving to the right.
$\leftarrow$ denotes plane waves moving to the left.
Superscript $^{(r)}$ denotes quantities on far right of a film.
Superscript $^{(\ell)}$ denotes quantities on far left of a film.
Numerical subscript $_m$ denotes index of thin film.

Analysis

Consider interface between first two film layers.
$\vec{E}_2^{(\ell)} = t_{12}\vec{E}_1^{(r)} + r_{21}\overleftarrow{E}_2^{(\ell)}$
$\overleftarrow{E}_1^{(r)} = t_{21}\overleftarrow{E}_2^{(\ell)} + r_{12}\vec{E}_1^{(r)}$
$\vec{E}_2^{(\ell)} = \frac{1}{t_{21}}\left[(t_{12}t_{21} - r_{21}r_{12})\vec{E}_1^{(r)} + r_{21}\overleftarrow{E}_1^{(r)}\right]$
$= \frac{1}{t_{21}}\left(\vec{E}_1^{(r)} + r_{21}\overleftarrow{E}_1^{(r)}\right)$ (using $t_{12}t_{21} - r_{21}r_{12} = 1$)
$\phi_m = n_mk_0a_m$ (phase shift through film m)
$\vec{E}_m^{(r)} = \vec{E}_m^{(\ell)}e^{i\phi_m}$ (L of film $m \to$ R of film m)
$\overleftarrow{E}_m^{(\ell)} = \overleftarrow{E}_m^{(r)}e^{i\phi_m}$ (R of film $m \to$ L of film m)

Matrix Formalism

$$\begin{pmatrix}\vec{E}_2^{(\ell)}\\ \overleftarrow{E}_2^{(\ell)}\end{pmatrix} = \frac{1}{t_{21}}\begin{pmatrix}1 & r_{21}\\ r_{21} & 1\end{pmatrix}\begin{pmatrix}\vec{E}_1^{(r)}\\ \overleftarrow{E}_1^{(r)}\end{pmatrix} \quad \text{(R of film 1} \to \text{L of film 2)}$$
$$\begin{pmatrix}\vec{E}_m^{(\ell)}\\ \overleftarrow{E}_m^{(\ell)}\end{pmatrix} = \frac{1}{t_{m,m-1}}\begin{pmatrix}1 & r_{m,m-1}\\ r_{m,m-1} & 1\end{pmatrix}\begin{pmatrix}\vec{E}_m^{(r)}\\ \overleftarrow{E}_m^{(r)}\end{pmatrix} \quad \text{(generally)}$$
$$\begin{pmatrix}\vec{E}_1^{(r)}\\ \overleftarrow{E}_1^{(r)}\end{pmatrix} = \frac{1}{t_{12}}\begin{pmatrix}1 & r_{12}\\ r_{12} & 1\end{pmatrix}\begin{pmatrix}\vec{E}_2^{(\ell)}\\ \overleftarrow{E}_2^{(\ell)}\end{pmatrix} \quad \text{(L of film 2} \to \text{R of film 1)}$$
$$\begin{pmatrix}\vec{E}_{m-1}^{(r)}\\ \overleftarrow{E}_{m-1}^{(r)}\end{pmatrix} = \frac{1}{t_{m-1,m}}\begin{pmatrix}1 & r_{m-1,m}\\ r_{m-1,m} & 1\end{pmatrix}\begin{pmatrix}\vec{E}_m^{(\ell)}\\ \overleftarrow{E}_m^{(\ell)}\end{pmatrix} \quad \text{(generally)}$$
$$\mathbf{M}_{m-1,m} \equiv \frac{1}{t_{m-1,m}}\begin{pmatrix}1 & r_{m-1,m}\\ r_{m-1,m} & 1\end{pmatrix} \quad \text{(transfer matrix)}$$
$$\begin{pmatrix}\vec{E}_m^{(\ell)}\\ \overleftarrow{E}_m^{(\ell)}\end{pmatrix} = \begin{pmatrix}e^{-i\phi_m} & 0\\ 0 & e^{i\phi_m}\end{pmatrix}\begin{pmatrix}\vec{E}_m^{(r)}\\ \overleftarrow{E}_m^{(r)}\end{pmatrix} \quad \text{(phase shift in film } m)$$
$$\mathbf{P}_m \equiv \begin{pmatrix}e^{-i\phi_m} & 0\\ 0 & e^{i\phi_m}\end{pmatrix} \quad \text{(phase shift matrix)}$$
$$\begin{pmatrix}\vec{E}_0^{(r)}\\ \overleftarrow{E}_0^{(r)}\end{pmatrix} = \mathbf{M}_{0,1}\mathbf{P}_1\mathbf{M}_{1,2}\mathbf{P}_2\cdots\mathbf{P}_N\mathbf{M}_{N,\text{out}}\begin{pmatrix}E_{\text{out}}\\ 0\end{pmatrix}$$
$$\equiv \mathbf{M}\begin{pmatrix}E_{\text{out}}\\ 0\end{pmatrix} \quad \text{(transfer through } N \text{ films)}$$
$$\begin{pmatrix}\vec{E}_0^{(r)}\\ \overleftarrow{E}_0^{(r)}\end{pmatrix} = \begin{pmatrix}M_{11} & M_{12}\\ M_{21} & M_{22}\end{pmatrix}\begin{pmatrix}E_{\text{out}}\\ 0\end{pmatrix} \quad \text{(by components)}$$
$t_f = \frac{E_{\text{out}}}{\vec{E}_0^{(r)}} = \frac{E_{\text{out}}}{E_{\text{in}}} = \frac{1}{M_{11}}$ (film's transmission coefficient)
$r_f = \frac{\overleftarrow{E}_0^{(r)}}{\vec{E}_0^{(r)}} = \frac{M_{21}}{M_{11}}$ (film's reflection coefficient)

Single Thin Film with Matrix Formalism

Consider a single film of width a and refractive index n_2 between a material with refractive index n_1.
Plane waves with amplitude E_{in} are normally incident on film.
$$\begin{pmatrix}\vec{E}_0^{(r)}\\ \overleftarrow{E}_0^{(r)}\end{pmatrix} = \mathbf{M}_{12}\mathbf{P}_2\mathbf{M}_{21}\begin{pmatrix}E_{\text{out}}\\ 0\end{pmatrix} \quad \text{(for normal incidence)}$$
$$\mathbf{M}_{12}\mathbf{P}_2\mathbf{M}_{21} = \frac{1}{t_{12}t_{21}}\begin{pmatrix}e^{-i\phi} + r_{12}r_{21}e^{i\phi} & r_{21}e^{-i\phi} + r_{12}e^{i\phi}\\ r_{12}e^{-i\phi} + r_{21}e^{i\phi} & r_{12}r_{21}e^{-i\phi} + e^{i\phi}\end{pmatrix}$$
$t_f = \frac{E_{\text{out}}}{E_{\text{in}}} = \frac{1-r_{12}^2}{e^{-i\phi}(1-r_{12}^2e^{2i\phi})}$ (transmission coefficient)
Let $\Phi \equiv 2\phi = 2n_2ak_0$ and $R \equiv |r_{12}|^2 = |r_{21}|^2$
$T_f = |t_f|^2 = \cdots = \frac{1}{1+\frac{4R^2}{(1-R)^2}\sin^2(\Phi/2)}$
Matrix formalism agrees with result in Transmittance of Thin Films.

Anti-Reflective Coating

Plane waves normally incident from air ($n = 1$) onto film of width a and $n = n_2$ pass into material with $n = n_3$.
Goal: find film parameters minimizing film reflectance R_f.
$$\begin{pmatrix}\vec{E}_0^{(r)}\\ \overleftarrow{E}_0^{(r)}\end{pmatrix} = \mathbf{M}_{12}\mathbf{P}_2\mathbf{M}_{23}\begin{pmatrix}E_{\text{out}}\\ 0\end{pmatrix} \quad \text{(for normal incidence)}$$
$R_f = |r_f| = \left|\frac{M_{12}}{M_{11}}\right|$
$$\mathbf{M}_{12}\mathbf{P}_2\mathbf{M}_{23} = \frac{1}{t_{12}t_{23}}\begin{pmatrix}e^{-i\phi} + r_{12}r_{23}e^{i\phi} & r_{23}e^{-i\phi} + r_{12}e^{i\phi}\\ r_{12}e^{-i\phi} + r_{23}e^{i\phi} & r_{12}r_{23}e^{-i\phi} + e^{i\phi}\end{pmatrix}$$
Let $\Phi \equiv 2\phi = 2n_2ak_0$ (twice phase shift through film)
$t_f = \frac{1}{M_{11}} = \frac{t_{12}t_{23}}{e^{-i\phi}(1+r_{12}r_{23}e^{i\Phi})}$ (transmission coefficient)
$T_f = \frac{n_3}{n_1}|t_f|^2 = \frac{t_{12}^2t_{23}^2}{1+r_{12}^2r_{23}^2+2r_{12}r_{23}\cos\Phi}\frac{n_3}{n_1}$ (transmittance)
$\Phi = (2m+1)\pi$ (minimum denominator $\implies$ maximum T_f)
$a_m = \frac{(2m+1)\pi}{2k_0n_2} = \frac{(2m+1)}{4}\frac{\lambda_0}{n_2}$ (condition for max. T_f)
$T_{\max} = \frac{t_{12}^2t_{23}^2}{(1-r_{12}r_{23})^2}\frac{n_3}{n_1}$ (when max T_f condition is met)
$R_f = |r_f|^2 = \left|\frac{M_{12}}{M_{11}}\right| = \frac{r_{12}^2+r_{23}^2+2r_{12}r_{23}\cos\Phi}{1+r_{12}^2r_{23}^2+2r_{12}r_{23}\cos\Phi}$ (reflectance)
$R_{\min} = \frac{(r_{12}-r_{23})^2}{(1-r_{12}r_{23})^2}$ (when max T_f condition is met)
$R_f = 0$ if $r_{12} = r_{23}$
$\frac{n_1-n_2}{n_1+n_2} = \frac{n_2-n_3}{n_2+n_3}$ ($r_{12} = r_{23}$ for TE and normal incidence)
$n_2 = \sqrt{n_1n_3}$ (condition for $R_f = 0$)
($R_f = 0$ only holds at the wavelength λ_0 and film thickness a required to give $\Phi = (2m+1)\pi$.)

Scattering

Goal: explain how light is affected by three-dimensional obstacles placed along the optical path.
Light attenuation in matter results from (i) scattering and (ii) absorption. We will consider only light attenuation from scattering.

Scattering Cross Section

Consider a scattering object with *geometric* cross section S.
$\sigma \equiv QS$ (the object's *scattering* cross section)
Q is the object's (dimensionless) scattering efficiency.
Interpretation: σ is the (hypothetical) cross-sectional area of an ideal black-body absorber that would cause equivalent attenuation from absorption as the scatterer does from scattering.

Exponential Attenuation From Scattering

Consider light of intensity j incident on a material cross section S and number density n_s of scatterers with SCS σ.
$\mathrm{d}P_0 = \sigma j$ (power dissipated by one scatterer)
$\mathrm{d}P_N = N\,\mathrm{d}P_0 = (n_s S\,\mathrm{d}z)\cdot(\sigma j)$ (power dissipated by N scatterers)
$\mathrm{d}P = -\,\mathrm{d}P_N$ (decrease in incident light's intensity)
$j(z) = j_0 e^{-\sigma n_s z} \equiv j_0 e^{-\mu z}$ (using $\mathrm{d}j = (\mathrm{d}P)/S$ and $\mu \equiv \sigma n_s$)
$\mu \equiv \sigma n_s$ is material's attenuation coefficient.
$j_0 = j(z)\big|_{z=0}$ is incident intensity.

Rayleigh Scattering

Rayleigh scattering is scattering of EM waves by particles with characteristic linear dimension R much less than the EM wave wavelength λ.
$R \ll \lambda \implies kR \ll 1$ (condition for Rayleigh scattering)
Approximation: E is constant throughout particle volume.
Principle: incident EM waves induce electric polarization of scattering particles, which emit dipole EM radiation.
$\mathbf{D} = \varepsilon_0\mathbf{E} + \mathbf{P}$ (in general)
Assumption: polarized material is linear: $\mathbf{D} = \varepsilon\varepsilon_0\mathbf{E}$.
Assumption: polarized material is nonmagnetic: $\mathbf{B} = \mu_0\mathbf{H}$.

Dipole Scattering

Consider spherical scattering particles with radius R and dielectric constant ε_2 in a medium with dielectric constant ε_1.
Let $\Delta\varepsilon \equiv \varepsilon_2 - \varepsilon_1$.
$\mathbf{E}_{\text{in}}(\mathbf{r},t) = E_{\text{in}}(\mathbf{r})e^{-i\omega t}\,\hat{\mathbf{e}}_{\text{in}}$ (ansatz for incident field)
$\mathbf{P} = \varepsilon_0\Delta\varepsilon\mathbf{E}_{\text{in}}$ (particle polarization relative to medium)
$\mathbf{p} = V\mathbf{P} = \frac{4\pi}{3}R^3\mathbf{P}$ (particle electric dipole moment)
$= \varepsilon_0\Delta\varepsilon\frac{4\pi}{3}R^2 E_{\text{in}}(\mathbf{r})e^{-i\omega t}\,\hat{\mathbf{e}}_{\text{in}}$
$\equiv \mathbf{p}_0 e^{-i\omega t}$ ($\mathbf{p}_0$ is dipole moment amplitude)

Dipole Antenna

Align coordinate system so that $\mathbf{p}_0 \parallel \hat{\mathbf{e}}_z$.
$\mathbf{E}_{\text{rad}}(\mathbf{r},t) = \frac{\omega^2 p_0}{4\pi\varepsilon_0 c_0^2 r}e^{i(kr-\omega t)}\sin\theta\,\hat{\mathbf{e}}_\theta$ (radiated dipole field)
$\mathbf{H}_{\text{rad}}(\mathbf{r},t) = \frac{\omega^2 p_0}{4\pi c_0 r}e^{i(kr-\omega t)}\sin\theta\,\hat{\mathbf{e}}_\varphi$ (assuming $E_0 = c_0 B_0$)
$\langle\mathbf{j}_{\text{rad}}\rangle = \langle\mathbf{S}_{\text{rad}}\rangle = \langle\mathbf{E}_{\text{rad}}\times\mathbf{H}_{\text{rad}}\rangle = \frac{\omega^4 p_0^2\sin^2\theta}{32\pi^2\varepsilon_0 c_0^3}\frac{\hat{\mathbf{e}}_r}{r^2} \propto \frac{1}{\lambda^4}\,\hat{\mathbf{e}}_r$
Conclusion: $\langle j_{\text{rad}}\rangle$ falls with 4-th power of incident wavelength.
$Q_s \sim \left(\frac{R}{\lambda}\right)^4$ (scattering efficiency for Rayleigh scattering)

Angular Dependence of Radiated Intensity

Goal: determine position/direction dependence of $\mathbf{j}_{\text{rad}}$.
Assume $\mathbf{k}_{\text{in}} \parallel \hat{\mathbf{e}}_z$ and $\mathbf{E}_{\text{in}} \parallel \hat{\mathbf{e}}_x$.
$\implies$ xy plane is equatorial plane for dipole radiation.
Observe scattered light with wave vector $\mathbf{k}_s$ at a detector.
Let θ_s (*scattering angle*) denote angle between $\mathbf{k}_s$ and $\hat{\mathbf{e}}_x$.
Assume incident light is unpolarized.
Decompose incident light into transverse and tangent polarizations relative to plane containing $\mathbf{k}_{\text{in}}$ and $\mathbf{k}_s$.
For transverse polarization: detector lies in equatorial plane $\implies$ $\theta = \pi/2$ and $\mathbf{j}_{\text{rad}}$ is constant with respect to θ_s.
For tangent polarization: $\theta = \pi/2 - \theta_s$ and $\mathbf{j}_{\text{rad}}$ depends on θ_s.
$j_{\text{rad}}^{(\text{total})} = j_{\text{rad}}^{\perp} + j_{\text{rad}}^{\parallel}$ (total radiated intensity)
$= j_{\text{rad}_0} + j_{\text{rad}_0}\sin^2\left(\frac{\pi}{2} - \beta\right)$
$= j_{\text{rad}_0}(1 + \cos^2\theta_s)$

Mie Scattering (in passing)

Mie scattering is scattering of EM waves by particles with characteristic linear dimension R of the order of the EM wave wavelength λ.
$R \sim \lambda \implies kR \sim 1$ (condition for Mie scattering)
Electric field phase varies throughout scatterer (b/c $R \sim \lambda$).
Analysis: compute electric field inside and outside scatterer; connect fields with boundary conditions.
$E(\mathbf{r},t) = E_0(r,\theta,\varphi)e^{i\omega t}$ (ansatz)
$\nabla^2\mathbf{E} + k^2\mathbf{E} = \mathbf{0}$ (Helmholtz equation)
$E_{lm}(r,\theta,\varphi) = E_0 e^{im\varphi}P_l^m(\cos\theta)z_l(kr)$ (eigenfunctions)
P_l^m are associated Legendre polynomials.
Z_l are spherical Bessel functions.
$E = \sum_{l,m} A_{lm}E_{lm}$ (general solution)

Coherence

Coherence is a well-defined relationship between the phases of EM waves at different points in space and time.
Perfectly coherent light has an electric field *exactly known at all points in space and time.*
$E(\mathbf{r}, t) = E_0(\mathbf{r}, t)e^{i\phi(\mathbf{r},t)}$ (perfect coherence; $E_0 \in \mathbb{R}$)

Temporal Coherence

Consider monochromatic EM waves of frequency ω with respect to time at a fixed position $\mathbf{r}_0$.
Principle: given field $E(\mathbf{r}, t) = E_0 e^{-i\omega t} e^{i\phi(\mathbf{r},t)}$ at some time t, determine field $E(\mathbf{r}_0, t+\tau) = E_0 e^{-i\omega(t+\tau)} e^{i\phi(\mathbf{r}_0,t+\tau)}$ at some later time $t+\tau$ where $\tau \in \mathbb{R}$.

Classes of Temporal Coherence

$\phi(\mathbf{r}_0, t+\tau) = f(\mathbf{r}_0, \tau, \phi(\mathbf{r}_0, t))$ for all τ (perfect coherence)
$\phi(\mathbf{r}_0, t+\tau) = \begin{cases} f(\mathbf{r}_0, \tau, \phi(\mathbf{r}_0, t)) & \tau < \tau_c \\ \text{unknown} & \tau > \tau_c \end{cases}$ (partial coherence)
$\phi(\mathbf{r}_0, t+\tau)$ unknown for all $\tau \in \mathbb{R}$ (incoherent light)
τ_c is *coherence time.*
$\tau_c \to \infty$ for perfect temporal coherence.
$\tau_c \sim 10^{-14}\,\mathrm{s}$ for a typical gas (e.g. neon-based) light.

Measuring Temporal Coherence

Principle: vary the mirror separation in a Michelson interferometer and observe when interferogram begins to fade.
Assumption: beam splitter has balanced $R = T = 0.5$.
Δl is distance of movable mirror from equilibrium position.
$\Delta L = 2\Delta l$ is difference in distance traveled between interfering beams at detector.
$\tau = \Delta L/c$ is separation in time, relative to mutual source, between interfering beams at detector.
$E_{\text{det}}(t) = E(t) + E(t+\tau)$ (electric field at detector)
$j_{\text{det}}(t) \propto |E_{\text{det}}(t)|^2$ (intensity at detector)
Observe detector for time T...
$\langle j_{\text{det}} \rangle \propto \frac{1}{T}\int_{-T/2}^{T/2} |E_{\text{det}}(t)|^2\,\mathrm{d}t$ (average intensity)
$\quad \propto 2\left\langle |E(t)|^2 \right\rangle + \frac{1}{T}\int_{-T/2}^{T/2} 2\,\mathrm{Re}\left[E(t)E^*(t+\tau)\right]\mathrm{d}t$
$G^{(1)}(\tau) \equiv \langle E(t)E^*(t+\tau)\rangle$ (autocorrelation function)
$\quad = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} E(t)E^*(t+\tau)\,\mathrm{d}t$
$\langle j_{\text{det}} \rangle = 2\left\langle |E(t)|^2 \right\rangle + 2\,\mathrm{Re}\,G^{(1)}(\tau)$ (for $T \gg \tau_c$)
Principle: estimate ΔL_c when $\langle j_{\text{det}} \rangle \to 2\left\langle |E(t)|^2 \right\rangle$.
$\tau_c = \Delta L_c/c$ (estimate of coherence time)

Measuring Temporal Coherence with Plane Waves

Situation as in Measuring Temporal Coherence.
$E(t) = E_0 e^{-i\omega t}$ (assume $E_0 \in \mathbb{R}$)
$E(t+\tau) = E_0 e^{-i\omega(t+\tau)}$
$G^{(1)}(\tau) = E_0^2 e^{i\omega\tau}$
$\langle j_{\text{det}} \rangle \propto 2\left\langle |E(t)|^2 \right\rangle + 2\,\mathrm{Re}\,G^{(1)}(\tau) = 2E_0^2(1+\cos\omega\tau)$
$\langle j_{\text{det}} \rangle = 2j_0(1+\cos\omega\tau)$ (alternate formulation)
$\tau = \Delta L/c$ (time shift between beams at detector)
$\omega\tau = \omega\Delta L/c = k\Delta L = \Delta\phi$ (phase shift btwn. beams at det.)
$\langle j_{\text{det}} \rangle = 2j_0(1+\cos\Delta\phi) = 4j_0\cos^2\frac{\Delta\phi}{2}$
$\langle j_{\text{det}} \rangle$ agrees with result from Michelson Interferometer.
$\langle \cos\Delta\Phi \rangle \to 0$ as $\Delta L \gg \tau_c c$
$\langle j_{\text{det}} \rangle \to 2j_0$ as $\Delta L \gg \tau_c c$

Wiener-Khinchin Theorem

The autocorrelation function $G^{(1)}(\tau)$ and power spectral density of a (sufficiently well-behaved) signal are Fourier transform pairs.
$S(\omega) \propto \int_{-\infty}^{\infty} G^{(1)}(\tau)e^{-i\omega\tau}\,\mathrm{d}\tau$
$G^{(1)}(\tau) \propto \int_{-\infty}^{\infty} S(\omega)(\tau)e^{+i\omega\tau}\,\mathrm{d}\tau$

Spatial Coherence

Consider monochromatic EM waves of frequency ω at fixed time t_0 with respect position $\mathbf{r}$.
Principle: given field $E(\mathbf{r}, t_0) = E_0 e^{-i\omega t_0} e^{i\phi(\mathbf{r},t_0)}$ at some position $\mathbf{r}$, determine field $E(\mathbf{r}+\Delta\mathbf{r}, t_0) = E_0 e^{-i\omega t_0} e^{i\phi(\mathbf{r}+\Delta\mathbf{r},t)}$ at some shifted position $\mathbf{r}+\Delta\mathbf{r}$.

Classes of Spatial Coherence

$\phi(\mathbf{r}+\Delta\mathbf{r}, t_0)$ known for all $\Delta\mathbf{r} \in \mathbb{R}^3$ (perfect coherence)
$\phi(\mathbf{r}+\Delta\mathbf{r}, t_0) = \begin{cases} \text{known} & \mathbf{r}+\Delta\mathbf{r} \in \Omega_c \\ \text{unknown} & \mathbf{r}+\Delta\mathbf{r} \notin \Omega_c \end{cases}$ (partial coherence)
$\phi(\mathbf{r}+\Delta\mathbf{r}, t_0)$ unknown for all $\Delta\mathbf{r} \in \mathbb{R}^3$ (incoherent light)
$\Omega_c \subset \mathbb{R}^3$ is a "coherence region" in position space centered on $\mathbf{r}$.
$\Omega_c \to \mathbb{R}^3$ for perfect spatial coherence.

Measuring Spatial Coherence

Principle: vary the separation between slits in a Young's double slit experiment and observe when interferogram begins to fade.
Similar to Young's Double-Slit Experiment
Difference: finite-size light source with characteristic linear dimension replaces replaces single slit and point source.
Consider two slits separated by distance D in xy plane.
z (optical) axis points from source to midpoint between slits.
Center of source is symmetrically placed between slits.
Let slit width run along x axis.
Work in xz plane; assume translational invariance along y axis.
L is characteristic linear dimension of light source.
θ_s is angle between optical axis and line from slits' midpoint to source's top.
θ_0 is angle between optical axis and line from slits' midpoint to observation point.

Measuring Spatial Coherence: Analysis

$\Delta\phi_s \approx kD\sin\theta_s$ (phase shift from source to slits between light from source center and light from source top)
$\Delta\phi_0 \approx kD\sin\theta_0$ (phase shift from slits to observation point between light from top and bottom slits)
$\theta_0 \approx \sin\theta_0 = \xi/z_0$ (assuming $z_0 \gg \xi$)
$\theta_s \approx \sin\theta_s = x_s/z_s$ (assuming $z_s \gg x_s$)
$\Delta\Phi = \Delta\phi_s + \Delta\phi_0$ (phase shift from source to obs. point)
$\langle j \rangle = 2j_0(1+\cos\Delta\Phi)$ (intensity at obs. point)
$\Delta\Phi_0 = \Delta\phi_0$ (for light from source's center with $x_s = 0$)
$\quad = kD\sin\theta_0$
$\theta_0 = m\pi,\ m \in \mathbb{N}$ (extrema for light from source's center)
Condition for vanishing interference: slit spacing D_c such that minima of light from source top at $x_s = L/2$ overlaps with maxima of light from source's center at $x_s = 0$.
$\Delta\phi_s = kD_c\sin\theta_s \approx kD_c\frac{L}{2z_s} = m\pi$ (condition for $\langle j \rangle \to 2j_0$)
$D_c = \frac{2\pi z_s}{kL} = \frac{z_s\lambda}{L}$ (setting $m = 1$)
Principle: set up double slit experiment with finite-sized light source and increase D until reaching D_c at which interference pattern disappears.

Application: Measuring Star Size

Principle: use a double-slit experiment with star as source and measure D_c at which interference pattern vanishes.
Assume distance z_s from star to Earth is known.
Assume wavelength λ of starlight is known.
$D_c = \frac{z_s\lambda}{L}$ (from Measuring Spatial Coherence: Analysis)
$L \sim 2R = \frac{z_s\lambda}{D_c}$ (estimate for star radius R)

Refractive Index

$n^2 = \varepsilon\mu$ (a material's refractive index)

Both ε and μ in general depend on the frequency ω of EM waves in the material.

$\varepsilon = \varepsilon(\omega)$ (general dispersion relation)

$\mu = \mu(\omega)$ (general dispersion relation)

Restriction: we consider only non-magnetic materials with $\mu = 1$.

Lorentz Model

Principle: Model material molecules (atoms) as a fixed sphere of positive charge (nucleus) connected to a mobile sphere of negative charge (electrons) by a classical spring.

Spheres' COM align in the absence of an external EM field.

External EM field causes negative sphere to oscillate.

Let x_0 denote equilibrium position of negative sphere.

Expand molecular potential about equilibrium x_0 to get...

$U(x) = U(x_0) + (x - x_0)\left(\frac{\mathrm{d}U}{\mathrm{d}x}\right)_{x_0} + \frac{(x-x_0)^2}{2}\left(\frac{\mathrm{d}^2U}{\mathrm{d}x^2}\right)_{x_0} + \mathcal{O}(x^3)$

$U(x) = U(x_0) + \frac{1}{2}k^2(x - x_0)^2$ (since $\left(\frac{\mathrm{d}U}{\mathrm{d}x}\right)_{x_0} = 0$)

$\quad = U(x_0) + \frac{1}{2}k^2x^2$ (if $x_0 = 0$)

Forces on (a Single) Electron

Align x axis with external electric field.

$\mathbf{E}(t) = E_0 e^{-i\omega t}\,\hat{\mathbf{e}}_x \equiv E(t)\,\hat{\mathbf{e}}_x$ (external electric field)

$\mathbf{F}_\mathrm{s} = -kx\,\hat{\mathbf{e}}_x$ (spring force)

$\mathbf{F}_\mathrm{d} = -\gamma m\dot{x}\,\hat{\mathbf{e}}_x$ (dissipative force; $m \equiv m_\mathrm{e}$)

$\mathbf{F}_\mathrm{e}(t) = -e_0\mathbf{E}(t) = -e_0E(t)\,\hat{\mathbf{e}}_x$ (electric force)

$m\ddot{x} = -kx - \gamma m\dot{x} - e_0E(t)$ (Newton's law for electron)

$\ddot{x} + \gamma\dot{x} + \omega_0^2x = -\frac{e_0}{m}E(t)$

$\omega_0^2 \equiv k/m$ is electron's natural oscillation frequency.

$x(t) = x_0(\omega)e^{-i\omega t}$ (ansatz for electron position)

$x_0 = -\frac{e_0}{m}\frac{E_0}{\omega_0^2-\omega^2-i\gamma\omega}$ (electron's oscillation amplitude)

$x(t) = -\frac{e_0}{m}\frac{E_0e^{-i\omega t}}{\omega_0^2-\omega^2-i\gamma\omega} = -\frac{e_0}{m}\frac{E(t)}{\omega_0^2-\omega^2-i\gamma\omega}$

Induced Polarization

Consider a non-magnetic ($\mu = 1$) linear ($\mathbf{D} = \varepsilon\varepsilon_0\mathbf{E}$) material exposed to an external electric field $\mathbf{E}(t) = E_0e^{-i\omega t}\,\hat{\mathbf{e}}_x$.

$\mathbf{p}(t) = -e_0x(t)\,\hat{\mathbf{e}}_x$ (induced dipole moment of one molecule)

$\quad = \frac{e_0^2}{m}\frac{\mathbf{E}(t)}{\omega_0^2-\omega^2-i\gamma\omega}$

n_e is the number density of microscopic molecular electric dipoles in the material.

$\mathbf{P} = n_\mathrm{e}\mathbf{p} = \frac{e_0^2n_\mathrm{e}}{m}\frac{\mathbf{E}(t)}{\omega_0^2-\omega^2-i\gamma\omega}$ (material's polarization)

$\mathbf{P} = \varepsilon_0(\varepsilon - 1)\mathbf{E}$ (in general, if $\mathbf{D} = \varepsilon_0\varepsilon_0\mathbf{E}$)

$\varepsilon = 1 + \frac{e_0^2n_\mathrm{e}}{m\varepsilon_0}\frac{1}{\omega_0^2-\omega^2-i\gamma\omega}$ (material's dielectric constant)

$\quad \equiv 1 + \frac{\omega_\mathrm{p}^2}{\omega_0^2-\omega^2-i\gamma\omega}$

$\omega_\mathrm{p} \equiv \frac{e_0^2n_\mathrm{e}}{m\varepsilon_0}$ is material's *plasma frequency.*

Refractive Index

$\mathcal{N}^2 = \varepsilon \in \mathbb{C}$ (material's refractive index assuming $\mu = 1$)

$\mathcal{N} = n_\mathrm{Re} + in_\mathrm{Im}$ (decomposition into Re and Im components)

$\mathcal{N}^2 = (n_\mathrm{Re} + in_\mathrm{Im})^2 = 1 + \frac{\omega_\mathrm{p}^2}{\omega_0^2-\omega^2-i\gamma\omega}$ (eq. for n_Re and n_Im)

$$= \underbrace{1 + \frac{\omega_\mathrm{p}^2(\omega_0^2-\omega^2)}{(\omega_0^2-\omega^2)^2+\gamma^2\omega^2}}_{\equiv\,\varepsilon_\mathrm{Re}} + i\underbrace{\frac{\gamma\omega\omega_\mathrm{p}^2}{(\omega_0^2-\omega^2)^2+\gamma^2\omega^2}}_{\equiv\,\varepsilon_\mathrm{Im}}$$

$n_\mathrm{Re}^2 = \frac{1}{2}\left(\varepsilon_\mathrm{Re} + \sqrt{\varepsilon_\mathrm{Re}^2 + \varepsilon_\mathrm{Im}^2}\right)$

$n_\mathrm{Im}^2 = \frac{1}{2}\left(-\varepsilon_\mathrm{Re} + \sqrt{\varepsilon_\mathrm{Re}^2 + \varepsilon_\mathrm{Im}^2}\right)$

Low-Density Material Approximation

Consider materials with low number density n_e (e.g. gases).

$|\mathcal{N}|^2 \approx 1$ (from experiment)

$\mathcal{N}^2 = 1 + \frac{\omega_\mathrm{p}^2}{\omega_0^2-\omega^2-i\gamma\omega}$ (for general linear materials with $\mu = 1$)

$\left|\frac{\omega_\mathrm{p}^2}{\omega_0^2-\omega^2-i\gamma\omega}\right| \ll 1$ (if $|\mathcal{N}|^2 \approx 1$)

Taylor expand square root in $\mathcal{N}$ and get...

$\mathcal{N} \approx 1 + \frac{\omega_\mathrm{p}^2}{2(\omega_0^2-\omega^2-i\gamma\omega)}$ (if $|\mathcal{N}|^2 \approx 1$)

$$= \underbrace{1 + \frac{1}{2}\frac{\omega_\mathrm{p}^2(\omega_0^2-\omega^2)}{(\omega_0^2-\omega^2)^2+\gamma^2\omega^2}}_{\equiv\,n_\mathrm{Re}} + \underbrace{\frac{1}{2}\frac{i\gamma\omega\omega_\mathrm{p}^2}{(\omega_0^2-\omega^2)^2+\gamma^2\omega^2}}_{\equiv\,in_\mathrm{Im}}$$

$n_\mathrm{Re} = 1 + \frac{1}{2}\frac{\omega_\mathrm{p}^2(\omega_0^2-\omega^2)}{(\omega_0^2-\omega^2)^2+\gamma^2\omega^2}$ (if $|\mathcal{N}|^2 \approx 1$)

$n_\mathrm{Im} = \frac{1}{2}\frac{i\gamma\omega\omega_\mathrm{p}^2}{(\omega_0^2-\omega^2)^2+\gamma^2\omega^2}$ (if $|\mathcal{N}|^2 \approx 1$)

$E(t) = E_0e^{i(k_0\mathcal{N}z-\omega t)}$ (EM waves in material)

$\quad = E_0e^{i(k_0n_\mathrm{Re}z-\omega t)}e^{-ik_0n_\mathrm{Im}z}$

n_Im is maximum at $\omega = \omega_0 \implies$ EM waves with frequency $\omega = \omega_0$ are maximally absorbed in the material.

$\frac{\mathrm{d}n_\mathrm{Re}}{\mathrm{d}\omega} > 0$ or $\frac{\mathrm{d}n_\mathrm{Re}}{\mathrm{d}\lambda} < 0$ (normal dispersion)

$\frac{\mathrm{d}n_\mathrm{Re}}{\mathrm{d}\omega} < 0$ or $\frac{\mathrm{d}n_\mathrm{Re}}{\mathrm{d}\lambda} > 0$ (anomalous dispersion)

Materials with Multiple Resonances

Real atoms/molecules have multiple resonance frequencies ω_{0_j}.

Each resonance has a corresponding damping coefficient γ_j.

$\mathcal{N}^2 = \varepsilon \to 1 + \sum_j\frac{f_j\omega_\mathrm{p}^2}{\omega_{0_j}^2-\omega^2-i\gamma_j\omega}$ (for multiple resonances)

f_j are dimensionless coefficients for strength of each resonance.

Transparent Materials Far From Resonance

Transparent materials have no damping of EM waves.

$\gamma \to 0$ (in transparent materials)

$n_\mathrm{Im} \to 0$ so $\mathcal{N}^2 \to n_\mathrm{Re}^2$ (because $\gamma \to 0$)

In the regime of normal dispersion (far from resonance)...

$n_\mathrm{Re}^2 = 1 + \sum_j\frac{f_j\omega_\mathrm{p}^2}{\omega_{0_j}^2-\omega^2}$ (approximation as $\gamma \to 0$)

$n_\mathrm{Re}^2 = 1 + \sum_j\frac{B_j}{\lambda^2-C_j}$ (Sellmeier equation)

$\sqrt{C_j}$ are resonant wavelengths (i.e. $C_j \iff \lambda_{0_j}^2$).

B_j are coefficients for strength of each resonance.

$n_\mathrm{Re}(\lambda) = A + \frac{B}{\lambda^2} + \frac{C}{\lambda^4} + \cdots$ (Cauchy equation)

$A, B, C, \ldots$ are empirically-determined coefficients.

Drude Model for Free Electrons

Model: nearly-free electron gas in a fixed lattice of positive charge.

$k \to 0$ (free electrons are not bound to atoms)

$\omega_0 \to 0$ (because $k \to 0$)

$\mathcal{N}^2 = 1 + \frac{\omega_\mathrm{p}^2}{\omega_0^2-\omega^2-i\gamma\omega}$ (for general linear materials with $\mu = 1$)

$\mathcal{N}^2 = 1 - \frac{\omega_\mathrm{p}^2}{\omega^2+i\gamma\omega}$ (refractive index of free electron gas)

$\mathcal{N}^2 \approx 1 - \frac{\omega_\mathrm{p}^2}{\omega^2}$ (for negligible electron damping $\gamma \to 0$)

$\mathcal{N} \approx \sqrt{1 - \frac{\omega_\mathrm{p}^2}{\omega^2}}$ (assuming $\gamma \to 0$)

$\implies \mathcal{N} \in \mathbb{R}$ for $\omega > \omega_\mathrm{p}$ and material is transparent

$\implies \mathcal{N} \in \mathbb{C}$ for $\omega < \omega_\mathrm{p}$ and EM waves attenuate in material

Dense Materials

Consider a non-magnetic ($\mu = 1$) linear ($\mathbf{D} = \varepsilon\varepsilon_0\mathbf{E}$) material exposed to an external electric field $\mathbf{E}_\mathrm{in}(t) = E_0e^{-i\omega t}\,\hat{\mathbf{e}}_x$.

Assume each molecule is closely surrounded by other molecules; $\implies$ the polarization of each molecule depends on the polarization of surrounding molecules.

$\mathbf{E}_\mathrm{tot} = \mathbf{E}_\mathrm{in} + \frac{\mathbf{P}}{3\varepsilon_0}$ (total electric field in material)

$\frac{\mathbf{P}}{3\varepsilon_0}$ is correction from internal polarization.

$\mathbf{P} = \varepsilon_0(\varepsilon - 1)\mathbf{E}_\mathrm{in}$ (polarization in general linear material)

$\mathbf{E}_\mathrm{tot} = \frac{\varepsilon+2}{3}\mathbf{E}_\mathrm{in}$ (using $\mathbf{P} = \varepsilon_0(\varepsilon - 1)\mathbf{E}_\mathrm{in}$)

$\mathbf{P} = \frac{\varepsilon+2}{3}\frac{e_0^2n_\mathrm{e}}{m}\frac{\mathbf{E}_\mathrm{in}(t)}{\omega_0^2-\omega^2-i\gamma\omega}$ (material's polarization)

$\frac{\varepsilon-1}{\varepsilon+2} = \frac{e_0^2 n_\mathrm{e}}{m\varepsilon_0}\frac{1}{\omega_0^2-\omega^2-i\gamma\omega}$ (material's dielectric constant) $\equiv \frac{1}{3}\frac{\omega_\mathrm{p}^2}{\omega_0^2-\omega^2-i\gamma\omega}$

$\omega_\mathrm{p}^2 \equiv \frac{e_0^2 n_\mathrm{e}}{m\varepsilon_0}$ is material's *plasma frequency.*

Optical Activity

Optical activity is a phenomenon in which the direction of linearly polarized light changes as the light passes through matter.
Consider linearly polarized light traveling in the $\hat{\mathbf{e}}_z$ direction normally incident on material of length L.
Let $\hat{\mathbf{e}}_x$ align with direction of incident light's polarization.

$\mathbf{J}_{\text{in}} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ (incident polarization)

$\mathbf{J}_{\text{out}} = \begin{pmatrix} \cos\Delta\phi \\ \sin\Delta\phi \end{pmatrix}$ (polarization on exiting matter)

Convention: ϕ is measured in xy plane relative to x axis by an observer looking towards source of incident EM waves.

Analysis: Optical Activity

Optical activity can be explained with a model in which LHC and RHC polarizations experience different refractive indices in the optically active material. First decompose linearly polarized incident light into RHC and LHC components:

$\mathbf{J}_{\text{in}} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ (incident polarization)

$= \frac{1}{2}\begin{pmatrix} 1 \\ i \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 1 \\ -i \end{pmatrix}$ (decomposed into RHC and LHC)

Model: in optically active materials, RHC- and LHC-polarized light experience difference refractive indices n_{RHC} and n_{LHC}.

$e^{in_{\text{RHC}}k_0L}$ (phase shift of RHC light in material)

$e^{in_{\text{LHC}}k_0L}$ (phase shift of LHC light in material)

$\mathbf{J}_{\text{out}} = \frac{1}{2}\begin{pmatrix} 1 \\ i \end{pmatrix} e^{in_{\text{LHC}}k_0L} + \frac{1}{2}\begin{pmatrix} 1 \\ -i \end{pmatrix} e^{in_{\text{RHC}}k_0L}$

$\bar{n} = \frac{1}{2}(n_{\text{RHC}} + n_{\text{LHC}})$ (average refractive index)

$\Delta n = n_{\text{RHC}} - n_{\text{LHC}}$ (difference of refractive indices)

$2\Delta\phi \equiv \Delta n k_0 L$ (phase shift between RHC and LHC)

$\mathbf{J}_{\text{out}} = \frac{e^{i\bar{n}k_0L}}{2}\left[\begin{pmatrix} 1 \\ i \end{pmatrix} e^{-i\Delta n k_0 L/2} + \begin{pmatrix} 1 \\ -i \end{pmatrix} e^{i\Delta n k_0 L/2}\right]$

$= e^{i\bar{n}k_0L}\begin{pmatrix} \cos\Delta\phi \\ \sin\Delta\phi \end{pmatrix}$

Conclusion: Transmitted light is linearly polarized at an angle $\Delta\phi$ relative to direction of incident polarization. The angle $\Delta\phi$ is proportional to the refractive index difference Δn and to the length L traveled through the optically active material.

The Faraday Effect

The Faraday effect is a phenomenon in which an external magnetic field applied to a dielectric material causes the material to become optically active for light traveling through the material in the direction of the field.
Consider a dielectric material exposed to a homogeneous external magnetic field $\mathbf{B}_0$; align coordinate system so $\mathbf{B}_0 = B_0\,\hat{\mathbf{e}}_z$.
Assume incident EM wave has wave vector $\mathbf{k} \parallel \hat{\mathbf{e}}_z \parallel \mathbf{B}_0$.

$\mathbf{E} = (E_x, 0, 0)$ (assume linear polarization)

$2\Delta\phi = k_0 L(n_{\text{RHC}} - n_{\text{LHC}})$ (general optical activity)

$\Delta\phi \equiv VL\frac{B_0}{\mu\mu_0}$ (model for Faraday effect)

V is material's *Verdet constant*.
$\Delta\phi$ is (approximately) proportional to external field B_0.

Sommerfeld's Analysis

Follows Arnold Sommerfeld's *Optics*, Section 20 on Magentic rotation in the plane of polarization.
Consider material in external homogeneous magnetic field $\mathbf{B}_0 = B_0\,\hat{\mathbf{e}}_z$. The equation of motion for an electron is...

$m\ddot{\mathbf{r}} + K\mathbf{r} = -e_0(\mathbf{E} + \dot{\mathbf{r}} \times \mathbf{B}_0)$

Divide by m, define $\omega_0^2 = K/m$ and rearrange to get...

$\ddot{\mathbf{r}} + \frac{e_0}{m}\dot{\mathbf{r}} \times \mathbf{B}_0 + \omega_0^2\mathbf{r} + \frac{e_0}{m}\mathbf{E} = \mathbf{0}$

Evaluate vector product and separate into components to get...

$$\ddot{x} + \frac{e_0}{m}\dot{y}B_0 + \omega_0^2 x = -\frac{e_0}{m}E_x$$
$$\ddot{y} - \frac{e_0}{m}\dot{x}B_0 + \omega_0^2 y = -\frac{e_0}{m}E_y.$$

Multiply y equation by $\pm i$ and add equations to get...

$$(\ddot{x} + i\ddot{y}) + \frac{e_0}{m}B_0(\dot{y} - i\dot{x}) + \omega_0^2(x + iy) = -\frac{e_0}{m}(E_x + iE_y)$$
$$(\ddot{x} - i\ddot{y}) + \frac{e_0}{m}B_0(\dot{y} + i\dot{x}) + \omega_0^2(x - iy) = -\frac{e_0}{m}(E_x - iE_y)$$

Rearrange the i factor in the e_0B_0/m terms to get

$$(\ddot{x} + i\ddot{y}) - i\frac{e_0}{m}B_0(\dot{x} + i\dot{y}) + \omega_0^2(x + iy) = -\frac{e_0}{m}(E_x + iE_y)$$
$$(\ddot{x} - i\ddot{y}) + i\frac{e_0}{m}B_0(\dot{x} - i\dot{y}) + \omega_0^2(x - iy) = -\frac{e_0}{m}(E_x - iE_y)$$

Define amplitudes $z_+ \equiv x+iy$, $z_- \equiv x-iy$ and field components $E_+ \equiv E_x + iE_y$, $E_- \equiv E_x - iE_y$ to get...

$$\ddot{z}_+ - i\frac{e_0}{m}B_0\dot{z}_+ + \omega_0^2 z_+ = -\frac{e_0}{m}E_+ \qquad (1)$$
$$\ddot{z}_- + i\frac{e_0}{m}B_0\dot{z}_- + \omega_0^2 z_- = -\frac{e_0}{m}E_-. \qquad (2)$$

For review, general linear polarization is decomposed into circular polarization as

$E_x = \frac{1}{2}(E_+ + E_-)$ $\qquad$ $E_y = \frac{1}{2}(E_+ - E_-)$

Physically: the transition to complex quantities corresponds to an analysis in terms of circular rather than linear polarization.

Assume ansatzes for circular polarizations of the form
$E_\pm = E_0 e^{i(k_\pm z - \omega t)}$
where $E_0 \in \mathbb{R}$ is field amplitude; note $|E_+| = |E_-| = E_0$.
Assume incident light is linearly polarized in the x direction:

$E_x = E_0 \cos\omega t$ $\qquad$ and $\qquad$ $E_y = 0$

Return to Equations 1 and 2 and use ansatzes of the form...

$z_\pm = z_0 e^{-i\omega t}$ (electron displacement)

Substitute into eq. of motion and solve for displacement to get

$z_\pm = \frac{e_0}{m}\frac{E_+}{\omega^2 \pm (e_0/m)B_0\omega - \omega_0^2}$

Consider material's polarization $\mathbf{P}$ and define complex polarizations of the form $P_\pm = P_x \pm iP_y$.
Electric dipole moment is related to electron amplitude by $\mathbf{p}(t) = -e_0\mathbf{r}(t)$, and polarization is related to electric dipole moment by $\mathbf{P} = n_{\text{e}}\mathbf{p}$, where n_{e} is the number density of microscopic electric dipoles in the material. Combining $\mathbf{P} = -n_{\text{e}}e_0\mathbf{r}$ with $z_\pm$ gives

$P_\pm = -\frac{e_0^2 n_{\text{e}}}{m}\frac{E_+}{\omega^2 \pm (e_0/m)B_0\omega - \omega_0^2}$

which is used (how?) to determine wave vectors

$k_\pm^2 = \frac{\omega^2}{c^2}\left(1 + \frac{e_0^2 n_{\text{e}}}{m}\frac{\mu_0 c^2}{\omega_0^2 \mp \frac{e_0}{m}B\omega - \omega^2}\right)$

Assume the linear relationship $\varepsilon = 1 + \frac{P}{\varepsilon_0 E}$ and combine with $E_\pm$ and $P_\pm$ to get the dielectric constants...

$\varepsilon_\pm = 1 - \frac{e_0^2 n_{\text{e}}}{\varepsilon_0 m}\frac{E_+}{\omega^2 \pm (e_0/m)B_0\omega - \omega_0^2}$

Finally assume $\mu = 1 \implies n_\pm^2 = \varepsilon_\pm$ and rearrange minus signs:

$n_\pm^2 = 1 + \frac{e_0^2 n_{\text{e}}}{\varepsilon_0 m}\frac{E_+}{\omega_0^2 \mp (e_0/m)B_0\omega - \omega^2}$

Tensor Analysis of the Faraday Effect

Follows Grant R. Fowles' *Introduction to Modern Optics 2nd Edition*, Section 6.9 on optical activity.

Wave Equation in Material

Goal: first derive a form of the wave equation in matter that will be useful for analyzing optical activity.
Begin with the macroscopic Maxwell equations...

$\nabla \times \mathbf{E} = -\mu_0\frac{\partial \mathbf{H}}{\partial t}$ and $\nabla \times \mathbf{H} = \varepsilon_0\frac{\partial \mathbf{E}}{\partial t} + \frac{\partial \mathbf{P}}{\partial t} + \mathbf{j}$

Take the curl of the $\nabla \times \mathbf{E}$ equation, combine with the $\nabla \times \mathbf{H}$ equation, and eliminate $\mathbf{H}$ to get the wave equation

$\nabla \times (\nabla \times \mathbf{E}) + \frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} = -\mu_0\frac{\partial^2 \mathbf{P}}{\partial t^2} - \mu_0\frac{\partial \mathbf{j}}{\partial t}$

Combine with the tensor relation $\mathbf{P} = \varepsilon_0\chi\mathbf{E}$, where χ is a material's susceptibility tensor, and assume $\frac{\partial \mathbf{j}}{\partial t} = \mathbf{0}$ to get...

$\nabla \times (\nabla \times \mathbf{E}) + \frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} = -\frac{1}{c^2}\chi\frac{\partial^2 \mathbf{E}}{\partial t^2}$

For plane waves of the form $\mathbf{E}(t) \propto e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}$ this becomes

$\mathbf{k} \times (\mathbf{k} \times \mathbf{E}) + \frac{\omega^2}{c^2}\mathbf{E} = -\frac{\omega^2}{c^2}\chi\mathbf{E}$.

In component form, this reads

$$\left(-k_y^2 - k_z^2 + \frac{\omega^2}{c^2}\right) E_x + k_x k_y E_y + k_x k_z E_z = -\frac{\omega^2}{c^2}\chi_{1j}E_j$$

$$k_y k_x E_x + \left(-k_x^2 - k_z^2 + \frac{\omega^2}{c^2}\right) E_y + k_y k_z E_z = -\frac{\omega^2}{c^2}\chi_{2j}E_j$$

$$k_z k_x E_x + k_z k_y E_y + \left(-k_x^2 - k_y^2 + \frac{\omega^2}{c^2}\right) E_z = -\frac{\omega^2}{c^2}\chi_{3j}E_j.$$

Susceptibility Tensor in Optically Active Material

The susceptibility tensor of an optically active takes the form...

$$\chi = \begin{pmatrix} \chi_{11} & i\chi_{12} & 0 \\ -i\chi_{12} & \chi_{11} & 0 \\ 0 & 0 & \chi_{33} \end{pmatrix} \qquad \text{(in an optically active material)}$$

To prove this is the correct susceptibility tensor in an optically active material, evaluate the x, y, and z components of the above-derived wave equation for the above χ and a wave propagating in the z direction with $\mathbf{k} = k\,\hat{\mathbf{e}}_z$ to get...

$$\left(-k^2 + \frac{\omega^2}{c^2}\right) E_x = -\frac{\omega^2}{c^2}(\chi_{11}E_x + i\chi_{12}E_y)$$

$$\left(-k^2 + \frac{\omega^2}{c^2}\right) E_y = -\frac{\omega^2}{c^2}(-i\chi_{12}E_x + \chi_{11}E_y)$$

$$\frac{\omega^2}{c^2}E_z = -\frac{\omega^2}{c^2}\chi_{33}E_z$$

The z component equation gives $E_z = 0$; require determinant of coefficients of x and y component equations be nonzero to get...

$$\det\begin{pmatrix} -k^2 + (\omega^2/c^2)(1+\chi_{11}) & i(\omega^2/c^2)\chi_{12} \\ -i(\omega^2/c^2)\chi_{12} & -k^2 + (\omega^2/c^2)(1+\chi_{11}) \end{pmatrix} = 0,$$

which leads to the dispersion relation...

$k = \frac{\omega}{c}\sqrt{1+\chi_{11} \pm \chi_{12}}$

Substitute this k into the x or y component equations to get...

$E_x = \pm iE_y$, corresponding to RHC and LHC polarized light.

The associated refractive indices are...

$n_{\mathrm{RHC}} = \sqrt{1+\chi_{11}+\chi_{12}}$ and $n_{\mathrm{LHC}} = \sqrt{1+\chi_{11}-\chi_{12}}$

The difference in refractive indices comes out to...

$$\begin{aligned}
\Delta n = n_{\mathrm{RHC}} - n_{\mathrm{LHC}} &= \sqrt{1+\chi_{11}+\chi_{12}} - \sqrt{1+\chi_{11}-\chi_{12}} \\
&= \sqrt{1+\chi_{11}}\left(\sqrt{1+\frac{\chi_{12}}{1+\chi_{11}}} - \sqrt{1-\frac{\chi_{12}}{1+\chi_{11}}}\right) \\
&\approx \sqrt{1+\chi_{11}}\left[1+\frac{\chi_{12}}{2(1+\chi_{11})} - \left(1-\frac{\chi_{12}}{2(1+\chi_{11})}\right)\right] \\
&= \frac{\chi_{12}}{\sqrt{1+\chi_{11}}}
\end{aligned}$$

$\Delta\phi = \Delta n\frac{\omega L}{2c} \approx \frac{\chi_{12}\omega L}{2c\sqrt{1+\chi_{11}}}$ (rotation of polarization)

Faraday Effect In Terms of Susceptibility Tensor

As in the Lorentz model, model dielectric material's molecules as mobile negative charges (i.e. electrons) bound to stationary positive charges (e.g. nuclei). Let $\mathbf{r}$ denote displacement of negative charge from equilibrium.

Ignoring (i) damping of electrons and (ii) the effect of the optical magnetic field, the equation of motion for an electron is...

$\ddot{\mathbf{r}} + K\mathbf{r} = -e_0(\mathbf{E} + \dot{\mathbf{r}} \times \mathbf{B}_0)$

$\ddot{\mathbf{r}} + \omega_0^2\mathbf{r} = -\frac{e_0}{m}(\mathbf{E} + \dot{\mathbf{r}} \times \mathbf{B}_0)$ (in terms of ω_0)

$\omega_0^2 = K/m$ is molecular resonance frequency.

Assume plane wave ansatzs $\mathbf{E} = \mathbf{E}_0 e^{-i\omega t}$ and $\mathbf{r} = \mathbf{r}_0 e^{-i\omega t}$; substitute ansatzes into equation of motion and simplify to get...

$-\omega^2\mathbf{r} + \omega_0^2\mathbf{r} = -\frac{e_0}{m}\mathbf{E} + i\frac{\omega e_0}{m}\mathbf{r} \times \mathbf{B}$

The material's polarization is $\mathbf{P} = -n_{\mathrm{e}}e_0\mathbf{r}$, where n_{e} is the number density of electrons in the material. In terms of $\mathbf{P}$ the electron equation of motion reads...

$-\frac{\mathbf{P}}{n_{\mathrm{e}}e_0}(-\omega^2 + \omega_0^2) = -\frac{e_0}{m}\mathbf{E} - i\frac{\omega}{mn_{\mathrm{e}}}\mathbf{P} \times \mathbf{B}$

$(-\omega^2 + \omega_0^2)\mathbf{P} = \frac{n_{\mathrm{e}}e_0^2}{m}\mathbf{E} + i\frac{\omega e_0}{m}\mathbf{P} \times \mathbf{B}$ (rearranged)

Assume $\mathbf{B} = B_0\,\hat{\mathbf{e}}_z$. and define cyclotron frequency $\omega_{\mathrm{c}} = \frac{e_0 B}{m}$.

Write equation in component form and solve for $\mathbf{P}$ to get

$$(-\omega^2 + \omega_0^2)P_x = \frac{n_{\mathrm{e}}e_0^2}{m}E_x + i\omega\omega_{\mathrm{c}}P_y$$

$$(-\omega^2 + \omega_0^2)P_y = \frac{n_{\mathrm{e}}e_0^2}{m}E_y - i\omega\omega_{\mathrm{c}}P_x$$

$$(-\omega^2 + \omega_0^2)P_z = \frac{n_{\mathrm{e}}e_0^2}{m}E_z.$$

Solve for $\mathbf{P}$'s components in terms of $\mathbf{E}$'s components to get

$$P_x = \frac{N(-w^2+\omega_0^2)}{(w^2-\omega_0^2)^2 - (\omega\omega_{\mathrm{c}})^2}E_x - \frac{N\cdot(-i\omega\omega_{\mathrm{c}})}{(w^2-\omega_0^2)^2 - (\omega\omega_{\mathrm{c}})^2}E_y$$

$$P_y = \frac{N\cdot(-i\omega\omega_{\mathrm{c}})}{(w^2-\omega_0^2)^2 - (\omega\omega_{\mathrm{c}})^2}E_x + \frac{N(-w^2+\omega_0^2)}{(w^2-\omega_0^2)^2 - (\omega\omega_{\mathrm{c}})^2}E_y$$

$$P_z = \frac{n_{\mathrm{e}}e_0^2}{(-w^2+\omega_0^2)m}E_z$$

Let χ denote material's susceptibility tensor; compare above components to general tensor relation $\mathbf{P} = \varepsilon_0\chi\mathbf{E}$, or

$$P_x = \varepsilon_0\left(\chi_{xx}E_x + \chi_{xy}E_y + \chi_{xy}E_z\right)$$

$$P_y = \varepsilon_0\left(\chi_{yx}E_x + \chi_{yy}E_y + \chi_{yy}E_z\right)$$

$$P_z = \varepsilon_0\left(\chi_{zx}E_x + \chi_{zy}E_y + \chi_{zy}E_z\right),$$

to conclude that...

$$\chi = \begin{pmatrix} \chi_{11} & +i\chi_{12} & 0 \\ -i\chi_{21} & \chi_{11} & 0 \\ 0 & 0 & \chi_{33}, \end{pmatrix}$$

where the components are...

$$\chi_{11} = \frac{n_{\mathrm{e}}e_0^2}{\varepsilon_0 m}\left(\frac{\omega_0^2 - \omega^2}{(\omega_0^2-\omega^2)^2 - (\omega\omega_{\mathrm{c}})^2}\right)$$

$$\chi_{12} = \frac{n_{\mathrm{e}}e_0^2}{\varepsilon_0 m}\left(\frac{\omega\omega_{\mathrm{c}}}{(\omega_0^2-\omega^2)^2 - (\omega\omega_{\mathrm{c}})^2}\right)$$

$$\chi_{33} = \frac{n_{\mathrm{e}}e_0^2}{\varepsilon_0 m}\left(\frac{1}{\omega_0^2 - \omega^2}\right)$$

The material's susceptibility tensor exactly matches the susceptibility tensor of an optically active material, showing the Faraday effect is analogous to magnetically-induced optical activity. The general optical activity result $\Delta\phi = \Delta n\frac{\omega L}{2c} \approx \frac{\chi_{12}\omega L}{2c\sqrt{1+\chi_{11}}}$ then lets one find the rotation of polarization in the Faraday effect.

Optically Anisotropic Materials

Optically anisotropic materials exhibit different optical properties for light traveling in different spatial directions.

Assumptions for Anisotropic Materials

We restrict our analysis to anisotropic material that are...

(i) magnetically isotropic (so $\mu = 1 \in \mathbb{R}$),

(ii) homogeneous: the material's properties are identical throughout the material (so $\epsilon \neq \epsilon(\mathbf{r})$ and $\mu \neq \mu(\mathbf{r})$),

(iii) charge-free: the material is free of net electric charge (so $\rho = 0$),

(iv) nonconducting: an electric field in the material does not establish electric currents (so $\mathbf{j} = \mathbf{0}$), and

(v) linear: $\mathbf{D} \propto \mathbf{E}$, $\mathbf{P} \propto \mathbf{E}$ and $\mathbf{B} \propto \mathbf{H}$.

Maxwell Equations in Matter Under Above Assumptions

$\nabla \cdot \mathbf{D} = 0$
$\nabla \cdot \mathbf{B} = 0$
$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$
$\nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t}$

Tensor Relations

$P_i = \varepsilon_0 \chi_{ij} E_j$ (in linear materials)
$\mathbf{P} = \varepsilon_0 \chi \mathbf{E}$ (in coordinate-free form)
$D_i = \varepsilon_0 \varepsilon_{ij} E_j$ (in linear materials)
$\mathbf{D} = \varepsilon \epsilon \mathbf{E}$ (in coordinate-free form)
χ is the rank-two susceptibility tensor.
ϵ is the rank-two dielectric tensor.
$\mathbf{P}$ and $\mathbf{E}$ are not parallel in anisotropic materials!
$\mathbf{D}$ and $\mathbf{E}$ are not parallel in anisotropic materials!

Refractive Index in Anisotropic Materials

Consider plane waves with wave vector $\mathbf{k}$ propagating through an anisotropic material with properties as in Assumptions for Anisotropic Materials.
Goal: find refractive index experienced by waves in the material as a function wave vector $\mathbf{k}$ and wave polarization.

Ansatzes for Anisotropic Materials

Assume plane-wave ansatzes for field quantities in the anisotropic material.
$\mathbf{E} = \mathbf{E}_0 e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)}$
$\mathbf{D} = \mathbf{D}_0 e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)}$
$\mathbf{B} = \mu_0 \mathbf{H} = \mathbf{B}_0 e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)}$ (assuming $\mu = 1$)
$\mathbf{S} = \mathbf{E} \times \mathbf{H} = \mathbf{S}_0 e^{i\mathbf{k}\cdot\mathbf{r} - \omega t}$

Directions of k and Field Quantities

In anisotropic materials for above assumptions and ansatzes...
$\mathbf{k} \perp \mathbf{D}$ (from $\nabla \cdot \mathbf{D} = 0 \implies i\mathbf{k} \cdot \mathbf{D}_0 = 0$)
$\mathbf{k} \perp \mathbf{B}$ (from $\nabla \cdot \mathbf{B} = 0 \implies i\mathbf{k} \cdot \mathbf{B}_0 = 0$)
$\mathbf{B} \perp \mathbf{E}$ (from $\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \implies i\mathbf{k} \times \mathbf{E}_0 = i\omega \mathbf{B}_0$)
$\mathbf{B} \perp \mathbf{D}$ (from $\nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t} \implies i\mathbf{k} \times \mathbf{B}_0 = -i\omega \mathbf{D}_0$)
$\mathbf{B} \parallel \mathbf{H}$ (from assumption $\mathbf{B} = \mu_0 \mathbf{H}$)
$\mathbf{S} \perp \mathbf{E}$ (from $\mathbf{S} = \mathbf{E} \times \mathbf{H}$)
$\mathbf{S} \perp \mathbf{H}$, $\mathbf{S} \perp \mathbf{B}$ (from $\mathbf{S} = \mathbf{E} \times \mathbf{H}$ and $\mathbf{B} \parallel \mathbf{H}$)
Conclusion: $\mathbf{k} \perp \mathbf{D} \perp \mathbf{B}$ but not $\mathbf{k} \perp \mathbf{E} \perp \mathbf{H}$.
In particular: $\mathbf{E} \cdot \mathbf{D} \neq 0$, $\mathbf{E} \cdot \mathbf{k} \neq 0$ and $\mathbf{S} \cdot \mathbf{k} \neq 0$.

Equation for E in Anisotropic Materials

Assumptions as in Assumptions for Anisotropic Materials.
Ansatzes as in Ansatzes for Anisotropic Materials.
$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$ (Maxwell equation)
$\nabla \times \mathbf{B} = \mu_0 \frac{\partial \mathbf{D}}{\partial t}$ (from $\mathbf{B} = \mu_0 \mathbf{H}$)
$\mathbf{D} = \varepsilon_0 \epsilon \mathbf{E}$ (linear tensor relation)

$$\nabla \times (\nabla \times \mathbf{E}) = -\frac{\partial (\nabla \times \mathbf{B})}{\partial t} = -\mu_0 \varepsilon_0 \frac{\partial^2 (\epsilon \mathbf{E})}{\partial t^2} = -\frac{1}{c_0^2} \frac{\partial^2 (\epsilon \mathbf{E})}{\partial t^2}$$
$$= \frac{\omega^2}{c_0^2} \epsilon \mathbf{E} = k_0^2 \epsilon \mathbf{E} \qquad \text{(using } k_0 = \omega / c_0\text{)}$$
$\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}$ (general vector identity)
$\nabla \times (\nabla \times \mathbf{E}) = -(\mathbf{k} \cdot \mathbf{E})\mathbf{k} + k^2 \mathbf{E}$ (using identity)
$(\mathbf{k} \cdot \mathbf{E})\mathbf{k} = k^2 \mathbf{E} - k_0^2 \epsilon \mathbf{E}$ (**equation for E in AM**)

Refractive Index

Simplification: perform all analyses in ϵ's system of principal axes.
Let $\hat{\mathbf{e}}_x$, $\hat{\mathbf{e}}_y$, and $\hat{\mathbf{e}}_z$ align with ϵ's principal axes.

$$\epsilon = \begin{pmatrix} \varepsilon_{xx} & 0 & 0 \\ 0 & \varepsilon_{yy} & 0 \\ 0 & 0 & \varepsilon_{zz} \end{pmatrix} \qquad \text{(in } \epsilon\text{'s system of principal axes)}$$

Convention: choose axes so $\varepsilon_{xx} \leq \varepsilon_{yy} \leq \varepsilon_{zz}$.
$(\mathbf{k} \cdot \mathbf{E})\mathbf{k} = k^2 \mathbf{E} - k_0^2 \epsilon \mathbf{E}$ (in linear AMs in general)
$(\mathbf{k} \cdot \mathbf{E}) k_x = \left(k^2 - k_0^2 \varepsilon_{xx}\right) E_x$ (in system of PA)
$(\mathbf{k} \cdot \mathbf{E}) k_y = \left(k^2 - k_0^2 \varepsilon_{yy}\right) E_y$ (in system of PA)
$(\mathbf{k} \cdot \mathbf{E}) k_z = \left(k^2 - k_0^2 \varepsilon_{zz}\right) E_z$ (in system of PA)
$n = k / k_0$ (refractive index)
$(\mathbf{k} \cdot \mathbf{E}) \left(\sum_j \frac{k_j^2}{k_0^2 (n^2 - \varepsilon_{jj})}\right) = \mathbf{k} \cdot \mathbf{E}$ (after rearranging, adding)
$\sum_j \frac{k_j^2}{k_0^2 (n^2 - \varepsilon_{jj})} = 1$ (canceling $\mathbf{k} \cdot \mathbf{E}$)
$\sum_j \frac{s_j^2}{n^2 - \varepsilon_{jj}} = \frac{1}{n^2}$ (using $\hat{\mathbf{s}} \equiv \mathbf{k}/|\mathbf{k}|$)
This equation has two positive solutions for n, called principal indices of refraction.
Conclusion: Light in an anisotropic material experiences *two* refractive indices depending on ε_{jj} (property of material) and $\hat{\mathbf{s}}$ (direction of light in material).

Index Ellipsoid

$\langle u_{\mathrm{EM}} \rangle = \frac{1}{2} \mathbf{D}_0 \cdot \mathbf{E}_0$ (energy density in linear material, $\mu = 1$)
$= \frac{1}{2\varepsilon_0} \mathbf{D}_0 (\epsilon^{-1} \mathbf{D}_0)$ (in terms of ϵ)

$$\epsilon^{-1} = \begin{pmatrix} 1/\varepsilon_{xx} & 0 & 0 \\ 0 & 1/\varepsilon_{yy} & 0 \\ 0 & 0 & 1/\varepsilon_{zz} \end{pmatrix} \qquad \text{(in system of principal axes)}$$

$2\varepsilon_0 \langle u_{\mathrm{EM}} \rangle = \frac{D_x^2}{\varepsilon_{xx}} + \frac{D_y^2}{\varepsilon_{yy}} + \frac{D_z^2}{\varepsilon_{zz}}$ (in system of PA)
$\mathbf{r} \equiv \frac{\mathbf{D}}{\sqrt{2\varepsilon_0 \langle u_{\mathrm{EM}} \rangle}}$ (new dimensionless variable)
$\implies \frac{x}{\varepsilon_{xx}} + \frac{y}{\varepsilon_{yy}} + \frac{z}{\varepsilon_{zz}} = 1$ (**index ellipsoid**)

Using the Index Ellipsoid

The index ellipsoid is used to graphically determine the principal refractive indices n_1, n_2 and principal polarizations $\mathbf{D}_1$, $\mathbf{D}_2$ for light with wave vector $\mathbf{k}$ in an anisotropic material with dielectric tensor eigenvalues ε_{jj}.

1. Construct a crystal's index ellipsoid using known ε_{jj}.

2. In the (dimensionless) space of the index ellipsoid, draw the unit vector $\hat{\mathbf{s}} = \mathbf{k}/|\mathbf{k}|$ of an incident wave vector.

3. Draw the plane that is (i) perpendicular to $\hat{\mathbf{s}}$ and (ii) passes through the ellipsoid's center (origin).
   Identify the ellipse formed by the intersection of the index ellipsoid and the thus-constructed plane.

4. The lengths of the ellipse's semi-major and semi-minor axes are n_1 and n_2; the corresponding directions of the major and minor axes give the directions of $\mathbf{D}_1$ and $\mathbf{D}_2$.

Wave Vector Surface

$(\mathbf{k}\cdot\mathbf{E})\mathbf{k} = (k^2\mathsf{I} - k_0^2\epsilon)\mathbf{E}$

Separate into components and rearrange to get...

$$\begin{aligned}
(k_0^2\varepsilon_{xx} - k_y^2 - k_z^2)E_x + k_xk_yE_y + k_xk_zE_z &= 0\\
k_xk_yE_x + (k_0\varepsilon_{yy} - k_x^2 - k_z^2)E_y + k_yk_zE_z &= 0\\
k_xk_zE_x + k_zk_yE_y + (k_0\varepsilon_{zz} - k_x^2 - k_y^2)E_z &= 0
\end{aligned}$$

In matrix form, this reads...

$$\begin{pmatrix} (k_0^2\varepsilon_{xx} - k_y^2 - k_z^2) & k_xk_y & k_xk_z\\ k_yk_x & (k_0^2\varepsilon_{yy} - k_x^2 - k_z^2) & k_yk_z\\ k_zk_x & k_zk_y & (k_0^2\varepsilon_{zz} - k_x^2 - k_y^2)\end{pmatrix}\begin{pmatrix}E_x\\E_y\\E_z\end{pmatrix} = \mathbf{0}$$

Let $\mathbf{K}$ denote the above wave vector matrix.

For a non-trivial solution for $\mathbf{E}$, we require...

$$0 \equiv \det\mathbf{K} = k_0^4 - k_0^2\left(\frac{k_x^2+k_y^2}{\varepsilon_{zz}} + \frac{k_x^2+k_z^2}{\varepsilon_{yy}} + \frac{k_y^2+k_z^2}{\varepsilon_{xx}}\right) + \left(\frac{k_x^2}{\varepsilon_{yy}\varepsilon_{zz}} + \frac{k_y^2}{\varepsilon_{xx}\varepsilon_{zz}} + \frac{k_z^2}{\varepsilon_{xx}\varepsilon_{yy}}\right)(k_x^2+k_y^2+k_z^2) \equiv 0$$

Uniaxial Material

Choose $n_x = n_y = n_0$ and $n_z = n_e$.

$$\det\mathbf{K} = \left(\frac{k_x^2}{n_o^2} + \frac{k_y^2}{n_o^2} + \frac{k_z^2}{n_o^2} - k_0^2\right)\left(\frac{k_x^2}{n_e^2} + \frac{k_y^2}{n_e^2} + \frac{k_z^2}{n_o^2} - k_0^2\right) \equiv 0$$

Example: Wave Vectors in the xy Plane

Assume $\mathbf{k} = (k_x, k_y, 0)$.

$$\mathbf{K} = \begin{pmatrix} k_0^2\varepsilon_{xx} - k_y^2 & k_xk_y & 0\\ k_xk_y & k_0^2\varepsilon_{yy} - k_x^2 & 0\\ 0 & 0 & k_0^2\varepsilon_{zz} - k_x^2 - k_y^2\end{pmatrix}$$

$\det\mathbf{K} = (k_0^2\varepsilon_{zz} - k_x^2 - k_y^2)\big[(k_0^2\varepsilon_{xx} - k_y^2)(k_0^2\varepsilon_{yy} - k_x^2) - k_x^2k_y^2\big]$

Let $\kappa_\alpha \equiv k_\alpha/k_0$ for $\alpha \in \{x,y,z\}$. (normalized components)

$\kappa_x^2 + \kappa_y^2 = \varepsilon_{zz}$ (circular solution)

$\frac{\kappa_x^2}{\varepsilon_{yy}} + \frac{\kappa_y^2}{\varepsilon_{xx}} = 1$ (elliptical solution)

Ellipse is inside circle. (assuming $\varepsilon_{xx} < \varepsilon_{yy} < \varepsilon_{zz}$)

Example: Wave Vectors in the xz Plane

Assume $\mathbf{k} = (k_x, 0, k_z)$.

$$\mathbf{K} = \begin{pmatrix} k_0^2\varepsilon_{xx} - k_z^2 & 0 & k_xk_z\\ 0 & k_0^2\varepsilon_{yy} - k_x^2 - k_z^2 & 0\\ k_xk_z & 0 & k_0^2\varepsilon_{zz} - k_x^2\end{pmatrix}$$

$\det\mathbf{K} = (k_0^2\varepsilon_{yy} - k_x^2 - k_z^2)\big[(k_0^2\varepsilon_{xx} - k_z^2)(k_0^2\varepsilon_{zz} - k_x^2) - k_x^2k_z^2\big]$

Let $\kappa_\alpha \equiv k_\alpha/k_0$ for $\alpha \in \{x,y,z\}$. (normalized components)

$\kappa_x^2 + \kappa_z^2 = \varepsilon_{yy}$ (circular solution)

$\frac{\kappa_x^2}{\varepsilon_{zz}} + \frac{\kappa_z^2}{\varepsilon_{xx}} = 1$ (elliptical solution)

Circle and ellipse intersect. (assuming $\varepsilon_{xx} < \varepsilon_{yy} < \varepsilon_{zz}$)

Example: Wave Vectors in the yz Plane

Assume $\mathbf{k} = (0, k_y, k_z)$.

$$\mathbf{K} = \begin{pmatrix} k_0^2\varepsilon_{xx} - k_y^2 - k_z^2 & 0 & 0\\ 0 & k_0^2\varepsilon_{yy} - k_z^2 & k_yk_z\\ 0 & k_yk_z & k_0^2\varepsilon_{zz} - k_y^2\end{pmatrix}$$

$\det\mathbf{K} = (k_0^2\varepsilon_{xx} - k_y^2 - k_z^2)\big[(k_0^2\varepsilon_{yy} - k_z^2)(k_0^2\varepsilon_{zz} - k_y^2) - k_y^2k_z^2\big]$

Let $\kappa_\alpha \equiv k_\alpha/k_0$ for $\alpha \in \{x,y,z\}$ (normalized components)

$\kappa_y^2 + \kappa_z^2 = \varepsilon_{xx}$ (circular solution)

$\frac{\kappa_y^2}{\varepsilon_{zz}} + \frac{\kappa_z^2}{\varepsilon_{yy}} = 1$ (elliptical solution)

Circle is inside ellipse. (assuming $\varepsilon_{xx} < \varepsilon_{yy} < \varepsilon_{zz}$)

Circular Solution

Consider only solutions for $\mathbf{k}$ confined to a 2D plane.

$\mathbf{E}$ perpendicular to plane containing $\mathbf{k}$.

$\mathbf{E}$'s direction is independent of light direction $\hat{\mathbf{s}}$.

Elliptical Solution

Consider only solutions for $\mathbf{k}$ confined to a 2D plane.

$\mathbf{E}$ lies in the plane containing $\mathbf{k}$.

$\mathbf{E}$'s direction depends on light direction $\hat{\mathbf{s}}$.

Optic Axis

Consider $\mathbf{k}$ confined to the xz plane. Physically: recall $\hat{\mathbf{e}}_x$ corresponds to crystal's smallest eigenvalue ε_{xx} and $\hat{\mathbf{e}}_z$ corresponds to crystal's largest eigenvalue ε_{zz}.

Definition: for light with direction $\hat{\mathbf{s}}$ parallel to the optic axis, the light's two principle polarizations (as determined by ε_{jj} and $\hat{\mathbf{s}}$) experience the same refractive index.

Result: both incident polarizations travel through the anisotropic material with equal phase velocity, so the light's incident polarization is preserved in the material.

In general anisotropic materials have two optic axes inclined at equal and opposite polar angles $\pm\theta_{\mathrm{oa}}$ relative to $\hat{\mathbf{e}}_z$.

Direction of Optic Axis

Consider $\mathbf{k}$ confined to the xz plane.

$k_x^2 + k_z^2 = k_0^2\varepsilon_{yy}$ (circular solution)

$k_x^2 + k_z^2 = |\mathbf{k}|^2 \implies k^2 = \varepsilon_{yy}k_0^2$ (for $\mathbf{k}$ in xz plane)

Compare to general relationship $k^2 = nk_0^2$...

$n_1 = \varepsilon_{yy}$ (1st principle refractive index)

$\frac{k_x^2}{\varepsilon_{zz}} + \frac{k_z^2}{\varepsilon_{xx}} = k_0^2$ (elliptical solution)

$\frac{(n_2s_xk_0)^2}{\varepsilon_{zz}} + \frac{(n_2s_zk_0)^2}{\varepsilon_{xx}} = k_0^2$ (using $\mathbf{k} = nk_0\,\hat{\mathbf{s}}$)

$\frac{1}{n_2} = \sqrt{\frac{s_x^2}{\varepsilon_{zz}} + \frac{s_z^2}{\varepsilon_{xx}}}$ (2nd principle refractive index)

Let θ denote angle between $\hat{\mathbf{s}}$ and z axis.

$\hat{\mathbf{s}} = (s_x, 0, s_z) = (\sin\theta, 0, \cos\theta)$ (in terms of θ)

$\frac{1}{n_2} = \sqrt{\frac{\sin^2\theta}{\varepsilon_{zz}} + \frac{\cos^2\theta}{\varepsilon_{xx}}}$ (in terms of θ)

$n_1 \equiv n_2 \implies \frac{1}{n_1^2} = \frac{1}{n_2^2}$ (along optic axis, by definition)

$\implies \frac{1}{\varepsilon_{yy}} = \frac{\sin^2\theta_{\mathrm{oa}}}{\varepsilon_{zz}} + \frac{\cos^2\theta_{\mathrm{oa}}}{\varepsilon_{xx}}$ (implicit equation for θ_{oa})

$\sin^2\theta_{\mathrm{oa}} = \frac{1}{\varepsilon_{zz} - \varepsilon_{xx}}\left(\varepsilon_{zz} - \frac{\varepsilon_{xx}\varepsilon_{zz}}{\varepsilon_{yy}}\right)$ (using $\cos^2 x = 1 - \sin^2 x$)

Direction of E Field for k in xy Plane

Assume $\mathbf{k} = (k_x, k_y, 0)$.

$$\begin{pmatrix} k_0^2\varepsilon_{xx} - k_y^2 & k_xk_y & 0\\ k_xk_y & k_0^2\varepsilon_{yy} - k_x^2 & 0\\ 0 & 0 & k_0^2\varepsilon_{zz} - k_x^2 - k_y^2\end{pmatrix}\begin{pmatrix}E_x\\E_y\\E_z\end{pmatrix} = \mathbf{0}$$

$E_z(k_0^2\varepsilon_{zz} - k_x^2 - k_y^2) = 0$ (from z component)

$k_x^2 + k_y^2 = k_0^2\varepsilon_{zz}$ (circular solution)

The z component equation $E_z(k_0^2\varepsilon_{zz} - k_x^2 - k_y^2) = 0$ is satisfied for arbitrary E_z for the circular solution, while $E_x, E_y \equiv 0$ to satisfy $\mathbf{KE} = \mathbf{0}$ for arbitrary $\mathbf{k}$.

$\mathbf{E}_1 \parallel \hat{\mathbf{e}}_z$ (first polarization; circular solution)

$\frac{E_y}{E_x} = \frac{1}{k_xk_y}\left(k_y^2 - k_0^2\varepsilon_{xx}\right)$ (from x component)

$\frac{E_x}{E_y} = \frac{1}{k_xk_y}\left(k_x^2 - k_0^2\varepsilon_{yy}\right)$ (from y component)

$\frac{k_x^2}{\varepsilon_{yy}} + \frac{k_y^2}{\varepsilon_{xx}} = k_0^2$ (elliptical solution)

$\frac{E_y}{E_x} = -\frac{\varepsilon_{xx}}{\varepsilon_{yy}}\frac{s_x}{s_y}$ (combining x/y comp. and ellip. solution)

The x and y component equations are satisfied for E_x and E_y satisfying $\frac{E_y}{E_x} = -\frac{\varepsilon_{xx}}{\varepsilon_{yy}}\frac{s_x}{s_y}$ while $E_z = 0$ to satisfy $\mathbf{KE} = \mathbf{0}$ for arbitrary $\mathbf{k}$.

$\mathbf{E}_2 = E_{2_x}\,\hat{\mathbf{e}}_x + E_{2_y}\,\hat{\mathbf{e}}_y$ (second polarization; elliptical solution)

Direction of E Field for k in xz Plane

Assume $\mathbf{k} = (k_x, 0, k_z)$.

$$\begin{pmatrix} k_0^2\varepsilon_{xx} - k_z^2 & 0 & k_xk_z\\ 0 & k_0^2\varepsilon_{yy} - k_x^2 - k_z^2 & 0\\ k_xk_z & 0 & k_0^2\varepsilon_{zz} - k_x^2\end{pmatrix}\begin{pmatrix}E_x\\E_y\\E_z\end{pmatrix} = \mathbf{0}$$

$E_y(k_0^2\varepsilon_{yy} - k_x^2 - k_z^2) = 0$ (from y component)

$k_x^2 + k_z^2 = k_0^2\varepsilon_{yy}$ (circular solution)

The y component equation $E_y(k_0^2\varepsilon_{yy} - k_x^2 - k_z^2) = 0$ is satisfied for arbitrary E_y for the circular solution, while $E_x, E_z \equiv 0$ to satisfy $\mathbf{KE} = \mathbf{0}$ for arbitrary $\mathbf{k}$.

$\mathbf{E}_1 \parallel \hat{\mathbf{e}}_y$ (first polarization; circular solution)

$\frac{E_z}{E_x} = \frac{1}{k_x k_z}\left(k_z^2 - k_0^2\varepsilon_{xx}\right)$ (from x component)

$\frac{E_x}{E_z} = \frac{1}{k_x k_z}\left(k_x^2 - k_0^2\varepsilon_{zz}\right)$ (from z component)

$\frac{k_x^2}{\varepsilon_{zz}} + \frac{k_z^2}{\varepsilon_{xx}} = k_0^2$ (elliptical solution)

$\frac{E_z}{E_x} = -\frac{\varepsilon_{xx}}{\varepsilon_{zz}}\frac{s_x}{s_z}$ (combining x/z comp. and ellip. solution)

The x and z component equations are satisfied for E_x and E_z satisfying $\frac{E_z}{E_x} = -\frac{\varepsilon_{xx}}{\varepsilon_{zz}}\frac{s_x}{s_z}$ while $E_y = 0$ to satisfy $\mathbf{KE} = \mathbf{0}$ for arbitrary $\mathbf{k}$.

$\mathbf{E}_2 = E_{2_x}\,\hat{\mathbf{e}}_x + E_{2_z}\,\hat{\mathbf{e}}_z$ (second polarization; elliptical solution)

Direction of E Field for k in yz Plane

Assume $\mathbf{k} = (0, k_x, k_z)$.

$$\begin{pmatrix} k_0^2\varepsilon_{xx} - k_y^2 - k_z^2 & 0 & 0 \\ 0 & k_0^2\varepsilon_{yy} - k_z^2 & k_y k_z \\ 0 & k_y k_z & k_0^2\varepsilon_{zz} - k_y^2 \end{pmatrix}\begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} = \mathbf{0}$$

$E_x(k_0^2\varepsilon_{xx} - k_y^2 - k_z^2) = 0$ (from x component)

$k_y^2 + k_z^2 = k_0^2\varepsilon_{xx}$ (circular solution)

The x component equation $E_x(k_0^2\varepsilon_{xx} - k_y^2 - k_z^2) = 0$ is satisfied for arbitrary E_x for the circular solution, while $E_y, E_z \equiv 0$ to satisfy $\mathbf{KE} = \mathbf{0}$ for arbitrary $\mathbf{k}$.

$\mathbf{E}_1 \parallel \hat{\mathbf{e}}_x$ (first polarization; circular solution)

$\frac{E_z}{E_y} = \frac{1}{k_y k_z}\left(k_z^2 - k_0^2\varepsilon_{yy}\right)$ (from y component)

$\frac{E_y}{E_z} = \frac{1}{k_y k_z}\left(k_y^2 - k_0^2\varepsilon_{zz}\right)$ (from z component)

$\frac{k_y^2}{\varepsilon_{zz}} + \frac{k_z^2}{\varepsilon_{yy}} = k_0^2$ (elliptical solution)

$\frac{E_z}{E_y} = -\frac{\varepsilon_{yy}}{\varepsilon_{zz}}\frac{s_y}{s_z}$ (combining y/z comp. and ellip. solution)

The y and z component equations are satisfied for E_y and E_z satisfying $\frac{E_z}{E_y} = -\frac{\varepsilon_{yy}}{\varepsilon_{zz}}\frac{s_y}{s_z}$ while $E_x = 0$ to satisfy $\mathbf{KE} = \mathbf{0}$ for arbitrary $\mathbf{k}$.

$\mathbf{E}_2 = E_{2_y}\,\hat{\mathbf{e}}_y + E_{2_z}\,\hat{\mathbf{e}}_z$ (second polarization; elliptical solution)

Angle Between E and D for k in xy Plane

Assume $\mathbf{k} = (k_x, k_y, 0)$.

$\mathbf{E}_1 \parallel \mathbf{D}_1$ (for circular solution)

Let θ be angle between $\hat{\mathbf{s}} = \mathbf{k}/|\mathbf{k}|$ and $\hat{\mathbf{e}}_y$.

$\hat{\mathbf{s}} = (\sin\theta, \cos\theta, 0)$ (expression for $\hat{\mathbf{s}}$)

$\mathbf{D}_2/|\mathbf{D}_2| = (-\cos\theta, 0, \sin\theta)$ (because $\mathbf{k} \perp \mathbf{D}$)

$\mathbf{E}_2 = E_{2_x}\,\hat{\mathbf{e}}_x + E_{2_y}\,\hat{\mathbf{e}}_y$ (elliptical solution)

$\frac{E_{2_y}}{E_{2_x}} = -\frac{\varepsilon_{xx}}{\varepsilon_{yy}}\frac{s_x}{s_y} = -\frac{\varepsilon_{xx}}{\varepsilon_{yy}}\frac{\sin\theta}{\cos\theta}$ (elliptical solution)

$\mathbf{E}_2 = \left(E_{2_x}, -\frac{\varepsilon_{xx}}{\varepsilon_{yy}}\frac{\sin\theta}{\cos\theta}E_{2_x}, 0\right)$

$\mathbf{E}_2/|\mathbf{E}_2| = \frac{1}{\sqrt{1+\frac{\varepsilon_{xx}^2}{\varepsilon_{yy}^2}\frac{\sin^2\theta}{\cos^2\theta}}}\left(1, 0, -\frac{\varepsilon_{xx}}{\varepsilon_{yy}}\frac{\sin\theta}{\cos\theta}\right)$

$\cos\gamma = \frac{\mathbf{E}\cdot\mathbf{D}}{|E||D|} = \frac{\varepsilon_{yy}\cos^2\theta + \varepsilon_{xx}\sin^2\theta}{\sqrt{\varepsilon_{yy}^2\cos^2\theta + \varepsilon_{xx}^2\sin^2\theta}}$ (angle btwn. $\mathbf{E}$ and $\mathbf{D}$)

Angle Between E and D for k in xz Plane

Assume $\mathbf{k} = (k_x, 0, k_z)$.

$\mathbf{E}_1 \parallel \mathbf{D}_1$ (for circular solution)

Let θ be angle between $\hat{\mathbf{s}} = \mathbf{k}/|\mathbf{k}|$ and $\hat{\mathbf{e}}_z$.

$\hat{\mathbf{s}} = (\sin\theta, 0, \cos\theta)$ (expression for $\hat{\mathbf{s}}$)

$\mathbf{D}_2/|\mathbf{D}_2| = (-\cos\theta, 0, \sin\theta)$ (because $\mathbf{k} \perp \mathbf{D}$)

$\mathbf{E}_2 = E_{2_x}\,\hat{\mathbf{e}}_x + E_{2_z}\,\hat{\mathbf{e}}_z$ (elliptical solution)

$\frac{E_{2_z}}{E_{2_x}} = -\frac{\varepsilon_{xx}}{\varepsilon_{zz}}\frac{s_x}{s_z} = -\frac{\varepsilon_{xx}}{\varepsilon_{zz}}\frac{\sin\theta}{\cos\theta}$ (elliptical solution)

$\mathbf{E}_2 = \left(E_{2_x}, 0, -\frac{\varepsilon_{xx}}{\varepsilon_{zz}}\frac{\sin\theta}{\cos\theta}E_{2_x}\right)$

$\mathbf{E}_2/|\mathbf{E}_2| = \frac{1}{\sqrt{1+\frac{\varepsilon_{xx}^2}{\varepsilon_{zz}^2}\frac{\sin^2\theta}{\cos^2\theta}}}\left(1, 0, -\frac{\varepsilon_{xx}}{\varepsilon_{zz}}\frac{\sin\theta}{\cos\theta}\right)$

$\cos\gamma = \frac{\mathbf{E}\cdot\mathbf{D}}{|E||D|} = \frac{\varepsilon_{zz}\cos^2\theta + \varepsilon_{xx}\sin^2\theta}{\sqrt{\varepsilon_{zz}^2\cos^2\theta + \varepsilon_{xx}^2\sin^2\theta}}$ (angle btwn. $\mathbf{E}$ and $\mathbf{D}$)

Angle Between E and D for k in yz Plane

Assume $\mathbf{k} = (0, k_y, k_z)$.

$\mathbf{E}_1 \parallel \mathbf{D}_1$ (for circular solution)

Let θ be angle between $\hat{\mathbf{s}} = \mathbf{k}/|\mathbf{k}|$ and $\hat{\mathbf{e}}_z$.

$\hat{\mathbf{s}} = (0, \sin\theta, \cos\theta)$ (expression for $\hat{\mathbf{s}}$)

$\mathbf{D}_2/|\mathbf{D}_2| = (0, -\cos\theta, \sin\theta)$ (because $\mathbf{k} \perp \mathbf{D}$)

$\mathbf{E}_2 = E_{2_y}\,\hat{\mathbf{e}}_y + E_{2_z}\,\hat{\mathbf{e}}_z$ (elliptical solution)

$\frac{E_{2_z}}{E_{2_y}} = -\frac{\varepsilon_{yy}}{\varepsilon_{zz}}\frac{s_y}{s_z} = -\frac{\varepsilon_{yy}}{\varepsilon_{zz}}\frac{\sin\theta}{\cos\theta}$ (elliptical solution)

$\mathbf{E}_2 = \left(E_{2_y}, 0, -\frac{\varepsilon_{yy}}{\varepsilon_{zz}}\frac{\sin\theta}{\cos\theta}E_{2_y}\right)$

$\mathbf{E}_2/|\mathbf{E}_2| = \frac{1}{\sqrt{1+\frac{\varepsilon_{yy}^2}{\varepsilon_{zz}^2}\frac{\sin^2\theta}{\cos^2\theta}}}\left(1, 0, -\frac{\varepsilon_{yy}}{\varepsilon_{zz}}\frac{\sin\theta}{\cos\theta}\right)$

$\cos\gamma = \frac{\mathbf{E}\cdot\mathbf{D}}{|E||D|} = \frac{\varepsilon_{zz}\cos^2\theta + \varepsilon_{yy}\sin^2\theta}{\sqrt{\varepsilon_{zz}^2\cos^2\theta + \varepsilon_{yy}^2\sin^2\theta}}$ (angle btwn. $\mathbf{E}$ and $\mathbf{D}$)

Optically Uniaxial Materials

Uniaxial materials have two equal dielectric tensor eigenvalues.

$n_x = n_y \equiv n_\mathrm{o}$ and $n_z \equiv n_\mathrm{e}$. (in uniaxial materials)

n_o is called *ordinary refractive index.*

n_e is called *extraordinary refractive index.*

Optic Axis

Let θ_oa denote angle between optic axis and z axis.

$\sin^2\theta_\mathrm{oa} = \frac{1}{\varepsilon_{zz}-\varepsilon_{xx}}\left(\varepsilon_{zz} - \frac{\varepsilon_{xx}\varepsilon_{zz}}{\varepsilon_{yy}}\right)$ (in general)

$\theta_\mathrm{oa} = 0$ (in uniaxial materials since $\varepsilon_{xx} = \varepsilon_{yy}$)

Conclusion: in uniaxial materials, both optic axes join into a single optic axis parallel to the z axis.

$\hat{\mathbf{e}}_\mathrm{oa} \parallel \hat{\mathbf{e}}_z$ (in uniaxial materials)

Wave Vector Surface

$(\mathbf{k}\cdot\mathbf{E})\mathbf{k} = (k^2\mathsf{I} - k_0^2\epsilon)\mathbf{E}$ (in general in anisotropic materials)

In matrix form in ϵ's system of principal axes, this reads...

$$\begin{pmatrix} (k_0^2\varepsilon_{xx} - k_y^2 - k_z^2) & k_x k_y & k_x k_z \\ k_y k_x & (k_0^2\varepsilon_{yy} - k_x^2 - k_z^2) & k_y k_z \\ k_z k_x & k_z k_y & (k_0^2\varepsilon_{zz} - k_x^2 - k_y^2) \end{pmatrix}\begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} = \mathbf{0}$$

Let $\mathbf{K}$ denote the above wave vector matrix.

For a non-trivial solution for $\mathbf{E}$, we require...

$$0 \equiv \det\mathbf{K} = k_0^4 - k_0^2\left(\frac{k_x^2+k_y^2}{\varepsilon_{zz}} + \frac{k_x^2+k_z^2}{\varepsilon_{yy}} + \frac{k_y^2+k_z^2}{\varepsilon_{xx}}\right) + \left(\frac{k_x^2}{\varepsilon_{yy}\varepsilon_{zz}} + \frac{k_y^2}{\varepsilon_{xx}\varepsilon_{zz}} + \frac{k_z^2}{\varepsilon_{xx}\varepsilon_{yy}}\right)(k_x^2 + k_y^2 + k_z^2) \equiv 0 \quad \text{(in general)}$$

In uniaxial materials, this simplifies to...

$\det\mathbf{K} = \left(\frac{k_x^2}{n_\mathrm{o}^2} + \frac{k_y^2}{n_\mathrm{o}^2} + \frac{k_z^2}{n_\mathrm{o}^2} - k_0^2\right)\left(\frac{k_x^2}{n_\mathrm{e}^2} + \frac{k_y^2}{n_\mathrm{e}^2} + \frac{k_z^2}{n_\mathrm{o}^2} - k_0^2\right) \equiv 0$

Without loss of generality (because of rotational symmetry about $\hat{\mathbf{e}}_z$) we may resolve an arbitrary $\mathbf{k}$ into a component $k_\parallel$ parallel to $\hat{\mathbf{e}}_z$ and a component $k_\perp$ perpendicular to $\hat{\mathbf{e}}_z$.

Let $k_x \equiv k_\perp$ and $k_z \equiv k_\parallel$.

$\det\mathbf{K} = \left(\frac{k_\perp^2}{n_\mathrm{o}^2} + \frac{k_\parallel^2}{n_\mathrm{o}^2} - k_0^2\right)\left(\frac{k_\perp^2}{n_\mathrm{e}^2} + \frac{k_\parallel^2}{n_\mathrm{o}^2} - k_0^2\right)$ ($\mathbf{k} = (k_\perp, 0, k_\parallel)$)

Ordinary Polarization

$k_\perp^2 + k_\parallel^2 = n_\mathrm{o}^2 k_0^2$ (solution to $\det\mathbf{K} = 0$)

Use $k^2 = k_\perp^2 + k_\parallel^2$ and compare to $k = n_1 k_0$ to get...

$n_1 = n_\mathrm{o}$ (ordinary refractive index)

$\mathbf{D}_1 \parallel \mathbf{E}_1$ (for ordinary polarization)

$\mathbf{S}_1 \parallel \mathbf{k}$ (for ordinary polarization)

n_1 and direction of $\mathbf{E}_1$ are independent of $\hat{\mathbf{s}}$.

Extraordinary Polarization

$\frac{k_\perp^2}{n_\mathrm{e}^2} + \frac{k_\parallel^2}{n_\mathrm{o}^2} = k_0^2$ (solution to $\det\mathbf{K} = 0$)

$\mathbf{k} = (k_\perp, 0, k_\parallel)$ (in general uniaxial materials)

$\hat{\mathbf{s}} = \mathbf{k}/|\mathbf{k}| = (s_\perp, 0, s_\parallel)$ (in general uniaxial materials)

Let θ denote angle between $\hat{\mathbf{e}}_z$ and $\hat{\mathbf{s}}$.

$\hat{\mathbf{s}} = (\sin\theta, 0, \cos\theta)$

$\frac{n_2^2\sin^2\theta}{n_\mathrm{e}^2} + \frac{n_2^2\cos^2\theta}{n_\mathrm{o}^2} = 1$ (using $k = n_2 k_0$)

$\frac{1}{n_2^2} = \frac{\sin^2\theta}{n_\mathrm{e}^2} + \frac{\cos^2\theta}{n_\mathrm{o}^2}$ (extraordinary refractive index)

$\mathbf{S}_2 \nparallel \mathbf{k}$; $\angle(\mathbf{S}_2, \mathbf{k}) = \angle(\mathbf{E}_2, \mathbf{D}_2)$

Angle Between E and D

Assume $\mathbf{k} = (k_\perp, 0, k_\parallel)$

$\mathbf{E}_1 \parallel \mathbf{D}_1$ (for ordinary polarization)

Let θ be angle between $\hat{\mathbf{s}} = \mathbf{k}/|\mathbf{k}|$ and $\hat{\mathbf{e}}_z$.

$\hat{\mathbf{s}} = (\sin\theta, 0, \cos\theta)$

$\mathbf{D}_2/|\mathbf{D}_2| = (-\cos\theta, 0, \sin\theta)$ (because $\mathbf{k} \perp \mathbf{D}$)

$\mathbf{E}_2 = E_2^\perp\,\hat{\mathbf{e}}_\perp + E_2^\parallel\,\hat{\mathbf{e}}_\parallel$ (extraordinary polarization)

$\frac{E_2^\parallel}{E_2^\perp} = -\frac{\varepsilon_\mathrm{o}}{\varepsilon_\mathrm{e}}\frac{s_\perp}{s_\parallel} = -\frac{\varepsilon_\mathrm{o}}{\varepsilon_\mathrm{e}}\frac{\sin\theta}{\cos\theta}$ (extraordinary polarization)

$\mathbf{E}_2 = \left(E_2^\perp, 0, -\frac{\varepsilon_\mathrm{o}}{\varepsilon_\mathrm{e}}\frac{\sin\theta}{\cos\theta}E_2^\perp\right)$

$\mathbf{E}_2/|\mathbf{E}_2| = \frac{1}{\sqrt{1+\frac{\varepsilon_\mathrm{o}^2}{\varepsilon_\mathrm{e}^2}\frac{\sin^2\theta}{\cos^2\theta}}}\left(1, 0, -\frac{\varepsilon_\mathrm{o}}{\varepsilon_\mathrm{e}}\frac{\sin\theta}{\cos\theta}\right)$

$\cos\gamma = \frac{\mathbf{E}\cdot\mathbf{D}}{|E||D|} = \frac{\varepsilon_\mathrm{e}\cos^2\theta+\varepsilon_\mathrm{o}\sin^2\theta}{\sqrt{\varepsilon_\mathrm{e}^2\cos^2\theta+\varepsilon_\mathrm{o}^2\sin^2\theta}}$ (angle btwn. **E** and **D**)

From Isotropic into Uniaxial Material

Consider EM plane waves incident at an angle θ_i from isotropic material with refractive index n_0 into uniaxial material with refractive indices n_o and n_e.

Review from Applying Boundary Conditions

$E_\mathrm{i}^\parallel + E_\mathrm{r}^\parallel = E_\mathrm{t}^\parallel$ for all $\mathbf{r} = (x, y, 0)$ in interface and for all t

$E_{\mathrm{i}_0}^\parallel e^{i\phi_\mathrm{i}} + E_{\mathrm{r}_0}^\parallel e^{i\phi_\mathrm{r}} = E_{\mathrm{t}_0}^\parallel e^{i\phi_\mathrm{t}}$ (for $x = y = z = 0$ and $t = 0$)

$\implies \phi_\mathrm{i} = \phi_\mathrm{r} = \phi_\mathrm{t} \equiv \phi$ (phases are equal)

$E_{\mathrm{i}_0}^\parallel e^{-i\omega_\mathrm{i}t}e^{i\phi} + E_{\mathrm{r}_0}^\parallel e^{-i\omega_\mathrm{r}t}e^{i\phi} = E_{\mathrm{t}_0}^\parallel e^{-i\omega_\mathrm{t}t}e^{i\phi}$ (for $\mathbf{r} = \mathbf{0}$)

$\implies \omega_\mathrm{i} = \omega_\mathrm{r} = \omega_\mathrm{t} \equiv \omega$ (frequencies are equal)

$E_{\mathrm{i}_0}^\parallel e^{i\mathbf{k}_\mathrm{i}\cdot\mathbf{r}}e^{i\phi} + E_{\mathrm{r}_0}^\parallel e^{i\mathbf{k}_\mathrm{r}\cdot\mathbf{r}}e^{i\phi} = E_{\mathrm{t}_0}^\parallel e^{i\mathbf{k}_\mathrm{t}\cdot\mathbf{r}}e^{i\phi}$ (for $t = 0$)

$\implies \mathbf{k}_\mathrm{i}\cdot\mathbf{r} = \mathbf{k}_\mathrm{r}\cdot\mathbf{r} = \mathbf{k}_\mathrm{t}\cdot\mathbf{r} = \text{constant}$

Geometrically: $\mathbf{k}_\mathrm{i}$, $\mathbf{k}_\mathrm{r}$ and $\mathbf{k}_\mathrm{t}$ lie in the same *plane of incidence*.

Convention: plane of incidence is xz plane for interface in xy plane.

Geometry

Let interface lie in xy plane.

Let plane of incidence lie in xz plane.

Let z axis point from isotropic to uniaxial material.

θ_i is angle of incidence.

θ_r is angle of reflection.

θ_t is angle of transmission.

All angles measured with respect to interface normal vector $\hat{\mathbf{n}}$.

$\mathbf{k}_\mathrm{i} = k_0 n_0(\sin\theta_\mathrm{i}, 0, \cos\theta_\mathrm{i})$ (incident wave vector)

$\mathbf{k}_{\mathrm{r}_{1,2}} = k_0 n_{1,2}(\sin\theta_\mathrm{r}, 0, -\cos\theta_\mathrm{r})$ (reflected wave vector)

$\mathbf{k}_{\mathrm{t}_{1,2}} = k_0 n_{1,2}(\sin\theta_{\mathrm{t}_{1,2}}, 0, \cos\theta_{\mathrm{t}_{1,2}})$ (transmitted wave vector)

Substitute $\mathbf{k}_\mathrm{i}$, $\mathbf{k}_\mathrm{t}$ into $\mathbf{k}_\mathrm{i}\cdot\mathbf{r} = \mathbf{k}_\mathrm{t}\cdot\mathbf{r}$; apply $\mathbf{r} = (x, y, 0)$

$\implies n_0\sin\theta_\mathrm{i} = n_1\sin\theta_{\mathrm{t}_1}$ (for ordinary ray)

$\implies n_0\sin\theta_\mathrm{i} = n_2(\theta_{\mathrm{t}_2})\sin\theta_{\mathrm{t}_2}$ (for extraordinary ray)

$n_2 = n_2(\theta_{\mathrm{t}_2})$ depends on direction of the uniaxial material's optic axis, the values of n_o and n_e, and on the direction of light in the material, given by the transmission angle θ_{t_2}.

Optic Axis Tangent to Boundary

Light is incident at θ_i from isotropic onto uniaxial material. Optic axis is tangent to the boundary plane.

Ordinary Polarization

$n_1 = n_\mathrm{o}$ (for ordinary polarization in general)

$n_0\sin\theta_\mathrm{i} = n_\mathrm{o}\sin\theta_{\mathrm{t}_1}$ (using $n_1 = n_\mathrm{o}$)

$\vartheta_1 = \pi/2 - \theta_{\mathrm{t}_1}$ (angle between ordinary ray and optic axis)

Extra Ordinary Polarization

θ_{t_2} is angle between EO ray and boundary normal

$\vartheta_2 = \pi/2 - \theta_{\mathrm{t}_2}$ (angle between EO ray and optic axis)

$\frac{1}{n_2^2} = \frac{\sin^2\vartheta_2}{n_\mathrm{e}^2} + \frac{\cos^2\vartheta_2}{n_\mathrm{o}^2}$ (EO refractive index)

$= \frac{\cos^2\theta_{\mathrm{t}_2}}{n_\mathrm{e}^2} + \frac{\sin^2\theta_{\mathrm{t}_2}}{n_\mathrm{o}^2}$ (using $\vartheta_2 = \pi/2 - \theta_{\mathrm{t}_2}$)

$\sin\theta_{\mathrm{t}_2} = \frac{n_0}{n_2}\sin\theta_\mathrm{i}$ (transmitted direction of EO ray)

$= \frac{n_0\sin\theta_\mathrm{i}}{\sqrt{n_\mathrm{e}^2+\left(1-\frac{n_\mathrm{e}^2}{n_\mathrm{o}^2}\right)n_0^2\sin^2\theta_\mathrm{i}}}$

Optic Axis Tangent to Boundary; Normal Incidence

Consider plane EM waves with wave vector $\mathbf{k}_0$ normally incident from isotropic material with refractive index n_0 on a uniaxial material of thickness L.

$\theta_\mathrm{i} = 0$ (normal incidence)

$\theta_{\mathrm{t}_1} = \theta_{\mathrm{t}_2} = 0$ (from $n_0\sin\theta_\mathrm{i} = n_{1,2}\sin\theta_{\mathrm{t}_{1,2}}$)

$\mathbf{k}_{\mathrm{t}_1} \parallel \mathbf{k}_{\mathrm{t}_2} \parallel \mathbf{k}_0$ (because $\theta_{\mathrm{t}_1} = \theta_{\mathrm{t}_2} = 0$)

$\mathbf{S}_1 \parallel \mathbf{S}_2 \parallel \mathbf{k}_{\mathrm{t}_{0,1,2}}$ (because optic axis tangent to boundary)

$n_1 = n_\mathrm{o}$ (for general ordinary polarization)

$n_2 = n_\mathrm{e}$ (only because $\theta_\mathrm{i} = 0$)

$n_1 \neq n_2 \implies v_1 \neq v_2$ (O and EO ray have different speeds)

$\phi_1 = n_1 k_0 L$ (phase accumulated by O ray in crystal)

$\phi_2 = n_2 k_0 L$ (phase accumulated by EO ray in crystal)

$\Delta\Phi = \phi_2 - \phi_1 = k_0 L(n_2 - n_1)$ (phase difference btwn. rays)

$= k_0 L(n_\mathrm{e} - n_\mathrm{o})$

Arbitrary Optic Axis; Normal Incidence

Consider plane EM waves with wave vector $\mathbf{k}_0$ normally incident from isotropic material with refractive index n_0 on a uniaxial material of thickness L.

$\theta_\mathrm{i} = 0$ (normal incidence)

$\theta_{\mathrm{t}_1} = \theta_{\mathrm{t}_2} = 0$ (from $n_0\sin\theta_\mathrm{i} = n_{1,2}\sin\theta_{\mathrm{t}_{1,2}}$)

$\mathbf{k}_{\mathrm{t}_1} \parallel \mathbf{k}_{\mathrm{t}_2} \parallel \mathbf{k}_0$ (because $\theta_{\mathrm{t}_1} = \theta_{\mathrm{t}_2} = 0$)

$\mathbf{S}_1 \parallel \mathbf{k}_{\mathrm{t}_1}$ (in general for ordinary polarization)

$\mathbf{S}_2 \nparallel \mathbf{k}_{\mathrm{t}_2}$ (because optic axis is angled relative to boundary)

Introduction to Lasers

Maxwell-Boltzmann Statistics
Assumptions: particles are non-interacting, in thermal equilibrium, and quantum effects are negligible.
We will use these assumptions to describe sparse gases.
Let $\beta \equiv 1/k_\mathrm{B}T$.
The average number $\langle N_i \rangle$ of particles with energy E_i in a system of N_tot particles with partition function Z is...
$\langle N_i \rangle = g_i \frac{N_\mathrm{tot}}{Z} e^{-\beta E_i}$ (Maxwell-Boltzmann statistics)
g_i is degeneracy of i-th energy level.
$N_\mathrm{tot} = \sum_i N_i$ (total number of particles in system)
$Z = \sum_i g_i e^{-\beta E_i}$ (partition function)
$\frac{\langle N_j \rangle}{\langle N_i \rangle} = \frac{g_j}{g_i} e^{-\beta(E_j - E_i)}$ (occupation ratio at different energies)
The average number $\langle N_\alpha \rangle$ of particles in the (potentially degenerate) α-th state is...
$\langle N_\alpha \rangle = \frac{N_\mathrm{tot}}{Z} e^{-\beta E_\alpha}$ (average occupation of α-th state)

Bose-Einstein Statistics
Assumptions: particles are non-interacting, indistinguishable bosons. Quantum effects are permitted.
We will use these assumptions to describe photons.
The average number $\langle N_i \rangle$ of particles with energy E_i in a system with chemical potential μ is...
$\langle N_i \rangle = \frac{g_i}{e^{\beta(E_i - \mu)}}$
g_i is degeneracy of i-th energy level.

Black-Body Cavity
Consider gas in a black-body cavity in thermal equilibrium at temperature T and admitting discrete quantum energy levels $E_0, E_1, \ldots$
The energy density per unit frequency w of black-body radiation emitted by the cavity walls is given by Planck's law:
$w(\omega)\,\mathrm{d}\omega = \frac{\hbar\omega^3}{\pi^2 c_0^3} \frac{\mathrm{d}\omega}{e^{\beta\hbar\omega} - 1}$ (energy density of BB radiation)
BB radiation interacts with gas atoms in cavity via:

1. spontaneous emission,
2. absorption, and
3. stimulated emission.

Spontaneous Emission
Consider a system with energy levels E_1 and $E_2 > E_1$.
$E_2 \to E_1 + \hbar\omega_{21}$ (spontaneous emission)
$\hbar\omega_{21} = E_2 - E_1$
$\frac{\mathrm{d}N_\mathrm{sp}}{\mathrm{d}t} = A_{21} N_2$ (rate of spontaneous emission)
N_2 is total number of atoms in cavity in state $|2\rangle$.
$A_{21} = \frac{P_\mathrm{dipole}}{\hbar\omega_{21}} = \frac{1}{\hbar\omega_{21}} \frac{\omega_{21}^4 e_0^2 (\langle 2|\mathbf{r}|1\rangle)^2}{3\pi\varepsilon_0 c_0^3}$
P_dipole is power from dipole radiation of emitted photons
Typically $A_{21} \sim 10^9\,\mathrm{Hz} \implies \tau_{21} \sim 1\,\mathrm{ns}$.
Stimulated emission photons are emitted isotropically.

Absorption
$E_1 + \hbar\omega_{21} \to E_2$ (absorption)
$\hbar\omega_{21} = E_2 - E_1$
$\frac{\mathrm{d}N_\mathrm{abs}}{\mathrm{d}t} = B_{12} w(\omega_{21}) N_1$ (rate of spontaneous emission)
N_1 is total number of atoms in cavity in state $|1\rangle$.

Stimulated Emission
$E_2 + \hbar\omega_{21} \to E_1 + 2\hbar\omega_{21}$
Atom in $|2\rangle$ interacts with incident photon $\hbar\omega_{21}$ and relaxes into E_1 by emitting photon $\hbar\omega_{21}$ identical to incident photon
$\frac{\mathrm{d}N_\mathrm{stim}}{\mathrm{d}t} = B_{21} w(\omega_{21}) N_2$
Important: frequency, direction, phase, etc... of emitted photon are identical to incident photon.
Result: stimulated emission produces two coherent photons.

Relationship Among Coefficients
Assumption: both E_1 and E_2 have equal degeneracy.
Assumption: transitions occur only btwn. states $|1\rangle$ and $|2\rangle$;
$\implies N_1 + N_2 \equiv N = \text{constant}$
$\frac{\mathrm{d}N_1}{\mathrm{d}t} = -\frac{\mathrm{d}N_2}{\mathrm{d}t}$ (because $N_1 + N_2 = \text{constant}$).
$\frac{\mathrm{d}N_1}{\mathrm{d}t} = \frac{\mathrm{d}N_\mathrm{sp}}{\mathrm{d}t} + \frac{\mathrm{d}N_\mathrm{stim}}{\mathrm{d}t} - \frac{\mathrm{d}N_\mathrm{abs}}{\mathrm{d}t}$
$\frac{\mathrm{d}N_2}{\mathrm{d}t} = \frac{\mathrm{d}N_\mathrm{abs}}{\mathrm{d}t} - \frac{\mathrm{d}N_\mathrm{sp}}{\mathrm{d}t} - \frac{\mathrm{d}N_\mathrm{stim}}{\mathrm{d}t}$
Assumption: gas and cavity are in thermal equilibrium.
$\frac{\mathrm{d}N_1}{\mathrm{d}t} = \frac{\mathrm{d}N_2}{\mathrm{d}t} = 0$ (assuming thermal equilibrium)
$A_{21}N_2 + B_{21}w(\omega_{21})N_2 - B_{12}w(\omega_{21})N_1 = 0$ (in th. eq.)
$\frac{N_1}{N_2} = e^{-\beta(E_2 - E_1)} = e^{+\beta\hbar\omega_{21}}$ (Boltzmann occupation ratio)
$\implies w(\omega_{21}) = \frac{A_{21}}{B_{12}\cdot(N_1/N_2) - B_{21}}$
$= \frac{A_{21}}{B_{12}\cdot e^{\beta\hbar\omega_{21}} - B_{21}}$
$w(\omega) = \frac{\hbar\omega^3}{\pi^2 c_0^3} \frac{1}{e^{\beta\hbar\omega} - 1}$ (general energy density of BB radiation)
Equate $w(\omega_{21})$ to general BB expression $w(\omega)|_{\omega=\omega_{21}}$ to get...
$\frac{A_{21}}{B(e^{\beta\hbar\omega_{21}} + 1)} = \frac{\hbar\omega_{21}^3}{\pi^2 c_0^3} \frac{1}{e^{\beta\hbar\omega_{21}} - 1}$ (letting $B_{12} = B_{21} \equiv B$)
$B_{21} = B_{21} \equiv B$ (to satisfy $w(\omega_{21}) = w(\omega)|_{\omega=\omega_{21}}$)
$A_{21}/B \equiv A/B = \frac{\hbar\omega_{21}^3}{\pi^2 c_0^3}$ (in thermal equilibrium)

Interactions in an Optical Resonator
$g(\omega)$ is spectral line shape for emission and absorption.
$\int_{-\infty}^{\infty} g(\omega)\,\mathrm{d}\omega = 1$
$\frac{\mathrm{d}N_\mathrm{abs}}{\mathrm{d}t} = B_{12}N_1 w(\omega_{21}) \to B_{12}N_1 \int_{-\infty}^{\infty} g(\omega)w(\omega)\,\mathrm{d}\omega$
$\frac{\mathrm{d}N_\mathrm{stim}}{\mathrm{d}t} = B_{21}N_2 w(\omega_{21}) \to B_{21}N_2 \int_{-\infty}^{\infty} g(\omega)w(\omega)\,\mathrm{d}\omega$
Assumption: optical resonator has a single resonance at ω_r.
Assumption: optical resonator's energy density spectral peak is much more narrow than characteristic width of $g(\omega)$;
$\implies g(\omega)$ is approximately constant relative to $w(\omega)$.
$\frac{\mathrm{d}N_\mathrm{abs}}{\mathrm{d}t} \approx B_{12}N_1 g(\omega_\mathrm{r}) \int_{-\infty}^{\infty} w(\omega)\,\mathrm{d}\omega = B_{12}N_1 g(\omega_\mathrm{r}) w_\mathrm{EM}$
$\frac{\mathrm{d}N_\mathrm{stim}}{\mathrm{d}t} \approx B_{21}N_2 g(\omega_\mathrm{r}) \int_{-\infty}^{\infty} w(\omega)\,\mathrm{d}\omega = B_{21}N_2 g(\omega_\mathrm{r}) w_\mathrm{EM}$
In practice: $g(\omega)$ is approximately Gaussian or Lorentzian and is approximated with a box function.

Optical Amplification

Principle: shine beam with power P_in on a material; material outputs beam with power $P_\mathrm{out} > P_\mathrm{in}$.
Consider thin plate of thickness $\mathrm{d}z$ and cross-sectional area S.
Assume light is incident on plate. The incident light's power...

- decreases from absorption, and
- increases from stimulated and spontaneous emission.

Assume light is normally incident on plate along z axis.
N quantities refer to atoms in optical resonator.
N' quantities refer to atoms in plate.
$\mathrm{d}P = \hbar\omega\left(\frac{\mathrm{d}N'_\mathrm{stim}}{\mathrm{d}t} - \frac{\mathrm{d}N'_\mathrm{abs}}{\mathrm{d}t} + \frac{\mathrm{d}N'_\mathrm{sp}}{\mathrm{d}t}\right)$ (power change in plate)
Assumption: neglect spontaneous emission $\frac{\mathrm{d}N'_\mathrm{sp}}{\mathrm{d}t}$ from plate.
Justification: we consider only light along axis of incident beam. All stimulated emission photons (which have same direction as incident photons) travel along beam axis, while only a negligible portion of spontaneous emission photons (emitted isotropically) will travel along beam axis.
$\mathrm{d}P \approx \hbar\omega\left(\frac{\mathrm{d}N'_\mathrm{stim}}{\mathrm{d}t} - \frac{\mathrm{d}N'_\mathrm{abs}}{\mathrm{d}t}\right)$ (neglecting $\frac{\mathrm{d}N'_\mathrm{sp}}{\mathrm{d}t}$)
$= \hbar\omega\left[N'_2 B g(\omega_\mathrm{r}) u_\mathrm{EM} - N'_1 B g(\omega_\mathrm{r}) u_\mathrm{EM}\right]$
$= \hbar\omega B g(\omega_\mathrm{r}) u_\mathrm{EM} (N'_2 - N'_1)$
$j = u_\mathrm{EM} c$ (energy current density in cavity)
$\mathrm{d}P = \frac{\hbar B j}{c} g(\omega_\mathrm{r})(N'_2 - N'_1)$ (in terms of j)
Keep in mind that $u_\mathrm{EM} = u_\mathrm{EM}(\omega_\mathrm{r})$ and so $j = j(\omega_\mathrm{r})$, i.e. j is a function of the resonator's resonance frequency ω_r.

Intermezzo: Some Useful Expressions
Let V denote volume of entire resonator cavity.
$N_1' = \frac{N_1}{V} S\,\mathrm{d}z$ (in terms of resonator number density N_1/V)
$N_2' = \frac{N_2}{V} S\,\mathrm{d}z$ (in terms of resonator number density N_2/V)
$\mathrm{d}P = \frac{\hbar B j}{c} g(\omega_\mathrm{r}) \frac{N_2 - N_1}{V} S\,\mathrm{d}z$ (in terms of N and V)
$\mathrm{d}j = \frac{\mathrm{d}P}{S} = \frac{\hbar B j}{c} g(\omega_\mathrm{r}) \frac{N_2 - N_1}{V}\,\mathrm{d}z$ (in terms of N and V)

Intermezzo: Absorption/Emission Cross Section
$\sigma(\omega) \equiv \frac{\hbar\omega g(\omega) B}{c}$ (absorption and emission cross section)
$\mathrm{d}j = \sigma(\omega_\mathrm{r}) \frac{N_2 - N_1}{V} j\,\mathrm{d}z$ (in terms of σ)
$\gamma(\omega) \equiv \sigma(\omega) \frac{N_2 - N_1}{V}$ (amplification constant)
$\mathrm{d}j = \gamma j\,\mathrm{d}z$ (in terms of γ)
$\gamma > 1 \implies N_2 > N_1$ (condition for amplification)
$N_2 > N_1$ is called *inverted occupation* and refers to a state with more occupation at higher energy than at lower energy; Inverted occupation it is inherently unstable and requires an external energy source to maintain.

Amplification in a Three State System
Inverted occupation is possible only in a system with three or more energy levels.
Consider a system with states $|0\rangle$, $|1\rangle$ and $|2\rangle$ and corresponding non-degenerate energies $E_0 < E_1 < E_2$.
$N = N_0 + N_1 + N_2$ (total number of particles in system)
Assumption: most particles are in ground state $|0\rangle$;
$\implies N_0 \gg N_1$ and $N_0 \gg N_2$
$N \approx N_0$ (assuming $N_0 \gg N_1, N_2$)
Goal: amplify EM waves with frequency $\hbar\omega_{21} = E_2 - E_1$.

Occupation Equations
Assume external pump mechanism (e.g. external light source) excites particles from $|0\rangle$ to $|2\rangle$.
Let N_p denote atoms "pumped" from $|0\rangle$ to $|2\rangle$.
$\frac{\mathrm{d}N_\mathrm{p}}{\mathrm{d}t} = rN_0$ (pumping dynamics, $r \in \mathbb{R}$)
$$\frac{\mathrm{d}N_2}{\mathrm{d}t} = \frac{\mathrm{d}N_\mathrm{p}}{\mathrm{d}t} - \frac{\mathrm{d}N_\mathrm{sp}^{(20)}}{\mathrm{d}t} - \frac{\mathrm{d}N_\mathrm{sp}^{(21)}}{\mathrm{d}t} + \frac{\mathrm{d}N_\mathrm{abs}^{(12)}}{\mathrm{d}t} - \frac{\mathrm{d}N_\mathrm{stim}^{(21)}}{\mathrm{d}t}$$
$$= rN_0 - A_{20}N_2 - A_{21}N_2 + B_{21}u_\mathrm{EM}(\omega_{21})g(\omega_{21})(N_1 - N_2)$$
$$\frac{\mathrm{d}N_1}{\mathrm{d}t} = -\frac{\mathrm{d}N_\mathrm{sp}^{(10)}}{\mathrm{d}t} + \frac{\mathrm{d}N_\mathrm{sp}^{(21)}}{\mathrm{d}t} - \frac{\mathrm{d}N_\mathrm{abs}^{(12)}}{\mathrm{d}t} + \frac{\mathrm{d}N_\mathrm{stim}^{(21)}}{\mathrm{d}t}$$
$$= -A_{10}N_1 + A_{21}N_2 - B_{21}u_\mathrm{EM}(\omega_{21})g(\omega_{21})(N_1 - N_2)$$
$$\frac{\mathrm{d}N_0}{\mathrm{d}t} = -\frac{\mathrm{d}N_\mathrm{p}}{\mathrm{d}t} + \frac{\mathrm{d}N_\mathrm{sp}^{(20)}}{\mathrm{d}t} + \frac{\mathrm{d}N_\mathrm{sp}^{(10)}}{\mathrm{d}t}$$
$$= -rN_0 + A_{20}N_2 + A_{10}N_1$$
Approximation: neglect spontaneous emission from $|2\rangle \to |0\rangle$ (i.e. let $A_{20} \approx 0$).
Approximation: $N_0 \approx N$. (most atoms in ground state)
$$\frac{\mathrm{d}N_2}{\mathrm{d}t} = rN - A_{21}N_2 + B_{21}u_\mathrm{EM}(\omega_{21})g(\omega_{21})(N_1 - N_2)$$
$$\frac{\mathrm{d}N_1}{\mathrm{d}t} = -A_{10}N_1 + A_{21}N_2 - B_{21}u_\mathrm{EM}(\omega_{21})g(\omega_{21})(N_1 - N_2)$$
$$\frac{\mathrm{d}N_0}{\mathrm{d}t} = -rN + A_{10}N_1$$

Equilibrium State
Consider an equilibrium state in which $\frac{\mathrm{d}N_0}{\mathrm{d}t} = \frac{\mathrm{d}N_1}{\mathrm{d}t} = \frac{\mathrm{d}N_2}{\mathrm{d}t} = 0$.
First set $\frac{\mathrm{d}N_1}{\mathrm{d}t} = 0$ and rearrange to get...
$\big[B_{21}u_\mathrm{EM}(\omega_{21})g(\omega_{21}) + A_{21}\big]N_2 = \big[B_{21}u_\mathrm{EM}(\omega_{21})g(\omega_{21}) + A_{10}\big]N_1$
$\frac{N_2 - N_1}{V} = \frac{A_{10} - A_{21}}{B_{21}u_\mathrm{EM}(\omega_{21})g(\omega_{21}) + A_{21}} \frac{N_1}{V}$ (after rearranging)
$N_1 = \frac{rN}{A_{10}}$ (from equation for $\frac{\mathrm{d}N_0}{\mathrm{d}t}$)
$\frac{N_2 - N_1}{V} = \frac{A_{10} - A_{21}}{B_{21}u_\mathrm{EM}(\omega_{21})g(\omega_{21}) + A_{21}} \frac{rN}{VA_{10}}$ (using $N_1 = \frac{rN}{A_{10}}$)
Conclusion: inverted occupation is possible if $A_{10} > A_{21}$.

Limit of $A_{10} \gg A_{21}$
Assume $A_{10} \gg A_{21}$.
Interpretation: use laser materials for which atoms stay in $|2\rangle$ for a long time, eventually relax to $|0\rangle$, then relax rapidly to $|0\rangle$.
$A_{10} - A_{21} \approx A_{10}$ (assuming $A_{10} \gg A_{21}$)
$$\frac{N_2 - N_1}{V} \approx \frac{1}{B_{21}u_\mathrm{EM}(\omega_{21})g(\omega_{21}) + A_{21}} \frac{rN}{V}$$
$$= \frac{rN}{VA_{21}} \left(\frac{1}{1 + \frac{B_{21}g(\omega_{21})}{A_{21}} u_\mathrm{EM}(\omega_{21})}\right)$$
$$= \frac{rN}{VA_{21}} \left(\frac{1}{1 + \frac{B_{21}g(\omega_{21})}{cA_{21}} j(\omega_{21})}\right)$$ (using $j = cu_\mathrm{EM}$)
$j_\mathrm{s}(\omega_{21}) \equiv \frac{A_{21}c}{B_{21}g(\omega_{21})}$ (saturation current)
Shorthand: $j \to j(\omega_{21})$ and $j_\mathrm{s} \to j_\mathrm{s}(\omega_{21})$
$\frac{N_2 - N_1}{V} = \frac{rN}{VA_{21}} \left(\frac{1}{1 + j/j_\mathrm{s}}\right)$ (in terms of j_s)

Lower-Power Amplification
$\sigma(\omega) \equiv \frac{\hbar\omega g(\omega) B}{c}$ (for review from earlier)
$\mathrm{d}j = \sigma(\omega_\mathrm{r}) \frac{N_2 - N_1}{V} j(\omega_\mathrm{r})\,\mathrm{d}z$ (for review from earlier)
Assume resonator frequency at $\omega_\mathrm{r} = \omega_{21}$.
Shorthand: abbreviate $j \to j(\omega_{21})$ and $j_\mathrm{s} \to j_\mathrm{s}(\omega_{21})$.
$\mathrm{d}j = \sigma(\omega_{21}) \frac{rN}{VA_{21}} \left(\frac{1}{1 + j/j_\mathrm{s}}\right) j\,\mathrm{d}z$ (energy current density in cavity)
$G \equiv \sigma(\omega_{21}) \frac{rN}{VA_{21}}$ (low power amplification coefficient)
$\mathrm{d}j = \frac{Gj}{1 + j/j_\mathrm{s}}\,\mathrm{d}z$ (in terms of G)

General Relationship
$\mathrm{d}j = \frac{Gj}{1 + j/j_\mathrm{s}}\,\mathrm{d}z$ (general differential equation)
$\frac{\mathrm{d}j}{\mathrm{d}j}\left(1 + \frac{j}{j_\mathrm{s}}\right) = G\,\mathrm{d}z$ (after rearranging)
$\int_{j_0}^{j} \frac{\mathrm{d}j'}{j'}\left(1 + \frac{j'}{j_\mathrm{s}}\right) = \int_0^z G\,\mathrm{d}z'$
$\ln\frac{j}{j_0} + \frac{j - j_0}{j_\mathrm{s}} = Gz$ (in principle, this equation defines $j(z)$)

Limit Cases
$\mathrm{d}j = \frac{Gj}{1 + j/j_\mathrm{s}}\,\mathrm{d}z$ (general differential equation)
$\mathrm{d}j \approx Gj\,\mathrm{d}z$ (if $j \ll j_\mathrm{s}$; low power limit)
$j = j_0 e^{Gz}$ (low power limit)
$\mathrm{d}j \approx Gj_\mathrm{s}\,\mathrm{d}z$ (if $j \gg j_\mathrm{s}$; high power limit)
$j = j_0 + Gj_\mathrm{s}z$ (high power limit)
$$Gj_\mathrm{s} = \left(\sigma(\omega_{21}) \frac{rN}{VA_{21}}\right) \cdot \frac{A_{21}c}{B_{21}g(\omega_{21})}$$ (from original definition)
$$= \frac{\hbar\omega_{21} g(\omega_{21}) B_{21}}{c} \cdot \frac{rN}{VA_{21}} \cdot \frac{A_{21}c}{B_{21}g(\omega_{21})}$$
$$= r\hbar\omega_{21}\frac{N}{V}$$
Conclusion: Gj_s is bounded above by number N of atoms in system.
$j = j_0 + r\hbar\omega_{21}\frac{N}{V}z$ (in terms of simplified Gj_s)

Laser

The basic components of a laser are...

1. an optical amplification module,
2. a pumping mechanism to achieve inverted occupation, and
3. an optical resonator to establish stimulated emission.

Let V denote resonator volume.
Let L_r denote resonator length.
Let L_a denote amplifier length.
Resonator has one mirror with $R = 1$ and one mirror with $R < 1$ to allow transmission of laser light.

Condition for Steady State Functionality
Consider one cycle of light from one resonator wall to the other and back (two passes of a wavefront through amplifier).
$U_\mathrm{EM} = u_\mathrm{EM}V = \frac{jV}{c}$ (EM energy in resonator)
$U_\mathrm{s} = \frac{j_\mathrm{s}V}{c}$ (EM energy at saturation current)
$\Delta j \approx Gj\frac{2L_\mathrm{a}}{1 + U/U_\mathrm{s}}$ (change in j after one cycle)
$\Delta U_\mathrm{a} \approx GU\frac{2L_\mathrm{a}}{1 + U/U_\mathrm{s}}$ (increase in U_EM from amplification)
$\Delta U_\mathrm{loss} \equiv -\Lambda U$ (decrease in U_EM from energy losses)
Λ is a constant encoding energy loss per cycle.
$\Delta U_\mathrm{a} + U_\mathrm{loss} = 0$ (condition for steady state)
$GU\frac{2L_\mathrm{a}}{1 + U/U_\mathrm{s}} = \Lambda U$ (steady state condition)
$U = U_\mathrm{s}\left(\frac{2L_\mathrm{a}G}{\Lambda} - 1\right)$ (after solving for U)
$G_\mathrm{th} \equiv \frac{\Lambda}{2L_\mathrm{a}}$ (threshold amplification constant)
$U = U_\mathrm{s}\left(\frac{G}{G_\mathrm{th}} - 1\right)$ (in terms of G_th)
Laser shines for $G > G_\mathrm{th}$.
Laser does not shine for $G < G_\mathrm{th}$.
$P_\mathrm{out} = \frac{T_\mathrm{out}}{2L_\mathrm{r}/c}U_\mathrm{s}\left(\frac{G}{G_\mathrm{th}} - 1\right)$ (outputted power)
T_out is transmittance of transmitting mirror.