Solved Exercises in Wave Optics

Notes from the Exercises component of the third-year undergraduate course *Optika* (Optics), led by dr. Andrej Petelin at the Faculty of Mathematics and Physics at the University of Ljubljana in the academic year 2020-21. The course covers wave optics at an undergraduate level. Credit for the material covered in these notes is due to dr. Petelin, while the voice, typesetting, and translation to English in this document are my own.

Disclaimer: This document will inevitably contain some mistakes—both simple typos and legitimate errors. Keep in mind that these are the notes of an undergraduate student in the process of learning the material himself, so take what you read with a grain of salt. If you find mistakes and feel like telling me, I will be grateful and happy to hear from you, even for the most trivial of errors. You can reach me by email, in English, Slovene, or Spanish, at ejmastnak@gmail.com.

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1 First Exercise Set

Technically, this is the second exercise set. The first, for lack of lecture material, was relegated to a review of introductory geometric optics, which I have left out.

1.1 An Optical Fiber with a Parabolic Refractive Index

Use the ray equation to analyze the propagation of light through a cylindrical optical fiber of radius a and a parabolic index of refraction given by

$$n(x,y) = n_0\sqrt{1-\alpha^2(x^2-y^2)},$$

where $a\alpha \ll 1$. You may restrict your analysis to the paraxial regime.

We begin by defining a coordinate system, which we choose such that the cylinder's longitudinal axis and the direction of light propagation both point along the z axis, while the cylinder's cross section is parallel to the xy plane. In general, the ray equation reads

$$\nabla n = \frac{\mathrm{d}}{\mathrm{d}s}\left(n\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}s}\right).$$

We will restrict our analysis to the xz plane. From the Pythagorean theorem, the distance $\mathrm{d}s$ travelled by a light ray during a displacement $\mathrm{d}x$ and $\mathrm{d}z$ is simply

$$\mathrm{d}s = \sqrt{\mathrm{d}x^2 + \mathrm{d}z^2}.$$

In the paraxial regime, in which light rays travel approximately parallel to the z axis and $\frac{\mathrm{d}x}{\mathrm{d}z} \ll 1$, the expression for $\mathrm{d}s$ simplifies to

$$\mathrm{d}s = \sqrt{\mathrm{d}x^2 + \mathrm{d}z^2} = \mathrm{d}z\sqrt{1+\frac{\mathrm{d}x^2}{\mathrm{d}z^2}} \approx \mathrm{d}z,$$

while the ray equation simplifies to

$$\nabla n = \frac{\mathrm{d}}{\mathrm{d}s}\left(n\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}s}\right) \longrightarrow \frac{\mathrm{d}n}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}z}\left(n\frac{\mathrm{d}x}{\mathrm{d}z}\right) = n\frac{\mathrm{d}^2x}{\mathrm{d}z^2}.$$

After dividing through by n, the equation governing light propagation is

$$\frac{\mathrm{d}^2x}{\mathrm{d}z^2} = \frac{1}{n}\frac{\mathrm{d}n}{\mathrm{d}x}.$$

Next, we substitute in the expression for n, set $y=0$, and evaluate $\frac{\mathrm{d}n}{\mathrm{d}x}$ to get

$$\frac{\mathrm{d}^2x}{\mathrm{d}z^2} = \frac{1}{n}\frac{\mathrm{d}}{\mathrm{d}x}\left(n_0\sqrt{1-\alpha^2x^2}\right) = \frac{n_0}{n}\left(\frac{-2\alpha^2x}{2\sqrt{1-\alpha^2x^2}}\right) = -\frac{n_0^2}{n^2}\alpha^2x,$$

where the last line uses $\sqrt{1-\alpha^2x^2} = \frac{n}{n_0}$. Finally, in the limit $\alpha \ll 1$, the equation reduces to

$$\frac{\mathrm{d}^2x}{\mathrm{d}z^2} = -\alpha^2x \implies x = x_0\sin(\alpha z).$$

1.2 A Light Ray Circumnavigating the Earth

Compute the gradient $\frac{\mathrm{d}n}{\mathrm{d}y}$ *of the index of refraction* n *with respect to vertical distance* y *for which a ray of light would circumnavigate the Earth's surface. The Earth's radius is* $R = 6400\,\mathrm{km}$.

We will work in a two-dimensional polar coordinate system coinciding with the earth's cross section, with the origin at the earth's center. In this coordinate system, the position vector reads

$$\mathbf{r} = (R+h)\sin\theta\,\hat{\mathbf{e}}_x + (R+h)\cos\theta\,\hat{\mathbf{e}}_y,$$

where θ is the polar angle from the North-South axis, h is the distance above the Earth's surface and $R+h$ is the distance from the Earth's center. The arc length $\mathrm{d}s$ corresponding to a displacement $\mathrm{d}\theta$ is

$$\mathrm{d}s = \sqrt{\mathrm{d}\mathbf{r}\cdot\mathrm{d}\mathbf{r}} = \sqrt{(R+h)^2\cdot(\sin^2\theta+\cos^2\theta)\cdot(\mathrm{d}\phi)^2} = (R+h)\,\mathrm{d}\theta.$$

In general, the ray equation reads

$$\nabla n = \frac{\mathrm{d}}{\mathrm{d}s}\left(n\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}s}\right).$$

At least to a first approximation, the atmosphere's index of refraction $n = n(r)$ depends only on height above the Earth and not the polar angle θ (and thus also not on $\mathrm{d}s$, since $\mathrm{d}s$ depends only $\mathrm{d}\theta$ for a ray circumnavigating the Earth). Assuming $n \neq n(s)$ and using $\mathrm{d}s = (R+h)\,\mathrm{d}\theta$, the ray equation simplifies to

$$\nabla n = n\frac{\mathrm{d}}{\mathrm{d}s}\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}s} = n\frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}s^2} = \frac{n}{(R+h)^2}\frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}\theta^2} = -\frac{n}{R+h}(\sin\theta\,\hat{\mathbf{e}}_x + \cos\theta\,\hat{\mathbf{e}}_y).$$

Along the North-South axis at the polar angle $\theta = 0$, the gradient simplifies to

$$\nabla n = -\frac{n}{R+h}(\sin\theta\,\hat{\mathbf{e}}_x + \cos\theta\,\hat{\mathbf{e}}_y) \longrightarrow \frac{\mathrm{d}n}{\mathrm{d}y} = -\frac{n}{R+h}.$$

Slight modification: find the temperature 10 meters above the ocean generating the gradient $\frac{\mathrm{d}n}{\mathrm{d}y}$ required for the ray to circumnavigate the Earth. Assume the entire path occurs over water. Note that temperature of water is colder than the air above the water.

We start at a height $y = 0$ above the water, and ask what is the corresponding n at $y + \Delta y = y + h$ and what is Δn to get

$$\Delta n = \frac{\mathrm{d}n}{\mathrm{d}y}\cdot\Delta y = -\frac{n}{R+h}\cdot h \approx -\frac{1}{6.4\cdot 10^6\,\mathrm{m}}\cdot 10\,\mathrm{m} \approx -1.56\cdot 10^{-6},$$

where we have made the approximation $h \ll R$ and used $n \approx 1$ for air. The corresponding temperature change comes from

$$\Delta n = -10^{-6}\,\mathrm{K}^{-1}\Delta T \implies \Delta T = 1.56\,\mathrm{K}.$$

Note that this is a physically realistic temperature change.

1.3 Transfer Matrix for a Circular Interface

Compute the transfer matrix $\mathbf{M}$ *encoding the transfer of light through a circular interface of radius* R *between two dielectric materials with refractive indices* n_1 *and* n_2.

Theory

We describe a light ray travelling along the z axis (the optical axis) through the yz plane with its vertical distance y above the z axis and its angle θ relative to the z axis. In the paraxial approximation $\frac{\mathrm{d}y}{\mathrm{d}z} \ll 1$, which just means the light ray always remains nearly parallel to the optical axis, we can relate a ray at some known point (y_1, θ_1) to another point (y_2, θ_2) with a transfer matrix $\mathbf{M}$ via

$$\begin{bmatrix} y_2 \\ \theta_2 \end{bmatrix} = \mathbf{M} \begin{bmatrix} y_1 \\ \theta_1 \end{bmatrix} \qquad \text{where} \qquad \mathbf{M} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}.$$

Solution

We assume the light travels along the positive z direction from region one into region two. We parameterize the light ray just before the interface with the displacement y_1 and angle θ_1, and aim to find the parameters y_2 and θ_2 an infinitesimal distance on the other side of the interface in material 2 in terms of y_1 and θ_1.

Since the two points are close together, the y coordinates are the equal, $y_1 = y_2$. To make it easier to recognize the coefficients A and B, we write this

$$y_2 = 1 \cdot y_1 + 0 \cdot \theta_1.$$

We then compare this expression to the general relationship $y_2 = Ay_1 + B\theta_1$, which immediately reveals $A = 1$ and $B = 0$.

To find C and D, we must find an expression for θ_2 in terms of y_1 and θ_1 matching the general relationship

$$\theta_2 = Cy_1 + D\theta_1,$$

from where we can read off the values of C and D. We begin with Snells law, i.e.

$$n_1 \sin\theta_1 = n_2 \sin\theta_2 \quad \xrightarrow{\theta \ll 1} \quad n_1\theta_1 = n_2\theta_2.$$

For light a distance y_1 above the z axis, the angles of incidence and refraction are given by

$$\phi_1 = \theta_1 + \frac{y_1}{R} \qquad \text{and} \qquad \phi_2 = \theta_2 + \frac{y_1}{R},$$

and the law of refraction reads

$$n_1\left(\theta_1 + \frac{y_1}{R}\right) = n_2\left(\theta_2 + \frac{y_1}{R}\right).$$

We then solve for θ_2 in terms of y_1 and θ_1 to get

$$\theta_2 = \left(\frac{n_1 - n_2}{n_2 R}\right) y_1 + \frac{n_1}{n_2}\theta_1.$$

In one place, the equations for y_2 and θ_2 are

$$\begin{aligned} y_2 &= 1 \cdot y_1 + 0 \cdot \theta_1 \\ \theta_2 &= \left(\frac{n_1 - n_2}{n_2 R}\right) y_1 + \frac{n_1}{n_2}\theta_1, \end{aligned}$$

and the corresponding transfer matrix is

$$\mathbf{M} = \begin{bmatrix} 1 & 0 \\ \frac{n_1-n_2}{n_2 R} & \frac{n_1}{n_2} \end{bmatrix}.$$

1.4 Transfer Matrix for a Thick Lens

Use the result of the previous problem to compute the transfer matrix for propagation of light through a thick lens of width d and radii R_1 and R_2.

We can decompose the passage through the lens into three components: crossing the left interface from region 1 (with index of refraction n_1) through the interface R_1 into the lens (region 2, with $n = n_2$), travelling a distance d through the lens, crossing the from the lens into region three with $n = n_3$.

The total transfer matrix is simply the product of the matrices for each of these three steps, and the passage through the lens is encoded by

$$\begin{bmatrix} y_2 \\ \theta_2 \end{bmatrix} = \mathbf{M}_3\mathbf{M}_2\mathbf{M}_1 \begin{bmatrix} y_1 \\ \theta_1 \end{bmatrix}.$$

We find the matrices $\mathbf{M}_1$ and $\mathbf{M}_3$ from the previous problem, making sure to substitute in the appropriate radius and refractive indexes. The results are

$$\mathbf{M}_1 = \begin{bmatrix} 1 & 0 \\ \frac{n_1-n_2}{n_2 R_1} & \frac{n_1}{n_2} \end{bmatrix} \qquad \text{and} \qquad \mathbf{M}_3 = \begin{bmatrix} 1 & 0 \\ \frac{n_3-n_2}{n_3 R_2} & \frac{n_2}{n_3} \end{bmatrix}.$$

The matrix $\mathbf{M}_2$ encodes translation through a medium with constant n, and reads

$$\mathbf{M}_2 = \begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix}.$$

The rest of this problem is just matrix multiplication; we start by computing

$$\mathbf{M}_2\mathbf{M}_1 = \begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \frac{n_1-n_2}{n_2 R_1} & \frac{n_1}{n_2} \end{bmatrix} = \begin{bmatrix} 1 + d\frac{n_1-n_2}{n_2 R_1} & d\frac{n_1}{n_2} \\ \frac{n_1-n_2}{n_2 R_1} & \frac{n_1}{n_2} \end{bmatrix}.$$

The full product is then

$$\mathbf{M}_3\mathbf{M}_2\mathbf{M}_1 \equiv \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \frac{n_3-n_2}{n_3 R_2} & \frac{n_2}{n_3} \end{bmatrix} \begin{bmatrix} 1 + d\frac{n_1-n_2}{n_2 R_1} & d\frac{n_1}{n_2} \\ \frac{n_1-n_2}{n_2 R_1} & \frac{n_1}{n_2} \end{bmatrix}.$$

We'll compute the result element by element; A and B are relatively simple:

$$A = 1 + d\frac{n_1 - n_2}{n_2 R_1} \qquad \text{and} \qquad B = d\frac{n_1}{n_2}.$$

For C we have

$$C = \frac{n_3 - n_2}{n_3 R_2}\left(1 + d\frac{n_1 - n_2}{n_2 R_1}\right) + \frac{n_2}{n_3}\frac{n_1 - n_2}{n_2 R_1},$$

while D reads

$$D = d\frac{n_1}{n_2}\frac{n_3 - n_2}{n_3 R_2} + \frac{n_1}{n_3}.$$

With A, B, C and D known we have solved the problem, even if the result is not particularly physically insightful.

1.5 Transfer Matrix for a Thin Lens

Use the result of the previous problem to find the transfer matrix for passage of light through a thin lens with index of refraction n surrounded by air with $n_{\text{air}} = 1$.

We begin by substituting $n_1 = n_3 = n_{\text{air}} = 1$ and $n_2 = n$ into the transfer matrix elements for a thick lens in the previous problem. The matrix elements A and B simplify to

$$A = 1 + d\frac{1-n}{nR_1} = 1 - d\frac{n-1}{nR_1} \qquad \text{and} \qquad B = \frac{d}{n},$$

while C and D reduce to

$$C = \frac{1-n}{R_2}\left(1 + d\frac{1-n}{nR_1}\right) + n\frac{1-n}{nR_1} \qquad \text{and} \qquad D = d\frac{1}{n_2}\frac{1-n}{R_2} + 1.$$

Next, we define the foci f_1 and f_2 with the relationships

$$\frac{1}{f_1} = \frac{n-1}{R_1} \qquad \text{and} \qquad \frac{1}{f_2} = \frac{n-1}{R_2},$$

in terms of which the matrix elements A, B, C and D simplify to

$$A = 1 - \frac{d}{nf_1} \qquad B = \frac{d}{n} \qquad D = 1 - \frac{d}{nf_2} \tag{1.1}$$

$$C = -\frac{1}{f_2}\left(1 - \frac{d}{nf_1}\right) - \frac{1}{f_1} = -\frac{1}{f_1} - \frac{1}{f_2} + \frac{d}{nf_1f_2}. \tag{1.2}$$

Next, we model a thin lens with $d \to 0$ which further simplifies things to

$$\mathbf{M} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -\frac{1}{f_1} - \frac{1}{f_2} & 1 \end{bmatrix} \equiv \begin{bmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{bmatrix},$$

where we have defined the total focus f according to

$$\frac{1}{f} \equiv \frac{1}{f_1} + \frac{1}{f_2} = \frac{n-1}{R_1} + \frac{n-1}{R_2}.$$

Finally, for a lens with equal radii $R_1 = R_2 \equiv R$, the focus becomes

$$\frac{1}{f} = \frac{2(n-1)}{R} \implies f = \frac{R}{2(n-1)},$$

and the transfer matrix for a thin lens simplifies to

$$\mathbf{M}_{\text{thin}} = \begin{bmatrix} 1 & 0 \\ -\frac{2(n-1)}{R} & 1 \end{bmatrix}.$$

Focus

Finally, we will establish why f is called the lens's focus by showing all rays incident on the lens and parallel to the optical axis are focused into a single point on the optical a distance f from the lens. The relationship defining the focus behavior is

$$\mathbf{M}\begin{bmatrix} y_1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ \theta_2 \end{bmatrix},$$

where $\theta_1 = 0$ corresponds to incident rays parallel to the optical axis, $y_2 = 0$ means all outgoing rays end up on the optical axis, and $\mathbf{M}$ is the to-be-determined transfer matrix encoding this behavior.

The matrix $\mathbf{M}$ corresponds to passage through the thin lens, known from the previous problem, followed a translation of to-be-determined distance z, which reads

$$\mathbf{M} = \mathbf{M}_{\text{trans}}(z) \cdot \mathbf{M}_{\text{thin}} = \begin{bmatrix} 1 & z \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{bmatrix} \equiv \begin{bmatrix} A & B \\ C & D \end{bmatrix}.$$

Our goal is to show that z equals the focus f found in the previous problem.

$$\begin{bmatrix} 1 - \frac{z}{f} & z \\ -\frac{1}{f} & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ \theta_2 \end{bmatrix},$$

Showing this is straightforward: we just multiply out and get

$$\left(1 - \frac{z}{f}\right) y_1 + 0 \cdot z = 0 \implies z = f.$$

2 Second Exercise Set

2.1 An Infinity-Corrected Microscope

Using optical transfer matrices, determine the magnification of a microscope formed of two lenses with foci f_1 and f_2 separated by distance d. Assume the sample observed by the microscope lies a distance f_1 from the first lens, while the detector lies a distance f_2 from the second lens.

Let y_1 denote the object point of the observed sample and let y_2 be the position at which the object appears on the detector screen. We describe the path of light from the sample to the detector using five individual transfer matrices. These are matrices correspond to:

1. Translation through air of distance f_1 from the sample to the first lens:
$$\mathbf{M}_1 = \begin{bmatrix} 1 & f_1 \\ 0 & 1 \end{bmatrix}.$$

2. Transfer through the first lens:
$$\mathbf{M}_2 = \begin{bmatrix} 1 & 0 \\ -\frac{1}{f_1} & 1 \end{bmatrix}.$$

3. A translation through air of distance d between the lenses:
$$\mathbf{M}_3 = \begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix}.$$

4. Transfer through the second lens.
$$\mathbf{M}_4 = \begin{bmatrix} 1 & 0 \\ -\frac{1}{f_2} & 1 \end{bmatrix}.$$

5. Translation through air of distance f_2 from the second lens to the detector:
$$\mathbf{M}_5 = \begin{bmatrix} 1 & f_2 \\ 0 & 1 \end{bmatrix}.$$

The total optical path from sample to detector reads

$$\mathbf{M}_{\mathrm{tot}} \begin{bmatrix} y_1 \\ \theta_1 \end{bmatrix} = \begin{bmatrix} y_2 \\ \theta_2 \end{bmatrix}, \qquad \text{where} \quad \mathbf{M}_{\mathrm{tot}} = \mathbf{M}_5\mathbf{M}_4\mathbf{M}_3\mathbf{M}_2\mathbf{M}_1 \equiv \begin{bmatrix} A & B \\ C & D \end{bmatrix}. \tag{2.1}$$

Multiplied out in terms of the matrix elements A, B, C, and D, the above matrix equation reads

$$Ay_1 + B\theta_1 = y_2 \qquad \text{and} \qquad Cy_1 + D\theta_1 = \theta_2.$$

Since magnification is defined as the ratio y_2/y_1, we only need the first equation involving y_2. More so, since we are mapping all light from the sample into a single focus point on the detector, we require that the final position y_2 from the optical axis is independent of the independent of the initial ray direction θ_1. Independence of the first equation on θ_1 holds only if $B = 0$, leaving use with

$$Ay_1 = y_2 \implies \frac{y_2}{y_1} = A.$$

In other words, we only need a single matrix element, A, to solve the problem.

We now turn to computing the total transfer matrix M_{tot} in Equation 2.1, beginning with the product of $\mathbf{M}_2$ and $\mathbf{M}_1$. This reads

$$\mathbf{M}_2\mathbf{M}_1 \equiv \mathbf{M}_{21} = \begin{bmatrix} 1 & 0 \\ -\frac{1}{f_1} & 1 \end{bmatrix} \begin{bmatrix} 1 & f_1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & f_1 \\ -\frac{1}{f_1} & -1+1 \end{bmatrix} = \begin{bmatrix} 1 & f_1 \\ -\frac{1}{f_1} & 0 \end{bmatrix}.$$

Next, we consider M_3, corresponding the translation between the lenses:

$$\mathbf{M}_3\mathbf{M}_{21} \equiv \mathbf{M}_{321} = \begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & f_1 \\ -\frac{1}{f_1} & 0 \end{bmatrix} = \begin{bmatrix} 1-\frac{d}{f_1} & f_1 \\ -\frac{1}{f_1} & 0 \end{bmatrix}.$$

We then evaluate the product of $\mathbf{M}_5$ and $\mathbf{M}_4$, which is

$$\mathbf{M}_5\mathbf{M}_4 \equiv \mathbf{M}_{54} = \begin{bmatrix} 1 & f_2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -\frac{1}{f_2} & 1 \end{bmatrix} = \begin{bmatrix} 0 & f_2 \\ -\frac{1}{f_2} & 1 \end{bmatrix}.$$

The final result is

$$\mathbf{M}_{\text{tot}} = \mathbf{M}_{54}\mathbf{M}_{321} = \begin{bmatrix} 0 & f_2 \\ -\frac{1}{f_2} & 1 \end{bmatrix} \begin{bmatrix} 1-\frac{d}{f_1} & f_1 \\ -\frac{1}{f_1} & 0 \end{bmatrix}.$$

As mentioned above, the magnification depends only on the A element, which is

$$A = 0 \cdot \left(1 - \frac{d}{f_1}\right) - \frac{f_2}{f_2} = -\frac{f_1}{f_2}.$$

Note also that $B = 0$, as required in the discussion a few lines up.

Using the just-derived result for A, the magnification is simply

$$\frac{y_1}{y_2} = A = -\frac{f_2}{f_1}.$$

Interpretation: the minus sign in the amplification means the picture is reflected about the optical axis. As a side note, the optical element in this problem is called an infinity-corrected microscope. For this element, at least to a first approximation, the amplification is independent of the distance d between the lenses.

2.2 Theory: Jones Calculus

Electromagnetic waves are completely described by the electric and magnetic fields $\mathbf{E}$ and $\mathbf{B}$ and a wave vector $\mathbf{k}$ encoding the direction of propagation through space. In homogeneous materials, the vectors $\mathbf{E}$, $\mathbf{B}$ and $\mathbf{k}$ are mutually orthogonal; this allows us to describe the field only in terms of $\mathbf{E}$ and $\mathbf{k}$, since the direction of $\mathbf{B}$ is determined once $\mathbf{E}$ and $\mathbf{k}$ are specified.

In homogeneous materials, an EM wave's electric field takes the general sinusoidal form

$$\mathbf{E}(\mathbf{r}, t) = \operatorname{Re}\left[\mathbf{E}_0 e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t + \delta)}\right].$$

If we set the global phase δ to zero and assume the wave propagates in the z direction, in which case the wave vector reads $\mathbf{k} = k\,\hat{\mathbf{e}}_z$, the electric field simplifies to

$$\mathbf{E} = \operatorname{Re}\left[\mathbf{E}_0 e^{i(kz - \omega t)}\right], \tag{2.2}$$

which is the form we will work with for most of this course.

The above electric field's *Jones vector* is simply the field's phase and amplitude

$$\mathbf{J} \equiv \begin{pmatrix} E_{0_x} e^{i\phi_x} \\ E_{0_y} e^{i\phi_y} \end{pmatrix},$$

where ϕ_x and ϕ_y are the phases of the field components E_x and E_y, respectively, while E_{0_x} and E_{0_y} are the respective field amplitudes. Because the EM wave propagates in the z direction, the electric field is entirely confined to the yz plane, allowing us to write the Jones vector with only two (and not three) components.

We will often work in terms of *normalized* Jones vectors, which read

$$\mathbf{J} \equiv \frac{1}{\sqrt{E_{0_x}^2 + E_{0_y}^2}} \begin{pmatrix} E_{0_x} e^{i\phi_x} \\ E_{0_y} e^{i\phi_y} \end{pmatrix},$$

Circular Polarization

We first write the electric field in Equation 2.2 in column vector form, which more closely resembles a Jones vector:

$$\mathbf{E}(z,t) = \begin{pmatrix} E_x(z,t) \\ E_y(z,t) \end{pmatrix} = \begin{pmatrix} E_{0_x} e^{i(kz-\omega t+\phi_x)} \\ E_{0_y} e^{i(kz-\omega t+\phi_x)} \end{pmatrix} = \begin{pmatrix} E_{0_x} e^{i\phi_x} \\ E_{0_y} e^{i\phi_y} \end{pmatrix} e^{i(kz-\omega t)}.$$

By definition, circular polarization occurs when the field components are out of phase by $\pm\pi/2$ and have equal amplitudes $E_{0_x} = E_{0_y}$. For convenience, we set the phases to $\phi_1 = 0$ and $\phi_2 = \pm\pi/2$ and define $E_{0_x} = E_{0_y} \equiv E_0$. The corresponding (normalized) Jones vector for circularly polarized light is then

$$\mathbf{J}_{\text{circ}} = \frac{1}{\sqrt{2}E_0} \begin{pmatrix} E_0 \cdot e^{i\cdot 0} \\ E_0 \cdot e^{\pm i\frac{\pi}{2}} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ \pm i \end{pmatrix}.$$

For the purposes of our course, we define the sign of right circularly polarized (RCP) light and left circularly polarized (LCP) light according to the convention

$$\mathbf{J}_{\text{LCP}} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix} \quad \text{and} \quad \mathbf{J}_{\text{RCP}} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix}. \tag{2.3}$$

See the lecture notes for a more thorough discussion of the definitions of RCP and LCP light, which may vary depending on the author.

2.3 Jones Matrices for Common Optical Elements

Calculate the polarization of light after passing through the following optical elements.

1. *a linear polarizer parallel to the x and y axes*
2. *a linear polarizer rotated by an angle $\pm 45°$ relative to the x axis.*
3. *a quarter waveplate with principle axes aligned with the x and y axes, where the y axis is the fast axis*
4. *a quarter waveplate with principle axes rotated by $\pm 45°$ relative to the x and y axes*
5. *two identical quarter waveplates, both rotated by $\pm 45°$ relative to the x and y axes*

2.3.1 Jones Matrix for a Linear Polarizer

A *linear polarizer* transmits the component of the incident electric field parallel to the polarizer's axis and blocks the component of the incident electric field perpendicular to the axis of polarization. A linear polarizer with axis of polarization parallel to the x axis has the Jones matrix

$$\mathbf{M}_{\text{lin}}^{(x)} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}. \tag{2.4}$$

We confirm this matrix produces the desired behavior of a linear polarizer by testing its effect on an arbitrary Jones vector (J_x, J_y). The result is

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} J_x \\ J_y \end{pmatrix} = \begin{pmatrix} J_x \\ 0 \end{pmatrix}.$$

As expected the polarizer passes the electric field component parallel to the x axis and block the field component perpendicular to the y axis.

A linear polarizer with axis of polarization parallel to the y axis has the Jones matrix

$$\mathbf{M}_{\text{lin}}^{(y)} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \tag{2.5}$$

When applied to an arbitrary vector (J_x, J_y), this matrix produces

$$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} J_x \\ J_y \end{pmatrix} = \begin{pmatrix} 0 \\ J_y \end{pmatrix},$$

which is the desired effect for a linear polarizer aligned with the y axis.

Note that the specific orientation of the x and y axes in space is completely arbitrary in these problems; the only requirement is that the electric field and polarizer's components are measured with respect to the *same* coordinate axes.

The Jones matrix for a linear polarizer whose polarizing axes are rotated by an angle 45° relative to the (x, y) axes reads

$$\mathbf{M}_{\text{lin}}^{(\pm 45)} = \frac{1}{2} \begin{pmatrix} 1 & \pm 1 \\ \pm 1 & 1 \end{pmatrix}.$$

Applied to an arbitrary Jones vector (J_x, J_y), the rotated linear polarizer produces

$$\frac{1}{2} \begin{pmatrix} 1 & \pm 1 \\ \pm 1 & 1 \end{pmatrix} \begin{pmatrix} J_x \\ J_y \end{pmatrix} = \frac{1}{2} \begin{pmatrix} J_x \pm J_y \\ \pm J_x + J_y \end{pmatrix}.$$

To get a feel for this result, we check the effect of a $\mathbf{M}_{\text{lin}}^{(+45)}$ on a Jones vector (J, J), parallel to the x' axis (i.e. the rotated polarizer's axis of polarization). The effect is

$$\frac{1}{2} \begin{pmatrix} 1 & +1 \\ +1 & 1 \end{pmatrix} \begin{pmatrix} J \\ J \end{pmatrix} = \frac{1}{2} \begin{pmatrix} J + J \\ J + J \end{pmatrix} = \begin{pmatrix} J \\ J \end{pmatrix}.$$

In other words, the entire vector is perfectly transmitted. This makes sense, since the vector (J, J) is parallel to the polarizer's axis. Meanwhile, the effect of a $\mathbf{M}_{\text{lin}}^{(-45)}$ matrix (rotated by $-45°$ relative to the x axis) on the same Jones vector (J, J) is

$$\frac{1}{2} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} J \\ J \end{pmatrix} = \frac{1}{2} \begin{pmatrix} J - J \\ J - J \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Evidently, the incident vector is completely blocked, since (J, J) is orthogonal to the rotated polarizer's axis of polarization.

2.3.2 Quarter Waveplate

Quick intro to birefringence: The component of light entering the waveplate with x polarization experiences refractive index n_x, while the component of light entering the waveplate with y polarization experiences a refractive index n_y. In short, the two polarization components experience different refractive indices.

A birefringent optical element of width z which acts on the x and y polarization components with refractive indices n_x and n_y, respectively, reads

$$\mathbf{M}_{\text{biref}} = \begin{pmatrix} e^{2\pi i \frac{z}{\lambda_0} n_x} & 0 \\ 0 & e^{2\pi i \frac{z}{\lambda_0} n_y} \end{pmatrix} \equiv \begin{pmatrix} e^{i\phi_x} & 0 \\ 0 & e^{i\phi_y} \end{pmatrix},$$

where λ_0 is the light's wavelength in vacuum.

A quarter-waveplate is such that the different in phases is precisely $\pi/2$, i.e.

$$\phi_y - \phi_x = \frac{2\pi}{\lambda_0} z \Delta n = \frac{\pi}{2}.$$

The name quarter-wave plate comes from the fact that either $\pi/2$ is a quarter of a revolution, or that the waveplate' width z is given by

$$z = \frac{1}{\Delta n} \frac{\lambda_0}{4},$$

i.e. (a scaled version of) the light's quarter wavelength. Keep in mind, however, that the defining feature of a quarter waveplate is introducing a phase shift of $\pi/2$ between the light's two polarization components.

Next, some vocabulary: in a birefringent material in which different polarization components experience different indices of refraction, the direction of polarization with a smaller index of refraction is called the fast axis, while the direction with a larger index of refraction is the slow axis. (Since light will travel faster in the direction with the smaller index of refraction). Put simply, the fast axis is just the axis with the smaller index of refraction.

For a fast axis in the y direction, we have

$$n_y < n_x \qquad c_y > c_x \qquad \phi_y < \phi_x.$$

The Jones matrix for a quarter waveplate with a fast y axis is

$$\mathbf{M}_{\text{qw}}^{(y)} = \begin{pmatrix} e^{i\phi_x} & 0 \\ 0 & e^{i\phi_y} \end{pmatrix} = e^{i\phi_x} \begin{pmatrix} 1 & 0 \\ 0 & e^{i(\phi_y - \phi_x)} \end{pmatrix} = e^{i\phi_x} \begin{pmatrix} 1 & 0 \\ 0 & e^{-i\frac{\pi}{2}} \end{pmatrix} = e^{i\phi_x} \begin{pmatrix} 1 & 0 \\ 0 & -i \end{pmatrix},$$

where $\phi_y - \phi_x = -\pi/2$ since $\phi_y < \phi_x$. So for a quarter waveplate, $|\phi_y - \phi_x| \equiv \pi/2$, while the sign in $\phi_y - \phi_x = \pm\pi/2$ is determined by choice of fast and slow axes.

Similarly, the Jones matrix for a quarter waveplate with a fast x axis is

$$\mathbf{M}_{\text{qw}}^{(x)} = \begin{pmatrix} e^{i\phi_x} & 0 \\ 0 & e^{i\phi_y} \end{pmatrix} = e^{i\phi_x} \begin{pmatrix} 1 & 0 \\ 0 & e^{i(\phi_y - \phi_x)} \end{pmatrix} = e^{i\phi_x} \begin{pmatrix} 1 & 0 \\ 0 & e^{+i\frac{\pi}{2}} \end{pmatrix} = e^{i\phi_x} \begin{pmatrix} 1 & 0 \\ 0 & +i \end{pmatrix}.$$

For our purposes we can drop phase coefficient $e^{i\phi_x}$, leaving us with

$$\mathbf{M}_{\text{qw}}^{(x)} = \begin{pmatrix} 1 & 0 \\ 0 & +i \end{pmatrix} \qquad \text{and} \qquad \mathbf{M}_{\text{qw}}^{(y)} = \begin{pmatrix} 1 & 0 \\ 0 & -i \end{pmatrix}.$$

2.3.3 Quarter Waveplate with Rotated Axes

First, for review, the coordinates (x', y') in a coordinate system S' whose x' axis is rotated by an angle θ relative to the x axis of a system S with coordinates (x, y) is given by

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \mathbf{R}(\theta)\begin{pmatrix} x \\ y \end{pmatrix}, \qquad \text{where} \quad \mathbf{R}(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}. \tag{2.6}$$

The identity $\mathbf{R}^\top = \mathbf{R}(-\theta) = \mathbf{R}^{-1}$ will be useful in the coming calculations.

To find the Jones matrix for a quarter-wave plate with the principle axes rotated by an angle of $\theta = \pi/4$ relative to (x, y) axes of a system S, we begin with the known quarter waveplate matrix $\mathbf{M}_{\text{qw}}$ from the previous section and compute

$$\mathbf{M}'_{\text{qw}} = \mathbf{R}^\top \mathbf{M}_{\text{qw}} \mathbf{R}.$$

To derive the above formula, we begin with the S' expression

$$\mathbf{M}'_{\text{qw}} \begin{pmatrix} x'_1 \\ y'_1 \end{pmatrix} = \begin{pmatrix} x'_2 \\ y'_2 \end{pmatrix}.$$

In S, in terms of the rotation matrix $\mathbf{R}$ in Equation 2.6, this relationship reads

$$\mathbf{M}_{\text{qw}} \mathbf{R} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \mathbf{R} \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}.$$

Finally, we multiply through from the left by $\mathbf{R}^{-1} = \mathbf{R}^\top$ to get

$$\mathbf{R}^\top \mathbf{M}_{\text{qw}} \mathbf{R} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}.$$

We then substitute in the expressions for $\mathbf{R}$ and $\mathbf{M}_{\text{qw}}$ and multiply out, which produces

$$\begin{aligned} \mathbf{M}'_{\text{qw}} = \mathbf{R}^\top \mathbf{M}_{\text{qw}} \mathbf{R} &= \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \\ &= \begin{pmatrix} \cos^2\theta + i\sin^2\theta & \cos\theta\sin\theta - i\sin\theta\cos\theta \\ \cos\theta\sin\theta - i\sin\theta\cos\theta & \sin^2\theta + i\cos^2\theta \end{pmatrix}. \end{aligned} \tag{2.7}$$

For a rotation by an angle $\theta = \pi/4$, the expression for $\mathbf{M}'_{\text{qw}}$ simplifies to

$$\mathbf{M}'_{\text{qw}} = \frac{1}{2}\begin{pmatrix} 1+i & 1-i \\ 1-i & 1+i \end{pmatrix}.$$

Next, using the polar notation $z = |z|e^{i\phi}$ for a complex number $z = x + iy$, where

$$|z| = \sqrt{x^2 + y^2} \qquad \text{and} \qquad \phi = \arctan\frac{x}{y},$$

the expression for $\mathbf{M}'_{\text{qw}}$ becomes

$$\mathbf{M}'_{\text{qw}} = \frac{\sqrt{2}}{2}\begin{pmatrix} e^{i\frac{\pi}{4}} & e^{-i\frac{\pi}{4}} \\ e^{-i\frac{\pi}{4}} & e^{i\frac{\pi}{4}} \end{pmatrix} = \frac{\sqrt{2}}{2} e^{i\frac{\pi}{4}} \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}.$$

We can then drop the global phase factor $e^{i\frac{\pi}{4}}$ to get the final expression

$$\mathbf{M}'_{\text{qw}} = \frac{\sqrt{2}}{2}\begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}. \tag{2.8}$$

Finally, we see how $\mathbf{M}'_{\text{qw}}$ transforms a y-polarized vector $(0, 1)$. The result is

$$\mathbf{M}'_{\text{qw}}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{\sqrt{2}}{2}\begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} -i \\ 1 \end{pmatrix} = \frac{-i}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix} \to \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix},$$

where we dropped the constant phase factor $-i$ from the last equality. We have seen the result $(1, i)$ before—this is left-circular polarized light.

Lesson: a quarter waveplate rotated by $\pi/4$ relative to the x and y axes transforms y-linearly polarized light into left circular polarized light.

2.3.4 Two Rotated Quarter Waveplates

To find the Jones matrix for two quarter waveplates, both with polarizing axes rotated by $\pi/4$ relative to the (x, y) axes and placed one immediately after the other, we square the Jones matrix $\mathbf{M}'_{\text{qw}}$ from the previous example. The resulting optical element is called a half waveplate, and (in our case with rotated axes) has the Jones matrix

$$\mathbf{M}'_{\text{hw}} = \mathbf{M}'_{\text{qw}} \cdot \mathbf{M}'_{\text{qw}} = \frac{1}{2}\begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}\begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 0 & -2i \\ -2i & 0 \end{pmatrix} \to \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

where we have factored out the two and dropped the phase factor $-i$ in the last equality.

We then check how this matrix transforms a y-polarized vector $(0, 1)$. The result is

$$\mathbf{M}'_{\text{hw}}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

The result is x-linearly polarized light. Lesson: half waveplate with axes rotated by $\pi/4$ relative to the (x, y) axes transforms y linearly polarized light into x linearly polarized light.

2.4 Identifying an Optical Filter

Linearly polarized light, with polarization at an angle α relative to the x axis, is incident on an optical filter with the Jones matrix

$$\mathbf{T} = \begin{pmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{pmatrix}.$$

Determine the polarization of the transmitted light and interpret the results.

We begin by writing the Jones vector for the incident light in the general form

$$\mathbf{J}_{\text{in}} = \cos\alpha\,\hat{\mathbf{e}}_x + \sin\alpha\,\hat{\mathbf{e}}_y = \begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix},$$

where we have simply decomposed the polarization into components $\cos\alpha$ and $\sin\alpha$ parallel and perpendicular to the x and y axes, respectively.

The optical filter $\mathbf{T}$ transforms the incident Jones vector as follows:

$$\begin{aligned}\mathbf{J}_{\text{out}} = \mathbf{T}\mathbf{J}_{\text{in}} &= \begin{pmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{pmatrix} \begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix} = \begin{pmatrix} \cos^2\theta\cos\alpha + \cos\theta\sin\theta\sin\alpha \\ \cos\theta\sin\theta\cos\alpha + \sin^2\theta\sin\alpha \end{pmatrix} \\ &= \begin{pmatrix} \cos\theta(\cos\theta\cos\alpha + \sin\theta\sin\alpha) \\ \sin\theta(\cos\theta\cos\alpha + \sin\theta\sin\alpha) \end{pmatrix}.\end{aligned}$$

We then reverse-engineer the general trigonometric identity

$$\cos(y - x) = \cos x\cos y + \sin y\sin y,$$

to get

$$\mathbf{J}_{\text{out}} = \begin{pmatrix} \cos\theta\cos(\theta - \alpha) \\ \sin\theta\cos(\theta - \alpha) \end{pmatrix} = \cos(\theta - \alpha)\begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}.$$

Interpretation: the transmitted light, which was incident with polarization α, is transmitted with polarization θ. In other words, the direction of transmitted polarization is independent of the incident polarization α. Note, however, that the angle α does affect the amplitude of the transmitted light. If $\theta = \alpha$, we have

$$\mathbf{J}_{\text{out}} = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}.$$

The optical filter $\mathbf{T}$ thus corresponds to a linear polarizer rotated at an angle θ relative to the (x, y) axes. We can confirm this by rotating a linear polarizer $\mathbf{M}_{\text{lin}}^{(x)}$ (given in Eq. 2.4) and checking if the result equals $\mathbf{T}$. The calculation reads

$$\begin{aligned}\mathbf{R}^\top\mathbf{M}_{\text{lin}}^{(x)}\mathbf{R} &= \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \\ &= \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} \cos\theta & \sin\theta \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \sin\theta\cos\theta & \sin^2\theta \end{pmatrix} = \mathbf{T}.\end{aligned} \tag{2.9}$$

2.5 An Optical Isolator

Show that the following sequence of optical elements:

1. *a linear polarizer with an axis of polarization in the y direction,*
2. *a quarter waveplate with principle axes rotated by $\pi/4$ relative to the (x, y) axis,*
3. *a mirror*

blocks all light passing through the linear polarizer on the return trip through the mirror.

To be clear: the problem involves the following situation: incident light hits the linear polarizer, passes through the waveplate, reflects from the mirror, passes back through the waveplate, and finally through the linear polarizer. Our goal is to show that no light makes it through the linear polarizer on the return trip from the mirror.

We begin by assembling the Jones matrices for the optical elements on the way to the mirror. From Equation 2.5, the matrix for a y linear polarizer is

$$\mathbf{M}_{\text{lin}}^{(y)} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$

while from Equation 2.8, the matrix for a waveplate with axes rotated by $\pi/4$ relative to the (x, y) axes, which we will call $\mathbf{M}_{\text{qw}}^{+}$, is

$$\mathbf{M}_{\text{qw}}^{+} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}.$$

Next, we multiply the linear polarizer and waveplate matrices to get

$$\mathbf{T}_0 \equiv \mathbf{M}_{\text{qw}}^{+} \mathbf{M}_{\text{lin}}^{(x)} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -i \\ 0 & 1 \end{pmatrix}.$$

We have to be careful with the mirror—when light hits the mirror, the sign of the optical axis is reversed, so the corresponding transfer matrix is

$$\mathbf{M}_{\text{mirror}} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Because the mirror reverses the optical axis, quarter waveplate on the return trip behaves as a waveplate rotated by an angle $-\pi/4$ relative the (x, y) axes; we will write the corresponding matrix $\mathbf{M}_{\text{qw}}^{-}$. From the general expression in Equation 2.7, the matrix is

$$\mathbf{M}_{\text{qw}}^{-} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}.$$

The linear polarizer is the same on the return trip, and the total transfer matrix encoding the journey to the mirror and back is

$$\begin{aligned} \mathbf{T}_{\text{tot}} &= \frac{1}{\sqrt{2}} \mathbf{M}_{\text{lin}}^{(y)} \mathbf{M}_{\text{qw}}^{-} \mathbf{M}_{\text{mirror}} \mathbf{T}_0 \\ &= \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \cdot \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -i \\ 0 & 1 \end{pmatrix} \\ &= \frac{1}{2} \begin{pmatrix} 0 & 0 \\ i & 1 \end{pmatrix} \begin{pmatrix} 0 & i \\ 0 & 1 \end{pmatrix} \\ &= \frac{1}{2} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}. \end{aligned}$$

As expected, the light is completely blocked on the return trip, and the combination of optical elements acts as a so-called *optical isolator*.

2.6 Jones Vector for Elliptically Polarized Light

Determine the Jones vector for elliptically polarized light with axes of polarization rotated by $\pi/4$ relative to the (x, y) axes, where the ellipse's major and minor axes (in the ellipse's system of principle axes) have lengths $2E_0$ and E_0, respectively.

We begin by writing the light's Jones vector in the ellipse's system of principle axes S', i.e. the system rotated by $\pi/4$ relative to the (x, y) axes.

We can derive the Jones vector for the above elliptic polarization from the general expression for a Jones vector, i.e.

$$\mathbf{J} = \frac{1}{\sqrt{E_{0_x}^2 + E_{0_y}^2}} \begin{pmatrix} E_{0_x} \\ E_{0_y} e^{i\delta} \end{pmatrix},$$

where δ is the phase shift between the x and y electric field components. In the ellipse's system of principle axes, $E_{0_x} = 2E_0$ and $E_{0_y} = E_0$, while, assuming right-handed elliptic polarization, the phase difference between the field components is $-\pi/2$. The corresponding Jones vector in the system of principle axes is then

$$\mathbf{J}' = \frac{E_0}{E_0\sqrt{2^2+1}} \begin{pmatrix} 1 \\ e^{-i\frac{\pi}{2}} \end{pmatrix} = \frac{1}{\sqrt{5}} \begin{pmatrix} 2 \\ -i \end{pmatrix}.$$

To find the Jones vector $\mathbf{J}$ in the S system, we simply rotate the S' vector $\mathbf{J}$ by an angle of $\pi/4$. Using an arbitrary angle θ for generality, the Jones vector $\mathbf{J}$ for light with right-handed elliptic polarization is

$$\mathbf{J} = \mathbf{R}^\top \mathbf{J}' = \frac{1}{\sqrt{5}} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} 2 \\ -i \end{pmatrix} = \frac{1}{\sqrt{5}} \begin{pmatrix} 2\cos\theta + i\sin\theta \\ 2\sin\theta - i\cos\theta \end{pmatrix}. \tag{2.10}$$

Finally, for our concrete case with an angle $\theta = \pi/4$, we have

$$\mathbf{J} = \frac{1}{\sqrt{5}} \begin{pmatrix} \sqrt{2} + i\frac{\sqrt{2}}{2} \\ \sqrt{2} - i\frac{\sqrt{2}}{2} \end{pmatrix} = \frac{1}{\sqrt{5}} \frac{\sqrt{2}}{2} \begin{pmatrix} 2+i \\ 2-i \end{pmatrix} = \frac{1}{\sqrt{10}} \begin{pmatrix} 2+i \\ 2-i \end{pmatrix}.$$

3 Third Exercise Set

3.1 Elliptical Polarization

Consider right-elliptically polarized (REP) light, where the ellipse's axes are rotated by $\pi/6$ respect to the x and y axes, while the ellipses major and minor axes, in the system of principle axes, are $2E_0$ and E_0, respectively. We send this elliptically-polarized light once through a horizontally-oriented linear polarizer and once through a vertically-oriented linear polarizer. Find the ratio of intensities of the transmitted light in the following ways:

1. *by rotating the polarized light's Jones vector to align with the linear polarizer's x and y axes*
2. *by rotating the two linear polarizers to align with the elliptical polarization's system of principle axes*

Version One: Rotating Jones Vector to Align with Polarizers

In Equation 2.10 in the last problem of the previous exercise set, we found the Jones vector for REP light, with the ellipse rotated by an angle θ relative to the (x, y) axes and with major and minor axes $2E_0$ and E_0, was (in the lab system of (x, y) axes)

$$\mathbf{J} = \mathbf{R}^\top(\theta)\mathbf{J}' = \frac{1}{\sqrt{5}}\begin{pmatrix} 2\cos\theta + i\sin\theta \\ 2\sin\theta - i\cos\theta \end{pmatrix}.$$

Using the result for $\theta = \pi/6$ leads to the Jones vector

$$\mathbf{J} = \frac{1}{\sqrt{5}}\begin{pmatrix} 2\frac{\sqrt{3}}{2} + i\frac{1}{2} \\ 1 - i\frac{\sqrt{3}}{2} \end{pmatrix}.$$

We first consider how this Jones vector transforms through a horizontal linear polarizer (with axis of polarization parallel to the x axis). A Jones matrix for an x linear polarizer preserves only the x component of an incident Jones vector, which means the vector $\mathbf{J}$ transforms as

$$\mathbf{J}_x = \frac{1}{\sqrt{5}}\begin{pmatrix} \sqrt{3} + \frac{i}{2} \\ 0 \end{pmatrix}.$$

Similarly, when passing through a y linear polarizer, which transmits only the y component of an incident Jones vector, we have

$$\mathbf{J}_y = \frac{1}{\sqrt{5}}\begin{pmatrix} 0 \\ 1 - i\frac{\sqrt{3}}{2} \end{pmatrix}.$$

Transmitted intensity is proportional to the squared magnitude of the transmitted Jones vector. Using the above $\mathbf{J}_x$ and $\mathbf{J}_y$ we have

$$I_x \propto |\mathbf{J}_x|^2 = \mathbf{J}_x\mathbf{J}_x^* = \frac{1}{5}\left(3 + \frac{1}{4}\right) = \frac{1}{5}\frac{13}{4}$$
$$I_y \propto |\mathbf{J}_y|^2 = \mathbf{J}_y\mathbf{J}_y^* = \frac{1}{5}\left(1 + \frac{3}{4}\right) = \frac{1}{5}\frac{7}{4}.$$

The ratio of transmitted intensities is thus

$$\frac{I_x}{I_y} = \frac{13}{7}.$$

Version Two: Rotating Polarizers Vector to Align with Elliptical Axes

We now solve the problem by rotating the linear polarizers to align with the ellipse's principle axes. From Equation 2.9, the Jones matrix for a linear polarizer rotated by an angle θ relative to the (x, y) axes is

$$\mathbf{T}' = \begin{pmatrix} \cos^2\theta & \sin\theta\cos\theta \\ \sin\theta\cos\theta & \sin^2\theta \end{pmatrix}.$$

We begin with the matrix $\mathbf{T}'_x$ for the rotated linear polarizer. Since we are rotating the polarizer from the lab frame (x, y) into the ellipse's frame (rather than rotating the ellipse into the lab frame) we must take the angle θ with a negative sign, i.e. $\theta = -\pi/6$ The matrix $\mathbf{T}'_x$ is thus

$$\mathbf{T}'_x = \begin{pmatrix} \frac{3}{4} & -\frac{\sqrt{3}}{4} \\ -\frac{\sqrt{3}}{4} & \frac{1}{4} \end{pmatrix}.$$

We now find the matrix $\mathbf{T}'_y$. Since the y axis makes an angle $\pi/2$ with the x axis, and we used a rotation angle $\theta = -\pi/6$ for the x axes, the appropriate rotation angle for the y axes is simply $-\pi/6 + \pi/2$. The corresponding transformed Jones matrix for the y linear polarizer is

$$\mathbf{T}'_y = \begin{pmatrix} \frac{1}{4} & \frac{\sqrt{3}}{4} \\ \frac{\sqrt{3}}{4} & \frac{3}{4} \end{pmatrix}.$$

Meanwhile, the REP light's Jones vector $\mathbf{J}'$ in the system of principle axes, where $\theta = 0$, is

$$\mathbf{J}' = \frac{1}{\sqrt{5}} \begin{pmatrix} 2\cos\theta + i\sin\theta \\ 2\sin\theta - i\cos\theta \end{pmatrix} = \frac{1}{\sqrt{5}} \begin{pmatrix} 2 \\ -i \end{pmatrix}.$$

Using the just-derived matrices for the rotated linear polarizers, the matrix $\mathbf{J}'$ transforms as

$$\mathbf{T}'_x\mathbf{J}' = \frac{1}{4\sqrt{5}} \begin{pmatrix} 6 + i\sqrt{3} \\ -2\sqrt{3} - i \end{pmatrix} \qquad \text{and} \qquad \mathbf{T}'_y\mathbf{J}' = \frac{1}{4\sqrt{5}} \begin{pmatrix} 2 - i\sqrt{3} \\ 2\sqrt{3} - 3i \end{pmatrix}.$$

As before, we find the transmitted intensities from the squared Jones vectors. For the x polarizer, we have

$$I_x \propto \left|\mathbf{J}'_x\right|^2 = \frac{1}{5}\frac{1}{16}(36 + 3 + 12 + 1)$$

and for y polarizer

$$I_y \propto \left|\mathbf{J}'_y\right|^2 = \frac{1}{5}\frac{1}{16}(4 + 3 + 12 + 9).$$

The ratio is

$$\frac{I_x}{I_y} = \frac{52}{28} = \frac{13}{7},$$

which is the same result, as it must be, from the earlier analysis in which we rotated the ellipse into the (x, y) lab system.

3.2 Orientation of Elliptical Polarization

The phase difference the x and y components of elliptically polarized light is $\delta = \pi/4$, while the ratio between the electric field components is $E_x/E_y = 2$. Find:

1. *the angle θ at which the ellipse is rotated relative to the (x, y) axes*
2. *the ratio of the ellipse's major and minor axes.*

We find the ellipse's angle θ relative to the (x, y) axes using the general formula

$$\tan 2\theta = \frac{2E_{0_x}E_{0_y}\cos\delta}{E_{0_x}^2 - E_{0_y}^2},$$

where E_{0_x} and E_{0_y} are the electric field amplitudes, while δ is the phase difference between the field components. In our case, we can write $E_{0_x} = 2E_{0_y}$, producing

$$\tan 2\theta = \frac{2\cdot(2E_{0_y})E_{0_y}\cos\frac{\pi}{4}}{E_{0_y}^2(2^2-1)} = \frac{2\cdot 2\cos\frac{\pi}{4}}{4-1} = \frac{4}{3}\frac{\sqrt{2}}{3} = \frac{2\sqrt{2}}{3}.$$

The angle θ is thus

$$\theta = \frac{1}{2}\arctan\frac{2\sqrt{2}}{3} = 0.38 = 21.7^\circ.$$

We find the ratio of axes lengths with the general formula

$$\frac{b}{a} = \frac{E_{0_y}\sin\delta\cos\theta}{E_{0_x}\cos\theta + E_{0_y}\cos\delta\sin\theta},$$

where δ is the phase difference between the field components, in our case equal to $\delta = \pi/4$. After cancelling electric field amplitudes, the ratio of axes lengths comes out to

$$\frac{b}{a} = \frac{\sin\frac{\pi}{4}\cos\theta}{2\cdot\cos\theta + \cos\frac{\pi}{4}\sin\theta} = \frac{\frac{\sqrt{2}}{2}\cos\theta}{2\cos\theta + \frac{\sqrt{2}}{2}\sin\theta}.$$

After dividing above and below by $(\sqrt{2})/2\cos\theta$, this comes out to

$$\frac{b}{a} = \frac{1}{\frac{4}{\sqrt{2}} + \tan\theta}.$$

Finally, we substitute in the earlier value of θ to get

$$\frac{b}{a} = \frac{1}{\frac{4}{\sqrt{2}} + \tan 21.7^\circ} \approx 0.31.$$

3.3 Theory: Electromagnetic Waves in Conductors

We provide a minimal review of theory on electromagnetic wave propagation in conductors; a more thorough treatment is given in lecture.

We begin with the ansatz for electromagnetic waves in conductors, which reads

$$\mathbf{E}(\mathbf{r}, t) = \mathbf{E}_0 e^{-i(\mathbf{K}\cdot\mathbf{r}+\omega t+\delta)}. \tag{3.1}$$

This is the same ansatz as for EM waves in free space, with one important exception: we allow for a complex wave vector $\mathbf{K} \in \mathbb{C}^3$.

For now, we will restrict our analysis to normally incident EM waves, in which case the magnitude of the complex wave vector evaluates to

$$\mathcal{K} = \mathcal{N}\frac{w}{c_0} \equiv \mathcal{N}k_0,$$

$\mathcal{N} \in \mathbb{C}$ is the material's complex refractive index, equal to

$$\mathcal{N}^2 = \varepsilon\mu + \frac{i\sigma_{\mathrm{E}}\mu}{\varepsilon_0\omega}, \tag{3.2}$$

where ε, μ and σ_{E} are the material's relative permittivity, relative permeability, and electric conductivity, respectively. Henceforth we will consider only non-magnetic materials with $\mu = 1$.

The next step in the analysis is to decompose the complex refractive index into real and imaginary components:

$$\mathcal{N} = n_{\mathrm{Re}} + in_{\mathrm{Im}}.$$

In this case, considering only propagation in the z direction (into the conductor) and dropping the phase term δ in Equation 3.1, the electric field magnitude comes out to

$$E = E_0 e^{-i(\mathcal{K}z+\omega t)} = E_0 e^{i\mathcal{N}k_0 z} e^{-i\omega t} = E_0 e^{in_{\mathrm{Re}}k_0 z} e^{-n_{\mathrm{Im}}k_0 z} e^{-i\omega t},$$

where the second term, $e^{-n_{\mathrm{Im}}k_0 z}$, represents attenuation of the electromagnetic waves with increasing distance z travelled through a conductive material. The characteristic length encoding the attenuation, called the *skin depth*, is

$$d = \frac{1}{n_{\mathrm{Im}}k_0}. \tag{3.3}$$

3.4 Skin Depth

Determine the skin depth in an indium tin oxide (ITO) electrode for light with a wavelength $\lambda = 500\,\mathrm{nm}$. *The resistivity of ITO is* $0.002\,\Omega\,\mathrm{m}$. *What fraction of normally incident light is transmitted through an electrode with a width of* $50\,\mathrm{nm}$*?*

From Equation 3.3, the skin depth in a conductor is given by

$$d = \frac{1}{n_{\mathrm{Im}}k_0},$$

where n_{Im} is the conductor's imaginary refractive index component. Finding the skin depth d thus reduces to finding the refractive index component n_{Im} using the electrode's known resistivity. We begin with Equation 3.2 for the complex refractive index $\mathcal{N}$ and assume the electrode is non-magnetic ($\mu = 1$), which gives

$$\mathcal{N}^2 = \varepsilon + \frac{i\sigma_{\mathrm{E}}}{\varepsilon_0\omega}.$$

We then combine this equation with the decomposition $\mathcal{N} = n_{\mathrm{Re}} + in_{\mathrm{Im}}$, producing

$$\mathcal{N}^2 = n_{\mathrm{Re}}^2 - n_{\mathrm{Im}}^2 + 2in_{\mathrm{Re}}n_{\mathrm{Im}} = \varepsilon + \frac{i\sigma_{\mathrm{E}}}{\varepsilon_0 w}.$$

We then equate the real and imaginary components on both sides of the equation, which gives

$$n_{\mathrm{Re}}^2 - n_{\mathrm{Im}}^2 = \varepsilon \qquad \text{and} \qquad 2n_{\mathrm{Re}}n_{\mathrm{Im}} = \frac{\sigma_{\mathrm{E}}}{\varepsilon_0\omega}.$$

Next, we solve the second equation for n_{Im}, substitute this into the first equation $n_{\mathrm{Re}}^2 - n_{\mathrm{Im}}^2$, and get a quadratic equation for the real component n_{Re}, which reads

$$n_{\mathrm{Re}}^4 - \left(\frac{\sigma_{\mathrm{E}}}{2\varepsilon_0\omega}\right)^2 = \varepsilon n_{\mathrm{Re}}^2.$$

Applying the quadratic formula leads to

$$n_{\mathrm{Re}}^2 = \frac{\varepsilon}{2} \pm \frac{1}{2}\sqrt{\varepsilon^2 + \frac{\sigma_{\mathrm{E}}^2}{\varepsilon_0^2\omega^2}}. \tag{3.4}$$

We choose the solution with a positive sign, which follows from the requirement that n_{Re} be a real number. From $n_{\mathrm{Re}}^2 - n_{\mathrm{Im}}^2 = \varepsilon$, the corresponding solution for n_{Im} is

$$n_{\mathrm{Im}}^2 = -\frac{\varepsilon}{2} + \frac{1}{2}\sqrt{\varepsilon^2 + \frac{\sigma_{\mathrm{E}}^2}{\varepsilon_0^2\omega^2}}. \tag{3.5}$$

Next, we make an approximation: in conductive materials with large σ_{E}, we can safely assume the relationship $\varepsilon \ll \frac{\sigma_{\mathrm{E}}}{\varepsilon_0\omega}$, which, after a few steps of algebra, leads to

$$n_{\mathrm{Im}} \approx \sqrt{\frac{\sigma_{\mathrm{E}}}{2\varepsilon_0\omega}}.$$

Using this expression for n_{Im}, the skin depth is

$$d = \frac{1}{k_0 n_{\mathrm{Im}}} = \frac{c_0}{\omega}\sqrt{\frac{2\varepsilon_0\omega}{\sigma_{\mathrm{E}}}} = \sqrt{\frac{2}{\sigma_{\mathrm{E}}\omega\mu_0}} = \sqrt{\frac{2\xi}{\omega\mu_0}},$$

where we have used $\varepsilon_0\mu_0c^2 = 1$ and $\sigma_{\mathrm{E}} = 1/\xi$, where ξ denotes the material's resistivity. It remains to substitute in numerical values. As an intermediate step, using the known wavelength $\lambda = 500\,\mathrm{nm}$, the frequency ω comes out to

$$\omega = \frac{2\pi c}{\lambda} = \frac{2\pi \cdot (3 \cdot 10^8\,\mathrm{m/s})}{500 \cdot 10^{-9}\,\mathrm{m}} = 3.77 \cdot 10^{15}\,\mathrm{s}^{-1}.$$

Using the just-computed frequency ω, the skin depth is

$$d = \sqrt{\frac{2 \cdot 0.002\,\Omega\,\mathrm{m}}{3.77 \cdot 10^{15}\,\mathrm{s}^{-1} \cdot 4\pi e - 7\,\mathrm{V\,s\,A^{-1}\,m^{-1}}}} = 930\,\mathrm{nm}.$$

Using the just-computed skin depth, the transmittance T for a 50 nm electrode is

$$T = |E|^2 = 1 \cdot \left|e^{-\frac{z}{d}}\right|^2 = e^{-2\frac{z}{d}} = e^{-2\frac{50\,\mathrm{nm}}{930\,\mathrm{nm}}} = 0.89.$$

Note that, in practice, the electrode would always be surrounded by a thin coating of protective material, which would decrease the transmittance value by reflecting incident EM waves. We have neglected this reflection in our simplified model.

3.5 Reflectance

Determine the reflectance of both blue and red light (assume wavelengths 400 nm and 700 nm, respectively) for normal incidence from air onto gold, silver, and aluminum. Use the real and imaginary components of the refractive index given in the table below; you may assume air has a refractive index $n = 1$.

	n_{Re} (400 nm)	n_{Im} (400 nm)	n_{Re} (700 nm)	n_{Im} (700 nm)
Au	1.4	1.9	0.14	3.8
Ag	0.08	1.9	0.08	4.6
Al	0.4	4.5	1.5	7.0

From the accompanying lecture notes, the Fresnel equation for the reflection coefficient for electromagnetic waves incident on a boundary between two non-conducting materials with refractive indices n_1 and n_2 is

$$r = \frac{n_1 \cos\alpha - n_2 \cos\beta}{n_1 \cos\alpha + n_2 \cos\beta},$$

where α and β denote the angles of incident and refraction, respectively.

This equation is fundamentally derived from the boundary conditions on the Maxwell equations along the interface between the two non-conductive materials. Without derivation, it turns out that in non-ideal conductors with finite conductivity, we can use the same Fresnel equation as above, as long as we allow for a complex index of refraction.

Similarly, Snell's law holds in conductors, just with complex index of refraction.

If we define the first material as air and the second as metal and write $n_1 = n_{\mathrm{air}} = 1$ and $n_2 \equiv n$, the Fresnel equation for normal incidence with $\alpha = 0$ simplifies to

$$r = \frac{1 - n\cos\beta}{1 + n\cos\beta}.$$

Formally, the relationship $\beta \equiv 0$ for normal incidence holds only in materials with a real refractive index, and for conductors (with complex n) we would find the refractive angle β with the generalized version of Snell's law. In our case, with $n_1 = 1$ and $\alpha = 0$, this reads

$$1 \cdot \sin 0 = 0 = (n_{\mathrm{Re}} + i n_{\mathrm{Im}}) \sin\beta.$$

We now have two solutions—the classic $\beta = 0$ and the possibility of $n_{\mathrm{Re}} = i n_{\mathrm{Im}}$. For lack of established theory from lectures at this point in the course, we will simply assume $\beta = 0$, but keep in mind that more exotic refractive behavior is possible in conductors.

Assuming $\beta = 0$ and writing $n = n_{\mathrm{Re}} + i n_{\mathrm{Im}}$, the reflection coefficient is simply

$$r = \frac{1 - n_{\mathrm{Re}} - i n_{\mathrm{Im}}}{1 + n_{\mathrm{Re}} + i n_{\mathrm{Im}}}.$$

We find the reflectance R from the squared modulus of r, which comes out to

$$R = |r|^2 = \frac{(1 - n_{\mathrm{Re}})^2 + n_{\mathrm{Im}}^2}{(1 + n_{\mathrm{Re}})^2 + n_{\mathrm{Im}}^2}. \tag{3.6}$$

Substituting in the values of n_{Re} and n_{Im} from the table given at the beginning of this exercise and computing the resulting value of R from Equation 3.6 leads to the numerical values

	R (400 nm)	R (400 nm)
Au	0.40	0.97
Ag	0.93	0.99
Al	0.93	0.89

Some lessons from these results are:

- Gold reflects blue light much more strongly than red light, and is useful for specialized applications requiring changing reflectance with wavelength.
- Silver has strong, relatively uniform reflectance across the visible spectrum, and is thus an excellent material for mirrors.
- Aluminum is a cheaper alternative to silver for mirrors in that aluminum also has uniform reflection across the visible spectrum, but with lower values of R.

4 Fourth Exercise Set

4.1 Reflectance for Good Conductors (for Normal Incidence)

Derive the dependence of reflectance on wavelength for normally incident light on a good conductor in the limit $\sigma_{\mathrm{E}} \gg \omega\varepsilon_0$. *You may assume* $n \gg 1$.

We begin with the TE reflection coefficient, which for review reads

$$r_s = \frac{n_1 \cos\theta_{\mathrm{i}} - n_2 \cos\theta_{\mathrm{t}}}{n_1 \cos\theta_{\mathrm{i}} + n_2 \cos\theta_{\mathrm{t}}}.$$

As in the previous exercise set, we assume $\theta_{\mathrm{i}} = \theta_{\mathrm{t}} = 0$ for normal incidence and write $n_1 = 1$ for air and $n_2 \equiv n = n_{\mathrm{Re}} + in_{\mathrm{Im}}$ since the refractive index in conductors is complex. The resulting expression for r_s is

$$r_s = \frac{1 - n_{\mathrm{Re}} - in_{\mathrm{Im}}}{1 + n_{\mathrm{Re}} + in_{\mathrm{Im}}}.$$

As in Equation 3.6, the corresponding reflectance is

$$R = |r|^2 = \frac{(1 - n_{\mathrm{Re}})^2 + n_{\mathrm{Im}}^2}{(1 + n_{\mathrm{Re}})^2 + n_{\mathrm{Im}}^2}. \tag{4.1}$$

Next, we recall from Equations 3.4 and 3.5 that in the non-magnetic limit $\mu \to 1$, the refractive index components n_{Re} and n_{Im} are given by

$$n_{\mathrm{Re}}^2 = \frac{\varepsilon}{2} + \frac{1}{2}\sqrt{\varepsilon^2 + \frac{\sigma_{\mathrm{E}}^2}{\varepsilon_0^2\omega^2}} \qquad \text{and} \qquad n_{\mathrm{Im}}^2 = -\frac{\varepsilon}{2} + \frac{1}{2}\sqrt{\varepsilon^2 + \frac{\sigma_{\mathrm{E}}^2}{\varepsilon_0^2\omega^2}}.$$

For a good conductor, for which $\frac{\sigma_{\mathrm{E}}}{\varepsilon_0\omega} \gg \varepsilon$, these components reduce further to

$$n_{\mathrm{Re}}^2 \approx \frac{1}{2}\frac{\sigma_{\mathrm{E}}}{\varepsilon_0\omega} \qquad \text{and} \qquad n_{\mathrm{Im}}^2 \approx \frac{1}{2}\frac{\sigma_{\mathrm{E}}}{\varepsilon_0\omega}. \tag{4.2}$$

Seeing that n_{Re} and n_{Im} are approximately equal, we introduce the shorter notation $n_{\mathrm{Re}} = n_{\mathrm{Im}} \equiv n$; we then substitute n into Equation 4.1, producing

$$R = \frac{(1-n)^2 + n^2}{(1+n)^2 + n^2} = \frac{1 - 2n + 2n^2}{1 + 2n + 2n^2}.$$

Further, assuming $n \gg 1$, we can drop the 1 term, and R simplifies to

$$R \approx \frac{-2 + 2n}{2 + 2n} = \frac{n-1}{n+1} \sim 1 - \frac{2}{n+1} \approx 1 - \frac{2}{n}$$

Finally, we substitute in n from Equation 4.2 to get

$$R = 1 - 2\sqrt{\frac{2\varepsilon_0\omega}{\sigma_{\mathrm{E}}}} = 1 - 4\sqrt{\frac{\pi\varepsilon_0\nu}{\sigma_{\mathrm{E}}}} = 1 - 4\sqrt{\frac{\pi\varepsilon_0 c}{\lambda\sigma}},$$

where we have used $\omega = 2\pi\nu$ and $\lambda\nu = c$. This equation gives a typical dependence of reflectance on wavelength for good conductors (for normally incident light).

4.2 Reflectance in Conductors for Angled Incidence

Light is incident from air onto aluminum at an angle of incidence $\theta_i = 30°$. *Determine the magnitude and direction of the complex wave vector in aluminum, and determine the corresponding reflectance for TE-polarized light. The real and imaginary components of aluminum's refractive index are* $n_{Re} = 0.4$ *and* $n_{Im} = 4.5$.

4.2.1 Part One: Magnitude and Direction of Complex Wave Vector

Aluminum is a conductor, and thus has a complex index of refraction, which we will denote by $\mathcal{N} \in \mathbb{C}$. More so, in the aluminum the wave vector and the transmitted angle θ_{t} used the Fresnel equations are also constant.

We begin by writing the wave vector in the aluminum in the form

$$\boldsymbol{\mathcal{K}} = \mathbf{k}_{\mathrm{Re}} + i\mathbf{k}_{\mathrm{Im}},$$

where the unusual notation $\boldsymbol{\mathcal{K}}$ is used to make clear that the transmitted aluminum wave vector is complex.

The electric field's phase information in a conductor is given by the plane wave ansatz

$$e^{i\boldsymbol{\mathcal{K}}\cdot\mathbf{r}} = e^{-i\mathbf{k}_{\mathrm{Re}}\cdot\mathbf{r}}e^{-\mathbf{k}_{\mathrm{Im}}\cdot\mathbf{r}}.$$

Conservation of phase along the border between the air and aluminum requires

$$e^{i\mathbf{k}_{\mathrm{i}}\mathbf{r}} = e^{i\mathbf{k}_{\mathrm{Re}}\cdot\mathbf{r}}e^{-\mathbf{k}_{\mathrm{Im}}\mathbf{r}} = e^{i\mathbf{k}_{\mathrm{Re}}\cdot\mathbf{r}-\mathbf{k}_{\mathrm{Im}}\mathbf{r}} \implies i\mathbf{k}_{\mathrm{i}}\cdot\mathbf{r} = i\mathbf{k}_{\mathrm{Re}}\cdot\mathbf{r} - \mathbf{k}_{\mathrm{Im}}\cdot\mathbf{r}$$

for all $\mathbf{r}$ in the boundary. This equality can hold only if

$$\mathbf{k}_{\mathrm{Im}}\cdot\mathbf{r} = 0$$

for all $\mathbf{r}$ in the boundary. In other words, $\mathbf{k}_{\mathrm{Im}}$ is normal to the boundary.

Meanwhile, from lecture, the complex wave vector in a conductor takes the form

$$\boldsymbol{\mathcal{K}}\cdot\boldsymbol{\mathcal{K}} = k_0^2\mathcal{N}^2,$$

where $k_0 = \omega/c_0$ is the light's wave number in vacuum, while $\mathcal{N} \in \mathbb{C}$ is the conductor's index of refraction. We then substitute in $\boldsymbol{\mathcal{K}} = \mathbf{k}_{\mathrm{Re}} + i\mathbf{k}_{\mathrm{Im}}$ to get

$$k_0^2\mathcal{N}^2 = \boldsymbol{\mathcal{K}}\cdot\boldsymbol{\mathcal{K}} = (\mathbf{k}_{\mathrm{Re}} + i\mathbf{k}_{\mathrm{Im}})\cdot(\mathbf{k}_{\mathrm{Re}} + i\mathbf{k}_{\mathrm{Im}}) = k_{\mathrm{Re}}^2 + 2i\mathbf{k}_{\mathrm{Im}}\cdot\mathbf{k}_{\mathrm{Re}} - k_{\mathrm{Im}}^2. \tag{4.3}$$

We now return to the condition $\mathbf{k}_{\mathrm{Im}}\cdot\mathbf{r} = 0$ along the boundary surface. Geometrically, this means that $\mathbf{k}_{\mathrm{Im}}$ is normal to the surface. It follows that the angle β between $\mathbf{k}_{\mathrm{Re}}$ and the surface normal $\hat{\mathbf{n}}$ equals the angle between $\mathbf{k}_{\mathrm{Re}}$ and $\mathbf{k}_{\mathrm{Im}}$, since $\mathbf{k}_{\mathrm{Im}}$ is parallel to $\hat{\mathbf{n}}$.

Since the angle between $\mathbf{k}_{\mathrm{Re}}$ and $\mathbf{k}_{\mathrm{Im}}$ is β, the dot product $\mathbf{k}_{\mathrm{Im}}\cdot\mathbf{k}_{\mathrm{Re}}$ may be written

$$\mathbf{k}_{\mathrm{Im}}\cdot\mathbf{k}_{\mathrm{Re}} = k_{\mathrm{Im}}k_{\mathrm{Re}}\cos\beta.$$

Using $\mathbf{k}_{\mathrm{Im}}\cdot\mathbf{k}_{\mathrm{Re}} = k_{\mathrm{Im}}k\cos\beta$, Equation 4.3 becomes

$$k_0^2\mathcal{N}^2 = k_{\mathrm{Re}}^2 + 2ik_{\mathrm{Im}}k_{\mathrm{Re}}\cos\beta - k_{\mathrm{Im}}^2.$$

Next, a slight trick—we use the general identity $\cos^2\beta + \sin^2\beta = 1$ to write

$$\begin{aligned}
k_0^2\mathcal{N}^2 &= k_{\mathrm{Re}}^2(\cos^2\beta + \sin^2\beta) + 2ik_{\mathrm{Im}}k_{\mathrm{Re}}\cos\beta - k_{\mathrm{Im}}^2\\
&= k_{\mathrm{Re}}^2\cos^2\beta + 2ik_{\mathrm{Im}}k_{\mathrm{Re}}\cos\beta - k_{\mathrm{Im}}^2 + k_{\mathrm{Re}}^2\sin^2\beta\\
&= (k_{\mathrm{Re}}\cos\beta + ik_{\mathrm{Im}})^2 + k_{\mathrm{Re}}^2\sin^2\beta.
\end{aligned} \tag{4.4}$$

QUESTION. We wrote law of refraction (or is this phase conservation?) in the form

$$k_{\mathrm{i}}\sin\alpha = k_{\mathrm{Re}}\sin\beta.$$

I'm cool with $n_1 \sin\alpha = n_2 \sin\beta$, but then is $k_\mathrm{i} = n_1 k_0$ and $k_\mathrm{t} = n_2 k_0$?

In any case, we end up with

$$k_0 \sin\alpha = k_\mathrm{Re} \sin\beta \implies k_\mathrm{Re}^2 \sin^2\beta = k_0^2 \sin^2\alpha,$$

which we substitute into Equation 4.4 and get

$$k_0^2 \mathcal{N}^2 = (k_\mathrm{Re}\cos\beta + ik_\mathrm{Im})^2 + k_0^2 \sin^2\alpha$$

After rearranging and combining like terms, we have

$$k_0^2 \left(\mathcal{N}^2 - \sin^2\alpha\right) = (k_\mathrm{Re}\cos\beta + ik_\mathrm{Im})^2$$

We then take the square root of the equation, giving

$$k_\mathrm{Re}\cos\beta + ik_\mathrm{Im} = k_0\sqrt{\mathcal{N}^2 - \sin^2\alpha}. \tag{4.5}$$

Equation 4.5 is satisfied only if the square root has both a real and imaginary component, which we write as

$$k_\mathrm{Re}\cos\beta = \operatorname{Re} k_0\sqrt{\mathcal{N}^2 - \sin^2\alpha} \qquad \text{and} \qquad k_\mathrm{Im} = \operatorname{Im} k_0\sqrt{\mathcal{N}^2 - \sin^2\alpha}.$$

From the data given in the problem instructions, we know

$$\mathcal{N} = n_\mathrm{Re} + in_\mathrm{Im} = 0.4 + 4.5i \qquad \text{and} \qquad \sin\alpha = \sin 30^\circ = 1/2.$$

Using a complex number calculator, the square root in Equation 4.5 evaluates to

$$\sqrt{\mathcal{N}^2 - \sin^2\alpha} = \sqrt{(0.4 + 4.5i) - 0.5} \approx 0.395 + 4.555i. \tag{4.6}$$

Note that (assuming no complex calculator) we could note that $|\mathcal{N}|^2$ is much larger than 0.5, which permits the approximation

$$\sqrt{\mathcal{N}^2 - \sin^2\alpha} = \sqrt{\mathcal{N}^2 - 0.5} \approx \sqrt{\mathcal{N}^2} = \mathcal{N} = 0.4 + 4.5i.$$

In any case, having computed the complex square root, Equation 4.5 results in

$$k_\mathrm{Re}\cos\beta = 0.395 \cdot k_0 \qquad \text{and} \qquad k_\mathrm{Im} = 4.555 \cdot k_0$$

We find the angle β from $k_0 \sin\alpha = k_\mathrm{Re}\sin\beta$. The calculation reads

$$\sin\beta = \frac{k_0}{k_\mathrm{Re}} \sin\alpha = \frac{k_0}{(0.395 \cdot k_0)/\cos\beta} \sin 30^\circ = \frac{\cos\beta}{0.395} \cdot \frac{1}{2};$$

after rearranging, we find β from

$$\tan\beta = \frac{1}{0.395} \cdot \frac{1}{2} \implies \beta \approx 51.7^\circ$$

With the angle β known, we can now find the magnitude k_Re, which comes out to

$$k_\mathrm{Re} = k_0 \frac{0.395}{\cos\beta} = k_0 \frac{0.395}{\cos 51.7^\circ} \approx 0.64 \cdot k_0.$$

4.2.2 Part Two: Reflectance for TE Polarization

We aim to find the reflectance R_s for TE polarization for the light incident from air onto aluminum at $\theta_\mathrm{i} = 30°$. We begin with the TE reflection coefficient, which reads

$$r_s = \frac{n_1 \cos\theta_\mathrm{i} - n_2 \cos\theta_\mathrm{t}}{n_1 \cos\theta_\mathrm{i} + n_2 \cos\theta_\mathrm{t}}.$$

Because aluminum is a conductor, the transmitted angle θ_t is potentially complex (and is not equal to the angle β used above[1]). Meanwhile, (since air is non-conducting) θ_i and α both refer to the same 30° angle of incidence.

In our case, for $n_1 = 1$ (air) and $\theta_\mathrm{i} = \alpha$, and $n_2 = \mathcal{N}$ the reflection coefficient reads

$$r_s = \frac{\cos\alpha - \mathcal{N}\cos\theta_\mathrm{t}}{\cos\alpha + \mathcal{N}\cos\theta_\mathrm{t}} = \frac{\cos\alpha - \mathcal{N}\sqrt{1-\sin^2\theta_\mathrm{t}}}{\cos\alpha + \mathcal{N}\sqrt{1-\sin^2\theta_\mathrm{t}}},$$

where the last line uses the identity $\cos^2 x + \sin^2 x = 1$. We find θ_t from the law of refraction $n_1 \sin\theta_\mathrm{i} = n_2 \sin\theta_\mathrm{t}$, which in our case reads

$$1 \cdot \sin\alpha = \mathcal{N}\sin\theta_\mathrm{t} \implies \sin\theta_\mathrm{t} = \frac{\sin\alpha}{\mathcal{N}}.$$

Using $\sin\theta_\mathrm{t} = \frac{\sin\alpha}{\mathcal{N}}$, the reflection coefficient r_s becomes

$$r_s = \frac{\cos\alpha - \sqrt{\mathcal{N}^2 - \sin^2\alpha}}{\cos\alpha + \sqrt{\mathcal{N}^2 - \sin^2\alpha}}.$$

From Equation 4.6, we know $\sqrt{\mathcal{N}^2 - \sin^2\alpha} \approx 0.395 + 4.555i$. Combined with the given incidence angle $\alpha = 30°$, the reflection coefficient is

$$r_s = \frac{\cos 30° - 0.395 - 4.555i}{\cos 30° + 0.395 + 4.555i} \approx \frac{0.471 - 4.55i}{1.261 + 4.555i},$$

and the corresponding reflectance is

$$R_s = |r_s|^2 \approx \frac{\sqrt{21.97}}{\sqrt{22.34}} \approx 0.992.$$

This large reflectance is typical for good conductors.

4.3 Circularly Polarized Light After Reflection From Glass

Right circular polarized (RCP) light is incident from air on onto glass at incidence angle $\theta_i = 30°$. Determine the polarization of the reflected light. Repeat the calculation for normal incidence, i.e. for $\theta_i = 90°$. Assume glass has a refractive index $n_{glass} = 1.5$.

Since the polarization of the incident light is not specified, we assume it contains a combination of both TE and TM polarizations. Our next step, then, is to compute the

[1] For review, the angle β denotes the angle between the normal to aluminum-air boundary and the real component $\mathbf{k}_\mathrm{Re}$ of the complex wave vector in the aluminum. Physically, since the real component $\mathbf{k}_\mathrm{Re}$ encodes phase information (while $\mathbf{k}_\mathrm{Im}$ encodes attenuation) the angle β is angle at which wave fronts of constant phase propagate through the conductor relative to the surface normal.

reflection coefficients both the light's TE-polarized and TM-polarized components. These are given by the Fresnel equations

$$r_s = -\frac{\sin(\theta_\mathrm{i} - \theta_\mathrm{t})}{\sin(\theta_\mathrm{i} + \theta_\mathrm{t})} \qquad \text{and} \qquad r_p = \frac{\tan(\theta_\mathrm{i} - \theta_\mathrm{t})}{\tan(\theta_\mathrm{i} + \theta_\mathrm{t})}$$

We find the angle of transmission θ_t form the known angle incident angle $\theta_\mathrm{i} = 30°$ and Snell's law. Assuming $n_\mathrm{air} = 1$, the calculation reads

$$\sin\theta_\mathrm{i} = n_\mathrm{glass}\sin\theta_\mathrm{t} \implies \theta_\mathrm{t} = \arcsin\frac{0.5}{1.5} \approx 19.5°.$$

Using $\theta_\mathrm{t} = 19.5°$, the TE and TM reflection coefficients are

$$r_s \approx -0.24 \qquad \text{and} \qquad r_p \approx 0.16.$$

The incident light is right-circular polarized; with reference to Equation 2.3, the Jones vector $\mathbf{J}_\mathrm{i}$ for the incident RCP light is

$$\mathbf{J}_\mathrm{i} = \mathbf{J}_\mathrm{RCP} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix}.$$

Our goal is to find the reflected Jones vector $\mathbf{J}_\mathrm{r}$. We find J_r using the known reflection coefficients r_s and r_p; the calculation reads

$$\mathbf{J}_\mathrm{r} = \begin{pmatrix} r_s & 0 \\ 0 & r_p \end{pmatrix}\mathbf{J}_\mathrm{i} = \frac{1}{\sqrt{2}}\begin{pmatrix} -0.24 & 0 \\ 0 & 0.16 \end{pmatrix}\begin{pmatrix} 1 \\ -i \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} -0.24 \\ 0.16i \end{pmatrix}.$$

In other words, the reflected light has general elliptical polarization without a deeper physical interpretation.

Normal Incidence

For normal incidence we have $\theta_\mathrm{i} = 0 \implies \theta_\mathrm{t} = 0$. The corresponding reflection coefficients are

$$r_s = \frac{n_1 - n_2}{n_1 + n_2} = \frac{1-n}{1+n} = -0.2 \qquad \text{and} \qquad r_p = \frac{n_2 - n_1}{n_1 + n_2} = \frac{n-1}{1+n} = 0.2,$$

while the reflected Jones vector is

$$\mathbf{J}_\mathrm{r} = \begin{pmatrix} r_s & 0 \\ 0 & r_p \end{pmatrix}\mathbf{J}_\mathrm{i} = \frac{1}{\sqrt{2}}\begin{pmatrix} -0.20 & 0 \\ 0 & 0.20 \end{pmatrix}\begin{pmatrix} 1 \\ -i \end{pmatrix} = -\frac{0.2}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix}.$$

The result (compare to Equation 2.3) is LCP light! Lesson: the direction of circular polarization changes during reflection for normal incidence on an optically denser material (where $n_2 > n_1$).

4.4 Brewster's Angle and Brewster Windows

How many Brewster windows must we place end-to-end such that unpolarized light incident on the series of windows is transmitted with a polarization ratio 1 : 100 *relative to the incident light?*

A Brewster window is a thick optical element angled relative to the direction of incidence light such that, after light passes through the window, the incident light's TM-polarized component is completely transmitted. For review, unpolarized light is light for which the direction of the electric field in space varies randomly with time.

Our first goal is to find the transmittance of a Brewster's window for both TM-polarized and TE-polarized light. First we will show that $T_p = 1$, as expected for Brewster's angle. Assume $\theta_\mathrm{i} = \theta_\mathrm{B}$, $n_1 = n_\mathrm{air} = 1$ and $n_2 \equiv n$.

The TM transmission coefficient for the first boundary, where light passes from air into the window, is

$$t_{p_1} = \frac{2n_1 \cos\theta_\mathrm{i}}{n_2\cos\theta_\mathrm{i} + n_1\cos\theta_\mathrm{t}} = \frac{2\cos\theta_\mathrm{B}}{n\cos\theta_\mathrm{B} + \cos\theta_\mathrm{t}}$$

From lecture, Brewster's angle is given by the relationship

$$\tan\theta_\mathrm{B} = \frac{n_2}{n_1} = \frac{n}{1} = n.$$

We then use the result $n = \tan\theta_\mathrm{B}$ in Snell's law, which reads

$$\sin\theta_\mathrm{B} = n\sin\theta_\mathrm{t} = \tan\theta_\mathrm{B}\sin\theta_\mathrm{t} = \frac{\sin\theta_\mathrm{B}}{\cos\theta_\mathrm{B}}\sin\theta_\mathrm{t} \implies \cos\theta_\mathrm{B} = \sin\theta_\mathrm{t} = \frac{\sin\theta_\mathrm{B}}{n}.$$

Using $\cos\theta_\mathrm{B} = \frac{\sin\theta_\mathrm{B}}{n}$ and $\cos^2 x + \sin^2 x = 1$, we then have

$$\cos\theta_\mathrm{B} = \frac{1}{n}\sin\theta_\mathrm{B} = \frac{1}{n}\sqrt{1-\cos^2\theta_\mathrm{B}} \implies \cos\theta_\mathrm{B} = \frac{1}{\sqrt{1+n^2}}$$

Next step is to find $\cos\theta_\mathrm{t}$. We begin with $\cos\theta_\mathrm{B} = \sin\theta_\mathrm{t}$ from above, and then use $\cos^2 x + \sin^2 x = 1$ to get

$$\cos\theta_\mathrm{B} = \sin\theta_\mathrm{t} \implies \sqrt{1-\sin^2\theta_\mathrm{B}} = \sqrt{1-\cos^2\theta_\mathrm{t}} \implies \cos\theta_\mathrm{t} = \sin\theta_\mathrm{B} = \sqrt{1-\cos^2\theta_\mathrm{B}}$$

Then substitute in the earlier-derived expression for $\cos\theta_\mathrm{B}$ to get

$$\cos\theta_\mathrm{t} = \sqrt{1 - \frac{1}{1+n^2}} = \frac{n}{\sqrt{1+n^2}}.$$

We then use these quantities to find t_{p_1} and get

$$t_{p_1} = \frac{2\frac{1}{\sqrt{1+n^2}}}{\frac{n}{\sqrt{1+n^2}} + \frac{n}{\sqrt{1+n^2}}} = \frac{1}{n}.$$

Next we compute t_{p_2} (for passage from the window into air) according to

$$t_{p_2} = \frac{2n\cos\theta_\mathrm{t}}{\cos\theta_\mathrm{t} + n\cos\theta_\mathrm{B}} = \frac{2n\frac{n}{\sqrt{1-n^2}}}{2\frac{n}{\sqrt{1+n^2}}} = n.$$

The total transmission coefficient, and the corresponding transmittance T_p, are

$$t_p = t_{p_1}\cdot t_{p_2} = 1 \implies T_p = |t_p|^2 = 1.$$

For TE

The TE transmission coefficient at the first boundary is

$$t_{s_1} = \frac{2n_1\cos\theta_\mathrm{i}}{n_1\cos\theta_\mathrm{i} + n_2\cos\theta_\mathrm{t}} = \frac{2\cos\theta_\mathrm{B}}{\cos\theta_\mathrm{B} + n\cos\theta_\mathrm{t}} = \frac{2}{1+n^2},$$

while t_s at the second boundary is

$$t_{s_2} = \frac{2n\cos\theta_\mathrm{t}}{n\cos\theta_\mathrm{t} + \cos\theta_\mathrm{B}} = \frac{2n^2}{1+n^2}.$$

The total transmission coefficient is

$$t_s = t_{s_1} \cdot t_{s_2} = \frac{4n^2}{(1+n^2)^2}.$$

The corresponding transmittance is, using $n = n_\mathrm{glass} \approx 1.5$

$$T_s = |t_s|^2 = \frac{16n^4}{(1+n^2)^4} \approx 0.73.$$

Ratio of TE and TM Transmittance

The problem requires enough Brewster's windows such that

$$\left(\frac{T_p}{T_s}\right)^n = \frac{100}{1}.$$

In terms of the just-derived expressions for T_p and T_s, this condition reads

$$\left(\frac{T_p}{T_s}\right)^n = \frac{1}{(0.73)^n} = 100 \implies (0.73)^n = 0.01.$$

Take the logarithm of both sides and get

$$n\log 0.73 = \log 0.01 \implies n = \frac{\log 0.01}{\log 0.73} = 14.5.$$

Rounding to the next integer number, we thus need 15 consecutive Brewster's windows to get a polarization ratio of 0.01.

4.5 Total Internal Reflection in Sapphire

Compute the phase difference between TE and TM polarizations in sapphire (refractive index $n = 1.77$) and determine the condition for which the phase difference between the reflected polarizations equals $\pi/4$.

Our first step is to find both the TE and TM reflection coefficients r_s and r_p. We begin with the general TM formula

$$r_p = \frac{n_2\cos\theta_\mathrm{i} - n_1\cos\theta_\mathrm{t}}{n_2\cos\theta_\mathrm{i} + n_1\cos\theta_\mathrm{t}}.$$

Next, we expression $\cos\theta_\mathrm{t}$ in terms of the incident angle θ_i using the law of refraction $n_1\sin\theta_\mathrm{i} = n_2\sin\theta_\mathrm{t}$ and the identity $\cos^2\theta_\mathrm{t} + \sin^2\theta_\mathrm{t} = 1$. This reads

$$\cos\theta_\mathrm{t} = \sqrt{1-\sin^2\theta_\mathrm{t}} = \sqrt{1-\left(\frac{n_1}{n_2}\right)^2\sin^2\theta_\mathrm{i}}.$$

During total internal reflection, the argument of square root is negative. We accommodated for the negative square root by writing

$$\cos\theta_\mathrm{t} = i\sqrt{\left(\frac{n_1}{n_2}\right)^2\sin^2\theta_\mathrm{i} - 1} \equiv i\kappa,$$

in terms of which the reflection coefficient r_p becomes

$$r_p = \frac{n_2 \cos\theta_\mathrm{i} - n_1 i\kappa}{n_2 \cos\theta_\mathrm{i} + n_1 i\kappa}.$$

When written in the complex form $r_p = |r_p|e^{i\phi}$, this expression simplifies to

$$r_p = |r_p|e^{i\phi} = \frac{n_2^2 \cos^2\theta_\mathrm{i} - n_1^2\kappa^2}{n_2^2 \cos^2\theta_\mathrm{i} - n_1^2\kappa^2} \cdot \frac{e^{i\phi_1}}{e^{i\phi_2}} = \frac{e^{i\phi_1}}{e^{i\phi_2}},$$

since the magnitudes of numerator and denominators are equal and cancel out. In other words, we only need to compute the phases ϕ_1 and ϕ_2 of numerator and denominator. These phases are

$$\phi_1 = \arctan\frac{-n_1\kappa}{n_2\cos\theta_\mathrm{i}} \qquad \text{and} \qquad \phi_2 = \arctan\frac{n_1\kappa}{n_2\cos\theta_\mathrm{i}} = -\phi_1,$$

and the corresponding (TM) phase difference is

$$\delta_p \equiv \phi_2 - \phi_1 = 2\arctan\frac{n_1\kappa}{n_2\cos\theta_\mathrm{i}}.$$

Finally, we rearrange this expression and substitute in κ to get

$$\tan\frac{\phi_p}{2} = \frac{n_1\kappa}{n_2\cos\theta_\mathrm{i}} = \frac{n_1}{n_2\cos\theta_\mathrm{i}}\sqrt{\left(\frac{n_1}{n_2}\right)^2 \sin^2\theta_\mathrm{i} - 1}.$$

TE

Analogous to TM, so written more concisely. Begin with

$$r_s = \frac{n_1 \cos\theta_\mathrm{i} - n_2 i\kappa}{n_1 \cos\theta_\mathrm{i} + n_2 i\kappa}.$$

Write in the complex form $r_s = |r_s|e^{i\phi}$, producing

$$r_s = |r_s|e^{i\phi} = \frac{n_1^2 \cos^2\theta_\mathrm{i} - n_2^2\kappa^2}{n_1^2 \cos^2\theta_\mathrm{i} - n_2^2\kappa^2} \cdot \frac{e^{i\phi_1}}{e^{i\phi_2}} = \frac{e^{i\phi_1}}{e^{i\phi_2}},$$

The complex phases of the numerator and denominator are

$$\phi_1 = \arctan\frac{-n_2\kappa}{n_1\cos\theta_\mathrm{i}} \qquad \text{and} \qquad \phi_2 = \arctan\frac{n_2\kappa}{n_1\cos\theta_\mathrm{i}} = -\phi_1,$$

and the corresponding (TE) phase difference is

$$\delta_s \equiv \phi_2 - \phi_1 = 2\arctan\frac{n_2\kappa}{n_1\cos\theta_\mathrm{i}}.$$

Finally, rearrange and substitute in κ to get

$$\tan\frac{\delta_s}{2} = \frac{n_2\kappa}{n_1\cos\theta_\mathrm{i}} = \frac{n_2}{n_1\cos\theta_\mathrm{i}}\sqrt{\left(\frac{n_1}{n_2}\right)^2 \sin^2\theta_\mathrm{i} - 1}.$$

Simplifications

Assume $n_2 = 1$ (for air) and define $n_1 \equiv n$. We then have

$$\tan\frac{\delta_p}{2} = \frac{n}{\cos\theta_{\mathrm{i}}}\sqrt{n^2\sin^2\theta_{\mathrm{i}} - 1} \qquad \text{and} \qquad \tan\frac{\delta_s}{2} = \frac{1}{n\cos\theta_{\mathrm{i}}}\sqrt{n^2\sin^2\theta_{\mathrm{i}} - 1}.$$

Completing the Solution

Our goal is to find the angle of incidence θ_{i} producing a phase difference $\delta_p - \delta_s = \pi/4$. We find this phase difference using the general trigonometric identity

$$\tan\frac{\delta_p - \delta_s}{2} = \frac{\tan\frac{\delta_p}{2} - \tan\frac{\delta_s}{2}}{1 + \tan\frac{\delta_p}{2}\tan\frac{\delta_s}{2}} =$$

Next note

$$\tan\frac{\delta_p - \delta_s}{2} = \frac{\tan\frac{\delta_p}{2} - \tan\frac{\delta_s}{2}}{1 + \tan\frac{\delta_p}{2}\tan\frac{\delta_s}{2}} = \frac{\cos\theta_{\mathrm{i}}}{n} \cdot \frac{\sqrt{n^2\sin^2\theta_{\mathrm{i}} - 1}(n^2 - 1)}{\cos^2\theta_{\mathrm{i}} + (n^2\sin^2\theta_{\mathrm{i}} - 1)}$$

Multiply above and below by n, then combine like n^2 in the denominator to get

$$= \frac{n\cos\theta_{\mathrm{i}}(n^2 - 1)\sqrt{n^2\sin^2\theta_{\mathrm{i}} - 1}}{n^2(\cos^2\theta_{\mathrm{i}} - 1) + n^4\sin^2\theta_{\mathrm{i}}} = \frac{n\cos\theta_{\mathrm{i}}(n^2 - 1)\sqrt{n^2\sin^2\theta_{\mathrm{i}} - 1}}{-n^2\sin^2\theta_{\mathrm{i}} + n^4\sin^2\theta_{\mathrm{i}}}$$

Cancel the $(n^2 - 1)$ term above and below to get

$$\tan\frac{\delta_p - \delta_s}{2} = \frac{1}{n}\frac{\cos\theta_{\mathrm{i}}\sqrt{n^2\sin^2\theta_{\mathrm{i}} - 1}}{\sin^2\theta_{\mathrm{i}}}$$

Next, for shorthand, define $x \equiv \sin^2\theta_{\mathrm{i}}$, and use $\cos\theta_{\mathrm{i}} = \sqrt{1 - x}$ to get

$$\tan\frac{\delta_p - \delta_s}{2} = \frac{1}{n}\frac{\sqrt{1 - x}\sqrt{n^2 x - 1}}{x}$$

For $\delta_p - \delta_s$ we require

$$\tan\frac{\delta_p - \delta_s}{2} = \tan\frac{\pi}{8} \implies \frac{1}{n}\frac{\sqrt{1 - x}\sqrt{n^2 x - 1}}{x} = \tan\frac{\pi}{8} \equiv A,$$

where we have defined $A \equiv \tan(\pi/8)$ for shorthand. We then multiply out, resulting in the quadratic equation.

$$(1 - x)(n^2 x - 1) = (Anx)^2 \implies (A^2 n^2 + n^2)x^2 - (n^2 + 1)x + 1 = 0.$$

Using the quadratic formula, the solutions for x are

$$x = \frac{1 + n^2 \pm \sqrt{(1 + n^2)^2 - 4A^2 n^2 - 4n^2}}{2(A^2 n^2 + n^2)}.$$

Substitute in numbers

$$A = \tan\frac{\pi}{8} = 0.414 \qquad n = 1.77.$$

Solutions are

$$x_1 = 0.774 \qquad \text{and} \qquad x_2 = 0.352$$

Using $x = \sin^2\theta_{\mathrm{i}}$ the corresponding angles of incidence should come out to

$$\theta_{\mathrm{i}_1} = 61.5^\circ \qquad \text{and} \qquad \theta_{\mathrm{i}_2} = 36.5^\circ.$$

4.6 Circular Polarization After TIR

In general, light is elliptically polarized after total internal reflection. Assuming the light is incident from air onto an optically denser material with refractive index $n > 1$, determine the smallest value of n for which light is circularly polarized after total internal reflection.

Reflected light will be circularly polarized when the phase difference between the TE and TM polarizations is $\pi/2$. In the notation of the previous problem, this occurs when

$$\delta_p - \delta_s = \pi/2.$$

We assume the incident light is linearly polarized.

Recall the polynomial equation

$$x = \frac{1 + n^2 \pm \sqrt{(1+n^2)^2 - 4A^2n^2 - 4n^2}}{2(A^2n^2 + n^2)},$$

For review, A is defined as

$$A = \tan \frac{\delta_p - \delta_s}{2}.$$

In our case, for $\delta_p - \delta_s = \pi/2$, we have $A = \tan \pi/4 = 1$.

Large values of n are difficult to attain experimentally, so we aim to find the smallest possible n producing a phase shift $\delta_p - \delta_s = \pi/2$. The smallest n occurs when the two solutions for $x_{1,2}$ are equal and degenerate, This happens when the square root's discriminant is zero, i.e. when

$$(1 + n^2)^2 - 4A^2n^2 - 4n^2 = 0.$$

Using $A = 1$, this simplifies to

$$1 + 2n^2 + n^4 - 4n^2 - 4n^2 = n^4 - 6n^2 + 1 \equiv u^2 - 6u + 1 = 0,$$

where we have defined $u \equiv n^2$. We find u with the quadratic formula

$$u = n^2 = \frac{6 \pm \sqrt{36 - 4}}{2} \approx 3 \pm 2.83.$$

We choose the solution with a plus sign (since $n^2 = 3 - 2.83 = .17$ corresponds to a nonphysical refractive index $n < 1$). The result is

$$n^2 \approx 5.83 \implies n \approx 2.4.$$

This is a large, but attainable index of refraction. For example, diamond has a refractive index $n_{\text{diamond}} \approx 2.417$ for visible light.

5 Fifth Exercise Set

5.1 Theory: Skin Depth and Total Internal Reflection

From lecture, the reflection coefficients for TE and TM polarizations for passage into optically less dense material for $\theta_\mathrm{i} > \theta_\mathrm{c}$ (i.e. for incident angles in the regime of total internal reflection) are given by

$$r_s \equiv \frac{n_1 \cos\theta_\mathrm{i} - in_2\kappa}{n_1 \cos\theta_\mathrm{i} + in_2\kappa} \qquad \text{and} \qquad r_p \equiv \frac{n_2 \cos\theta_\mathrm{i} - in_1\kappa}{n_2 \cos\theta_\mathrm{i} + in_1\kappa},$$

where the constant κ is defined as

$$\kappa \equiv \sqrt{\left(\frac{n_1 \sin\theta_\mathrm{i}}{n_2}\right)^2 - 1} = \sqrt{\left(\frac{\sin\theta_\mathrm{i}}{\sin\theta_\mathrm{c}}\right)^2 - 1}.$$

Note that for $\theta_\mathrm{i} > \theta_\mathrm{c}$, the complex quantity $i\kappa$ replaces the $\cos\theta_\mathrm{t}$ term, which is the real-valued equivalent for $\theta_\mathrm{i} < \theta_\mathrm{c}$.

We begin by considering TE-polarized light, in which case the transmitted wave vector (Figure 4.13b, page 65) is given by

$$\mathbf{k}_\mathrm{t} = k_0 n_2 (\sin\theta_\mathrm{t}, 0, \cos\theta_\mathrm{t}).$$

We then use the law of refraction to write the x component as $n_2 \sin\theta_\mathrm{t} = n_1 \sin\theta_\mathrm{i}$, and replace $\cos\theta_\mathrm{t}$ with the complex generalization $i\kappa$ to get

$$\mathbf{k}_\mathrm{t} = k_0 (n_1 \sin\theta_\mathrm{i}, 0, in_2\kappa).$$

Using the above expression for κ_t, we write the transmitted electric field $\mathbf{E}_\mathrm{t}$ as

$$\begin{aligned}\mathbf{E}_\mathrm{t} &= \mathbf{E}_{0_\mathrm{t}} e^{i(\mathbf{k}_\mathrm{t}\cdot\mathbf{r} - \omega t)} = \mathbf{E}_{0_\mathrm{t}} e^{ik_0(n_1 \sin\theta_\mathrm{i} x + in_2\kappa z)} e^{-i\omega t} \\ &\equiv \mathbf{E}_{0_\mathrm{t}} e^{ik_0 n_1 \sin\theta_\mathrm{i} x} e^{-\xi z} e^{-i\omega t},\end{aligned}$$

where we have defined the attenuation constant

$$\xi \equiv k_0 n_2 \kappa. \tag{5.1}$$

The exponentially attenuating transmitted electric field $\mathbf{E}_\mathrm{t}$ (attenuating due to the term $e^{-\xi z}$) is called the *evanescent field.* In real form, the evanescent field reads

$$\mathbf{E}_\mathrm{t} = \mathbf{E}_{0_\mathrm{t}} e^{-\xi z} \cos(k_0 n_1 \sin\theta_\mathrm{i} x - \omega t).$$

Finally, noting that for TE polarization the electric field is tangent to the boundary surface, and thus parallel to the y axis in our coordinate system, we may write

$$\mathbf{E}_\mathrm{t} = E_{0_\mathrm{t}} e^{ik_0 n_1 \sin\theta_\mathrm{i} x} e^{-\xi z} e^{-i\omega t}\, \hat{\mathbf{e}}_y, \qquad \mathbf{E}_\mathrm{t} = E_{0_\mathrm{t}} e^{-\xi z} \cos(k_0 n_1 \sin\theta_\mathrm{i} x - \omega t)\, \hat{\mathbf{e}}_y,$$

Skin Depth

The skin depth is defined as the inverse of the attenuation constant ξ in Equation 5.1, which, written out in full, reads

$$\xi = k_0 n_2 \kappa = k_0 n_2 \sqrt{\left(\frac{\sin\theta_\mathrm{i}}{\sin\theta_\mathrm{c}}\right)^2 - 1}.$$

The skin depth d is thus

$$d \equiv \frac{1}{\xi} = \frac{1}{k_0 n_2}\left(\frac{\sin^2\theta_\mathrm{i}}{\sin^2\theta_\mathrm{c}} - 1\right)^{-1/2} = \frac{\lambda_0}{2\pi}\frac{1}{\sqrt{n_1^2\sin^2\theta_\mathrm{i} - n_2^2}}, \tag{5.2}$$

where we have simplified the bracket term using the definition of the critical angle $\sin\theta_\mathrm{c} = \frac{n_2}{n_1}$, and used $k_0 = (2\pi)/\lambda_0$, where λ_0 denotes the light's wavelength in vacuum.

5.2 Skin Depth for TIR in Diamond

Monochromatic light with wavelength $\lambda = 600\,\mathrm{nm}$ is incident on diamond, which has refractive index $n = 2.417$ for $\lambda = 600\,\mathrm{nm}$. Determine the skin depth for the angles of incidence $\theta_\mathrm{i} \in \{24.5°, 25°, 50°\}$, and sketch the dependence of skin depth on the angle of incidence θ_i.

For orientation, we first compute the critical angle for passage of light from diamond to air. Assuming $n_2 = n_\mathrm{air} = 1$, the critical angle is

$$\sin\theta_\mathrm{c} = \frac{n_2}{n_1} = \frac{1}{2.417} \implies \theta_\mathrm{c} = 24.44°.$$

In other words the first incidence angle $\theta_\mathrm{i} = 24.5°$ is intentionally just above the critical angle.

From Equation 5.2, the formula for skin depth (for TE polarization) is

$$d = \frac{1}{k_0 n_2 \kappa} = \frac{\lambda}{2\pi n_2}\left(\frac{\sin^2\theta_\mathrm{i}}{\sin^2\theta_\mathrm{c}} - 1\right)^{-1/2}.$$

Substituting in $\theta_\mathrm{i} = 24.5°$, $\lambda = 600\,\mathrm{nm}$ and $n_2 = n_\mathrm{air} \approx 1$ results in

$$d(24.5°) = \frac{600\,\mathrm{nm}}{2\pi\cdot 1}\left(\frac{\sin^2 24.5°}{(1/2.417)^2} - 1\right)^{-1/2} \approx 1.4\,\mathrm{\mu m}$$

Analogous calculations for the remaining critical angles produce

$$d(25°) \approx 458\,\mathrm{nm} \qquad \text{and} \qquad d(50°) \approx 61.3\,\mathrm{nm}.$$

In other words, the skin depth decreases with increasing angle of incidence.

5.3 Fraunhofer Diffraction for a Rectangular Aperture

Derive the diffraction pattern produced by a rectangular aperture of width a and height b in the far-field regime of Fraunhofer diffraction.

Let R_0 denote the distance between the diffracting and observation planes; let (x, y) be the coordinates in the diffracting plane and (ξ, η) be the coordinates in the observation plane. Finally, we define the wave vector-like quantities

$$\kappa_x = \frac{k\xi}{R_0} \qquad \text{and} \qquad \kappa_y = \frac{k\eta}{R_0}.$$

From lecture, we then quote the Fraunhofer diffraction formula

$$E(\kappa_x, \kappa_y, R_0) = \frac{iE_0}{\lambda}\frac{e^{ikR_0}}{R_0}\iint f(x,y)e^{-i\kappa_x x}e^{-i\kappa_y y}\,\mathrm{d}x\,\mathrm{d}y. \tag{5.3}$$

Note that Fraunhofer diffraction holds only in the regime

$$F = \frac{d^2}{\lambda z} \ll 1.$$

A rectangular aperture is described by the aperture function

$$f(x,y) = \begin{cases} 1 & x \in \left(-\frac{a}{2}, \frac{a}{2}\right), y \in \left(-\frac{b}{2}, \frac{b}{2}\right) \\ 0 & \text{otherwise.} \end{cases}$$

We then substitute the limits imposed by the aperture function into the diffraction integral in Equation 5.3 to get

$$E(\kappa_x, \kappa_y, R_0) = \frac{iE_0}{\lambda} \frac{e^{ikR_0}}{R_0} \int_{-a/2}^{a/2} \mathrm{d}x \int_{-b/2}^{b/2} \mathrm{d}y \, e^{-i\kappa_x x} e^{-i\kappa_y y}. \tag{5.4}$$

The integral over x in Equation 5.4 evaluates to

$$\begin{aligned} I_x(a/2) \equiv \int_{-a/2}^{a/2} e^{-i\kappa_x x} \, \mathrm{d}x &= -\frac{1}{i\kappa_x} e^{-i\kappa_x x}\Big|_{-a/2}^{a/2} = -\frac{1}{i\kappa_x}\left(e^{-i\kappa_x \frac{a}{2}} - e^{i\kappa_x \frac{a}{2}}\right) \\ &= \frac{2}{\kappa_x} \sin\left(\frac{\kappa_x a}{2}\right). \end{aligned}$$

While the above answer is perfectly correct, we will write it in a more compact form by multiplying above and below by a, producing

$$I_x(a/2) = a\frac{2}{a\kappa_x} \sin\left(\frac{\kappa_x}{2} a\right) = a \cdot \frac{\sin\left(\frac{\kappa_x a}{2}\right)}{\frac{\kappa_x a}{2}} = a \cdot \operatorname{sinc}\left(\frac{\kappa_x a}{2}\right),$$

where we have introduced the sinc function, defined as $\operatorname{sinc} x = \frac{\sin x}{x}$.

The integral over y in Equation 5.4 is identical in form to the integral over x; we just have to replace d with a and κ_x with κ_y. The result is

$$I_y(b/2) \equiv \int_{-b/2}^{b/2} e^{-i\kappa_y y} \, \mathrm{d}y = b \cdot \operatorname{sinc}\left(\frac{\kappa_y b}{2}\right).$$

The total electric field is then

$$E = \frac{iE_0 e^{ikR_0}}{\lambda R_0} \cdot I_x(a/2) \cdot I_y(b/2) = \frac{iE_0 e^{ikR_0}}{\lambda R_0} ab \operatorname{sinc}\left(\frac{\kappa_x a}{2}\right) \operatorname{sinc}\left(\frac{\kappa_y b}{2}\right)$$

The corresponding intensity, using $I = |E|^2$, is

$$I = I_0 \operatorname{sinc}^2\left(\frac{\kappa_x a}{2}\right) \operatorname{sinc}^2\left(\frac{\kappa_y b}{2}\right),$$

where we have grouped all constant terms into the coefficient I_0.

Next, we define the diffracting angles θ_x and θ_y according to

$$\begin{aligned} \sin\theta_x &= \frac{\xi}{R_0} \implies \kappa_x = \frac{k\xi}{R_0} = k\sin\theta_x \\ \sin\theta_y &= \frac{\eta}{R_0} \implies \kappa_y = \frac{k\eta}{R_0} = k\sin\theta_y. \end{aligned}$$

We then consider the diffraction pattern $I(\theta_x, 0)$. Geometrically, $I(\theta_x, 0)$ is just the diffraction pattern along the line $\eta = 0$ the ξ axis in the observation plane. We have (using the limit $\operatorname{sinc} x \to 1$ as $x \to 0$)

$$I(\theta_x, 0) = I_0 \operatorname{sinc}^2\left(\frac{ka\sin\theta_x}{2}\right).$$

The minima of $\operatorname{sinc}^2 x$ occur when $\operatorname{sinc} x = 0$, which occurs when the argument x is an integer multiple of π. In our case, for $I_{\theta_x,0}$, this occurs when

$$\frac{ka\sin\theta_x}{2} = n\pi \implies \theta_x = \sin^{-1}\left(\frac{2n\pi}{ka}\right).$$

The intensity maxima, meanwhile, do not have a simple closed-form solution. In practice, they are computed numerically.

5.4 Fraunhofer Diffraction for a Diffraction Grating

Using the far-field Fraunhofer approximation, determine the structure factor, grating factor, and characteristic diffraction pattern of a diffraction grating.

We begin with the Fraunhofer diffraction integral and write the integral for one slit, translated along the x axis by the distance x_0. We will assume the y dimension in a diffraction grating is much larger than the x dimension (width) of a slit; essentially, we can treat the slit as infinitely high along the y dimension. Because the problem's resulting translational symmetry with respect to the y coordinate, it suffices to consider only the diffraction integral along the x coordinate.

At the origin, the Fraunhofer diffraction integral reads

$$E_0(\kappa_x, \kappa_y) = \alpha \int f(x, y) e^{-i\kappa_x x}\, \mathrm{d}x, \tag{5.5}$$

where α is a proportionality constant. Translated by x_0, the diffraction integral changes to

$$E(\kappa_x, \kappa_y) = e^{-i\kappa_x x_0} E_0(\kappa_x, \kappa_y),$$

where E_0 is the field at the origin (computed above). Lesson: translation by a distance x_0 only contributes the phase factor $e^{-i\kappa_x x_0}$.

Let D denote the distance between the center of two apertures and d denote the width of a single slit. For convenience, assume an even number N of slits, and place the origin along the x axis at the center of the diffraction grating. We take the origin to occur in solid material, and not within an aperture, so that the center of the first left and right aperture occur at $x = \pm D/2$.

We find the field of many slits by summing the contributions from individual slits: we sum over $N/2$ slits on the right of the origin and $N/2$ slits on the left of the origin to get

$$E = \sum_j^{N/2} E_0 e^{-i\kappa_x D j} e^{-i\frac{\kappa_x D}{2}} + \sum_j^{N/2} E_0 e^{+i\kappa_x D j} e^{+i\frac{\kappa_x D}{2}},$$

where the constants $e^{\pm i\frac{\kappa_x \pm D}{2}}$ occur because the first apertures are shifted by $x = \pm D/2$ relative to the origin. The two sums above are geometric series of the form

$$\sum_{j=0}^{N} x^j = \frac{1-x^N}{1-x},$$

and evaluate to

$$\begin{aligned}
E &= E_0 e^{-i\frac{\kappa_x D}{2}} \left(\frac{1 - e^{-i\frac{\kappa_x DN}{2}}}{1 - e^{-i\kappa_x D}} \right) + E_0 e^{+i\frac{\kappa_x D}{2}} \left(\frac{1 - e^{+i\frac{\kappa_x DN}{2}}}{1 - e^{+i\kappa_x D}} \right) \\
&= E_0 \cdot \frac{1 - e^{-i\frac{\kappa_x DN}{2}}}{e^{+i\frac{\kappa_x D}{2}} - e^{-i\frac{\kappa_x D}{2}}} + E_0 \cdot \frac{1 - e^{+i\frac{\kappa_x DN}{2}}}{e^{-i\frac{\kappa_x D}{2}} - e^{+i\frac{\kappa_x D}{2}}} \\
&= 2E_0 \cdot \frac{e^{+i\frac{\kappa_x DN}{2}} - e^{-i\frac{\kappa_x DN}{2}}}{e^{-i\frac{\kappa_x D}{2}} - e^{+i\frac{\kappa_x D}{2}}} = 2E_0 \frac{\sin\frac{\kappa_x DN}{2}}{\sin\frac{\kappa_x D}{2}}.
\end{aligned}$$

We call the E_0 term (from Equation 5.5) the diffraction grating's *structure factor*, while the ratio of sine terms is called the *grating factor*. The structure factor is the diffraction pattern associated with a single thin slit, while the grating factor arises from the grating's periodicity. For a single thin slit of width d, the E_0 structure factor is a sinc function $E_0 \propto \operatorname{sinc}(\kappa_x d/2)$, and so the diffraction pattern for a diffraction grating comes out to

$$E = 2E_0 \frac{\sin\frac{\kappa_x DN}{2}}{\sin\frac{\kappa_x D}{2}} \propto \underbrace{\operatorname{sinc}\frac{\kappa_x d}{2}}_{\text{structure}} \cdot \underbrace{\frac{\sin\frac{\kappa_x DN}{2}}{\sin\frac{\kappa_x D}{2}}}_{\text{grating}}. \tag{5.6}$$

6 Sixth Exercise Set

6.1 Resolution of a Diffraction Grating

How many diffraction grating slits N are needed to resolve sodium's spectral lines, which occur at the wavelengths $\lambda_1 = 589.0\,\mathrm{nm}$ and $\lambda_2 = 589.6\,\mathrm{nm}$?

We begin with the diffraction grating diffraction pattern from Equation 6.1, which for review reads

$$E \propto \underbrace{\operatorname{sinc}\frac{\kappa_x d}{2}}_{\text{structure}} \cdot \underbrace{\frac{\sin\frac{\kappa_x DN}{2}}{\sin\frac{\kappa_x D}{2}}}_{\text{grating}},$$

where D is the distance between slits and d is the width of a slit. We then square the electric field to get the intensity

$$I(\theta) \propto |E|^2 = I_0 \operatorname{sinc}^2\frac{\kappa_x d}{2} \cdot \frac{\sin^2\frac{\kappa_x DN}{2}}{\sin^2\frac{\kappa_x D}{2}}.$$

For shorthand, we will simplify the arguments of the structure and grating factors by defining

$$\sigma(\theta) \equiv \frac{kd}{2}\sin\theta \qquad \text{and} \qquad \Gamma(\theta) \equiv \frac{kD}{2}\sin\theta$$

("sigma" for "structure" and "gamma" for "grating"), in terms of which $I(\theta)$ reads

$$I(\theta) = I_0 \operatorname{sinc}^2\sigma \cdot \left(\frac{\sin\Gamma N}{\sin\Gamma}\right)^2.$$

We will assume $N \gg 1$ for the number of slits.

Although it is difficult to describe in words and better to draw a picture, a diffraction grating's characteristic diffraction pattern $I(\theta)$ has to main contributions:

- a slowly-oscillating envelope due to the structure factor $\operatorname{sinc}^2\sigma$, and
- a rapidly-oscillating signal within the envelope, due to the grating factor $\frac{\sin^2(\Gamma N)}{\sin^2\Gamma}$.

The grating factor has maxima when its denominator $\sin^2\Gamma$ equals zero, i.e. when

$$\sin^2\Gamma = 0 \implies \Gamma = \frac{kD}{2}\sin\theta = m\pi \implies \sin\theta_m = \frac{2\pi m}{kD} = \frac{m\lambda}{D}, \tag{6.1}$$

where we have used $k = 2\pi/\lambda$. The maximum angles θ_m will have different values for the different sodium wavelengths λ_1 and λ_2.

We aim to find the change in intensity-maximum angle $\Delta\theta_m$ corresponding to a change in wavelength $\Delta\lambda$. We find this relationship by taking the differential of Equation 6.1, which produces

$$\cos\theta\Delta\theta = \frac{m}{D}\Delta\lambda \implies \Delta\theta = \frac{m}{D\cos\theta}\Delta\lambda. \tag{6.2}$$

Next, we need to determine a sensible condition for resolution of the two wavelengths. First consider the sum of the two wavelengths near the maxima—since the sum of the two wavelength's contributions is what the detector "sees". The two peaks are resolved when the peaks are separated more than by width of a given peak.

We will define the characteristic width of a diffraction peak as the angular spacing over which the diffraction pattern $I(\theta)$ first falls to zero from its central maximum at $\theta = 0$. The diffraction pattern's zeros occur when the grating factor equals zero, which occurs for

$$\sin^2(\Gamma N) = 0 \implies \Gamma N = \frac{kDN}{2}\sin\theta = n\pi \implies \sin\theta_n = \frac{s\pi m}{kDN} = \frac{m\lambda}{DN},$$

where we have used n to index the minima angles θ_n at which $I(\theta) = 0$.

Assume a maximum diffraction peak at n. In this case the first minimum occurs at $n+1$. The idea of minimum at $n+1$ gives us the difference

$$(n+1)\pi - n\pi = \frac{kDN}{2}(\sin\theta_{\mathrm{min}} - \sin\theta_{\mathrm{max}})$$

We assume $\theta_{\mathrm{min}} \approx \theta_{\mathrm{max}}$, which gives us $\sin\theta_{\mathrm{min}} - \sin\theta_{\mathrm{max}} \approx \Delta$. We then have

$$\pi = \frac{kD}{2}N\Delta[\sin\theta] = \frac{kDN}{2}\cos\theta\Delta\theta$$

Note this is all sort of hand-wavy. The idea is to get an expression for the width of a peak as

$$\Delta\theta = \frac{2\pi}{kDN\cos\theta} = \frac{\lambda}{ND\cos\theta_{\mathrm{max}}}.$$

Equate to the expression for $\Delta\theta$ in Equation 6.2 and get

$$\frac{\lambda}{ND\cos\theta_{\mathrm{max}}} = \frac{m}{D\cos\theta_{\mathrm{max}}}\,\mathrm{d}\lambda \implies \frac{\lambda}{\Delta\lambda} = mN.$$

We can now answer the question: how many slits are needed for resolving sodium. Note that we will consider the first-order diffraction peaks at $m = 1$ (this is just the first θ_{max}); note that spread increases but intensity falls for larger m. Anyway, we have

$$\frac{\lambda}{\Delta\lambda} = \frac{590\,\mathrm{nm}}{0.6\,\mathrm{nm}} \approx 1000.$$

For λ we use a central wavelength somewhere between the two wavelengths. Its exact value isn't super important, it is important that $\Delta\lambda$ is exact.

6.2 Theory: Fresnel Diffraction

In Fresnel diffraction, as in Fraunhofer diffraction, we consider a diffracting aperture in the (x, y) plane and an observation screen a distance z_0 from the diffracting plane; the observation screen is parallel to the (x, y) plane with coordinates (ξ, η).

In addition, in Fresnel diffraction we consider a source point $S = (x', y', -z_0')$, where z_0' is the distance between the source point and diffracting plane.

For Fresnel diffraction, we write the electric field at a point (x, y) in the aperture in the form

$$E_0(x, y) \equiv \frac{\widetilde{E}_0}{R'}e^{ikR'}, \ \widetilde{E}_0 = \text{constant},$$

where the constant $\widetilde{E}_0$ has units of electric field times length.

The distance from the source point to a point (x, y) in the diffracting aperture is

$$R' = \sqrt{(x'-x)^2 + (y'-y)^2 + z_0'^2},$$

while the distance from the aperture to a point (ξ, η) in the observation plane, like for Fraunhofer diffraction, is

$$R = \sqrt{(\xi - x)^2 + (\eta - y)^2 + z_0^2}.$$

Fresnel diffraction is the approximation of R up to the quadratic term $x^2 + y^2$, i.e.

$$R \approx R_0 \left(1 - \frac{\xi x}{R_0^2} - \frac{\eta y}{R_0^2} + \frac{x^2 + y^2}{2R_0^2}\right) \qquad \text{(Fresnel approximation).}$$

In the regime of Fresnel diffraction, the electric field at a point (ξ, η) on the observation screen reads

$$E(\xi, \eta) = \frac{\widetilde{E}_0}{i\lambda} \frac{e^{ik(z_0' + z_0)}}{R_0' R_0} \iint f(x, y) e^{\frac{ik}{2z_0'}\left[(x - x')^2 + (y - y')^2\right]} e^{\frac{ik}{2z_0}\left[(x - \xi)^2 + (y - \eta)^2\right]} \,\mathrm{d}x \,\mathrm{d}y, \tag{6.3}$$

where $f(x, y)$ is the aperture function, defined as for Fraunhofer diffraction by

$$f(x, y) = \begin{cases} 1 & (x, y) \in \text{aperture} \\ 0 & \text{otherwise.} \end{cases}$$

6.3 Fresnel Zones

Light is normally incident on a circular aperture of radius a. The source, center of the aperture, and observation point are colinear; the distances between the source and aperture and between the aperture and observation point are z_0' and z_0, respectively. Determine the intensity at the observation point...

1. *if the aperture's radius is $a_1 = \sqrt{\lambda L}$,*
2. *if the aperture's radius is $a_2 = \sqrt{2\lambda L}$,*
3. *and if we place a circular screen of radius a_1 within the aperture of radius a_2;*

where $\frac{1}{L} = \frac{1}{z_0'} + \frac{1}{z_0}$.

Since we assume the source and observation point lie on the optical axis, in which case $(x', y') = (\xi', \eta') = (0, 0)$, $R_0 = z_0$ and $R_0' = z_0'$, the Fresnel diffraction pattern in Equation 6.10 simplifies considerably to

$$\begin{aligned} E &= \frac{\widetilde{E}_0}{i\lambda} \frac{e^{ikz_0'}}{z_0'} \frac{e^{ikz_0}}{z_0} \iint f(x, y) e^{i\frac{k}{2z_0'}(x^2 + y^2)} e^{i\frac{k}{2z_0}(x^2 + y^2)} \,\mathrm{d}x \,\mathrm{d}y \\ &= \frac{\widetilde{E}_0}{i\lambda} \frac{e^{ikz_0'}}{z_0'} \frac{e^{ikz_0}}{z_0} \iint f(x, y) e^{i\frac{k}{2L}(x^2 + y^2)} \,\mathrm{d}x \,\mathrm{d}y. \end{aligned}$$

For shorthand, we group the constant coefficients into the single constant

$$C \equiv \widetilde{E}_0 \frac{e^{ikz_0'}}{z_0'} \frac{e^{ikz_0}}{z_0}, \tag{6.4}$$

in terms of which the diffracted electric field reads

$$E = \frac{C}{i\lambda} \iint f(x, y) e^{i\frac{k}{2L}(x^2 + y^2)} \,\mathrm{d}x \,\mathrm{d}y. \tag{6.5}$$

Next, to take advantage of the aperture's circular symmetry, we transition to circular coordinates (ρ, ϕ), where

$$x = \rho\cos\phi \qquad y = \rho\sin\phi \qquad x^2 + y^2 = \rho^2 \qquad \mathrm{d}x\,\mathrm{d}y = \rho\,\mathrm{d}\phi\,\mathrm{d}\rho.$$

In terms of polar coordinates, Equation 6.5 simplifies further to

$$E = \frac{C}{i\lambda}\int_0^{2\pi}\mathrm{d}\phi\int_0^{a_1} e^{\frac{ik}{2L}\rho^2}\rho\,\mathrm{d}\rho = \frac{2\pi i C}{\lambda}\int_0^{a_1} e^{\frac{ik}{2L}\rho^2}\rho\,\mathrm{d}\rho.$$

Next, we introduce the new variable u given by

$$u = \frac{k\rho^2}{2L} \implies \mathrm{d}u = \frac{k\rho}{L}\,\mathrm{d}\rho, \tag{6.6}$$

in terms of which the diffracted electric field becomes

$$E = \frac{2\pi C}{i\lambda}\frac{L}{k}\int_0^{\frac{ka_1^2}{2L}} e^{iu}\,\mathrm{d}u = \frac{CL}{i}\int_0^{\frac{ka_1^2}{2L}} e^{iu}\,\mathrm{d}u = -CL(e^{i\pi} - 1) = 2CL,$$

where, referring to $a_1 = \sqrt{\lambda L}$, we have substituted in $\frac{ka_1^2}{2L} = \frac{k\lambda L}{2L} = \frac{2\pi}{\lambda}\frac{\lambda}{2} = \pi$.

Next, we consider the case when the aperture's radius is $a_2 = \sqrt{2\lambda L}$. Re-using the previous results with a_2 instead of a_1 results in

$$E = \frac{CL}{i}\int_0^{\frac{ka_2^2}{2L}} e^{iu}\,\mathrm{d}u = -CL(e^{2\pi i} - 1) = -CL(1-1) = 0.$$

Finally, we consider an annular aperture with inner radius a_1 and outer radius a_2. In this case, the diffracted electric field is

$$E = \frac{CL}{i}\int_{\frac{ka_1^2}{2L}}^{\frac{ka_2^2}{2L}} e^{iu}\,\mathrm{d}u = -CL(e^{2\pi i} - e^{i\pi}) = -CL(1+1) = -2CL.$$

6.4 A Fresnel Lens

Light from a source of spherical waves produces an electric field E_0 at an observation point when passing unobstructed from source to observation point. Show that when a Fresnel lens centered on the optical axis is placed between the source and observation point, the electric field at the observation point is $E_n = 2(-1)^n E_0$. The radii of successive Fresnel zones are given by the parameters $a_2 = \sqrt{(n+1)\lambda L}$ and $a_1 = \sqrt{n\pi L}$. Show that an even (or odd) sequence of such apertures acts as a lens with focus $f = L = (a_2^2 - a_1^2)/\lambda$.

First, we aim to find the electric field E_0 in absence of the aperture. Since the source emits spherical waves this electric field is simply

$$E_0 = \frac{\widetilde{E}_0 e^{ik(z_0' + z_0)}}{z_0' + z_0}, \tag{6.7}$$

where $z_0' + z_0$ is the distance from the source to observation point.

We now turn to the electric field with the Fresnel lens aperture. We begin by recalling the dimensionless variable u from Equation 6.6 in the previous exercise, which for review reads

$$u = \frac{k\rho^2}{2L}$$

where ρ is the radius of a Fresnel zone. The values of u for the radii a_1 and a_2 given in this problem's instructions are

$$u(a_2) = \frac{k}{2L}\big[(n+1)\lambda L\big] = \frac{2\pi}{2\lambda L}\big[(n+1)\lambda L\big] = (n+1)\pi \quad \text{and} \quad u(a_1) = \cdots = n\pi.$$

The corresponding diffracted electric field, again reusing the results of the previous exercise, is

$$\begin{aligned} E_n &= -CLe^{iu}\Big|_{n\pi}^{(n+1)\pi} = -CL\left(e^{in\pi}e^{i\pi} - e^{in\pi}\right) = -CLe^{in\pi}(e^{i\pi} - 1) = 2CLe^{in\pi} \\ &= (-1)^n \cdot 2CL. \end{aligned} \tag{6.8}$$

The explicit expression for C, defined in Equation 6.4, and the length L are

$$C \equiv \widetilde{E}_0 \frac{e^{ikz_0'}}{z_0'}\frac{e^{ikz_0}}{z_0} \qquad \text{and} \qquad L = \frac{z_0 z_0'}{z_0 + z_0'},$$

in terms of which the diffracted electric field with a Fresnel lens aperture reads

$$E_n = (-1)^n \cdot 2CL = (-1)^n \cdot 2\frac{\widetilde{E}_0 e^{ik(z_0'+z_0)}}{z_0 + z_0'}.$$

By comparing to the spherical-wave electric field E_0 in Equation 6.7, we see that the diffracted electric field can indeed be written in the form

$$E_n = (-1)^n \cdot 2E_0,$$

as required in the problem's instructions.

Fresnel Lenses

We now consider why using only even or only odd Fresnel zones causes the diffracting aperture to act as a lens. Using even n as an example, the sum of the first N even Fresnel zones produces an electric field

$$E = \sum_{n \text{ even}} E_n = \sum_{n \text{ even}} \big[(-1)^n \cdot 2E_0\big] = 2NE_0$$

where E_0 is the spherical wave electric field (in the absence of an aperture) given in Equation 6.7. The corresponding intensity is

$$I = 4N^2\left|\widetilde{E}_0\right|^2$$

Lesson: the electric field, and thus intensity, become increasingly concentrated at the observation point along the optical axis with increasing N. But this is just the behavior of a lens, which focuses an inputted plane wave into a singular point of high intensity.

Next, we note that we could get an even better Fresnel lens (i.e. even larger value of E or I at the observation point) if, instead of covering up alternating Fresnel zones, we could just flip the phase of alternating Fresnel zones. The result is a factor 4 instead of a factor 2 in the E expression, and thus a factor of 16 instead of 4 in the intensity expression.

In practice, such an apparatus is realized by using different refractive indices n_1 and n_2 in the alternating Fresnel zones, such that their difference, together with the width d of the lens, satisfies

$$k_1 d - k_2 d = \pi \implies n_1 2\pi\lambda d - 2\pi\lambda n_2 d = \pi \implies d = \frac{1}{2\lambda(n_1 - n_2)},$$

where we have used $k_i = 2\pi n_i \lambda$.

6.5 Fresnel Diffraction for a Rectangular Aperture

Compute the diffraction pattern of a rectangular aperture in the regime of Fresnel refraction. Express the result in terms of the Fresnel integrals.

The Fresnel diffraction equation, the source and observation points lie on the optical axis and thus $(x', y') = (\xi, \eta) = (0, 0)$, reads

$$E(z_0) = \frac{\widetilde{E}_0}{i\lambda}\frac{e^{ik(z_0'+z_0)}}{R_0' R_0}\int_{x_1}^{x_2} \mathrm{d}x \int_{y_1}^{y_2} \mathrm{d}y\, e^{\frac{ik(x^2+y^2)}{2L}},$$

where $L^{-1} = (z_0')^{-1} + (z_0)^{-1}$. We then define the new variables

$$u = \sqrt{\frac{k}{\pi L}}x \qquad \text{and} \qquad v = \sqrt{\frac{k}{\pi L}}y,$$

in terms of which the above integral reads

$$E(z_0) \propto \frac{\widetilde{E}_0}{i\lambda}\int_{u_1}^{u_2} e^{\frac{i\pi u^2}{2}}\,\mathrm{d}u \int_{v_1}^{v_2} e^{\frac{i\pi v^2}{2}}\,\mathrm{d}v. \tag{6.9}$$

Next, using Euler's formula $e^{ix} = \cos x = i \sin x$, we note the similarity of the above integrals to the Fresnel integrals C and S, defined as

$$\int_0^w e^{i\frac{\pi}{2}x^2}\,\mathrm{d}x = \int_0^w \cos\left(\frac{\pi}{2}x^2\right)\mathrm{d}x + i\int_0^w \sin\left(\frac{\pi}{2}x^2\right)\mathrm{d}x \equiv C(w) + iS(w).$$

In terms of the Fresnel integrals, the electric field in Equation 6.9 reads

$$E \propto \big[C(u) + iS(u)\big]_{u_1}^{u_2}\big[C(v) + iS(v)\big]_{v_1}^{v_2}. \tag{6.10}$$

As a concrete example, we consider a rectangular aperture defined by the coordinates $x_1 = -0.2\,\mathrm{mm}$ and $x_2 = 0.2\,\mathrm{mm}$ and $y_1 = -0.2\,\mathrm{mm}$ and $y_2 = 0.2\,\mathrm{mm}$, monochromatic light with wavelength $\lambda = 633\,\mathrm{nm}$, and take the source, aperture and observation point to be separated by distances $z_0 = z_0' = 1\,\mathrm{m}$. Our value of L is

$$L = \frac{z_0 z_0'}{z_0 + z_0'} = 0.5\,\mathrm{m}.$$

Next, we double-check the Fresnel diffraction approximation is valid for our problem's dimensions. The relevant Fresnel number is

$$F = \frac{a^2}{\lambda L} = \frac{(0.4\,\mathrm{mm})^2}{633\,\mathrm{nm}\cdot 0.5\,\mathrm{m}} \approx 0.5.$$

Since this result is less than one, the Fresnel approximation is valid, (but Fraunhofer diffraction, which requires $F \ll 1$, would not be).

Our goal is to compute the ratio of electric field amplitudes E/E_0, where E_0 denotes the hypothetical electric field in the absence of an aperture. We find E_0 by sending the aperture size to infinity, in which case Equation 6.10 reads

$$E_0 \propto \Big[C(u) + iS(u)\Big]_{-\infty}^{\infty}\Big[C(v) + iS(v)\Big]_{-\infty}^{\infty}.$$

Using the large-argument limits of the Fresnel integrals, which are

$$\lim_{x\to\infty} C(x) = S(x) = \frac{1}{2} \qquad \text{and} \qquad \lim_{x\to-\infty} C(-x) = S(-x) = -\frac{1}{2},$$

the no-aperture field E_0 reduces to

$$E_0 \propto (1+i)^2.$$

The ratio of electric fields E/E_0 is then

$$\frac{E}{E_0} = \frac{\big[C(u) + iS(u)\big]_{u_1}^{u_2}\big[C(u) + iS(u)\big]_{v_1}^{v_2}}{(1+i)^2}$$

The dimensionless variables u and v corresponding to the coordinates $x_1 = y_1 = -0.2\,\text{mm}$, and $x_2 = y_2 = -0.2\,\text{mm}$ are

$$u_1 = \sqrt{\frac{k}{\pi L}}x_1 = \sqrt{\frac{2}{\lambda L}}x_1 = \sqrt{\frac{2}{633\,\text{nm}\cdot 0.5\,\text{m}}} \cdot -0.2\,\text{mm} = -0.05028$$

Analogous calculations for the remaining coordinates produce

$$v_1 = -0.05028 \qquad u_2 = 0.05028 \qquad v_2 = 0.05028.$$

Using a numerical computation program, such as Wolfram Mathematica, produces

$$C(0.5028) = 0.495 \qquad \text{and} \qquad S(0.5028) = 0.0658.$$

Since the Fresnel integrals are odd and the integral limits are symmetric, the electric field ratio reduces to

$$\frac{E}{E_0} = 4\frac{(0.495 + i\cdot 0.0658)^2}{(1+i)^2}.$$

The corresponding intensity, using $I \propto |E|^2$, is

$$\frac{I}{I_0} = 16\frac{\big|(0.495 + i\cdot 0.0658)^2\big|^2}{|(1+i)^2|^2}.$$

7 Seventh Exercise Set

7.1 Thin Slit Interference

We illuminate two thin slits, separated by a distance $D = 0.54\,\mathrm{mm}$, with monochromatic light of wavelength $\lambda = 600\,\mathrm{nm}$. How far away from the slits along the optical axis should we place an observation screen such that interference pattern maxima are separated by a distance $\xi = 1\,\mathrm{mm}$?

Let z_0 denote the distance between the slits and the observation screen; we assume $z_0 \gg D$ so that the two light beams travelling from the two slits to the observation point are essentially parallel.

Interference maxima occur when the distance between optical path lengths from the two slits to the observation point obeys

$$2\pi m = k_0 D \sin\theta \approx k_0 D\theta, \tag{7.1}$$

where θ is the angle between the slits and observation point.

The distance ξ between interference maxima on the observation screen is found from the geometric relation

$$\tan\theta = \frac{\xi}{z_0} \implies \xi = z_0 \tan\theta \approx z_0\theta.$$

We find the angle θ from the interference condition in Equation 7.1, where we assume a first-order maximum, i.e. $m = 1$. We then have

$$\theta = \frac{2\pi}{k_0 D} = \frac{\lambda}{D} \implies z_0 = \frac{\xi}{\theta} = \frac{D\xi}{\lambda} = \frac{1\,\mathrm{mm} \cdot 0.54\,\mathrm{mm}}{600\,\mathrm{nm}} = 0.9\,\mathrm{m}.$$

7.2 Double Slit Experiment

We begin with the same double slit set-up from the previous exercise, with two thin slits separated by $D = 0.54\,\mathrm{mm}$, and then place a glass plate of width $h = 10\,\mathrm{\mu m}$ and refractive index $n = 1.5$ in front of one of the slits. (i) Determine the shift $\Delta\xi$ of the interference pattern maxima on the observation screen. (ii) Determine the glass plate width for which the maxima will shift by $1\,\mathrm{mm}$ relative to their position in the previous exercise, without the glass plate.

Part One: Maxima Shift for Given Plate Thickness

Conceptually: the interference pattern shifts because the light passing through the slit covered with glass travels through a longer optical path than the light passing through the unobstructed slit. As a result, the difference in optical path length between the light from the two slits to the observation point changes.

The constructive interference condition generalizes to

$$\begin{aligned} 2\pi m &= k_0 D \sin\theta + k_0 \cdot (n_{\text{glass}} - n_{\text{air}})h \\ &= k_0 D \sin\theta + k_0(n-1)h \\ &\equiv k_0 D \sin\theta + \Delta k h. \end{aligned}$$

We set $m = 1$, corresponding to a first order maximum, and get

$$2\pi - \Delta k h = k_0 D \sin\theta. \tag{7.2}$$

Assuming the presence of the single glass plate will only cause a small shift in interference maxima, we write the angle θ as

$$\theta = \theta_0 + \Delta\theta,$$

where θ_0 is the angle between slit and interference maxima when both slits are unobstructed (equivalently, when $d = 0$). Equation 7.2 then reads

$$2\pi - \Delta k h = k_0 D \sin(\theta_0 + \Delta\theta) \approx k_0 D \cdot (\theta_0 + \Delta\theta).$$

Recalling $\theta_0 = 2\pi/(k_0 D)$ from the previous exercise, we find

$$2\pi - \Delta k h = k_0 D \cdot \left(\frac{2\pi}{k_0 D} + \Delta\theta\right) \implies \theta = -\frac{\Delta k h}{k_0 D} = -\frac{k_0 h \cdot (n-1)}{k_0 D}$$

The corresponding shift $\Delta\xi$ between the interference maxima is

$$\Delta\xi \approx \Delta\theta \cdot z_0 = -\frac{h \cdot (n-1)}{D} z_0 = -\frac{10\,\mathrm{\mu m} \cdot (1.5-1)}{0.54\,\mathrm{mm}} \cdot 0.9\,\mathrm{m} \approx 8.3\,\mathrm{mm}.$$

Part Two: Plate Thickness for Given Maxima Shift

We now assume the interference maxima when using a glass plate shift by $\Delta\xi = 1\,\mathrm{mm}$ relative to the reference position without a glass plate. To find the corresponding glass plate thickness h, we begin with

$$\Delta\xi = -z_0 \frac{(n-1) \cdot h}{D},$$

which we rearrange to get

$$h = -\frac{\Delta\xi \cdot D}{z_0(n-1)} = -\frac{(1\,\mathrm{mm}) \cdot (0.54\,\mathrm{mm})}{(0.9\,\mathrm{m}) \cdot (1.5-1)} = 1.2\,\mathrm{\mu m}.$$

7.3 Fresnel Biprism

We illuminate a glass Fresnel biprism with refractive index $n = 1.5$ with monochromatic plane wave light of wavelength $\lambda = 633\,\mathrm{nm}$. Determine the acute angle α in the biprism for which interference maxima on a distance observation screen are separated by a distance $\xi = 0.5\,\mathrm{mm}$. You may assume $\alpha \ll 1$.

Our first step is to compute interference pattern on observation screen. The optical field on the observation screen is the sum of the light from the prism's upper and lower faces. Assuming these have equal amplitude E_0, the field at the observation point is

$$E = E_0 e^{i(\mathbf{k}_1 \cdot \mathbf{r} - \omega t)} + E_0 e^{i(\mathbf{k}_2 \cdot \mathbf{r} - \omega t)}$$

We will work in planar Cartesian coordinate system in which the incident plane wave travels along the z axis while the observation screen and flat side of the Fresnel prism align with the z axis. In this case the light from the upper and lower biprism faces have wave vectors $\mathbf{k}_1 = (-k_x, 0, k_y)$ and $\mathbf{k}_2 = (+k_x, 0, k_y)$, and the optical field at the observation point is

$$E(x, z) = E_0 \left(e^{-ik_x x} + e^{+ik_x x}\right) e^{i(k_z z - \omega t)}$$

The corresponding intensity at the observation point is

$$I \propto |E|^2 = |E_0|^2 \cos^2(k_x x).$$

Intensity maxima occur when $\cos(k_x x) = \pm 1$, i.e. when $k_x x = \pi$. Assuming α is the prism's acute angle and β is the angle of refraction of light passing through the two angled prism faces into air, the component k_x is

$$k_x = k_0 \sin(\beta - \alpha) = \frac{2\pi}{\lambda} \sin(\beta - \alpha).$$

The condition for interference maxima is then

$$k_x x = \pi \implies 2\pi \frac{x}{\lambda} \sin(\beta - \alpha) = \pi. \tag{7.3}$$

We find the angle β from Snell's law; assuming α and β are small, we have

$$n_{\text{glass}} \sin\alpha = n_{\text{air}} \sin\beta \implies n \sin\alpha = \sin\beta \quad \xrightarrow{\alpha,\beta \ll 1} \quad n\alpha \approx \beta.$$

Using $\beta \approx n\alpha$, Equation 7.3 becomes

$$\frac{2\pi x}{\lambda} \sin(n\alpha - \alpha) = \pi \implies \sin\big(\alpha(n-1)\big) \approx \alpha \cdot (n-1) = \frac{\lambda}{2x}.$$

We then solve for the prism angle α to get

$$\alpha = \frac{\lambda}{2x \cdot (n-1)} = \frac{\lambda}{2x(n-1)} = \frac{0.633\,\mu\text{m}}{2 \cdot 500\,\mu\text{m} \cdot 0.5} \approx 1.3 \cdot 10^{-3}\,\text{rad}.$$

7.4 Plano-Convex Lens on a Flat Surface

We place a plano-convex lens with refractive index $n = 1.5$ on a perfectly flat table, and illuminate the lens's flat surface from above with monochromatic plane wave light with wavelength $\lambda = 546\,\text{nm}$. (i) Determine the position of the resulting interference maximum rings (Newton rings) as a function of the lens's radius of curvature. (ii) Determine the lens's focus if the tenth maximum ring occurs at a distance $1.02\,\text{mm}$ from the lens's center.

First Question

Assuming the plano-convex lens has rotational symmetry about its central axis, we work in a planar Cartesian coordinate system through a lens cross section and passing through the lens's center. We let the x axis align with the flat table, the z axis with the lens's longitudinal axis, the origin with the contact point between lens and surface, and let $h(x)$ denote the distance between the flat table and the lens's bottom surface.

Interference occurs between light reflecting (i) directly off the lens's bottom surface and (ii) light passing through the lens's bottom surface, reflecting off the flat table, and passing up through the lens.

The phase shift $\Delta\phi$ between these two interfering light sources occurs from the extra distance traveled by the second source between the lens, to the table, and back; this phase shift is

$$\Delta\phi = 2h(x)k_0 n_{\text{air}} = 2k_0 h(x).$$

However (assuming the incident light is TE-polarized) the passage passage of light returning from optically less dense air back into the lens introduces an additional phase shift π, so the full phase difference between interference light sources is

$$\Delta\phi = 2k_0 h(x) + \pi.$$

We compute the distance $h(x)$ between lens and table using the Pythagorean theorem, which gives

$$(R-h)^2 + x^2 = R^2,$$

where R is the lens's radius of curvature. We then multiply out and assume $h \ll x$. Neglecting the second-order h^2 term, the result is

$$x^2 \approx 2Rh \implies h(x) = \frac{x^2}{2R}.$$

The condition for constructive interference between the two interfering light sources is

$$2\pi m = \Delta\phi = 2k_0 h(x) + \pi = 4\pi \frac{h(x)}{\lambda} + \pi$$

We then rearrange to get

$$\frac{\lambda}{2}\left(m - \frac{1}{2}\right) = h(x) = \frac{x^2}{2R}$$

The desired relationship between the position of an interference ring and the lens's radius of curvature is then

$$x^2 = \lambda R \cdot \left(m - \frac{1}{2}\right). \tag{7.4}$$

Second Question

To answer question (ii), we first solve Equation 7.4 for the lens's radius of curvature using the given values of m and x; the result is

$$R = \frac{x^2}{\lambda\left(m - \frac{1}{2}\right)} = \frac{(1.02\,\mathrm{mm})^2}{(546\,\mathrm{nm}) \cdot \left(10 - \frac{1}{2}\right)} \approx 20\,\mathrm{cm}.$$

We then find the lens's focus with the general biconvex lens formula

$$\frac{1}{f} = (n-1)\left(\frac{1}{R_1} + \frac{1}{R_2}\right).$$

For our plano-convex lens $R_1 = R$ on the curve surface and $R_2 \to \infty$ on the flat surface, leaving

$$f = \frac{R}{n-1} = \frac{20\,\mathrm{cm}}{1.5-1} = 40\,\mathrm{cm}.$$

7.5 Theory: Fabry-Perot Interferometer

Derive the phase difference $\Delta\phi$ between the light rays exiting a Fabry-Perot interferometer in which light enters at an angle of incidence α.

A Fabry-Perot interferometer consists of two surfaces separated by a thin film of width d and refractive index n. The reflectance of both surfaces is $R \sim 1$. Light enters the film at an angle of incidence α and forms the same zig-zag pattern inside the thin film that was seen in lecture.

We first consider one cycle in which a light ray enters the interferometer through the left surface, reflects from right surface, reflects from left surface, returns to right surface and exits film. We then aim to find the phase shift between two neighboring transmitted rays.

Let y be the "vertical" distance between points at which adjacent transmitted rays exit the film. And then x is hard to explain—see picture provided in vaje.

Using the Pythagorean theorem and applying basic geometrical reasoning to the path traced out by a light ray in the film, the extra distance travelled by the second ray in the interferometer is

$$\ell = 2\sqrt{\left(\frac{y}{2}\right)^2 + d^2}.$$

Assign phases ϕ_1 and ϕ_2 to the two transmitted waves. These are

$$\phi_1 = xk_0 \qquad \text{and} \qquad \phi_2 = \ell k_0 = 2k_0\sqrt{\left(\frac{y}{2}\right)^2 + d^2}.$$

Note that the second wave also undergoes two reflections in the interferometer, which introduce phases shifts of π and $-\pi$, and thus cancel.

Another expression for ϕ_1 and ϕ_2 in terms of trigonometric functions is

$$\phi_2 = 2\frac{d}{\cos\beta}nk_0 \qquad \text{and} \qquad \phi_1 = y\sin\alpha k_0 = yn\sin\beta k_0.$$

Meanwhile $y = 2\tan\beta d$. Use this to get

$$\phi_1 = 2\frac{\sin\beta}{\cos\beta}\sin\beta dnk_0.$$

The phase difference between neighboring rays exiting the interferometer is then

$$\Delta\phi = \phi_2 - \phi_1 = 2dnk_0\left(\frac{1}{\cos\beta} - \frac{\sin^2\beta}{\cos\beta}\right) = 2dnk_0\cos\beta. \tag{7.5}$$

8 Eighth Exercise Set

8.1 Resolving the Hydrogen Doublet with a Fabry-Perot Interferometer

Consider a glass Fabry-Perot interferometer with an index of refraction $n = 1.5$, surfaces with reflectance $R = 0.95$, and an adjustable width d. Determine the width h such that the interferometer can resolve the hydrogen α doublet lines, which are centered at $\lambda_0 = 656.28\,\mathrm{mm}$ and separated by $\Delta\lambda = 0.016\,\mathrm{mm}$.

We assume the Fabry-Perot interferometer is surrounded by air on both sides—we call the air "region 1" and the glass interferometer film "region 2". Quoting from lecture, if we illuminate one side of the interferometer with plane wave light of amplitude E_0 at an incidence angle α, the transmitted light emerging from the interferometer is given by

$$E_{\mathrm{out}} = E_0 \frac{t_{12}t_{21}}{1 - r_{21}^2 e^{i\phi}}, \quad \text{where } \phi = 2k_0 nd\cos\alpha,$$

where t_{ij} and r_{ij} denote the transmission and reflection coefficients for light passing from region i into region j.

Our goal is to determine how the interferometer width d affects the interferometer's resolution. Our first step is to find the interferometer's transmittance; quoting from lecture, this is

$$T = \frac{1}{1 + \frac{4R}{(1-R)^2}\sin^2(\phi/2)}, \quad \text{where } R = |r_{21}|^2.$$

We often define an interferometer *finesse constant* F given by

$$F = \frac{4R}{(1-R)^2} \implies T = \frac{1}{1 + F\sin^2(\phi/2)}. \tag{8.1}$$

The transmittance T, via the phase term ϕ (Equation 7.5), depends on the incident light's wavelength λ. If we draw $T(\lambda)$ for the two closely-spaced hydrogen doublet lines, we get two closely-spaced peaks at the wavelengths λ_1 and λ_2. These two peaks must be separated enough (or, alternatively, the transmission peaks must be narrow enough) to resolve the two doublet lines.

We define the following criterion for resolution:

> The two $T(\lambda)$ peaks are resolved if they are still separated in λ space when each individual peak falls to half of its maximum value.

Setting $T = 1/2$ Equation 8.1 and simplifying produces the condition

$$F\sin^2\frac{\phi}{2} = 1 \implies \frac{\phi}{2} = \arcsin\frac{1}{\sqrt{F}}.$$

We then write ϕ in terms of wavelength using Equation 7.5 and assume normal incidence, which produces

$$\frac{\phi}{2} = k_0 nd = \frac{2\pi}{\lambda}nd = \arcsin\frac{1}{\sqrt{F}}. \tag{8.2}$$

We now consider how the peak of the interferometer's $T(\lambda)$ curve changes with wavelength λ. We first note $\lambda = c/\nu$, which we use to get

$$\mathrm{d}\lambda = -\frac{c}{\nu^2}\,\mathrm{d}\nu \implies \frac{\mathrm{d}\lambda}{\lambda} = -\frac{\mathrm{d}\nu}{\nu}.$$

We then write Equation 8.2 in terms of frequency, which produces

$$\frac{\phi}{2} = \arcsin \frac{1}{\sqrt{F}} = k_0 nd = \frac{2\pi}{\lambda} nd = \frac{\omega nd}{c} \implies \arcsin \frac{1}{\sqrt{F}} = \frac{\Delta\omega nd}{c}, \tag{8.3}$$

where the frequency spacing $\Delta\omega$ must agree with the width $\Delta\lambda$ between the doublet lines. Next, we note that for $F \gg 1$, corresponding to large interferometer reflectance R, we can make the approximation

$$\arcsin \frac{1}{\sqrt{F}} \approx \frac{1}{\sqrt{F}},$$

which we substitute into Equation 8.3 to get

$$\arcsin \frac{1}{\sqrt{F}} \approx \frac{1}{\sqrt{F}} = \frac{\Delta\omega nd}{c} = \frac{nd}{c}\frac{\omega}{\lambda}\Delta\lambda.$$

We then solve for the desired interferometer width d, which should come out to

$$d = \frac{1}{2\pi n\sqrt{F}}\frac{\lambda^2}{\Delta\lambda} = \cdots = 73\,\mu\text{m}.$$

This is the width of glass layer at which the Fabry-Perot interferometer will work at high enough resolution to resolve hydrogen doublet.

Finally, in passing, we note that in conductors where $k \in \mathbb{C}$, an Fabry-Perot interferometer transmittance generalizes to

$$T = \left(1 - \frac{A}{1-R}\right)^2 \frac{1}{1 + \frac{4R}{(1-R)^2}\sin^2(\phi/2)},$$

where A is of order one but for good film application we can achieve $A \ll 1$ and in this case the conductor correction term is negligible.

8.2 Theory: Matrix Formalism for Thin Films

We now summarize the matrix formalism for thin films from lecture. Consider plane-wave light with electric field amplitude E_0 normally incidence along the x axis on a series of thin films with refractive indices $n_1, n_2, \ldots$ The x axis points from left to right in our drawing on paper.

Notation:

- Unprimed: transmitted light (moving in the $+x$ direction)
- Primed: reflected light (moving in the $-x$ direction)
- Tilde: quantities on the far right of a film region
- No tilde: quantities on the far left of a film region

The electric fields on either side of the boundary between the $(m-1)$-th and m-th films are related by

$$\begin{pmatrix} \widetilde{E}_{m-1} \\ \widetilde{E}'_{m-1} \end{pmatrix} = \frac{1}{t_{m-1,m}} \begin{pmatrix} 1 & r_{m-1,m} \\ r_{m-1,m} & 1 \end{pmatrix} \begin{pmatrix} E_m \\ E'_m \end{pmatrix},$$

while the additional phase accumulated by light passing through the m-th film is accounted for by the relationship

$$\begin{pmatrix} E_m \\ E'_m \end{pmatrix} = \begin{pmatrix} e^{-i\delta_m} & 0 \\ 0 & e^{i\delta_m} \end{pmatrix} \begin{pmatrix} \widetilde{E}_m \\ \widetilde{E}'_m \end{pmatrix} \equiv \mathbf{P}_m \begin{pmatrix} \widetilde{E}_m \\ \widetilde{E}'_m \end{pmatrix}, \quad \text{where } \delta_m = n_m k_0 d_m.$$

The transfer matrix encoding passage of normally incident light through a series of N thin films (note the matrix is constructed from right to left, i.e. transforming from transmitted light to incident light) is

$$\mathbf{M}_{\text{tot}} \equiv \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} = \mathbf{M}_{\text{in},1}\mathbf{P}_1\mathbf{M}_{12}\mathbf{P}_2 \cdots \mathbf{P}_{\text{N}}\mathbf{M}_{N,\text{out}},$$

and so the incident and transmitted light is related by

$$\mathbf{M}_{\text{tot}} \begin{pmatrix} E_{\text{out}} \\ E'_{\text{out}} \end{pmatrix} = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \begin{pmatrix} E_{\text{out}} \\ E'_{\text{out}} \end{pmatrix} = \begin{pmatrix} E_{\text{in}} \\ E'_{\text{in}} \end{pmatrix}.$$

We then recognize that $E'_{\text{out}} = 0$ (the transmitted light moves only in the $+x$ direction), which we use to find the film's transmission coefficient t:

$$E_{\text{in}} = M_{11} E_{\text{out}} \implies t = \frac{E_{\text{out}}}{E_{\text{in}}} = \frac{1}{M_{11}}. \tag{8.4}$$

Similarly, we find the film's reflection coefficient r from

$$r = \frac{E'_{\text{in}}}{E_{\text{in}}} = \frac{M_{21} E_{\text{out}}}{M_{11} E_{\text{out}}} = \frac{M_{21}}{M_{11}}. \tag{8.5}$$

8.3 Perfect Transmittance Through a Thin Film

Consider the following sequence of materials: (i) air, (ii) a thin film of to-be-determined refractive index n_2 and width d_2, and (iii) glass with refractive index $n_3 = 1.54$. Determine smallest possible thickness d_2 and refractive index n_2 for which the three-material sequence perfectly transmits normally-incident light of wavelength $\lambda = 540\,\text{nm}$.

Assuming the air and glass layers are thick compared to the thin film, we can represent the passage of light from air through film and into glass with the transfer matrix

$$\mathbf{M} = \mathbf{M}_{12}\mathbf{P}\mathbf{M}_{23} = \mathbf{M}_{12} \begin{pmatrix} e^{-i\delta} & 0 \\ 0 & e^{+i\delta} \end{pmatrix} \mathbf{M}_{23}, \quad \text{where } \delta = n_2 k_0 d.$$

The matrix $\mathbf{M}_{12}$ encodes passage from air to film, $\mathbf{P}$ the phase accumulated by light passing through the film, and $\mathbf{M}_{23}$ the passage from film to glass. In components, the matrices $\mathbf{M}_{12}$ and $\mathbf{M}_{23}$ read

$$\mathbf{M}_{12} = \frac{1}{t_{12}} \begin{pmatrix} 1 & r_{12} \\ r_{12} & 1 \end{pmatrix} \quad \text{and} \quad \mathbf{M}_{23} = \frac{1}{t_{23}} \begin{pmatrix} 1 & r_{23} \\ r_{23} & 1 \end{pmatrix},$$

and the product $\mathbf{M}_{12}\mathbf{P}\mathbf{M}_{23}$ comes out to

$$\mathbf{M} = \mathbf{M}_{12}\mathbf{P}\mathbf{M}_{23} = \frac{1}{t_{12}t_{23}} \begin{pmatrix} 1 & r_{12} \\ r_{12} & 1 \end{pmatrix} \begin{pmatrix} e^{-i\delta} & e^{-i\delta} r_{23} \\ r_{23} e^{i\delta} & e^{i\delta} \end{pmatrix}.$$

Since we are ultimately interested in the film's transmittance and reflectance, and thus need the coefficients r and t (Equations 8.4 and 8.5), we only need the matrix elements M_{11} and M_{21}. These come out to

$$M_{11} = \frac{1}{t_{12}t_{23}}\left(e^{-i\delta} + r_{12}r_{23}e^{i\delta}\right) \qquad \text{and} \qquad M_{21} = \frac{1}{t_{12}t_{23}}\left(r_{12}e^{-i\delta} + r_{23}e^{i\delta}\right),$$

from which we find

$$t = \frac{1}{M_{11}} = \frac{t_{12}t_{23}}{\left(e^{-i\delta} + r_{12}r_{23}e^{i\delta}\right)} \qquad \text{and} \qquad r = \frac{M_{21}}{M_{11}} = \frac{\left(r_{12}e^{-i\delta} + r_{23}e^{i\delta}\right)}{\left(e^{-i\delta} + r_{12}r_{23}e^{i\delta}\right)}. \tag{8.6}$$

The condition for perfect transmission of light through the film is formulated as $T = 1$ and $R = 0$. Assuming normal incidence, T and R are given by

$$T = |t|^2\frac{n_3}{n_1} \qquad \text{and} \qquad R = |r|^2.$$

Using the above expression for t, we find the film's transmittance T is

$$\begin{aligned} T &= \frac{n_3}{n_1}\frac{\left|t_{12}t_{23}\right|^2}{\left|e^{-i\delta} + r_{12}r_{23}e^{i\delta}\right|^2} = \frac{n_3}{n_1}\frac{\left|t_{12}t_{23}\right|^2}{\left|(1 + r_{12}r_{23}e^{2i\delta})\right|^2} \\ &= \frac{n_3}{n_1}\frac{\left|t_{12}t_{23}\right|^2}{1 + \left|r_{12}r_{23}\right|^2 + 2r_{12}r_{23}\cos 2\delta}. \end{aligned}$$

The denominator is minimized, and hence T maximized, when $\cos 2\delta = -1$ or $2\delta = (2m-1)\pi$. Recalling $\delta = k_0 n_2 d$ and choosing $m = 1$, which will produce the smallest possible film thickness d, the maximum transmittance condition reads

$$2k_0 n_2 d = \pi \implies d = \frac{\pi}{2k_0 n_2} = \frac{\lambda}{4n_2}. \tag{8.7}$$

Meanwhile, the film's reflectance, assuming real reflection coefficients $r_{12}, r_{23} \in \mathbb{R}$, is given by

$$R = |r|^2 = \frac{r_{12}^2 + r_{23}^2 + 2r_{12}r_{23}\cos 2\delta}{1 + r_{12}r_{23}^2 + 2r_{12}r_{23}\cos 2\delta}, \tag{8.8}$$

Applying the maximum-transmittance condition $\cos 2\delta = -1$, the reflectance becomes

$$\frac{r_{12}^2 + r_{23}^2 - 2r_{12}r_{23}}{1 + r_{12}r_{23}^2 - 2r_{12}r_{23}} = \frac{(r_{12} - r_{23})^2}{1 + r_{12}r_{23}^2 - 2r_{12}r_{23}},$$

from which we see that $R = 0$, and thus $T = 1$, when $r_{12} = r_{23}$.

We express r_{12} and r_{23} using the Fresnel equations in the simple case of normal incidence; the coefficients read

$$r_{12} = \frac{n_1 - n_2}{n_1 + n_2} \qquad \text{and} \qquad r_{23} = \frac{n_2 - n_3}{n_2 + n_3}. \tag{8.9}$$

We then apply the maximum transmittance condition $r_{12} = r_{23}$ to get

$$\frac{n_1 - n_2}{n_1 + n_2} = \frac{n_2 - n_3}{n_2 + n_3} \implies (n_1 - n_2)(n_2 + n_3) = (n_2 - n_3)(n_1 + n_2)$$

After multiplying out and cancelling common terms, the result is

$$n_2^2 = n_1 n_3 \implies n_2 = \sqrt{n_1 n_3} = \sqrt{n_3} = \sqrt{1.54} \approx 1.24.$$

where we have used $n_1 = n_{\text{air}} = 1$. From Equation 8.7, the corresponding film thickness for maximum transmission at the given wavelength $\lambda = 540\,\text{nm}$ is

$$d = \frac{\lambda}{4n_2} = \frac{540\,\text{nm}}{4 \cdot 1.24} \approx 109\,\text{nm}.$$

8.4 Maximizing Transmittance Through a Thin Film

Similarly to the previous problem, we consider a (i) air, (ii) thin film, (iii) glass sequence, where in this case the film is made of magnesium fluoride with refractive index $n_2 = 1.38$, while the glass has refractive index $n_3 = 1.5$. Determine the smallest possible film thickness d such that the air-film-glass interface transmits green light of wavelength $\lambda = 550\,\mathrm{nm}$ with maximum possible transmittance. Is perfect transmittance possible?

The air-film-glass interface is identical to the previous problem, so we can reuse the previous problem's results. Using Equation 8.7, which encodes the maximum-transmittance condition $\cos 2\delta = -1$, together with our problem's given wavelength and thin film refractive index, we find maximum transmittance occurs at a film thickness

$$d = \frac{\lambda}{4n_2} = \frac{550\,\mathrm{nm}}{4 \cdot 1.38} \approx 100\,\mathrm{nm}.$$

From Equation 8.8 (assuming real reflection coefficients) the film's reflectance R is

$$R = \frac{r_{12}^2 + r_{23}^2 + 2r_{12}r_{23}\cos 2\delta}{1 + r_{12}^2 r_{23}^2 + 2r_{12}r_{23}\cos 2\delta},$$

which under the maximum-transmittance condition $\cos 2\delta = -1$ becomes

$$R = \frac{(r_{12} - r_{23})^2}{1 + r_{12}^2 r_{23}^2 - 2r_{12}r_{23}}. \tag{8.10}$$

Using Equation 8.9 to compute r_{12} and r_{23}, we find

$$\begin{aligned} r_{12} &= \frac{n_1 - n_2}{n_1 + n_2} = \frac{1 - 1.38}{2.38} \approx -0.16 \\ r_{23} &= \frac{n_2 - n_3}{n_2 + n_3} = \frac{1.38 - 1.5}{2.88} \approx 0.042. \end{aligned}$$

Substituting these coefficients into Equation 8.10 produces

$$R = \frac{(r_{12} - r_{23})^2}{1 + r_{12}^2 r_{23}^2 - 2r_{12}r_{23}} \approx 0.01 \implies T \approx 0.99.$$

In other words, perfect transmittance is not possible for the given wavelength $\lambda = 550\,\mathrm{nm}$ at the thin film refractive index $n_2 = 1.38$, but we can come very close.

9 Ninth Exercise Set

9.1 A Multi-Film Dielectric Mirror

A dielectric mirror consists of 17 thin films with alternating refractive indices $n_a = 1.38$ and $n_b = 2.32$, applied to single glass layer with refractive index $n_3 = 1.52$. Compute the mirror's reflectance for normally-incident light of wavelength λ if the a and b films are $\lambda/(4n_a)$ and $\lambda/(4n_b)$ thick, respectively.

Structure: air, a, b, a, b, ..., a, b, a, glass. The total transfer matrix for passage of light through the air-film-glass sequence is

$$\mathbf{M}_{\text{tot}} = \mathbf{M}_{1a}\mathbf{P}_a\mathbf{M}_{ab}\mathbf{P}_b\mathbf{M}_{ba}\cdots\mathbf{M}_{ba}\mathbf{P}_a\mathbf{M}_{a3} \equiv \mathbf{M}_{1a}\mathbf{M}_N\mathbf{P}_a\mathbf{M}_{a3}, \tag{9.1}$$

where we have defined

$$\mathbf{M}_N \equiv \mathbf{P}_a\mathbf{M}_{ab}\mathbf{P}_b\mathbf{M}_{ba}\cdots\mathbf{P}_a\mathbf{M}_{ab}\mathbf{P}_b\mathbf{M}_{ba}.$$

Passage from air into the first a film is given by

$$\mathbf{M}_{1a} = \frac{1}{t_{1a}}\begin{pmatrix} 1 & r_{1a} \\ r_{1a} & 1 \end{pmatrix}; \qquad r_{1a} = \frac{1-n_a}{1+n_a}, \quad t_{1a} = 1 + r_{1a} = \frac{2}{1+n_a}.$$

Passage from the last a film into glass given by

$$\mathbf{M}_{a3} = \frac{1}{t_{a3}}\begin{pmatrix} 1 & r_{a3} \\ r_{a3} & 1 \end{pmatrix}; \qquad r_{a3} = \frac{n_a - n_3}{n_a + n_3}, \qquad t_{a3} = 1 + r_{a3} = \frac{2n_a}{n_a + n_3}$$

In terms of refractive indices, $\mathbf{M}_{1a}$ and $\mathbf{M}_{a3}$ read

$$\mathbf{M}_{1a} = \frac{1}{2}\begin{pmatrix} 1+n_a & 1-n_a \\ 1-n_a & 1+n_a \end{pmatrix} \qquad \text{and} \qquad \mathbf{M}_{a3} = \frac{1}{2}\begin{pmatrix} n_a+n_3 & n_a-n_3 \\ n_a-n_3 & n_a+n_3 \end{pmatrix}.$$

The phase matrix for passage through an a or b film is

$$\mathbf{P}_{a,b} = \begin{pmatrix} e^{-i\delta_{a,b}} & 0 \\ 0 & e^{i\delta_{a,b}} \end{pmatrix} = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}, \tag{9.2}$$

where the last equality follows from

$$\delta_{ab} = n_{a,b}k_0 d_{a,b} = n_{a,b}\frac{2\pi}{\lambda}\frac{\lambda}{4n_{a,b}} = \frac{\pi}{2} \implies e^{\pm i\delta_{a,b}} = \pm i.$$

Finally, the transfer matrices for ab and ba boundaries read

$$\mathbf{M}_{ab} = \frac{1}{t_{ab}}\begin{pmatrix} 1 & r_{ab} \\ r_{ab} & 1 \end{pmatrix} \qquad \text{and} \qquad \mathbf{M}_{ba} = \frac{1}{t_{ba}}\begin{pmatrix} 1 & r_{ba} \\ r_{ba} & 1 \end{pmatrix}.$$

For compactness, we note $r_{ba} = -r_{ab}$ and introduce the new notation $r_{ab} \equiv r$, giving

$$r_{ba} = -r_{ab} = -r, \qquad t_{ab} = 1 + r, \qquad t_{ba} = 1 - r.$$

The ab and ba interface transfer matrices then read

$$\mathbf{M}_{ab} = \frac{1}{1+r}\begin{pmatrix} 1 & r \\ r & 1 \end{pmatrix} \qquad \text{and} \qquad \mathbf{M}_{ba} = \frac{1}{1-r}\begin{pmatrix} 1 & -r \\ -r & 1 \end{pmatrix}.$$

We then define the matrix $\mathbf{M}_{1/N}$ for passage through a single ab sequence as

$$\mathbf{M}_{1/N} = \mathbf{P}_a \mathbf{M}_{ab} \mathbf{P}_b \mathbf{M}_{ba},$$

which, factoring out i terms from the phase matrices, comes out to

$$\begin{aligned}\mathbf{M}_{1/N} &= -i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \frac{1}{1+r} \begin{pmatrix} 1 & r \\ r & 1 \end{pmatrix} (-i) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \frac{1}{1-r} \begin{pmatrix} 1 & -r \\ -r & 1 \end{pmatrix} \\ &= -\frac{1}{1-r^2} \begin{pmatrix} 1 & r \\ -r & -1 \end{pmatrix} \begin{pmatrix} 1 & -r \\ r & -1 \end{pmatrix} \\ &= -\frac{1}{1-r^2} \begin{pmatrix} 1+r^2 & -2r \\ -2r & 1+r^2 \end{pmatrix}.\end{aligned}$$

Our next step is to compute the product

$$\mathbf{M}_N = \big(\mathbf{M}_{1/N}\big)^N.$$

To make the multiplication easier, we will first diagonalize $\mathbf{M}_{1/N}$ in the form

$$\mathbf{M}_{1/N} = \mathbf{S}\mathbf{D}\mathbf{S}^{-1}. \tag{9.3}$$

We find $\mathbf{M}_{1/N}$'s eigenvalues from the solutions to the characteristic polynomial

$$\det\big(\mathbf{M}_{1/N} - \lambda \mathbf{I}\big) = \begin{vmatrix} 1+r^2-\lambda & -2r \\ -2r & 1+r^2-\lambda \end{vmatrix} = (1+r^2-\lambda)^2 - 4r^2 = 0,$$

which as the solution

$$\lambda_\pm = 1 + r^2 \pm 2r = (1 \pm r)^2.$$

Without full derivation, the corresponding eigenvectors and transformation matrix are, in the eigenvalue order λ_- and λ_+, equal to

$$\mathbf{S} = \big(\mathbf{s}_- \ \ \mathbf{s}_+\big) = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \implies \mathbf{S}^{-1} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}. \tag{9.4}$$

The diagonal matrix in Equation 9.3 is then

$$\mathbf{D} = \frac{1}{r^2-1}\begin{pmatrix} \lambda_- & 0 \\ 0 & \lambda_+ \end{pmatrix} = \frac{1}{r^2-1}\begin{pmatrix} (1-r)^2 & 0 \\ 0 & (1+r)^2 \end{pmatrix},$$

the matrix product for $\mathbf{M}_N$ is

$$\mathbf{M}_N = \big(\mathbf{M}_{1/N}\big)^N = \big(\mathbf{S}\mathbf{D}\mathbf{S}^{-1}\big)^N = \mathbf{S}\mathbf{D}^N\mathbf{S}^{-1},$$

while the transfer matrix for the entire air-film-glass sequence (Equation 9.1) is

$$\mathbf{M}_{\text{tot}} = \mathbf{M}_{1a}\mathbf{S}\mathbf{D}^N\mathbf{S}^{-1}\mathbf{P}_a\mathbf{M}_{a3}.$$

It remains to compute the matrix products. Beginning with $\mathbf{D}^N$, we have

$$\begin{aligned}\mathbf{D}^N &= \frac{1}{(r^2-1)^N}\begin{pmatrix} (1-r)^{2N} & 0 \\ 0 & (1+r^2)^{2N} \end{pmatrix} = \cdots = \begin{pmatrix} \left(\frac{r-1}{r+1}\right)^N & 0 \\ 0 & \left(\frac{r+1}{r-1}\right)^N \end{pmatrix} \\ &\equiv \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}, \quad \text{where } A \equiv \left(\frac{r-1}{r+1}\right)^N \quad \text{and} \quad B = \frac{1}{A}.\end{aligned} \tag{9.5}$$

where we have defined the constants A and B for compactness. We then return to computing $\mathbf{M}_{\text{tot}}$, which in terms of $\mathbf{D}^N$ reads

$$\begin{aligned}\mathbf{M}_{\text{tot}} &= \mathbf{M}_{1a}\mathbf{S}\mathbf{D}^N\mathbf{S}^{-1}\mathbf{P}_a\mathbf{M}_{a3}\\ &= \frac{1}{2}\mathbf{M}_{1a}\mathbf{S}\begin{pmatrix} A & A \\ B & -B\end{pmatrix}\mathbf{P}_a\mathbf{M}_{a3}\\ &= \frac{1}{4}\mathbf{M}_{1a}\begin{pmatrix} A+B & A-B \\ A-B & A+B\end{pmatrix}\mathbf{P}_a\mathbf{M}_{a3},\end{aligned}$$

where we have used the expressions for $\mathbf{S}$ and $\mathbf{S}^{-1}$ in Equation 9.4. Next, using the expression for $\mathbf{M}_{1a}$ in Equation 9.1, we have

$$\mathbf{M}_{\text{tot}} = \frac{1}{8}\begin{pmatrix} 1+n_a & 1-n_a \\ 1-n_a & 1+n_a\end{pmatrix}\begin{pmatrix} A+B & A-B \\ A-B & A+B\end{pmatrix}\mathbf{P}_a\mathbf{M}_{a3}.$$

Next we have the intermediate calculations

$$\mathbf{P}_a\mathbf{M}_{a3} = (-i)\begin{pmatrix} n_a+n_3 & n_a-n_3 \\ n_a-n_3 & -n_a-n_3\end{pmatrix}$$

$$\begin{pmatrix} 1+n_a & 1-n_a \\ 1-n_a & 1+n_a\end{pmatrix}\begin{pmatrix} A+B & A-B \\ A-B & A+B\end{pmatrix} = \begin{pmatrix} 2A+2n_aB & 2A-2n_aB \\ 2A-2n_aB & 2A+2n_aB\end{pmatrix},$$

in terms of which $\mathbf{M}_{\text{tot}}$ is

$$\mathbf{M}_{\text{tot}} = -\frac{i}{8}\begin{pmatrix} 2A+2n_aB & 2A-2n_aB \\ 2A-2n_aB & 2A+2n_aB\end{pmatrix}\begin{pmatrix} n_a+n_3 & n_a-n_3 \\ n_a-n_3 & -n_a-n_3\end{pmatrix}.$$

Before making the final multiplication, we recall that to find reflectance R, we only need matrix elements M_{11} and M_{21}, since

$$R = |r|^2 = \left|\frac{M_{21}}{M_{11}}\right|^2.$$

The required matrix elements come out to

$$\mathrm{M}_{11} = 4An_3 + 4n_a^2B \qquad \text{and} \qquad \mathrm{M}_{21} = 4An_3 - 4n_a^2B,$$

and, after cancelling the common factor of 4, the corresponding reflectance is

$$R = \left|\frac{M_{21}}{M_{11}}\right|^2 = \frac{\left|An_3 - n_a^2B\right|^2}{\left|An_3 - n_a^2B\right|^2}.$$

We then recall the original definitions $A = \left(\frac{r-1}{r+1}\right)^N$ (Eq. 9.5) and $r = r_{ab} = \frac{n_a-n_b}{n_a+n_b}$ to get

$$A = \left(\frac{r-1}{r+1}\right)^N = \left(-\frac{n_b}{n_a}\right)^N = \left(-\frac{2.32}{1.38}\right)^{17} \approx 6850 \gg 1.$$

Correspondingly $B = 1/A \ll 1$, from which we can conclude

$$R = \frac{\left|An_3 - n_a^2B\right|^2}{\left|An_3 - n_a^2B\right|^2} \approx \left|\frac{An_3}{An_3}\right|^2 \approx 1.$$

A more thorough calculation using the exact values of A and B should produce $R = 0.9992$.

Keep in mind that this entire derivation rests on the films acting as quarter wave plates, i.e. introducing a phase shift $\pi/2$, which leads to the simplified phase matrix expressions in Equation 9.2.

9.2 Theory: Skin Depth of a Metallic Mirror

Derive the transmittance of a metallic mirror in the limit cases for which (i) the metal's width is small compared to its skin depth (ii) the metal's width is large compared to its skin depth

We model a metallic mirror as thin film of metal with refractive index n_2 and width d applied to glass, analogous to the film-coated glass in Exercise 8.3. We begin by recalling the transmission coefficient for the air-film-glass equation from Equation 8.6 in Exercise 8.3, which reads

$$t = \frac{t_{12}t_{23}}{e^{-i\delta}(1 + r_{12}r_{23}e^{2i\delta})},$$

where $\delta = n_2 k_0 d$ is the phase accumulated by light passing through the metal film. Assuming normal incident, the metal mirror's corresponding transmittance is

$$T = |t|^2 \frac{n_{\text{glass}}}{n_{\text{air}}} = n_{\text{g}}|t|^2,$$

where we have written $n_{\text{glass}} \equiv n$ for shorthand and used $n_{\text{air}} = 1$.

Unlike in Equation 8.6, we now consider a metallic film, which will have a complex refractive index; we write this as

$$n_2 = n_{\text{Re}} + in_{\text{Im}} \implies \delta = n_{\text{Re}}k_0 d + in_{\text{Im}}k_0 d \equiv \phi + i\kappa.$$

Using this expression for δ, the mirror's transmission coefficient is then

$$t = \frac{t_{12}t_{23}}{e^{-i\phi}e^{\kappa}(1 + r_{12}r_{23}e^{2i\phi}e^{-2\kappa})}.$$

Then, using a slight trick, we add and subtract $r_{12}r_{23}$ from the denominator and get

$$t = \frac{t_{12}t_{23}}{e^{-i\phi}e^{\kappa}(1 + r_{12}r_{23} + r_{12}r_{23}e^{2i\phi}e^{-2\kappa}) - r_{12}r_{23}},$$

and then define

$$x \equiv r_{12}r_{23}(e^{2i\phi}e^{-2\kappa} - 1) \implies t = \frac{t_{12}t_{23}}{e^{-i\phi}e^{\kappa}(1 + r_{12}r_{23} + x)}.$$

The corresponding transmittance is

$$T = n|t|^2 = \frac{n|t_{12}t_{23}|^2 e^{-2\kappa}}{|1 + r_{12}r_{23} + x|^2}.$$

Limit Case: Small Film Width

The condition of small film width relative to skin depth reads

$$d \ll \frac{1}{k_0 n_{\text{Im}}},$$

in which case $\phi, \kappa \approx 0$ which implies $x \approx 0$. The mirror's transmittance is then

$$T \approx \frac{n|t_{12}t_{23}|^2}{|1 + r_{12}r_{23}|^2} e^{-2\kappa} \equiv T_0 e^{-2\kappa},$$

where T_0 is the transmittance if there were only glass and no metal film.

Limit Case: Large Film Width

The condition of small film width relative to skin depth reads

$$d \gg 1/(k_0 n_{\mathrm{Im}}),$$

in which case $\kappa = n_{\mathrm{Im}} k_0 d \gg 1$ and thus $x \approx -r_{12} r_{23}$. In this case the mirror's transmittance is

$$T \approx n |t_{12} t_{23}|^2 e^{-2\kappa}.$$

9.3 Theory: Autocorrelation and Power Spectral Density

Briefly reviewing from lecture: the electric field at a Michelson interferometer using two beams, in which one travels an additional distance $2x$ from source to detector, where x is the shift of the interferometer's movable mirror relative to the equilibrium reference position, is

$$E_{\mathrm{d}} = E(t) + E(t+\tau), \quad \text{where } \tau = \frac{2x}{c_0}.$$

If $I_0(t) = |E(t)|^2$ denotes the intensity of a single beam, the intensity at the detector is given by

$$I_{\mathrm{d}} = |E_{\mathrm{d}}|^2 = 2I_0 + 2\operatorname{Re} G^{(1)}(\tau), \tag{9.6}$$

where $G^{(1)}$ is the autocorrelation function

$$G^{(1)}(\tau) = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} E(t) E^*(t+\tau) \,\mathrm{d}t. \tag{9.7}$$

Finally, we quote that the power spectral density $S(\omega)$ at the detector and autocorrelation function $G^{(1)}(\tau)$ are related by the Wiener-Khinchin according to

$$G^{(1)}(\tau) = \int_{-\infty}^{\infty} S(\omega) e^{i\omega\tau} \,\mathrm{d}\omega \qquad \text{and} \qquad S(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} G^{(1)}(\tau) e^{-i\omega\tau} \,\mathrm{d}\tau.$$

In other words, the power spectral density is the Fourier transform of the autocorrelation function.

9.4 Michelson Interferometer

We wish to use a Michelson interferometer to observe sodium's doublet lines, which have wavelengths $\lambda_1 = 588.995\,\mathrm{nm}$ and $\lambda_2 = 589.592\,\mathrm{nm}$. How far must we move the interferometer's movable mirror to resolve the doublet lines?

Letting $\omega_0 = \omega_1 - \omega_2$ denote the doublet's central frequency, and $\Delta\omega = |_1 - \omega_2|$, we can model sodium doublet's power spectral density as

$$S(\omega) = \frac{I}{2} \left\{ \delta\Big(\omega - \big(\omega_0 - \tfrac{\Delta\omega}{2}\big)\Big) + \delta\Big(\omega - \big(\omega_0 + \tfrac{\Delta\omega}{2}\big)\Big) \right\},$$

which is just two delta functions centered at the two spectral line frequencies.

$$S(\omega) = \frac{1}{2} I_0 \Big[\delta(\omega - \omega_0) + \delta\big(\omega - (\omega_0 + \Delta\omega)\big)\Big]$$

where $\Delta\omega$ is spacing between spectrum lines.

The spectrum's autocorrelation function is

$$G^{(1)}(\tau) = \int_{-\infty}^{\infty} S(\omega)e^{i\omega\tau}\,\mathrm{d}\omega = \frac{I_0}{2}\left(e^{i\left(\omega_0 - \frac{\Delta\omega}{2}\right)\tau} + e^{i\left(\omega_0 + \frac{\Delta\omega}{2}\right)\tau}\right),$$

which we use in Equation 9.6 to find the intensity at the detector:

$$I_{\mathrm{det}} = 2I_0 + 2\,\mathrm{Re}\,G^{(1)}(\tau) = 2I_0 + I_0\,\mathrm{Re}\left(e^{i\left(\omega_0 - \frac{\Delta\omega}{2}\right)\tau} + e^{i\left(\omega_0 + \frac{\Delta\omega}{2}\right)\tau}\right).$$

We then take the real component of each complex exponent to get

$$I_{\mathrm{det}} = 2I_0 + I_0 \cdot \Big\{ \cos\left(\left[\omega_0 - \tfrac{\Delta\omega}{2}\right]\tau\right) + \cos\left(\left[\omega_0 + \tfrac{\Delta\omega}{2}\right]\tau\right) \Big\}.$$

Finally, we apply the identity $\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$ to get

$$I_{\mathrm{det}} = 2I_0\left(1 + \cos\omega_0\tau\cos\frac{\Delta\omega\tau}{2}\right).$$

Interpretation: the detector intensity $I_{\mathrm{det}}(\tau)$ is amplitude modulated with large frequency envelope $\cos(\omega_0\tau)$ and faster oscillations in between, from $\cos\frac{\Delta\Omega\tau}{2}$.

Our condition for resolution on is to use a mirror translation for which we capture a complete period of the $I_{\mathrm{det}}(\tau)$ interference pattern—for example the distance (in τ space) between two of I_{det}'s minima. The minima occur when $\cos\frac{\Delta\omega\tau}{2} = 0$, from which we get the condition

$$\cos\frac{\Delta\omega\tau}{2} = 0 \implies \frac{\Delta\omega\tau}{2} = \frac{\pi}{2} \implies \Delta\omega\tau = \pi \implies \frac{2\Delta\omega x}{c_0} = \pi.$$

We then solve for the required mirror translation to get

$$\Delta x = \frac{\pi c_0}{2\Delta\omega} = \frac{\pi c_0}{2} \cdot \frac{1}{2\pi c_0\left(\frac{1}{\lambda_1} - \frac{1}{\lambda_2}\right)} = \frac{1}{4}\frac{\lambda_2 - \lambda_1}{\lambda_1\lambda_2} \approx 1.45\,\mu\mathrm{m}.$$

10 Tenth Exercise Set

10.1 Spectrum and Autocorrelation of a Gaussian Pulse

We model a short laser pulse with the Gaussian pulse

$$E(t) = E_0 e^{-\pi\left(\frac{t}{t_c}\right)^2} e^{-i\omega_0 t}.$$

Find this electric field's autocorrelation function and power spectral density.

Autocorrelation function

From Equation 9.7, the electric field $E(t)$'s autocorrelation function is

$$G^{(1)}(\tau) = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} E(t)E^*(t+\tau)\,\mathrm{d}t.$$

We will work in terms of the normalized autocorrelation function

$$\begin{aligned} g(\tau) &\equiv \frac{G^{(1)}(\tau)}{G^{(1)}(0)} = \lim_{T\to\infty} \frac{\int_{-T/2}^{T/2} E(t)E^*(t+\tau)\,\mathrm{d}t}{\int_{-T/2}^{T/2} |E(t)|^2\,\mathrm{d}t} \\ &\equiv \frac{1}{C} \lim_{T\to\infty} \int_{-T/2}^{T/2} E(t)E^*(t+\tau)\,\mathrm{d}t, \end{aligned}$$

where for shorthand we have abbreviated the denominator as

$$C = \lim_{T\to\infty} \int_{-T/2}^{T/2} |E(t)|^2\,\mathrm{d}t = |E_0|^2 \lim_{T\to\infty} \int_{-T/2}^{T/2} e^{-2\pi(t/t_c)^2}\,\mathrm{d}t.$$

Using the general integral identity

$$\int_{-\infty}^{\infty} e^{-ax^2}\,\mathrm{d}x = \sqrt{\frac{\pi}{a}}, \tag{10.1}$$

with $a = (2\pi)/(t_c^2)$, the normalization constant comes out to

$$C = |E_0|^2 \sqrt{\pi \cdot \frac{t_c^2}{2\pi}} = |E_0|^2 \frac{t_c}{\sqrt{2}}.$$

The Gaussian laser pulse's normalized autocorrelation function is then

$$\begin{aligned} g(\tau) &= \frac{1}{C} \lim_{T\to\infty} \int_{-T/2}^{T/2} E_0 e^{-\pi\left(\frac{t}{t_c}\right)^2} e^{-i\omega_0 t} \cdot E_0^* e^{i\omega_0 (t+\tau)} e^{-\pi\left(\frac{t+\tau}{t_c}\right)^2}\,\mathrm{d}t \\ &= \frac{|E_0|^2}{C} \lim_{T\to\infty} \int_{-T/2}^{T/2} e^{-\pi\left(\frac{t}{t_c}\right)^2} e^{-\pi\left(\frac{t+\tau}{t_c}\right)^2} e^{i\omega_0\tau}\,\mathrm{d}t \\ &= \frac{\sqrt{2}}{t_c} \lim_{T\to\infty} \int_{-T/2}^{T/2} e^{-\frac{2\pi}{t_c^2}\left(t^2 + t\tau + \frac{\tau}{2}\right)} e^{i\omega_0\tau}\,\mathrm{d}t, \end{aligned}$$

where the last line uses $C = \left|E_0^2\right| \cdot t_c/\sqrt{2}$. We evaluate the resulting integral by completing the square in the exponent:

$$\begin{aligned} g(\tau) &= \frac{\sqrt{2}}{t_c} \lim_{T\to\infty} \int_{-T/2}^{T/2} e^{-\frac{2\pi}{t_c^2}\left(t^2+t\tau+\frac{\tau}{2}\right)} e^{i\omega_0\tau}\,\mathrm{d}t \\ &= \frac{\sqrt{2}}{t_c} \lim_{T\to\infty} \int_{-T/2}^{T/2} e^{-\frac{2\pi}{t_c^2}\left(t^2+t\tau+\frac{\tau}{4}\right)} e^{-\frac{2\pi\tau^2}{4t_c^2}} e^{i\omega_0\tau}\,\mathrm{d}t \\ &= \frac{\sqrt{2}}{t_c} e^{-\frac{\pi\tau^2}{2t_c^2}} e^{i\omega_0\tau} \lim_{T\to\infty} \int_{-T/2}^{T/2} e^{-\frac{2\pi}{t_c^2}\left(t+\frac{\tau}{2}\right)^2}\,\mathrm{d}t. \end{aligned}$$

We then introduce the new variable $x = t + \tau/2$ and evaluate the intergral to get

$$\begin{aligned} g(\tau) &= \frac{\sqrt{2}}{t_c} e^{-\frac{\pi\tau^2}{2t_c^2}} e^{i\omega_0\tau} \lim_{T\to\infty} \int_{-T/2}^{T/2} e^{-\frac{2\pi}{t_c^2}\left(t+\frac{\tau}{2}\right)^2}\,\mathrm{d}t \\ &= \frac{\sqrt{2}}{t_c} e^{-\frac{\pi\tau^2}{2t_c^2}} e^{i\omega_0\tau} \int_{-\infty}^{\infty} e^{-\frac{2\pi}{t_c^2}x^2}\,\mathrm{d}x \\ &= \frac{\sqrt{2}}{t_c} e^{-\frac{\pi\tau^2}{2t_c^2}} e^{i\omega_0\tau} \cdot \frac{t_c}{\sqrt{2}} \\ &= e^{-\frac{\pi\tau^2}{2t_c^2}} e^{i\omega_0\tau}. \end{aligned}$$

where the second-to-last equality uses Equation 10.1 with $a = 2\pi/t_c^2$. Note the result: the autocorrelation function of a Gaussian pulse is again a Gaussian pulse.

Note that, using Equation 10.1 with $a = \pi/t_c^2$, we have

$$\int_{-\infty}^{\infty} |g(\tau)|^2\,\mathrm{d}\tau = \int_{-\infty}^{\infty} e^{-\frac{\pi\tau^2}{t_c^2}} = \sqrt{\pi \cdot \frac{t_c^2}{\pi}} = t_c^2.$$

In fact, the expression $\int_{-\infty}^{\infty} |g(\tau)|^2\,\mathrm{d}\tau$ is often used as the *definition* of a signal's coherence time.

Power spectral density

By the Wiener–Khinchin theorem, a (sufficiently well-behaved) signal's autocorrelation function $G^{(1)}(\tau)$ and power spectral density $S(\omega)$ are Fourier transform pairs, i.e.

$$S(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} G^{(1)}(\tau) e^{-i\omega\tau}\,\mathrm{d}\tau.$$

The corresponding normalized expression is

$$s(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} g(\tau) e^{-i\omega\tau}\,\mathrm{d}\tau.$$

Using the just-derived expression for the Gaussian pulse's $g(\tau)$, the expression for $s(\omega)$ comes out to

$$s(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\frac{\pi\tau^2}{2t_c^2}} e^{i\omega_0\tau} e^{-i\omega\tau}\,\mathrm{d}\tau.$$

We again evaluate the integral by completing the square in the exponent. The process is straightforward but the algebra is rather messy:

$$\begin{aligned} -\frac{\pi}{2t_c^2}\tau^2 + i(\omega_0 - \omega)\tau &= -\frac{\pi}{2t_c^2}\left[\tau^2 + 2i(\omega-\omega_0)\frac{t_c^2}{\pi}\tau\right] \\ &= -\frac{\pi}{2t_c^2}\left(\tau + \frac{i(\omega-\omega_0)t_c^2}{\pi}\right)^2 - \frac{\pi}{2t_c^2} \cdot \left(-\frac{i(\omega-\omega_0)t_c^2}{\pi}\right)^2. \end{aligned}$$

The power spectral density then comes out to

$$\begin{aligned} s(\omega) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\frac{\pi}{2t_c^2}\left(\tau+i(\omega-\omega_0)\frac{t_c^2}{\pi}\right)^2} e^{-\frac{\pi}{2t_c^2}(\omega_0-\omega)^2\frac{(t_c^2)^2}{\pi^2}}\,\mathrm{d}\tau \\ &= \frac{1}{2\pi}e^{-(\omega_0-\omega)^2\frac{t_c^2}{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{\pi}{2t_c^2}\left(\tau+i(\omega-\omega_0)\frac{t_c^2}{\pi}\right)^2}\,\mathrm{d}\tau. \end{aligned}$$

Again using Equation 10.1, the integral over τ comes out to

$$s(\omega) = \frac{t_c}{\sqrt{2}\pi}e^{-\frac{t_c^2}{2\pi}(\omega_0-\omega)^2} = \frac{t_c}{\sqrt{2}\pi}e^{-\frac{t_c^2}{2\pi}(\omega-\omega_0)^2}$$

Note that the integral of the normalized power spectral density over all frequencies comes out to unity:

$$\int_{-\infty}^{\infty} s(\omega)\,\mathrm{d}\omega = \frac{t_c}{\sqrt{2}\pi}\int_{-\infty}^{\infty} e^{-\frac{t_c^2}{2\pi}(\omega-\omega_0)^2}\,\mathrm{d}\omega = \frac{t_c}{\sqrt{2}\pi}\cdot\sqrt{\pi\cdot\frac{2\pi}{t_c^2}} = 1.$$

10.2 Spectrum and Autocorrelation of an Exponential Pulse

Find the autocorrelation function and power spectral density of the exponential pulse

$$E = \begin{cases} E_0 e^{-t/t_c} e^{-i\omega_0 t} & t \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

The pulse's autocorrelation function is

$$G^{(1)}(\tau) = |E_0|^2 \lim_{T\to\infty}\frac{1}{T}\int_0^{T/2} e^{-t/t_c}e^{-\frac{(t+\tau)}{t_c}}e^{i\omega_0\tau}\,\mathrm{d}t.$$

The corresponding normalized autocorrelation function is

$$g(\tau) = \frac{G(\tau)}{G(0)}.$$

The normalization factor $G(0)$ comes out to

$$T\cdot G(0) = |E_0|^2\lim_{T\to\infty}\int_0^T e^{-(t/t_c)^2}\,\mathrm{d}t = \frac{1}{2}\cdot\sqrt{\pi}\cdot t_c|E_0|^2,$$

and so (after cancelling T),

$$\begin{aligned} g(\tau) &= \frac{2}{\sqrt{\pi}\cdot t_c}\lim_{T\to\infty}\int_0^{-T/2} e^{-t/t_c}e^{-\frac{(t+\tau)}{t_c}}e^{i\omega_0\tau}\,\mathrm{d}t \\ &= \frac{2}{\sqrt{\pi}\cdot t_c}e^{-\tau/t_c}e^{i\omega_0\tau}\int_0^{\infty} e^{-2t/t_c}\,\mathrm{d}t \\ &= \frac{2}{\sqrt{\pi}\cdot t_c}e^{-\tau/t_c}e^{i\omega_0\tau}\int_0^{\infty} e^{-2t/t_c}\,\mathrm{d}t \\ &= \frac{2}{\sqrt{\pi}\cdot t_c}e^{-\tau/t_c}e^{i\omega_0\tau}\cdot\left[-\frac{t_c}{2}e^{-2t/t_c}\right]_0^{\infty} \\ &= \frac{1}{\sqrt{\pi}}e^{-\tau/t_c}e^{i\omega_0\tau}. \end{aligned}$$

Note: this differs by a factor of $\sqrt{\pi}$ from the "official" result from exercises, which as quoted as

$$g(\tau) = e^{-\tau/t_c} e^{i\omega_0 \tau}.$$

Quoting from exercises, the exponential pulse's normalized power spectral density should come out to

$$s(\omega) = \frac{1}{2\pi} \frac{\gamma^2}{(\omega - \omega_0)^2 + \gamma^2},$$

where $\gamma = 1/\tau_c$, but at the time of writing I did not derive this result myself.

11 Eleventh Exercise Set

11.1 Theory: Lorentz Oscillator Model and the Refractive Index

Review from lecture: using the Lorentz oscillator model of an electron connected to a positive atomic nucleus via a spring described by resonant frequency ω_0 and damping constant γ, the induced electric polarization $\mathbf{P}_\mathrm{e}$ of a material with electric dipole number density ρ_n in the presence of an external electric field of amplitude $\mathbf{E}_0$ and frequency ω is

$$\mathbf{P}_\mathrm{e} = \frac{e_0^2 \rho_n}{m_\mathrm{e}} \frac{\mathbf{E}_0 e^{-i\omega t}}{(\omega_0^2 - \omega^2) - i\gamma\omega}$$

Assuming the material is non-magnetic and the displacement and electric field are related by $\mathbf{E} = \varepsilon\varepsilon_0 \mathbf{D} \implies \mathbf{P}_\mathrm{e} = \varepsilon_0(\varepsilon - 1)\mathbf{E}$, the material's (in general complex) index of refraction $\mathcal{N}$ and dielectric constant ε are related by

$$\varepsilon = \mathcal{N}^2 = 1 + \frac{\omega_\mathrm{p}^2}{(\omega_0^2 - \omega^2) - i\gamma\omega}, \quad \text{where } \omega_\mathrm{p}^2 \equiv \frac{\rho_n e_0^2}{m\varepsilon_0},$$

where we have written m as the electron mass for conciseness.

The above formula applies to materials in which the electron-nucleus interaction is described by a single resonance frequency ω_0. In materials with multiple resonances, each with their own frequency ω_{0_j} and damping coefficient γ_j, the dielectric constant generalizes to

$$\varepsilon = \mathcal{N}^2 = 1 + \sum_j \frac{f_j \omega_{\mathrm{p}_j}^2}{\omega_{0_j}^2 - \omega^2 - i\gamma_j\omega},$$

where the f_j are dimensionless quantities encoding the strength of each resonance.

When analyzing a material with multiple resonances, but one for which we are interested in only a few resonances and neglect the others, we use the Sellmeier formula

$$\mathcal{N}^2(\lambda) = A + \sum_k \frac{G_k \lambda^2}{\lambda^2 - \lambda_{0_k}^2}, \tag{11.1}$$

where the G_k are the (in practice empirically-determined) Sellmeier constants associated with each resonance. Note that the Sellmeier formula is conventionally written in terms of wavelength instead of frequency ω, while the non-unit constant A accounts for corrections associated with the neglected resonances.

The Cauchy formula is an approximation of the Sellmeier formula for λ far from any resonances, and models a material's refractive index as

$$\mathcal{N} = C_1 + \frac{C_2}{\lambda^2} + \frac{C_3}{\lambda^4} + \cdots \tag{11.2}$$

11.2 Constants in the Cauchy Formula

(i) Derive expressions for the first two coefficients C_1 and C_2 in the Cauchy formula for the index of refraction in terms of the two Sellmeier formula coefficients A and G for a material with a single resonance λ_0 when exposed to an external electric field with wavelength $\lambda \gg \lambda_0$.

(ii) Compute the values of C_1 and C_2 for toluene, which has $A = 1$, $G = 1.17577$ and $\lambda_0 = 0.135\,\mu\mathrm{m}$; then compute the refractive index in toluene for light of wavelength $\lambda = 500\,\mathrm{nm}$ and $\lambda = 800\,\mathrm{nm}$.

Using the Cauchy formula (Equation 11.2) up to the second coefficient C_2 and the Sellmeier formula (Equation 11.1) for a material with a single resonance, we have

$$\mathcal{N} = C_1 + \frac{C_2}{\lambda^2} \qquad \text{and} \qquad \mathcal{N}^2 = A + \frac{G\lambda^2}{\lambda^2 - \lambda_0^2} = A + \frac{G}{1 - (\lambda_0^2/\lambda^2)}.$$

We the square the Cauchy formula, producing

$$\mathcal{N}^2 = C_1^2 + 2\frac{C_1 C_2}{\lambda^2} + \frac{C_2^2}{\lambda^4};$$

we then equate this expression to the Sellmeier formula to get

$$C_1^2 + 2\frac{C_1 C_2}{\lambda^2} + \frac{C_2^2}{\lambda^4} = A + \frac{G}{1 - (\lambda_0^2/\lambda^2)}.$$

Finally, we expand the Sellmeier formula on the RHS in the limit $\lambda \gg \lambda_0$ to get

$$C_1^2 + 2\frac{C_1 C_2}{\lambda^2} + \frac{C_2^2}{\lambda^4} \approx A + G\left(1 + \frac{\lambda_0^2}{\lambda^2}\right) = (A + G) + G\frac{\lambda_0^2}{\lambda^2}.$$

Comparing the coefficients for each power of λ, we find

$$C_1^2 = A + G \implies C_1 = \sqrt{A + G}$$

and

$$2C_1 C_2 = G\lambda_0^2 \implies C_2 = \frac{G\lambda_0^2}{2C_1} = \frac{G\lambda_0^2}{2\sqrt{A + G}}.$$

The index of refraction in this approximation is then

$$\mathcal{N} = C_1 + \frac{C_2}{\lambda^2} = \sqrt{A + G} + \frac{G\lambda_0^2}{2\sqrt{A + G}}\frac{1}{\lambda^2}$$

Concrete Calculation for Toluene

For toluene, for which we are given the values $A = 1$, $G = 1.17477$, and $0.135\,\mu\text{m}$ the Cauchy coefficients C_1 and C_2 read

$$C_1 = \sqrt{1 + 1.17477} \approx 1.475, C_2 = \frac{G\lambda_0^2}{2C_1} = \frac{1.17477 \cdot (0.135\,\mu\text{m})^2}{2 \cdot 1.475} \approx 7.25\,\text{nm}^2.$$

The corresponding refractive indexes at external electric field wavelengths $\lambda = 500\,\text{nm}$ and $\lambda = 800\,\text{nm}$ come out to $n \approx 1.503$ and $n \approx 1.486$, respectively.

Note, however, that the approximation $\lambda \gg \lambda_0$ does not even remotely apply for $\lambda_0 = 0.135\,\mu\text{m}$, so the above calculations are probably non-physical.

11.3 Optically Active Materials and Circular Dichroism

Derive (i) the Jones matrix for passage of light through an optically active material and (ii) the matrix describing the optical phenomenon of circular dichroism. Finally (iii) write the matrix describing passage of light through a material that is both optically active and exhibits circular dichroism.

11.3.1 Optically Active Material

In optically active materials, right-circular-polarized and left-circular-polarized electromagnetic waves experience different indices of refraction, i.e. $n_{\text{lcp}} \neq n_{\text{rcp}}$.

We consider linearly-polarized incident light, which then decompose into LCP and RCP components via

$$\mathbf{J}_{\text{in}} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 \\ i \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 1 \\ -i \end{pmatrix}$$

In an optically active material, the two polarizations experience different refractive indices, and after passing through and optically-active material of width d the output Jones vector reads

$$\mathbf{J}_{\text{out}} = \frac{1}{2}\begin{pmatrix} 1 \\ i \end{pmatrix} e^{i n_{\text{lcp}} k_0 d} + \frac{1}{2}\begin{pmatrix} 1 \\ -i \end{pmatrix} e^{i n_{\text{rcp}} k_0 d}.$$

We proceed by defining an average refractive index $\bar{n}$ and difference Δn given by

$$\bar{n} \equiv \frac{n_{\text{lcp}} + n_{\text{rcp}}}{2} \qquad \text{and} \qquad \Delta n \equiv n_{\text{rcp}} - n_{\text{lcp}},$$

in terms of which n_{lcp} and n_{rcp} read

$$n_{\text{rcp}} = \bar{n} + \frac{\Delta n}{2} \qquad \text{and} \qquad n_{\text{lcp}} = \bar{n} - \frac{\Delta n}{2}.$$

In terms of $\bar{n}$ and Δn, the output Jones vector reads

$$\mathbf{J}_{\text{out}} = \frac{1}{2}\begin{pmatrix} e^{ik_0\bar{n}d}e^{-ik_0\frac{\Delta n}{2}d} + e^{ik_0\bar{n}d}e^{ik_0\frac{\Delta n}{2}d} \\ ie^{ik_0\bar{n}d}e^{-ik_0\frac{\Delta n}{2}d} - ie^{ik_0\bar{n}d}e^{ik_0\frac{\Delta n}{2}d}. \end{pmatrix} = \frac{e^{ik_0\bar{n}d}}{2}\begin{pmatrix} e^{-ik_0\frac{\Delta n}{2}d} + e^{ik_0\frac{\Delta n}{2}d} \\ ie^{-ik_0\frac{\Delta n}{2}d} - ie^{ik_0\frac{\Delta n}{2}d} \end{pmatrix}$$

We then convert to sines and cosines, in terms of which $\mathbf{J}_{\text{out}}$ reads

$$\mathbf{J}_{\text{out}} = e^{ik_0\bar{n}d}\begin{pmatrix} \cos k_0\frac{\Delta n}{2}d \\ \sin k_0\frac{\Delta n}{2}d \end{pmatrix}.$$

The corresponding Jones matrix encoding the optically-active material's effect on linearly-polarized incident light is just the rotation matrix

$$\mathbf{R}_{\text{OA}} = e^{ik_0\bar{n}d}\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \quad \text{where } \theta \equiv k_0\frac{\Delta n}{2}d, \tag{11.3}$$

which we can verify obeys $\mathbf{J}_{\text{out}} = \mathbf{R}_{\text{OA}}\mathbf{J}_{\text{in}}$.

11.3.2 Material With Circular Dichroism

In circularly dichroic materials, right-circular-polarized and left-circular-polarized electromagnetic waves are absorbed by different amounts. We can formulate circular dichroism as RCP and LCP wave experiencing different complex refractive indices, which motivates introducing the quantities

$$\kappa_{\text{lcp}} \equiv n_{\text{Im}}^{(\text{lcp})}k \qquad \text{and} \qquad \kappa_{\text{rcp}} \equiv n_{\text{Im}}^{(\text{rcp})}k.$$

As above for optically active materials consider linearly-polarized incident light, which then decompose into LCP and RCP components via

$$\mathbf{J}_{\text{in}} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 \\ i \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 1 \\ -i \end{pmatrix}.$$

In a circularly dichroic material, the two polarizations experience different complex refractive indices, and after passing through and optically-active material of width d the output Jones vector reads

$$\mathbf{J}_{\text{out}} = \frac{1}{2}\begin{pmatrix}1\\ i\end{pmatrix} e^{-\kappa_{\text{lcp}} d} + \frac{1}{2}\begin{pmatrix}1\\ -i\end{pmatrix} e^{-\kappa_{\text{rcp}} d}.$$

Following an analogous procedure as for optically active materials, we then define

$$\bar{\kappa} = \frac{\kappa_{\text{lcp}} + \kappa_{\text{rcp}}}{2} \qquad \text{and} \qquad \Delta\kappa \equiv \kappa_{\text{rcp}} - \kappa_{\text{lcp}},$$

in terms of which κ_{lcp} and κ_{rcp} read

$$\kappa_{\text{rcp}} = \bar{\kappa} + \frac{\Delta\kappa}{2} \qquad \text{and} \qquad \kappa_{\text{lcp}} = \bar{\kappa} - \frac{\Delta\kappa}{2}.$$

In terms of $\Delta\kappa$ and $\bar{\kappa}$, J_{out} reads

$$\mathbf{J}_{\text{out}} = \frac{e^{-\bar{\kappa}\cdot d}}{2}\begin{pmatrix} e^{+\frac{\Delta\kappa d}{2}} + e^{\frac{\Delta\kappa d}{2}} \\ ie^{+\frac{\Delta\kappa d}{2}} - ie^{-\frac{\Delta\kappa d}{2}}\end{pmatrix} = e^{-\bar{\kappa} d}\begin{pmatrix}\cosh\frac{\Delta\kappa d}{2}\\ i\sinh\frac{\Delta k d}{2}\end{pmatrix}.$$

Finally, using $\sinh x = -i\sin(ix)$ and $\cosh x = \cos(ix)$ we write $\mathbf{J}_{\text{out}}$ in terms of sine and cosine as

$$\mathbf{J}_{\text{out}} = e^{-\bar{\kappa}\cdot d}\begin{pmatrix}\cos\frac{i\Delta\kappa d}{2}\\ \sin\frac{i\Delta\kappa d}{2}\end{pmatrix}.$$

The corresponding Jones matrix encoding the circularly-dichroic material's effect on linearly-polarized incident light is the rotation matrix

$$\mathbf{R}_{\text{CD}} = e^{-\bar{\kappa} d}\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}, \quad \text{where } \theta \equiv i\frac{\Delta\kappa d}{2}.$$

Interpretation: comparing to Equation 11.3, we see that in the Jones formalism a circularly dichroic material has a similar effect to an optically active material, except that a CD material rotates an incident Jones vector by a *complex* angle.

11.3.3 Optically-Active Material with Circular Dichroism

In this case LCP and RCP light experiences both different real and imaginary components of the refractive index.

$$n_{\text{Re}}^{(\text{lcp})} \neq n_{\text{Re}}^{(\text{rcp})} \qquad \text{and} \qquad n_{\text{Im}}^{(\text{lcp})} \neq n_{\text{Im}}^{(\text{rcp})}$$

Solution: we model a material with both phenomena as a circularly dichroic material placed in from of an optically active material. This allows us to multiply the transformation matrices for the optically active and circularly dichroic materials.

However, instead of actually multiplying out, we recognize that the product of two rotation matrices is the same as a single rotation matrix by the sum of the two angles. The result is then

$$\mathbf{R}_{\text{both}} = \mathbf{R}_{\text{OA}}\mathbf{R}_{\text{CD}} = e^{ik_0\bar{n}d}e^{-\bar{\kappa}d}\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}, \quad \text{where } \theta = \frac{\Delta n_{\text{Re}} k_0 d}{2} + i\frac{\Delta\kappa d}{2}.$$

11.4 Theory: The Magneto-Optic (Faraday) Effect

Derive the refractive indices felt by both RCP and LCP electromagnetic waves during the magneto-optic (Faraday) effect.

Review from lecture, in an external electromagnetic field, electron described by the Lorentz oscillator model with a spring constant k obeys the equation of motion

$$m\ddot{\mathbf{r}} = -k\mathbf{r} - e_0(\mathbf{E} + \dot{\mathbf{r}} \times \mathbf{B}).$$

We will consider a homogeneous magnetic field $\mathbf{B} = (0, 0, B_0)$, in which case, after evaluating the cross product, the electron's equation of motion becomes

$$\ddot{\mathbf{r}} = -\omega_0^2 \mathbf{r} - \frac{e_0}{m}\left(\mathbf{E} + \dot{y}B_0\,\hat{\mathbf{e}}_x - \dot{x}B_0\,\hat{\mathbf{e}}_y\right),$$

where we have introduced the resonance frequency $\omega_0^2 = k/m$. We separately consider x, y and z components and get

$$\begin{aligned}\ddot{x} + \omega_0^2 x &= -\frac{e_0}{m}\left(E_x + \dot{y}B_0\right),\\ \ddot{y} + \omega_0^2 y &= -\frac{e_0}{m}\left(E_y - \dot{x}B_0\right).\end{aligned}$$

We solve the above equations with a plane wave ansatz $\propto e^{-i\omega t}$ for x, y, E with the same frequency, which results in the algebraic equations

$$(\omega_0^2 - \omega^2)x_0 = -\frac{e_0}{m}\left(E_x^{(0)} - i\omega y_0 B_0\right), \tag{11.4}$$

$$(\omega_0^2 - \omega^2)y_0 = -\frac{e_0}{m}\left(E_y^{(0)} + i\omega x_0 B_0\right). \tag{11.5}$$

We then introduce the complex variables $z_0^{\pm} = x_0 \pm iy_0$ and write the LCP and RCP electric field polarizations as

$$E_{\text{lcp}} \equiv E_x^{(0)} + iE_y^{(0)} \qquad \text{and} \qquad E_{\text{rcp}} \equiv E_x^{(0)} - iE_y^{(0)},$$

which we then combine with Equations 11.4 and 11.5 to get

$$\begin{aligned}(\omega_0^2 - \omega^2)z_0^{+} &= -\frac{e_0}{m}\left(E_{\text{lcp}} - \omega B_0 z_0^{+}\right)\\ (\omega_0^2 - \omega^2)z_0^{-} &= -\frac{e_0}{m}\left(E_{\text{rcp}} + \omega B_0 z_0^{-}\right).\end{aligned}$$

From here we can solve for amplitudes of electron oscillation; these are

$$\begin{aligned}z_0^{+} &= -\frac{e_0}{m}\frac{E_{\text{lcp}}}{\omega_0^2 - \omega^2 - \omega\Omega},\\ z_0^{-} &= -\frac{e_0}{m}\frac{E_{\text{rcp}}}{\omega_0^2 - \omega^2 + \omega\Omega},\end{aligned}$$

where we have defined the cyclotron frequency $\Omega = \frac{e_0 B_0}{m}$. Using the just-derived amplitudes of electron oscillation, the material's polarization is

$$\begin{aligned}P_{\text{e}}^{(\text{lcp})} &= -e_0\rho_n z_0^{+} = -\frac{e_0}{m}\frac{E_{\text{lcp}}}{\omega_0^2 - \omega^2 - \omega\Omega},\\ P_{\text{e}}^{(\text{rcp})} &= -e_0\rho_n z_0^{-} = -\frac{e_0}{m}\frac{E_{\text{rcp}}}{\omega_0^2 - \omega^2 + \omega\Omega}.\end{aligned}$$

Finally, comparing to the general relation $P_{\mathrm{e}} = \varepsilon_0(\varepsilon - 1)E$, we find the dielectric indices felt by RCP and LCP electromagnetic waves in the magneto-optic effect are

$$n_{\mathrm{lcp}}^2 \approx \varepsilon_{\mathrm{lcp}} = 1 + \frac{e_0^2}{\varepsilon_0 m}\frac{\rho_n}{\omega_0^2 - \omega^2 - \omega\Omega}, \tag{11.6}$$

$$n_{\mathrm{rcp}}^2 \approx \varepsilon_{\mathrm{rcp}} = 1 + \frac{e_0^2}{\varepsilon_0 m}\frac{\rho_n}{\omega_0^2 - \omega^2 + \omega\Omega}. \tag{11.7}$$

11.5 Rotation of Polarization in the Magneto-Optic Effect

Derive an expression for the rotation of polarization induced by the magneto-optic effect.

Recall from Equation 11.3 in Exercise 11.3 that when linearly polarized light is incident on a optically active material of thickness d, the transmitted LCP and RCP polarizations are rotated relative to each other by an angle

$$\theta = \frac{k_0 \Delta n d}{2}, \quad \text{where } \Delta n = n_{\mathrm{rcp}} - n_{\mathrm{lcp}}. \tag{11.8}$$

Interpretation is that a Faraday effect element of thickness d acts as an optical element that rotates an incident Jones vector by the above angle θ.

Next, we solve the above expressions for the magneto-optic effect refractive indices n_{lcp}^2 and n_{rcp}^2 in Equations 11.6 and 11.7 of the previous exercise in the regime $\Omega \ll \omega$ (corresponding to small magnetic field) to get the expressions

$$\begin{aligned} n_{\mathrm{rcp}} &\approx 1 + \frac{1}{2}\frac{\omega_{\mathrm{p}}^2}{\omega_0^2 - \omega^2 + \omega\Omega} \\ n_{\mathrm{lcp}} &\approx 1 + \frac{1}{2}\frac{\omega_{\mathrm{p}}^2}{\omega_0^2 - \omega^2 - \omega\Omega}. \end{aligned}$$

We then rewrite n_{lcp} and n_{rcp} as

$$\begin{aligned} n_{\mathrm{rcp}} &= 1 + \frac{1}{2}\frac{\omega_{\mathrm{p}}^2}{\omega_0^2 - \omega^2} \cdot \frac{1}{1 + \frac{\omega\Omega}{\omega_0^2 - \omega^2}} \\ n_{\mathrm{lcp}} &= 1 + \frac{1}{2}\frac{\omega_{\mathrm{p}}^2}{\omega_0^2 - \omega^2} \cdot \frac{1}{1 - \frac{\omega\Omega}{\omega_0^2 - \omega^2}}. \end{aligned}$$

We then note that for $\Omega \ll \omega$ we can approximate the second terms with

$$\frac{1}{1 \pm \frac{\omega\Omega}{\omega_0^2 - \omega^2}} \approx \left(1 \mp \frac{\omega\Omega}{\omega_0^2 - \omega^2}\right),$$

in terms of which n_{rcp} and n_{lcp} become

$$\begin{aligned} n_{\mathrm{rcp}} &= 1 + \frac{1}{2}\frac{\omega_{\mathrm{p}}^2}{\omega_0^2 - \omega^2} \cdot \left(1 - \frac{\omega\Omega}{\omega_0^2 - \omega^2}\right) \\ n_{\mathrm{lcp}} &= 1 + \frac{1}{2}\frac{\omega_{\mathrm{p}}^2}{\omega_0^2 - \omega^2} \cdot \left(1 + \frac{\omega\Omega}{\omega_0^2 - \omega^2}\right). \end{aligned}$$

The difference of refractive indices needed in Equation 11.8 is then

$$\Delta n = n_{\mathrm{rcp}} - n_{\mathrm{lcp}} = -\frac{1}{2}\frac{\omega_{\mathrm{p}}^2}{\omega_0^2 - \omega^2} \cdot \frac{2\omega\Omega}{\omega_0^2 - \omega^2} = -\frac{\omega_{\mathrm{p}}^2 \Omega\omega}{(\omega_0^2 - \omega^2)^2}.$$

From here we can read off from Equation 11.8 that the transmitted LCP and RCP polarizations of an electromagnetic wave passing through a material of thickness d in which the wave experiences the magneto-optic effect are rotated relative to each other by an angle

$$\theta = \frac{k_0 \Delta n d}{2} = -\frac{1}{2} k_0 d \frac{\omega_\mathrm{p}^2 \omega \Omega}{(\omega_0^2 - \omega^2)^2},$$

where ω_p^2 and Ω are given by

$$\Omega = \frac{e_0 B_0}{m_\mathrm{e}} \qquad \text{and} \qquad \omega_\mathrm{p}^2 = \frac{\rho_n e_0^2}{m \varepsilon_0}.$$

For the sake of completeness, substituting in Ω, ω_p^2 and $k_0 = \omega/c$ produces

$$\theta = \frac{\rho_n e_0^3 \omega^2}{2 m_\mathrm{e}^2 \epsilon_0 c (\omega_0^2 - \omega^2)^2} B_0 d.$$

12 Twelfth Exercise Set

12.1 Index Ellipsoid and Wave Vector Surface

Compute and sketch the index ellipsoid and wave vector surface for propagation of electromagnetic waves through a biaxial crystal with refractive index eigenvalues n_{xx}, n_{yy} and n_{zz} in the xz, xy and yz planes.

12.1.1 Index Ellipsoid

We begin by quoting a result form lecture: in a non-magnetic ($\mu = 1$) optically anisotropic material, the wave equation for the electric field $\mathbf{E}$ generalizes to

$$\left(k^2 \mathsf{I} - k_0^2 \epsilon\right) \mathbf{E} = (\mathbf{k} \cdot \mathbf{E})\mathbf{k}, \tag{12.1}$$

where ϵ is a second-rank tensor called the *dielectric tensor* (a generalization of the dielectric constant ε in isotropic materials) and I is the identity tensor. Importantly, in anisotropic materials an electromagnetic wave's wave vector $\mathbf{k}$ is in general not parallel to the electric field $\mathbf{E}$ as in isotropic materials, so that $\mathbf{k} \cdot \mathbf{E} \neq 0$. Note that setting $\mathbf{k} \perp \mathbf{E} \implies \mathbf{k} \cdot \mathbf{E} = 0$ and $\epsilon \to \varepsilon = n^2$ as in an isotropic material recovers the familiar relation $k = nk_0$.

In the dielectric tensor's system of principal axes, ϵ takes the simple diagonal form

$$\epsilon = \begin{pmatrix} \varepsilon_{xx} & 0 & 0 \\ 0 & \varepsilon_{yy} & 0 \\ 0 & 0 & \varepsilon_{zz} \end{pmatrix},$$

where ε_{xx}, ε_{yy} and ε_{zz} are the tensor's eigenvalues. By convention, the eigenvalues are arranged (or the principle axes labeled) such that $\varepsilon_{xx} < \varepsilon_{yy} < \varepsilon_{zz}$. We will work in the dielectric tensor's system of principal axes and thus assume ϵ is diagonal for the remainder of this course.

Without derivation, we quote from lecture that in the dielectric tensor's system of principal axes, Equation 12.1 is equivalent to the matrix equation

$$\begin{pmatrix} k_y^2 + k_z^2 - k_0^2 \varepsilon_{xx} & -k_x k_y & -k_x k_z \\ -k_x k_y & k_x^2 + k_z^2 - k_0^2 \varepsilon_{yy} & -k_y k_z \\ -k_z k_x & -k_y k_z & k_x^2 + k_y^2 - k_0^2 \varepsilon_{zz} \end{pmatrix} \mathbf{E} \equiv \mathbf{M}\mathbf{E} = \mathbf{0}.$$

As for any homogeneous system of equations, for this equation to have a non-trivial solution, i.e. such that $\mathbf{E} \neq \mathbf{0}$, we require

$$\det \mathbf{M} = \begin{vmatrix} k_y^2 + k_z^2 - k_0^2 \varepsilon_{xx} & -k_x k_y & -k_x k_z \\ -k_x k_y & k_x^2 + k_z^2 - k_0^2 \varepsilon_{yy} & -k_y k_z \\ -k_z k_x & -k_y k_z & k_x^2 + k_y^2 - k_0^2 \varepsilon_{zz} \end{vmatrix} = 0. \tag{12.2}$$

The resulting 3×3 determinant is rather tedious to work with, so we often work in either the xy, xz or yz planes, in which $\mathbf{M}$ and thus $\det \mathbf{M}$ simplify considerably.

12.1.2 Solution in the xy Plane

We begin by working in the xy plane, in which an arbitrary wave vector $\mathbf{k}$ reads $\mathbf{k} = (k_x, k_y, 0)$, in which case $\mathbf{M}$ simplifies to

$$\mathbf{M} = \begin{pmatrix} k_y^2 - k_0^2 \varepsilon_{xx} & -k_x k_y & 0 \\ -k_x k_y & k_x^2 - k_0^2 \varepsilon_{yy} & 0 \\ 0 & 0 & k_x^2 + k_y^2 - k_0^2 \varepsilon_{zz} \end{pmatrix}. \tag{12.3}$$

Expanding about the third row, the determinant $\det \mathbf{M}$ the comes out to

$$\begin{aligned}\det \mathbf{M} &= (k_x^2 + k_y^2 - k_0^2\varepsilon_{zz})\Big[(k_y^2 - k_0^2\varepsilon_{xx})(k_x^2 - k_0^2\varepsilon_{yy}) - k_x^2k_y^2\Big]\\ &= (k_x^2 + k_y^2 - k_0^2\varepsilon_{zz})\big(k_0^4\varepsilon_{xx}\varepsilon_{yy} - k_0^2k_x^2\varepsilon_{xx} - k_0^2k_y^2\varepsilon_{yy}\big).\end{aligned}$$

The solutions to the equation $\det \mathbf{M} = 0$ are then

$$k_x^2 + k_y^2 = k_0^2\varepsilon_{zz} \qquad \text{and} \qquad k_0^2k_x^2\varepsilon_{xx} + k_0^2k_y^2\varepsilon_{yy} = k_0^4\varepsilon_{xx}\varepsilon_{yy}.$$

After dividing through by k_0 and introducing the normalized vector components $\kappa_\alpha = k_\alpha/k_0$, the above equation reduces to

$$\kappa_x^2 + \kappa_y^2 = \varepsilon_{zz} \qquad \text{and} \qquad \frac{\kappa_x^2}{\varepsilon_{yy}} + \frac{\kappa_y^2}{\varepsilon_{xx}} = 1, \tag{12.4}$$

which we recognize as the equations of a circle and ellipse in $\mathbf{k}$ space (under the initial assumption $\mathbf{k} = (k_x, k_y, 0)$). The values of $\mathbf{k}$ satisfying these equations correspond to non-trivial solutions for the electric field $\mathbf{E}$ in the anisotropic material described by the dielectric tensor ϵ.

Plotting the two solutions in the $\kappa_x\kappa_y$ plane produces a circle of radius $\sqrt{\varepsilon_{zz}}$ and an ellipse with major and minor axes reaching $\pm\sqrt{\varepsilon_{yy}}$ on the κ_x axis and $\pm\sqrt{\varepsilon_{xx}}$ on the κ_y axis, respectively. Since the dielectric tensor's eigenvalues are arranged to obey $\varepsilon_{xx} < \varepsilon_{yy} < \varepsilon_{zz}$, the ellipse is fully contained within the circle.

12.1.3 Solution in the xz Plane

We now consider wave vectors the xz plane, in which an arbitrary wave vector $\mathbf{k}$ reads $\mathbf{k} = (k_x, 0, k_z)$ and $\mathbf{M}$ simplifies to

$$\mathbf{M} = \begin{pmatrix} k_z^2 - k_0^2\varepsilon_{xx} & 0 & -k_xk_z \\ 0 & k_x^2 + k_z^2 - k_0^2\varepsilon_{yy} & 0 \\ -k_zk_x & 0 & k_x^2 - k_0^2\varepsilon_{zz} \end{pmatrix}. \tag{12.5}$$

Instead of computing the determinant by hand (e.g. by expanding about the second row), we can just permute the previous section's results via $y \to z$ and $z \to y$ to get

$$\kappa_x^2 + \kappa_z^2 = \varepsilon_{yy} \qquad \text{and} \qquad \frac{\kappa_x^2}{\varepsilon_{zz}} + \frac{\kappa_z^2}{\varepsilon_{xx}} = 1. \tag{12.6}$$

Plotting the two solutions in the $\kappa_x\kappa_z$ plane produces a circle of radius $\sqrt{\varepsilon_{yy}}$ and an ellipse with major and minor axes reaching $\pm\sqrt{\varepsilon_{zz}}$ on the κ_x axis and $\pm\sqrt{\varepsilon_{xx}}$ on the κ_z axis, respectively. Because $\varepsilon_{xx} < \varepsilon_{yy} < \varepsilon_{zz}$ the ellipse and circle intersect at four points in the xz plane.

12.1.4 Solution in the yz Plane

Finally, we consider wave vectors the yz plane, in which an arbitrary wave vector $\mathbf{k}$ reads $\mathbf{k} = (0, k_y, k_z)$ and $\mathbf{M}$ simplifies to

$$\mathbf{M} = \begin{pmatrix} k_y^2 + k_z^2 - k_0^2\varepsilon_{xx} & 0 & 0 \\ 0 & k_z^2 - k_0^2\varepsilon_{yy} & -k_yk_z \\ 0 & -k_yk_z & k_y^2 - k_0^2\varepsilon_{zz} \end{pmatrix}. \tag{12.7}$$

To avoid computing the determinant by hand, we permute the last section's results in the xz plane by switching $x \to y$ and $y \to z$. The result is

$$\kappa_y^2 + \kappa_z^2 = \varepsilon_{xx} \qquad \text{and} \qquad \frac{\kappa_y^2}{\varepsilon_{zz}} + \frac{\kappa_z^2}{\varepsilon_{yy}} = 1. \tag{12.8}$$

Plotting the two solutions in the $\kappa_y\kappa_z$ plane produces a circle of radius $\sqrt{\varepsilon_{xx}}$ and an ellipse with major and minor axes reaching $\pm\sqrt{\varepsilon_{zz}}$ on the κ_y axis and $\pm\sqrt{\varepsilon_{yy}}$ on the κ_z axis, respectively. Because $\varepsilon_{xx} < \varepsilon_{yy} < \varepsilon_{zz}$ circle is fully contained within the ellipse.

Polarization

First consider xy plane solution. Each solution corresponds to a given refractive index and a different polarization.

One polarization is tangent to ellipse and one is perpendicularly "outward".

One polarization points along the κ_z axis.

This polarization is the solution for which, recall $\mathbf{M}$, we have

$$k_x^2 + k_y^2 - k_0^2\varepsilon_{zz} = 0.$$

The corresponding vector (in the bottom right ε_{zz} position) is $\hat{\mathbf{z}}$, and so $\mathbf{E} = (0, 0, E_0)$.

Similarly, the solution $\kappa_x^2 + \kappa_z^2 = \varepsilon_{yy}$ in the xz plane corresponds to a polarization $\mathbf{E} = (0, E_0, 0)$

Similarly, the solution $\kappa_y^2 + \kappa_z^2 = \varepsilon_{xx}$ in the yz plane corresponds to a polarization $\mathbf{E} = (E_0, 0, 0)$

That covers the polarizations perpendicular to the given plane in which we found $\mathbf{k}$ solutions, which corresponds to circular $\mathbf{k}$ solutions.

The second class of polarization must be perpendicular to the first.

This class of polarizations is tangent to the elliptical solution, and lies in the same plane in which the $\mathbf{k}$ solutions were found.

12.2 Electric Field Direction and Wave Vector Surface

Show that in anisotropic materials the electric field $\mathbf{E}$ *is tangent to the wave vector surface (the set of* $\mathbf{k}$ *obtained by solving Equation 12.2).*

12.2.1 Solution in the xy Plane

We will begin in the xy plane, in which case $\mathbf{k} = (k_x, k_y, 0)$. We begin by normalizing matrix $\mathbf{M}$ in Equation 12.3 by k_0^2 to get

$$\widetilde{\mathbf{M}} \equiv \frac{\mathbf{M}}{k_0^2} = \begin{pmatrix} \kappa_y^2 - \varepsilon_{xx} & -\kappa_x\kappa_y & 0 \\ -\kappa_x\kappa_y & \kappa_x^2 - \varepsilon_{yy} & 0 \\ 0 & 0 & \kappa_x^2 + \kappa_y^2 - \varepsilon_{zz} \end{pmatrix} \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} = \mathbf{0}.$$

The x component (first matrix row) of the above matrix equation produces

$$\left(\kappa_y^2 - \varepsilon_{xx}\right) E_x - \kappa_x\kappa_y E_y = 0 \implies \frac{E_y}{E_x} = \frac{\kappa_y^2 - \varepsilon_{xx}}{\kappa_x\kappa_y}.$$

We then recall the second, elliptical solution to Equation 12.2 for $\mathbf{k}$ in the xy plane; from Equation 12.4, this solution is

$$\frac{\kappa_y^2}{\varepsilon_{xx}} + \frac{\kappa_x^2}{\varepsilon_{yy}} = 1 \implies \kappa_y^2 = \varepsilon_{xx} - \kappa_x^2 \frac{\varepsilon_{xx}}{\varepsilon_{yy}}.$$

We then substitute this expression for κ_y into the above ratio E_y/E_x to get

$$\frac{E_y}{E_x} = \frac{\kappa_y^2 - \varepsilon_{xx}}{\kappa_x \kappa_y} = -\frac{\varepsilon_{xx}}{\varepsilon_{yy}} \frac{\kappa_x}{\kappa_y}. \tag{12.9}$$

Note that the ratio E_y/E_x gives the tangent of the angle of the electric field in the xy plane relative to the x axis.

Our goal is to show the electric field solution $\mathbf{E} = (E_x, E_y, 0)$ corresponding to the elliptical solution of Equation 12.2 is tangent to the ellipse, which for review reads

$$1 = \frac{\kappa_x^2}{\varepsilon_{yy}} + \frac{\kappa_y^2}{\varepsilon_{xx}}.$$

The tangent line to this elliptical solution is given by the derivative $\frac{\mathrm{d}\kappa_y}{\mathrm{d}\kappa_x}$, which we find by implicitly differentiating with respect to κ_x to get

$$0 = \frac{2\kappa_x}{\varepsilon_{yy}} + 2\frac{\kappa_y}{\varepsilon_{xx}} \frac{\mathrm{d}\kappa_y}{\mathrm{d}\kappa_x} \implies \frac{\mathrm{d}\kappa_y}{\mathrm{d}\kappa_x} = -\frac{\varepsilon_{xx}}{\varepsilon_{yy}} \frac{\kappa_x}{\kappa_y}.$$

Since the tangent to the elliptical solution equals the electric field ratio in Equation 12.9, we conclude that, for the elliptical solution to Equation 12.2 for wave vectors $\mathbf{k} = (k_x, k_y, 0)$, the corresponding electric field is tangent to the ellipse.

12.2.2 Solution in the xz Plane

We now consider the xz plane, in which case $\mathbf{k} = (k_x, 0, k_z)$. We begin by normalizing matrix $\mathbf{M}$ in Equation 12.5 by k_0^2 to get

$$\widetilde{\mathbf{M}} \equiv \frac{\mathbf{M}}{k_0^2} = \begin{pmatrix} \kappa_z^2 - \varepsilon_{xx} & 0 & -k_x k_z \\ 0 & \kappa_x^2 + \kappa_z^2 - \varepsilon_{yy} & 0 \\ -\kappa_z \kappa_x & 0 & \kappa_x^2 - \varepsilon_{zz} \end{pmatrix} \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} = \mathbf{0}.$$

The x component (first matrix row) of the above matrix equation produces

$$\left(\kappa_z^2 - \varepsilon_{xx}\right) E_x - \kappa_x \kappa_z E_z = 0 \implies \frac{E_z}{E_x} = \frac{\kappa_z^2 - \varepsilon_{xx}}{\kappa_x \kappa_z}.$$

We then recall the second, elliptical solution to Equation 12.2 for $\mathbf{k}$ in the xz plane; from Equation 12.6, this solution is

$$\frac{\kappa_z^2}{\varepsilon_{xx}} + \frac{\kappa_x^2}{\varepsilon_{zz}} = 1 \implies \kappa_z^2 = \varepsilon_{xx} - \kappa_x^2 \frac{\varepsilon_{xx}}{\varepsilon_{zz}}.$$

We then substitute this expression for κ_z into the above ratio E_z/E_x to get

$$\frac{E_z}{E_x} = \frac{\kappa_z^2 - \varepsilon_{xx}}{\kappa_x \kappa_z} = -\frac{\varepsilon_{xx}}{\varepsilon_{zz}} \frac{\kappa_x}{\kappa_z}. \tag{12.10}$$

Note that the ratio E_z/E_x gives the tangent of the angle of the electric field in the xz plane relative to the x axis.

Our goal is to show the electric field solution $\mathbf{E} = (E_x, 0, E_z)$ corresponding to the elliptical solution of Equation 12.2 is tangent to the ellipse, which for review reads

$$1 = \frac{\kappa_x^2}{\varepsilon_{zz}} + \frac{\kappa_z^2}{\varepsilon_{xx}}.$$

The tangent line to this elliptical solution is given by the derivative $\frac{\mathrm{d}\kappa_z}{\mathrm{d}\kappa_x}$, which we find by implicitly differentiating with respect to κ_x to get

$$0 = \frac{2\kappa_x}{\varepsilon_{zz}} + 2\frac{\kappa_z}{\varepsilon_{xx}}\frac{\mathrm{d}\kappa_z}{\mathrm{d}\kappa_x} \implies \frac{\mathrm{d}\kappa_z}{\mathrm{d}\kappa_x} = -\frac{\varepsilon_{xx}}{\varepsilon_{zz}}\frac{\kappa_x}{\kappa_z}.$$

Since the tangent to the elliptical solution equals the electric field ratio in Equation 12.10, we conclude that, for the elliptical solution to Equation 12.2 for wave vectors $\mathbf{k} = (k_x, 0, k_z)$, the corresponding electric field is tangent to the ellipse.

12.2.3 Solution in the yz Plane

We now consider the xz plane, in which case $\mathbf{k} = (k_x, 0, k_z)$. We begin by normalizing matrix $\mathbf{M}$ in Equation 12.7 by k_0^2 to get

$$\widetilde{\mathbf{M}} \equiv \frac{\mathbf{M}}{k_0^2} = \begin{pmatrix} \kappa_y^2 + \kappa_z^2 - \varepsilon_{xx} & 0 & 0 \\ 0 & \kappa_z^2 - \varepsilon_{yy} & -\kappa_y\kappa_z \\ 0 & -\kappa_y\kappa_z & \kappa_y^2 - \varepsilon_{zz} \end{pmatrix} \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} = \mathbf{0}.$$

The y component (second matrix row) of the above matrix equation produces

$$\left(\kappa_z^2 - \varepsilon_{yy}\right) E_y - \kappa_y\kappa_z E_z = 0 \implies \frac{E_z}{E_y} = \frac{\kappa_z^2 - \varepsilon_{yy}}{\kappa_y\kappa_z}.$$

We then recall the second, elliptical solution to Equation 12.2 for $\mathbf{k}$ in the yz plane; from Equation 12.8, this solution is

$$\frac{\kappa_z^2}{\varepsilon_{yy}} + \frac{\kappa_y^2}{\varepsilon_{zz}} = 1 \implies \kappa_z^2 = \varepsilon_{yy} - \kappa_y^2\frac{\varepsilon_{yy}}{\varepsilon_{zz}}.$$

We then substitute this expression for κ_z into the above ratio E_z/E_y to get

$$\frac{E_z}{E_y} = \frac{\kappa_z^2 - \varepsilon_{yy}}{\kappa_y\kappa_z} = -\frac{\varepsilon_{yy}}{\varepsilon_{zz}}\frac{\kappa_y}{\kappa_z}. \tag{12.11}$$

Note that the ratio E_z/E_y gives the tangent of the angle of the electric field in the yz plane relative to the y axis.

Our goal is to show the electric field solution $\mathbf{E} = (0, E_y, E_z)$ corresponding to the elliptical solution of Equation 12.2 is tangent to the ellipse, which for review reads

$$1 = \frac{\kappa_y^2}{\varepsilon_{zz}} + \frac{\kappa_z^2}{\varepsilon_{yy}}.$$

The tangent line to this elliptical solution is given by the derivative $\frac{\mathrm{d}\kappa_z}{\mathrm{d}\kappa_y}$, which we find by implicitly differentiating with respect to κ_y to get

$$0 = \frac{2\kappa_y}{\varepsilon_{zz}} + 2\frac{\kappa_z}{\varepsilon_{yy}}\frac{\mathrm{d}\kappa_z}{\mathrm{d}\kappa_y} \implies \frac{\mathrm{d}\kappa_z}{\mathrm{d}\kappa_y} = -\frac{\varepsilon_{yy}}{\varepsilon_{zz}}\frac{\kappa_y}{\kappa_z}.$$

Since the tangent to the elliptical solution equals the electric field ratio in Equation 12.11, we conclude that, for the elliptical solution to Equation 12.2 for wave vectors $\mathbf{k} = (0, k_y, k_z)$, the corresponding electric field is tangent to the ellipse.

12.3 Principal Polarizations in a Biaxial Material

Topaz is a optically biaxial crystal with refractive index eigenvalues $n_{xx} = 1.619$, $n_{yy} = 1.620$ and $n_{zz} = 1.627$. We illuminate the crystal such that the incident light's wave vector lies in the crystal's xz plane and makes an angle $30°$ degrees with the crystal's z axis.

Analyze the two principal polarizations for the above incident light: compute the refractive indices, directions of the $\mathbf{E}$ and $\mathbf{D}$ fields, the angle between the $\mathbf{E}$ and $\mathbf{D}$ fields, and the angle between the $\mathbf{E}$ and $\mathbf{D}$ fields and the Poynting vector $\mathbf{S}$. Finally, determine the angle between the two optic axes.

(Suggested exercise: repeat the following analysis in xy and yz planes.) The incident wave vector lies in the xz plane, so we begin with the xz plane solutions to Equation 12.2, which we found in Equation 12.6 of Exercise 12.1. These are

$$\kappa_x^2 + \kappa_z^2 = \varepsilon_{yy} \qquad \text{and} \qquad \frac{\kappa_x^2}{\varepsilon_{zz}} + \frac{\kappa_z^2}{\varepsilon_{xx}} = 1.$$

Using the given refractive indices and $\varepsilon_{\alpha\alpha} = n_{\alpha\alpha}^2$, we then compute

$$\varepsilon_{xx} = 1.619^2 = 2.6211, \qquad \varepsilon_{yy} = 1.620^2 = 2.6244, \qquad \varepsilon_{zz} = 1.627^2 = 2.6471.$$

Light is incident such that wave vector makes an angle $30°$ degrees with z axis.

Draw this as κ_x, κ_z plane diagram with vector $\mathbf{k}$ in this plane making, well, an angle $\theta = 30°$ with κ_z axis.

Circular solution: we first recall $\kappa_\alpha = k_\alpha/k_0$, in terms of which the circular solution reads

$$\kappa_x^2 + \kappa_z^2 = \varepsilon_{yy} \implies k_x^2 + k_z^2 = \varepsilon_{yy}k_0^2.$$

But, since $k_y = 0$ in the xz plane, the LHS is just k^2. We then use the general relationship $k = nk_0$ to solve for the refractive index n; this reads:

$$k^2 = n^2k_0^2 = \varepsilon_{yy}k_0^2 \implies n^2 = \varepsilon_{yy} = n_{yy}^2 \implies n_1 = n_{yy} = 1.620. \tag{12.12}$$

Elliptical solution: using $k_\alpha = k_\alpha/k_0$, the elliptical solution reads

$$\frac{\kappa_x^2}{\varepsilon_{zz}} + \frac{\kappa_z^2}{\varepsilon_{xx}} = 1 \implies \frac{k_x^2}{\varepsilon_{zz}} + \frac{k_z^2}{\varepsilon_{xx}} = k_0^2.$$

We are given that the incident wave vector makes an angle $30°$ with the z axis, so $k_x = nk_0 \sin\theta$ and $k_y = nk_0 \cos\theta$, which we substitute in above to get

$$\frac{n^2 \sin^2\theta}{\varepsilon_{zz}} + \frac{n^2\cos^2\theta}{\varepsilon_{xx}} = 1$$

We then solve for n and substitute in numerical values to get

$$n_2 = \sqrt{\frac{\varepsilon_{xx}\varepsilon_{zz}}{\varepsilon_{xx}\sin^2\theta + \varepsilon_{zz}\cos^2\theta}} \approx 1.621. \tag{12.13}$$

Electric Field Solutions

For the circular solution, both the **E** and **D** fields are perpendicular to the xz wave vector plane, and point along the y axis.

For the second, elliptical solution, the electric field **E** is tangent to the ellipse in the xz plane.

At the point at which the wave vector $\mathbf{k} = k(\sin 30°, 0, \cos 30°)$ intersects the ellipse, the **E** field lies in the xz plane tangent to the ellipse, while the **D** field also lies in the xz plane but is perpendicular to **k**.

Since **k** is given as making an angle 30° degrees with the z axis and **D** is perpendicular to **k**, we conclude that **D** makes an angle $-60°$ with the z axis.

We find the direction of the **E** field for the elliptical solution using the result in Equation 12.10, which was also derived for **k** in the xz plane. For review, Equation 12.10 reads

$$\frac{E_z}{E_x} = -\frac{\varepsilon_{xx}}{\varepsilon_{zz}}\frac{\kappa_x}{\kappa_z},$$

and gives the tangent of **E** relative to the x axis (or the cotangent of **E** relative to the z axis). We then recognize that κ_x/κ_z is the tangent of **k** relative to the z axis—since the angle between **k** and $\hat{\mathbf{z}}$ is known to be 30°, we can solve for the angle ϕ between **E** and z axis via

$$\cot\phi = \frac{E_z}{E_x} = -\frac{\varepsilon_{xx}}{\varepsilon_{zz}}\frac{k_x}{k_z} = -\frac{\varepsilon_{xx}}{\varepsilon_{zz}}\tan 30° \approx -0.5716.$$

From here we find the angle ϕ between **E** and the z axis is

$$\cot\phi = -0.5716 \implies \phi \approx -60.244°.$$

Note that, for the elliptical solution, **E** is very nearly parallel to **D** (which we said above makes an angle $-60°$ with the z axis), but not exactly so—this is a consequence of electromagnetic waves' behavior in anisotropic materials.

Poynting Vector Direction

For the circular solution, in which **E** and **D** are parallel and perpendicular to the xz plane, the Poynting vector **S** is parallel to **k**.

In the second elliptical solution, we begin with the definition

$$\mathbf{S} = \mathbf{E} \times \mathbf{H}.$$

The magnetic **H** field is perpendicular to **E** and points out of the page along the y axis. Thus **S**, which is by construction $\mathbf{S} = \mathbf{E} \times \mathbf{H}$ perpendicular to both **E** and **H**, will lie in the xz plane and point perpendicularly to the electric field **E** for the elliptical solution.

Since **E** makes an angle $\phi = -60.244°$ with the z axis, **S** will make an angle $-60.244° + 90° = 30.244°$ with the z axis.

Optic Axis

Definition: The optic axis in an anisotropic material is the axis along which incident light whose (wave vector?) aligns with the optic axis experiences the same refractive index regardless of the incident light's polarization. Basically, an anisotropic material behaves like an isotropic material for light incident along the optic axis.

An anisotropic material with three different refractive index eigenvalues has two optic axis and is thus called a biaxial material. The optic axes point along the lines connecting the origin to the intersections of the two wave vector surfaces in that material. In our two-dimensional planar analyses above, the optic axes point from the origin to the points of intersection between the elliptical and circular wave vector solutions.

An anisotropic material with two unique dielectric tensor eigenvalues has a single optic axis, and is called a uniaxial material. (A material with three equal eigenvalues is an optically isotropic material.)

In uniaxial materials, assuming the convention $\varepsilon_{xx} = \varepsilon_{yy} \neq \varepsilon_{zz}$, the optic axis points along the z axis (or more generally, along the dielectric tensor's principal axis corresponding to the unique eigenvalue ε_{zz}).

In the convention in which $\varepsilon_{xx} < \varepsilon_{yy} < \varepsilon_{zz}$, the optic axes always lie in xz plane. More generally, the optic axes lie in the plane spanned by the dielectric tensor's principal axes corresponding to the largest and smallest refractive index eigenvalues.

Our goal is to find the angle between the optic axes—we do this by finding the angle θ between $\mathbf{k}$ and z axis for which the two anisotropic refractive indices n_1 and n_2 in Equation 12.12 and 12.13 are equal. For review, n_1 and n_2 are given by

$$n_1 = n_{yy} \qquad \text{and} \qquad n_2 = \sqrt{\frac{\varepsilon_{xx}\varepsilon_{zz}}{\varepsilon_{xx}\sin^2\theta + \varepsilon_{yy}\cos^2\theta}},$$

so the angle, say ϑ, made by the optic axes with respect to the z axis is the solution to the equation

$$n_{yy} = \sqrt{\frac{\varepsilon_{xx}\varepsilon_{zz}}{\varepsilon_{xx}\sin^2\vartheta + \varepsilon_{yy}\cos^2\vartheta}},$$

Using the trigonometric identity $\cos^2\vartheta = 1 - \sin^2\vartheta$ and rearranging produces

$$\begin{aligned}
\sin^2\vartheta &= \frac{1}{\varepsilon_{zz} - \varepsilon_{xx}}\left(\varepsilon_{zz} - \frac{\varepsilon_{xx}\varepsilon_{zz}}{n_{yy}^2}\right) = \frac{1}{\varepsilon_{zz} - \varepsilon_{xx}}\left(\varepsilon_{zz} - \frac{\varepsilon_{xx}\varepsilon_{zz}}{\varepsilon_{yy}}\right)\\
&= \tfrac{1}{2.6471 - 2.6211}\left(2.6471 - \tfrac{(2.6211)\cdot(2.6471)}{2.6244}\right)\\
&\approx 0.1280.
\end{aligned}$$

We then solve for ϑ to get

$$\sin^2\vartheta = 0.128 \implies \vartheta = \arcsin\sqrt{0.128} \approx 20.96^\circ.$$

The optic axes are symmetrically spaced about the z axis, so the angle between the optic axes is just twice the angle between the optic axis and the z axis; i.e.

$$\theta_{\text{oa}} = 2\vartheta \approx 41.93^\circ.$$

12.4 Principal Polarizations in a Uniaxial Material

(i) Derive expressions for the two principal polarizations in an optically uniaxial material, including the angle between the **E** *and* **D** *field.*

(ii) Compute the angle between **E** *and* **D** *in calcite, which has refractive index eigenvalues* $n_\perp = 1.658$ *and* $n_\parallel = 1.486$*, for light incident at an angle* $30°$ *relative to the crystal's optic axis.*

An optically uniaxial material has a dielectric tensor with two equal eigenvalues; by convention we label the eigenvalues so that $\varepsilon_{xx} = \varepsilon_{yy} \neq \varepsilon_{zz}$. A uniaxial material's optical properties are invariant with respect to rotation about the z axis, i.e. the dielectric tensor's principal axis corresponding to the unique eigenvalue ε_{zz}, and uniaxial materials have a single optic axis aligned with the z axis.

In uniaxial material we must consider both the case $\varepsilon_{zz} > \varepsilon_{xx} = \varepsilon_{yy}$ (optically positive materials) and the case $\varepsilon_{zz} < \varepsilon_{xx} = \varepsilon_{yy}$ (optically positive materials); recall that in biaxial materials we have only $\varepsilon_{xx} < \varepsilon_{yy} < \varepsilon_{zz}$.

ε_{zz} is $\varepsilon_\parallel$ is optic axis is dielectric tensor's principal axis corresponding to the unique eigenvalue.

$\varepsilon_{xx} = \varepsilon_{yy}$ are labelled $\varepsilon_\perp$, because they correspond to principal axes perpendicular to the uniaxial material's optic axis.

Refractive Indices

We first consider the case $\varepsilon_\parallel > \varepsilon_\perp$ (for the sake of completeness, even though calcite has $\varepsilon_\parallel < \varepsilon_\perp$) and consider a wave vector $\mathbf{k} = (k_x, 0, k_y)$ in the xz plane.

From Equation 12.6, the solutions to Equation 12.2 in the xz plane read

$$\kappa_x^2 + \kappa_z^2 = \varepsilon_\perp \qquad \text{and} \qquad \frac{\kappa_x^2}{\varepsilon_\parallel} + \frac{\kappa_z^2}{\varepsilon_\perp} = 1,$$

where we have made the substitutions $\varepsilon_{xx} = \varepsilon_{yy} \equiv \varepsilon_\perp$ and $\varepsilon_{zz} \equiv \varepsilon_\parallel$ for a uniaxial material. Plotting the two solutions in the $\kappa_x \kappa_z$ plane produces a circle of radius $\sqrt{\varepsilon_\perp}$ and an ellipse with major and minor axes reaching $\pm\sqrt{\varepsilon_\parallel}$ on the κ_x axis and $\pm\sqrt{\varepsilon_\perp}$ on the κ_z axis, respectively.

If $\varepsilon_\parallel > \varepsilon_\perp$, the ellipse contains the circle, while if $\varepsilon_\parallel < \varepsilon_\perp$, the circle contains the ellipse.

Note that the elliptical and circular solutions intersect only along the z axis at $\pm\sqrt{\varepsilon_\perp}$—this single pair of intersection points is why a uniaxial material has only a single optic axis.

Circular solution: using $\kappa_\alpha = k_\alpha / k_0$, the circular solution reads

$$\kappa_x^2 + \kappa_z^2 = \varepsilon_\perp \implies k_x^2 + k_z^2 = \varepsilon_\perp k_0^2.$$

Since $k_y = 0$ in the xz plane, the LHS is just k^2. We then use the general relationship $k = nk_0$ to solve for the refractive index n; this reads:

$$k^2 = n^2 k_0^2 = \varepsilon_\perp k_0^2 \implies n^2 = \varepsilon_\perp = n_\perp^2 \implies n = n_\perp. \tag{12.14}$$

Elliptical solution: again using $\kappa_\alpha = k_\alpha / k_0$, the elliptical solution reads

$$\frac{\kappa_x^2}{\varepsilon_\parallel} + \frac{\kappa_z^2}{\varepsilon_\perp} = 1 \implies \frac{k_x^2}{\varepsilon_\parallel} + \frac{k_z^2}{\varepsilon_\perp} = k_0^2.$$

Assuming the incident wave vector makes an angle 30° with the optic axis (the z axis), we wave $k_x = nk_0 \sin\theta$ and $k_y = nk_0 \cos\theta$, which we substitute in to get

$$\frac{n^2 \sin^2\theta}{\varepsilon_\parallel} + \frac{n^2 \cos^2\theta}{\varepsilon_\perp} = 1 \implies \frac{1}{n^2} = \frac{\sin^2\theta}{n_\parallel^2} + \frac{\cos^2\theta}{n_\perp^2}, \tag{12.15}$$

Notation: $n_\perp \equiv n_\mathrm{o}$ (ordinary) and $n_\parallel \equiv n_\mathrm{e}$ (extraordinary). Note that n_o and n_e are the dielectric tensor eigenvalues, and are a property of the uniaxial material.

Light propagating parallel to the optic axis (corresponding to the circular solution) experiences a refractive index n_o.

Ordinary polarization: light with ordinary polarization experiences refractive index equal to n_o regardless of direction of light propagation through the anisotropic material (Equation 12.14).

Extraordinary polarization: light with extraordinary polarization experiences a direction-dependent refractive index between n_o and n_e given by Equation 12.15.

Angle Between E and D

The **D** field. For the circular solution **D** is perpendicular to the xz plane. For the elliptical solution, **D**, even in anisotropic materials, is always perpendicular to **k**, and thus has direction

$$\mathbf{D} = D_0(-\cos\theta, 0, \sin\theta).$$

Electric field: For the circular solution **E** is perpendicular to the xz plane and parallel to **E**. For the elliptical solution, the direction of **E** relative to the optical axis is given by Equation 12.10, which reads

$$\frac{E_z}{E_x} = -\frac{\varepsilon_{xx}}{\varepsilon_{zz}}\frac{\kappa_x}{\kappa_z} \equiv -\frac{\varepsilon_\perp}{\varepsilon_\parallel}\frac{\kappa_x}{\kappa_z},$$

and gives the tangent of **E** relative to the x axis (or the cotangent of **E** relative to the z axis). Meanwhile, κ_x/κ_z is the tangent of **k** relative to the z axis. Letting ϕ denote the angle between **E** and the z axis, we have

$$\cot\phi = \frac{E_z}{E_x} = -\frac{\varepsilon_\perp}{\varepsilon_\parallel}\tan\theta \implies \phi = \operatorname{arccot}\left(-\frac{\varepsilon_\perp}{\varepsilon_\parallel}\tan\theta\right).$$

We then substitute in numerical values to get

$$\phi = \operatorname{arccot}\left(-\frac{\varepsilon_\perp}{\varepsilon_\parallel}\tan\theta\right) = \operatorname{arccot}\left(-\frac{1.658}{1.486}\tan 30^\circ\right) \approx 122.789^\circ.$$

Subtract off 180° (since arccot only gives positive angles) to get $\phi \approx -57.211^\circ$. The angle between **D** and **E**, which we'll call γ, is then

$$\gamma = (-60^\circ) - (-57.211^\circ) \approx 2.789^\circ.$$

Alternatively, rearrange and get

$$\frac{E_z}{E_x} = -\frac{\varepsilon_\perp}{\varepsilon_\parallel}\tan\theta \implies \varepsilon_\parallel E_z\cos\theta = -\varepsilon_\perp E_x \sin\theta$$

Write electric field as

$$\begin{aligned}
\mathbf{E} &= (E_x, 0, E_z) = \left(E_x, 0, -E_x \frac{\varepsilon_\perp}{\varepsilon_\parallel} \frac{\sin\theta}{\cos\theta}\right) = E_x \left(1, 0, -\frac{\varepsilon_\perp}{\varepsilon_\parallel} \frac{\sin\theta}{\cos\theta}\right) \\
&= \frac{E_x}{\varepsilon_\parallel \cos\theta} (\varepsilon_\parallel \cos\theta, 0, -\varepsilon_\perp \sin\theta). \\
&\equiv E_0 (\varepsilon_\parallel \cos\theta, 0, -\varepsilon_\perp \sin\theta).
\end{aligned}$$

We then compute γ from the geometric relationship

$$\cos\gamma = \frac{\mathbf{E}\cdot\mathbf{D}}{|\mathbf{E}||\mathbf{D}|} = \frac{\varepsilon_\parallel \cos^2\theta + \varepsilon_\perp \sin^2\theta}{\sqrt{\varepsilon_\parallel^2 \cos^2\theta + \varepsilon_\perp^2 \sin^2\theta}} \implies \gamma \approx 2.789^\circ. \tag{12.16}$$

13 Thirteenth Exercise Set

13.1 Refraction Into a Uniaxial Crystal

We consider a titanium dioxide crystal (a uniaxial material with $n_\mathrm{o} = 2.616$ and $n_\mathrm{e} = 2.903$) which is cut such that the optical axis is perpendicular to the crystal's surface.

Light is incident on the crystal's surface at an incidence angle $60°$; determine the angles at which the transmitted light travels through the crystal.

First decompose incident light into ordinary and extraordinary electric field polarizations. Suppose light is incident at angle α. Incident light refracts at the crystal surface according to Snell's law. Let β_o and β_x denote the angles of refraction at which the incident light's ordinary and extraordinary polarizations propagate through the crystal.

Ordinary Polarization

The incident light's ordinary polarization has an electric field perpendicular to the incident wave vector $\mathbf{k}$ (out of the page in our drawing of $\mathbf{k}$ in the xz plane)

The incident ordinary polarization experiences a direction-independent refractive index equal to n_o, as in Equation 12.14 (which uses the notation $n_\perp$ instead of n_o).

Snell's law for the incident polarization reads

$$\sin\alpha = n_\mathrm{o} \sin\beta_\mathrm{o}.$$

Solving for β_o produces

$$\beta_\mathrm{o} = \arcsin\left(\frac{\sin\alpha}{n_\mathrm{o}}\right) = \arcsin\left(\frac{\sin 60°}{2.616}\right) \approx 19.332°.$$

Extraordinary Polarization

TLDR: Snell's law for the extraordinary ray reads

$$\sin\alpha = n_\mathrm{x} \sin\beta_\mathrm{x}.$$

The problem reduces to finding n_x to find β_x.

Important conceptually: the incident light's extraordinary polarization does not experience refractive index exactly n_e, but a direction-dependent refractive index $n_\mathrm{x} \in [n_\mathrm{o}, n_\mathrm{e}]$ given by Equation 12.15, which in this problem's notation reads

$$\frac{1}{n_\mathrm{x}^2} = \frac{\sin^2\theta_\mathrm{x}}{n_\mathrm{e}^2} + \frac{\cos^2\theta_\mathrm{x}}{n_\mathrm{o}^2},$$

where θ_x is the angle between the crystal's optical axis and the direction of the extraordinary ray's wave vector in the crystal (after refraction from air). In our problem, in which the crystal's optical axis aligns with the normal to the crystal's surface, we have $\theta_\mathrm{x} = \beta_\mathrm{x}$; however if the normal to the crystal and the crystal's optical axis were rotated relative to each other by an angle φ, then $\theta_\mathrm{x} = \beta_\mathrm{x} - \varphi$.

Solve for n_x and use $\theta_\mathrm{x} = \beta_\mathrm{x}$ to get

$$n_\mathrm{x} = \sqrt{\frac{n_\mathrm{e}^2 n_\mathrm{o}^2}{n_\mathrm{o}^2 \sin^2\beta_\mathrm{x} + n_\mathrm{e}^2 \cos^2\beta_\mathrm{x}}}.$$

This is the refractive index felt by the extraordinary ray inside the uniaxial crystal; we then substitute n_x into Snell's law to find the incident extraordinary polarization's angle of refraction:

$$\sin\alpha = n_\mathrm{x}\sin\beta_\mathrm{x} = \frac{n_\mathrm{e} n_\mathrm{o}\sin\beta_\mathrm{x}}{\sqrt{n_\mathrm{o}^2\sin^2\beta_\mathrm{x} + n_\mathrm{e}^2\cos^2\beta_\mathrm{x}}}.$$

It remains to solve the above equation for β_x. We first square both sides and use $\cos^2\theta = 1 - \sin^2\theta$, then solve for $\sin^2\beta_\mathrm{x}$ to get

$$\sin^2\beta_\mathrm{x} = \frac{n_\mathrm{e}^2\sin^2\alpha}{n_\mathrm{e}^2 n_\mathrm{o}^2 + (n_\mathrm{e}^2 - n_\mathrm{o}^2)\sin^2\alpha}.$$

We then solve for $\sin\beta_\mathrm{x}$ and substitute in given numerical values to get

$$\sin\beta_\mathrm{x} = \frac{n_\mathrm{e}\sin\alpha}{\sqrt{n_\mathrm{e}^2 n_\mathrm{o}^2 + (n_\mathrm{e}^2 - n_\mathrm{o}^2)\sin^2\alpha}} \approx 0.3277 \implies \beta_\mathrm{x} \approx 19.128^\circ.$$

Note that the extraordinary angle of refraction $\beta_\mathrm{x} = 19.128^\circ$ is similar, but not equal, to the ordinary angle of refraction $\beta_\mathrm{o} = 19.332^\circ$ found above.

Poynting Vector and Electric Field Direction

For ordinary polarization, we have $\mathbf{k}_\mathrm{o} \parallel \mathbf{S}_\mathrm{o}$. For extraordinary polarization, we use $\mathbf{S} = \mathbf{E}\times\mathbf{H}$. The $\mathbf{H}$ field points "out of the page", $\mathbf{E}$ is tangent to the ellipse solution

$$\frac{k_\perp^2}{\varepsilon_\parallel} + \frac{k_\parallel^2}{\varepsilon_\perp} = k_0^2$$

at the point where $\mathbf{k}_\mathrm{x}$ (which makes the just-computed angle β_x with the optical axis) intersects the ellipse. Since $\mathbf{S}$ is perpendicular to both $\mathbf{H}$ and $\mathbf{E}$, we conclude that $\mathbf{S}_\mathrm{x}$ lies in the "plane of the paper" (the same plane as $\mathbf{k}_\mathrm{x}$ and $\mathbf{E}_\mathrm{x}$) and is normal to the ellipse (perpendicular to $\mathbf{E}_\mathrm{x}$) at the point where $\mathbf{k}_\mathrm{x}$ intersects the ellipse.

Next find angle γ between $\mathbf{E}_\mathrm{x}$ and $\mathbf{D}_\mathrm{x}$. Recall $\mathbf{D}$ is always perpendicular to $\mathbf{k}$, even in anisotropic materials.

Recall Equation 12.16:

$$\cos\gamma = \frac{\mathbf{E}\cdot\mathbf{D}}{|\mathbf{E}||\mathbf{D}|} = \frac{\varepsilon_\parallel\cos^2\theta + \varepsilon_\perp\sin^2\theta}{\sqrt{\varepsilon_\parallel^2\cos^2\theta + \varepsilon_\perp^2\sin^2\theta}}.$$

However, because $\mathbf{D}$ is perpendicular to $\mathbf{k}$ and $\mathbf{E}$ is perpendicular to $\mathbf{S}$, the angle between $\mathbf{k}$ and $\mathbf{S}$ will equal the angle between $\mathbf{E}$ and $\mathbf{D}$, i.e. γ is also the angle between $\mathbf{k}_\mathrm{x}$ and $\mathbf{S}_\mathrm{x}$. Using $\theta \to \beta_\mathrm{x}$, this comes out to

$$\cos\gamma \approx 0.9982 \implies \gamma = \arccos 0.9982 \approx 3.399^\circ.$$

The Poynting vector for extraordinary polarization thus propagates through the crystal at an angle

$$\beta_{\mathrm{S}_\mathrm{x}} = \beta_\mathrm{x} - \gamma = 19.128^\circ - 3.399^\circ \approx 15.73^\circ.$$

TODO: Version II Optical axis parallel to boundary plane.

13.2 Shift After Transmission Through a Uniaxial Material I

A calcite crystal of width $d = 2\,\mathrm{cm}$ (a uniaxial material with refractive indices $n_\mathrm{e} = 1.486$ and $n_\mathrm{o} = 1.658$) is oriented so that the crystal's optic axis is perpendicular to plane of incidence. Light is incident on the crystal at an angle $40°$ degrees relative to the normal with the crystal surface.

Determine the spacing Δx between the transmitted ordinary and extraordinary rays.

Calcite crystal with $n_\mathrm{o} = 1.658$ and $n_\mathrm{e} = 1.486$. Note that $n_\mathrm{o} > n_\mathrm{e}$. Procedure similar to earlier, just roles of n_e and n_o are reversed. Optic axis is perpendicular to plane of incidence. Optically negative material has ellipse within circle.

Goals is to find shift between transmitted waves. For material of thickness d we have

$$d \tan \beta_\mathrm{o} = x_\mathrm{o} \qquad \text{and} \qquad d \tan \beta_{S_x} = x_\mathrm{x}.$$

For $d = 7\,\mathrm{cm}$ we have $\Delta x = x_\mathrm{x} - x_\mathrm{o} = 2.3\,\mathrm{mm}$. Note that characteristic width beam of light used must smaller than Δx to note a difference polarizations.

13.3 Shift After Transmission Through a Uniaxial Material II

We again consider a calcite crystal of width $d = 2\,\mathrm{cm}$ (refractive indices $n_\mathrm{e} = 1.486$ and $n_\mathrm{o} = 1.658$), this time oriented so that the crystal's optic axis makes an angle $\varphi = 45°$ degrees with the normal to the crystal's surface. Light is normally incident on the crystal's surface; determine the spacing Δx between the transmitted ordinary and extraordinary rays.

We first consider refraction of incident light at the air-crystal interface. Since the problem involves normally incident light, the angle of refraction is $\beta = 0$ for both the incident light's ordinary and extraordinary polarizations.

Incident light's ordinary polarization experiences refractive index n_o. Incident light's extraordinary polarization experiences refractive index n_x given by

$$\frac{1}{n_\mathrm{x}^2} = \frac{\sin^2\theta_\mathrm{x}}{n_\mathrm{e}^2} + \frac{\cos^2\theta_\mathrm{x}}{n_\mathrm{o}^2},$$

where θ_x is the angle between the crystal's optic axis and the direction of the extraordinary ray's wave vector in the crystal. Our problem has no refraction at the crystal surface, so the angle between the transmitted extraordinary polarization and the optic axis equals the angle between the incident extraordinary polarization and the optic axis, i.e. $\theta_\mathrm{x} = \varphi$.

More generally, we would have $\theta_\mathrm{x} = \beta_\mathrm{x} + \varphi$, where β_x is the incident extraordinary polarization's angle of refraction at the crystal surface.

Find angle γ between $\mathbf{E}$ and $\mathbf{D}$ field using Equation 12.16 which, for extra-ordinary polarization with $\theta \to \theta_\mathrm{x}$, reads

$$\cos\gamma = \frac{\mathbf{E}\cdot\mathbf{D}}{|\mathbf{E}||\mathbf{D}|} = \frac{\varepsilon_\| \cos^2\theta_\mathrm{x} + \varepsilon_\perp \sin^2\theta_\mathrm{x}}{\sqrt{\varepsilon_\|^2\cos^2\theta_\mathrm{x} + \varepsilon_\perp^2\sin^2\theta_\mathrm{x}}}.$$

As mentioned above, in this problem $\theta_\mathrm{x} = \varphi = 45°$. Writing $\varepsilon_\| = n_\mathrm{e}^2$, $\varepsilon_\perp = n_\mathrm{o}^2$ and noting $\cos 45° = \sin 45° = 1/\sqrt{2}$, we have

$$\cos\gamma = \frac{\frac{1}{2}(n_\mathrm{e}^2 + n_\mathrm{o}^2)}{\sqrt{\frac{1}{2}}\sqrt{n_\mathrm{e}^4 + n_\mathrm{o}^4}} = \frac{1}{\sqrt{2}}\frac{n_\mathrm{e}^2 + n_\mathrm{o}^2}{\sqrt{n_\mathrm{e}^4 + n_\mathrm{o}^4}}.$$

Substituting in the given values $n_e = 1.486$ and $n_o = 1.658$ produces

$$\cos\gamma \approx 0.9941 \implies \gamma \approx 6.226°.$$

As mentioned at the end of Exercise 13.1, γ is also the angle between the extraordinary Poynting vector and extraordinary wave vector.

Shift between transmitted ordinary and extraordinary polarizations is the angle between their Poynting vectors. Because of light is normally incident on crystal, both ordinary and extraordinary rays have the same wave vector, which points along the normal direction of incident. The ordinary Poynting vector is parallel to the ordinary wave vector, while the extraordinary Poynting vector makes the just-computed angle γ with the extraordinary wave vector. Ordinary and extraordinary wave vectors are parallel, so angle between ordinary and extraordinary Poynting vectors is just γ, and the shift between the outputted polarizations is

$$\Delta x = d\tan\gamma = (2\,\mathrm{cm}) \cdot \tan 6.226° \approx 2.18\,\mathrm{mm}.$$

13.4 A Quartz Quarter-Waveplate

We wish to use a thin quartz plate ($n_o = 1.544$ and $n_e = 1.553$) to convert linearly polarized incident light into circularly polarized light. Determine the plate orientation and minimum plate thickness for which normally incident linearly polarized light of wavelength $\lambda = 500\,\mathrm{nm}$ will be circularly polarized after transmission.

The change in polarization from linear to circular occurs because of birefringent effects. The stronger the crystal's birefringence, the thinner the crystal needs to be. Thus, for minimum thickness, we orient the crystal so that its optical axis is perpendicular to the normal to the crystal's surface (i.e. normal to the direction of incidence), which will produce maximum birefringent effects.

For review from way back in Section 2.3.2, a quarter-waveplate works by introducing a phase shift of $\pi/2$ between the incident light's two polarizations.

In the context of anisotropic materials, we consider ordinary and extraordinary polarization, so creating a quarter-waveplate quartz plate essentially reduces to introducing a phase shift of $\pi/2$ between the incident light's ordinary and extraordinary polarizations. The phases accumulated by each polarization when passing through the plate are

$$\phi_o = k_0 n_o d \qquad \text{and} \qquad \phi_x = k_0 n_x d,$$

so the quarter-waveplate condition reads

$$\Delta\phi = \phi_x - \phi_o = \frac{\pi}{2} \implies \frac{\pi}{2} = \Delta\phi = k_0 d \cdot (n_x - n_o) = \frac{2\pi}{\lambda} d \cdot (n_x - n_o).$$

We then solve for the plate thickness d to get

$$d(n_x - n_o) = \frac{\lambda}{4} \implies d = \frac{\lambda}{4}\frac{1}{n_x - n_o}.$$

The incident ordinary polarization feels a refractive index n_o, while the extraordinary polarization feels a refractive index n_x give by

$$\frac{1}{n_x^2} = \frac{\sin^2\theta_x}{n_e^2} + \frac{\cos^2\theta_x}{n_o^2},$$

where θ_x is the angle between the crystal's optic axis and the direction of the extraordinary ray's wave vector in the crystal. For this problems simplified case of normal incidence and an optical axis perpendicular to the surface normal we have $\theta_x = 90°$ and so

$$\frac{1}{n_x^2} = \frac{1}{n_e^2} + \frac{0}{n_o^2} \implies n_x = n_e,$$

i.e. the extraordinary ray feels a refractive index exactly equal to n_e.

With n_x known, we solve for crystal thickness d to get

$$d = \frac{\lambda}{4(n_x - n_o)} = \frac{500\,\text{nm}}{4(1.553 - 1.544)} \approx 13.89\,\text{mm}.$$

13.5 Glan-Taylor Prism

We wish to use a two quartz prisms ($n_o = 1.544$ and $n_e = 1.553$) separated along their long faces by a thin gap of air as a polarizer. (This instrument is called a Glan-Taylor prism.) Determine the acute prism angle at which incident unpolarized light is transmitted with only a single polarization.

Working principle: use total internal reflection at the quartz-air gap interface to block one polarization.

First, we must choose which incident polarization we which to block. Naturally, we desire that the passed polarization exits with maximum possible transmission. This is possible for TM polarization incident on the quartz-air gap interface at Brewster's angle.

If we draw the prism in cross section in the plane of the paper, in which the plane of the paper aligns with the plane of incidence, then TE polarization points normal to the plane of the paper and TM polarization lies in the plane of the paper.

The material should have a refractive index n_{TE} for which total internal reflection will occur for TE polarization, while we simultaneously require $n_{\text{TM}} < n_{\text{TE}}$ so that TM polarization will not be internally reflected.

For maximum birefringent effects, we place the optic axis perpendicular to the direction of incidence, which is out of the page for our geometry.

TE polarization is parallel to the optic axis and corresponds to extraordinary polarization, while TM polarization, which is perpendicular to the optic axis, corresponds to ordinary polarization.

Thus $n_{\text{TM}} < n_{\text{TE}}$ corresponds to $n_o < n_x$. We assume the incident light is normally incident and perpendicular to the optical axis, in which case $n_x = n_e$.

We then determine the critical angle for total internal reflection from an optically denser material into air via the general formula

$$\sin\theta_c = \frac{1}{n}.$$

The critical angles for extraordinary (TE) and ordinary (TM) polarization are

$$\sin\theta_c = \frac{1}{n_e} = \frac{1}{1.553} \implies \theta_c^{(e)} \approx 40.084°.$$
$$\sin\theta_c = \frac{1}{n_o} = \frac{1}{1.544} \implies \theta_c^{(o)} \approx 40.366°.$$

The prism must be cut (very precisely!) at an angle such that light is incident on the glass-air gap interface at an angle between $\theta_c^{(e)}$ and $\theta_c^{(o)}$, for which extraordinary polarization is blocked but ordinary polarization passes through.

In practice, a Glan-Taylor prism would use calcite, which has a larger difference in refractive indices and thus a larger tolerance in spacing between $\theta_c^{(e)}$ and $\theta_c^{(o)}$.