Contents

Miscellaneous Useful Material

Constants

$N_A = 6.022 \cdot 10^{23}\,\text{mol}^{-1}$

$\alpha \approx 1/137$

$m_e c^2 \approx 511\,\text{keV}$

$m_p c^2 \approx 938\,\text{keV}$

$m_n c^2 \approx 940\,\text{MeV}$

$m_\pi \approx 140\,\text{MeV}$

$m_K \approx 500\,\text{MeV}$

$K_{BB} \approx 0.3\,\text{MeV}\,\text{g}^{-1}\,\text{cm}^2$

$(\beta\gamma)_{\text{MIP}} \approx 3.5, \qquad \beta_{\text{MIP}} \approx 0.96$

$1\,\text{torr} = 1\,\text{mmHg} \approx 133\,\text{Pa}$

$k_B T\big|_{T=300\,\text{K}} \approx 0.025\,\text{eV}$

Some Relationships from Special Relativity

$\beta \equiv v/c \qquad \gamma \equiv 1/\sqrt{1-\beta^2}$

$E^2 = m^2 c^4 + p^2 c^2 \qquad E = \gamma m c^2 \qquad E = T + mc^2$

$\gamma\beta = \frac{pc}{mc^2} \quad \gamma^2 = 1 + (\beta\gamma)^2 \quad \beta^2 = 1 - 1/\gamma^2$

$\beta^2 = \frac{p^2 c^2}{m^2 c^4 + p^2 c^2}$

$mc^2 = \frac{p^2 c^2 - T^2}{2T}$

Some Relationships from Chemistry

The number density n_a (n_e) of atoms (electrons) in material with density ρ, molar mass M_m and atomic number Z is...

$n_a = \frac{\rho N_A}{M_m} \qquad \text{and} \qquad n_e = Z n_a$

Molar mass M_m and relative atomic mass A_r are related by...

$M_m = A_r M_u$, where $M_u \equiv 1\,\text{g}\,\text{mol}^{-1}$

$n_a \approx \frac{\rho N_A}{A M_u}$ (if $A_r \approx A$)

<u>Ideal Gas</u>

$V_m \approx 22.4\,\text{L}\,\text{mol}^{-1}$ (molar volume of ideal gas at STP)

$p_0 = 1\,\text{atm} \approx 101\,\text{kPa}$ (atmospheric pressure)

$n = \frac{N_A}{V_m}\frac{p}{p_0}$ (number density of ideal gas molecules)

Statistics

Binomial distribution: the probability of n events occurring over the course of N trials, where the probability of an event occurring in a single trial is p, is given by

$P(n|N,p) = \binom{N}{n} p^n (1-p)^{N-n} = \frac{N!}{n!(N-n)!} p^n (1-p)^{N-n}$

Poisson distribution: the probability of n independent random events occuring in the time interval T, where the probability for an event per unit time is λ, is

$P(n|\lambda, T) = \frac{(\lambda T)^n}{n!} e^{-\lambda T}$, or...

$P(n|\mu) = \frac{\mu^n e^{-\mu}}{n!}$,   where $\mu \equiv \lambda T$

$\langle X \rangle = \sigma_X^2 = \lambda$ (if X is Poisson distributed with mean μ)

Error function and *standard normal distribution* CDF

$\Phi(x) = \frac{1}{2}\left[1 + \text{erf}\left(\frac{x}{\sqrt{2}}\right)\right]$

Statistical significance in signal/background classification

$n = \frac{N_{\text{sig}}}{\sigma_{\text{bg}}}$
N_{sig} is number of counted signal events
σ_b is fluctuatio in background events

$\overline{N}_{\text{sig}} = st \qquad \overline{N}_{\text{bg}} = bt$ (if rates s, b are known)

Interaction of Particles and Matter

Scattering Cross Section

Beam with flux F of incident particles on target; $\frac{dN_s}{d\Omega}$ is number of particles scattered into solid angle $d\Omega$ per unit time.

$\frac{d\sigma}{d\Omega} = \frac{1}{F}\frac{dN_s}{d\Omega} \qquad \sigma_{\text{tot}}(E) = \iint_\Omega \frac{d\sigma}{d\Omega}\,d\Omega$

$N_s(\Omega) = F S n_t \frac{d\sigma}{d\Omega}\delta x \qquad N_s = F S n_t \sigma_{\text{tot}}\delta x$

$dP_{\text{scat}} = n_t \sigma_{\text{tot}}\,dx$ (probability for scattering in region dx)

$P(x + dx) = P(x) \cdot (1 - dP_{\text{scat}}) = P(x)(1 - n_t \sigma_{\text{tot}}\,dx)$

$P(x) = e^{-n_t \sigma_{\text{tot}} x}$ (probability for not scattering up to x)

$1\,\text{b} = 1 \cdot 10^{-24}\,\text{cm}^2 = 1 \cdot 10^{-28}\,\text{m}^2$

Charged Particles in Matter

Charged ionizing particle (IP) of mass m and valence Z_p travels through material with atomic number Z_m and density ρ.
Assume IP is heavy ($m \gg m_e$)
Energy loss occurs primarily because of inelastic collisions of IP with electrons in the material.

Bethe-Bloch Formula

Valid for $\beta\gamma \sim (0.5, 10^3)$

$-\frac{dE}{dx} = \frac{4\pi}{m_e c^2} \cdot \frac{n_e Z_p^2}{\beta^2} \cdot \left(\frac{e_0^2}{4\pi\epsilon_0}\right)^2 \ln\left[\left(\frac{2 m_e c^2 \gamma^2 \beta^2}{Z_m I_0}\right) - \beta^2\right]$

$\qquad = K \cdot \frac{\rho Z_m}{A} \cdot \frac{Z_p^2}{\beta^2}\left[\ln\left(\frac{2 m_e c^2 \beta^2 \gamma^2}{Z_m I_0}\right) - \beta^2\right]$

$K \approx 0.3\,\text{MeV}\,\text{g}^{-1}\,\text{cm}^2 \qquad I_0 \sim 10\,\text{eV}$

Small β approximation (e.g. for $\beta\gamma \lesssim 1$) produces...

$-\frac{dE}{dx} \sim \beta^{-2} \sim T^{-1} \implies \frac{dT}{dx} = -\frac{k}{T}, \qquad k = -T_0 \frac{dT}{dx}\Big|_{T=T_0}$

In Polyatomic Substances

Example: for H_2O_4, $i \in \{H, O\}$, e.g. $a_H = 2$, $a_O = 4$

$Z \to Z_{\text{eff}} = \sum_i a_i Z_i$

$A \to A_{\text{eff}} = \sum_i a_i A_i$

$\ln I \to \ln I_{\text{eff}} = \sum_i \frac{a_i Z_i \ln I_i}{Z_{\text{eff}}}$

$\left(-\frac{dE}{dx}\right)_{\text{tot}} = \sum_i w_i \frac{dE}{dx}\Big|_i, \quad w_i = \frac{a_i A_i}{\sum_j a_j A_j}$

$\left(-\frac{dE}{dx}\right)_{\text{tot}} = K \cdot \frac{\rho_{\text{tot}} Z_{\text{eff}}}{A_{\text{eff}}} \cdot \frac{Z_p^2}{\beta^2}\left[\ln\left(\frac{2 m_e c^2 \beta^2 \gamma^2}{I_{\text{eff}}}\right) - \beta^2\right]$

Photons in Matter

Three processes: photoelectric effect, Compton scattering, and pair production

$\sigma_\gamma = \sigma_{\text{pe}} + Z \cdot \sigma_{\text{C}} + \sigma_{\text{pair}}$

$\sigma_{\text{pe}} \sim \frac{Z^n}{E_\gamma^{7/2}}, \quad n \lesssim [4, 5]$

$j(x) = j_0 e^{-\mu x} \quad \mu_\gamma = n_{\text{a}} \sigma_\gamma = \frac{\rho N_{\text{A}}}{M_{\text{m}}} \sigma_\gamma \quad \lambda_\gamma = 1/\mu_\gamma$

$\mu_{\text{tot}} = \sum_i w_i \mu_i = \sum_i \left(\frac{A_i}{\sum_j A_j} \right) \mu_i \quad$ (polyatomic substances)

Compton Scattering

Incident and scattered γ energies: E_γ, E'_γ; θ scattering angle

$\frac{E'_\gamma}{E_\gamma} = \frac{1}{1 + \alpha(1 - \cos\theta)}, \qquad \alpha \equiv \frac{E_\gamma}{m_e c^2}$

$\frac{d\sigma}{d\Omega} = \frac{r_e^2}{2} \left(\frac{E'_\gamma}{E_\gamma} \right)^2 \left[\frac{E'_\gamma}{E_\gamma} + \frac{E_\gamma}{E'_\gamma} - \sin^2\theta \right]$

$\sigma_{\text{C}} = \frac{8\pi r_e^2}{3} \left[\frac{1 - 2\alpha + 1.2\alpha^2}{(1 + 2\alpha)^2} \right], \quad r_e = \frac{1}{4\pi\epsilon_0} \frac{e_0}{m_e c^2} \sim 2.8 \,\text{fm}$

$\frac{d\sigma_{\text{C}}}{dT} = \frac{\pi r_0^2}{m_e c^2 \alpha^2} \left[2 + \frac{s^2}{\alpha^2(1-s)^2} + \frac{s}{1-s} \left(s - \frac{2}{\alpha} \right) \right], \quad s = T/E_\gamma$

Particle Detectors

Energy Resolution

For a particle depositing energy E_{dep} and producing N ion pairs in a detector with Fano factor F...

$\mathcal{R} \equiv \frac{\sigma_{E_{\text{dep}}}}{E_{\text{dep}}} = \frac{\sigma_N}{N}$

Particle passes through detector: $\sigma_N = \sqrt{N}$

Particle stops inside detector: $\sigma_N = \sqrt{FN}$

$N = \frac{E_{\text{dep}}}{w_i} \implies \mathcal{R} = \sqrt{\frac{w_i}{E_{\text{dep}}}} \quad \text{or} \quad \mathcal{R} = \sqrt{\frac{F w_i}{E_{\text{dep}}}}$

Ionization-Based Detectors

Parallel-Plate Ionization Cell

Consider a parallel-plate cell with pressure p, spacing d, potential difference U and constant electric field $E = U/d$.

$dW = qE\,dx = \frac{qU}{d}\,dx \qquad$ (work on a charge q)

$dW_{\text{C}} = CU\,dU \qquad$ (change in capacitor energy)

$dW = dW_{\text{C}} \implies dU = \frac{q}{C} \frac{dx}{d}$

$v_{\text{d}} = \frac{E\mu}{p} \qquad$ (drift velocity, mobility)

$\Delta U(t) = \frac{q}{C} \frac{\mu}{pd} Et \qquad$ (before all ions reach electrodes)

$\Delta U = \frac{Q}{C} \qquad$ (when total charge Q reaches electrodes)

Multiplication Factor

For an incident particle freeing N_0 primary ions, which in turn free an average of N secondary ions...

$M \equiv \frac{N}{N_0} \qquad$ (multiplication factor)

λ is electron mean free path for ionizing collisions

$\alpha \equiv 1/\lambda$ is probability for ionization per distance traveled

$dN = N\alpha\,dx \implies N(x) = N_0 e^{\alpha x} \qquad$ (for N initial electrons)

$M(x) \equiv N/N_0 = e^{\alpha x} \quad \text{or} \quad M = \exp\left(\int_{x_1}^{x_2} \alpha(x)\,dx \right)$

In a cell at pressure p with electric field E...

$\alpha = Ape^{-\frac{Bp}{E}} \qquad$ (Townsend discharge model; A, B given)

Cylindrical Ionization Chamber

For a cylindrical chamber with outer radius R and anode wire radius r_0 at voltage U_0....

$E(r) = \frac{U_0}{\ln(R/r_0)} \frac{1}{r} \quad \phi(r) = -\frac{U_0}{\ln(R/r_0)} \ln\frac{r}{r_0} \quad C = \frac{2\pi\epsilon_0 L}{\ln(R/r_0)}$

$v_{\text{d}} = E\mu \qquad$ (drift velocity v_{d}, mobility μ)

Signal detection delay t_{sig} between ionization event at $r = r^*$ and primary electrons reaching anode wire is...

$t_{\text{sig}} = \frac{\ln(R/r_0)R^2}{2\mu_e U_0} \left[\left(\frac{r^*}{R} \right)^2 - \left(\frac{r_0}{R} \right)^2 \right] \approx \frac{\ln(R/r_0)}{2\mu_e U_0} (r^*)^2$

Only secondary positive ions contribute appreciably to signal

$U(t) = -\frac{Q_i}{4\pi\epsilon_0 L} \ln\left[1 + \frac{\mu_i C U_0}{\pi\epsilon_0 L r_0^2} \cdot (t - t_{\text{sig}}) \right] \equiv -\frac{Q_i}{4\pi\epsilon_0 L} \ln\left(1 + \frac{t - t_{\text{sig}}}{t_0} \right)$

$U(t) = -\frac{N_s e_0}{4\pi\epsilon_0 L} \begin{cases} \approx 0 & t < t_{\text{sig}} \\ \ln\left(1 + \frac{t - t_{\text{sig}}}{t_0} \right) & t_{\text{sig}} < t < t_{\text{sig}} + t_{\text{ion}}. \end{cases}$

$t_0 \equiv \frac{\pi\epsilon_0 L r_0^2}{\mu_i C U_0}, \qquad t_{\text{ion}} \approx \frac{\ln(R/r_0)}{2\mu_i U_0} R^2$

$N_s = M N_p = M \frac{E_{\text{dep}}}{w_i}$

Measuring Momentum

Use a central drift chamber with beamline axis $\hat{\mathbf{z}}$ and magnetic field $\boldsymbol{B} \approx B\,\hat{\mathbf{e}}_z$

For particle of charge q with trajectory curvature radius R...

$\frac{mv_{\text{T}}^2}{R} = qv_{\text{T}}B \implies p_{\text{T}} = qBR \qquad$ (very simplified)

$p_{\text{T}}c \approx (0.3qBR) \,\text{GeV} \dots$
... if q is measured in e_0, B in tesla and R in meters

Momentum resolution $\sigma_{p_{\text{T}}}$ if trajectory resolution is σ_x...

$\frac{\sigma_{p_{\text{T}}}}{p_{\text{T}}} \approx \frac{\sqrt{96}\sigma_x}{qBL^2} \cdot p_{\text{T}} \qquad$ (three points on trajectory)

$\frac{\sigma_{p_{\text{T}}}}{p_{\text{T}}} \approx \frac{\sigma_x}{qBL^2} \cdot \sqrt{\frac{720}{N+4}} \cdot p_{\text{T}} \qquad$ (N points on trajectory)

L is characteristic length of cylindrical drift chamber

Semiconducting Detectors

E_{v} is top of valence band

E_{c} is bottom of conduction band

$E_{\text{g}} \equiv E_{\text{c}} - E_{\text{v}}$ is band gap

$f(E) = \frac{1}{e^{\beta(E-\mu)} + 1} \qquad$ (Fermi-Dirac distribution)

$g_{\text{c}}(E) \approx \frac{1}{2\pi^2} \left(\frac{2m_{\text{c}}^*}{\hbar^2} \right)^{3/2} \sqrt{|E - E_{\text{c}}|}$

$g_{\text{v}}(E) \approx \frac{1}{2\pi^2} \left(\frac{2m_{\text{v}}^*}{\hbar^2} \right)^{3/2} \sqrt{|E - E_{\text{v}}|}$

$n_{\text{c}} = \frac{1}{4} \left(\frac{2m_{\text{c}}^* k_B T}{\pi\hbar^2} \right)^{3/2} e^{-\beta(E_{\text{c}} - \mu)} \equiv N_{\text{c}}(T) e^{-\beta(E_{\text{c}} - \mu)}$

$p_{\text{v}} = \frac{1}{4} \left(\frac{2m_{\text{v}}^* k_B T}{\pi\hbar^2} \right)^{3/2} e^{-\beta(\mu - E_{\text{v}})} \equiv P_{\text{v}}(T) e^{-\beta(\mu - E_{\text{v}})}$

In instrinsic SC: $n_{\text{c}} = p_{\text{v}} \equiv n_{\text{i}} \implies n_{\text{i}}^2 = N_{\text{c}} P_{\text{v}} e^{-\beta E_{\text{g}}}$

$n_{\text{i}} = \frac{1}{4} \left(\frac{2k_B T \sqrt{m_{\text{e}}^* m_{\text{h}}^*}}{\pi\hbar^2} \right)^{3/2} e^{-\frac{\beta E_{\text{g}}}{2}}$

Resistivity, Conductivty, Current Density

Consider conductor of conductivity σ_{E} with number density n of charge carriers q and mobility μ moving at drift velocity v_{d} under external electric field E

$j = \sigma_{\text{E}} E \quad \text{and} \quad j = nq v_{\text{d}}$

$v_{\text{d}} = \mu E$

$\rho_{\text{E}} \equiv \frac{1}{\sigma_{\text{E}}}; \qquad \rho_{\text{E}} = \frac{1}{nq\mu} \qquad \sigma_{\text{E}} = nq\mu$

$j = e_0 n_{\text{i}} (\mu_{\text{e}} + \mu_{\text{h}}) E \qquad$ (in instrinsic SC)

$j_{\text{n}} \approx e_0 N_{\text{d}} \mu_{\text{e}} E, \quad j_{\text{p}} \approx e_0 N_{\text{a}} \mu_{\text{h}} E \qquad$ (in doped SC)

$$\sigma_{\mathrm{n}} \approx e_0 N_{\mathrm{d}}\mu_{\mathrm{e}}, \qquad \sigma_{\mathrm{p}} \approx e_0 N_{\mathrm{a}}\mu_{\mathrm{h}} \qquad \text{(in doped SC)}$$

p-n Junction

Join p- and n-type SCs with dopant densities N_{a} and N_{d}

Depletion region spans $x \in (-x_{\mathrm{p}}, x_{\mathrm{n}})$

$$N_{\mathrm{a}}x_{\mathrm{p}} = N_{\mathrm{d}}x_{\mathrm{n}} \qquad \text{(conservation of charge)}$$

$$\frac{\partial^2 \phi}{\partial x^2} = -\frac{\rho(x)}{\epsilon \epsilon_0} \qquad \text{(Poisson equation for potential)}$$

$$\rho(x) \approx \begin{cases} -e_0 N_{\mathrm{a}} & x \in (-x_{\mathrm{p}}, 0) \\ e_0 N_{\mathrm{d}} & x \in (0, x_{\mathrm{n}}) \end{cases}$$

$$\frac{\mathrm{d}\phi}{\mathrm{d}x} \approx \begin{cases} \frac{e_0 N_{\mathrm{a}}}{\epsilon \epsilon_0}(x + x_{\mathrm{p}}) & x \in (-x_{\mathrm{p}}, 0) \\ -\frac{e_0 N_{\mathrm{d}}}{\epsilon \epsilon_0}(x - x_{\mathrm{n}}) & x \in (0, x_{\mathrm{n}}). \end{cases}$$

$$\phi(-x_{\mathrm{p}}) \equiv 0\,\mathrm{V}, \quad V_0 \equiv \phi(x_{\mathrm{n}}) - \phi(-x_{\mathrm{p}}) = \phi(x_{\mathrm{n}})$$

$$V_0 = \frac{e_0}{2\epsilon \epsilon_0}\left(N_{\mathrm{d}}x_{\mathrm{n}}^2 + N_{\mathrm{a}}x_{\mathrm{p}}^2\right)$$

$$\phi(x) = \begin{cases} \frac{e_0 N_{\mathrm{a}}}{2\epsilon \epsilon_0}(x + x_{\mathrm{p}})^2 & x \in (-x_{\mathrm{p}}, 0) \\ V_0 - \frac{e_0 N_{\mathrm{d}}}{2\epsilon \epsilon_0}(x - x_{\mathrm{n}})^2 & x \in (0, x_{\mathrm{n}}) \end{cases}$$

$$x_{\mathrm{n}}^2 = \frac{2\epsilon \epsilon_0 V_0}{e_0 N_{\mathrm{d}}\left(1 + \frac{N_{\mathrm{d}}}{N_{\mathrm{a}}}\right)} \qquad x_{\mathrm{p}}^2 = \frac{2\epsilon \epsilon_0 V_0}{e_0 N_{\mathrm{d}}\left(1 + \frac{N_{\mathrm{a}}}{N_{\mathrm{d}}}\right)}$$

$$d_{\mathrm{pn}} = x_{\mathrm{n}} + x_{\mathrm{p}} = \sqrt{\frac{2\epsilon \epsilon_0 V_0}{e_0}\frac{N_{\mathrm{a}} + N_{\mathrm{d}}}{N_{\mathrm{a}}N_{\mathrm{d}}}}$$

$$d_{\mathrm{pn}} \approx x_{\mathrm{n}} \approx \sqrt{\frac{2\epsilon \epsilon_0 V_0}{e_0 N_{\mathrm{d}}}} \qquad \text{(if } N_{\mathrm{a}} \gg N_{\mathrm{d}})$$

$$d_{\mathrm{pn}} \approx x_{\mathrm{p}} \approx \sqrt{\frac{2\epsilon \epsilon_0 V_0}{e_0 N_{\mathrm{a}}}} \qquad \text{(if } N_{\mathrm{d}} \gg N_{\mathrm{a}})$$

$$d_{\mathrm{pn}}^{(\mathrm{b})} = d_{\mathrm{pn}}^{(0)}\sqrt{1 + \frac{V_{\mathrm{b}}}{V_0}} \qquad \text{(with reverse bias voltage } V_{\mathrm{b}})$$

Approximate Expressions Depletion Region Width

$$\sigma_{\mathrm{n}} \approx e_0 N_{\mathrm{d}}\mu_{\mathrm{e}}, \ \sigma_{\mathrm{p}} \approx e_0 N_{\mathrm{a}}\mu_{\mathrm{h}} \implies \rho_{\mathrm{n}} \approx \frac{1}{e_0 N_{\mathrm{d}}\mu_{\mathrm{e}}}, \ \rho_{\mathrm{p}} \approx \frac{1}{e_0 N_{\mathrm{a}}\mu_{\mathrm{h}}}$$

$$d_{\mathrm{pn}} \approx \sqrt{2\epsilon \epsilon_0 \rho_{\mathrm{n}}\mu_{\mathrm{e}} V_0} \qquad \text{(if } N_{\mathrm{a}} \gg N_{\mathrm{d}})$$

$$d_{\mathrm{pn}} \approx \sqrt{2\epsilon \epsilon_0 \rho_{\mathrm{p}}\mu_{\mathrm{h}} V_0} \qquad \text{(if } N_{\mathrm{d}} \gg N_{\mathrm{a}})$$

Using $\epsilon_{\mathrm{Si}} \approx 12$ and $\epsilon_{\mathrm{Ge}} \approx 16$ we get...

$$d_{\mathrm{Si}} \approx 0.53\sqrt{\rho_{\mathrm{n}}V_0}\cdot\mathrm{\mu m} \qquad \text{(if } N_{\mathrm{a}} \gg N_{\mathrm{d}})$$

$$d_{\mathrm{Si}} \approx 0.32\sqrt{\rho_{\mathrm{p}}V_0}\cdot\mathrm{\mu m} \qquad \text{(if } N_{\mathrm{d}} \gg N_{\mathrm{a}})$$

$$d_{\mathrm{Ge}} \approx 1.00\sqrt{\rho_{\mathrm{n}}V_0}\cdot\mathrm{\mu m} \qquad \text{(if } N_{\mathrm{a}} \gg N_{\mathrm{d}})$$

$$d_{\mathrm{Ge}} \approx 0.65\sqrt{\rho_{\mathrm{p}}V_0}\cdot\mathrm{\mu m} \qquad \text{(if } N_{\mathrm{d}} \gg N_{\mathrm{a}})$$

... assuming V_0 in volts and ρ in $\Omega\,\mathrm{cm}$

Signal Dynamics in a p-n Semiconducting Detector

Shift coordinate system so that $x_{\mathrm{p}} \equiv 0$

Let x_0 denote initial position of electron-hole pair

$$\tau_{\mathrm{h}} \equiv \frac{\epsilon \epsilon_0}{e_0 \mu_{\mathrm{h}} N_{\mathrm{a}}}, \qquad \tau_{\mathrm{e}} \equiv \frac{\mu_{\mathrm{h}}}{\mu_{\mathrm{e}}}\tau_{\mathrm{h}}, \qquad t_{\mathrm{e}} = \tau_{\mathrm{h}}\frac{\mu_{\mathrm{h}}}{\mu_{\mathrm{e}}}\cdot\ln\frac{d_{\mathrm{pn}}}{x_0}$$

$$Q_{\mathrm{e}}(t) = +\frac{e_0}{d_{\mathrm{pn}}}x_0\left(1 - e^{\frac{\mu_{\mathrm{e}}}{\mu_{\mathrm{h}}}\frac{t}{\tau_{\mathrm{h}}}}\right) \qquad \text{(for } t < t_{\mathrm{e}})$$

$$Q_{\mathrm{h}}(t) = -\frac{e_0}{d_{\mathrm{pn}}}x_0\left(1 - e^{-t/\tau_{\mathrm{h}}}\right)$$

$$I_{\mathrm{e}}(t) = \frac{\mathrm{d}Q_{\mathrm{e}}}{\mathrm{d}t} = -\frac{e_0}{d_{\mathrm{pn}}}\frac{x_0}{\tau_{\mathrm{h}}}\frac{\mu_{\mathrm{e}}}{\mu_{\mathrm{h}}}e^{\frac{\mu_{\mathrm{e}}}{\mu_{\mathrm{h}}}\frac{t}{\tau_{\mathrm{h}}}} \qquad \text{(for } t < t_{\mathrm{e}})$$

$$I_{\mathrm{h}}(t) = \frac{\mathrm{d}Q_{\mathrm{h}}}{\mathrm{d}t} = \frac{e_0}{d_{\mathrm{pn}}}\frac{x_0}{\tau_{\mathrm{h}}}e^{-t/\tau_{\mathrm{h}}}$$

$$I_0^{\mathrm{h}} \equiv \frac{e_0}{d_{\mathrm{pn}}}\frac{x_0}{\tau_{\mathrm{h}}}, \qquad I_0^{\mathrm{e}} \equiv -\frac{e_0}{d_{\mathrm{pn}}}\frac{x_0}{\tau_{\mathrm{e}}}$$

$$U_{\mathrm{e}}(t) = \frac{I_0^{\mathrm{e}}R}{1 + (RC)/\tau_{\mathrm{e}}}\begin{cases} e^{t/\tau_{\mathrm{e}}} - e^{-\frac{t}{RC}} & t < t_{\mathrm{e}} \\ \left(e^{t_{\mathrm{e}}/\tau_{\mathrm{e}}} - e^{-\frac{t_{\mathrm{e}}}{RC}}\right)e^{-\frac{(t-t_{\mathrm{e}})}{RC}} & t > t_{\mathrm{e}}. \end{cases}$$

$$U_{\mathrm{h}}(t) = \frac{I_0^{\mathrm{h}}R}{1 - (RC)/\tau_{\mathrm{h}}}\left(e^{-t/\tau_{\mathrm{h}}} - e^{-\frac{t}{RC}}\right),$$

Limit Cases of Electron Signal

$$U_{\mathrm{e}}(t) \approx I_0^{\mathrm{e}}R\begin{cases} e^{t/\tau_{\mathrm{e}}} - e^{-\frac{t}{RC}} & t < t_{\mathrm{e}} \\ \left(e^{t_{\mathrm{e}}/\tau_{\mathrm{e}}} - e^{-\frac{t_{\mathrm{e}}}{RC}}\right)e^{-\frac{(t-t_{\mathrm{e}})}{RC}} & t > t_{\mathrm{e}} \end{cases} \qquad (RC \ll \tau_{\mathrm{e}})$$

$$U_{\mathrm{e}}(t) = \frac{I_0^{\mathrm{e}}\tau_{\mathrm{e}}}{C}\left(e^{t_{\mathrm{e}}/\tau_{\mathrm{e}}} - 1\right)e^{-\frac{(t-t_{\mathrm{e}})}{RC}} = \frac{Q_{\mathrm{e}}(t_{\mathrm{e}})}{C}e^{-\frac{(t-t_{\mathrm{e}})}{RC}} \qquad (RC \gg \tau_{\mathrm{e}})$$

Position Measurement

Consider parallel silicon microstrips separated by *pitch p*

$$\sigma_x = \frac{p}{\sqrt{12}} \qquad \text{(when using one strip to measure position)}$$

$$\overline{x} = \frac{\sum_i Q_i x_i}{\sum_i Q_i} \qquad \text{(using multiple strips to measure position)}$$

$$\sigma_{\overline{x}}^2 \propto p^2 \frac{\sum_j \sigma_{Q_j}^2}{\left(\sum_i Q_i\right)^2} = p^2 \frac{(\text{noise})^2}{(\text{signal})^2} = \frac{p^2}{\mathrm{SNR}^2}$$

Q_j is charge on j-th strip
$\sigma_{Q_j}^2$ is resolution of charge on j-th strip

Scintillating Detectors

Consider scintillator with time constant τ, emitting $Y \equiv \frac{\mathrm{d}N}{\mathrm{d}E}$ photons per unit absorbed energy and photodetector with efficiency η and multiplication factor M

$$\eta \equiv E_{\mathrm{scint}}/E_{\mathrm{dep}}, \quad E_{\mathrm{scint}} = N_{\mathrm{scint}}h\nu = hc/\lambda \qquad \text{(efficiency)}$$

$$N(t) = N_0 e^{-t/\tau} \qquad \text{(number of scintillation photons)}$$

We assume a fast photodetector, so $I(t)$ follows $N(t)$, i.e.

$$I(t) = I_0 e^{-t/\tau} \qquad \text{(photodetector current)}$$

$$Q = \eta e_0 M Y E_{\mathrm{dep}} \qquad \text{(photodetector charge)}$$

$$Q = \int_0^\infty (t)\,\mathrm{d}t = I_0 \tau \implies I_0 \tau = \eta e_0 M Y E_{\mathrm{dep}}$$

$$U(t) = \frac{I_0 R}{1 - (RC)/\tau}\left(e^{-t/\tau} - e^{-\frac{t}{RC}}\right)$$

$$U(t) \approx \frac{I_0 \tau}{C}e^{-t/RC} = \frac{Q}{C}e^{-t/(RC)} \qquad (RC \gg \tau)$$

$$U(t) \approx R I_0 e^{-t/\tau} = R I(t) \qquad (RC \ll \tau)$$

Fluctuations in Photomultipliers

X is the number of secondary electrons reaching PMT anode as a result of one initial cathode photoelectron

n is the number of initial cathode photoelectrons

S is the sum of all secondary electrons reaching PMT anode

n is Poisson-distributed with mean λ

$$\langle S \rangle = \lambda \langle X \rangle$$

$$\sigma_S^2 = \lambda \langle X^2 \rangle\left(1 + \frac{\sigma_X^2}{\langle X \rangle^2}\right) \equiv F\lambda\langle X^2 \rangle$$

Neutron Detection

In a material with scattering center density n_s and neutron cross section σ... $\lambda = \frac{1}{n_s \sigma}$

In a material of width d with neutron MFP λ, probability for one neutron interaction is... $P = \int_0^d \frac{1}{\lambda} e^{-\frac{x}{\lambda}}\, dx$

Post-Scattering Energy Distribution of Fast Neutrons

Consider fast neutron with initial energy $E \gg k_B T$ scattering from a nucleus with mass number A at angle θ

Assume isotropic scattering $\frac{d\sigma}{d\Omega} = \frac{\sigma_{\text{tot}}}{4\pi}$

$\alpha \equiv \frac{(A-1)^2}{(A+1)^2}$

$\frac{E'}{E} = \frac{A^2 + 2A\cos\theta + 1}{(A+1)^2}$

$E'_{\max} = E, \quad E'_{\min} = \alpha E$ (bounds on E')

$$\frac{dP}{dE'} = \begin{cases} 0 & E' < \alpha E \\ \frac{1}{(1-\alpha)E} & E' \in (\alpha E, E) \\ 0 & E' > E. \end{cases}$$ (distribution of E')

Slowing Neutrons to Thermal Energy

Goal: slow neutron from $E_0 \gg k_B T$ to $E_T \sim k_B T$

$\xi \equiv \left\langle \ln \frac{E_0}{E'} \right\rangle \implies \ln \frac{E'}{E_0} = -\xi \implies E' = E_0 e^{-\xi}$

$\xi = 1 + \frac{\alpha}{1-\alpha} \ln \alpha = 1 + \frac{(A-1)^2}{2A} \ln \frac{A-1}{A+1}$

$\xi \approx \frac{2}{A+2/3}$ (for heavy nuclei, $A \gtrsim 10$)

$\overline{\xi} \equiv \frac{\sum_i \sigma_i \xi_i}{\sum_i \sigma_i}$ (polyatomic materials)

$E'_N = e^{-N\xi} E_0$ (energy after N-th collision)

$N = \frac{1}{\xi} \ln \frac{E_0}{E_T}$ (collisions to reach energy E_T)

Cherenkov Radiation

Consider particle with charge $z = q/e_0$ moving along x axis in material with refractive index n at speed $v > c/n$

$\cos\theta_C = \frac{1}{n\beta} \implies \theta_C = \cos^{-1}\frac{1}{n\beta}$ (Cherenkov angle)

$\beta > 1/n \quad \text{or} \quad pc > \frac{mc^2}{\sqrt{1-(1/n^2)}}$ (thresholds for radiation)

$\frac{d^2 E}{dx\, d\omega} = z^2 \frac{\alpha\hbar\omega}{c} \sin^2\theta_C$

$\frac{d^2 N}{dx\, d\omega} = \frac{z^2\alpha}{c}\sin^2\theta_C = \frac{z^2\alpha}{c}\left(1 - \frac{1}{(n\beta)^2}\right)$

$\frac{d^2 N}{dx\, d\lambda} = \frac{2\pi z^2\alpha}{\lambda^2}\sin^2\theta_C = \frac{2\pi z^2\alpha}{\lambda^2}\left(1 - \frac{1}{(n\beta)^2}\right)$

Cherenkov Detectors

Consider a detector sensitive to radiation in the range $\lambda_{\min}, \lambda_{\max}$ with efficiency $\eta(\lambda)$

$N_{\text{det}} = d \int_{\lambda_{\min}}^{\lambda_{\max}} \eta(\lambda) \frac{d^2 N}{dx\, d\lambda}\, d\lambda$

$N_C \propto \sin^2\theta_C = \left(1 - \frac{1}{(\beta n)^2}\right) \implies N_C \to N_{\max}$ as $\beta \to 1$

$\langle N \rangle = \frac{N_{\max}}{1-1/n^2}\left(1 - \frac{1}{\beta^2 n^2}\right) \equiv a\left(1 - \frac{1}{\beta^2 n^2}\right)$

$\implies \beta = \frac{1}{n\sqrt{1-\left(\langle N\rangle/a\right)}}$