

# RATIONAL INTERPOLATION OF THE EXPONENTIAL FUNCTION

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**ABSTRACT.** Let  $m, n$  be nonnegative integers and  $B^{(m+n)}$  be a set of  $m+n+1$  real interpolation points (not necessarily distinct). Let  $R_{m,n} = P_{m,n}/Q_{m,n}$  be the unique rational function with  $\deg P_{m,n} \leq m$ ,  $\deg Q_{m,n} \leq n$ , that interpolates  $e^x$  in the points of  $B^{(m+n)}$ . If  $m = m_\nu$ ,  $n = n_\nu$  with  $m_\nu + n_\nu \rightarrow \infty$ , and  $m_\nu/n_\nu \rightarrow \lambda$  as  $\nu \rightarrow \infty$ , and the sets  $B^{(m+n)}$  are uniformly bounded, we show that

$$P_{m,n}(z) \rightarrow e^{\lambda z/(1+\lambda)}, \quad Q_{m,n}(z) \rightarrow e^{-z/(1+\lambda)}$$

locally uniformly in the complex plane  $\mathbb{C}$ , where the normalization  $Q_{m,n}(0) = 1$  has been imposed. Moreover, for any compact set  $K \subset \mathbb{C}$  we obtain sharp estimates for the error  $|e^z - R_{m,n}(z)|$  when  $z \in K$ . These results generalize properties of the classical Padé approximants. Our convergence theorems also apply to best (real)  $L_p$  rational approximants to  $e^x$  on a finite real interval.

**1. Introduction.** The study of Padé approximants to the exponential function  $e^z$  was initiated by C. Hermite [8] and continued by his student H. Padé [12] [13] [14]. Given a pair  $(m, n)$  of nonnegative integers, the Padé approximant of type  $(m, n)$  to  $e^z$  is the unique rational function

$$R_{m,n}^0 = P_{m,n}^0/Q_{m,n}^0, \quad \text{with} \quad \deg P_{m,n}^0 = m, \quad \deg Q_{m,n}^0 = n, \quad Q_{m,n}^0(0) = 1,$$

that satisfies

$$(1.1) \quad e^z - R_{m,n}^0(z) = O(z^{m+n+1}) \quad \text{as } z \rightarrow 0;$$

that is,  $R_{m,n}^0$  interpolates  $e^z$  in the origin taken of multiplicity  $m+n+1$ . (The superscript 0 is used to emphasize that the interpolation points are all at  $z = 0$ ). Unlike Padé approximants to most other functions, it is possible to write simple explicit formulas for  $P_{m,n}^0$  and  $Q_{m,n}^0$ :

$$(1.2) \quad P_{m,n}^0(z) = \sum_{j=0}^m \frac{(m+n-j)! m! z^j}{(m+n)! j! (m-j)!}, \quad Q_{m,n}^0(z) = \sum_{j=0}^n \frac{(m+n-j)! n! (-z)^j}{(m+n)! j! (n-j)!}.$$

The properties of these approximants form a classical subject that has application to number theory (Hermite's proof of the transcendence of  $e$  and Lindemann's proof of the transcendence of  $\pi$ ; cf. [21]), the stability of numerical methods for solving differential equations (cf. [9], [23]), and continued fraction representations for the efficient calculation of the exponential.

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Padé proved (cf. also [15]) that any sequence  $\{R_{m,n}^0\}$  for which  $m+n \rightarrow \infty$  converges to  $e^z$  locally uniformly in the complex plane. (In particular, the zeros and poles of such Padé approximants all tend to infinity). Moreover, if  $m = m_\nu$  and  $n = n_\nu$ , where  $m_\nu + n_\nu \rightarrow \infty$  and  $m_\nu/n_\nu \rightarrow \lambda$  as  $\nu \rightarrow \infty$ , then the Padé numerators  $P_{m_\nu, n_\nu}^0$  and Padé denominators  $Q_{m_\nu, n_\nu}^0$  themselves converge:

$$\lim_{\nu \rightarrow \infty} P_{m_\nu, n_\nu}^0(z) = e^{\lambda z/(1+\lambda)}, \quad \lim_{\nu \rightarrow \infty} Q_{m_\nu, n_\nu}^0(z) = e^{-z/(1+\lambda)}.$$

The location of the zeros and poles of the  $R_{m,n}^0$  has also been a subject of substantial interest (cf. [20], [23]) for they display an elegant behavior of relevance to the stability of numerical methods as applied to the test differential equation  $y'(x) = \tau y(x)$ .

It is therefore somewhat surprising that the study of so-called *multi-point Padé approximants* to  $e^z$ , that is, rational functions that interpolate  $e^z$  in some fixed triangular scheme of points, has till now only received little attention. In the case when interpolation points  $\{x_k^{(2n)}\}_{k=0}^{2n}$  all lie in some fixed interval of the real axis of length  $\alpha$  with  $\alpha < 2$ , P. Borwein [4], [5] has obtained estimates for the error

$$e^x - R_{n,n}(x),$$

where  $R_{n,n}(x)$  interpolates  $e^x$  in  $\{x_k^{(2n)}\}_{k=0}^{2n}$ . However, his estimates (which are given only for the diagonal case  $m = n$ ), while sharp up to a multiplicative constant, were only shown to hold for  $x$  in the same interval as the interpolation points and the multiplicative constants involved become unbounded as the length  $\alpha$  of the interval tends to 2.

The purpose of the present work is to analyze the behavior of multi-point Padé approximants to  $e^z$  for any triangular scheme of real interpolation points that belong to some fixed interval (of any finite length). For such points we establish a generalization of the convergence properties of the classical Padé approximants which have all interpolation points at  $z = 0$ . Furthermore, we obtain sharp (up to multiplicative constants) error estimates for these multi-point interpolants which are valid at every point  $z$  in the complex plane  $\mathbb{C}$ . Also, we apply our results to the study of best (real)  $L_p$  rational approximants to  $e^x$  on a finite real interval.

**2. Statements of main results.** We list our main results first, deferring their proofs to the next section.

**THEOREM 2.1.** *Let  $B^{(m+n)} := \{x_k^{(m+n)}\}_{k=0}^{m+n}$ ,  $m = m_\nu$ ,  $n = n_\nu$  be a triangular sequence of (not necessarily distinct) real interpolation points contained in the interval  $[-\rho, \rho]$  such that*

$$(2.1) \quad \lim_{\nu \rightarrow \infty} m_\nu + n_\nu = \infty,$$

*and denote by  $R_{m,n} = P_{m,n}/Q_{m,n}$  the rational function of type  $(m,n)$  that interpolates  $e^z$  in  $B^{(m+n)}$ . Then*

$$\lim_{\nu \rightarrow \infty} R_{m_\nu, n_\nu}(z) = e^z$$

locally uniformly in  $\mathbb{C}$ . If in addition  $(m, n)$  is a ray sequence, that is,

$$(2.2) \quad \lim_{\nu \rightarrow \infty} \frac{m_\nu}{n_\nu} = \lambda \quad (0 \leq \lambda \leq +\infty),$$

we have as  $\nu \rightarrow \infty$

$$(2.3) \quad P_{m_\nu, n_\nu}(z) \rightarrow e^{\lambda z/(1+\lambda)} \quad \text{and} \quad Q_{m_\nu, n_\nu}(z) \rightarrow e^{-z/(1+\lambda)}$$

locally uniformly in  $\mathbb{C}$ , where  $Q_{m_\nu, n_\nu}$  is normalized so that  $Q_{m_\nu, n_\nu}(0) = 1$ .

The above statement tacitly assumes that  $R_{m_\nu, n_\nu}$  has no pole at zero. We shall see in Lemma 2.4 that this eventually becomes true as  $\nu$  increases to infinity. In the case where  $m_\nu > \rho - 1$ , the nonvanishing of  $Q_{m_\nu, n_\nu}$  at zero also follows from Proposition 2.8.

For simplicity, we shall usually omit the subscript  $\nu$  in the sequel, writing  $m$  instead of  $m_\nu$  and  $n$  instead of  $n_\nu$ . The convergence asserted in Theorem 2.1 can be further estimated as follows.

**THEOREM 2.2.** *Let  $B^{(m+n)}$ ,  $m = m_\nu$ ,  $n = n_\nu$  and  $R_{m, n}$  be as in Theorem 2.1, where it is assumed that (2.1) and (2.2) hold. For  $K \subset \mathbb{C}$  a compact set, define*

$$c_0 = \min_{z \in K} |e^z|, \quad c_1 = \max_{z \in K} |e^z|,$$

and put

$$C_0 = e^{-\rho} c_0^{2/(1+\lambda)}, \quad C_1 = e^{\rho} c_1^{2/(1+\lambda)}.$$

Then, for any positive real number  $\alpha < 1$ , there exists a positive integer  $L$  such that the rational interpolants  $R_{m, n}(z)$  to  $e^z$  satisfy for all  $z \in K$

$$(2.4) \quad \alpha C_0 \prod_{k=0}^{m+n} |z - x_k^{(m+n)}| \leq \frac{(m+n)! (m+n+1)!}{m! n!} |e^z - R_{m, n}(z)| \leq \frac{C_1}{\alpha} \prod_{k=0}^{m+n} |z - x_k^{(m+n)}|,$$

as soon as  $m+n \geq L$ .

From Theorem 2.2 we shall deduce absolute error bounds which do not depend on a particular ray sequence, namely:

**THEOREM 2.3.** *For  $K \subset \mathbb{C}$  a compact set, let  $c_0$  and  $c_1$  be as in Theorem 2.2. Define  $m_0$  to be  $c_0^2$  if  $c_0 \leq 1$  and to be 1 otherwise. In a symmetric manner, let  $m_1$  be  $c_1^2$  if  $c_1 \geq 1$  and 1 otherwise. Then, for any positive real number  $\alpha < 1$ , there exists a positive integer  $L$  depending only on  $\rho$ ,  $K$ , and  $\alpha$  such that any rational interpolant  $R_{m, n}(z)$  of type  $(m, n)$  to  $e^z$  in  $m+n+1$  points of  $[-\rho, \rho]$  satisfies for all  $z \in K$*

$$(2.5) \quad \alpha e^{-\rho} m_0 \prod_{k=0}^{m+n} |z - x_k^{(m+n)}| \leq \frac{(m+n)! (m+n+1)!}{m! n!} |e^z - R_{m, n}(z)| \leq \frac{e^{\rho} m_1}{\alpha} \prod_{k=0}^{m+n} |z - x_k^{(m+n)}|,$$

as soon as  $m+n \geq L$ . Inequality (2.5) is sharp, in the sense that it would not hold with a larger constant than  $e^{-\rho} m_0$  on the left nor a smaller constant than  $e^{\rho} m_1$  on the right.

Let us comment about the sharpness of the bounds; if  $K$  consists solely of the complex number  $z = \sigma + it$  and if we choose  $\rho = 0$ , that is, if  $x_k^{(m+n)} = 0$  for all  $k, m$  and  $n$ , we get from (2.4) by letting  $\alpha$  tend to 1:

$$(2.6) \quad |e^z - R_{m,n}(z)| = \frac{m! n!}{(m+n)! (m+n+1)!} |z|^{m+n+1} e^{2\sigma/(1+\lambda)} (1 + o(1)),$$

which is the correct asymptotic for the well-known Padé approximants (see [6], equation (5.5), p. 138).

If, for some real  $\xi$ , we replace  $z$  by  $z - \xi$  in (2.6) and then multiply by  $e^\xi$ , we obtain

$$(2.7) \quad |e^z - e^\xi R_{m,n}(z - \xi)| \sim \frac{m! n!}{(m+n)! (m+n+1)!} |z - \xi|^{m+n+1} e^{2\sigma/(1+\lambda)} e^{\xi(\lambda-1)/(1+\lambda)}.$$

Noting that  $e^\xi R_{m,n}(z - \xi)$  is the Padé approximant to  $e^z$  at  $\xi$ , we see from (2.7) that any estimate of the error which is uniform with respect to the location of the interpolation points over  $[-\rho, \rho]$  must include some exponential factor of  $\rho$  from above and the reciprocal factor from below as soon as  $\lambda \neq 1$ . This shows, in Theorem 2.2, that the term  $e^\rho$  appearing in  $C_0$  and  $C_1$ , though possibly not optimal, is not a pure artefact of our approach. We also notice that the estimates (2.4) cannot be improved upon when  $\lambda = 0$  or  $\lambda = \infty$ , because they coincide then with the upper bound of the Padé estimates (2.7) when  $\xi$  ranges over  $[-\rho, \rho]$ . Since  $m_0$  and  $m_1$ , in Theorem 2.3, are respectively equal to the lower and upper bound of  $c_0^{2/(1+\lambda)}$  and  $c_1^{2/(1+\lambda)}$  when  $\lambda$  ranges over  $[0, \infty]$ , the preceding remark accounts for the sharpness of (2.5).

The proof of Theorem 2.1 relies on the following lemma, which may be of independent interest.

**LEMMA 2.4.** *Let  $s_n(z)$  be a sequence of monic polynomials with  $\deg s_n = n$  and assume that their zeros all lie in some disk  $|z| \leq \rho$ . Let  $q_{m,n}(z)$  be the monic polynomial of degree  $n$  such that*

$$(I + D)^{m+1} q_{m,n} = s_n, \quad m \in \mathbf{N},$$

where  $D$  denotes differentiation. Then the following assertions hold true:

- (i) all the zeros of  $q_{m,n}$  tend to infinity as  $m+n$  tends to infinity;
- (ii) if  $m+n$  is large enough, so that  $q_{m,n}(0) \neq 0$  by (i), the family  $\{q_{m,n}/q_{m,n}(0)\}$  is normal in the complex plane;
- (iii) if  $m = m_\nu$ ,  $n = n_\nu$  satisfy (2.1) and (2.2), then

$$(2.8) \quad \lim_{\nu \rightarrow \infty} q_{m,n}(z)/q_{m,n}(0) = e^{-z/(1+\lambda)}$$

locally uniformly in the complex plane.

Note that the rational interpolant  $P_{m,n}/Q_{m,n}$  considered in Theorem 2.1 is such that the ratio

$$\frac{Q_{m,n}(z)e^z - P_{m,n}(z)}{q_{m+n+1}(z)},$$

is holomorphic in the whole complex plane, where  $q_{m+n+1}$  is defined as

$$q_{m+n+1}(z) = \prod_{k=0}^{m+n} (z - x_k^{(m+n)}).$$

We choose to denote this as follows:

$$(2.9) \quad Q_{m,n}(z)e^z - P_{m,n}(z) = O(q_{m+n+1}(z)).$$

We now perform a normalization which will be important in the sequel. We denote by  $\tilde{P}_{m,n}$  and  $\tilde{Q}_{m,n}$  the polynomials which are obtained upon dividing  $P_{m,n}$  and  $Q_{m,n}$  by the leading coefficient of the latter. Hence,  $\tilde{Q}_{m,n}$  is monic. We still have

$$(2.10) \quad \tilde{Q}_{m,n}(z)e^z - \tilde{P}_{m,n}(z) = O(q_{m+n+1}(z)).$$

Differentiating this equality  $(m+1)$  times and using the fact that the zeros of  $q_{m+n+1}$  are all real, we deduce from Rolle's theorem that there exists a monic polynomial  $\pi_n$  of degree  $n$  whose roots all lie in  $[-\rho, \rho]$  such that

$$e^z(I+D)^{m+1}\tilde{Q}_{m,n}(z) = O(\pi_n(z)).$$

As the polynomials  $\tilde{Q}_{m,n}$  and  $\pi_n$  are both monic and  $\deg \tilde{Q}_{m,n} \leq n$  while  $e^z$  does not vanish, we obtain

$$(2.11) \quad (I+D)^{m+1}\tilde{Q}_{m,n} = \pi_n,$$

which is the equation we met in Lemma 2.4. Note that (2.11) implies that the degree of  $\tilde{Q}_{m,n}$  is equal to  $n$  which is the well-known normality of the exponential function (cf. [16], Section 5.1). We remark also that  $\pi_n(z) = z^n$  corresponds to the case where  $\tilde{Q}_{m,n}$  is the (monic) denominator of the classical Padé approximant of type  $(m, n)$ .

The proof of Theorem 2.2 will require further estimates on the leading and constant coefficients of  $\tilde{P}_{m,n}$  and  $\tilde{Q}_{m,n}$  respectively, which are gathered in the following lemma. To state the lemma, we need to keep track of the interpolation scheme, and we shall use a superscript for this purpose. Hence,  $\tilde{Q}_{m,n}^B$  refers for example to the denominator of the function interpolating  $e^z$  at the points of  $B := B^{(m+n)}$ , while  $\tilde{Q}_{m,n}^{-B}$  refers to the denominator of the function interpolating  $e^z$  at the negatives of the points of  $B$ .

LEMMA 2.5. *Let  $m = m_\nu$ ,  $n = n_\nu$  satisfy (2.1) and (2.2).*

(i) *If  $s_n, q_{m,n}$  are as in Lemma 2.4, then, for any real number  $\eta < 1$ , we have*

$$(2.12) \quad \eta e^{-\rho/(1+\lambda)} \leq (-1)^n \frac{m!}{(m+n)!} q_{m,n}(0)$$

as soon as  $\nu$  is large enough.

(ii) *Let  $\tilde{P}_{m,n}$  and  $\tilde{Q}_{m,n}$  be as in (2.10) and define  $\tilde{p}_{m,n}$  to be the leading coefficient of  $\tilde{P}_{m,n}$ . For any real number  $0 < \eta < 1$ , we have*

$$(2.13) \quad \eta e^{-\rho/(1+\lambda)} \leq (-1)^n \frac{m!}{(m+n)!} \tilde{Q}_{m,n}^B(0) \leq \frac{1}{\eta} e^{\rho/(1+\lambda)}$$

as soon as  $\nu$  is large enough. In the same vein, for any  $0 < \eta' < 1$ , we have

$$(2.14) \quad \eta' \frac{(-1)^n \tilde{Q}_{m,n}^B(0)}{(-1)^m \tilde{Q}_{n,m}^{-B}(0)} \leq (-1)^n \tilde{P}_{m,n}^B \leq \frac{1}{\eta'} \frac{(-1)^n \tilde{Q}_{m,n}^B(0)}{(-1)^m \tilde{Q}_{n,m}^{-B}(0)}$$

and also the absolute estimate

$$(2.15) \quad \eta' e^{-\rho} \leq (-1)^n \frac{m!}{n!} \tilde{P}_{m,n}^B \leq \frac{e^\rho}{\eta'},$$

as soon as  $\nu$  is large enough.

Here again, arguing by contradiction, these inequalities can easily be made independent of a particular ray sequence so that (2.13) would hold uniformly with respect to  $m+n$  and  $|m/n - \lambda|$ , while (2.15) would hold uniformly with respect to  $m+n$ . It can be shown that these estimates are optimal, but we shall not need this. Inequality (2.15) was included for the sake of symmetry, but it is actually (2.14) which is used to prove Theorem 2.2.

We now draw some consequences of the above results. From Theorem 2.1, we will deduce

**COROLLARY 2.6.** *Let  $r_{m,n}^* = r_{m,n}^*([a, b], p) = p_{m,n}^*/q_{m,n}^*$  be a best  $L_p$  real rational approximant of type  $(m, n)$  to  $e^x$  on the finite real interval  $[a, b]$ , with  $1 \leq p \leq \infty$ . Assume that  $m = m_\nu$ ,  $n = n_\nu$  satisfy (2.1). Then, as  $\nu \rightarrow \infty$ , all zeros and poles of  $r_{m,n}^*$  tend to infinity and*

$$r_{m,n}^*(z) \rightarrow e^z$$

locally uniformly in  $\mathbb{C}$ . If, in addition, (2.2) holds, then, normalizing  $q_{m,n}^*$  so that  $q_{m,n}^*(0) = 1$ , we have

$$p_{m,n}^*(z) \rightarrow e^{\lambda z/(1+\lambda)}, \quad q_{m,n}^*(z) \rightarrow e^{-z/(1+\lambda)}$$

locally uniformly in  $\mathbb{C}$ .

The proof will rely on the fact that  $r_{m,n}^*$  interpolates  $e^x$  at  $m+n+1$  points in  $[a, b]$ . As a corollary to Theorem 2.2 we shall be able to describe the asymptotic behavior of these interpolation points. To achieve this, the following definition is needed. If  $Q$  is a polynomial of degree  $n$  with zeros  $z_1, z_2, \dots, z_n$ , the *normalized zero distribution* associated with  $Q$  is defined by

$$\mu(Q) := \frac{1}{n} \sum_{k=1}^n \delta_{z_k},$$

where  $\delta_{z_k}$  denotes the unit point mass at  $z_k$ .

**COROLLARY 2.7.** *Let  $r_{m,n}^*$  be defined as in Corollary 2.6 and let  $q_{m+n+1}^*$  be the monic polynomial whose roots are the interpolation points of  $r_{m,n}^*$  to  $e^x$ . As  $m+n \rightarrow \infty$ , the zero distribution  $\mu(q_{m+n+1}^*)$  tends in the weak-star topology to the equilibrium distribution  $\mu_{[a,b]}$  on  $[a, b]$  whose associated measure is the arcsine measure  $d\mu_{[a,b]} = (1/\pi) dx / \sqrt{(x-a)(b-x)}$ .*

From the proof of (i) of Lemma 2.4, we also obtain generalizations of known results about the location of zeros and poles of Padé approximants to the exponential function (cf. [18]). In particular, we mention the following result.

**PROPOSITION 2.8.** *With the notation and assumptions of Lemma 2.4, all the zeros of  $q_{m,n}$  lie on or outside the circle  $|z| = m + 1 - \rho$ .*

*With the notation and assumptions of Theorem 2.1, if  $n \geq 2$  and  $m \geq n + \rho - 2$ , then all the poles of  $R_{m,n}$  lie in the right half-plane. Hence the coefficients of  $Q_{m,n}$  have alternating signs in this case.*

*In particular, if  $\rho \leq 2$ , then all the poles of  $R_{n,n}$  lie in the right half-plane and all its zeros lie in the left half-plane for  $n = 1, 2, \dots$  so the coefficients of  $Q_{n,n}$  have alternating signs, while those of  $P_{n,n}$  have constant sign.*

*Furthermore, if  $\rho \leq 1$ , then all the poles of  $R_{n-1,n}$  lie in the right half-plane and all its zeros lie in the half-plane  $\operatorname{Re}(z) < -2$ . Consequently, the coefficients of  $Q_{n-1,n}$  have alternating signs, while those of  $P_{n-1,n}$  have constant sign.*

Note that it follows from the first assertion of this proposition that all the zeros and poles of  $R_{n,n}$  lie on or outside the circle  $|z| = n + 1 - \rho$  when  $n \geq \rho - 1$ , and all the zeros and poles of  $R_{n-1,n}$  lie on or outside the circle  $|z| = n - \rho$  when  $n \geq \rho$ . Here, the assertion on the zeros is of course obtained by symmetry, upon changing  $z$  into  $-z$ .

Finally, the convergence result of Theorem 2.1 can be slightly extended by adjoining to the exponential function a rational factor:

**THEOREM 2.9.** *Let  $\{\alpha_j\}_{j=1}^k, \{\beta_j\}_{j=1}^l$  be two sets of nonzero complex numbers and let  $B^{(m+n)}$ ,  $m = m_\nu$ ,  $n = n_\nu$  be as in Theorem 2.1 with the additional requirement that the  $B^{(m+n)}$  do not contain any point in  $\{1/\beta_j\}_{j=1}^l$ . If (2.1) holds and  $m \geq k$ ,  $n \geq l$ , then for  $\nu$  large enough, the rational function  $R_{m,n} = P_{m,n}/Q_{m,n}$  of type  $(m, n)$  that interpolates the function  $e^z \prod_{j=1}^k (1 + \alpha_j z) / \prod_{j=1}^l (1 - \beta_j z)$  in  $B^{(m+n)}$  exists and is normal, that is,  $\deg P_{m,n} = m$  and  $\deg Q_{m,n} = n$ . Moreover,*

$$(2.16) \quad \lim_{\nu \rightarrow \infty} R_{m_\nu, n_\nu}(z) = e^z \frac{\prod_{j=1}^k (1 + \alpha_j z)}{\prod_{j=1}^l (1 - \beta_j z)}$$

*locally uniformly in  $\mathbb{C} \setminus \cup_{j=1}^l \{1/\beta_j\}$ . If, in addition,  $m$  and  $n$  satisfy (2.2), then, as  $\nu \rightarrow \infty$ ,*

$$(2.17) \quad P_{m_\nu, n_\nu}(z) \rightarrow e^{\lambda z/(1+\lambda)} \prod_{j=1}^k (1 + \alpha_j z), \quad Q_{m_\nu, n_\nu}(z) \rightarrow e^{-z/(1+\lambda)} \prod_{j=1}^l (1 - \beta_j z)$$

*locally uniformly in  $\mathbb{C}$ , where  $Q_{m,n}$  is normalized so that  $Q_{m,n}(0) = 1$ .*

The proof of Theorem 2.9 is but an easy extension to the multipoint case of a classical property in Padé approximation, namely the separate convergence to some entire functions of the numerators and the denominators of a sequence of Padé approximants is preserved under multiplication by a rational function (cf. [2], Lemma 18.3). A famous result of Arms and Edrei [1] asserts that for ray sequences of Padé approximants, that

is, when  $\rho = 0$ , Theorem 2.9 still holds if the rational factor is replaced by an infinite product, provided the  $\alpha_j$ 's and  $\beta_j$ 's are positive. At present, it is not known whether this holds true for a nonzero  $\rho$ .

**3. Proofs.** We first recall a classical theorem of Padé already mentioned in the introduction (*cf.* [15], equations (11)–(12), Satz 7, Section 75, pp. 434–436):

**THEOREM (PADÉ).** *Let  $R_{m,n}^0 = P_{m,n}^0/Q_{m,n}^0$  denote the Padé approximant of type  $(m, n)$  to  $e^z$ , normalized so that  $Q_{m,n}^0(0) = 1$ . Then, as  $m + n$  tends to infinity,*

$$R_{m,n}^0(z) \rightarrow e^z$$

*locally uniformly in  $\mathbb{C}$ . Moreover, if  $m/n$  tends to a limit  $\lambda$ , with  $0 \leq \lambda \leq \infty$ , then*

$$P_{m,n}^0(z) \rightarrow e^{\lambda z/(1+\lambda)} \quad \text{and} \quad Q_{m,n}^0(z) \rightarrow e^{-z/(1+\lambda)}$$

*locally uniformly in  $\mathbb{C}$ .*

In the above statement, it is understood that the approximation error  $|R_{m,n}^0(z) - e^z|$  can be made uniformly small on any given compact set, no matter what  $m$  and  $n$  are provided  $m + n$  is large enough. We shall also appeal to two other results that are particularly suited for our purposes. The first is a theorem of Szegő (*cf.* [11], Theorem 16.1, p. 65):

**THEOREM (SZEGŐ).** *Let*

$$f(z) = \sum_{k=0}^n \binom{n}{k} A_k z^k, \quad g(z) = \sum_{k=0}^n \binom{n}{k} B_k z^k$$

*and*

$$h(z) = \sum_{k=0}^n \binom{n}{k} A_k B_k z^k.$$

*If all the zeros of  $f(z)$  lie in a circular region  $A$ , then every zero  $\gamma$  of  $h(z)$  has the form  $\gamma = -\alpha\beta$ , where  $\alpha$  is a suitably chosen point in  $A$  and  $\beta$  is a zero of  $g(z)$ .*

The second result that we need is Walsh's theorem (*cf.* [11], Theorem 18.1, p. 81):

**THEOREM (WALSH).** *Let*

$$f(z) = \sum_{j=0}^n a_j z^j, \quad g(z) = \sum_{j=0}^n b_j z^j = b_n \prod_{j=1}^n (z - \beta_j),$$

*and*

$$h(z) = \sum_{j=0}^n (n-j)! b_{n-j} f^{(j)}(z).$$

*If all the zeros of  $f(z)$  lie in a circular region  $A$ , then all the zeros of  $h(z)$  lie in the point set  $C$  consisting of  $n$  circular regions obtained by translating  $A$  in the amount and direction of the vectors  $\beta_j$ .*

As to the overall organization, we shall begin with the proof of Lemma 2.4 and deduce from it Theorem 2.1. We then proceed with Lemma 2.5 in order to establish Theorem 2.2, from which Theorem 2.3 will follow. The remaining results are proved in their order of appearance.



PROOF OF LEMMA 2.4. The polynomial  $q_{m,n}$  is given by

$$q_{m,n} = (I + D)^{-(m+1)} s_n.$$

Writing the Taylor expansion

$$(1+x)^{-(m+1)} = \sum_{j=0}^{\infty} (-1)^j \binom{m+j}{m} x^j,$$

we get

$$(3.1) \quad q_{m,n} = \sum_{j=0}^n (-1)^j \binom{m+j}{m} s_n^{(j)},$$

where we have used the fact that  $s_n$  is a polynomial of degree  $n$ . Thus, if  $q_{m,n}^0$  denotes the monic Padé denominator:

$$(I + D)^{m+1} q_{m,n}^0 = z^n,$$

we have

$$(3.2) \quad q_{m,n}^0(z) = \sum_{j=0}^n (-1)^j \binom{m+j}{m} \frac{n!}{(n-j)!} z^{n-j}.$$

Let us apply Walsh's theorem with  $f(z) = s_n(z)$  and  $h(z) = q_{m,n}(z)$ . By (3.1)

$$h(z) = \sum_{j=0}^n (-1)^j \binom{m+j}{m} s_n^{(j)}(z) = \sum_{j=0}^n (n-j)! b_{n-j} s_n^{(j)}(z),$$

where

$$b_{n-j} := \frac{(-1)^j}{(n-j)!} \binom{m+j}{m}.$$

We must then take in the theorem (cf. (3.2))

$$g(z) = \sum_{j=0}^n b_{n-j} z^{n-j} = \frac{1}{n!} q_{m,n}^0(z).$$

From Padé's theorem, we know that the zeros of  $q_{m,n}^0$  tend to infinity with  $m+n$  since they are the poles of  $R_{m,n}^0$ . But as soon as the zeros of  $q_{m,n}^0$  have modulus greater than, say,  $d$ , then all the zeros of  $q_{m,n}$  have modulus greater than  $d - \rho$  by Walsh's theorem. This proves (i).

To establish (ii), we let

$$s_n(z) = \sum_{k=0}^n \alpha_k^{(n)} z^k, \quad \alpha_n^{(n)} = 1,$$

so the fact that the moduli of the zeros of  $s_n$  are not greater than  $\rho$  implies

$$(3.3) \quad |\alpha_k^{(n)}| \leq \binom{n}{k} \rho^{n-k}.$$

From (3.1), we get

$$(3.4) \quad q_{m,n}(z) = \sum_{k=0}^n \alpha_k^{(n)} \sum_{j=0}^k (-1)^j \binom{m+j}{m} \frac{k!}{(k-j)!} z^{k-j},$$

or equivalently

$$(3.5) \quad q_{m,n}(z) = \sum_{i=0}^n \left( \sum_{k=i}^n (-1)^{k-i} \binom{m+k-i}{m} \alpha_k^{(n)} k! \right) \frac{z^i}{i!}.$$

We first normalize this polynomial by the constant term of  $q_{m,n}^0$  which is (cf. (3.2))

$$(-1)^n \binom{m+n}{m} n! = (-1)^n \frac{(m+n)!}{m!};$$

thus, we set

$$(3.6) \quad \hat{q}_{m,n} := (-1)^n \frac{m!}{(m+n)!} q_{m,n}.$$

Assuming that the variable  $z$  lies in some closed disk  $D(0, M)$  of radius  $M$  and center zero and using (3.3), we get from (3.4)

$$\begin{aligned} |\hat{q}_{m,n}(z)| &\leq \frac{m!}{(n+m)!} \sum_{k=0}^n \sum_{j=0}^k \frac{n! \rho^{n-k}}{k! (n-k)!} \frac{(m+j)! k!}{m! j! (k-j)!} M^{k-j} \\ &= \sum_{k=0}^n \sum_{j=0}^k \frac{n! (j+m)!}{(n-k)! j! (k-j)!} \frac{\rho^{n-k} M^{k-j}}{(n+m)!} \\ &= \sum_{k=0}^n \frac{\rho^{n-k}}{(n-k)!} \frac{n!}{(n+m)!} \sum_{l=0}^k \frac{(m+k-l)! M^l}{(k-l)! l!} \\ &\leq \sum_{k=0}^n \frac{1}{(n-k)!} \left( \frac{n\rho}{n+m} \right)^{n-k} \sum_{l=0}^k \frac{(m+k-l) \cdots (k-l+1) M^l}{(k+m) \cdots (k+1) l!}. \end{aligned}$$

As

$$\frac{(m+k-l) \cdots (k-l+1)}{(m+k) \cdots (k+1)} = \prod_{s=1}^m \frac{(k-l+s)}{(k+s)} \leq 1, \quad 0 \leq l \leq k,$$

we have

$$(3.7) \quad |\hat{q}_{m,n}(z)| \leq \sum_{k=0}^n \frac{1}{(n-k)!} \left( \frac{n\rho}{n+m} \right)^{n-k} \sum_{l=0}^k \frac{M^l}{l!} \leq e^M e^{\frac{n\rho}{n+m}}.$$

Hence  $|\hat{q}_{m,n}(z)| \leq e^{M+\rho}$ , so the family  $\{\hat{q}_{m,n}\}$  is normal in the complex plane.

From (3.7), we see that the modulus of the constant term of  $\hat{q}_{m,n}$  is bounded from above by  $e^\rho$ . We now prove that it is also bounded away from zero, provided  $m+n$  is large enough.

In view of Padé's theorem, we may assume that the zeros of  $q_{m,n}^0$  have modulus greater than, say,  $2\rho$  as soon as  $m+n > L$ . We now compute  $\hat{q}_{m,n}(0)$  which we shall for simplicity rename as  $\tau_{m,n}$ . From (3.5) and (3.6) we have

$$(3.8) \quad \tau_{m,n} := (-1)^n \frac{m!}{(m+n)!} \sum_{k=0}^n (-1)^k \alpha_k^{(n)} \binom{m+k}{m} k! = 1 + \frac{(-1)^n}{(m+n)!} \sum_{k=0}^{n-1} (-1)^k \alpha_k^{(n)} (m+k)!.$$

Let us define

(3.9)

$$h_{m,n}(z) := 1 + \frac{(-1)^n}{(m+n)!} \sum_{k=0}^{n-1} (-1)^k \alpha_k^{(n)} (m+k)! z^{n-k} = 1 + \sum_{j=1}^n \alpha_{n-j}^{(n)} (-1)^j \frac{(m+n-j)!}{(m+n)!} z^j,$$

so that  $\tau_{m,n} = h_{m,n}(1)$ . The family  $\{h_{m,n}\}$  is uniformly bounded on any compact set in  $\mathbb{C}$ , for if  $|z| \leq M$  we have, thanks to (3.3):

$$|h_{m,n}(z)| \leq 1 + \sum_{j=1}^n \frac{n! \rho^j}{(n-j)! j!} \frac{(m+n-j)!}{(m+n)!} M^j = 1 + \sum_{j=1}^n \frac{(\rho M)^j}{j!} \frac{(n+m-j) \cdots (n-j+1)}{(n+m) \cdots (n+1)},$$

so that  $|h_{m,n}(z)| \leq e^{\rho M}$ . Hence  $\{h_{m,n}\}$  is normal in the complex plane. Let  $H$  be any limit function of this family. We claim that  $H$  is never zero on the closed unit disk. To see this, introduce the reciprocal polynomial of  $s_n$ :

$$(3.10) \quad f_n(z) := \sum_{j=0}^n \alpha_{n-j}^{(n)} z^j = \sum_{j=0}^n \binom{n}{j} \frac{\alpha_{n-j}^{(n)}}{\binom{n}{j}} z^j,$$

whose zeros all lie in  $|z| \geq 1/\rho$ . Next set (cf. (3.2))

$$(3.11) \quad g_{m,n}(z) := \sum_{j=0}^n \binom{n}{j} (-1)^j \frac{(m+n-j)!}{(m+n)!} z^j = (-1)^n \frac{m!}{(m+n)!} q_{m,n}^0(z).$$

When  $m+n > L$ , we know that the zeros of  $q_{m,n}^0$  lie in  $|z| > 2\rho$ . Applying Szegő's theorem to  $f_n$  and  $g_{m,n}$ , we conclude that the zeros of  $h_{m,n}$  all lie in  $|z| > 2$ . By a classical theorem of Hurwitz, the limit function  $H(z)$  is either identically zero or never vanishes in  $\{|z| < 2\}$ . But  $H(0) = 1$  since  $h_{m,n}(0) = 1$  for all  $m, n$ . This proves the claim.

Consider now the doubly-indexed sequence  $\{\tau_{m,n}\}_{m+n>L}$ . If it were not bounded away from zero, there would be a subsequence  $\tau_{m_k, n_k} = h_{m_k, n_k}(1)$  that tends to 0 when  $k$  tends to infinity. From the sequence  $h_{m_k, n_k}(z)$ , we could extract a subsequence converging locally uniformly on  $\mathbb{C}$  to some function  $H$ . But then  $H(1) = 0$ , contradicting the previous claim. Hence, there exists a positive constant  $C$  such that

$$(3.12) \quad |\hat{q}_{m,n}(0)| = \frac{m!}{(m+n)!} |q_{m,n}(0)| > C > 0 \quad \text{for } m+n > L.$$

From this, we deduce that the family  $\{\hat{q}_{m,n}/\hat{q}_{m,n}(0)\}_{m+n>L} = \{q_{m,n}/q_{m,n}(0)\}_{m+n>L}$  is in turn normal. Thus (ii) is proved.

We now turn to the proof of (iii), where we assume that  $m = m_\nu$ ,  $n = n_\nu$  satisfy (2.1) and (2.2). For brevity, we set

$$w_{m,n}(z) := \frac{q_{m,n}(z)}{q_{m,n}(0)} \quad \text{for } m+n > L, \text{ and } u := \frac{1}{1+\lambda}.$$

We know from the preceding discussion that the family  $w_{m,n}$  is well-defined and normal, and we are going to prove that it converges to  $e^{-z/(1+\lambda)}$  by showing the latter to be the only possible limit function. Extracting a subsequence if necessary, we may as well assume that  $w_{m,n}$  converges locally uniformly in  $\mathbf{C}$  to some entire function

$$g(z) = \sum_{s=0}^{\infty} A_s z^s.$$

Since  $A_0 = 1$  by construction, the relation  $g(z) \equiv e^{-uz}$  is equivalent to the following identity:

$$(s+1)A_{s+1} + uA_s = 0, \quad s \geq 0.$$

As  $A_s$  is given by the Cauchy integral formula, it is equal to the limit of the  $s$ -th Taylor coefficient at zero of  $w_{m,n}$  as  $\nu$  tends to infinity. Thus, writing

$$w_{m,n}(z) = \sum_{k=0}^n (w_{m,n})_k z^k,$$

it suffices to show that  $u(w_{m,n})_s + (s+1)(w_{m,n})_{s+1}$  tends to zero, for each fixed  $s$ , as  $\nu$  tends to infinity. Finally, since  $(w_{m,n})_s = (q_{m,n})_s / q_{m,n}(0)$ , it is enough to prove (cf. (3.12)) that

$$X_{m,n} := \frac{m!}{(n+m)!} [u(q_{m,n})_s + (s+1)(q_{m,n})_{s+1}]$$

tends to zero for fixed  $s$  as  $\nu$  goes to infinity. Now, we get from (3.4)

$$\begin{aligned} X_{m,n} &= \frac{m!}{(n+m)!} \left[ u\alpha_s^{(n)} + u \sum_{k=s+1}^n \alpha_k^{(n)} (-1)^{k-s} \binom{k-s+m}{m} \frac{k!}{s!} \right. \\ &\quad \left. + (s+1) \sum_{k=s+1}^n \alpha_k^{(n)} (-1)^{k-s-1} \binom{k-s-1+m}{m} \frac{k!}{(s+1)!} \right] \\ &= \frac{m!}{(n+m)!} \left[ u\alpha_s^{(n)} + \sum_{k=s+1}^n \alpha_k^{(n)} (-1)^{k-s} \frac{(k-s-1+m)! k!}{(k-s-1)! s! m!} \frac{(u-1)(k-s)+um}{k-s} \right]. \end{aligned}$$

Using (3.3), we have

$$\begin{aligned} |X_{m,n}| &\leq \frac{m!}{(n+m)!} \frac{n! \rho^{n-s}}{s! (n-s)!} |u| \\ &\quad + \frac{1}{(n+m)!} \sum_{k=s+1}^n \frac{n! \rho^{n-k}}{(n-k)!} \frac{(k-s-1+m)!}{(k-s-1)! s!} \frac{|(u-1)(k-s)+um|}{(k-s)}. \end{aligned}$$

The first term in the right-hand side of this inequality is

$$\frac{1}{s!} \frac{n(n-1) \cdots 1}{(m+n) \cdots (m+1)} \frac{|u| \rho^{n-s}}{(n-s)!} \leq \frac{|u| \rho^{n-s}}{s! (n-s)!},$$

which tends to zero if  $n$  tends to  $\infty$ . If, however,  $n$  remains bounded, then necessarily  $\lambda = \infty$  so that  $u = 0$  and this term is identically zero.

As for the second term, we treat the cases  $\lambda \neq +\infty$  and  $\lambda = +\infty$  separately.

THE CASE  $\lambda \neq +\infty$ . The second term may be rewritten as

$$(3.13) \quad \sum_{k=s+1}^n \frac{\rho^{n-k}}{(n-k)! s!} \frac{|(u-1)(k-s)+um|}{(k-s)} \frac{(k-s-1+m) \cdots (k-s)}{(n+m) \cdots (n+1)}.$$

The last fraction in this expression is equal to

$$\prod_{r=1}^m \frac{(k-s+r-1)}{(n+r)} \leq \prod_{r=1}^m \frac{(n+r-1)}{(n+r)} < 1$$

so that (3.13) is less than

$$\sum_{k=s+1}^n \frac{\rho^{n-k}}{(n-k)! s!} \frac{|(u-1)(k-s)+um|}{(k-s)}.$$

Choose  $N \leq n-s-1$  and decompose this expression as two sums:

$$\frac{1}{s!} \left[ \sum_{k=s+1}^{n-N} + \sum_{k=n-N+1}^n \right].$$

The integer  $s$  still being fixed, we show that the bracketed expression tends to zero. Setting  $p = n-k$  we write it as

$$(3.14) \quad \sum_{p=N}^{n-s-1} \frac{\rho^p}{p!} \frac{|(u-1)(n-p-s)+um|}{(n-p-s)} + \sum_{p=0}^{N-1} \frac{\rho^p}{p!} \frac{|(u-1)(n-p-s)+um|}{(n-p-s)}.$$

Since for  $p < n-s$

$$\frac{1}{n-p-s} = \frac{n}{n} \frac{1}{n-p-s} = \frac{1}{n} \left[ 1 + \frac{p+s}{n-p-s} \right] \leq \frac{1}{n} [1+p+s],$$

the first summand is less than

$$\sum_{p=N}^{n-s-1} (1+p+s) \frac{\rho^p}{p!} \frac{|(u-1)(n-p-s)+um|}{n} \leq \sum_{p=N}^{n-s-1} (1+p+s) \frac{\rho^p}{p!} \left( |u-1| + |u| \frac{m}{n} \right).$$

Since  $|u-1| + |u|m/n$  is bounded, this last sum is the tail of some convergent series so we can make it less than any  $\epsilon > 0$  by choosing  $N$  large enough, which is possible since  $\lambda < \infty$  implies  $n \rightarrow \infty$ . The integer  $N$  now being fixed, letting  $\nu$  tend to infinity, the second sum in (3.14) tends to

$$\sum_{p=0}^{N-1} \frac{\rho^p}{p!} |(u-1) + u\lambda|$$

which is equal to zero by definition of  $u$ .

THE CASE  $\lambda = +\infty$ . In this case,  $u$  equals zero. Thus, we have

$$|X_{m,n}| \leq \sum_{k=s+1}^n \frac{\rho^{n-k}}{(n-k)! s!} \frac{(k-s-1+m) \cdots (k-s)}{(n+m) \cdots (n+1)}.$$

If  $s \geq n$ , this already yields  $X_{m,n} = 0$ . Otherwise, the last fraction is not greater than

$$\frac{(n+m-s-1) \cdots (n-s)}{(n+m) \cdots (n+1)} = \frac{n \cdots (n-s)}{(n+m) \cdots (n+m-s)}$$

which tends to zero as  $\nu$  tends to  $+\infty$  for each factor  $(n-k)/(n+m-k)$  is less than 1 and tends to zero since  $m/n$  goes to  $+\infty$ . As the sum

$$\sum_{k=s+1}^n \frac{\rho^{n-k}}{(n-k)! s!}$$

is bounded,  $X_{m,n}$  itself tends to zero. We have thus obtained (2.8).  $\blacksquare$

PROOF OF THEOREM 2.1. We shall first prove the convergence of  $P_{m,n}$  and  $Q_{m,n}$  when  $(m, n)$  is a ray sequence and then deduce from this the convergence of  $R_{m,n}$  whenever  $m+n \rightarrow \infty$ . Therefore, we begin by assuming that (2.2) holds.

Applying Lemma 2.4 to (2.11), we deduce that  $\tilde{Q}_{m,n}(0)$  is nonzero as soon as  $m+n$  is large enough and moreover that

$$\lim_{\nu \rightarrow \infty} \tilde{Q}_{m,n}(z) / \tilde{Q}_{m,n}(0) = e^{-z/(1+\lambda)}$$

locally uniformly in  $\mathbb{C}$ . From the normalization of  $Q_{m,n}$ , we have  $\tilde{Q}_{m,n} / \tilde{Q}_{m,n}(0) = Q_{m,n}$ , and thus

$$(3.15) \quad \lim_{\nu \rightarrow \infty} Q_{m,n}(z) = e^{-z/(1+\lambda)},$$

which gives the desired limit for  $Q_{m,n}$ . From the defining equation

$$(3.16) \quad Q_{m,n}(z)e^z - P_{m,n}(z) = O(q_{m+n+1}(z)),$$

we see upon dividing by  $e^z$  and changing  $z$  into  $-z$  that

$$(3.17) \quad P_{m,n}(-z)e^z - Q_{m,n}(-z) = O(q_{m+n+1}(-z)).$$

We then deduce again from Lemma 2.4 that

$$(3.18) \quad \lim_{\nu \rightarrow \infty} P_{m,n}(z) / P_{m,n}(0) = e^{z/(1+1/\lambda)} = e^{\lambda z/(1+\lambda)}$$

locally uniformly in  $\mathbb{C}$ , where it should be observed that  $P_{m,n}(0) \neq 0$  as soon as  $\nu$  is large enough. It remains to show that  $P_{m,n}(0)$  tends to 1. Suppose this is not the case. Then there exists an  $\epsilon > 0$  and a subsequence  $(m', n')$  such that

$$(3.19) \quad |P_{m',n'}(0) - 1| \geq \epsilon.$$

Extracting a subsequence from  $(m', n')$  if necessary, we may assume that some sequence  $y_{m', n'}$  of interpolation points converges to a limit point  $y$  in  $[-\rho, \rho]$ . Equation (3.16) evaluated at  $y_{m', n'}$  leads to

$$Q_{m', n'}(y_{m', n'})e^{y_{m', n'}} = P_{m', n'}(y_{m', n'}).$$

In the limit, we get from (3.15) and (3.18) that

$$e^{\lambda y/(1+\lambda)} = \lim_{\nu \rightarrow \infty} P_{m', n'}(0) \cdot e^{\lambda y/(1+\lambda)}$$

which contradicts (3.19). This establishes (2.3), from which the local uniform convergence of  $R_{m, n}$  to  $e^z$ , when (2.2) holds, is immediate.

Finally, if (2.2) does not hold, the convergence of the rational interpolants  $R_{m, n}$  to  $e^z$  locally uniformly in  $\mathbb{C}$  remains true because if not, by extracting a subsequence  $R_{m', n'}$  such that  $m'/n'$  tends to some number in  $[0, +\infty]$  (i.e.  $(m', n')$  is a ray sequence), we contradict the first part of the proof. ■

PROOF OF LEMMA 2.5. We first establish (2.12) which by (3.6) is equivalent to

$$\hat{q}_{m, n}(0) \geq \eta e^{-\rho/(1+\lambda)}.$$

Recall from the definitions (3.8) and (3.9) that

$$\hat{q}_{m, n}(0) = \tau_{m, n} = h_{m, n}(1).$$

We have already proved, using Szegő's theorem as applied to the families  $f_n$  and  $g_{m, n}$  defined in (3.10) and (3.11) that  $h_{m, n}$  will have no zeros in  $[-1, 1]$  as soon as  $m + n > L$ . In this case,  $\tau_{m, n}$  will be positive. Now, let us replace the family  $\{h_{m, n}(z)\}$  above by the family  $\{h_{m, n}(z) - \epsilon\}$  where  $\epsilon$  is some positive real number less than 1. As the constant term of  $f_n$  is 1, Szegő's theorem can be applied as before to the families  $\{f_n\}$  and  $\{g_{m, n} - \epsilon\}$ . Since  $g_{m, n}(z)$  is just the Padé denominator normalized with unit constant term, we know from Padé's theorem that  $g_{m, n}$  tends to  $\exp(-z/(1+\lambda))$ . Therefore, we know that  $g_{m, n} - \epsilon$  tends to  $\exp(-z/(1+\lambda)) - \epsilon$ .

If  $\lambda \neq \infty$ , the set of zeros of this last function is

$$\{z = -(1+\lambda)\log \epsilon + (1+\lambda)2ik\pi, k \in \mathbb{Z}\}.$$

Thus, for any positive real number  $\alpha$ , if  $\nu$  is large enough, the polynomial  $g_{m, n} - \epsilon$  will have no zeros in  $|z| \leq -(1+\lambda)(\log \epsilon + \alpha)$  and, by Szegő's theorem,  $h_{m, n} - \epsilon$  will have no zeros in  $|z| \leq -(1+\lambda)(\log \epsilon + \alpha)/\rho$ . But  $\tau_{m, n} - \epsilon = h_{m, n}(1) - \epsilon$  on one hand and  $h_{m, n}(0) - \epsilon = 1 - \epsilon > 0$  on the other hand. Hence  $\tau_{m, n} - \epsilon = \hat{q}_{m, n}(0) - \epsilon$  will be positive if  $h_{m, n} - \epsilon$  has no zero in  $|z| \leq 1$ . This will be the case as soon as  $\nu$  is large enough, provided

$$1 \leq -(1+\lambda)(\log \epsilon + \alpha)/\rho$$

which is easily seen to be satisfied iff

$$\epsilon \leq \eta e^{-\rho/(1+\lambda)}$$

where  $\eta = e^{-\alpha}$ . This gives (2.12) for the case  $\lambda \neq \infty$ .

If  $\lambda = \infty$ ,  $g_{m,n} - \epsilon$  tends to  $1 - \epsilon$ . Therefore, this polynomial will have no zeros in any given compact set of the complex plane as soon as  $\nu$  is large enough, and Szegő's theorem shows that this property is also satisfied by the polynomial  $h_{m,n} - \epsilon$ . Thus, since  $\epsilon < 1$ ,  $\tau_{m,n} - \epsilon$  will be positive for large  $\nu$ . This gives (2.12) for the case  $\lambda = \infty$  also, and the proof of (i) is complete.

That the left inequality in (2.13) holds for large  $\nu$  follows from (i) above and (2.11); that the right inequality also holds for large  $\nu$  is a consequence of (3.7) where we set  $M = 0$ , recalling that  $n/(n+m) \rightarrow 1/(1+\lambda)$ . From (3.17), we see that

$$\frac{\tilde{P}_{m,n}^B(-z)}{(-1)^m \tilde{P}_{m,n}^B} = \tilde{Q}_{n,m}^{-B}(z);$$

thus

$$(3.20) \quad (-1)^m \tilde{P}_{m,n}^B = \frac{\tilde{P}_{m,n}^B(0)}{\tilde{Q}_{n,m}^{-B}(0)}$$

as soon as  $\tilde{Q}_{n,m}^{-B}(0) \neq 0$ , that is, as soon as  $\nu$  is large enough by Theorem 2.1. For any  $0 < \eta' < 1$ , by the same theorem, we have for sufficiently large  $\nu$

$$(3.21) \quad \eta' \leq \frac{\tilde{P}_{m,n}^B(0)}{\tilde{Q}_{n,m}^{-B}(0)} \leq \frac{1}{\eta'}.$$

From (2.13), we see that  $(-1)^n \tilde{Q}_{m,n}^B(0)$  and  $(-1)^m \tilde{Q}_{n,m}^{-B}(0)$  are positive for large  $\nu$ . Upon multiplying (3.21) by the positive quantity  $(-1)^{m+n} \tilde{Q}_{m,n}^B(0)/\tilde{Q}_{n,m}^{-B}(0)$  and in view of (3.20), we get (2.14). Finally, multiplying the latter by  $m!/n!$  and using (2.13) yields (2.15). This completes the proof of (ii). ■

**PROOF OF THEOREM 2.2.** For the sake of simplicity, as a first step, we shall treat the diagonal case and write  $R_l = P_l/Q_l$  for the multipoint interpolant of type  $(l, l)$ . As several interpolation schemes enter into the proof, we shall keep track of them by using a superscript as in Lemma 2.5. Following P. B. Borwein (cf. [5]), we consider for each positive  $n$  the triangular interpolation scheme  $C_n$  whose  $(2n+2k)$ -th row is obtained by adding to the set  $B^{(2n)} = \{x_i^{(2n)}\}_{i=0}^{2n}$  the point zero with multiplicity  $2k$  for  $k \geq 0$ , while the first  $2n-1$  rows can be chosen arbitrarily in  $[-\rho, \rho]$ . This defines a family of interpolation schemes indexed by  $n$ , and it is important to notice that  $B^{(2n)} = C_n^{(2n)}$  for each  $n$ . By Theorem 2.1 as applied to  $C_n$ , we have that for any  $z \in \mathbb{C}$ ,

$$(3.22) \quad e^z - R_n^B(z) = \sum_{k=0}^{\infty} [R_{n+k+1}^{C_n}(z) - R_{n+k}^{C_n}(z)].$$



From the interpolation conditions and upon checking degrees, we get the following factorization:

$$(3.23) \quad D_k(z) := R_{n+k+1}^{C_n}(z) - R_{n+k}^{C_n}(z) = \frac{\beta_{k+1} z^{2k} \prod_{i=0}^{2n} (z - x_i^{(2n)})}{\tilde{Q}_{n+k}^{C_n}(z) \tilde{Q}_{n+k+1}^{C_n}(z)},$$

where  $\beta_{k+1}$  is the leading coefficient of  $\tilde{P}_{n+k+1}^{C_n} \tilde{Q}_{n+k}^{C_n} - \tilde{P}_{n+k}^{C_n} \tilde{Q}_{n+k+1}^{C_n}$ . As the polynomials  $\tilde{Q}_{n+k}^{C_n}$  and  $\tilde{Q}_{n+k+1}^{C_n}$  are monic, we have

$$(3.24) \quad \beta_{k+1} = \tilde{P}_{n+k+1}^{C_n} - \tilde{P}_{n+k}^{C_n}.$$

We derive first the upper estimate in (2.4). By equation (2.14) in Lemma 2.5, we know that for any  $0 < \eta' < 1$  we have

$$(3.25) \quad (-1)^l \tilde{P}_l^{C_n} \leq \frac{1}{\eta'} \frac{\tilde{Q}_l^{C_n}(0)}{\tilde{Q}_l^{-C_n}(0)}$$

for  $l$  large enough. Applying Theorem 2.1 to the scheme  $C_n$  which is evidently ray with  $\lambda = 1$ , we also see that for any  $0 < \delta < 1$  and  $l$  large enough

$$(3.26) \quad |\tilde{Q}_l^{C_n}(z)| \geq \frac{\delta |\tilde{Q}_l^{C_n}(0)|}{\sqrt{c_1}}.$$

Using the lower estimate in (2.13) for the schemes  $C_n$  and  $-C_n$  successively, we further obtain that for any  $0 < \eta < 1$

$$(3.27) \quad \eta e^{-\rho/2} \leq (-1)^l \frac{l!}{(2l)!} \tilde{Q}_l^{C_n}(0),$$

$$(3.28) \quad \eta e^{-\rho/2} \leq (-1)^l \frac{l!}{(2l)!} \tilde{Q}_l^{-C_n}(0),$$

for  $l$  large.

To summarize, we know that when  $n$  is fixed and  $l = n + k$  is large enough, (3.25), (3.26), (3.27), and (3.28) hold true. *We claim there exists  $n_0$  such that these inequalities are valid for any scheme  $C_n$  and all  $k \geq 0$ , as soon as  $n$  is larger than  $n_0$ .*

Indeed, assume the contrary. Then, we can find a sequence  $n' + k'$  with  $n' \rightarrow \infty$  and  $k' \geq 0$  such that the interpolant  $R_{n'+k'}^{C_{n'+k'}}$  constantly violates one of the inequalities. But the scheme obtained by selecting for each pair of indices  $(n', k')$ , the row  $C_{n'}^{(2n'+2k')}$  is again a ray sequence (with  $\lambda = 1$ ) of interpolation points in  $[-\rho, \rho]$  to which our analysis can be applied. *This proves the claim by contradiction.*

Now, from (3.25) and (3.26) where we substitute  $n + k$  and then  $n + k + 1$  for  $l$ , we get in view of (3.23)

$$|D_k(z)| \leq \frac{c_1}{\eta' \delta^2} \left[ \left| \frac{\tilde{Q}_{n+k+1}^{C_n}(0)}{\tilde{Q}_{n+k+1}^{-C_n}(0)} \right| + \left| \frac{\tilde{Q}_{n+k}^{C_n}(0)}{\tilde{Q}_{n+k}^{-C_n}(0)} \right| \right] \frac{1}{|\tilde{Q}_{n+k}^{C_n}(0) \tilde{Q}_{n+k+1}^{C_n}(0)|},$$

that is,

$$|D_k(z)| \leq \frac{c_1}{\eta' \delta^2} \left[ \left| \frac{1}{\tilde{Q}_{n+k}^{C_n}(0) \tilde{Q}_{n+k+1}^{-C_n}(0)} \right| + \left| \frac{1}{\tilde{Q}_{n+k+1}^{C_n}(0) \tilde{Q}_{n+k}^{-C_n}(0)} \right| \right]$$

as soon as  $n > n_0$  and  $k \geq 0$ . Making use of (3.27) and (3.28), we now obtain

$$|D_k(z)| \leq 2 \frac{e^\rho c_1}{\eta' \delta^2 \eta^2} \frac{(n+k+1)! (n+k)!}{(2n+2k+2)! (2n+2k)!} |z|^{2k} \prod_{i=0}^{2n} |z - x_i^{(2n)}|,$$

as soon as  $n$  is large enough and for all  $k$ . Note that

$$\begin{aligned} \frac{(n+k+1)! (n+k+2)!}{(2n+2k+2)! (2n+2k+4)!} |z|^{2k+2} & \bigg/ \frac{(n+k)! (n+k+1)!}{(2n+2k)! (2n+2k+2)!} |z|^{2k} \leq \frac{|z|^2}{16(n+k)^2} \\ & \leq \frac{M^2}{16(n+k)^2}, \end{aligned}$$

where  $M = \max_{z \in K} |z|$ . Suppose that  $n$  is so large that

$$\frac{M^2}{16n^2} \leq 1.$$

Then, with  $\alpha' = \eta' \eta^2 \delta^2 < 1$ ,

(3.29)

$$|e^z - R_n^B(z)| \leq \sum_{k=0}^{\infty} |D_k(z)| \leq \frac{2e^\rho c_1}{\alpha'} \frac{n! (n+1)!}{(2n)! (2n+2)!} \left[ 1 + \sum_{k=1}^{\infty} \frac{M^2}{16(n+k)^2} \right] \prod_{i=0}^{2n} |z - x_i^{(2n)}|.$$

As  $\sum_k 1/(n+k)^2$  is the tail of a convergent series, the quantity

$$\left[ 1 + \sum_{k=1}^{\infty} \frac{M^2}{16(n+k)^2} \right] / \alpha'$$

can be made arbitrarily close to 1 by choosing  $\eta$ ,  $\eta'$ , and  $\delta$  close enough to 1 and  $n$  sufficiently large. Hence, we get that for any  $\alpha < 1$  and  $n$  large enough

$$|e^z - R_n^B(z)| \leq \frac{c_1 e^\rho}{\alpha} \frac{n! n!}{(2n)! (2n+1)!} \prod_{i=0}^{2n} |z - x_i^{(2n)}|,$$

which is our claimed upper estimate for  $\lambda = 1$ .

Let us now proceed with the lower estimate in (2.4). We have

$$|e^z - R_n^B(z)| \geq |D_0(z)| - \sum_{k=1}^{\infty} |D_k(z)|.$$

From (2.13) we observe that for any  $0 < \eta < 1$ ,  $l$  large,

$$(3.30) \quad \left| \frac{l!}{(2l)!} \tilde{Q}_l^{C_n}(0) \right| \leq \frac{e^{\rho/2}}{\eta},$$

$$(3.31) \quad \left| \frac{l!}{(2l)!} \tilde{Q}_l^{-C_n}(0) \right| \leq \frac{e^{\rho/2}}{\eta}.$$

By Theorem 2.1, for any  $0 < \delta < 1$ ,  $l$  large and  $z \in K$ , we have

$$(3.32) \quad |\tilde{Q}_l^{C_n}(z)| \leq \frac{1}{\delta \sqrt{c_0}} |\tilde{Q}_l^{C_n}(0)|.$$

Moreover, from (2.14), we get for any  $0 < \eta' < 1$  and  $l = n + k$  large

$$(3.33) \quad \eta' \frac{\tilde{Q}_l^{C_n}(0)}{\tilde{Q}_l^{-C_n}(0)} \leq (-1)^l \tilde{p}_l^{C_n}.$$

Reasoning as before, these estimates can be made uniform with respect to  $n$  when the latter is sufficiently large and they yield

$$(3.34) \quad \begin{aligned} |D_0(z)| &\geq 2\eta'\eta^2\delta^2c_0e^{-\rho} \frac{n!(n+1)!}{(2n)!(2n+2)!} \prod_{i=0}^{2n} |z - x_i^{(2n)}| \\ &= \eta'\eta^2\delta^2c_0e^{-\rho} \frac{n!n!}{(2n)!(2n+1)!} \prod_{i=0}^{2n} |z - x_i^{(2n)}|. \end{aligned}$$

On the other hand, we know that for large  $n$  (cf. (3.29))

$$\sum_{k=1}^{\infty} |D_k(z)| \leq \frac{c_1e^{\rho}}{\alpha'} \frac{n!n!}{(2n)!(2n+1)!} \left[ \sum_{k=1}^{\infty} \frac{M^2}{16(n+k)^2} \right] \prod_{i=0}^{2n} |z - x_i^{(2n)}|.$$

As the term in (3.34) is dominant compared to the above summation, we get that for any  $\alpha < 1$  and  $n$  large enough

$$|e^z - R_n^B(z)| \geq \alpha c_0 e^{-\rho} \frac{n!n!}{(2n)!(2n+1)!} \prod_{i=0}^{2n} |z - x_i^{(2n)}|,$$

thereby establishing (2.4) in the diagonal case.

We shall now briefly describe the general case where the rational fraction is of type  $(m, n)$  with  $m = m_\nu$ ,  $n = n_\nu$  satisfying (2.1) and (2.2). We shall again, as in equality (3.22), decompose the error  $e^z - R_{m,n}^B(z)$  as a sum of differences by introducing schemes  $C_{m,n}$  whose  $(m + k_1) + (n + k_2)$ -rows are obtained by adding the point zero with multiplicity  $k_1 + k_2$  to the set  $B^{(m+n)}$ . However, we now consider two distinct types of differences; the first is obtained by adding 1 to the degree of the numerator and the second by adding 1 to the degree of the denominator, namely

$$(3.35) \quad \frac{P_{l+1,l'}}{Q_{l+1,l'}} - \frac{P_{l,l'}}{Q_{l,l'}} = \frac{\tilde{P}_{l+1,l'}\tilde{Q}_{l,l'} - \tilde{P}_{l,l'}\tilde{Q}_{l+1,l'}}{\tilde{Q}_{l,l'}\tilde{Q}_{l+1,l'}}$$

and

$$(3.36) \quad \frac{P_{l,l'+1}}{Q_{l,l'+1}} - \frac{P_{l,l'}}{Q_{l,l'}} = \frac{\tilde{P}_{l,l'+1}\tilde{Q}_{l,l'} - \tilde{P}_{l,l'}\tilde{Q}_{l,l'+1}}{\tilde{Q}_{l,l'}\tilde{Q}_{l,l'+1}}.$$

In our sum of differences, we choose alternatively differences of the first or the second type in such a way that the quotient of the degrees of the numerator and denominator

tends to the limit  $\lambda$  (intuitively, this simply means that we approximate a line of slope  $\lambda$  by a step function). The factorization (3.23) can also be performed in (3.35) and (3.36) but here the leading coefficient of the numerator is equal to  $\tilde{p}_{m+1,n}$  in the first case and  $-\tilde{p}_{m,n}$  in the second case. From Theorem 2.1, we get for any  $0 < \delta < 1$  that for  $l + l'$  large enough, and  $z \in K$ ,

$$(3.37) \quad \delta \frac{|\tilde{Q}_{l,l'}^{C_{m,n}}(0)|}{c_1^{1/(1+\lambda)}} \leq |\tilde{Q}_{l,l'}^{C_{m,n}}(z)| \leq \frac{1}{\delta} \frac{|\tilde{Q}_{l,l'}^{C_{m,n}}(0)|}{c_0^{1/(1+\lambda)}}.$$

As the analog of (3.22), we have now differences of the first type

$$\frac{P_{m+k_1+1,n+k_2}^{C_{m,n}}}{Q_{m+k_1+1,n+k_2}^{C_{m,n}}} - \frac{P_{m+k_1,n+k_2}^{C_{m,n}}}{Q_{m+k_1,n+k_2}^{C_{m,n}}}$$

or of the second type

$$\frac{P_{m+k_1,n+k_2+1}^{C_{m,n}}}{Q_{m+k_1,n+k_2+1}^{C_{m,n}}} - \frac{P_{m+k_1,n+k_2}^{C_{m,n}}}{Q_{m+k_1,n+k_2}^{C_{m,n}}}.$$

The reader can check, using (2.13) for  $C_{m,n}$  and  $-C_{m,n}$  successively, and also using (2.14), (3.37), and finally reasoning as before to make these estimates uniform with respect to  $m + n$ , that the same upper bound for both differences can be obtained for  $m + n$  large enough and all  $k_1, k_2$ , namely:

$$\frac{1}{\eta' \eta^2 \delta^2} e^{\rho} c_1^{2/(1+\lambda)} \frac{(m+k_1)!(n+k_2)!}{(n+m+k_1+k_2)!(n+m+k_1+k_2+1)!} |z|^{k_1+k_2} \prod_{i=0}^{m+n} |z - x_i^{(m+n)}|.$$

Note that if  $k_1$  is increased by 1, since the quotient  $(m+k_1)/(n+k_2)$  tends to  $\lambda$ , we have for any  $\delta' > 1$  and  $m+n$  large

$$\begin{aligned} & \frac{(m+k_1+1)!(n+k_2)! |z|^{k_1+k_2+1}}{(n+m+k_1+k_2+1)!(n+m+k_1+k_2+2)!} \bigg/ \frac{(m+k_1)!(n+k_2)! |z|^{k_1+k_2}}{(n+m+k_1+k_2)!(n+m+k_1+k_2+1)!} \\ & \leq \frac{(m+k_1+1)|z|}{(n+m+k_1+k_2+1)(n+m+k_1+k_2+2)} \leq \frac{\delta' M}{(1+1/\lambda)(n+m+k_1+k_2+1)}, \end{aligned}$$

whereas if  $k_2$  is increased by 1, we have

$$\begin{aligned} & \frac{(m+k_1)!(n+k_2+1)! |z|^{k_1+k_2+1}}{(n+m+k_1+k_2+1)!(n+m+k_1+k_2+2)!} \bigg/ \frac{(m+k_1)!(n+k_2)! |z|^{k_1+k_2}}{(n+m+k_1+k_2)!(n+m+k_1+k_2+1)!} \\ & \leq \frac{(n+k_2+1)|z|}{(n+m+k_1+k_2+1)(n+m+k_1+k_2+2)} \leq \frac{\delta' M}{(1+\lambda)(n+m+k_1+k_2+1)}. \end{aligned}$$

Thus, there exists a constant  $C > 0$  such that the two previous quotients are less than  $C/(n+m+k_1+k_2+1)$ . Suppose that  $n+m$  is so large that

$$\frac{C}{n+m} \leq 1.$$

Then with  $\alpha' = \eta' \eta^2 \delta^2 < 1$ ,

$$|e^z - R_{m,n}^B(z)| \leq \frac{e^\rho}{\alpha'} \frac{c_1^{2/(1+\lambda)} m! n!}{(m+n)! (m+n+1)!} \left[ 1 + \frac{C}{n+m+1} + \sum_{k=1}^{\infty} \frac{C^2}{(n+m+k)^2} \right] \prod_{i=0}^{m+n} |z - x_i^{(m+n)}|$$

so that for any  $\alpha < 1$  and  $n+m$  large enough,

$$|e^z - R_{m,n}^B(z)| \leq c_1^{2/(1+\lambda)} \frac{e^\rho}{\alpha} \frac{m! n!}{(m+n)! (m+n+1)!} \prod_{i=0}^{m+n} |z - x_i^{(m+n)}|.$$

The lower estimate can be obtained in a similar way from the complementary inequalities in (2.13), (2.14), and (3.37), yielding for any  $0 < \alpha < 1$

$$|e^z - R_{m,n}^B(z)| \geq \alpha c_0^{2/(1+\lambda)} e^{-\rho} \frac{m! n!}{(m+n)! (m+n+1)!} \prod_{i=0}^{m+n} |z - x_i^{(m+n)}|,$$

as soon as  $m+n$  is large enough. This completes the proof of Theorem 2.2. ■

**PROOF OF THEOREM 2.3.** First observe that Theorem 2.2 implies a seemingly stronger statement which does not involve a particular ray sequence, namely:

*There exist  $L$  and  $\epsilon$ , both depending only on  $\rho$ ,  $K$ ,  $\alpha$ , and  $\lambda$  such that any interpolant of type  $(m, n)$  to  $e^z$  in  $m+n+1$  points of  $[-\rho, \rho]$  satisfies (2.4) as soon as  $m+n > L$  and  $|\lambda - m/n| < \epsilon$  (resp.  $n/m < \epsilon$  if  $\lambda = \infty$ ).*

Indeed, assuming the contrary, we could find some interpolation scheme  $B^{(m+n)}$  satisfying (2.1) and (2.2) but such that (2.4) does not hold however large  $m+n$ , thereby contradicting Theorem 2.2. The above statement really means that to each  $\lambda$  in  $[0, \infty]$ , we can attach a neighborhood  $\mathcal{V}_\lambda$  and an integer  $L_\lambda$  such that (2.4) holds for any interpolant of type  $(m, n)$  such that  $m/n \in \mathcal{V}_\lambda$  and  $m+n > L_\lambda$ . But then, (2.5) also holds because it is weaker than (2.4). Covering  $[0, \infty]$  with a finite number of  $\mathcal{V}_\lambda$ 's and defining  $L$  to be the supremum of the corresponding  $L_\lambda$ 's, we conclude that (2.5), which does not depend on  $\lambda$ , holds for any value of  $m/n$  provided  $m+n > L$ . Finally, the sharpness of (2.5) was treated in the remarks following the statement of Theorem 2.3. ■

**PROOF OF COROLLARY 2.6.** In view of Theorem 2.1, all we have to show is that  $r_{m,n}^*$  interpolates  $e^x$  at  $m+n+1$  points on  $[a, b]$ , counting multiplicities. For  $p = \infty$ , we observe that for each pair  $(k, l)$  of nonnegative integers, every function that is a linear combination of  $1, x, \dots, x^k, e^x, xe^x, \dots, x^l e^x$  has at most  $k+l+1$  zeros in  $[a, b]$ , except the zero function. Hence  $e^x$  is hypernormal (cf. [16], Section 5.1) which implies that  $e^x - r_{m,n}^*(x)$  has on  $[a, b]$  an alternation set of  $m+n+2$  points and so  $m+n+1$  (distinct) zeros in  $(a, b)$ .

For  $p = 2$  the desired interpolation property appears in [7] and, in fact, holds for any continuous function. For other values of  $p$ , the authors could not find a reference so we include an argument which is valid for  $1 \leq p < \infty$  and any function sharing with the exponential function the normality property in real interpolation.

Let  $\mathbf{P}_m[t]$  denote the space of real polynomials of degree at most  $m$  and  $\mathbf{P}_n^+[t]$  denote the set of polynomials of degree at most  $n$  with no roots in  $[a, b]$ . Let  $r_{m,n} = p_{m,n}/q_{m,n}$  be

a rational function with  $p_{m,n} \in \mathbf{P}_m[t]$  and  $q_{m,n} \in \mathbf{P}_n^+[t]$ . By definition of the  $L_p$ -norm, we have

$$(3.38) \quad \|e^x - r_{m,n}\|_p^p = \int_a^b |e^t - r_{m,n}(t)|^p dt.$$

We set  $\deg p_{m,n}^* = k$  and  $\deg q_{m,n}^* = l$ , with  $0 \leq k \leq m$ ,  $0 \leq l \leq n$ , and we assume without loss of generality that  $p_{m,n}^*$  and  $q_{m,n}^*$  are coprime. Writing  $|e^t - r_{m,n}(t)|^p$  as  $[(e^t - r_{m,n}(t))^2]^{p/2}$  in (3.38) and differentiating under the integral sign with respect to the coefficients of  $p_{m,n}$ , we get for  $r_{m,n} = r_{m,n}^*$  that

$$(3.39) \quad - \int_a^b p |e^t - r_{m,n}^*(t)|^{p-2} (e^t - r_{m,n}^*(t)) \frac{t^i}{q_{m,n}^*(t)} dt = 0, \quad 0 \leq i \leq m,$$

while differentiating with respect to the coefficients of  $q_{m,n}$  yields

$$(3.40) \quad \int_a^b p |e^t - r_{m,n}^*(t)|^{p-2} (e^t - r_{m,n}^*(t)) \frac{p_{m,n}^*(t)^j}{(q_{m,n}^*(t))^2} dt = 0, \quad 0 \leq j \leq n.$$

Note that we actually used the fact that  $e^t - r_{m,n}^*(t)$  is almost everywhere nonzero to justify differentiation under the integral sign when  $p = 1$ . Combining linearly (3.39) and (3.40) and taking into account the coprimeness of  $p_{m,n}^*$  and  $q_{m,n}^*$  implies that

$$(3.41) \quad \int_a^b p |e^t - r_{m,n}^*(t)|^{p-2} (e^t - r_{m,n}^*(t)) \frac{\mathbf{P}_s[t]}{(q_{m,n}^*(t))^2} dt = 0$$

with  $s = \max(m+l, n+k)$ . From (3.41), we get that  $s < k+l+1$  for, by hypernormality,  $e^t - r_{m,n}^*(t)$  cannot change sign more than  $k+l+1$  times on  $[a, b]$  (counting multiplicities). This implies  $k = m$  and  $l = n$ , i.e. the problem is normal, and forces  $e^t - r_{m,n}^*(t)$  to change sign  $m+n+1$  times on  $[a, b]$  as desired. ■

**PROOF OF COROLLARY 2.7.** Let  $\|\cdot\|_p$ ,  $1 \leq p \leq \infty$ , denote the  $L_p$  norm on the interval  $[a, b]$ . Let  $q_l^\infty$  denote the monic Chebyshev polynomial of degree  $l$  for the interval  $[a, b]$ . Its  $L_\infty$  norm realizes the minimum  $t_l([a, b])$  of  $L_\infty$  norms over  $[a, b]$  of monic polynomials of degree  $l$ . We define the *Chebyshev constant* for  $[a, b]$ :

$$\text{cheb}([a, b]) := \lim_{l \rightarrow \infty} [t_l([a, b])]^{1/l} = \lim_{l \rightarrow \infty} \|q_l^\infty\|_\infty^{1/l}.$$

It is well known (cf. [22]) that

$$\text{cheb}([a, b]) = (b - a)/4 > 0.$$

We write  $r_{m,n}^\infty$  for the rational fraction of type  $(m, n)$  that interpolates  $e^x$  at the roots of  $q_{m+n+1}^\infty$ . We recall that  $r_{m,n}^*$  denotes a best  $L_p$  rational approximant to  $e^x$  on  $[a, b]$  and  $q_{m+n+1}^*$  the monic polynomial whose roots are the interpolation points of  $r_{m,n}^*$  to  $e^x$ . Choosing

some  $\alpha < 1$ , we deduce from (2.5) that for  $m + n$  large enough

$$\begin{aligned} \alpha e^{-\rho} m_0 \|q_{m+n+1}^*\|_p &\leq \frac{(m+n)! (m+n+1)!}{m! n!} \|e^z - r_{m,n}^*(z)\|_p \\ &\leq \frac{(m+n)! (m+n+1)!}{m! n!} \|e^z - r_{m,n}^\infty(z)\|_p \\ &\leq \frac{(m+n)! (m+n+1)!}{m! n!} (b-a)^{1/p} \|e^z - r_{m,n}^\infty(z)\|_\infty \\ &\leq (b-a)^{1/p} \frac{e^\rho m_1}{\alpha} \|q_{m+n+1}^\infty\|_\infty. \end{aligned}$$

Taking  $(m+n+1)$ -th roots, we obtain

$$\lim_{m+n \rightarrow \infty} \|q_{m+n+1}^*\|_p^{1/(m+n+1)} \leq \text{cheb}([a, b]).$$

However, asymptotically,  $\|q_{m+n+1}^*\|_p^{1/(m+n+1)} \sim \|q_{m+n+1}^\infty\|_\infty^{1/(m+n+1)}$  (cf. [10]) so that

$$(3.42) \quad \lim_{m+n \rightarrow \infty} \|q_{m+n+1}^*\|_\infty^{1/(m+n+1)} = \text{cheb}([a, b])$$

as by definition, the left-hand term in (3.42) cannot be less than the Chebyshev constant. But then, we can apply a theorem of Blatt, Saff and Simkani (cf. [3] or [17], Theorem 5.1) asserting that when  $E \subset \mathbb{C}$  is compact,  $\text{cheb}(E) > 0$ , and  $E$  does not contain or surround a set with nonempty (2-dimensional) interior and if  $p_n$  is a sequence of monic polynomials of respective degrees  $n$  that satisfies

$$\lim_{n \rightarrow \infty} \|p_n\|_\infty^{1/n} = \text{cheb}(E),$$

then the normalized zero distribution  $\mu(p_n)$  tends in the weak-star sense to the equilibrium distribution  $\mu_E$ . ■

**PROOF OF PROPOSITION 2.8.** With the notation of Lemma 2.4, the fact that the zeros of  $q_{m,n}$  lie in  $|z| \geq m+1-\rho$  is obtained by appealing to the following result (cf. [19], Corollary 3.1 p. 351):

*The Padé denominator  $q_{m,n}^0$  is zero-free in the parabolic region*

$$\{z = x + iy \in \mathbb{C} : y^2 \leq 4(m+1)(m+1-x) \text{ and } x < m+1\}.$$

Thus, all the zeros of  $q_{m,n}^0$  lie in  $|z| \geq m+1$ . As we saw in the proof of (i) of Lemma 2.4,  $q_{m,n}$  is obtained from  $s_n$  and  $q_{m,n}^0/n!$  in the way described by Walsh's theorem. Hence all the zeros of  $q_{m,n}$  lie in  $|z| \geq m+1-\rho$ .

We now recall Theorem 2.4 of [18]:

*If*

$$1 < m < 3n + 4,$$

*then all the zeros of the Padé approximant of type  $(m, n)$  for  $e^z$  lie in the half-plane*

$$\text{Re}(z) < m - n - 2.$$

The symmetry between the numerator and the denominator of the Padé approximant for  $e^z$  implies the corresponding assertion about poles:

If

$$1 < n < 3m + 4,$$

then all the poles of the Padé approximant of type  $(m, n)$  for  $e^z$  lie in the half-plane

$$\operatorname{Re}(z) > m - n + 2.$$

Consequently, using Walsh's theorem as at the beginning of the proof, the poles of  $R_{m,n}$  lie in the half-plane

$$\operatorname{Re}(z) > m - n - \rho + 2,$$

and thus in the right half-plane if  $m \geq n + \rho - 2$ . But it is easily checked that this last condition implies  $n < 3m + 4$ , thereby establishing the second assertion of the proposition. Using the symmetry between the denominator and the numerator, the remaining assertions concerning  $R_{n,n}$  and  $R_{n-1,n}$  become particular cases of the previous result. ■

PROOF OF THEOREM 2.9. Set

$$f(z) = e^z, \quad f_1(z) = e^z(1 + \alpha_1 z), \quad f_2(z) = e^z / (1 - \beta_1 z).$$

The proof relies essentially on the connection between multipoint interpolants of  $f(z)$ ,  $f_1(z)$  and  $f_2(z)$ . Let  $B^{(m+n)} = \{x_k^{(m+n)}\}_{k=0}^{m+n}$  and set  $y := x_{m+n}^{(m+n)}$  to simplify notation. Let

$$q_{m+n}(z) = \prod_{k=0}^{m+n-1} (z - x_k^{(m+n)}), \quad q_{m+n+1}(z) = \prod_{k=0}^{m+n} (z - x_k^{(m+n)}) = q_{m+n}(z)(z - y).$$

Assume that we have the three following interpolation relations:

$$(3.43) \quad Q_{m,n}^f f - P_{m,n}^f = O(q_{m+n+1}),$$

$$(3.44) \quad Q_{m,n}^{f_1} f_1 - P_{m,n}^{f_1} = O(q_{m+n+1})$$

$$(3.45) \quad Q_{m-1,n}^f f - P_{m-1,n}^f = O(q_{m+n}).$$

We first show that in case such  $P_{m,n}^{f_1}$  and  $Q_{m,n}^{f_1}$  exist, we must have  $\deg P_{m,n}^{f_1} = m$  and  $\deg Q_{m,n}^{f_1} = n$  at least for  $m + n$  large enough. Indeed, as the function  $f(z) = e^z$  is normal, we know that

$$\deg P_{m,n}^{f_1} + \deg Q_{m,n}^{f_1} \geq m + n - 1.$$

If equality holds,  $P_{m,n}^{f_1} / ((1 + \alpha_1 z)Q_{m,n}^{f_1})$  is a Padé approximant to  $e^z$ ; but this contradicts Theorem 2.1 for  $m + n$  large, as the pole  $-1/\alpha_1$  does not tend to infinity.



Multiplying (3.45), by  $\tau_{m,n}(z-y)$ , where  $\tau_{m,n}$  is some number to be adjusted, and subtracting from (3.44), we obtain

$$(3.46) \quad [(1 + \alpha_1 z)Q_{m,n}^f - \tau_{m,n}(z-y)Q_{m-1,n}^f]f + [\tau_{m,n}(z-y)P_{m-1,n}^f - P_{m,n}^f] = O(q_{m+n+1}).$$

Choosing the constant  $\tau_{m,n}$  so that the degree of  $[(1 + \alpha_1 z)Q_{m,n}^f - \tau_{m,n}(z-y)Q_{m-1,n}^f]$  is  $n$ , we get by uniqueness of a normalized rational interpolant of type  $(m, n)$ :

$$(3.47) \quad (1 + \tau_{m,n}y)Q_{m,n}^f = (1 + \alpha_1 z)Q_{m,n}^f - \tau_{m,n}(z-y)Q_{m-1,n}^f,$$

$$(3.48) \quad (1 + \tau_{m,n}y)P_{m,n}^f = -\tau_{m,n}(z-y)P_{m-1,n}^f + P_{m,n}^f.$$

Setting  $z = -1/\alpha_1$  in (3.47), we get

$$(3.49) \quad (1 + \tau_{m,n}y)Q_{m,n}^f(-1/\alpha_1) = \tau_{m,n}(1/\alpha_1 + y)Q_{m-1,n}^f(-1/\alpha_1).$$

We begin by assuming that (2.2) holds. Note that in this case, for  $m+n$  large enough, a rational interpolant to  $f_1$  always exists. Indeed, choose  $\tau_{m,n}$  such that (3.49) is satisfied. This is possible because the interpolation point  $y$  is bounded while by Theorem 2.1,  $Q_{m,n}^f(-1/\alpha_1)$  and  $Q_{m-1,n}^f(-1/\alpha_1)$  converge to the same nonzero value. Then, define  $Q_{m,n}^f$  and  $P_{m,n}^f$  by (3.47) and (3.48) respectively. From (3.46), (3.43), and (3.45), we get (3.44); i.e.  $P_{m,n}^f/Q_{m,n}^f$  interpolates  $f_1$  in  $B^{(m+n)}$ .

Using the convergence of multipoint interpolant denominators to  $e^{-z/(1+\lambda)}$  given in Theorem 2.1 and the nonvanishing of the exponential function, we deduce from (3.49) that

$$\lim_{\nu \rightarrow \infty} \tau_{m,n} = \alpha_1.$$

By taking the limit in (3.47), and simplifying we get

$$\lim_{\nu \rightarrow \infty} Q_{m,n}^f = (1 + \alpha_1 z) \lim_{\nu \rightarrow \infty} Q_{m,n}^f - \alpha_1 z \lim_{\nu \rightarrow \infty} Q_{m-1,n}^f$$

or

$$\lim_{\nu \rightarrow \infty} Q_{m,n}^f = \lim_{\nu \rightarrow \infty} Q_{m,n}^f = e^{-z/(1+\lambda)},$$

locally uniformly in  $\mathbb{C}$ . By taking the limit in (3.48), we get

$$\lim_{\nu \rightarrow \infty} P_{m,n}^f = -\alpha_1 z \lim_{\nu \rightarrow \infty} P_{m-1,n}^f + \lim_{\nu \rightarrow \infty} P_{m,n}^f$$

or

$$\lim_{\nu \rightarrow \infty} P_{m,n}^{f_1} = (1 + \alpha_1 z) \lim_{\nu \rightarrow \infty} P_{m,n}^f = (1 + \alpha_1 z) e^{\lambda z/(1+\lambda)}.$$

Repeating our argument, we prove the convergence theorem for any  $k$  if  $l = 0$ . To settle the case  $l > 0$ , we leave it to the reader to check that it is possible to use the same type of relations between multipoint interpolants of functions  $f(z)$  and  $f_2(z)$ , and then combine the two cases. Another possibility is to use duality, remarking that

$$\left[ e^{-z} \frac{\prod_{j=1}^k (1 - \alpha_j z)}{\prod_{j=1}^l (1 + \beta_j z)} \right]^{-1} = \left[ \frac{1}{e^{-z} \prod_{j=1}^k (1 - \alpha_j z)} \right] \prod_{j=1}^l (1 + \beta_j z)$$

and the function in the brackets plays the role of  $f(z)$  in the preceding argument. This establishes (2.17) from which (2.16), when (2.2) holds, is immediate.

If (2.2) does not hold, an argument similar to the one given at the end of the proof of Theorem 2.1 shows that for  $m + n$  large enough, a rational interpolant  $R_{m,n}$  to  $e^z \prod_{j=1}^k (1 + \alpha_j z) / \prod_{j=1}^l (1 - \beta_j z)$  exists and that (2.16) remains true. ■

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