Electromagnetism Exercises Notes

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About

These are my notes from the *Exercises* (problem-solving) portion of the course *Elektromagnetno Polje* (Electromagnetic Field), given to third-year physics students at the Faculty of Math and Physics in Ljubljana, Slovenia. The exercises are led by Asst. Prof. Martin Klanšek, who has curated the problem sets and leads us through the solutions. I am making the notes freely available in case they might help others taking a similar course, but credit should go to Prof. Klanšek for the problem sets.

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1 Electrostatics

1.1 First

1.1.1 Background Theory

Electric field of a charge distribution.

$$\boldsymbol{E}(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\tilde{\boldsymbol{r}})\,\mathrm{d}^3\tilde{\boldsymbol{r}}}{|\boldsymbol{r}-\tilde{\boldsymbol{r}}|^2}\frac{\boldsymbol{r}-\tilde{\boldsymbol{r}}}{|\boldsymbol{r}-\tilde{\boldsymbol{r}}|}$$

where ρ is volume charge density and $\tilde{\boldsymbol{r}}$ is a dummy variable for integration over the position vector $\boldsymbol{r}$. Think of $\rho(\tilde{\boldsymbol{r}})\,\mathrm{d}^3\tilde{\boldsymbol{r}}$ as an infinitesimal element of charge at the position $\tilde{\boldsymbol{r}}$, while $\boldsymbol{r}-\tilde{\boldsymbol{r}}$ represents the position of each charge. The term $\frac{\boldsymbol{r}-\tilde{\boldsymbol{r}}}{|\boldsymbol{r}-\tilde{\boldsymbol{r}}|}$ just gives the direction of each charge element.

1.1.2 Charged Disk

A charged disk has surface charge density σ and radius a. What is the disk's electric field $E(z)$ along an axis through the disk's center and normal to the disk? Investigate the limit behavior of $E(z)$ for small and large z.

- Break the disk into infinitesimal concentric rings and integrate over the contributions $\mathrm{d}E$ of each ring; r and $\mathrm{d}r$ represent the radius and thickness of a ring, respectively.

  First break each ring into infinitesimal segments with area $r\,\mathrm{d}\phi\,\mathrm{d}r$. Along the perpendicular z axis through the disk's center each, segment contributes

  $$\mathrm{d}E_1 = \frac{1}{4\pi\epsilon_0}\frac{\sigma r\,\mathrm{d}\phi\,\mathrm{d}r}{z^2+r^2}$$

  Notice the term $r\,\mathrm{d}\phi\,\mathrm{d}r$ must have units of area to produce a charge when multiplied by the surface charge density σ. The term z^2+r^2 is just the square of the distance $\sqrt{z^2+r^2}$ from the charge element to the z axis.

- Recognize the circular symmetry of each ring: both the x and y components of the electric field symmetrically cancel along the z axis, so the electric field only has a z component. Relate the magnitudes of $\mathrm{d}E$ and $\mathrm{d}E_1$ along the z axis with similar triangles to get the contribution $\mathrm{d}E$ of a ring of radius r at the point $(0,0,z)$:

  $$\frac{\mathrm{d}E}{\mathrm{d}E_1} = \frac{z}{\sqrt{z^2+r^2}} \implies \mathrm{d}E = \frac{z}{(z^2+r^2)^{3/2}}\frac{\sigma r\,\mathrm{d}\phi\,\mathrm{d}r}{4\pi\epsilon_0}$$

- Integrate over the contributions $\mathrm{d}E$ to find $E(z)$:

  $$E(z) = \int \mathrm{d}E = \int_0^{2\pi}\int_0^a \mathrm{d}\phi\,\mathrm{d}r\frac{\sigma z r}{4\pi\epsilon_0(z^2+r^2)^{3/2}} = \frac{\sigma z}{2\epsilon_0}\int_0^a \frac{r\,\mathrm{d}r}{(z^2+r^2)^{3/2}}$$

  Solve the integral with the substitution $u = z^2+r^2$, $\mathrm{d}u = 2r\,\mathrm{d}r$:

  $$\begin{aligned} E(z) &= \int_{z^2}^{z^2+a^2}\frac{\mathrm{d}u}{u^{3/2}} = -\frac{\sigma z}{2\epsilon_0}\left(\frac{1}{\sqrt{z^2+a^2}} - \frac{1}{\sqrt{z^2}}\right) \\ &= \frac{\sigma}{2\epsilon_0}\left(1-\frac{z}{\sqrt{z^2+a^2}}\right) = \frac{\sigma}{2\epsilon_0}\left(1-\frac{1}{1+\frac{a^2}{z^2}}\right) \end{aligned}$$

- For $z \ll a$ (very close to the disk), $1+\frac{a^2}{z^2} \to \infty$ and $\frac{1}{1+\frac{a^2}{z^2}} \to 0$, leaving
$$E(z) \to \frac{\sigma}{2\epsilon_0} \qquad (z \ll a)$$
which is the electric field of an infinite charged plane.

- For $z \gg a$ (very far from the disk), $\frac{a^2}{z^2} \ll 1$, and we use the Taylor approximation $(1+x)^p \approx 1+px$ for $x \ll 1$:
$$E(z) = \frac{\sigma}{2\epsilon_0}\left[1-\left(1+\frac{a^2}{z^2}\right)^{-1/2}\right] \approx \frac{\sigma}{2\epsilon_0}\left[1-\left(1-\frac{a^2}{2z^2}\right)\right] = \frac{\sigma a^2}{4\epsilon_0 z^2}$$
Multiply above and below by π to match this to the expression for a point charge:
$$E(z) = \frac{\pi a^2 \sigma}{4\pi\epsilon_0 z^2} = \frac{\sigma S}{4\pi\epsilon_0 z^2} = \frac{q}{4\pi\epsilon_0 z^2} \qquad (z \gg a)$$
where $q = \sigma S$ is the disk's charge.

1.1.3 Charged Plate with a Slit

We take an infinitely large charged rectangular plate with surface charged density σ and remove a slit of width a from the plate. Determine the electric field E in the plate perpendicular to the plate and passing through the center of the slit as a function the orthogonal distance z from the plate. Investigate the limit behavior of $E(z)$ for small and large z.

- Break the plate into thin ribbons and integrate over the contributions $\mathrm{d}E$ of each ribbon; r and $\mathrm{d}r$ represent the orthogonal distance from the slit and the thickness of each ribbon, respectively.

- Find the electric field of a ribbon a distance r from the slit along the z axis using Gauss's law with a cylindrical surface. For a cylinder of radius R and length l, Gauss's law reads
$$\oint \boldsymbol{E}\cdot \mathrm{d}\boldsymbol{S} = 2\pi R l E = \frac{q_{\mathrm{enc}}}{\epsilon_0} \implies E(R) = \frac{q_{\mathrm{enc}}}{2\pi\epsilon_0 l R}$$
Applied to the ribbon, the enclosed charged q_{enc} is the ribbon's infinitesimal charge $\mathrm{d}q_1 = \sigma\,\mathrm{d}S_1 = \sigma l\,\mathrm{d}r$, while the cylinder's radius R is the distance from the ribbon to the z axis: $R = \sqrt{z^2+r^2}$, so the contribution $\mathrm{d}E_1$ of one ribbon is
$$\mathrm{d}E_1 = \frac{\mathrm{d}q_1}{2\pi\epsilon_0 l\sqrt{z^2+r^2}} = \frac{\sigma l\,\mathrm{d}r}{2\pi\epsilon_0 l\sqrt{z^2+r^2}} = \frac{\sigma\,\mathrm{d}r}{2\pi\epsilon_0\sqrt{z^2+r^2}}$$

- Because of mirror-image symmetry, both the x and y components of the electric field cancel, leaving only the z component $\mathrm{d}E_z$. Relate $\mathrm{d}E_z$ and $\mathrm{d}E_1$ using similar triangles
$$\frac{\mathrm{d}E_z}{\mathrm{d}E_1} = \frac{z}{\sqrt{z^2+r^2}} \implies \mathrm{d}E_z = \frac{\sigma z\,\mathrm{d}r}{2\pi\epsilon_0(z^2+r^2)}$$

- Find the total electric field along the z axis by integrating over the contributions $\mathrm{d}E_z$ of all the ribbons. Because of mirror symmetry, you can calculate only the contribution of e.g. the right plane and multiple the result by two.
$$E(z) = \int \mathrm{d}E_z = 2\int_{a/2}^{\infty}\left(\frac{\sigma z\,\mathrm{d}r}{2\pi\epsilon_0(z^2+r^2)}\right) = \frac{\sigma z}{\pi\epsilon_0}\int_{a/2}^{\infty}\frac{\mathrm{d}r}{z^2+r^2}$$
$$= \frac{\sigma z}{\pi\epsilon_0}\left[\frac{1}{z}\arctan\frac{r}{z}\right]_{a/2}^{\infty} = \frac{\sigma}{\pi\epsilon_0}\left[\frac{\pi}{2} - \arctan\left(\frac{a}{2z}\right)\right]$$

- In the limit $z \gg a$ (very far from the slit), $\arctan \frac{a}{2z} \to 0$ and the electric field along the z axis is
$$E(z) = \frac{\sigma}{2\epsilon_0} \qquad (z \gg a)$$
which is the field of an infinite sheet of charge.

- In the limit $z \ll a$ (very close to the slit), we have $\frac{a}{2z} \to \infty$. Use the asymptotic expansion $\arctan x \approx \frac{\pi}{2} - \frac{1}{x}$ for large x to get
$$E(z) \approx \frac{\sigma}{\pi\epsilon_0}\left[\frac{\pi}{2} - \left(\frac{\pi}{2} - \frac{2z}{a}\right)\right] = \frac{2\sigma}{\pi\epsilon_0}\frac{z}{a}$$
In this case the electric field scales linearly as $E \sim z$.

1.2 Second

1.2.1 Theory

Start with Gauss's law in differential form
$$\nabla \cdot \cdot \boldsymbol{E} = \frac{\rho}{\epsilon_0}$$
and
$$\boldsymbol{E} = -\nabla U$$
where U is electric potential. Plug into Gauss's law:
$$-\nabla \cdot \nabla U = \frac{\rho}{\epsilon_0} \implies \nabla^2 U = -\frac{\rho}{\epsilon_0}$$
The idea is: if we know charge distribution ρ, we can find electric field potential U, and we can then find electric field $\boldsymbol{E}$.
As a simple example for a point charge: the charge distribution is a delta function.
$$\rho(\boldsymbol{r}) = q\delta(\boldsymbol{r})$$
then plugging in gives
$$\nabla^2 U(\boldsymbol{r}) = -\frac{q}{\epsilon_0}\delta(\boldsymbol{r})$$
This the Poisson equation for a point charge—in general anything with $\nabla^2 f(\boldsymbol{r}) = g(\boldsymbol{r})$ is a Poisson equation.
We will solve this with a Fourier transform! **The Fourier Transform**
Think of it as an expansion over a basis of plane waves of the form $e^{i\boldsymbol{k}\cdot\boldsymbol{r}}$.
We just need some kind of weight to determine how much each wave contributes.
$$U(\boldsymbol{r}) = \int \mathrm{d}^3 k U(\boldsymbol{k}) e^{i\boldsymbol{k}\cdot\boldsymbol{r}}$$
where $U(\boldsymbol{k})$ is the amplitude of a plane wave with wave vector $\boldsymbol{k}$. And how to find $U(\boldsymbol{k})$? Use a dot product with a basis function $e^{-i\tilde{\boldsymbol{k}}\cdot\boldsymbol{r}}$, as usual! Because basis functions are orthogonal. So dot and integrate both sides over $\boldsymbol{r}$
$$\int U(\boldsymbol{r}) e^{-i\tilde{\boldsymbol{k}}\cdot\boldsymbol{r}}\,\mathrm{d}^3 r = \int\int \mathrm{d}^3 k\,\mathrm{d}^3 r U(\boldsymbol{k}) e^{i(\boldsymbol{k}-\tilde{\boldsymbol{k}})\cdot\boldsymbol{r}}$$

Integral over $\boldsymbol{r}$ is relatively simple. It is a delta function!? I guess because the waves cancel out over all space except at the origin, where they constructively interfere to infinity. So

$$\int U(\boldsymbol{r})e^{-i\tilde{\boldsymbol{k}}\cdot\boldsymbol{r}}\,\mathrm{d}^3r = (2\pi)^3\int U(\boldsymbol{k})\delta(\boldsymbol{k}-\tilde{\boldsymbol{k}})\,\mathrm{d}^3k = (2\pi)^3U(\tilde{\boldsymbol{k}})$$

The delta function suppresses the integral everywhere besides at $(\boldsymbol{k}-\tilde{\boldsymbol{k}})$. And that's how you get the amplitude $U(\tilde{\boldsymbol{k}})$ for a wave vector $\tilde{\boldsymbol{k}}$.

$$U(\tilde{\boldsymbol{k}}) = \frac{1}{(2\pi)^3}\int U(\boldsymbol{r})e^{-i\boldsymbol{k}\cdot\boldsymbol{r}}\,\mathrm{d}^3r$$

This is also an inverse Fourier transform of sorts, to get $U(\boldsymbol{k})$ from $U(\boldsymbol{r})$.
Recipe: Take the function you're working with e.g. U, expand it over a basis of plane waves, find the amplitude of each plane wave. Take the plane waves back into the Fourier transform and find $U(\boldsymbol{r})$.
First of Two More Notes: What happens if we insert gradient into the Fourier transform? Remember it acts only on $\boldsymbol{r}$. So

$$\nabla U(\boldsymbol{r}) = \int \mathrm{d}^3kU(\boldsymbol{k})\nabla e^{i\boldsymbol{k}\cdot\boldsymbol{r}}$$

Evaluate the gradient over (x, y, z) components. The result is

$$\nabla e^{i\boldsymbol{k}\cdot\boldsymbol{r}} = \begin{bmatrix} ikx \\ iky \\ ikz \end{bmatrix} e^{i(k_1x+k_2y+k_3z)} = ike^{i\boldsymbol{k}\cdot\boldsymbol{r}}$$

So:

$$\nabla U(\boldsymbol{r}) = \int \mathrm{d}^3kU(\boldsymbol{k})\nabla e^{i\boldsymbol{k}\cdot\boldsymbol{r}} = \int \mathrm{d}^3kU(\boldsymbol{k})i\boldsymbol{k}e^{i\boldsymbol{k}\cdot\boldsymbol{r}}$$

which has the form of a Fourier transform! This time of the function $U(\boldsymbol{k})i\boldsymbol{k}$.
The interpretation is that ∇ transforms to $i\boldsymbol{k}$ under the Fourier transform. Analogously, ∇^2 would transform into $(i\boldsymbol{k})^2 = ik^2$.
Second of Two More Notes: What happens if we insert $\delta(\boldsymbol{r}$ into the Fourier transform? Let $\delta(\boldsymbol{k})$ denote the amplitude in the expansion of $\delta(\boldsymbol{r})$. Like $U(\boldsymbol{k})$ for $U(\boldsymbol{r})$. Well, using the inverse Fourier transform the integral properties of the delta function

$$\delta(\boldsymbol{k}) = \frac{1}{(2\pi)^3}\delta(\boldsymbol{r})e^{-i\boldsymbol{k}\cdot\boldsymbol{r}}\,\mathrm{d}^3r = \frac{1}{(2\pi)^3}e^{-i\boldsymbol{k}\cdot 0} = \frac{1}{(2\pi)^3}$$

So now we have the machinery we need. The goal is to solve the Poisson equation for point particle.

1.2.2 Poisson Equation for a Point Particle

The Poisson equation for a point particle is

$$\nabla^2U(\boldsymbol{r}) = -\frac{q}{\epsilon_0}\delta(\boldsymbol{r})$$

Solve equation for $U(\boldsymbol{r})$.

- The plan is to transform into $\boldsymbol{k}$ space, solve for $U(\boldsymbol{k})$, then transform back to $U(\boldsymbol{r})$. First, take the Fourier transform of both sides. Remember the identities for how ∇^2 becomes $-k^2$ and $\delta(\boldsymbol{r})$ becomes $\frac{1}{(2\pi)^3}$.

$$-k^2 U(\boldsymbol{k}) = -\frac{q}{\epsilon_0}\frac{1}{(2\pi)^3} \implies U(\boldsymbol{k}) = \frac{q}{(2\pi)^3 \epsilon_0 k^2}$$

- Next, find $U(\boldsymbol{r})$ using a second Fourier transform

$$U(\boldsymbol{r}) = \int \mathrm{d}^3 k \frac{q e^{i\boldsymbol{k}\cdot\boldsymbol{r}}}{k^2 \epsilon_0 (2\pi)^3} = \frac{q}{(2\pi)^3 \epsilon_0} \int \frac{e^{i\boldsymbol{k}\cdot\boldsymbol{r}}}{k^2}\,\mathrm{d}^3 k$$

- Now, how to solve the integral? Suppose θ is the angle between $\boldsymbol{r}$ and $\boldsymbol{k}$. To simplify the dot product. And then integrate in spherical coordinates:

$$U(\boldsymbol{r}) = \frac{q}{(2\pi)^3 \epsilon_0} \int_0^{2\pi} \int_{-1}^{1} \int_0^{\infty} \mathrm{d}\phi\, \mathrm{d}[\cos\theta] k^2\, \mathrm{d}k \frac{e^{ikr\cos\theta}}{k^2}$$

So integrate over ϕ, which is simple and gives 2π:

$$U(\boldsymbol{r}) = \frac{q}{(2\pi)^2 \epsilon_0} \int_{-1}^{1} \int_0^{\infty} e^{i\cos\theta kr}\, \mathrm{d}[\cos\theta]\, \mathrm{d}k$$

Integrate first over θ, (to avoid $e^{i\cos\theta\cdot\infty}$ from the upper k limit). Recognize the sine function the difference of exponents!

$$U(\boldsymbol{r}) = \frac{q}{(2\pi)^2 \epsilon_0} \int_0^{\infty} \frac{e^{i\cos\theta kr}}{ikr}\bigg|_{\theta=-1}^{1} \mathrm{d}k = \frac{q}{(2\pi)^2 \epsilon_0} \int_0^{\infty} \frac{e^{ikr} - e^{-ikr}}{ikr}\, \mathrm{d}k = \frac{q}{(2\pi)^2 \epsilon_0} \int_0^{\infty} \frac{2\sin(kr)}{kr}\, \mathrm{d}k$$

The integral of the sinc function is

$$\int_0^{\infty} \frac{\sin x}{x}\, \mathrm{d}x = \frac{\pi}{2}$$

And applying this integral gives

$$U(\boldsymbol{r}) = \frac{2q}{(2\pi)^2 \epsilon_0} \frac{\pi}{2r} = \frac{q}{4\pi\epsilon_0 r}$$

which is the electric potential of a point charge. Cool! We derived it directly from Maxwell's equations.

1.2.3 Interlude: More Theory

- Next, plug $U(\boldsymbol{r})$ into the Poisson equation. The result, after moving constants around, is

$$\nabla^2 \frac{1}{r} = -4\pi\delta(\boldsymbol{r})$$

This will be useful in the next problems.

- Another consideration. We solved the Poisson equation for the simple case $\rho(\boldsymbol{r}) = \delta(\boldsymbol{r})$. Can we use this result to solve the general case $\rho = \rho(\boldsymbol{r})$? We could, if we expand $\rho(\boldsymbol{r})$ over a basis of delta functions, as follows:

$$\rho(\boldsymbol{r}) = \int \mathrm{d}^3\tilde{r}\rho(\tilde{\boldsymbol{r}})\delta(\boldsymbol{r} - \tilde{\boldsymbol{r}})$$

  In this case, the solution of $U(\boldsymbol{r})$ to the Poisson equation is

$$U(\boldsymbol{r}) = \int \mathrm{d}^3\tilde{r}\rho(\tilde{\boldsymbol{r}})\frac{1}{4\pi\epsilon_0|\boldsymbol{r} - \tilde{\boldsymbol{r}}|} = \frac{1}{4\pi\epsilon_0}\int \frac{\mathrm{d}^3\tilde{r}\rho(\tilde{\boldsymbol{r}})}{|\boldsymbol{r} - \tilde{\boldsymbol{r}}|}$$

  Well that's cool. By solving the Poisson equation for a delta function and then expanding an arbitrary $\rho(\boldsymbol{r})$ in terms of the delta function, we got the equation to the Poisson equation for any $\rho(\boldsymbol{r})$. And we can then get electric field using

$$\boldsymbol{E} = -\nabla U = \frac{1}{4\pi\epsilon_0}\int \frac{\mathrm{d}^3\tilde{r}\rho(\tilde{\boldsymbol{r}})}{|\boldsymbol{r} - \tilde{\boldsymbol{r}}|^2}\frac{\boldsymbol{r} - \tilde{\boldsymbol{r}}}{|r - \tilde{\boldsymbol{r}}|}$$

  which agrees with the result from the previous Exercises.

1.2.4 Electric Field of a Hydrogen Atom

The Hydrogen atom has the electric potential

$$U(\boldsymbol{r}) = \frac{q}{4\pi\epsilon_0}\frac{e^{-\alpha r}}{r}\left(1 + \frac{\alpha r}{2}\right), \qquad \alpha = \frac{2}{r_B}$$

Find the charge density $\rho(\boldsymbol{r})$ generates this potential.

- Use the Poisson equation, which connects U and ρ

$$\nabla^2 U(\boldsymbol{r}) = -\frac{\rho(\boldsymbol{r})}{\epsilon_0}$$

  Calculate the Laplacian of our $U(\boldsymbol{r})$ and work in spherical coordinates, since the potential is spherically symmetric (depends only on r). As a review, when acting on a function that depends only on r, ∇^2 in spherical coordinates reads

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)$$

- Applying ∇^2 to $U(r)$, after some uncomplicated but rather tedious differentiation, leads to

$$\nabla^2 U(\boldsymbol{r}) = \frac{q\alpha^3}{8\pi\epsilon_0}e^{-\alpha r}$$

  Rearranging the Poisson equation then gives

$$\rho(\boldsymbol{r}) = -\frac{q\alpha^3}{8\pi}e^{-\alpha r}$$

- Note that the charge density is negative, which corresponds to the negatively charged electron cloud. Inserting the definition of $\alpha = \frac{2}{r_B}$ gives

$$\rho(\boldsymbol{r}) = -\frac{q}{\pi r_B^3}e^{-\frac{2r}{r_B}}$$

Another interpretation: $e^{-\frac{2r}{r_B}}$ is equivalent to $\left(e^{-\frac{r}{r_B}}\right)^2$, which is the square of the hydrogen atom's ground state wave function. The square of the wave function is probability, and multiplying the probability by $\frac{q}{r_B^3}$ gives a charge density.

- And why does the charge of the proton not contribute? Because the proton occurs at the origin, which corresponds to a charge density singularity at the origin. Terms in $U(r)$ with $\frac{1}{r}$ generate the singularity. And we just glossed over this when applying ∇^2.

  We can resolve this problem with a special case from the limit

  $$\lim_{r\to 0} U(\boldsymbol{r}) = \frac{q}{4\pi\epsilon_0 r}$$

  We would then have to solve the Poisson equation for this potential. But we already know this potential: it is the potential for a point charge and corresponds to a charge density

  $$\rho(\boldsymbol{r}) = q\delta(\boldsymbol{r})$$

  The correct total result for the hydrogen atom is the sum of the electron cloud result and the charge density of the nucleus.

  $$\rho(\boldsymbol{r}) = q\delta(\boldsymbol{r}) - \frac{q\alpha^3}{8\pi}e^{-\alpha r}$$

  Lesson: Be careful when working with the Poisson equation when $U(r)$ has singularities!

1.3 Third

1.3.1 Theory

- Review from last time for potential from Poisson equation

  $$\nabla^2 U(\boldsymbol{r}) = -\frac{\rho(\boldsymbol{r})}{\epsilon_0} \qquad \text{where} \qquad \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

  Often $\rho(\boldsymbol{r}) = 0$ in places we're solving for the electric potential.

  In this case $\nabla^2 U(\boldsymbol{r}) = 0$. This is called a Laplace equation.

1.3.2 Perpendicular Ribbon in a Parallel-Plate Capacitor

A parallel plate capacitor means large plate size compared to the distance between them. Distance between them is a. Place a metal ribbon in the space between the plates—almost touching both plates with a little bit of air/insulation between ribbon ends and plates. Ground both plates and set potential of ribbon to U_0. What is $U(\boldsymbol{r})$ inside the capacitor?

- First, decide on a coordinate system. Choose Cartesian coordinates since the problem is rectangular. Put the origin at the bottom of the ribbon—y axis along ribbon, x axis along plates and z axis "out of the page".

  Note that the problem is independent of z (translation symmetry), so we only need $U(x, y)$. More so, the problem has reflection symmetry, so we can find the solution on only one side of the ribbon (one half of the x axis) and reflect the solution about the y axis.

- The space between the capacitor plates is empty—there is no charge, and we use the Laplace equation for the space between the plates.

$$\nabla^2 U(x,y) = 0$$

Charge can occur only along the ribbon or on the capacitor plates.

- Next, determine boundary conditions for $U(x,y)$ so the equation has a unique solution. The problem's boundaries are the ribbon and edges of the capacitor plates.

  On the bottom plate, $U(x,0) = 0$. On the upper plate, $U(x,a) = 0$. Both are zero because the plates are grounded. For the ribbon $U(0,y) = U_0$. And we need one more boundary—infinity. We require only that $U(x \to \infty, y)$ is bounded, i.e. that U does not diverge at ∞.

- First, attempt solving the problem with separation of variables: $U(x,y) = X(x)Y(y)$—this approach tends to work well with symmetric problems. Plugging this ansatz into the Laplace equation and evaluating ∇^2 gives

$$X^{''}Y + XY^{''} = 0 \implies \frac{X''}{X} = -\frac{Y''}{Y}$$

The separation of variables is successful—we were able to get only x and only x on different sides of the equation.

- As usual, set the equation equal to a separation constant κ^2 and get two equations

$$X'' - \kappa^2 X = 0 \qquad \text{and} \qquad Y'' + \kappa^2 Y = 0$$

Both equations have simple solutions! The equations for X and Y are solved by exponential and sinusoidal functions, respectively.

$$X(x) = Ae^{\kappa x} + Be^{-\kappa x} \qquad \text{and} \qquad Y(y) = C\sin(\kappa y) + D\cos(\kappa y)$$

We then plug the expressions for X and Y back into the ansatz:

$$U(x,y) = X(x)Y(y) = \left(Ae^{\kappa x} + Be^{-\kappa x}\right)\left(C\sin(\kappa y) + D\cos(\kappa y)\right)$$

- We find the coefficients A, B, C and D using the boundary conditions.
  - Start with the most powerful condition, that $U(x \to \infty, y)$ is bounded. This conditions implies $A = 0$ to suppress the divergent exponential function e^{κ}.
  - Then, use the next two simplest conditions, the ones requiring $U(x,y) = 0$. Starting with $U(x,0) = 0$ gives

$$0 \equiv U(x,0) = 1 \cdot (0 + D) \implies D = 0$$

    With both $A = D = 0$, we're left at this point with only

$$U(x,y) = Be^{-\kappa x} \cdot C\sin(\kappa y)$$

  - Next, applying $U(x,a) = 0$ gives

$$0 \equiv U(x,a) = Be^{-\kappa x}C\sin(\kappa a) \equiv Fe^{-\kappa x}\sin(\kappa a)$$

Note that we've joined the product of two constants into one constant $F = BC$.
We have two options: either $F = 0$ or $\sin(\kappa a) = 0$. The option $F = 0$ gives the trivial solution $U(x, y) = 0$. The non-trivial solution comes from

$$\sin(\kappa a) = 0 \implies \kappa a = n\pi, \quad n1, 2, 3, \ldots$$

Note that $n = 0$ leads to a trivial solution $U(x, y) = 0$, which we reject.
Respecting the quantization of κ and F by the index n, the general solution at this point is the linear superposition

$$U(x, y) = \sum_{n=1}^{\infty} F_n e^{-\kappa_n x} \sin(\kappa_n y)$$

– To find F_n, we use the last boundary condition $U(0, y) = U_0$.

$$U_0 = \sum_{n=1}^{\infty} F_n \sin(\kappa_n y) = \sum_{n=1}^{\infty} F_n \sin\Big(\frac{n\pi y}{a}\Big)$$

where we've used $\kappa_n = \frac{n\pi}{a}$. This is a Fourier expansion of the constant U_0 over sine functions.
We find the coefficients by taking the inner product of both sides of the equation (I think on the vector space $L(0, a)$), which amounts to multiplying both sides by $\sin \frac{m\pi y}{a}$ and integrating both sides over y from 0 to a.
The left side U_0 becomes

$$U_0 \int_0^a \sin\Big(\frac{m\pi y}{a}\Big)\, \mathrm{d}y = -\frac{U_0 a}{m\pi} \cos\Big(\frac{m\pi y}{a}\Big)\Big|_0^a = \frac{U_0 a}{m\pi}\big[1 - (-1)^m\big]$$

And the right hand side, with the sum, we switch the sum and integral to get

$$\sum_{n=1}^{\infty} F_n \int_0^a \sin\Big(\frac{n\pi y}{a}\Big) \sin\Big(\frac{m\pi y}{a}\Big)\, \mathrm{d}y = \sum_{n=1}^{\infty} F_n \delta_{mn} \int_0^a \sin^2\Big(\frac{m\pi y}{a}\Big)\, \mathrm{d}y = \frac{F_m a}{2}$$

Because of the orthogonality of the sine functions, the integral is zero for $m \neq n$.
Only the case $m = n$ gives a non-zero result.
Equating the two sides gives the desired expression for F_m:

$$\frac{U_0 a}{m\pi}\big[1 - (-1)^m\big] = \frac{F_m a}{2} \implies F_m = \frac{2U_0}{m\pi}\big[1 - (-1)^m\big]$$

- With F_m known, the final result for $U(x, y)$ is then

$$U(x, y) = \frac{2U_0}{\pi} \sum_{n=1}^{\infty} \frac{1 - (-1)^n}{n} \exp\Big(-\frac{n\pi x}{a}\Big) \sin\Big(\frac{n\pi y}{a}\Big)$$

Some limit cases: for $x \gg a$, the exponent terms very small, and we can neglect all terms in the series except the first term $e^{-\frac{\pi x}{a}}$ with $n = 1$. The result is

$$U(x, y) = \frac{4U_0}{\pi} \exp\Big(-\frac{\pi x}{a}\Big) \sin\Big(\frac{\pi y}{a}\Big)$$

A separate case for which we can find a nice analytic solution is in the center of the capacitor at $y = \frac{a}{2}$. The solution reads

$$U(x, \tfrac{a}{2}) = \frac{2U_0}{\pi} \sum_{n=1}^{\infty} \frac{1 - (-1)^n}{n} \exp\Big(-\frac{n\pi x}{a}\Big) \sin\Big(\frac{n\pi}{2}\Big)$$

Instead of finding $U(x, \frac{a}{2})$, we'll find the electric field $\boldsymbol{E}(x, \frac{a}{2})$. Because of reflection symmetry across the line $y = \frac{a}{2}$, the electric field cannot have a y component—$\boldsymbol{E}$ only has an x component. We'll find $E_x(x)$ from the potential:

$$E_x(x) = -\frac{\partial}{\partial x} U(x, \tfrac{a}{2}) = -\frac{2U_0}{a} \sum_{n=1}^{\infty} \big[1 - (-1)^n\big] \exp\Big(-\frac{n\pi x}{a}\Big) \sin\Big(\frac{n\pi}{2}\Big)$$

Next, note that

$$\big[1 - (-1)^n\big] \sin\Big(\frac{n\pi}{2}\Big) = \begin{cases} 0 & n \text{ even} \\ 2 & n = 1, 5, 9, \dots \\ -2 & n = 3, 7, 11, \dots \end{cases}$$

The sum simplifies to

$$E_x(x) = \frac{4U_0}{a} \Big[e^{-\frac{\pi x}{a}} - e^{-\frac{3\pi x}{a}} + e^{-\frac{5\pi x}{a}} \mp \dots\Big] = \frac{4U_0}{a} e^{-\frac{\pi x}{a}} \left[1 - e^{-\frac{2\pi x}{a}} + \Big(e^{-\frac{2\pi x}{a}}\Big)^2 \mp \dots\right]$$

which is a geometric series in $e^{-\frac{2\pi x}{a}}$. The result is

$$E_x(x) = \frac{4U_0}{a} \frac{e^{-\frac{\pi x}{a}}}{1 + e^{-\frac{2\pi x}{a}}} = \frac{4U_0}{a} \frac{1}{e^{\frac{\pi x}{a}} + e^{-\frac{\pi x}{a}}} = \frac{2U_0}{a \cosh\big(\frac{\pi x}{a}\big)}$$

1.3.3 A Halved Conducting Cylinder

Imagine a long cylinder of radius a cut in half along a plane running along the cylinder's longitudinal axis. We separate the two cylinder halves by a small amount (so they are insulated) and apply a potential difference U_0 between the two halves. The halved cylinder acts as a capacitor. Find the electric potential U inside the cylinder.

- Decide on a coordinate system: our problem has cylindrical symmetry, so cylindrical coordinates are the natural choice. Let the z axis run along the cylinder's longitudinal axis. Because of translational symmetry along the z axis, U is independent of z.

- There is no charge inside the cylinder, so we get a Laplace equation

  $$\nabla^2 U(r, \phi) = 0$$

  In cylindrical coordinates (when independent of z), the Laplacian operator reads **TODO: record in eq sheet**

  $$\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \phi^2}$$

  In our case,

  $$\nabla^2 U(r, \phi) = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial U}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 U}{\partial \phi^2} = 0$$

- Again, we separate variables with the ansatz $U(r, \phi) = R(r)\Phi(\phi)$. Plugging this into the Laplace equation gives

$$\Phi\frac{1}{r}(rR')' + \frac{R}{r^2}\Phi'' = \Phi\left(\frac{R'}{r} + R''\right) + \frac{R}{r^2}\Phi'' = 0$$

Note that $r' = 1$. Dividing through by Φ and rearranging gives

$$\frac{rR'}{R} + r^2\frac{R''}{R} = -\frac{\Phi''}{\Phi}$$

- Following the usual separation procedure, we set both sides equal to the separation constant m^2. The equations for Φ and R read

$$\Phi'' + m^2\Phi = 0 \qquad \text{and} \qquad r^2R'' + rR' - m^2R = 0$$

The solution for Φ is sinusoidal:

$$\Phi(\phi) = A\sin(m\phi) + B\cos(m\phi)$$

Now, our cylindrical problem is periodic in ϕ with period 2π—this just means the cylinder repeats after on revolution. Periodicity in ϕ is possible only if m takes on integer values, so we can immediately index the solutions for Φ with

$$\Phi_m(\phi) = A_m\sin(m\phi) + B_m\cos(m\phi), \quad m = 1, 2, 3, \ldots$$

We only use positive integers because the odd/even symmetry of sin and cos means negative integers give the same result as positive one—solving for negative m would be redundant. We reject $m = 0$ because this solution leads to $\Phi'' = 0$, meaning Φ is a linear function. But a linear function can't be periodic in ϕ, so we reject $m = 0$.

The second equation for R is solved with powers of r. The result is

$$R_m(r) = C_mr^m + D_mr^{-m}$$

- The general solution is the linear superposition **TODO: record this general solution in eq sheet**

$$U(r, \phi) = \sum_{m=1}^{\infty}\Phi_m(\phi)R_m(m) = \sum_{m=1}^{\infty}\left(A_m\sin(m\phi) + B_m\cos(m\phi)\right)\left(C_mr^m + D_mr^{-m}\right)$$

Note: we ended the problem at this point (out of time) and in the fourth set of exercises. I'm completing the problem in this section to maintain continuity.

- To find a solution specific to our problem, we apply boundary conditions—in our case, that the capacitor halves have a potential difference U_0 between them. It is best to write this potential difference in the symmetric form

$$U(a, \phi) = \begin{cases} \frac{U_0}{2} & \phi \in (0, \pi) \\ -\frac{U_0}{2} & \phi \in (\pi, 2\pi) \end{cases}$$

There is another condition—that U does not diverge at $r = 0$. This condition implies the D_m coefficients are zero, because the D_mr^{-m} term diverges at $r = 0$.

Observation: the main boundary condition is an odd function of ϕ. This implies that only odd (sine) terms can appear in the final solution. This allows us to set the A_m coefficients equal to zero to eliminate the cosine terms. We are left with

$$U(r,\phi) = \sum_{m=1}^{\infty} F_m r^m \sin(m\phi)$$

where we have defined $B_m C_m \equiv F_m$.

- Applying the main boundary condition gives

$$U(a,\phi) = \sum_{m=1}^{\infty} F_m a^m \sin(m\phi)$$

To solve this, we take the scalar product of the equation with $\sin(n\phi)$:

$$\int_0^{2\pi} U(a,\phi)\sin(n\phi)\,\mathrm{d}\phi = \int_0^{2\pi}\sum_{m=1}^{\infty} F_m a^m \sin(m\phi)\sin(n\phi)\,\mathrm{d}\phi$$

Plugging in the step values of $U(a,\phi)$ gives

$$\frac{U_0}{2}\int_0^{\pi}\sin(n\phi)\,\mathrm{d}\phi - \frac{U_0}{2}\int_{\pi}^{2\pi}\sin(n\phi)\,\mathrm{d}\phi = \int_0^{2\pi}\sum_{m=1}^{\infty} F_m a^m \sin(m\phi)\sin(n\phi)\,\mathrm{d}\phi$$

First, we solve the left-hand side

$$\begin{aligned}\frac{U_0}{2}\left[-\frac{1}{n}\cos(n\phi)\Big|_0^{\pi} + \frac{1}{n}\cos(n\phi)\Big|_{\pi}^{2\pi}\right] &= \frac{U_0}{2n}\left[-\cos(\pi n) + 1 + 1 - \cos(n\pi)\right]\\ &= \frac{U_0}{n}\left(1-(-1)^n\right)\end{aligned}$$

where we have written the $\cos(n\pi)$ terms in terms of $(-1)^n$. **TODO record the cosine identity**.

- Next, we solve the right-hand side. Switching the order of integration and summation gives **TODO: record the simple result**

$$\sum_{m=1}^{\infty} F_m a^m \int_0^{2\pi}\sin(m\phi)\sin(n\phi)\,\mathrm{d}\phi = \sum_{m=1}^{\infty} F_m a^m\left(\frac{2\pi}{2}\delta_{mn}\right) = F_n a^n \pi$$

Combining the left and right sides gives

$$F_n = \frac{U_0}{\pi}\frac{1-(-1)^n}{na^n}$$

So, the solution for $U(r,\phi)$ is

$$U(r,\phi) = \sum_{n=1}^{\infty}\frac{U_0}{\pi}\frac{1-(-1)^n}{na^n} r^n\sin(n\phi) = \sum_{n=1}^{\infty}\frac{U_0}{\pi}\left(\frac{r}{a}\right)^n\frac{1-(-1)^n}{n}\sin(n\phi)$$

- Next, some limiting cases. It will be easier to work in terms of electric field instead of potential. We will find the electric field in the two planes parallel and perpendicular to the slit between the capacitor halves.

First, in the perpendicular (vertical) plane. The field points from high to low potential, so from the top half of the capacitor to the bottom half. In this plane we can work with just one coordinate r, which represents the vertical distance from the cylinder's center. Note that $\phi = \frac{\pi}{2}$. The component E_r we're after is

$$\begin{aligned} E_r &= -\frac{\partial}{\partial r} U(r,\phi)\Big|_{\phi=\frac{\pi}{2}} = -\frac{U_0}{\pi}\sum_{n=1}^{\infty} n\left(\frac{r}{a}\right)^{n-1}\frac{1}{a}\frac{1-(-1)^n}{n}\sin(n\phi)\Big|_{\phi=\frac{\pi}{2}} \\ &= -\frac{U_0}{\pi a}\sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^{n-1}\frac{1-(-1)^n}{n}\sin\left(\frac{n\pi}{2}\right) \end{aligned}$$

The sum simplifies considerably when you realize

$$\frac{1-(-1)^n}{n}\sin\left(\frac{n\pi}{2}\right) = \begin{cases} 0 & n \text{ even} \\ 2 & n = 1, 5, \ldots \\ -2 & n = 3, 7, \ldots \end{cases}$$

We can then write the field as a geometric series

$$E_r = \frac{-2U_0}{\pi a}\left[1 - \left(\frac{r}{a}\right)^2 + \left(\frac{r}{a}\right)^4 \mp \cdots\right] = \frac{-2U_0}{\pi a}\frac{1}{1+\left(\frac{r}{a}\right)^2}$$

Note that E_r is largest at $r=0$, decreases monotonically with increasing r, and falls to half of its maximum value at $r=a$.

- In a field parallel to the slit, we would set $\phi = 0$. This plane is perpendicular to the vertical plane, which used the radial component E_r, so for the parallel plane we work with the ϕ component E_ϕ.

$$E_\phi = -\frac{1}{r}\frac{\partial}{\partial\phi}U(r,\phi)\Big|_{\phi=0}$$

As before, the sum simplifies considerably to a geometric series. The result turns out to be

$$E_\phi = -\frac{2U_0}{\pi a}\frac{1}{1-\left(\frac{r}{a}\right)^2}$$

Note that E_ϕ diverges at $r=a$. This is a consequence of the very small slit spacing between the capacitor halves at $r=a$; schematically have $E = \frac{U_0}{d} \to \infty$ as $d \to 0$.

1.4 Fourth

1.4.1 Theory

1.4.2 Conducting Cylinder, Continued

First, we completed the cylindrical capacitor problem from the previous set of exercises, starting with the general solution. I wrote the solution in the previous exercise set.

$$U(r,\phi) = \sum_{m=1}^{\infty}\Phi_m(\phi)R_m(m) = \sum_{m=1}^{\infty}\left(A_m\sin(m\phi) + B_m\cos(m\phi)\right)\left(C_m r^m + D_m r^{-m}\right)$$

1.4.3 Conducting Sphere in a Uniform Electric Field

A sphere of radius a placed in a uniform electric field E_0 pointing in the e.g. z direction. Find the electric field U inside and outside the sphere.

- Use spherical coordinates to take advantage of spherical symmetry. This means we want $U(r, \phi, \theta)$. Because of the problem's rotational symmetry, the solution will be independent of ϕ. We need only $U(r, \theta)$.

- The sphere is at a constant potential because it is a conductor. We'll set $U = 0$ inside the sphere. In the space around the sphere, we solve the Laplace equation

  $$\nabla^2 U(r, \theta) = 0$$

  We would separate variables into $U = R(r)\Theta(\theta)$ as usual. The result is

  $$U(r, \theta) = \sum_{l=0}^{\infty} \left[A_l r^l + B_l r^{-(l+1)}\right] P_l(\cos\theta)$$

  where P_l are the Legendre polynomials. **TODO: record the Legendre polynomials**. **TODO: record this general solution**.

- On to the boundary conditions. On the surface we'll set $U(a, \theta)0$. And at infinity, we use the boundary condition

  $$U(r \to \infty, 0) = -E_0 z = -E_0 r \cos\theta$$

  This is the potential of a uniform electric field (at infinity, the potential from the sphere is negligible). This potential is chosen so that

  $$-\frac{\partial U}{\partial z} = E_0$$

  i.e. that the potential at infinity recovers the uniform electric field E_0.

- We'll start with the second boundary condition at $r \to \infty$. Applying the condition to the general solution gives

  $$\sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta) = -E_0 r \cos\theta$$

  Note that the $r^{-(l+1)}$ terms vanish as $r \to \infty$. The entire series equals a single term. Evidently, we get the $\cos\theta$ term from $P_1(\cos\theta) = \cos\theta$ ie. $P_1(x) = x$. So only the $l = 1$ term in the series is non-zero. The non-zero term is

  $$A_1 r \cos\theta = -E_0 r \cos\theta \implies A_1 = -E_0$$

  so we have $A_l = -E_0 \delta_{l1}$. The solution for $U(r, \theta)$ simplifies to

  $$U(r, \theta) = -E_0 r \cos\theta + \sum_{l=1}^{\infty} B_l r^{-(l+1)} P_l(\cos\theta)$$

- Next, the second boundary condition: $U(a,\theta) = 0$. Substituting the condition into the intermediate solution gives

$$E_0 a \cos\theta = \sum_{l=1}^{\infty} B_l a^{-(l+1)} P_l(\cos\theta)$$

Again, the entire series equals only a single term. Again, this will be the $l = 1$ term corresponding to $P_1(\cos\theta) = \cos\theta$. For $l \neq 1$ we have $B_l = 0$. The $l = 1$ term gives

$$E_0 a \cos\theta = B_1 a^{-2} \cos\theta \implies B_1 = E_0 a^3$$

With A_l and B_l known for all l, the final result is

$$U(r,\theta) = -E_0 \cos\theta + \frac{E_0 a^3}{r^2} \cos\theta$$

The first term, $-E_0 \cos\theta$, is the potential of the uniform external field E_0. The second term comes from the sphere. In fact, this second term has the same form as the potential of an electric dipole!

Limiting cases are discussed in the next exercise set.

1.5 Fourth Vaje

1.5.1 Conducting Sphere in a Uniform Electric Field (continued)

- Where we left off last time, we had found the potential due to the sphere was

$$U(r,\theta) = -E_0 \cos\theta + \frac{E_0 a^3}{r^2} \cos\theta,$$

and identified the sphere's contribution $\frac{E_0 a^3}{r^2} \cos\theta$ corresponded to the potential of an electric dipole. Our next step is to explore the sphere's dipole behavior.

- We consider an infinitesimal element of the sphere's surface at the angle θ carrying charge $\mathrm{d}q$, on the upper hemisphere with positive charge and θ. Recall the electric field, as for any conductor, is perpendicular to the surface.

We write Gauss's law for the small surface element, which is simple because the electric field is perpendicular to the surface

$$\int \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{S} = E_\perp \,\mathrm{d}S = \frac{\mathrm{d}q}{\epsilon_0} \implies \frac{\mathrm{d}e}{\mathrm{d}S} = \sigma = \epsilon_0 E_\perp$$

This equality gives us an expression for σ in terms of the electric field $E_\perp$ perpendicular to the surface. We can find $E_\perp$ from the potential:

$$E_\perp = -\frac{\partial U}{\partial r}\bigg|_{r=a} = E_0 \cos\theta + 2E_0 \cos\theta \implies \sigma = 3\epsilon_0 E_0 \cos\theta$$

The charge density's dependence on σ quantitatively demonstrates the sphere's dipole-like charge distribution.

- With the charge density σ in hand, we can find the sphere's electric dipole moment via

$$\boldsymbol{p}_e = \int \tilde{\boldsymbol{r}} \,\mathrm{d}q$$

We qualitatively expect $\boldsymbol{p}_e$ to point upward (from the negative to the positively charged hemisphere), and confirm this analytically. By spherical symmetry, only the z component of $\boldsymbol{p}_e$ is non-zero; this is

$$p_{e_z} = \int \tilde{z}\,\mathrm{d}q = \int (a\cos\theta)\cdot(\sigma\,\mathrm{d}S) = \int (a\cos\theta)\cdot(3\epsilon_0 E_0)\cdot\mathrm{d}S$$

To find $\mathrm{d}S$, we find the area of a small band of width $\mathrm{d}a$ around the sphere's surface. The band's area is $2\pi r\,\mathrm{d}a = 2\pi(a\sin\theta)(a\,\mathrm{d}\theta)$. The dipole moment p_{e_z} is then

$$\begin{aligned} p_{e_z} &= \int_0^\pi (a\cos\theta)\cdot(3\epsilon_0 E_0)\cdot 2\pi(a\sin\theta)(a\,\mathrm{d}\theta) = 6a^3\pi\epsilon_0 E_0 \int_{-1}^{1} \cos^2\theta\,\mathrm{d}[\cos\theta] \\ &= 6a^3\pi\epsilon_0 E_0 \left[\frac{1}{3}\cos^3\theta\right]_{-1}^{1} = 4\pi\epsilon_0 E_0 a^3 \end{aligned}$$

1.5.2 Electric Dipole in a Conducting Spherical Shell

We place an electric dipole with dipole moment $\boldsymbol{p}_e$ in the center of an empty spherical conducting shell of radius a. What is the electric potential inside the shell?

- We use spherical coordinates, which are best suited to the problem's spherical symmetry. By rotational symmetry, the potential depends only on the coordinates r and θ, not ϕ. Besides at the center, the charge density inside the sphere is zero, so we solve the Laplace equation
$$\nabla^2 U(r,\theta) = 0$$
The general solution is
$$U(r,\theta) = \sum_{l=0}^{\infty} \left(A_l r^l + B_l r^{-(l+1)}\right) P_l(\cos\theta)$$
where P_l are the Legendre polynomials.

- We then apply boundary conditions to find a solution specific to our problem. First, the potential on the shell's surface is constant, since the shell is a conductor. For convenience, we'll set $U(a,\theta) = 0$. The second boundary condition concerns the dipole at the sphere's center. Namely, the potential approaches the potential of an electric dipole near the sphere's center. Quantitatively, this condition reads
$$U(r\to 0,\theta) = \frac{p_e\cos\theta}{4\pi\epsilon_0 r^2}$$

- We start with the simpler second boundary condition, (the boundary $r\to 0$ eliminates the r^l-dependent term). Inserted into the general solution, the second condition reads
$$U(r\to 0,\theta) = \sum_{l=0}^{\infty} B_l r^{-(l+1)} P_l(\cos\theta) = \frac{p_e\cos\theta}{4\pi\epsilon_0} r^{-2}$$
Note that the entire series sums to only a single term; for this to work, only the $l=1$ term in the series can be non-zero, leaving
$$B_1 r^{-2}\cos\theta = \frac{p_e\cos\theta}{4\pi\epsilon_0} r^{-2} \implies B_1 = \frac{p_e}{4\pi\epsilon_0} \qquad \text{and} \qquad B_{l\neq 1} = 0$$

Note the use of $P_1(x) = x$. The intermediate solution at this stage is

$$U(r,\theta) = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta) + \frac{p_e}{4\pi\epsilon_0} r^{-2}\cos\theta$$

We apply the second boundary condition $U(a,\theta) = 0$ to get

$$\sum_{l=0}^{\infty} A_l a^l P_l(\cos\theta) = -\frac{p_e}{4\pi\epsilon_0} a^{-2}\cos\theta$$

As before, only the $l = 1$ term can be non-zero to satisfy the equality. The result is

$$A_1 a\cos\theta = -\frac{p_e}{4\pi\epsilon_0} a^{-2}\cos\theta \implies A_1 = -\frac{p_e}{4\pi\epsilon_0 a^3} \qquad \text{and} \qquad A_{l\neq 1} = 0$$

With the coefficients A_l and B_l known, the solution for $U(r,\theta)$ is

$$U(r,\theta) = \left[\frac{p_e}{4\pi\epsilon_0 r^2} - \frac{p_e}{4\pi\epsilon_0 a^3} r\right]\cos\theta = \frac{p_e\cos\theta}{4\pi\epsilon_0}\left[\frac{1}{r^2} - \frac{r}{a^3}\right]$$

The $\frac{1}{r^2}$ term is the dipole's contribution. The $\frac{r}{a^3}$ comes from the charge induced on the conducting shell.

- The induced term is worth a closer look, noting that $r\cos\theta = z$.

$$U_{\text{induced}} = -\frac{p_e}{4\pi\epsilon_0 a} r\cos\theta = -\frac{p_e}{4\pi\epsilon_0 a} z$$

In particular, the associated electric field is

$$E_{\text{induced}} = -\frac{\partial}{\partial z} U_{\text{induced}} = \frac{p_e}{4\pi\epsilon_0 a}$$

In other words, the electric field generated by the induced charge is constant! The uniform field also tells us about the charge distribution on the sphere's surface: to create a uniform field in the z direction, the shell must have a dipole-like charge distribution, with positive charge on the lower hemisphere and negative charge on the upper hemisphere.

- Next, we're interested in the analytic expression for the surface charge density σ. We consider a small surface element $\mathrm{d}S$, and consider the total electric field at that surface. The electric field must be perpendicular to the surface, since the shell is a conductor. Gauss's law applied to the surface element reads

$$-E_\perp \, \mathrm{d}S = \frac{\mathrm{d}q}{\epsilon_0} \implies \sigma \equiv \frac{\mathrm{d}q}{\mathrm{d}S} = -\epsilon_0 E_\perp$$

Note the minus sign, indicating the field's electric flux leaving the surface element from inside the shell. We find an expression for $E_\perp$ from U:

$$E_\perp = -\frac{\partial U}{\partial r}\bigg|_{r=a} = +\frac{p_e\cos\theta}{4\pi\epsilon_0}\left[\frac{2}{r^3} + \frac{1}{a^3}\right]_{r=a} = \frac{3p_e}{4\pi\epsilon_0 a^3}\cos\theta$$

The associated surface charge density is

$$\sigma = -\epsilon_0 E_\perp = -\frac{3p_e}{4\pi a^3}\cos\theta$$

1.5.3 Point Charge Above a Conducting Plane

Consider a positive point charge q a distance d above a large, grounded conducting plane. What is the electric potential in space due to the charge-plane system?

- We will use a trick called the *method of images* to solve the problem. Namely, we imagine a negative point charge $-q$ a distance d below the plane—a mirror image of the original positive charge. The resulting charge distribution is an electric dipole.

  Note: placing an imaginary negative charge a distance d below the plane does not change the field above the plane due to the positive charge.

- Considering both points, the potential at an arbitrary position $\boldsymbol{r}$ from the origin is

$$U(\boldsymbol{r}) = \frac{q}{4\pi\epsilon_0}\frac{1}{|\boldsymbol{r}-\boldsymbol{d}|} - \frac{q}{4\pi\epsilon_0}\frac{1}{|\boldsymbol{r}+\boldsymbol{d}|}$$

  where the vector $\boldsymbol{d}$ points perpendicularly up from the plane toward the positive charge. Introducing an angle θ between $\boldsymbol{d}$ and $\boldsymbol{r}$, we have

$$|\boldsymbol{r} \pm \boldsymbol{d}| = \sqrt{r^2 + d^2 \pm 2rd\cos\theta}$$

- Next, we're interested in the surface charge density $\sigma(\rho)$ on the plane where ρ is the radial distance from the origin. As usual, we start with Gauss's law for a small surface element of the plane:

$$-E_\perp \,\mathrm{d}S = \frac{\mathrm{d}q}{\epsilon_0} \implies \sigma \equiv \frac{\mathrm{d}q}{\mathrm{d}S} = -\sigma E_\perp$$

  To find $E_\perp$, we differentiate U with respect to the vertical coordinate z. First, we introduce z into the expression $|\boldsymbol{r} \pm \boldsymbol{d}|$

$$|\boldsymbol{r} \pm \boldsymbol{d}| = \sqrt{r^2 + d^2 \pm 2rd\cos\theta} = \sqrt{\rho^2 + z^2 + d^2 \pm 2dz}$$

  We then have

$$U(\rho, z) = \frac{q}{4\pi\epsilon_0}\left[\frac{1}{\sqrt{\rho^2 + z^2 + d^2 - 2dz}} - \frac{1}{\sqrt{\rho^2 + z^2 + d^2 + 2dz}}\right]$$

  We then find $E_\perp$ and then σ with

$$\begin{aligned}\sigma = -\epsilon_0 E_\perp = -\epsilon_0 \frac{\partial U}{\partial z}\bigg|_{z=0} &= -\frac{q}{4\pi}\left[\frac{d}{(\rho^2+d^2)^{3/2}} + \frac{d}{(\rho^2+d^2)^{3/2}}\right] \\ &= -\frac{q}{2\pi}\frac{d}{(\rho^2+d^2)^{3/2}}\end{aligned}$$

- With surface charge density σ known, we then ask what is the total charge on the plane. Integrating the plane over rings with area $\mathrm{d}S = 2\pi\rho\,\mathrm{d}\rho$, we have

$$q_{\text{plane}} = \int \sigma \,\mathrm{d}S = -\int_0^\infty \frac{qd}{2\pi(\rho^2+d^2)^{3/2}}(2\pi\rho\,\mathrm{d}\rho)$$

  In terms of the new variable $u = \rho^2 + d^2$, the integral evaluates to

$$q_{\text{plane}} = -\frac{qd}{2}\int_{d^2}^\infty \frac{\mathrm{d}u}{u^{3/2}} = qd\left[\frac{1}{u^{1/2}}\right]_{d^2}^\infty = -q$$

- Summary of what we did: recognize that the field above the plane from the positive charge looks like half the field of an electric dipole. Since we know the solution for a dipole, instead of solving the charge-plane system, we solve the (imaginary) two-charge system, which gives the same field above the plane anyway. We then reuse the upper half of the dipole solution for the single-charge plane system, and set the field below the plane equal to zero. The basic idea is: the field above the plane is the same for both the positive-charge plane system and for a dipole system, so we can use either approach to solve for the field above the plane.

- Next, we ask what is the electrostatic force on the point charge above the plane? First, some theory:

Theory: Electrostatic Force

$$\boldsymbol{F}_e = \epsilon_0 \oint \left[\boldsymbol{E}(\boldsymbol{E}\cdot\hat{\boldsymbol{n}}) - \frac{1}{2}E^2\hat{\boldsymbol{n}}\right] \mathrm{d}S$$

$\hat{\boldsymbol{n}}$ is the normal to the surface. This is the electrostatic force on a point charge in an electric field...? Like with Gauss's law, a good choice of the integration surface, usually taking advantage of the problem's symmetries, tends to simplify the problem. Alternatively, if the electric field vanishes at infinity, we choose a surface that closes at infinity.

- Back to our problem: we choose an infinite surface whose base runs along the plane, then turns upward and closes at infinity to enclose the upper half of space above the plane. The field in this case is the same $E_\perp$ calculated above:

  $$E_\perp = \frac{q}{2\pi\epsilon_0}\frac{d}{(\rho^2+d^2)^{3/2}}$$

  For the part of the surface running parallel to the plane, the normal to the surface $\hat{\boldsymbol{n}}$ points perpendicularly into the plane, parallel to the electric field. The force equation for the bottom half of the surface reads

  $$\boldsymbol{F}_e = \epsilon_0 \oint \left[E^2\hat{\boldsymbol{n}} - \frac{1}{2}E^2\hat{\boldsymbol{n}}\right] \mathrm{d}S = \frac{\epsilon_0}{2}\hat{\boldsymbol{n}}\int E_\perp^2 \,\mathrm{d}S$$

  In fact, the contribution from the upper half of the surface is zero—the upper half extends to infinity, where the electric field vanishes. We only need to integrate over the bottom of the surface, running parallel to the plane. Writing $\mathrm{d}S = 2\pi\rho\,\mathrm{d}\rho$ and substituting in the expression for $E_\perp$, the force reads

  $$\begin{aligned}\boldsymbol{F}_e &= \hat{\boldsymbol{n}}\frac{\epsilon_0}{2}\int_0^\infty \frac{q^2}{4\pi^2\epsilon_0^2}\frac{d^2}{(\rho^2+d^2)^3}2\pi\rho\,\mathrm{d}\rho = \frac{q^2d^2}{4\pi\epsilon_0}\hat{\boldsymbol{n}}\int_0^\infty \frac{\rho}{(\rho^2+d^2)^3}\,\mathrm{d}\rho\\ &= \frac{q^2d^2}{8\pi\epsilon_0}\hat{\boldsymbol{n}}\int_{d^2}^\infty \frac{\mathrm{d}u}{u^3} = -\frac{q^2d^2}{16\pi\epsilon_0}\hat{\boldsymbol{n}}\left[\frac{1}{u^2}\right]_{d^2}^\infty = \frac{q^2}{16\pi\epsilon_0 d^2}\hat{\boldsymbol{n}}\end{aligned}$$

  The force points in the direction of $\hat{\boldsymbol{n}}$—downward into the plane. A final note: if we write

  $$\boldsymbol{F}_e = \frac{q^2}{4\pi\epsilon_0(2d)^2}\hat{\boldsymbol{n}}$$

  the force takes the form of the electric force between a positive and negative charge separated by a distance $2d$—the same situation we used in the method of images above.

1.6 Sixth Exercise Set

1.6.1 Theory

- Recall that we defined electrostatic force on a region of space V as

$$\boldsymbol{F}_e = \epsilon_0 \oint_{\partial V} \left[\boldsymbol{E}(\boldsymbol{E} \cdot \hat{\boldsymbol{n}}) - \frac{1}{2} \boldsymbol{E} \cdot \boldsymbol{n} \right] \mathrm{d}S$$

1.6.2 Force on a Conducting Spherical Shell

We place a conducting sphere of radius a in a homogeneous electric field E_0. Find the electrostatic force on the upper half of the sphere.

- Suppose the field points in the z direction. Recall from the previous exercise set that the potential from the sphere and electric field is

$$U(r, \theta) = -E_0 r \cos\theta + \frac{E_0 a^3}{r^2} \cos\theta$$

  Qualitatively, there are two main contributions to the force on the sphere: an upwards contribution in the positive z direction from the external electric field, and a downward contribution in the negative z direction from the negative charge accumulated on the bottom half of the sphere.

- We are interested in the force on the upper half of the sphere—the next step is to choose a surface around the sphere's upper half that will simplify the force calculation. Recall the field points perpendicularly out of the conducting sphere's surface at all points.

  With this perpendicular field in mind, choose a surface that tightly hugs the sphere's upper half—in this case, the field and normal to the surface $\hat{\boldsymbol{n}}$ are parallel at all points outside the sphere. In the hemisphere plane inside the sphere, there is no field at all. These to facts simplify the dot product $\boldsymbol{E} \cdot \hat{\boldsymbol{n}}$ in the force equation.

- We then have $\boldsymbol{E} \cdot \hat{\boldsymbol{n}} = E$ and $\boldsymbol{E}(\boldsymbol{E} \cdot \hat{\boldsymbol{n}}) = E^2 \hat{\boldsymbol{n}}$. The contribution to the force on along the sphere's outside surface is

$$\boldsymbol{F}_e = \epsilon_0 \int \frac{1}{2} E^2 \hat{\boldsymbol{n}} \, \mathrm{d}S$$

  The contribution from the hemisphere plane through the sphere zero, since $\boldsymbol{E} = 0$ inside the sphere.

- Next, we find the magnitude E on the sphere's surface from the potential $U(r, \theta)$. The field points radially outwards, so we differentiate U with respect to r to get

$$E - \frac{\partial U}{\partial r}\bigg|_{r=a} = E_0 \cos\theta + 2E_0 \cos\theta = 3E_0 \cos\theta$$

  Inserting E into the force equation gives

$$\boldsymbol{F}_e = \epsilon_0 \int \frac{1}{2} \big(3E_0 \cos\theta\big)^2 \hat{\boldsymbol{n}} \, \mathrm{d}S$$

In spherical coordinates, the unit normal $\hat{\boldsymbol{n}}$ to the sphere's surface is $\hat{\boldsymbol{n}} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$. The surface element $\mathrm{d}S$ at the surface $r = a$ is (just like in the previous exercise sets) $\mathrm{d}S = a^2\,\mathrm{d}\phi\sin\theta\,\mathrm{d}\theta$. The force on the sphere's upper half is then

$$\boldsymbol{F}_e = \frac{9\epsilon_0 E_0^2}{2}\int_{\theta=0}^{\pi/2}\int_{\phi=0}^{2\pi}\cos^2\theta\begin{bmatrix}\sin\theta\cos\phi\\ \sin\theta\sin\phi\\ \cos\theta\end{bmatrix}(a^2\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi)$$

Both the x and y components will be zero—integrating $\cos\phi$ and $\sin\phi$ over a full period 2π give zero, while the ϕ contribution to the z component is 2π. We make this explicit with

$$\boldsymbol{F}_e = \frac{9\epsilon_0 E_0^2}{2}\int_{\theta=0}^{\pi/2}\cos^2\theta\begin{bmatrix}0\\ 0\\ 2\pi\cos\theta\end{bmatrix}(a^2\sin\theta\,\mathrm{d}\theta)$$

The non-zero z component F_z is

$$\begin{aligned}F_z &= \frac{9\epsilon_0 E_0^2}{2}\int_{\theta=0}^{\pi/2}2\pi\cos^3\theta(a^2\sin\theta\,\mathrm{d}\theta) = 9\pi\epsilon_0 E_0^2 a^2\int_{\theta=0}^{\pi/2}\cos^3\theta(\sin\theta\,\mathrm{d}\theta)\\ &= 9\pi\epsilon_0 E_0^2 a^2\int_0^1\cos^3\theta\,\mathrm{d}[\cos\theta] = \frac{9\pi\epsilon_0 a^2 E_0^2}{4}\end{aligned}$$

The vector force can be written simply as

$$\boldsymbol{F} = \frac{9\pi\epsilon_0 a^2 E_0^2}{4}\hat{\boldsymbol{z}}$$

In other words, the force on the upper half points upward in the positive z direction.

1.6.3 Point Charge Between Two Conducting Plates

We place two large conducting plates at a right angle to each other, so that the plates come close together but just barely do not touch. We then place a point charge q along the line bisecting the right angle between the plates, at a perpendicular distance a from each plate. Both plates are grounded. What is the electric potential in the region bounded by the plates at large distances from the plates' intersection?

- Assume $r = 0$ along the line connecting the two plates. For a single plate, we could solve the problem with the method of images—see the previous exercise set. With two plates we proceed analogously, with a mirror image for each plate. Because of the two reflections from the two plates, we end up with three imaginary charges plus the one original one in a quadrupole arrangement. (This is hard to describe in words, it is best to see a picture). For large r, the charge arrangement will have the field of an electric quadrupole. Solving the problem thus reduces to a multipole expansion to the quadrupole term.

Theory: Multipole Expansion

- The multipole expansion of U to quadrupole order, using the Einstein summation convention, is

$$U(r) = \frac{1}{4\pi\epsilon_0}\left[\frac{q}{r} + \frac{p_i r_i}{r^3} + \frac{Q_{ij}r_i r_j}{r^5}\right]$$

where $\boldsymbol{p}$ is the electric dipole moment and $\mathbf{Q}$ is the quadrupole moment tensor. *Note*: we could think of the charge q is a scalar monopole moment, creating a logical progression from scalar monopole moment to vector dipole moment to tensor quadrupole moment.

- We find the dipole moment with

$$\boldsymbol{p} = \int \mathrm{d}^3\tilde{r}\rho(\tilde{\boldsymbol{r}})\tilde{\boldsymbol{r}}$$

  We find the quadrupole moment by components:

$$Q_{ij} = \int \mathrm{d}^3\tilde{r}\rho(\tilde{\boldsymbol{r}})\big(3\tilde{r}_i\tilde{r}_j - \delta_{ij}\tilde{r}^2\big)$$

  The discrete analog a configuration of N charges reads

$$Q_{ij} = \sum_{n=1}^{N} q_n\left(3r_{n_i}r_{n_j} - \delta_{ij}r_n^2\right)$$

  Note that both definitions produces a symmetric tensor. Also important: the tensor's trace—the sum of the diagonal elements is zero:

$$\operatorname{tr}\mathbf{Q} = \sum_n q_n\left[3x_n^2 - r_n^2 + 3z_n^2 - r_n^2 + 3z_n^2 - r_n^2\right] = \sum_n q_n\left[3r_n^2 - 3r_n^2\right] = 0$$

Back to Our Problem

- For our imaginary quadrupole configuration of four charges, the total charge, and thus the monopole moment, is zero. Analogously, the total dipole moment of the arrangement, which consists of two positive and two negative charges, is zero—the two dipoles cancel each other out.

  From the three terms in our multipole expansion of $U(r)$, only the quadrupole term remains. We just have to calculate the quadrupole tensor Q_{ij}. We label the four charges in the imaginary quadrupole configuration as 1, 2, 3, and 4, where 1 is the original positive charge in the upper right corner, 2 is the negative image charge in the upper left corner, 3 is the positive image charge in the lower left corner, and 4 is the negative image charge in the lower right corner.

- Using the discrete formula for Q_{ij}, the first component Q_{xx} is

$$\begin{aligned} Q_{xx} &= \sum_{n=1}^{N} q_n\left(3x_n^2 - \delta_{ij}r_n^2\right) = q\left(3a^2 - 2a^2\right) + (-q)\left(3a^2 - 2a^2\right) \\ &= q\left(3a^2 - 2a^2\right) + (-q)\left(3a^2 - 2a^2\right) = 0 \end{aligned}$$

  The other diagonal terms Q_{yy} and Q_{zz} will analogously sum to zero.

  All off-diagonal terms with a z component are also zero, since the charges lie in a plane with $z = 0$. We thus have $Q_{xz} = Q_{zx} = Q_{yz} = Q_{zy} = 0$. We have just two terms left calculate: Q_{xy} and Q_{yx}. By the tensor's symmetry, the two are equal, so we really only have one term:

$$\begin{aligned} Q_{xy} &= \sum_{n=1}^{N} q_n\left(3x_ny_n - 0\cdot r_n^2\right) = 3qa^2 + 3(-q)(-a^2) + 3qa^2 + 3(-q)(-a^2) \\ &= 12qa^2 \end{aligned}$$

The quadrupole tensor is

$$\mathbf{Q} = \begin{bmatrix} 0 & 12qa^2 & 0 \\ 12qa^2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

As expected, the tensor is symmetric with trace $\operatorname{tr}\mathbf{Q} = 0$.

- Recall the quadrupole expansion of $U(r)$:

$$U(r) = \frac{1}{4\pi\epsilon_0}\left[\frac{q}{r} + \frac{p_i r_i}{r^3} + \frac{Q_{ij} r_i r_j}{r^5}\right]$$

In our case, with $q = 0$ and $\boldsymbol{p} = 0$, we have

$$\begin{aligned} U(r) &= \frac{1}{4\pi\epsilon_0}\left[\frac{12qa^2xy}{r^5} + \frac{12qa^2yx}{r^5} + 0 + \cdots + 0\right] = \frac{6qa^2}{4\pi\epsilon_0}\frac{xy}{r^5} \\ &= \frac{6qa^2}{4\pi\epsilon_0}\frac{\cos\phi\sin\phi\sin^2\theta}{r^3} \end{aligned}$$

The second line uses the spherical coordinates $x = r\cos\phi\sin\theta$ and $y = r\sin\phi\sin\theta$. In fact, the expression for $U(r)$ takes the exact same form as the wave function of a d electron orbital (angular momentum quantum number $l = 2$) in a hydrogen atom. The equipotential surfaces of $U(r)$ have the same spatial distribution as the d_{xy} orbitals for a hydrogen wave function.

2 Magnetostatics

2.1 Seventh Exercise Set

2.1.1 Theory

- We will need to use two more Maxwell equations for magnetostatics. The first is

$$\nabla \cdot \boldsymbol{B} = 0$$

This equation rules out the possibility of magnetic monopoles and allows $\boldsymbol{B}$ to be written as the curl of a vector potential
A as

$$\boldsymbol{B} = \nabla \times \boldsymbol{A}$$

The second Maxwell equation is

$$\nabla \times \boldsymbol{B} = \mu_0\left(\boldsymbol{j} + \epsilon_0\frac{\partial \boldsymbol{E}}{\partial t}\right)$$

For static situations with a constant electric field this simplifies to

$$\nabla \times \boldsymbol{B} = \mu_0\boldsymbol{j}$$

Substituting the expression $\boldsymbol{B} = \nabla \times \boldsymbol{A}$ into the static second Maxwell equation produces

$$\nabla \times \big(\nabla \times \boldsymbol{A}\big) = \nabla \cdot \big(\nabla \cdot \boldsymbol{A}\big) - \nabla^2\boldsymbol{A} = \mu_0\boldsymbol{j}$$

The magnetic vector potential is defined only up to a constant; we usually choose $\boldsymbol{A}$ so that $\nabla \cdot \boldsymbol{A} = 0$. In this convention, we have

$$\nabla^2 \boldsymbol{A} = -\mu_0 \boldsymbol{j}$$

Which is a vector analog of the Poisson equation $\nabla^2 U = -\frac{\rho}{\epsilon_0}$ from electrostatics. Similarly to how the electrostatic potential U at a point $\boldsymbol{r}$ in region of space with charge density ρ is found with

$$U = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\tilde{\boldsymbol{r}})\, \mathrm{d}^3\tilde{r}}{|\boldsymbol{r} - \tilde{r}|}$$

we find the magnetic potential at a point $\boldsymbol{r}$ in region of space with magnetic flux $\boldsymbol{j}$ with

$$\boldsymbol{A} = \frac{\mu_0}{4\pi} \int \frac{\boldsymbol{j}(\tilde{\boldsymbol{r}})\, \mathrm{d}^3\tilde{r}}{|\boldsymbol{r} - \tilde{r}|}$$

We often encounter problems involving one-dimensional conductors, in which $\boldsymbol{j}$ is non-zero only along the conductor. In this case, the expression for $\boldsymbol{j}(\tilde{\boldsymbol{r}})\, \mathrm{d}^3\tilde{r}$ simplifies to

$$\boldsymbol{j}(\tilde{r})\, \mathrm{d}^3\tilde{r} = \boldsymbol{j}(\tilde{\boldsymbol{r}})\, \mathrm{d}\tilde{S}\, \mathrm{d}\tilde{l} = \big(I\hat{\boldsymbol{t}}\big)\, \mathrm{d}\tilde{l} = I\, \mathrm{d}\tilde{\boldsymbol{l}}$$

where $\hat{\boldsymbol{t}}$ is the unit normal vector tangent to the conductor, I is the current through the conductor at the point $\tilde{\boldsymbol{r}}$, $\mathrm{d}\tilde{l}$ is a small distance element along the conductor's length, and $\mathrm{d}\tilde{\boldsymbol{l}}$ is length element $\mathrm{d}\tilde{l}$ with the direction of $\hat{\boldsymbol{t}}$. The magnetic vector potential for a one-dimensional conductor then simplifies to

$$\boldsymbol{A}(\boldsymbol{r}) = \frac{\mu_0 I}{4\pi} \int \frac{\mathrm{d}\tilde{\boldsymbol{l}}}{|\boldsymbol{r} - \tilde{\boldsymbol{r}}|}$$