

# Linear Algebra Review

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$A \in \mathbb{C}^{n \times n}$  is Hermitian if  $A = A^\dagger$

$A \in \mathbb{R}^{n \times n}$  is Hermitian if  $A = A^T$

$O: V \rightarrow V$  Hermitian if  $\langle Ox, y \rangle = \langle x, Oy \rangle$  for all  $x, y \in V$

Hermitian operators have real eigenvalues, orthogonal eigenfunctions and correspond to observable quantities

$U \in \mathbb{C}^{n \times n}$  unitary if  $UU^\dagger = U^\dagger U = I \implies U^{-1} = U^\dagger$

$O \in \mathbb{R}^{n \times n}$  orthogonal if  $OO^T = O^T O = I \implies O^{-1} = O^T$

## Commutator

$$[A, B] = AB - BA$$

$$[A, B] = 0 \iff AB = BA \iff B = ABA^{-1} = A^{-1}BA$$

$$[B, A] = -[A, B] \quad [AB, C] = A[B, C] + [A, C]B$$

$$\{A, B\} = AB + BA$$

# Quantum Mechanics in 1D

## Some Basic Operators

$$p_x \rightarrow -i\hbar \frac{\partial}{\partial x} \quad \mathbf{p} \rightarrow -i\hbar \nabla$$

## Uncertainty Principle

$$(\Delta A \Delta B)^2 \geq (\frac{1}{2} |\langle [A, B] \rangle|)^2$$

$$\Delta x \Delta p \geq \frac{\hbar}{2} \quad \Delta E \Delta t \geq \frac{\hbar}{2}$$

## Expectation Values, etc...

$$\langle \mathcal{O} \rangle = \langle \psi | \mathcal{O} | \psi \rangle \equiv \langle \psi | \mathcal{O} \psi \rangle \equiv \int \psi^* \mathcal{O} \psi \, dx$$

$$\langle \psi | \mathcal{O} \psi \rangle = \langle \mathcal{O}^* \psi | \psi \rangle \text{ and } \mathcal{O} = \mathcal{O}^* \implies \langle \psi | \mathcal{O} \psi \rangle = \langle \mathcal{O} \psi | \psi \rangle$$

$$\langle \psi_m | \psi_n \rangle = \int \psi_m^* \psi_n \, dx$$

$$\langle \mathcal{O} \psi | \mathcal{O} \psi \rangle = \|\mathcal{O} \psi\|^2$$

$$\langle A^\dagger \rangle = \langle A \rangle^* \implies \langle A \rangle + \langle A^\dagger \rangle = \langle A \rangle + \langle A \rangle^* = 2 \operatorname{Re} \langle A \rangle$$

## Uncertainties

$$\Delta x(t) = \sqrt{\langle x^2, t \rangle - \langle x, t \rangle^2} \quad \Delta p(t) = \sqrt{\langle p^2, t \rangle - \langle p, t \rangle^2}$$

$$(\Delta A)^2 = \langle A^2 \rangle - \langle A \rangle^2 = \langle (A - \langle A \rangle)^2 \rangle$$

## Basis Expansion

$$\psi(x, 0) = \int c(k) \frac{e^{ikx}}{\sqrt{2\pi}} \, dk \implies c(k) = \int \psi(x, 0) \frac{e^{-ikx}}{\sqrt{2\pi}} \, dx$$

Given orthonormal basis  $\langle \psi_n | \psi_m \rangle = \delta_{nm}$ :

$$\psi = \sum_n c_n \psi_n \quad c_n = \int \psi_n^* \psi \, dx \equiv \langle \psi_n(x) | \psi(x, 0) \rangle$$

## Observation

Expand  $|\psi\rangle$  in basis of operator being observed

Possible outcomes are eigenvalues of each basis state

Probability is square of eigenstate coefficient

$\psi$  then collapses to eigenfunction of observed eigenvalue

## Free Particle

$$\psi_k(x) = \frac{e^{ikx}}{\sqrt{2\pi}} \quad E_k = \frac{\hbar^2 k^2}{2m}$$

$$\frac{1}{2\pi} \int e^{i(\tilde{k}-k)x} \, dx \equiv \delta(\tilde{k}-k)$$

## Infinite Potential Well

$$V(|x| < \frac{a}{2}) = 0 \quad V(|x| > \frac{a}{2}) \rightarrow \infty$$

$$\psi_n(x) = \sqrt{\frac{2}{a}} \sin \left[ \frac{n\pi}{a} \left( x - x_c + \frac{a}{2} \right) \right] \quad E_n = \frac{n^2 \pi^2 \hbar^2}{2ma^2}$$

$$\text{If } x \in [0, a], \quad \psi_n(x) = \sqrt{\frac{2}{a}} \begin{cases} \sin \frac{n\pi x}{a} & n \text{ even} \\ \cos \frac{n\pi x}{a} & n \text{ odd} \end{cases}$$

## Finite Potential Well

$$V(|x| < \frac{a}{2}) = 0 \quad V(|x| > \frac{a}{2}) = V_0$$

$$\kappa^2 = \frac{2mE}{\hbar^2} \quad k^2 = \frac{2m(V_0-E)}{\hbar^2}, \quad E > 0$$

$$\text{Sym: } \kappa = k \tan \left( \frac{ka}{2} \right) \quad \text{Asym: } \kappa = -k \cot \left( \frac{ka}{2} \right)$$

$$u = ka \quad u_0^2 = \frac{2mV_0 a^2}{\hbar^2} \quad \kappa^2 = \frac{u_0^2 - u^2}{a^2}$$

$$\tan\left(\frac{u}{2}\right) = \sqrt{\frac{u_0^2}{u^2} - 1} \quad \cot\left(\frac{u}{2}\right) = -\sqrt{\frac{u_0^2}{u^2} - 1}$$

## Delta Function Well

$$V(x) = -\lambda \delta(x)$$

$$\text{Boundary Condition: } \psi'(0_+) - \psi'(0_-) = -\frac{2m\lambda\psi(0)}{\hbar^2}$$

$$\psi(x) = \sqrt{\kappa} e^{-\kappa|x|}, \quad E_0 = \frac{m\lambda^2}{2\hbar^2}, \quad \kappa = \frac{m\lambda}{\hbar^2}$$

## Scattering

$$j(x) = \frac{\hbar}{2mi} [\psi^* \psi' - \psi'^* \psi] = \frac{\hbar}{m} \operatorname{Im} \{ \psi^* \psi' \}$$

Regions 1 and 3 free, region 2 with  $E > V = V(x)$

$$\psi_2 = Cf(x) + Dg(x)$$

$$\psi_1 = \frac{A_1}{\sqrt{v_1}} e^{ik_1 x} + \frac{B_1}{\sqrt{v_1}} e^{-ik_1 x} \quad \psi_3 = \frac{A_3}{\sqrt{v_3}} e^{-ik_3 x} + \frac{B_3}{\sqrt{v_3}} e^{ik_3 x}$$

$$v \equiv \frac{\hbar k}{m}, \quad k^2 = \frac{2mE}{\hbar^2}$$

$$j_{1,3} = |A_{1,3}|^2 - |B_{1,3}|^2$$

$$\begin{bmatrix} B_1 \\ B_3 \end{bmatrix} = \begin{bmatrix} r & \tilde{t} \\ t & \tilde{r} \end{bmatrix} \begin{bmatrix} A_1 \\ A_3 \end{bmatrix} \iff \mathbf{B} = \mathbf{S} \mathbf{A}$$

$$T = |t|^2 \quad R = |r|^2 \quad \tilde{T} = |\tilde{t}|^2 \quad \tilde{R} = |\tilde{r}|^2$$

$$T + R = 1 \text{ and } \tilde{T} + \tilde{R} = 1$$

$$j_{\text{in}} = \mathbf{A}^\dagger \mathbf{A} \quad j_{\text{out}} = \mathbf{B}^\dagger \mathbf{B} \quad j_{\text{in}} \equiv j_{\text{out}}$$

$$j_{\text{in}} = j_{\text{out}} \implies (\mathbf{A}^\dagger \mathbf{S}^\dagger)(\mathbf{S} \mathbf{A}) \implies \mathbf{S}^\dagger \mathbf{S} = \mathbf{I}$$

## Delta Function Scattering

$$H = \frac{p^2}{2m} + \lambda \delta(x), \quad E > 0$$

$$r = \tilde{r} = -\frac{i\kappa}{k+i\kappa}, \quad t = \tilde{t} = \frac{k}{k+i\kappa}$$

$$k^2 = \frac{2mE}{\hbar^2} \quad \kappa = \frac{m\lambda}{\hbar^2}$$

$$\mathbf{S} = \begin{bmatrix} r & t \\ t & r \end{bmatrix} = \frac{1}{k+i\kappa} \begin{bmatrix} -i\kappa & k \\ k & -i\kappa \end{bmatrix}$$

## Harmonic Oscillator

$$H = \frac{p^2}{2m} + \frac{kx^2}{2} = \frac{p^2}{2m} + \frac{m\omega}{2} x^2, \quad \omega = \sqrt{\frac{k}{m}}$$

$$H |n\rangle = \hbar\omega \left( n + \frac{1}{2} \right) |n\rangle, \quad n = 0, 1, 2, \dots$$

$$a = \frac{1}{\sqrt{2}} \left( \frac{x}{x_0} + i \frac{p}{p_0} \right) \quad a^\dagger = \frac{1}{\sqrt{2}} \left( \frac{x}{x_0} - i \frac{p}{p_0} \right)$$

$$[a, a^\dagger] = 1 \quad [a^\dagger, a] = -1$$

$$a |n\rangle = \sqrt{n} |n-1\rangle \quad a^\dagger |n\rangle = \sqrt{n+1} |n+1\rangle$$

$$x_0 = \sqrt{\frac{\hbar}{m\omega}} \quad p_0 = \frac{\hbar}{x_0}$$

$$x = \frac{x_0}{\sqrt{2}} (a + a^\dagger) \quad p = \frac{p_0}{\sqrt{2}i} (a - a^\dagger) \quad H = \hbar\omega \left( a^\dagger a + \frac{1}{2} \right)$$

$$a^\dagger a |n\rangle = n |n\rangle \quad |n\rangle = \frac{(a^\dagger)^n}{\sqrt{n!}} |0\rangle$$

$$\text{Ehrenfest: } \frac{d}{dt} \langle x, t \rangle = \frac{\langle p, t \rangle}{m}, \quad \frac{d}{dt} \langle p, t \rangle = \left\langle -\frac{dV}{dx} \right\rangle = -k \langle x, t \rangle$$

## Coherent States

$$a |z\rangle = z |z\rangle, \quad z \in \mathbb{C} \quad \sum_n c_n a |n\rangle = \sum_n c_n z |n\rangle$$

$$c_{n+1} \sqrt{n+1} = c_n z \implies c_n = \frac{z^n}{\sqrt{n!}} c_0$$

$$|z\rangle = \sum_n c_n |n\rangle = c_0 \sum_n \frac{z^n}{\sqrt{n!}} |n\rangle, \quad c_0 = e^{-\frac{|z|^2}{2}}$$

$$|z\rangle = e^{-\frac{|z|^2}{2}} e^{za^\dagger} |0\rangle$$

$$|z, t\rangle = e^{-i\frac{\omega}{2}t} |z e^{-i\omega t}\rangle \quad z(t) = z e^{-i\omega t}$$

$$\langle z | (a^\dagger)^n a^m | z \rangle = (z^*)^n z^m$$

$$\langle x \rangle = \sqrt{2} x_0 \operatorname{Re} z \quad \langle x^2 \rangle = x_0^2 \left( 2 \operatorname{Re}^2 z + \frac{1}{2} \right)$$

$$\langle p \rangle = \sqrt{2} p_0 \operatorname{Im} z \quad \langle p^2 \rangle = p_0^2 \left( 2 \operatorname{Im}^2 z + \frac{1}{2} \right)$$

$$\langle H \rangle = \frac{\hbar\omega}{2} + \hbar\omega |z|^2 \quad \langle H^2 \rangle = \hbar^2 \omega^2 (|z|^4 + 2|z|^2 + \frac{1}{4})$$

## 2D Isotropic Oscillator

$$k_x = k_y \equiv k \quad \omega_x = \omega_y \equiv \omega = \sqrt{\frac{k}{m}}$$

$$V(\mathbf{r}) = \frac{1}{2}k(x^2 + y^2) = \frac{1}{2}kr^2$$

$$H|n_x n_y\rangle = \hbar\omega(n_x + n_y + 1)|n_x n_y\rangle$$

### Gaussian Wave Packet

$$\psi(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\langle x \rangle)^2}{4\sigma^2}\right) e^{-i\frac{\langle p \rangle}{\hbar}x}$$

$$\Delta x(0) = \sigma \implies \langle x^2, 0 \rangle = \sigma^2 + \langle x, 0 \rangle^2$$

$$\Delta p(0) = \frac{\hbar}{2\Delta x(0)} = \frac{\hbar}{2\sigma} \implies \langle p^2, 0 \rangle = \frac{\hbar^2}{4\sigma^2} + \langle p, 0 \rangle^2$$

### Time Evolution

Expand  $\psi(x, 0)$  over eigenfunctions  $\psi_n(x)$  solving

$$H\psi_n(x) = E_n\psi_n(x) \text{ to get } \psi(x, 0) = \sum_n c_n \psi_n(x).$$

$$\text{Find } \psi(x, t) \text{ with } \psi(x, t) = \sum_n c_n e^{-i\frac{E_n}{\hbar}t} \psi_n(x).$$

$$|\psi, t\rangle \equiv \sum_n c_n e^{-i\frac{E_n}{\hbar}t} |n\rangle = e^{-i\frac{H}{\hbar}t} |\psi, 0\rangle$$

### Time-Dependent Expectation Value

$$\text{Heisenberg approach: } \langle \mathcal{O}, t \rangle = \langle \psi, 0 | \mathcal{O}(t) | \psi, 0 \rangle$$

$$\text{Schrödinger approach: } \langle \mathcal{O}, t \rangle = \langle \psi, t | \mathcal{O} | \psi, t \rangle$$

### Time-Dependent Operator

$$\mathcal{O}(t) = e^{i\frac{H}{\hbar}t} \mathcal{O} e^{-i\frac{H}{\hbar}t} \quad \frac{d}{dt} \mathcal{O}(t) = \frac{i}{\hbar} [H, \mathcal{O}](t)$$

$$\mathcal{O}(t)|_{t=0} = e^{i\frac{H}{\hbar} \cdot 0} \mathcal{O} e^{-i\frac{H}{\hbar} \cdot 0} = \mathcal{O}$$

$$(A \cdot B)(t) = A(t)B(t)$$

### Angular Momentum

$$L_z = xp_y - yp_x \rightarrow -\hbar \frac{\partial}{\partial \phi}$$

$$L_z |lm\rangle = m\hbar |lm\rangle \quad L^2 |lm\rangle = l(l+1)\hbar^2 |lm\rangle$$

$$m = -l, -l+1, \dots, l-1, l$$

$$\langle \mathbf{r} | lm \rangle = Y_l^m(\vartheta, \phi) = (-1)^m \sqrt{\frac{2l+1}{4\pi} \frac{(l-m)!}{(l+m)!}} P_l^m(\cos \vartheta) e^{im\phi}$$

$$L_{\pm} |lm\rangle = \hbar \sqrt{l(l+1-m(m\pm 1))} |l, m\pm 1\rangle$$

$$\mathbf{L} \cdot \hat{\mathbf{e}} |\psi\rangle = \hbar l |\psi\rangle \implies \mathbf{L} \parallel \hat{\mathbf{e}}$$

$$L_+ \equiv L_x + iL_y \quad L_- \equiv L_x - iL_y$$

$$L_+ |lm\rangle = \hbar \sqrt{l(l+1-m(m+1))} |l, m+1\rangle$$

$$L_- |lm\rangle = \hbar \sqrt{l(l+1-m(m-1))} |l, m-1\rangle$$

$$L_x = \frac{L_+ + L_-}{2} \quad L_y = \frac{L_+ - L_-}{2i}$$

$$\langle L_x, t \rangle = \text{Re} \langle L_+, t \rangle \quad \langle L_y, t \rangle = \text{Im} \langle L_+, t \rangle \quad \boldsymbol{\mu} = \gamma \mathbf{L}$$

### Spin

$$S^2 = |sm_s\rangle = \hbar^2 s(s+1) |sm_s\rangle$$

$$S_z |sm_s\rangle = \hbar m_s |sm_s\rangle$$

$$|\frac{1}{2} \frac{1}{2}\rangle \equiv |\uparrow\rangle \text{ and } |\frac{1}{2} - \frac{1}{2}\rangle \equiv |\downarrow\rangle$$

$$S_+ = S_x + iS_y$$

$$S_- = S_x - iS_y$$

$$S_x = \frac{S_+ + S_-}{2}$$

$$S_y = \frac{S_+ - S_-}{2i}$$

$$S_+ |sm_s\rangle = \hbar \sqrt{s(s+1) - m_s(m_s+1)} |s, m_s+1\rangle$$

$$S_- |sm_s\rangle = \hbar \sqrt{s(s+1) - m_s(m_s-1)} |s, m_s-1\rangle$$

$$\mathbf{S} \cdot \hat{\mathbf{e}} |\psi_s\rangle = \frac{\hbar}{2} |\psi_s\rangle \text{ implies}$$

$$\mathbf{S} \parallel \hat{\mathbf{e}} = (\cos \varphi \sin \theta, \sin \varphi \sin \theta, \cos \theta)$$

$$|\psi_s\rangle = \cos \frac{\theta}{2} |\uparrow\rangle + \sin \frac{\theta}{2} e^{i\varphi} |\downarrow\rangle$$

$$\text{Useful: } \mathbf{k} = (k_x, k_y) \text{ to } k_x + ik_y = ke^{i\phi}, k_x - ik_y = ke^{-i\phi}$$

### Spin Matrices

$$\mathbf{S} = \frac{\hbar}{2} \boldsymbol{\sigma} \quad \boldsymbol{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$$

$$\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \quad \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\{\sigma_a, \sigma_b\} = 2\delta_{ab}I$$

### Two-Particle Spin System

Product basis:  $|s_1 m_1 s_2 m_2\rangle$ .

Total angular momentum basis:  $|sm_s s_1 s_2\rangle$

$$|s_1 - s_2| \leq s \leq s_1 + s_2 \quad m_s = -s, -s+1, \dots, s-1, s.$$

### Time Reversal

#### TODO

For spin  $s = 1/2$  particles  $T = i\sigma_y K$ ;  $K : \psi \mapsto \psi^*$

### Nondegenerate Perturbations

$$H = H_0 + H' \quad H_0 |n_0\rangle = E_n^{(0)} |n_0\rangle$$

$$E_n = E_n^{(0)} + \langle n_0 | H' | n_0 \rangle + \sum_{m \neq n} \frac{|\langle m_0 | H' | n_0 \rangle|^2}{E_n^{(0)} - E_m^{(0)}}$$

$$|n\rangle = |n_0\rangle + \sum_{m \neq n} \frac{|\langle m_0 | H' | n_0 \rangle|^2}{E_n^{(0)} - E_m^{(0)}} |m_0\rangle$$

### Degenerate Perturbations

$$H = H_0 + H' \quad H_0 |\psi\rangle = E^{(0)} |\psi\rangle$$

$E^{(0)}$  is  $N$ -times degenerate...

$$\dots \text{ and } |\psi\rangle = \sum_i^N c_i |i\rangle \quad H |i\rangle = E^{(0)} |i\rangle \quad \forall i$$

$$\text{Create matrix } \mathbf{P} = \begin{pmatrix} \langle 1 | H' | 1 \rangle & \dots & \langle 1 | H' | N \rangle \\ \langle 2 | H' | 1 \rangle & \dots & \langle 2 | H' | N \rangle \\ \vdots & \ddots & \vdots \\ \langle N | H' | 1 \rangle & \dots & \langle N | H' | N \rangle \end{pmatrix}$$

Diagonalize  $\mathbf{P}$ ; find eigenvalues  $\lambda_j$  and eigenvectors  $\mathbf{u}_j$ .

$$E_j = E^{(0)} + \lambda_j \quad \text{and} \quad |\psi_j\rangle = \sum_{k=1}^N u_{jk} |k\rangle$$

### Time-Dependent Perturbation

$$H(t) = H_0 + H'(t)$$

$$H_0 |n\rangle = E_n |n\rangle$$

$$i \frac{\partial}{\partial t} |\psi, t\rangle = H(t) |\psi, t\rangle$$

$$|\psi, t\rangle = \sum_n c_n(t) |n, t\rangle \quad |n, t\rangle = |n\rangle e^{-i\frac{E_n}{\hbar}t}$$

$$c_n(t) \approx c_n(t_0) - \frac{i}{\hbar} \int_{t_0}^t \langle n, t' | H'(t') | m, t' \rangle c_m(t_0) dt'$$

### Useful Matrix Element Techniques

$$[\mathcal{O}, L_z] = 0 \implies \langle nlm | \mathcal{O} | n'l'm' \rangle = \langle nlm | \mathcal{O} | n'l'm' \rangle \delta_{mm'}$$

$$\{\mathcal{O}, \mathcal{P}\} = 0 \implies \langle nlm | \mathcal{O} | n'l'm' \rangle = 0 \text{ if } (-1)^{l'} + (-1)^l \neq 0$$