Linear Algebra Review

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$A \in \mathbb{C}^{n\times n}$ is *Hermitian* if $A = A^\dagger$
$A \in \mathbb{R}^{n\times n}$ is *Hermitian* if $A = A^T$
$O : V \to V$ *Hermitian* if $\langle Ox, y\rangle = \langle x, Oy\rangle$ for all $x, y \in V$
Hermitian operators have real eigenvalues, orthogonal eigenfunctions and correspond to observable quantities
$U \in \mathbb{C}^{n\times n}$ *unitary* if $UU^\dagger = U^\dagger U = I \Longrightarrow U^{-1} = U^\dagger$
$O \in \mathbb{R}^{n\times n}$ *orthogonal* if $OO^T = O^T O = I \Longrightarrow O^{-1} = O^T$

Commutator

$[A, B] = AB - BA$
$[A, B] = 0 \Longleftrightarrow AB = BA \Longleftrightarrow B = ABA^{-1} = A^{-1}BA$
$[B, A] = -[A, B] \qquad [AB, C] = A[B, C] + [A, C]B$
$\{A, B\} = AB + BA$

Quantum Mechanics in 1D

Some Basic Operators

$p_x \to -i\hbar\frac{\partial}{\partial x} \quad \boldsymbol{p} \to -i\hbar\nabla$

Uncertainty Principle

$(\Delta A \Delta B)^2 \geq \left(\frac{1}{2}|\langle [A, B]\rangle|\right)^2$
$\Delta x \Delta p \geq \frac{\hbar}{2} \quad \Delta E \Delta t \geq \frac{\hbar}{2}$

Expectation Values, etc...

$\langle \mathcal{O}\rangle = \langle\psi|\,\mathcal{O}\,|\psi\rangle \equiv \langle\psi|\mathcal{O}\psi\rangle \equiv \int \psi^* \mathcal{O}\psi \,\mathrm{d}x$
$\langle\psi|\mathcal{O}\psi\rangle = \langle\mathcal{O}^*\psi|\psi\rangle$ and $\mathcal{O} = \mathcal{O}^* \Longrightarrow \langle\psi|\mathcal{O}\psi\rangle = \langle\mathcal{O}\psi|\psi\rangle$
$\langle\psi_m|\psi_n\rangle = \int \psi_m^*\psi_n \,\mathrm{d}x$
$\langle\mathcal{O}\psi|\mathcal{O}\psi\rangle = \|\mathcal{O}\psi\|^2$
$\langle A^\dagger\rangle = \langle A\rangle^* \Longrightarrow \langle A\rangle + \langle A^\dagger\rangle = \langle A\rangle + \langle A\rangle^* = 2\,\mathrm{Re}\,\langle A\rangle$

Uncertainties

$\Delta x(t) = \sqrt{\langle x^2, t\rangle - \langle x, t\rangle^2} \quad \Delta p(t) = \sqrt{\langle p^2, t\rangle - \langle p, t\rangle^2}$
$(\Delta A)^2 = \langle A^2\rangle - \langle A\rangle^2 = \langle (A - \langle A\rangle)\rangle$

Basis Expansion

$\psi(x, 0) = \int c(k)\frac{e^{ikx}}{\sqrt{2\pi}}\,\mathrm{d}k \Longrightarrow c(k) = \int \psi(x, 0)\frac{e^{-ikx}}{\sqrt{2\pi}}\,\mathrm{d}x$
Given orthonormal basis $\langle\psi_n|\psi_m\rangle = \delta_{nm}$:
$\psi = \sum_n c_n\psi_n \qquad c_n = \int \psi_n^*\psi\,\mathrm{d}x \equiv \langle\psi_n(x)|\psi(x, 0)\rangle$

Observation

Expand $|\psi\rangle$ in basis of operator being observed
Possible outcomes are eigenvalues of each basis state
Probability is square of eigenstate coefficient
ψ then collapses to eigenfunction of observed eigenvalue

Free Particle

$\psi_k(x) = \frac{e^{ikx}}{\sqrt{2\pi}} \qquad E_k = \frac{\hbar^2 k^2}{2m}$
$\frac{1}{2\pi}\int e^{i(\tilde{k}-k)x}\,\mathrm{d}x \equiv \delta(\tilde{k} - k)$

Infinite Potential Well

$V(|x| < \frac{a}{2}) = 0 \qquad V(|x| > \frac{a}{2}) \to \infty$
$\psi_n(x) = \sqrt{\frac{2}{a}}\sin\left[\frac{n\pi}{a}\left(x - x_c + \frac{a}{2}\right)\right] \qquad E_n = \frac{n^2\pi^2\hbar^2}{2ma^2}$
If $x \in [0, a]$, $\quad \psi_n(x) = \sqrt{\frac{2}{a}}\begin{cases}\sin\frac{n\pi x}{a} & n \text{ even}\\ \cos\frac{n\pi x}{a} & n \text{ odd}\end{cases}$

Finite Potential Well

$V(|x| < \frac{a}{2}) = 0 \qquad V(|x| > \frac{a}{2}) = V_0$
$\kappa^2 = \frac{2mE}{\hbar^2} \qquad k^2 = \frac{2m(V_0 - E)}{\hbar^2}, \quad E > 0$
Sym: $\kappa = k\tan\left(\frac{ka}{2}\right) \qquad$ Asym: $\kappa = -k\cot\left(\frac{ka}{2}\right)$
$u = ka \qquad u_0^2 = \frac{2mV_0a^2}{\hbar^2} \qquad \kappa^2 = \frac{u_0^2 - u^2}{a^2}$
$\tan\left(\frac{u}{2}\right) = \sqrt{\frac{u_0^2}{u^2} - 1} \qquad \cot\left(\frac{u}{2}\right) = -\sqrt{\frac{u_0^2}{u^2} - 1}$

Delta Function Well

$V(x) = -\lambda\delta(x)$
Boundary Condition: $\psi'(0_+) - \psi'(0_-) = -\frac{2m\lambda\psi(0)}{\hbar^2}$
$\psi(x) = \sqrt{\kappa}e^{-\kappa|x|}, \quad E_0 = \frac{m\lambda^2}{2\hbar^2}, \quad \kappa = \frac{m\lambda}{\hbar^2}$

Scattering

$j(x) = \frac{\hbar}{2mi}\left[\psi^*\psi' - \psi'^*\psi\right] = \frac{\hbar}{m}\,\mathrm{Im}\,\{\psi^*\psi'\}$
Regions 1 and 3 free, region 2 with $E > V = V(x)$
$\psi_2 = Cf(x) + Dg(x)$
$\psi_1 = \frac{A_1}{\sqrt{v_1}}e^{ik_1x} + \frac{B_1}{\sqrt{v_1}}e^{-ik_1x} \qquad \psi_3 = \frac{A_3}{\sqrt{v_3}}e^{-ik_3x} + \frac{B_3}{\sqrt{v_3}}e^{ik_3x}$

$v \equiv \frac{\hbar k}{m}, \qquad k^2 = \frac{2mE}{\hbar^2}$
$j_{1,3} = |A_{1,3}|^2 - |B_{1,3}|^2$
$\begin{bmatrix}B_1\\B_3\end{bmatrix} = \begin{bmatrix}r & \tilde{t}\\ t & \tilde{r}\end{bmatrix}\begin{bmatrix}A_1\\A_3\end{bmatrix} \Longleftrightarrow \boldsymbol{B} = \mathbf{S}\boldsymbol{A}$
$T = |t|^2 \quad R = |r|^2 \qquad \tilde{T} = |\tilde{t}|^2 \quad \tilde{R} = |\tilde{r}|^2$
$T + R = 1$ and $\tilde{T} + \tilde{R} = 1$
$j_{\text{in}} = \boldsymbol{A}^\dagger\boldsymbol{A} \qquad j_{\text{out}} = \boldsymbol{B}^\dagger\boldsymbol{B} \qquad j_{\text{in}} \equiv j_{\text{out}}$
$j_{\text{in}} = j_{\text{out}} \Longrightarrow (\boldsymbol{A}^\dagger\mathbf{S}^\dagger)(\mathbf{S}\boldsymbol{A}) \Longrightarrow \boldsymbol{S}^\dagger\mathbf{S} = \mathbf{I}$

Delta Function Scattering

$H = \frac{p^2}{2m} + \lambda\delta(x), \quad E > 0$
$r = \tilde{r} = -\frac{i\kappa}{k + i\kappa}, \quad t = \tilde{t} = \frac{k}{k + i\kappa}$
$k^2 = \frac{2mE}{\hbar^2} \qquad \kappa = \frac{m\lambda}{\hbar^2}$
$\mathbf{S} = \begin{bmatrix}r & t\\ t & r\end{bmatrix} = \frac{1}{k + i\kappa}\begin{bmatrix}-i\kappa & k\\ k & -i\kappa\end{bmatrix}$

Harmonic Oscillator

$H = \frac{p^2}{2m} + \frac{kx^2}{2} = \frac{p^2}{2m} + \frac{m\omega}{2}x^2, \qquad \omega = \sqrt{\frac{k}{m}}$
$H|n\rangle = \hbar\omega\left(n + \frac{1}{2}\right)|n\rangle, \quad n = 0, 1, 2, \ldots$
$a = \frac{1}{\sqrt{2}}\left(\frac{x}{x_0} + i\frac{p}{p_0}\right) \qquad a^\dagger = \frac{1}{\sqrt{2}}\left(\frac{x}{x_0} - i\frac{p}{p_0}\right)$
$[a, a^\dagger] = 1 \qquad [a^\dagger, a] = -1$
$a|n\rangle = \sqrt{n}\,|n - 1\rangle \qquad a^\dagger|n\rangle = \sqrt{n + 1}\,|n + 1\rangle$
$x_0 = \sqrt{\frac{\hbar}{m\omega}} \qquad p_0 = \frac{\hbar}{x_0}$
$x = \frac{x_0}{\sqrt{2}}(a + a^\dagger) \quad p = \frac{p_0}{\sqrt{2}i}(a - a^\dagger) \quad H = \hbar\omega\left(a^\dagger a + \frac{1}{2}\right)$
$a^\dagger a|n\rangle = n|n\rangle \qquad |n\rangle = \frac{(a^\dagger)^n}{\sqrt{n!}}|0\rangle$
Ehrenfest: $\frac{\mathrm{d}}{\mathrm{d}t}\langle x, t\rangle = \frac{\langle p, t\rangle}{m}$, $\frac{\mathrm{d}}{\mathrm{d}t}\langle p, t\rangle = \left\langle -\frac{\mathrm{d}V}{\mathrm{d}x}\right\rangle = -k\langle x, t\rangle$

Coherent States

$a|z\rangle = z|z\rangle$, $z \in \mathbb{C} \qquad \sum_n c_n a|n\rangle = \sum_n c_n z|n\rangle$
$c_{n+1}\sqrt{n + 1} = c_n z \Longrightarrow c_n = \frac{z^n}{\sqrt{n!}}c_0$
$|z\rangle = \sum_n c_n|n\rangle = c_0\sum_n \frac{z^n}{\sqrt{n!}}|n\rangle, \quad c_0 = e^{-\frac{|z|^2}{2}}$
$|z\rangle = e^{-\frac{|z|^2}{2}}e^{za^\dagger}|0\rangle$
$|z, t\rangle = e^{-i\frac{\omega}{2}t}\left|ze^{-i\omega t}\right\rangle \qquad z(t) = ze^{-i\omega t}$
$\langle z|(a^\dagger)^n a^m|z\rangle = (z^*)^n z^m$
$\langle x\rangle = \sqrt{2}x_0\,\mathrm{Re}\,z \qquad \langle x^2\rangle = x_0^2\left(2\,\mathrm{Re}^2\,z + \frac{1}{2}\right)$
$\langle p\rangle = \sqrt{2}p_0\,\mathrm{Im}\,z \qquad \langle p^2\rangle = p_0^2\left(2\,\mathrm{Im}^2\,z + \frac{1}{2}\right)$
$\langle H\rangle = \frac{\hbar\omega}{2} + \hbar\omega|z|^2 \quad \langle H^2\rangle = \hbar^2\omega^2\left(|z|^4 + 2|z|^2 + \frac{1}{4}\right)$

2D Isotropic Oscillator

$k_x = k_y \equiv k \quad \omega_x = \omega_y \equiv \omega = \sqrt{\frac{k}{m}}$
$V(\boldsymbol{r}) = \frac{1}{2}k(x^2+y^2) = \frac{1}{2}kr^2$
$H\left|n_x n_y\right\rangle = \hbar\omega(n_x+n_y+1)\left|n_x n_y\right\rangle$

Gaussian Wave Packet
$\psi(x) = \frac{1}{\sqrt[4]{2\pi\sigma^2}}\exp\left(\frac{(x-\langle x\rangle)^2}{4\sigma^2}\right)e^{-i\frac{\langle p\rangle}{\hbar}x}$
$\Delta x(0) = \sigma \implies \left\langle x^2, 0\right\rangle = \sigma^2 + \langle x, 0\rangle^2$
$\Delta p(0) = \frac{\hbar}{2\Delta x(0)} = \frac{\hbar}{2\sigma} \implies \left\langle p^2, 0\right\rangle = \frac{\hbar^2}{4\sigma^2} + \langle p, 0\rangle^2$

Time Evolution
Expand $\psi(x,0)$ over eigenfunctions $\psi_n(x)$ solving $H\psi_n(x) = E_n\psi_n(x)$ to get $\psi(x,0) = \sum_n c_n\psi_n(x)$.
Find $\psi(x,t)$ with $\psi(x,t) = \sum_n c_n e^{-\frac{iE_n}{\hbar}t}\psi_n(x)$.
$|\psi, t\rangle \equiv \sum_n c_n e^{-i\frac{E_n}{\hbar}t}|n\rangle = e^{-i\frac{H}{\hbar}t}|\psi, 0\rangle$

Time-Dependent Expectation Value
Heisenberg approach: $\langle\mathcal{O}, t\rangle = \langle\psi, 0|\,\mathcal{O}(t)\,|\psi, 0\rangle$
Schrödinger approach: $\langle\mathcal{O}, t\rangle = \langle\psi, t|\,\mathcal{O}\,|\psi, t\rangle$

Time-Dependent Operator
$\mathcal{O}(t) = e^{i\frac{H}{\hbar}t}\mathcal{O}e^{-i\frac{H}{\hbar}t} \qquad \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{O}(t) = \frac{i}{\hbar}[H,\mathcal{O}](t)$
$\mathcal{O}(t)\big|_{t=0} = e^{i\frac{H}{\hbar}\cdot 0}\mathcal{O}e^{-i\frac{H}{\hbar}\cdot 0} = \mathcal{O}$
$(A\cdot B)(t) = A(t)B(t)$

Angular Momentum
$L_z = xp_y - yp_x \to -\hbar\frac{\partial}{\partial\phi}$
$L_z|lm\rangle = m\hbar|lm\rangle \qquad L^2|lm\rangle = l(l+1)\hbar^2|lm\rangle$
$m = -l, -l+1, \ldots, l-1, l$
$\langle\mathbf{r}|lm\rangle = Y_l^m(\vartheta,\phi) = (-1)^m\sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}P_{lm}(\cos\vartheta)e^{im\phi}$

$L_\pm|lm\rangle = \hbar\sqrt{l(l+1-m(m\pm 1))}\,|l, m\pm 1\rangle$

$\boldsymbol{L}\cdot\hat{\boldsymbol{e}}|\psi\rangle = \hbar l|\psi\rangle \implies \boldsymbol{L}\parallel\hat{\boldsymbol{e}}$
$L_+ \equiv L_x + iL_y \qquad L_- \equiv L_+^\dagger = L_x - iL_y$
$L_+|lm\rangle = \hbar\sqrt{l(l+1)-m(m+1)}\,|l, m+1\rangle$
$L_-|lm\rangle = \hbar\sqrt{l(l+1)-m(m-1)}\,|l, m-1\rangle$
$L_x = \frac{L_+ + L_-}{2} \qquad L_y = \frac{L_+ - L_-}{2i}$
$\langle L_x, t\rangle = \operatorname{Re}\langle L_+, t\rangle \qquad \langle L_y, t\rangle = \operatorname{Im}\langle L_+, t\rangle \;\; \boldsymbol{\mu} = \gamma\boldsymbol{L}$

Spin
$S^2 = |sm_s\rangle = \hbar^2 s(s+1)|sm_s\rangle$
$S_z|sm_s\rangle = \hbar m_s|sm_s\rangle$
$\left|\frac{1}{2}\frac{1}{2}\right\rangle \equiv |\uparrow\rangle$ and $\left|\frac{1}{2} -\frac{1}{2}\right\rangle \equiv |\downarrow\rangle$

$S_+ = S_x + iS_y \qquad S_- = S_x - iS_y$
$S_x = \frac{S_+ + S_-}{2} \qquad S_y = \frac{S_+ - S_-}{2i}$

$S_+|sm_s\rangle = \hbar\sqrt{s(s+1)-m_s(m_s+1)}\,|s, m_s+1\rangle$
$S_-|sm_s\rangle = \hbar\sqrt{s(s+1)-m_s(m_s-1)}\,|s, m_s-1\rangle$
$\boldsymbol{S}\cdot\hat{\boldsymbol{e}}|\psi_s\rangle = \frac{\hbar}{2}|\psi_s\rangle$ implies $\boldsymbol{S}\parallel\hat{\boldsymbol{e}}$ where
$\hat{\boldsymbol{e}} = (\cos\varphi\sin\theta, \sin\varphi\sin\theta, \cos\theta)$
$|\psi_s\rangle = \cos\frac{\theta}{2}|\uparrow\rangle + \sin\frac{\theta}{2}e^{i\varphi}|\downarrow\rangle$
Useful: $\boldsymbol{k} = (k_x, k_y)$ to $k_x + ik_y = ke^{i\phi}$, $k_x - ik_y = ke^{-i\phi}$

Spin Matrices
$\boldsymbol{S} = \frac{\hbar}{2}\boldsymbol{\sigma} \qquad \boldsymbol{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$
$\sigma_x = \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix} \qquad \sigma_y = \begin{bmatrix}0 & -i\\ i & 0\end{bmatrix} \qquad \sigma_z = \begin{bmatrix}1 & 0\\ 0 & -i\end{bmatrix}$
$\{\sigma_a, \sigma_b\} = 2\delta_{ab}I$

Two-Particle Spin System
Product basis: $|s_1 m_1 s_2 m_2\rangle$.
Total angular momentum basis: $|sm_s s_1 s_2\rangle$
$|s_1 - s_2| \leq s \leq s_1 + s_2 \qquad m_s = -s, -s+1, \ldots, s-1, s.$

Time Reversal
$\langle\mathcal{T}\psi_1|\mathcal{T}\psi_2\rangle = \langle\psi_1|\psi_2\rangle^*$
For spin $s = 1/2$ particles $T = i\sigma_y K$; $K : \psi \mapsto \psi^*$

Nondegenerate Perturbations
$H = H_0 + H' \qquad H_0|n_0\rangle = E_n^{(0)}|n_0\rangle$
$E_n = E_n^{(0)} + \langle n_0|H'|n_0\rangle + \sum_{m\neq n}\frac{|\langle m_0|H'|n_0\rangle|^2}{E_n^{(0)} - E_m^{(0)}},$
$|n\rangle = |n_0\rangle + \sum_{m\neq n}\frac{|\langle m_0|H'|n_0\rangle|^2}{E_n^{(0)} - E_m^{(0)}}|m_0\rangle$

Degenerate Perturbations
$H = H_0 + H' \qquad H_0|\psi\rangle = E^{(0)}|\psi\rangle$
$E^{(0)}$ is N-times degenerate...
... and $|\psi\rangle = \sum_i^N c_i|i\rangle \qquad H|i\rangle = E^{(0)}|i\rangle \;\; \forall\, i$
Create matrix $\mathbf{P} = \begin{pmatrix}\langle 1|H'|1\rangle & \cdots & \langle 1|H'|N\rangle\\ \langle 2|H'|1\rangle & \cdots & \langle 2|H'|N\rangle\\ \vdots & \ddots & \vdots\\ \langle N|H'|1\rangle & \cdots & \langle N|H'|N\rangle\end{pmatrix}$
Diagonalize $\mathbf{P}$; find eigenvalues λ_j and eigenvectors $\boldsymbol{u}_j$.
$E_j = E^{(0)} + \lambda_j$ and $|\psi_j\rangle = \sum_{k=1}^N u_{jk}|k\rangle$

Time-Dependent Perturbation
$H(t) = H_0 + H'(t)$
$H_0|n\rangle = E_n|n\rangle$
$i\frac{\partial}{\partial t}|\psi, t\rangle = H(t)|\psi, t\rangle$
$|\psi, t\rangle = \sum_n c_n(t)|n, t\rangle \qquad |n, t\rangle = |n\rangle\, e^{-i\frac{E_n}{\hbar}t}$
$c_n(t) \approx c_n(t_0) - \frac{i}{\hbar}\int_{t_0}^t \langle n, t'|H'(t')|m, t'\rangle\, c_m(t_0)\,\mathrm{d}t'$

Useful Matrix Element Techniques
$[\mathcal{O}, L_z] = 0 \implies \langle nlm|\mathcal{O}|n'l'm'\rangle = \langle nlm|\mathcal{O}|n'l'm'\rangle\,\delta_{mm'}$
$\{\mathcal{O}, \mathcal{P}\} = 0 \implies \langle nlm|\mathcal{O}|n'l'm'\rangle = 0$ if $(-1)^{l'} + (-1)^l \neq 0$