Electromagnetic Field Lecture Notes

Elijan Mastnak

Winter Semester 2020-2021

About These Notes

These are my lecture notes from the course *Elektromagnetno polje* (Electromagnetic Field), a required course for third-year physics students at the Faculty of Math and Physics in Ljubljana, Slovenia. The course covers electro- and magnetostatics, quasistatic fields, dynamic electromagnetic fields, the Maxwell equations in free space and in matter, and concludes with brief chapters on the integration of electromagnetism with the Hamiltonian formalism and special relativity. The exact material herein is specific to the physics program at the University of Ljubljana, but the content is fairly standard for a late-undergraduate course in electromagnetism. I am making the notes publicly available in the hope that they might help others learning the same material—the most recent version can be found on GitHub.

Navigation: For easier document navigation, the table of contents is "clickable", meaning you can jump directly to a section by clicking the colored section names in the table of contents. Unfortunately, the *clickable links do not work in most online or mobile PDF viewers*; you have to download the file first.

On Authorship: These content of these notes is far from original. The lectures were given by Professor Miha Ravnik and, to the best of my knowledge, draw from the content of the textbook *Elektromagnetno polje* by Rudolf Podgornik and Andrej Vilfan, published by FMF's publishing house DMFA in 2012. Accordingly, credit for the material in these notes goes to Professors Ravnik, Podgornik and Vilfan; I have merely typeset the notes, translated to English and provided an additional explanation or two where I saw fit.

Disclaimer: Mistakes—both trivial typos and legitimate errors—are likely. Keep in mind that these are the notes of an undergraduate student in the process of learning the material himself—take what you read with a grain of salt. If you find mistakes and feel like telling me, by GitHub pull request, email or some other means, I'll be happy to hear from you, even for the most trivial of errors.

Contents

1 Electrostatics

1.1 TODO: Material from First Lecture

1.2 Electric Potential and Charge Density

- Electric potential ϕ is a scalar quantity, defined in terms of electric field $\boldsymbol{E}$ as

$$\boldsymbol{E}(\boldsymbol{r}) = -\nabla\phi(\boldsymbol{r}).$$

As a side note, we remark that it will be useful in the coming sections to memorize the identity

$$\nabla\frac{1}{\boldsymbol{r}} = -\frac{\boldsymbol{r}}{r^3}.$$

- As an example, we find the electric potential of a point charge, which as electric field

$$\boldsymbol{E}(\boldsymbol{r}) = \frac{q}{4\pi\epsilon_0}\frac{\boldsymbol{r}}{r^3}.$$

Using the above gradient identity $\nabla\frac{1}{\boldsymbol{r}} = -\frac{\boldsymbol{r}}{r^3}$,the electric potential ϕ satisfying the equality $\boldsymbol{E}(\boldsymbol{r}) - \nabla\phi(\boldsymbol{r})$ is

$$\phi = \frac{q}{4\pi\epsilon_0}\frac{1}{r} + \phi_0.$$

Note that ϕ is determined only up to a constant ϕ_0.

- The property of ϕ being determined only up to a constant is a simple introduction to gauge theory—more on this in the sections on relativistic electromagnetism. Choosing a condition such that ϕ is precisely determined is called fixing a gauge. For electric potential, the gauge is simply a constant, but in more complicated situations, the gauge can be a variable field.

1.2.1 The Superposition Principle

- Consider a set of point charges $\{q_i\}$. We then place another charge q in the system at the position $\boldsymbol{r}$. Via the superposition principle, the force acting on the additional charge q is the vector sum

$$\boldsymbol{F}(\boldsymbol{r}) = \frac{q}{4\pi\epsilon_0}\sum_i \frac{(\boldsymbol{r} - \boldsymbol{r}_i)}{|\boldsymbol{r} - \boldsymbol{r}_i|^3}q_i.$$

- Because the superposition principle holds for force, it also holds for electric field, and the electric field felt by the additional charge is thus

$$\boldsymbol{E}(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0}\sum_i \frac{(\boldsymbol{r} - \boldsymbol{r}_i)}{|\boldsymbol{r} - \boldsymbol{r}_1|^3}q_i.$$

Similarly, electric potential felt by the additional charge obeys the superposition principle and is equal to

$$\phi(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0}\sum_i \frac{q_i}{|\boldsymbol{r} - \boldsymbol{r}_1|}.$$

The superposition principle is useful for finding the electric field and potential of complicated charge distributions.

1.2.2 Charge Density

- At the fundamental atomic level, charge is discrete. However, on a macroscopic level, it is more convenient to work in terms of charge density. The charge density of a discrete charge distribution $\{q_i\}$ is

$$\rho(\boldsymbol{r}) = \sum_i q_i \delta^3(\boldsymbol{r} - \boldsymbol{r}_i),$$

  where we note that the delta function has units length^{-3}. We find the distribution's total charge by integrating the charge density over the volume V containing the charges:

$$\iiint_V \rho(\boldsymbol{r})\,\mathrm{d}^3\boldsymbol{r} = \sum_i q_i \iiint_V \delta^3(\boldsymbol{r} - \boldsymbol{r}_i)\,\mathrm{d}^3\boldsymbol{r} = \sum_i q_i = q_{\text{total}}$$

  Meanwhile, the charge density of a continuous charge distribution is

$$\rho(\boldsymbol{r}) = \frac{\mathrm{d}q}{\mathrm{d}V} \implies q_{\text{total}} = \iiint_V \rho(\boldsymbol{r})\,\mathrm{d}^3\boldsymbol{r}$$

- The electric force, electric field and electric potential of a charge distribution with charge density $\rho(\boldsymbol{r})$ contained in the volume V are

$$\begin{aligned}
\boldsymbol{F} &= \iiint_V \rho(\boldsymbol{r})\boldsymbol{E}(\boldsymbol{r})\,\mathrm{d}^3\boldsymbol{r}\\
\boldsymbol{E}(\boldsymbol{r}) &= \frac{1}{4\pi\epsilon_0}\iiint_V \frac{\rho(\tilde{\boldsymbol{r}})(\boldsymbol{r} - \tilde{\boldsymbol{r}})}{|\boldsymbol{r} - \tilde{\boldsymbol{r}}|^3}\,\mathrm{d}^3\tilde{\boldsymbol{r}}\\
\phi(\boldsymbol{r}) &= \frac{1}{4\pi\epsilon_0}\iiint_V \frac{\rho(\tilde{\boldsymbol{r}})}{|\boldsymbol{r} - \tilde{\boldsymbol{r}}|}\,\mathrm{d}^3\tilde{\boldsymbol{r}}
\end{aligned}$$

1.2.3 Charge Density Examples

Point Charge

- The charge density of a point charge located at the position $\boldsymbol{r}_0$ is

$$\rho(\boldsymbol{r}) = q\delta^3(\boldsymbol{r} - \boldsymbol{r}_0).$$

Electric Dipole

- The charge density of an electric dipole dipole with positive and negative charges at $\boldsymbol{r}_+$ and $\boldsymbol{r}_-$, respectively, is

$$\rho(\boldsymbol{r}) = q\delta^3(\boldsymbol{r} - \boldsymbol{r}_+) - q\delta^3(\boldsymbol{r} - \boldsymbol{r}_-).$$

  Alternatively, if the dipole's center occurs at $\boldsymbol{r}_0$ and the positive and negative charges occur at $\boldsymbol{r}_0 + \delta\boldsymbol{r}$ and $\boldsymbol{r}_0 - \delta\boldsymbol{r}$, the dipole's charge density is

$$\rho(\boldsymbol{r}) = q\delta^3(\boldsymbol{r} - \boldsymbol{r}_0 - \delta\boldsymbol{r}) - q\delta^3(\boldsymbol{r} - \boldsymbol{r}_0 + \delta\boldsymbol{r}).$$

  Next, we consider the limit case $|\delta\boldsymbol{r}| \ll |\boldsymbol{r}_0|$, i.e. when the distance δr between the dipole charges is small compared to the distance r_0 to the dipole. We expand the charge density to get

$$\rho(\boldsymbol{r}) \approx -q\delta\boldsymbol{r}\cdot\nabla\big(\delta^3(\boldsymbol{r} - \boldsymbol{r}_0)\big) - q\delta\boldsymbol{r}\cdot\nabla\big(\delta^3(\boldsymbol{r} - \boldsymbol{r}_0)\big) = -2q\delta\boldsymbol{r}\cdot\nabla\big(\delta^3(\boldsymbol{r} - \boldsymbol{r}_0)\big).$$

The term $2q\delta\boldsymbol{r}$ is the dipole moment $\boldsymbol{p}_e$, and the dipole's charge density is then

$$\rho(\boldsymbol{r}) = -\boldsymbol{p}_e \cdot \nabla\big(\delta^3(\boldsymbol{r}-\boldsymbol{r}_0)\big).$$

Next, we make the auxiliary calculation

$$\begin{aligned}\nabla\cdot\big(\boldsymbol{p}_e\delta^3(\boldsymbol{r}-\boldsymbol{r}_0)\big) &= (\nabla\cdot\boldsymbol{p}_e)\delta^3(\boldsymbol{r}-\boldsymbol{r}_0) + \boldsymbol{p}_e\cdot\nabla\big(\delta^3(\boldsymbol{r}-\boldsymbol{r}_0)\big)\\ &= 0 + \boldsymbol{p}_e\cdot\nabla\big(\delta^3(\boldsymbol{r}-\boldsymbol{r}_0)\big) \implies\\ \boldsymbol{p}_e\cdot\nabla\big(\delta^3(\boldsymbol{r}-\boldsymbol{r}_0)\big) &= -\nabla\cdot\big(\boldsymbol{p}_e\delta^3(\boldsymbol{r}-\boldsymbol{r}_0)\big).\end{aligned}$$

where we've used $\nabla\cdot\boldsymbol{p}_e = 0$. The dipole's charge density is then

$$\rho(\boldsymbol{r}) = -\nabla\cdot\big(\boldsymbol{p}_e\delta^3(\boldsymbol{r}-\boldsymbol{r}_0)\big) \equiv -\nabla\cdot\boldsymbol{P}(\boldsymbol{r}).$$

where $\boldsymbol{P}$ is electric polarization. The relationship

$$\rho(\boldsymbol{r}) = -\nabla\cdot\boldsymbol{P}(\boldsymbol{r}).$$

holds for any charge distribution with the form of an electric dipole.

Surface Charge Distribution

- We analyze a surface charge distribution in terms of surface charge density $\sigma(\boldsymbol{r})$.

  $$\rho(\boldsymbol{r}) = \sigma(\boldsymbol{r})\delta(z-z_0).$$

  where we assume the surface occurs at $z = z_0$. Recall the delta function has units length^{-1}, so units are consistent.

Spherical Charge Distribution

- The charge density of a uniform spherical charge distribution of radius a is

  $$\rho(\boldsymbol{r}) = \rho_0 H(a-r)\begin{cases}\rho_0 & r<a\\ 0 & r>a\end{cases}$$

  where H is the Heaviside function.

- The polarization of a spherically-distributed dipole within a sphere of radius a is

  $$\boldsymbol{P}(\boldsymbol{r}) = \frac{\mathrm{d}\boldsymbol{p}_e}{\mathrm{d}V} = \begin{cases}\boldsymbol{P}_0 & r<a\\ 0 & r>a\end{cases} = \boldsymbol{P}_0 H(a-r).$$

  The corresponding charge density is

  $$\begin{aligned}\rho(\boldsymbol{r}) &= -\nabla\cdot\boldsymbol{P} = -\nabla\cdot\big[\boldsymbol{P}_0 H(a-r)\big] = -(\nabla\cdot\boldsymbol{P}_0)H(a-r) - \boldsymbol{P}_0\nabla H(a-r)\\ &= 0 - \boldsymbol{P}_0\delta(a-r)\cdot\left(-\frac{\boldsymbol{r}}{r}\right) = \boldsymbol{P}_0\frac{\boldsymbol{r}}{r}\delta(a-r).\end{aligned}$$

  Note that the charge density occurs only at $r = a$ (at the sphere's surface) this is because internal charges from internal electric dipoles within the sphere cancel each other out.

1.3 Gauss's Law

1.3.1 Integral Form of Gauss's Law

- Consider a distribution of charges $\{q_i\}$ at the positions $\{\boldsymbol{r}_i\}$. We enclose the charges with a sphere of radius r with normal vector $\hat{\boldsymbol{n}}$. In general, the electric flux through a closed surface containing the charges is

$$\Phi_{\mathrm{E}} = \oiint_S \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{S} = \oiint_S \boldsymbol{E} \cdot \hat{\boldsymbol{n}} \, \mathrm{d}S = \oiint_S E \, \mathrm{d}S \cos\theta,$$

where θ is the angle between the electric field and normal vector to the surface.

- In our case, for the collection of charges $\{q_i\}$, the electric flux is

$$\begin{aligned}\Phi_{\mathrm{E}} &= \oiint_S \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{S} = \sum_i \frac{q_i}{4\pi\epsilon_0} \oiint_S \frac{\cos\theta_i}{\left|\boldsymbol{r} - \boldsymbol{r}_i\right|^2} \, \mathrm{d}S \\ &= \frac{1}{4\pi\epsilon_0} \left[\oiint_S \frac{q_1 \cos\theta_1}{\left|\boldsymbol{r} - \boldsymbol{r}_2\right|^2} \, \mathrm{d}S + \oiint_S \frac{q_2 \cos\theta_2}{\left|\boldsymbol{r} - \boldsymbol{r}_2\right|^2} \, \mathrm{d}S + \cdots \right],\end{aligned}$$

where θ_i represent the angle between each particle's $\boldsymbol{E}$ field and $\hat{\boldsymbol{n}}$.

- Next, note that for charges at the origin, the angle θ_i is zero. We choose the origin so that $\boldsymbol{r}_1 = 0$. Then $\theta_1 = 0$, $\cos\theta_1 = 1$ and the first integral is (using $\mathrm{d}S = \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\phi$)

$$\oiint_S \frac{q_1}{r^2} \, \mathrm{d}S = \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \frac{q_1}{r^2} r^2 \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\phi = 4\pi q_1.$$

The idea is to play the same game for each integral, switching the origin one at a time and arguing that the value of the integral is the same regardless of the choice of origin. The end result is Gauss's law:

$$\Phi_{\mathrm{E}} = \oiint_S \boldsymbol{E} \cdot \mathrm{d}\mathbf{S} = \frac{1}{4\pi\epsilon_0} \left[4\pi q_1 + 4\pi q_2 + \cdots\right] = \frac{Q}{\epsilon_0},$$

where $Q = \sum_i q_i$. In terms of charge density, this relationship reads

$$\Phi_{\mathrm{E}} = \oiint_{\partial V} \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{S} = \frac{1}{\epsilon_0} \iiint_V \rho(\boldsymbol{r}) \, \mathrm{d}^3\boldsymbol{r},$$

where ∂V denotes the boundary surface enclosing the charges in the volume V.

- Finally, note that if the charges occur at the volume's surface, then $|\boldsymbol{r}_i| = r$, giving a singularity in the integral denominators. Because of this singularity, Gauss's law is thus not valid for a surface that does not fully enclose the charges.

1.3.2 Differential Form of Gauss's Law

- To find the differential form of Gauss's law, we start with the integral form

$$\oiint_{\mathrm{S}} \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{S} = \frac{1}{\epsilon_0} \iiint_V \rho(\boldsymbol{r}) \, \mathrm{d}^3\boldsymbol{r}.$$

We then recall the divergence theorem, which reads

$$\oiint_{\partial V} \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{S} = \iiint_V \nabla \cdot \boldsymbol{E} \, \mathrm{d}^3\boldsymbol{r}.$$

Applied to Gauss's law, the divergence theorem gives

$$\oiint_{\partial V} \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{S} = \iiint_V \nabla \cdot \boldsymbol{E} \, \mathrm{d}^3\boldsymbol{r} = \frac{1}{\epsilon_0} \iiint_V \rho(\boldsymbol{r}) \, \mathrm{d}^3\boldsymbol{r}.$$

We then equate the integrands in the last equality to get the differential form of Gauss's law:

$$\nabla \cdot \boldsymbol{E} = \frac{\rho}{\epsilon_0}.$$

1.4 The General Expression for Electric Potential

1.4.1 The Poisson and Laplace Equations for Electric Potential

- The Poisson equation is used to find the electrostatic potential $\phi(\boldsymbol{r})$ of an arbitrary charge distribution $\rho(\boldsymbol{r})$. To derive the Poisson equation, we substitute $E = -\nabla\phi$ into the differential form of Gauss's law to get

  $$\frac{\rho}{\epsilon_0} \equiv \nabla \cdot \boldsymbol{E} = \nabla \cdot \big[- \nabla \phi \big] = -\nabla^2 \phi.$$

  The result is the Poisson equation

  $$\nabla^2 \phi = -\frac{\rho}{\epsilon_0}.$$

  In empty space (i.e. in the absence of electric charge) charge density is simply $\rho(\boldsymbol{r}) = 0$, and the Poisson equation reduces to the Laplace equation

  $$\nabla^2 \phi = 0.$$

  In principle, the Laplace equation is always solved with the trivial solution $\phi(\boldsymbol{r}) = 0$, but we are usually interested in nontrivial results arising from the boundary conditions of various charge distributions.

1.4.2 Green's Function for the Poisson Equation

- We find the general solution of the Poisson equation using a Green's function, i.e. we want to find the Green's function G for the Poisson equation

  $$\nabla^2 \phi = -\frac{\rho}{\epsilon_0}$$

  We find the solution using the convolution

  $$\phi(\boldsymbol{r}) = \iiint_V G(\boldsymbol{r} - \tilde{\boldsymbol{r}}) \rho(\tilde{\boldsymbol{r}}) \, \mathrm{d}^3\tilde{\boldsymbol{r}}$$

  where G is the Green's function for the equation. Note that we have assumed a priori that the Green's function exists, which is not necessarily trivial in general cases.

- We then substitute the convolution ansatz for ϕ into the Poisson equation to get

  $$\nabla^2 \phi(\boldsymbol{r}) = \nabla^2 \left[\iiint_V G(\boldsymbol{r} - \tilde{\boldsymbol{r}}) \rho(\tilde{\boldsymbol{r}}) \, \mathrm{d}^3\tilde{\boldsymbol{r}} \right] = \iiint_V \nabla^2 G(\boldsymbol{r} - \tilde{\boldsymbol{r}}) \rho(\tilde{\boldsymbol{r}}) \, \mathrm{d}^3\tilde{\boldsymbol{r}} = -\frac{\rho(\boldsymbol{r})}{\epsilon_0}$$

  Note that the Laplacian ∇^2 applies only to the coordinates of $\boldsymbol{r}$ and not to $\tilde{\boldsymbol{r}}$.

We want to find the G satisfying the above equality. For the equality to hold, the Green's function must obey

$$\nabla^2 G(\boldsymbol{r} - \tilde{\boldsymbol{r}}) = -\frac{\delta^3(\boldsymbol{r} - \tilde{\boldsymbol{r}})}{\epsilon_0},$$

which reproduces the desired equality

$$\iiint_V \nabla^2 G(\boldsymbol{r} - \tilde{\boldsymbol{r}})\rho(\tilde{\boldsymbol{r}})\,\mathrm{d}^3\tilde{\boldsymbol{r}} = \iiint_V \left(-\frac{\delta^3(\boldsymbol{r} - \tilde{\boldsymbol{r}})}{\epsilon_0}\right)\rho(\tilde{\boldsymbol{r}})\,\mathrm{d}^3\tilde{\boldsymbol{r}} = -\frac{\rho(\boldsymbol{r})}{\epsilon_0}.$$

As a side note, we point out the similarity of the expression

$$\nabla^2 G(\boldsymbol{r} - \tilde{\boldsymbol{r}}) = -\frac{\delta^3(\boldsymbol{r} - \tilde{\boldsymbol{r}})}{\epsilon_0}$$

to the charge density of a point charge. In fact, it turns out that the Green's function (up to the charge factor q) is the solution to the Poisson equation for a point charge.

- Point charge aside, we find the Green's function for an arbitrary charge distribution by transforming to Fourier space, which simplifies the Laplace operator.

  As in auxiliary calculation, the Green's function in Fourier space is

$$G(\boldsymbol{r} - \tilde{\boldsymbol{r}}) = \iiint \frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^3} e^{i\boldsymbol{k}(\boldsymbol{r}-\tilde{\boldsymbol{r}})} G(\boldsymbol{k}),$$

while the delta function in Fourier space is

$$\delta(\boldsymbol{r} - \tilde{\boldsymbol{r}}) = \iiint \frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^3} e^{i\boldsymbol{k}(\boldsymbol{r}-\tilde{\boldsymbol{r}})} \cdot 1.$$

Substituting these two results into the Poisson equation gives

$$\nabla^2 \left(\iiint \frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^3} e^{i\boldsymbol{k}(\boldsymbol{r}-\tilde{\boldsymbol{r}})} G(\boldsymbol{k})\right) = -\frac{1}{\epsilon_0}\left(\iiint \frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^3} e^{i\boldsymbol{k}(\boldsymbol{r}-\tilde{\boldsymbol{r}})}\right).$$

We write the equation in terms of a single integral to get

$$\iiint \frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^3}\left[\nabla^2 e^{i\boldsymbol{k}(\boldsymbol{r}-\tilde{\boldsymbol{r}})} G(\boldsymbol{k}) + \frac{1}{\epsilon_0} e^{i\boldsymbol{k}(\boldsymbol{r}-\tilde{\boldsymbol{r}})}\right] = 0.$$

We then factor out the exponential term to get

$$\iiint \frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^3}\left[-k^2 G(\boldsymbol{k}) + \frac{1}{\epsilon_0}\right] e^{i\boldsymbol{k}(\boldsymbol{r}-\tilde{\boldsymbol{r}})} = 0.$$

For this equality to hold for all $\boldsymbol{k}$ and $\boldsymbol{r}$, the quantity in brackets must be zero, which gives us for the expression for the Green's function in $\boldsymbol{k}$ space:

$$-k^2 G(\boldsymbol{k}) + \frac{1}{\epsilon_0} \equiv 0 \implies G(\boldsymbol{k}) = \frac{1}{\epsilon_0 k^2}.$$

- We now aim to transform $G(\boldsymbol{k})$ from $\boldsymbol{k}$ space back into position space via

$$G(\boldsymbol{r} - \tilde{\boldsymbol{r}}) = \iiint \frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^3} e^{i\boldsymbol{k}(\boldsymbol{r}-\tilde{\boldsymbol{r}})} G(\boldsymbol{k}) = \iiint \frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^3} e^{i\boldsymbol{k}(\boldsymbol{r}-\tilde{\boldsymbol{r}})}\left(\frac{1}{\epsilon_0 k^2}\right).$$

We perform the integral over $\boldsymbol{k}$ by converting to spherical coordinates

$$\begin{aligned} G(\boldsymbol{r}-\tilde{\boldsymbol{r}}) &= \frac{1}{\epsilon_0}\int_0^\infty \int_0^\pi \int_0^{2\pi} \mathrm{d}\phi \sin\theta \,\mathrm{d}\theta k^2 \,\mathrm{d}k \frac{1}{(2\pi)^3} e^{i\boldsymbol{k}(\boldsymbol{r}-\tilde{\boldsymbol{r}})} \frac{1}{k^2} \\ &= \frac{1}{\epsilon_0}\frac{1}{(2\pi)^2}\int_0^\infty \int_{-1}^1 \mathrm{d}[\cos\theta] e^{ik|\boldsymbol{r}-\tilde{\boldsymbol{r}}|\cos\theta} \,\mathrm{d}k \\ &= \frac{1}{4\pi\epsilon_0|\boldsymbol{r}-\tilde{\boldsymbol{r}}|}. \end{aligned}$$

where the last equality relies on the integral $\int_0^\infty \frac{\sin x}{x} = \frac{\pi}{2}$. Note that the k^2 terms in the first line cancel, which is what makes the conversion back to position space feasible.

- Using the just-derived Green's function, the general solution to the Poisson equation for electric potential $\phi(\boldsymbol{r})$ is

$$\phi(\boldsymbol{r}) = \phi(\boldsymbol{r}) = \iiint_V G(\boldsymbol{r}-\tilde{\boldsymbol{r}})\rho(\tilde{\boldsymbol{r}}) \,\mathrm{d}^3\tilde{\boldsymbol{r}} = \iiint_V \frac{\rho(\tilde{\boldsymbol{r}})}{4\pi\epsilon_0|\boldsymbol{r}-\tilde{\boldsymbol{r}}|} \,\mathrm{d}^3\tilde{\boldsymbol{r}}.$$

This result is important: it means that as long as we know a charge distribution charge density $\rho(\boldsymbol{r})$, we can find the corresponding electric potential $\phi(\boldsymbol{r})$.

- Note that the general solution for $\phi(\boldsymbol{r})$ is a generalized form of the electric potential of a point particle of charge q, for which the electric potential is

$$\phi(\boldsymbol{r}) = \frac{q}{4\pi\epsilon_0 r}.$$

This similarity indicates the Poisson equation and its general solution is consistent with Coulomb's law and the potential of a point particle.

- Finally, with $\phi(\boldsymbol{r})$ known, we can find the electric field of a charge distribution $\rho(\boldsymbol{r})$ from the the gradient of the electric potential:

$$\boldsymbol{E} = -\nabla\phi(\boldsymbol{r}) = -\nabla \iiint \frac{\rho(\tilde{\boldsymbol{r}})}{4\pi\epsilon_0|\boldsymbol{r}-\tilde{\boldsymbol{r}}|} \,\mathrm{d}^3\tilde{\boldsymbol{r}} = \iiint \frac{\rho(\tilde{\boldsymbol{r}}) \,\mathrm{d}^3\tilde{\boldsymbol{r}}}{4\pi\epsilon_0} \frac{\boldsymbol{r}-\tilde{\boldsymbol{r}}}{|\boldsymbol{r}-\tilde{\boldsymbol{r}}|^3}$$

Note that the gradient acts only on $\boldsymbol{r}$ and not on $\tilde{\boldsymbol{r}}$. As for electric potential, the general solution for the electric field of charge distribution is a generalization of Coulomb's law for a point charge.

1.5 In Passing: Earnshaw's Theorem and the Thomson Problem

- Earnshaw's theorem states that a collection of point charges cannot exist in *stable* equilibrium due solely to the electrostatic interaction between the charges.

- At a hypothetical point of stable electrostatic equilibrium in free space, all electrostatic force lines would locally point towards the point. In this case the electric force's divergence is nonzero, which implies

$$0 \neq \nabla\cdot\boldsymbol{F} = \nabla\cdot(q\boldsymbol{E}) = -\nabla\cdot(q\nabla\phi) = -q\nabla^2\phi \neq 0.$$

In other words, stable electrostatic equilibrium results in $\nabla^2\phi = 0$, which violates the Laplace equation.

- More mathematically, in free space (with zero charge density), electric potential obeys the Laplace equation and is thus a harmonic function with no minima or maxima; at most, electric potential in free space can have saddle points, which correspond to points of *unstable* electrostatic equilibrium.

- Note that the Earnshaw theorem only applies in the regime of classical electromagnetism, and matter is nonetheless stable, despite the Earnshaw theorem, because of the existence of quantum mechanical effects such as stationary orbital states and the Pauli exclusion principle.

The Thomson Problem

- The electrostatic energy of a charge distribution can have minima if the electric charges are fixed. The Thomson problem involves finding the distribution in space of a collection of point charges such that the charges' collective electrostatic energy is minimized.

- The problem is important in chemistry and biology, when investigating molecular bonds. We mention the problem because of its historical significance but don't investigate it further.

1.6 Electrostatic Energy

1.6.1 Electrostatic Energy in an External Field

- Consider a charge q in a known external electric field $\boldsymbol{E}$. Naturally, we cannot create an electric field out of thin air, but for this section we ignore the charges responsible for the electric field and take the field's existence for granted.

- In the electric field, the particle experiences a force $\boldsymbol{F} = q\boldsymbol{E}$. We interpret electrostatic energy as the potential energy needed to keep the particle at rest, or alternatively, the work against the electrostatic force needed to bring the particle to its position from infinity. The differential of work is

$$\mathrm{d}W = -\boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r} = -q\boldsymbol{E} \cdot \mathrm{d}\boldsymbol{r} = q\nabla\phi \cdot \mathrm{d}\boldsymbol{r}.$$

Use the work-energy theorem, the total work W and electrostatic energy by W_E are related by

$$W = \int_{(1)}^{(2)} \mathrm{d}W = W_\mathrm{E}^{(2)} - W_\mathrm{E}^{(1)} = q\int_{(1)}^{(2)} \nabla\phi \, \mathrm{d}\boldsymbol{r} = q\big(\phi^{(2)} - \phi^{(1)}\big).$$

From the equality $W_\mathrm{E}^{(2)} - W_\mathrm{E}^{(1)} = q\big(\phi^{(2)} - \phi^{(1)}\big)$, the electrostatic energy of a point charge in an external electric field is thus

$$W_\mathrm{E} = q\phi,$$

where ϕ is the electric potential arising from the *external* field.

- For a continuous charge distribution $\rho(\boldsymbol{r})$, the appropriate generalization is

$$W_\mathrm{E} = \iiint_V \rho(\boldsymbol{r})\phi(\boldsymbol{r}) \, \mathrm{d}^3\boldsymbol{r},$$

where ϕ is again the electric potential arising from the external field only (and has nothing to do with the charge distribution $\rho(\boldsymbol{r})$), while V is the region of space containing ρ.

1.6.2 Total Electrostatic Energy

- In the previous section, we ignored the "background" charges creating an external field. We now consider the total electrostatic energy of charge distribution in an electric field with respect to both the charge distribution and the charges creating the external field.

  To do this, we consider a charge distribution $\rho(\boldsymbol{r})$ that generates a potential $\phi(\boldsymbol{r})$, and ask what is the associated electrostatic energy needed to assemble the charge distribution $\rho(\boldsymbol{r})$ from empty space. To make this easier, we introduce parameter $\alpha \in [0, 1]$, which continuously "turns on" our charge distribution.

- We now consider the change in energy $\mathrm{d}W_{\mathrm{E}}$ from adding additional charge density $\mathrm{d}\rho = \rho(\boldsymbol{r})\,\mathrm{d}\alpha$ to the charge we have already assembled. The existing charges create an external field like in the previous problem, where the associated electrostatic energy is $W_{\mathrm{E}} = \int \tilde{\rho}(\boldsymbol{r})\tilde{\phi}(\boldsymbol{r})\,\mathrm{d}^3\boldsymbol{r}$. The corresponding differential $\mathrm{d}W_{\mathrm{E}}$ is

$$\begin{aligned}\mathrm{d}W_{\mathrm{E}} &= \iiint_V \mathrm{d}\tilde{\rho}(\boldsymbol{r})\tilde{\phi}(\boldsymbol{r})\,\mathrm{d}^3\boldsymbol{r} = \iiint_V \rho(\boldsymbol{r})\,\mathrm{d}\alpha\big[\alpha\phi(\boldsymbol{r})\big]\,\mathrm{d}^3\boldsymbol{r}\\ &= \alpha\,\mathrm{d}\alpha \iiint_V \rho(\boldsymbol{r})\phi(\boldsymbol{r})\,\mathrm{d}^3\boldsymbol{r}.\end{aligned}$$

  where we have applied the linearity of the Poisson equation, i.e. $\nabla^2[\alpha\phi] = -\frac{\alpha\rho}{\epsilon_0}$.

  We find the total energy by integrating over α from 0 to 1:

$$W_{\mathrm{E}} = \mathrm{d}W_{\mathrm{E}} = \int_0^1 \left[\alpha \iiint_V \rho(\boldsymbol{r})\phi(\boldsymbol{r})\,\mathrm{d}^3\boldsymbol{r}\right]\mathrm{d}\alpha = \frac{1}{2}\iiint_V \rho(\boldsymbol{r})\phi(\boldsymbol{r})\,\mathrm{d}^3\boldsymbol{r}.$$

  This final result is sometimes referred to as the electrostatic energy of an electric field, and refers to the energy of the charge distribution ρ creating the electric potential ϕ.

  Although this result is quantitatively similar to the electrostatic energy of charge distribution in an external field, differing only by a factor 1/2, the two energies are fundamentally different quantities.

  In this section, the charge distribution $\rho(\boldsymbol{r})$ creates the electric potential ϕ, and in the previous section, the charge density ρ was unrelated to the electric potential ϕ, whose existence we took for granted and otherwise ignored.

- Next, we will derive an expression for the just-derived electrostatic energy of an electric field in terms of only the charge density creating the field. We substitute in the Green's function representation of electric potential, i.e.

$$\phi(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0}\iint_V \frac{\rho(\tilde{\boldsymbol{r}})}{|\boldsymbol{r} - \tilde{\boldsymbol{r}}|}\,\mathrm{d}^3\tilde{\boldsymbol{r}},$$

  into the electric field energy W_{E} to get

$$W_{\mathrm{E}} = \frac{1}{2}\iiint_V \rho(\boldsymbol{r})\phi(\boldsymbol{r})\,\mathrm{d}^3\boldsymbol{r} = \frac{1}{8\pi\epsilon_0}\iiint_V \left[\iiint_V \frac{\rho(\boldsymbol{r})\rho(\tilde{\boldsymbol{r}})}{|\boldsymbol{r} - \tilde{\boldsymbol{r}}|}\,\mathrm{d}^3\tilde{\boldsymbol{r}}\right]\mathrm{d}^3\boldsymbol{r},$$

  where we stress that ρ is the charge distribution creating the electric field.

1.6.3 Electric Field Energy In Terms of Electric Field

- Next, we will derive and an expression total electric field energy in terms of only the electric field $\boldsymbol{E}$. We start with
$$W_{\mathrm{E}} = \frac{1}{2}\iiint_V \rho(\boldsymbol{r})\phi(\boldsymbol{r})\,\mathrm{d}^3\boldsymbol{r}$$
and use Gauss's law $\boldsymbol{E} = \frac{\rho}{\epsilon_0}$ to get
$$W_{\mathrm{E}} = \frac{1}{2}\iiint_V \epsilon_0(\nabla\cdot\boldsymbol{E})\phi(\boldsymbol{r})\,\mathrm{d}^3\boldsymbol{r}.$$
Next we use a reverse-engineered form of the vector calculus identity
$$\nabla\cdot(f\boldsymbol{g}) = \nabla f\cdot\boldsymbol{g} + f(\nabla\cdot\boldsymbol{g}) \implies \epsilon_0(\nabla\cdot\boldsymbol{E})\phi(\boldsymbol{r}) = \nabla\cdot(\phi\boldsymbol{E}) - \nabla\phi\cdot\boldsymbol{E},$$
followed by the divergence theorem, to get
$$\begin{aligned} W_{\mathrm{E}} &= \frac{\epsilon_0}{2}\iiint_V \left[\nabla\cdot(\phi\boldsymbol{E}) - \nabla\phi\cdot\boldsymbol{E}\right]\mathrm{d}^3\boldsymbol{r} = \frac{\epsilon_0}{2}\oiint_{\partial V}\phi\boldsymbol{E}\,\mathrm{d}\boldsymbol{S} - \frac{\epsilon_0}{2}\iiint_V(\nabla\phi)\boldsymbol{E}\,\mathrm{d}^3\boldsymbol{r} \\ &= \frac{\epsilon_0}{2}\oiint_{\partial V}\phi\boldsymbol{E}\,\mathrm{d}\boldsymbol{S} - \frac{\epsilon_0}{2}\iiint_V(-\boldsymbol{E})\boldsymbol{E}\,\mathrm{d}^3\boldsymbol{r}, \end{aligned}$$
where we've used $\nabla\phi = -\boldsymbol{E}$ in the second integral.

  It turns out that the first integral is zero. Here's why: first, every electric potential arises from a charge distribution. The electric potential falls as $U \propto r^{-1}$ with increasing distance r from the charge distribution, and the field falls as $\boldsymbol{E} \sim r^{-2}$, while surface area grows as $\boldsymbol{S} \propto r^{-2}$. The integrand $\phi\boldsymbol{E}\cdot\mathrm{d}\boldsymbol{S}$ in the first integral thus grows as r^{-1} and vanishes as $r \to \infty$.

  What remains is the familiar formula
$$W_{\mathrm{E}} = \frac{\epsilon_0}{2}\iiint_V E^2\,\mathrm{d}^3\boldsymbol{r}.$$
Interpretation: This is the energy associated with the electric field $\boldsymbol{E}$ resulting from assembling a charge distribution $\rho(\boldsymbol{r})$ in the region V in what was originally empty space.

1.7 Electrostatic Force and the Stress-Energy Tensor

1.7.1 Electric Force in Terms of of Electric Field

- Consider a body with charge distribution $\rho(\boldsymbol{r})$ in an external electric field $\boldsymbol{E}_{\mathrm{ext}}$. We are interested in the force acting on the charge distribution due to the total electric field $\boldsymbol{E}$, accounting for both the external field $\boldsymbol{E}_{\mathrm{ext}}$ and the field $\boldsymbol{E}_\rho$ created by the charge distribution $\rho(\boldsymbol{r})$, and we want to write this force in terms of only the total electric field $\boldsymbol{E}$.

- We find the electric force on the charge distribution by generalizing $\boldsymbol{F} = q\boldsymbol{E}_{\mathrm{ext}}$. Without proof, we note that the field $\boldsymbol{E}_\rho$ created by the charge distribution $\rho(\boldsymbol{r})$ does not contribute to the electric force on the charged body—this makes sense, since a body cannot create a net force on itself. As a result, we can write
$$\boldsymbol{F} = \iiint_V \rho(\boldsymbol{r})\boldsymbol{E}_{\mathrm{ext}}(\boldsymbol{r})\,\mathrm{d}^3\boldsymbol{r} = \iiint_V \rho(\boldsymbol{r})\boldsymbol{E}(\boldsymbol{r})\,\mathrm{d}^3\boldsymbol{r}$$
where V is the region of space containing the charge distribution.

- Next, we use Gauss's law $\rho(\boldsymbol{r}) = \epsilon_0 \boldsymbol{E}_\rho$, to write the electric force in the form

$$\boldsymbol{F} = \epsilon_0 \iiint_V (\nabla \cdot \boldsymbol{E}_\rho) \boldsymbol{E}(\boldsymbol{r}) \, \mathrm{d}^3 \boldsymbol{r} \stackrel{?}{=} \epsilon_0 \iiint_V (\nabla \cdot \boldsymbol{E}) \boldsymbol{E}(\boldsymbol{r}) \, \mathrm{d}^3 \boldsymbol{r}$$

  Note that the field in the $\nabla \cdot \boldsymbol{E}_\rho$ term arises from the charge distribution $\rho(\boldsymbol{r})$ and is different from $\boldsymbol{E}$. We resolved this (seems hand-wavy to me) by saying that $\rho(\boldsymbol{r})$ now includes both the charge distribution of the body and the distance charge distribution responsible for the external field $\boldsymbol{E}_{\mathrm{ext}}$. As I understood it, the argument is that because the external charge distribution is far away and thus negligible compared to the localized distribution corresponding to the charged body, we can safely incorporate ρ_{ext} into ρ.

  In any case, we proceed with the expression

$$\boldsymbol{F} = \epsilon_0 \iiint_V (\nabla \cdot \boldsymbol{E}) \boldsymbol{E}(\boldsymbol{r}) \, \mathrm{d}^3 \boldsymbol{r},$$

  where $\boldsymbol{E}$ is the total electric field.

- The integral currently runs over the entire body's volume. In practice, it is hard to measure the electric field inside a body, so our next step is to convert to an integral over the body's surface. We apply the vector analysis identity

$$\nabla \cdot (\boldsymbol{E} \otimes \boldsymbol{E}) = \boldsymbol{E} \cdot (\nabla \boldsymbol{E}) + (\boldsymbol{E} \cdot \nabla) \boldsymbol{E},$$

  in terms of which the electric force reads

$$\begin{aligned} \boldsymbol{F} &= \epsilon_0 \iiint_V \left[\nabla \cdot (\boldsymbol{E} \otimes \boldsymbol{E}) - (\boldsymbol{E} \cdot \nabla) \boldsymbol{E} \right] \mathrm{d}^3 \boldsymbol{r} \\ &= \epsilon_0 \oiint_{\partial V} \boldsymbol{E} (\boldsymbol{E} \, \mathrm{d} \boldsymbol{S}) - \epsilon_0 \iiint_V (\boldsymbol{E} \cdot \nabla) \boldsymbol{E} \, \mathrm{d}^3 \boldsymbol{r}, \end{aligned}$$

  where the last equality uses the divergence theorem.

- Next, we introduce a normal vector $\hat{\boldsymbol{n}}$ to the body's surface to get

$$\boldsymbol{F} = \epsilon_0 \oiint_{\partial V} \boldsymbol{E} (\boldsymbol{E} \cdot \hat{\boldsymbol{n}}) \, \mathrm{d} S - \epsilon_0 \iiint_V (\boldsymbol{E} \cdot \nabla) \boldsymbol{E} \, \mathrm{d}^3 \boldsymbol{r}.$$

  Next we use the vector identity $\frac{1}{2} \nabla (E^2) = (\boldsymbol{E} \cdot \nabla) \boldsymbol{E} + \boldsymbol{E} \times (\nabla \times \boldsymbol{E})$, combined with the electrostatic Maxwell equation $\nabla \times \boldsymbol{E} = 0$ (which doesn't hold in the presence of a time-varying magnetic field) to get

$$\boldsymbol{F} = \epsilon_0 \oiint_{\partial V} \boldsymbol{E} (\boldsymbol{E} \cdot \hat{\boldsymbol{n}}) \, \mathrm{d} S - \frac{\epsilon_0}{2} \iiint_V \nabla E^2 \, \mathrm{d}^3 \boldsymbol{r}.$$

  Finally, we apply the scalar field corollary of the divergence theorem to the integral of ∇E^2 over the volume V to get

$$\boldsymbol{F} = \epsilon_0 \oiint_{\partial V} \boldsymbol{E} (\boldsymbol{E} \cdot \hat{\boldsymbol{n}}) \, \mathrm{d} S - \frac{\epsilon_0}{2} \oiint_{\partial V} E^2 \hat{\boldsymbol{n}} \, \mathrm{d} S = \epsilon_0 \oiint_{\partial V} \left[\boldsymbol{E} (\boldsymbol{E} \cdot \hat{\boldsymbol{n}}) - \frac{E^2}{2} \hat{\boldsymbol{n}} \right] \mathrm{d} S.$$

  As desired, the integral runs only over the body's surface ∂V, and the electric force is expressed only in terms of the total electric field $\boldsymbol{E}$.

- Finally, we note that the information about the body's charge distribution ρ is hidden in the body's contribution to the total electric field, since we converted $\rho(\boldsymbol{r})$ to electric field using Gauss's law.

1.7.2 The Electrostatic Stress-Energy Tensor

- We can write the above expression for electrostatic force using a rank-two tensor called the electrostatic stress-energy tensor. By components, in terms of the stress tensor T_{ik}, the force is
$$F_i = \oiint_{\partial V} \mathrm{T}_{ik}\hat{n}_k \,\mathrm{d}S,$$
where the components of the stress tensor are defined as
$$\mathrm{T}_{ik} = \epsilon_0 \left(E_i E_k - \frac{1}{2} E^2 \delta_{ik} \right).$$

- We can also write the electrostatic force as a volume integral:
$$F_i = \oiint_{\partial V} \mathrm{T}_{ik}\hat{n}_k \,\mathrm{d}S = \iiint_V \frac{\partial \mathrm{T}_{ik}}{\partial r_k} \,\mathrm{d}^3\boldsymbol{r}.$$
The quantity $\frac{\partial \mathrm{T}_{ik}}{\partial r_k}$ represents electrostatic force density $f(\boldsymbol{r})$ in the direction $\hat{\boldsymbol{r}}_i$:
$$f_i(\boldsymbol{r}) = \frac{\partial \mathrm{T}_{ik}}{\partial r_k}.$$
Note that stress-energy tensors are widely used (i.e. not just in electromagnetism) to describe the force density associated with a field, for example in hydrodynamics and continuum mechanics.

1.8 The Multipole Expansion in Electrostatics

1.8.1 Multipole Expansion of Electric Potential

- We begin by approximating an arbitrary charge distribution with n-poles (e.g. monopoles, dipoles, quadrupoles, etc...).

  Consider a charge distribution with density $\rho(\boldsymbol{s})$ localized in the region of space V_0. Our goal is to find an expression for the electric potential $\phi(\boldsymbol{r})$ far from the charge distribution, written in terms of a multipole expansion of $\rho(\boldsymbol{s})$.

  Note that the charge distribution $\rho(\boldsymbol{s})$ must be localized in space for the idea of "far away" to make sense—we take "far away" to be around an order of magnitude larger than a characteristic distance of the region V_0, although smaller distances may be used in practice with high-order multipole expansions.

- We begin by writing the electric potential $\phi(\boldsymbol{r})$ in the form
$$\phi(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0} \iiint_{V_0} \frac{\rho(\boldsymbol{s})}{|\boldsymbol{r} - \boldsymbol{s}|} \,\mathrm{d}^3\boldsymbol{s},$$
and then expanding integrand in the limit $|\boldsymbol{r}| \gg |\boldsymbol{s}|$. To second order, the expansion reads
$$\begin{aligned} \frac{1}{|\boldsymbol{r} - \boldsymbol{s}|} &= \frac{1}{|\boldsymbol{r}|} - \boldsymbol{s} \cdot \nabla \left(\frac{1}{|\boldsymbol{r}|} \right) + \frac{1}{2|\boldsymbol{r}|} \boldsymbol{s}^T \mathbf{H} \boldsymbol{s} + \cdots \\ &= \frac{1}{|\boldsymbol{r}|} + \frac{\boldsymbol{s} \cdot \boldsymbol{r}}{|\boldsymbol{r}|^3} + \frac{1}{2|\boldsymbol{r}|} \boldsymbol{s}^T \mathbf{H} \boldsymbol{s} + \cdots, \end{aligned}$$
where we have used the identity $\nabla \frac{1}{|\boldsymbol{r}|} = \frac{\boldsymbol{r}}{|\boldsymbol{r}^3|}$.

- We then substitute the first-order expansion into the expression for $\phi(\boldsymbol{r})$ to get

$$\begin{aligned}\phi(\boldsymbol{r}) &= \frac{1}{4\pi\epsilon_0|\boldsymbol{r}|}\iiint_{V_0}\rho(\boldsymbol{s})\,\mathrm{d}^3\boldsymbol{s} + \frac{1}{4\pi\epsilon_0}\frac{\boldsymbol{r}}{|\boldsymbol{r}|^3}\iiint_{V_0}\boldsymbol{s}\rho(\boldsymbol{s})\,\mathrm{d}^3\boldsymbol{s}\\ &= \frac{q}{4\pi\epsilon_0 r} + \frac{\boldsymbol{r}\cdot\boldsymbol{p}_e}{4\pi\epsilon_0 r^3},\end{aligned}$$

where we have written the integrals in terms of the charge q and dipole moment $\boldsymbol{p}_e$, which are defined as

$$q = \iiint_{V_0}\rho(\boldsymbol{s})\,\mathrm{d}^3\boldsymbol{s} \qquad \text{and} \qquad \boldsymbol{p}_e = \iiint_{V_0}\boldsymbol{s}\rho(\boldsymbol{s})\,\mathrm{d}^3\boldsymbol{s}.$$

- Without derivation, a higher-order expansion of the electrostatic potential reads

$$\phi(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0}\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{4\pi}{(2l+1)}\frac{Q_{lm}}{r^{l+1}}Y_l^m(\theta,\phi).$$

In other words, the multipole expansion is an expansion of $\phi(\boldsymbol{r})$ in the basis of the spherical harmonics Y_l^m. We find the multipole moments Q_{lm} with

$$Q_{lm} = \iiint_{V_0}\rho(\boldsymbol{s})s^l Y_l^m(\theta,\phi)\,\mathrm{d}^3\boldsymbol{s}.$$

1.8.2 Multipole Expansion of Electrostatic Energy

- We now aim to express the electrostatic energy of a charge distribution in an external field in terms of a multipole expansion of the charge distribution. As before, we consider a charge distribution localized in the region V_0, centered around the position $\boldsymbol{r}_0$. We then place this charge distribution in an *external* potential $\phi(\boldsymbol{r})$. We stress that the external potential $\phi(\boldsymbol{r})$ is unrelated to the charge distribution ρ.

- Recall the general form of electrostatic energy for a charge distribution in an external field is

$$W_{\mathrm{E}} = \iiint_V \rho(\boldsymbol{r})\phi(\boldsymbol{r})\,\mathrm{d}^3\boldsymbol{r}.$$

In our case, we assume the majority of the charge in the distribution is concentrated around the distribution's "center" $\boldsymbol{r}_0$, in which case the majority of the distribution's electrostatic energy is also concentrated around $\boldsymbol{r}_0$.

- We then expand the electric potential about $\boldsymbol{r}_0$ according to

$$\phi(\boldsymbol{r}) = \phi(\boldsymbol{r}_0) + (\boldsymbol{r}-\boldsymbol{r}_0)\nabla\phi(\boldsymbol{r}_0) + \cdots,$$

where ∇ acts on $\boldsymbol{r}$. Substituting this expression for $\phi(\boldsymbol{r})$ into the general expression for energy W_{E} gives

$$\begin{aligned}W_{\mathrm{E}} &= \iiint_V \mathrm{d}^3\boldsymbol{r}\rho(\boldsymbol{r})\big[\phi(\boldsymbol{r}_0) + (\boldsymbol{r}-\boldsymbol{r}_0)\nabla\phi(\boldsymbol{r}_0) + \cdots\big]\\ &= \phi(\boldsymbol{r}_0)\iiint_V \mathrm{d}^3\boldsymbol{r}\rho(\boldsymbol{r}) + \nabla\phi(\boldsymbol{r}_0)\iiint_V \mathrm{d}^3\boldsymbol{r}\rho(\boldsymbol{r})(\boldsymbol{r}-\boldsymbol{r}_0)\\ &= q\phi(\boldsymbol{r}_0) - \boldsymbol{E}(\boldsymbol{r}_0)\cdot\boldsymbol{p}_e + \cdots.\end{aligned}$$

where we have substituted in the charge q and electric dipole $\boldsymbol{p}_e$ defined in the previous section. Note also the use of $\nabla\phi(\boldsymbol{r}_0) = -\boldsymbol{E}(\boldsymbol{r}_0)$.

The two terms in the above expression for W_{E} are, respectively, the energy of a point charge and the energy of an electric dipole in an external electric field. The dipole term indicates that the energy of dipole in an external field is minimized when the dipole is parallel to the external field.

1.8.3 Multipole Expansion of Force in an External Field

- Again, we consider a charge distribution ρ localized in the region V_0, centered around the position $\boldsymbol{r}_0$. We place the distribution in an *external* potential $\phi(\boldsymbol{r})$ corresponding to an electric field $\boldsymbol{E}(\boldsymbol{r})$. As before, the electric field and potential are unrelated to the charge distribution ρ.

  Continuing the pattern of previous sections, we aim to find an expression for the electrostatic force on the charge distribution ρ in terms of a multipole expansion of the charge distribution.

- We find force from the gradient of potential energy via $\boldsymbol{F} = -\nabla W_{\mathrm{E}} \iff \mathrm{d}W_{\mathrm{E}} = -\boldsymbol{F}\cdot \mathrm{d}\boldsymbol{r}$. The electrostatic potential energy is known from two sections and reads

$$W_{\mathrm{E}} = q\phi(\boldsymbol{r}_0) - \boldsymbol{E}(\boldsymbol{r}_0)\cdot\boldsymbol{p}_e + \cdots .$$

- We begin by differentiating W_{E} to get $\mathrm{d}W_{\mathrm{E}}$, which reads

$$\mathrm{d}W_{\mathrm{E}} = q\nabla\phi(\boldsymbol{r}_0)\,\mathrm{d}\boldsymbol{r} - \nabla\big[\boldsymbol{p}_e\cdot\boldsymbol{E}(\boldsymbol{r}_0)\big]\,\mathrm{d}\boldsymbol{r} \equiv -\boldsymbol{F}\cdot\mathrm{d}\boldsymbol{r}.$$

  Next, we cancel the common factor $\mathrm{d}\boldsymbol{r}$ and substitute in the identity $\nabla\phi(\boldsymbol{r}_0) = -\boldsymbol{E}(\boldsymbol{r}_0)$, along with the vector identity

$$\nabla\big[\boldsymbol{p}_e\cdot\boldsymbol{E}(\boldsymbol{r}_0)\big] = \boldsymbol{p}_e\times(\nabla\times\boldsymbol{E}(\boldsymbol{r}_0)) + (\boldsymbol{p}_e\cdot\nabla)\boldsymbol{E}(\boldsymbol{r}_0).$$

  The electrostatic force then simplifies to

$$\boldsymbol{F} = q\boldsymbol{E}(\boldsymbol{r}_0) + \boldsymbol{p}_e\times(\nabla\times\boldsymbol{E}(\boldsymbol{r}_0)) + (\boldsymbol{p}_e\cdot\nabla)\boldsymbol{E}(\boldsymbol{r}_0).$$

- Finally, we substitute in the electrostatics identity $\nabla\times\boldsymbol{E} = 0$. The final expression for electrostatic force is then

$$\boldsymbol{F} = q\boldsymbol{E} + (\boldsymbol{p}_e\cdot\nabla)\boldsymbol{E}(\boldsymbol{r}_0).$$

  The two terms in this expression are the electrostatic force on a monopole (point charge) and a dipole, respectively.

1.8.4 Multipole Expansion of Torque in an External Field

- Finally, we will find an expression for the electrostatic torque on a charge distribution in an external electric field in terms of a dipole expansion of the charge distribution. As before, we consider a charge distribution ρ centered at the position $\boldsymbol{r}_0$ in the localized region of space V_0.

- To find the electrostatic torque, we consider a small rotation $\mathrm{d}\boldsymbol{\phi}$ of the charge distribution in space. We will relate energy and torque with the general relationship

$$\mathrm{d}W_{\mathrm{E}} = -\boldsymbol{M} \cdot \mathrm{d}\boldsymbol{\phi}.$$

  We begin with the dipole expansion of the charge distribution's electrostatic energy, which reads

$$W_{\mathrm{E}} = q\phi(\boldsymbol{r}_0) - \boldsymbol{E}(\boldsymbol{r}_0) \cdot \boldsymbol{p}_e + \cdots .$$

  We then take the total differential of this expression to find $\mathrm{d}W_{\mathrm{E}}$. The term $q\phi(\boldsymbol{r}_0)$ is a scalar quantity and vanishes, and the result is

$$\mathrm{d}W_{\mathrm{E}} = -\,\mathrm{d}\boldsymbol{p}_e \cdot \boldsymbol{E}(\boldsymbol{r}_0).$$

- The differential $\mathrm{d}\boldsymbol{p}_e$ can be written as the infinitesimal rotation $\mathrm{d}\boldsymbol{p}_e = \mathrm{d}\boldsymbol{\phi} \times \boldsymbol{p}_e$, in which case $\mathrm{d}W_{\mathrm{E}}$ becomes

$$\mathrm{d}W_{\mathrm{E}} = -(\mathrm{d}\boldsymbol{\phi} \times \boldsymbol{p}_e) \cdot \boldsymbol{E}(\boldsymbol{r}_0).$$

  This expression for $\mathrm{d}W_{\mathrm{E}}$ contains a scalar triple product, which we rearrange to get

$$\mathrm{d}W_{\mathrm{E}} = -\,\mathrm{d}\boldsymbol{\phi} \cdot (\boldsymbol{p}_e \times \boldsymbol{E}) = -\,\mathrm{d}\boldsymbol{\phi} \cdot \boldsymbol{M}.$$

  The resulting torque on the charge distribution is thus

$$\boldsymbol{M}(\boldsymbol{r}_0) = \boldsymbol{p}_e \times \boldsymbol{E}(\boldsymbol{r}_0),$$

  which is the expression for the torque on an electric dipole in an external electric field. Note that the dipole's potential energy $\mathrm{d}W_{\mathrm{E}}$ is minimized when the dipole and external field are parallel.

2 Magnetostatics

In this chapter we consider only static magnetic fields that do not change with respect to time, hence the name magnetostatics. Magnetic fields are created by electric currents or time-varying electric fields. In this chapter we will consider current sources of magnetic fields—we will consider magnetic fields generated by time-varying electric fields when studying electrodynamics.

2.1 Fundamental Relationships in Magnetostatics

2.1.1 Magnetic Force Between Parallel Wires

- We begin with a simple expression for magnetic force. Consider two parallel wires carrying currents I_1 and I_2. If the currents flow in the same direction, the force between the wires is attractive, and vice versa.

  We work in a planar cross section, and assume the wires occur at the positions $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$. In this case, the force between the wires is

  $$\boldsymbol{F} = \frac{\mu_0}{2\pi} \frac{L I_1 I_2}{|\boldsymbol{r}_2 - \boldsymbol{r}_1|} \frac{\boldsymbol{r}_2 - \boldsymbol{r}_1}{|\boldsymbol{r}_2 - \boldsymbol{r}_1|}$$

  where L is the length of the wires.

  Important: We take this relationship as an empirical law—we have not derived it from theory. This expression for the magnetic force between parallel wires can be considered a magnetic analog for Coulomb's law for the electric force between static electric charges.

2.1.2 Ampere's Law

- We now consider the magnetic force between two conductors (wires) that are not necessarily parallel. We parameterize the position along each conductors with the following quantities:

  - l_1 and l_2 are the distance travelled along the first and second conductor, respectively
  - $\mathrm{d}\boldsymbol{l}_1$ and $\mathrm{d}\boldsymbol{l}_2$ are infinitesimal segements of each conductor. They point tangent to the conductor in the direction of the electric current.
  - $\boldsymbol{r}(l_1)$ and $\boldsymbol{r}(l_2)$ are the positions along the conductor

- The force on the first wire due to the second conduwire ctor, considering two infinitesimal pieces of each wire, is

  $$\mathrm{d}^2\boldsymbol{F} = \frac{\mu_0}{4\pi} \frac{(I_1\,\mathrm{d}\boldsymbol{l}_1)\cdot(I_2\,\mathrm{d}\boldsymbol{l}_2)}{|\boldsymbol{r}(l_2) - \boldsymbol{r}(l_1)|^2} \frac{\boldsymbol{r}(l_2) - \boldsymbol{r}(l_1)}{|\boldsymbol{r}(l_2) - \boldsymbol{r}(l_1)|}.$$

  Note that this is a double differential, since it involves the product of two differential segments $\mathrm{d}\boldsymbol{l}_1$ and $\mathrm{d}\boldsymbol{l}_2$. This expression may look daunting, be it is really just a generalization of the magnetic force between parallel wires given above.

- In integral form, the force on the first conductor C_1 due to the current in C_2 is written in terms of the line integrals over both C_1 and C_2. The expression reads

  $$\boldsymbol{F} = \frac{\mu_0 I_1 I_2}{4\pi} \oint_{C_1} \oint_{C_2} \frac{\mathrm{d}\boldsymbol{l}_1 \cdot \mathrm{d}\boldsymbol{l}_2}{|\boldsymbol{r}(l_2) - \boldsymbol{r}(l_1)|^2} \frac{\boldsymbol{r}(l_2) - \boldsymbol{r}(l_1)}{|\boldsymbol{r}(l_2) - \boldsymbol{r}(l_1)|}.$$

To find the force on C_2 due to C_1, we just reverse the role of $\boldsymbol{r}(l_1)$ and $\boldsymbol{r}(l_2)$.

- Finally, without derivation, we note that the above expression for the force between two conductors can be written in the equivalent vector-product form

$$\boldsymbol{F} = -\frac{\mu_0 I_1 I_2}{4\pi} \oint_{C_1} \oint_{C_2} \frac{\mathrm{d}\boldsymbol{l}_1 \times \left[\mathrm{d}\boldsymbol{l}_2 \times (\boldsymbol{r}(l_2) - \boldsymbol{r}(l_1))\right]}{|\boldsymbol{r}(l_2) - \boldsymbol{r}(l_1)|^3}.$$

This expression will useful when analyzing magnetic flux density.

2.1.3 Electric Current and its Magnitude

We dedicate a brief section to review electric current, since we will be working with electric current constantly in magnetostatics.

- Electric current is the motion of charged particles—for the time being, we will limit ourselves to currents along wire-like conductors. Electric current is defined as

$$I = \frac{\mathrm{d}q}{\mathrm{d}t}$$

and is a scalar quantity. In magnetostatics, we consider only time-independent currents for which $\frac{\mathrm{d}I}{\mathrm{d}t} = 0$.

- Some typical orders of magnitude for electric current are given in the table below

Current in...	Typical order of magnitude
a cell membrane channel	1 pA to 10 pA
a nerve impulse	1 µA
common household devices	1 A
superconducting magnets at the LHC	12 kA
lightning	10 kA to 200 kA
the Earth's core	1 GA

2.2 Magnetic Flux Density

2.2.1 The Biot-Savart Law for Magnetic Field

- We now aim to convert the expression for the magnetic force between conductors into an expression for magnetic field, just like we used Coulomb's law for the electric force between charges to derive an expression for electric field.

  We begin with the vector-product form of the force between two conductors C_1 and C_2, which reads

$$\boldsymbol{F} = -\frac{\mu_0 I_1 I_2}{4\pi} \oint_{C_1} \oint_{C_2} \frac{\mathrm{d}\boldsymbol{l}_1 \times \left[\mathrm{d}\boldsymbol{l}_2 \times (\boldsymbol{r}(l_2) - \boldsymbol{r}(l_1))\right]}{|\boldsymbol{r}(l_2) - \boldsymbol{r}(l_1)|^3}.$$

  We then rearrange this expression into the form

$$\boldsymbol{F} = \oint_{C_1} I_1 \,\mathrm{d}\boldsymbol{l}_1 \times \left(\frac{\mu_0 I_2}{4\pi} \oint_{C_2} \frac{\mathrm{d}\boldsymbol{l}_2 \times \left(\boldsymbol{r}(l_1) - \boldsymbol{r}(l_2)\right)}{|\boldsymbol{r}(l_1) - \boldsymbol{r}(l_2)|^3}\right),$$

- Comparing this expression for force to $\boldsymbol{F} = \oint_{C_1} I_1 \, \mathrm{d}\boldsymbol{l}_1 \times \boldsymbol{B}$ motivates the definition

$$\boldsymbol{B}(\boldsymbol{r}(l_1)) \equiv \frac{\mu_0 I_2}{4\pi} \oint_{C_2} \frac{\mathrm{d}\boldsymbol{l}_2 \times \big(\boldsymbol{r}(l_1) - \boldsymbol{r}(l_2)\big)}{|\boldsymbol{r}(l_1) - \boldsymbol{r}(l_2)|^3}.$$

  This is the magnetic field generated by the conductor C_2 carrying current I_2. Note that this expression is essentially the familiar Biot-Savart law.

  Finally, note that the expression for the magnetic force on the the conductor C_1, which reads

$$\boldsymbol{F} = \oint I_1 \, \mathrm{d}\boldsymbol{l}_1 \times \boldsymbol{B}(\boldsymbol{r}(l_1)),$$

  is just the integral form of the familiar expression $\boldsymbol{F} = I(\boldsymbol{l} \times \boldsymbol{B})$.

- Some typically values of magnetic field are given in the table below

Phenomenon	Magnetic Field
brain activity	1 fT
intergalactic magnetic fields	1 pT to 10 pT
heart activity	100 pT
earth's magnetic field	20 µT to 70 µT
iron ferromagnet	100 mT
sunspot	1 T
simple particle accelerator	10 T
neutron star	10^6 T to 10^{11} T
atomic nucleus	1 TT

2.2.2 Magnetic Field Lines

- In terms of the arc length parameter s, magnetic field lines $\boldsymbol{r}(s)$ are closed curves related to magnetic field $\boldsymbol{B}$ by

$$\dot{\boldsymbol{r}}(s) = \frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}s} = \frac{\boldsymbol{B}(\dot{\boldsymbol{r}}(s))}{|\boldsymbol{B}(\boldsymbol{r}(s))|}.$$

  Note that all magnetic field lines are closed curves—there are no magnetic monopoles.

2.2.3 Magnetic Circulation

- Magnetic circulation Γ_{m} is defined as the line integral

$$\Gamma_m = \oint_C \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{r} \neq 0.$$

  Magnetic circulation is nonzero because magnetic field lines are closed. In contrast, electric circulation can be zero for static fields, since electric field lines are not necessarily closed.

  The statement that all magnetic field lines are closed is equivalent to stating that magnetic monopoles do not exist (have not been observed) in nature.

- Example: the magnetic circulation circular wire of radius R in a magnetic field $\boldsymbol{B}$ is

$$\Gamma_m = \oint_C \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{r} = B \cdot (2\pi R) \neq 0.$$

- Writing magnetic circulation as a surface integral using Stokes' theorem leads to the conclusion
$$0 \neq \oint_C \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{r} = \iint_S \nabla \times \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{S} \implies \nabla \times \boldsymbol{B} \neq 0.$$

2.2.4 Magnetic Flux

- The magnetic flux through a surface S is defined as
$$\Phi_M = \iint_S \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{S}.$$
In the above expression, S is an arbitrary surface. However, if S is a closed surface, we can immediately write
$$\oiint_S \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{S} = 0$$
This is a consequence of magnetic field lines being closed. This result differs from it electric analog, Gauss's law, which reads $\Phi_E = Q/\epsilon_0 \neq 0$.

- In integral form, using the divergence theorem, the magnetic flux through a closed surface is
$$0 \equiv \Phi_M = \oiint_{\partial V} \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{S} = \iiint_V \nabla \cdot \boldsymbol{B} \, \mathrm{d}V.$$
In differential form the above equation is equivalent to the requirement
$$\nabla \cdot \boldsymbol{B} = 0,$$
which is a special case of a Maxwell equation for magnetostatics. The relationship $\nabla \cdot \boldsymbol{B} = 0$ is another manifestation of the absence of magnetic monopoles in nature.

2.3 Electric Current Density

- Electric current density allows us to generalize currents in a wire-like conductor to currents in a three-dimensional region of space, which will give us a much more general formalism for describing magnetic fields.

  Current density is defined in terms of the relationship
$$I = \iint_S \boldsymbol{j} \cdot \mathrm{d}\boldsymbol{S} = \iint_S \boldsymbol{j} \cdot \hat{\boldsymbol{n}} \, \mathrm{d}S,$$
where I is the current through the planar cross-sectional surface S. Note that I is scalar and $\boldsymbol{j}$ is a vector quantity. Since current density is a vector quantity, it can have arbitrary direction in space and is not tied to a conductor.

2.3.1 Examples of Current Densities

- Our first exqmple is the current density of continuous charge distribution. We begin by considering the current
$$\mathrm{d}I = \boldsymbol{j} \cdot \mathrm{d}\boldsymbol{S}$$
through some surface $\mathrm{d}\boldsymbol{S}$. By definition of current, we have
$$\mathrm{d}I = \mathrm{d}\left[\frac{\mathrm{d}q}{\mathrm{d}t}\right] = \mathrm{d}\left(\frac{\rho(\boldsymbol{r}) \, \mathrm{d}V}{\mathrm{d}t}\right) = \mathrm{d}\left(\frac{\rho(\boldsymbol{r}) \, \mathrm{d}S \, \mathrm{d}x}{\mathrm{d}t}\right) = \rho \frac{(\boldsymbol{v} \cdot \hat{\boldsymbol{n}}) \, \mathrm{d}S \, \mathrm{d}t}{\mathrm{d}t} = \rho v \, \mathrm{d}S,$$

where $(\boldsymbol{v} \cdot \hat{\boldsymbol{n}})$ is the charge carrier velocity normal to $\mathrm{d}\boldsymbol{S}$. The above implies

$$\boldsymbol{j}(\boldsymbol{r}, t) = \rho(\boldsymbol{r})\boldsymbol{v}(\boldsymbol{r}),$$

where $\boldsymbol{v}$ is the velocity field of the charge distribution at position $\boldsymbol{r}$—which we have assumed is uniformly distributed.

- Next, the current density of a one-dimensional conductor embedded in a plane at the origin is

$$\boldsymbol{j}(\boldsymbol{r}) = I\delta^2(\boldsymbol{\rho})\hat{\boldsymbol{l}},$$

where $\hat{\boldsymbol{l}}$ is the unit vector tangent to the conductor and parallel to the current.

- The current density of a point charge with velocity $\boldsymbol{v}$ is

$$\boldsymbol{j}(\boldsymbol{r}) = q\delta^3(\boldsymbol{r} - \boldsymbol{r}(t))\boldsymbol{v}(t).$$

- The current density of charge distributed through a planar surface at $z = z_0$ and moving with velocity $\boldsymbol{v}$ is

$$\boldsymbol{j}(\boldsymbol{r}) = \sigma\delta(z - z_0)\boldsymbol{v},$$

where σ is surface charge density.

2.4 Ampere's Law

- Ampere's law will allow us to explicitly connect a magnetic field to its current sources. We begin by considering a current loop C' parameterized by
  - the arc length l'
  - the infinitesimal arc segment $\mathrm{d}\boldsymbol{l}'$ pointing tangent to the conductor in the direction of the electric current
  - the position $\boldsymbol{r}(l')$ along the conductor.
- We then consider enclosing a portion of the loop C' with another loop C parameterized by the analogously defined quantities $\mathrm{d}\boldsymbol{l}$ and $\boldsymbol{r}(l)$.

  We being the analysis by finding the magnetic circulation along the curve C. Using the Biot-Savart to express $\boldsymbol{B}$, the circulation Γ_{M} is

$$\Gamma_{\mathrm{M}} = \oint_C \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{l} = \oint_C \left[-\frac{\mu_0 I}{4\pi} \oint_{C'} \mathrm{d}\boldsymbol{l}' \times \nabla \left(\frac{1}{|\boldsymbol{r}(l) - \boldsymbol{r}(l')|} \right) \right] \cdot \mathrm{d}\boldsymbol{l},$$

  where we've written the integrand in the second equality with a reverse-engineered gradient.

- The integrand then takes the form of a scalar triple product, which we can rearrange to get

$$\Gamma_{\mathrm{M}} = -\frac{\mu_0 I}{4\pi} \oint_C \oint_{C'} (\mathrm{d}\boldsymbol{l} \times \mathrm{d}\boldsymbol{l}') \cdot \nabla \left(\frac{1}{|\boldsymbol{r}(l) - \boldsymbol{r}(l')|} \right).$$

  Note that $\mathrm{d}\boldsymbol{l} \times \mathrm{d}\boldsymbol{l}'$ corresponds to a surface element—we'll call it $\mathrm{d}\boldsymbol{S}$. We then calculate the gradient and evaluate the dot product to get

$$\begin{aligned}\Gamma_{\mathrm{M}} &= -\frac{\mu_0 I}{4\pi} \oint \mathrm{d}\boldsymbol{S} \cdot \nabla \left(\frac{1}{|\boldsymbol{r}(l) - \boldsymbol{r}(l')|} \right) \\ &= +\frac{\mu_0 I}{4\pi} \oint \frac{\mathrm{d}S \cos\theta}{|\boldsymbol{r}(l) - \boldsymbol{r}(l')|^2},\end{aligned}$$

  where θ is the angle between $\mathrm{d}\boldsymbol{S}$ and the gradient.

- Next, we recognize the integrand is precisely the differential of the solid angle element $\mathrm{d}\Omega$. In terms of $\mathrm{d}\Omega$, the magnetic circulation becomes

$$\Gamma_{\mathrm{M}} = \frac{\mu_0 I}{4\pi} \oint \mathrm{d}\Omega = \frac{\mu_0 I}{4\pi} \cdot 4\pi = \mu_0 I.$$

  This result is Ampere's law, which, in terms of magnetic circulation, reads

$$\Gamma_{\mathrm{M}} = \mu_0 I.$$

  In words, Ampere's law states that the magnetic circulation in a loop (our C) enclosing a current-carrying loop with current I (our C') is proportional to the current I through the C'.

- Finally, we will combine Ampere's law with the definition of magnetic circulation. To review, the two relationships are

$$\Gamma_{\mathrm{M}} = \oint \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{S} = \iint \nabla \times \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{S} \qquad \text{and} \qquad \Gamma_{\mathrm{M}} = \mu_0 I = \mu_0 \iint \boldsymbol{j} \cdot \mathrm{d}\boldsymbol{S}.$$

  Combining the two relationships produces the important result

$$\nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{j}.$$

  This a Maxwell equation for magnetostatics and is often called Ampere's law in itself.

2.5 The Magnetic Vector Potential

- Like for electric field, it is often convenient to analyze magnetic fields in terms of a potential rather than the field itself; our goal in this section is to find an expression for the magnetic potential.

- Recall that magnetic field lines are closed, which results in $\nabla \times \boldsymbol{B} \neq 0$. The equality $\nabla \times \boldsymbol{B} \neq 0$ prohibits the introduction of a scalar potential in the style $\boldsymbol{B} = -\nabla\phi$.

  The reason is straightforward: the curl of a gradient is always zero. If we were to take the curl of a hypothetical equality $\boldsymbol{B} = -\nabla\phi$, we would get $\nabla \times \boldsymbol{B} = -\nabla \times \left[\nabla\phi\right] = 0$, which contradicts $\nabla \times \boldsymbol{B} \neq 0$.

- As a solution, we turn to the relationship $\nabla \cdot \boldsymbol{B} = 0$ (which corresponds to the fact that magnetic field lines are closed). Recall from vector calculus that the divergence of the curl of a vector field is always zero, which means we can write $\boldsymbol{B}$ as the curl of a vector field:

$$\nabla \cdot \boldsymbol{B} \equiv \nabla \cdot (\nabla \times \boldsymbol{A}) = 0 \implies \boldsymbol{B} = \nabla \times \boldsymbol{A}.$$

  The relationship $\boldsymbol{B} = \nabla \times \boldsymbol{A}$ gives us an implicit definition of the magnetic vector potential $\boldsymbol{A}$. Again, we stress that this definition rests on the identity $\nabla \cdot \boldsymbol{B} = 0$.

- Magnetic flux is written in terms of $\boldsymbol{A}$ using Stokes' theorem according to

$$\Phi_{\mathrm{M}} \equiv \iint_S \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{S} = \iint_S (\nabla \times \boldsymbol{A}) \cdot \mathrm{d}\boldsymbol{S} = \oint_{\partial S} \boldsymbol{A} \cdot \mathrm{d}\boldsymbol{r}.$$

  In other words, the magnetic flux through a surface equals the circulation of the magnetic potential $\boldsymbol{A}$ around the surface's boundary ∂S.

2.5.1 Example: Magnetic Vector Potential of an Inductor

- We consider a long, straight inductor, which has the simple magnetic field: inside the inductor, the magnetic field is approximately homogeneous and obeys $\boldsymbol{B}(\boldsymbol{r}) = \boldsymbol{B}_0$. Outside the inductor, the magnetic field is zero. We choose our coordinate system so that $\boldsymbol{B}_0 = (0, 0, B_0)$.

- The magnetic vector potential is defined by $\boldsymbol{B} = \nabla \times \boldsymbol{A}$. Without derivation, the correct expression for of $\boldsymbol{A}$ inside the inductor is

$$\boldsymbol{A} = \frac{1}{2}\boldsymbol{B}_0 \times \boldsymbol{r}.$$

Since there is not a standard inverse-curl operation, one would have to play around a bit to derive this result—we simply quote it to save time on vector calculus acrobatics. Note that although $\boldsymbol{B}$ is constant, $\boldsymbol{A}$ has a much more complicated dependence on position.

- Outside the inductor, we know $\boldsymbol{B} = 0$. What about $\boldsymbol{A}$? We assume inductor has radius a and consider a loop just hugging the outside of our inductor and bounding the planar surface S, which aligns with the inductor's circular cross section.

The magnetic flux through the surface S is then

$$\Phi_{\mathrm{M}} = \iint_S \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{S} = \boldsymbol{B}_0(\pi a^2).$$

Note that only the porition of S inside the inductor (for $r < a$) contributes non-zero magnetic flux, since $\boldsymbol{B} = 0$ outside the inductor.

Next, we recall the general definition of flux in terms of vector potential, which reads

$$\Phi_{\mathrm{M}} = \iint_S \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{S} = \oint_{\partial S} \boldsymbol{A} \cdot \mathrm{d}\boldsymbol{r}.$$

Comparing this general definition to our just-derived result $\Phi_{\mathrm{M}} = \pi a^2 \boldsymbol{B}_0$ gives

$$\oint_{\partial S} \boldsymbol{A} \cdot \mathrm{d}\boldsymbol{r} = \Phi_{\mathrm{M}} = \boldsymbol{B}_0(\pi a^2) \neq 0 \implies \boldsymbol{A} \neq 0.$$

In other words, the vector potential $\boldsymbol{A}$ is non-zero outside the inductor, even though $\boldsymbol{B}$ is zero in the same region.

- It turns out (again without derviation) that the vector potential outside the inductor is

$$\boldsymbol{A} = C\boldsymbol{B}_0 \times \frac{\boldsymbol{r}}{r^2},$$

where C is a constant. Using this expression for $\boldsymbol{A}$, the magnetic flux Φ_{M} through the loop ∂S comes out to

$$\Phi_{\mathrm{M}} = \oint_{\partial S} \boldsymbol{A} \cdot \mathrm{d}\boldsymbol{r} = \oint_{\partial S} C \left(\boldsymbol{B}_0 \times \frac{\boldsymbol{r}}{r^2}\right) \mathrm{d}\boldsymbol{r} = \cdots = 2\pi C B_0$$

The result of the integral is quoted without derivation. Combining the above result with the earlier equality $\Phi_M = B_0 \pi a^2$ gives

$$B_0 \pi a^2 = \Phi_M = 2\pi C B_0 \implies C = \frac{a^2}{2}.$$

The magnetic vector potential outside the inductor is thus

$$\boldsymbol{A} = \frac{a^2}{2}\boldsymbol{B}_0 \times \frac{\boldsymbol{r}}{r^2}.$$

Once again, we note the complexity of $\boldsymbol{A}$ outside the inductor relative to the trivial magnetic field $\boldsymbol{B} = 0$. Finally, we note that $\boldsymbol{A}$ changes continuously across the inductor boundary at $r = a$, which we can see by comparing the expressions for $\boldsymbol{A}$ inside and outside the inductor.

2.5.2 Gauges and the Magnetic Potential of an Inductor

- Consider the inductor example above. Even for a simple magnetic field, the corresponding magnetic potential $\boldsymbol{A}$ turned out to be quite complicated. However, we can often simplify the magnetic vector potential using gauge transformation.

- For more convenient calculations, we are free to define a new vector potential using the transform

$$\boldsymbol{A}' = \boldsymbol{A} + \nabla\zeta(\boldsymbol{r}).$$

  We can safely add the term $\nabla\zeta(\boldsymbol{r})$ because the curl of the gradient of a scalar field is always zero, and won't affect the end result for $\boldsymbol{B}$, which is the quantity actually measured in experiment.

  In other words, both $\boldsymbol{A}$ and $\boldsymbol{A}'$ correspond to the same magnetic field $\boldsymbol{B}$:

$$\boldsymbol{B} = \nabla \times \boldsymbol{A} = \nabla \times \boldsymbol{A}'.$$

- For the specific case of a long, straight inductor, we can simplify the expression for $\boldsymbol{A}$ by choosing

$$\zeta(\boldsymbol{r}) = -\frac{B_0 a^2}{2}\arctan\frac{y}{x}.$$

  For this choice of ζ, the vector potential outside the inductor simplifies to

$$\begin{aligned}\boldsymbol{A}' = \boldsymbol{A} + \nabla\zeta(\boldsymbol{r}) &= \frac{a^2}{2}\boldsymbol{B}_0 \nabla \times \frac{\boldsymbol{r}}{r^2} - \nabla\left(\frac{\boldsymbol{B}_0 a^2}{2}\arctan\frac{y}{x}\right)\\ &= \frac{\boldsymbol{B}_0 a^2}{2}\frac{2\pi}{a}\delta(\phi - \pi)\hat{\boldsymbol{e}}_\phi.\end{aligned}$$

  In this case, $\boldsymbol{A}'$ is zero everywhere besides along the curve $\phi = \pi$, corresponding to the negative portion of the x axis.

- To summarize, gauge transforms allow us to simplify the expression for vector potential $\boldsymbol{A}$ without changing the value of the magnetic field $\boldsymbol{B}$, which is the physically relevant quantity measured in experiment.

2.6 Magnetic Force

- Recall from previous sections that the magnetic force on a wire-like conductor C in the presence of an external magnetic field $\boldsymbol{B}$ is

$$\boldsymbol{F} = \int_C I\,\mathrm{d}\boldsymbol{l} \times \boldsymbol{B}(\boldsymbol{r}(l)),$$

where the direction of $\mathrm{d}\boldsymbol{l}$ aligns with the direction of current. If the field is homogeneous (same direction and same magnitude in all space) and the conductor is straight, the force simplifies to

$$\boldsymbol{F} = I\boldsymbol{l} \times \boldsymbol{B}.$$

- More generally, we can describe magnetic force in terms of current density $\boldsymbol{j}$, which is related to current via

$$I\,\mathrm{d}\boldsymbol{l} = \boldsymbol{j}\,\mathrm{d}^3\boldsymbol{r}.$$

In terms of current density, the magnetic force on a current distribution contained in the region V is

$$\boldsymbol{F} = \iiint_V \boldsymbol{j} \times \boldsymbol{B}\,\mathrm{d}^3\boldsymbol{r}.$$

- Next, we consider a the magnetic force on a moving charge particle, for which current density and charge are related via

$$\boldsymbol{j}(\tilde{\boldsymbol{r}}) = \rho(\tilde{\boldsymbol{r}})\boldsymbol{v} = q\delta^3(\tilde{\boldsymbol{r}} - \boldsymbol{r}(t))\boldsymbol{v},$$

where $\boldsymbol{r}(t)$ is the particle's trajectory and $\boldsymbol{v}$ is the particle's velocity. Substituting this expression for $\boldsymbol{j}$ into the expression for magnetic force leads to

$$\begin{aligned}\boldsymbol{F} &= \iiint_V \boldsymbol{j} \times \boldsymbol{B}\,\mathrm{d}^3\boldsymbol{r} = \iiint_V q\delta^3(\tilde{\boldsymbol{r}} - \boldsymbol{r}(t))\boldsymbol{v} \times \boldsymbol{B}\,\mathrm{d}^3\tilde{\boldsymbol{r}}\\ &= q\boldsymbol{v} \times \boldsymbol{B},\end{aligned}$$

which is the familiar Lorentz force on a charged particle in a magnetic field.

2.7 The Magnetic Analog of the Poisson Equation

- Next, we aim to find an equation for magnetic vector potential analogous to the Poisson equation for electric potential.

We begin with Ampere's law in terms of $\boldsymbol{A}$, which leads to

$$\mu_0\boldsymbol{j} = \nabla \times \boldsymbol{B} = \nabla \times (\nabla \times \boldsymbol{A}) = \nabla(\nabla \cdot \boldsymbol{A}) - \nabla^2\boldsymbol{A}.$$

It is possible to make the $\nabla \cdot \boldsymbol{A}$ term vanish. We do this with Helmholtz's theorem, which tells us that we can write an arbitrary vector field in the form

$$\boldsymbol{A} = \boldsymbol{A}_1 + \boldsymbol{A}_2,$$

where $\nabla \cdot \boldsymbol{A}_1 = 0$ and $\nabla \times \boldsymbol{A}_2 = 0$, i.e. $\boldsymbol{A}$ can be written as the sum of a solenoidal and irrotational vector field.

- We then introduce a gauge transformation of $\boldsymbol{A}$ such that $\boldsymbol{A}_2 = 0$. We are free to set $\boldsymbol{A}_2 = 0$ because $\nabla \times \boldsymbol{A}_2 = 0$ and thus has zero contribution to $\boldsymbol{B} = \nabla \times \boldsymbol{A}$, regardless of what value we choose for A_2. Substituting $\boldsymbol{A}_2 = 0$ into the Helmholtz composition gives $\boldsymbol{A} = \boldsymbol{A}_1$. We then take the divergence of this equality and apply $\nabla \cdot \boldsymbol{A}_1 = 0$ to get

$$\boldsymbol{A} = \boldsymbol{A}_1 \implies \nabla \cdot \boldsymbol{A} = \nabla \cdot \boldsymbol{A}_1 \equiv 0 \implies \nabla \cdot \boldsymbol{A} = 0.$$

Finally, we substitute $\nabla \cdot \boldsymbol{A} = 0$ into Ampere's law to get the desired equation

$$\mu_0\boldsymbol{j} = 0 - \nabla^2\boldsymbol{A} \implies \nabla^2\boldsymbol{A} = -\mu_0\boldsymbol{j}.$$

Note the similarity to the Poisson equation for electric potential.

- Formally, we would solve this equation using a vector Green's function, similarly to how we solved the electrostatic Poisson equation with a Green's function. However, given the close similarity between the magnetostatic and electrostatic Poisson equations, we can simply guess the solution for $\boldsymbol{A}$ with reference to the solution for ϕ. Without derivation, the result comes out to be

$$\boldsymbol{A}(\boldsymbol{r}) = \frac{\mu_0}{4\pi} \iiint_V \frac{\boldsymbol{j}(\tilde{\boldsymbol{r}})}{|\boldsymbol{r} - \tilde{\boldsymbol{r}}|} \, \mathrm{d}^3\tilde{\boldsymbol{r}},$$

where V is the region of space with non-zero current density $\boldsymbol{j}$. This is the general expression for magnetic vector potential $\boldsymbol{A}$.

- The above general solution for $\boldsymbol{A}$ is closely related to the Biot-Savart law. To show this, we take the curl of $\boldsymbol{A}(\boldsymbol{r})$ with respect to $\boldsymbol{r}$ to get

$$\begin{aligned}\boldsymbol{B}(\boldsymbol{r}) = \nabla \times \boldsymbol{A}(\boldsymbol{r}) &= \nabla \times \left(\frac{\mu_0}{4\pi} \iiint_V \frac{\boldsymbol{j}(\tilde{\boldsymbol{r}})}{|\boldsymbol{r} - \tilde{\boldsymbol{r}}|} \, \mathrm{d}^3\tilde{\boldsymbol{r}} \right) \\ &= \frac{\mu_0}{4\pi} \iiint_V \frac{\boldsymbol{j} \times (\boldsymbol{r} - \tilde{\boldsymbol{r}})}{|\boldsymbol{r} - \tilde{\boldsymbol{r}}|^3} \, \mathrm{d}^3\tilde{\boldsymbol{r}},\end{aligned}$$

which is the Biot-Savart law for a current distribution $\boldsymbol{j}$. The Biot-Savart law and the magnetic Poisson equation are closely related, and it is difficult to say which is more fundamental. In the coming chapters, we will sidestep this dilemma by using the Maxwell equations as the foundation of electromagnetism.

2.8 Magnetic Energy

- Our analysis of magnetostatic energy will proceed analogously to the analysis of electrostatic energy, in that we will first consider the magnetic energy of a current distribution an external magnetic field (whose source we take for granted and otherwise ignore), and then use that result to find the total energy of current distribution in a magnetic field where we will include the field's source.

- We begin by considering a current loop C carrying current I in an external magnetic field $\boldsymbol{B}$. We stress that we don't consider the source of the field $\boldsymbol{B}$—we simply take its existence for granted.

  Note also that we will work in the quasi-static regime of electromagnetism meaning we take currents to be nonzero but constant with respect to time, ie. $I \neq 0$ and $\dot{I} = 0$.

  Finally, for use in the next bullet, recall that the magnetic force on a current carrying-conductor is

$$\boldsymbol{F} = \oint_C I \, \mathrm{d}\boldsymbol{l} \times \boldsymbol{B} = I \oint_C \hat{\boldsymbol{t}} \times \boldsymbol{B} \, \mathrm{d}l,$$

  where $\hat{\boldsymbol{t}}$ is the tangent to the conducting loop pointing in the direction of current.

- We then consider moving the loop by the infinitesimal displacement $\mathrm{d}\boldsymbol{r}$—in which case we do work W against the magnetic field force, which is related to the desired magnetic energy W_B by the work-energy theorem. The work W associated with the displacement $\mathrm{d}\boldsymbol{r}$ is

$$\begin{aligned}W = -\boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r} &= -I \oint_C \hat{\boldsymbol{t}} \times \boldsymbol{B} \, \mathrm{d}l \cdot \mathrm{d}\boldsymbol{r} \\ &= -I \oint_C \mathrm{d}\boldsymbol{r} \cdot (\hat{\boldsymbol{t}} \times \boldsymbol{B}) \, \mathrm{d}l.\end{aligned}$$

The last result is a scalar triple product, which we rearrange to get

$$W = -I \oint_C (\mathrm{d}\boldsymbol{r} \times \hat{\boldsymbol{t}}) \cdot \boldsymbol{B} \, \mathrm{d}l.$$

Next, we recognize that quantity $(\mathrm{d}\boldsymbol{r} \times \hat{\boldsymbol{t}}) \, \mathrm{d}l$ is the surface element $\mathrm{d}\boldsymbol{S}$ of the loop during the displacement $\mathrm{d}\boldsymbol{r}$. In terms of $\mathrm{d}\boldsymbol{S}$, the work reads

$$W = -I \iint_S \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{S} \equiv -I\Phi_{\mathrm{M}}.$$

We stress that the magnetic flux in the above expression is the flux arising from the external field $\boldsymbol{B}$ only, and is unrelated to the current distribution.

Note also that if $\boldsymbol{B}$ is homogeneous and the quantity $\boldsymbol{B} \cdot \boldsymbol{S}$ does not change through time, the magnetic flux Φ_{M} and thus the magnetic work W is zero.

- Next, we use $\boldsymbol{B} = \nabla \times \boldsymbol{A}$ to write the magnetic work in terms of the magnetic potential:

  $$W = -I \iint_S \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{S} = -I \iint_S (\nabla \times \boldsymbol{A}) \cdot \mathrm{d}\boldsymbol{S}$$

  The last term is the surface integral of a curl quantity, which we can rewrite as an integral over the surface boundary ∂S using Stokes' theorem. But which surface boundary—this is important! Assume the initial conductor before the displacement $\mathrm{d}\boldsymbol{r}$ is described by the curve C_1, and that after the displacement $\mathrm{d}\boldsymbol{r}$ the conductor is described by the curve C_2. In this case, the appropriate surface is the surface between the curves C_1 and C_2, and the integral for W becomes

  $$W = -I \iint_S (\nabla \times \boldsymbol{A}) \cdot \mathrm{d}\boldsymbol{S} = -I \oint_{C_2} \boldsymbol{A} \cdot \mathrm{d}\boldsymbol{r} + I \oint_{C_1} \boldsymbol{A} \cdot \mathrm{d}\boldsymbol{r},$$

  which is the difference of the line integrals over the two curves.

- Next, we generalize our expression from a current loop C to a generalized current distribution described by the current density $\boldsymbol{j}$.

  First, an intermediate step: we use the identity $I \, \mathrm{d}\boldsymbol{r} = \boldsymbol{j} \, \mathrm{d}^3\boldsymbol{r}$ to write

  $$I \oint_C \boldsymbol{A} \cdot \mathrm{d}\boldsymbol{r} = \iiint_V \boldsymbol{j}(\boldsymbol{r})\boldsymbol{A}(\boldsymbol{r}) \cdot \mathrm{d}^3\boldsymbol{r}$$

  where V is the region of space containing the non-zero current density $\boldsymbol{j}$. Using this relantionship, the work energy theorem for magnetic energy then reads

  $$W \equiv W_{\mathrm{B}}^{(1)} - W_{\mathrm{B}}^{(2)} = -\iiint_{V_2} \boldsymbol{j}_2(\boldsymbol{r})\boldsymbol{A}(\boldsymbol{r}) \cdot \mathrm{d}^3\boldsymbol{r} + \iiint_{V_1} \boldsymbol{j}_1(\boldsymbol{r})\boldsymbol{A}(\boldsymbol{r}) \cdot \mathrm{d}^3\boldsymbol{r}$$

  where the (2) terms correspond to quantities after the displacement $\mathrm{d}\boldsymbol{r}$ and the (1) terms to quantities before the displacement $\mathrm{d}\boldsymbol{r}$. The magnetostatic energy W_{M} of a charge distribution $\boldsymbol{j}$ in an *external* magnetic field $\boldsymbol{B}$ is thus

  $$W_{\mathrm{B}} = -\iiint_V \boldsymbol{j}(\boldsymbol{r})\boldsymbol{A}(\boldsymbol{r}) \, \mathrm{d}^3\boldsymbol{r},$$

  and the corresponding magnetic energy density w_{B} is

  $$w_{\mathrm{B}}(\boldsymbol{r}) = -\boldsymbol{j}(\boldsymbol{r})\boldsymbol{A}(\boldsymbol{r}).$$

2.8.1 Magnetic Field Energy as a Function of Current

- Using the general solution to the magnetic Poisson equation, the magnetic vector potential $\boldsymbol{A}$ associated with the *external* magnetic field $\boldsymbol{B}$ is

$$\boldsymbol{A}(\boldsymbol{r}) = \frac{\mu_0}{4\pi} \iiint_{\tilde{V}} \frac{\tilde{\boldsymbol{j}}(\boldsymbol{r})}{|\boldsymbol{r} - \tilde{\boldsymbol{r}}|} \, \mathrm{d}^3\tilde{\boldsymbol{r}}$$

  where we have used $\tilde{j}$ to denote the current distribution creating the external magnetic field $\boldsymbol{B}$, which we otherwise took for granted in the previous section, while $\tilde{V}$ denotes the region of space containing $\tilde{j}$.

- We then substitute this expression for $\boldsymbol{A}$ into magnetic field energy W_B to get

$$W_\mathrm{B} = -\frac{\mu_0}{4\pi} \iiint_V \iiint_{\tilde{V}} \frac{\boldsymbol{j}(\boldsymbol{r})\tilde{\boldsymbol{j}}(\tilde{\boldsymbol{r}}) \, \mathrm{d}^3\boldsymbol{r} \, \mathrm{d}^3\tilde{\boldsymbol{r}}}{|\boldsymbol{r} - \tilde{\boldsymbol{r}}|}$$

2.8.2 TODO:

Last hour of 6th lecture missing. Topics missing include:

- Total energy in a magnetic field
- Energy in a magnetic field, included energy needed to maintain currents

2.8.3 Magnetic Force and the Stress-Energy Tensor

2.8.4 Magnetic Force as a Function of Magnetic Field

- Consider a current distribution $\boldsymbol{j}(\boldsymbol{r})$ consisting of an arbitrary system of closed current loops. We then place the current distribution in an external magnetic field $\boldsymbol{B}$ and consider the total magnetic field, i.e. the sum of the external magnetic field and the field generated by the current distribution. Our goal in this section is to find the force acting on the current distribution written in terms of only the external field.

- In general, the magnetic force on a current distribution is

$$\boldsymbol{F}_\mathrm{M} = \iiint_V \boldsymbol{j} \times \boldsymbol{B} \, \mathrm{d}^3\boldsymbol{r},$$

  where V is the region of space containing the current distribution $\boldsymbol{j}$ and $\boldsymbol{B}$ is the total magnetic field, including the "internal" field generated by the current distribution itself.

  Note, however, (quoted and not proven) that the contribution of the internal field to the magnetic force integrates to zero. This makes sense—the current distribution shouldn't be able to generate a force on itself.

  For a simple current loop, we could show that the internal field contributes zero force by substituting in the Biot-Savart law for $\boldsymbol{B}$ and applying $\boldsymbol{j} \times \boldsymbol{j} = 0$.

- Next, we write $\boldsymbol{j}$ in terms of $\boldsymbol{B}$ using Ampere's law $\nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{j}$. The resulting expression for $\boldsymbol{F}_\mathrm{M}$ is

$$\boldsymbol{F}_\mathrm{M} = \iiint_V \boldsymbol{j} \times \boldsymbol{B} \, \mathrm{d}^3\boldsymbol{r} = \frac{1}{\mu_0} \iiint (\nabla \times \boldsymbol{B}) \times \boldsymbol{B} \, \mathrm{d}^3\boldsymbol{r}.$$

Next, we use the general vector identity

$$\boldsymbol{B} \times (\nabla \times \boldsymbol{B}) = \frac{1}{2}\nabla B^2 - \nabla \cdot (\boldsymbol{B} \otimes \boldsymbol{B}) + \boldsymbol{B}(\nabla \cdot \boldsymbol{B}),$$

which simplifies in our case because $\nabla \cdot \boldsymbol{B} = 0$ (absence of magnetic monopoles). The expression for $\boldsymbol{F}_\mathrm{M}$ becomes

$$\boldsymbol{F}_\mathrm{M} = \frac{1}{\mu_0} \iiint_V \nabla \cdot (\boldsymbol{B} \otimes \boldsymbol{B}) \, \mathrm{d}^3\boldsymbol{r} - \frac{1}{2\mu_0} \iiint_V \nabla B^2 \, \mathrm{d}^3\boldsymbol{r}.$$

- Next, we want to write $\boldsymbol{F}_\mathrm{M}$ in terms of a surface integral—this is easier to measure than a volume integral. We do this with the divergence theorem:

$$\boldsymbol{F}_\mathrm{M} = \frac{1}{\mu_0} \oiint_{\partial V} \left(\boldsymbol{B} \otimes \boldsymbol{B} - \frac{1}{2} B^2 \mathbf{I} \right) \mathrm{d}\boldsymbol{S},$$

where $\mathbf{I}$ is the identity matrix and ∂V is the surface of the region containing the original current distribution $\boldsymbol{j}$. Note that the quantity $\boldsymbol{B}$, which is the total field, now implicitly contains the current distribution $\boldsymbol{j}$.

2.8.5 Magnetostatic Stress Tensor

- We write the magnetostatic force in terms of the magnetostatic stress tensor, using Einstein summation notation, as

$$F_{\mathrm{M}_i} = \oiint_{\partial V} \mathrm{T}_{ik} \hat{n}_k \, \mathrm{d}S,$$

where $\hat{n}_k$ is the normal to the surface ∂V and $\mathbf{T}$ is the (3×3) magnetostatic stress tensor with components given by

$$T_{ik} = \frac{1}{\mu_0} \left(B_i B_k - \frac{1}{2} B^2 \delta_{ik} \right).$$

- We can also write the force as a volume integral in the form

$$F_{\mathrm{M}_i} = \iiint_V \frac{\partial \mathrm{T}_{ik}}{\partial x_k} \, \mathrm{d}^3\boldsymbol{r} \equiv \iiint_V f_{\mathrm{M}_i} \, \mathrm{d}^3\boldsymbol{r},$$

where we have introduced the volume force density f_M, motivated by the fact that $\frac{\partial \mathrm{T}_{ik}}{\partial x_k}$ has units $\mathrm{N\,m^{-3}}$.

2.9 The Multipole Expansion of Magnetic Field

- We begin with the multipole expansion of the magnetic vector potential $\boldsymbol{A}$—we consider the behavior of $\boldsymbol{A}(\boldsymbol{r})$ far from the source of $\boldsymbol{A}$. The general expression for $\boldsymbol{A}$, from the solution to the magnetic Poisson equation, is

$$\boldsymbol{A}(\boldsymbol{r}) = \frac{\mu_0}{4\pi} \iiint_V \frac{\boldsymbol{j}(\boldsymbol{s})}{|\boldsymbol{r} - \boldsymbol{s}|} \, \mathrm{d}^3\boldsymbol{s},$$

where V is the region of space containing the current density $\boldsymbol{j}$.

- We then expand $\boldsymbol{A}$ in the regime of $|\boldsymbol{r}| \gg |\boldsymbol{s}|$, using only the first-order expansion:
$$\frac{1}{|\boldsymbol{r}-\boldsymbol{s}|} \approx \frac{1}{r} - (\boldsymbol{s}\cdot\nabla)\frac{1}{r} + \cdots = \frac{1}{r} + \frac{\boldsymbol{s}\cdot\boldsymbol{r}}{r^3} + \cdots .$$
Using this expansion, the magnetic potential becomes
$$\boldsymbol{A}(\boldsymbol{r}) \approx \frac{\mu_0}{4\pi r}\iiint_V \boldsymbol{j}(\boldsymbol{s})\,\mathrm{d}^3\boldsymbol{s} + \frac{\mu_0}{4\pi r^3}\iiint_V (\boldsymbol{r}\cdot\boldsymbol{s})\boldsymbol{j}(\boldsymbol{s})\,\mathrm{d}^3\boldsymbol{s}.$$
The first term in the above expansion is called the monopole term—note that the monopole is a vector quantity (as opposed to the electric monopole, which corresponds to a point charge, which is a scalar quantity).

  The second term—the dipole term—is a tensor (as opposed to the electric field dipole, which is a vector). In general, the vector quantities arising in the magnetic multipole expansion have one more index than the analogous terms in the electric multipole expansion because $\boldsymbol{A}$ is a vector field while electric potential ϕ is a scalar field.

- **The Monopole Term:** The monopole term is zero, which corresponds to the fact that in magnetostatics all current loops contributing the current density $\boldsymbol{j}$ are closed, meaning $\nabla\cdot\boldsymbol{j} = 0$. The result is
$$\iiint_V \boldsymbol{j}(\boldsymbol{s})\,\mathrm{d}^3\boldsymbol{s} = 0.$$
Note, however, that $\nabla\cdot\boldsymbol{j} = 0$ only applies in magnetostatics.

- **The Dipole Term:** Since the monopole term is zero, the multiple expression for $\boldsymbol{A}(\boldsymbol{r})$ simplifies to
$$\boldsymbol{A}(\boldsymbol{r}) = \frac{\mu_0}{4\pi r^3}\iiint_V (\boldsymbol{r}\cdot\boldsymbol{s})\boldsymbol{j}(\boldsymbol{s})\,\mathrm{d}^3\boldsymbol{s}.$$
Without derivation, we state that we can use the divergence theorem and some vector calculus acrobatics to write $A(\boldsymbol{r})$ in the form
$$\boldsymbol{A}(\boldsymbol{r}) = \frac{\mu_0}{4\pi}\frac{\boldsymbol{m}\times\boldsymbol{r}}{r^3},$$
where we have defined the magnetic dipole moment
$$\boldsymbol{m} = \frac{1}{2}\iiint_V \boldsymbol{s}\times\boldsymbol{j}(\boldsymbol{s})\,\mathrm{d}^3\boldsymbol{s}.$$
As a side note, the dipole term can also be written in the form
$$\boldsymbol{A}(\boldsymbol{r}) = \frac{\mu_0}{4\pi}\nabla\times\frac{\boldsymbol{m}}{|\boldsymbol{r}|}$$

2.9.1 The Magnetic Field of a Magnetic Dipole

- We now have everything we need to find the magnetic field of a magnetic dipole. We start with the just-derived expression for magnetic vector potential, which reads
$$\boldsymbol{A}(\boldsymbol{r}) = \frac{\mu_0}{4\pi}\frac{\boldsymbol{m}\times\boldsymbol{r}}{r^3}.$$
Using $\boldsymbol{A}$, we find the magnetic field $\boldsymbol{B} \equiv \nabla\times\boldsymbol{A}$ via
$$\boldsymbol{B} = \nabla\times\boldsymbol{A} = \nabla\times\left(\frac{\mu_0}{4\pi}\frac{\boldsymbol{m}\times\boldsymbol{r}}{r^3}\right) = \frac{\mu_0}{4\pi}\frac{3\boldsymbol{r}(\boldsymbol{r}\cdot\boldsymbol{m}) - \boldsymbol{m}r^2}{r^5}.$$

2.9.2 Ampere Equivalence for a Circular Current Loop

- We start with the magnetic dipole moment of a circular current loop of radius a and carrying current I. From introductory electromagnetism, the magnetic dipole moment of a circular current loop is

$$\hat{\boldsymbol{m}} = IS\hat{\boldsymbol{n}},$$

where $\hat{\boldsymbol{n}}$ is the normal to the current loop. However, we will now formally derive this result.

- Using the formalism of the magnetic dipole expansion, we find the circular loop's magnetic dipole moment via

$$\boldsymbol{m} = \frac{1}{2}\iiint_V \boldsymbol{s} \times \boldsymbol{j}(\boldsymbol{s})\,\mathrm{d}^3\boldsymbol{s} = \frac{1}{2}\oint \boldsymbol{s} \times (I\,\mathrm{d}\boldsymbol{s}).$$

We then transition to polar coordinates in which $\boldsymbol{s} \times \mathrm{d}\boldsymbol{s} = a\hat{\boldsymbol{e}}_r \times \hat{\boldsymbol{e}}_\phi$ to get

$$\boldsymbol{m} = \frac{1}{2}I\oint (a\hat{\boldsymbol{e}}_r) \times (\hat{\boldsymbol{e}}_\phi)\,\mathrm{d}l = \frac{aI}{2}\oint \hat{\boldsymbol{e}}_z\,\mathrm{d}l = \pi a^2 I\hat{\boldsymbol{e}}_z = IS\hat{\boldsymbol{e}}_z.$$

The result $\boldsymbol{m} = IS\hat{\boldsymbol{e}}_z$ is the known expression for a circular loop's magnetic dipole moment.

- Ampere equivalence refers to the following: a circular current-carrying loop is equivalent to a magnetic dipole in an external magnetic field.

2.9.3 Multipole Expansion of Magnetic Energy

- We consider an arbitrary system of current carrying loops with current density $\boldsymbol{j}$ localized in space around the position $\boldsymbol{r}_0$—a current distribution *must* be localized to use a multipole expansion.

  We assume the current distribution occurs in an external magnetic field described with the vector potential $\boldsymbol{A}$.

- The current distribution's magnetic field energy in the *external* magnetic field is

$$W_\mathrm{B} = -\iiint_V \boldsymbol{j}(\boldsymbol{r}) \cdot \boldsymbol{A}(\boldsymbol{r})\,\mathrm{d}^3\boldsymbol{r},$$

where V is the region containing the localized current distribution $\boldsymbol{j}$.

- We then expand the magnetic potential about $\boldsymbol{r}_0$ to get

$$\boldsymbol{A}(\boldsymbol{r}) \approx \boldsymbol{A}(\boldsymbol{r}_0) + \big[(\boldsymbol{r} - \boldsymbol{r}_0) \cdot \nabla_0\big]\boldsymbol{A}(\boldsymbol{r}_0) + \cdots,$$

where the gradient acts on $\boldsymbol{r}_0$. In terms of this expansion, the magnetic energy is then

$$\begin{aligned} W_\mathrm{B} &= -\iiint_V \boldsymbol{j}(\boldsymbol{r})(\boldsymbol{r}) \cdot \boldsymbol{A}(\boldsymbol{r}_0)\,\mathrm{d}^3\boldsymbol{r} - \iiint \boldsymbol{j}(\boldsymbol{r})\big[(\boldsymbol{r} - \boldsymbol{r}_0) \cdot \nabla_0\big]\boldsymbol{A}(\boldsymbol{r})\,\mathrm{d}^3\boldsymbol{r} \\ &= -\boldsymbol{A}(\boldsymbol{r}_0)\iiint \boldsymbol{j}(\boldsymbol{r})\,\mathrm{d}^3\boldsymbol{r} - \iiint \boldsymbol{j}(\boldsymbol{r})\big[(\boldsymbol{r} - \boldsymbol{r}_0) \cdot \nabla_0\big]\boldsymbol{A}(\boldsymbol{r}_0)\,\mathrm{d}^3\boldsymbol{r}. \end{aligned}$$

The first term is the monopole term and integrates to zero. The second term is more complicated—we will handle it by components in the next bullet point.

- First, we note that
$$\nabla_0 \boldsymbol{A}(\boldsymbol{r}_0) = \frac{\partial \boldsymbol{A}(\boldsymbol{r}_0)_j}{\partial \boldsymbol{r}_{0_i}}.$$
Note that this term depends only on $\boldsymbol{r}_0$, and thus can be moved outside the integral over $\boldsymbol{r}$ in the energy expression.

  Next, we apply tensor symmetrization (justified in the next section), which allows us to write
$$\iiint_V \boldsymbol{j}(\boldsymbol{r})_j (\boldsymbol{r} - \boldsymbol{r}_0)_i \, \mathrm{d}^3\boldsymbol{r} = -\iiint \boldsymbol{j}(\boldsymbol{r})_i (\boldsymbol{r} - \boldsymbol{r}_0)_j \, \mathrm{d}^3\boldsymbol{r}$$
Basically, the indexes i and j of the components of the vectors $\boldsymbol{j}$ and $(\boldsymbol{r} - \boldsymbol{r}_0)$ are switched, with the addition of a minus sign. We use this symmetry to write
$$\iiint_V \boldsymbol{j}(\boldsymbol{r})_j (\boldsymbol{r} - \boldsymbol{r}_0)_i \, \mathrm{d}^3\boldsymbol{r} = \frac{1}{2} \iiint_V \left[\boldsymbol{j}(\boldsymbol{r})_j (\boldsymbol{r} - \boldsymbol{r}_0)_i - \boldsymbol{j}(\boldsymbol{r})_i (\boldsymbol{r} - \boldsymbol{r}_0)_j\right] \mathrm{d}^3\boldsymbol{r}$$
The symmetry that permits the index switching comes from a generalized form of the divergence theorem—we postpone a derivation to the next section.

- Putting the pieces together, the magnetic energy is
$$W_{\mathrm{B}} = -\frac{1}{2} \frac{\partial \boldsymbol{A}(\boldsymbol{r}_0)_j}{\partial \boldsymbol{r}_{0_i}} \iiint_V \left[\boldsymbol{j}(\boldsymbol{r})_j (\boldsymbol{r} - \boldsymbol{r}_0)_i - \boldsymbol{j}(\boldsymbol{r})_i (\boldsymbol{r} - \boldsymbol{r}_0)_j\right] \mathrm{d}^3\boldsymbol{r}$$
Back in vector form, the magnetic energy in the external field reads
$$W_{\mathrm{B}} = -\frac{1}{2} \iiint_V \Big\{\boldsymbol{j}(\boldsymbol{r})\big[(\boldsymbol{r} - \boldsymbol{r}_0) \cdot \nabla_0\big] \boldsymbol{A}(\boldsymbol{r}_0) - (\boldsymbol{r} - \boldsymbol{r}_0)(\boldsymbol{j} \cdot \nabla_0) \boldsymbol{A}(\boldsymbol{r}_0)\Big\} \, \mathrm{d}^3\boldsymbol{r}.$$

- Next, to make a subsequent step clearer, we define the following vectors:
$$\boldsymbol{a} \equiv (\boldsymbol{r} - \boldsymbol{r}_0) \qquad \boldsymbol{b} \equiv \boldsymbol{j}(\boldsymbol{r}) \qquad \boldsymbol{c} \equiv \nabla_0 \qquad \boldsymbol{d} \equiv \boldsymbol{A}(\boldsymbol{r}_0).$$
In this notation, the magnetic energy is
$$\begin{aligned} W_{\mathrm{B}} &= -\frac{1}{2} \iiint_V \Big[\boldsymbol{b}(\boldsymbol{a} \cdot \boldsymbol{c})\boldsymbol{d} - \boldsymbol{a}(\boldsymbol{b} \cdot \boldsymbol{c})\boldsymbol{d}\Big] \, \mathrm{d}^3\boldsymbol{r} \\ &= -\frac{1}{2} \iiint_V \Big[(\boldsymbol{a} \cdot \boldsymbol{c})(\boldsymbol{b} \cdot \boldsymbol{d}) - (\boldsymbol{a} \cdot \boldsymbol{d})(\boldsymbol{b} \cdot \boldsymbol{c})\Big] \, \mathrm{d}^3\boldsymbol{r}. \end{aligned}$$
We then use the vector identity[1]
$$(\boldsymbol{a} \times \boldsymbol{b}) \cdot (\boldsymbol{c} \times \boldsymbol{d}) = (\boldsymbol{a} \cdot \boldsymbol{c})(\boldsymbol{b} \cdot \boldsymbol{d}) - (\boldsymbol{a} \cdot \boldsymbol{d})(\boldsymbol{b} \cdot \boldsymbol{c})$$
to rewrite the magnetic field energy in the form
$$\begin{aligned} W_{\mathrm{B}} &= -\frac{1}{2} \iiint_V \Big\{\big[(\boldsymbol{r} - \boldsymbol{r}_0) \times \boldsymbol{j}\big] \cdot \big[\nabla_0 \times \boldsymbol{A}(\boldsymbol{r}_0)\big]\Big\} \, \mathrm{d}^3\boldsymbol{r} \\ &= -\frac{1}{2} \big[\nabla_0 \times \boldsymbol{A}(\boldsymbol{r}_0)\big] \cdot \iiint_V \big[(\boldsymbol{r} - \boldsymbol{r}_0) \times \boldsymbol{j}\big] \, \mathrm{d}^3\boldsymbol{r}, \end{aligned}$$
where we have moved the $\boldsymbol{r}_0$-dependent term $\nabla_0 \times \boldsymbol{A}$ out of the integral over $\boldsymbol{r}$.

- Recognizing $\boldsymbol{B} = \nabla \times \boldsymbol{A}$ and the magnetic dipole in the integrand, the result is
$$\begin{aligned} W_{\mathrm{B}} &= -\frac{1}{2} \big[\nabla_0 \times \boldsymbol{A}(\boldsymbol{r}_0)\big] \cdot \iiint_V \big[(\boldsymbol{r} - \boldsymbol{r}_0) \times \boldsymbol{j}\big] \, \mathrm{d}^3\boldsymbol{r} \\ &= -\boldsymbol{B}(\boldsymbol{r}_0) \cdot \boldsymbol{m}, \end{aligned}$$
which is the familiar expression for the energy of a magnetic dipole in an external magnetic field.

[1] This is the Binet–Cauchy identity in three dimensions.

2.9.4 More on Tensor Symmetrization

- The general idea of tensor symmetrization is to write a complicated tensor as a sum of tensors with a simpler structure. We consider a generic vector field $\boldsymbol{A}$ (not necessarily magnetic vector potential) start with the divergence theorem, in the familiar form
$$\oiint_{\partial V} \boldsymbol{A} \cdot \mathrm{d}\boldsymbol{S} = \iiint_V \nabla \cdot \boldsymbol{A} \, \mathrm{d}^3\boldsymbol{r}.$$
However, the divergence theorem can be generalized. In our case, applied to the third-rank tensor $\boldsymbol{r} \otimes \boldsymbol{r} \otimes \boldsymbol{A}$, the divergence theorem, in component form, reads
$$\oiint_{\partial V} r_i r_j A_k \hat{n}_k \, \mathrm{d}S = \iiint_V \big[r_i A_j + r_j A_i + r_i r_j (\nabla \cdot \boldsymbol{A})\big] \, \mathrm{d}^3\boldsymbol{r}.$$

- We now apply this generalized divergence theorem to the previous section involving the multipole expansion of magnetic energy, and take $\boldsymbol{A} = \boldsymbol{j}$, where $\boldsymbol{j}$ is the current density of our current distribution. The result is
$$\oiint_{\partial V} r_i r_j j_k \hat{n}_k \, \mathrm{d}S = \iiint_V \big[r_i j_j + r_j j_i + r_i r_j (\nabla \cdot \boldsymbol{j})\big] \, \mathrm{d}^3\boldsymbol{r}.$$
In our magnetostatic situation with closed current loops we have $\nabla \cdot \boldsymbol{j} = 0$, and the above expression simplifies to
$$\oiint_{\partial V} r_i r_j j_k \hat{n}_k \, \mathrm{d}S = \iiint_V \big[r_i j_j + r_j j_i\big] \, \mathrm{d}^3\boldsymbol{r}.$$

- Next, we note that on the current distribution's border ∂V, the current density is $\boldsymbol{j} = 0$ (since $\boldsymbol{j}$ is zero outside of V and must change continuously across the border). Note that the simplification $\boldsymbol{j}\big|_{\partial V} = 0$ is possible only because we assumed our charge distribution is localized, as is reasonable for a magnetic dipole.

  From $\boldsymbol{j}\big|_{\partial V} = 0$ follows
$$\oiint_{\partial V} r_i r_j j_k \hat{n}_k \, \mathrm{d}S = 0 \implies \iiint_V \big[r_i j_j + r_j j_i\big] \, \mathrm{d}^3\boldsymbol{r} = 0.$$
The integral in the last equality can be zero for all $\boldsymbol{r}$ only if
$$r_i j_j = -r_j j_i.$$
In other words, the tensor $\boldsymbol{r} \otimes \boldsymbol{r} \otimes \boldsymbol{j}$ is symmetric, which justifies the use of tensor symmetrization in the derivation of a magnetic dipole's magnetic energy in an external magnetic field in the previous section.

2.9.5 Force on a Magnetic Dipole in an External Magnetic Field

- In this section, we aim to find an expression for the force on a magnetic dipole in an external magnetic field. We begin with the general relationship between force and energy:
$$\mathrm{d}W_{\mathrm{M}} = -\boldsymbol{F}_{\mathrm{M}} \cdot \mathrm{d}\boldsymbol{r},$$
and take $\mathrm{d}\boldsymbol{r}$ to be a small displacement of a current distribution $\boldsymbol{j}$ in an *external* magnetic field.

- Combining the expression for a magnetic dipole's energy, i.e. $W_{\mathrm{M}} = -\boldsymbol{B}(\boldsymbol{r}) \cdot \boldsymbol{m}$, with the relationship $\boldsymbol{F}_{\mathrm{M}} = -\nabla W_{\mathrm{M}}$ gives

$$\boldsymbol{F}_{\mathrm{M}} = -\nabla W_{\mathrm{M}} = \nabla\big[\boldsymbol{m} \cdot \boldsymbol{B}(\boldsymbol{r})\big].$$

  Combining this result with $\mathrm{d}W_{\mathrm{M}} = -\boldsymbol{F}_{\mathrm{M}} \cdot \mathrm{d}\boldsymbol{r}$ gives

$$\mathrm{d}W_{\mathrm{M}} = -\nabla\big[\boldsymbol{m} \cdot \boldsymbol{B}(\boldsymbol{r})\big] \cdot \mathrm{d}\boldsymbol{r}.$$

  The gradient evaluates to

$$\nabla\big[\boldsymbol{m} \cdot \boldsymbol{B}(\boldsymbol{r})\big] = \boldsymbol{m} \times (\nabla \times \boldsymbol{B}) + (\boldsymbol{m} \cdot \nabla)\boldsymbol{B},$$

  which results in

$$\mathrm{d}W_{\mathrm{B}} = -\big[\boldsymbol{m} \times (\nabla \times \boldsymbol{B}) + (\boldsymbol{m} \cdot \nabla)\boldsymbol{B}\big] \cdot \mathrm{d}\boldsymbol{r}.$$

- We know from Ampere's law that $\nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{j}$. But careful here! So far, when evaluating energy and force, we have considered only the external magnetic field, and not the contribution of the current distribution $\boldsymbol{j}$ to the total magnetic field. However, the magnetic field in the expression $\boldsymbol{m} \cdot \boldsymbol{B}$ and thus $\nabla \times \boldsymbol{B}$ refers only to the external field.

  To make this distinction clear, we write Ampere's law for our problem as

$$\nabla \times \boldsymbol{B} = \mu_0 \tilde{\boldsymbol{j}},$$

  where $\tilde{\boldsymbol{j}}$ is the current distribution generating the external field $\boldsymbol{B}$—note that the $\tilde{\boldsymbol{j}}$ generating the external field is unrelated to the current $\boldsymbol{j}$ for which we are calculating magnetic force.

- We now make an approximation—we assume the current distribution $\tilde{\boldsymbol{j}}$ generating the external field is far from the current distribution $\boldsymbol{j}$ for which we are calculating magnetic force. This assumption implies $|\tilde{\boldsymbol{j}}| \ll |\boldsymbol{j}|$ near the localized region of space containing $\boldsymbol{j}$. Since $\tilde{\boldsymbol{j}}$ is negligible compared to $\boldsymbol{j}$, we can approximate

$$\nabla \times \boldsymbol{B} = \mu_0 \tilde{\boldsymbol{j}} \approx 0.$$

- Using $\nabla \times \boldsymbol{B} \approx 0$ leads to

$$\nabla\big[\boldsymbol{m} \cdot \boldsymbol{B}(\boldsymbol{r})\big] = \boldsymbol{m} \times (\nabla \times \boldsymbol{B}) + (\boldsymbol{m} \cdot \nabla)\boldsymbol{B} \approx (\boldsymbol{m} \cdot \nabla)\boldsymbol{B},$$

  and thus

$$\mathrm{d}W_{\mathrm{M}} = -\nabla\big[\boldsymbol{m} \cdot \boldsymbol{B}(\boldsymbol{r})\big] \cdot \mathrm{d}\boldsymbol{r} \approx -\big[(\boldsymbol{m} \cdot \nabla)\boldsymbol{B}\big] \cdot \mathrm{d}\boldsymbol{r}.$$

  Comparing this to the general force-energy relation $\mathrm{d}W_{\mathrm{B}} = -\boldsymbol{F}_{\mathrm{M}}\,\mathrm{d}\boldsymbol{r}$ produces

$$\boldsymbol{F}_{\mathrm{M}} = (\boldsymbol{m} \cdot \nabla)\boldsymbol{B}(\boldsymbol{r}).$$

  In other words, the magnetic force on a magnetic dipole in an external magnetic field is the directional derivative of $\boldsymbol{B}$ in the direction of $\boldsymbol{m}$.

2.9.6 Torque on a Magnetic Dipole in an External Magnetic Field

- Finally, we will find an expression for the torque on a magnetic dipole in an external magnetic field.

  We begin with the general torque-energy relation

$$\mathrm{d}W_{\mathrm{M}} = -\boldsymbol{M}_{\mathrm{B}} \cdot \mathrm{d}\boldsymbol{\phi}$$

- A small rotation of a magnetic dipole in an external magnetic field reads

$$\mathrm{d}\boldsymbol{m} = \mathrm{d}\boldsymbol{\phi} \times \boldsymbol{m},$$

  and the corresponding change in the dipole's energy because of this rotation, using $W_{\mathrm{M}} = -\boldsymbol{m} \cdot \boldsymbol{B}$, is

$$\mathrm{d}W_{\mathrm{M}} = -\,\mathrm{d}\boldsymbol{m} \cdot \mathrm{d}\boldsymbol{B} = -(\mathrm{d}\boldsymbol{\phi} \times \boldsymbol{m}) \cdot \boldsymbol{B}.$$

  This is a scalar triple product, which we can rearrange to get

$$\mathrm{d}W_{\mathrm{M}} = -\,\mathrm{d}\boldsymbol{\phi}(\boldsymbol{m} \times \boldsymbol{B}).$$

  Comparing this expression to the general relation $\mathrm{d}W_{\mathrm{M}} = -\boldsymbol{M}_{\mathrm{M}} \cdot \mathrm{d}\boldsymbol{\phi}$ produces the desired expression for magnetic torque:

$$\boldsymbol{M}_{\mathrm{M}} = \boldsymbol{m} \times \boldsymbol{B}.$$

  This is the torque on a magnetic dipole in an external magnetic field. We are familiar with this result from introductory electromagnetism, and now we have derived it from fundamental principles.

3 Quasi-Static Electromagnetic Fields

By convention, the quasi-static approximation to electromagnetic fields adds the consideration of magnetic induction to magnetostatics.

3.1 Electromagnetic Induction

- We begin by recalling Lenz's law from introductory electromagnetism:

  > The change in magnetic flux through a current loop induces an electric current that opposes the magnetic flux inducing the current.

  Note that the law in this form is qualitative. We will instead use Maxwell's formulation of Lenz's law, which states that the electric circulation Γ_{E} in a current loop and the magnetic flux Φ_{M} through the loop are observed to obey the relationship

  $$\Gamma_{\mathrm{E}} = -\frac{\mathrm{d}}{\mathrm{d}t}\Phi_{\mathrm{M}}.$$

  We stress that this result comes from experiment. We have not derived it from fundamental principles, it is simply the observed physical behavior.

- In integral form, the quantitative formulation of Lenz's law reads

  $$\oint_{\partial S} \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{r} = -\frac{\mathrm{d}}{\mathrm{d}t}\int_S \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{S}.$$

  We then apply Stokes' theorem to transform this equation to the differential form

  $$\int \nabla \times \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{S} = -\int \frac{\partial \boldsymbol{B}}{\partial t} \cdot \mathrm{d}\boldsymbol{S},$$

  where we have assumed the shape of the current loop is constant through time. The above equation implies

  $$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}.$$

  This is important result is one of the Maxwell equations for electromagnetism. Note that $\boldsymbol{E}$ and $\boldsymbol{B}$ are coupled, and that this equation does not include constants—only $\boldsymbol{E}$ and $\boldsymbol{B}$ and their derivatives.

3.1.1 Maxwell Magnetic Field Impulse

- We now combine the just-derived Maxwell equation with $\boldsymbol{B} = \nabla \times \boldsymbol{A}$ to get

  $$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t} = -\nabla \times \frac{\partial \boldsymbol{A}}{\partial t},$$

  which leads to to interesting relationship

  $$\boldsymbol{E} = -\frac{\partial \boldsymbol{A}}{\partial t}.$$

  This relationship holds as long as magnetic field lines are closed, which is what allowed us to write $\boldsymbol{B} = \nabla \times \boldsymbol{A}$. However, this expression for $\boldsymbol{E}$ is not complete—$\boldsymbol{E}$ is undetermined up to the gradient of a scalar field. We will correct this shortly.

- We can combine the above expression for $\boldsymbol{E}$ with Newton's law and the electrostatic force to get

$$\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} = \boldsymbol{F} = q\boldsymbol{E} = -\frac{\partial(q\boldsymbol{A})}{\partial t}.$$

  In other words, we've derived the relationship $\boldsymbol{p} = q\boldsymbol{A}$—note that $q\boldsymbol{A}$ behaves as a momentum. This quantity is often called canonical momentum, while the familiar expression $\boldsymbol{p} = m\boldsymbol{v}$ is kinetic momentum.

3.2 The Quasistatic Maxwell Equations

- In the form we have used them so far (in the quasi-static scope of electrostatics and magnetostatics), the four Maxwell equations are

$$\begin{aligned} \nabla \cdot \boldsymbol{E} &= \frac{\rho}{\epsilon_0} \qquad & \nabla \times \boldsymbol{E} &= -\frac{\partial \boldsymbol{B}}{\partial t} \\ \nabla \cdot \boldsymbol{B} &= 0 \qquad & \nabla \times \boldsymbol{B} &= \mu_0 \boldsymbol{j}. \end{aligned}$$

  We note that these equations hold only in the quasistatic regime in which all current loops are closed, i.e. $\nabla \cdot \boldsymbol{j} = 0$.

  The above Maxwell equations completely describe quasistatic electromagnetism. They do not, however, describe all of electromagnetism—the equations $\nabla \cdot \boldsymbol{j} = 0$ and $\nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{j}$ must be generalized.

3.2.1 The Electromagnetic Potentials for Quasistatic Fields

- We now return to the incomplete relationship between $\boldsymbol{E}$ and $\boldsymbol{A}$. More generally, our goal in this section is to write $\boldsymbol{E}$ and $\boldsymbol{B}$ in terms of the fundamental electromagnetic potentials ϕ and $\boldsymbol{A}$.

- Recall that the definition of magnetic potential via $\boldsymbol{B} = \nabla \times \boldsymbol{A}$ was possible because of the relationship $\nabla \cdot \boldsymbol{B} = 0$, which is always satisfied if $\boldsymbol{B}$ is written as the curl of a vector field—the divergence of a curl of a vector field is always zero. The expression $\boldsymbol{B} = \nabla \times \boldsymbol{A}$ thus holds in dynamic as well as quasistatic situations, and we don't need to generalize if further.

- We now ask how to write $\boldsymbol{E}$ in terms of ϕ and $\boldsymbol{A}$.

  In electrostatics, we began with $\nabla \times \boldsymbol{E} = 0$. Using the law of induction, we then generalized this relationship in the quasistatic regime to

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t} = -\nabla \times \frac{\partial \boldsymbol{A}}{\partial t}.$$

  We then rewrite the above equation for $\nabla \times \boldsymbol{E}$ in the form

$$\nabla \times \left(\boldsymbol{E} + \frac{\partial \boldsymbol{A}}{\partial t} \right) = 0.$$

  Recall that the curl of the gradient of a scalar field is always zero. This means that the above expression in parentheses is determined only up to the gradient of a scalar field. This implies

$$\boldsymbol{E} + \frac{\partial \boldsymbol{A}}{\partial t} = -\nabla\phi \qquad \text{or} \qquad \boldsymbol{E} = -\nabla\phi - \frac{\partial \boldsymbol{A}}{\partial t}$$

In the quasistatic regime, the complete expression for electric field is thus

$$\boldsymbol{E} = -\nabla\phi - \frac{\partial \boldsymbol{A}}{\partial t}.$$

We note that the Maxwell equations $\nabla \cdot \boldsymbol{B} = 0$ and $\nabla \times \boldsymbol{E} = -\frac{\partial B}{\partial t}$ allow us to write $\boldsymbol{E}$ and $\boldsymbol{B}$ in terms of the fundamental electromagnetic potentials $\boldsymbol{A}$ and ϕ in the quasistatic regime.

3.3 Conductors and Ohm's Law

- At a basic level, conductors are materials permitting the motion of charged particles through the material. The charged particles are typically electrons, holes, or ions.
- Conductors obey Ohm's law, which we will write in the general form

  $$\boldsymbol{j} = \sigma_{\mathrm{E}}\boldsymbol{E},$$

  where $\boldsymbol{j}$ is the current density in the material, $\boldsymbol{E}$ is the electric field in the material, and σ_{E} is electric conductivity.
- We take a material containing mobile charge carriers as our model of a conductor.

  Important: Cnductors are electrically neutral! The charges in the material my move around, but the conductor as a whole is neutral.

  When we turn on an external electric field—positive charges move in the direction of the external field and negative charges opposite the external field.

  The effect is that the free charges in the conductor rearrange in a configuration that cancels out the external field. The total electric field in the conductor at equilibrium, after the charge redistribution, is thus zero–if it weren't zero, the charges would keep moving! At equilibrium, we have $\boldsymbol{j} = 0$ and $\boldsymbol{E} = 0$ within the conductor.
- Continuing with the example of an conductor in an external electric field at equilibrium, we recall that $\boldsymbol{E}$ and ρ are related by Gauss's law via

  $$\nabla \cdot \boldsymbol{E} = \frac{\rho}{\epsilon_0}.$$

  Because $\boldsymbol{E} = 0$ at equilibrium, it follows from Gauss's law that ρ is also zero at equilibrium. In other words, there is zero volume charge density in a conductor at equilibrium.
- If the charges are not within the conductor, they must then be at the conductor's surface.

  We let $\hat{\boldsymbol{n}}$ and σ denote the normal to the surface and the surface charge density, respectively. Near the conductor's surface we must have

  $$\boldsymbol{E} \cdot \hat{\boldsymbol{n}} = \frac{\sigma}{\epsilon_0},$$

  which means $\boldsymbol{E}$ is perpendicular to the conductor's surface (to be discussed further).

- *Keep in mind*: Assume we increase the external electric field. As we increase the external field, the conductor must naturally redistribute more and more charge to its surface to cancel out the external field.

  In principle, at some point, there might not be enough charge left in the material to cancel out the external field. In this course, however, we will assume conductors always have enough available charge pairs (e.g. electrons and holes) to cancel out an external magnetic field.

- Review: a conductor at equilbrium obeys $\boldsymbol{E} = \boldsymbol{j} = 0$, which implies $\rho = 0$, implying the conductor's charge is concentrated at the surface. This induced surface charge cancels out the external field such that the total electric field inside the conductor is zero.

- For the conductor to be in equilibrium, the electric field must always be normal to the surface (i.e. $\boldsymbol{E} \times \hat{\boldsymbol{n}} = 0$). If the electric field at the surface had a tangential component, charges would move along the surface in the tangent direction, and the conductor wouldn't be in equilibrium.

  Since $\boldsymbol{E}$ is perpendicular to the conductor's surface at equilibrium, the surface is an equipotential surface and obeys

  $$\phi_2 - \phi_1 = \int \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{l} = \int \boldsymbol{E} \cdot \hat{\boldsymbol{t}} \,\mathrm{d}l = \int 0 \,\mathrm{d}l = 0 \implies \phi \text{ constant.}$$

  where $\hat{\boldsymbol{t}}$ is the tangent to the surface—$\boldsymbol{E} \cdot \hat{\boldsymbol{t}} = 0$ since $\boldsymbol{E}$ is normal to the surface.

3.3.1 The Relaxation Time Constant of a Conductor

- We now ask how quickly a conductor can react to an external electric field, i.e. how quickly charges redistribute in response to the external field to create equilibrium.

- We begin with the continuity equation, which reads

  $$\nabla \cdot \boldsymbol{j} + \frac{\partial \rho}{\partial t} = 0.$$

  Note that this is a generalization of the earlier expression $\nabla \cdot j = 0$ to electrodynamic situations involving time-varying charge distributions.

- We then substitute Ohm's law $\boldsymbol{j} = \sigma_{\mathrm{E}} \boldsymbol{E}$ into the continuity equation to get

  $$\nabla \cdot (\sigma_{\mathrm{E}} \boldsymbol{E}) + \frac{\partial \rho}{\partial t} = 0.$$

  Next, we write $\boldsymbol{E}$ in terms of ρ using Gauss's law $\nabla \cdot \boldsymbol{E} = \frac{\rho}{\epsilon_0}$, which produces

  $$\frac{\partial \rho}{\partial t} + \frac{\sigma_{\mathrm{E}}}{\epsilon_0} \rho = 0.$$

  The general solution for ρ is exponential:

  $$\rho(\boldsymbol{r}, t) = \rho(\boldsymbol{r}, 0) e^{-\frac{t}{\tau}}, \quad \text{where } \tau = \frac{\epsilon_0}{\sigma_{\mathrm{E}}}.$$

  The larger the conductivity σ_{E}, the smaller the characteristic response time τ, and the sooner the conductor reaches equilibrium when placed in an external electric field.

- As an example, for iron, we have $\tau \approx 8.85 \times 10^{-19}\,\mathrm{s}$. In other words, the time constant is incredibly small—on a macroscopic scale, the conductor reaches equilibrium essentially instantly.

3.3.2 The Drude Model of Conductivity

- We will now derive Ohm's law from fundamental principles, using a simple model of microscopic charge carriers called the Drude model.

  We begin with Newton's law for a charge carrier moving through a conductor in an electric field. We consider two forces on the charge carrier:

  1. a dissipative velocity-dependent force $-m\gamma \boldsymbol{v}(t)$, where γ is a damping constant,
  2. the accelerating electric force $q\boldsymbol{E}(t)$.

  In terms of these two forces, Newton's law for the particle in the conductor reads

$$m\frac{\mathrm{d}v}{\mathrm{d}t} = -m\gamma \boldsymbol{v}(t) + q\boldsymbol{E}(t).$$

- When $\boldsymbol{E} = 0$, the solution is $\boldsymbol{v}(t) = \boldsymbol{v}_0 e^{-\gamma t}$. In this case, the particle's velocity exponentially decays with time, which means particles in the conductor rapidly stop moving. This makes sense—electric current is not observed to flow in the absence of an external electric field.

  Energy interpretation: the particle's kinetic energy dissipates into thermal energy, which increases the conductor's internal energy.

- When $\boldsymbol{E} \neq 0$, we use the ansatz (which is quoted, not derived or explained further)

$$\boldsymbol{v}(t) = \frac{q}{m}\int_{-\infty}^{t} e^{-\gamma(t-\tilde{t})}\boldsymbol{E}(\tilde{t})\,\mathrm{d}\tilde{t},$$

  where $\tilde{t}$ is a placeholder integration variable.

- Recall that current density obeys

$$\boldsymbol{j} = \rho\boldsymbol{v} = nq\boldsymbol{v},$$

  where q is the charge of a single charge carrier, n is the number density of charge carriers in the conductor, and $\boldsymbol{v}$ is the charge carriers' drift velocity. Substituting the ansatz for charge carrier velocity $\boldsymbol{v}$ into the equation for $\boldsymbol{j}$ gives

$$\boldsymbol{j} = \frac{nq^2}{m}\int_{-\infty}^{t} e^{-\gamma(t-\tilde{t})}\boldsymbol{E}(\tilde{t})\,\mathrm{d}\tilde{t}.$$

- For a constant field, $\boldsymbol{E}(t) = \boldsymbol{E}_0$, we can easily solve the integral to get Ohm's law

$$\boldsymbol{j} = \frac{nq^2}{m\gamma}\boldsymbol{E}_0.$$

  The Drude model thus gives a prediction for conductivity

$$\sigma_{\mathrm{E}} = \frac{nq^2}{m\gamma}$$

  Note that the Drude model gives an expression for conductivity in terms of fundamental quantities, rather than taking conductivity itself to be fundamental.

- Some numerical values of conductivity (measured in siemens $\mathrm{S} = \Omega^{-1}$) are given in the table below

Material	Conductivity
aluminum	$3.7 \times 10^{7}\,\mathrm{S\,m^{-1}}$
iron	$9.9 \times 10^{6}\,\mathrm{S\,m^{-1}}$
$YBa_2Ca_3O_7$, $T > 92\,\mathrm{K}$	$1 \times 10^{6}\,\mathrm{S\,m^{-1}}$
$YBa_2Ca_3O_7$, $T < 92\,\mathrm{K}$	$\infty\,\mathrm{S\,m^{-1}}$
glass, $T = 300\,\mathrm{K}$	$1 \times 10^{-15}\,\mathrm{S\,m^{-1}}$
glass, $T = 1000\,\mathrm{K}$	$1 \times 10^{-7}\,\mathrm{S\,m^{-1}}$

The material $YBa_2Ca_3O_7$ is a superconductor at low temperatures, which examplains $\sigma_{\mathrm{E}} = \infty$. Note also that σ_{E} is in general strongly temperature dependent, as seen in the case of glass and, more extermely, the superconductor $YBa_2Ca_3O_7$.

3.3.3 Energy Dissipation

- In general a charge distribution experiences two forces—the electric and magnetic forces. The magnetic force $\boldsymbol{F}_{\mathrm{M}} = q\boldsymbol{v} \times \boldsymbol{B}$ is perpendicular to velocity and cannot do work, and we thus expect that the magnetic force cannot contribute to dissipative forces.

- More formally, the dissipative power associated with a charge carrier in terms of electromagnetic force density $\boldsymbol{f}$ is

$$P = \int \boldsymbol{v} \cdot \boldsymbol{f}\,\mathrm{d}^3\boldsymbol{r} = \int \frac{\boldsymbol{j}}{\rho}\left(\rho\boldsymbol{E} + \boldsymbol{j} \times \boldsymbol{B}\right)\mathrm{d}^3\boldsymbol{r},$$

  where we have substituted in $\boldsymbol{v} = \boldsymbol{j}/\rho$. The second integrand contains the term $\boldsymbol{j} \cdot (\boldsymbol{j} \times \boldsymbol{B}) = 0$, which is why the magnetic force cannot dissipate power. The dissipated power thus reduces to

$$P = \int_V \boldsymbol{j} \cdot \boldsymbol{E}\,\mathrm{d}^3\boldsymbol{r}.$$

  This expression represents the power dissipated because of charge carrier motion associated with the current density $\boldsymbol{j}$ in the electric field $\boldsymbol{E}$. Note that this analysis assumes the charge carriers obey $\boldsymbol{v} = \boldsymbol{j}/\rho$.

  Finally, we note that in a classic scenario $\boldsymbol{j} \propto \boldsymbol{E}$ and thus $P \propto \boldsymbol{j} \cdot \boldsymbol{E} \propto E^2$.

3.3.4 Capacitance

- Our goal in this section is to develop a general expression for a conductor's capacitance. Capacitance describes how much charge can accumulate on a capacitor at a given potential difference. A conductor's capacitance depends on its geometry (and also permittivity, but we leave that for later).

- We consider N conductors indexed by $i = 1, \ldots, N$. For example, for a parallel-plate capacitor with two plates, we would have $N = 2$.

  In a more general situation, however, we can have many "conductors" (e.g. in batteries) contributing to a cumulative capacitance.

- The surface of any conductor is an equipotential surface, which we write for our system of $i = 1, \ldots, N$ conductors in the form

$$\phi(\boldsymbol{r})\big|_{\partial V_i)} = k_i,$$

where the k_i are constants encoding the constant potential on the i-th conductor's surface.

- Next, we consider the total electric energy associated with the collection of conductors, which is

$$W_{\mathrm{E}} = \frac{1}{2} \iiint_V \rho(\boldsymbol{r})\phi(\boldsymbol{r}) \, \mathrm{d}^3\boldsymbol{r},$$

where $\rho(\boldsymbol{r})$ is the volume charge density in the conductor collection. However, since all charges in the conductors occur at the surface, we can change the expression for W_{E} to a surface integral of the form

$$W_{\mathrm{E}} = \frac{\phi}{2} \iint_S \sigma(\boldsymbol{r}) \, \mathrm{d}\boldsymbol{S},$$

where ϕ is moved out of the integral because it is constant along the surface. Writen in terms of the individual contributions of each conductor, the energy becomes

$$W_{\mathrm{E}} = \frac{1}{2} \sum_i \phi_i \iint_S \sigma_i(\boldsymbol{r}) \, \mathrm{d}\boldsymbol{S}_i = \frac{1}{2} \sum_i \phi_i q_i$$

where q_i is the charge on the ith conductor's surface. The total electric field energy in the conductor collection is thus

$$W_{\mathrm{E}} = \frac{1}{2} \sum_i \phi_i q_i.$$

- We will now find the same total electric field energy with a different approach. We will then equate the two expressions for W_{E} to get a generalized expression for capacitance. As before, we begin with the definition

$$W_{\mathrm{E}} = \frac{1}{2} \iiint_V \rho(\boldsymbol{r})\phi(\boldsymbol{r}) \, \mathrm{d}^3\boldsymbol{r}.$$

Our goal is to write the integrand purely in terms of charge q. We begin by writing ϕ in terms of ρ using the Poisson equation, which gives

$$\phi(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0} \iiint_V \frac{\rho(\tilde{\boldsymbol{r}})}{|\boldsymbol{r} - \tilde{\boldsymbol{r}}|} \, \mathrm{d}^3\tilde{\boldsymbol{r}}.$$

Using this expression for ϕ, the electric field energy is thus

$$W_{\mathrm{E}} = \frac{1}{8\pi\epsilon_0} \iiint_V \left(\iiint_V \frac{\rho(\boldsymbol{r})\rho(\tilde{\boldsymbol{r}})}{|\boldsymbol{r} - \tilde{\boldsymbol{r}}|} \, \mathrm{d}^3\tilde{\boldsymbol{r}} \right) \mathrm{d}^3\boldsymbol{r}.$$

Because the conductor charge is on the surface only, we re-write the integral in terms surface charge density and surface elements, which gives

$$W_{\mathrm{E}} = \frac{1}{8\pi\epsilon_0} \sum_{i,k} \iint_S \iint_S \frac{\sigma_i \sigma_k \, \mathrm{d}S_i \, \mathrm{d}S_k}{|\boldsymbol{r}_i - \boldsymbol{r}_k|},$$

where $\mathrm{d}S_i$ and $\mathrm{d}S_k$ are the surface elements of the ith and kth conductors, and $\boldsymbol{r}_i$ and $\boldsymbol{r}_k$ are the position vectors on the ith and kth surfaces.

Finally, we multiply above and below by $q_i q_k$ to get

$$W_{\mathrm{E}} = \frac{1}{2} \sum_{i,k} \frac{1}{4\pi\epsilon_0 q_i q_k} q_i q_k \iint_S \iint_S \frac{\sigma_i \sigma_k \, \mathrm{d}S_i \, \mathrm{d}S_k}{|\boldsymbol{r}_i - \boldsymbol{r}_k|}.$$

- Next, wuate the above expression for W_E to the earlier W_E, which results in the general expression for capacitance
$$\frac{1}{C_{ik}} = \frac{1}{4\pi\epsilon_0 q_i q_k}\iint_S \iint_S \frac{\sigma_i \sigma_k \,\mathrm{d}S_i \,\mathrm{d}S_k}{|\boldsymbol{r}_i - \boldsymbol{r}_k|}$$
In terms of capacitance, the system's total electric field energy is
$$W_\mathrm{E} = \frac{1}{2}\sum_i \phi_i q_i = \frac{1}{2}\sum_{i,k}(C_{ik})^{-1} q_i q_k.$$

- Note that the expression for capacitance C_{ik} is consistent with capacitance depending only on geometry (and not charge): the $q_i q_k$ in the denominator cancel with the $\sigma_i \sigma_k$ after the charge densities are integrated over the surfaces $\mathrm{d}S_i$ and $\mathrm{d}S_k$. This exact cancellation works out since the expression $|\boldsymbol{r}_i - \boldsymbol{r}_k|$ "normalizes" the geometric properties of the potentially varying surface charge densities.

- Summary: Capacitance relates the charge on a given conductor to the potential difference across the conductor. Capacitance depends only on geometry. For a generalized collection of N conductors, capacitance, charge, and electric potential are related by
$$\phi_i = \sum_k (C^{-1})_{ik} q_k \qquad \text{where} \qquad q_k = C_{ki}\phi_i.$$
Note that these expressions generalize capacitance C to a $N \times N$ rank-two tensor. Finally, note that the charge on a given conductor depends on the capacitance and potential difference of all of the other conductors in the collection—this is a consequence of the principle of superposition.

3.3.5 Inductance

- Inductance relates magnetic flux and electric current. Like capacitance, inductance depends only on an inductor's geometry. In this section, we will derive general expression for the inductance of a generalized system of inductors, i.e. N current-carrying loops carrying currents $I_1, I_2, \ldots, I_N$.

- We first find the system's total magnetic field energy. We begin with the definition
$$W_\mathrm{M} = \frac{1}{2}\iiint_V \boldsymbol{j}\cdot\boldsymbol{A}\,\mathrm{d}^3\boldsymbol{r},$$
where we integrate over the volume with non-zero current density $\boldsymbol{j}$. We then rewrite the current in the form $\boldsymbol{j}\,\mathrm{d}^3\boldsymbol{r} = I\,\mathrm{d}\boldsymbol{l}$ and consider all N loops together to get
$$W_\mathrm{M} = \frac{1}{2}\sum_i I_i \oint_{C_i} \boldsymbol{A}\cdot\mathrm{d}\boldsymbol{l}_i$$
where the current I_i in the i-th conductor is constant along the conductor's curve C_i, and thus moves out of the integral.

- Finally, we write the integrand in terms of magnetic flux, for which we use Stokes's theorem:
$$\begin{aligned} W_\mathrm{M} &= \frac{1}{2}\sum_i I_i \oint_{C_i}\boldsymbol{A}\cdot\mathrm{d}\boldsymbol{l}_i = \frac{1}{2}\sum_i I_i \iint_{S_i}\nabla\times\boldsymbol{A}\cdot\mathrm{d}\boldsymbol{S}_i \\ &= \frac{1}{2}\sum_i I_i \iint_{S_i}\boldsymbol{B}\cdot\mathrm{d}\boldsymbol{S}_i = \frac{1}{2}\sum_i I_i \Phi_{\mathrm{M}_i}. \end{aligned}$$

- We will now find the same magnetic field energy with a second approach. We will then equate the two results for W_{M} to get a generalized expression for inductance. As before, we begin with the general definition

$$W_{\mathrm{M}} = \frac{1}{2} \iiint_V \boldsymbol{j} \cdot \boldsymbol{A} \, \mathrm{d}^3 \boldsymbol{r}.$$

  We then express $\boldsymbol{A}$ in terms of $\boldsymbol{j}$ via

$$\boldsymbol{A}(\boldsymbol{r}) = \frac{\mu_0}{4\pi} \iiint_V \frac{\boldsymbol{j}(\tilde{\boldsymbol{r}})}{|\boldsymbol{r} - \tilde{\boldsymbol{r}}|} \, \mathrm{d}^3 \tilde{\boldsymbol{r}},$$

  and substitute this expression for $\boldsymbol{A}(\boldsymbol{r})$ into W_{M} to get

$$W_{\mathrm{M}} = \frac{\mu_0}{8\pi} \iiint_V \left(\iiint_V \frac{\boldsymbol{j}(\boldsymbol{r}) \boldsymbol{j}(\tilde{\boldsymbol{r}})}{|\boldsymbol{r} - \tilde{\boldsymbol{r}}|} \, \mathrm{d}^3 \tilde{\boldsymbol{r}} \right) \mathrm{d}^3 \boldsymbol{r}.$$

  Finally, we convert $\boldsymbol{j} \, \mathrm{d}^3 \boldsymbol{r}$ to current, $I \, \mathrm{d}\boldsymbol{l}$, and sum over all loop pairs to get

$$W_{\mathrm{M}} = \frac{\mu_0}{8\pi} \sum_{i,k} I_i I_k \oint_{C_i} \oint_{C_k} \frac{\mathrm{d}\boldsymbol{l}_i \, \mathrm{d}\boldsymbol{l}_k}{|\boldsymbol{r}(l_1) - \boldsymbol{r}(l_k)|},$$

  where C_i and C_k are the space curves associated with the ith and kth conducting loops.

- We then equate the two expressions for W_{M} to get

$$\frac{1}{2} \sum_i I_i \Phi_{\mathrm{M}_i} = \frac{\mu_0}{8\pi} \sum_{i,k} I_i I_k \oint_{C_i} \oint_{C_k} \frac{\mathrm{d}\boldsymbol{l}_i \, \mathrm{d}\boldsymbol{l}_k}{|\boldsymbol{r}(l_1) - \boldsymbol{r}(l_k)|}.$$

  This relationship motivates the definition of inductance L_{ik} as

$$L_{ik} = \frac{\mu_0}{4\pi} \oint_{C_i} \oint_{C_k} \frac{\mathrm{d}\boldsymbol{l}_i \, \mathrm{d}\boldsymbol{l}_k}{|\boldsymbol{r}(l_1) - \boldsymbol{r}(l_k)|},$$

  in terms of which the relationship between the two forms of W_{M} simplifies to

$$\sum_i I_i \Phi_{\mathrm{M}_i} = \sum_{i,k} I_i I_k L_{ik}.$$

- A few final remarks on inductance:
  - Note that L_{ik} depends only on geometric quantities and is thus consistent with inductance depending only on the loops' geometry.
  - The rank-two tensor L_{ik}'s diagonal terms L_{ii} are called self-inductance, while the off-diagonal terms correspond to mutual inductance between different current loops.
  - In general, the magnetic flux through the ith loop is

$$\Phi_{\mathrm{M}_i} = \sum_i L_{ik} I_k.$$

3.3.6 The Skin Effect

- Qualitatively, the skin effect is summarized as follows:

  > At high frequencies, alternative current tends to run mostly along a conductor's surface. More formally, the current density is largest at a conductor's surface for high frequency alternating currents.

  Of course, we will want to perform a more quantitative analysis of the skin effect, for which we begin with the Maxwell equations

$$\nabla \cdot \boldsymbol{E} = \frac{\rho}{\epsilon_0} = 0 \qquad \text{and} \qquad \nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t},$$

  where volume charge density is $\rho = 0$ in a conductor. For magnetic fields, we have

$$\nabla \cdot \boldsymbol{B} = 0 \qquad \text{and} \qquad \nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{j} = \mu_0 \sigma_{\mathrm{E}} \boldsymbol{E}$$

  where we have substituted in Ohm's law $\boldsymbol{j} = \sigma \boldsymbol{E}$. We then the curl of $\nabla \times \boldsymbol{E}$ to get

$$\nabla \times \big[\nabla \times \boldsymbol{E}\big] = -\frac{\partial}{\partial t}\big[\nabla \times \boldsymbol{B}\big] = -\mu_0 \sigma_{\mathrm{E}} \frac{\partial \boldsymbol{E}}{\partial t}.$$

  We then apply curl to $\nabla \times \boldsymbol{B}$ to get

$$\nabla \times (\nabla \times \boldsymbol{B}) = \mu_0 \sigma_{\mathrm{E}} \nabla \times \boldsymbol{E} = -\mu_0 \sigma_{\mathrm{E}} \frac{\partial \boldsymbol{B}}{\partial t}.$$

  Note the symmetry of the two equations for $\boldsymbol{E}$ and $\boldsymbol{B}$.

- Next, we use the general vector calculus identity

$$\nabla \times (\nabla \times \boldsymbol{F}) = \nabla(\nabla \cdot \boldsymbol{F}) - \nabla^2 \boldsymbol{F},$$

  where $\boldsymbol{F}$ is a vector field. We apply this identity to both double curl equations, together with $\nabla \cdot \boldsymbol{E} = \nabla \cdot \boldsymbol{B} = 0$, to replace the double curl with the Laplacian:

$$\nabla^2 \boldsymbol{E} = \mu_0 \sigma_{\mathrm{E}} \frac{\partial \boldsymbol{E}}{\partial t} \qquad \text{and} \qquad \nabla^2 \boldsymbol{B} = \mu_0 \sigma_{\mathrm{E}} \frac{\partial \boldsymbol{B}}{\partial t}.$$

  These are diffusion equations for $\boldsymbol{E}$ and $\boldsymbol{B}$—their perfect symmetry is a consequence of $\rho = 0$ for a conductor and $\boldsymbol{j} = \sigma_{\mathrm{E}} \boldsymbol{E}$ from Ohm's law.

- We solve the diffusion equations with separation of variables using the ansatzes

$$\boldsymbol{E}(\boldsymbol{r}, t) = \boldsymbol{E}(\boldsymbol{r}) e^{-i\omega t} \qquad \text{and} \qquad \boldsymbol{B}(\boldsymbol{r}, t) = \boldsymbol{B}(\boldsymbol{r}) e^{-i\omega t}.$$

  These ansatzes produce

$$\nabla^2 \boldsymbol{E} = k^2 \boldsymbol{E} \qquad \text{and} \qquad \nabla^2 \boldsymbol{B} = k^2 \boldsymbol{B},$$

  where $k^2 = -i\omega\mu_0\sigma_{\mathrm{E}}$. The value of k, using the complex number identity $\sqrt{i} = \frac{1-i}{\sqrt{2}}$, is

$$k = \frac{\sqrt{2}}{2}(1 - i)\sqrt{\omega \mu_0 \sigma_{\mathrm{E}}}.$$

  In one positional dimension, the position solutions are

$$\begin{Bmatrix} \boldsymbol{E}(z) \\ \boldsymbol{B}(z) \end{Bmatrix} \sim e^{-kz} = \exp\left(-\frac{\sqrt{2}}{2}\sqrt{\omega \mu_0 \sigma_{\mathrm{E}}} \cdot z\right) \exp\left(i\frac{\sqrt{2}}{2}\sqrt{\omega \mu_0 \sigma_{\mathrm{E}}} \cdot z\right).$$

Qualitatively: the $\sim e^{iz}$ term describes oscillation, while the $\sim e^{-z}$ term describes exponential decay, with characteristic decay length $z_0 = \sqrt{\frac{2}{\omega\mu_0\sigma}}$. Note that z_0 decreases with increasing ω, so the electric field (and thus $\boldsymbol{j}$ via Ohm's law) decays more rapidly with position at increasing frequencies.

- As an example, at $\omega = 50\,\mathrm{Hz}$, the decay distance of copper is $z_0 \sim 1\,\mathrm{cm}$. This distance is much larger than the diameter of most wires, so the skin effect is negligible.

  Meanwhile, for copper at $\omega = 750\,\mathrm{MHz}$ (which is a typical radio frequency for a `Cat-6` Ethernet cable) we have $z_0 \sim 1\,\mathrm{nm}$. Only the very outside of the conducting wire carries a signal. Keep in mind that a nanometer is only of order 5-10 atomic radii. At higher frequencies we thus "run out" of atoms to carry current. We solve this skin-effect induced problem by using optical cables instead of metal conductors.

3.3.7 Electromagnetic Field Geometry in the Skin Effect

- We now consider a long, straight wire, which we model as a cylinder with longitudinal axis in the z direction. Our goal is to solve the earlier diffusion equations for $\boldsymbol{E}$ and $\boldsymbol{B}$. We use a cylindrical basis and cylindrical coordinates (r, ϕ, z).

- The current runs in the z direction, so $\boldsymbol{j} = (0, 0, j_z)$, which corresponds to an electric field $\boldsymbol{E} = (0, 0, E_z)$. Because of rotational symmetry about the polar angle ϕ, the electric field E_z depends only on r:

  $$E_z(\boldsymbol{r}, t) = E_z(r)e^{-i\omega t}.$$

  The magnetic field points only tangent to the cylinder's circular cross section, i.e. in the $\hat{\boldsymbol{e}}_\phi$ direction in cylindrical coordinates. The expression for magnetic field is thus

  $$B_\phi(\boldsymbol{r}, t) = B_\phi(r)e^{-i\omega t}.$$

- Next, recall the diffusion equations for $\boldsymbol{E}$ and $\boldsymbol{B}$ were

  $$\nabla^2 \boldsymbol{E} = \mu_0\sigma_{\mathrm{E}}\frac{\partial \boldsymbol{E}}{\partial t} \qquad \text{and} \qquad \nabla^2 \boldsymbol{B} = \mu_0\sigma_{\mathrm{E}}\frac{\partial \boldsymbol{B}}{\partial t}.$$

  We evaluate the Laplacian operators in cylindrical coordinates to get

  $$\begin{aligned} \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial E_z}{\partial r}\right) &= -i\mu_0\omega\sigma_{\mathrm{E}}E_z \\ \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial B_\phi}{\partial r}\right) - \frac{B_\phi}{r^2} &= -i\mu_0\omega\sigma_{\mathrm{E}}B_\phi. \end{aligned}$$

  As before, we define $k^2 = -i\mu_0\omega\sigma_{\mathrm{E}}$.

- Next, recall the Maxwell equation $\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}$. In our cylindrical basis and coordinate system, this equation implies $\boldsymbol{E}$ and $\boldsymbol{B}$ are related according to

  $$i\omega B_\phi = (\nabla \times \boldsymbol{E})_\phi = -\frac{\partial E_z}{\partial r}.$$

  If we solve for $E_z(r)$, the solution turns out to be

  $$E_z(r) = AJ_0(kr) \qquad \text{and} \qquad B_\phi = iA\frac{k}{\omega}J_1(kr),$$

  where A is an unknown constant, $k = \frac{\sqrt{2}}{2}(1-i)\sqrt{\omega\mu_0\sigma_{\mathrm{E}}}$, and J_0 and J_1 are the Bessel functions.

3.3.8 Electric Current the Skin Effect (continued)

- Recall from the previous lecture that the electric and magnetic field in a cylindrical conductor, using the spherical coordinate (r, ϕ, z), were

$$\boldsymbol{E} = (0, 0, E_z), \qquad E_z(r) = AJ_0(kr)$$
$$\boldsymbol{B} = (0, B_\phi, 0), \qquad B_\phi = iA\frac{k}{\omega}J_1(kr),$$

  where $k = \frac{\sqrt{2}}{2}(1-i)\sqrt{\omega\mu_0\sigma_\mathrm{E}}$, A is an unknown constant, and the J_0 and J_1 are the Bessel functions.

  We will now find the electric current in the conductor. We start with current density, which we find with Ohm's law according to

$$\boldsymbol{j} = \sigma_\mathrm{E}\boldsymbol{E} = \sigma_\mathrm{E} A J_0(kr)\hat{\boldsymbol{e}}_z.$$

  The total current through the conductor is thus

$$I = \iint_S \boldsymbol{j} \cdot \mathrm{d}\boldsymbol{S} = \iint_S (\boldsymbol{j} \cdot \hat{\boldsymbol{n}})\,\mathrm{d}S = \sigma_\mathrm{E} \int_0^a E_z(r)(2\pi r\,\mathrm{d}r).$$

- Next, without derivattion, we can write the electric field component E_z in cylindrical coordinates in the form

$$E_z(r) r = \frac{1}{-i\mu_0\sigma_\mathrm{E}\omega}\frac{\partial}{\partial r}\left(r\frac{\partial E_z(r)}{\partial r}\right).$$

  We then subsitute this expression for E_z in the expression for I, which eliminiates the integral and leaves us with

$$I = \frac{2\pi a}{-i\omega\mu_0}\frac{\partial E_z}{\partial r}\bigg|_0^a = \frac{2\pi a}{\mu_0}B_\phi(a),$$

  where the last equality follows from $\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}$ and $\frac{\partial E_z}{\partial r} = -i\omega B_\phi$.

  To summarize: we can express the electric current through the wire in terms of the radial derivative of $\boldsymbol{E}$ or in terms of the ϕ component of $\boldsymbol{B}$.

4 Maxwell's Equations

4.1 Charge Conservation and the Continuity Equation

- The Maxwell equations relate the fields $\boldsymbol{E}(\boldsymbol{r},t)$ and $\boldsymbol{B}(\boldsymbol{r},t)$ to their sources, which are charges for electric field and currents for magnetic field. More generally, these sources are analyzed in terms of charge density $\rho(\boldsymbol{r},t)$ and current density $\boldsymbol{j}(\boldsymbol{r},t)$.

- Note that the fields $\boldsymbol{E}$ and $\boldsymbol{B}$ are expressed in terms of their divergence and curl. The mathematical background is the Helmholtz decomposition theorem, which states that a vector field is uniquely determined by its divergence and curl, as discussed in the section gauge transformations of the magnetic vector potential.

- Interesting interpretation: the Helmholtz theorem (i.e. mathematical theory) tells us that $\boldsymbol{E}$ and $\boldsymbol{B}$ are fully determined by the expressions

$$\begin{aligned} \nabla \cdot \boldsymbol{E} &= \text{something} \qquad & \nabla \cdot \boldsymbol{B} &= \text{something} \\ \nabla \times \boldsymbol{E} &= \text{something} \qquad & \nabla \times \boldsymbol{B} &= \text{something}. \end{aligned}$$

  It is then up to physics to determine what these "something" terms are. Determining the "something" terms and thus completing the Maxwell equations is the subject of this chapter.

4.1.1 Charge Conservation

- We begin by considering a volume V containing a charge density $\rho(\boldsymbol{r},t)$. In general, the volume can exchange charge with its surroundings, and we encode the flow of charge in and out with current density $\boldsymbol{j}$.

- The total charge q in the volume V is

$$q(t) = \iiint_V \rho(\boldsymbol{r},t)\,\mathrm{d}^3\boldsymbol{r},$$

  while the change in charge over time is

$$\frac{\mathrm{d}q}{\mathrm{d}t} = -\iint_{\partial V} \boldsymbol{j}(\boldsymbol{r},t)\hat{\boldsymbol{n}}\,\mathrm{d}S,$$

  where $\hat{\boldsymbol{n}}$ is the normal to the surface ∂—we assume $\hat{\boldsymbol{n}}$ points out of the region V. The charge's time derivative is then

$$\frac{\mathrm{d}q}{\mathrm{d}t} = \frac{\partial}{\partial t}\iiint_V \rho(\boldsymbol{r},t)\,\mathrm{d}^3\boldsymbol{r} = \iiint_V \frac{\partial}{\partial t}\rho(\boldsymbol{r},t)\,\mathrm{d}^3\boldsymbol{r} \equiv I = -\iint_{\partial V} \boldsymbol{j}(\boldsymbol{r},t)\,\mathrm{d}\boldsymbol{S}.$$

  We rewrite the surface integral with the divergence theorem, which produces

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot \boldsymbol{j}.$$

  This important result the continuity equation. It generalizes the electrostatic relationship $\nabla \cdot \boldsymbol{j} = 0$.

- *Important*: the time dependence in the continuity equations introduces the possibility that, as long as $\frac{\partial \rho}{\partial t}$ is nonzero, we have $\nabla \cdot \boldsymbol{j} \neq 0$, meaning that current loops are not closed—they could simply end, and charge could accumulate at their ends.

4.1.2 Displacement Current

- First, we recall the (incomplete) Maxwell equations from the chapter on quasi-static fields:

$$\nabla \cdot \boldsymbol{E} = \frac{\rho}{\epsilon_0} \qquad \nabla \cdot \boldsymbol{B} = 0$$
$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t} \qquad \nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{j}.$$

  The last equation, $\nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{j}$, is based on the assumption that all current loops are closed. This holds in the quasi-static regime, but is in contradiction with the continuity equation for more general dynamic situations. To explicitly show this contradiction, we take the equation's divergence, which produces

$$\mu_0 \nabla \cdot \boldsymbol{j} = \nabla \cdot (\nabla \times \boldsymbol{B}) = 0 \implies \nabla \cdot \boldsymbol{j} = 0,$$

  where we have used the fact that the divergence of a curl is always zero.

- Evidently, we must generalize the equation $\nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{j}$ if the continuity equation is to hold. We make this generalization by introducing a displacement current. This produces

$$\nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{j} + \epsilon_0 \mu_0 \frac{\partial \boldsymbol{E}}{\partial t},$$

  where $\epsilon_0 \mu_0 \frac{\partial \boldsymbol{E}}{\partial t}$ is displacement current. The displacement current is created by time-dependent electric fields.

  Now that we have added the displacement current term, taking the divergence of the last Maxwell equation gives

$$\mu_0 \nabla \cdot j + \epsilon_0 \mu_0 \frac{\partial}{\partial t}(\nabla \cdot \boldsymbol{E}) = 0 \implies \nabla \cdot \boldsymbol{j} = -\frac{\partial \rho}{\partial t},$$

  where we've used $\nabla \cdot \boldsymbol{E} = \frac{\rho}{\epsilon_0}$. This result agrees with the continuity equation, as it must.

4.1.3 Maxwell's Equations

- In one place, the complete set of Maxwell's equations is

$$\nabla \cdot \boldsymbol{E} = \frac{\rho}{\epsilon_0} \qquad \nabla \cdot \boldsymbol{B} = 0$$
$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t} \qquad \nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{j} + \epsilon_0 \mu_0 \frac{\partial \boldsymbol{E}}{\partial t}$$

  The divergence equations relate fields to their sources, while the curl equations (sometimes called the kinematic Maxwell equations) encode the relationship between $\boldsymbol{E}$ and $\boldsymbol{B}$. These equations completely describe the entirety of classical electromagnetism, and thanks to the introduction of displacement current to the $\nabla \times \boldsymbol{B}$ equation, also the continuity equation $\nabla \cdot \boldsymbol{j} + \frac{\partial \rho}{\partial t} = 0$, meaning the Maxwell equations preserve conservation of charge.

- In the following sections, we will show that Maxwell's equations, in addition to conserving charge, also conserve electromagnetic generalizations of momentum, angular momentum and electromagnetic field energy, as any physically complete theory must.

4.2 Conservation Laws and Maxwell's Equations

4.2.1 Conservation of Energy

- We begin with the continuity equation for energy. First, we cross-multiply the third and fourth Maxwell equations, which produces

$$\boldsymbol{B}\cdot(\nabla\times\boldsymbol{E}) = -\boldsymbol{B}\cdot\frac{\partial\boldsymbol{B}}{\partial t}$$
$$\boldsymbol{E}\cdot(\nabla\times\boldsymbol{B}) = \mu_0\boldsymbol{j}\cdot\boldsymbol{E} + \mu_0\epsilon_0\boldsymbol{E}\cdot\frac{\partial\boldsymbol{E}}{\partial t}.$$

We then subtract the two equations to get

$$\mu_0\epsilon_0\boldsymbol{E}\frac{\partial\boldsymbol{E}}{\partial t} + \boldsymbol{B}\frac{\partial\boldsymbol{B}}{\partial t} = \boldsymbol{E}\cdot(\nabla\times\boldsymbol{B}) - \boldsymbol{B}\cdot(\nabla\times\boldsymbol{E}) - \mu_0\boldsymbol{j}\boldsymbol{E}.$$

Next, we divide by μ_0 and rewrite the right hand side with a reverse-engineered "divergence product rule" to get

$$\epsilon_0\boldsymbol{E}\frac{\partial\boldsymbol{E}}{\partial t} + \frac{1}{\mu_0}\boldsymbol{B}\frac{\partial\boldsymbol{B}}{\partial t} = -\frac{1}{\mu_0}\nabla\cdot(\boldsymbol{E}\times\boldsymbol{B}) - \boldsymbol{j}\boldsymbol{E}.$$

We now turn our attention to the left hand side, and reverse-engineer the time derivative to get

$$\frac{\partial}{\partial t}\left(\frac{\epsilon_0}{2}\boldsymbol{E}^2 + \frac{1}{2\mu_0}\boldsymbol{B}^2\right) = -\frac{1}{\mu_0}\nabla\cdot(\boldsymbol{E}\times\boldsymbol{B}) - \boldsymbol{j}\boldsymbol{E}.$$

The left hand side is exactly the electromagnetic field energy density w from earlier sections, which reads

$$w = \frac{\epsilon_0}{2}\boldsymbol{E}^2 + \frac{1}{2\mu_0}\boldsymbol{B}^2.$$

- In terms of energy density w, the energy continuiuty equation becomes

$$\frac{\partial w}{\partial t} + \nabla\cdot\mathbf{S} - \boldsymbol{j}\cdot\boldsymbol{E},$$

where we have introduced the Poynting vector[2]

$$\mathbf{S} \equiv \frac{1}{\mu_0}\boldsymbol{E}\times\boldsymbol{B}.$$

The Poynting vector corresponds to energy current density, i.e. power per unit cross-sectional area or simply intensity. Finally, we note that the term $\boldsymbol{j}\cdot\boldsymbol{E}$ corresponds to Ohmic (dissipative) energy losses, which we showed in the energy dissipation section of the chapter on quasi-static electromagnetic fields.

---

[2] Choosing a good notation for the Poynting vector is somewhat troublesome. The symbol $\boldsymbol{P}$ is used at FMF, but this interferes with both polarization and potentially electric power. I have used the more conventional symbol $\mathbf{S}$, which can lead to confusion when integrating over vector surface elements $\mathrm{d}\boldsymbol{S}$. I have tried to mitigate this confusion by always writing the Poynting vector in upright font $\mathbf{S}$, while surface elements are always italic, as in $\boldsymbol{S}$.

- Putting the pieces together, conservation of electromagnetic energy is encoded in the equation
$$\frac{\partial w}{\partial t} + \nabla \cdot \mathbf{S} - \boldsymbol{j} \cdot \boldsymbol{E}.$$
In integral form for a region of space V, using the divergence theorem, the equation reads
$$\frac{\partial}{\partial t} \iiint_V w \, \mathrm{d}^3 \boldsymbol{r} = - \oint\!\!\!\oint_{\partial V} \mathbf{S} \cdot \hat{\boldsymbol{n}} \, \mathrm{d}\boldsymbol{S} - \iiint_V \boldsymbol{j} \cdot \boldsymbol{E} \, \mathrm{d}^3 \boldsymbol{r}.$$
Interpretation: in a given volume V, electromagnetic energy changes either because of energy flow through the surface, encoded by $\mathbf{S}$, or because of dissipative losses inside the volume, encoded by $\boldsymbol{j} \cdot \boldsymbol{E}$.

4.2.2 Conservation of Momentum and Maxwell's Equations

- Next, we aim to find a continuity equation encoding conservation of momentum. We begin by considering the time derivative
$$\frac{\partial}{\partial t} \epsilon_0 (\boldsymbol{E} \times \boldsymbol{B}).$$
Evaluating the derivative with the product rule produces
$$\frac{\partial}{\partial t} \epsilon_0 (\boldsymbol{E} \times \boldsymbol{B}) = \epsilon_0 \left[\frac{\partial \boldsymbol{E}}{\partial t} \times \boldsymbol{B} + \boldsymbol{E} \times \frac{\partial B}{\partial t} \right].$$
We then substitute in the third and fourth Maxwell equation to replace the time derivatives, which leads to
$$= \epsilon_0 \left[\frac{1}{\epsilon_0 \mu_0} (\nabla \times \boldsymbol{B}) \times \boldsymbol{B} - \frac{1}{ee} \boldsymbol{j} \times \boldsymbol{B} - \boldsymbol{E} \times (\nabla \times \boldsymbol{E}) \right].$$

- Next, some vector calculus acrobatics (only quoted and not proven) leads us to the expression
$$\frac{\partial}{\partial t} \big[\epsilon_0 (\boldsymbol{E} \times \boldsymbol{B}) \big] = \nabla \left[\epsilon_0 \boldsymbol{E} \otimes \boldsymbol{E} - \frac{1}{2} \epsilon_0 \boldsymbol{E}^2 + \frac{1}{\mu_0} \boldsymbol{B} \otimes \boldsymbol{B} - \frac{1}{2\mu_0} \boldsymbol{B}^2 \right] - \big[\rho \boldsymbol{E} - \boldsymbol{j} \times \boldsymbol{B} \big].$$
The large brackets contain the electromagnetic stress tensor. Meanwhile, the very last term is the electromagnetic (or Lorentz) force density. Note that in general (e.g. from basic mechanics) the time derivative of momentum is force. This implies the quantity $\epsilon_0 (\boldsymbol{E} \times \boldsymbol{B})$ on the left hand side of the equation is some form of momentum.

- Collected in one place, these terms are:
  - electromagnetic momentum $\boldsymbol{p}_{\mathrm{em}} = \epsilon_0 (\boldsymbol{E} \times \boldsymbol{B})$,
  - the electromagnetic stress tensor
  $$\mathrm{T}_{ik} = \epsilon_0 \big(E_i E_k - \frac{1}{2} E^2 \delta_{ik} \big) + \frac{1}{\mu_0} \left(B_i B_k + \frac{1}{2} B^2 \delta_{ik} \right),$$
  - and the Lorentz force density $\boldsymbol{f}_{\mathrm{em}} = \rho \boldsymbol{E} + \boldsymbol{j} \times \boldsymbol{B}$.

In this notation, the complicated momentum relationship between $\boldsymbol{E}$ and $\boldsymbol{B}$ becomes

$$\frac{\partial p_{\mathrm{em}_i}}{\partial t} - \frac{\partial \mathrm{T}_{ik}}{\partial x_k} + f_{\mathrm{em}_i} = 0.$$

This is the Cauchy continuity equation for electromagnetic momentum—we note that a similar Cauchy equation holds for momentum in other transport situations.

- In integral form for a volume V (using the divergence theorem), the momentum continuity equation reads

$$\frac{\partial}{\partial t}\iiint_V p_{\mathrm{em}_i}\,\mathrm{d}^3\boldsymbol{r} = \oiint_{\partial V} \mathrm{T}_{ik}\hat{n}_k\,\mathrm{d}\boldsymbol{S} - \iiint_V f_{\mathrm{em}_i}\,\mathrm{d}^3\boldsymbol{r}.$$

Interpretation: electromagnetic momentum in a region of space V can change because of of momentum flux through the surface (encoded by $\mathrm{T_{ik}}$) and because of the Lorentz force acting on the entire volume (encoded by $\boldsymbol{f}_{\mathrm{em}}$).

4.2.3 Angular Momentum and Maxwell's Equations

- Next, we aim to find a continuity equation encoding conservation angular momentum.

We begin with the just-derived continuity equation for electromagnetic momentum

$$\frac{\partial p_{\mathrm{em}_i}}{\partial t} - \frac{\partial \mathrm{T}_{ik}}{\partial x_k} + f_{\mathrm{em}_i} = 0.$$

From basic mechanics, momentum $\boldsymbol{p}$ and angular momentum $\boldsymbol{L}$ are related by $\boldsymbol{L} = \boldsymbol{r}\times\boldsymbol{p}$. With this relationship in mind, we will try to introduce a cross product into the momentum continuity equation, which will produce an equation for angular momentum.

- We start by multiplying the momentum continuity equation by x_j to get

$$\frac{\partial x_j p_{\mathrm{em}_i}}{\partial t} = x_j\frac{\partial \mathrm{T}_{ik}}{\partial x_k} - x_j f_{\mathrm{em}_i} = \frac{\partial}{\partial x_k}(x_j \mathrm{T}_{ik}) - \mathrm{T}_{ij} - x_j f_{\mathrm{em}_i}.$$

To create a cross product, we then multiply by the Levi-Civita tensor ϵ_{lji} to get

$$\frac{\partial}{\partial t}(\epsilon_{lji}x_j p_{\mathrm{em}_i}) = \frac{\partial}{\partial x_k}(\epsilon_{lji}x_j\mathrm{T}_{ik}) - \epsilon_{lji}\mathrm{T}_{ij} - \epsilon_{lji}x_j f_{\mathrm{em}_i}.$$

Note that T_{ij} is symmetric and ϵ_{lji} is antisymmetric, so the term $\epsilon_{lji}\mathrm{T}_{ij}$ always sums to zero because $\mathrm{T}_{ij} = \mathrm{T}_{ji}$. The quantity $\epsilon_{lji}x_j p_{\mathrm{em}_i}$ is now a proper cross product, in component form, of the quantities $\boldsymbol{r}$ and $\boldsymbol{p}_{\mathrm{em}}$, and thus corresponds to electromagnetic angular momentum $\boldsymbol{L}_{\mathrm{em}}$.

- In component form, the terms in the above equation are:
  - electromagnetic angular momentum $L_{\mathrm{em}_l} = \epsilon_{lji}x_j p_{\mathrm{em}_i}$,
  - and electromagnetic torque density $m_{\mathrm{em}_l} = \epsilon_{lji}x_j f_i$.

In terms of L_{em_l} and m_{em_l}, the continuity equation for electromagnetic angular momentum is

$$\frac{\partial L_{\mathrm{em}_l}}{\partial t} - \frac{\partial(\epsilon_{lji}x_j\mathrm{T}_{ik})}{\partial x_k} + m_{\mathrm{em}_l} = 0.$$

- In integral form for a region of space V, the continuity equation reads

$$\frac{\partial}{\partial t} \iiint_V L_{\mathrm{em}_l} \, \mathrm{d}^3 \boldsymbol{r} = \oiint_{\partial V} \epsilon_{lji} x_j \mathrm{T}_{ik} \hat{n}_k \, \mathrm{d}\boldsymbol{S} - \iiint_V m_{\mathrm{em}_l} \, \mathrm{d}^3 \boldsymbol{r}.$$

  Interpretation: the electromagnetic angular momentum in a region of space can change because of surface flux, encoded by by T_{ik}, or because of electromagnetic torque acting on the entire volume, encoded by the electromagnetic torque $\boldsymbol{m}_{\mathrm{em}}$, which results from the Lorentz force.

5 Electromagnetic Field in Matter

In this chapter, we analyze how a material responds to the presence of an external electromagnetic field, and discuss how Maxwell's equations change and why they change.

5.1 Electric Field in Matter

5.1.1 Bound Charge

- We will consider atoms as a fundamental building block of matter. All atoms carry electrons and protons, so charge is thus present in all matter. Often, the charge cannot move from the atoms or the crystal structure—charge that cannot move is called bound charge and has a generally complex dependence on the material's microscopic structure.

  Free charge can freely move in space—since we have so far only considered Maxwell's equations in free space, the charge we have been working with up to now was free charge. However, the ρ in Maxwell's equations is technically total charge density—the sum of both bound and free charge.

- We define the bound charge in a material as

$$\rho_{\mathrm{b}} = \left\langle \sum_i q_i \delta^3(\boldsymbol{r} - \boldsymbol{r}_i) \right\rangle,$$

  where sum runs over all charges bound in the material's crystal lattice and the brackets denote an average.

  The average is calculated over "hydrodynamic volume"—this allows for a slightly continuous distribution of charge; think of the particles spreading around their delta functions into Gauss functions. The idea of a "hydrodynamic volume average" is to allow for microscopic variation around the discrete charge positions in material, which thus allows us to treat bound charge as a continuous quantity.

- In terms of bound charge, the first Maxwell equation becomes

$$\nabla \cdot \boldsymbol{E}(\boldsymbol{r}, t) = \frac{\rho(\boldsymbol{r}, t)}{\epsilon_0} = \frac{\rho_{\mathrm{f}}(\boldsymbol{r}, t)}{\epsilon_0} + \frac{\rho_{\mathrm{b}}(\boldsymbol{r}, t)}{\epsilon_0},$$

  where ρ_{f} is free charge. Note that we are treating charge as a continuous quantity, using a charge field, on a macroscopic scale. Keep in mind, however, that charge is fundamentally discrete, and that a field approach can differ to the results of quantum mechanics, where the charges are treated as discrete.

5.1.2 Electric Polarization

- Bound charge and electric polarization are related according to

$$\rho_{\mathrm{b}} = -\nabla \cdot \boldsymbol{P}.$$

  We have already derived this relationship in the electric dipole section of the electrostatics chapter, but at that point we had not yet introduced the concept of bound charge.

- Substituting the relationship $\rho_{\mathrm{b}} = -\nabla \cdot \boldsymbol{P}$ into the Mawell equation

$$\nabla \cdot \boldsymbol{E}(\boldsymbol{r}, t) = \frac{\rho_{\mathrm{f}}(\boldsymbol{r}, t)}{\epsilon_0} + \frac{\rho_{\mathrm{b}}(\boldsymbol{r}, t)}{\epsilon_0}$$

and rearranging produces the relationship

$$\nabla \cdot (\epsilon_0 \boldsymbol{E} + \boldsymbol{P}) = \rho_{\mathrm{f}}.$$

Note that the bound charge is contained in the $\boldsymbol{E}$ field. This just-derived relationship motivates the introduction of the $\boldsymbol{D}$ field:

$$\boldsymbol{D} = \epsilon_0 \boldsymbol{E} + \boldsymbol{P} \implies \nabla \cdot \boldsymbol{D} = \rho_{\mathrm{f}}.$$

The $\boldsymbol{D}$ field arises only from free charge (as encoded by $\nabla \cdot \boldsymbol{D} = \rho_{\mathrm{f}}$), which is encoded within the polarization term $\boldsymbol{P}$ (recall that $\boldsymbol{E}$ contains bound charge).

Example: Homogeneous Polarization

- As an example, we consider homogeneous polarization $\boldsymbol{P} = (0, 0, P_0)$ in a long and wide rectangular material whose height, which is much smaller than both the length and width, aligns with the z axis and varies from $z = 0$ to $z = h$. The bound charge in the material is

$$\rho_{\mathrm{b}} = -\nabla \cdot \boldsymbol{P} = -\frac{\partial P_z}{\partial z},$$

where we have used the material's long and wide geometry to approximate the divergence operator with $\nabla \cdot \to \frac{\partial}{\partial x}$. Since $\boldsymbol{P}$ is homogeneous, the derivative is nonzero only at the boundary between material and free space, which leads to

$$\rho_{\mathrm{b}} = -\nabla \cdot \boldsymbol{P} = -\frac{\partial P_z}{\partial z} = -P_z \delta(x) + P_z \delta(z - h).$$

In other words, the bound charge is confined to the planes at $z = 0$ and $z = h$, i.e. the material's surface.

In general, in a material with homogeneous polarization, bound charge occurs only along the boundaries.

5.1.3 The Constitive Relation for Electric Field in Matter

- Polarization in matter depends on the $\boldsymbol{D}$ field according to the general relationship

$$\boldsymbol{P} = \boldsymbol{P}(\boldsymbol{D}),$$

which we call the constitutive relation. In general, the constitutive relationship is nonlinear and may be quite complicated. However, in an isotropic, homogeneous material, we can expand the relation to first order in $\boldsymbol{D}$ to get

$$\boldsymbol{P}(\boldsymbol{D}) \approx \chi_{\mathrm{E}} \boldsymbol{D} + \mathcal{O}(\boldsymbol{D}^2),$$

where we have introduced electric susceptibility χ_{E}. Electric susceptibility is often defined in terms of electric permittivity ϵ (the dielectric constant) via

$$\chi_{\mathrm{E}} = 1 - \frac{1}{\epsilon}.$$

- In terms of χ_{E} and ϵ, the linear constitutive relation reads

$$\boldsymbol{D} = \epsilon_0 \boldsymbol{E} + \boldsymbol{P} \approx \epsilon_0 \boldsymbol{E} + \chi_{\mathrm{E}} \boldsymbol{D} = \epsilon_0 \boldsymbol{E} + \left(1 - \frac{1}{\epsilon}\right) \boldsymbol{D}$$

  Thus, with a little algebra, the $\boldsymbol{E}$ and $\boldsymbol{D}$ fields are related to first order by

$$\boldsymbol{D} = \epsilon \epsilon_0 \boldsymbol{E}.$$

  In terms of the relationship $\boldsymbol{D} = \epsilon\epsilon_0 \boldsymbol{E}$, the relationship between polarization and the $\boldsymbol{E}$ field is

$$\boldsymbol{P} = \boldsymbol{D} - \epsilon_0 \boldsymbol{E} \approx \epsilon_0 (\epsilon - 1) \boldsymbol{E}.$$

  Important: we stress that the relationships between $\boldsymbol{E}$, $\boldsymbol{D}$ and $\boldsymbol{P}$ given in this last bullet point are not general. They hold only for an isotropic, homogeneous material in the regime of a first-order approximation of the constitutive relation $\boldsymbol{P} = \boldsymbol{P}(\boldsymbol{D})$.

5.1.4 Dielectrics and Conductors

- Dielectric have a finite dielectric constant ϵ (also called the dielectric function) and can store electric field energy. In ideal dielectrics, ϵ is constant. More generally, the dielectric constant is a function of electric field frequency.

- Conductors have an infinite value of ϵ and shadow (cancel out) external electric fields within their volume. In an external electric field, charge occurs only on the conductor's surface, as discussed in more detail in the section on conductors.

5.2 Magnetic Field in Matter

- As before, we consider the model to be the fundamental building block of matter, and model an atom as a positive nucleus with classically "orbiting" electrons, which give rise to the localized microscopic currents. More accurately, we can view microscopic bound currents in material arising from quantum mechanical fluctuations of charge carriers within atoms. In any case, we assume the presence of microscopic currents on an atomic scale within matter.

- In the absence of an external magnetic field, these microscopic internal currents have no macroscopic effect—the "hydrodynamic volume average" of the bound currents is zero. In the presence of an external magnetic field, however, the microscopic bound currents can give rise to macroscopic effects.

5.2.1 Bound Currents

- Analogously to our definition of bound charge, we define the bound current density $\boldsymbol{j}_{\mathrm{b}}$ in a material as

$$\boldsymbol{j}_{\mathrm{b}}(\boldsymbol{r}, t) = \left\langle \sum_i \boldsymbol{j}_i \delta^3(\boldsymbol{r} - \boldsymbol{r}_i) \right\rangle,$$

  where the sum runs over the positions $\boldsymbol{r}_i$ of atoms or molecules in the material and the brackets denote a hydrodynamic average, which allows us to treat the bound current density as a continuous quantity (even though on a quantum level the microscopic currents are technically discrete).

- Total current density is then the some of free and bound current density, i.e.

$$\boldsymbol{j}(\boldsymbol{r}, t) = j_{\mathrm{f}}(\boldsymbol{r}, t) + j_{\mathrm{b}}(\boldsymbol{r}, t).$$

In terms of free and bound current density, the third Maxwell equation becomes

$$\nabla \times \boldsymbol{B}(\boldsymbol{r}, t) = \mu_0 \boldsymbol{j}_{\mathrm{f}}(\boldsymbol{r}, t) + \mu_0 \boldsymbol{j}_{\mathrm{b}}(\boldsymbol{r}, t) + \epsilon_0 \mu_0 \frac{\partial \boldsymbol{E}(\boldsymbol{r}, t)}{\partial t}.$$

5.2.2 Magnetization

- We usually analyze bound current in terms of magnetization (analogous to analyzing bound charge in terms of electric polarization). We define magnetization $\boldsymbol{M}$ with the relationship

$$\boldsymbol{j}_{\mathrm{b}} = \nabla \times \boldsymbol{M} + \frac{\partial \boldsymbol{P}}{\partial t}.$$

Note that polarization and magnetization are coupled—which means that bound currents depend on the time derivative of $\boldsymbol{P}$ as well as on magnetization.

- The time derivative of $\boldsymbol{P}$ is included in the above relationship between bound current and magnetization to satisfy the continuity equation

$$\nabla \cdot \boldsymbol{j}_{\mathrm{b}} + \frac{\partial \rho_{\mathrm{b}}}{\partial t} = 0.$$

To show this, we substitute the expression for $\boldsymbol{j}_{\mathrm{b}}$ into the continuiuty equation to get

$$\nabla \cdot \boldsymbol{j}_{\mathrm{b}} + \frac{\partial \rho_{\mathrm{b}}}{\partial t} = \nabla \cdot (\nabla \times \boldsymbol{M}) + \nabla \cdot \frac{\partial \boldsymbol{P}}{\partial t} - \frac{\partial \nabla \cdot \boldsymbol{P}}{\partial t} = 0.$$

Since the divergence of a curl quantity is zero, we have $\nabla \cdot (\nabla \times \boldsymbol{M}) = 0$ and thus

$$\nabla \cdot \boldsymbol{j}_{\mathrm{b}} + \frac{\partial \rho_{\mathrm{b}}}{\partial t} = 0 + \nabla \cdot \frac{\partial \boldsymbol{P}}{\partial t} - \frac{\partial \nabla \cdot \boldsymbol{P}}{\partial t} = 0,$$

which satisfies the continuity equation $\nabla \cdot \boldsymbol{j}_{\mathrm{b}} + \frac{\partial \rho_{\mathrm{b}}}{\partial t} = 0$.

- In terms of bound current and magnetization, the third Maxwell equation (skipping a few steps of algebra) becomes

$$\nabla \times \left(\frac{\boldsymbol{B}}{\mu_0} - \boldsymbol{M} \right) = \boldsymbol{j}_{\mathrm{f}} + \frac{\partial \boldsymbol{D}}{\partial t}.$$

This expression motivates the definition of magnetic field strength $\boldsymbol{H}$ as

$$\boldsymbol{H} = \frac{\boldsymbol{B}}{\mu_0} - \boldsymbol{M} \implies \boldsymbol{B} = \mu_0(\boldsymbol{H} + \boldsymbol{M}).$$

Note that $\boldsymbol{H}$ arises only from free currents and free charge, analogously to how the $\boldsymbol{D}$ field arises from only free charge. In terms of $\boldsymbol{H}$, the Maxwell equation reads

$$\nabla \times \boldsymbol{H} = \boldsymbol{j}_{\mathrm{f}} + \frac{\partial \boldsymbol{D}}{\partial t}.$$

Example: Homogeneous Magnetization

- Consider a cuboid of base $a \times b$ and height h with homogeneous magnetization $\boldsymbol{M} = (0, 0, M_0)$ in the z direction.

  The bound current corresponding to this homogeneous magnetization is

  $$\boldsymbol{j}_{\mathrm{b}} = \nabla \times \boldsymbol{M} + \frac{\partial \boldsymbol{P}}{\partial t} = \nabla \times \boldsymbol{M} + 0 = \left(\frac{\partial M_z}{\partial y}, -\frac{\partial M_z}{\partial x}, 0\right).$$

  The derivatives are nonzero only at the borders, which produces

  $$\boldsymbol{j}_{\mathrm{b}} = \cdots = \big[M_0 \delta(y) - M_0 \delta(y - b), -M_0 \delta(x) + M_0 \delta(x - a), 0\big]$$

  Lesson: the bound currents in a material with homogeneous magnetization occur only on the material's lateral surface enclosing the axis of magnetization.

  The surface currents shadow the magnetic field in the material, analogously to how surface charge shadows electric field in conductors.

5.2.3 The Constitive Relation for Magnetic Field in Matter

- Magnetization depends on external magnetic field $\boldsymbol{H}$ according to the general relationship

  $$\boldsymbol{M} = \boldsymbol{M}(\boldsymbol{H}),$$

  which we call the constitutive relation. In general, the constitutive relationship is nonlinear and may be quite complicated. However, in an isotropic, homogeneous material, we can expand the relation to first order in $\boldsymbol{H}$ to get

  $$\boldsymbol{M}(\boldsymbol{H}) \approx \chi_{\mathrm{M}} \boldsymbol{H} + \mathcal{O}(\boldsymbol{H}^2),$$

  where we have introduced magnetic susceptibility χ_{M}. Magnetic susceptibility is often defined in terms of magnetic permeability μ via

  $$\chi_{\mathrm{M}} = \mu - 1.$$

  For most materials χ_{M} is close to zero and μ is close to one. Magnetic permeability μ is the magnetic analog of electric permittivity ϵ, i.e. the dielectric constant.

- In terms of χ_{M} and μ, the linear constitutive relation reads

  $$\boldsymbol{H} = \frac{\boldsymbol{B}}{\mu_0} - \boldsymbol{M} \approx \frac{\boldsymbol{B}}{\mu_0} - \chi_{\mathrm{M}} \boldsymbol{H} = \frac{\boldsymbol{B}}{\mu_0} - (\mu - 1)\boldsymbol{H}.$$

  Thus, to first order, the $\boldsymbol{B}$ and $\boldsymbol{H}$ fields are related by:

  $$\boldsymbol{H} = \frac{\boldsymbol{B}}{\mu \mu_0} \implies \boldsymbol{B} = \mu \mu_0 \boldsymbol{H}.$$

  In terms of the relationship $\boldsymbol{B} = \mu \mu_0 \boldsymbol{H}$, the relationship between magnetization and the $\boldsymbol{B}$ field is

  $$\boldsymbol{M} = \left(1 - \frac{1}{\mu}\right) \frac{\boldsymbol{B}}{\mu_0}.$$

  Important: we stress that the relationships between $\boldsymbol{B}$, $\boldsymbol{H}$ and $\boldsymbol{M}$ given in this last bullet point are not general. They hold only for an isotropic, homogeneous material in the regime of a first-order approximation to the constitutive relation $\boldsymbol{M} = \boldsymbol{M}(\boldsymbol{H})$.

5.2.4 Magnetization and Magnetic Dipole Density

- Recall that electric polarization corresponds to the volume density of electric dipole moments in a material. In this section, we will derive an analogous relationship between magnetization $\boldsymbol{M}$ and density of magnetic dipoles.

- We begin with Maxwell's equation

$$\nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{j}_\mathrm{f} + \mu_0 (\nabla \times \boldsymbol{M})$$

We then introduce magnetic potential via $\boldsymbol{B} = \nabla \times \boldsymbol{A}$ to get

$$\nabla \times \big[\nabla \times \boldsymbol{A}\big] = \mu_0 \boldsymbol{j}_\mathrm{f} + \mu_0 (\nabla \times \boldsymbol{M}),$$

and apply the vector calculus identity $\nabla \times \big[\nabla \times \boldsymbol{A}\big] = \nabla(\nabla \cdot \boldsymbol{A}) - \nabla^2 \boldsymbol{A}$ to get

$$\nabla(\nabla \cdot \boldsymbol{A}) - \nabla^2 \boldsymbol{A} = \mu_0 \boldsymbol{j}_\mathrm{f} + \mu_0 (\nabla \times \boldsymbol{M})$$

- **TODO:** Sketched derivation: One converts the double curl to divergence of $\boldsymbol{A}$, writes the equation for $\boldsymbol{A}$, find the solution for $\boldsymbol{A}$. Compare the result to the magnetic potential of a magnetic dipole. The result is that $\boldsymbol{M}$ is the volume density of magnetic dipole moments in a material.

- Recall, via the Ampere equivalence relation, that we can consider a magnetic dipole as a circular current loop, so it makes sense that magnetic dipoles are related to magnetization and thus bound electric current density.

5.2.5 Classifying Materials By Their Magnetic Field Response

- We classify materials based on the value of their magnetic susceptibility χ_M.

- Ferromagnetic materials have a permanent nonzero magnetization independent of the external magnetic field. However, the magnetization in ferromagnetic materials has a strong temperature dependence, and ferromagnetic materials demagnetize above the so-called Curie temperature (more on this in solid state physics).

- Diamagnetic materials have a magnetization only in the presence of an external electric field. Typical values of susceptibility are $\chi_\mathrm{M} \lesssim 0$, i.e. slightly less than zero.

  A superconductor is an ideal diamagnetic and has $\chi_\mathrm{M} = -1$—magnetic field is completely expelled from the material.

- Paramagnetic materials have $\mu > 1$, and an external magnetic field increase the magnetization in the material. Paramagentic materials behave like magnets in an external magnetic field, but demagnetize in the absence of an external field.

5.3 Maxwell's Equations in Matter

- The complete set of Maxwell's equations in materials is is

$$\begin{aligned} \nabla \cdot \boldsymbol{D} &= \rho_\mathrm{f} & \qquad \nabla \cdot \boldsymbol{B} &= 0 \\ \nabla \times \boldsymbol{E} &= -\frac{\partial \boldsymbol{B}}{\partial t} & \qquad \nabla \times \boldsymbol{H} &= \boldsymbol{j}_\mathrm{f} + \frac{\partial D}{\partial t}. \end{aligned}$$

Note that to complete the description of electomagnetism in matter, we must also consider in the constitutive relationships

$$\boldsymbol{P} = \boldsymbol{P}(\boldsymbol{D}) \qquad \text{and} \qquad \boldsymbol{M} = \boldsymbol{M}(\boldsymbol{H}).$$

In isotropic, homogeneous materials, we can expand the constitutive relations to first order in $\boldsymbol{D}$ and $\boldsymbol{H}$ to produce the relationships

$$\boldsymbol{D} = \epsilon\epsilon_0 \boldsymbol{E} \qquad \text{and} \qquad \boldsymbol{B} = \mu\mu_0 \boldsymbol{H}$$

between the field flux densitities ($\boldsymbol{D}$ and $\boldsymbol{B}$) and their strengths ($\boldsymbol{E}$ and $\boldsymbol{H}$). In anisotropic materials, ϵ and μ generalize to rank-two tensors.

5.4 Conservation Laws in Matter

5.4.1 Conservation of Energy

- Recall from the chapter on Maxwell's equations (in free space) that electromagnetic energy conservation is encoded by the Poynting theorem

$$\frac{\partial w}{\partial t} + \nabla \cdot \mathbf{S} + \boldsymbol{j} \cdot \boldsymbol{E} = 0,$$

where $\mathbf{S}$ is the Poynting vector, $\boldsymbol{j} \cdot \boldsymbol{E}$ encodes Ohmic losses, and w is electromagnetic energy density. This expression still holds in matter, except that we generalize the definition of energy density to

$$w = \int_0^{\boldsymbol{D}} \boldsymbol{E}(\boldsymbol{D}') \, \mathrm{d}\boldsymbol{D}' + \int_0^{\boldsymbol{B}} \boldsymbol{H}(\boldsymbol{B}') \, \mathrm{d}\boldsymbol{B}',$$

and define the Poynting vector as $\mathbf{S} = \boldsymbol{E} \times \boldsymbol{H}$. Note if we have a linear constitutive relation i.e. $\boldsymbol{E} \propto \boldsymbol{D}$ and $\boldsymbol{H} \propto \boldsymbol{B}$, the expression for w simplifies to an expression involving quadratic forms, just like for w in free space.

5.4.2 Electric Field Energy in Matter

- We now consider how electric field energy in a region of empty space changes when we place an dielectric material (with $\epsilon \neq 1$) in the region.

- First, using W to denote electric field energy and letting the subscript zero denote vacuum, we integrate the generalized expression for energy density in matter to get

$$W - W_0 = \iiint_V (w - w_0) \, \mathrm{d}^3\boldsymbol{r} = \iiint_V \left[\int \boldsymbol{E}(\boldsymbol{D}) \, \mathrm{d}\boldsymbol{D} - \int \boldsymbol{E}_0(\boldsymbol{D}_0) \, \mathrm{d}\boldsymbol{D}_0 \right] \mathrm{d}^3\boldsymbol{r}.$$

We then assume a linear constitutive relation, which implies

$$\boldsymbol{E} \propto \boldsymbol{D} \implies \int \boldsymbol{E}(\boldsymbol{D}) \, \mathrm{d}\boldsymbol{D} \propto \frac{\boldsymbol{D}^2}{2}.$$

- Next, we substitute the relation $\int \boldsymbol{E}(\boldsymbol{D}) \, \mathrm{d}\boldsymbol{D} \propto \frac{\boldsymbol{D}^2}{2}$ into the energy difference, more the factor $1/2$ otuside the integral, and rewrite on $\boldsymbol{D}$ in terms of $\boldsymbol{E}$ to get

$$W - W_0 = \frac{1}{2} \iiint_V \boldsymbol{E} \cdot \boldsymbol{D} \, \mathrm{d}^3\boldsymbol{r} - \frac{1}{2} \iiint_V \boldsymbol{E}_0 \cdot \boldsymbol{D}_0 \, \mathrm{d}^3\boldsymbol{r}.$$

Next, a trick: using reverse-engineered multiplication, we rewrite the above expression in the algebraically equivalent form

$$W - W_0 = \frac{1}{2}\iiint_V (\boldsymbol{E}\cdot\boldsymbol{D}_0 - \boldsymbol{E}_0\cdot\boldsymbol{D})\,\mathrm{d}^3\boldsymbol{r} + \frac{1}{2}\iiint_V (\boldsymbol{E}+\boldsymbol{E}_0)\cdot(\boldsymbol{D}-\boldsymbol{D}_0)\,\mathrm{d}^3\boldsymbol{r}.$$

It turns out the second term is zero, which we prove in the following bullet points.

- To show the aforementioned second term is zero, we first consider the sum $\boldsymbol{E}+\boldsymbol{E}_0$, which we rewrite in terms of electric potential as $\boldsymbol{E}+\boldsymbol{E}_0 = -\nabla\phi$. Using this expression for $\boldsymbol{E}$ and some vector calculus acrobatics, the second term becomes

$$\begin{aligned} I &\equiv \frac{1}{2}\iiint_V (\boldsymbol{E}+\boldsymbol{E}_0)\cdot(\boldsymbol{D}-\boldsymbol{D}_0)\,\mathrm{d}^3\boldsymbol{r} = \frac{1}{2}\iiint_V (-\nabla\phi)(\boldsymbol{D}-\boldsymbol{D}_0)\,\mathrm{d}^3\boldsymbol{r} \\ &= -\frac{1}{2}\iiint_V \left[\nabla\cdot(\phi\cdot(\boldsymbol{D}-\boldsymbol{D}_0)) - \nabla\cdot(\boldsymbol{D}-\boldsymbol{D}_0)\phi\right]\mathrm{d}^3\boldsymbol{r}. \end{aligned}$$

  Next, recall that the $\boldsymbol{D}$ field arises from free charge via $\nabla\cdot\boldsymbol{D} = \rho_\mathrm{f}$. Since placing a dielectric material into a region of space changes only bound charge and not free charge, the free charge in the region of space is the same before and after filling it with dielectric material, i.e. $\rho_\mathrm{f} = \rho_{\mathrm{f}_0}$, which implies $\nabla\cdot(\boldsymbol{D}-\boldsymbol{D}_0) = 0$ and thus

$$I = -\frac{1}{2}\iiint_V \nabla\cdot(\phi\cdot(\boldsymbol{D}-\boldsymbol{D}_0))\,\mathrm{d}^3\boldsymbol{r}.$$

  The remaining term is a divergence integrated over the region of space V containing the material, which we rewrite with the divergence theorem to get

$$I = -\frac{1}{2}\oiint_{\partial V}\phi\cdot(\boldsymbol{D}-\boldsymbol{D}_0)\,\mathrm{d}S.$$

  But along the region's surface $\boldsymbol{D} = \boldsymbol{D}_0$, so we are integrating $\phi\cdot 0$ over the surface, which is obviously zero. The result is the quoted expression $I = 0$.

- The difference in electric field energies, using the linear constitutive relation to relate $\boldsymbol{E}$ and $\boldsymbol{D}$ along with the just-derived result $I = 0$, is then

$$\begin{aligned} W - W_0 &= \frac{1}{2}\int_V (\boldsymbol{E}\cdot\boldsymbol{D}_0 - \boldsymbol{E}_0\cdot\boldsymbol{D})\,\mathrm{d}^3\boldsymbol{r} \\ &= -\frac{1}{2}\int_V \epsilon_0(\epsilon-1)\boldsymbol{E}_0\boldsymbol{E}\,\mathrm{d}^3\boldsymbol{r} \\ &= -\frac{1}{2}\int_V \boldsymbol{P}\cdot\boldsymbol{E}_0\,\mathrm{d}^3\boldsymbol{r}, \end{aligned}$$

  where $\boldsymbol{P}$ is the electric polarization in the dielectric material and $\boldsymbol{E}_0$ is the electric field that would be present in the region of space in the absence of the material.

5.4.3 Magnetic Field Energy in Matter

- We now consider how magnetic field energy in a region of empty space changes when we place an magnetically active material with $\mu \neq 1$ inside the material. The derivation is analogous to the above derivation for electric field energy, so we will proceed at a faster pace.

- Assuming a linear constitutive relation for the magnetic field, and using W to denote magnetic field energy, the difference magnetic field energies before and after placing the magnetically active material in the region is

$$\begin{aligned}
W - W_0 &= \frac{1}{2} \iiint_V \boldsymbol{H} \cdot \boldsymbol{B} \, \mathrm{d}^3\boldsymbol{r} - \frac{1}{2} \iiint_V \boldsymbol{H}_0 \cdot \boldsymbol{B}_0 \, \mathrm{d}^3\boldsymbol{r} \\
&= \frac{1}{2} \iiint_V \big(\boldsymbol{B}\boldsymbol{H}_0 - \boldsymbol{B}_0\boldsymbol{H}\big) \, \mathrm{d}^3\boldsymbol{r} + \frac{1}{2} \iiint_V (\boldsymbol{B} + \boldsymbol{B}_0) \cdot (\boldsymbol{H} - \boldsymbol{H}_0) \, \mathrm{d}^3\boldsymbol{r}.
\end{aligned}$$

  If we assume free currents in the region of space do not change after adding the magnetically active material (just like we assumed free charge didn't change when adding a dielectric material), we have $\nabla \times (\boldsymbol{H} - \boldsymbol{H}_0)$.

  Sketched procedure: write $\boldsymbol{B} + \boldsymbol{B}_0$ as $\nabla \times \boldsymbol{A}$, which changes the integrands to a set of curl terms, and apply $\nabla \times (\boldsymbol{H} - \boldsymbol{H}_0)$.

  Next, change the the remaining non-zero integral to a surface integral and apply $\boldsymbol{A}(\boldsymbol{H} - \boldsymbol{H}_0) = \boldsymbol{A}(0)$ along the surface where $\boldsymbol{H} = \boldsymbol{H}_0$. This produces

$$\begin{aligned}
W - W_0 &= \frac{1}{2} \iiint_V \big(\boldsymbol{B}\boldsymbol{H}_0 - \boldsymbol{B}_0\boldsymbol{H}\big) \, \mathrm{d}^3\boldsymbol{r} \\
&= \frac{1}{2} \iiint_V \mu_0(\mu - 1)\boldsymbol{H}_0\boldsymbol{H} \, \mathrm{d}^3\boldsymbol{r} \\
&= \frac{1}{2} \iiint_V \boldsymbol{M} \cdot \boldsymbol{B}_0 \, \mathrm{d}^3\boldsymbol{r}.
\end{aligned}$$

  Lesson: the change in magnetic field energy that results from placing a magnetically active material in empty space permeated by the magnetic field $\boldsymbol{B}_0$ is proportional to the original field $\boldsymbol{B}_0$ and the magnetization $\boldsymbol{M}$ induced in the magnetically active material. Keep in mind that this result holds only for a linear constitutive relation.

5.4.4 Electromagnetic Momentum Conservation in Matter

- We now turn our attention to electromagnetic momentum conservation in matter. We begin with the continuity equation for electromagnetic momentum, assume a linear constitutive relation, and write electromagnetic momentum density as

$$\boldsymbol{p}_{\mathrm{em}} = \boldsymbol{D} \times \boldsymbol{B}.$$

  Recall that in free space we had defined EM momentum as $\boldsymbol{p}_{\mathrm{em}} = \epsilon_0 \boldsymbol{E} \times \boldsymbol{B}$.

- In terms of the $\boldsymbol{D}$ field, the Cauchy continuity equation in matter reads

$$\frac{\partial p_{\mathrm{em}_i}}{\partial t} - \frac{\partial \mathrm{T}_{ik}}{\partial x_k} + f_{\mathrm{em}_i} = 0,$$

  where f_{em_i} is electromagnetic force density and T_{ik} is the electromagnetic stress tensor, which is defined in matter as

$$\mathrm{T}_{ik} = E_i D_k - \frac{1}{2}(\boldsymbol{D} \cdot \boldsymbol{E})\delta_{ik} + B_i H_i - \frac{1}{2}(\boldsymbol{B} \cdot \boldsymbol{H})\delta_{ik}$$

- **Note:** For a general constitutive relation, the expression $\frac{\partial \boldsymbol{p}_{\mathrm{em}}}{\partial t}$ cannot be written in terms of a divergence of a vector field. As a result, it becomes impossible to write the surface term involving T_{ik} in closed form. As a result, the electromagnetic stress tensor does not generally exist (we don't know how to write it) for a general constitutive relation.

Physical interpretation: the stress tensor is related to the expression for the force on a body in terms of a field on the body's surface. If we can't write the stress tensor, it means the expression for force involves interactions within the body's volume than cannot be written as surface interactions.

5.5 Nonhomogeneous Material in an Electromagnetic Field

- In a nonhomogeneous material, $\epsilon = \epsilon(\boldsymbol{r})$ and $\mu = \mu(\boldsymbol{r})$ vary throughout the material. In this section, we consider how such a material reacts to the presence of an electromagnetic field field.

- If we move the material within the field, the value of ϵ and μ change relative to the field $\boldsymbol{E}$ and $\boldsymbol{B}$, so the electromagnetic force on the material changes. We consider only the electric field in this section, and then consider the magnetic field in the following section.

- Assuming we can assign the field an electrostatic stress tensor, the electric force on the material can be written in terms of the tensor as

$$F_{\mathrm{E}_i} = \iiint_V \frac{\partial \mathrm{T}_{ik}}{\partial x_k} \, \mathrm{d}^3\boldsymbol{r}.$$

- First, we consider the electrostatic term

$$\frac{\partial \mathrm{T}_{ik}}{\partial x_k} = \frac{\partial}{\partial x_k}\left[E_i D_k - \frac{1}{2}(\boldsymbol{E}\cdot\boldsymbol{D})\delta_{ik}\right].$$

We then recall $\epsilon = \epsilon(\boldsymbol{r})$ and assume a linear constitutive relation $\boldsymbol{D} = \epsilon\epsilon_0\boldsymbol{E}$, which leads to

$$\frac{\partial \mathrm{T}_{ik}}{\partial x_k} = E_i\frac{\partial D_k}{\partial x_k} + E_k\frac{\partial D_i}{\partial x_k} - \frac{1}{2}\frac{\partial \epsilon}{\partial x_i}\epsilon_0 E^2 - \frac{1}{2}\epsilon\epsilon_0\frac{\partial E^2}{\partial x_i}.$$

Next, we rewrite the term $\frac{\partial E^2}{\partial x_i}$ according to

$$\frac{1}{2}\nabla E^2 = \boldsymbol{E}\times(\nabla\times\boldsymbol{E}) + (\boldsymbol{E}\cdot\nabla)\boldsymbol{E},$$

which leads to a vector expression for electric force:

$$\begin{aligned}\boldsymbol{F}_{\mathrm{E}} = &\iiint_V \left[\boldsymbol{E}(\nabla\cdot\boldsymbol{D}) + (\boldsymbol{E}\cdot\nabla)\boldsymbol{D} - \boldsymbol{D}\times(\nabla\times\boldsymbol{E}) - (\boldsymbol{E}\cdot\nabla)\boldsymbol{D}\right]\mathrm{d}^3\boldsymbol{r}\\ &- \frac{1}{2}\iiint_V (\nabla\epsilon(\boldsymbol{r}))\epsilon_0 E^2 \,\mathrm{d}^3\boldsymbol{r}.\end{aligned}$$

- Next, we consider a simplified case of the above result. We make two assumptions:

  1. We assume the $\nabla\cdot\boldsymbol{D}\approx 0$, which corresponds to working in the absence of appreciable free charge (recall $\nabla\cdot\boldsymbol{D}\equiv\rho_{\mathrm{f}}$). The approximation $\nabla\cdot\boldsymbol{D}\approx 0$ often gives satisfactory results, but is of course not valid in general.
  2. We also assume $\nabla\times\boldsymbol{E}=0$, which corresponds to working in a system without induction, i.e. in the absence of a time-varying magnetic field (recall the relationship $\nabla\times\boldsymbol{E} = -\frac{\partial\boldsymbol{B}}{\partial t}$).

Under these assumptions, the first integral vanishes and the force on the nonhomogeneous dielectric simplifies to

$$\boldsymbol{F}_{\mathrm{E}} = -\frac{1}{2}\int_V (\nabla\epsilon(\boldsymbol{r}))\epsilon_0 E^2 \,\mathrm{d}^3\boldsymbol{r}.$$

In other words, $\boldsymbol{F}$ at a given point depends on the gradient of ϵ.

TODO: Force on a Nonhomogeneous Magnetic Material

- The analysis of the magnetic force on a nonhomogeneous material with $\mu = \mu(\boldsymbol{r})$ is analogous to the above derivation for electric force on a material with $\epsilon = \epsilon(\boldsymbol{r})$, so we only briefly skecthed it in lecture. The procedure reads: assume a linear constitutive relation $\boldsymbol{H} = \mu\mu_0\boldsymbol{B}$ and write the magnetic stress-energy tensor in component form. Write the term $\frac{\partial^2 B}{\partial x_i^2}$ and magnetic force in vector form. Assume $\nabla\times\boldsymbol{H} = 0$ (absence of free currents) and use $\nabla\cdot\boldsymbol{B} = 0$ to get

$$\boldsymbol{F}_{\mathrm{M}} = -\frac{1}{2}\iiint_V (\nabla\mu(\boldsymbol{r}))\mu_0 H^2 \,\mathrm{d}^3\boldsymbol{r}.$$

5.6 Boundary Conditions For Maxwell's Equations

Consider a boundary between two materials with different electromagnetic properties, i.e. different values of ϵ and μ, where each material is assigned a surface normal vector $\hat{\boldsymbol{n}}$ point into the material. Our goal in this section is to find the appropriate boundary conditions on the $\boldsymbol{E}$, $\boldsymbol{D}$, $\boldsymbol{H}$, and $\boldsymbol{B}$ fields at the interface between the two materials.

5.6.1 Boundary Condition for the B Field

- Let the magnetic field in the first and second materials be $\boldsymbol{B}_1$ and $\boldsymbol{B}_2$, respectively. We begin with the Maxwell equation

$$\nabla\cdot\boldsymbol{B} = 0 \implies \iiint_V \nabla\cdot\boldsymbol{B}\,\mathrm{d}^3\boldsymbol{r} = 0,$$

which must hold for any region of space V.

- We then consider a cylinder of infinitesimal height $\mathrm{d}\boldsymbol{l}$ just enclosing the boundary between the two materials (a so-called Gaussian pillbox). We integrate over the cylindrical volume to get

$$\begin{aligned} 0 \equiv \iiint_V \nabla\cdot\boldsymbol{B}\,\mathrm{d}^3\boldsymbol{r} &= \oiint_{\partial V} \boldsymbol{B}\cdot\mathrm{d}\boldsymbol{S} \\ &= \iint_{S_1} \boldsymbol{B}_1\cdot\hat{\boldsymbol{n}}_1\,\mathrm{d}S + \iint_{S_2} \boldsymbol{B}_2\cdot\hat{\boldsymbol{n}}_2\,\mathrm{d}S + \iint_{S_3} \boldsymbol{B}_3\cdot\hat{\boldsymbol{n}}\,\mathrm{d}S \end{aligned}$$

where S_3 is the cylinder's lateral surface area.

- We send the cylinder height $\mathrm{d}\boldsymbol{l}$ to zero, since we're interested only in the boundary between the two materials. The third integral over the lateral surface area vanishes in the limit $\mathrm{d}\boldsymbol{l}\to 0$, leaving

$$\iint_{S_1} \boldsymbol{B}_1\cdot\hat{\boldsymbol{n}}_1\,\mathrm{d}S + \iint_{S_2} \boldsymbol{B}_2\cdot\hat{\boldsymbol{n}}_2\,\mathrm{d}S = 0.$$

At the boundary, the $\boldsymbol{B}$ field thus obeys the relationship

$$\boldsymbol{B}_1 \cdot \hat{\boldsymbol{n}}_1 + \boldsymbol{B}_2 \cdot \hat{\boldsymbol{n}}_2 = 0 \implies B_1^{\perp} - B_2^{\perp} = 0,$$

where $B^{\perp}$ denotes the magnetic field component perpendicular to the surface. Note the change in sign of the component expression relative to the vector expression, which occurs because the normal vector $\hat{\boldsymbol{n}}_1$ points in the opposite direction as $\hat{\boldsymbol{n}}_2$.

5.6.2 Boundary Condition for the $\boldsymbol{D}$ Field

- We consider the same interface between two materials as before and let $\boldsymbol{D}_1$ and $\boldsymbol{D}_2$ denote the $\boldsymbol{D}$ fields in the first and second material, respectively. We now base our analysis on the Maxwell equation

$$\nabla \cdot \boldsymbol{D} = \rho_{\mathrm{f}}.$$

- As before, we consider a cylinder with infinitesimal height $\mathrm{d}\hat{\boldsymbol{l}}$ enclosing the boundary between the two materials. In integral form, the Maxwell equation reads

$$\iiint_V \nabla \cdot \boldsymbol{D} \, \mathrm{d}^3\boldsymbol{r} = \iiint_V \rho_{\mathrm{f}} \, \mathrm{d}^3\boldsymbol{r}.$$

We use the divergence theorem to write the left-hand integral as an integral over the cylinder surface, which gives

$$\iiint_V \rho_{\mathrm{f}} \, \mathrm{d}^3\boldsymbol{r} = \oiint_{\partial V} \boldsymbol{D} \cdot \mathrm{d}\boldsymbol{S} = \iint_{S_1} \boldsymbol{D}_1 \cdot \hat{\boldsymbol{n}}_1 \, \mathrm{d}S + \iint_{S_2} \boldsymbol{D}_2 \cdot \hat{\boldsymbol{n}}_2 \, \mathrm{d}S + \iint_{S_3} \boldsymbol{D}_3 \cdot \hat{\boldsymbol{n}} \, \mathrm{d}S,$$

where S_3 again denotes the cylinder's lateral surface area.

- We then send the cylinder height to zero, in which case the integral $\iiint_V \rho_{\mathrm{f}} \, \mathrm{d}^3\boldsymbol{r}$ becomes a surface integral over the cylinder's cross-sectional area and the volume charge density ρ_{f} approaches the surface charge density σ_{f}, leaving us with

$$\iint_{S_1} \boldsymbol{D}_1 \cdot \hat{\boldsymbol{n}}_1 \, \mathrm{d}S + \iint_{S_2} \boldsymbol{D}_2 \cdot \hat{\boldsymbol{n}}_2 \, \mathrm{d}S = \iint_S \sigma_{\mathrm{f}} \, \mathrm{d}S.$$

Along the boundary between the materials, the $\boldsymbol{D}$ field must thus obey

$$D_1^{\perp} - D_2^{\perp} = \sigma_{\mathrm{f}}.$$

5.6.3 Boundary Condition for the $\boldsymbol{E}$ Field

- We now consider the boundary between two materials with electric fields $\boldsymbol{E}_1$ and $\boldsymbol{E}_2$, respectively. This time around, we use the Maxwell equation

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}.$$

We then integrate this equation over the boundary and apply Stoke's theorem to get

$$-\frac{\partial}{\partial t} \iint_S \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{S} = \iint_S \nabla \times \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{S} = \oint_{\partial S} \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{s},$$

where S is a rectangular surface of length l and height h whose boundary ∂S encloses the interface between the two materials. The loop's length runs parallel to the interface while the height is perpendicular to the interface, so that decreasing the height $\mathrm{d}h$ causes the loop to approach the interface between the materials. The tangents to the loop (along its lengths) are $\hat{\boldsymbol{t}}_1 = -\hat{\boldsymbol{t}}_2$ in the two regions, respectively.

- Next, we split the closed line integral over the rectangular surface's boundary ∂S into integrals over the loop's length and height to get

$$-\frac{\partial}{\partial t}\iint_S \boldsymbol{B}\cdot \mathrm{d}\boldsymbol{S} = \int_{l_1}\boldsymbol{E}\cdot \mathrm{d}\boldsymbol{l} + \int_{l_2}\boldsymbol{E}\cdot \mathrm{d}\boldsymbol{l} + \int_{h_1}\boldsymbol{E}\cdot \mathrm{d}\boldsymbol{h} + \int_{h_2}\boldsymbol{E}\cdot \mathrm{d}\boldsymbol{h}.$$

We then send the surface's height to zero, i.e. $\mathrm{d}h \to 0$, in which case the integrals over height vanish, leaving us with

$$\begin{aligned}-\frac{\partial}{\partial t}\iint_S \boldsymbol{B}\cdot \mathrm{d}\boldsymbol{S} &= \int_{l_1}\boldsymbol{E}\cdot \mathrm{d}\boldsymbol{l} + \int_{l_2}\boldsymbol{E}\cdot \mathrm{d}\boldsymbol{l} + 0 + 0\\ &= \int_l (\boldsymbol{E}_1\cdot \hat{\boldsymbol{t}}_1 + \boldsymbol{E}_2\cdot \hat{\boldsymbol{t}}_2)\,\mathrm{d}l,\end{aligned}$$

where we have written the line elements in terms of the tangent vectors $\hat{\boldsymbol{t}}_1$ and $\hat{\boldsymbol{t}}_2$. However, we can write the surface element $\mathrm{d}S$ in the form $\mathrm{d}S = \mathrm{d}l\,\mathrm{d}h$, which shows that $\mathrm{d}S$ also vanishes[3] in the limit $\mathrm{d}h \to 0$, leaving us with

$$\int_l (\boldsymbol{E}_1\cdot \hat{\boldsymbol{t}}_1 + \boldsymbol{E}_2\cdot \hat{\boldsymbol{t}}_2)\,\mathrm{d}l.$$

It follows that at the boundary between the two materials, the $\boldsymbol{E}$ field must obey

$$E_1^{\|} - E_2^{\|} = 0,$$

where $E^{\|}$ denotes the component of the electric field parallel to the boundary between the two materials. Alternatively, in terms of the normal to the surface, we can write the above condition as

$$(\hat{\boldsymbol{n}}_1 \times \boldsymbol{E}_1) - (\hat{\boldsymbol{n}}_2 \times \boldsymbol{E}_2) = 0.$$

This expression is useful, since the normal to the material interface is always well-defined, while the tangent is more ambiguous.

5.6.4 Boundary Condition for the $\boldsymbol{H}$ Field

- We now consider an interface between two materials with $\boldsymbol{H}$ fields $\boldsymbol{H}_1$ and $\boldsymbol{H}_2$, and base our analysis on the Maxwell equation

$$\nabla \times \boldsymbol{H} = \boldsymbol{j} + \frac{\partial D}{\partial t}.$$

In an analogous procedure as for $\boldsymbol{E}$, we introduce a small planar surface S in the interface between the two materials with height h perpendicular to the interface and length l, defined by the tangent vectors $\hat{\boldsymbol{t}}_1$ and $\hat{\boldsymbol{t}}_2$, parallel to the interface.

- Using Stokes' theorem, the integral of the above Maxwell equation is

$$\iint_S \boldsymbol{j}\cdot \mathrm{d}\boldsymbol{S} + \frac{\partial}{\partial t}\iint_S \boldsymbol{D}\cdot \mathrm{d}\boldsymbol{S} = \iint_S \nabla\times\boldsymbol{H}\cdot \mathrm{d}\boldsymbol{S} = \oint_{\partial S}\boldsymbol{H}\cdot \mathrm{d}\boldsymbol{s}.$$

[3]The fact that the surface integral vanishes should make intuitive sense; as $\mathrm{d}h \to 0$ the rectangle approaches a line and there is no surface left to integrate over!

- We then perform a geometrically identical procedure as in the analysis of the $\boldsymbol{E}$ field boundary condition, in which we send the loop height to zero, i.e. $\mathrm{d}h \to 0$ and thus $\mathrm{d}S = \mathrm{d}l\,\mathrm{d}h \to 0$. In this case, the $\boldsymbol{D}$-dependent term vanishes, while the line integral over ∂S keeps only the portions over the loop's length, leaving us with

$$\iint_S \boldsymbol{j} \cdot \mathrm{d}\boldsymbol{S} = \iint_S \boldsymbol{j} \cdot \hat{\boldsymbol{n}}_S \, \mathrm{d}S = \int (\boldsymbol{H} \cdot \hat{\boldsymbol{t}}_1 + \boldsymbol{H}_2 \cdot \hat{\boldsymbol{t}}_2) \, \mathrm{d}l.$$

- Note that we have kept the surface integral containing $\boldsymbol{j}$. The integrand contains $\boldsymbol{j} \cdot \hat{\boldsymbol{n}}_S$, i.e. the component of $\boldsymbol{j}$ parallel to the normal of the Stokes' theorem-derived integration surface $\hat{\boldsymbol{S}}$. Keep in mind that $\hat{\boldsymbol{n}}_S$ is normal to the vectors $\hat{\boldsymbol{n}}_1$ and $\hat{\boldsymbol{n}}_2$, which are the normal vectors to the interface between the two materials.

  As $\mathrm{d}h \to 0$, the current density $\boldsymbol{j}$ aligns with $\hat{\boldsymbol{n}}_S$, and represents a "surface current density" with units $\mathrm{A\,m^{-1}}$, which we will denote $\boldsymbol{k}$; the direction of $\boldsymbol{k}$ is determined by the right hand rule in the direction of the tangent vectors $\hat{\boldsymbol{t}}_1$ and $\hat{\boldsymbol{t}}_1$ round the loop. In terms of $\boldsymbol{k}$ in the limit $\mathrm{d}h \to 0$, the integral of $\boldsymbol{j}$ over the surface becomes

$$\iint_S \boldsymbol{j} \cdot \hat{\boldsymbol{n}}_S \, \mathrm{d}S \to \int \boldsymbol{k} \cdot \mathrm{d}\boldsymbol{l},$$

  which produces the relationship

$$\int \boldsymbol{k} \cdot \mathrm{d}\boldsymbol{l} = \int (\boldsymbol{H} \cdot \hat{\boldsymbol{t}}_1 + \boldsymbol{H}_2 \cdot \hat{\boldsymbol{t}}_2) \, \mathrm{d}l.$$

  Thus, at the interface between the two materials, the $\boldsymbol{H}$ field must obey the boundary condition

$$H_1^{\|} - H_1^{\|} = k,$$

  where k is the magnitude of the surface current density. Alternatively, in terms of the normal vectors to the materials, the boundary condition reads

$$(\hat{\boldsymbol{n}}_1 \times \boldsymbol{H}_1) - (\hat{\boldsymbol{n}}_2 \times \boldsymbol{H}_2) = \boldsymbol{k}$$

  where $\hat{\boldsymbol{k}} = k\hat{\boldsymbol{n}}_S$.

6 Frequency Dependence of the Dielectric Function

As mentioned in the previous chapter, the dielectric function is generally a function of frequency, $\epsilon = \epsilon(\omega)$. As a side note, although it is really a subject of optics, we mention that the dielectric function and index of refraction in matter are related according to

$$\epsilon = \epsilon(\omega) \qquad \text{and} \qquad \epsilon(\omega) = n^2(\omega)$$

6.1 Frequency-Dependent Dielectric Function

- So far, we have analyzed electromagnetism exclusively in the time domain. To analyze the dielectric fucntion's frequency dependence, it is naturally more convient to work in the frequency domain—we transform between time and frequency space with the Fourier transform.
- We begin by assuming a linear constitutive relation, which gives us the time-domain relationship
$$\boldsymbol{D}(\boldsymbol{r}, t) = \epsilon(t)\epsilon_0\boldsymbol{E}(\boldsymbol{r}, t).$$
We then Fourier-transform this relationship to the frequency domain to get
$$\boldsymbol{D}(\boldsymbol{r}, \omega) = \epsilon(\omega)\epsilon_0\boldsymbol{E}(\boldsymbol{r}, \omega).$$
Note that, under the Fourier transform, $\epsilon(\omega)$ is a complex number, and the real and imaginary components both have important implications for the behavior of electromagnetic waves in matter.
- The real component $\operatorname{Re}(\epsilon)$ is associated with reflection and refraction and encodes the classical index of refraction, as in the context of the law of refraction. Meanwhile, the imaginary component $\operatorname{Im}(\epsilon)$ encodes the absorption of electromagnetic waves in the material—a large imaginary component corresponds to high absorption.

6.2 Kramers-Kronig Relations

- The Kramers-Kronig relations relate the real and imaginary components of ϵ. In other words, if one knows one component of ϵ, one can then use the Kramers-Kronig relations to find the other component.
- Without derivation, the Kramers-Kronig relations read:
$$\begin{aligned}\operatorname{Re}(\epsilon) &= 1 + \frac{2}{\pi}\mathcal{P}\int_0^\infty \frac{\omega' \operatorname{Im}\big[\ln(\epsilon(\omega'))\big]}{\omega'^2 - \omega^2}\,\mathrm{d}\omega' \\ \operatorname{Im}(\epsilon) &= -\frac{2\omega}{\pi}\mathcal{P}\int_0^\infty \frac{\omega' \operatorname{Re}\big[\ln(\epsilon(\omega'))\big] - 1}{\omega'^2 - \omega^2}\,\mathrm{d}\omega'.\end{aligned}$$
Note that the real and imaginary components of ϵ are coupled, and that each component is given in terms of an integral of the other component.
The symbol $\mathcal{P}$ denotes the integral's Cauchy principle value and is defined as
$$\mathcal{P}\int_{-\infty}^\infty \frac{g(\omega')}{\omega' - \omega}\,\mathrm{d}\omega' = \lim_{\epsilon\to 0}\left[\int_{-\infty}^{\omega-\epsilon} \frac{g(\omega')}{\omega' - \omega}\,\mathrm{d}\omega' + \int_{\omega+\epsilon}^\infty \frac{g(\omega')}{\omega' - \omega}\,\mathrm{d}\omega'\right].$$
Put simply, the principle value is a way to avoid the pole in the left-hand integral at $\omega' = \omega$. We note in passing that the Kramers-Kronig relations are derived from the definition of the Cauchy principle value, followed by Hilbert transforms, followed by the Plemelj equations, but this is beyond the scope of this course.

6.3 Dissipation of Electromagnetic Energy and the Imaginary Dielectric Component

- As long as a material has nonzero $\mathrm{Im}(\epsilon)$, then electromagnetic energy dissipates when passing through the material.
- We begin our analysis by writing the complex number ϵ in terms of amplitude and phase, in which case the relationship $\boldsymbol{D} = \epsilon\epsilon_0\boldsymbol{E}$ implies that a nonzero imaginary component $\mathrm{Im}\,\epsilon \neq 0$ results in the $\boldsymbol{E}$ and $\boldsymbol{D}$ fields falling out of phase in matter.
- We will consider only the electric component of electromagnetic energy density. Neglecting the magnetic component is generally acceptable because ϵ tends to vary with frequency across multiple order of magnitude, while μ tends to be of order one. As a result, only ϵ and thus electric energy vary appreciably with frequency.
- Assuming a linear constitutive relation, the EM power density in material is

  $$\frac{\partial w}{\partial t} = \boldsymbol{E}\frac{\partial \boldsymbol{D}}{\partial t}.$$

  We then integrate the above expression to get the change in energy over time:

  $$w(2) - w(1) = \int_{t_1}^{t_2} \frac{\partial w}{\partial t}\,\mathrm{d}t = \int_{t_1}^{t_2} \boldsymbol{E}\frac{\partial \boldsymbol{D}}{\partial t}\,\mathrm{d}t.$$

- Next, we Fourier-transform $\boldsymbol{E}$ and $\boldsymbol{D}$ to the frequency domains to get

  $$\boldsymbol{E}(t) = \frac{1}{2\pi}\int \boldsymbol{E}(\omega)e^{-i\omega t}\,\mathrm{d}\omega \qquad \text{and} \qquad \boldsymbol{D}(t) = \frac{1}{2\pi}\int \boldsymbol{D}(\omega)e^{-i\omega t}\,\mathrm{d}\omega.$$

  We then substitute these transformations into the energy difference and evaluating the derivative of $\boldsymbol{D}$, which produces

  $$w(2) - w(1) = \frac{1}{(2\pi)^2}\int_{t_1}^{t_2}\left[\int \boldsymbol{E}(\omega)e^{-i\omega t}\,\mathrm{d}\omega \cdot \int -i\omega'\boldsymbol{D}(\omega')e^{-i\omega' t}\,\mathrm{d}\omega'\right]\mathrm{d}t.$$

  Next, we substitute in the linear constitutive relation

  $$\boldsymbol{D}(\omega') = \epsilon(\omega')\epsilon_0\boldsymbol{E}(\omega'),$$

  which leads to

  $$w(2) - w(1) = \frac{\epsilon_0}{(2\pi)^2}\int \boldsymbol{E}(\omega)\,\mathrm{d}\omega \int (-i\omega')\epsilon(\omega')\boldsymbol{E}(\omega')\,\mathrm{d}\omega' \int_{t_1}^{t_2} e^{-i(\omega+\omega')t}\,\mathrm{d}t.$$

- Next, we send the integration limits of the time integral to $t_{1,2} \to \pm\infty$, in which case the time integral becomes the delta function $2\pi\delta(\omega + \omega')$, leaving

  $$w(2) - w(1) = \frac{\epsilon_0}{2\pi}\int \boldsymbol{E}(\omega)\,\mathrm{d}\omega \int (-i\omega')\epsilon(\omega')\boldsymbol{E}(\omega')\,\mathrm{d}\omega'\delta(\omega + \omega').$$

  Sketched derivation from here forward: the presence of the delta function means the frequency integrals give non-zerp contributions only when $\omega = \omega'$. Combining this fact with the complex number identity $i\epsilon(-\omega) - i\epsilon(\omega) = 2\,\mathrm{Im}\,\epsilon(\omega)$ leads to

  $$w(2) - w(1) = \frac{\epsilon_0}{2\pi}\int \omega\,\mathrm{Im}\,\epsilon(\omega)\big|\boldsymbol{E}(\omega)\big|^2\,\mathrm{d}\omega.$$

Finally, we integrate over volume to get electric energy from energy density, and write the $\boldsymbol{E}$ field's position dependence explicitly to get

$$W(2) - W(1) = \frac{\epsilon_0}{2\pi} \int \omega \operatorname{Im} \epsilon(\omega) \, \mathrm{d}\omega \iiint_V \left| \boldsymbol{E}(\boldsymbol{r}, \omega) \right|^2 \mathrm{d}^3 \boldsymbol{r}.$$

This is the desired end result for the change in electromagnetic energy as it passes through material over the course of two distance times $t_{1,2} \to \pm\infty$.

Interpretation: if $\operatorname{Im}(\epsilon) \neq 0$, then electromagnetic energy in a material dissipates with time, and the dissipation is proportional to $\operatorname{Im}(\epsilon)$.

6.4 Basic Models for the Dielectric Constant's Frequency Dependence

6.4.1 Equation of Motion for Bound Charge

- We begin with a simple, classical equation of motion for a bound charge q, which reads
$$m \frac{\mathrm{d}^2 \boldsymbol{r}}{\mathrm{d}t^2} = -m\omega_0^2 \boldsymbol{r} - m\gamma\dot{\boldsymbol{r}} + q\boldsymbol{E}(t),$$
where $\boldsymbol{r}$ is the charge's position. The charge is bound within the dielectric's crystal lattice by a harmonic potential encoded by the harmonic frequency ω_0, and the second term encodes a velocity-dependent dissipative force.

- We then take the Fourier transform of the equation of motion. The time derivatives in Fourier space simplify to multiplication by $i\omega$, and the result is
$$\begin{aligned} -m\omega^2 \boldsymbol{r}(\omega) &= -m\omega_0^2 \boldsymbol{r}(\omega) + i\omega\gamma \boldsymbol{r}(\omega) + q\boldsymbol{E}(\omega), \\ \boldsymbol{r}(\omega) &= \frac{q}{m} \frac{\boldsymbol{E}(\omega)}{(\omega_0^2 - \omega^2) - i\gamma\omega}. \end{aligned}$$
Note that this function takes the shape of a resonance curve.

- We now want to derive expression ϵ, and begin by introducing polarization
$$\boldsymbol{P}(\omega) = nq\boldsymbol{r}(\omega),$$
where n is the volume charge density of electric dipoles contributing to the polarization. We then substitute the earlier expression for $\boldsymbol{r}(\omega)$ into the expression for polarization, which gives
$$\boldsymbol{P}(\omega) = \frac{q^2 n}{m} \frac{\boldsymbol{E}(\omega)}{\omega_0^2 - \omega^2 - i\gamma\omega}.$$
Finally, we use $\boldsymbol{P}(\omega) = \epsilon_0 \big[\epsilon(\omega) - 1\big] \boldsymbol{E}(\omega)$ to get the desired relationship for ϵ,
$$\epsilon_0 \big[\epsilon(\omega) - 1\big] = \frac{q^2 n}{m} \frac{1}{\omega_0^2 - \omega^2 - i\gamma\omega}.$$
Again, we note that $\epsilon(\omega)$ has resonance behavior. The resonance frequency ω_0 is determined by the nature of the harmonic potential binding the bound charges.

- We often work with the above expression for ϵ in one of three limit cases, which are called:

- Debye relaxation,
- Lorentz relaxation,
- and plasma relaxation.

Since electromagnetic frequencies vary over many orders of magnitude, it makes sense that we might have different dissipative and binding mechanism at different frequencies. We note also that real materials, we often have multiple source of harmonic-like potentials, which results in multiple eigenfrequencies ω_{0_i} and multiple modes of dissipation (multiple γ_i).

6.4.2 Debye Relaxation

- Debye relaxation is relevant at low frequencies, e.g. in the range $\omega \sim 10^7\,\mathrm{Hz}$ to $10^9\,\mathrm{Hz}$, and is commonly used to model the relaxation of electric dipoles in matter.

  At low frequencies we can neglect the term proportional to ω^2, which case the expression for polarization simplifies to

$$\boldsymbol{P}(\omega) = \frac{q^2 n}{m\omega_0^2}\frac{\boldsymbol{E}(\omega)}{1 - i\omega\tau},$$

  where the Debye relaxation time τ is defined as $\tau = \frac{\gamma}{\omega_0}$.

- Using this expression for polarization, the dielectric constant in the Debye regime is

$$\epsilon(\omega) = 1 + \frac{(\epsilon(0) - 1)(1 + i\omega\tau)}{1 + \omega^2\tau^2},$$

  where $\epsilon(0)$ is the dielectric constant at $\omega = 0$ and we have assumed the relationship

$$\epsilon_0\big[\epsilon(0) - 1\big] = \frac{q^2 n}{m\omega_0^2}.$$

  The real component of ϵ decays to 0 at large frequencies in a sort of half-sigmoid curve. The imaginary component grows to a maximum and then falls to zero at large frequencies.

6.4.3 Lorentz Relaxation

- The Lorentz regime is not an approximation at all—it accounts for all frequency terms in the general expression for $\epsilon(\omega)$. Lorentz relaxation applies to frequencies in the range $10^{12}\,\mathrm{Hz}$ to $10^{16}\,\mathrm{Hz}$, which includes the range of visible light, and is used, for example, to model the oscillation of molecules.

- In the regime of Lorentz relaxation, polarization retains the general expression

$$\boldsymbol{P}(\omega) = \frac{q^2 n}{m}\frac{\boldsymbol{E}(\omega)}{(\omega_0^2 - \omega^2) - i\gamma\omega},$$

  while the corresponding dielectric function is given by the relationship

$$\epsilon_0\big[\epsilon(\omega) - 1\big] = \frac{q^2 n}{m}\frac{1}{\omega_0^2 - \omega^2 - i\gamma\omega}.$$

In terms of the dielectric constant at $\omega = 0$, the dielectric function reads

$$\epsilon(\omega) = 1 + \frac{\big[\epsilon(0) - 1\big]\omega_0^2}{\omega_0^2 - \omega^2 - i\gamma\omega}.$$

The real component has two extrema and a node at the resonance frequency; in general form, it resembles the first excited quantum state of a finite potential well, although the functional dependence is of course different.

The imaginary component has a single extrema at the resonance frequency, and decays to zero at $\omega \to \pm\infty$.

6.4.4 Plasma Relaxation

- Plasma relaxation applies to very high frequencies, where we can neglect all ω-dependent terms in the equation of motion except the accelerating electric force.
- In the plasma regime, the polarization reads

$$\boldsymbol{P}(\omega) = -\frac{q^2 n}{m}\frac{\boldsymbol{E}(\omega)}{\omega^2},$$

and the corresponding dielectric constant is

$$\epsilon(\omega) = 1 - \frac{\omega_\mathrm{p}^2}{\omega^2} \qquad \text{where} \qquad \omega_\mathrm{p} = \frac{nq^2}{m\epsilon_0}.$$

- The plasma model corresponds to electrons behaving as free particles in material, and begins to apply at frequencies approximately greater than 10^{16} Hz.

6.4.5 Example: Dielectric Function in Water

- As mentioned in the introduction to this section, real materials have multiple dissipative and binding mechanics, and thus multiple eigenfrequencies ω_{0_i} that contribute differently at different electric field frequencies. The resulting dielectric function is quite complex, as seen in the following example of water.
- We model water with a single Debye relaxation term for low electric field frequencies and fully eleven Lorentz relaxation terms to encode various binding mechanisms at larger frequencies. The result is the phenomenolgical relationship

$$\epsilon(i\omega) = 1 + \frac{d}{1 + \omega\tau} + \sum_{i=1}^{11} \frac{f_i}{\omega_{0_i}^2 + g_i\omega + \omega^2},$$

where d, τ, ω_{0_i}, g_i, and f_i are phenomenologically-determined constants found with measurements and fitting.

The first term uses the Debye model and the further 11 terms use the Lorentz model—note that we need 12 different modes in total to describe the frequency dependence of water's dielectric function.

7 Hamiltonian Formalism for the Electromagnetic Field

7.1 Review of the Hamiltonian Formalism

Before analyzing the electromagnetic field, we briefly review the Hamiltonian formalism from classical mechanics. Note, however, that this is only a brief review—it is assumed the reader is already familiar with the Hamiltonian formalism.

7.1.1 The Lagrange Equations

- In a few sentences, we summarize the Lagrangian formalism by considering a system of particles, introducing a Lagrangian function L describing the system's kinetic and potential energy; and then defining action S as the integral of the Lagrange function over time. The minimum of the action, via the least action principle, then encodes the system's physical state.

- As a concrete example, the Lagrangian function for a point particle might read

  $$L = \frac{1}{2}m\dot{\boldsymbol{r}}^2 - V(\boldsymbol{r}),$$

  and the corresponding action is for the particle moving along the trajectory $\boldsymbol{r}(t)$ with velocity $\dot{\boldsymbol{r}}(t)$ is

  $$S = \int L(\boldsymbol{r}(t), \dot{\boldsymbol{r}}(t), t)\,\mathrm{d}t.$$

  We then vary the action and require $\delta S = 0$. The result is the Euler-Lagrange equations

  $$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\boldsymbol{r}}}\right) - \frac{\partial L}{\partial \boldsymbol{r}} = 0,$$

  which are equivalent to Newton's second law but often more convenient for analyzing systems in classical mechanics.

7.1.2 The Hamilton Equations

- In the Hamiltonian formalism we introduce the generalized momenta

  $$\boldsymbol{p} = \frac{\partial L}{\partial \dot{\boldsymbol{r}}}$$

  and the Hamiltonian function

  $$H(\boldsymbol{r}(t), \boldsymbol{p}(t), t) = \dot{\boldsymbol{r}}\boldsymbol{p} - L(\boldsymbol{r}, \dot{\boldsymbol{r}}, t).$$

- For a single particle, the Hamiltonian typically reads

  $$H = \frac{\boldsymbol{p}^2}{2m} + V(\boldsymbol{r}).$$

- The equations of motion in the Hamiltonian formalism are

  $$\dot{\boldsymbol{r}}(t) = \frac{\partial H}{\partial \boldsymbol{p}} \qquad \text{and} \qquad \dot{\boldsymbol{p}}(t) = -\frac{\partial H}{\partial \boldsymbol{r}}.$$

7.2 Lagrangian Function of a Charged Particle in an EM Field

- We consider a particle of charge q in an external electric field $\boldsymbol{E}(\boldsymbol{r}, t)$ and magnetic field $\boldsymbol{B}(\boldsymbol{r}, t)$. The particle expriences the Lorentz force

  $$\boldsymbol{F} = q(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}).$$

  We then substitute the Lorentz force to get

  $$m\ddot{\boldsymbol{r}} = q\boldsymbol{E} + q\boldsymbol{v} \times \boldsymbol{B} = -q\nabla\phi - q\frac{\partial \boldsymbol{A}}{\partial t} + q\boldsymbol{v} \times \big(\nabla \times \boldsymbol{A}\big),$$

  where we stress the need to write the electric field in terms of both ϕ and $\boldsymbol{A}$.

- Next, we apply the curl vector identity

  $$\boldsymbol{v} \times \big(\nabla \times \boldsymbol{A}\big) = \nabla(\boldsymbol{v} \cdot \boldsymbol{A}) - (\boldsymbol{v} \cdot \nabla)\boldsymbol{A},$$

  in terms of which Newton's second law becomes

  $$m\ddot{\boldsymbol{r}} = -q\nabla\phi - q\frac{\partial \boldsymbol{A}}{\partial t} + q\nabla(\boldsymbol{v} \cdot \boldsymbol{A}) - q(\boldsymbol{v} \cdot \nabla)\boldsymbol{A}.$$

  Finally, we join the time and directional derivative of $\boldsymbol{A}$ into a material derivative, i.e. $q\frac{\partial \boldsymbol{A}}{\partial t} + q(\boldsymbol{v} \cdot \nabla)\boldsymbol{A} = q\frac{\mathrm{d}\boldsymbol{A}}{\mathrm{d}t}$, which simplifies the above equation to

  $$m\ddot{\boldsymbol{r}} = -q\nabla\phi + q\nabla(\boldsymbol{v} \cdot \boldsymbol{A}) - q\frac{\mathrm{d}\boldsymbol{A}}{\mathrm{d}t}.$$

- We will now rewrite the above equation of motion into a form matching the Euler-Lagrange equation, which will reveal the Lagrange function. The first step is

  $$\frac{\mathrm{d}}{\mathrm{d}t}\left(m\dot{\boldsymbol{r}} + q\boldsymbol{A}\right) = -\nabla(q\phi - q\boldsymbol{v} \cdot \boldsymbol{A}) \equiv -\frac{\partial}{\partial \boldsymbol{r}}(q\phi - q\dot{\boldsymbol{r}} \cdot \boldsymbol{A}).$$

  We then further reverse-engineer the time derivative on the left-hand side to get

  $$\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\partial}{\partial \dot{\boldsymbol{r}}}\left(\frac{1}{2}m\dot{\boldsymbol{r}}^2 + q\boldsymbol{A} \cdot \dot{\boldsymbol{r}}\right)\right] = -\frac{\partial}{\partial \boldsymbol{r}}(q\phi - q\dot{\boldsymbol{r}} \cdot \boldsymbol{A}).$$

  Comparing this expression to the Lagrange equation $\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\boldsymbol{r}}}\right) = \frac{\partial L}{\partial \boldsymbol{r}}$ motivates the following definition for the Lagrangian of a particle in an EM field:

  $$L(\boldsymbol{r}(t), \dot{\boldsymbol{r}}(t), t) \equiv \frac{1}{2}m\dot{\boldsymbol{r}}^2 - q\phi(\boldsymbol{r}(t)) + q\dot{\boldsymbol{r}} \cdot \boldsymbol{A}(\boldsymbol{r}(t), t).$$

  Note that even though the $q\phi$ term doesn't explicitly appear in the left-hand side of the Lagrange equation and the $\frac{1}{2}m\dot{\boldsymbol{r}}^2$ term doesn't explicitly appear in the right-hand side, these terms would vanish under the derivatives $\frac{\partial}{\partial \dot{\boldsymbol{r}}}$ and $\frac{\partial}{\partial \boldsymbol{r}}$, respectively, which ends up satisfying the Lagrange equations.

  Interpretation: The Lagrangian L has three contributions: kinetic energy, coupling of q and ϕ, and coupling of q, $\dot{\boldsymbol{r}}$ and $\boldsymbol{A}$. Note also that the Lagrangian is written in terms of field potentials and not the fields themselves.

7.3 Hamiltonian Function for a Charged Particle in an EM Field

- We find the Hamiltonian H for a charged particle in an EM field using the just-derived Lagrangian L with the general relationship

$$H(\boldsymbol{p}, \boldsymbol{r}, t) = \dot{\boldsymbol{r}}\dot{\boldsymbol{p}} - L.$$

First, the appropriate generalized momentum $\boldsymbol{p}$ is

$$\boldsymbol{p} = \frac{\partial L}{\partial \dot{\boldsymbol{r}}} = m\dot{\boldsymbol{r}} + q\boldsymbol{A},$$

while the corresponding time derivative of position is

$$\dot{\boldsymbol{r}} = \frac{\boldsymbol{p} - q\boldsymbol{A}}{m}.$$

Note that the quantity $q\boldsymbol{A}$ corresponds to momentum in the Hamiltonian formalism.

- In terms of $\boldsymbol{p}$, $\dot{\boldsymbol{r}}$ and L, the charged particle's Hamiltonian is then

$$H = \frac{1}{m}(\boldsymbol{p} - q\boldsymbol{A})\boldsymbol{p} - \frac{1}{2m}(\boldsymbol{p} - q\boldsymbol{A})^2 - q\boldsymbol{A}\left(\frac{\boldsymbol{p} - q\boldsymbol{A}}{m}\right) + q\phi.$$

Finally, we combine like terms to get

$$H = \frac{1}{2m}(\boldsymbol{p}(t) - q\boldsymbol{A}(\boldsymbol{r}, t))^2 + q\phi(\boldsymbol{r}, t).$$

Note that the $\boldsymbol{p} = m\dot{\boldsymbol{r}} + q\boldsymbol{A}$ is often called canonical momentum while $m\dot{\boldsymbol{r}} = \boldsymbol{p} - q\boldsymbol{A}$ is called kinetic momentum. Keep in mind that despite all the complicated analysis up to now the Lagrange and Hamiltonian functions fundamentally arise from Newton's second law and the Lorentz force on a charged particle in an electromagnetic field.

7.4 Hamiltonian Formalism for a Continuous Charge Distribution

7.4.1 Schwarzschild Invariant

- Up to now, we had consider only a point particle; we now aim to find the Lagrangian for a continuous distribution of charged particles in an external electromagnetic field, which we will describe in terms of the potentials ϕ and $\boldsymbol{A}$.

- Recall that for a single a particle, the Lagrangian reads

$$L = \frac{1}{2}m\dot{\boldsymbol{r}}^2 - q\phi(\boldsymbol{r}, t) + q\dot{\boldsymbol{r}} \cdot \boldsymbol{A}(\boldsymbol{r}, t) \equiv \frac{1}{2}m\dot{\boldsymbol{r}}^2 + L_{\text{ext}},$$

where $L_{\text{ext}} = -q\phi(\boldsymbol{r}, t) + q\dot{\boldsymbol{r}} \cdot \boldsymbol{A}(\boldsymbol{r}, t)$ encodes the coupling of the particle to the external electromagnetic field.

We then generalize the field-coupled Lagrangian to a charge distribution $\rho(\boldsymbol{r}, t)$ via

$$L_{\text{ext}} = -\iiint_V \rho(\boldsymbol{r}, t)\phi(\boldsymbol{r}, t)\, \mathrm{d}^3\boldsymbol{r} + \iiint_V \boldsymbol{j}(\boldsymbol{r}, t) \cdot \boldsymbol{A}(\boldsymbol{r}, t)\, \mathrm{d}^3\boldsymbol{r},$$

where we stress that L_{ext} encodes only the external EM field-coupled portion of the total Lagrangian L (and leaves out the kinetic energy term $\frac{1}{2}m\dot{\boldsymbol{r}}^2$).

Finally, we introduce the Lagrangian density $\mathcal{L}_{\text{ext}}$, also called the Schwarzschild invariant, via

$$\mathcal{L}_{\text{ext}} = -\rho(\boldsymbol{r}, t)\phi(\boldsymbol{r}, t) + \boldsymbol{j}(\boldsymbol{r}, t) \cdot \boldsymbol{A}(\boldsymbol{r}, t).$$

Note that $\mathcal{L}_{\text{ext}}$ encodes coupling with an *external* field—it does not account for electromagnetic field of the charged particle itself (analogous to the considerations of external versus total electromagnetic field energy in the chapters on electrostatics and magnetostatics).

7.4.2 Lagrangian Function for the Total EM Field

- We now aim to derive a Lagrangian encoding the "total" field—both the external field and the field arising from the charge distribution itself. Interpretation: we consider the charge distribution as a field source in itself, and then allow for the distribution to occur in an additional external field.

  Motivatived by this separation in to internal and external field terms, we write the total electromagnetic Lagrangian in the form

  $$L_{\text{EM}} = \iiint_V \mathcal{L}_{\text{EM}}(\boldsymbol{r}, t)\,\mathrm{d}^3\boldsymbol{r} = \iiint_V \mathcal{L}_{\text{int}}(\boldsymbol{r}, t)\,\mathrm{d}^3\boldsymbol{r} + \iiint_V \mathcal{L}_{\text{ext}}(\boldsymbol{r}, t)\,\mathrm{d}^3\boldsymbol{r},$$

  where $\mathcal{L}_{\text{DP}} = -\rho(\boldsymbol{r}, t)\phi(\boldsymbol{r}, t) + \boldsymbol{j}(\boldsymbol{r}, t) \cdot \boldsymbol{A}(\boldsymbol{r}, t)$, derived in the previous section, represents the charge distribution's coupling to the external EM field, while $\mathcal{L}_{\text{int}}$ (internal), which we derive in this section, represents the contribution of the field generated by the charge distribution itself. Without derivation, the correct expression for $\mathcal{L}_{\text{int}}$ is

  $$\mathcal{L}_{\text{P}}(\boldsymbol{r}, t) = \frac{1}{2}\epsilon_0 E^2(\boldsymbol{r}, t) - \frac{1}{2\mu_0} B^2(\boldsymbol{r}, t).$$

  We note, again without proof, that precisely this choice of $\mathcal{L}_{\text{int}}$ leads to the correct form of Maxwell's equations. Note also that the above expression is precisely electromagnetic energy density contained in a field. In other words, $\mathcal{L}_{\text{int}}$ qualitatively represents the electromagnetic energy density of the charge distribution's internal field, which is an intuitively appropriate quantity for the $\mathcal{L}_{\text{int}}$ term.

- In terms of $\mathcal{L}_{\text{int}}$ and $\mathcal{L}_{\text{ext}}$, the total electromagnetic Lagrangian density $\mathcal{L}_{\text{EM}}$ is then

  $$\mathcal{L}(\boldsymbol{r}, t) = \frac{1}{2}\epsilon_0 E^2(\boldsymbol{r}, t) - \frac{1}{2\mu_0} B^2(\boldsymbol{r}, t) - \rho(\boldsymbol{r}, t)\phi(\boldsymbol{r}, t) + \boldsymbol{j}(\boldsymbol{r}, t) \cdot \boldsymbol{A}(\boldsymbol{r}, t),$$

  where $\boldsymbol{E}$ and $\boldsymbol{B}$ are defined in terms of the funamental potentials by the relationships $\boldsymbol{E} = -\nabla\phi - \frac{\partial \boldsymbol{A}}{\partial t}$ and $\boldsymbol{B} = \nabla \times \boldsymbol{A}$.

7.4.3 Euler-Lagrange and Riemann-Lorenz Equations

- The action S associated with the above total electromagnetic Lagrangian is

  $$S \equiv \int L(\boldsymbol{r}, t)\,\mathrm{d}t = \int \left[\iiint_V \mathcal{L}_{\text{EM}}(\phi(\boldsymbol{r}, t), \boldsymbol{A}(\boldsymbol{r}, t))\,\mathrm{d}^3\boldsymbol{r}\right]\mathrm{d}t.$$

The Euler-Lagrange equations for the above action read

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\partial \mathcal{L}}{\partial\left(\frac{\partial \phi}{\partial t}\right)}\right] + \nabla\left[\frac{\partial \mathcal{L}}{\partial(\nabla\phi)}\right] - \frac{\partial \mathcal{L}}{\partial \phi} = 0$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\partial \mathcal{L}}{\partial\left(\frac{\partial A_i}{\partial t}\right)}\right] + \nabla\left[\frac{\partial \mathcal{L}}{\partial(\nabla A_i)}\right] - \frac{\partial \mathcal{L}}{\partial A_i} = 0,$$

where we have written $\mathcal{L}_{\mathrm{EM}} \to \mathcal{L}$ for conciseness.

- Without proof, the above Euler-Lagrange equations lead to the Riemann-Lorenz equations

$$\nabla^2\phi - \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} = -\frac{\rho}{\epsilon_0} \qquad \text{and} \qquad \nabla^2 A_i - \frac{1}{c^2}\frac{\partial^2 A_i}{\partial t^2} = -\mu_0 j_i.$$

These two equations encode the fully time-dependent form of the Maxwell equations in terms of the fundamental field potentials ϕ and $\boldsymbol{A}$. In passing, we note that the equations are derived using the Lorenz gauge.

The Riemann-Lorenz equations conclude the generalization of the Lagrange-Hamiltonian formalism from a single charged particle to a continuous charge distribution.

8 Introduction to Special Relativity

8.1 EM Fields and the Lorentz Transformation

- We consider two systems S and S', where S' moves at near-light speed $v \lesssim c$ in the x' direction relative to S. System S contains the fields $\boldsymbol{E}$ and $\boldsymbol{B}$ and has the coordinates (x, y, z, t), while system S' contains the fields $\boldsymbol{E}'$ and $\boldsymbol{B}'$ and has coordinates (x', y', z', t').

- In the theory of special relativity, we transform between the systems S and S' using the Lorentz transformations, which read

$$\begin{aligned} x' &= \gamma(x - \beta ct) \qquad & y' &= y \\ ct' &= \gamma(ct - \beta x) \qquad & z' &= z \end{aligned}$$

  where $\beta = v/c$ and $\gamma = \left(1 - \beta^2\right)^{-1/2}$.

- In this section, we aim to find the transformations for $\boldsymbol{E}$ and $\boldsymbol{B}$ from S to S' corresponding to the above Lorentz transformations. The transformation rests on the following condition:

  > We assume the Maxwell equations for $\boldsymbol{E}$ and $\boldsymbol{B}$ in S have the same form as the Maxwell equations for $\boldsymbol{E}'$ and $\boldsymbol{B}'$ in S'.

  We note that although this assumption makes intuitive sense—and it indeed leads to a correct result—a it is formally not a trivial assumption, and we do not discuss the theoretical framework for making it.

- For simplicity, we make another assumption: we neglect (or assume an absence of) electromagnetic field sources in the frame S', which allows use to write the first two Maxwell equations as

$$\nabla' \cdot \boldsymbol{E}' = 0 \qquad \text{and} \qquad \nabla' \cdot \boldsymbol{B}' = 0.$$

  The remaining two Maxwell equations are

$$\nabla' \times \boldsymbol{E}' = -\frac{\partial \boldsymbol{B}'}{\partial t'} \qquad \text{and} \qquad \nabla' \times \boldsymbol{B}' = \frac{1}{c^2}\frac{\partial \boldsymbol{E}'}{\partial t'},$$

  where we have used $\epsilon_0 \mu_0 = 1/c^2$.

- As an intermediate step, we write the Maxwell equations in Cartisian components, starting with the equation for $\nabla \times \boldsymbol{B}'$, which reads

$$\begin{aligned} \frac{\partial E'_{z'}}{\partial y'} - \frac{\partial E'_{y'}}{\partial z'} &= -\frac{\partial B'_{x'}}{\partial t'} \\ \frac{\partial E'_{x'}}{\partial z'} - \frac{\partial E'_{z'}}{\partial x'} &= -\frac{\partial B'_{y'}}{\partial t'} \\ \frac{\partial E'_{y'}}{\partial x'} - \frac{\partial E'_{x'}}{\partial y'} &= -\frac{\partial B'_{z'}}{\partial t'}. \end{aligned}$$

  Meanwhile, the equation for $\nabla' \cdot \boldsymbol{E}'$ reads

$$\frac{\partial E'_{x'}}{\partial x'} + \frac{\partial E'_{y'}}{\partial y'} + \frac{\partial E'_{z'}}{\partial z'} = 0.$$

- According to the Loretnz transformations, the coordinate derivatives transform as

$$\frac{\partial}{\partial x'} = \gamma\left(\frac{\partial}{\partial x} + \beta\frac{\partial}{\partial(ct)}\right) \qquad \text{and} \qquad \frac{\partial}{\partial(ct')} = \gamma\left(\frac{\partial}{\partial(ct)} + \beta\frac{\partial}{\partial x}\right).$$

  Note the presence of plus (and not minus) signs in the transformations, which is related to contravariant vector notation and is beyond the scope of our limited treatment of special relativity. The transformations of the y and z derivatives are simpler:

$$\frac{\partial}{\partial y'} = \frac{\partial}{\partial y} \qquad \text{and} \qquad \frac{\partial}{\partial z'} = \frac{\partial}{\partial z}.$$

  We then apply these transformations to the component-form Maxwell equations. The $\nabla' \times \boldsymbol{B}'$ equation transforms to

$$\begin{aligned}
&\frac{\partial E'_{z'}}{\partial y} - \frac{\partial E'_{y'}}{\partial z} = -\gamma v\frac{\partial B'_{x'}}{\partial x} - \gamma\frac{\partial B'_{x'}}{\partial t}\\
&\frac{\partial E'_{x'}}{\partial z} - \left(\gamma\frac{\partial E'_{z'}}{\partial x} + \gamma\frac{v}{c^2}\frac{\partial E'_{z'}}{\partial t}\right) = -\gamma v\frac{\partial B'_{y'}}{\partial x} - \gamma\frac{\partial B'_{y'}}{\partial t}\\
&\left(\gamma\frac{\partial E'_{y'}}{\partial x} + \gamma\frac{v}{c^2}\frac{\partial E'_{y'}}{\partial t}\right) - \frac{\partial E'_{x'}}{\partial y} = -\gamma v\frac{\partial B'_{z'}}{\partial x} - \gamma\frac{\partial B'_{z'}}{\partial t},
\end{aligned}$$

  while divergence equation $\nabla' \cdot \boldsymbol{E}' = 0$ becomes

$$\gamma\frac{\partial E'_{x'}}{\partial x} + \gamma\frac{v}{c^2}\frac{\partial E'_{x'}}{\partial t} + \frac{\partial E'_{y'}}{\partial y} + \frac{\partial E'_{z'}}{\partial t} = 0.$$

- We would then perform analogous transformations of the equations $\nabla' \cdot \boldsymbol{B}' = 0$ and $\nabla' \times \boldsymbol{E}' = -\frac{\partial B'}{\partial t'}$, which we leave out for lack of time. The end result is a system of equations relating the components of the electromagnetic field.

  The next question is: what combination of primed field quantities (i.e. which combination of components $E'_{x,y,z}$ and $B'_{x,y,z}$) should we take so that, when substituted into the above conditions, we recover the Maxwell equations in terms of the S frame field quantities $\boldsymbol{E}$ and $\boldsymbol{B}$.

  Without proof, it turns out the correct transformations are

$$\begin{aligned}
E_x &= E'_{x'} & B_x &= B'_{x'}\\
E_y &= \gamma(E'_{y'} + vB'_{z'}) & B_y &= \gamma\left(B'_{y'} - \tfrac{v}{c^2}E'_{z'}\right)\\
E_z &= \gamma(E'_{z'} - vB'_{y'}) & B_z &= \gamma\left(B'_{z'} + \tfrac{v}{c^2}E'_{y'}\right).
\end{aligned}$$

  Note that the x components are preserved and the y and z components are mixed up, which is opposite the transformations of the (x, y, z, t) coordinates themselves.

Implications

- We now consider some implications of the above transformations for the $\boldsymbol{E}$ and $\boldsymbol{B}$ fields. First, we calculate $\boldsymbol{E} \cdot \boldsymbol{B}$, which comes out to

$$\begin{aligned}
\boldsymbol{E} \cdot \boldsymbol{B} &= E_xB_x + E_yB_y + E_zB_z\\
&= E'_{x'}B'_{x'} + \gamma^2\left(1 - \frac{v^2}{c^2}\right)E'_{y'}B'_{y'} + \gamma^2\left(1 - \frac{v^2}{c^2}\right)E'_{z'}B'_{z'}.
\end{aligned}$$

Note that $\gamma^2(1 - \frac{v^2}{c^2}) = 1$ which simplifies the above relationship to

$$\boldsymbol{E} \cdot \boldsymbol{B} = \boldsymbol{E}' \cdot \boldsymbol{B}'$$

In other words, the dot product $\boldsymbol{E} \cdot \boldsymbol{B}$ is Lorentz-invariant (i.e. angles are preserved) with respect to the Lorentz transformations of the electromagnetic field.

- Next, we consider the quantity $E^2 - c^2 B^2$, which is essentially electromagnetic, as long as we multiply through by $\epsilon_0/2$. Writing the quantities E^2 and B^2 in component form and applying the electromagnetic field transformations leads to

$$E^2 - c^2 B^2 = E'^2 - c^2 B'^2,$$

which corresponds to conservation of energy.

- Summary: In free space, the solutions to the Maxwell equations in S' have the same form as in S, which follows from preservation of angles under the invariance relationship $\boldsymbol{E} \cdot \boldsymbol{B} = \boldsymbol{E}' \cdot \boldsymbol{B}'$. The relative magnitudes of the $\boldsymbol{E}'$ and $\boldsymbol{B}'$ fields however, change relative to $\boldsymbol{E}$ and $\boldsymbol{B}$.

8.2 Minkowski Space

- For the remainder of this chapter, we will work in terms of four-vectors in four-dimensional Minkowski space, where we introduce time coordinate ct to the familiar Euclidean position coordinates x, y and z. We will work with the $(+++-)$ metric, which is preserved in special relativity.

- The fundamental element of Minkowski space is the position four-vector, which we write in the form

$$x_\mu = (x, y, z, ct).$$

In our convention, Greek letter indices (e.g. μ) run over all four components, while Latin indices (e.g. i) run over the position components only. The index μ occuring in the subscript denotes that x_μ is a covariant vector. An index above, as in x^μ, represents a contravariant vector and reads

$$x^\mu = (x, y, z, -ct).$$

- Importantly, the Lorentz transformation preserves the square of the four-vector:

$$x_\mu x^\mu = x^2 + y^2 + z^2 - c^2 t^2 = x'_\mu x'^\mu.$$

This expression is a generalization of the Euclidean dot product to four-dimensional Minkowski space—the use of the minus sign before the time coordinate corresponds to the $(+++-)$ metric.

8.2.1 Current Density Four-Vector

- We now consider the generalization of current density to Minkowski space.

We begin by requiring conservation of total charge across all systems, i.e.

$$q = \iiint_V \rho \, \mathrm{d}^3\boldsymbol{r} = \iiint_{V'} \rho' \, \mathrm{d}^3\boldsymbol{r}'.$$

Working in components and applying the Loretnz transformations of the coordinates leads to the expression

$$q = \iiint_V \rho \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z = \iiint_{V'} \rho' \, \mathrm{d}x' \, \mathrm{d}y' \, \mathrm{d}z' = \iiint_V \rho' \gamma \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z,$$

which implies $\rho' = \rho/\gamma$. In other words, the quantity $\frac{\rho}{\gamma}$ is Lorentz-invariant, and not simply ρ.

- We now turn to the current density four-vector. We start with the classical exprssion $\boldsymbol{j} = \rho \boldsymbol{v}$. Using the velocity four-vector u_μ and the just-derived expression $\rho \to \rho/\gamma$, the classical expression generalizes to the four vector

$$j_\mu = \frac{\rho}{\gamma} u_\mu = \frac{\rho}{\gamma} (\gamma \boldsymbol{v}, \gamma c) = \rho(\boldsymbol{v}, c) = (\boldsymbol{j}, c\rho),$$

  Note that neither γ nor β occur in the current density four-vector.

- Since j_μ is independent of γ and β it is a well-defined four vector and obeys the familiar four-vector transformation rules:

$$\begin{aligned} j'_{x'} &= \gamma(j_x - \beta c\rho) \qquad & j'_{y'} &= j_y \\ j'_{z'} &= j_z & c\rho' &= \gamma(c\rho - \beta j_x). \end{aligned}$$

  More so, j_μ is Lorentz invariant, i.e.

$$j_\mu j^\mu = j'_\mu j'^\mu = \boldsymbol{j} \cdot \boldsymbol{j} - c^2 \rho^2.$$

  Interpretation: Like for the electromagnetic field components $\boldsymbol{E}$ and $\boldsymbol{B}$, the relative magnitudes of $\boldsymbol{j}$ and ρ are mixed in the transformation between frames of reference. In other words, the relative composition of the current density four vector in terms of current density $\boldsymbol{j}$ and charge density ρ depends on the system in which the four vector is measured. However, the quantity $\boldsymbol{j} \cdot \boldsymbol{j} - c^2 \rho^2$ is preserved across all systems.

  Note that the two field sources $\boldsymbol{j}$ and ρ are now coupled, and it makes sense to speak of a total electromagnetic field instead of the independent fields $\boldsymbol{E}$ and $\boldsymbol{B}$.

8.2.2 EM Field Four Vectors

- We will now write the four-vector encoding the electromagnetic field potential. First, we recall the Riemann-Lorenz equations for $\boldsymbol{A}$ and ϕ, which we write in terms of the d'Alembert box operator $\Box$ as

$$\begin{aligned} \Box^2 \phi &= \nabla^2 \phi - \frac{1}{c^2} \frac{\partial^2 \phi}{\partial t^2} = -\frac{\rho}{\epsilon_0} \\ \Box^2 \boldsymbol{A} &= \nabla^2 \boldsymbol{A} - \frac{1}{c^2} - \frac{1}{c^2} \frac{\partial^2 \boldsymbol{A}}{\partial t^2} = -\mu_0 \boldsymbol{j}. \end{aligned}$$

  We then combine the above two equations and introduce the current density four vector $j_\mu = (\boldsymbol{j}, c\rho)$ to get the electromagnetic potential four vector:

$$A_\mu = \left(\boldsymbol{A}, \frac{\phi}{c}\right) \qquad \text{and} \qquad A^\mu = \left(\boldsymbol{A}, -\frac{\phi}{c}\right).$$

  In terms of the EM potential four vector, the Riemann-Lorenz equations can be written in the Lorentz-invariant form

$$\Box^2 A_\mu = -\mu_0 j_\mu \qquad \text{and} \qquad \Box^2 A^\mu = -\mu_0 j^\mu,$$

  where $\Box$ again denotes the d'Alembert box operator.

- Since A_μ, like j_μ, is independent of γ and β, it is a well-defined four vector and obeys the usual Lorentz transformations:

$$\begin{aligned} A'_{x'} &= \gamma\left(A_x - \beta\frac{\phi}{c}\right) & \frac{\phi'}{c} &= \gamma\left(\frac{\phi}{c} - \beta A_x\right) \\ A'_{y'} &= A_y & A'_{z'} &= A_z \end{aligned}$$

More so, the EM potential four vector obeys the Lorentz invariance relation

$$A_\mu A^\mu = A'_\mu A'^\mu = \boldsymbol{A}\cdot\boldsymbol{A} - \frac{\phi^2}{c^2}.$$

Analogously to j_μ, the relative magnitudes of the components $\boldsymbol{A}$ and ϕ are mixed in the transformation of A_μ between different frames of reference, while the invariant quantity $\boldsymbol{A}\cdot\boldsymbol{A} - \frac{\phi^2}{c^2}$ is preserved.

8.2.3 Covariant EM Field Tensor

- For the purposes of this course, we take covariant to denote a quantitiy that, like x_μ, j_μ or A_μ, is manifestly invariant under the Lorentz transformations. Our goal in this section is to find a covariant expression for the electromagnetic fields $\boldsymbol{E}$ and $\boldsymbol{B}$.

- We begin with the expressions for $\boldsymbol{E}$ and $\boldsymbol{B}$ in terms of their potentials, i.e.:

$$\boldsymbol{B} = \nabla\times\boldsymbol{A} \qquad \text{and} \qquad \boldsymbol{E} = -\nabla\phi - \frac{\partial\boldsymbol{A}}{\partial t}.$$

Next, we introduce a four vector derivative, which we define according to

$$\frac{\partial}{\partial x_\mu} = \left(\frac{\partial}{\partial\boldsymbol{r}}, \frac{\partial}{\partial(ct)}\right).$$

In terms of $\boldsymbol{A} = \left(\boldsymbol{A}, \frac{\phi}{c}\right)$ and $\frac{\partial}{\partial x_\mu}$, we now have contravariant four-vector expressions for all of the terms in the right hand side of the equations for $\boldsymbol{E}$ and $\boldsymbol{B}$. With these four vector quantities in mind, we then define the antisymmetric EM field tensor

$$\mathrm{F}_{\mu\nu} = \frac{\partial A_\nu}{\partial x^\mu} - \frac{\partial A_\mu}{\partial x^\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu.$$

Without formal proof, we note that this tensor is Lorentz-invariant, although this should make intuitive sense—it is constructed from the invariant quantity A_μ.

- In terms of components, the the electromagnetic field tensor reads

$$\mathrm{F}_{\mu\nu} = \begin{bmatrix} 0 & B_z & -B_y & -\frac{E_x}{c} \\ -B_z & 0 & B_x & -\frac{E_y}{c} \\ B_y & -B_x & 0 & -\frac{E_z}{c} \\ \frac{E_x}{c} & \frac{E_y}{c} & \frac{E_z}{c} & 0 \end{bmatrix},$$

where the components of the tensor are the components of the $\boldsymbol{E}$ and $\boldsymbol{B}$ fields. In four dimensions, the $\boldsymbol{E}$ and $\boldsymbol{B}$ fields thus become a single unified electromagnetic field, written in terms of a tensor.

- The covariant electromagnetic action associated with the EM tensor is

$$S = \frac{1}{c}\int\left[-\frac{1}{4}\mathrm{F}^{\mu\nu}\,\mathrm{F}_{\mu\nu} + A^\mu j_\mu\right]\mathrm{d}^4 x_\lambda.$$

This action is the basis for quantizing the electromagnetic field in more advanced physics. This concludes our course.