

Electromagnetism Exercise Notes

Elijan Mastnak

Winter Semester 2020-2021

About These Notes

These are my notes are from the Exercises portion of the class *Elektromagnetno Polje* (Electromagnetic Field), given to third-year physics students at the Faculty of Math and Physics in Ljubljana, Slovenia. The exact problem sets herein are specific to the physics program at the University of Ljubljana, but the content is fairly standard for a late-undergraduate electromagnetism course. I am making the notes publicly available in the hope that they might help others learning the same material.

Navigation: For easier document navigation, the table of contents is “clickable”, meaning you can jump directly to a section by clicking the colored section names in the table of contents. Unfortunately, *the clickable links do not work in most online or mobile PDF viewers*; you have to download the file first.

On Authorship: The exercises are led by Asst. Prof. Martin Klanjšek, who has curated the problem sets and guides us through the solutions. Accordingly, credit for the problems in these notes goes to Prof. Klanjšek. I take credit for nothing beyond typesetting the notes, translating to English, and perhaps an additional explanation or two where I saw fit. The original exercise in Slovene (without solutions) can be found on the [course website](#).

Disclaimer: Mistakes—both trivial typos and legitimate errors—are likely. Keep in mind that these are the notes of an undergraduate student in the process of learning the material himself—take what you read with a grain of salt. If you find mistakes and feel like telling me, by [GitHub](#) pull request, [email](#) or some other means, I’ll be happy to hear from you, even for the most trivial of errors.

Contents

1	Electrostatics	4
1.1	First Exercise Set	4
1.1.1	Background Theory: Electric Field of a Charge Distribution	4
1.1.2	Charged Disk	4
1.1.3	Charged Plate with a Slit	5
1.2	Second Exercise Set	6
1.2.1	Theory: The Poisson Equation and the Fourier Transform	6
1.2.2	Poisson Equation for a Point Particle	7
1.2.3	Theory Interlude: Electric Field of a General Charge Distribution	9
1.2.4	Electric Field of a Hydrogen Atom	9
1.3	Third Exercise Set	10
1.3.1	Theory: Laplace Equation	10
1.3.2	Perpendicular Ribbon in a Parallel-Plate Capacitor	10
1.3.3	A Halved Conducting Cylinder	13
1.4	Fourth Exercise Set	15
1.4.1	A Halved Conducting Cylinder (continued)	15
1.4.2	Conducting Sphere in a Uniform Electric Field	17
1.5	Fifth Exercise Set	18
1.5.1	Conducting Sphere in a Uniform Electric Field (continued)	18
1.5.2	Electric Dipole in a Conducting Spherical Shell	19
1.5.3	Point Charge Above a Conducting Plane	21
1.6	Sixth Exercise Set	23
1.6.1	Theory: Electrostatic Force	23
1.6.2	Force on a Conducting Spherical Shell	23
1.6.3	Point Charge Between Two Conducting Plates	24
2	Magnetostatics and Electrodynamics	27
2.1	Seventh Exercise Set	27
2.1.1	Theory: Magnetic Vector Potential and the Biot-Savart Law	27
2.1.2	Magnetic Field of a Circular Current Loop	28
2.1.3	Magnetic Field of a Rotating Charged Disk	29
2.2	Eighth Exercise Set	31
2.2.1	Magnetic Force in a Coaxial Cable	31
2.2.2	Tension in a Toroidal Inductor	33
2.2.3	Resistance of a Thin Conducting Plate	34
2.3	Ninth Exercise Set	35
2.3.1	Mutual Inductance	35
2.3.2	Cabrera Experiment—Magnetic Monopoles	37
2.4	Tenth Exercise Set	40
2.4.1	Skin Effect in a Ribbon-Like Conductor	40
2.4.2	Theory: Conservation of Electromagnetic Energy	42
2.4.3	Power in a Coaxial and Cylindrical Conductor	43
2.4.4	Cylindrical Conductor with a Slit	45
2.5	Eleventh Exercise Set (rough draft)	46
2.5.1	A Radially Polarized Sphere	46
2.5.2	A Polarized Halved Sphere	47
2.5.3	Theory: Dielectric and Displacement Field	49
2.5.4	Dielectric Plate in a Parallel-Plate Capacitor	50

2.6	Twelfth Exercise Set (rough draft)	51
2.6.1	Point Dipole in a Spherical Dielectric Cavity	51
2.6.2	Dielectric Constant of Cold Plasma	54
3	Electromagnetic Wave Propagation	57
3.1	Thirteenth Exercise Set (rough draft)	57
3.1.1	Theory: Electromagnetic Wave Propagation in Waveguides	57
3.1.2	Waveguide of Two Parallel Plates	59

1 Electrostatics

1.1 First Exercise Set

1.1.1 Background Theory: Electric Field of a Charge Distribution

The electric field of a spatial charge distribution with volume charge density $\rho(\mathbf{r})$ is

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\tilde{\mathbf{r}}) d^3\tilde{\mathbf{r}}}{|\mathbf{r} - \tilde{\mathbf{r}}|^2} \frac{\mathbf{r} - \tilde{\mathbf{r}}}{|\mathbf{r} - \tilde{\mathbf{r}}|}$$

where $\tilde{\mathbf{r}}$ is a dummy variable for integration. Think of $\rho(\tilde{\mathbf{r}}) d^3\tilde{\mathbf{r}}$ as an infinitesimal element of charge at the position $\tilde{\mathbf{r}}$, while $\mathbf{r} - \tilde{\mathbf{r}}$ represents the position of each charge. The term $\frac{\mathbf{r} - \tilde{\mathbf{r}}}{|\mathbf{r} - \tilde{\mathbf{r}}|}$ just gives the direction of each charge element.

1.1.2 Charged Disk

A charged disk has surface charge density σ and radius a . What is the disk's electric field $E(z)$ along an axis through the disk's center and normal to the disk? Investigate the limit behavior of $E(z)$ for small and large z .

- Break the disk into infinitesimal concentric rings and integrate over the contributions dE of each ring; r and dr represent the radius and thickness of a ring, respectively. First break each ring into infinitesimal segments with area $r d\phi dr$. Along the perpendicular z axis through the disk's center each, segment contributes

$$dE_1 = \frac{1}{4\pi\epsilon_0} \frac{\sigma r d\phi dr}{z^2 + r^2}$$

Notice the term $r d\phi dr$ must have units of area to produce a charge when multiplied by the surface charge density σ . The term $z^2 + r^2$ is just the square of the distance $\sqrt{z^2 + r^2}$ from the charge element to the z axis.

- Recognize the circular symmetry of each ring: both the x and y components of the electric field symmetrically cancel along the z axis, so the electric field only has a z component. Relate the magnitudes of dE and dE_1 along the z axis with similar triangles to get the contribution dE of a ring of radius r at the point $(0, 0, z)$:

$$\frac{dE}{dE_1} = \frac{z}{\sqrt{z^2 + r^2}} \implies dE = \frac{z}{(z^2 + r^2)^{3/2}} \frac{\sigma r d\phi dr}{4\pi\epsilon_0}$$

- Integrate over the contributions dE to find $E(z)$:

$$E(z) = \int dE = \int_0^{2\pi} \int_0^a d\phi dr \frac{\sigma z r}{4\pi\epsilon_0 (z^2 + r^2)^{3/2}} = \frac{\sigma z}{2\epsilon_0} \int_0^a \frac{r dr}{(z^2 + r^2)^{3/2}}$$

Solve the integral with the substitution $u = z^2 + r^2$, $du = 2r dr$:

$$\begin{aligned} E(z) &= \int_{z^2}^{z^2+a^2} \frac{du}{u^{3/2}} = -\frac{\sigma z}{2\epsilon_0} \left(\frac{1}{\sqrt{z^2+a^2}} - \frac{1}{\sqrt{z^2}} \right) \\ &= \frac{\sigma}{2\epsilon_0} \left(1 - \frac{z}{\sqrt{z^2+a^2}} \right) = \frac{\sigma}{2\epsilon_0} \left(1 - \frac{1}{1 + \frac{a^2}{z^2}} \right) \end{aligned}$$

- For $z \ll a$ (very close to the disk), $1 + \frac{a^2}{z^2} \rightarrow \infty$ and $\frac{1}{1 + \frac{a^2}{z^2}} \rightarrow 0$, leaving

$$E(z) \rightarrow \frac{\sigma}{2\epsilon_0} \quad (z \ll a)$$

which is the electric field of an infinite charged plane.

- For $z \gg a$ (very far from the disk), $\frac{a^2}{z^2} \ll 1$, and we use the Taylor approximation $(1 + x)^p \approx 1 + px$ for $x \ll 1$:

$$E(z) = \frac{\sigma}{2\epsilon_0} \left[1 - \left(1 + \frac{a^2}{z^2} \right)^{-1/2} \right] \approx \frac{\sigma}{2\epsilon_0} \left[1 - \left(1 - \frac{a^2}{2z^2} \right) \right] = \frac{\sigma a^2}{4\epsilon_0 z^2}$$

Multiply above and below by π to match this to the expression for a point charge:

$$E(z) = \frac{\pi a^2 \sigma}{4\pi \epsilon_0 z^2} = \frac{\sigma S}{4\pi \epsilon_0 z^2} = \frac{q}{4\pi \epsilon_0 z^2} \quad (z \gg a)$$

where $q = \sigma S$ is the disk's charge.

1.1.3 Charged Plate with a Slit

We take an infinitely large charged rectangular plate with surface charged density σ and remove a slit of width a from the plate. Determine the electric field E in the plane perpendicular to the plate and passing through the center of the slit as a function of the vertical distance z from the plate. Investigate the limit behavior of $E(z)$ for small and large z .

- Break the plate into thin ribbons and integrate over the contributions dE of each ribbon; r and dr represent the orthogonal distance from the slit and the thickness of each ribbon, respectively.
- Find the electric field of a ribbon a distance r from the slit along the z axis using Gauss's law with a cylindrical surface. For a cylinder of radius R and length l , Gauss's law reads

$$\oint \mathbf{E} \cdot d\mathbf{S} = 2\pi R l E = \frac{q_{\text{enc}}}{\epsilon_0} \implies E(R) = \frac{q_{\text{enc}}}{2\pi \epsilon_0 l R}$$

Applied to the ribbon, the enclosed charged q_{enc} is the ribbon's infinitesimal charge $dq_1 = \sigma dS_1 = \sigma l dr$, while the cylinder's radius R is the distance from the ribbon to the z axis: $R = \sqrt{z^2 + r^2}$, so the contribution dE_1 of one ribbon is

$$dE_1 = \frac{dq_1}{2\pi \epsilon_0 l \sqrt{z^2 + r^2}} = \frac{\sigma l dr}{2\pi \epsilon_0 l \sqrt{z^2 + r^2}} = \frac{\sigma dr}{2\pi \epsilon_0 \sqrt{z^2 + r^2}}$$

- Because of mirror-image symmetry, both the x and y components of the electric field cancel, leaving only the z component dE_z . Relate dE_z and dE_1 using similar triangles

$$\frac{dE_z}{dE_1} = \frac{z}{\sqrt{z^2 + r^2}} \implies dE_z = \frac{\sigma z dr}{2\pi \epsilon_0 (z^2 + r^2)}$$

- Find the total electric field along the z axis by integrating over the contributions dE_z of all the ribbons. Because of mirror symmetry, you can calculate only the contribution of e.g. the right plane and multiple the result by two.

$$\begin{aligned} E(z) &= \int dE_z = 2 \int_{a/2}^{\infty} \left(\frac{\sigma z dr}{2\pi \epsilon_0 (z^2 + r^2)} \right) = \frac{\sigma z}{\pi \epsilon_0} \int_{a/2}^{\infty} \frac{dr}{z^2 + r^2} \\ &= \frac{\sigma z}{\pi \epsilon_0} \left[\frac{1}{z} \arctan \frac{r}{z} \right]_{a/2}^{\infty} = \frac{\sigma}{\pi \epsilon_0} \left[\frac{\pi}{2} - \arctan \left(\frac{a}{2z} \right) \right] \end{aligned}$$

- In the limit $z \gg a$ (very far from the slit), $\arctan \frac{a}{2z} \rightarrow 0$ and the electric field along the z axis is

$$E(z) = \frac{\sigma}{2\epsilon_0} \quad (z \gg a)$$

which is the field of an infinite sheet of charge.

- In the limit $z \ll a$ (very close to the slit), we have $\frac{a}{2z} \rightarrow \infty$. Use the asymptotic expansion $\arctan x \approx \frac{\pi}{2} - \frac{1}{x}$ for large x to get

$$E(z) \approx \frac{\sigma}{\pi\epsilon_0} \left[\frac{\pi}{2} - \left(\frac{\pi}{2} - \frac{2z}{a} \right) \right] = \frac{2\sigma}{\pi\epsilon_0} \frac{z}{a}$$

In this case the electric field scales linearly as $E \sim z$.

1.2 Second Exercise Set

1.2.1 Theory: The Poisson Equation and the Fourier Transform

- Start with Gauss's law in differential form and the relationship between \mathbf{E} and U

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \quad \text{and} \quad \mathbf{E} = -\nabla U$$

Substitute $\mathbf{E} = -\nabla U$ into Gauss's law to get

$$\nabla \cdot [-\nabla U] = \frac{\rho}{\epsilon_0} \implies \nabla^2 U = -\frac{\rho}{\epsilon_0}$$

The last equality takes the form of a *Poisson equation*.¹ It relates charge density ρ to electric potential U . In other words, if we know a spatial charge distribution ρ , we can find the electric field potential U thus the electric field \mathbf{E} with $\mathbf{E} = -\nabla U$.

- As a simple example we start with a point charge. The charge distribution is

$$\rho(\mathbf{r}) = q\delta(\mathbf{r}) \implies \nabla^2 U = -\frac{\rho}{\epsilon_0} = -\frac{q}{\epsilon_0}\delta(\mathbf{r})$$

This is the Poisson equation for a point charge. We will solve the equation with a Fourier transform.

- First, a review of the Fourier transform. Think of it as an expansion over a basis of plane waves of the form $e^{i\mathbf{k}\cdot\mathbf{r}}$. Where $U(\mathbf{k})$ plays the role of a weight function determining how much each wave contributes to the expansion

$$U(\mathbf{r}) = \int d^3k U(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{r}}$$

where $U(\mathbf{k})$ is the amplitude of the plane wave with wave vector \mathbf{k} . To find $U(\mathbf{k})$ we take the inner product of both sides of the above equation with the basis function $e^{-i\tilde{\mathbf{k}}\cdot\mathbf{r}}$.

$$\int U(\mathbf{r}) e^{-i\tilde{\mathbf{k}}\cdot\mathbf{r}} d^3r = \int \int d^3k d^3r U(\mathbf{k}) e^{i(\mathbf{k}-\tilde{\mathbf{k}})\cdot\mathbf{r}}$$

The integrals over \mathbf{r} on the right-hand side is in fact a delta function, because the orthogonal plane waves cancel out over all space except at the origin, where they constructively interfere to infinity. So

$$\int U(\mathbf{r}) e^{-i\tilde{\mathbf{k}}\cdot\mathbf{r}} d^3r = (2\pi)^3 \int U(\mathbf{k}) \delta(\mathbf{k}-\tilde{\mathbf{k}}) d^3k = (2\pi)^3 U(\tilde{\mathbf{k}})$$

¹In general any equation of the form $\nabla^2 f(\mathbf{r}) = g(\mathbf{r})$ is called a Poisson equation.

The delta function suppresses the integral everywhere besides at $(\mathbf{k} - \tilde{\mathbf{k}})$. And that's how you get the amplitude $U(\tilde{\mathbf{k}})$ for a wave vector $\tilde{\mathbf{k}}$.

$$U(\tilde{\mathbf{k}}) = \frac{1}{(2\pi)^3} \int U(\mathbf{r}) e^{-i\tilde{\mathbf{k}} \cdot \mathbf{r}} d^3r$$

This is also an inverse Fourier transform of sorts, to get $U(\mathbf{k})$ from $U(\mathbf{r})$.

Recipe: Fourier transform the Poisson equation from \mathbf{r} into \mathbf{k} space (where the Laplacian operator ∇^2 simplifies to $-ik^2$ under \mathcal{F}) and solve for the amplitude $U(\mathbf{k})$ of each plane wave $e^{-i\mathbf{k} \cdot \mathbf{r}}$. Substitute $U(\mathbf{k})$ into the Fourier transform

$$U(\mathbf{r}) = \int d^3k U(\mathbf{k}) e^{i\mathbf{k} \cdot \mathbf{r}}$$

and evaluate the integral—usually in spherical coordinates—to find $U(\mathbf{r})$.

- Here is the first of two more notes. What happens if we insert gradient into the Fourier transform? Remember the gradient operator acts only on \mathbf{r} . So

$$\nabla U(\mathbf{r}) = \int d^3k U(\mathbf{k}) \nabla e^{i\mathbf{k} \cdot \mathbf{r}}$$

Evaluating the gradient over (x, y, z) components gives

$$\nabla e^{i\mathbf{k} \cdot \mathbf{r}} = \begin{bmatrix} ikx \\ iky \\ ikz \end{bmatrix} e^{i(k_1x + k_2y + k_3z)} = i\mathbf{k} e^{i\mathbf{k} \cdot \mathbf{r}}$$

So:

$$\nabla U(\mathbf{r}) = \int d^3k U(\mathbf{k}) \nabla e^{i\mathbf{k} \cdot \mathbf{r}} = \int d^3k U(\mathbf{k}) i\mathbf{k} e^{i\mathbf{k} \cdot \mathbf{r}}$$

which has the form of a Fourier transform! This time of the function $U(\mathbf{k})i\mathbf{k}$.

The interpretation is that the gradient operator ∇ transforms to multiplication by $i\mathbf{k}$ under the Fourier transform. Analogously, the Laplacian ∇^2 would transform into multiplication by $(i\mathbf{k})^2 = -k^2$.

- And the second of the two notes: What happens if we insert $\delta(\mathbf{r})$ into the Fourier transform? Let $\delta(\mathbf{k})$ denote the amplitude in the expansion of $\delta(\mathbf{r})$, analogous to $U(\mathbf{k})$ for $U(\mathbf{r})$. Using the inverse Fourier transform and the integral properties of the delta function produces

$$\delta(\mathbf{k}) = \frac{1}{(2\pi)^3} \int \delta(\mathbf{r}) e^{-i\mathbf{k} \cdot \mathbf{r}} d^3r = \frac{1}{(2\pi)^3} e^{-i\mathbf{k} \cdot 0} = \frac{1}{(2\pi)^3}$$

1.2.2 Poisson Equation for a Point Particle

The Poisson equation for a point particle is

$$\nabla^2 U(\mathbf{r}) = -\frac{q}{\epsilon_0} \delta(\mathbf{r})$$

Solve the equation for $U(\mathbf{r})$.

- The plan is to transform into \mathbf{k} space, solve for $U(\mathbf{k})$, then transform back to $U(\mathbf{r})$. First, take the Fourier transform of both sides—use the Fourier transform identities $\nabla^2 \rightarrow -k^2$ and $\delta(\mathbf{r}) \rightarrow \frac{1}{(2\pi)^3}$ from the theory section.

$$-k^2 U(\mathbf{k}) = -\frac{q}{\epsilon_0} \frac{1}{(2\pi)^3} \implies U(\mathbf{k}) = \frac{q}{(2\pi)^3 \epsilon_0 k^2}$$

- Next, find $U(\mathbf{r})$ using a second Fourier transform

$$U(\mathbf{r}) = \int d^3k \frac{q e^{i\mathbf{k} \cdot \mathbf{r}}}{k^2 \epsilon_0 (2\pi)^3} = \frac{q}{(2\pi)^3 \epsilon_0} \int \frac{e^{i\mathbf{k} \cdot \mathbf{r}}}{k^2} d^3k$$

- Now, how to solve the integral? Suppose θ is the angle between \mathbf{r} and \mathbf{k} . To simplify the dot product. And then integrate in spherical coordinates:

$$U(\mathbf{r}) = \frac{q}{(2\pi)^3 \epsilon_0} \int_0^{2\pi} d\phi \int_{-1}^1 d[\cos \theta] \int_0^\infty dk k^2 \frac{e^{i k r \cos \theta}}{k^2}$$

So integrate over ϕ , which is simple and gives 2π :

$$U(\mathbf{r}) = \frac{q}{(2\pi)^2 \epsilon_0} \int_{-1}^1 \int_0^\infty e^{i \cos \theta k r} d[\cos \theta] dk$$

Integrate first over θ , (to avoid $e^{i \cos \theta \cdot \infty}$ from the upper k limit). Recognize the sine function the difference of exponents!

$$U(\mathbf{r}) = \frac{q}{(2\pi)^2 \epsilon_0} \int_0^\infty \left. \frac{e^{i \cos \theta k r}}{i k r} \right|_{\theta=-1}^1 dk = \frac{q}{(2\pi)^2 \epsilon_0} \int_0^\infty \frac{e^{i k r} - e^{-i k r}}{i k r} dk = \frac{q}{(2\pi)^2 \epsilon_0} \int_0^\infty \frac{2 \sin(kr)}{kr} dk$$

The integral of the sinc function is

$$\int_0^\infty \frac{\sin x}{x} dx = \frac{\pi}{2}$$

And applying this integral gives

$$U(\mathbf{r}) = \frac{2q}{(2\pi)^2 \epsilon_0} \frac{\pi}{2r} = \frac{q}{4\pi \epsilon_0 r}$$

which is the electric potential of a point charge. Cool! We derived it directly from Maxwell's equations.

- Finally, substitute our result for $U(\mathbf{r})$ into the Poisson equation for a point charge. The result is

$$\nabla^2 \frac{q}{4\pi \epsilon_0 r} = -\frac{q}{\epsilon_0} \delta(\mathbf{r}) \implies \nabla^2 \frac{1}{r} = -4\pi \delta(\mathbf{r}),$$

which will be useful in the next problems.

1.2.3 Theory Interlude: Electric Field of a General Charge Distribution

- We just solved the Poisson equation for the simple case $\rho(\mathbf{r}) = \delta(\mathbf{r})$. Can we use this result to solve the general case $\rho = \rho(\mathbf{r})$? The answer is yes, if we expand $\rho(\mathbf{r})$ over a basis of delta functions, as follows:

$$\rho(\mathbf{r}) = \int d^3\tilde{\mathbf{r}} \rho(\tilde{\mathbf{r}}) \delta(\mathbf{r} - \tilde{\mathbf{r}})$$

In this case, the solution of $U(\mathbf{r})$ to the Poisson equation is

$$U(\mathbf{r}) = \int d^3\tilde{\mathbf{r}} \rho(\tilde{\mathbf{r}}) \frac{1}{4\pi\epsilon_0 |\mathbf{r} - \tilde{\mathbf{r}}|} = \frac{1}{4\pi\epsilon_0} \int \frac{d^3\tilde{\mathbf{r}} \rho(\tilde{\mathbf{r}})}{|\mathbf{r} - \tilde{\mathbf{r}}|}$$

Well that's cool. By solving the Poisson equation for a delta function and then expanding an arbitrary $\rho(\mathbf{r})$ in terms of the delta function, we got the equation to the Poisson equation for any $\rho(\mathbf{r})$. And we can then get electric field using

$$\mathbf{E} = -\nabla U = \frac{1}{4\pi\epsilon_0} \int \frac{d^3\tilde{\mathbf{r}} \rho(\tilde{\mathbf{r}})}{|\mathbf{r} - \tilde{\mathbf{r}}|^2} \frac{\mathbf{r} - \tilde{\mathbf{r}}}{|\mathbf{r} - \tilde{\mathbf{r}}|}$$

which agrees with the equation quoted in the previous exercise set.

1.2.4 Electric Field of a Hydrogen Atom

The hydrogen atom has the electric potential

$$U(\mathbf{r}) = \frac{q}{4\pi\epsilon_0} \frac{e^{-\alpha r}}{r} \left(1 + \frac{\alpha r}{2}\right), \quad \alpha = \frac{2}{r_B}$$

Find the charge density $\rho(\mathbf{r})$ that generates this potential.

- Use the Poisson equation, which connects U and ρ

$$\nabla^2 U(\mathbf{r}) = -\frac{\rho(\mathbf{r})}{\epsilon_0}$$

Calculate the Laplacian of our $U(\mathbf{r})$ and work in spherical coordinates, since the potential is spherically symmetric (depends only on r). As a review, when acting on a function that depends only on r , ∇^2 in spherical coordinates reads

$$\nabla^2 = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial}{\partial r} \right)$$

- Applying ∇^2 to $U(r)$, after some straightforward but rather tedious differentiation, leads to

$$\nabla^2 U(\mathbf{r}) = \frac{q\alpha^3}{8\pi\epsilon_0} e^{-\alpha r}$$

Rearranging the Poisson equation then gives

$$\rho(\mathbf{r}) = -\frac{q\alpha^3}{8\pi} e^{-\alpha r}$$

- Note that the charge density is negative, which corresponds to the negatively charged electron cloud. Inserting the definition of $\alpha = \frac{2}{r_B}$ gives

$$\rho(\mathbf{r}) = -\frac{q}{\pi r_B^3} e^{-\frac{2r}{r_B}}$$

Another interpretation: $e^{-\frac{2r}{r_B}}$ is equivalent to $\left(e^{-\frac{r}{r_B}}\right)^2$, which is the square of the hydrogen atom's ground state wave function. The square of the wave function is probability, and multiplying the probability by $\frac{q}{r_B^3}$ gives a charge density.

- And why does the charge of the proton not contribute? Because the proton occurs at the origin, which corresponds to a charge density singularity at the origin. Terms in $U(r)$ with $\frac{1}{r}$ generate the singularity. And we just glossed over this when applying ∇^2 .

We can resolve this problem with a special case from the limit

$$\lim_{r \rightarrow 0} U(r) = \frac{q}{4\pi\epsilon_0 r}$$

We would then have to solve the Poisson equation for this potential. But we already know this potential: it is the potential for a point charge and corresponds to a charge density

$$\rho(\mathbf{r}) = q\delta(\mathbf{r})$$

The correct total result for the hydrogen atom is the sum of the electron cloud result and the charge density of the nucleus.

$$\rho(\mathbf{r}) = q\delta(\mathbf{r}) - \frac{q\alpha^3}{8\pi} e^{-\alpha r}$$

Lesson: Be careful when working with the Poisson equation if $U(r)$ has singularities!

1.3 Third Exercise Set

1.3.1 Theory: Laplace Equation

Just a quick review from the last exercise set: the Poisson equation used to solve for the electric field potential generated by a charge density ρ is

$$\nabla^2 U(\mathbf{r}) = -\frac{\rho(\mathbf{r})}{\epsilon_0} \quad \text{where} \quad \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

Often $\rho(\mathbf{r}) = 0$ in places we're solving for the electric potential. In this case $\nabla^2 U(\mathbf{r}) = 0$. This equation is called a *Laplace equation*.

1.3.2 Perpendicular Ribbon in a Parallel-Plate Capacitor

We place a long, thin conducting ribbon between the plates of a large parallel-plate capacitor, perpendicularly to the plates; the ribbon almost touches both plates with a little bit of air/insulation between the ribbon edges and the plates. The ribbon height and distance between the plates is a . We ground both plates and set the ribbon potential to U_0 . What is the electric potential inside the capacitor?

- First, decide on a coordinate system. Choose Cartesian coordinates since the problem has rectangular symmetry. I will try to describe the axes, but you basically need to see a picture. Here's my attempt: we drew a 2D projection on lined paper where the capacitor plates run from left to right along the page and are separated vertically by the distance a ; the ribbon is a vertical line between the top and bottom plates. In this case, the y axis runs vertically upward along the ribbon (connecting the top and bottom plates), the x axis runs left to right along the page, and the z axis points out of the page.

Note that the problem is independent of z (by translation symmetry in the z direction), so we only need $U(x, y)$. More so, the problem has reflection symmetry, so we can find the solution on only one side of the ribbon (one half of the x axis) and reflect the solution about the y axis.

- The space between the capacitor plates is empty—there is no charge, and we use the Laplace equation for the space between the plates.

$$\nabla^2 U(x, y) = 0$$

Charge can occur only along the ribbon or on the capacitor plates.

- Next, determine boundary conditions for $U(x, y)$ so the equation has a unique solution. The problem's boundaries are the ribbon and edges of the capacitor plates.

On the bottom plate, $U(x, 0) = 0$. On the upper plate, $U(x, a) = 0$. Both are zero because the plates are grounded. For the ribbon $U(0, y) = U_0$. And we need one more boundary—infinity. We require only that $U(x \rightarrow \infty, y)$ is bounded, i.e. that U does not diverge at ∞ .

- First, attempt solving the problem with separation of variables: $U(x, y) = X(x)Y(y)$ —this approach tends to work well with symmetric problems. Plugging this ansatz into the Laplace equation and evaluating ∇^2 gives

$$X''Y + XY'' = 0 \implies \frac{X''}{X} = -\frac{Y''}{Y}$$

The separation of variables is successful—we were able to get only x and only y on different sides of the equation.

- As usual, set the equation equal to a separation constant κ^2 and get two equations

$$X'' - \kappa^2 X = 0 \quad \text{and} \quad Y'' + \kappa^2 Y = 0$$

Both equations have simple solutions! The equations for X and Y are solved by exponential and sinusoidal functions, respectively.

$$X(x) = Ae^{\kappa x} + Be^{-\kappa x} \quad \text{and} \quad Y(y) = C \sin(\kappa y) + D \cos(\kappa y)$$

We then substitute the expressions for X and Y back into the ansatz $U = XY$:

$$U(x, y) = X(x)Y(y) = (Ae^{\kappa x} + Be^{-\kappa x})(C \sin(\kappa y) + D \cos(\kappa y))$$

- We find the coefficients A, B, C and D using the boundary conditions.
 - Start with the most powerful condition, that $U(x \rightarrow \infty, y)$ is bounded. This condition implies $A = 0$ to suppress the divergent exponential function e^κ .

- Then, use the next two simplest conditions, the ones requiring $U(x, y) = 0$. Starting with $U(x, 0) = 0$ gives

$$0 \equiv U(x, 0) = 1 \cdot (0 + D) \implies D = 0$$

With both $A = D = 0$, we're left at this point with only

$$U(x, y) = Be^{-\kappa x} \cdot C \sin(\kappa y)$$

- Next, applying $U(x, a) = 0$ gives

$$0 \equiv U(x, a) = Be^{-\kappa x} C \sin(\kappa a) \equiv Fe^{-\kappa x} \sin(\kappa a)$$

Note that we've joined the product of two constants into one constant $F = BC$. We have two options: either $F = 0$ or $\sin(\kappa a) = 0$. The option $F = 0$ gives the trivial solution $U(x, y) = 0$. The non-trivial solution comes from

$$\sin(\kappa a) = 0 \implies \kappa a = n\pi, \quad n = 1, 2, 3, \dots$$

Note that $n = 0$ leads to a trivial solution $U(x, y) = 0$, which we reject.

Respecting the quantization of κ and F by the index n , the general solution at this point is the linear superposition

$$U(x, y) = \sum_{n=1}^{\infty} F_n e^{-\kappa_n x} \sin(\kappa_n y)$$

- To find F_n , we use the last boundary condition $U(0, y) = U_0$.

$$U_0 = \sum_{n=1}^{\infty} F_n \sin(\kappa_n y) = \sum_{n=1}^{\infty} F_n \sin\left(\frac{n\pi y}{a}\right)$$

where we've used $\kappa_n = \frac{n\pi}{a}$. This is a Fourier expansion of the constant U_0 over sine functions.

We find the coefficients by taking the inner product of both sides of the equation (I think on the vector space $L(0, a)$), which amounts to multiplying both sides by $\sin\left(\frac{m\pi y}{a}\right)$ and integrating both sides over y from 0 to a .

The left side U_0 becomes

$$U_0 \int_0^a \sin\left(\frac{m\pi y}{a}\right) dy = -\frac{U_0 a}{m\pi} \cos\left(\frac{m\pi y}{a}\right) \Big|_0^a = \frac{U_0 a}{m\pi} [1 - (-1)^m]$$

And the right hand side, with the sum, we switch the sum and integral to get

$$\sum_{n=1}^{\infty} F_n \int_0^a \sin\left(\frac{n\pi y}{a}\right) \sin\left(\frac{m\pi y}{a}\right) dy = \sum_{n=1}^{\infty} F_n \delta_{mn} \int_0^a \sin^2\left(\frac{m\pi y}{a}\right) dy = \frac{F_m a}{2}$$

Because of the orthogonality of the sine functions, the integral is zero for $m \neq n$. Only the case $m = n$ gives a non-zero result.

Equating the two sides gives the desired expression for F_m :

$$\frac{U_0 a}{m\pi} [1 - (-1)^m] = \frac{F_m a}{2} \implies F_m = \frac{2U_0}{m\pi} [1 - (-1)^m]$$

- With F_m known, the final result for $U(x, y)$ is then

$$U(x, y) = \frac{2U_0}{\pi} \sum_{n=1}^{\infty} \frac{1 - (-1)^n}{n} \exp\left(-\frac{n\pi x}{a}\right) \sin\left(\frac{n\pi y}{a}\right)$$

Some limit cases: for $x \gg a$, the exponent terms very small, and we can neglect all terms in the series except the first term $e^{-\frac{\pi x}{a}}$ with $n = 1$. The result is

$$U(x, y) = \frac{4U_0}{\pi} \exp\left(-\frac{\pi x}{a}\right) \sin\left(\frac{\pi y}{a}\right)$$

- A separate case for which we can find a nice analytic solution is in the center of the capacitor at $y = \frac{a}{2}$. The solution reads

$$U(x, \frac{a}{2}) = \frac{2U_0}{\pi} \sum_{n=1}^{\infty} \frac{1 - (-1)^n}{n} \exp\left(-\frac{n\pi x}{a}\right) \sin\left(\frac{n\pi}{2}\right)$$

Instead of finding $U(x, \frac{a}{2})$, we'll find the electric field $\mathbf{E}(x, \frac{a}{2})$. Because of reflection symmetry across the line $y = \frac{a}{2}$, the electric field cannot have a y component— \mathbf{E} only has an x component. We'll find $E_x(x)$ from the potential:

$$E_x(x) = -\frac{\partial}{\partial x} U(x, \frac{a}{2}) = -\frac{2U_0}{a} \sum_{n=1}^{\infty} [1 - (-1)^n] \exp\left(-\frac{n\pi x}{a}\right) \sin\left(\frac{n\pi}{2}\right)$$

Next, note that

$$[1 - (-1)^n] \sin\left(\frac{n\pi}{2}\right) = \begin{cases} 0 & n \text{ even} \\ 2 & n = 1, 5, 9, \dots \\ -2 & n = 3, 7, 11, \dots \end{cases}$$

The sum simplifies to

$$E_x(x) = \frac{4U_0}{a} \left[e^{-\frac{\pi x}{a}} - e^{-\frac{3\pi x}{a}} + e^{-\frac{5\pi x}{a}} \mp \dots \right] = \frac{4U_0}{a} e^{-\frac{\pi x}{a}} \left[1 - e^{-\frac{2\pi x}{a}} + \left(e^{-\frac{2\pi x}{a}}\right)^2 \mp \dots \right]$$

which is a geometric series in $e^{-\frac{2\pi x}{a}}$. The result is

$$E_x(x) = \frac{4U_0}{a} \frac{e^{-\frac{\pi x}{a}}}{1 + e^{-\frac{2\pi x}{a}}} = \frac{4U_0}{a} \frac{1}{e^{\frac{\pi x}{a}} + e^{-\frac{\pi x}{a}}} = \frac{2U_0}{a \cosh\left(\frac{\pi x}{a}\right)}$$

1.3.3 A Halved Conducting Cylinder

Consider a long cylinder of radius a cut in half along a plane running along the cylinder's longitudinal axis. We separate the two cylinder halves by an arbitrarily small amount (so the halves are insulated) and apply a potential difference U_0 between the two halves. The halved cylinder acts as a capacitor. Find the electric potential inside the cylinder.

- Decide on a coordinate system: we use cylindrical coordinates because our problem has cylindrical symmetry. Let the z axis run along the cylinder's longitudinal axis. Because of translational symmetry along the z axis, U is independent of z .

- There is no charge inside the cylinder, so we get a Laplace equation

$$\nabla^2 U(r, \phi) = 0$$

In cylindrical coordinates (when acting on a function independent of z), the Laplacian operator reads

$$\nabla^2 = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2}{\partial \phi^2}$$

In our case,

$$\nabla^2 U(r, \phi) = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial U}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 U}{\partial \phi^2} = 0$$

- Again, we separate variables with the ansatz $U(r, \phi) = R(r)\Phi(\phi)$. Plugging this into the Laplace equation gives

$$\Phi \frac{1}{r} (rR')' + \frac{R}{r^2} \Phi'' = \Phi \left(\frac{R'}{r} + R'' \right) + \frac{R}{r^2} \Phi'' = 0$$

Note that $r' = 1$. Dividing through by Φ and rearranging gives

$$\frac{rR'}{R} + r^2 \frac{R''}{R} = -\frac{\Phi''}{\Phi}$$

- Following the usual separation procedure, we set both sides equal to the separation constant m^2 . The equations for Φ and R read

$$\Phi'' + m^2 \Phi = 0 \quad \text{and} \quad r^2 R'' + rR' - m^2 R = 0$$

The solution for Φ is sinusoidal:

$$\Phi(\phi) = A \sin(m\phi) + B \cos(m\phi)$$

Now, our cylindrical problem is periodic in ϕ with period 2π —this just means the cylinder repeats after one revolution. Periodicity in ϕ is possible only if m takes on integer values, so we can immediately index the solutions for Φ with

$$\Phi_m(\phi) = A_m \sin(m\phi) + B_m \cos(m\phi), \quad m = 1, 2, 3, \dots$$

We only use positive integers because the odd/even symmetry of sin and cos means negative integers give the same result as positive one—solving for negative m would be redundant. We reject $m = 0$ because this solution leads to $\Phi'' = 0$, meaning Φ is a linear function. But a linear function can't be periodic in ϕ , so we reject $m = 0$.

The second equation for R is solved with powers of r . The result is

$$R_m(r) = C_m r^m + D_m r^{-m}$$

- The general solution is the linear superposition

$$U(r, \phi) = \sum_{m=1}^{\infty} \Phi_m(\phi) R_m(m) = \sum_{m=1}^{\infty} (A_m \sin(m\phi) + B_m \cos(m\phi)) (C_m r^m + D_m r^{-m})$$

Note: We ended the problem at this point (ran out of time) and continued in the fourth exercise set.

1.4 Fourth Exercise Set

1.4.1 A Halved Conducting Cylinder (continued)

- We left off last time with the general solution

$$U(r, \phi) = \sum_{m=1}^{\infty} \Phi_m(\phi) R_m(m) = \sum_{m=1}^{\infty} (A_m \sin(m\phi) + B_m \cos(m\phi)) (C_m r^m + D_m r^{-m})$$

for the potential inside the cylinder. To find a solution specific to our problem, we apply boundary conditions. We already applied the periodic boundary condition $U(r, \phi) = U(r, \phi + 2\pi)$, which required m be integer-valued.

A second boundary condition requires the capacitor halves have a potential difference U_0 between them. It is best to write this potential difference in the symmetric form

$$U(a, \phi) = \begin{cases} \frac{U_0}{2} & \phi \in (0, \pi) \\ -\frac{U_0}{2} & \phi \in (\pi, 2\pi) \end{cases}$$

There is another condition—that U does not diverge at $r = 0$. This condition implies the D_m coefficients are zero, because the $D_m r^{-m}$ term diverges at $r = 0$.

Observation: the second boundary condition is an odd function of ϕ . This implies that only odd (sine) terms can appear in the final solution. This allows us to set the A_m coefficients equal to zero to eliminate the cosine terms. We are left with

$$U(r, \phi) = \sum_{m=1}^{\infty} F_m r^m \sin(m\phi)$$

where we have defined $B_m C_m \equiv F_m$.

- Applying the second boundary condition at $r = a$ gives

$$U(a, \phi) = \sum_{m=1}^{\infty} F_m a^m \sin(m\phi)$$

To solve this, we take the scalar product of the equation with $\sin(n\phi)$:

$$\int_0^{2\pi} U(a, \phi) \sin(n\phi) d\phi = \int_0^{2\pi} \sum_{m=1}^{\infty} F_m a^m \sin(m\phi) \sin(n\phi) d\phi$$

Plugging in the step values of $U(a, \phi)$ gives

$$\frac{U_0}{2} \int_0^{\pi} \sin(n\phi) d\phi - \frac{U_0}{2} \int_{\pi}^{2\pi} \sin(n\phi) d\phi = \int_0^{2\pi} \sum_{m=1}^{\infty} F_m a^m \sin(m\phi) \sin(n\phi) d\phi$$

First, we solve the left-hand side

$$\begin{aligned} \frac{U_0}{2} \left[-\frac{1}{n} \cos(n\phi) \Big|_0^{\pi} + \frac{1}{n} \cos(n\phi) \Big|_{\pi}^{2\pi} \right] &= \frac{U_0}{2n} [-\cos(\pi n) + 1 + 1 - \cos(n\pi)] \\ &= \frac{U_0}{n} (1 - (-1)^n) \end{aligned}$$

where we've used the identity $\cos(n\pi) = (-1)^n$.

- Next, we solve the right-hand side. Switching the order of integration and summation gives

$$\sum_{m=1}^{\infty} F_m a^m \int_0^{2\pi} \sin(m\phi) \sin(n\phi) d\phi = \sum_{m=1}^{\infty} F_m a^m \left(\frac{2\pi}{2} \delta_{mn} \right) = F_n a^n \pi$$

Combining the left and right sides gives

$$F_n = \frac{U_0 [1 - (-1)^n]}{\pi n a^n}$$

So, the solution for $U(r, \phi)$ is

$$U(r, \phi) = \sum_{n=1}^{\infty} \left[\frac{U_0 [1 - (-1)^n]}{\pi n a^n} \right] r^n \sin(n\phi) = \frac{U_0}{\pi} \sum_{n=1}^{\infty} \left(\frac{r}{a} \right)^n \frac{[1 - (-1)^n]}{n} \sin(n\phi)$$

- Next, some limiting cases. It will be easier to work in terms of electric field instead of potential. We will find the electric field in the two planes parallel and perpendicular to the slit between the capacitor halves.

First, in the perpendicular (vertical) plane. The field points from high to low potential, so from the top half of the capacitor to the bottom half. In this plane we can work with just one coordinate r , which represents the vertical distance from the cylinder's center. Note that $\phi = \frac{\pi}{2}$. The component E_r we're after is

$$\begin{aligned} E_r &= -\frac{\partial}{\partial r} U(r, \phi) \Big|_{\phi=\frac{\pi}{2}} = -\frac{U_0}{\pi} \sum_{n=1}^{\infty} n \left(\frac{r}{a} \right)^{n-1} \frac{1}{a} \frac{[1 - (-1)^n]}{n} \sin(n\phi) \Big|_{\phi=\frac{\pi}{2}} \\ &= -\frac{U_0}{\pi a} \sum_{n=1}^{\infty} \left(\frac{r}{a} \right)^{n-1} \frac{[1 - (-1)^n]}{n} \sin\left(\frac{n\pi}{2}\right) \end{aligned}$$

The sum simplifies considerably when you realize

$$\frac{1 - (-1)^n}{n} \sin\left(\frac{n\pi}{2}\right) = \begin{cases} 0 & n \text{ even} \\ 2 & n = 1, 5, \dots \\ -2 & n = 3, 7, \dots \end{cases}$$

We can then write the field as a geometric series

$$E_r = \frac{-2U_0}{\pi a} \left[1 - \left(\frac{r}{a} \right)^2 + \left(\frac{r}{a} \right)^4 \mp \dots \right] = \frac{-2U_0}{\pi a} \frac{1}{1 + \left(\frac{r}{a} \right)^2}$$

Note that E_r is largest at $r = 0$, decreases monotonically with increasing r , and falls to half of its maximum value at $r = a$.

- In a field parallel to the slit, we would set $\phi = 0$. This plane is perpendicular to the vertical plane, which used the radial component E_r , so for the parallel plane we work with the ϕ component E_ϕ .

$$E_\phi = -\frac{1}{r} \frac{\partial}{\partial \phi} U(r, \phi) \Big|_{\phi=0}$$

As before, the sum simplifies considerably to a geometric series. The result turns out to be

$$E_\phi = -\frac{2U_0}{\pi a} \frac{1}{1 - \left(\frac{r}{a} \right)^2}$$

Note that E_ϕ diverges at $r = a$. This is a consequence of the very small slit spacing between the capacitor halves at $r = a$; schematically have $E = \frac{U_0}{d} \rightarrow \infty$ as $d \rightarrow 0$.

1.4.2 Conducting Sphere in a Uniform Electric Field

A conducting sphere of radius a is placed in a uniform electric field E_0 pointing in the z direction. Find the electric field U inside and outside the sphere.

- Use spherical coordinates to take advantage of spherical symmetry. This means we want $U(r, \phi, \theta)$. Because of the problem's rotational symmetry, the solution will be independent of ϕ . We need only $U(r, \theta)$.
- The sphere is at a constant potential because it is a conductor. We'll set $U = 0$ inside the sphere. In the space around the sphere, we solve the Laplace equation

$$\nabla^2 U(r, \theta) = 0$$

We then separate variables via $U = R(r)\Theta(\theta)$, which leads to the general solution

$$U(r, \theta) = \sum_{l=0}^{\infty} \left[A_l r^l + B_l r^{-(l+1)} \right] P_l(\cos \theta)$$

where P_l are the Legendre polynomials.

- On to the boundary conditions. On the surface we'll set $U(a, \theta) = 0$. And at infinity, we use the boundary condition

$$U(r \rightarrow \infty, \theta) = -E_0 z = -E_0 r \cos \theta$$

This is the potential of a uniform electric field (at infinity, the potential from the sphere is negligible). This potential is chosen so that

$$-\frac{\partial U}{\partial z} = E_0$$

i.e. that the potential at infinity recovers the uniform electric field E_0 .

- We'll start with the second boundary condition at $r \rightarrow \infty$. Applying the condition to the general solution gives

$$\sum_{l=0}^{\infty} A_l r^l P_l(\cos \theta) = -E_0 r \cos \theta$$

Note that the $r^{-(l+1)}$ terms vanish as $r \rightarrow \infty$. The entire series equals a single term proportional to $\cos \theta$ and the equality holds if only the $l = 1$ term in the series is non-zero, which generates a corresponding $\cos \theta$ term from $P_1(\cos \theta) = \cos \theta$ i.e. $P_1(x) = x$. The $l = 1$ term is

$$A_1 r \cos \theta = -E_0 r \cos \theta \implies A_1 = -E_0$$

so we have $A_l = -E_0 \delta_{l1}$. The solution for $U(r, \theta)$ simplifies to

$$U(r, \theta) = -E_0 r \cos \theta + \sum_{l=1}^{\infty} B_l r^{-(l+1)} P_l(\cos \theta)$$

- Next, the second boundary condition: $U(a, \theta) = 0$. Substituting the condition into the intermediate solution gives

$$E_0 a \cos \theta = \sum_{l=1}^{\infty} B_l a^{-(l+1)} P_l(\cos \theta)$$

Again, the entire series equals only a single term. Again, this will be the $l = 1$ term corresponding to $P_1(\cos \theta) = \cos \theta$. For $l \neq 1$ we have $B_l = 0$. The $l = 1$ term gives

$$E_0 a \cos \theta = B_1 a^{-2} \cos \theta \implies B_1 = E_0 a^3$$

With A_l and B_l known for all l , the final result is

$$U(r, \theta) = -E_0 r \cos \theta + \frac{E_0 a^3}{r^2} \cos \theta$$

The first term, $-E_0 r \cos \theta$, is the potential of the uniform external field E_0 . The second term comes from the sphere. In fact, this second term has the same form as the potential of an electric dipole!

Limiting cases are discussed in the next exercise set.

1.5 Fifth Exercise Set

1.5.1 Conducting Sphere in a Uniform Electric Field (continued)

- Where we left off last time, we had found the potential due to the sphere was

$$U(r, \theta) = -E_0 \cos \theta + \frac{E_0 a^3}{r^2} \cos \theta,$$

and identified the sphere's contribution $\frac{E_0 a^3}{r^2} \cos \theta$ corresponded to the potential of an electric dipole. Our next step is to explore the sphere's dipole behavior.

- We consider an infinitesimal element of the sphere's surface at the angle θ carrying charge dq , on the upper hemisphere with positive charge and θ . Recall the electric field, as for any conductor, is perpendicular to the surface.

We write Gauss's law for the small surface element, which is simple because the electric field is perpendicular to the surface

$$E_{\perp} dS = \frac{dq}{\epsilon_0} \implies \frac{de}{dS} = \sigma = \epsilon_0 E_{\perp}$$

This equality gives us an expression for σ in terms of the electric field E_{\perp} perpendicular to the surface. We can find E_{\perp} from the potential:

$$E_{\perp} = -\left. \frac{\partial U}{\partial r} \right|_{r=a} = E_0 \cos \theta + 2E_0 \cos \theta \implies \sigma = 3\epsilon_0 E_0 \cos \theta$$

The charge density's dependence on θ quantitatively demonstrates the sphere's dipole-like charge distribution.

- With charge density σ known, we can find the sphere's electric dipole moment via

$$\mathbf{p}_e = \int \tilde{\mathbf{r}} dq$$

We qualitatively expect \mathbf{p}_e to point upward (from the negative to the positively charged hemisphere), and confirm this analytically. By spherical symmetry, only the z component of \mathbf{p}_e is non-zero; this is

$$p_{ez} = \int \tilde{z} dq = \int (a \cos \theta) \cdot (\sigma dS) = \int (a \cos \theta) \cdot (3\epsilon_0 E_0 \cos \theta) \cdot dS$$

To find dS , we find the area of a small band of width da around the sphere's surface. The band's area is $2\pi r da = 2\pi(a \sin \theta)(a d\theta)$. The dipole moment p_{ez} is then

$$\begin{aligned} p_{ez} &= \int_0^\pi (3\epsilon_0 E_0 a \cos^2 \theta) \cdot 2\pi(a \sin \theta)(a d\theta) = 6a^3 \pi \epsilon_0 E_0 \int_{-1}^1 \cos^2 \theta d[\cos \theta] \\ &= 6a^3 \pi \epsilon_0 E_0 \left[\frac{1}{3} \cos^3 \theta \right]_{-1}^1 = 4\pi \epsilon_0 E_0 a^3 \end{aligned}$$

1.5.2 Electric Dipole in a Conducting Spherical Shell

We place an electric dipole with dipole moment \mathbf{p} in the center of a conducting spherical shell of radius a . What is the electric potential inside the shell?

- We use spherical coordinates, which are best suited to the problem's spherical symmetry. By rotational symmetry, the potential depends only on the coordinates r and θ , not ϕ . Besides at the center, the charge density inside the sphere is zero, so we solve the Laplace equation

$$\nabla^2 U(r, \theta) = 0$$

The general solution is

$$U(r, \theta) = \sum_{l=0}^{\infty} \left(A_l r^l + B_l r^{-(l+1)} \right) P_l(\cos \theta)$$

where P_l are the Legendre polynomials.

- We then apply boundary conditions to find a solution specific to our problem. First, the potential on the shell's surface is constant, since the shell is a conductor. For convenience, we'll set $U(a, \theta) = 0$. The second boundary condition concerns the dipole at the sphere's center. Namely, the potential approaches the potential of an electric dipole near the sphere's center. Quantitatively, this condition reads

$$U(r \rightarrow 0, \theta) = \frac{p_e \cos \theta}{4\pi \epsilon_0 r^2}$$

- We start with the simpler second boundary condition, (the boundary $r \rightarrow 0$ eliminates the r^l -dependent term). Inserted into the general solution, the second condition reads

$$U(r \rightarrow 0, \theta) = \sum_{l=0}^{\infty} B_l r^{-(l+1)} P_l(\cos \theta) = \frac{p_e \cos \theta}{4\pi \epsilon_0} r^{-2}$$

Note that the entire series sums to only a single term; for this to work, only the $l = 1$ term in the series can be non-zero, leaving

$$B_1 r^{-2} \cos \theta = \frac{p_e \cos \theta}{4\pi \epsilon_0} r^{-2} \implies B_1 = \frac{p_e}{4\pi \epsilon_0} \quad \text{and} \quad B_{l \neq 1} = 0$$

Note the use of $P_1(\cos \theta) = \cos \theta$. The intermediate solution at this stage is

$$U(r, \theta) = \sum_{l=0}^{\infty} A_l r^l P_l(\cos \theta) + \frac{p_e}{4\pi\epsilon_0} r^{-2} \cos \theta$$

We apply the second boundary condition $U(a, \theta) = 0$ to get

$$\sum_{l=0}^{\infty} A_l a^l P_l(\cos \theta) = -\frac{p_e}{4\pi\epsilon_0} a^{-2} \cos \theta$$

As before, only the $l = 1$ term can be non-zero to satisfy the equality. The result is

$$A_1 a \cos \theta = -\frac{p_e}{4\pi\epsilon_0} a^{-2} \cos \theta \implies A_1 = -\frac{p_e}{4\pi\epsilon_0 a^3} \quad \text{and} \quad A_{l \neq 1} = 0$$

With the coefficients A_l and B_l known, the solution for $U(r, \theta)$ is

$$U(r, \theta) = \left[\frac{p_e}{4\pi\epsilon_0 r^2} - \frac{p_e}{4\pi\epsilon_0 a^3} r \right] \cos \theta = \frac{p_e \cos \theta}{4\pi\epsilon_0} \left[\frac{1}{r^2} - \frac{r}{a^3} \right]$$

The $\frac{1}{r^2}$ term is the dipole's contribution. The $\frac{r}{a^3}$ comes from the charge induced on the conducting shell.

- The induced term is worth a closer look, noting that $r \cos \theta = z$.

$$U_{\text{induced}} = -\frac{p_e}{4\pi\epsilon_0 a^3} r \cos \theta = -\frac{p_e}{4\pi\epsilon_0 a^3} z$$

In particular, the associated electric field is

$$E_{\text{induced}} = -\frac{\partial}{\partial z} U_{\text{induced}} = \frac{p_e}{4\pi\epsilon_0 a^3}$$

In other words, the electric field generated by the induced charge is constant! The uniform field also tells us about the charge distribution on the sphere's surface: to create a uniform field in the z direction, the shell must have a dipole-like charge distribution, with positive charge on the lower hemisphere and negative charge on the upper hemisphere.

- Next, we're interested in the analytic expression for the surface charge density σ . We consider a small surface element dS , and consider the total electric field at that surface. The electric field must be perpendicular to the surface, since the shell is a conductor. Gauss's law applied to the surface element reads

$$-E_{\perp} dS = \frac{dq}{\epsilon_0} \implies \sigma \equiv \frac{dq}{dS} = -\epsilon_0 E_{\perp}$$

Note the minus sign, indicating the field's electric flux leaving the surface element from inside the shell. We find an expression for E_{\perp} from U :

$$E_{\perp} = -\frac{\partial U}{\partial r} \Big|_{r=a} = +\frac{p_e \cos \theta}{4\pi\epsilon_0} \left[\frac{2}{r^3} + \frac{1}{a^3} \right]_{r=a} = \frac{3p_e}{4\pi\epsilon_0 a^3} \cos \theta$$

The associated surface charge density is

$$\sigma = -\epsilon_0 E_{\perp} = -\frac{3p_e}{4\pi a^3} \cos \theta$$

1.5.3 Point Charge Above a Conducting Plane

Consider a positive point charge q a distance d above a large, grounded conducting plane. What is the electric potential in space due to the charge-plane system?

- We will use a trick called the *method of images* to solve the problem. Namely, we imagine a negative point charge $-q$ a distance d below the plane—a mirror image of the original positive charge. The resulting charge distribution is an electric dipole.

Note: placing an imaginary negative charge a distance d below the plane does not change the field above the plane due to the positive charge.

- Considering both points, the potential at an arbitrary position \mathbf{r} from the origin is

$$U(\mathbf{r}) = \frac{q}{4\pi\epsilon_0} \frac{1}{|\mathbf{r} - \mathbf{d}|} - \frac{q}{4\pi\epsilon_0} \frac{1}{|\mathbf{r} + \mathbf{d}|}$$

where the vector \mathbf{d} points perpendicularly up from the plane toward the positive charge. Introducing an angle θ between \mathbf{d} and \mathbf{r} , we have

$$|\mathbf{r} \pm \mathbf{d}| = \sqrt{r^2 + d^2 \pm 2rd \cos \theta}$$

The expression for $U(\mathbf{r})$ for the two charges is then simply

$$U_2(\mathbf{r}) = \frac{q}{4\pi\epsilon_0} \left[\frac{1}{\sqrt{r^2 + d^2 \pm 2rd \cos \theta}} - \frac{1}{\sqrt{r^2 + d^2 \pm 2rd \cos \theta}} \right]$$

For the original configuration of a single positive charge a distance d above the plane, the potential above the plane agrees with U_2 , while the potential below the plane, where there is in reality no charge, is zero. The correct expression for the single charge $+q$ is then

$$U(\mathbf{r}) = \begin{cases} U_2(\mathbf{r}) & \text{above the plane} \\ 0 & \text{below the plane} \end{cases}$$

- Next, we're interested in the surface charge density $\sigma(\rho)$ on the plane where ρ is the radial distance in the plane from the origin. As usual, we start with Gauss's law for a small surface element of the plane:

$$-E_{\perp} dS = \frac{dq}{\epsilon_0} \implies \sigma \equiv \frac{dq}{dS} = -\epsilon_0 E_{\perp}$$

To find E_{\perp} , we differentiate U with respect to the vertical coordinate z . First, we introduce z into the expression $|\mathbf{r} \pm \mathbf{d}|$

$$|\mathbf{r} \pm \mathbf{d}| = \sqrt{r^2 + d^2 \pm 2rd \cos \theta} = \sqrt{\rho^2 + z^2 + d^2 \pm 2dz}$$

We then have

$$U(\rho, z) = \frac{q}{4\pi\epsilon_0} \left[\frac{1}{\sqrt{\rho^2 + z^2 + d^2 - 2dz}} - \frac{1}{\sqrt{\rho^2 + z^2 + d^2 + 2dz}} \right]$$

We then find E_{\perp} and then σ with

$$\begin{aligned} \sigma &= -\epsilon_0 E_{\perp} = -\epsilon_0 \left. \frac{\partial U}{\partial z} \right|_{z=0} = -\frac{q}{4\pi} \left[\frac{d}{(\rho^2 + d^2)^{3/2}} + \frac{d}{(\rho^2 + d^2)^{3/2}} \right] \\ &= -\frac{q}{2\pi} \frac{d}{(\rho^2 + d^2)^{3/2}} \end{aligned}$$

- With surface charge density σ known, we then ask what is the total charge on the plane. Integrating the plane over rings with area $dS = 2\pi\rho d\rho$, we have

$$q_{\text{plane}} = \int \sigma dS = - \int_0^\infty \frac{qd}{2\pi(\rho^2 + d^2)^{3/2}} (2\pi\rho d\rho)$$

In terms of the new variable $u = \rho^2 + d^2$, the integral evaluates to

$$q_{\text{plane}} = -\frac{qd}{2} \int_{d^2}^\infty \frac{du}{u^{3/2}} = qd \left[\frac{1}{u^{1/2}} \right]_{d^2}^\infty = -q$$

- Summary of what we did: recognize that the field above the plane from the positive charge looks like half the field of an electric dipole. Since we know the solution for a dipole, instead of solving the charge-plane system, we solve the (imaginary) two-charge system, which gives the same field above the plane anyway. We then reuse the upper half of the dipole solution for the single-charge plane system, and set the field below the plane equal to zero. The basic idea is: the field above the plane is the same for both the positive-charge plane system and for a dipole system, so we can use either approach to solve for the field above the plane.
- Next, we ask what is the electrostatic force on the point charge above the plane? First, some theory:

Theory: Electrostatic Force

The total electrostatic force \mathbf{F} acting on the charges enclosed in a region of space V permeated with an electric field \mathbf{E} is

$$\mathbf{F} = \epsilon_0 \oint_{\partial V} \left[\mathbf{E}(\mathbf{E} \cdot \hat{\mathbf{n}}) - \frac{1}{2} E^2 \hat{\mathbf{n}} \right] dS$$

where $\hat{\mathbf{n}}$ is the normal to the surface ∂V enclosing the charges. Like with Gauss's law, a good choice of the boundary surface, usually taking advantage of the problem's symmetries, tends to simplify the problem. Alternatively, if the electric field vanishes at infinity, we choose a surface that closes at infinity.

- Back to our problem: we choose an infinite surface whose base runs along the plane, then turns upward and closes at infinity to enclose the upper half of space above the plane. The field in this case is the same E_\perp calculated above:

$$E_\perp = \frac{q}{2\pi\epsilon_0} \frac{d}{(\rho^2 + d^2)^{3/2}}$$

For the part of the surface running parallel to the plane, the normal to the surface $\hat{\mathbf{n}}$ points perpendicularly into the plane, parallel to the electric field. The force equation for the bottom half of the surface reads

$$\mathbf{F}_e = \epsilon_0 \oint \left[E^2 \hat{\mathbf{n}} - \frac{1}{2} E^2 \hat{\mathbf{n}} \right] dS = \frac{\epsilon_0}{2} \hat{\mathbf{n}} \int E_\perp^2 dS$$

In fact, the contribution from the upper half of the surface is zero—the upper half extends to infinity, where the electric field vanishes. We only need to integrate over

the bottom of the surface, running parallel to the plane. Writing $dS = 2\pi\rho d\rho$ and substituting in the expression for E_\perp , the force reads

$$\begin{aligned}\mathbf{F}_e &= \hat{\mathbf{n}} \frac{\epsilon_0}{2} \int_0^\infty \frac{q^2}{4\pi^2\epsilon_0^2} \frac{d^2}{(\rho^2 + d^2)^3} 2\pi\rho d\rho = \frac{q^2 d^2}{4\pi\epsilon_0} \hat{\mathbf{n}} \int_0^\infty \frac{\rho}{(\rho^2 + d^2)^3} d\rho \\ &= \frac{q^2 d^2}{8\pi\epsilon_0} \hat{\mathbf{n}} \int_{d^2}^\infty \frac{du}{u^3} = -\frac{q^2 d^2}{16\pi\epsilon_0} \hat{\mathbf{n}} \left[\frac{1}{u^2} \right]_{d^2}^\infty = \frac{q^2}{16\pi\epsilon_0 d^2} \hat{\mathbf{n}}\end{aligned}$$

The force points in the direction of $\hat{\mathbf{n}}$ —downward into the plane. A final note: if we write

$$\mathbf{F}_e = \frac{q^2}{4\pi\epsilon_0(2d)^2} \hat{\mathbf{n}}$$

the force takes the form of the electric force between a positive and negative charge separated by a distance $2d$ —the same situation we used in the method of images above.

1.6 Sixth Exercise Set

1.6.1 Theory: Electrostatic Force

Recall the total electrostatic force \mathbf{F} acting on the charges enclosed in a region of space V permeated with an electric field \mathbf{E} is

$$\mathbf{F}_e = \epsilon_0 \oint_{\partial V} \left[\mathbf{E}(\mathbf{E} \cdot \hat{\mathbf{n}}) - \frac{1}{2} E^2 \hat{\mathbf{n}} \right] dS$$

where $\hat{\mathbf{n}}$ is the normal to the surface ∂V enclosing the charges.

1.6.2 Force on a Conducting Spherical Shell

We place a conducting sphere of radius a in a homogeneous electric field E_0 . Find the electrostatic force on the upper half of the sphere.

- Suppose the field points in the z direction. Recall from the previous exercise set that the potential from the sphere and electric field is

$$U(r, \theta) = -E_0 r \cos \theta + \frac{E_0 a^3}{r^2} \cos \theta$$

Qualitatively, there are two main contributions to the force on the sphere: an upwards contribution in the positive z direction from the external electric field, and a downward contribution in the negative z direction from the negative charge accumulated on the bottom half of the sphere.

- We are interested in the force on the upper half of the sphere—the next step is to choose a surface around the sphere’s upper half that will simplify the force calculation. Recall the field points perpendicularly out of the conducting sphere’s surface at all points.

With this perpendicular field in mind, choose a surface that tightly hugs the sphere’s upper half—in this case, the field and normal to the surface $\hat{\mathbf{n}}$ are parallel at all points outside the sphere. In the hemisphere plane inside the sphere, there is no field at all. These two facts simplify the dot product $\mathbf{E} \cdot \hat{\mathbf{n}}$ in the force equation.

- We then have $\mathbf{E} \cdot \hat{\mathbf{n}} = E$ and $\mathbf{E}(\mathbf{E} \cdot \hat{\mathbf{n}}) = E^2 \hat{\mathbf{n}}$. The contribution to the force on along the sphere's outside surface is

$$\mathbf{F}_e = \epsilon_0 \int \frac{1}{2} E^2 \hat{\mathbf{n}} dS$$

The contribution from the hemisphere plane through the sphere zero, since $\mathbf{E} = 0$ inside the sphere.

- Next, we find the magnitude E on the sphere's surface from the potential $U(r, \theta)$. The field points radially outwards, so we differentiate U with respect to r to get

$$E = \left. \frac{\partial U}{\partial r} \right|_{r=a} = E_0 \cos \theta + 2E_0 \cos \theta = 3E_0 \cos \theta$$

Inserting E into the force equation gives

$$\mathbf{F}_e = \epsilon_0 \int \frac{1}{2} (3E_0 \cos \theta)^2 \hat{\mathbf{n}} dS$$

In spherical coordinates, the unit normal $\hat{\mathbf{n}}$ to the sphere's surface is $\hat{\mathbf{n}} = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta)$. The surface element dS at the surface $r = a$ is (just like in the previous exercise sets) $dS = a^2 d\phi \sin \theta d\theta$. The force on the sphere's upper half is then

$$\mathbf{F}_e = \frac{9\epsilon_0 E_0^2}{2} \int_{\theta=0}^{\pi/2} \int_{\phi=0}^{2\pi} \cos^2 \theta \begin{bmatrix} \sin \theta \cos \phi \\ \sin \theta \sin \phi \\ \cos \theta \end{bmatrix} (a^2 \sin \theta d\theta d\phi)$$

Both the x and y components will be zero—integrating $\cos \phi$ and $\sin \phi$ over a full period 2π give zero, while the ϕ contribution to the z component is 2π . We make this explicit with

$$\mathbf{F}_e = \frac{9\epsilon_0 E_0^2}{2} \int_{\theta=0}^{\pi/2} \cos^2 \theta \begin{bmatrix} 0 \\ 0 \\ 2\pi \cos \theta \end{bmatrix} (a^2 \sin \theta d\theta)$$

The non-zero z component F_z is

$$\begin{aligned} F_z &= \frac{9\epsilon_0 E_0^2}{2} \int_{\theta=0}^{\pi/2} 2\pi \cos^3 \theta (a^2 \sin \theta d\theta) = 9\pi\epsilon_0 E_0^2 a^2 \int_{\theta=0}^{\pi/2} \cos^3 \theta (\sin \theta d\theta) \\ &= 9\pi\epsilon_0 E_0^2 a^2 \int_0^1 \cos^3 \theta d[\cos \theta] = \frac{9\pi\epsilon_0 a^2 E_0^2}{4} \end{aligned}$$

The vector force can be written simply as

$$\mathbf{F} = \frac{9\pi\epsilon_0 a^2 E_0^2}{4} \hat{\mathbf{z}}$$

In other words, the force on the upper half points upward in the positive z direction.

1.6.3 Point Charge Between Two Conducting Plates

We place two large conducting plates at a right angle to each other, so that the plates come close together but just barely do not touch. We then place a point charge q along the line bisecting the right angle between the plates, at a perpendicular distance a from each plate. Both plates are grounded. What is the electric potential in the region bounded by the plates at large distances from the plates' intersection?

- Assume $r = 0$ along the line connecting the two plates. For a single plate, we could solve the problem with the method of images—see the previous exercise set. With two plates we proceed analogously, with a mirror image for each plate. Because of the two reflections from the two plates, we end up with three imaginary charges plus the one original one in a quadrupole arrangement. (This is hard to describe in words, it is best to see a picture). For large r , the charge arrangement will have the field of an electric quadrupole. Solving the problem thus reduces to a multipole expansion of the electric potential to the quadrupole term.

Theory: Multipole Expansion

- The multipole expansion of U to quadrupole order, using the Einstein summation convention, is

$$U(r) = \frac{1}{4\pi\epsilon_0} \left[\frac{q}{r} + \frac{p_i r_i}{r^3} + \frac{Q_{ij} r_i r_j}{r^5} \right]$$

where \mathbf{p} is the electric dipole moment and \mathbf{Q} is the quadrupole moment tensor. *Note:* we could think of the charge q is a scalar monopole moment, creating a logical progression from scalar monopole moment to vector dipole moment to tensor quadrupole moment.

- We find the dipole moment with

$$\mathbf{p} = \int d^3\tilde{r} \rho(\tilde{\mathbf{r}}) \tilde{\mathbf{r}}$$

We find the quadrupole moment by components:

$$Q_{ij} = \int d^3\tilde{r} \rho(\tilde{\mathbf{r}}) (3\tilde{r}_i \tilde{r}_j - \delta_{ij} \tilde{r}^2)$$

The discrete analog a configuration of N charges reads

$$Q_{ij} = \sum_{n=1}^N q_n (3r_{ni} r_{nj} - \delta_{ij} r_n^2)$$

Note that both definitions produces a symmetric tensor. Also important: the tensor's trace—the sum of the diagonal elements is zero:

$$\text{tr } \mathbf{Q} = \sum_n q_n [3x_n^2 - r_n^2 + 3z_n^2 - r_n^2 + 3z_n^2 - r_n^2] = \sum_n q_n [3r_n^2 - 3r_n^2] = 0$$

Back to Our Problem

- For our imaginary quadrupole configuration of four charges, the total charge, and thus the monopole moment, is zero. Analogously, the total dipole moment of the arrangement, which consists of two positive and two negative charges, is zero—the two dipoles cancel each other out.

From the three terms in our multipole expansion of $U(r)$, only the quadrupole term remains. We just have to calculate the quadrupole tensor Q_{ij} . We label the four charges in the imaginary quadrupole configuration as 1, 2, 3, and 4, where 1 is the original positive charge in the upper right corner, 2 is the negative image charge in the upper left corner, 3 is the positive image charge in the lower left corner, and 4 is the negative image charge in the lower right corner.

- Using the discrete formula for Q_{ij} , the first component Q_{xx} is

$$\begin{aligned} Q_{xx} &= \sum_{n=1}^N q_n (3x_n^2 - \delta_{ij} r_n^2) = q (3a^2 - 2a^2) + (-q) (3a^2 - 2a^2) \\ &= q (3a^2 - 2a^2) + (-q) (3a^2 - 2a^2) = 0 \end{aligned}$$

The other diagonal terms Q_{yy} and Q_{zz} will analogously sum to zero.

All off-diagonal terms with a z component are also zero, since the charges lie in a plane with $z = 0$. We thus have $Q_{xz} = Q_{zx} = Q_{yz} = Q_{zy} = 0$. We have just two terms left calculate: Q_{xy} and Q_{yx} . By the tensor's symmetry, the two are equal, so we really only have one term:

$$\begin{aligned} Q_{xy} &= \sum_{n=1}^N q_n (3x_n y_n - 0 \cdot r_n^2) = 3qa^2 + 3(-q)(-a^2) + 3qa^2 + 3(-q)(-a^2) \\ &= 12qa^2 \end{aligned}$$

The quadrupole tensor is

$$\mathbf{Q} = \begin{bmatrix} 0 & 12qa^2 & 0 \\ 12qa^2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

As expected, the tensor is symmetric with trace $\text{tr } \mathbf{Q} = 0$.

- Recall the quadrupole expansion of $U(r)$:

$$U(r) = \frac{1}{4\pi\epsilon_0} \left[\frac{q}{r} + \frac{p_i r_i}{r^3} + \frac{Q_{ij} r_i r_j}{r^5} \right]$$

In our case, with $q = 0$ and $\mathbf{p} = 0$, we have

$$\begin{aligned} U(r) &= \frac{1}{4\pi\epsilon_0} \left[\frac{12qa^2 xy}{r^5} + \frac{12qa^2 yx}{r^5} + 0 + \dots + 0 \right] = \frac{6qa^2}{\pi\epsilon_0} \frac{xy}{r^5} \\ &= \frac{6qa^2}{\pi\epsilon_0} \frac{\cos \phi \sin \phi \sin^2 \theta}{r^3} \end{aligned}$$

The second line uses the spherical coordinates $x = r \cos \phi \sin \theta$ and $y = r \sin \phi \sin \theta$. In fact, the expression for $U(r)$ takes the exact same form as the wave function of a d electron orbital (angular momentum quantum number $l = 2$) in a hydrogen atom. The equipotential surfaces of $U(r)$ have the same spatial distribution as the d_{xy} orbitals for a hydrogen wave function.

2 Magnetostatics and Electrodynamics

2.1 Seventh Exercise Set

2.1.1 Theory: Magnetic Vector Potential and the Biot-Savart Law

- We will need to use two more Maxwell equations for magnetostatics. The first is

$$\nabla \cdot \mathbf{B} = 0$$

This equation rules out the possibility of magnetic monopoles and allows \mathbf{B} to be written as the curl of a vector potential

\mathbf{A} as

$$\mathbf{B} = \nabla \times \mathbf{A}$$

The second Maxwell equation is

$$\nabla \times \mathbf{B} = \mu_0 \left(\mathbf{j} + \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right)$$

For static situations with a constant electric field this simplifies to

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{j}$$

Substituting the expression $\mathbf{B} = \nabla \times \mathbf{A}$ into the static second Maxwell equation produces

$$\nabla \times (\nabla \times \mathbf{A}) = \nabla \cdot (\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} = \mu_0 \mathbf{j}$$

- The magnetic vector potential is defined only up to a constant; we usually choose \mathbf{A} so that $\nabla \cdot \mathbf{A} = 0$. In this convention, we have

$$\nabla^2 \mathbf{A} = -\mu_0 \mathbf{j}$$

where \mathbf{j} is the current density vector. This equation is a vector analog of the Poisson equation $\nabla^2 U = -\frac{\rho}{\epsilon_0}$ from electrostatics. Similarly to how the electrostatic potential U at a point \mathbf{r} in region of space with charge density ρ is found with

$$U = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\tilde{\mathbf{r}}) d^3\tilde{\mathbf{r}}}{|\mathbf{r} - \tilde{\mathbf{r}}|}$$

The magnetic potential at a point \mathbf{r} in region of space with current density \mathbf{j} is

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{j}(\tilde{\mathbf{r}}) d^3\tilde{\mathbf{r}}}{|\mathbf{r} - \tilde{\mathbf{r}}|}$$

- In problems involving one-dimensional conductors, where \mathbf{j} is non-zero only along the conductor, the expression for $\mathbf{j}(\tilde{\mathbf{r}}) d^3\tilde{\mathbf{r}}$ simplifies to

$$\mathbf{j}(\tilde{\mathbf{r}}) d^3\tilde{\mathbf{r}} = \mathbf{j}(\tilde{\mathbf{r}}) d\tilde{S} d\tilde{l} = I \hat{\mathbf{t}} d\tilde{l}$$

where $\hat{\mathbf{t}}$ is the unit normal vector tangent to the conductor, I is the current through the conductor at the point $\tilde{\mathbf{r}}$ and $d\tilde{l}$ is a small distance element along the conductor's length. The magnetic vector potential for a one-dimensional conductor carrying a current I then simplifies to

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0 I}{4\pi} \int \frac{\hat{\mathbf{t}} d\tilde{l}}{|\mathbf{r} - \tilde{\mathbf{r}}|}$$

- Recall $\mathbf{B} = \nabla \times \mathbf{A}$. Taking the curl of the general expression for \mathbf{A} in terms of current density \mathbf{j} gives general expression for the magnetic field

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{j}(\tilde{\mathbf{r}}) \times (\mathbf{r} - \tilde{\mathbf{r}})}{|\mathbf{r} - \tilde{\mathbf{r}}|^3} d^3\tilde{\mathbf{r}}$$

This is a general form of the Biot-Savart law for the magnetic field \mathbf{B} of a current distribution.

2.1.2 Magnetic Field of a Circular Current Loop

A closed conducting loop of radius a carries current I . What is the magnetic vector potential far from the conducting loop?

- Our starting point is the vector potential of a one-dimensional conductor from the theory section, i.e.

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0 I}{4\pi} \int \frac{\hat{\mathbf{t}} dl}{|\mathbf{r} - \tilde{\mathbf{r}}|}$$

We need expressions for \mathbf{r} , $\tilde{\mathbf{r}}$ and $\hat{\mathbf{t}}$.

Assume the loop lies in the x, y plane with the z axis normal to the loop. For mathematical convenience, we rotate the x, y plane so \mathbf{r} lies in the x, z plane (i.e. $\phi = 0$); this just gives us one less non-zero component to work with, since the $\sin \phi$ term in the y component of \mathbf{r} is zero. In polar coordinates \mathbf{r} reads

$$\mathbf{r} = (r \sin \theta, 0, r \cos \theta)$$

where θ is the angle between \mathbf{r} and the z axis.

The integration variable $\tilde{\mathbf{r}}$, which runs over the current loop in the x, y plane, reads

$$\tilde{\mathbf{r}} = (a \cos \tilde{\phi}, a \sin \tilde{\phi}, 0)$$

Where $\tilde{\phi}$ is the azimuthal angle between $\tilde{\mathbf{r}}$ and the x axis.

Finally, the expression for $\hat{\mathbf{t}}$, the tangent to the current loop, is

$$\hat{\mathbf{t}} = (-\sin \tilde{\phi}, \cos \tilde{\phi}, 0)$$

The small distance element is $d\tilde{l} = a d\tilde{\phi}$.

$$\begin{aligned} |\mathbf{r} - \tilde{\mathbf{r}}| &= \sqrt{(r \sin \theta - a \cos \tilde{\phi})^2 + a^2 \sin^2 \tilde{\phi} + r^2 \cos^2 \theta} \\ &= \sqrt{r^2 + a^2 - 2ra \sin \theta \cos \tilde{\phi}} \\ &\approx \sqrt{r^2 - 2ra \sin \theta \cos \tilde{\phi}} = r \sqrt{1 - \frac{2a}{r} \sin \theta \cos \tilde{\phi}} \end{aligned}$$

where the last line uses $r \gg a$ (recall we're interested in the solution far from the conducting loop). We now have, again using $a \ll r \implies \frac{a}{r} \ll 1$,

$$\frac{1}{|\mathbf{r} - \tilde{\mathbf{r}}|} = \frac{1}{r} \left(1 - \frac{2a}{r} \sin \theta \cos \tilde{\phi} \right)^{-1/2} \approx \frac{1}{r} \left(1 + \frac{1}{2} \frac{2a}{r} \sin \theta \cos \tilde{\phi} \right)$$

Substituting the expressions for $\frac{1}{|\mathbf{r}-\tilde{\mathbf{r}}|}$, $\hat{\mathbf{t}}$ and $d\tilde{l}$ into expression for \mathbf{A} gives

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0 I}{4\pi} \int \frac{\hat{\mathbf{t}} d\tilde{l}}{|\mathbf{r}-\tilde{\mathbf{r}}|} = \frac{\mu_0 I}{4\pi} \frac{a}{r} \int_0^{2\pi} d\tilde{\phi} \begin{bmatrix} -\sin \tilde{\phi} \\ \cos \tilde{\phi} \\ 0 \end{bmatrix} \left(1 + \frac{1}{2} \frac{2a}{r} \sin \theta \cos \tilde{\phi} \right)$$

- The integrals conveniently simplify, since we are integrating sinusoidal terms over an entire period. Only the integral of $\cos^2 \tilde{\phi}$ in the y component is nonzero. We end up with $A_x = A_z = 0$ and

$$A_y(r) = \frac{\mu_0 I}{4\pi} \frac{a}{r} \int_0^{2\pi} \frac{a \sin \theta}{r} \cos^2 \tilde{\phi} d\tilde{\phi} = \frac{\mu_0 I}{4} \frac{a^2}{r^2} \sin \theta$$

or, in vector form,

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0 I}{4} \frac{a^2}{r^2} \sin \theta \hat{\mathbf{y}} = \frac{\mu_0 I}{4} \frac{a^2}{r^2} \hat{\mathbf{z}} \times \hat{\mathbf{r}}$$

The second expression is preferable: the first, in terms of $\hat{\mathbf{y}}$, holds only with x, y plane rotated so $\phi = 0$ and \mathbf{r} lies in the x, z plane. The second, which uses $\sin \theta \hat{\mathbf{y}} = \hat{\mathbf{z}} \times \hat{\mathbf{r}}$, holds for any orientation of the x, y plane.

Finally, using $I\pi a^2 = |\mathbf{m}|$ (where \mathbf{m} is the loop's magnetic dipole moment, which points in the direction of the loop's normal), the expression for \mathbf{A} simplifies to

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0 \mathbf{m} \times \hat{\mathbf{r}}}{4\pi r^2} = \frac{\mu_0 \mathbf{m} \times \mathbf{r}}{4\pi r^3}$$

This is the same form of magnetic vector potential as for a magnetic dipole. In other words, a circular current loop behaves as magnetic dipole at long distances.

2.1.3 Magnetic Field of a Rotating Charged Disk

Consider a charged, insulating disk with uniform surface charge density σ and radius a . The disk rotates uniformly about an axis through its center with constant angular speed ω . Find the magnetic field along the axis of rotation.

- We choose a coordinate system so the disk lies in the x, y plane and the rotation axis coincides with the z axis, so $\boldsymbol{\omega} = (0, 0, \omega)$. Start with the general Biot-Savart law

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{j}(\tilde{\mathbf{r}}) \times (\mathbf{r} - \tilde{\mathbf{r}})}{|\mathbf{r} - \tilde{\mathbf{r}}|^3} d^3 \tilde{\mathbf{r}}$$

and, like in the previous problem, find expressions for each vector quantity in the equation. The expression for \mathbf{r} along the z axis is simply $\mathbf{r} = (0, 0, z)$, while the expression for $\tilde{\mathbf{r}}$, which lies in the disk in the x, y plane, is

$$\tilde{\mathbf{r}} = (\tilde{r} \cos \tilde{\phi}, \tilde{r} \sin \tilde{\phi}, 0)$$

The difference of \mathbf{r} and $\tilde{\mathbf{r}}$ and its magnitude is

$$\mathbf{r} - \tilde{\mathbf{r}} = (-\tilde{r} \cos \tilde{\phi}, -\tilde{r} \sin \tilde{\phi}, z) \quad \text{and} \quad |\mathbf{r} - \tilde{\mathbf{r}}| = \sqrt{\tilde{r}^2 + z^2}$$

Finally, the current density \mathbf{j} , which points tangent to the disk's rotation, is

$$\mathbf{j} = j(-\sin \tilde{\phi}, \cos \tilde{\phi}, 0)$$

- Using the expressions for our vector quantities, the cross product $\mathbf{j}(\tilde{\mathbf{r}}) \times (\mathbf{r} - \tilde{\mathbf{r}})$ is

$$\mathbf{j}(\tilde{\mathbf{r}}) \times (\mathbf{r} - \tilde{\mathbf{r}}) = j \begin{bmatrix} -\sin \tilde{\phi} \\ \cos \tilde{\phi} \\ 0 \end{bmatrix} \begin{bmatrix} -\tilde{r} \cos \tilde{\phi} \\ -\tilde{r} \sin \tilde{\phi} \\ z \end{bmatrix} = j \begin{bmatrix} z \cos \tilde{\phi} \\ z \sin \tilde{\phi} \\ \tilde{r} \end{bmatrix}$$

Substituting $\mathbf{j}(\tilde{\mathbf{r}}) \times (\mathbf{r} - \tilde{\mathbf{r}})$ and $|\mathbf{r} - \tilde{\mathbf{r}}|$ into the Biot-Savart law gives

$$\mathbf{B}(z) = \frac{\mu_0}{4\pi} \int \frac{j d^3 \tilde{\mathbf{r}}}{(\tilde{r}^2 + z^2)^{3/2}} \begin{bmatrix} z \cos \tilde{\phi} \\ z \sin \tilde{\phi} \\ \tilde{r} \end{bmatrix}$$

Next, we write $j d^3 \tilde{\mathbf{r}} = j d\tilde{S} d\tilde{l} = j d\tilde{S}(\tilde{r} d\tilde{\phi}) = dI \tilde{r} d\tilde{\phi}$. Note that the product $j d\tilde{S}$ is the current element dI in the surface element $d\tilde{S}$. The current dI at the radius \tilde{r} on the disk rotating with period $t_0 = \frac{2\pi}{\omega}$ is

$$dI = \frac{dq}{t_0} = \frac{dq}{2\pi} \omega = \frac{(\sigma 2\pi \tilde{r} d\tilde{r})}{2\pi} \omega = \sigma \omega \tilde{r} d\tilde{r}$$

Substituting $j d^3 \tilde{\mathbf{r}} = dI \tilde{r} d\tilde{\phi} = \sigma \omega \tilde{r}^2 d\tilde{r} d\tilde{\phi}$ into the Biot-Savart law gives

$$\mathbf{B}(z) = \frac{\mu_0}{4\pi} \int_0^a d\tilde{r} \int_0^{2\pi} d\tilde{\phi} \frac{\sigma \omega \tilde{r}^2}{(\tilde{r}^2 + z^2)^{3/2}} \begin{bmatrix} z \cos \tilde{\phi} \\ z \sin \tilde{\phi} \\ \tilde{r} \end{bmatrix}$$

The first and second components of \mathbf{B} contain integrals \cos and \sin terms over a full period—the result is zero. After integrating over $\tilde{\phi}$, the magnetic field simplifies to

$$\mathbf{B}(z) = \frac{\mu_0}{4\pi} \int_0^a d\tilde{r} \frac{\sigma \omega \tilde{r}^2}{(\tilde{r}^2 + z^2)^{3/2}} \begin{bmatrix} 0 \\ 0 \\ 2\pi \tilde{r} \end{bmatrix}$$

Only the z component of \mathbf{B} is non-zero; it is

$$B_z(z) = \frac{\mu_0}{2} \int_0^a d\tilde{r} \frac{\sigma \omega \tilde{r}^3}{(\tilde{r}^2 + z^2)^{3/2}}$$

In terms of the new variable $u = \tilde{r}^2 + z^2$, the integral evaluates to

$$\begin{aligned} B_z(z) &= \frac{\mu_0}{2} \frac{\sigma \omega}{2} \int_{z^2}^{z^2+a^2} \frac{u - z^2}{u^{3/2}} du = \frac{\mu_0 \sigma \omega}{4} \left[2u^{1/2} + 2z^2 u^{-1/2} \right]_{z^2}^{z^2+a^2} \\ &= \frac{\mu_0 \sigma \omega}{2} \left(\sqrt{z^2 + a^2} - z + \frac{z^2}{\sqrt{z^2 + a^2}} - z \right) \\ &= \frac{\mu_0 \sigma \omega}{2} \left(\frac{2z^2 + a^2}{\sqrt{z^2 + a^2}} - 2z \right) \end{aligned}$$

The magnetic field along the z axis is thus $\mathbf{B} = (0, 0, B_z)$ with B_z as above.

- Next, we consider the limit case $z \gg a$, far from the rotating disk. Expanding the square root to fourth order in the small quantity $\frac{a}{z}$, multiplying out and simplifying like terms gives

$$\begin{aligned} B_z &= \frac{\mu_0 \sigma \omega}{2} \left(\frac{2z^2 + a^2}{z \sqrt{1 + \frac{a^2}{z^2}}} - 2z \right) \approx \frac{\mu_0 \sigma \omega}{2} \left[\left(2z + \frac{a^2}{z} \right) \left(1 - \frac{a^2}{2z^2} + \frac{3a^4}{8z^4} \right) - 2z \right] \\ &= \frac{\mu_0 \sigma \omega}{2} \left[2z + \frac{a^2}{z} - \frac{a^2}{z} - \frac{1}{2} \frac{a^4}{z^3} + \frac{3}{4} \frac{a^4}{z^3} + \frac{3}{8} \frac{a^6}{z^5} - 2z \right] = \frac{\mu_0 \sigma \omega}{2} \left[\frac{1}{4} \frac{a^4}{z^3} + \frac{3}{8} \frac{a^6}{z^5} \right] \end{aligned}$$

Neglecting the highest-order $\frac{a^6}{z^5}$ term gives the simple result

$$B_z \approx \frac{\mu_0 \sigma \omega}{8} \frac{a^4}{z^3}, \quad z \gg a$$

In other words, far from the disk, the magnetic field falls off as z^{-3} , just like the field of a magnetic dipole.

- Next, we will try to write the magnetic field in the form $B_z \propto \frac{|\mathbf{m}|}{z^3}$ where \mathbf{m} is the disk's magnetic dipole moment. Integrating over concentric rings with area S carrying current dI , the disk's magnetic dipole moment is

$$|\mathbf{m}| = \int S dI = \int_0^a (\pi \tilde{r}^2) \cdot (\sigma \omega \tilde{r} d\tilde{r}) = \pi \sigma \omega \int_0^a \tilde{r}^3 d\tilde{r} = \frac{\pi}{4} \sigma \omega a^4$$

Comparing this expression for $|\mathbf{m}|$ to the similar expression for B_z leads to

$$B_z \approx \frac{\mu_0 \sigma \omega}{8} \frac{a^4}{z^3} = \frac{\mu_0 |\mathbf{m}|}{2\pi z^3}$$

which is in the desired form $B_z \propto \frac{|\mathbf{m}|}{z^3}$. The general form for the magnetic field of a magnetic dipole is

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \frac{3(\mathbf{m} \cdot \mathbf{r})\mathbf{r} - \mathbf{m}r^2}{r^5}$$

In fact, this expression is equivalent to our result $B_z(z) = \frac{\mu_0 |\mathbf{m}|}{2\pi z^3}$. Since \mathbf{m} and \mathbf{r} both point along the z axis, their dot product is $\mathbf{m} \cdot \mathbf{r} = |\mathbf{m}|r$. Along the z axis, $\mathbf{r} = (0, 0, z)$ and the general expression for the dipole magnetic field simplifies to

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \frac{3|\mathbf{m}|z^2 - |\mathbf{m}|z^2}{z^5} \hat{\mathbf{z}} = \frac{\mu_0 |\mathbf{m}|}{2\pi z^3} \hat{\mathbf{z}},$$

in agreement with our expression for B_z for $z \gg a$.

2.2 Eighth Exercise Set

Theory: Magnetic Force

The magnetic force \mathbf{F} on the matter contained in the region of space enclosed in the region V and permeated by the magnetic field \mathbf{B} is

$$\mathbf{F} = \frac{1}{\mu_0} \oint_{\partial V} \left[\mathbf{B}(\mathbf{B} \cdot \hat{\mathbf{n}}) - \frac{1}{2} \mathbf{B}^2 \hat{\mathbf{n}} \right] dS$$

where $\hat{\mathbf{n}}$ is the normal vector to the region's boundary ∂V .

2.2.1 Magnetic Force in a Coaxial Cable

A long coaxial cable consists of a thin inner wire and outer sheath with radius a . The inner wire carries a current I , and we connect the sheath to the inner wire at the cable's ends so that the current I returns along the outer sheath in the opposite direction as the current along the inner wire. Find the magnetic force per unit length on the outer sheath.

- There are two contributions to the magnetic force on the sheath: the repulsive, radially outward force between the sheath and the inner wire and an attractive “surface tension” force distributed across the sheath's surface, which carries uniformly distributed current I (think of the sheath as a collection of parallel conducting wires, which attract each other).

- Consider one-half of the sheath, which forms a semicircular cross section. There are downward forces F_1 at end of the semicircle and an upward magnetic force F_m acting on the top of the sheath. In equilibrium, the forces are related by $F_m = 2F_1$.
- First, we find the magnetic field in the conductor. Inside the sheath, Ampere's law with a circular path encircling the inner wire reads

$$\oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 I \implies B(2\pi r) = \mu_0 I \implies B = \frac{\mu_0 I}{2\pi r}$$

Outside the conductor, the net current enclosed by a circular path encircling both the outer sheath and inner wire is zero, and the Ampere's law reads

$$\int \mathbf{B} \cdot d\mathbf{l} = \mu_0(+I - I) = 0 \implies B = 0$$

In other words, there is no magnetic field outside the conductor.

- With the coaxial cable's magnetic field known, we now find the magnetic force \mathbf{F}_m on a semi-circular half of the outer sheath.

$$\mathbf{F}_m = \frac{1}{\mu_0} \oint_{\partial V} \left[\mathbf{B}(\mathbf{B} \cdot \hat{\mathbf{n}}) - \frac{1}{2} B^2 \hat{\mathbf{n}} \right] dS$$

We choose an integration surface tightly hugging the half-sheath and work in cylindrical coordinates r, ϕ . Outside the sheath, the magnetic field is zero and there is no contribution to \mathbf{F}_m . Inside the sheath, the magnetic field is tangent to the semicircle, so $\mathbf{B} \cdot \hat{\mathbf{n}} = 0$, and the force integral reads

$$\mathbf{F}_m = \frac{1}{\mu_0} \int \left[0 - \frac{1}{2} B^2 \hat{\mathbf{n}} \right] dS$$

The magnetic field at along the sheath (where $r = a$) is constant and equal to $B = \frac{\mu_0 I}{2\pi a}$ and can be moved outside the integral. The normal vector in cylindrical coordinates reads

$$\hat{\mathbf{n}} = (-\cos \phi, -\sin \phi, 0)$$

while the surface element is $dS = a l d\phi$. The force integral reads

$$\mathbf{F}_m = -\frac{1}{2} \frac{\mu_0 I^2}{4\pi^2 a^2} \int_0^\pi \begin{bmatrix} -\cos \phi \\ -\sin \phi \\ 0 \end{bmatrix} l a d\phi$$

The x component with $\cos \phi$ integrates to zero over $\phi \in [0, \pi]$. Only the y component is nonzero, and the vector force reads

$$\mathbf{F}_m = +\frac{1}{2} \frac{\mu_0 I^2}{4\pi^2 a^2} (2al) \hat{\mathbf{y}} = \frac{\mu_0 I^2 l}{4\pi^2 a} \hat{\mathbf{y}}$$

- The forces F_1 at the two ends of the semicircle are then

$$F_1 = \frac{F_m}{2} = \frac{\mu_0 I^2 l}{8\pi^2 a} \implies \frac{F}{l} = \frac{\mu_0 I^2}{8\pi^2 a}$$

2.2.2 Tension in a Toroidal Inductor

A toroidal inductor with N coils radius r_1 and cross-sectional radius r_2 carries a current I . Find the tension force on a single coil if $r_1 \gg r_2$.

- First, we find the magnetic field inside the inductor with Ampere's law using a closed circular path of radius $r \in (r_1, r_1 + r_2)$ in the inductor's equatorial plane. Ampere's law reads

$$B_{\text{in}}(2\pi r) = \mu_0(NI) \implies B_{\text{in}} = \frac{\mu_0 NI}{2\pi r}$$

Outside the inductor, with a circular path of radius $r > r_1 + r_2$ in the inductor's equatorial plane, Ampere's law reads

$$B_{\text{out}}(2\pi r) = \mu_0(NI + N(-I)) = 0 \implies B_{\text{out}} = 0$$

- Consider a semicircular half of a single inductor coil. As in the previous problem, the magnetic force with magnitude F_m acts upwards on the top of the semicircle, while two "surface tension" forces $F_1 = \frac{F_m}{2}$ act downward at the semicircle's two ends. As usual, we find the magnetic force \mathbf{F}_m using

$$\mathbf{F}_m = \frac{1}{\mu_0} \oint_{\partial V} \left[\mathbf{B}(\mathbf{B} \cdot \hat{\mathbf{n}}) - \frac{1}{2} B^2 \hat{\mathbf{n}} \right] dS$$

For the integration surface (awkward to describe, best to see a picture) we choose a surface that looks like a coin cut in half, basically a thin three-dimensional extension of a semicircle enclosing the coil's semicircular upper half. We split the surface into four parts: the left and right semicircular faces, the circular ribbon along the surface's outer radius, and the thin rectangular plane along the surface's bottom.

The circular upper ribbon occurs just outside the inductor coil (where $B = 0$) and thus does not contribute the magnetic force. The left and right semicircular faces have equal magnitude and opposite-sign contributions, since the normal to the surface changes sign for each face.

- Only the rectangular plane has a nonzero contribution to \mathbf{F}_m . Under the assumption $r_1 \gg r_2$, the magnetic field along the plane simplifies to

$$B = \int_{r_1}^{r_1+r_2} \frac{\mu_0 NI}{2\pi \tilde{r}} d\tilde{r} \approx \frac{\mu_0 NI}{2\pi r_1}$$

\mathbf{B} points along the toroid's longitudinal axis (into the plane of a cross-sectional coil) and is perpendicular to the normal $\hat{\mathbf{n}}$ to the planar integration surface, so $\hat{\mathbf{B}} \cdot \hat{\mathbf{n}} = 0$. The magnetic force simplifies to

$$\begin{aligned} \mathbf{F}_m &= \frac{1}{\mu_0} \left[0 - \frac{1}{2} B^2 \hat{\mathbf{n}} \right] dS = -\frac{1}{2\mu_0} \left(\frac{\mu_0 NI}{2\pi r_1} \right)^2 \hat{\mathbf{n}} \int dS \\ &= -\frac{1}{2\mu_0} \left(\frac{\mu_0 NI}{2\pi r_1} \right)^2 \hat{\mathbf{n}} \left(r_2 \frac{2\pi r_1}{N} \right) = -\frac{\mu_0 NI^2 r_2}{2\pi r_1} \hat{\mathbf{n}} \end{aligned}$$

$\hat{\mathbf{n}}$ points downward, so $-\hat{\mathbf{n}}$ and thus \mathbf{F}_m point upward and pull the coil apart. The magnitude of the tension on the coil is then

$$F_1 = \frac{F_m}{2} = \frac{\mu_0 NI^2 r_2}{4\pi r_1}$$

2.2.3 Resistance of a Thin Conducting Plate

The conducting plate is half of a thin annulus with inner radius r_1 , outer radius r_2 , thickness h and resistivity σ with electrodes at each end. Find the plates electric resistance R when we establish a potential difference U_0 between the electrodes at the plate's ends.

- We first find the current density in the conducting plate using Ohm's law

$$\mathbf{j} = \sigma \mathbf{E}$$

where \mathbf{j} and \mathbf{E} are the current density and electric field in the conductor. We write \mathbf{j} in terms of the continuity equation

$$\nabla \cdot \mathbf{j} + \frac{\partial \rho}{\partial t} = 0$$

For a stationary charge distribution, we have $\nabla \cdot \mathbf{j} = 0$. We then take the divergence of both sides of the equation $\mathbf{j} = \sigma \mathbf{E}$ and apply $\mathbf{E} = -\nabla U$ to get

$$\nabla \cdot \mathbf{j} = \sigma \nabla \cdot \mathbf{E} = -\sigma \nabla^2 U = \nabla \cdot \mathbf{j} = 0 \implies \nabla^2 U = 0$$

We end up with a Laplace equation for U in the conductor. The plan is to find $U(\mathbf{r})$, then $\mathbf{E}(\mathbf{r})$, then \mathbf{j} via $\mathbf{j} = \sigma \mathbf{E}$, then I and finally resistance with $R = \frac{U}{I}$.

- The general solution of the Laplace equation in cylindrical coordinates (including the $m = 0$ term) is

$$U(r, \phi) = \sum_{m=1}^{\infty} [A_m \cos(m\phi) + B_m \sin(m\phi)] \cdot [C_m r^m + D_m r^{-m}] + (a\phi + b)(c \ln r + d)$$

To find a unique solution, we need boundary conditions for our particular problem. At the first electrode at $\phi = 0$, the electric potential is constant; we'll set $U(r, 0) = 0$ for convenience. At the second electrode, the potential is $U(r, \pi) = -U_0$ to make a potential difference U_0 between electrodes (we choose $-U_0$ so the current runs in the direction of increasing ϕ).

Along the annulus's semicircular boundaries the electric field must be tangent to the surface to satisfy $\nabla \cdot \mathbf{j} = 0$. Along these surfaces the radial component of both \mathbf{j} and \mathbf{E} is zero; $E_r = 0$ gives the boundary condition

$$E_r = - \left. \frac{\partial U}{\partial r} \right|_{r_1, r_2} = 0$$

- The general solution for $U(r, \phi)$ satisfies the boundary condition $\left. \frac{\partial U}{\partial r} \right|_{r_1, r_2} = 0$ for all ϕ only if $C_m = D_m = c = 0$ (try finding $\left. \frac{\partial U}{\partial r} \right|_{r_1, r_2}$ to see for yourself). With $C_m = D_m = c = 0$, the expression for $U(r, \phi)$ simplifies to

$$U(r, \phi) = (a\phi + b)d \equiv \tilde{a}\phi + \tilde{b}$$

The boundary condition $U(r, 0) = 0$ gives

$$U(r, 0) \equiv 0 + \tilde{b} \implies \tilde{b} = 0$$

The final boundary condition $U(r, \pi) = -U_0$ gives

$$U(r, \pi) \equiv -U_0 \implies \tilde{a} = -\frac{U_0}{\pi} \implies U(r, \phi) = -\frac{U_0}{\pi}\phi$$

In other words, the electric potential is a linear function of ϕ .

- With $U(r, \phi)$ known, we find the tangential electric field E_t with

$$E_t = -\frac{1}{r} \frac{\partial U}{\partial \phi} = \frac{U_0}{\pi r},$$

from which we find the tangential current density j_t with

$$j_t = \sigma E_t = \frac{\sigma U_0}{\pi r}$$

We find the total current I from the current density via

$$I = \int j_t dS = \int_{r_1}^{r_2} \left(\frac{\sigma U_0}{\pi r} \right) \cdot h dr = \frac{\sigma U_0 h}{\pi} \ln \frac{r_2}{r_1}$$

where h is the plate's thickness. The electrical resistance is then

$$R = \frac{U_0}{I} = \frac{\pi}{\sigma h \ln \frac{r_2}{r_1}}$$

2.3 Ninth Exercise Set

Theory: Inductance

- Inductance is the proportionality constant between magnetic flux Φ and current I . For a single object, e.g. a current loop, the relationship between Φ and I reads

$$\Phi_1 = L_{11} I_1$$

where the quantity L_{11} is called self-inductance.

- For a system of two current-carrying loops, where the current through one loop generates a magnetic field and thus magnetic flux through the second loop, and vice versa, the relationships between Φ and I read

$$\Phi_1 = L_{12} I_2 \quad \text{and} \quad \Phi_2 = L_{21} I_1$$

where the quantities L_{12} and L_{21} are the loops' mutual inductances. Without derivation, we state that $L_{12} = L_{21}$ for reasons of symmetry.

2.3.1 Mutual Inductance

Consider two long, parallel wires of length l separated by a distance $d \ll l$ and connected at their endpoints to form a long conducting loop. We place a square frame with side length $\sqrt{2}d$ between the wires and supply the frame with an alternating current source $I_1 = I_0 \sin \omega t$. Find the system's mutual inductivity and the induced current in the wires. Neglect Ohmic losses, and assume $a \ll d$ where a is the wires' radius.

Finding Mutual Inductance

- We label the frame as object 1 and the wires as object 2 and introduce a coordinate y separating the parallel wires so that the bottom wire occurs at $y = 0$ and the top wire at $y = d$. To find the system's mutual inductance L_{12} , we consider the hypothetical magnetic flux Φ_1 through the frame due to a current I_2 in the parallel wires. We then find L_{12} with

$$L_{12} = \frac{\Phi_1}{I_2}$$

- If the parallel wires carry a current I_2 (in opposite directions, since they form a closed loop), the corresponding magnetic field is

$$B = \frac{\mu_0 I_2}{2\pi} \left(\frac{1}{y} + \frac{1}{d-y} \right) = \frac{\mu_0 I_2}{2\pi} \frac{d}{y(d-y)}$$

- We find magnetic flux with $\Phi = \int \mathbf{B} \cdot d\mathbf{S}$; the dot product drops because \mathbf{B} is parallel to the frame's cross section $d\mathbf{S}$. The square frame's surface element is $dS = 2y dy$ (the frame's width is $2y$), and the magnetic flux through the square frame is

$$\begin{aligned} \Phi_1 &= \int B dS = 2 \int_0^{d/2} \frac{\mu_0 I_2 d}{2\pi} \frac{2y dy}{y(d-y)} = -\frac{2\mu_0 I_2 d}{\pi} \ln(d-y) \Big|_0^{d/2} \\ &= \frac{2\mu_0 I_2 d}{\pi} \left[\ln d - \ln \frac{d}{2} \right] = \frac{2\mu_0 d \ln 2}{\pi} I_2 \end{aligned}$$

Note the use of symmetry—we integrate only from 0 to $d/2$ and multiply by two.

The mutual inductance—the proportionality between Φ and I_2 is

$$L_{12} = \frac{\Phi_1}{I_2} = \frac{2 \ln 2}{\pi} \mu_0 d$$

Note that L_{12} depends only on system's geometry.

Induced Current in the Parallel Wires

- Recall the square frame carries an alternating current

$$I_1(t) = I_{10} \sin \omega t,$$

so we expect the induced current $I_2(t)$ in the parallel wires to alternate with the same frequency ω and a general form

$$I_2(t) = I_{20} \sin(\omega t)$$

We will find the ratio of current amplitudes I_{20}/I_{10} .

- We'll solve the problem as follows: use the frame current I_1 to find the magnetic flux Φ_2 through the parallel wires, then use Φ_2 to find induced voltage U_2 in the wires, and finally use U_2 to find the induced current I_2 .

We find Φ_2 from $\Phi_2 = L_{21} I_1$ and the symmetry relation $L_{12} = L_{21}$, i.e.

$$\Phi_2 = L_{21} I_1 = L_{12} I_1$$

where L_{12} was found the first part of the problem. We then find U_2 with

$$U_{12} = -\dot{\Phi}_2 = -L_{12} \dot{I}_1$$

- Next, we find I_2 from U_2 from the general circuit equation

$$U = RI + L_s \dot{I},$$

which relates the voltage in a loop of resistance R and self-inductance L_s to the current I through the loop. In our case, neglecting resistance, we have

$$U_{12} \approx L_{22} \dot{I}_2$$

where L_{22} is the parallel wire's self-inductance. Substituting U_{12} into the earlier expression $U_{12} = -L_{12}\dot{I}_1$ gives

$$-L_{12}\dot{I}_1 = L_{22}\dot{I}_2$$

If we assume both I_1 and I_2 are sinusoidal with amplitudes I_{10} and I_{20} , the above reduces to

$$\frac{I_{20}}{I_{10}} = \frac{L_{12}}{L_{22}}$$

What about the minus sign?

- To find the current amplitude ratio, we just need to find the parallel wires self-inductance L_{22} using $\Phi_2 = L_{22}I_2$

If we send a hypothetical current I_2 through the wires, the corresponding magnetic field is, as before,

$$B = \frac{\mu_0 I_2}{2\pi} \left(\frac{1}{y} + \frac{1}{d-y} \right)$$

The field and surface are parallel, so $\mathbf{B} \cdot d\mathbf{S} = B dS$, and the magnetic flux is

$$\Phi_2 = \int B dS = \frac{\mu_0 I_2 l d}{2\pi} \int_a^{d-a} \left(\frac{1}{y} + \frac{1}{d-y} \right) dy$$

where a is the wire's radius. The integral evaluates to

$$\Phi_2 = \frac{\mu_0 I_2 l}{2\pi} \left[\ln \frac{d-a}{a} - \ln \frac{a}{d-a} \right] = \frac{\mu_0 I_2 l}{\pi} \ln \frac{d-a}{a}$$

Applying $d \gg a$ we have

$$\Phi_2 = \frac{\mu_0 I_2 l}{\pi} \ln \frac{d}{a}$$

The self-inductance—the proportionality constant between Φ_2 and I_2 —is thus

$$L_{22} = \frac{\mu_0 l}{\pi} \ln \frac{d}{a}$$

- With L_{22} known, we return to the current amplitude ratio to get

$$\frac{I_{20}}{I_{10}} = \frac{\frac{2 \ln 2}{\pi} \mu_0 d}{\frac{\mu_0 l}{\pi} \ln \frac{d}{a}} = \frac{2 \ln 2}{\frac{l}{d} \ln \frac{d}{a}}$$

To get a better feel for the numbers involved, if we assume $l/d = d/a = 10$, we have

$$\frac{I_{20}}{I_{10}} \approx 0.06$$

2.3.2 Cabrera Experiment—Magnetic Monopoles

Consider a superconducting current loop with non-zero inductance L , radius a and a built-in ammeter. Assume a magnetic monopole passes through the loop along the axis of symmetry. Find the resulting current pulse in the loop.

- First, the magnetic field of a hypothetical monopole is

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0 g}{4\pi} \frac{\mathbf{r}}{r^3}$$

where g is the “magnetic charge”, with units A m. Our plan is to find the magnetic flux through the loop, use this to find voltage induced in the loop, use the induced voltage to find current.

- Let $d(t)$ be the monopole’s perpendicular distance from the loop’s center, and let ρ denote radial distance from the current loop’s center (in the plane of the loop).

The magnetic field magnitude a distance r from the monopole is

$$B = \frac{\mu_0}{4\pi} \frac{g}{r^2} = \frac{\mu_0 g}{4\pi} \frac{1}{d^2 + \rho^2}$$

We find the magnetic field component B_\perp perpendicular to the current loop with similar triangles:

$$B_\perp = \frac{d}{\sqrt{d^2 + \rho^2}} B = \frac{\mu_0 g d}{4\pi} \frac{1}{(\rho^2 + d^2)^{3/2}}$$

- The magnetic flux Φ through the loop, using $dS = 2\pi\rho d\rho$, is

$$\begin{aligned} \Phi &= \int B_\perp dS = \frac{\mu_0 g d}{2} \int_0^a \frac{\rho d\rho}{(\rho^2 + d^2)^{3/2}} = \frac{\mu_0 g d}{4} (-2) \frac{1}{\sqrt{\rho^2 + d^2}} \Big|_0^a \\ &= \frac{\mu_0 g}{2} \left(1 - \frac{d}{\sqrt{a^2 + d^2}} \right) \end{aligned}$$

- We assume the monopole moves with constant speed v and passes through the loop’s center at $t = 0$. On the left of the loop (and thus for negative time), the monopole’s perpendicular distance from the loop is

$$d(t) = -vt$$

The time-dependent flux on the left of the loop is thus

$$\Phi(t) = \frac{\mu_0 g}{2} \left(1 + \frac{vt}{\sqrt{a^2 + v^2 t^2}} \right)$$

On the right of loop (and for positive time), the distance from the loop is $d = vt$, and thus the magnetic flux through the loop is

$$\Phi(t) = -\frac{\mu_0 g}{2} \left(1 - \frac{vt}{\sqrt{a^2 + v^2 t^2}} \right)$$

Note the additional minus sign, since on the right of the loop, once the particle passes through, the magnetic flux points in the opposite direction.

Note that $\Phi(t)$ increases from 0 at $t \rightarrow -\infty$ to a maximum value of $\frac{\mu_0 g}{2}$ as $t \rightarrow 0$ from the left. As the particle passes through the loop at $t = 0$, Φ jumps discontinuously to $-\frac{\mu_0 g}{2}$, and then decreases back to 0 as $t \rightarrow \infty$.

- Next, we find the induced voltage in the loop. We start with a modified Maxwell equation:

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} - \mu_0 \mathbf{j}_m$$

where \mathbf{j}_m is magnetic current density and accounts for the possible existence of magnetic monopoles. We then use this modified Maxwell equation to re-derive the law of induction in the presence of magnetic monopoles. We integrate the equation over the loop's surface to get

$$\iint_S (\nabla \times \mathbf{E}) \cdot d\mathbf{S} = -\frac{\partial}{\partial t} \iint_S \mathbf{B} \cdot d\mathbf{S} - \mu_0 \iint_S \mathbf{j}_m \cdot d\mathbf{S}$$

and apply Stokes' law to get

$$\oint \mathbf{E} \cdot d\mathbf{l} = U_i = -\dot{\Phi} - \mu_0 I_m$$

where I_m is magnetic current through the loop's cross section and U_i is the induced voltage in the loop. The magnetic current through the loop is nonzero only at the singular instant when the magnetic monopole passes through, which we model with the delta function:

$$I_m = g\delta(t)$$

where $\delta(t)$ has units s^{-1} .

- If we neglect resistive and capacitive effects, the loop's circuit equation reads $U_i = L\dot{I}$. Substituting in $U_i = -\dot{\Phi} - \mu_0 I_m$ gives

$$U_i = U_i = -\dot{\Phi} - \mu_0 I_m = -\dot{\Phi} - \mu_0 g\delta(t) = L\dot{I}$$

We then integrate this equation over time to get

$$\begin{aligned} L \int_{-\infty}^t \dot{I} d\tilde{t} &= -\int_{-\infty}^t \dot{\Phi} d\tilde{t} - \mu_0 g \int_{-\infty}^t \delta(\tilde{t}) d\tilde{t} \\ LI(t) \Big|_{-\infty}^t &= -\Phi(\tilde{t}) \Big|_{-\infty}^t - \mu_0 g H(t) \end{aligned}$$

where $H(t)$ is the Heaviside step function. Since $I(t \rightarrow -\infty) = \Phi(t \rightarrow -\infty) = 0$ (when the monopole is infinitely far from the loop), the equation reduces to

$$LI(t) = -\Phi(t) - \mu_0 g H(t)$$

- Remember that Φ changes from $\frac{\mu_0 g}{2}$ to $-\frac{\mu_0 g}{2}$ as the monopole passes through the loop at $t = 0$. The discontinuity in Φ is exactly balanced by the Heaviside step function activating with magnitude $\mu_0 g$ at $t = 0$, and the effect is that $LI(t)$ and thus the current through the loop is as a continuous, measurable quantity.
- If we use the Dirac quantization of magnetic charge, which reads

$$\frac{\mu_0 g e_0}{2} = h \implies \mu_0 g = \frac{2h}{e_0},$$

we get a numerical result for the quantity $\mu_0 g$, and a theoretically expected value of $LI(t)$.

2.4 Tenth Exercise Set

2.4.1 Skin Effect in a Ribbon-Like Conductor

Consider a long conducting ribbon of conductivity σ , width a , height b and length l where $l \gg b \gg a$. We place ideal electrodes at each end of the ribbon, and connect the electrodes to an alternating voltage source with frequency ω . Find the dependence of the conductor's impedance on frequency and investigate the high and low-frequency limits.

- Our plan is to solve for the electric field $\mathbf{E}(t)$ in the ribbon, use \mathbf{E} to find the potential difference U , and finally use U to find the ribbon's impedance. We start with the relevant Maxwell equations, which are

$$\begin{aligned}\nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t} & \nabla \cdot \mathbf{E} &= 0 \\ \nabla \times \mathbf{B} &= \mu_0 \mathbf{j} + \epsilon_0 \mu_0 \frac{\partial \mathbf{E}}{\partial t} & \nabla \cdot \mathbf{B} &= 0\end{aligned}$$

Note that $\nabla \cdot \mathbf{E} = 0$ because the conductor is neutral—no electric field escapes.

We make a quasi-static approximation to simplify the equations, neglecting the displacement current $\epsilon_0 \mu_0 \frac{\partial \mathbf{E}}{\partial t}$ to get

$$\nabla \times \mathbf{B} \approx \mu_0 \mathbf{j}$$

We take the curl of the first Maxwell equation to get

$$\nabla \times (\nabla \times \mathbf{E}) = \nabla(\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E} = -\frac{\partial}{\partial t} \nabla \times \mathbf{B}$$

We then apply $\nabla \cdot \mathbf{E} = 0$ and $\nabla \times \mathbf{B} = \mu_0 \mathbf{j} = \mu_0(\sigma \mathbf{E})$ to get

$$-\nabla^2 \mathbf{E} = -\frac{\partial}{\partial t} \mu_0 \sigma \mathbf{E}$$

- Since the ribbon is attached to an alternating voltage source, the electric field $\mathbf{E} = \mathbf{E}(t)$ reads

$$\mathbf{E}(t) = \mathbf{E}_0 e^{i\omega t} \implies \frac{\partial}{\partial t} \mathbf{E}(t) = i\omega \mathbf{E}(t)$$

Finally, we substitute the $\frac{\partial \mathbf{E}}{\partial t}$ into the earlier equation $-\nabla^2 \mathbf{E} = -\frac{\partial}{\partial t} \mu_0 \sigma \mathbf{E}$ to get

$$\nabla^2 \mathbf{E} - i\omega \mu_0 \sigma \mathbf{E} \equiv \nabla^2 \mathbf{E} - k^2 \mathbf{E} = 0$$

where we have defined the amplitude

$$k^2 \equiv i\omega \mu_0 \sigma \implies k = \sqrt{i\omega \mu_0 \sigma} = \frac{1+i}{\sqrt{2}} \sqrt{\omega \mu_0 \sigma}$$

- Next, we define coordinate system whose x axis aligns with the ribbon's width a and whose y axis aligns with the ribbon's height. The conductor's length corresponds to the z axis. We choose the origin so that $x \in [-a/2, a/2]$, meaning the ribbon's center occurs at $x = 0$.

With respect to this coordinate system, we can then simplify the Laplacian $\nabla^2 \mathbf{E}$. Since the ribbon's width a is much smaller than the height and length, the Laplacian's derivatives with respect to y and z are negligible, i.e.

$$\nabla^2 \mathbf{E} = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) \mathbf{E} \approx \frac{\partial^2 \mathbf{E}}{\partial x^2}$$

The alternating voltage is applied along the conductor's length—along the z axis—so the electric field reads $\mathbf{E} = E_z(x)\hat{\mathbf{z}}$. Plugging all of these simplifications into the amplitude equation gives

$$\nabla^2 \mathbf{E} - k^2 \mathbf{E} \approx \frac{\partial^2}{\partial x^2} [E_z(x)\hat{\mathbf{z}}] - k^2 E_z(x)\hat{\mathbf{z}} = E_z''(x) - k^2 E_z(x) = 0$$

- The solutions to the equation

$$E_z'' - k^2 E_z = 0$$

can be written either as exponents or hyperbolic functions; we will use hyperbolic functions, which are best suited to the problem's reflection symmetry about the y axis. The general solution is

$$E_z(x) = A \cosh kx + B \sinh kx$$

However, the problem's reflection symmetry means E_z will have only the even component \cosh , and the solution simplifies to

$$E_z(x) = A \cosh kx$$

- We find A from the problem's boundary conditions. Assuming $E_z(\pm a/2) = E_0$, we have

$$A \cosh\left(\pm \frac{ka}{2}\right) = A \cosh \frac{ka}{2} = E_0 \implies A = \frac{E_0}{\cosh \frac{ka}{2}}$$

With the constant A known, the solution for $E_z(x)$ is then

$$E_z(x) = A \cosh kx = E_0 \frac{\cosh kx}{\cosh \frac{ka}{2}}$$

Next, we draw some qualitative sketches of $E(x)$ with k as a parameter. The resulting curves show that as k (and thus frequency ω) increases, $E(x)$ becomes concentrated near the ribbon's outer surfaces $x = \pm a/2$. This is a qualitative demonstration of the skin effect, where electric field and current become concentrated along a conductor's surface at high frequencies.

Potential Difference, Current and Impedance

- Next, with $E_z(x)$ known, the potential difference across the ribbon's is simply

$$U_z(x) = E_z(x)l$$

where l is the conductor length.

- With potential difference known, we find the current through the conductor with

$$I = \int j \, dS = \int (\sigma E_z) \, dS = \frac{\sigma E_0}{\cosh \frac{ka}{2}} \int_{-a/2}^{a/2} \cosh(kx) (b \, dx)$$

where we have written the surface element $dS = b \, dx$ in terms of the conductor's height b . The integral evaluates to

$$\begin{aligned} I &= \frac{\sigma E_0 b}{\cosh \frac{ka}{2}} \int_{-a/2}^{a/2} \cosh(kx) \, dx = \frac{\sigma E_0 b}{\cosh \frac{ka}{2}} \cdot \frac{2}{k} \cdot \sinh kx \Big|_{x=0}^{a/2} \\ &= \frac{2\sigma E_0 b}{k} \tanh \frac{ka}{2} \end{aligned}$$

Next, we introduce the dimensionless quantity $\kappa = \frac{ka}{2}$ and substitute in $U_0 = E_0 l$ to get

$$I = \frac{\sigma U_0 b a}{l} \frac{2}{ka} \tanh \frac{ka}{2} = \frac{\sigma U_0 b a}{l} \frac{\tanh \kappa}{\kappa} = \frac{U_0}{R_0} \frac{\tanh \kappa}{\kappa}$$

where, in the last equality, we have substituted in the conductor's static resistance

$$R_0 = \frac{l}{\sigma S}$$

In terms of R_0 , the conductor's impedance is then

$$Z \equiv \frac{U_0}{I} = R_0 \frac{\kappa}{\tanh \kappa}$$

High and Low Frequency Limits

- In the low frequency limit, k and thus κ are small ($\kappa \ll 1$) and we expand the $\tanh x \approx$ function to get

$$Z \approx R_0 \frac{\kappa}{\kappa} = R_0$$

In other words, the ribbon's impedance approaches the static resistance R_0 .

- Finally, for large frequencies and thus $\kappa \gg 1$, we use the asymptotic expansion $\tanh x \approx 1$ to get

$$Z \approx R_0 \kappa = \frac{R_0 a}{2} k = \frac{R_0 a}{2} \sqrt{\frac{\sigma \mu_0 \omega}{2}} (1 + i)$$

The real component $\text{Re } Z$ corresponds to resistance $R(\omega)$, while the imaginary component $\text{Im } Z$ corresponds to reactance. Reactance is out of phase with resistance by $\pi/2$. At high frequencies, the resistance is

$$R(\omega) = \frac{r_0 a}{2} \sqrt{\frac{\sigma \mu_0 \omega}{2}}$$

Note the relationship $R \propto \sqrt{\omega}$, meaning resistance increases with frequency, since the electric field is more concentrated at the conductor edges (the “skin”) and the current has a smaller effective cross section and thus larger resistance.

2.4.2 Theory: Conservation of Electromagnetic Energy

- Conservation of electromagnetic energy is written as the energy balance

$$\frac{\partial w}{\partial t} + \nabla \cdot \mathbf{S} + \mathbf{j} \cdot \mathbf{E} = 0$$

where w is electromagnetic energy density

$$w = \frac{1}{2} \left(\epsilon_0 E^2 + \frac{1}{\mu_0} B^2 \right)$$

and \mathbf{S} is the Poynting vector, defined as

$$\mathbf{S} = \frac{1}{\mu_0} \mathbf{E} \times \mathbf{B} = \mathbf{E} \times \mathbf{H}$$

- In integral form for a region of space V , the energy equation reads

$$\frac{\partial}{\partial t} \int_V w \, dV + \int_{\partial V} \mathbf{S} \cdot d\mathbf{S} + \int_V \mathbf{j} \cdot \mathbf{E} \, dV = 0$$

where we have used the divergence theorem to convert the Poynting vector term to a surface integral.

- The w term corresponds to the changing electromagnetic field energy within the region V
- The \mathbf{S} term encodes energy flow (power) through the surface
- The $\mathbf{j} \cdot \mathbf{E}$ term corresponds to Ohmic energy losses within the region.

2.4.3 Power in a Coaxial and Cylindrical Conductor

Find the electromagnetic energy flow (power) through:

1. The cross section of a coaxial cable carrying current I , of length l , with potential difference U between the inner and outer conductor. Neglect resistive losses.
2. The lateral surface long, straight cylindrical conductor of radius a , length l , conductivity σ and carrying a current I with a potential difference U between the cylinder ends.

Lossless Coaxial Conductor

- In a coaxial cable, the magnetic field \mathbf{B} points tangent to the circular cross section (perpendicular to the radial direction), while the electric field \mathbf{E} points radially outward. We assume positive charge accumulates on the inner conductor and negative charge on the outer conductor.

We find the magnetic field with Ampere's law using a loop around the inner conductor. The result is

$$\mu_0 I = B \cdot 2\pi r \implies B = \frac{\mu_0 I}{2\pi r}$$

Meanwhile, we find electric field with Gauss's law, using a cylinder enclosing the inner conductor:

$$Q = \epsilon_0 E \cdot 2\pi r l \implies E = \frac{Q}{2\pi \epsilon_0 l r}$$

- The induced charge Q and potential difference between the inner and outer conductor are related by

$$U = \int_a^b E \, dr = \frac{Q}{2\pi \epsilon_0 l} \ln \frac{b}{a}$$

In terms of U , the electric field is thus

$$E = \frac{Q}{2\pi \epsilon_0 l r} = \frac{U}{r \ln \frac{b}{a}}$$

- The Poynting vector points in the direction $\mathbf{E} \times \mathbf{B}$, along the conductor's longitudinal axis. In our case we have $\mathbf{E} \perp \mathbf{B}$, so the Poynting vector magnitude S is

$$S = \frac{1}{\mu_0} E B = \frac{1}{\mu_0} \left(\frac{\mu_0 I}{2\pi r} \right) \left(\frac{U}{r \ln \frac{b}{a}} \right) = \frac{UI}{2\pi \ln \frac{b}{a} r^2}$$

- The electromagnetic power through the cross section is thus

$$\begin{aligned} P &= \int \mathbf{S} \cdot d\mathbf{S} = \int S dS = \int_a^b \left(\frac{UI}{2\pi \ln \frac{b}{a} r^2} \right) (2\pi r dr) \\ &= \frac{UI}{\ln \frac{b}{a}} \int_a^b \frac{dr}{r} = \frac{UI}{\ln \frac{b}{a}} \ln \frac{b}{a} = UI \end{aligned}$$

where $\mathbf{S} \cdot d\mathbf{S} = S dS$ because \mathbf{S} is parallel to coaxial cable's cross section.

Cylindrical Conductor

- As for the coaxial cable, the cylindrical conductor's magnetic field is tangent to the conductor's circular cross section. Meanwhile, the electric field points in the direction of the potential difference—along the conductor's longitudinal axis. The Poynting vector $\mathbf{S} = \mathbf{E} \times \mathbf{B}$ thus along the radial direction. We choose I to flow so that \mathbf{S} points radially inward towards the conductor's center (as opposed to radially outward for an opposite current).
- The electric field in the conductor, which arises from the potential difference U between the conductor's ends, is simply

$$E = \frac{U}{l}$$

where l is the conductor's length.

We find the conductor's magnetic field with Ampere's law, using a circular loop centered along the conductor's longitudinal axis. Assuming current is uniformly distributed across the cross section, the magnetic field is

$$B \cdot (2\pi r) = \mu_0 I \left(\frac{r}{a} \right)^2 \implies B = \frac{\mu_0 I}{2\pi a^2} r$$

- Since \mathbf{E} and \mathbf{B} is perpendicular, the Poynting vector is

$$\mathbf{S} = \frac{1}{\mu_0} \mathbf{E} \mathbf{B} = \frac{1}{\mu_0} \left(\frac{U}{L} \right) \left(\frac{\mu_0 I}{2\pi a^2} r \right) = \frac{UI}{2\pi a^2 l} r$$

Note that S increases linearly with r .

- We find electromagnetic power by integrating S over the cylinder's lateral surface (not the cross section!). The power through a lateral surface at radius r is

$$P(r) = \int_{S_{\text{lat}}} \mathbf{S} \cdot d\mathbf{S} = \int_{S_{\text{lat}}} S dS = \left(\frac{UI}{2\pi a^2 l} r \right) (2\pi r l) = \frac{UI r^2}{a^2}$$

The power through the entire conductor's lateral surface occurs at $r = a$, which produces the familiar result

$$P = \frac{UI a^2}{a^2} = UI$$

- Next, we consider the conductor's energy balance

$$-\frac{\partial}{\partial t} \int_V w dV = \int_{\partial V} \mathbf{S} \cdot d\mathbf{S} + \int_V \mathbf{j} \cdot \mathbf{E} dV = P + \int_V \mathbf{j} \cdot \mathbf{E} dV$$

Because the situation is stationary we have $\frac{\partial w}{\partial t} = 0$, and thus

$$P + \int_V \mathbf{j} \cdot \mathbf{E} dV \equiv P + P_{\text{loss}} = 0 \implies P_{\text{loss}} = -P = -UI$$

Note that Ohmic losses amount to $P_{\text{loss}} = -UI$, which is the familiar expression from e.g. high school physics. The energy balance reads

$$P + \int \mathbf{j} \cdot \mathbf{E} dV = P + P_{\text{loss}} = UI - UI = 0$$

and energy is conserved, as it must be.

2.4.4 Cylindrical Conductor with a Slit

Consider a long cylindrical conductor with radius a carrying a current I . We make a narrow slit of width $d \ll r$ through the conductor's cross section. Find the energy flow through the slit's lateral surface.

- Because of the current I , positive charge accumulates on one surface of the slit and negative charge on the other. Because $d \ll r$, we can thus treat the two surfaces of the slit as parallel-plate capacitor.
- The charge accumulating on the slit is $q(t) = It$, and, using the model of a parallel-plate capacitor, the corresponding electric field is

$$E(t) = \frac{\sigma(t)}{\epsilon_0} = \frac{1}{\epsilon_0} \frac{q(t)}{S} = \frac{I}{\epsilon_0 S} t$$

- Next, we find the magnetic field in the slit. The time-varying changing electric field $E(t)$ creates displacement current, and the Maxwell equation for $\nabla \times \mathbf{B}$ reads

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{j} + \epsilon_0 \mu_0 \frac{\partial \mathbf{E}}{\partial t} = \epsilon_0 \mu_0 \frac{\partial \mathbf{E}}{\partial t}$$

Note that $\mu_0 \mathbf{j} = 0$ inside the slit, since there's no free current. We integrate the equation over the conductor's cross-sectional surface S to get

$$\int_S \nabla \times \mathbf{B} \cdot d\mathbf{S} = \oint \mathbf{B} \cdot d\mathbf{l} = \epsilon_0 \mu_0 \frac{\partial}{\partial t} \int_S \mathbf{E} \cdot d\mathbf{S} = \epsilon_0 \mu_0 \frac{\partial E}{\partial t} \cdot (\pi r^2)$$

We evaluate the line integral over the surface's boundary and substitute in electric field to get

$$\oint \mathbf{B} \cdot d\mathbf{l} = B \cdot (2\pi r) = \epsilon_0 \mu_0 \frac{\partial E(t)}{\partial t} \implies B(r) = \frac{\mu_0 I}{2S} r$$

where $S = \pi r^2$ is the cross-sectional surface area. The magnetic field points tangent to the cylindrical lateral, just like the magnetic field of a conductor, except that free current is replaced by displacement current.

- Since \mathbf{E} points along the conductor's longitudinal axis and \mathbf{B} is tangent to the lateral surface, $\mathbf{S} = \mathbf{E} \times \mathbf{B}$ points along the radial direction. Because $\mathbf{B} \perp \mathbf{S}$, the power through the lateral surface is simply

$$P = \int_{S_{\text{lat}}} \mathbf{S} \cdot d\mathbf{S} = \frac{1}{\mu_0} \int_{S_{\text{lat}}} E B dS = \frac{1}{\mu_0} \int_{S_{\text{lat}}} \left(\frac{I}{\epsilon_0 S} t \right) \left(\frac{\mu_0 I}{2S} r \right) dS$$

We set $r = a$ at the lateral surface to get

$$P = \frac{I^2 a}{2\epsilon_0 S^2} t(2\pi a d) = \frac{I^2 d}{\epsilon_0 S} t$$

where $S = \pi a^2$ is cross sectional area at $r = a$.

- Finally we consider the conductor's energy balance. If we neglect Ohmic losses, the energy balance reads

$$\frac{\partial}{\partial t} \int_V w \, dV + \int_{\partial V} \mathbf{S} \cdot d\mathbf{S} = \frac{\partial W_{\text{EM}}}{\partial t} + P = 0$$

where $W_{\text{EM}} = W_{\text{E}} + W_{\text{B}}$ is the sum of magnetic and electric field energy. Note that $B = B(r)$ and thus magnetic energy W_{B} are independent of time. It follows that

$$\frac{\partial W_{\text{EM}}}{\partial t} = \frac{\partial W_{\text{E}}}{\partial t} + 0$$

Using $W_{\text{E}} = w_{\text{E}} V$, the electric field energy in the slit is

$$W_{\text{E}} = w_{\text{E}} V = \left(\frac{1}{2} \epsilon_0 E^2 \right) \cdot (Sd) = \frac{\epsilon_0}{2} \frac{I^2}{\epsilon_0^2 S^2} S d t^2 = \frac{I^2 d}{2\epsilon_0 S} t^2$$

In terms of W_{E} , the time derivative of total magnetic field energy is thus

$$\frac{\partial W_{\text{EM}}}{\partial t} = \frac{\partial W_{\text{E}}}{\partial t} = \frac{\partial}{\partial t} \frac{I^2 d}{2\epsilon_0 S} t^2 = \frac{I^2 d}{\epsilon_0 S} t = P$$

Up to a negative sign (depending on the direction of current), $\frac{\partial W_{\text{EM}}}{\partial t}$ and P are equal, indicating that energy is conserved, as it must be.

2.5 Eleventh Exercise Set (rough draft)

2.5.1 A Radially Polarized Sphere

Consider a radially polarized sphere—polarization points in the radial direction and grows with radius as $\mathbf{P} = k\mathbf{r}$ where k is a known constant. Find volume density of bound charges, surface density of bound charges, and the total bound charge.

- We relate polarization and volume charge density as

$$\rho_{\text{b}} = -\nabla \cdot \mathbf{P}$$

We also know Gauss's law

$$\rho = \epsilon_0 \nabla \cdot \mathbf{E}$$

which accounts for all charges, including both bound and external charges.

- We now relate ρ and σ . Consider a surface element of the sphere. With normal $\hat{\mathbf{n}}$. We use the polarization in integral form.

$$\int dV \rho_{\text{b}} = - \int \nabla \cdot \mathbf{P} \, dV \implies q_{\text{b}} = - \int \mathbf{P} \cdot d\mathbf{S} = -\mathbf{P} \cdot d\mathbf{S}|_{\text{in}}^{\text{out}} = -0 - (-\mathbf{P} \cdot \hat{\mathbf{n}} S)$$

We then get surface charge density as

$$\sigma_{\text{b}} = \mathbf{P} \cdot \hat{\mathbf{n}}$$

- Start with ρ_b , with known form of \mathbf{P} :

$$\rho_b = -\nabla \cdot \mathbf{P} = -k \nabla \cdot \mathbf{r} = -k \cdot 3 = -3k$$

Note $\nabla \cdot \mathbf{r} = 3$. Note the volume charge density is negative; ρ_b is negative because all dipoles in the sphere have their negative pole towards the sphere's center and the positive poles radially outward. The middle dipoles cancel, and we're schematically left with negative charges in the center.

- Next: surface charge density

$$\sigma_b = \mathbf{P} \cdot \hat{\mathbf{n}} = k\mathbf{r} \cdot \hat{\mathbf{n}} = kr \Big|_{r=a} = ka$$

This is positive, which makes sense—the dipoles point outwards (see above bullet point).

- Next, we find total charge. We expect this to be zero; all dipoles should cancel out. We have (integrate both volume and surface)

$$q_b = \int \rho_b dV + \int \sigma_b dS = -3k \frac{4}{3} \pi a^3 + ka 4\pi a^2 = 0$$

- Next we consider electric field. Note use of spherical symmetry—Gauss's law:

$$\epsilon_0 E 4\pi r^2 = \rho_b V = \rho_b \frac{4}{3} \pi r^3 \implies E(r) = \frac{\rho_b}{3\epsilon_0} r$$

The electric field points radially outwards:

$$\mathbf{E}(r) = \frac{\rho_b}{3\epsilon_0} \mathbf{r}$$

Plug in value of $\rho_b = -3k$ to get

$$\mathbf{E} = \frac{-k}{\epsilon_0} \mathbf{r} \implies \mathbf{E} = -\frac{\mathbf{P}}{\epsilon_0}$$

Note the linear relationship between electric field and polarization. Comes from the relationship $\rho_b = -\nabla \cdot \mathbf{P}$ and Gauss's law $\rho = \nabla \cdot (\epsilon_0 \mathbf{E})$. Because there are no external charge $\rho = \rho_b$, which is the reason for $\mathbf{E} = -\frac{\mathbf{P}}{\epsilon_0}$

2.5.2 A Polarized Halved Sphere

Consider a sphere of radius a with homogeneous (uniform) polarization \mathbf{P} pointing in the e.g. z direction. Find the electric field inside and outside the sphere.

We then cut the sphere in half in a plane perpendicular to the direction of polarization and slightly separate the halves to form a capacitor. Find the electric field in the slit between the halves.

- First, find the typical quantities related to polarization. First, volume density of bound charges:

$$\rho_b = -\nabla \cdot \mathbf{P} = 0,$$

since the divergence of the homogeneous field \mathbf{P} is zero. In other words, there are no bound charges in the sphere's volume.

- Next, $\sigma_b = \mathbf{P} \cdot \hat{\mathbf{n}}$ we have

$$\sigma_b = \mathbf{P} \cdot \hat{\mathbf{n}} = P \cos \theta$$

where θ is the angle between the normal $\hat{\mathbf{n}}$ and polarization \mathbf{P} . Note that the surface charge density is nonzero.

- We proceed with the Poisson equation, which simplifies to the Laplace equation where $\rho_b = 0$, which is everywhere but the surface:

$$\nabla^2 U(\mathbf{r}) = 0$$

We account for the surface with boundary conditions.

The general solution for $U(\mathbf{r})$ is

$$U(r) = \sum_{l=0}^{\infty} (A_l r^l + B_l r^{-(l+1)}) P_l(\cos \theta)$$

Note that our charge distribution depends only on $\cos \theta$. So our solution can include only $\cos \theta$ -dependent terms, which is only $l = 1$. Our general solution is

$$U(\mathbf{r}) = (A_1 r + B_1 r^{-2}) \cos \theta$$

Inside and outside we have

$$U(\mathbf{r}) = \begin{cases} A_1 r \cos \theta & r < a \\ B_1 r^{-2} \cos \theta & r > a \end{cases}$$

to avoid divergence at $r \rightarrow 0$ and $r \rightarrow \infty$.

We need to find A_1 and B_1 with the boundary conditions.

- First boundary condition: the electric potential must be continuous at the boundary (discontinuous potential would imply infinite charge, which is nonphysical). This continuity condition at $r = a$ requires

$$A_1 a = \frac{B_1}{a^2} \implies A_1 a^3 = B_1$$

Next, we consider the electric field, which we can relate to surface charge density, for the second boundary condition. Electric field is perpendicular to the surface. From Gauss's law we have

$$q = \epsilon_0 S (E_{\text{out}}^{\perp} - E_{\text{in}}^{\perp}) \implies \sigma_b = \epsilon_0 (E_{\text{out}}^{\perp} - E_{\text{in}}^{\perp})$$

We know σ_b and use $E_{\text{in}}^{\perp} = -\frac{\partial U}{\partial r} \Big|_{r=a} = \frac{2B_1}{a^3} \cos \theta$

$$\sigma_b = P \cos \theta = \epsilon_0 \left[\frac{2B_1}{a^3} + A_1 \right] \cos \theta$$

We insert the first boundary condition into the second to get

$$P \cos \theta = \epsilon_0 (2A_1 + A_1) \cos \theta \implies P = 3\epsilon_0 A_1$$

and thus

$$A_1 = \frac{P}{3\epsilon_0} \quad \text{and} \quad B_1 = \frac{Pa^3}{3\epsilon_0}$$

The electric potential is then

$$U(\mathbf{r}) = \begin{cases} \frac{P}{3\epsilon_0} r \cos \theta & r < a \\ \frac{Pa^3}{3\epsilon_0} \frac{1}{r^2} \cos \theta & r > a \end{cases}$$

- We now discuss the solution. The 3 is called the depolarization factor (it was 1 in the previous solution where we had $U = \frac{P}{\epsilon_0} \dots$). Next, note that $r \cos \theta = z$. $r \cos \theta = z$ means the field inside the sphere is homogeneous:

$$E_z = -\frac{\partial U}{\partial z} = -\frac{P}{3\epsilon_0} \implies \mathbf{E} = -\frac{\mathbf{P}}{3\epsilon_0}$$

where we note that \mathbf{P} points in the z direction. The field is negative because with our choice of polarization, positive charges are in the positive z direction and negative charges in the negative z direction.

- For the field outside the sphere we have the field of an electric dipole, which is expected; the sphere is polarized like an electric dipole. Think of the cosine term as from the dot product of \mathbf{p} and \mathbf{r} .

Halved Sphere; Electric Field in the Slit

- Cut the sphere in half perpendicular to the polarization. We know have bound charges on the cut surfaces as well. The bound charges on the top half's bottom surface are negative, and vice-versa. Note the relationship $\sigma_b = \mathbf{P} \cdot \hat{\mathbf{n}}$ is preserved.
- Think of the slit as a parallel-plate capacitor. We know the field of a parallel plate capacitor:

$$\tilde{\mathbf{E}} = \frac{\tilde{\sigma}}{\epsilon_0} = \frac{\mathbf{P}}{\epsilon_0}$$

where tilde accounts for the flat surface.

- But we must also consider the hemispherical surfaces. These have

$$\mathbf{E}_{\text{in}} = -\frac{\mathbf{P}}{3\epsilon_0}$$

And the total field in the slit is

$$\mathbf{E} = \frac{\mathbf{P}}{\epsilon_0} - \frac{\mathbf{P}}{3\epsilon_0} = \frac{2\mathbf{P}}{3\epsilon_0}$$

2.5.3 Theory: Dielectric and Displacement Field

- Stress that the previous two problems involved ferro-electric materials—the polarization was built in to the material. We now consider a dielectric material; the polarization occurs only in the presence of an external field.
- We recall the equations $\rho_b = -\nabla \cdot \mathbf{P}$ and Gauss's law $\rho = \epsilon_0 \nabla \cdot \mathbf{E}$. We subtract the equations to get

$$\rho_f = \rho - \rho_b = \nabla \cdot (\epsilon_0 \mathbf{E} + \mathbf{P}) \equiv \nabla \cdot \mathbf{D}$$

Where the free charge density ρ_f is the sum of total charge density and the bound charge density. The electric displacement field is $\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P}$. \mathbf{D} is useful because it depends only on free charge.

- For small fields, we make the approximation $\mathbf{D} \propto \mathbf{E}$ where \mathbf{D} and \mathbf{E} are linearly dependent. We have the approximate relationship

$$\mathbf{D} = \epsilon_0 \boldsymbol{\chi} \mathbf{E}$$

where $\boldsymbol{\chi}$ is the permittivity tensor.

2.5.4 Dielectric Plate in a Parallel-Plate Capacitor

Consider a parallel-plate capacitor with plate separation d , plate surface area S and a dielectric insulator. The dielectric tensor has components ϵ_1 and ϵ_2 , and ϵ_2 is angled from the y axis by an angle ϕ . Find the capacitance. 2D coordinate system.

- Capacitance is defined as

$$C = \frac{q}{U}$$

Assume we put a potential difference between the plates, and determine the accumulated charge q on the plates.

- Because $\boldsymbol{\chi}$ is angled, \mathbf{E} and \mathbf{D} are not parallel. However, \mathbf{E} is perpendicular to the plates. We now consider why.

We use the following Maxwell equation:

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

In our static case we have $\nabla \times \mathbf{E} = 0$. In integral form we have

$$\oint \mathbf{E} \cdot d\mathbf{l} = 0$$

where we make a rectangular path thinly hugging one capacitor plate. Because the path thinly hugs the plate, the integration registers only the component of \mathbf{E} parallel to the plates:

$$E_{\text{out}}^{(\parallel)} \cdot l = E_{\text{int}}^{(\parallel)} \cdot l = 0 \implies E_{\text{out}}^{(\parallel)} = E_{\text{in}}^{(\parallel)}$$

Now, the field outside the capacitor is zero; we have

$$E_{\text{out}}^{(\parallel)} = 0 \implies E_{\text{in}}^{(\parallel)} = 0$$

And $E_{\text{in}}^{(\parallel)} = 0$ implies the electric field has only a component perpendicular to the capacitor plates.

- We now need to write the tensor in the principle axis system

$$\tilde{\boldsymbol{\chi}} = \begin{pmatrix} \epsilon_1 & 0 \\ 0 & \epsilon_2 \end{pmatrix}$$

In our system, rotated by ϕ relative to the principle axis system, we have rotate the principle axes tensor by rotation matrices:

$$\begin{aligned} \boldsymbol{\chi} &= \mathbf{R} \tilde{\boldsymbol{\chi}} \mathbf{R} = \begin{pmatrix} \cos \phi & \sin \phi \\ -\sin \phi & \cos \phi \end{pmatrix} \begin{pmatrix} \epsilon_1 & 0 \\ 0 & \epsilon_2 \end{pmatrix} \begin{pmatrix} \cos \phi & -\sin \phi \\ \sin \phi & \cos \phi \end{pmatrix} \\ &= \begin{pmatrix} \epsilon_1 \cos^2 \phi + \epsilon_2 \sin^2 \phi & (\epsilon_2 - \epsilon_1) \sin \phi \cos \phi \\ (\epsilon_2 - \epsilon_1) \sin \phi \cos \phi & \epsilon_1 \sin^2 \phi + \epsilon_2 \cos^2 \phi \end{pmatrix} \end{aligned}$$

Note that $\boldsymbol{\chi}$ is symmetric, as it must be.

- Note that \mathbf{E} has only a y (perpendicular) component. Take advantage of this to write (from $\mathbf{D} = \epsilon_0 \chi \mathbf{E}$).

$$\begin{pmatrix} D_x \\ D_y \end{pmatrix} = \epsilon_0 \begin{pmatrix} \epsilon_1 \cos^2 \phi + \epsilon_2 \sin^2 \phi & (\epsilon_2 - \epsilon_1) \sin \phi \cos \phi \\ (\epsilon_2 - \epsilon_1) \sin \phi \cos \phi & \epsilon_1 \sin^2 \phi + \epsilon_2 \cos^2 \phi \end{pmatrix} \begin{pmatrix} 0 \\ E \end{pmatrix}$$

where we have $E_y = E$.

- Now onto $C = \frac{q}{U}$. We get U from E . We get q from the y component of D_y , which points perpendicular to the capacitor plates. We would use Gauss's law with \mathbf{D} , enclosing a capacitor plate, to get (recall $\rho - \rho_b = \nabla \cdot \mathbf{D}$). So Gauss in terms of \mathbf{D} reads

$$q_f = \int \rho_f dV = \int \mathbf{D} \cdot d\mathbf{S}$$

which we use for our capacitor plate to get (note the D_y is parallel to $d\mathbf{S}$)

$$q_f = D_y S$$

- We turn to the matrix equation to get

$$D_y = \epsilon_0 (\epsilon_1 \sin^2 \phi + \epsilon_2 \cos^2 \phi) E$$

We then have from Gauss that $D_y = \frac{q_f}{S}$ so

$$\frac{q_f}{S} = \epsilon_0 (\epsilon_1 \sin^2 \phi + \epsilon_2 \cos^2 \phi) \frac{U}{d}$$

where we wrote E in terms of U as $E = \frac{U}{d}$.

- The capacitance is then

$$C = \frac{q_f}{U} = \frac{\epsilon_0 S}{d} (\epsilon_1 \sin^2 \phi + \epsilon_2 \cos^2 \phi) = C_0 (\epsilon_1 \sin^2 \phi + \epsilon_2 \cos^2 \phi)$$

where $C_0 = \frac{\epsilon_0 S}{d}$ is the capacitance of an empty capacitor. Note that $\phi = 0$ (a nicely aligned permittivity tensor) implies $C = C_0 \epsilon_2$.

2.6 Twelfth Exercise Set (rough draft)

2.6.1 Point Dipole in a Spherical Dielectric Cavity

Spherical cavity of radius a inside a dielectric material with permittivity ϵ . Place a dipole with dipole momentum \mathbf{p} in the cavity. What is electric potential inside and outside the cavity? And find the resulting effective dipole moment \mathbf{p}' in the dielectric.

- Assume \mathbf{p} points in the z direction.
- We solve the Laplace equation

$$\nabla^2 U(\mathbf{r}) = 0$$

in spherical coordinates. The general solution is

$$U(r) = \sum_{l=0}^{\infty} (A_l r^l + B_l r^{-(l+1)}) P_l(\cos \theta)$$

Boundary conditions: we consider the spherical cavity's boundary, and the center of the cavity. The potential at the center should approach the dipole potential:

$$U(r \rightarrow 0) = U_{\text{dipole}} = \frac{p_e \cos \theta}{4\pi\epsilon_0 r^2}$$

This implies only the $l = 1$ term with $\cos \theta$ can exist in the general solution for U .

Outside only terms with A survive to avoid divergence at small r .

Outside only terms with B terms survive to avoid divergence at large r .

- We have (reusing earlier results from similar problems) a general solution

$$U(r, \theta) = \begin{cases} \frac{p_e \cos \theta}{4\pi\epsilon_0 r^2} + A_1 r \cos \theta & r < a \\ \frac{B_1}{r^2} \cos \theta & r > a \end{cases}$$

- More boundary conditions: U is continuous at $r = a$. We get (canceling $\cos \theta$)

$$\frac{p_e}{4\pi\epsilon_0 a^2} + A_1 a = \frac{B_1}{a^2}$$

- And a final boundary condition using the electric field: we work with the electric field tangent to the cavity's surface. From Maxwell:

$$\nabla \times \mathbf{E} = 0 \rightarrow \oint \mathbf{E} \cdot d\mathbf{l} = 0$$

Which implies \mathbf{E}_{\parallel} is equal on each side of the cavity.

We differentiate with respect to θ to get E_{\parallel}

$$E_{\parallel} = -\frac{1}{r} \frac{\partial U}{\partial \theta}$$

Applying the continuity condition of \mathbf{E}_{\parallel} at the boundary gives

$$-\frac{p_e}{4\pi\epsilon_0 a^2} \sin \theta - A_1 \sin \theta = \frac{B_1}{a^2} \sin \theta$$

This is the same result as the previous boundary condition requiring continuity of U at the boundary.

- We need another boundary condition: We use the \mathbf{D} field and thus consider only free charges. We use Gauss's law for \mathbf{D} , which requires that D_{\perp} is the same at both sides of the boundary. In general

$$D_{\perp} = -\epsilon_0 \epsilon \frac{\partial U}{\partial r}$$

and we evaluate this at a and apply continuity at the boundary to get

$$+2\frac{p_e}{4\pi\epsilon_0 a^3} - A_1 = 2\epsilon \frac{B_1}{a^3}$$

- Continuity of U gives

$$\frac{p_e}{4\pi\epsilon_0 a^3} + A_1 = \frac{B_1}{a^3}$$

Adding this to the D_\perp equation gives

$$3\frac{p_e}{4\pi\epsilon_0 a^3} = \frac{B_1}{a^3}(1 + 2\epsilon)$$

Which gives us

$$B_1 = \frac{3}{1 + 2\epsilon} \frac{p_e}{4\pi\epsilon_0}$$

- From which we can find A_1 with

$$A_1 = \frac{B_1}{a^3} - \frac{p_e}{4\pi\epsilon_0 a^3} = \frac{p_e}{4\pi\epsilon_0 a^3} \left(\frac{3}{1 + 2\epsilon} - 1 \right) = \frac{2(1 - \epsilon)}{1 + 2\epsilon} \frac{p_e}{4\pi\epsilon_0 a^3}$$

- Our specific solution for $U(r, \theta)$ is then

$$U(r, \theta) = \frac{p_e}{4\pi\epsilon_0} \cos \theta \begin{cases} \frac{1}{r^2} - \frac{2(r-1)}{1+2\epsilon} \frac{r}{a^3} & r < a \\ \frac{3}{1+2\epsilon} \frac{1}{r^2} & r > a \end{cases}$$

For $r < a$ the first $1/r^2$ term is the dipole term. The second $r < a$ contains a $z = r \cos \theta$ term, which corresponds to a homogeneous field inside the cavity.

For $r > a$ we have another $1/r^2$ term—again like a dipole. But note the larger factor $\frac{3}{1+2\epsilon}$, which would reduce to the usual 1 for relative permittivity $\epsilon = 1$. This is the so-called effective dipole term in the problem instructions. Outside we thus have the effective dipole moment

$$p'_e = \frac{3}{1 + 2\epsilon} p_e$$

- We could find surface density of bound charges with Gauss's law and

$$\sigma_b = \epsilon_0 (E_\perp^{\text{out}} - E_\perp^{\text{in}})$$

Alternatively, we could us

$$\sigma_b = \mathbf{P} \cdot \hat{\mathbf{n}}$$

where \mathbf{P} is electric polarization. Note that $\hat{\mathbf{n}}$ points from the dielectric material into the cavity, and \mathbf{P} points out of the cavity. Thus

$$\mathbf{P} \cdot \hat{\mathbf{n}} = -P_\perp$$

We could then use

$$\mathbf{D} = \epsilon_0 \epsilon \mathbf{E} = \epsilon_0 \mathbf{E} + \mathbf{P}$$

And thus

$$\mathbf{P} = \epsilon_0(\epsilon - 1)\mathbf{E}$$

which gives

$$\sigma_b = -\epsilon_0(\epsilon - 1)\mathbf{E}_\perp|_{r=a}$$

2.6.2 Dielectric Constant of Cold Plasma

Find the relative permittivity ϵ of a cold plasma.

- Recall plasma is ionized gas—electrons leave their atoms, and we end up with positive ions from atomic cores and negative free electrons. Since atoms are much more massive than electrons, we assume the atoms are at rest as a first approximation.
- We assume an oscillatory electric field inside the plasma:

$$\mathbf{E} = \mathbf{E}_0 e^{i(kz - \omega t)}$$

Assume atoms have charge q and electrons have charge $-q$. The force on a representative free electron is then

$$\mathbf{F} = -q\mathbf{E} = -q\mathbf{E}_0 e^{i(kz - \omega t)} \equiv m\mathbf{a} = m\ddot{\mathbf{r}}$$

where we have introduced the coordinate \mathbf{r} to measure the electron's displacement. Since force is oscillatory, we have

$$\mathbf{r} = \mathbf{r}_0 e^{i(kz - \omega t)}$$

and thus

$$\mathbf{F} = -m\omega^2 \mathbf{r}_0 e^{i(kz - \omega t)} = -q\mathbf{E}_0 e^{i(kz - \omega t)}$$

which implies

$$m\omega^2 \mathbf{r}_0 = q\mathbf{E}_0 \implies \mathbf{r}_0 = \frac{q}{m\omega^2} \mathbf{E}_0$$

This amplitude of displacement of the negative electron from the positive atom creates a dipole moment with amplitude

$$\mathbf{p}_0 = -q\mathbf{r}_0 = -\frac{q^2}{m\omega^2} \mathbf{E}_0$$

The total polarization in the substance is then

$$\mathbf{P}_0 = n\mathbf{p}_0 = -\frac{nq^2}{m\omega^2} \mathbf{E}_0$$

where n is the number density of electric dipoles in the material.

We now have a relationship between \mathbf{P} and \mathbf{E} , which we can connect with permittivity:

$$\mathbf{P}_0 = \epsilon_0(\epsilon - 1)\mathbf{E}_0$$

From which we find the dielectric constant (relative permittivity)

$$\epsilon = 1 - \frac{nq^2}{m\epsilon_0\omega^2}$$

We note that $\frac{nq^2}{m\epsilon_0}$ has units of frequency, and we define

$$\omega_p = \sqrt{\frac{nq^2}{m\epsilon_0}}$$

which describes the oscillation of electrons if displaced from the positive atomic ions. We then have

$$\epsilon = 1 - \frac{\omega_p^2}{\omega^2}$$

- Note that for field frequencies $\omega > \omega_p$ we have $\epsilon > 0$ and for $\omega < \omega_p$ we have $\epsilon < 0$. For material with $\epsilon < 0$, EM radiation cannot propagate through the material—discussed more shortly.

Maxwell Equation

- We want to find the wave equation. In material, we have

$$\nabla \cdot \mathbf{E} = 0 \quad \text{and} \quad \nabla \times \mathbf{E} = -\mu_0 \epsilon \frac{\partial \mathbf{H}}{\partial t}$$

and also

$$\nabla \cdot \mathbf{H} = 0 \quad \text{and} \quad \nabla \times \mathbf{H} = \epsilon_0 \epsilon \frac{\partial \mathbf{E}}{\partial t}$$

note that there is no current density \mathbf{j} in our material, which simplifies the last equation—there is only displacement current.

- Apply curl to the first lines to get

$$\nabla \times (\nabla \times \mathbf{E}) = -\mu_0 \epsilon \frac{\partial}{\partial t} \left(\epsilon_0 \epsilon \frac{\partial \mathbf{E}}{\partial t} \right)$$

$$\nabla(\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E} = -\mu_0 \epsilon \epsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2}$$

Because $\nabla \cdot \mathbf{E} = 0$ we get

$$\nabla^2 \mathbf{E} - \mu_0 \epsilon \epsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2} = 0$$

If we insert the earlier oscillatory ansatz $\mathbf{E} = \mathbf{E}_0 e^{i(kz - \omega t)}$ we get

$$-k^2 \mathbf{E} - \mu_0 \epsilon \epsilon_0 (-\omega^2) \mathbf{E} = 0 \implies \mathbf{E}(\mu_0 \epsilon \epsilon_0 \omega^2 - k^2) = 0$$

Which gives us

$$\mu_0 \epsilon \epsilon_0 \omega^2 - k^2 = 0 \implies \frac{\omega^2}{k^2} = \frac{1}{\epsilon_0 \mu_0} \frac{1}{\epsilon} = \frac{c_0^2}{\epsilon}$$

And next, substituting in $\epsilon = 1 - \frac{\omega_p^2}{\omega^2}$, we have

$$c_0^2 = \frac{\omega^2}{k^2} \left(1 - \frac{\omega_p^2}{\omega^2} \right) = \frac{\omega^2}{k^2} - \frac{\omega_p^2}{k^2}$$

And then rearrange to get

$$k^2 c_0^2 = \omega^2 - \omega_p^2 \implies \omega = \sqrt{\omega_p^2 + c_0^2 k^2}$$

This is a dispersion relation for electric field in a material—it relates the frequency ω and wave-number k of the electric field. Note that the relation is nonlinear.

- For electric field in a vacuum, we have $\epsilon = 1$; in this case we have the linear dispersion relation

$$\frac{\omega^2}{k^2} = \frac{c_0^2}{\epsilon} = c_0^2 \implies \omega = c_0 k$$

- Back to the dispersion relation in material:

$$\omega = \sqrt{\omega_p^2 + c_0^2 k^2}$$

For large k we get roughly $\omega = c_0 k$.

For small k we get $\omega \rightarrow \omega_p$.

A more accurate expansion for small ω would give

$$\omega = \omega_p \left(1 + \frac{c_0^2}{\omega_p^2} k^2 \right)^{1/2} \approx \omega_p \left[1 + \frac{c_0^2}{2\omega_p^2} k^2 \right]$$

- Note that for small k and small frequencies, we have $\omega \sim k^2$, which means the photons making up the electric field behave like particles with mass, which leads to the concept of effective mass, like in solid state physics.
- Note also that the dispersion relation prohibits $\omega < \omega_p$ since at the minimum possible k , i.e. $k = 0$, frequency is $\omega = \omega_p$. The prohibited region of $\omega < \omega_p$ corresponds to $\epsilon < 0$ —this connection validates the previous assertion that EM radiation cannot exist in materials with $\epsilon < 0$.
- We more thoroughly consider $\epsilon < 0$. Rearranging the equation

$$\frac{\omega^2}{k^2} = \frac{c_0^2}{\epsilon}$$

leads to

$$k = \frac{\omega}{c_0} \sqrt{\epsilon} = \frac{\omega}{c_0} i \sqrt{\epsilon_R}$$

Note that k is imaginary because $\epsilon < 0$. Substituting this k into our wave ansatz for \mathbf{E} gives

$$\mathbf{E} = \mathbf{E}_0 e^{i(kz - \omega t)} = \mathbf{E}_0 e^{-\frac{\omega}{c_0} \sqrt{\epsilon_R} z} e^{-i\omega t}$$

The position-dependent term is $\propto e^{-z}$, which means that an electromagnetic wave will exponentially decay with position in a material with $\epsilon < 0$.

- Next, we use the dispersion relation to find the group and phase velocity of the electromagnetic waves in the plasma. The phase velocity is

$$v_p = \frac{\omega_p}{k} = \frac{\sqrt{\omega_p^2 + c_0^2 k^2}}{k} = \sqrt{c_0^2 + \frac{\omega_p^2}{k^2}}$$

Note that $v_p > c_0$ —the phase velocity is greater than the speed of light in vacuum.

- The group velocity—the quantity relevant to the universal speed limit of c_0 —is

$$v_g = \frac{\partial \omega}{\partial k} = \frac{2c_0^2 k}{2\sqrt{\omega_p^2 + c_0^2 k^2}} = \frac{c_0}{\sqrt{1 + \frac{\omega_p^2}{c_0^2 k^2}}}$$

As expected, $v_g < c_0$, in agreement with special relativity. Note also $v_p v_g = c_0^2$.

3 Electromagnetic Wave Propagation

3.1 Thirteenth Exercise Set (rough draft)

3.1.1 Theory: Electromagnetic Wave Propagation in Waveguides

- Assume empty waveguides with $\epsilon = 1$ —we can use the wave equation

$$\nabla^2 \mathbf{E} - \frac{1}{c_0^2} \frac{\partial^2 \mathbf{E}}{\partial t^2} = 0$$

- We assume bounded waveguide geometry, e.g. a pipe-like waveguide along the z axis. We use the following ansatz for electric field:

$$\mathbf{E}(\mathbf{r}, t) = \mathbf{E}(\boldsymbol{\rho}) e^{i(kz - \omega t)}$$

where $\boldsymbol{\rho}$ is the position within the waveguide's planar cross section, i.e. the xy plane perpendicular to $\hat{\mathbf{z}}$ along the direction of EM propagation.

- We write the Laplacian in two components to get

$$\nabla^2 = \frac{\partial^2}{\partial z^2} + \nabla_{\perp}^2 = -k^2 + \nabla_{\perp}^2$$

where $\nabla_{\perp}^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ corresponds to differentiation in the xy plane/the waveguide's cross section.

We write the time derivative as

$$\frac{\partial^2}{\partial t^2} = -\omega^2$$

The wave equation in a waveguide then simplifies to

$$\left[\nabla_{\perp}^2 + \left(\frac{\omega^2}{c_0^2} - k^2 \right) \right] \mathbf{E} = 0$$

where $\mathbf{E} = \mathbf{E}(\boldsymbol{\rho})$. However, because there is no time derivative, we can use the same equation for $\mathbf{E} = \mathbf{E}(\mathbf{r}, t)$.

- \mathbf{E} has three components, so the wave equation in vector form really has 3 equations for each component.

Analogously, \mathbf{H} has three components, and thus three associated equations.

However, we can express \mathbf{H} in terms of \mathbf{E} , which simplifies the equations involved.

Relationship Between \mathbf{E} and \mathbf{H}

- We start with the Maxwell equations

$$\nabla \times \mathbf{E} = -\mu_0 \frac{\partial \mathbf{H}}{\partial t} \quad \text{and} \quad \nabla \times \mathbf{H} = \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

In components, the equations read

$$\begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix} \times \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix} = \begin{bmatrix} \frac{\partial E_z}{\partial y} - ikE_y & ikE_x - \frac{\partial E_z}{\partial x} \\ \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} \\ \end{bmatrix} = i\mu_0 \omega \begin{bmatrix} H_x \\ H_y \\ H_z \end{bmatrix}$$

where we have used $\frac{\partial}{\partial z} = ik$ and $\frac{\partial}{\partial t} = -i\omega$

- The second equation reads

$$\begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix} \times \begin{bmatrix} H_x \\ H_y \\ H_z \end{bmatrix} = \begin{bmatrix} \frac{\partial H_z}{\partial y} - ikH_y ikH_x - \frac{\partial H_z}{\partial x} \\ \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} \\ \frac{\partial H_z}{\partial y} - \frac{\partial H_x}{\partial y} \end{bmatrix} = -i\epsilon_0\omega \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix}$$

We have six component equations.

- The first and fifth equations both contain H_x and E_y .

They read

$$\frac{\partial E_z}{\partial y} - ikE_y = i\mu_0\omega H_x \quad \text{and} \quad ikH_x - \frac{\partial H_y}{\partial x} = -i\epsilon_0\omega E_y$$

We combine to get

$$k \frac{\partial E_x}{\partial y} - ik^2 E_y = \mu_0\omega \left(\frac{\partial H_z}{\partial x} - i\epsilon_0\omega E_y \right)$$

then

$$k \frac{\partial E_z}{\partial y} - \mu_0\omega \frac{\partial H_z}{\partial x} = iE_y (k^2 - \omega^2\mu_0\epsilon_0) = iE_y \left(k^2 - \frac{\omega^2}{c^2} \right)$$

We find E_y in terms of the z components with

$$E_y = i \frac{k \frac{\partial E_z}{\partial y} - \mu_0\omega \frac{\partial H_z}{\partial x}}{\frac{\omega^2}{c^2} - k^2}$$

- Similarly, the second and fourth component equations contain E_x and H_y in terms of E_z and H_z .

Without derivation (analogous to the above) the other equations for the components E_x, E_y and H_x, H_y are

$$\begin{aligned} H_x &= i \frac{k \frac{\partial H_z}{\partial x} - \omega\epsilon_0 \frac{\partial E_z}{\partial y}}{\frac{\omega^2}{c^2} - k^2} \\ E_x &= i \frac{k \frac{\partial E_z}{\partial x} + \omega\mu_0 \frac{\partial H_z}{\partial y}}{\frac{\omega^2}{c^2} - k^2} \\ H_y &= i \frac{k \frac{\partial H_z}{\partial y} + \omega\epsilon_0 \frac{\partial E_z}{\partial x}}{\frac{\omega^2}{c^2} - k^2} \end{aligned}$$

Note that these equations apply only in Cartesian coordinates.

- Next, we consider boundary conditions. Consider a boundary wall of the waveguide. Encircle a small portion of the boundary wall with a surface to use Stoke's law. The wall contains $\rho(z, t)$ and $j(z, t)$.

We use the other two equations:

$$\nabla \times \mathbf{E} = -\mu_0 \frac{\partial \mathbf{H}}{\partial t} \quad \text{and} \quad \nabla \cdot \mathbf{H} = 0$$

When we send the loop to zero thickness, the first equation gives

$$\nabla \times \mathbf{E} \rightarrow 0 \implies E_{\parallel} = 0$$

while the second equation gives

$$H_{\perp} \rightarrow 0$$

- Finally, we find E_z and H_z from the waveguide wave equation

$$\left[\nabla_{\perp}^2 + \left(\frac{w^2}{c^2} - k^2 \right) \right] \begin{Bmatrix} E_z \\ H_z \end{Bmatrix} = 0$$

We consider solutions with two orthogonal components: one with $E_z = 0$ and $H_z \neq 0$ and the second with $H_z = 0$ and $E_z \neq 0$. The general solution will be a linear combination of the two solutions.

The first method is called transverse electric (TE), since the electric field is perpendicular (transverse) to the direction of EM wave propagation (and $E_z = 0$).

The first method is called transverse magnetic (TM), since the magnetic field is perpendicular (transverse) to the direction of EM wave propagation (and $H_z = 0$).

3.1.2 Waveguide of Two Parallel Plates

Consider a waveguide consisting to two large parallel conducting plates separated by a distance a

- Let x be vertical (up down in paper), z left right in paper (direction of wave propagation) and y out of the page.

In this case $\frac{\partial}{\partial y} = 0$ because of translational symmetry along y .

- For TM we have $E_z \neq 0$ and $H_z = 0$.

First what remaining components are non-zero. $H_y \neq 0$. $E_y = 0$. $H_x = 0$ and $E_x \neq 0$. Noting that $\frac{\partial}{\partial y} = 0$.

\mathbf{H} points out of the page along y , while \mathbf{E} points in the xz plane.

- Next, we find E_z using

$$\left[\nabla_{\perp}^2 + \left(\frac{w^2}{c^2} + k^2 \right) \right] E_z \equiv [\nabla_{\perp}^2 + \kappa^2] E_z$$

Since $\frac{\partial}{\partial y} = 0$ this becomes

$$\left(\frac{\partial^2}{\partial x^2} + \kappa^2 \right) E_z = 0$$

The solution is oscillation. We can write the general solution as

$$E_z(x) = A \sin \kappa x + B \cos \kappa x$$

- Boundary conditions $E_{\parallel} = 0$. The parallel components to waveguide surface are z and y . We already know $E_y = 0$, so at the boundary surfaces we have

$$E_{\perp} = E_z(0) = E_z(a) = 0$$

The condition $E_z(0) = 0$ implies $B = 0$ and thus

$$E_z = A \sin \kappa x$$

While $E_z(a) = 0$ implies $\kappa a = n\pi$, so

$$\kappa = \frac{n\pi}{a}, \quad n = 1, 2, 3, \dots$$

Note that $n = 0$ gives a trivial solution

The TM solution for E_z is then

$$E_z(x) = A \sin \frac{n\pi x}{a}$$

Dispersion Relation

- We want to find the relationship between ω and k .

We use the E_z wave equation and the definition of κ to get

$$\kappa^2 = \frac{\omega^2}{c^2} - k^2 = \left(\frac{n\pi}{a}\right)^2$$

which implies

$$\omega = c_0 \sqrt{k^2 + \left(\frac{n\pi}{a}\right)^2}$$

At large k , this approaches the free space relation $\omega = c_0 k$

- Note also that frequencies below $\frac{n\pi}{a}$ are unattainable—in other words, there is a minimum possible frequency at which EM waves can propagate through the waveguide.
- We call the frequency band spanning the frequencies between $n = 1$ and $n = 2$ the wave band. The bandwidth is the difference between the $n = 1$ and $n = 2$ frequencies.

$$\Delta\omega = \frac{c_0\pi}{a}$$

Impedance

- The impedance for TM waves is defined in terms of the fields as

$$Z \equiv \frac{E_{\perp}}{H_{\parallel}}$$

In our case for the parallel-plate waveguide this reads

$$Z = \frac{E_x}{H_y}$$

Using the equations for the components E_x and H_y , we have

$$Z = \frac{ik \frac{\partial E_z}{\partial x}}{i\omega\epsilon_0 \frac{\partial E_z}{\partial x}} = \frac{k}{\omega\epsilon_0}$$

We use the dispersion relation to get

$$Z = \frac{\sqrt{\frac{\omega^2}{c_0^2} - \left(\frac{n\pi}{a}\right)^2}}{\omega\epsilon_0} = \frac{1}{c_0\epsilon_0} \sqrt{1 - \left(\frac{n\pi c_0}{a}\right)^2 \frac{1}{\omega^2}}$$

The coefficient is

$$\frac{1}{c_0\epsilon_0} = \frac{\sqrt{\epsilon_0\mu_0}}{\epsilon_0} = \sqrt{\frac{\mu_0}{\epsilon_0}} \equiv Z_0$$

which is the impedance of free space.

- We also define the frequency

$$\frac{n\pi c_0}{a} \equiv \omega_n$$

which is the frequency of each brach in the dispersion relation.

In terms of ω_n then the impedance is

$$Z(\omega) = Z_0 \sqrt{1 - \left(\frac{\omega_n}{\omega}\right)^2}$$

For large ω , the impedance approaches Z_0 .

Ratio of Electric Field Amplitudes

- Find the ratio of amplitudes $\frac{E_{z0}}{E_{x0}}$.

First

$$\frac{E_z}{E_x} = \frac{E_z}{ik \frac{\partial E_z}{\partial x}} \kappa^2$$

We know $E_z = A \sin \kappa x$ from earlier, and get

$$\frac{E_{z0}}{E_{x0}} = \frac{\kappa}{k} =$$

We can interpret κ as a wave vector in the x direction, while k is the wave vector along the direction of propagation z .

TE Propagation Mode

- This is analogous to TM, so only sketched. In this case $E_z = 0$ and we find H_z .
- The wave equation is solved with

$$H_z = A \sin \kappa x + B \cos \kappa x$$

- Boundary equations are now different though— $H_\perp = 0$. The perpendicular component is H_x . We turn to the component equations to get

$$H_x = i \frac{k \frac{\partial H_z}{\partial x}}{\kappa^2}$$

Since $H_x = 0$ at the boundary it follows that

$$\left. \frac{\partial H_z}{\partial x} \right|_{x=0,a} = 0$$

The boundary condition is now in terms of a derivative—note that $\frac{\partial H_z}{\partial x}$ is a “normal” derivative; since in points in the direction x , which is normal to the waveguide’s boundary.

Note that in general, the boundary condition in TE mode involves the normal derivative of H_z with respect to the waveguide boundary surface.

- Taking the derivative of H_z and evaluating the boundary condition gives

$$H_z(x) = B \cos \frac{n\pi x}{a}, \quad n = 1, 2, 3, \dots$$

Note that $n = 0$ produces $H_z = B$ which is not a trivial solution in itself, but the other field components involve derivatives of H_z and are trivial, so we again reject $n = 0$.

- Since κ is the same as in TM mode, the dispersion relation in TE is the same. The only notable difference in TE mode is the boundary condition involving the normal derivative of H_z .

And also impedance turns out to be

$$Z(\omega) = \frac{Z_0}{\sqrt{1 - \left(\frac{\omega_n}{\omega}\right)^2}}$$

Generalization: waveguide with a rectangular cross section

- In this case,

$$\nabla_{\perp}^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

and $\frac{\partial}{\partial y}$ is not zero. We then have

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \kappa^2 \right) E_x(x, y) = 0$$

We use separation of variables: $E_x(x, y) = X(x)Y(y)$. We get

$$X''Y + XY'' + \kappa^2 XY = 0 \implies \frac{X''}{X} + \frac{Y''}{Y} + \kappa^2 = 0$$

The x and y terms must be constant for the equation to hold, so we define

$$\frac{X''}{X} = -\kappa_x^2 \quad \text{and} \quad \frac{Y''}{Y} = -\kappa_y^2$$

- The equations then read

$$X'' + \kappa_x^2 X = 0 \quad \text{and} \quad Y'' + \kappa_y^2 Y = 0$$

We solve the equations like before and get sinusoidal solutions.

- Boundary conditions now involve a and b

$$\kappa_x = \frac{n\pi}{a} \quad \text{and} \quad \kappa_y = \frac{m\pi}{b}, \quad n, m = 1, 2, 3, \dots$$

Note $\kappa^2 = \kappa_x^2 + \kappa_y^2$.

- Dispersion relation is

$$\omega = c_0 \sqrt{k^2 + \left(\frac{n\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2}$$

The lowest branch now depends on the ratio of a and b .

- Finally, note that because of both two indices n and m , in TE mode with cosine solutions one of either n or m can be zero.

In TM mode both indices are $n, m = 1, 2, 3, \dots$