# Vector Analysis

### Line Integrals

$\int_\Gamma f\,\mathrm{d}s = \int_I f(\boldsymbol{r})\|\dot{\boldsymbol{r}}\|\,\mathrm{d}t \quad \oint_\Gamma \boldsymbol{F}\cdot\mathrm{d}\boldsymbol{s}$ (circulation)

$\int_{\boldsymbol{a}}^{\boldsymbol{b}} \boldsymbol{\Phi}\cdot\mathrm{d}\boldsymbol{s} = \phi(\boldsymbol{b}) - \phi(\boldsymbol{a}) \quad \int_\Gamma \boldsymbol{F}\cdot\mathrm{d}\boldsymbol{s} = \int_I \boldsymbol{F}(\boldsymbol{r})\cdot\dot{\boldsymbol{r}}\,\mathrm{d}t$

### Surface Integral Formulas

$\iint_{\mathcal{S}} f\,\mathrm{d}S = \iint_D f(\boldsymbol{r})\sqrt{EG-F^2}\,\mathrm{d}u\,\mathrm{d}v$

$\iint_{\mathcal{S}} \boldsymbol{F}\cdot\mathrm{d}\boldsymbol{S} =$ flux through $\mathcal{S}$

$\iint_{\mathcal{S}} \boldsymbol{F}\cdot\mathrm{d}\boldsymbol{S} = \iint_{\mathcal{S}} (\boldsymbol{F}\cdot\boldsymbol{n})\,\mathrm{d}S$

$= \iint_D \boldsymbol{F}(\boldsymbol{r})\cdot(\boldsymbol{r}_u\times\boldsymbol{r}_v)\,\mathrm{d}u\,\mathrm{d}v$

### Vector Operator Formulas

$\boldsymbol{\nabla} f(x,y,z) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$

$\frac{\partial f}{\partial \boldsymbol{n}} = \left\langle \boldsymbol{\nabla} f, \frac{\boldsymbol{n}}{\|\boldsymbol{n}\|}\right\rangle$

$\boldsymbol{\nabla}\cdot\boldsymbol{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$

$\boldsymbol{\nabla}\times\boldsymbol{F} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)$

$\Delta f = \boldsymbol{\nabla}\cdot(\boldsymbol{\nabla} f) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$

### Important Theorems

$\oint_{\partial D} \boldsymbol{F}\cdot\mathrm{d}\boldsymbol{s} = \iint_D (Y_x - X_y)\,\mathrm{d}x\,\mathrm{d}y$ (Green)

$\oiint_{\partial\Omega} \boldsymbol{F}\cdot\mathrm{d}\boldsymbol{S} = \iiint_\Omega \boldsymbol{\nabla}\cdot\boldsymbol{F}\,\mathrm{d}V$ (Gauss)

$\oint_{\partial\mathcal{S}} \boldsymbol{F}\cdot\mathrm{d}\boldsymbol{s} = \iint_{\mathcal{S}} \boldsymbol{\nabla}\times\boldsymbol{F}\cdot\mathrm{d}\boldsymbol{S}$ (Stokes)

## Parameterizations of Common Shapes

### Sphere

$\Omega = \left\{(x,y,z)\subset\mathbb{R}^3 \,\middle|\, x^2+y^2+z^2 \le a^2\right\}$

$\boldsymbol{r}(\phi,\theta,\rho) = (\rho\cos\theta\cos\phi, \rho\cos\theta\sin\phi, \rho\sin\theta)$

$\mathcal{S} = \left\{(x,y,z)\subset\mathbb{R}^3 \,\middle|\, x^2+y^2+z^2 = a^2\right\}$

$\boldsymbol{r}(\phi,\theta) = (a\cos\theta\cos\phi, a\cos\theta\sin\phi, a\sin\theta)$

$\phi\in[0,2\pi]$, $\theta\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$, $\rho\in[0,a] \quad J = r^2\cos\theta$

### Surface of Torus

$\boldsymbol{r}(\phi,\theta) = ((b\cos\theta+a)\cos\phi, (b\cos\theta+a)\sin\phi, b\sin\theta)$

$\phi\in[0,2\pi]$, $\theta\in[0,2\pi]$

### Surface of Cone

$\mathcal{S} = \left\{(x,y,z)\in\mathbb{R}^3 \,|\, z^2 = x^2+y^2,\ z\in[0,1]\right\}$

$\boldsymbol{r}(\rho,\phi) = (\rho\cos\phi, \rho\sin\phi, \rho)$

$\phi\in[0,2\pi]$, $\rho\in[0,1]$

### Surface of Cylinder

$\mathcal{S} = \left\{(x,y,z)\in\mathbb{R}^3 \,|\, x^2+y^2 = a^2,\ z\in[z_{min}, z_{max}]\right\}$

$\boldsymbol{r}(\phi,z) = (a\cos\phi, a\sin\phi, z)$

$\phi\in[0,2\pi]$, $z\in[z_{min}, z_{max}]$

# Differential Equations

## First-Order Differential Equations

### Linear DE

$y' + p(x)y = q(x)$

$P(x) \equiv \int p(x)\,\mathrm{d}x \quad C(x) = \int q(x)e^{P(x)}$

$y = C(x)e^{-P(x)}$

### Bernoulli DE

$u(x)y' + v(x)y = w(x)y^\alpha$

new var: $z = y^{1-\alpha} \quad z' = (1-\alpha)y^{-\alpha}y'$

$\frac{u(x)}{(1-\alpha)}z' + v(x)z = w(x) \implies$ LDE for $z$

### Riccati DE

$y' = uy^2 + vy + w$

guess solution $y_1$ new var: $z = y - y_1$

$z' - (2uy_1 + v)z = uz^2 \implies$ Bernoulli DE for $z$

### Homogeneous DE

$y' = f(x,y); \quad f(tx,ty) = f(x,y)$

$g(z) \equiv f(1,z) - z \implies \frac{\mathrm{d}z}{g(z)} = \frac{\mathrm{d}x}{x}$ separable DE for $z$

$y(x) = xz(x)$

### Exact DE

$P(x,y)\,\mathrm{d}x + Q(x,y)\,\mathrm{d}y = 0; \quad P_y = Q_x$

solve with separation of variables

### Solving with Integrating Factors

$P(x,y)\,\mathrm{d}x + Q(x,y)\,\mathrm{d}y = 0; \quad P_y \ne Q_x$

$$\mu = \begin{cases} \mu(x) = \exp\left(\int \frac{P_y - Q_x}{Q}\,\mathrm{d}x\right); & \frac{P_y - Q_x}{Q} \text{ independent of } y \\ \mu(y) = \exp\left(\int \frac{Q_x - P_y}{P}\,\mathrm{d}x\right); & \frac{Q_x - P_y}{P} \text{ independent of } x \end{cases}$$

$\mu P(x,y)\,\mathrm{d}x + \mu Q(x,y)\,\mathrm{d}y = 0$

$u_x = \mu P(x,y) \qquad u_y = \mu\tilde{Q}(x,y)$

$$u = u(x,y) = \begin{cases} \int \mu P(x,y)\,\mathrm{d}x + C(y) + D \\ \int \mu Q(x,y)\,\mathrm{d}y + C(x) + D \end{cases}$$

### Parametric Solution Case I

$F(x,y') = 0$

find parameterizations for $x(t)$ and $y'(t)$

$y(t) = \int \dot{x}(t)y'(t)\,\mathrm{d}t \implies$ param. solution $x(t), y(t)$

### Parametric Solution Case II

$F(y,y') = 0$

find parameterizations for $y(t)$ and $y'(t)$

$x(t) = \int \frac{\dot{y}(t)}{y'(t)}\,\mathrm{d}t \implies$ param. solution $x(t), y(t)$

### Clairaut DE

$y = xy' + f(y')$

general solution: $y_g = Cx + f(C)$

singular solution: $x + f'(y_s') = 0 \implies$ parametric or separable

### Calculus of Variations

$L = L(x,y') \implies L_{y'} = C$

$L = L(y,y') \implies L - y'L_{y'} = C$

## Systems of LDEs with Constant Coefficients

### Homogeneous and Diagonalizable

$\dot{\boldsymbol{x}} = \mathbf{A}\boldsymbol{x}(t); \quad \mathbf{A}$ diagonalizable

$\mathbf{D} = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n) \quad \mathbf{P} = [v_1, v_2, \ldots, v_n]$

$\mathbf{P}e^{t\mathbf{D}}\boldsymbol{c} = c_1e^{\lambda_1 t}\boldsymbol{v_1} + \cdots + c_ne^{\lambda_n t}\boldsymbol{v_n}$

### Homogeneous and Non-Diagonalizable

$\dot{\boldsymbol{x}} = \mathbf{A}\boldsymbol{x}(t); \quad \mathbf{A}$ non-diagonalizable

$\lambda$ multiplicity 1 $\implies$ one solution: $ce^{\lambda t}\boldsymbol{v}$

$\lambda$ multiplicity $r > 1 \implies r$ solutions:

$c_1e^{\lambda t}\boldsymbol{v}^{(1)}, \quad c_2e^{\lambda t}\left(\boldsymbol{v}^{(2)} + t\boldsymbol{v}^{(1)}\right), \ldots,$

$c_re^{\lambda t}\left(\boldsymbol{v}^{(r)} + t\boldsymbol{v}^{(r-1)} + \frac{t^2}{2}\boldsymbol{v}^{(r-2)} + \cdots + \frac{t^{r-1}}{(r-1)!}\boldsymbol{v}^{(1)}\right)$

$\boldsymbol{v}^{(r)} \in \ker(\mathbf{A} - \lambda\mathbf{I})^r \setminus \ker(\mathbf{A} - \lambda\mathbf{I})^{r-1}$

$\boldsymbol{v}^{(i-1)} = \mathbf{A}\boldsymbol{v}^{(i)}; \quad \boldsymbol{v}^{(1)}$ eigenvector for $\lambda$

### Non-Homogeneous

$\dot{\boldsymbol{x}}(t) = \mathbf{A}\boldsymbol{x}(t) + \boldsymbol{b}(t)$

$\boldsymbol{x}_h$ solution to homogeneous system $\dot{\boldsymbol{x}}(t) = \mathbf{A}\boldsymbol{x}(t)$

$\boldsymbol{x}(t) = (x_1(t), \ldots, x_n(t)); \quad \mathbf{X}(t) = [x_1(t)\cdots x_n(t)]$

$\boldsymbol{x}_p(t) = \boldsymbol{c}(t)\mathbf{X}(t); \quad \boldsymbol{c}(t) = \int \mathbf{X}^{-1}(t)\boldsymbol{b}(t)\,\mathrm{d}t$

general solution: $\boldsymbol{x}_g = \boldsymbol{x}_h + \boldsymbol{x}_p$

### Complex Solutions of Systems of LDEs

$\lambda, \lambda^* \in \mathbb{C}$ complex conjugate eigenvalue pair

$\boldsymbol{v}, \tilde{\boldsymbol{v}} \in \mathbb{C}^n$ complex eigenvectors for $\lambda, \lambda^*$

complex solutions: $ce^{\lambda t}\boldsymbol{v}, \tilde{c}e^{\lambda^* t}\tilde{\boldsymbol{v}}$

equivalent real solutions:

$\mathrm{Re}(c_1e^{\lambda t}\boldsymbol{v}), \mathrm{Im}(c_2e^{\lambda t}\boldsymbol{v})$ OR $\mathrm{Re}(c_1e^{\lambda^* t}\tilde{\boldsymbol{v}}), \mathrm{Im}(c_2e^{\lambda^* t}\tilde{\boldsymbol{v}})$

# Higher-Order Differential Equations

### Homogeneous LDE with Constant Coefficients

$a_n y^{(n)} + \cdots + a_1 y' + a_0 y = 0$
$p(\lambda) = a_n \lambda^n + \cdots + a_1 \lambda + a_0$
$\lambda$ zero, multiplicity 1 : one solution $y = ce^{\lambda x}$
$\lambda$ zero, multiplicity $k$ : $k$ solutions
$y_1 = c_1 e^{\lambda x}, y_2 = c_2 x e^{\lambda x}, \ldots, y_k = c_k x^{k-1} e^{\lambda x}$

### Homogeneous Cauchy-Euler

$a_n x^n y^{(n)} + \cdots + a_1 x y' + a_0 y = 0$
$p(\lambda) = a_0 + a_1 \lambda + a_2 \lambda(\lambda-1) + \cdots + a_n \lambda(\lambda-1)\cdots(\lambda-n+1)$
$\lambda$ zero, multiplicity 1 : one solution $y = cx^\lambda$
$\lambda$ zero, multiplicity $k$ : $k$ solutions
$y_1 = c_1 x^\lambda, y_2 = c_2 \ln|x| x^\lambda, \ldots, y_k = c_k \ln|x|^{k-1} x^\lambda$

### Non-Homogeneous LDE

$a_n y^{(n)} + \cdots + a_1 y' + a_0 y = b(x)$
$y_h$ solution to homog. eq. $a_n y^{(n)} + \cdots + a_1 y' + a_0 y = 0$

$$\mathbf{Y}(x) = \begin{bmatrix} y_1 & \cdots & y_n \\ \vdots & \ddots & \vdots \\ y_1^{(n-1)} & \cdots & y_n^{(n-1)} \end{bmatrix} \quad \boldsymbol{b}(x) = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \frac{b(x)}{a_n} \end{bmatrix}$$

$y_p = \boldsymbol{y}(x) \cdot \boldsymbol{c}(x); \quad \boldsymbol{c}(x) = \int \mathbf{Y}^{-1}(x) \boldsymbol{b}(x)\,\mathrm{d}x$
general solution: $\boldsymbol{x}_g = \boldsymbol{x}_h + \boldsymbol{x}_p$

### Special Case I: Non-Homogeneous LDE

$a_n y^{(n)} + \cdots + a_1 y' + a_0 y = P(x) e^{\lambda_0 x}$; $P$ polynomial degree $p$
$y_h$ solution to homog. eq. $a_n y^{(n)} + \cdots + a_1 y' + a_0 y = 0$
ansatz: $y_p = x^k Q(x) e^{\lambda_0 x}$
$k$ multiplicity of $\lambda_0$ as zero of char. poly $p(\lambda)$
$Q(x) = a_p x^p + a_{p-1} x^{p-1} + \cdots + a_1 x + a_0$
general solution: $\boldsymbol{x}_g = \boldsymbol{x}_h + \boldsymbol{x}_p$

### Special Case II: Non-Homogeneous LDE

$a_n y^{(n)} + \cdots + a_1 y' + a_0 y = P(x) e^{ax} \cos(bx)$
$a_n y^{(n)} + \cdots + a_1 y' + a_0 y = P(x) e^{ax} \sin(bx)$
$y_h$ solution to homog. eq. $a_n y^{(n)} + \cdots + a_1 y' + a_0 y = 0$
$\tilde{y}_p$ particular solution to equation:
$a_n y^{(n)} + \cdots + a_1 y' + a_0 y = P(x) e^{(a+ib)x}$

$$y_p = \begin{cases} \mathrm{Re}(\tilde{y}_p) & \text{non-homogeneous term is cosine} \\ \mathrm{Im}(\tilde{y}_p) & \text{non-homogeneous term is sine} \end{cases}$$

general solution: $\boldsymbol{x}_g = \boldsymbol{x}_h + \boldsymbol{x}_p$

### Non-Homogeneous Cauchy-Euler

$a_n x^n y^{(n)} + \cdots + a_1 x y' + a_0 y = b(x)$
$y_h$ solution to homog. eq. $a_n x^n y^{(n)} + \cdots + a_1 x y' + a_0 y = 0$
$\mathbf{Y}(x)$ fundamental matrix; $\boldsymbol{b}(x) = \left(0, \ldots, 0, \frac{b(x)}{a_n x^n}\right) \in \mathbb{R}^n$
$y_p = \boldsymbol{y}(x) \cdot \boldsymbol{c}(x); \quad \boldsymbol{c}(x) = \int \mathbf{Y}^{-1}(x) \boldsymbol{b}(x)\,\mathrm{d}x$
general solution: $\boldsymbol{x}_g = \boldsymbol{x}_h + \boldsymbol{x}_p$

### Complex Solutions of Higher-Order LDEs

$\lambda, \lambda^* \in \mathbb{C}$ zeros of char. poly; complex conjugate pair
complex solutions: $ce^{\lambda x}, \tilde{c}e^{\lambda^* x}$
equivalent real solutions:
$\mathrm{Re}(ce^{\lambda x}), \mathrm{Im}(ce^{\lambda x})$ OR $\mathrm{Re}(e^{\lambda^* x}), \mathrm{Im}(e^{\lambda^* x})$

# Calculus of Variations

$I(y) = \int_a^b L(x, y, y')\,\mathrm{d}x$
$\frac{\partial L}{\partial y} = \frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{\partial L}{\partial y'}\right]$ and $\left[\frac{\partial L}{\partial y'}\mu\right]_a^b = 0 \implies I$ extrema at $y$
$L = L(x, y) \implies \frac{\partial L}{\partial y} = 0$
$L = L(x, y') \implies \frac{\partial L}{\partial y'} = C$
$L = L(y, y') \implies L - y' \frac{\partial L}{\partial y'} = C$

### Path from $A$ to $B$ on Surface $\mathcal{S}$

1. Parameterize path with $\boldsymbol{r} : J \to \mathbb{R}^3$, one variable $x, y, z$ unknown, $J \subset \mathbb{R}$ interval from $A$ to $B$.
2. Calculate $\boldsymbol{r}'$ and its magnitude $\|\boldsymbol{r}'\| = \sqrt{\boldsymbol{r}_x'^2 + \boldsymbol{r}_y'^2 + \boldsymbol{r}_z'^2}$
3. Find extrema of functional $\ell = \int_J \|\boldsymbol{r}'\|$.