Linear Algebra Review

Linear Algebra Review
$A \in \mathbb{C}^{n \times n}$ is *Hermitian* if $A = A^\dagger$
$A \in \mathbb{R}^{n \times n}$ is *Hermitian* if $A = A^T$
$O : V \to V$ *Hermitian* if $\langle Ox, y \rangle = \langle x, Oy \rangle$ for all $x, y \in V$
Hermitian operators have real eigenvalues, orthogonal eigenfunctions and correspond to observable quantities
$U \in \mathbb{C}^{n \times n}$ *unitary* if $UU^\dagger = U^\dagger U = I \implies U^{-1} = U^\dagger$
$O \in \mathbb{R}^{n \times n}$ *orthogonal* if $OO^T = O^T O = I \implies O^{-1} = O^T$

Commutator
$[A, B] = AB - BA$
$[A, B] = 0 \iff AB = BA \iff B = ABA^{-1} = A^{-1}BA$
$[B, A] = -[A, B] \qquad [AB, C] = A[B, C] + [A, C]B$
$\{A, B\} = AB + BA$

Quantum Mechanics in 1D

Some Basic Operators
$p_x \to -i\hbar \frac{\partial}{\partial x} \qquad \boldsymbol{p} \to -i\hbar \nabla$

Uncertainty Principle
$(\Delta A \Delta B)^2 \geq \left(\frac{1}{2} |\langle [A, B] \rangle| \right)^2$
$\Delta x \Delta p \geq \frac{\hbar}{2} \qquad \Delta E \Delta t \geq \frac{\hbar}{2}$

Expectation Values, etc...
$\langle \mathcal{O} \rangle = \langle \psi | \mathcal{O} | \psi \rangle \equiv \langle \psi | \mathcal{O}\psi \rangle \equiv \int \psi^* \mathcal{O}\psi \, dx$
$\langle \psi | \mathcal{O}\psi \rangle = \langle \mathcal{O}^* \psi | \psi \rangle$ and $\mathcal{O} = \mathcal{O}^* \implies \langle \psi | \mathcal{O}\psi \rangle = \langle \mathcal{O}\psi | \psi \rangle$
$\langle \psi_m | \psi_n \rangle = \int \psi_m^* \psi_n \, dx$
$\langle \mathcal{O}\psi | \mathcal{O}\psi \rangle = \|\mathcal{O}\psi\|^2$
$\langle A^\dagger \rangle = \langle A \rangle^* \implies \langle A \rangle + \langle A^\dagger \rangle = \langle A \rangle + \langle A \rangle^* = 2 \operatorname{Re} \langle A \rangle$

Uncertainties
$\Delta x(t) = \sqrt{\langle x^2, t \rangle - \langle x, t \rangle^2} \qquad \Delta p(t) = \sqrt{\langle p^2, t \rangle - \langle p, t \rangle^2}$
$(\Delta A)^2 = \langle A^2 \rangle - \langle A \rangle^2 = \langle (A - \langle A \rangle) \rangle$

Basis Expansion
$\psi(x, 0) = \int c(k) \frac{e^{ikx}}{\sqrt{2\pi}} \, dk \implies c(k) = \int \psi(x, 0) \frac{e^{-ikx}}{\sqrt{2\pi}} \, dx$
Given orthonormal basis $\langle \psi_n | \psi_m \rangle = \delta_{nm}$:
$\psi = \sum_n c_n \psi_n \qquad c_n = \int \psi_n^* \psi \, dx \equiv \langle \psi_n(x) | \psi(x, 0) \rangle$

Observation
Expand $|\psi\rangle$ in basis of operator being observed
Possible outcomes are eigenvalues of each basis state
Probability is square of eigenstate coefficient
ψ then collapses to eigenfunction of observed eigenvalue

Free Particle
$\psi_k(x) = \frac{e^{ikx}}{\sqrt{2\pi}} \qquad E_k = \frac{\hbar^2 k^2}{2m}$
$\frac{1}{2\pi} \int e^{i(\tilde{k} - k)x} \, dx \equiv \delta(\tilde{k} - k)$

Infinite Potential Well
$V(|x| < \frac{a}{2}) = 0 \qquad V(|x| > \frac{a}{2}) \to \infty$
$\psi_n(x) = \sqrt{\frac{2}{a}} \sin\left[\frac{n\pi}{a}\left(x - x_c + \frac{a}{2} \right) \right] \qquad E_n = \frac{n^2 \pi^2 \hbar^2}{2ma^2}$

If $x \in [0, a]$, $\quad \psi_n(x) = \sqrt{\frac{2}{a}} \begin{cases} \sin \frac{n\pi x}{a} & n \text{ even} \\ \cos \frac{n\pi x}{a} & n \text{ odd} \end{cases}$

Finite Potential Well
$V(|x| < \frac{a}{2}) = 0 \qquad V(|x| > \frac{a}{2}) = V_0$

$\kappa^2 = \frac{2mE}{\hbar^2} \qquad k^2 = \frac{2m(V_0 - E)}{\hbar^2}, \quad E > 0$
Sym: $\kappa = k \tan\left(\frac{ka}{2} \right) \qquad$ Asym: $\kappa = -k \cot\left(\frac{ka}{2} \right)$
$u = ka \qquad u_0^2 = \frac{2mV_0 a^2}{\hbar^2} \qquad \kappa^2 = \frac{u_0^2 - u^2}{a^2}$
$\tan\left(\frac{u}{2} \right) = \sqrt{\frac{u_0^2}{u^2} - 1} \qquad \cot\left(\frac{u}{2} \right) = -\sqrt{\frac{u_0^2}{u^2} - 1}$

Delta Function Well
$V(x) = -\lambda \delta(x)$
Boundary Condition: $\psi'(0_+) - \psi'(0_-) = -\frac{2m\lambda\psi(0)}{\hbar^2}$
$\psi(x) = \sqrt{\kappa} e^{-\kappa|x|}, \qquad E_0 = \frac{m\lambda^2}{2\hbar^2}, \qquad \kappa = \frac{m\lambda}{\hbar^2}$

Delta Function Scattering
$\mathbf{S} = \begin{bmatrix} r & t \\ t & r \end{bmatrix} = \frac{1}{k + i\kappa} \begin{bmatrix} -i\kappa & k \\ k & -i\kappa \end{bmatrix}$

TODO Scattering

Harmonic Oscillator
$H = \frac{p^2}{2m} + \frac{kx^2}{2} = \frac{p^2}{2m} + \frac{m\omega}{2} x^2, \qquad \omega = \sqrt{\frac{k}{m}}$
$H |n\rangle = \hbar\omega \left(n + \frac{1}{2} \right) |n\rangle, \quad n = 0, 1, 2, \dots$
$a = \frac{1}{\sqrt{2}} \left(\frac{x}{x_0} + i\frac{p}{p_0} \right) \qquad a^\dagger = \frac{1}{\sqrt{2}} \left(\frac{x}{x_0} - i\frac{p}{p_0} \right)$
$[a, a^\dagger] = 1 \qquad [a^\dagger, a] = -1$
$a |n\rangle = \sqrt{n} |n-1\rangle \qquad a^\dagger |n\rangle = \sqrt{n+1} |n+1\rangle$
$x_0 = \sqrt{\frac{\hbar}{m\omega}} \qquad p_0 = \frac{\hbar}{x_0}$
$x = \frac{x_0}{\sqrt{2}}(a + a^\dagger) \qquad p = \frac{p_0}{\sqrt{2}i}(a - a^\dagger) \qquad H = \hbar\omega\left(a^\dagger a + \frac{1}{2} \right)$
$a^\dagger a |n\rangle = n |n\rangle \qquad |n\rangle = \frac{(a^\dagger)^n}{\sqrt{n!}} |0\rangle$
Ehrenfest: $\frac{d}{dt} \langle x, t \rangle = \frac{\langle p, t \rangle}{m}, \frac{d}{dt} \langle p, t \rangle = \left\langle -\frac{dV}{dx} \right\rangle = -k \langle x, t \rangle$

Coherent States
$a |z\rangle = z |z\rangle, z \in \mathbb{C} \qquad \sum_n c_n a |n\rangle = \sum_n c_n z |n\rangle$
$c_{n+1} \sqrt{n+1} = c_n z \implies c_n = \frac{z^n}{\sqrt{n!}} c_0$
$|z\rangle = \sum_n c_n |n\rangle = c_0 \sum_n \frac{z^n}{\sqrt{n!}} |n\rangle, \qquad c_0 = e^{-\frac{|z|^2}{2}}$
$|z\rangle = e^{-\frac{|z|^2}{2}} e^{za^\dagger} |0\rangle$
$|z, t\rangle = e^{-i\frac{\omega}{2}t} |ze^{-i\omega t}\rangle \qquad z(t) = ze^{-i\omega t}$
$\langle z| (a^\dagger)^n a^m |z\rangle = (z^*)^n z^m$
$\langle x \rangle = \sqrt{2} x_0 \operatorname{Re} z \qquad \langle x^2 \rangle = x_0^2 \left(2\operatorname{Re}^2 z + \frac{1}{2} \right)$
$\langle p \rangle = \sqrt{2} p_0 \operatorname{Im} z \qquad \langle p^2 \rangle = p_0^2 \left(2\operatorname{Im}^2 z + \frac{1}{2} \right)$
$\langle H \rangle = \frac{\hbar\omega}{2} + \hbar\omega |z|^2 \qquad \langle H^2 \rangle = \hbar^2 \omega^2 \left(|z|^4 + 2|z|^2 + \frac{1}{4} \right)$

2D Isotropic Oscillator
$k_x = k_y \equiv k \qquad \omega_x = \omega_y \equiv \omega = \sqrt{\frac{k}{m}}$
$V(\boldsymbol{r}) = \frac{1}{2} k(x^2 + y^2) = \frac{1}{2} kr^2$
$H |n_x n_y\rangle = \hbar\omega(n_x + n_y + 1) |n_x n_y\rangle$

Gaussian Wave Packet
$\psi(x) = \frac{1}{\sqrt[4]{2\pi\sigma^2}} \exp\left(\frac{(x - \langle x \rangle)^2}{4\sigma^2} \right) e^{-i\frac{\langle p \rangle}{\hbar} x}$
$\Delta x(0) = \sigma \implies \langle x^2, 0 \rangle = \sigma^2 + \langle x, 0 \rangle^2$
$\Delta p(0) = \frac{\hbar}{2\Delta x(0)} = \frac{\hbar}{2\sigma} \implies \langle p^2, 0 \rangle = \frac{\hbar^2}{4\sigma^2} + \langle p, 0 \rangle^2$

Time Evolution
Expand $\psi(x, 0)$ over eigenfunctions $\psi_n(x)$ solving $H\psi_n(x) = E_n \psi_n(x)$ to get $\psi(x, 0) = \sum_n c_n \psi_n(x)$.
Find $\psi(x, t)$ with $\psi(x, t) = \sum_n c_n e^{-\frac{iE_n}{\hbar}t} \psi_n(x)$.
$|\psi, t\rangle \equiv \sum_n c_n e^{-i\frac{E_n}{\hbar}t} |n\rangle = e^{-i\frac{H}{\hbar}t} |\psi, 0\rangle$

Time-Dependent Expectation Value
Heisenberg approach: $\langle \mathcal{O}, t \rangle = \langle \psi, 0| \mathcal{O}(t) |\psi, 0\rangle$
Schrödinger approach: $\langle \mathcal{O}, t \rangle = \langle \psi, t| \mathcal{O} |\psi, t\rangle$

Time-Dependent Operator

$\mathcal{O}(t) = e^{i\frac{H}{\hbar}t}\mathcal{O}e^{-i\frac{H}{\hbar}t}$ $\qquad \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{O}(t) = \frac{i}{\hbar}[H, \mathcal{O}](t)$

$\mathcal{O}(t)\big|_{t=0} = e^{i\frac{H}{\hbar}\cdot 0}\mathcal{O}e^{-i\frac{H}{\hbar}\cdot 0} = \mathcal{O}$

$(A \cdot B)(t) = A(t)B(t)$

Angular Momentum

$L_z = xp_y - yp_x \to -\hbar\frac{\partial}{\partial\phi}$

$L_z\,|lm\rangle = m\hbar\,|lm\rangle$ $\qquad L^2\,|lm\rangle = l(l+1)\hbar^2\,|lm\rangle$

$m = -l, -l+1, \ldots, l-1, l$

$\langle\mathbf{r}|lm\rangle = Y_l^m(\vartheta, \phi) = (-1)^m\sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}P_{lm}(\cos\vartheta)e^{im\phi}$

$L_\pm\,|lm\rangle = \hbar\sqrt{l(l+1-m(m\pm 1))}\,|l, m\pm 1\rangle$

$\boldsymbol{L}\cdot\hat{\boldsymbol{e}}\,|\psi\rangle = \hbar l\,|\psi\rangle \implies \boldsymbol{L}\parallel\hat{\boldsymbol{e}}$

$L_+ \equiv L_x + iL_y$ $\qquad L_- \equiv L_+^\dagger = L_x - iL_y$

$L_+\,|lm\rangle = \hbar\sqrt{l(l+1) - m(m+1)}\,|l, m+1\rangle$

$L_-\,|lm\rangle = \hbar\sqrt{l(l+1) - m(m-1)}\,|l, m-1\rangle$

$L_x = \frac{L_+ + L_-}{2}$ $\qquad L_y = \frac{L_+ - L_-}{2i}$

$\langle L_x, t\rangle = \mathrm{Re}\,\langle L_+, t\rangle$ $\qquad \langle L_y, t\rangle = \mathrm{Im}\,\langle L_+, t\rangle$ $\boldsymbol{\mu} = \gamma\boldsymbol{L}$

Spin

$S^2 = |sm_s\rangle = \hbar^2 s(s+1)\,|sm_s\rangle$

$S_z\,|sm_s\rangle = \hbar m_s\,|sm_s\rangle$

$\left|\frac{1}{2}\frac{1}{2}\right\rangle \equiv |\uparrow\rangle$ and $\left|\frac{1}{2}-\frac{1}{2}\right\rangle \equiv |\downarrow\rangle$

$S_+ = S_x + iS_y$ $\qquad S_- = S_x - iS_y$

$S_x = \frac{S_+ + S_-}{2}$ $\qquad S_y = \frac{S_+ - S_-}{2i}$

$S_+\,|sm_s\rangle = \hbar\sqrt{s(s+1) - m_s(m_s+1)}\,|s, m_s+1\rangle$

$S_-\,|sm_s\rangle = \hbar\sqrt{s(s+1) - m_s(m_s-1)}\,|s, m_s-1\rangle$

$\boldsymbol{S}\cdot\hat{\boldsymbol{e}}\,|\psi_s\rangle = \frac{\hbar}{2}\,|\psi_s\rangle$ implies

$\quad \boldsymbol{S}\parallel\hat{\boldsymbol{e}} = (\cos\varphi\sin\theta, \sin\varphi\sin\theta, \cos\theta)$

$\quad |\psi_s\rangle = \cos\frac{\theta}{2}\,|\uparrow\rangle + \sin\frac{\theta}{2}e^{i\varphi}\,|\downarrow\rangle$

Useful: $\boldsymbol{k} = (k_x, k_y)$ to $k_x + ik_y = ke^{i\phi}$, $k_x - ik_y = ke^{-i\phi}$

Spin Matrices

$\boldsymbol{S} = \frac{\hbar}{2}\boldsymbol{\sigma}$ $\qquad \boldsymbol{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$

$\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ $\quad \sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$ $\quad \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -i \end{bmatrix}$

$\{\sigma_a, \sigma_b\} = 2\delta_{ab}I$

Two-Particle Spin System

Product basis: $|s_1 m_1 s_2 m_2\rangle$.

Total angular momentum basis: $|sm_s s_1 s_2\rangle$

$|s_1 - s_2| \le s \le s_1 + s_2$ $\qquad m_s = -s, -s+1, \ldots, s-1, s.$

Time Reversal
TODO

For spin $s = 1/2$ particles $T = i\sigma_y K$; $K: \psi \mapsto \psi^*$

Nondegenerate Perturbations

$H = H_0 + H'$ $\qquad H_0\,|n_0\rangle = E_n^{(0)}\,|n_0\rangle$

$E_n = E_n^{(0)} + \langle n_0|H'|n_0\rangle + \sum_{m\neq n}\frac{|\langle m_0|H'|n_0\rangle|^2}{E_n^{(0)} - E_m^{(0)}},$

$|n\rangle = |n_0\rangle + \sum_{m\neq n}\frac{|\langle m_0|H'|n_0\rangle|^2}{E_n^{(0)} - E_m^{(0)}}\,|m_0\rangle$

Degenerate Perturbations

Theory: Degenerate Perturbation Theory

$H = H_0 + H'$ $\qquad H_0\,|\psi\rangle = E^{(0)}\,|\psi\rangle$

$E^{(0)}$ is N-times degenerate...

$\quad$ ... and $|\psi\rangle = \sum_i^N c_i\,|i\rangle$ $\qquad H\,|i\rangle = E^{(0)}\,|i\rangle \ \forall\, i$

Create matrix $\mathbf{P} = \begin{pmatrix} \langle 1|H'|1\rangle & \cdots & \langle 1|H'|N\rangle \\ \langle 2|H'|1\rangle & \cdots & \langle 2|H'|N\rangle \\ \vdots & \ddots & \vdots \\ \langle N|H'|1\rangle & \cdots & \langle N|H'|N\rangle \end{pmatrix}$

Diagonalize $\mathbf{P}$; find eigenvalues λ_j and eigenvectors $\boldsymbol{u}_j$.

$E_j = E^{(0)} + \lambda_j$ $\qquad$ and $\qquad |\psi_j\rangle = \sum_{k=1}^N u_{j_k}\,|k\rangle$