Linear Algebra Review

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$A \in \mathbb{C}^{n\times n}$ is *Hermitian* if $A = A^\dagger$
$A \in \mathbb{R}^{n\times n}$ is *Hermitian* if $A = A^T$
$O : V \to V$ *Hermitian* if $\langle Ox, y\rangle = \langle x, Oy\rangle$ for all $x, y \in V$
Hermitian operators have real eigenvalues, orthogonal eigenfunctions and correspond to observable quantities
$U \in \mathbb{C}^{n\times n}$ *unitary* if $UU^\dagger = U^\dagger U = I \Longrightarrow U^{-1} = U^\dagger$
$O \in \mathbb{R}^{n\times n}$ *orthogonal* if $OO^T = O^T O = I \Longrightarrow O^{-1} = O^T$

Commutator

$[A, B] = AB - BA$
$[A, B] = 0 \Longleftrightarrow AB = BA \Longleftrightarrow B = ABA^{-1} = A^{-1}BA$
$[B, A] = -[A, B] \qquad [AB, C] = A[B, C] + [A, C]B$
$\{A, B\} = AB + BA$

Quantum Mechanics in 1D

Some Basic Operators

$p_x \to -i\hbar\frac{\partial}{\partial x} \quad \boldsymbol{p} \to -i\hbar\nabla$

Uncertainty Principle

$(\Delta A \Delta B)^2 \geq \left(\frac{1}{2}|\langle[A, B]\rangle|\right)^2$
$\Delta x \Delta p \geq \frac{\hbar}{2} \quad \Delta E \Delta t \geq \frac{\hbar}{2}$

Expectation Values, etc...

$\langle \mathcal{O}\rangle = \langle\psi|\,\mathcal{O}\,|\psi\rangle \equiv \langle\psi|\mathcal{O}\psi\rangle \equiv \int \psi^*\mathcal{O}\psi\,\mathrm{d}x$
$\langle\psi|\mathcal{O}\psi\rangle = \langle\mathcal{O}^*\psi|\psi\rangle$ and $\mathcal{O} = \mathcal{O}^* \Longrightarrow \langle\psi|\mathcal{O}\psi\rangle = \langle\mathcal{O}\psi|\psi\rangle$
$\langle\psi_m|\psi_n\rangle = \int \psi_m^*\psi_n\,\mathrm{d}x$
$\langle\mathcal{O}\psi|\mathcal{O}\psi\rangle = \|\mathcal{O}\psi\|^2$
$\langle A^\dagger\rangle = \langle A\rangle^* \Longrightarrow \langle A\rangle + \langle A^\dagger\rangle = \langle A\rangle + \langle A\rangle^* = 2\,\mathrm{Re}\,\langle A\rangle$

Uncertainties

$\Delta x(t) = \sqrt{\langle x^2, t\rangle - \langle x, t\rangle^2} \quad \Delta p(t) = \sqrt{\langle p^2, t\rangle - \langle p, t\rangle^2}$
$(\Delta A)^2 = \langle A^2\rangle - \langle A\rangle^2 = \langle(A - \langle A\rangle)\rangle$

Basis Expansion

$\psi(x, 0) = \int c(k)\frac{e^{ikx}}{\sqrt{2\pi}}\,\mathrm{d}k \Longrightarrow c(k) = \int \psi(x, 0)\frac{e^{-ikx}}{\sqrt{2\pi}}\,\mathrm{d}x$
Given orthonormal basis $\langle\psi_n|\psi_m\rangle = \delta_{nm}$:
$\psi = \sum_n c_n\psi_n \qquad c_n = \int \psi_n^*\psi\,\mathrm{d}x \equiv \langle\psi_n(x)|\psi(x, 0)\rangle$

Observation

Expand $|\psi\rangle$ in basis of operator being observed
Possible outcomes are eigenvalues of each basis state
Probability is square of eigenstate coefficient
ψ then collapses to eigenfunction of observed eigenvalue

Free Particle

$\psi_k(x) = \frac{e^{ikx}}{\sqrt{2\pi}} \qquad E_k = \frac{\hbar^2 k^2}{2m}$
$\frac{1}{2\pi}\int e^{i(\tilde{k}-k)x}\,\mathrm{d}x \equiv \delta(\tilde{k} - k)$

Infinite Potential Well

$V(|x| < \frac{a}{2}) = 0 \qquad V(|x| > \frac{a}{2}) \to \infty$
$\psi_n(x) = \sqrt{\frac{2}{a}}\sin\left[\frac{n\pi}{a}\left(x - x_c + \frac{a}{2}\right)\right] \qquad E_n = \frac{n^2\pi^2\hbar^2}{2ma^2}$
If $x \in [0, a]$, $\quad \psi_n(x) = \sqrt{\frac{2}{a}}\begin{cases}\sin\frac{n\pi x}{a} & n \text{ even}\\ \cos\frac{n\pi x}{a} & n \text{ odd}\end{cases}$

Finite Potential Well

$V(|x| < \frac{a}{2}) = 0 \qquad V(|x| > \frac{a}{2}) = V_0$
$\kappa^2 = \frac{2mE}{\hbar^2} \qquad k^2 = \frac{2m(V_0 - E)}{\hbar^2}, \quad E > 0$
Sym: $\kappa = k\tan\left(\frac{ka}{2}\right) \qquad$ Asym: $\kappa = -k\cot\left(\frac{ka}{2}\right)$
$u = ka \qquad u_0^2 = \frac{2mV_0a^2}{\hbar^2} \qquad \kappa^2 = \frac{u_0^2 - u^2}{a^2}$
$\tan\left(\frac{u}{2}\right) = \sqrt{\frac{u_0^2}{u^2} - 1} \qquad \cot\left(\frac{u}{2}\right) = -\sqrt{\frac{u_0^2}{u^2} - 1}$

Delta Function Well

$V(x) = -\lambda\delta(x)$
Boundary Condition: $\psi'(0_+) - \psi'(0_-) = -\frac{2m\lambda\psi(0)}{\hbar^2}$
$\psi(x) = \sqrt{\kappa}e^{-\kappa|x|}, \quad E_0 = \frac{m\lambda^2}{2\hbar^2}, \quad \kappa = \frac{m\lambda}{\hbar^2}$

Delta Function Scattering

$\mathbf{S} = \begin{bmatrix} r & t\\ t & r\end{bmatrix} = \frac{1}{k + i\kappa}\begin{bmatrix} -i\kappa & k\\ k & -i\kappa\end{bmatrix}$

TODO Scattering

Harmonic Oscillator

$H = \frac{p^2}{2m} + \frac{kx^2}{2} = \frac{p^2}{2m} + \frac{m\omega}{2}x^2, \qquad \omega = \sqrt{\frac{k}{m}}$
$H\,|n\rangle = \hbar\omega\left(n + \frac{1}{2}\right)|n\rangle, \quad n = 0, 1, 2, \ldots$
$a = \frac{1}{\sqrt{2}}\left(\frac{x}{x_0} + i\frac{p}{p_0}\right) \qquad a^\dagger = \frac{1}{\sqrt{2}}\left(\frac{x}{x_0} - i\frac{p}{p_0}\right)$
$\left[a, a^\dagger\right] = 1 \qquad \left[a^\dagger, a\right] = -1$
$a\,|n\rangle = \sqrt{n}\,|n - 1\rangle \qquad a^\dagger\,|n\rangle = \sqrt{n + 1}\,|n + 1\rangle$
$x_0 = \sqrt{\frac{\hbar}{m\omega}} \qquad p_0 = \frac{\hbar}{x_0}$
$x = \frac{x_0}{\sqrt{2}}(a + a^\dagger) \quad p = \frac{p_0}{\sqrt{2}i}(a - a^\dagger) \quad H = \hbar\omega\left(a^\dagger a + \frac{1}{2}\right)$
$a^\dagger a\,|n\rangle = n\,|n\rangle \qquad |n\rangle = \frac{(a^\dagger)^n}{\sqrt{n!}}\,|0\rangle$
Ehrenfest: $\frac{\mathrm{d}}{\mathrm{d}t}\langle x, t\rangle = \frac{\langle p, t\rangle}{m}$, $\frac{\mathrm{d}}{\mathrm{d}t}\langle p, t\rangle = \left\langle -\frac{\mathrm{d}V}{\mathrm{d}x}\right\rangle = -k\,\langle x, t\rangle$

Coherent States

$a\,|z\rangle = z\,|z\rangle$, $z \in \mathbb{C} \qquad \sum_n c_n a\,|n\rangle = \sum_n c_n z\,|n\rangle$
$c_{n+1}\sqrt{n + 1} = c_n z \Longrightarrow c_n = \frac{z^n}{\sqrt{n!}}c_0$
$|z\rangle = \sum_n c_n\,|n\rangle = c_0\sum_n \frac{z^n}{\sqrt{n!}}\,|n\rangle, \quad c_0 = e^{-\frac{|z|^2}{2}}$
$|z\rangle = e^{-\frac{|z|^2}{2}}e^{za^\dagger}\,|0\rangle$
$|z, t\rangle = e^{-i\frac{\omega}{2}t}\left|ze^{-i\omega t}\right\rangle \qquad z(t) = ze^{-i\omega t}$
$\langle z|\,(a^\dagger)^n a^m\,|z\rangle = (z^*)^n z^m$
$\langle x\rangle = \sqrt{2}x_0\,\mathrm{Re}\,z \qquad \langle x^2\rangle = x_0^2\left(2\,\mathrm{Re}^2 z + \frac{1}{2}\right)$
$\langle p\rangle = \sqrt{2}p_0\,\mathrm{Im}\,z \qquad \langle p^2\rangle = p_0^2\left(2\,\mathrm{Im}^2 z + \frac{1}{2}\right)$
$\langle H\rangle = \frac{\hbar\omega}{2} + \hbar\omega|z|^2 \qquad \langle H^2\rangle = \hbar^2\omega^2\left(|z|^4 + 2|z|^2 + \frac{1}{4}\right)$

2D Isotropic Oscillator

$k_x = k_y \equiv k \quad \omega_x = \omega_y \equiv \omega = \sqrt{\frac{k}{m}}$
$V(\boldsymbol{r}) = \frac{1}{2}k(x^2 + y^2) = \frac{1}{2}kr^2$
$H\,|n_x n_y\rangle = \hbar\omega(n_x + n_y + 1)\,|n_x n_y\rangle$

Gaussian Wave Packet

$\psi(x) = \frac{1}{\sqrt[4]{2\pi\sigma^2}}\exp\left(\frac{(x - \langle x\rangle)^2}{4\sigma^2}\right)e^{-i\frac{\langle p\rangle}{\hbar}x}$
$\Delta x(0) = \sigma \Longrightarrow \langle x^2, 0\rangle = \sigma^2 + \langle x, 0\rangle^2$
$\Delta p(0) = \frac{\hbar}{2\Delta x(0)} = \frac{\hbar}{2\sigma} \Longrightarrow \langle p^2, 0\rangle = \frac{\hbar^2}{4\sigma^2} + \langle p, 0\rangle^2$

Time Evolution

Expand $\psi(x, 0)$ over eigenfunctions $\psi_n(x)$ solving $H\psi_n(x) = E_n\psi_n(x)$ to get $\psi(x, 0) = \sum_n c_n\psi_n(x)$.
Find $\psi(x, t)$ with $\psi(x, t) = \sum_n c_n e^{-\frac{iE_n}{\hbar}t}\psi_n(x)$.
$|\psi, t\rangle \equiv \sum_n c_n e^{-i\frac{E_n}{\hbar}t}\,|n\rangle = e^{-i\frac{H}{\hbar}t}\,|\psi, 0\rangle$

Time-Dependent Expectation Value

Heisenberg approach: $\langle\mathcal{O}, t\rangle = \langle\psi, 0|\,\mathcal{O}(t)\,|\psi, 0\rangle$
Schrödinger approach: $\langle\mathcal{O}, t\rangle = \langle\psi, t|\,\mathcal{O}\,|\psi, t\rangle$

Time-Dependent Operator

$\mathcal{O}(t) = e^{i\frac{H}{\hbar}t}\mathcal{O}e^{-i\frac{H}{\hbar}t} \qquad \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{O}(t) = \frac{i}{\hbar}[H,\mathcal{O}](t)$

$\mathcal{O}(t)\big|_{t=0} = e^{i\frac{H}{\hbar}\cdot 0}\mathcal{O}e^{-i\frac{H}{\hbar}\cdot 0} = \mathcal{O}$

$(A\cdot B)(t) = A(t)B(t)$

Angular Momentum

$L_z = xp_y - yp_x \to -\hbar\frac{\partial}{\partial\phi}$

$L_z\left|lm\right\rangle = m\hbar\left|lm\right\rangle \qquad L^2\left|lm\right\rangle = l(l+1)\hbar^2\left|lm\right\rangle$

$m = -l, -l+1, \dots, l-1, l$

$\langle\mathbf{r}|lm\rangle = Y_l^m(\vartheta,\phi) = (-1)^m\sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}P_{lm}(\cos\vartheta)e^{im\phi}$

$L_\pm\left|lm\right\rangle = \hbar\sqrt{l(l+1-m(m\pm 1))}\left|l,m\pm 1\right\rangle$

$\boldsymbol{L}\cdot\hat{\boldsymbol{e}}\left|\psi\right\rangle = \hbar l\left|\psi\right\rangle \implies \boldsymbol{L}\parallel\hat{\boldsymbol{e}}$

$L_+ \equiv L_x + iL_y \qquad L_- \equiv L_+^\dagger = L_x - iL_y$

$L_+\left|lm\right\rangle = \hbar\sqrt{l(l+1)-m(m+1)}\left|l,m+1\right\rangle$

$L_-\left|lm\right\rangle = \hbar\sqrt{l(l+1)-m(m-1)}\left|l,m-1\right\rangle$

$L_x = \frac{L_+ + L_-}{2} \qquad L_y = \frac{L_+ - L_-}{2i}$

$\langle L_x, t\rangle = \mathrm{Re}\langle L_+, t\rangle \qquad \langle L_y, t\rangle = \mathrm{Im}\langle L_+, t\rangle \; \boldsymbol{\mu} = \gamma\boldsymbol{L}$

Spin

$S^2 = \left|sm_s\right\rangle = \hbar^2 s(s+1)\left|sm_s\right\rangle$

$S_z\left|sm_s\right\rangle = \hbar m_s\left|sm_s\right\rangle$

$\left|\frac{1}{2}\frac{1}{2}\right\rangle \equiv \left|\uparrow\right\rangle$ and $\left|\frac{1}{2} - \frac{1}{2}\right\rangle \equiv \left|\downarrow\right\rangle$

$S_+ = S_x + iS_y \qquad S_- = S_x - iS_y$

$S_x = \frac{S_+ + S_-}{2} \qquad S_y = \frac{S_+ - S_-}{2i}$

$S_+\left|sm_s\right\rangle = \hbar\sqrt{s(s+1)-m_s(m_s+1)}\left|s,m_s+1\right\rangle$

$S_-\left|sm_s\right\rangle = \hbar\sqrt{s(s+1)-m_s(m_s-1)}\left|s,m_s-1\right\rangle$

$\boldsymbol{S}\cdot\hat{\boldsymbol{e}}\left|\psi_s\right\rangle = \frac{\hbar}{2}\left|\psi_s\right\rangle$ implies

$\boldsymbol{S}\parallel\hat{\boldsymbol{e}} = (\cos\varphi\sin\theta, \sin\varphi\sin\theta, \cos\theta)$

$\left|\psi_s\right\rangle = \cos\frac{\theta}{2}\left|\uparrow\right\rangle + \sin\frac{\theta}{2}e^{i\varphi}\left|\downarrow\right\rangle$

Spin Matrices

$\boldsymbol{S} = \frac{\hbar}{2}\boldsymbol{\sigma} \qquad \boldsymbol{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$

$\sigma_x = \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix} \quad \sigma_y = \begin{bmatrix}0 & -i\\ i & 0\end{bmatrix} \quad \sigma_z = \begin{bmatrix}1 & 0\\ 0 & -i\end{bmatrix}$

$\{\sigma_a, \sigma_b\} = 2\delta_{ab}I$

Two-Particle Spin System

Product basis: $|s_1 m_1 s_2 m_2\rangle$.

Total angular momentum basis: $|s m_s s_1 s_2\rangle$

$|s_1 - s_2| \le s \le s_1 + s_2 \qquad m_s = -s, -s+1, \dots, s-1, s.$

Time Reversal

TODO

For spin $s = 1/2$ particles $T = i\sigma_y K$; $K : \psi \mapsto \psi^*$

Nondegenerate Perturbations

$H = H_0 + H' \qquad H_0\left|n_0\right\rangle = E_n^{(0)}\left|n_0\right\rangle$

$E_n = E_n^{(0)} + \langle n_0|H'|n_0\rangle + \sum_{m\neq n}\frac{|\langle m_0|H'|n_0\rangle|^2}{E_n^{(0)} - E_m^{(0)}},$

$\left|n\right\rangle = \left|n_0\right\rangle + \sum_{m\neq n}\frac{|\langle m_0|H'|n_0\rangle|^2}{E_n^{(0)} - E_m^{(0)}}\left|m_0\right\rangle$

Degenerate Perturbations

Theory: Degenerate Perturbation Theory

$H = H_0 + H' \qquad H_0\left|\psi\right\rangle = E^{(0)}\left|\psi\right\rangle$

$E^{(0)}$ is N-times degenerate...

... and $\left|\psi\right\rangle = \sum_i^N c_i\left|i\right\rangle \qquad H\left|i\right\rangle = E^{(0)}\left|i\right\rangle \;\forall\, i$

Create matrix $\mathbf{P} = \begin{pmatrix}\langle 1|H'|1\rangle & \cdots & \langle 1|H'|N\rangle\\ \langle 2|H'|1\rangle & \cdots & \langle 2|H'|N\rangle\\ \vdots & \ddots & \vdots\\ \langle N|H'|1\rangle & \cdots & \langle N|H'|N\rangle\end{pmatrix}$

Diagonalize $\mathbf{P}$; find eigenvalues λ_j and eigenvectors $\boldsymbol{u}_j$.

$E_j = E^{(0)} + \lambda_j$ and $\left|\psi_j\right\rangle = \sum_{k=1}^N u_{jk}\left|k\right\rangle$

where $\boldsymbol{u}_j$ is the perturbation matrix's eigenvector corresponding to λ_j, and u_{j_k} is $\boldsymbol{u}_j$'s k-th component.