# Optics Lecture Notes

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### About These Notes

These are my lecture notes from the course *Optika* (Optics), an elective course offered to third-year physics students at the Faculty of Math and Physics in Ljubljana, Slovenia. The exact material herein is specific to the physics program at the University of Ljubljana, but the content is fairly standard for an undergraduate course in wave optics. I am making the notes publicly available in the hope that they might help others learning similar material—the most up-to-date version can be found on GitHub.

*Navigation*: For easier document navigation, the table of contents is "clickable", meaning you can jump directly to a section by clicking the colored section names in the table of contents. Unfortunately, the *clickable links do not work in most online or mobile PDF viewers*; you have to download the file first.

*On Content*: The material herein is far from original—it comes almost exclusively from lecture notes by Professors Irena Drevenšek Olenik and Mojca Vilfan at the University of Ljubljana. I have merely typset the notes and translated to English

*Disclaimer:* Mistakes—both trivial typos and legitimate errors—are likely. Keep in mind that these are the notes of an undergraduate student in the process of learning the material himself—take what you read with a grain of salt. If you find mistakes and feel like telling me, by GitHub pull request, email or some other means, I'll be happy to hear from you, even for the most trivial of errors.

# Contents

# 1 Review of Geometrical Optics

Geometrical optics is the study of light, in which the light is modelled as rays that:

- propagate in straight-line paths as they travel through a homogeneous medium,
- bend, and in particular circumstances may split in two, at the interface between two optically dissimilar media,
- follow curved paths in a medium in which the refractive index changes
- may be absorbed or reflected.

This description is taken from the Wikipedia page on Geometrical optics.

**Overview of Chapter Material:**

- Fermat's principle and the ray equation
- The ABCD ray transfer matrices
- Fundamental optical instruments (mirrors, lenses, etc...)

## 1.1 Fermat's Principle

### 1.1.1 Introductory Remarks: Speed of Light and Index of Refraction

- Before beginning a quantitative treatment of Fermat's principle, we first introduce two important concepts: the speed of light and the index of refraction. We denote the speed of light in matter by $c$ and the speed of light in vacuum by $c_0$. The speed of light in vacuum is a universal constant—to four significant figures, its value is $c_0 = 2.997\,\mathrm{m\,s^{-1}}$. The light speeds $c_0$ and $c$ in vacuum and in matter, respectively, are related by
$$c = \frac{c_0}{n},$$
where $n$ is the material's index of refraction. The index of refraction is a property of matter and in general may be dependent on both position $\boldsymbol{r}$ in the material and the frequency $\omega$ of the light passing through, i.e. $n = n(\boldsymbol{r}, \omega)$.

- Some representative values of the index of refraction for visible light are given in the table below:

| Material | Value of $n$ |
|---|---|
| Vacuum | 1 |
| Water | 1.3 |
| Glass | 1.4 to 1.9 |
| Diamond | 2.9 |

Keep in mind that the index of refraction is frequency-dependent, and that these values may differ for frequencies outside the visible spectrum.

### 1.1.2 Fermat's Principle

- Qualitatively, Fermat's principle states:

  > Light travels between any two points in space along the path minimizing the travel time between the two points.

  More quantitatively, Fermat's principle is formulated as a least action principle, as discussed immediately below.

- Consider light travelling through material with index of refraction $n$. We parameterize the light's path through the material with the arc length parameter $s$, and divide the path into many infinitesimal elements $\mathrm{d}s$.

- The light travels a distance $\mathrm{d}s$ in the time $\mathrm{d}t$, and the two quantities are related according to

  $$\mathrm{d}t = \frac{\mathrm{d}s}{c} = \frac{\mathrm{d}s}{(c_0/n)} = \frac{n}{c_0}\,\mathrm{d}s.$$

  Fermat's theorem states that the light takes the path minimizing the quantity

  $$\int_{(1)}^{(2)} \mathrm{d}t = \frac{1}{c_0}\int_{(1)}^{(2)} n\,\mathrm{d}s.$$

  Since $c_0$ is a constant, Fermat's theorem is equivalent to the requirement

  $$S \equiv \int_{(1)}^{(2)} n\,\mathrm{d}s = \min,$$

  where we have defined the *optical path length* $S$, which is just the total distance travelled by the light between the points 1 and 2, scaled by the index of refraction $n$. When written as a minimizing condition on the optical path length $S$, Fermat's theorem is sometimes called the optical least action principle, and is analogous to the least action principle in Lagrangian mechanics.

### 1.1.3 Example: Deriving the Law of Refraction

- The law of refraction applies to light incident on an interface between two materials with different indexes of refraction. If the light, originally in material 1 with index of refraction $n_1$, is incident at an angle $\alpha$ on material 2 with index of refraction $n_2$, then the angle of refraction $\beta$ is given by

  $$n_1 \sin\alpha = n_2 \sin\beta \implies \beta = \arcsin\left(\frac{n_1}{n_2}\sin\alpha\right).$$

- To derive the law of refraction, we first consider the schematic shown in Figure TODO. Our goal is to find the path between point 1 in material 1 and point 2 in material 2 minimizing the optical path $S$. Assuming the index of refraction is constant in each region, the optical length $S$ is

  $$S \equiv \int_{(1)}^{(2)} n\,\mathrm{d}s = n_1\int_{\mathrm{I}} \mathrm{d}s + n_2\int_{\mathrm{II}} \mathrm{d}s = n_1 s_1 + n_2 s_2,$$

where the integral subscripts I and II refer to the paths between materials 1 and 2, respectively. In terms of the coordinates $x$ and $z$, the optical path length reads

$$S = n_1\sqrt{x_1^2 + z_1^2} + n_2\sqrt{x_2^2 + z_2^2} = n_1\sqrt{z_1^2 + x_1^2} + n_2\sqrt{z_2^2 + (x - x_1)^2},$$

where we have introduced the total vertical distance $x = x_1 + x_2$.

- We then differentiate $S$ with respect to $x_1$ to find the value of $x_1$ minimizing the path length; the derivative reads

$$0 \equiv \frac{\mathrm{d}S}{\mathrm{d}x_1} = \frac{2n_1x_1}{2\sqrt{z_1^2 + x_1^2}} + \frac{2\cdot(-1)\cdot(x - x_1)}{2\sqrt{z_2^2 + (x - x_1)^2}} \tag{1}$$

With reference to Figure TODO, we write the angles of incidence and refraction in terms of the $x$ and $z$ coordinates in the form

$$\sin\alpha = \frac{x_1}{\sqrt{x_1^2 + z_1^2}} \qquad \text{and} \qquad \sin\beta = \frac{x_2}{\sqrt{x_2^2 + z_2^2}} = \frac{(x - x_1)}{\sqrt{(x - x_1)^2 + z_2^2}}.$$

In terms of the angles $\alpha$ and $\beta$, Equation 1, after simplifying and rearranging, simplifies considerably to

$$n_1 \sin\alpha = n_2 \sin\beta,$$

which is the familiar law of refraction. Interpretation: precisely the behavior encoded by the law of refraction satisfies Fermat's principle of least optical action.

## 1.2 The Ray Equation

- We now consider the behavior of light travelling between materials in which the index of refraction is not constant; instead, we have $n = n(\boldsymbol{r}) = n(x, y, z)$. In this case, light's behavior is determined by the *ray equation*, which reads

$$\boldsymbol{\nabla} n = \frac{\mathrm{d}}{\mathrm{d}s}\left(n\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}s}\right).$$

Our goal in this section is to derive the behavior of light in this non-homogeneous material by combining the Euler-Lagrange equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\boldsymbol{r}}}\right) - \frac{\partial L}{\partial \boldsymbol{r}} = 0 \tag{2}$$

with the principle of least optical action.

### 1.2.1 Derivation of the Ray Equation

- Working in Cartesian coordinates, we first write the optical path length $S$ as

$$S = \int_{(1)}^{(2)} n(x, y, z)\,\mathrm{d}s = \int_{(1)}^{(2)} n(x, y, z)|\dot{\boldsymbol{r}}|\,\mathrm{d}t = \int_{(1)}^{(2)} n(x, y, z)\sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}\,\mathrm{d}t,$$

where the dot denotes differentiation with respect to time, and we have used the identity

$$\mathrm{d}s = |\dot{\boldsymbol{r}}|\,\mathrm{d}t = \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}\,\mathrm{d}t.$$

We then compare the optical path $S$ to the general Lagrangian action

$$S = \int_{(1)}^{(2)} L(\boldsymbol{r}, \dot{\boldsymbol{r}}) \,\mathrm{d}t,$$

which motivates the definition of the *optical Lagrangian* as

$$L(\boldsymbol{r}, \dot{\boldsymbol{r}}) = n(\boldsymbol{r})|\dot{\boldsymbol{r}}|.$$

In Cartesian coordinates, the optical Lagrangian reads

$$L(x, y, z, \dot{x}, \dot{y}, \dot{z}) = n(x, y, z)\sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}.$$

- We then substitute the just-derived optical Lagrangian into the Euler-Lagrange equation (Eq. 2). Beginning with the $x$ coordinate, the E-L equation reads

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{n\dot{x}}{\sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}}\right) = \frac{\partial n}{\partial x}\sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}.$$

We then multiply the equation through by $\frac{\mathrm{d}t}{\mathrm{d}s}$ to get

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{n\dot{x}}{\sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}}\right)\frac{\mathrm{d}t}{\mathrm{d}s} = \frac{\partial n}{\partial x}\sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}\,\frac{\mathrm{d}t}{\mathrm{d}s}.$$

- Next, we apply the chain rule on the left hand side, which allows to eliminate the differential $\mathrm{d}t$ and then write $\dot{x} \equiv \frac{\mathrm{d}x}{\mathrm{d}t}$; on the right hand side we use the earlier identity $\mathrm{d}s = \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}\,\mathrm{d}t$, which results in

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\mathrm{d}x}{\mathrm{d}t}\frac{n}{\sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}}\right) = \frac{\partial n}{\partial x}\frac{\mathrm{d}s}{\mathrm{d}s} = \frac{\partial n}{\partial x}.$$

One more application of the identity $\mathrm{d}s = \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}\,\mathrm{d}t$ on the left hand side simplifies things to

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(n\frac{\mathrm{d}x}{\mathrm{d}s}\right) = \frac{\partial n}{\partial x}.$$

- An identical procedure for the coordinates $y$ and $z$ produces the analogous results

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(n\frac{\mathrm{d}y}{\mathrm{d}s}\right) = \frac{\partial n}{\partial y} \qquad \text{and} \qquad \frac{\mathrm{d}}{\mathrm{d}s}\left(n\frac{\mathrm{d}z}{\mathrm{d}s}\right) = \frac{\partial n}{\partial z}.$$

Combining the equations for $x$, $y$ and $z$ into a single vector equation produces

$$\boldsymbol{\nabla} n = \frac{\mathrm{d}}{\mathrm{d}s}\left(n\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}s}\right),$$

which is the ray equation quoted at the beginning of this subsection.

### 1.2.2 Example: The Ray Equation in Homogeneous Matter

- Consider light travelling through a homogeneous material—in the context of optics, this means a material with constant index of refraction $n$. In this case $\boldsymbol{\nabla} n = 0$, and the ray equation simplifies to

$$0 = \frac{\mathrm{d}}{\mathrm{d}s}\left(n\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}s}\right) = n\frac{\mathrm{d}}{\mathrm{d}s}\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}s} = \frac{\mathrm{d}^2\boldsymbol{r}}{\mathrm{d}s^2}.$$

- The equation $\boldsymbol{r}''(s) = 0$ is solved by the linear ray

$$\boldsymbol{r}(s) = \boldsymbol{a}_0 + \boldsymbol{a}_1 s,$$

where the vector $a_0$ represents an initial point and the vector $\boldsymbol{a}_1$ represents the direction of ray propagation.

### 1.2.3 The Ray Equation and the Paraxial Approximation

- In optics, we often encounter situations in which the direction of light propagation deviates only slightly from some given direction in space. We call this direction the *optical axis*, and usually choose it to align with the $z$ axis.

- For simplicity, we consider a two-dimensional system in which light moves in the $xz$ plane, shown in Figure TODO, and assume the index of refraction is a single-variable function of $x$, i.e. $n = n(x)$.

  Qualitatively, the *paraxial approximation* assumes that the light ray deviates only slightly from the $z$ axis (the optical axis) as it moves through the $xz$ plane. Mathematically, the approximation reads

$$\frac{\mathrm{d}x}{\mathrm{d}z} \ll 1. \tag{3}$$

- Next, we introduce an angle $\theta$ between the tangent to the light ray's path and the optical axis, defined as

$$\theta = \frac{\mathrm{d}x}{\mathrm{d}z}.$$

  In the regime of the paraxial approximation, the angle $\theta$ obeys

$$\sin\theta \approx \tan\theta \approx \theta.$$

- Using the paraxial approximation (Eq. 3), an infinitesimal arc length $\mathrm{d}s$ in our two-dimensional system reads

$$\mathrm{d}s = \sqrt{(\mathrm{d}x)^2 + (\mathrm{d}z)^2} = \sqrt{1 + \left(\frac{\mathrm{d}x}{\mathrm{d}z}\right)^2}\,\mathrm{d}z \approx \mathrm{d}z.$$

- In terms of the just-derived paraxial result $\mathrm{d}s \approx \mathrm{d}z$, for a two-dimensional system of the form shown in Figure TODO, the ray equation simplifies to

$$\boldsymbol{\nabla} n = \frac{\mathrm{d}}{\mathrm{d}s}\left(n(\boldsymbol{r})\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}s}\right) \xrightarrow{\text{parax.}} \frac{\mathrm{d}x}{\mathrm{d}s} = \frac{\mathrm{d}}{\mathrm{d}z}\left(n(x)\frac{\mathrm{d}x}{\mathrm{d}z}\right) = n(x)\frac{\mathrm{d}^2x}{\mathrm{d}z^2}.$$

  Finally, we divide through by $n(x)$ to write the equation in the final form

$$\frac{\mathrm{d}^2x}{\mathrm{d}z^2} = \frac{1}{n(x)}\frac{\mathrm{d}n}{\mathrm{d}x}. \tag{4}$$

### 1.2.4 Example: A Material with a Parabolic Refractive Dependence

- Consider a two-dimensional material in which the index of refraction depends on the vertical distance $x$ from the optical according to the parabolic relationship

$$n(x) = n_0 \left(1 - \frac{\alpha^2 x^2}{2}\right),$$

where we assume $\alpha x \ll 1$. We now aim to determine the trajectory of a light ray through this material in the regime of the paraxial approximation.

- We begin by substituting the index of refraction into the planar paraxial ray equation (Eq. 4), evaluate the derivative, and apply the identity $\alpha x \ll 1$ to get

$$\frac{\mathrm{d}^2 x}{\mathrm{d}z^2} = \frac{1}{n(x)} \frac{\mathrm{d}n}{\mathrm{d}x} = -\frac{n_0 \alpha^2 x}{n_0 \left(1 - \frac{\alpha^2 x^2}{2}\right)} \approx -\frac{n_0 \alpha^2 x}{n_0} = -\alpha^2 x.$$

The resulting differential equation, i.e. $x''(z) = -\alpha^2 x$, is solved by the ansatz

$$x(z) = A\cos(\alpha z) + B\sin(\alpha z).$$

If we assume that at $z = 0$ the light ray occurs a distance $x_0$ above the optical axis at an angle $\theta_0$ to the optical axis, i.e.

$$x(0) = x_0 \qquad \text{and} \qquad \left(\frac{\mathrm{d}x}{\mathrm{d}z}\right)_{z=0} = \theta_0,$$

the solution for the light's trajectory through the material is

$$x(z) = x_0 \cos(\alpha z) + \frac{\theta_0}{\alpha}\sin(\alpha z).$$

In other words, in a two-dimensional material where $n$ has parabolic dependence on $x$, the light ray traces out a sinusoidal curve as it moves through the material.

## 1.3 The Optical Transfer Matrices

- In this section, we will continue working the two-dimensional, paraxial regime shown in Figure TODO, where the path of a light ray through the $xz$ plane is described by the distance $x$ from the optical $z$ axis and the angle $\theta$ between the path's tangent and the optical axis. Our goal in this section is to derive a formalism relating a light ray at two different points in space separated by an optical medium or optical element, such as a lens.

- We begin by considering the generic light ray shown in Figure TODO. Given the position $x_1$ and direction $\theta_1$ at point 1, it is possible to write the position $x_2$ and direction $\theta_2$ at a later point in the trajectory as the linear combinations

$$\begin{aligned} x_2 &= Ax_1 + B\theta_1 \\ \theta_2 &= Cx_1 + D\theta_1. \end{aligned} \tag{5}$$

In matrix form, we can write the relationship between the values $x_1$ and $\theta_1$ and $x_2$ and $\theta_2$ as

$$\begin{bmatrix} x_2 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} x_1 \\ \theta_1 \end{bmatrix} \equiv \mathbf{M} \begin{bmatrix} x_1 \\ \theta_1 \end{bmatrix}, \tag{6}$$

where we have defined the *optical transfer matrix*

$$\mathbf{M} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}.$$

In general, a transfer matrix encodes the passage of a light ray between media with different optical properties.

### 1.3.1 Example: Transfer Matrix for Translation Through a Homogeneous Material

- As a first example, we consider a homogeneous material with constant index of refraction $n$. We assume the light ray has the known coordinates $x_1$ and $\theta_1$ at the point $z_1$, as shown in Figure TODO; our goal is to find the position $x_2$ and direction $\theta_2$ after a translation of length $L$ along the $z$ axis to the point $z_2$.

- The material has a constant index of refraction, so the direction $\theta$ is unchanged after the translation, i.e. $\theta_1 = \theta_2$. Meanwhile, the distance $x_2$ from the $z$ axis changes as

$$x_2 = x_1 + (z_2 - z_1)\theta_1 = x_1 + L\theta_2.$$

To reveal the appropriate transfer matrix, we first write the system of equations

$$\begin{aligned} x_2 &= 1 \cdot x_1 + L \cdot \theta_1 \\ \theta_2 &= 0 \cdot x_1 + 1 \cdot \theta_1. \end{aligned}$$

We then compare this system of equations to the general form in Equations 5 and 6, which motivates the definition of the homogeneous translation's transfer matrix

$$\mathbf{M} = \begin{bmatrix} 1 & L \\ 0 & 1 \end{bmatrix}.$$

### 1.3.2 Example: Transfer Matrix for Passage Between Materials Through a Straight Interface

- Assume light in material 1 is incident at an angle $\theta_1$ on a straight boundary between materials 1 and 2, with indexes of refraction $n_1$ and $n_2$, respectively, as shown in Figure TODO

- We describe the light's position just before and just after passing through the interface with two infinitesimally close points. In other words, the $x$ coordinates before and after crossing the boundary are equal,

$$x_1 = x_2.$$

Meanwhile, because the two materials have different indexes of refraction, the light ray's initial direction $\theta_1$ changes after passing through the interface according to the law of refraction

$$n_1 \sin\theta_1 = n_2 \sin\theta_2 \quad \xrightarrow{\theta \ll 1} \quad n_1\theta_1 = n_2\theta_2,$$

where have used the small angle approximation $\sin\theta \approx \theta$.

- Written as a system of equations, the coordinates $x$ and $\theta$ on either side of the interface are related by

$$x_2 = 1 \cdot x_1 + 0 \cdot \theta_1$$
$$\theta_2 = 0 \cdot x_1 + \frac{n_1}{n_2} \cdot \theta_1.$$

The corresponding transfer matrix is thus

$$\mathbf{M} = \begin{bmatrix} 1 & 0 \\ 1 & \frac{n_1}{n_2} \end{bmatrix}.$$

Note that in the case $n_1 = n_2$, the matrix reduces to the identity matrix. Interpreted physically, this just represents the logical fact that a light ray stays the same when passing through an interface between optically identical materials.

### 1.3.3 Example: Transfer Matrix for Passage Between Materials Through a Curved Interface

- Consider two materials with indexes of refraction $n_1$ and $n_2$, separated by a curved boundary of radius $R$,[1] as shown in Figure TODO.

- As in the previous example for a straight interface, we choose two infinitesimally close points on either side of the boundary, which results in the relationship

$$x_1 = x_2.$$

- Finding the relationship between the directions $\theta_1$ and $\theta_2$ takes a little more work. We begin by introducing three angles:

  1. an angle $\phi$ between the optical axis and the normal to the boundary,
  2. an angle of incidence $\alpha = \theta_1 + \phi$ between the normal to the boundary and the incident ray in material 1, and
  3. an angle of refraction $\beta = \theta_2 + \phi$ between the normal to the boundary and the refracted ray in material 2.

  With reference to the geometry of Figure TODO, we see that the angle $\phi$ is defined via

$$\sin\phi = \frac{x_1}{R} \approx \phi,$$

  where the last equality assumes $\phi$ is small, i.e. that the normal to the boundary is close the optical axis. In the paraxial regime, where $\theta \ll 1$, the law of refraction reads

$$n_1\alpha = n_2\beta \implies n_1(\theta_1 + \phi) = n_2(\theta_2 + \phi),$$

  from which we solved for the angle $\theta_2$ according to

$$\theta_2 = \frac{n_1 - n_2}{n_2}\phi + \frac{n_1}{n_2}\theta_1 = \frac{n_1 - n_2}{n_2}\frac{x_1}{R} + \frac{n_1}{n_2}\theta_1,$$

  where the last line uses the geometric identity $\phi = (x_1)/R$.

[1] By convention, the radius $R$ is positive if the boundary is convex with respect to the direction of increasing $z$, as in Figure TODO, and negative if the boundary is concave with respect to increasing $z$.

- Using the just-derived expression for $\theta^2$, the system of equations relating $x$ and $\theta$ on either side of the boundary is

$$\begin{aligned} x_2 &= 1 \cdot x_1 &&+ 0 \cdot \theta_1 \\ \theta_2 &= \tfrac{n_1 - n_2}{n_2} \cdot \tfrac{x_1}{R} &&+ \tfrac{n_1}{n_2} \cdot \theta_1, \end{aligned}$$

and the corresponding transfer matrix is

$$\mathbf{M} = \begin{bmatrix} 1 & 0 \\ \frac{n_1 - n_2}{n_2} \frac{1}{R} & \frac{n_1}{n_2} \end{bmatrix},$$

Note that in the limit $R \to \infty$, which corresponds geometrically to a straight boundary, the transfer matrix approaches the result in Example 1.3.2, which should make sense.

### 1.3.4 Concluding Remarks on the Transfer Matrix

#### The Determinant of a Transfer Matrix

- In general, the determinant of a transfer matrix encoding the passage of light between two regions equals the ratio of the indexes of refraction in the two materials.
- As an example, the determinant of the transfer matrix in Example 1.3.1, is

$$\det \begin{bmatrix} 1 & L \\ 0 & 1 \end{bmatrix} = 1,$$

which corresponds to the relationship $n_1 = n_2$ in that example.

- Analogously, the determinants of the transfer matrices in Examples 1.3.2 and 1.3.3, corresponding to passage between materials with refractive indexes $n_1$ and $n_2$, are

$$\det \begin{bmatrix} 1 & 0 \\ 1 & \frac{n_1}{n_2} \end{bmatrix} = \det \begin{bmatrix} 1 & 0 \\ \frac{n_1 - n_2}{n_2} \frac{1}{R} & \frac{n_1}{n_2} \end{bmatrix} = \frac{n_1}{n_2}.$$

#### Transfer Matrices For Multiple Boundaries

- So far, we have considered only the passage of light through a single boundary. Conveniently, the cumulative transfer matrix for passage of light between multiple boundaries is simply the product of the individual transfer matrices. In equation for, the total transfer matrix encoding a light ray's transformation through $N$ boundaries, each with the individual transfer matrix $\mathbf{M}_i$, where $i = 1, \ldots, N$, simply

$$\mathbf{M}_{\text{total}} = \mathbf{M}_N \mathbf{M}_{n-1} \cdots \mathbf{M}_2 \mathbf{M}_1.$$

Note, however, that the order of multiplication is important, since matrix multiplication is not commutative. The transfer matrix for the first boundary, i.e. the boundary the light hits first, is furthest to the right.

- In terms of the transfer matrix formalism, the light ray coordinates $x_N$ and $\theta_N$ and $x_1$ and $\theta_1$ on either side of series of $N$ boundaries are related by

$$\begin{bmatrix} x_n \\ \theta_n \end{bmatrix} = \mathbf{M}_n \mathbf{M}_{n-1} \cdots \mathbf{M}_2 \mathbf{M}_1 \begin{bmatrix} x_1 \\ \theta_1 \end{bmatrix} = \mathbf{M}_{\text{total}} \begin{bmatrix} x_1 \\ \theta_1 \end{bmatrix}.$$

## 1.4 Lenses: TODO