

# C H A P T E R   1 1

## Vectors and the Geometry of Space

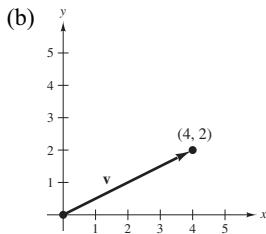
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# C H A P T E R 11

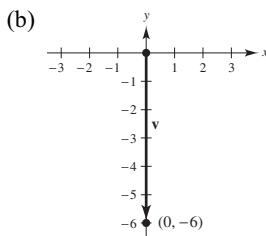
## Vectors and the Geometry of Space

### Section 11.1 Vectors in the Plane

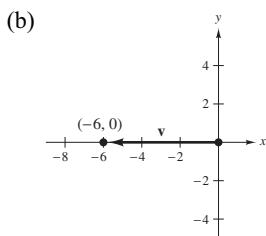
1. (a)  $\mathbf{v} = \langle 5 - 1, 4 - 2 \rangle = \langle 4, 2 \rangle$



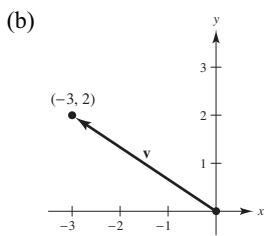
2. (a)  $\mathbf{v} = \langle 3 - 3, -2 - 4 \rangle = \langle 0, -6 \rangle$



3. (a)  $\mathbf{v} = \langle -4 - 2, -3 - (-3) \rangle = \langle -6, 0 \rangle$



4. (a)  $\mathbf{v} = \langle -1 - 2, 3 - 1 \rangle = \langle -3, 2 \rangle$



5.  $\mathbf{u} = \langle 5 - 3, 6 - 2 \rangle = \langle 2, 4 \rangle$

$\mathbf{v} = \langle 3 - 1, 8 - 4 \rangle = \langle 2, 4 \rangle$

$\mathbf{u} = \mathbf{v}$

6.  $\mathbf{u} = \langle 1 - (-4), 8 - 0 \rangle = \langle 5, 8 \rangle$

$\mathbf{v} = \langle 7 - 2, 7 - (-1) \rangle = \langle 5, 8 \rangle$

$\mathbf{u} = \mathbf{v}$

7.  $\mathbf{u} = \langle 6 - 0, -2 - 3 \rangle = \langle 6, -5 \rangle$

$\mathbf{v} = \langle 9 - 3, 5 - 10 \rangle = \langle 6, -5 \rangle$

$\mathbf{u} = \mathbf{v}$

8.  $\mathbf{u} = \langle 11 - (-4), -4 - (-1) \rangle = \langle 15, -3 \rangle$

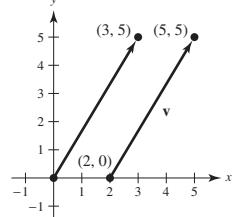
$\mathbf{v} = \langle 25 - 0, 10 - 13 \rangle = \langle 15, -3 \rangle$

$\mathbf{u} = \mathbf{v}$

9. (b)  $\mathbf{v} = \langle 5 - 2, 5 - 0 \rangle = \langle 3, 5 \rangle$

(c)  $\mathbf{v} = 3\mathbf{i} + 5\mathbf{j}$

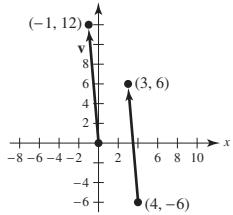
(a), (d)



10. (b)  $\mathbf{v} = \langle 3 - 4, 6 - (-6) \rangle = \langle -1, 12 \rangle$

(c)  $\mathbf{v} = -\mathbf{i} + 12\mathbf{j}$

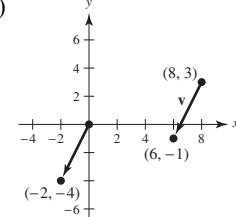
(a), (d)



11. (b)  $\mathbf{v} = \langle 6 - 8, -1 - 3 \rangle = \langle -2, -4 \rangle$

(c)  $\mathbf{v} = -2\mathbf{i} - 4\mathbf{j}$

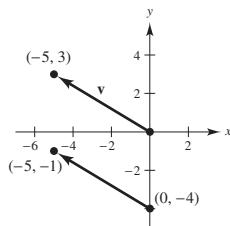
(a), (d)



12. (b)  $\mathbf{v} = \langle -5 - 0, -1 - (-4) \rangle = \langle -5, 3 \rangle$

(c)  $\mathbf{v} = -5\mathbf{i} + 3\mathbf{j}$

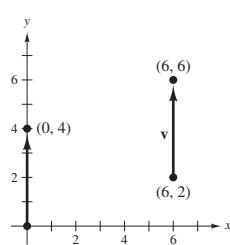
(a) and (d).



13. (b)  $\mathbf{v} = \langle 6 - 6, 6 - 2 \rangle = \langle 0, 4 \rangle$

(c)  $\mathbf{v} = 4\mathbf{j}$

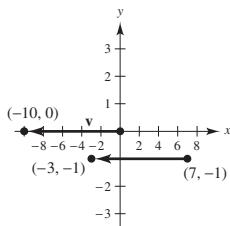
(a) and (d).



14. (b)  $\mathbf{v} = \langle -3 - 7, -1 - (-1) \rangle = \langle -10, 0 \rangle$

(c)  $\mathbf{v} = -10\mathbf{i}$

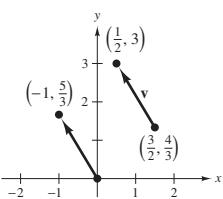
(a) and (d).



15. (b)  $\mathbf{v} = \left\langle \frac{1}{2} - \frac{3}{2}, 3 - \frac{4}{3} \right\rangle = \left\langle -1, \frac{5}{3} \right\rangle$

(c)  $\mathbf{v} = -\mathbf{i} + \frac{5}{3}\mathbf{j}$

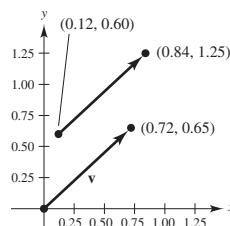
(a) and (d)



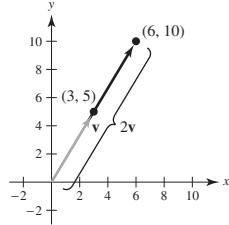
16. (b)  $\mathbf{v} = \langle 0.84 - 0.12, 1.25 - 0.60 \rangle = \langle 0.72, 0.65 \rangle$

(c)  $\mathbf{v} = 0.72\mathbf{i} + 0.65\mathbf{j}$

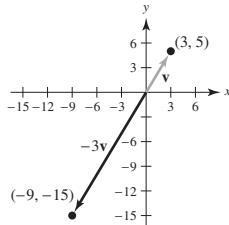
(a) and (d).



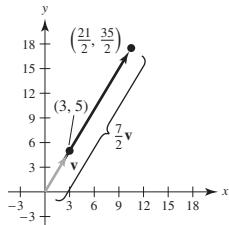
17. (a)  $2\mathbf{v} = 2\langle 3, 5 \rangle = \langle 6, 10 \rangle$



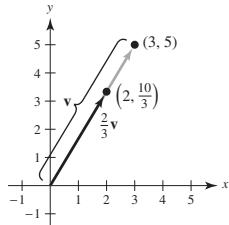
(b)  $-3\mathbf{v} = \langle -9, -15 \rangle$



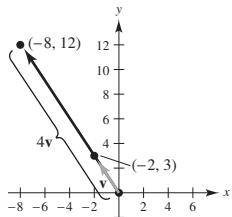
(c)  $\frac{7}{2}\mathbf{v} = \left\langle \frac{21}{2}, \frac{35}{2} \right\rangle$



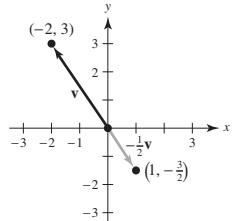
(d)  $\frac{2}{3}\mathbf{v} = \left\langle 2, \frac{10}{3} \right\rangle$



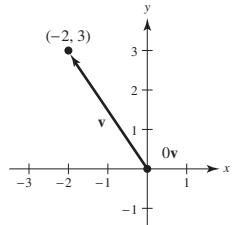
18. (a)  $4\mathbf{v} = 4\langle -2, 3 \rangle = \langle -8, 12 \rangle$



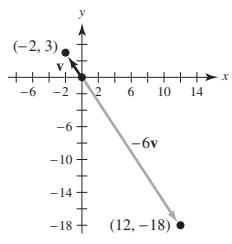
(b)  $-\frac{1}{2}\mathbf{v} = \left\langle 1, -\frac{3}{2} \right\rangle$



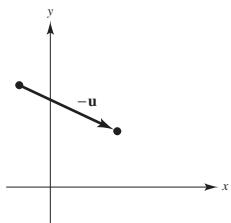
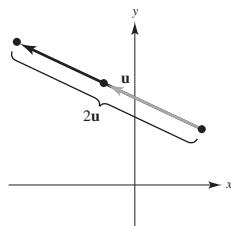
(c)  $0\mathbf{v} = \langle 0, 0 \rangle$



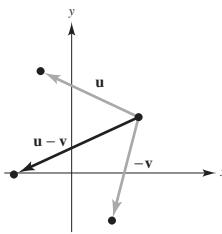
(d)  $-6\mathbf{u} = \langle 12, -18 \rangle$



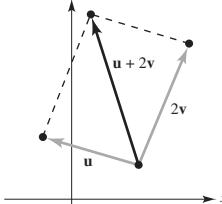
19.

20. Twice as long as given vector  $\mathbf{u}$ .

21.



22.



23. (a)  $\frac{2}{3}\mathbf{u} = \frac{2}{3}\langle 4, 9 \rangle = \left\langle \frac{8}{3}, 6 \right\rangle$

(b)  $\mathbf{v} - \mathbf{u} = \langle 2, -5 \rangle - \langle 4, 9 \rangle = \langle -2, -14 \rangle$

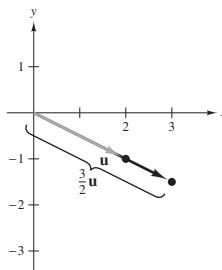
(c)  $2\mathbf{u} + 5\mathbf{v} = 2\langle 4, 9 \rangle + 5\langle 2, -5 \rangle = \langle 18, -7 \rangle$

24. (a)  $\frac{2}{3}\mathbf{u} = \frac{2}{3}\langle -3, -8 \rangle = \langle -2, -\frac{16}{3} \rangle$

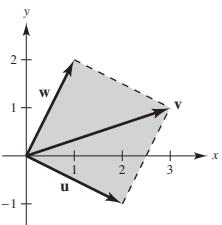
(b)  $\mathbf{v} - \mathbf{u} = \langle 8, 25 \rangle - \langle -3, -8 \rangle = \langle 11, 33 \rangle$

(c)  $2\mathbf{u} + 5\mathbf{v} = 2\langle -3, -8 \rangle + 5\langle 8, 25 \rangle = \langle 34, 109 \rangle$

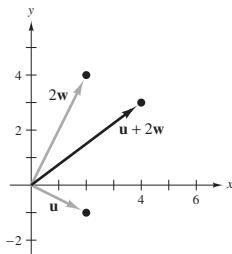
25.  $\mathbf{v} = \frac{3}{2}(2\mathbf{i} - \mathbf{j}) = 3\mathbf{i} - \frac{3}{2}\mathbf{j} = \left\langle 3, -\frac{3}{2} \right\rangle$



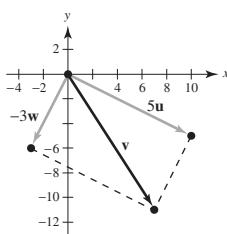
26.  $\mathbf{v} = (2\mathbf{i} - \mathbf{j}) + (\mathbf{i} + 2\mathbf{j}) = 3\mathbf{i} + \mathbf{j} = \langle 3, 1 \rangle$



27.  $\mathbf{v} = (2\mathbf{i} - \mathbf{j}) + 2(\mathbf{i} + 2\mathbf{j}) = 4\mathbf{i} + 3\mathbf{j} = \langle 4, 3 \rangle$



28.  $\mathbf{v} = 5\mathbf{u} - 3\mathbf{w} = 5\langle 2, -1 \rangle - 3\langle 1, 2 \rangle = \langle 7, -11 \rangle$



29.  $u_1 - 4 = -1$

$u_2 - 2 = 3$

$$\begin{aligned} Q &= (3, 5) \\ &\text{Terminal point} \end{aligned}$$

$u_1 = 3$

$u_2 = 5$

$$Q = (3, 5)$$

30.  $u_1 - 5 = 4$

$u_2 - 3 = -9$

$$Q = (9, -6)$$

$u_1 = 9$

$u_2 = -6$

Terminal point

31.  $\|\mathbf{v}\| = \sqrt{0 + 7^2} = 7$

32.  $\|\mathbf{v}\| = \sqrt{(-3)^2 + 0} = 3$

33.  $\|\mathbf{v}\| = \sqrt{4^2 + 3^2} = 5$

34.  $\|\mathbf{v}\| = \sqrt{12^2 + (-5)^2} = 13$

35.  $\|\mathbf{v}\| = \sqrt{6^2 + (-5)^2} = \sqrt{61}$

36.  $\|\mathbf{v}\| = \sqrt{(-10)^2 + 3^2} = \sqrt{109}$

37.  $\mathbf{v} = \langle 3, 12 \rangle$

$\|\mathbf{v}\| = \sqrt{3^2 + 12^2} = \sqrt{153}$

$$\begin{aligned} \mathbf{u} &= \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{\langle 3, 12 \rangle}{\sqrt{153}} = \left\langle \frac{3}{\sqrt{153}}, \frac{12}{\sqrt{153}} \right\rangle \\ &= \left\langle \frac{\sqrt{17}}{17}, \frac{4\sqrt{17}}{17} \right\rangle \text{ unit vector} \end{aligned}$$

38.  $\mathbf{v} = \langle -5, 15 \rangle$

$\|\mathbf{v}\| = \sqrt{25 + 225} = \sqrt{250} = 5\sqrt{10}$

$$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{\langle -5, 15 \rangle}{5\sqrt{10}} = \left\langle -\frac{\sqrt{10}}{10}, \frac{3\sqrt{10}}{10} \right\rangle \text{ unit vector}$$

39.  $\mathbf{v} = \left\langle \frac{3}{2}, \frac{5}{2} \right\rangle$

$\|\mathbf{v}\| = \sqrt{\left(\frac{3}{2}\right)^2 + \left(\frac{5}{2}\right)^2} = \frac{\sqrt{34}}{2}$

$$\begin{aligned} \mathbf{u} &= \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{\left\langle \frac{3}{2}, \frac{5}{2} \right\rangle}{\frac{\sqrt{34}}{2}} = \left\langle \frac{3}{\sqrt{34}}, \frac{5}{\sqrt{34}} \right\rangle \\ &= \left\langle \frac{3\sqrt{34}}{34}, \frac{5\sqrt{34}}{34} \right\rangle \text{ unit vector} \end{aligned}$$

40.  $\mathbf{v} = \langle -6.2, 3.4 \rangle$

$\|\mathbf{v}\| = \sqrt{(-6.2)^2 + (3.4)^2} = \sqrt{50} = 5\sqrt{2}$

$$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{\langle -6.2, 3.4 \rangle}{5\sqrt{2}} = \left\langle -\frac{31\sqrt{2}}{50}, \frac{17\sqrt{2}}{50} \right\rangle \text{ unit vector}$$

41.  $\mathbf{u} = \langle 1, -1 \rangle, \mathbf{v} = \langle -1, 2 \rangle$

(a)  $\|\mathbf{u}\| = \sqrt{1+1} = \sqrt{2}$

(b)  $\|\mathbf{v}\| = \sqrt{1+4} = \sqrt{5}$

(c)  $\mathbf{u} + \mathbf{v} = \langle 0, 1 \rangle$

$\|\mathbf{u} + \mathbf{v}\| = \sqrt{0+1} = 1$

(d)  $\frac{\mathbf{u}}{\|\mathbf{u}\|} = \frac{1}{\sqrt{2}} \langle 1, -1 \rangle$

$\left\| \frac{\mathbf{u}}{\|\mathbf{u}\|} \right\| = 1$

(e)  $\frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{1}{\sqrt{5}} \langle -1, 2 \rangle$

$\left\| \frac{\mathbf{v}}{\|\mathbf{v}\|} \right\| = 1$

(f)  $\frac{\mathbf{u} + \mathbf{v}}{\|\mathbf{u} + \mathbf{v}\|} = \langle 0, 1 \rangle$

$\left\| \frac{\mathbf{u} + \mathbf{v}}{\|\mathbf{u} + \mathbf{v}\|} \right\| = 1$

**42.**  $\mathbf{u} = \langle 0, 1 \rangle, \mathbf{v} = \langle 3, -3 \rangle$

- $\|\mathbf{u}\| = \sqrt{0 + 1} = 1$
- $\|\mathbf{v}\| = \sqrt{9 + 9} = 3\sqrt{2}$
- $\mathbf{u} + \mathbf{v} = \langle 3, -2 \rangle$   
 $\|\mathbf{u} + \mathbf{v}\| = \sqrt{9 + 4} = \sqrt{13}$
- $\frac{\mathbf{u}}{\|\mathbf{u}\|} = \langle 0, 1 \rangle$   
 $\left\| \frac{\mathbf{u}}{\|\mathbf{u}\|} \right\| = 1$
- $\frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{1}{3\sqrt{2}} \langle 3, -3 \rangle$   
 $\left\| \frac{\mathbf{v}}{\|\mathbf{v}\|} \right\| = 1$
- $\frac{\mathbf{u} + \mathbf{v}}{\|\mathbf{u} + \mathbf{v}\|} = \frac{1}{\sqrt{13}} \langle 3, -2 \rangle$   
 $\left\| \frac{\mathbf{u} + \mathbf{v}}{\|\mathbf{u} + \mathbf{v}\|} \right\| = 1$

**43.**  $\mathbf{u} = \left\langle 1, \frac{1}{2} \right\rangle, \mathbf{v} = \langle 2, 3 \rangle$

- $\|\mathbf{u}\| = \sqrt{1 + \frac{1}{4}} = \frac{\sqrt{5}}{2}$
- $\|\mathbf{v}\| = \sqrt{4 + 9} = \sqrt{13}$
- $\mathbf{u} + \mathbf{v} = \left\langle 3, \frac{7}{2} \right\rangle$   
 $\|\mathbf{u} + \mathbf{v}\| = \sqrt{9 + \frac{49}{4}} = \frac{\sqrt{85}}{2}$
- $\frac{\mathbf{u}}{\|\mathbf{u}\|} = \frac{2}{\sqrt{5}} \left\langle 1, \frac{1}{2} \right\rangle$   
 $\left\| \frac{\mathbf{u}}{\|\mathbf{u}\|} \right\| = 1$
- $\frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{1}{\sqrt{13}} \langle 2, 3 \rangle$   
 $\left\| \frac{\mathbf{v}}{\|\mathbf{v}\|} \right\| = 1$
- $\frac{\mathbf{u} + \mathbf{v}}{\|\mathbf{u} + \mathbf{v}\|} = \frac{2}{\sqrt{85}} \left\langle 3, \frac{7}{2} \right\rangle$   
 $\left\| \frac{\mathbf{u} + \mathbf{v}}{\|\mathbf{u} + \mathbf{v}\|} \right\| = 1$

**44.**  $\mathbf{u} = \langle 2, -4 \rangle, \mathbf{v} = \langle 5, 5 \rangle$

- $\|\mathbf{u}\| = \sqrt{4 + 16} = 2\sqrt{5}$
- $\|\mathbf{v}\| = \sqrt{25 + 25} = 5\sqrt{2}$
- $\mathbf{u} + \mathbf{v} = \langle 7, 1 \rangle$   
 $\|\mathbf{u} + \mathbf{v}\| = \sqrt{49 + 1} = 5\sqrt{2}$
- $\frac{\mathbf{u}}{\|\mathbf{u}\|} = \frac{1}{2\sqrt{5}} \langle 2, -4 \rangle$   
 $\left\| \frac{\mathbf{u}}{\|\mathbf{u}\|} \right\| = 1$
- $\frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{1}{5\sqrt{2}} \langle 5, 5 \rangle$   
 $\left\| \frac{\mathbf{v}}{\|\mathbf{v}\|} \right\| = 1$
- $\frac{\mathbf{u} + \mathbf{v}}{\|\mathbf{u} + \mathbf{v}\|} = \frac{1}{5\sqrt{2}} \langle 7, 1 \rangle$   
 $\left\| \frac{\mathbf{u} + \mathbf{v}}{\|\mathbf{u} + \mathbf{v}\|} \right\| = 1$

**45.**  $\mathbf{u} = \langle 2, 1 \rangle$

$$\|\mathbf{u}\| = \sqrt{5} \approx 2.236$$

$$\mathbf{v} = \langle 5, 4 \rangle$$

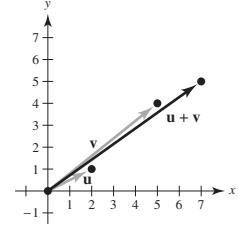
$$\|\mathbf{v}\| = \sqrt{41} \approx 6.403$$

$$\mathbf{u} + \mathbf{v} = \langle 7, 5 \rangle$$

$$\|\mathbf{u} + \mathbf{v}\| = \sqrt{74} \approx 8.602$$

$$\|\mathbf{u} + \mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|$$

$$\sqrt{74} \leq \sqrt{5} + \sqrt{41}$$



**46.**  $\mathbf{u} = \langle -3, 2 \rangle$

$$\|\mathbf{u}\| = \sqrt{13} \approx 3.606$$

$$\mathbf{v} = \langle 1, -2 \rangle$$

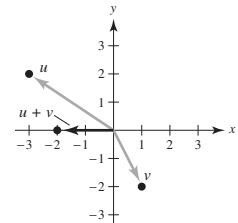
$$\|\mathbf{v}\| = \sqrt{5} \approx 2.236$$

$$\mathbf{u} + \mathbf{v} = \langle -2, 0 \rangle$$

$$\|\mathbf{u} + \mathbf{v}\| = 2$$

$$\|\mathbf{u} + \mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|$$

$$2 \leq \sqrt{13} + \sqrt{5}$$



**47.**  $\frac{\mathbf{u}}{\|\mathbf{u}\|} = \frac{1}{3} \langle 0, 3 \rangle = \langle 0, 1 \rangle$

$$6 \left( \frac{\mathbf{u}}{\|\mathbf{u}\|} \right) = 6 \langle 0, 1 \rangle = \langle 0, 6 \rangle$$

$$\mathbf{v} = \langle 0, 6 \rangle$$

48.  $\frac{\mathbf{u}}{\|\mathbf{u}\|} = \frac{1}{\sqrt{2}} \langle 1, 1 \rangle$

$$4 \left( \frac{\mathbf{u}}{\|\mathbf{u}\|} \right) = 2\sqrt{2} \langle 1, 1 \rangle$$

$$\mathbf{v} = \langle 2\sqrt{2}, 2\sqrt{2} \rangle$$

49.  $\frac{\mathbf{u}}{\|\mathbf{u}\|} = \frac{1}{\sqrt{5}} \langle -1, 2 \rangle = \left\langle -\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \right\rangle$

$$5 \left( \frac{\mathbf{u}}{\|\mathbf{u}\|} \right) = 5 \left\langle -\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \right\rangle = \langle -\sqrt{5}, 2\sqrt{5} \rangle$$

$$\mathbf{v} = \langle -\sqrt{5}, 2\sqrt{5} \rangle$$

50.  $\frac{\mathbf{u}}{\|\mathbf{u}\|} = \frac{1}{2\sqrt{3}} \langle \sqrt{3}, 3 \rangle$

$$2 \left( \frac{\mathbf{u}}{\|\mathbf{u}\|} \right) = \frac{1}{\sqrt{3}} \langle \sqrt{3}, 3 \rangle$$

$$\mathbf{v} = \langle 1, \sqrt{3} \rangle$$

51.  $\mathbf{v} = 3[(\cos 0^\circ)\mathbf{i} + (\sin 0^\circ)\mathbf{j}] = 3\mathbf{i} = \langle 3, 0 \rangle$

52.  $\mathbf{v} = 5[(\cos 120^\circ)\mathbf{i} + (\sin 120^\circ)\mathbf{j}]$

$$= -\frac{5}{2}\mathbf{i} + \frac{5\sqrt{3}}{2}\mathbf{j} = \left\langle -\frac{5}{2}, \frac{5\sqrt{3}}{2} \right\rangle$$

53.  $\mathbf{v} = 2[(\cos 150^\circ)\mathbf{i} + (\sin 150^\circ)\mathbf{j}]$

$$= -\sqrt{3}\mathbf{i} + \mathbf{j} = \langle -\sqrt{3}, 1 \rangle$$

54.  $\mathbf{v} = 4[(\cos 3.5^\circ)\mathbf{i} + (\sin 3.5^\circ)\mathbf{j}]$

$$\approx 3.9925\mathbf{i} + 0.2442\mathbf{j}$$

$$= \langle 3.9925, 0.2442 \rangle$$

55.  $\mathbf{u} = (\cos 0^\circ)\mathbf{i} + (\sin 0^\circ)\mathbf{j} = \mathbf{i}$

$$\mathbf{v} = 3(\cos 45^\circ)\mathbf{i} + 3(\sin 45^\circ)\mathbf{j} = \frac{3\sqrt{2}}{2}\mathbf{i} + \frac{3\sqrt{2}}{2}\mathbf{j}$$

$$\mathbf{u} + \mathbf{v} = \left\langle \frac{2+3\sqrt{2}}{2}, \frac{3\sqrt{2}}{2} \right\rangle$$

56.  $\mathbf{u} = 4(\cos 0^\circ)\mathbf{i} + 4(\sin 0^\circ)\mathbf{j} = 4\mathbf{i}$

$$\mathbf{v} = 2(\cos 30^\circ)\mathbf{i} + 2(\sin 30^\circ)\mathbf{j} = \mathbf{i} + \sqrt{3}\mathbf{j}$$

$$\mathbf{u} + \mathbf{v} = 5\mathbf{i} + \sqrt{3}\mathbf{j} = \langle 5, \sqrt{3} \rangle$$

57.  $\mathbf{u} = 2(\cos 4)\mathbf{i} + 2(\sin 4)\mathbf{j}$

$$\mathbf{v} = (\cos 2)\mathbf{i} + (\sin 2)\mathbf{j}$$

$$\mathbf{u} + \mathbf{v} = (2 \cos 4 + \cos 2)\mathbf{i} + (2 \sin 4 + \sin 2)\mathbf{j}$$

$$= \langle 2 \cos 4 + \cos 2, 2 \sin 4 + \sin 2 \rangle$$

58.  $\mathbf{u} = 5[\cos(-0.5)]\mathbf{i} + 5[\sin(-0.5)]\mathbf{j}$

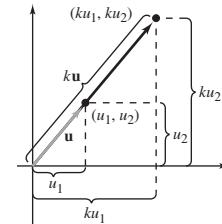
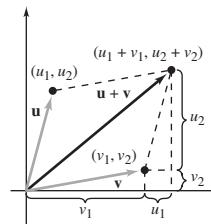
$$= 5(\cos 0.5)\mathbf{i} - 5(\sin 0.5)\mathbf{j}$$

$$\mathbf{v} = 5(\cos 0.5)\mathbf{i} + 5(\sin 0.5)\mathbf{j}$$

$$\mathbf{u} + \mathbf{v} = 10(\cos 0.5)\mathbf{i} = \langle 10 \cos 0.5, 0 \rangle$$

59. Answers will vary. *Sample answer:* A scalar is a real number such as 2. A vector is represented by a directed line segment. A vector has both magnitude and direction. For example  $\langle \sqrt{3}, 1 \rangle$  has direction  $\frac{\pi}{6}$  and a magnitude of 2.

60. See page 766:



61. (a) Vector. The velocity has both magnitude and direction.

- (b) Scalar. The price is a number.

62. (a) Scalar. The temperature is a number.

- (b) Vector. The weight has magnitude and direction.

**For Exercises 63–68,**

$$au + bw = a(\mathbf{i} + 2\mathbf{j}) + b(\mathbf{i} - \mathbf{j}) = (a + b)\mathbf{i} + (2a - b)\mathbf{j}.$$

63.  $\mathbf{v} = 2\mathbf{i} + \mathbf{j}$ . So,  $a + b = 2$ ,  $2a - b = 1$ . Solving simultaneously, you have  $a = 1$ ,  $b = 1$ .

64.  $\mathbf{v} = 3\mathbf{j}$ . So,  $a + b = 0$ ,  $2a - b = 3$ . Solving simultaneously, you have  $a = 1$ ,  $b = -1$ .

65.  $\mathbf{v} = 3\mathbf{i}$ . So,  $a + b = 3$ ,  $2a - b = 0$ . Solving simultaneously, you have  $a = 1$ ,  $b = 2$ .

66.  $\mathbf{v} = 3\mathbf{i} + 3\mathbf{j}$ . So,  $a + b = 3$ ,  $2a - b = 3$ . Solving simultaneously, you have  $a = 2$ ,  $b = 1$ .

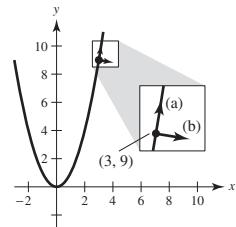
67.  $\mathbf{v} = \mathbf{i} + \mathbf{j}$ . So,  $a + b = 1$ ,  $2a - b = 1$ . Solving simultaneously, you have  $a = \frac{2}{3}$ ,  $b = \frac{1}{3}$ .

68.  $\mathbf{v} = -\mathbf{i} + 7\mathbf{j}$ . So,  $a + b = -1$ ,  $2a - b = 7$ . Solving simultaneously, you have  $a = 2$ ,  $b = -3$ .

69.  $f(x) = x^2, f'(x) = 2x, f'(3) = 6$

(a)  $m = 6$ . Let  $\mathbf{w} = \langle 1, 6 \rangle, \|\mathbf{w}\| = \sqrt{37}$ , then  $\pm \frac{\mathbf{w}}{\|\mathbf{w}\|} = \pm \frac{1}{\sqrt{37}} \langle 1, 6 \rangle$ .

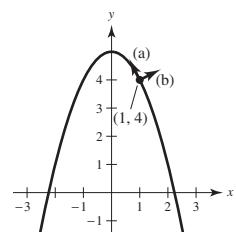
(b)  $m = -\frac{1}{6}$ . Let  $\mathbf{w} = \langle -6, 1 \rangle, \|\mathbf{w}\| = \sqrt{37}$ , then  $\pm \frac{\mathbf{w}}{\|\mathbf{w}\|} = \pm \frac{1}{\sqrt{37}} \langle -6, 1 \rangle$ .



70.  $f(x) = -x^2 + 5, f'(x) = -2x, f'(1) = -2$

(a)  $m = -2$ . Let  $\mathbf{w} = \langle 1, -2 \rangle, \|\mathbf{w}\| = \sqrt{5}$ , then  $\pm \frac{\mathbf{w}}{\|\mathbf{w}\|} = \pm \frac{1}{\sqrt{5}} \langle 1, -2 \rangle$ .

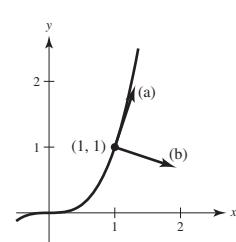
(b)  $m = \frac{1}{2}$ . Let  $\mathbf{w} = \langle 2, 1 \rangle, \|\mathbf{w}\| = \sqrt{5}$ , then  $\pm \frac{\mathbf{w}}{\|\mathbf{w}\|} = \pm \frac{1}{\sqrt{5}} \langle 2, 1 \rangle$ .



71.  $f(x) = x^3, f'(x) = 3x^2 = 3$  at  $x = 1$ .

(a)  $m = 3$ . Let  $\mathbf{w} = \langle 1, 3 \rangle, \|\mathbf{w}\| = \sqrt{10}$ , then  $\frac{\mathbf{w}}{\|\mathbf{w}\|} = \pm \frac{1}{\sqrt{10}} \langle 1, 3 \rangle$ .

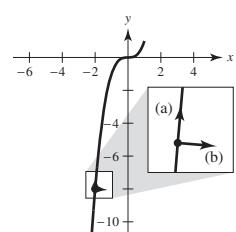
(b)  $m = -\frac{1}{3}$ . Let  $\mathbf{w} = \langle 3, -1 \rangle, \|\mathbf{w}\| = \sqrt{10}$ , then  $\frac{\mathbf{w}}{\|\mathbf{w}\|} = \pm \frac{1}{\sqrt{10}} \langle 3, -1 \rangle$ .



72.  $f(x) = x^3, f'(x) = 3x^2 = 12$  at  $x = -2$ .

(a)  $m = 12$ . Let  $\mathbf{w} = \langle 1, 12 \rangle, \|\mathbf{w}\| = \sqrt{145}$ , then  $\frac{\mathbf{w}}{\|\mathbf{w}\|} = \pm \frac{1}{\sqrt{145}} \langle 1, 12 \rangle$ .

(b)  $m = -\frac{1}{12}$ . Let  $\mathbf{w} = \langle 12, -1 \rangle, \|\mathbf{w}\| = \sqrt{145}$ , then  $\frac{\mathbf{w}}{\|\mathbf{w}\|} = \pm \frac{1}{\sqrt{145}} \langle 12, -1 \rangle$ .

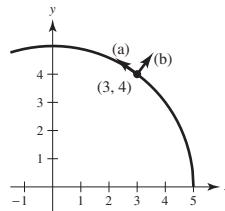


73.  $f(x) = \sqrt{25 - x^2}$

$f'(x) = \frac{-x}{\sqrt{25 - x^2}} = \frac{-3}{4}$  at  $x = 3$ .

(a)  $m = -\frac{3}{4}$ . Let  $\mathbf{w} = \langle -4, 3 \rangle, \|\mathbf{w}\| = 5$ , then  $\frac{\mathbf{w}}{\|\mathbf{w}\|} = \pm \frac{1}{5} \langle -4, 3 \rangle$ .

(b)  $m = \frac{4}{3}$ . Let  $\mathbf{w} = \langle 3, 4 \rangle, \|\mathbf{w}\| = 5$ , then  $\frac{\mathbf{w}}{\|\mathbf{w}\|} = \pm \frac{1}{5} \langle 3, 4 \rangle$ .

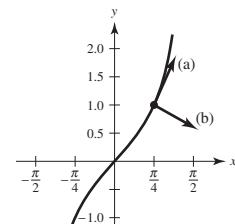


74.  $f(x) = \tan x$

$f'(x) = \sec^2 x = 2$  at  $x = \frac{\pi}{4}$

(a)  $m = 2$ . Let  $\mathbf{w} = \langle 1, 2 \rangle, \|\mathbf{w}\| = \sqrt{5}$ , then  $\frac{\mathbf{w}}{\|\mathbf{w}\|} = \pm \frac{1}{\sqrt{5}} \langle 1, 2 \rangle$ .

(b)  $m = -\frac{1}{2}$ . Let  $\mathbf{w} = \langle -2, 1 \rangle, \|\mathbf{w}\| = \sqrt{5}$ , then  $\frac{\mathbf{w}}{\|\mathbf{w}\|} = \pm \frac{1}{\sqrt{5}} \langle -2, 1 \rangle$ .



75.  $\mathbf{u} = \frac{\sqrt{2}}{2}\mathbf{i} + \frac{\sqrt{2}}{2}\mathbf{j}$

$$\mathbf{u} + \mathbf{v} = \sqrt{2}\mathbf{j}$$

$$\mathbf{v} = (\mathbf{u} + \mathbf{v}) - \mathbf{u} = -\frac{\sqrt{2}}{2}\mathbf{i} + \frac{\sqrt{2}}{2}\mathbf{j} = \left\langle -\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \right\rangle$$

76.  $\mathbf{u} = 2\sqrt{3}\mathbf{i} + 2\mathbf{j}$

$$\mathbf{u} + \mathbf{v} = -3\mathbf{i} + 3\sqrt{3}\mathbf{j}$$

$$\mathbf{v} = (\mathbf{u} + \mathbf{v}) - \mathbf{u} = (-3 - 2\sqrt{3})\mathbf{i} + (3\sqrt{3} - 2)\mathbf{j} \\ = \langle -3 - 2\sqrt{3}, 3\sqrt{3} - 2 \rangle$$

77. (a)–(c) Programs will vary.

(d) Magnitude  $\approx 63.5$

Direction  $\approx -8.26^\circ$

80.  $\|\mathbf{F}_1\| = 2, \theta_{\mathbf{F}_1} = -10^\circ$

$$\|\mathbf{F}_2\| = 4, \theta_{\mathbf{F}_2} = 140^\circ$$

$$\|\mathbf{F}_3\| = 3, \theta_{\mathbf{F}_3} = 200^\circ$$

$$\|\mathbf{R}\| = \|\mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3\| \approx 4.09$$

$$\theta_{\mathbf{R}} = \theta_{\mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3} \approx 163.0^\circ$$

81.  $\mathbf{F}_1 + \mathbf{F}_2 = (500 \cos 30^\circ \mathbf{i} + 500 \sin 30^\circ \mathbf{j}) + (200 \cos(-45^\circ) \mathbf{i} + 200 \sin(-45^\circ) \mathbf{j}) = (250\sqrt{3} + 100\sqrt{2})\mathbf{i} + (250 - 100\sqrt{2})\mathbf{j}$

$$\|\mathbf{F}_1 + \mathbf{F}_2\| = \sqrt{(250\sqrt{3} + 100\sqrt{2})^2 + (250 - 100\sqrt{2})^2} \approx 584.6 \text{ lb}$$

$$\tan \theta = \frac{250 - 100\sqrt{2}}{250\sqrt{3} + 100\sqrt{2}} \Rightarrow \theta \approx 10.7^\circ$$

82. (a)  $180(\cos 30^\circ \mathbf{i} + \sin 30^\circ \mathbf{j}) + 275\mathbf{i} \approx 430.88\mathbf{i} + 90\mathbf{j}$

Direction:  $\alpha \approx \arctan\left(\frac{90}{430.88}\right) \approx 0.206 \text{ } (\approx 11.8^\circ)$

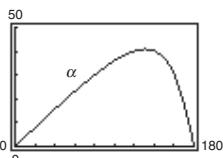
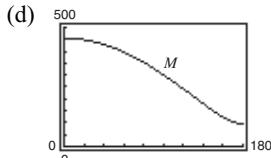
Magnitude:  $\sqrt{430.88^2 + 90^2} \approx 440.18 \text{ newtons}$

(b)  $M = \sqrt{(275 + 180 \cos \theta)^2 + (180 \sin \theta)^2}$

$$\alpha = \arctan\left[\frac{180 \sin \theta}{275 + 180 \cos \theta}\right]$$

(c)

$\theta$	$0^\circ$	$30^\circ$	$60^\circ$	$90^\circ$	$120^\circ$	$150^\circ$	$180^\circ$
$M$	455	440.2	396.9	328.7	241.9	149.3	95
$\alpha$	$0^\circ$	$11.8^\circ$	$23.1^\circ$	$33.2^\circ$	$40.1^\circ$	$37.1^\circ$	0



(e)  $M$  decreases because the forces change from acting in the same direction to acting in the opposite direction as  $\theta$  increases from  $0^\circ$  to  $180^\circ$ .

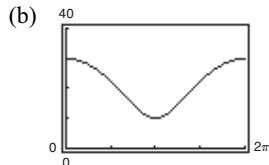
83.  $\mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 = (75 \cos 30^\circ \mathbf{i} + 75 \sin 30^\circ \mathbf{j}) + (100 \cos 45^\circ \mathbf{i} + 100 \sin 45^\circ \mathbf{j}) + (125 \cos 120^\circ \mathbf{i} + 125 \sin 120^\circ \mathbf{j})$   
 $= \left(\frac{75}{2}\sqrt{3} + 50\sqrt{2} - \frac{125}{2}\right)\mathbf{i} + \left(\frac{75}{2} + 50\sqrt{2} + \frac{125}{2}\sqrt{3}\right)\mathbf{j}$   
 $\|\mathbf{R}\| = \|\mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3\| \approx 228.5 \text{ lb}$   
 $\theta_{\mathbf{R}} = \theta_{\mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3} \approx 71.3^\circ$

84.  $\mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 = [400(\cos(-30^\circ)\mathbf{i} + \sin(-30^\circ)\mathbf{j})] + [280(\cos(45^\circ)\mathbf{i} + \sin(45^\circ)\mathbf{j})] + [350(\cos(135^\circ)\mathbf{i} + \sin(135^\circ)\mathbf{j})]$   
 $= [200\sqrt{3} + 140\sqrt{2} - 175\sqrt{2}]\mathbf{i} + [-200 + 140\sqrt{2} + 175\sqrt{2}]\mathbf{j}$   
 $\|\mathbf{R}\| = \sqrt{(200\sqrt{3} - 35\sqrt{2})^2 + (-200 + 315\sqrt{2})^2} \approx 385.2483 \text{ newtons}$   
 $\theta_{\mathbf{R}} = \arctan\left(\frac{-200 + 315\sqrt{2}}{200\sqrt{3} - 35\sqrt{2}}\right) \approx 0.6908 \approx 39.6^\circ$

85. (a) The forces act along the same direction.  $\theta = 0^\circ$ .  
(b) The forces cancel out each other.  $\theta = 180^\circ$ .  
(c) No, the magnitude of the resultant can not be greater than the sum.

86.  $\mathbf{F}_1 = \langle 20, 0 \rangle, \mathbf{F}_2 = 10\langle \cos \theta, \sin \theta \rangle$

(a)  $\|\mathbf{F}_1 + \mathbf{F}_2\| = \|\langle 20 + 10 \cos \theta, 10 \sin \theta \rangle\|$   
 $= \sqrt{400 + 400 \cos \theta + 100 \cos^2 \theta + 100 \sin^2 \theta}$   
 $= \sqrt{500 + 400 \cos \theta}$



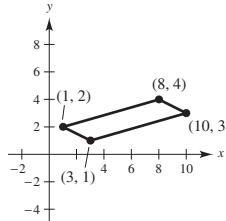
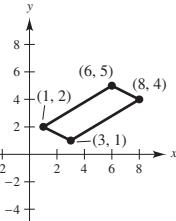
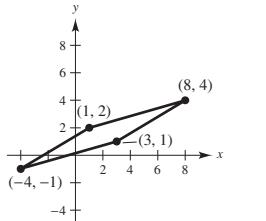
- (c) The range is  $10 \leq \|\mathbf{F}_1 + \mathbf{F}_2\| \leq 30$ .

The maximum is 30, which occur at  $\theta = 0$  and  $\theta = 2\pi$ .

The minimum is 10 at  $\theta = \pi$ .

- (d) The minimum of the resultant is 10.

87.  $(-4, -1), (6, 5), (10, 3)$



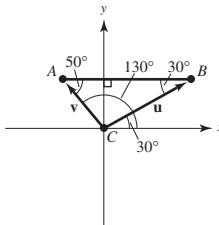
88.  $\mathbf{u} = \langle 7 - 1, 5 - 2 \rangle = \langle 6, 3 \rangle$

$\frac{1}{3}\mathbf{u} = \langle 2, 1 \rangle$

$P_1 = (1, 2) + (2, 1) = (3, 3)$

$P_2 = (1, 2) + 2(2, 1) = (5, 4)$

89.  $\mathbf{u} = \overrightarrow{CB} = \|\mathbf{u}\|(\cos 30^\circ \mathbf{i} + \sin 30^\circ \mathbf{j})$   
 $\mathbf{v} = \overrightarrow{CA} = \|\mathbf{v}\|(\cos 130^\circ \mathbf{i} + \sin 130^\circ \mathbf{j})$



Vertical components:  $\|\mathbf{u}\| \sin 30^\circ + \|\mathbf{v}\| \sin 130^\circ = 3000$

Horizontal components:  $\|\mathbf{u}\| \cos 30^\circ + \|\mathbf{v}\| \cos 130^\circ = 0$

Solving this system, you obtain

$$\|\mathbf{u}\| \approx 1958.1 \text{ pounds}$$

$$\|\mathbf{v}\| \approx 2638.2 \text{ pounds}$$

90.  $\theta_1 = \arctan\left(\frac{24}{20}\right) \approx 0.8761 \text{ or } 50.2^\circ$

$$\theta_2 = \arctan\left(\frac{24}{-10}\right) + \pi \approx 1.9656 \text{ or } 112.6^\circ$$

$$\mathbf{u} = \|\mathbf{u}\|(\cos \theta_1 \mathbf{i} + \sin \theta_1 \mathbf{j})$$

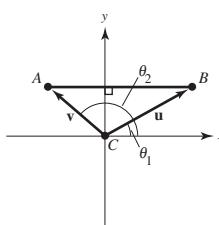
$$\mathbf{v} = \|\mathbf{v}\|(\cos \theta_2 \mathbf{i} + \sin \theta_2 \mathbf{j})$$

Vertical components:  $\|\mathbf{u}\| \sin \theta_1 + \|\mathbf{v}\| \sin \theta_2 = 5000$

Horizontal components:  $\|\mathbf{u}\| \cos \theta_1 + \|\mathbf{v}\| \cos \theta_2 = 0$

Solving this system, you obtain

$$\|\mathbf{u}\| \approx 2169.4 \text{ and } \|\mathbf{v}\| \approx 3611.2.$$



93.  $\mathbf{u} = 900(\cos 148^\circ \mathbf{i} + \sin 148^\circ \mathbf{j})$

$$\mathbf{v} = 100(\cos 45^\circ \mathbf{i} + \sin 45^\circ \mathbf{j})$$

$$\begin{aligned} \mathbf{u} + \mathbf{v} &= (900 \cos 148^\circ + 100 \cos 45^\circ) \mathbf{i} + (900 \sin 148^\circ + 100 \sin 45^\circ) \mathbf{j} \\ &\approx -692.53 \mathbf{i} + 547.64 \mathbf{j} \end{aligned}$$

$$\theta \approx \arctan\left(\frac{547.64}{-692.53}\right) \approx -38.34^\circ; 38.34^\circ \text{ North of West}$$

$$\|\mathbf{u} + \mathbf{v}\| \approx \sqrt{(-692.53)^2 + (547.64)^2} \approx 882.9 \text{ km/h}$$

91. Horizontal component =  $\|\mathbf{v}\| \cos \theta$   
 $= 1200 \cos 6^\circ \approx 1193.43 \text{ ft/sec}$

Vertical component =  $\|\mathbf{v}\| \sin \theta$   
 $= 1200 \sin 6^\circ \approx 125.43 \text{ ft/sec}$

92. To lift the weight vertically, the sum of the vertical components of  $\mathbf{u}$  and  $\mathbf{v}$  must be 100 and the sum of the horizontal components must be 0.

$$\mathbf{u} = \|\mathbf{u}\|(\cos 60^\circ \mathbf{i} + \sin 60^\circ \mathbf{j})$$

$$\mathbf{v} = \|\mathbf{v}\|(\cos 110^\circ \mathbf{i} + \sin 110^\circ \mathbf{j})$$

So,  $\|\mathbf{u}\| \sin 60^\circ + \|\mathbf{v}\| \sin 110^\circ = 100$ , or

$$\|\mathbf{u}\| \left(\frac{\sqrt{3}}{2}\right) + \|\mathbf{v}\| \sin 110^\circ = 100.$$

And  $\|\mathbf{u}\| \cos 60^\circ + \|\mathbf{v}\| \cos 110^\circ = 0$  or

$$\|\mathbf{u}\| \left(\frac{1}{2}\right) + \|\mathbf{v}\| \cos 110^\circ = 0.$$

Multiplying the last equation by  $(\sqrt{3})$  and adding to the first equation gives

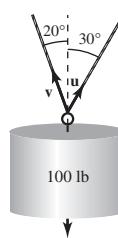
$$\|\mathbf{u}\|(\sin 110^\circ - \sqrt{3} \cos 110^\circ) = 100 \Rightarrow \|\mathbf{v}\| \approx 65.27 \text{ lb.}$$

Then,  $\|\mathbf{u}\| \left(\frac{1}{2}\right) + 65.27 \cos 110^\circ = 0$  gives

$$\|\mathbf{u}\| \approx 44.65 \text{ lb.}$$

(a) The tension in each rope:  $\|\mathbf{u}\| = 44.65 \text{ lb}$ ,  
 $\|\mathbf{v}\| = 65.27 \text{ lb}$

(b) Vertical components:  $\|\mathbf{u}\| \sin 60^\circ \approx 38.67 \text{ lb}$ ,  
 $\|\mathbf{v}\| \sin 110^\circ \approx 61.33 \text{ lb}$



94.  $\mathbf{u} = 400\mathbf{i}$  (plane)

$$\mathbf{v} = 50(\cos 135^\circ \mathbf{i} + \sin 135^\circ \mathbf{j}) = -25\sqrt{2}\mathbf{i} + 25\sqrt{2}\mathbf{j}$$
 (wind)

$$\mathbf{u} + \mathbf{v} = (400 - 25\sqrt{2})\mathbf{i} + 25\sqrt{2}\mathbf{j} \approx 364.64\mathbf{i} + 35.36\mathbf{j}$$

$$\tan \theta = \frac{35.36}{364.64} \Rightarrow \theta \approx 5.54^\circ$$

Direction North of East:  $\approx$  N  $84.46^\circ$ E

Speed:  $\approx 336.35$  mi/h

95. True

96. True

97. True

98. False

$$a = b = 0$$

99. False

$$\|\alpha\mathbf{i} + \beta\mathbf{j}\| = \sqrt{2}|\alpha|$$

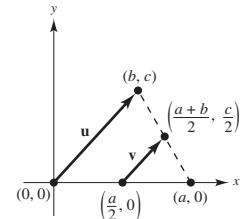
100. True

101.  $\|\mathbf{u}\| = \sqrt{\cos^2 \theta + \sin^2 \theta} = 1$ ,

$$\|\mathbf{v}\| = \sqrt{\sin^2 \theta + \cos^2 \theta} = 1$$

102. Let the triangle have vertices at  $(0, 0)$ ,  $(a, 0)$ , and  $(b, c)$ . Let  $\mathbf{u}$  be the vector joining  $(0, 0)$  and  $(b, c)$ , as indicated in the figure. Then  $\mathbf{v}$ , the vector joining the midpoints, is

$$\begin{aligned}\mathbf{v} &= \left( \frac{a+b}{2} - \frac{a}{2} \right) \mathbf{i} + \frac{c}{2} \mathbf{j} \\ &= \frac{b}{2} \mathbf{i} + \frac{c}{2} \mathbf{j} \\ &= \frac{1}{2}(b\mathbf{i} + c\mathbf{j}) = \frac{1}{2}\mathbf{u}.\end{aligned}$$



103. Let  $\mathbf{u}$  and  $\mathbf{v}$  be the vectors that determine the parallelogram, as indicated in the figure. The two diagonals are  $\mathbf{u} + \mathbf{v}$  and  $\mathbf{v} - \mathbf{u}$ . So,

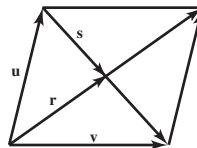
$$\mathbf{r} = x(\mathbf{u} + \mathbf{v}), \mathbf{s} = 4(\mathbf{v} - \mathbf{u}).$$

$$\mathbf{u} = \mathbf{r} - \mathbf{s}$$

$$= x(\mathbf{u} + \mathbf{v}) - y(\mathbf{v} - \mathbf{u}) = (x + y)\mathbf{u} + (x - y)\mathbf{v}.$$

So,  $x + y = 1$  and  $x - y = 0$ . Solving you have

$$x = y = \frac{1}{2}.$$



104.  $\mathbf{w} = \|\mathbf{u}\|\mathbf{v} + \|\mathbf{v}\|\mathbf{u}$

$$= \|\mathbf{u}\|[\|\mathbf{v}\|\cos \theta_v \mathbf{i} + \|\mathbf{v}\|\sin \theta_v \mathbf{j}] + \|\mathbf{v}\|[\|\mathbf{u}\|\cos \theta_u \mathbf{i} + \|\mathbf{u}\|\sin \theta_u \mathbf{j}]$$

$$= \|\mathbf{u}\|\|\mathbf{v}\|[(\cos \theta_u + \cos \theta_v) \mathbf{i} + (\sin \theta_u + \sin \theta_v) \mathbf{j}]$$

$$= 2\|\mathbf{u}\|\|\mathbf{v}\| \left[ \cos\left(\frac{\theta_u + \theta_v}{2}\right) \cos\left(\frac{\theta_u - \theta_v}{2}\right) \mathbf{i} + \sin\left(\frac{\theta_u + \theta_v}{2}\right) \cos\left(\frac{\theta_u - \theta_v}{2}\right) \mathbf{j} \right]$$

$$\tan \theta_w = \frac{\sin\left(\frac{\theta_u + \theta_v}{2}\right) \cos\left(\frac{\theta_u - \theta_v}{2}\right)}{\cos\left(\frac{\theta_u + \theta_v}{2}\right) \cos\left(\frac{\theta_u - \theta_v}{2}\right)} = \tan\left(\frac{\theta_u + \theta_v}{2}\right)$$

So,  $\theta_w = (\theta_u + \theta_v)/2$  and  $\mathbf{w}$  bisects the angle between  $\mathbf{u}$  and  $\mathbf{v}$ .

105. The set is a circle of radius 5, centered at the origin.

$$\|\mathbf{u}\| = \|\langle x, y \rangle\| = \sqrt{x^2 + y^2} = 5 \Rightarrow x^2 + y^2 = 25$$

- 106.** Let  $x = v_0 t \cos \alpha$  and  $y = v_0 t \sin \alpha - \frac{1}{2} g t^2$ .

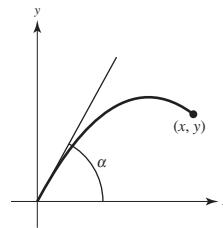
$$\begin{aligned} t &= \frac{x}{v_0 \cos \alpha} \Rightarrow y = v_0 \sin \alpha \left( \frac{x}{v_0 \cos \alpha} \right) - \frac{1}{2} g \left( \frac{x}{v_0 \cos \alpha} \right)^2 \\ &= x \tan \alpha - \frac{g}{2v_0^2} x^2 \sec^2 \alpha \\ &= x \tan \alpha - \frac{gx^2}{2v_0^2} (1 + \tan^2 \alpha) \\ &= \frac{v_0^2}{2g} - \frac{gx^2}{2v_0^2} - \frac{gx^2}{2v_0^2} \tan^2 \alpha + x \tan \alpha - \frac{v_0^2}{2g} \\ &= \frac{v_0^2}{2g} - \frac{gx^2}{2v_0^2} \left[ \tan^2 \alpha - 2 \tan \alpha \left( \frac{v_0^2}{gx} \right) + \frac{v_0^4}{g^2 x^2} \right] \\ &= \frac{v_0^2}{2g} - \frac{gx^2}{2v_0^2} \left( \tan \alpha - \frac{v_0^2}{gx} \right)^2 \end{aligned}$$

If  $y \leq \frac{v_0^2}{2g} - \frac{gx^2}{2v_0^2}$ , then  $\alpha$  can be chosen to hit the point  $(x, y)$ . To hit  $(0, y)$ : Let  $\alpha = 90^\circ$ . Then

$$y = v_0 t - \frac{1}{2} g t^2 = \frac{v_0^2}{2g} - \frac{v_0^2}{2g} \left( \frac{g}{v_0} t - 1 \right)^2, \text{ and you need } y \leq \frac{v_0^2}{2g}.$$

The set H is given by  $0 \leq x$ ,  $0 < y$  and  $y \leq \frac{v_0^2}{2g} - \frac{gx^2}{2v_0^2}$

**Note:** The parabola  $y = \frac{v_0^2}{2g} - \frac{gx^2}{2v_0^2}$  is called the “parabola of safety.”

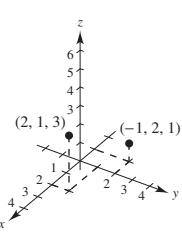


## Section 11.2 Space Coordinates and Vectors in Space

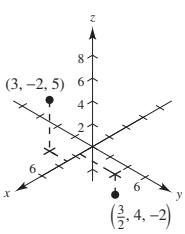
1.  $A(2, 3, 4)$   
 $B(-1, -2, 2)$

2.  $A(2, -3, -1)$   
 $B(-3, 1, 4)$

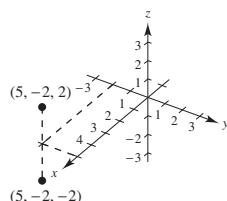
3.



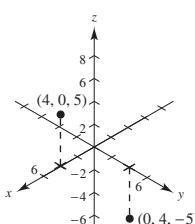
4.



5.



6.



7.  $x = -3, y = 4, z = 5: (-3, 4, 5)$

8.  $x = 7, y = -2, z = -1: (7, -2, -1)$

9.  $y = z = 0, x = 12: (12, 0, 0)$

10.  $x = 0, y = 3, z = 2: (0, 3, 2)$

11. The  $z$ -coordinate is 0.
12. The  $x$ -coordinate is 0.
13. The point is 6 units above the  $xy$ -plane.
14. The point is 2 units in front of the  $xz$ -plane.
15. The point is on the plane parallel to the  $yz$ -plane that passes through  $x = -3$ .
16. The point is on the plane parallel to the  $xy$ -plane that passes through  $z = -5/2$ .
17. The point is to the left of the  $xz$ -plane.
18. The point is in front of the  $yz$ -plane.
19. The point is on or between the planes  $y = 3$  and  $y = -3$ .
20. The point is in front of the plane  $x = 4$ .
21. The point  $(x, y, z)$  is 3 units below the  $xy$ -plane, and below either quadrant I or III.
22. The point  $(x, y, z)$  is 4 units above the  $xy$ -plane, and above either quadrant II or IV.
23. The point could be above the  $xy$ -plane and so above quadrants II or IV, or below the  $xy$ -plane, and so below quadrants I or III.
24. The point could be above the  $xy$ -plane, and so above quadrants I and III, or below the  $xy$ -plane, and so below quadrants II or IV.
25.  $d = \sqrt{(-4 - 0)^2 + (2 - 0)^2 + (7 - 0)^2} = \sqrt{16 + 4 + 49} = \sqrt{69}$
26.  $d = \sqrt{(2 - (-2))^2 + (-5 - 3)^2 + (-2 - 2)^2} = \sqrt{16 + 64 + 16} = \sqrt{96} = 4\sqrt{6}$
27.  $d = \sqrt{(6 - 1)^2 + (-2 - (-2))^2 + (-2 - 4)^2} = \sqrt{25 + 0 + 36} = \sqrt{61}$
28.  $d = \sqrt{(4 - 2)^2 + (-5 - 2)^2 + (6 - 3)^2} = \sqrt{4 + 49 + 9} = \sqrt{62}$
29.  $A(0, 0, 4), B(2, 6, 7), C(6, 4, -8)$
- $$\begin{aligned}|AB| &= \sqrt{2^2 + 6^2 + 3^2} = \sqrt{49} = 7 \\ |AC| &= \sqrt{6^2 + 4^2 + (-12)^2} = \sqrt{196} = 14 \\ |BC| &= \sqrt{4^2 + (-2)^2 + (-15)^2} = \sqrt{245} = 7\sqrt{5} \\ |BC|^2 &= 245 = 49 + 196 = |AB|^2 + |AC|^2\end{aligned}$$

Right triangle

30.  $A(3, 4, 1), B(0, 6, 2), C(3, 5, 6)$
- $$\begin{aligned}|AB| &= \sqrt{9 + 4 + 1} = \sqrt{14} \\ |AC| &= \sqrt{0 + 1 + 25} = \sqrt{26} \\ |BC| &= \sqrt{9 + 1 + 16} = \sqrt{26}\end{aligned}$$
- Because  $|AC| = |BC|$ , the triangle is isosceles.
31.  $A(-1, 0, -2), B(-1, 5, 2), C(-3, -1, 1)$
- $$\begin{aligned}|AB| &= \sqrt{0 + 25 + 16} = \sqrt{41} \\ |AC| &= \sqrt{4 + 1 + 9} = \sqrt{14} \\ |BC| &= \sqrt{4 + 36 + 1} = \sqrt{41}\end{aligned}$$
- Because  $|AB| = |BC|$ , the triangle is isosceles.
32.  $A(4, -1, -1), B(2, 0, -4), C(3, 5, -1)$
- $$\begin{aligned}|AB| &= \sqrt{4 + 1 + 9} = \sqrt{14} \\ |AC| &= \sqrt{1 + 36 + 0} = \sqrt{37} \\ |BC| &= \sqrt{1 + 25 + 9} = \sqrt{35}\end{aligned}$$
- Neither
33. The  $z$ -coordinate is changed by 5 units:  
 $(0, 0, 9), (2, 6, 12), (6, 4, -3)$
34. The  $y$ -coordinate is changed by 3 units:  
 $(3, 7, 1), (0, 9, 2), (3, 8, 6)$
35.  $\left(\frac{5 + (-2)}{2}, \frac{-9 + 3}{2}, \frac{7 + 3}{2}\right) = \left(\frac{3}{2}, -3, 5\right)$
36.  $\left(\frac{4 + 8}{2}, \frac{0 + 8}{2}, \frac{-6 + 20}{2}\right) = (6, 4, 7)$
37. Center:  $(0, 2, 5)$
- Radius: 2  
 $(x - 0)^2 + (y - 2)^2 + (z - 5)^2 = 4$
38. Center:  $(4, -1, 1)$
- Radius: 5  
 $(x - 4)^2 + (y + 1)^2 + (z - 1)^2 = 25$
39. Center:  $\frac{(2, 0, 0) + (0, 6, 0)}{2} = (1, 3, 0)$
- Radius:  $\sqrt{10}$   
 $(x - 1)^2 + (y - 3)^2 + (z - 0)^2 = 10$

**40.** Center:  $(-3, 2, 4)$

$$r = 3$$

(tangent to  $yz$ -plane)

$$(x + 3)^2 + (y - 2)^2 + (z - 4)^2 = 9$$

**41.**  $x^2 + y^2 + z^2 - 2x + 6y + 8z + 1 = 0$

$$(x^2 - 2x + 1) + (y^2 + 6y + 9) + (z^2 + 8z + 16) = -1 + 1 + 9 + 16$$

$$(x - 1)^2 + (y + 3)^2 + (z + 4)^2 = 25$$

Center:  $(1, -3, -4)$

Radius: 5

**42.**  $x^2 + y^2 + z^2 + 9x - 2y + 10z + 19 = 0$

$$\left(x^2 + 9x + \frac{81}{4}\right) + (y^2 - 2y + 1) + (z^2 + 10z + 25) = -19 + \frac{81}{4} + 1 + 25$$

$$\left(x + \frac{9}{2}\right)^2 + (y - 1)^2 + (z + 5)^2 = \frac{109}{4}$$

Center:  $\left(-\frac{9}{2}, 1, -5\right)$

Radius:  $\frac{\sqrt{109}}{2}$

**43.**  $9x^2 + 9y^2 + 9z^2 - 6x + 18y + 1 = 0$

$$x^2 + y^2 + z^2 - \frac{2}{3}x + 2y + \frac{1}{9} = 0$$

$$\left(x^2 - \frac{2}{3}x + \frac{1}{9}\right) + (y^2 + 2y + 1) + z^2 = -\frac{1}{9} + \frac{1}{9} + 1$$

$$\left(x - \frac{1}{3}\right)^2 + (y + 1)^2 + (z - 0)^2 = 1$$

Center:  $\left(\frac{1}{3}, -1, 0\right)$

Radius: 1

**44.**  $4x^2 + 4y^2 + 4z^2 - 24x - 4y + 8z - 23 = 0$

$$(x^2 - 6x + 9) + (y^2 - y + \frac{1}{4}) + (z^2 + 2z + 1) = \frac{23}{4} + 9 + \frac{1}{4} + 1$$

$$(x - 3)^2 + \left(y - \frac{1}{2}\right)^2 + (z + 1)^2 = 16$$

Center:  $\left(3, \frac{1}{2}, -1\right)$

Radius: 4

**45.**  $x^2 + y^2 + z^2 \leq 36$

Solid sphere of radius 6 centered at origin.

**46.**  $x^2 + y^2 + z^2 > 4$

Set of all points in space outside the ball of radius 2 centered at the origin.

47.  $x^2 + y^2 + z^2 < 4x - 6y + 8z - 13$

$$(x^2 - 4x + 4) + (y^2 + 6y + 9) + (z^2 - 8z + 16) < 4 + 9 + 16 - 13$$

$$(x - 2)^2 + (y + 3)^2 + (z - 4)^2 < 16$$

Interior of sphere of radius 4 centered at  $(2, -3, 4)$ .

48.  $x^2 + y^2 + z^2 > -4x + 6y - 8z - 13$

$$(x^2 + 4x + 4) + (y^2 - 6y + 9) + (z^2 + 8z + 16) > -13 + 4 + 9 + 16$$

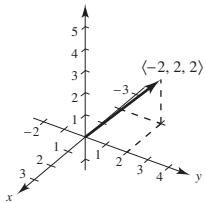
$$(x + 2)^2 + (y - 3)^2 + (z + 4)^2 > 16$$

Set of all points in space outside the ball of radius 4 centered at  $(-2, 3, -4)$ .

49. (a)  $\mathbf{v} = \langle 2 - 4, 4 - 2, 3 - 1 \rangle = \langle -2, 2, 2 \rangle$

(b)  $\mathbf{v} = -2\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$

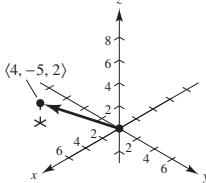
(c)



50. (a)  $\mathbf{v} = \langle 4 - 0, 0 - 5, 3 - 1 \rangle = \langle 4, -5, 2 \rangle$

(b)  $\mathbf{v} = 4\mathbf{i} - 5\mathbf{j} + 2\mathbf{k}$

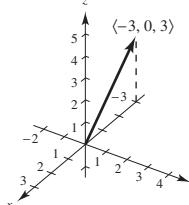
(c)



51. (a)  $\mathbf{v} = \langle 0 - 3, 3 - 3, 3 - 0 \rangle = \langle -3, 0, 3 \rangle$

(b)  $\mathbf{v} = -3\mathbf{i} + 3\mathbf{k}$

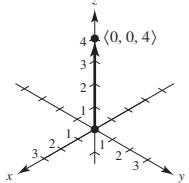
(c)



52. (a)  $\mathbf{v} = \langle 2 - 2, 3 - 3, 4 - 0 \rangle = \langle 0, 0, 4 \rangle$

(b)  $\mathbf{v} = 4\mathbf{k}$

(c)



53.  $\langle 4 - 3, 1 - 2, 6 - 0 \rangle = \langle 1, -1, 6 \rangle$

$$\|\langle 1, -1, 6 \rangle\| = \sqrt{1 + 1 + 36} = \sqrt{38}$$

$$\text{Unit vector: } \frac{\langle 1, -1, 6 \rangle}{\sqrt{38}} = \left\langle \frac{1}{\sqrt{38}}, \frac{-1}{\sqrt{38}}, \frac{6}{\sqrt{38}} \right\rangle$$

54.  $\langle -1 - 4, 7 - (-5), -3 - 2 \rangle = \langle -5, 12, -5 \rangle$

$$\|\langle -5, 12, -5 \rangle\| = \sqrt{25 + 144 + 25} = \sqrt{194}$$

$$\text{Unit vector: } \frac{\langle -5, 12, -5 \rangle}{\sqrt{194}} = \left\langle \frac{-5}{\sqrt{194}}, \frac{12}{\sqrt{194}}, \frac{-5}{\sqrt{194}} \right\rangle$$

55.  $\langle -5 - (-4), 3 - 3, 0 - 1 \rangle = \langle -1, 0, -1 \rangle$

$$\|\langle -1, 0, -1 \rangle\| = \sqrt{1 + 1} = \sqrt{2}$$

$$\text{Unit vector: } \frac{\langle -1, 0, -1 \rangle}{\sqrt{2}} = \left\langle \frac{-1}{\sqrt{2}}, 0, \frac{-1}{\sqrt{2}} \right\rangle$$

56.  $\langle 2 - 1, 4 - (-2), -2 - 4 \rangle = \langle 1, 6, -6 \rangle$

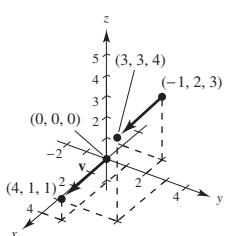
$$\|\langle 1, 6, -6 \rangle\| = \sqrt{1 + 36 + 36} = \sqrt{73}$$

$$\text{Unit vector: } \frac{\langle 1, 6, -6 \rangle}{\sqrt{73}} = \left\langle \frac{1}{\sqrt{73}}, \frac{6}{\sqrt{73}}, \frac{-6}{\sqrt{73}} \right\rangle$$

57. (b)  $\mathbf{v} = \langle 3 - (-1), 3 - 2, 4 - 3 \rangle = \langle 4, 1, 1 \rangle$

(c)  $\mathbf{v} = 4\mathbf{i} + \mathbf{j} + \mathbf{k}$

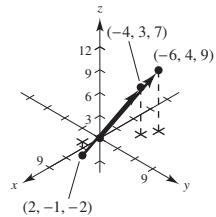
(a), (d)



58. (b)  $\mathbf{v} = \langle -4 - 2, 3 - (-1), 7 - (-2) \rangle = \langle -6, 4, 9 \rangle$

(c)  $\mathbf{v} = 6\mathbf{i} + 4\mathbf{j} + 9\mathbf{k}$

(a), (d)



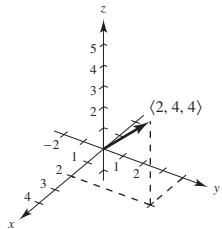
59.  $(q_1, q_2, q_3) - (0, 6, 2) = (3, -5, 6)$

$Q = (3, 1, 8)$

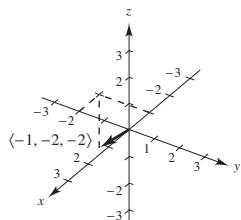
60.  $(q_1, q_2, q_3) - (0, 2, \frac{5}{2}) = (1, -\frac{2}{3}, \frac{1}{2})$

$Q = (1, -\frac{4}{3}, 3)$

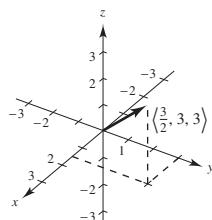
61. (a)  $2\mathbf{v} = \langle 2, 4, 4 \rangle$



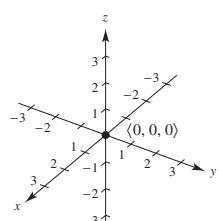
(b)  $-\mathbf{v} = \langle -1, -2, -2 \rangle$



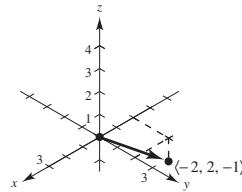
(c)  $\frac{3}{2}\mathbf{v} = \left\langle \frac{3}{2}, 3, 3 \right\rangle$



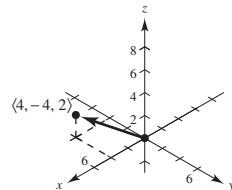
(d)  $0\mathbf{v} = \langle 0, 0, 0 \rangle$



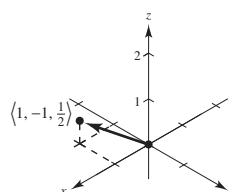
62. (a)  $-\mathbf{v} = \langle -2, 2, -1 \rangle$



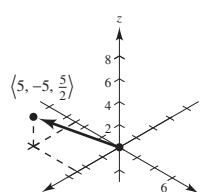
(b)  $2\mathbf{v} = \langle 4, -4, 2 \rangle$



(c)  $\frac{1}{2}\mathbf{v} = \langle 1, -1, \frac{1}{2} \rangle$



(d)  $\frac{5}{2}\mathbf{v} = \left\langle 5, -5, \frac{5}{2} \right\rangle$



63.  $\mathbf{z} = \mathbf{u} - \mathbf{v} = \langle 1, 2, 3 \rangle - \langle 2, 2, -1 \rangle = \langle -1, 0, 4 \rangle$

64.  $\mathbf{z} = \mathbf{u} - \mathbf{v} + 2\mathbf{w}$

$$= \langle 1, 2, 3 \rangle - \langle 2, 2, -1 \rangle + \langle 8, 0, -8 \rangle = \langle 7, 0, -4 \rangle$$

65.  $\mathbf{z} = 2\mathbf{u} + 4\mathbf{v} - \mathbf{w}$

$$= \langle 2, 4, 6 \rangle + \langle 8, 8, -4 \rangle - \langle 4, 0, -4 \rangle = \langle 6, 12, 6 \rangle$$

66.  $\mathbf{z} = 5\mathbf{u} - 3\mathbf{v} - \frac{1}{2}\mathbf{w}$

$$= \langle 5, 10, 15 \rangle - \langle 6, 6, -3 \rangle - \langle 2, 0, -2 \rangle$$

$$= \langle -3, 4, 20 \rangle$$

67.  $2\mathbf{z} - 3\mathbf{u} = 2(z_1, z_2, z_3) - 3\langle 1, 2, 3 \rangle = \langle 4, 0, -4 \rangle$

$$2z_1 - 3 = 4 \Rightarrow z_1 = \frac{7}{2}$$

$$2z_2 - 6 = 0 \Rightarrow z_2 = 3$$

$$2z_3 - 9 = -4 \Rightarrow z_3 = \frac{5}{2}$$

$$\mathbf{z} = \left\langle \frac{7}{2}, 3, \frac{5}{2} \right\rangle$$

68.  $2\mathbf{u} + \mathbf{v} - \mathbf{w} + 3\mathbf{z} = 2\langle 1, 2, 3 \rangle + \langle 2, 2, -1 \rangle - \langle 4, 0, -4 \rangle + 3\langle z_1, z_2, z_3 \rangle = \langle 0, 0, 0 \rangle$

$$\langle 0, 6, 9 \rangle + \langle 3z_1, 3z_2, 3z_3 \rangle = \langle 0, 0, 0 \rangle$$

$$0 + 3z_1 = 0 \Rightarrow z_1 = 0$$

$$6 + 3z_2 = 0 \Rightarrow z_2 = -2$$

$$9 + 3z_3 = 0 \Rightarrow z_3 = -3$$

$$\mathbf{z} = \langle 0, -2, -3 \rangle$$

69. (a) and (b) are parallel because

$$\langle -6, -4, 10 \rangle = -2\langle 3, 2, -5 \rangle \text{ and}$$

$$\langle 2, \frac{4}{3}, -\frac{10}{3} \rangle = \frac{2}{3}\langle 3, 2, -5 \rangle.$$

70. (b) and (d) are parallel because

$$-\mathbf{i} + \frac{4}{3}\mathbf{j} - \frac{3}{2}\mathbf{k} = -2\left(\frac{1}{2}\mathbf{i} - \frac{2}{3}\mathbf{j} + \frac{3}{4}\mathbf{k}\right) \text{ and}$$

$$\frac{3}{4}\mathbf{i} - \mathbf{j} + \frac{9}{8}\mathbf{k} = \frac{3}{2}\left(\frac{1}{2}\mathbf{i} - \frac{2}{3}\mathbf{j} + \frac{3}{4}\mathbf{k}\right).$$

71.  $\mathbf{z} = -3\mathbf{i} + 4\mathbf{j} + 2\mathbf{k}$

(a) is parallel because  $-6\mathbf{i} + 8\mathbf{j} + 4\mathbf{k} = 2\mathbf{z}$ .

72.  $\mathbf{z} = \langle -7, -8, 3 \rangle$

(b) is parallel because  $(-z)\mathbf{z} = \langle 14, 16, -6 \rangle$ .

73.  $P(0, -2, -5), Q(3, 4, 4), R(2, 2, 1)$

$$\overline{PQ} = \langle 3, 6, 9 \rangle$$

$$\overline{PR} = \langle 2, 4, 6 \rangle$$

$$\langle 3, 6, 9 \rangle = \frac{3}{2}\langle 2, 4, 6 \rangle$$

So,  $\overline{PQ}$  and  $\overline{PR}$  are parallel, the points are collinear.

74.  $P(4, -2, 7), Q(-2, 0, 3), R(7, -3, 9)$

$$\overline{PQ} = \langle -6, 2, -4 \rangle$$

$$\overline{PR} = \langle 3, -1, 2 \rangle$$

$$\langle 3, -1, 2 \rangle = -\frac{1}{2}\langle -6, 2, -4 \rangle$$

So,  $\overline{PQ}$  and  $\overline{PR}$  are parallel. The points are collinear.

75.  $P(1, 2, 4), Q(2, 5, 0), R(0, 1, 5)$

$$\overline{PQ} = \langle 1, 3, -4 \rangle$$

$$\overline{PR} = \langle -1, -1, 1 \rangle$$

Because  $\overline{PQ}$  and  $\overline{PR}$  are not parallel, the points are not collinear.

76.  $P(0, 0, 0), Q(1, 3, -2), R(2, -6, 4)$

$$\overline{PQ} = \langle 1, 3, -2 \rangle$$

$$\overline{PR} = \langle 2, -6, 4 \rangle$$

Because  $\overline{PQ}$  and  $\overline{PR}$  are not parallel, the points are not collinear.

77.  $A(2, 9, 1), B(3, 11, 4), C(0, 10, 2), D(1, 12, 5)$

$$\overline{AB} = \langle 1, 2, 3 \rangle$$

$$\overline{CD} = \langle 1, 2, 3 \rangle$$

$$\overline{AC} = \langle -2, 1, 1 \rangle$$

$$\overline{BD} = \langle -2, 1, 1 \rangle$$

Because  $\overline{AB} = \overline{CD}$  and  $\overline{AC} = \overline{BD}$ , the given points form the vertices of a parallelogram.

78.  $A(1, 1, 3), B(9, -1, -2), C(11, 2, -9), D(3, 4, -4)$

$$\overline{AB} = \langle 8, -2, -5 \rangle$$

$$\overline{DC} = \langle 8, -2, -5 \rangle$$

$$\overline{AD} = \langle 2, 3, -7 \rangle$$

$$\overline{BC} = \langle 2, 3, -7 \rangle$$

Because  $\overline{AB} = \overline{DC}$  and  $\overline{AD} = \overline{BC}$ , the given points form the vertices of a parallelogram.

79.  $\mathbf{v} = \langle 0, 0, 0 \rangle$

$$\|\mathbf{v}\| = 0$$

80.  $\mathbf{v} = \langle 1, 0, 3 \rangle$

$$\|\mathbf{v}\| = \sqrt{1 + 0 + 9} = \sqrt{10}$$

81.  $\mathbf{v} = 3\mathbf{j} - 5\mathbf{k} = \langle 0, 3, -5 \rangle$

$$\|\mathbf{v}\| = \sqrt{0 + 9 + 25} = \sqrt{34}$$

82.  $\mathbf{v} = 2\mathbf{i} + 5\mathbf{j} - \mathbf{k} = \langle 2, 5, -1 \rangle$

$$\|\mathbf{v}\| = \sqrt{4 + 25 + 1} = \sqrt{30}$$

83.  $\mathbf{v} = \mathbf{i} - 2\mathbf{j} - 3\mathbf{k} = \langle 1, -2, -3 \rangle$

$$\|\mathbf{v}\| = \sqrt{1 + 4 + 9} = \sqrt{14}$$

84.  $\mathbf{v} = -4\mathbf{i} + 3\mathbf{j} + 7\mathbf{k} = \langle -4, 3, 7 \rangle$

$$\|\mathbf{v}\| = \sqrt{16 + 9 + 49} = \sqrt{74}$$

85.  $\mathbf{v} = \langle 2, -1, 2 \rangle$

$$\|\mathbf{v}\| = \sqrt{4 + 1 + 4} = 3$$

(a)  $\frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{1}{3} \langle 2, -1, 2 \rangle$

(b)  $-\frac{\mathbf{v}}{\|\mathbf{v}\|} = -\frac{1}{3} \langle 2, -1, 2 \rangle$

86.  $\mathbf{v} = \langle 6, 0, 8 \rangle$

$$\|\mathbf{v}\| = \sqrt{36 + 0 + 64} = 10$$

(a)  $\frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{1}{10} \langle 6, 0, 8 \rangle$

(b)  $-\frac{\mathbf{v}}{\|\mathbf{v}\|} = -\frac{1}{10} \langle 6, 0, 8 \rangle$

87.  $\mathbf{v} = \langle 3, 2, -5 \rangle$

$$\|\mathbf{v}\| = \sqrt{9 + 4 + 25} = \sqrt{38}$$

(a)  $\frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{1}{\sqrt{38}} \langle 3, 2, -5 \rangle$

(b)  $-\frac{\mathbf{v}}{\|\mathbf{v}\|} = -\frac{1}{\sqrt{38}} \langle 3, 2, -5 \rangle$

88.  $\mathbf{v} = \langle 8, 0, 0 \rangle$

$$\|\mathbf{v}\| = 8$$

(a)  $\frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{1}{8} \langle 1, 0, 0 \rangle$

(b)  $-\frac{\mathbf{v}}{\|\mathbf{v}\|} = -\frac{1}{8} \langle -1, 0, 0 \rangle$

89. (a)–(d) Programs will vary.

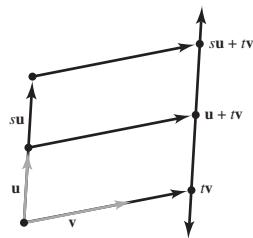
(e)  $\mathbf{u} + \mathbf{v} = \langle 4, 7.5, -2 \rangle$

$$\|\mathbf{u} + \mathbf{v}\| \approx 8.732$$

$$\|\mathbf{u}\| \approx 5.099$$

$$\|\mathbf{v}\| \approx 9.019$$

90. The terminal points of the vectors  $t\mathbf{u}$ ,  $\mathbf{u} + t\mathbf{v}$  and  $s\mathbf{u} + t\mathbf{v}$  are collinear.



91.  $\|c\mathbf{v}\| = \|c(2\mathbf{i} + 2\mathbf{j} - \mathbf{k})\| = \sqrt{4c^2 + 4c^2 + c^2} = 7$

$$\sqrt{9c^2} = 7$$

$$9c^2 = 49$$

$$c = \pm \frac{7}{3}$$

92.  $\|c\mathbf{u}\| = \|c(-\mathbf{i} + 2\mathbf{j} + 3\mathbf{k})\| = \sqrt{c^2 + 4c^2 + 9c^2} = 4$

$$\sqrt{14c^2} = 4$$

$$14c^2 = 16$$

$$c = \pm \frac{8}{7}$$

93.  $\mathbf{v} = 10 \frac{\mathbf{u}}{\|\mathbf{u}\|} = 10 \frac{\langle 0, 3, 3 \rangle}{3\sqrt{2}}$

$$= 10 \left\langle 0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle = \left\langle 0, \frac{10}{\sqrt{2}}, \frac{10}{\sqrt{2}} \right\rangle$$

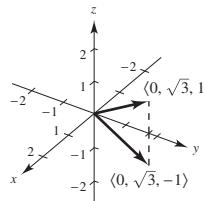
94.  $\mathbf{v} = 3 \frac{\mathbf{u}}{\|\mathbf{u}\|} = 3 \frac{\langle 1, 1, 1 \rangle}{\sqrt{3}}$

$$= 3 \left\langle \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right\rangle = \left\langle \frac{3}{\sqrt{3}}, \frac{3}{\sqrt{3}}, \frac{3}{\sqrt{3}} \right\rangle$$

95.  $\mathbf{v} = \frac{3}{2} \frac{\mathbf{u}}{\|\mathbf{u}\|} = \frac{3}{2} \frac{\langle 2, -2, 1 \rangle}{3} = \frac{3}{2} \left\langle \frac{2}{3}, \frac{-2}{3}, \frac{1}{3} \right\rangle = \left\langle 1, -1, \frac{1}{2} \right\rangle$

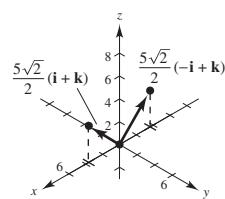
96.  $\mathbf{v} = 7 \frac{\mathbf{u}}{\|\mathbf{u}\|} = 7 \frac{\langle -4, 6, 2 \rangle}{2\sqrt{14}} = \left\langle \frac{-14}{\sqrt{14}}, \frac{21}{\sqrt{14}}, \frac{7}{\sqrt{14}} \right\rangle$

97.  $\mathbf{v} = 2[\cos(\pm 30^\circ)\mathbf{j} + \sin(\pm 30^\circ)\mathbf{k}]$   
 $= \sqrt{3}\mathbf{j} \pm \mathbf{k} = \langle 0, \sqrt{3}, \pm 1 \rangle$



98.  $\mathbf{v} = 5(\cos 45^\circ \mathbf{i} + \sin 45^\circ \mathbf{k}) = \frac{5\sqrt{2}}{2}(\mathbf{i} + \mathbf{k})$  or

$$\mathbf{v} = 5(\cos 135^\circ \mathbf{i} + \sin 135^\circ \mathbf{k}) = \frac{5\sqrt{2}}{2}(-\mathbf{i} + \mathbf{k})$$



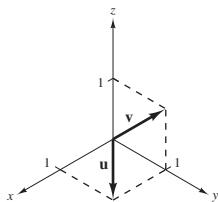
99.  $\mathbf{v} = \langle -3, -6, 3 \rangle$

$$\frac{2}{3}\mathbf{v} = \langle -2, -4, 2 \rangle$$

$$(4, 3, 0) + (-2, -4, 2) = (2, -1, 2)$$

100.  $\mathbf{v} = \langle 5, 6, -3 \rangle$   
 $\frac{2}{3}\mathbf{v} = \left\langle \frac{10}{3}, 4, -2 \right\rangle$   
 $(1, 2, 5) + \left\langle \frac{10}{3}, 4, -2 \right\rangle = \left\langle \frac{13}{3}, 6, 3 \right\rangle$

101. (a)



(b)  $\mathbf{w} = a\mathbf{u} + b\mathbf{v} = a\mathbf{i} + (a+b)\mathbf{j} + b\mathbf{k} = 0$

$a = 0, a+b = 0, b = 0$

So,  $a$  and  $b$  are both zero.

(c)  $a\mathbf{i} + (a+b)\mathbf{j} + b\mathbf{k} = \mathbf{i} + 2\mathbf{j} + \mathbf{k}$

$a = 1, a+b = 2, b = 1$

$\mathbf{w} = \mathbf{u} + \mathbf{v}$

(d)  $a\mathbf{i} + (a+b)\mathbf{j} + b\mathbf{k} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$

$a = 1, a+b = 2, b = 3$

Not possible

102. A sphere of radius 4 centered at  $(x_1, y_1, z_1)$ .

$$\begin{aligned}\|\mathbf{v}\| &= \| \langle x - x_1, y - y_1, z - z_1 \rangle \| \\ &= \sqrt{(x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2} = 4 \\ (x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2 &= 16\end{aligned}$$

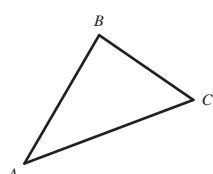
103.  $x_0$  is directed distance to  $yz$ -plane. $y_0$  is directed distance to  $xz$ -plane. $z_0$  is directed distance to  $xy$ -plane.

104.  $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$

105.  $(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = r^2$

106. Two nonzero vectors  $\mathbf{u}$  and  $\mathbf{v}$  are parallel if  $\mathbf{u} = c\mathbf{v}$  for some scalar  $c$ .

107.



$\overline{AB} + \overline{BC} = \overline{AC}$

So,  $\overline{AB} + \overline{BC} + \overline{CA} = \overline{AC} + \overline{CA} = \mathbf{0}$

108.  $\|\mathbf{r} - \mathbf{r}_0\| = \sqrt{(x - 1)^2 + (y - 1)^2 + (z - 1)^2} = 2$   
 $(x - 1)^2 + (y - 1)^2 + (z - 1)^2 = 4$

This is a sphere of radius 2 and center  $(1, 1, 1)$ .109. (a) The height of the right triangle is  $h = \sqrt{L^2 - 18^2}$ .The vector  $\overrightarrow{PQ}$  is given by

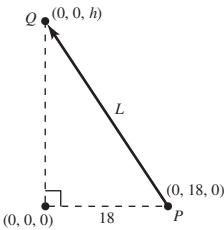
$\overrightarrow{PQ} = \langle 0, -18, h \rangle$

The tension vector  $\mathbf{T}$  in each wire is

$\mathbf{T} = c\langle 0, -18, h \rangle$  where  $ch = \frac{24}{3} = 8$ .

So,  $\mathbf{T} = \frac{8}{h}\langle 0, -18, h \rangle$  and

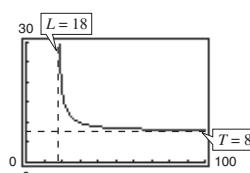
$$\begin{aligned}T &= \|\mathbf{T}\| = \frac{8}{h}\sqrt{18^2 + h^2} \\ &= \frac{8}{\sqrt{L^2 - 18^2}}\sqrt{18^2 + (L^2 - 18^2)} \\ &= \frac{8L}{\sqrt{L^2 - 18^2}}, L > 18.\end{aligned}$$



(b)

$L$	20	25	30	35	40	45	50
$T$	18.4	11.5	10	9.3	9.0	8.7	8.6

(c)

 $x = 18$  is a vertical asymptote and  $y = 8$  is a horizontal asymptote.

(d)  $\lim_{L \rightarrow 18^+} \frac{8L}{\sqrt{L^2 - 18^2}} = \infty$

$\lim_{L \rightarrow \infty} \frac{8L}{\sqrt{L^2 - 18^2}} = \lim_{L \rightarrow \infty} \frac{8}{\sqrt{1 - (18/L)^2}} = 8$

(e) From the table,  $T = 10$  implies  $L = 30$  inches.110. As in Exercise 109(c),  $x = a$  will be a vertical asymptote. So,  $\lim_{r_0 \rightarrow a^-} T = \infty$ .

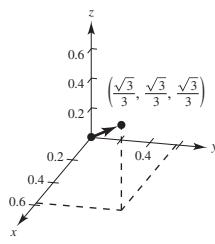
111. Let  $\alpha$  be the angle between  $\mathbf{v}$  and the coordinate axes.

$$\mathbf{v} = (\cos \alpha)\mathbf{i} + (\cos \alpha)\mathbf{j} + (\cos \alpha)\mathbf{k}$$

$$\|\mathbf{v}\| = \sqrt{3} \cos \alpha = 1$$

$$\cos \alpha = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$

$$\mathbf{v} = \frac{\sqrt{3}}{3}(\mathbf{i} + \mathbf{j} + \mathbf{k}) = \frac{\sqrt{3}}{3}\langle 1, 1, 1 \rangle$$



112.  $550 = \|c(75\mathbf{i} - 50\mathbf{j} - 100\mathbf{k})\|$

$$302,500 = 18,125c^2$$

$$c^2 = 16.689655$$

$$c \approx 4.085$$

$$\mathbf{F} \approx 4.085(75\mathbf{i} - 50\mathbf{j} - 100\mathbf{k})$$

$$\approx 306\mathbf{i} - 204\mathbf{j} - 409\mathbf{k}$$

114. Let A lie on the  $y$ -axis and the wall on the  $x$ -axis. Then  $A = (0, 10, 0)$ ,  $B = (8, 0, 6)$ ,  $C = (-10, 0, 6)$  and

$$\overline{AB} = \langle 8, -10, 6 \rangle, \overline{AC} = \langle -10, -10, 6 \rangle.$$

$$\|AB\| = 10\sqrt{2}, \|AC\| = 2\sqrt{59}$$

$$\text{Thus, } \mathbf{F}_1 = 420 \frac{\overline{AB}}{\|AB\|}, \mathbf{F}_2 = 650 \frac{\overline{AC}}{\|AC\|}$$

$$\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2 \approx \langle 237.6, -297.0, 178.2 \rangle + \langle -423.1, -423.1, 253.9 \rangle \approx \langle -185.5, -720.1, 432.1 \rangle$$

$$\|\mathbf{F}\| \approx 860.0 \text{ lb}$$

115.  $d(AP) = 2d(BP)$

$$\sqrt{x^2 + (y+1)^2 + (z-1)^2} = 2\sqrt{(x-1)^2 + (y-2)^2 + z^2}$$

$$x^2 + y^2 + z^2 + 2y - 2z + 2 = 4(x^2 + y^2 + z^2 - 2x - 4y + 5)$$

$$0 = 3x^2 + 3y^2 + 3z^2 - 8x - 18y + 2z + 18$$

$$-6 + \frac{16}{9} + 9 + \frac{1}{9} = \left( x^2 - \frac{8}{3}x + \frac{16}{9} \right) + (y^2 - 6y + 9) + \left( z^2 + \frac{2}{3}z + \frac{1}{9} \right)$$

$$\frac{44}{9} = \left( x - \frac{4}{3} \right)^2 + (y - 3)^2 + \left( z + \frac{1}{3} \right)^2$$

Sphere; center:  $\left( \frac{4}{3}, 3, -\frac{1}{3} \right)$ , radius:  $\frac{2\sqrt{11}}{3}$

113.  $\overline{AB} = \langle 0, 70, 115 \rangle, \mathbf{F}_1 = C_1 \langle 0, 70, 115 \rangle$

$$\overline{AC} = \langle -60, 0, 115 \rangle, \mathbf{F}_2 = C_2 \langle -60, 0, 115 \rangle$$

$$\overline{AD} = \langle 45, -65, 115 \rangle, \mathbf{F}_3 = C_3 \langle 45, -65, 115 \rangle$$

$$\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 = \langle 0, 0, 500 \rangle$$

$$\text{So: } -60C_2 + 45C_3 = 0$$

$$70C_1 - 65C_3 = 0$$

$$115(C_1 + C_2 + C_3) = 500$$

Solving this system yields  $C_1 = \frac{104}{69}$ ,  $C_2 = \frac{28}{23}$ , and

$$C_3 = \frac{112}{69}. \text{ So:}$$

$$\|\mathbf{F}_1\| \approx 202.919 \text{ N}$$

$$\|\mathbf{F}_2\| \approx 157.909 \text{ N}$$

$$\|\mathbf{F}_3\| \approx 226.521 \text{ N}$$

### Section 11.3 The Dot Product of Two Vectors

1.  $\mathbf{u} = \langle 3, 4 \rangle, \mathbf{v} = \langle -1, 5 \rangle$

(a)  $\mathbf{u} \cdot \mathbf{v} = 3(-1) + 4(5) = 17$

(b)  $\mathbf{u} \cdot \mathbf{u} = 3(3) + 4(4) = 25$

(c)  $\|\mathbf{u}\|^2 = 3^2 + 4^2 = 25$

(d)  $(\mathbf{u} \cdot \mathbf{v})\mathbf{v} = 17\langle -1, 5 \rangle = \langle -17, 85 \rangle$

(e)  $\mathbf{u} \cdot (2\mathbf{v}) = 2(\mathbf{u} \cdot \mathbf{v}) = 2(17) = 34$

2.  $\mathbf{u} = \langle 4, 10 \rangle, \mathbf{v} = \langle -2, 3 \rangle$

(a)  $\mathbf{u} \cdot \mathbf{v} = 4(-2) + 10(3) = 22$

(b)  $\mathbf{u} \cdot \mathbf{u} = 4(4) + 10(10) = 116$

(c)  $\|\mathbf{u}\|^2 = 4^2 + 10^2 = 116$

(d)  $(\mathbf{u} \cdot \mathbf{v})\mathbf{v} = 22\langle -2, 3 \rangle = \langle -44, 66 \rangle$

(e)  $\mathbf{u} \cdot (2\mathbf{v}) = 2(\mathbf{u} \cdot \mathbf{v}) = 2(22) = 44$

3.  $\mathbf{u} = \langle 6, -4 \rangle, \mathbf{v} = \langle -3, 2 \rangle$

(a)  $\mathbf{u} \cdot \mathbf{v} = 6(-3) + (-4)(2) = -26$

(b)  $\mathbf{u} \cdot \mathbf{u} = 6(6) + (-4)(-4) = 52$

(c)  $\|\mathbf{u}\|^2 = 6^2 + (-4)^2 = 52$

(d)  $(\mathbf{u} \cdot \mathbf{v})\mathbf{v} = -26\langle -3, 2 \rangle = \langle 78, -52 \rangle$

(e)  $\mathbf{u} \cdot (2\mathbf{v}) = 2(\mathbf{u} \cdot \mathbf{v}) = 2(-26) = -52$

4.  $\mathbf{u} = \langle -4, 8 \rangle, \mathbf{v} = \langle 7, 5 \rangle$

(a)  $\mathbf{u} \cdot \mathbf{v} = -4(7) + 8(5) = 12$

(b)  $\mathbf{u} \cdot \mathbf{u} = (-4)(-4) + 8(8) = 80$

(c)  $\|\mathbf{u}\|^2 = (-4)^2 + 8^2 = 80$

(d)  $(\mathbf{u} \cdot \mathbf{v})\mathbf{v} = 12\langle 7, 5 \rangle = \langle 84, 60 \rangle$

(e)  $\mathbf{u} \cdot (2\mathbf{v}) = 2(\mathbf{u} \cdot \mathbf{v}) = 2(12) = 24$

5.  $\mathbf{u} = \langle 2, -3, 4 \rangle, \mathbf{v} = \langle 0, 6, 5 \rangle$

(a)  $\mathbf{u} \cdot \mathbf{v} = 2(0) + (-3)(6) + (4)(5) = 2$

(b)  $\mathbf{u} \cdot \mathbf{u} = 2(2) + (-3)(-3) + 4(4) = 29$

(c)  $\|\mathbf{u}\|^2 = 2^2 + (-3)^2 + 4^2 = 29$

(d)  $(\mathbf{u} \cdot \mathbf{v})\mathbf{v} = 2\langle 0, 6, 5 \rangle = \langle 0, 12, 10 \rangle$

(e)  $\mathbf{u} \cdot (2\mathbf{v}) = 2(\mathbf{u} \cdot \mathbf{v}) = 2(2) = 4$

6.  $\mathbf{u} = \mathbf{i}, \mathbf{v} = \mathbf{i}$

(a)  $\mathbf{u} \cdot \mathbf{v} = 1$

(b)  $\mathbf{u} \cdot \mathbf{u} = 1$

(c)  $\|\mathbf{u}\|^2 = 1$

(d)  $(\mathbf{u} \cdot \mathbf{v})\mathbf{v} = \mathbf{i}$

(e)  $\mathbf{u} \cdot (2\mathbf{v}) = 2(\mathbf{u} \cdot \mathbf{v}) = 2$

7.  $\mathbf{u} = 2\mathbf{i} - \mathbf{j} + \mathbf{k}, \mathbf{v} = \mathbf{i} - \mathbf{k}$

(a)  $\mathbf{u} \cdot \mathbf{v} = 2(1) + (-1)(0) + 1(-1) = 1$

(b)  $\mathbf{u} \cdot \mathbf{u} = 2(2) + (-1)(-1) + (1)(1) = 6$

(c)  $\|\mathbf{u}\|^2 = 2^2 + (-1)^2 + 1^2 = 6$

(d)  $(\mathbf{u} \cdot \mathbf{v})\mathbf{v} = \mathbf{v} = \mathbf{i} - \mathbf{k}$

(e)  $\mathbf{u} \cdot (2\mathbf{v}) = 2(\mathbf{u} \cdot \mathbf{v}) = 2$

8.  $\mathbf{u} = 2\mathbf{i} + \mathbf{j} - 2\mathbf{k}, \mathbf{v} = \mathbf{i} - 3\mathbf{j} + 2\mathbf{k}$

(a)  $\mathbf{u} \cdot \mathbf{v} = 2(1) + 1(-3) + (-2)(2) = -5$

(b)  $\mathbf{u} \cdot \mathbf{u} = 2(2) + 1(1) + (-2)(-2) = 9$

(c)  $\|\mathbf{u}\|^2 = 2^2 + 1^2 + (-2)^2 = 9$

(d)  $(\mathbf{u} \cdot \mathbf{v})\mathbf{v} = -5(\mathbf{i} - 3\mathbf{j} + 2\mathbf{k}) = -5\mathbf{i} + 15\mathbf{j} - 10\mathbf{k}$

(e)  $\mathbf{u} \cdot (2\mathbf{v}) = 2(\mathbf{u} \cdot \mathbf{v}) = 2(-5) = -10$

9.  $\frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \|\mathbf{v}\|} = \cos \theta$

$\mathbf{u} \cdot \mathbf{v} = (8)(5) \cos \frac{\pi}{3} = 20$

10.  $\frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \|\mathbf{v}\|} = \cos \theta$

$\mathbf{u} \cdot \mathbf{v} = (40)(25) \cos \frac{5\pi}{6} = -500\sqrt{3}$

11.  $\mathbf{u} = \langle 1, 1 \rangle, \mathbf{v} = \langle 2, -2 \rangle$

$\cos \theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \|\mathbf{v}\|} = \frac{0}{\sqrt{2}\sqrt{8}} = 0$

$\theta = \frac{\pi}{2}$

12.  $\mathbf{u} = \langle 3, 1 \rangle, \mathbf{v} = \langle 2, -1 \rangle$

$\cos \theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \|\mathbf{v}\|} = \frac{5}{\sqrt{10}\sqrt{5}} = \frac{1}{\sqrt{2}}$

$\theta = \frac{\pi}{4}$

13.  $\mathbf{u} = 3\mathbf{i} + \mathbf{j}$ ,  $\mathbf{v} = -2\mathbf{i} + 4\mathbf{j}$

$$\cos \theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \|\mathbf{v}\|} = \frac{-2}{\sqrt{10} \sqrt{20}} = \frac{-1}{5\sqrt{2}}$$

$$\theta = \arccos\left(-\frac{1}{5\sqrt{2}}\right) \approx 98.1^\circ$$

14.  $\mathbf{u} = \cos\left(\frac{\pi}{6}\right)\mathbf{i} + \sin\left(\frac{\pi}{6}\right)\mathbf{j} = \frac{\sqrt{3}}{2}\mathbf{i} + \frac{1}{2}\mathbf{j}$

$$\mathbf{v} = \cos\left(\frac{3\pi}{4}\right)\mathbf{i} + \sin\left(\frac{3\pi}{4}\right)\mathbf{j} = -\frac{\sqrt{2}}{2}\mathbf{i} + \frac{\sqrt{2}}{2}\mathbf{j}$$

$$\cos \theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \|\mathbf{v}\|}$$

$$= \frac{\sqrt{3}}{2}\left(-\frac{\sqrt{2}}{2}\right) + \frac{1}{2}\left(\frac{\sqrt{2}}{2}\right) = \frac{\sqrt{2}}{4}(1 - \sqrt{3})$$

$$\theta = \arccos\left[\frac{\sqrt{2}}{4}(1 - \sqrt{3})\right] = 105^\circ$$

15.  $\mathbf{u} = \langle 1, 1, 1 \rangle$ ,  $\mathbf{v} = \langle 2, 1, -1 \rangle$

$$\cos \theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \|\mathbf{v}\|} = \frac{2}{\sqrt{3}\sqrt{6}} = \frac{\sqrt{2}}{3}$$

$$\theta = \arccos\frac{\sqrt{2}}{3} \approx 61.9^\circ$$

16.  $\mathbf{u} = 3\mathbf{i} + 2\mathbf{j} + \mathbf{k}$ ,  $\mathbf{v} = 2\mathbf{i} - 3\mathbf{j}$

$$\cos \theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \|\mathbf{v}\|} = \frac{3(2) + 2(-3) + 0}{\|\mathbf{u}\| \|\mathbf{v}\|} = 0$$

$$\theta = \frac{\pi}{2}$$

17.  $\mathbf{u} = 3\mathbf{i} + 4\mathbf{j}$ ,  $\mathbf{v} = 2\mathbf{j} + 3\mathbf{k}$

$$\cos \theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \|\mathbf{v}\|} = \frac{-8}{5\sqrt{13}} = \frac{-8\sqrt{13}}{65}$$

$$\theta = \arccos\left(-\frac{8\sqrt{13}}{65}\right) \approx 116.3^\circ$$

18.  $\mathbf{u} = 2\mathbf{i} - 3\mathbf{j} + \mathbf{k}$ ,  $\mathbf{v} = \mathbf{i} - 2\mathbf{j} + \mathbf{k}$

$$\cos \theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \|\mathbf{v}\|} = \frac{9}{\sqrt{14}\sqrt{6}} = \frac{9}{2\sqrt{21}} = \frac{3\sqrt{21}}{14}$$

$$\theta = \arccos\left(\frac{3\sqrt{21}}{14}\right) \approx 10.9^\circ$$

19.  $\mathbf{u} = \langle 4, 0 \rangle$ ,  $\mathbf{v} = \langle 1, 1 \rangle$

$\mathbf{u} \neq c\mathbf{v} \Rightarrow$  not parallel

$\mathbf{u} \cdot \mathbf{v} = 4 \neq 0 \Rightarrow$  not orthogonal

Neither

20.  $\mathbf{u} = \langle 2, 18 \rangle$ ,  $\mathbf{v} = \left\langle \frac{3}{2}, -\frac{1}{6} \right\rangle$

$\mathbf{u} \neq c\mathbf{v} \Rightarrow$  not parallel

$\mathbf{u} \cdot \mathbf{v} = 0 \Rightarrow$  orthogonal

21.  $\mathbf{u} = \langle 4, 3 \rangle$ ,  $\mathbf{v} = \left\langle \frac{1}{2}, -\frac{2}{3} \right\rangle$

$\mathbf{u} \neq c\mathbf{v} \Rightarrow$  not parallel

$\mathbf{u} \cdot \mathbf{v} = 0 \Rightarrow$  orthogonal

22.  $\mathbf{u} = -\frac{1}{3}(\mathbf{i} - 2\mathbf{j})$ ,  $\mathbf{v} = 2\mathbf{i} - 4\mathbf{j}$

$\mathbf{u} = -\frac{1}{6}\mathbf{v} \Rightarrow$  parallel

23.  $\mathbf{u} = \mathbf{j} + 6\mathbf{k}$ ,  $\mathbf{v} = \mathbf{i} - 2\mathbf{j} - \mathbf{k}$

$\mathbf{u} \neq c\mathbf{v} \Rightarrow$  not parallel

$\mathbf{u} \cdot \mathbf{v} = -8 \neq 0 \Rightarrow$  not orthogonal

Neither

24.  $\mathbf{u} = -2\mathbf{i} + 3\mathbf{j} - \mathbf{k}$ ,  $\mathbf{v} = 2\mathbf{i} + \mathbf{j} - \mathbf{k}$

$\mathbf{u} \neq c\mathbf{v} \Rightarrow$  not parallel

$\mathbf{u} \cdot \mathbf{v} = 0 \Rightarrow$  orthogonal

25.  $\mathbf{u} = \langle 2, -3, 1 \rangle$ ,  $\mathbf{v} = \langle -1, -1, -1 \rangle$

$\mathbf{u} \neq c\mathbf{v} \Rightarrow$  not parallel

$\mathbf{u} \cdot \mathbf{v} = 0 \Rightarrow$  orthogonal

26.  $\mathbf{u} = \langle \cos \theta, \sin \theta, -1 \rangle$ ,

$$\mathbf{v} = \langle \sin \theta, -\cos \theta, 0 \rangle$$

$\mathbf{u} \neq c\mathbf{v} \Rightarrow$  not parallel

$\mathbf{u} \cdot \mathbf{v} = 0 \Rightarrow$  orthogonal

27. The vector  $\langle 1, 2, 0 \rangle$  joining  $(1, 2, 0)$  and  $(0, 0, 0)$  is

perpendicular to the vector  $\langle -2, 1, 0 \rangle$  joining

$(-2, 1, 0)$  and  $(0, 0, 0)$ :  $\langle 1, 2, 0 \rangle \cdot \langle -2, 1, 0 \rangle = 0$

The triangle has a right angle, so it is a right triangle.

28. Consider the vector  $\langle -3, 0, 0 \rangle$  joining  $(0, 0, 0)$  and

$(-3, 0, 0)$ , and the vector  $\langle 1, 2, 3 \rangle$  joining  $(0, 0, 0)$  and

$(1, 2, 3)$ :  $\langle -3, 0, 0 \rangle \cdot \langle 1, 2, 3 \rangle = -3 < 0$

The triangle has an obtuse angle, so it is an obtuse triangle.

29.  $A(2, 0, 1)$ ,  $B(0, 1, 2)$ ,  $C\left(-\frac{1}{2}, \frac{3}{2}, 0\right)$

$$\overrightarrow{AB} = \langle -2, 1, 1 \rangle \quad \overrightarrow{BA} = \langle 2, -1, -1 \rangle$$

$$\overrightarrow{AC} = \left\langle -\frac{5}{2}, \frac{3}{2}, -1 \right\rangle \quad \overrightarrow{CA} = \left\langle \frac{5}{2}, -\frac{3}{2}, 1 \right\rangle$$

$$\overrightarrow{BC} = \left\langle -\frac{1}{2}, \frac{1}{2}, -2 \right\rangle \quad \overrightarrow{CB} = \left\langle \frac{1}{2}, -\frac{1}{2}, 2 \right\rangle$$

$$\overrightarrow{AB} \cdot \overrightarrow{AC} = 5 + \frac{3}{2} - 1 > 0$$

$$\overrightarrow{BA} \cdot \overrightarrow{BC} = -1 - \frac{1}{2} + 2 > 0$$

$$\overrightarrow{CA} \cdot \overrightarrow{CB} = \frac{5}{4} + \frac{3}{4} + 2 > 0$$

The triangle has three acute angles, so it is an acute triangle.

30.  $A(2, -7, 3), B(-1, 5, 8), C(4, 6, -1)$

$$\overrightarrow{AB} = \langle -3, 12, 5 \rangle \quad \overrightarrow{BA} = \langle 3, -12, -5 \rangle$$

$$\overrightarrow{AC} = \langle 2, 13, -4 \rangle \quad \overrightarrow{CA} = \langle -2, -13, 4 \rangle$$

$$\overrightarrow{BC} = \langle 5, 1, -9 \rangle \quad \overrightarrow{CB} = \langle -5, -1, 9 \rangle$$

$$\overrightarrow{AB} \cdot \overrightarrow{AC} = -6 + 156 - 20 > 0$$

$$\overrightarrow{BA} \cdot \overrightarrow{BC} = 15 - 12 + 45 > 0$$

$$\overrightarrow{CA} \cdot \overrightarrow{CB} = 10 + 13 + 36 > 0$$

The triangle has three acute angles, so it is an acute triangle.

31.  $\mathbf{u} = \mathbf{i} + 2\mathbf{j} + 2\mathbf{k}, \|\mathbf{u}\| = 3$

$$\cos \alpha = \frac{1}{3}$$

$$\cos \beta = \frac{2}{3}$$

$$\cos \gamma = \frac{2}{3}$$

$$\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = \frac{1}{9} + \frac{4}{9} + \frac{4}{9} = 1$$

34.  $\mathbf{u} = \langle a, b, c \rangle, \|\mathbf{u}\| = \sqrt{a^2 + b^2 + c^2}$

$$\cos \alpha = \frac{a}{\sqrt{a^2 + b^2 + c^2}}$$

$$\cos \beta = \frac{b}{\sqrt{a^2 + b^2 + c^2}}$$

$$\cos \gamma = \frac{c}{\sqrt{a^2 + b^2 + c^2}}$$

$$\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = \frac{a^2}{a^2 + b^2 + c^2} + \frac{b^2}{a^2 + b^2 + c^2} + \frac{c^2}{a^2 + b^2 + c^2} = 1$$

35.  $\mathbf{u} = \langle 3, 2, -2 \rangle \|\mathbf{u}\| = \sqrt{17}$

$$\cos \alpha = \frac{3}{\sqrt{17}} \Rightarrow \alpha \approx 0.7560 \text{ or } 43.3^\circ$$

$$\cos \beta = \frac{2}{\sqrt{17}} \Rightarrow \beta \approx 1.0644 \text{ or } 61.0^\circ$$

$$\cos \gamma = \frac{-2}{\sqrt{17}} \Rightarrow \gamma \approx 2.0772 \text{ or } 119.0^\circ$$

36.  $\mathbf{u} = \langle -4, 3, 5 \rangle \|\mathbf{u}\| = \sqrt{50} = 5\sqrt{2}$

$$\cos \alpha = \frac{-4}{5\sqrt{2}} \Rightarrow \alpha \approx 2.1721 \text{ or } 124.4^\circ$$

$$\cos \beta = \frac{3}{5\sqrt{2}} \Rightarrow \beta \approx 1.1326 \text{ or } 64.9^\circ$$

$$\cos \gamma = \frac{5}{5\sqrt{2}} = \frac{1}{\sqrt{2}} \Rightarrow \gamma \approx \frac{\pi}{4} \text{ or } 45^\circ$$

32.  $\mathbf{u} = \langle 5, 3, -1 \rangle \|\mathbf{u}\| = \sqrt{35}$

$$\cos \alpha = \frac{5}{\sqrt{35}}$$

$$\cos \beta = \frac{3}{\sqrt{35}}$$

$$\cos \gamma = \frac{-1}{\sqrt{35}}$$

$$\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = \frac{25}{35} + \frac{9}{35} + \frac{1}{35} = 1$$

33.  $\mathbf{u} = \langle 0, 6, -4 \rangle, \|\mathbf{u}\| = \sqrt{52} = 2\sqrt{13}$

$$\cos \alpha = 0$$

$$\cos \beta = \frac{3}{\sqrt{13}}$$

$$\cos \gamma = -\frac{2}{\sqrt{13}}$$

$$\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 0 + \frac{9}{13} + \frac{4}{13} = 1$$

37.  $\mathbf{u} = \langle -1, 5, 2 \rangle \|\mathbf{u}\| = \sqrt{30}$

$$\cos \alpha = \frac{-1}{\sqrt{30}} \Rightarrow \alpha \approx 1.7544 \text{ or } 100.5^\circ$$

$$\cos \beta = \frac{5}{\sqrt{30}} \Rightarrow \beta \approx 0.4205 \text{ or } 24.1^\circ$$

$$\cos \gamma = \frac{2}{\sqrt{30}} \Rightarrow \gamma \approx 1.1970 \text{ or } 68.6^\circ$$

38.  $\mathbf{u} = \langle -2, 6, 1 \rangle \|\mathbf{u}\| = \sqrt{41}$

$$\cos \alpha = \frac{-2}{\sqrt{41}} \Rightarrow \alpha \approx 1.8885 \text{ or } 108.2^\circ$$

$$\cos \beta = \frac{6}{\sqrt{41}} \Rightarrow \beta \approx 0.3567 \text{ or } 20.4^\circ$$

$$\cos \gamma = \frac{1}{\sqrt{41}} \Rightarrow \gamma \approx 1.4140 \text{ or } 81.0^\circ$$

39.  $\mathbf{F}_1: C_1 = \frac{50}{\|\mathbf{F}_1\|} \approx 4.3193$

$\mathbf{F}_2: C_2 = \frac{80}{\|\mathbf{F}_2\|} \approx 5.4183$

$$\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2$$

$$\approx 4.3193\langle 10, 5, 3 \rangle + 5.4183\langle 12, 7, -5 \rangle$$

$$= \langle 108.2126, 59.5246, -14.1336 \rangle$$

$$\|\mathbf{F}\| \approx 124.310 \text{ lb}$$

$$\cos \alpha \approx \frac{108.2126}{\|\mathbf{F}\|} \Rightarrow \alpha \approx 29.48^\circ$$

$$\cos \beta \approx \frac{59.5246}{\|\mathbf{F}\|} \Rightarrow \beta \approx 61.39^\circ$$

$$\cos \gamma \approx \frac{-14.1336}{\|\mathbf{F}\|} \Rightarrow \gamma \approx 96.53^\circ$$

40.  $\mathbf{F}_1: C_1 = \frac{300}{\|\mathbf{F}_1\|} \approx 13.0931$

$\mathbf{F}_2: C_2 = \frac{100}{\|\mathbf{F}_2\|} \approx 6.3246$

$$\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2$$

$$\approx 13.0931\langle -20, -10, 5 \rangle + 6.3246\langle 5, 15, 0 \rangle$$

$$= \langle -230.239, 36.062, 65.4655 \rangle$$

$$\|\mathbf{F}\| \approx 242.067 \text{ lb}$$

$$\cos \alpha \approx \frac{-230.239}{\|\mathbf{F}\|} \Rightarrow \alpha \approx 162.02^\circ$$

$$\cos \beta \approx \frac{-36.062}{\|\mathbf{F}\|} \Rightarrow \beta \approx 98.57^\circ$$

$$\cos \gamma \approx \frac{65.4655}{\|\mathbf{F}\|} \Rightarrow \gamma \approx 74.31^\circ$$

41.  $\overline{OA} = \langle 0, 10, 10 \rangle$

$$\cos \alpha = \frac{0}{\sqrt{0^2 + 10^2 + 10^2}} = 0 \Rightarrow \alpha = 90^\circ$$

$$\cos \beta = \cos \gamma = \frac{10}{\sqrt{0^2 + 10^2 + 10^2}}$$

$$= \frac{1}{\sqrt{2}} \Rightarrow \beta = \gamma = 45^\circ$$

42.  $\mathbf{F}_1 = C_1(0, 10, 10)$

$$\|\mathbf{F}_1\| = 200 = C_1 10\sqrt{2} \Rightarrow C_1 = 10\sqrt{2} \text{ and}$$

$$\mathbf{F}_1 = \langle 0, 100\sqrt{2}, 100\sqrt{2} \rangle$$

$$\mathbf{F}_2 = C_2\langle -4, -6, 10 \rangle$$

$$\mathbf{F}_3 = C_3\langle 4, -6, 10 \rangle$$

$$\mathbf{F} = \langle 0, 0, w \rangle$$

$$\mathbf{F} + \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 = \mathbf{0}$$

$$-4C_2 + 4C_3 = 0 \Rightarrow C_2 = C_3$$

$$100\sqrt{2} - 6C_2 - 6C_3 = 0 \Rightarrow C_2 = C_3 = \frac{25\sqrt{2}}{3} \text{ N}$$

$$W = 10C_2 + 10C_3 + 100\sqrt{2} = \frac{800\sqrt{2}}{3}$$

43.  $\mathbf{u} = \langle 6, 7 \rangle, \mathbf{v} = \langle 1, 4 \rangle$

$$\begin{aligned} \text{(a)} \quad \mathbf{w}_1 &= \text{proj}_{\mathbf{v}} \mathbf{u} = \left( \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2} \right) \mathbf{v} \\ &= \frac{6(1) + 7(4)}{1^2 + 4^2} \langle 1, 4 \rangle \\ &= \frac{34}{17} \langle 1, 4 \rangle = \langle 2, 8 \rangle \end{aligned}$$

$$\text{(b)} \quad \mathbf{w}_2 = \mathbf{u} - \mathbf{w}_1 = \langle 6, 7 \rangle - \langle 2, 8 \rangle = \langle 4, -1 \rangle$$

44.  $\mathbf{u} = \langle 9, 7 \rangle, \mathbf{v} = \langle 1, 3 \rangle$

$$\begin{aligned} \text{(a)} \quad \mathbf{w}_1 &= \text{proj}_{\mathbf{v}} \mathbf{u} = \left( \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2} \right) \mathbf{v} \\ &= \frac{9(1) + 7(3)}{1 + 3^2} \langle 1, 3 \rangle \\ &= \frac{30}{10} \langle 1, 3 \rangle = \langle 3, 9 \rangle \end{aligned}$$

$$\text{(b)} \quad \mathbf{w}_2 = \mathbf{u} - \mathbf{w}_1 = \langle 9, 7 \rangle - \langle 3, 9 \rangle = \langle 6, -2 \rangle$$

45.  $\mathbf{u} = 2\mathbf{i} + 3\mathbf{j} = \langle 2, 3 \rangle$ ,  $\mathbf{v} = 5\mathbf{i} + \mathbf{j} = \langle 5, 1 \rangle$

$$\begin{aligned} \text{(a)} \quad \mathbf{w}_1 &= \text{proj}_{\mathbf{v}} \mathbf{u} = \left( \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2} \right) \mathbf{v} \\ &= \frac{2(5) + 3(1)}{5^2 + 1} \langle 5, 1 \rangle \\ &= \frac{13}{26} \langle 5, 1 \rangle = \left\langle \frac{5}{2}, \frac{1}{2} \right\rangle \end{aligned}$$

$$\text{(b)} \quad \mathbf{w}_2 = \mathbf{u} - \mathbf{w}_1 = \langle 2, 3 \rangle - \left\langle \frac{5}{2}, \frac{1}{2} \right\rangle = \left\langle -\frac{1}{2}, \frac{5}{2} \right\rangle$$

46.  $\mathbf{u} = 2\mathbf{i} - 3\mathbf{j} = \langle 2, -3 \rangle$ ,  $\mathbf{v} = 3\mathbf{i} + 2\mathbf{j} = \langle 3, 2 \rangle$

$$\begin{aligned} \text{(a)} \quad \mathbf{w}_1 &= \text{proj}_{\mathbf{v}} \mathbf{u} = \left( \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2} \right) \mathbf{v} \\ &= \frac{2(3) + (-3)(2)}{3^2 + 2^2} \langle 3, 2 \rangle \\ &= 0 \langle 3, 2 \rangle = \langle 0, 0 \rangle \end{aligned}$$

$$\text{(b)} \quad \mathbf{w}_2 = \mathbf{u} - \mathbf{w}_1 = \langle 2, -3 \rangle$$

47.  $\mathbf{u} = \langle 0, 3, 3 \rangle$ ,  $\mathbf{v} = \langle -1, 1, 1 \rangle$

$$\begin{aligned} \text{(a)} \quad \mathbf{w}_1 &= \text{proj}_{\mathbf{v}} \mathbf{u} = \left( \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2} \right) \mathbf{v} \\ &= \frac{0(-1) + 3(1) + 3(1)}{1+1+1} \langle -1, 1, 1 \rangle \\ &= \frac{6}{3} \langle -1, 1, 1 \rangle = \langle -2, 2, 2 \rangle \end{aligned}$$

$$\text{(b)} \quad \mathbf{w}_2 = \mathbf{u} - \mathbf{w}_1 = \langle 0, 3, 3 \rangle - \langle -2, 2, 2 \rangle = \langle 2, 1, 1 \rangle$$

48.  $\mathbf{u} = \langle 8, 2, 0 \rangle$ ,  $\mathbf{v} = \langle 2, 1, -1 \rangle$

$$\begin{aligned} \text{(a)} \quad \mathbf{w}_1 &= \text{proj}_{\mathbf{v}} \mathbf{u} = \left( \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2} \right) \mathbf{v} \\ &= \frac{8(2) + 2(1) + 0(-1)}{2^2 + 1 + 1} \langle 2, 1, -1 \rangle \\ &= \frac{18}{6} \langle 2, 1, -1 \rangle = \langle 6, 3, -3 \rangle \end{aligned}$$

$$\text{(b)} \quad \mathbf{w}_2 = \mathbf{u} - \mathbf{w}_1 = \langle 8, 2, 0 \rangle - \langle 6, 3, -3 \rangle = \langle 2, -1, 3 \rangle$$

49.  $\mathbf{u} = 2\mathbf{i} + \mathbf{j} + 2\mathbf{k} = \langle 2, 1, 2 \rangle$

$$\mathbf{v} = 3\mathbf{j} + 4\mathbf{k} = \langle 0, 3, 4 \rangle$$

$$\begin{aligned} \text{(a)} \quad \mathbf{w}_1 &= \text{proj}_{\mathbf{v}} \mathbf{u} = \left( \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2} \right) \mathbf{v} \\ &= \frac{2(0) + 1(3) + 2(4)}{3^2 + 4^2} \langle 0, 3, 4 \rangle \\ &= \frac{11}{25} \langle 0, 3, 4 \rangle = \left\langle 0, \frac{33}{25}, \frac{44}{25} \right\rangle \end{aligned}$$

$$\text{(b)} \quad \mathbf{w}_2 = \mathbf{u} - \mathbf{w}_1 = \langle 2, 1, 2 \rangle - \left\langle 0, \frac{33}{25}, \frac{44}{25} \right\rangle = \left\langle 2, -\frac{8}{25}, \frac{6}{25} \right\rangle$$

50.  $\mathbf{u} = \mathbf{i} + 4\mathbf{k} = \langle 1, 0, 4 \rangle$

$$\begin{aligned} \mathbf{v} &= 3\mathbf{i} + 2\mathbf{k} = \langle 3, 0, 2 \rangle \\ \text{(a)} \quad \mathbf{w}_1 &= \text{proj}_{\mathbf{v}} \mathbf{u} = \left( \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2} \right) \mathbf{v} \\ &= \frac{1(3) + 4(2)}{3^2 + 2^2} \langle 3, 0, 2 \rangle \\ &= \frac{11}{13} \langle 3, 0, 2 \rangle = \left\langle \frac{33}{13}, 0, \frac{22}{13} \right\rangle \end{aligned}$$

$$\begin{aligned} \text{(b)} \quad \mathbf{w}_2 &= \mathbf{u} - \mathbf{w}_1 = \langle 1, 0, 4 \rangle - \left\langle \frac{33}{13}, 0, \frac{22}{13} \right\rangle \\ &= \left\langle -\frac{20}{13}, 0, \frac{30}{13} \right\rangle \end{aligned}$$

51.  $\mathbf{u} \cdot \mathbf{v} = \langle u_1, u_2, u_3 \rangle \cdot \langle v_1, v_2, v_3 \rangle = u_1 v_1 + u_2 v_2 + u_3 v_3$

52. The vectors  $\mathbf{u}$  and  $\mathbf{v}$  are orthogonal if  $\mathbf{u} \cdot \mathbf{v} = 0$ . The angle  $\theta$  between  $\mathbf{u}$  and  $\mathbf{v}$  is given by  $\cos \theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \|\mathbf{v}\|}$ .

53. (a) and (b) are defined. (c) and (d) are not defined because it is not possible to find the dot product of a scalar and a vector or to add a scalar to a vector.

54. See page 786. Direction cosines of  $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$  are

$\cos \alpha = \frac{v_1}{\|\mathbf{v}\|}$ ,  $\cos \beta = \frac{v_2}{\|\mathbf{v}\|}$ ,  $\cos \gamma = \frac{v_3}{\|\mathbf{v}\|}$ .  $\alpha$ ,  $\beta$ , and  $\gamma$  are the direction angles. See Figure 11.26.

55. See figure 11.29, page 787.

56. (a)  $\left( \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2} \right) \mathbf{v} = \mathbf{u} \Rightarrow \mathbf{u} = c\mathbf{v} \Rightarrow \mathbf{u}$  and  $\mathbf{v}$  are parallel.

(b)  $\left( \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2} \right) \mathbf{v} = \mathbf{0} \Rightarrow \mathbf{u} \cdot \mathbf{v} = 0 \Rightarrow \mathbf{u}$  and  $\mathbf{v}$  are orthogonal.

$$\begin{aligned} \text{57. Yes, } \left\| \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2} \mathbf{v} \right\| &= \left\| \frac{\mathbf{v} \cdot \mathbf{u}}{\|\mathbf{u}\|^2} \mathbf{u} \right\| \\ |\mathbf{u} \cdot \mathbf{v}| \frac{\|\mathbf{v}\|}{\|\mathbf{v}\|^2} &= |\mathbf{v} \cdot \mathbf{u}| \frac{\|\mathbf{u}\|}{\|\mathbf{u}\|^2} \\ \frac{1}{\|\mathbf{v}\|} &= \frac{1}{\|\mathbf{u}\|} \\ \|\mathbf{u}\| &= \|\mathbf{v}\| \end{aligned}$$

58. (a) Orthogonal,  $\theta = \frac{\pi}{2}$

(b) Acute,  $0 < \theta < \frac{\pi}{2}$

(c) Obtuse,  $\frac{\pi}{2} < \theta < \pi$

59.  $\mathbf{u} = \langle 3240, 1450, 2235 \rangle$

$$\mathbf{v} = \langle 1.35, 2.65, 1.85 \rangle$$

$$\begin{aligned}\mathbf{u} \cdot \mathbf{v} &= 3240(1.35) + 1450(2.65) + 2235(1.85) \\ &= \$12,351.25\end{aligned}$$

This represents the total amount that the restaurant earned on its three products.

60.  $\mathbf{u} = \langle 3240, 1450, 2235 \rangle$

$$\mathbf{v} = \langle 1.35, 2.65, 1.85 \rangle$$

Increase prices by 4%:  $1.04\mathbf{v}$

$$\begin{aligned}\text{New total amount: } 1.04(\mathbf{u} \cdot \mathbf{v}) &= 1.04(12,351.25) \\ &= \$12,845.30\end{aligned}$$

61. (a)–(c) Programs will vary.

62.  $\|\mathbf{u}\| \approx 9.165$

$$\|\mathbf{v}\| \approx 5.745$$

$$\theta = 90^\circ$$

63. Programs will vary.

64.  $\left\langle -\frac{21}{26}, \frac{63}{26}, \frac{42}{13} \right\rangle$

65. Because  $\mathbf{u}$  and  $\mathbf{v}$  are parallel,  $\text{proj}_{\mathbf{v}} \mathbf{u} = \mathbf{u}$

66. Because  $\mathbf{u}$  and  $\mathbf{v}$  are perpendicular,  $\text{proj}_{\mathbf{v}} \mathbf{u} = \mathbf{0}$

67. Answers will vary. *Sample answer:*

$$\mathbf{u} = -\frac{1}{4}\mathbf{i} + \frac{3}{2}\mathbf{j}. \text{ Want } \mathbf{u} \cdot \mathbf{v} = 0.$$

$\mathbf{v} = 12\mathbf{i} + 2\mathbf{j}$  and  $-\mathbf{v} = -12\mathbf{i} - 2\mathbf{j}$  are orthogonal to  $\mathbf{u}$ .

68. Answers will vary. *Sample answer:*

$$\mathbf{u} = 9\mathbf{i} - 4\mathbf{j}. \text{ Want } \mathbf{u} \cdot \mathbf{v} = 0.$$

$$\mathbf{v} = 4\mathbf{i} + 9\mathbf{j} \text{ and } -\mathbf{v} = -4\mathbf{i} - 9\mathbf{j}$$

are orthogonal to  $\mathbf{u}$ .

69. Answers will vary. *Sample answer:*

$$\mathbf{u} = \langle 3, 1, -2 \rangle. \text{ Want } \mathbf{u} \cdot \mathbf{v} = 0.$$

$$\mathbf{v} = \langle 0, 2, 1 \rangle \text{ and } -\mathbf{v} = \langle 0, -2, -1 \rangle \text{ are orthogonal to } \mathbf{u}.$$

70. Answers will vary. *Sample answer:*

$$\mathbf{u} = \langle 4, -3, 6 \rangle. \text{ Want } \mathbf{u} \cdot \mathbf{v} = 0$$

$$\mathbf{v} = \langle 0, 6, 3 \rangle \text{ and } -\mathbf{v} = \langle 0, -6, -3 \rangle$$

are orthogonal to  $\mathbf{u}$ .

71. (a) Gravitational Force  $\mathbf{F} = -48,000\mathbf{j}$

$$\mathbf{v} = \cos 10^\circ \mathbf{i} + \sin 10^\circ \mathbf{j}$$

$$\mathbf{w}_1 = \frac{\mathbf{F} \cdot \mathbf{v}}{\|\mathbf{v}\|^2} \mathbf{v} = (\mathbf{F} \cdot \mathbf{v})\mathbf{v}$$

$$= (-48,000)(\sin 10^\circ)\mathbf{v}$$

$$\approx -8335.1(\cos 10^\circ \mathbf{i} + \sin 10^\circ \mathbf{j})$$

$$\|\mathbf{w}_1\| \approx 8335.1 \text{ lb}$$

(b)  $\mathbf{w}_2 = \mathbf{F} - \mathbf{w}_1$

$$= -48,000\mathbf{j} + 8335.1(\cos 10^\circ \mathbf{i} + \sin 10^\circ \mathbf{j})$$

$$= 8208.5\mathbf{i} - 46,552.6\mathbf{j}$$

$$\|\mathbf{w}_2\| \approx 47,270.8 \text{ lb}$$

72.  $\overline{OA} = \langle 10, 5, 20 \rangle, \mathbf{v} = \langle 0, 0, 1 \rangle$

$$\text{proj}_{\mathbf{v}} \overline{OA} = \frac{20}{1^2} \langle 0, 0, 1 \rangle = \langle 0, 0, 20 \rangle$$

$$\|\text{proj}_{\mathbf{v}} \overline{OA}\| = 20$$

73.  $\mathbf{F} = 85 \left( \frac{1}{2}\mathbf{i} + \frac{\sqrt{3}}{2}\mathbf{j} \right)$

$$\mathbf{v} = 10\mathbf{i}$$

$$W = \mathbf{F} \cdot \mathbf{v} = 425 \text{ ft-lb}$$

74.  $\mathbf{F} = 25(\cos 20^\circ \mathbf{i} + \sin 20^\circ \mathbf{j})$

$$\mathbf{v} = 50\mathbf{i}$$

$$W = \mathbf{F} \cdot \mathbf{v} = 1250 \cos 20^\circ \approx 1174.6 \text{ ft-lb}$$

75.  $\mathbf{F} = 1600(\cos 25^\circ \mathbf{i} + \sin 25^\circ \mathbf{j})$

$$\mathbf{v} = 2000\mathbf{i}$$

$$W = \mathbf{F} \cdot \mathbf{v} = 1600(2000)\cos 25^\circ$$

$$\approx 2,900,184.9 \text{ Newton meters (Joules)}$$

$$\approx 2900.2 \text{ km-N}$$

76.  $\overline{PQ} = 40\mathbf{i}$

$$\mathbf{F} = 100 \cos 25^\circ \mathbf{i}$$

$$W = \mathbf{F} \cdot \overline{PQ} = 4000 \cos 25^\circ \approx 3625.2 \text{ Joules}$$

77. False.

For example, let  $\mathbf{u} = \langle 1, 1 \rangle, \mathbf{v} = \langle 2, 3 \rangle$  and

$$\mathbf{w} = \langle 1, 4 \rangle. \text{ Then } \mathbf{u} \cdot \mathbf{v} = 2 + 3 = 5 \text{ and}$$

$$\mathbf{u} \cdot \mathbf{w} = 1 + 4 = 5.$$

78. True

$$\mathbf{w} \cdot (\mathbf{u} + \mathbf{v}) = \mathbf{w} \cdot \mathbf{u} + \mathbf{w} \cdot \mathbf{v} = 0 + 0 = 0 \text{ so, } \mathbf{w} \text{ and } \mathbf{u} + \mathbf{v} \text{ are orthogonal.}$$

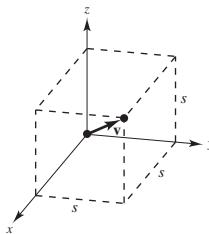
79. Let  $s$  = length of a side.

$$\mathbf{v} = \langle s, s, s \rangle$$

$$\|\mathbf{v}\| = s\sqrt{3}$$

$$\cos \alpha = \cos \beta = \cos \gamma = \frac{s}{s\sqrt{3}} = \frac{1}{\sqrt{3}}$$

$$\alpha = \beta = \gamma = \arccos\left(\frac{1}{\sqrt{3}}\right) \approx 54.7^\circ$$



80.  $\mathbf{v}_1 = \langle s, s, s \rangle$

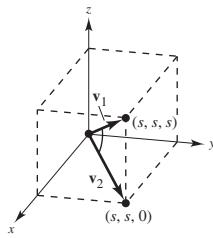
$$\|\mathbf{v}_1\| = s\sqrt{3}$$

$$\mathbf{v}_2 = \langle s, s, 0 \rangle$$

$$\|\mathbf{v}_2\| = s\sqrt{2}$$

$$\cos \theta = \frac{2\sqrt{2}}{2\sqrt{3}} = \frac{\sqrt{6}}{3}$$

$$\theta = \arccos \frac{\sqrt{6}}{3} \approx 35.26^\circ$$



81. (a) The graphs  $y_1 = x^2$  and  $y_2 = x^{1/3}$  intersect at  $(0, 0)$  and  $(1, 1)$ .

$$(b) y'_1 = 2x \text{ and } y'_2 = \frac{1}{3x^{2/3}}.$$

At  $(0, 0), \pm \langle 1, 0 \rangle$  is tangent to  $y_1$  and  $\pm \langle 0, 1 \rangle$  is tangent to  $y_2$ .

$$\text{At } (1, 1), y'_1 = 2 \text{ and } y'_2 = \frac{1}{3}.$$

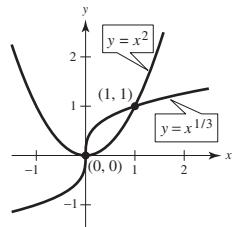
$\pm \frac{1}{\sqrt{5}}\langle 1, 2 \rangle$  is tangent to  $y_1$ ,  $\pm \frac{1}{\sqrt{10}}\langle 3, 1 \rangle$  is tangent to  $y_2$ .

- (c) At  $(0, 0)$ , the vectors are perpendicular ( $90^\circ$ ).

At  $(1, 1)$ ,

$$\cos \theta = \frac{\frac{1}{\sqrt{5}}\langle 1, 2 \rangle \cdot \frac{1}{\sqrt{10}}\langle 3, 1 \rangle}{(1)(1)} = \frac{5}{\sqrt{50}} = \frac{1}{\sqrt{2}}$$

$$\theta = 45^\circ$$



82. (a) The graphs  $y_1 = x^3$  and  $y_2 = x^{1/3}$  intersect at  $(-1, -1)$ ,  $(0, 0)$  and  $(1, 1)$ .

$$(b) y'_1 = 3x^2 \text{ and } y'_2 = \frac{1}{3x^{2/3}}.$$

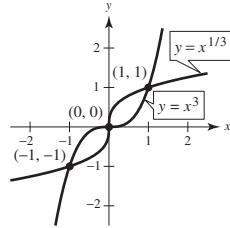
At  $(0, 0)$ ,  $\pm \langle 1, 0 \rangle$  is tangent to  $y_1$  and  $\pm \langle 0, 1 \rangle$  is tangent to  $y_2$ .

$$\text{At } (1, 1), y'_1 = 3 \text{ and } y'_2 = \frac{1}{3}.$$

$\pm \frac{1}{\sqrt{10}}\langle 1, 3 \rangle$  is tangent to  $y_1$ ,  $\pm \frac{1}{\sqrt{10}}\langle 3, 1 \rangle$  is tangent to  $y_2$ .

$$\text{At } (-1, -1), y'_1 = 3 \text{ and } y'_2 = \frac{1}{3}.$$

$\pm \frac{1}{\sqrt{10}}\langle 1, 3 \rangle$  is tangent to  $y_1$ ,  $\pm \frac{1}{\sqrt{10}}\langle 3, 1 \rangle$  is tangent to  $y_2$ .



- (c) At  $(0, 0)$ , the vectors are perpendicular ( $90^\circ$ ).

At  $(1, 1)$ ,

$$\cos \theta = \frac{\frac{1}{\sqrt{10}}\langle 1, 3 \rangle \cdot \frac{1}{\sqrt{10}}\langle 3, 1 \rangle}{(1)(1)} = \frac{6}{10} = \frac{3}{5}.$$

$$\theta \approx 0.9273 \text{ or } 53.13^\circ$$

By symmetry, the angle is the same at  $(-1, -1)$ .

83. (a) The graphs of  $y_1 = 1 - x^2$  and  $y_2 = x^2 - 1$  intersect at  $(1, 0)$  and  $(-1, 0)$ .

$$(b) y'_1 = -2x \text{ and } y'_2 = 2x.$$

$$\text{At } (1, 0), y'_1 = -2 \text{ and } y'_2 = 2. \pm \frac{1}{\sqrt{5}}\langle 1, -2 \rangle$$

is tangent to  $y_1$ ,  $\pm \frac{1}{\sqrt{5}}\langle 1, 2 \rangle$  is tangent to  $y_2$ .

At  $(-1, 0), y'_1 = 2$  and  $y'_2 = -2. \pm \frac{1}{\sqrt{5}}\langle 1, 2 \rangle$  is tangent to  $y_1$ ,  $\pm \frac{1}{\sqrt{5}}\langle 1, -2 \rangle$  is tangent to  $y_2$ .

$$(c) \text{At } (1, 0), \cos \theta = \frac{1}{\sqrt{5}}\langle 1, -2 \rangle \cdot \frac{-1}{\sqrt{5}}\langle 1, -2 \rangle = \frac{3}{5}.$$

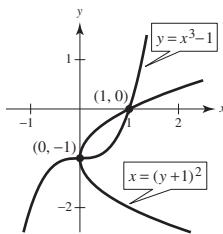
$$\theta \approx 0.9273 \text{ or } 53.13^\circ$$

By symmetry, the angle is the same at  $(-1, 0)$ .

84. (a) To find the intersection points, rewrite the second equation as  $y + 1 = x^3$ . Substituting into the first equation

$$(y + 1)^2 = x \Rightarrow x^6 = x \Rightarrow x = 0, 1.$$

There are two points of intersection,  $(0, -1)$  and  $(1, 0)$ , as indicated in the figure.



- (b) First equation:

$$(y + 1)^2 = x \Rightarrow 2(y + 1)y' = 1 \Rightarrow y' = \frac{1}{2(y + 1)}$$

At  $(1, 0)$ ,  $y' = \frac{1}{2}$ .

Second equation:  $y = x^3 - 1 \Rightarrow y' = 3x^2$ . At  $(1, 0)$ ,  $y' = 3$ .

$\pm \frac{1}{\sqrt{5}} \langle 2, 1 \rangle$  unit tangent vectors to first curve,

$\pm \frac{1}{\sqrt{10}} \langle 1, 3 \rangle$  unit tangent vectors to second curve

At  $(0, 1)$ , the unit tangent vectors to the first curve are  $\pm \langle 0, 1 \rangle$ , and the unit tangent vectors to the second curve are  $\pm \langle 1, 0 \rangle$ .

- (c) At  $(1, 0)$ ,

$$\cos \theta = \frac{1}{\sqrt{5}} \langle 2, 1 \rangle \cdot \frac{1}{\sqrt{10}} \langle 1, 3 \rangle = \frac{5}{\sqrt{50}} = \frac{1}{\sqrt{2}}.$$

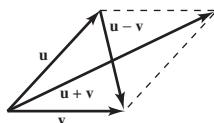
$$\theta \approx \frac{\pi}{4} \text{ or } 45^\circ$$

At  $(0, -1)$  the vectors are perpendicular,  $\theta = 90^\circ$ .

85. In a rhombus,  $\|\mathbf{u}\| = \|\mathbf{v}\|$ . The diagonals are  $\mathbf{u} + \mathbf{v}$  and  $\mathbf{u} - \mathbf{v}$ .

$$\begin{aligned} (\mathbf{u} + \mathbf{v}) \cdot (\mathbf{u} - \mathbf{v}) &= (\mathbf{u} + \mathbf{v}) \cdot \mathbf{u} - (\mathbf{u} + \mathbf{v}) \cdot \mathbf{v} \\ &= \mathbf{u} \cdot \mathbf{u} + \mathbf{v} \cdot \mathbf{u} - \mathbf{u} \cdot \mathbf{v} - \mathbf{v} \cdot \mathbf{v} \\ &= \|\mathbf{u}\|^2 - \|\mathbf{v}\|^2 = 0 \end{aligned}$$

So, the diagonals are orthogonal.



86. If  $\mathbf{u}$  and  $\mathbf{v}$  are the sides of the parallelogram, then the diagonals are  $\mathbf{u} + \mathbf{v}$  and  $\mathbf{u} - \mathbf{v}$ , as indicated in the figure.

the parallelogram is a rectangle.

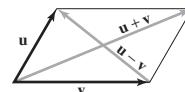
$$\Leftrightarrow \mathbf{u} \cdot \mathbf{v} = 0$$

$$\Leftrightarrow 2\mathbf{u} \cdot \mathbf{v} = -2\mathbf{u} \cdot \mathbf{v}$$

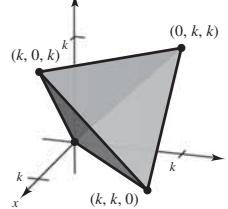
$$\Leftrightarrow (\mathbf{u} + \mathbf{v}) \cdot (\mathbf{u} + \mathbf{v}) = (\mathbf{u} - \mathbf{v}) \cdot (\mathbf{u} - \mathbf{v})$$

$$\Leftrightarrow \|\mathbf{u} + \mathbf{v}\|^2 = \|\mathbf{u} - \mathbf{v}\|^2$$

$\Leftrightarrow$  The diagonals are equal in length.



87. (a)



$$(b) \text{ Length of each edge: } \sqrt{k^2 + k^2 + 0^2} = k\sqrt{2}$$

$$(c) \cos \theta = \frac{k^2}{(k\sqrt{2})(k\sqrt{2})} = \frac{1}{2}$$

$$\theta = \arccos\left(\frac{1}{2}\right) = 60^\circ$$

$$(d) \vec{r}_1 = \langle k, k, 0 \rangle - \left\langle \frac{k}{2}, \frac{k}{2}, \frac{k}{2} \right\rangle = \left\langle \frac{k}{2}, \frac{k}{2}, -\frac{k}{2} \right\rangle$$

$$\vec{r}_2 = \langle 0, 0, 0 \rangle - \left\langle \frac{k}{2}, \frac{k}{2}, \frac{k}{2} \right\rangle = \left\langle -\frac{k}{2}, -\frac{k}{2}, -\frac{k}{2} \right\rangle$$

$$\cos \theta = \frac{-\frac{k^2}{4}}{\left(\frac{k}{2}\right)^2 \cdot 3} = -\frac{1}{3}$$

$$\theta = 109.5^\circ$$

88.  $\mathbf{u} = \langle \cos \alpha, \sin \alpha, 0 \rangle, \mathbf{v} = \langle \cos \beta, \sin \beta, 0 \rangle$

The angle between  $\mathbf{u}$  and  $\mathbf{v}$  is  $\alpha - \beta$ . (Assuming that  $\alpha > \beta$ ). Also,

$$\begin{aligned} \cos(\alpha - \beta) &= \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \|\mathbf{v}\|} \\ &= \frac{\cos \alpha \cos \beta + \sin \alpha \sin \beta}{(1)(1)} \\ &= \cos \alpha \cos \beta + \sin \alpha \sin \beta. \end{aligned}$$

$$\begin{aligned}
 89. \|\mathbf{u} - \mathbf{v}\|^2 &= (\mathbf{u} - \mathbf{v}) \cdot (\mathbf{u} - \mathbf{v}) \\
 &= (\mathbf{u} - \mathbf{v}) \cdot \mathbf{u} - (\mathbf{u} - \mathbf{v}) \cdot \mathbf{v} \\
 &= \mathbf{u} \cdot \mathbf{u} - \mathbf{v} \cdot \mathbf{u} - \mathbf{u} \cdot \mathbf{v} + \mathbf{v} \cdot \mathbf{v} \\
 &= \|\mathbf{u}\|^2 - \mathbf{u} \cdot \mathbf{v} - \mathbf{u} \cdot \mathbf{v} + \|\mathbf{v}\|^2 \\
 &= \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - 2\mathbf{u} \cdot \mathbf{v}
 \end{aligned}$$

$$\begin{aligned}
 90. \mathbf{u} \cdot \mathbf{v} &= \|\mathbf{u}\| \|\mathbf{v}\| \cos \theta \\
 |\mathbf{u} \cdot \mathbf{v}| &= |\|\mathbf{u}\| \|\mathbf{v}\| \cos \theta| \\
 &= \|\mathbf{u}\| \|\mathbf{v}\| |\cos \theta| \\
 &\leq \|\mathbf{u}\| \|\mathbf{v}\| \text{ because } |\cos \theta| \leq 1.
 \end{aligned}$$

$$\begin{aligned}
 91. \|\mathbf{u} + \mathbf{v}\|^2 &= (\mathbf{u} + \mathbf{v}) \cdot (\mathbf{u} + \mathbf{v}) \\
 &= (\mathbf{u} + \mathbf{v}) \cdot \mathbf{u} + (\mathbf{u} + \mathbf{v}) \cdot \mathbf{v} \\
 &= \mathbf{u} \cdot \mathbf{u} + \mathbf{v} \cdot \mathbf{u} + \mathbf{u} \cdot \mathbf{v} + \mathbf{v} \cdot \mathbf{v} \\
 &= \|\mathbf{u}\|^2 + 2\mathbf{u} \cdot \mathbf{v} + \|\mathbf{v}\|^2 \\
 &\leq \|\mathbf{u}\|^2 + 2\|\mathbf{u}\|\|\mathbf{v}\| + \|\mathbf{v}\|^2 \leq (\|\mathbf{u}\| + \|\mathbf{v}\|)^2
 \end{aligned}$$

So,  $\|\mathbf{u} + \mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|$ .

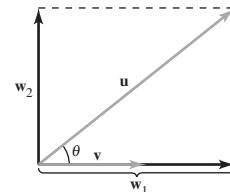
92. Let  $\mathbf{w}_1 = \text{proj}_{\mathbf{v}} \mathbf{u}$ , as indicated in the figure. Because  $\mathbf{w}_1$  is a scalar multiple of  $\mathbf{v}$ , you can write  $\mathbf{u} = \mathbf{w}_1 + \mathbf{w}_2 = c\mathbf{v} + \mathbf{w}_2$ .

Taking the dot product of both sides with  $\mathbf{v}$  produces

$$\begin{aligned}
 \mathbf{u} \cdot \mathbf{v} &= (c\mathbf{v} + \mathbf{w}_2) \cdot \mathbf{v} = c\mathbf{v} \cdot \mathbf{v} + \mathbf{w}_2 \cdot \mathbf{v} \\
 &= c\|\mathbf{v}\|^2, \text{ because } \mathbf{w}_2 \text{ and } \mathbf{v} \text{ are orthogonal.}
 \end{aligned}$$

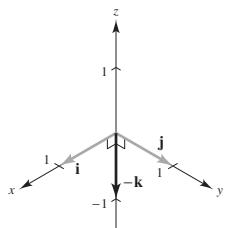
So,  $\mathbf{u} \cdot \mathbf{v} = c\|\mathbf{v}\|^2 \Rightarrow c = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2}$  and

$$\mathbf{w}_1 = \text{proj}_{\mathbf{v}} \mathbf{u} = c\mathbf{v} = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2} \mathbf{v}.$$

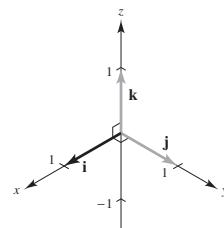


## Section 11.4 The Cross Product of Two Vectors in Space

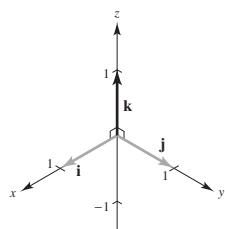
$$1. \mathbf{j} \times \mathbf{i} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{vmatrix} = -\mathbf{k}$$



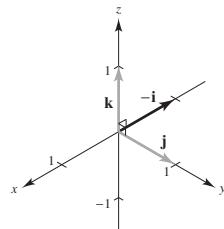
$$3. \mathbf{j} \times \mathbf{k} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = \mathbf{i}$$



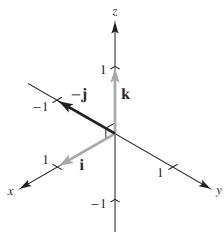
$$2. \mathbf{i} \times \mathbf{j} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = \mathbf{k}$$



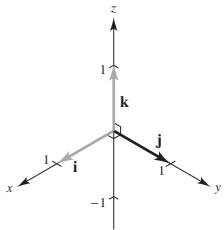
$$4. \mathbf{k} \times \mathbf{j} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{vmatrix} = -\mathbf{i}$$



5.  $\mathbf{i} \times \mathbf{k} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{vmatrix} = -\mathbf{j}$



6.  $\mathbf{k} \times \mathbf{i} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{vmatrix} = \mathbf{j}$



7. (a)  $\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & 4 & 0 \\ 3 & 2 & 5 \end{vmatrix} = 20\mathbf{i} + 10\mathbf{j} - 16\mathbf{k}$

(b)  $\mathbf{v} \times \mathbf{u} = -(\mathbf{u} \times \mathbf{v}) = -20\mathbf{i} - 10\mathbf{j} + 16\mathbf{k}$

(c)  $\mathbf{v} \times \mathbf{v} = \mathbf{0}$

8. (a)  $\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 0 & 5 \\ 2 & 3 & -2 \end{vmatrix} = -15\mathbf{i} + 16\mathbf{j} + 9\mathbf{k}$

(b)  $\mathbf{v} \times \mathbf{u} = -(\mathbf{u} \times \mathbf{v}) = 15\mathbf{i} - 16\mathbf{j} - 9\mathbf{k}$

(c)  $\mathbf{v} \times \mathbf{v} = \mathbf{0}$

9. (a)  $\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 7 & 3 & 2 \\ 1 & -1 & 5 \end{vmatrix} = 17\mathbf{i} - 33\mathbf{j} - 10\mathbf{k}$

(b)  $\mathbf{v} \times \mathbf{u} = -(\mathbf{u} \times \mathbf{v}) = -17\mathbf{i} + 33\mathbf{j} + 10\mathbf{k}$

(c)  $\mathbf{v} \times \mathbf{v} = \mathbf{0}$

10. (a)  $\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -2 & -2 \\ 1 & 5 & 1 \end{vmatrix} = 8\mathbf{i} - 5\mathbf{j} + 17\mathbf{k}$

(b)  $\mathbf{v} \times \mathbf{u} = -(\mathbf{u} \times \mathbf{v}) = -8\mathbf{i} + 5\mathbf{j} + 17\mathbf{k}$

(c)  $\mathbf{v} \times \mathbf{v} = \mathbf{0}$

11.  $\mathbf{u} = \langle 12, -3, 0 \rangle, \mathbf{v} = \langle -2, 5, 0 \rangle$

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 12 & -3 & 0 \\ -2 & 5 & 0 \end{vmatrix} = 54\mathbf{k} = \langle 0, 0, 54 \rangle$$

$$\mathbf{u} \cdot (\mathbf{u} \times \mathbf{v}) = 12(0) + (-3)(0) + 0(54) = 0 \Rightarrow \mathbf{u} \perp \mathbf{u} \times \mathbf{v}$$

$$\mathbf{v} \cdot (\mathbf{u} \times \mathbf{v}) = -2(0) + 5(0) + 0(54) = 0 \Rightarrow \mathbf{v} \perp \mathbf{u} \times \mathbf{v}$$

12.  $\mathbf{u} = \langle -1, 1, 2 \rangle, \mathbf{v} = \langle 0, 1, 0 \rangle$

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 1 & 2 \\ 0 & 1 & 0 \end{vmatrix} = -2\mathbf{i} - \mathbf{k} = \langle -2, 0, -1 \rangle$$

$$\mathbf{u} \cdot (\mathbf{u} \times \mathbf{v}) = (-1)(-2) + (1)(0) + (2)(-1) = 0 \Rightarrow \mathbf{u} \perp \mathbf{u} \times \mathbf{v}$$

$$\mathbf{v} \cdot (\mathbf{u} \times \mathbf{v}) = (0)(-2) + (1)(0) + (0)(-1) = 0 \Rightarrow \mathbf{v} \perp \mathbf{u} \times \mathbf{v}$$

13.  $\mathbf{u} = \langle 2, -3, 1 \rangle, \mathbf{v} = \langle 1, -2, 1 \rangle$

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -3 & 1 \\ 1 & -2 & 1 \end{vmatrix} = -\mathbf{i} - \mathbf{j} - \mathbf{k} = \langle -1, -1, -1 \rangle$$

$$\mathbf{u} \cdot (\mathbf{u} \times \mathbf{v}) = 2(-1) + (-3)(-1) + (1)(-1) = 0 \Rightarrow \mathbf{u} \perp \mathbf{u} \times \mathbf{v}$$

$$\mathbf{v} \cdot (\mathbf{u} \times \mathbf{v}) = 1(-1) + (-2)(-1) + (1)(-1) = 0 \Rightarrow \mathbf{v} \perp \mathbf{u} \times \mathbf{v}$$

14.  $\mathbf{u} = \langle -10, 0, 6 \rangle, \mathbf{v} = \langle 5, -3, 0 \rangle$

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -10 & 0 & 6 \\ 5 & -3 & 0 \end{vmatrix} = 18\mathbf{i} + 30\mathbf{j} + 30\mathbf{k} = \langle 18, 30, 30 \rangle$$

$$\mathbf{u} \cdot (\mathbf{u} \times \mathbf{v}) = -10(18) + 0(30) + 6(30) = 0$$

$$\Rightarrow \mathbf{u} \perp \mathbf{u} \times \mathbf{v}$$

$$\mathbf{v} \cdot (\mathbf{u} \times \mathbf{v}) = 5(18) - 3(30) + 0(30) = 0$$

$$\Rightarrow \mathbf{v} \perp \mathbf{u} \times \mathbf{v}$$

15.  $\mathbf{u} = \mathbf{i} + \mathbf{j} + \mathbf{k}, \mathbf{v} = 2\mathbf{i} + \mathbf{j} - \mathbf{k}$

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 1 \\ 2 & 1 & -1 \end{vmatrix} = -2\mathbf{i} + 3\mathbf{j} - \mathbf{k} = \langle -2, 3, -1 \rangle$$

$$\mathbf{u} \cdot (\mathbf{u} \times \mathbf{v}) = 1(-2) + 1(3) + 1(-1) = 0 \Rightarrow \mathbf{u} \perp \mathbf{u} \times \mathbf{v}$$

$$\mathbf{v} \cdot (\mathbf{u} \times \mathbf{v}) = 2(-2) + 1(3) + (-1)(-1) = 0 \Rightarrow \mathbf{v} \perp \mathbf{u} \times \mathbf{v}$$

$$(-\mathbf{v}) \times \mathbf{u} = -(\mathbf{v} \times \mathbf{u}) = \mathbf{u} \times \mathbf{v}$$

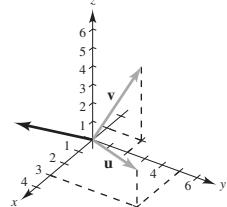
16.  $\mathbf{u} = \mathbf{i} + 6\mathbf{j}$ ,  $\mathbf{v} = -2\mathbf{i} + \mathbf{j} + \mathbf{k}$

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 6 & 0 \\ -2 & 1 & 1 \end{vmatrix} = 6\mathbf{i} - \mathbf{j} + 13\mathbf{k}$$

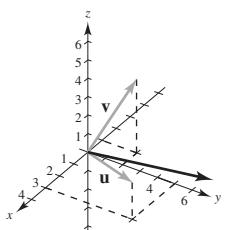
$$\mathbf{u} \cdot (\mathbf{u} \times \mathbf{v}) = 1(6) + 6(-1) = 0 \Rightarrow \mathbf{u} \perp (\mathbf{u} \times \mathbf{v})$$

$$\mathbf{v} \cdot (\mathbf{u} \times \mathbf{v}) = -2(6) + 1(-1) + 1(13) = 0 \Rightarrow \mathbf{v} \perp (\mathbf{u} \times \mathbf{v})$$

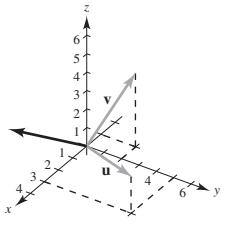
17.



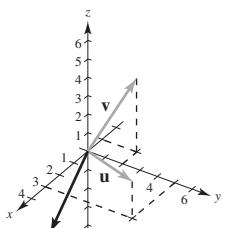
18.



19.



20.



21.  $\mathbf{u} = \langle 4, -3.5, 7 \rangle$ ,  $\mathbf{v} = \langle 2.5, 9, 3 \rangle$

$$\mathbf{u} \times \mathbf{v} = \langle -73.5, 5.5, 44.75 \rangle$$

$$\frac{\mathbf{u} \times \mathbf{v}}{\|\mathbf{u} \times \mathbf{v}\|} = \left\langle -\frac{2.94}{\sqrt{11.8961}}, \frac{0.22}{\sqrt{11.8961}}, \frac{1.79}{\sqrt{11.8961}} \right\rangle$$

22.  $\mathbf{u} = \langle -8, -6, 4 \rangle$

$$\mathbf{v} = \langle 10, -12, -2 \rangle$$

$$\mathbf{u} \times \mathbf{v} = \langle 60, 24, 156 \rangle$$

$$\frac{\mathbf{u} \times \mathbf{v}}{\|\mathbf{u} \times \mathbf{v}\|} = \frac{1}{36\sqrt{22}} \langle 60, 24, 156 \rangle$$

$$= \left\langle \frac{5}{3\sqrt{22}}, \frac{2}{3\sqrt{22}}, \frac{13}{3\sqrt{22}} \right\rangle$$

23.  $\mathbf{u} = \langle -3, 2, -5 \rangle$ ,  $\mathbf{v} = \langle 0.4, -0.8, 0.2 \rangle$

$$\mathbf{u} \times \mathbf{v} = \langle -3.6, -1.4, 1.6 \rangle$$

$$\frac{\mathbf{u} \times \mathbf{v}}{\|\mathbf{u} \times \mathbf{v}\|} = \left\langle -\frac{1.8}{\sqrt{4.37}}, -\frac{0.7}{\sqrt{4.37}}, \frac{0.8}{\sqrt{4.37}} \right\rangle$$

24.  $\mathbf{u} = \left\langle 0, 0, \frac{7}{10} \right\rangle$ ,  $\mathbf{v} = \left\langle \frac{3}{2}, 0, \frac{31}{5} \right\rangle$

$$\mathbf{u} \times \mathbf{v} = \left\langle 0, \frac{21}{20}, 0 \right\rangle$$

$$\frac{\mathbf{u} \times \mathbf{v}}{\|\mathbf{u} \times \mathbf{v}\|} = \langle 0, 1, 0 \rangle$$

25. Programs will vary.

26.  $\mathbf{u} \times \mathbf{v} = \langle -50, 40, -34 \rangle$

$$\|\mathbf{u} \times \mathbf{v}\| \approx 72.498$$

27.  $\mathbf{u} = \mathbf{j}$

$$\mathbf{v} = \mathbf{j} + \mathbf{k}$$

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{vmatrix} = \mathbf{i}$$

$$A = \|\mathbf{u} \times \mathbf{v}\| = \|\mathbf{i}\| = 1$$

28.  $\mathbf{u} = \mathbf{i} + \mathbf{j} + \mathbf{k}$

$$\mathbf{v} = \mathbf{j} + \mathbf{k}$$

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{vmatrix} = -\mathbf{j} + \mathbf{k}$$

$$A = \|\mathbf{u} \times \mathbf{v}\| = \|-\mathbf{j} + \mathbf{k}\| = \sqrt{2}$$

29.  $\mathbf{u} = \langle 3, 2, -1 \rangle$

$$\mathbf{v} = \langle 1, 2, 3 \rangle$$

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 2 & -1 \\ 1 & 2 & 3 \end{vmatrix} = \langle 8, -10, 4 \rangle$$

$$A = \|\mathbf{u} \times \mathbf{v}\| = \|\langle 8, -10, 4 \rangle\| = \sqrt{180} = 6\sqrt{5}$$

30.  $\mathbf{u} = \langle 2, -1, 0 \rangle$

$\mathbf{v} = \langle -1, 2, 0 \rangle$

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & 0 \\ -1 & 2 & 0 \end{vmatrix} = \langle 0, 0, 3 \rangle$$

$A = \|\mathbf{u} \times \mathbf{v}\| = \|\langle 0, 0, 3 \rangle\| = 3$

31.  $A(0, 3, 2), B(1, 5, 5), C(6, 9, 5), D(5, 7, 2)$

$\overline{AB} = \langle 1, 2, 3 \rangle$

$\overline{DC} = \langle 1, 2, 3 \rangle$

$\overline{BC} = \langle 5, 4, 0 \rangle$

$\overline{AD} = \langle 5, 4, 0 \rangle$

Because  $\overline{AB} = \overline{DC}$  and  $\overline{BC} = \overline{AD}$ , the figure  $ABCD$  is a parallelogram.

$\overline{AB}$  and  $\overline{AD}$  are adjacent sides

$$\overline{AB} \times \overline{AD} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 3 \\ 5 & 4 & 0 \end{vmatrix} = \langle -12, 15, -6 \rangle$$

$A = \|\overline{AB} \times \overline{AD}\| = \sqrt{144 + 225 + 36} = 9\sqrt{5}$

32.  $A(2, -3, 1), B(6, 5, -1), C(7, 2, 2), D(3, -6, 4)$

$\overline{AB} = \langle 4, 8, -2 \rangle$

$\overline{DC} = \langle 4, 8, -2 \rangle$

$\overline{BC} = \langle 1, -3, 3 \rangle$

$\overline{AD} = \langle 1, -3, 3 \rangle$

Because  $\overline{AB} = \overline{DC}$  and  $\overline{BC} = \overline{AD}$ , the figure  $ABCD$  is a parallelogram.

$\overline{AB}$  and  $\overline{AD}$  are adjacent sides

$$\overline{AB} \times \overline{AD} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & 8 & -2 \\ 1 & -3 & 3 \end{vmatrix} = \langle 18, -14, -20 \rangle$$

$A = \|\overline{AB} \times \overline{AD}\| = \sqrt{324 + 196 + 400} = 2\sqrt{230}$

33.  $A(0, 0, 0), B(1, 0, 3), C(-3, 2, 0)$

$\overline{AB} = \langle 1, 0, 3 \rangle, \overline{AC} = \langle -3, 2, 0 \rangle$

$$\overline{AB} \times \overline{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 3 \\ -3 & 2 & 0 \end{vmatrix} = \langle -6, -9, 2 \rangle$$

$A = \frac{1}{2} \|\overline{AB} \times \overline{AC}\| = \frac{1}{2} \sqrt{36 + 81 + 4} = \frac{11}{2}$

34.  $A(2, -3, 4), B(0, 1, 2), C(-1, 2, 0)$

$\overline{AB} = \langle -2, 4, -2 \rangle, \overline{AC} = \langle -3, 5, -4 \rangle$

$$\overline{AB} \times \overline{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & 4 & -2 \\ -3 & 5 & -4 \end{vmatrix} = -6\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}$$

$A = \frac{1}{2} \|\overline{AB} \times \overline{AC}\| = \frac{1}{2} \sqrt{44} = \sqrt{11}$

35.  $A(2, -7, 3), B(-1, 5, 8), C(4, 6, -1)$

$\overline{AB} = \langle -3, 12, 5 \rangle, \overline{AC} = \langle 2, 13, -4 \rangle$

$$\overline{AB} \times \overline{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -3 & 12 & 5 \\ 2 & 13 & -4 \end{vmatrix} = \langle -113, -2, -63 \rangle$$

$A = \frac{1}{2} \|\overline{AB} \times \overline{AC}\| = \frac{1}{2} \sqrt{16,742}$

36.  $A(1, 2, 0), B(-2, 1, 0), C(0, 0, 0)$

$\overline{AB} = \langle -3, -1, 0 \rangle, \overline{AC} = \langle -1, -2, 0 \rangle$

$$\overline{AB} \times \overline{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -3 & -1 & 0 \\ -1 & -2 & 0 \end{vmatrix} = 5\mathbf{k}$$

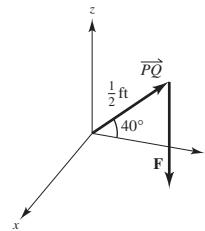
$A = \frac{1}{2} \|\overline{AB} \times \overline{AC}\| = \frac{5}{2}$

37.  $\mathbf{F} = -20\mathbf{k}$

$\overline{PQ} = \frac{1}{2}(\cos 40^\circ \mathbf{j} + \sin 40^\circ \mathbf{k})$

$$\overline{PQ} \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & \cos 40^\circ/2 & \sin 40^\circ/2 \\ 0 & 0 & -20 \end{vmatrix} = -10 \cos 40^\circ \mathbf{i}$$

$\|\overline{PQ} \times \mathbf{F}\| = 10 \cos 40^\circ \approx 7.66 \text{ ft-lb}$

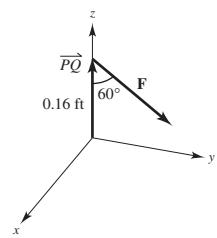


38.  $\mathbf{F} = -2000(\cos 30^\circ \mathbf{j} + \sin 30^\circ \mathbf{k}) = -1000\sqrt{3}\mathbf{j} - 1000\mathbf{k}$

$\overline{PQ} = 0.16\mathbf{k}$

$$\overline{PQ} \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 0.16 \\ 0 & -1000\sqrt{3} & -1000 \end{vmatrix} = 160\sqrt{3}\mathbf{i}$$

$\|\overline{PQ} \times \mathbf{F}\| = 160\sqrt{3} \text{ ft-lb}$



39. (a) Place the wrench in the  $xy$ -plane, as indicated in the figure.

The angle from  $\overline{AB}$  to  $\mathbf{F}$  is  $30^\circ + 180^\circ + \theta = 210^\circ + \theta$

$$\|\overline{OA}\| = 18 \text{ inches} = 1.5 \text{ feet}$$

$$\overline{OA} = 1.5[\cos(30^\circ)\mathbf{i} + \sin(30^\circ)\mathbf{j}] = \frac{3\sqrt{3}}{4}\mathbf{i} + \frac{3}{4}\mathbf{j}$$

$$\mathbf{F} = 56[\cos(210^\circ + \theta)\mathbf{i} + \sin(210^\circ + \theta)\mathbf{j}]$$

$$\overline{OA} \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{3\sqrt{3}}{4} & \frac{3}{4} & 0 \\ 56 \cos(210^\circ + \theta) & 56 \sin(210^\circ + \theta) & 0 \end{vmatrix}$$

$$= [42\sqrt{3} \sin(210^\circ + \theta) - 42 \cos(210^\circ + \theta)]\mathbf{k}$$

$$= [42\sqrt{3}(\sin 210^\circ \cos \theta + \cos 210^\circ \sin \theta) - 42(\cos 210^\circ \cos \theta - \sin 210^\circ \sin \theta)]\mathbf{k}$$

$$= [42\sqrt{3}\left(-\frac{1}{2} \cos \theta - \frac{\sqrt{3}}{2} \sin \theta\right) - 42\left(-\frac{\sqrt{3}}{2} \cos \theta + \frac{1}{2} \sin \theta\right)]\mathbf{k} = (-84 \sin \theta)\mathbf{k}$$

$$\|\overline{OA} \times \mathbf{F}\| = 84 \sin \theta, 0 \leq \theta \leq 180^\circ$$

$$(b) \text{ When } \theta = 45^\circ, \|\overline{OA} \times \mathbf{F}\| = 84 \frac{\sqrt{2}}{2} = 42\sqrt{2} \approx 59.40$$

$$(c) \text{ Let } T = 84 \sin \theta$$

$$\frac{dT}{d\theta} = 84 \cos \theta = 0 \text{ when } \theta = 90^\circ.$$

This is reasonable. When  $\theta = 90^\circ$ , the force is perpendicular to the wrench.

$$40. (a) AC = 15 \text{ inches} = \frac{5}{4} \text{ feet}$$

$$BC = 12 \text{ inches} = 1 \text{ foot}$$

$$\overline{AB} = -\frac{5}{4}\mathbf{j} + \mathbf{k}$$

$$\mathbf{F} = -180(\cos \theta \mathbf{j} + \sin \theta \mathbf{k})$$

$$(b) \overline{AB} \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & -\frac{5}{4} & 1 \\ 0 & -180 \cos \theta & -180 \sin \theta \end{vmatrix}$$

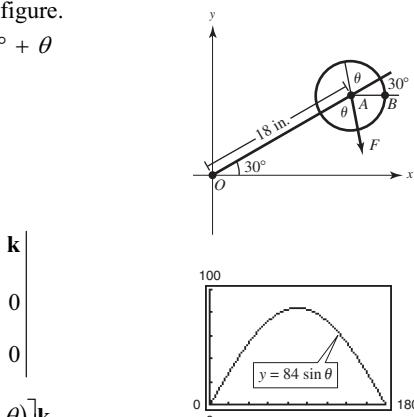
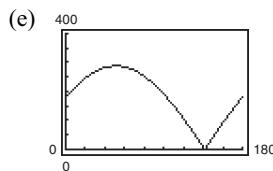
$$= (225 \sin \theta + 180 \cos \theta)\mathbf{i}$$

$$\|\overline{AB} \times \mathbf{F}\| = |225 \sin \theta + 180 \cos \theta|$$

$$(c) \text{ When } \theta = 30^\circ, \|\overline{AB} \times \mathbf{F}\| = 225\left(\frac{1}{2}\right) + 180\left(\frac{\sqrt{3}}{2}\right) \approx 268.38$$

$$(d) \text{ If } T = |225 \sin \theta + 180 \cos \theta|, T = 0 \text{ for } 225 \sin \theta = -180 \cos \theta \Rightarrow \tan \theta = -\frac{4}{5} \Rightarrow \theta \approx 141.34^\circ.$$

For  $0 < \theta < 141.34$ ,  $T'(\theta) = 225 \cos \theta - 180 \sin \theta = 0 \Rightarrow \tan \theta = \frac{5}{4} \Rightarrow \theta \approx 51.34^\circ$ .  $\overline{AB}$  and  $\mathbf{F}$  are perpendicular.



From part (d), the zero is  $\theta \approx 141.34^\circ$ , when the vectors are parallel.

$$41. \mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1$$

$$42. \mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = -1$$

$$43. \mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \begin{vmatrix} 2 & 0 & 1 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 6$$

$$44. \mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \begin{vmatrix} 2 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 2 & 2 \end{vmatrix} = 0$$

$$45. \mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{vmatrix} = 2$$

$$V = |\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})| = 2$$

$$46. \mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \begin{vmatrix} 1 & 3 & 1 \\ 0 & 6 & 6 \\ -4 & 0 & -4 \end{vmatrix} = -72$$

$$V = |\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})| = 72$$

$$47. \mathbf{u} = \langle 3, 0, 0 \rangle$$

$$\mathbf{v} = \langle 0, 5, 1 \rangle$$

$$\mathbf{w} = \langle 2, 0, 5 \rangle$$

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \begin{vmatrix} 3 & 0 & 0 \\ 0 & 5 & 1 \\ 2 & 0 & 5 \end{vmatrix} = 75$$

$$V = |\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})| = 75$$

$$48. \mathbf{u} = \langle 0, 4, 0 \rangle$$

$$\mathbf{v} = \langle -3, 0, 0 \rangle$$

$$\mathbf{w} = \langle -1, 1, 5 \rangle$$

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \begin{vmatrix} 0 & 4 & 0 \\ -3 & 0 & 0 \\ -1 & 1 & 5 \end{vmatrix} = -4(-15) = 60$$

$$V = |\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})| = 60$$

49.  $\mathbf{u} \times \mathbf{v} = \mathbf{0} \Rightarrow \mathbf{u}$  and  $\mathbf{v}$  are parallel.

$\mathbf{u} \cdot \mathbf{v} = \mathbf{0} \Rightarrow \mathbf{u}$  and  $\mathbf{v}$  are orthogonal.

So,  $\mathbf{u}$  or  $\mathbf{v}$  (or both) is the zero vector.

$$50. (a) \mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = (\mathbf{v} \times \mathbf{w}) \cdot \mathbf{u} \quad (b)$$

$$= \mathbf{w} \cdot (\mathbf{u} \times \mathbf{v}) = (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} \quad (c)$$

$$= \mathbf{v} \cdot (\mathbf{w} \times \mathbf{u}) = (\mathbf{u} \times \mathbf{w}) \cdot \mathbf{v} \quad (d)$$

$$= \mathbf{v} \cdot (\mathbf{w} \times \mathbf{u}) = (\mathbf{w} \times \mathbf{u}) \cdot \mathbf{v} \quad (h)$$

$$(e) \mathbf{u} \cdot (\mathbf{w} \times \mathbf{v}) = \mathbf{w} \cdot (\mathbf{v} \times \mathbf{u}) \quad (f)$$

$$= \mathbf{w} \cdot (\mathbf{v} \times \mathbf{u}) = (-\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} \quad (g)$$

So,  $a = b = c = d = h$  and  $e = f = g$

$$51. \mathbf{u} \times \mathbf{v} = \langle u_1, u_2, u_3 \rangle \cdot \langle v_1, v_2, v_3 \rangle$$

$$= (u_2 v_3 - u_3 v_2) \mathbf{i} - (u_1 v_3 - u_3 v_1) \mathbf{j} + (u_1 v_2 - u_2 v_1) \mathbf{k}$$

52. See Theorem 11.8, page 794.

53. The magnitude of the cross product will increase by a factor of 4.

54. From the vectors for two sides of the triangle, and compute their cross product.

$$\langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle \times \langle x_3 - x_1, y_3 - y_1, z_3 - z_1 \rangle$$

55. False. If the vectors are ordered pairs, then the cross product does not exist.

56. False. In general,  $\mathbf{u} \times \mathbf{v} = -(\mathbf{v} \times \mathbf{u})$

57. False. Let  $\mathbf{u} = \langle 1, 0, 0 \rangle$ ,  $\mathbf{v} = \langle 1, 0, 0 \rangle$ ,  $\mathbf{w} = \langle -1, 0, 0 \rangle$ .

Then,  $\mathbf{u} \times \mathbf{v} = \mathbf{u} \times \mathbf{w} = \mathbf{0}$ , but  $\mathbf{v} \neq \mathbf{w}$ .

58. True

$$59. \mathbf{u} = \langle u_1, u_2, u_3 \rangle, \mathbf{v} = \langle v_1, v_2, v_3 \rangle, \mathbf{w} = \langle w_1, w_2, w_3 \rangle$$

$$\begin{aligned} \mathbf{u} \times (\mathbf{v} + \mathbf{w}) &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 + w_1 & v_2 + w_2 & v_3 + w_3 \end{vmatrix} \\ &= [u_2(v_3 + w_3) - u_3(v_2 + w_2)]\mathbf{i} - [u_1(v_3 + w_3) - u_3(v_1 + w_1)]\mathbf{j} + [u_1(v_2 + w_2) - u_2(v_1 + w_1)]\mathbf{k} \\ &= (u_2 v_3 - u_3 v_2)\mathbf{i} - (u_1 v_3 - u_3 v_1)\mathbf{j} + (u_1 v_2 - u_2 v_1)\mathbf{k} + (u_2 w_3 - u_3 w_2)\mathbf{i} - (u_1 w_3 - u_3 w_1)\mathbf{j} + (u_1 w_2 - u_2 w_1)\mathbf{k} \\ &= (\mathbf{u} \times \mathbf{v}) + (\mathbf{u} \times \mathbf{w}) \end{aligned}$$

**60.**  $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ ,  $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ ,  $c$  is a scalar:

$$\begin{aligned} (\mathbf{c}\mathbf{u}) \times \mathbf{v} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ cu_1 & cu_2 & cu_3 \\ v_1 & v_2 & v_3 \end{vmatrix} \\ &= (cu_2v_3 - cu_3v_2)\mathbf{i} - (cu_1v_3 - cu_3v_1)\mathbf{j} + (cu_1v_2 - cu_2v_1)\mathbf{k} \\ &= c[(u_2v_3 - u_3v_2)\mathbf{i} - (u_1v_3 - u_3v_1)\mathbf{j} + (u_1v_2 - u_2v_1)\mathbf{k}] = c(\mathbf{u} \times \mathbf{v}) \end{aligned}$$

**61.**  $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$

$$\mathbf{u} \times \mathbf{u} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ u_1 & u_2 & u_3 \end{vmatrix} = (u_2u_3 - u_3u_2)\mathbf{i} - (u_1u_3 - u_3u_1)\mathbf{j} + (u_1u_2 - u_2u_1)\mathbf{k} = \mathbf{0}$$

$$62. \mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

$$\begin{aligned} (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} &= \mathbf{w} \cdot (\mathbf{u} \times \mathbf{v}) = \begin{vmatrix} w_1 & w_2 & w_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} \\ &= w_1(u_2v_3 - v_2u_3) - w_2(u_1v_3 - v_1u_3) + w_3(u_1v_2 - v_1u_2) \\ &= u_1(v_2w_3 - w_2v_3) - u_2(v_1w_3 - w_1v_3) + u_3(v_1w_2 - w_1v_2) = \mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) \end{aligned}$$

$$63. \mathbf{u} \times \mathbf{v} = (u_2v_3 - u_3v_2)\mathbf{i} - (u_1v_3 - u_3v_1)\mathbf{j} + (u_1v_2 - u_2v_1)\mathbf{k}$$

$$(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{u} = (u_2v_3 - u_3v_2)u_1 + (u_3v_1 - u_1v_3)u_2 + (u_1v_2 - u_2v_1)u_3 = \mathbf{0}$$

$$(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{v} = (u_2v_3 - u_3v_2)v_1 + (u_3v_1 - u_1v_3)v_2 + (u_1v_2 - u_2v_1)v_3 = \mathbf{0}$$

So,  $\mathbf{u} \times \mathbf{v} \perp \mathbf{u}$  and  $\mathbf{u} \times \mathbf{v} \perp \mathbf{v}$ .

**64.** If  $\mathbf{u}$  and  $\mathbf{v}$  are scalar multiples of each other,  $\mathbf{u} = cv$  for some scalar  $c$ .

$$\mathbf{u} \times \mathbf{v} = (cv) \times \mathbf{v} = c(\mathbf{v} \times \mathbf{v}) = c(\mathbf{0}) = \mathbf{0}$$

If  $\mathbf{u} \times \mathbf{v} = \mathbf{0}$ , then  $\|\mathbf{u}\|\|\mathbf{v}\| \sin \theta = 0$ . (Assume  $\mathbf{u} \neq \mathbf{0}, \mathbf{v} \neq \mathbf{0}$ .) So,  $\sin \theta = 0, \theta = 0$ , and  $\mathbf{u}$  and  $\mathbf{v}$  are parallel. So,  $\mathbf{u} = cv$  for some scalar  $c$ .

$$65. \|\mathbf{u} \times \mathbf{v}\| = \|\mathbf{u}\|\|\mathbf{v}\| \sin \theta$$

If  $\mathbf{u}$  and  $\mathbf{v}$  are orthogonal,  $\theta = \pi/2$  and  $\sin \theta = 1$ . So,  $\|\mathbf{u} \times \mathbf{v}\| = \|\mathbf{u}\|\|\mathbf{v}\|$ .

$$66. \mathbf{u} = \langle a_1, b_1, c_1 \rangle, \mathbf{v} = \langle a_2, b_2, c_2 \rangle, \mathbf{w} = \langle a_3, b_3, c_3 \rangle$$

$$\mathbf{v} \times \mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = (b_2c_3 - b_3c_2)\mathbf{i} - (a_2c_3 - a_3c_2)\mathbf{j} + (a_2b_3 - a_3b_2)\mathbf{k}$$

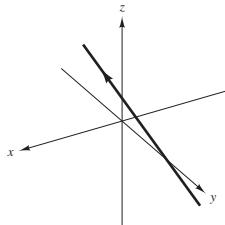
$$\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & b_1 & c_1 \\ (b_2c_3 - b_3c_2) & (a_3c_2 - a_2c_3) & (a_2b_3 - a_3b_2) \end{vmatrix}$$

$$\begin{aligned} \mathbf{u} \times (\mathbf{v} \times \mathbf{w}) &= [b_1(a_2b_3 - a_3b_2) - c_1(a_3c_2 - a_2c_3)]\mathbf{i} - [a_1(a_2b_3 - a_3b_2) - c_1(b_2c_3 - b_3c_2)]\mathbf{j} \\ &\quad + [a_1(a_3c_2 - a_2c_3) - b_1(b_2c_3 - b_3c_2)]\mathbf{k} \\ &= [a_2(a_1a_3 + b_1b_3 + c_1c_3) - a_3(a_1a_2 + b_1b_2 + c_1c_2)]\mathbf{i} + [b_2(a_1a_3 + b_1b_3 + c_1c_3) - b_3(a_1a_2 + b_1b_2 + c_1c_2)]\mathbf{j} \\ &\quad + [c_2(a_1a_3 + b_1b_3 + c_1c_3) - c_3(a_1a_2 + b_1b_2 + c_1c_2)]\mathbf{k} \\ &= (a_1a_3 + b_1b_3 + c_1c_3)\langle a_2, b_2, c_2 \rangle - (a_1a_2 + b_1b_2 + c_1c_2)\langle a_3, b_3, c_3 \rangle = (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\mathbf{w} \end{aligned}$$

## Section 11.5 Lines and Planes in Space

1.  $x = 1 + 3t, y = 2 - t, z = 2 + 5t$

(a)



(b) When  $t = 0, P = (1, 2, 2)$ . When  $t = 3, Q = (10, -1, 17)$ .

$$\overrightarrow{PQ} = \langle 9, -3, 15 \rangle$$

The components of the vector and the coefficients of  $t$  are proportional because the line is parallel to  $\overrightarrow{PQ}$ .

(c)  $y = 0$  when  $t = 2$ . So,  $x = 7$  and  $z = 12$ .

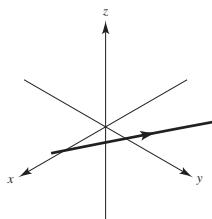
Point:  $(7, 0, 12)$

$x = 0$  when  $t = -\frac{1}{3}$ . So,  $y = \frac{7}{3}$  and  $z = \frac{1}{3}$ . Point:  $(0, \frac{7}{3}, \frac{1}{3})$

$z = 0$  when  $t = -\frac{2}{5}$ . So,  $x = -\frac{1}{5}$  and  $y = \frac{12}{5}$ . Point:  $(-\frac{1}{5}, \frac{12}{5}, 0)$

2.  $x = 2 = 3t, y = 2, z = 1 - t$

(a)



(b) When  $t = 0, P = (2, 2, 1)$ . When  $t = 2$ ,

$$Q = (-4, 2, -1)$$

$$\overrightarrow{PQ} = \langle -6, 0, -2 \rangle$$

The components of the vector and the coefficients of  $t$  are proportional because the line is parallel to  $\overrightarrow{PQ}$ .

(c)  $z = 0$  when  $t = 1$ . So,  $x = -1$  and  $y = 2$ .

Point:  $(-1, 2, 0)$

$x = 0$  when  $t = \frac{2}{3}$ . So,  $y = 2$  and  $z = \frac{1}{3}$

Point:  $(0, 2, \frac{1}{3})$

3.  $x = -2 + t, y = 3t, z = 4 + t$

(a)  $(0, 6, 6)$ : For  $x = 0 = -2 + t$ , you have

$$t = 2. \text{ Then } y = 3(2) = 6 \text{ and}$$

$$z = 4 + 2 = 6. \text{ Yes, } (0, 6, 6) \text{ lies on the line.}$$

(b)  $(2, 3, 5)$ : For  $x = 2 = -2 + t$ , you have

$$t = 4. \text{ Then } y = 3(4) = 12 \neq 3. \text{ No, } (2, 3, 5) \text{ does not lie on the line.}$$

4.  $\frac{x - 3}{2} = \frac{y - 7}{8} = z + 2$

(a)  $(7, 23, 0)$ : Substituting, you have

$$\frac{7 - 3}{2} = \frac{23 - 7}{8} = 0 + 2 \\ 2 = 2 = 2$$

Yes,  $(7, 23, 0)$  lies on the line.

(b)  $(1, -1, -3)$ : Substituting, you have

$$\frac{1 - 3}{2} = \frac{-1 - 7}{8} = -3 + 2 \\ -1 = -1 = -1$$

Yes,  $(1, -1, -3)$  lies on the line.

5. Point:  $(0, 0, 0)$

Direction vector:  $\langle 3, 1, 5 \rangle$

Direction numbers: 3, 1, 5

(a) Parametric:  $x = 3t, y = t, z = 5t$

(b) Symmetric:  $\frac{x}{3} = \frac{y}{1} = \frac{z}{5}$

6. Point:  $(0, 0, 0)$

Direction vector:  $\mathbf{v} = \left\langle -2, \frac{5}{2}, 1 \right\rangle$

Direction numbers: -4, 5, 2

(a) Parametric:  $x = -4t, y = 5t, z = 2t$

(b) Symmetric:  $\frac{x}{-4} = \frac{y}{5} = \frac{z}{2}$

**7.** Point:  $(-2, 0, 3)$ Direction vector:  $\mathbf{v} = \langle 2, 4, -2 \rangle$ Direction numbers:  $2, 4, -2$ (a) Parametric:  $x = -2 + 2t, y = 4t, z = 3 - 2t$ (b) Symmetric:  $\frac{x+2}{2} = \frac{y}{4} = \frac{z-3}{-2}$ **8.** Point:  $(-3, 0, 2)$ Direction vector:  $\mathbf{v} = \langle 0, 6, 3 \rangle$ Direction numbers:  $0, 2, 1$ (a) Parametric:  $x = -3, y = 2t, z = 2 + t$ (b) Symmetric:  $\frac{y}{2} = z - 2, x = -3$ **9.** Point:  $(1, 0, 1)$ Direction vector:  $\mathbf{v} = 3\mathbf{i} - 2\mathbf{j} + \mathbf{k}$ Direction numbers:  $3, -2, 1$ (a) Parametric:  $x = 1 + 3t, y = -2t, z = 1 + t$ (b) Symmetric:  $\frac{x-1}{3} = \frac{y}{-2} = \frac{z-1}{1}$ **10.** Point:  $(-3, 5, 4)$ Direction numbers:  $3, -2, 1$ (a) Parametric:  $x = -3 + 3t, y = 5 - 2t, z = 4 + t$ (b) Symmetric:  $\frac{x+3}{3} = \frac{y-5}{-2} = z - 4$ **11.** Points:  $(5, -3, -2), \left(-\frac{2}{3}, \frac{2}{3}, 1\right)$ Direction vector:  $\mathbf{v} = \frac{17}{3}\mathbf{i} - \frac{11}{3}\mathbf{j} - 3\mathbf{k}$ Direction numbers:  $17, -11, -9$ 

(a) Parametric:

$$x = 5 + 17t, y = -3 - 11t, z = -2 - 9t$$

(b) Symmetric:  $\frac{x-5}{17} = \frac{y+3}{-11} = \frac{z+2}{-9}$ **12.** Points:  $(0, 4, 3), (-1, 2, 5)$ Direction vector:  $\langle 1, 2, -2 \rangle$ Direction numbers:  $1, 2, -2$ (a) Parametric:  $x = t, y = 4 + 2t, z = 3 - 2t$ (b) Symmetric:  $x = \frac{y-4}{2} = \frac{z-3}{-2}$ **13.** Points:  $(7, -2, 6), (-3, 0, 6)$ Direction vector:  $\langle -10, 2, 0 \rangle$ Direction numbers:  $-10, 2, 0$ (a) Parametric:  $x = 7 - 10t, y = -2 + 2t, z = 6$ (b) Symmetric: Not possible because the direction number for  $z$  is 0. But, you could describe the line as  $\frac{x-7}{10} = \frac{y+2}{-2}, z = 6$ .**14.** Points:  $(0, 0, 25), (10, 10, 0)$ Direction vector:  $\langle 10, 10, -25 \rangle$ Direction numbers:  $2, 2, -5$ (a) Parametric:  $x = 2t, y = 2t, z = 25 - 5t$ (b) Symmetric:  $\frac{x}{2} = \frac{y}{2} = \frac{z-25}{-5}$ **15.** Point:  $(2, 3, 4)$ Direction vector:  $\mathbf{v} = \mathbf{k}$ Direction numbers:  $0, 0, 1$ Parametric:  $x = 2, y = 3, z = 4 + t$ **16.** Point:  $(-4, 5, 2)$ Direction vector:  $\mathbf{v} = \mathbf{j}$ Direction numbers:  $0, 1, 0$ Parametric:  $x = -4, y = 5 + t, z = 2$ **17.** Point:  $(2, 3, 4)$ Direction vector:  $\mathbf{v} = 3\mathbf{i} + 2\mathbf{j} - \mathbf{k}$ Direction numbers:  $3, 2, -1$ Parametric:  $x = 2 + 3t, y = 3 + 2t, z = 4 - t$ **18.** Point  $(-4, 5, 2)$ Direction vector:  $\mathbf{v} = -\mathbf{i} + 2\mathbf{j} + \mathbf{k}$ Direction numbers:  $-1, 2, 1$ Parametric:  $x = -4 - t, y = 5 + 2t, z = 2 + t$ **19.** Point:  $(5, -3, -4)$ Direction vector:  $\mathbf{v} = \langle 2, -1, 3 \rangle$ Direction numbers:  $2, -1, 3$ Parametric:  $x = 5 + 2t, y = -3 - t, z = -4 + 3t$ **20.** Point:  $(-1, 4, -3)$ Direction vector:  $\mathbf{v} = 5\mathbf{i} - \mathbf{j}$ Direction numbers:  $5, -1, 0$ Parametric:  $x = -1 + 5t, y = 4 - t, z = -3$

**21.** Point:  $(2, 1, 2)$

Direction vector:  $\langle -1, 1, 1 \rangle$

Direction numbers:  $-1, 1, 1$

Parametric:  $x = 2 - t, y = 1 + t, z = 2 + t$

**22.** Point:  $(-6, 0, 8)$

Direction vector:  $\langle -2, 2, 0 \rangle$

Direction numbers:  $-2, 2, 0$

Parametric:  $x = -6 - 2t, y = 2t, z = 8$

**23.** Let  $t = 0$ :  $P = (3, -1, -2)$  (other answers possible)

$\mathbf{v} = \langle -1, 2, 0 \rangle$  (any nonzero multiple of  $\mathbf{v}$  is correct)

**24.** Let  $t = 0$ :  $P = (0, 5, 4)$  (other answers possible)

$\mathbf{v} = \langle 4, -1, 3 \rangle$  (any nonzero multiple of  $\mathbf{v}$  is correct)

**25.** Let each quantity equal 0:

$P = (7, -6, -2)$  (other answers possible)

$\mathbf{v} = \langle 4, 2, 1 \rangle$  (any nonzero multiple of  $\mathbf{v}$  is correct)

**26.** Let each quantity equal 0:

$P = (-3, 0, 3)$  (other answers possible)

$\mathbf{v} = \langle 5, 8, 6 \rangle$  (any nonzero multiple of  $\mathbf{v}$  is correct)

**27.**  $L_1: \mathbf{v} = \langle -3, 2, 4 \rangle$        $(6, -2, 5)$  on line

$L_2: \mathbf{v} = \langle 6, -4, -8 \rangle$        $(6, -2, 5)$  on line

$L_3: \mathbf{v} = \langle -6, 4, 8 \rangle$        $(6, -2, 5)$  not online

$L_4: \mathbf{v} = \langle 6, 4, -6 \rangle$       not parallel to  $L_1, L_2$ , nor  $L_3$

$L_1$  and  $L_2$  are identical.  $L_1 = L_2$  and is parallel to  $L_3$ .

**28.**  $L_1: \mathbf{v} = \langle 2, -6, -2 \rangle$        $(3, 0, 1)$  on line

$L_2: \mathbf{v} = \langle 2, -1, 3 \rangle$        $(1, -1, 0)$  on line

$L_3: \mathbf{v} = \langle 2, -10, -4 \rangle$        $(-1, 3, 1)$  on line

$L_4: \mathbf{v} = \langle 2, -1, 3 \rangle$        $(5, 1, 8)$  on line

$L_2$  and  $L_4$  are parallel, not identical, because  $(1, -1, 0)$  is not on  $L_4$ .

**29.**  $L_1: \mathbf{v} = \langle 4, -2, 3 \rangle$        $(8, -5, -9)$  on line

$L_2: \mathbf{v} = \langle 2, 1, 5 \rangle$

$L_3: \mathbf{v} = \langle -8, 4, -6 \rangle$        $(8, -5, -9)$  on line

$L_4: \mathbf{v} = \langle -2, 1, 1.5 \rangle$

$L_1$  and  $L_3$  are identical.

**30.**  $L_1: \mathbf{v} = \langle 2, 1, 2 \rangle$        $(3, 2, -2)$  on line

$L_2: \mathbf{v} = \langle 4, 2, 4 \rangle$        $(1, 1, -3)$  on line

$L_3: \mathbf{v} = \langle 1, \frac{1}{2}, 1 \rangle$        $(-2, 1, 3)$  on line

$L_4: \mathbf{v} = \langle 2, 4, -1 \rangle$        $(3, -1, 2)$  on line

$L_1, L_2$  and  $L_3$  have same direction.

$(3, 2, -2)$  is not on  $L_2$  nor  $L_3$

$(1, 1, -3)$  is not on  $L_3$

So, the three lines are parallel, not identical.

**31.** At the point of intersection, the coordinates for one line equal the corresponding coordinates for the other line.

So,

(i)  $4t + 2 = 2s + 2$ , (ii)  $3 = 2s + 3$ , and

(iii)  $-t + 1 = s + 1$ .

From (ii), you find that  $s = 0$  and consequently, from (iii),  $t = 0$ . Letting  $s = t = 0$ , you see that equation (i) is satisfied and so the two lines intersect. Substituting zero for  $s$  or for  $t$ , you obtain the point  $(2, 3, 1)$ .

$$\mathbf{u} = 4\mathbf{i} - \mathbf{k} \quad (\text{First line})$$

$$\mathbf{v} = 2\mathbf{i} + 2\mathbf{j} + \mathbf{k} \quad (\text{Second line})$$

$$\cos \theta = \frac{|\mathbf{u} \cdot \mathbf{v}|}{\|\mathbf{u}\| \|\mathbf{v}\|} = \frac{8 - 1}{\sqrt{17} \sqrt{9}} = \frac{7}{3\sqrt{17}} = \frac{7\sqrt{17}}{51}$$

**32.** By equating like variables, you have

(i)  $-3t + 1 = 3s + 1$ , (ii)  $4t + 1 = 2s + 4$ , and

(iii)  $2t + 4 = -s + 1$ .

From (i) you have  $s = -t$ , and consequently from (ii),  $t = \frac{1}{2}$  and from (iii),  $t = -3$ . The lines do not intersect.

**33.** Writing the equations of the lines in parametric form you have

$$x = 3t \quad y = 2 - t \quad z = -1 + t$$

$$x = 1 + 4s \quad y = -2 + s \quad z = -3 - 3s.$$

For the coordinates to be equal,  $3t = 1 + 4s$  and  $2 - t = -2 + s$ . Solving this system yields  $t = \frac{17}{7}$  and  $s = \frac{11}{7}$ . When using these values for  $s$  and  $t$ , the  $z$  coordinates are not equal. The lines do not intersect.

34. Writing the equations of the lines in parametric form you have

$$\begin{aligned}x &= 2 - 3t & y &= 2 + 6t & z &= 3 + t \\x &= 3 + 2s & y &= -5 + s & z &= -2 + 4s.\end{aligned}$$

By equating like variables, you have

$$2 - 3t = 3 + 2s, 2 + 6t = -5 + s, 3 + t = -2 + 4s.$$

So,  $t = -1, s = 1$  and the point of intersection is

$$(5, -4, 2).$$

$$\mathbf{u} = \langle -3, 6, 1 \rangle \quad (\text{First line})$$

$$\mathbf{v} = \langle 2, 1, 4 \rangle \quad (\text{Second line})$$

$$\cos \theta = \frac{|\mathbf{u} \cdot \mathbf{v}|}{\|\mathbf{u}\| \|\mathbf{v}\|} = \frac{4}{\sqrt{46} \sqrt{21}} = \frac{4}{\sqrt{966}} = \frac{2\sqrt{966}}{483}$$

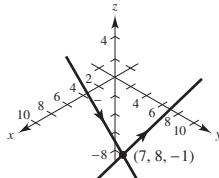
35.  $x = 2t + 3 \quad x = -2s + 7$

$$y = 5t - 2 \quad y = s + 8$$

$$z = -t + 1 \quad z = 2s - 1$$

Point of intersection:  $(7, 8, -1)$

Note:  $t = 2$  and  $s = 0$

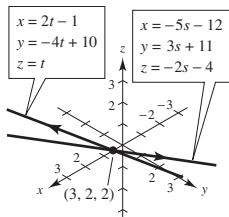


36.  $x = 2t - 1 \quad x = -5s - 12$

$$y = -4t + 10 \quad y = 3s + 11$$

$$z = t \quad z = -2s - 4$$

Point of intersection:  $(3, 2, 2)$



37.  $4x - 3y - 6z = 6$

(a)  $P(0, 0, -1), Q(0, -2, 0), R(3, 4, -1)$

$$\overline{PQ} = \langle 0, -2, 1 \rangle, \overline{PR} = \langle 3, 4, 0 \rangle$$

(b)  $\overline{PQ} \times \overline{PR} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & -2 & 1 \\ 3 & 4 & 0 \end{vmatrix} = \langle -4, 3, 6 \rangle$

The components of the cross product are proportional to the coefficients of the variables in the equation. The cross product is parallel to the normal vector.

38.  $2x + 3y + 4z = 4$

$$P(0, 0, 1), Q(2, 0, 0), R(3, 2, -2)$$

(a)  $\overline{PQ} = \langle 2, 0, -1 \rangle, \overline{PR} = \langle 3, 2, -3 \rangle$

(b)  $\overline{PQ} \times \overline{PR} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 0 & -1 \\ 3 & 2 & -3 \end{vmatrix} = \langle 2, 3, 4 \rangle$

The components of the cross product are proportional (for this choice of  $P, Q$ , and  $R$ , they are the same) to the coefficients of the variables in the equation. The cross product is parallel to the normal vector.

39.  $x + 2y - 4z - 1 = 0$

(a)  $(-7, 2, -1): (-7) + 2(2) - 4(-1) - 1 = 0$

Point is in plane

(b)  $(5, 2, 2): 5 + 2(2) - 4(2) - 1 = 0$

Point is in plane

40.  $2x + y + 3z - 6 = 0$

(a)  $(3, 6, -2): 2(3) + 6 + 3(-2) - 6 = 0$

Point is in plane

(b)  $(-1, 5, -1): 2(-1) + 5 + 3(-1) - 6 = -6 \neq 0$

Point is not in plane

41. Point:  $(1, 3, -7)$

Normal vector:  $\mathbf{n} = \mathbf{j} = \langle 0, 1, 0 \rangle$

$$0(x - 1) + 1(y - 3) + 0(z - (-7)) = 0$$

$$y - 3 = 0$$

42. Point:  $(0, -1, 4)$

Normal vector:  $\mathbf{n} = \mathbf{k} = \langle 0, 0, 1 \rangle$

$$0(x - 0) + 0(y + 1) + 1(z - 4) = 0$$

$$z - 4 = 0$$

43. Point:  $(3, 2, 2)$

Normal vector:  $\mathbf{n} = 2\mathbf{i} + 3\mathbf{j} - \mathbf{k}$

$$2(x - 3) + 3(y - 2) - 1(z - 2) = 0$$

$$2x + 3y - z = 10$$

44. Point:  $(0, 0, 0)$

Normal vector:  $\mathbf{n} = -3\mathbf{i} + 2\mathbf{k}$

$$-3(x - 0) + 0(y - 0) + 2(z - 0) = 0$$

$$-3x + 2z = 0$$

**45.** Point:  $(-1, 4, 0)$

Normal vector:  $\mathbf{v} = \langle 2, -1, -2 \rangle$

$$\begin{aligned} 2(x+1) - 1(y-4) - 2(z-0) &= 0 \\ 2x - y - 2z + 6 &= 0 \end{aligned}$$

**46.** Point:  $(3, 2, 2)$

Normal vector:  $\mathbf{v} = 4\mathbf{i} + \mathbf{j} - 3\mathbf{k}$

$$\begin{aligned} 4(x-3) + (y-2) - 3(z-2) &= 0 \\ 4x + y - 3z &= 8 \end{aligned}$$

**47.** Let  $\mathbf{u}$  be the vector from  $(0, 0, 0)$  to

$$(2, 0, 3): \mathbf{u} = \langle 2, 0, 3 \rangle$$

Let  $\mathbf{u}$  be the vector from  $(0, 0, 0)$  to

$$(-3, -1, 5): \mathbf{v} = \langle -3, -1, 5 \rangle$$

$$\text{Normal vectors: } \mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 0 & 3 \\ -3 & -1 & 5 \end{vmatrix} = \langle 3, -19, -2 \rangle$$

$$\begin{aligned} 3(x-0) - 19(y-0) - 2(z-0) &= 0 \\ 3x - 19y - 2z &= 0 \end{aligned}$$

**48.** Let  $\mathbf{u}$  be the vector from  $(3, -1, 2)$  to  $(2, 1, 5)$ :

$$\mathbf{u} = \langle -1, 2, 3 \rangle$$

Let  $\mathbf{u}$  be the vector from  $(3, -1, 2)$  to  $(1, -2, -2)$ :

$$\mathbf{v} = \langle -2, -1, -4 \rangle$$

Normal vector:

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 2 & 3 \\ -2 & -1 & -4 \end{vmatrix} = \langle -5, -10, 5 \rangle = -5\langle 1, 2, -1 \rangle$$

$$\begin{aligned} 1(x-3) + 2(y+1) - (z-2) &= 0 \\ x + 2y - z + 1 &= 0 \end{aligned}$$

**49.** Let  $\mathbf{u}$  be the vector from  $(1, 2, 3)$  to

$$(3, 2, 1): \mathbf{u} = 2\mathbf{i} - 2\mathbf{k}$$

Let  $\mathbf{v}$  be the vector from  $(1, 2, 3)$  to

$$(-1, -2, 2): \mathbf{v} = -2\mathbf{i} - 4\mathbf{j} - \mathbf{k}$$

Normal vector:

$$\left(\frac{1}{2}\mathbf{u}\right) \times (-\mathbf{v}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & -1 \\ 2 & 4 & 1 \end{vmatrix} = 4\mathbf{i} - 3\mathbf{j} + 4\mathbf{k}$$

$$\begin{aligned} 4(x-1) - 3(y-2) + 4(z-3) &= 0 \\ 4x - 3y + 4z &= 10 \end{aligned}$$

**50.**  $(1, 2, 3)$ , Normal vector:  $\mathbf{v} = \mathbf{i}$ ,  $1(x-1) = 0, x = 1$

**51.**  $(1, 2, 3)$ , Normal vector:  $\mathbf{v} = \mathbf{k}$ ,  $1(z-3) = 0, z = 3$

**52.** The plane passes through the three points  $(0, 0, 0), (0, 1, 0), (\sqrt{3}, 0, 1)$ .

The vector from  $(0, 0, 0)$  to  $(0, 1, 0)$ :  $\mathbf{u} = \mathbf{j}$

The vector from  $(0, 0, 0)$  to  $(\sqrt{3}, 0, 1)$ :  $\mathbf{v} = \sqrt{3}\mathbf{i} + \mathbf{k}$

$$\text{Normal vector: } \mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & 0 \\ \sqrt{3} & 0 & 1 \end{vmatrix} = \mathbf{i} - \sqrt{3}\mathbf{k}$$

$$x = \sqrt{3}z = 0$$

**53.** The direction vectors for the lines are

$$\mathbf{u} = -2\mathbf{i} + \mathbf{j} + \mathbf{k}, \mathbf{v} = -3\mathbf{i} + 4\mathbf{j} - \mathbf{k}$$

$$\text{Normal vector: } \mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & 1 & 1 \\ -3 & 4 & -1 \end{vmatrix} = -5(\mathbf{i} + \mathbf{j} + \mathbf{k})$$

Point of intersection of the lines:  $(-1, 5, 1)$

$$(x+1) + (y-5) + (z-1) = 0$$

$$x + y + z = 5$$

**54.** The direction of the line is  $\mathbf{u} = 2\mathbf{i} - \mathbf{j} + \mathbf{k}$ . Choose any point on the line,  $[(0, 4, 0)$ , for example], and let  $\mathbf{v}$  be the vector from  $(0, 4, 0)$  to the given point  $(2, 2, 1)$ :

$$\mathbf{v} = 2\mathbf{i} - 2\mathbf{j} + \mathbf{k}$$

$$\text{Normal vector: } \mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & 1 \\ 2 & -2 & 1 \end{vmatrix} = \mathbf{i} - 2\mathbf{k}$$

$$(x-2) - 2(z-1) = 0$$

$$x - 2z = 0$$

**55.** Let  $\mathbf{v}$  be the vector from  $(-1, 1, -1)$  to

$$(2, 2, 1): \mathbf{v} = 3\mathbf{i} + \mathbf{j} + 2\mathbf{k}$$

Let  $\mathbf{n}$  be a vector normal to the plane  $2x - 3y + z = 3$ :  $\mathbf{n} = 2\mathbf{i} - 3\mathbf{j} + \mathbf{k}$

Because  $\mathbf{v}$  and  $\mathbf{n}$  both lie in the plane  $P$ , the normal vector to  $P$  is

$$\mathbf{v} \times \mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 1 & 2 \\ 2 & -3 & 1 \end{vmatrix} = 7\mathbf{i} - \mathbf{j} - 11\mathbf{k}$$

$$7(x-2) + 1(y-2) - 11(z-1) = 0$$

$$7x + y - 11z = 5$$

56. Let  $\mathbf{v}$  be the vector from  $(3, 2, 1)$  to

$$(3, 1, -5): \mathbf{v} = -\mathbf{j} - 6\mathbf{k}$$

Let  $\mathbf{n}$  be the normal to the given plane:

$$\mathbf{n} = 6\mathbf{i} + 7\mathbf{j} + 2\mathbf{k}$$

Because  $\mathbf{v}$  and  $\mathbf{n}$  both lie in the plane  $P$ , the normal vector to  $P$  is:

$$\begin{aligned}\mathbf{v} \times \mathbf{n} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & -1 & -6 \\ 6 & 7 & 2 \end{vmatrix} = 40\mathbf{i} - 36\mathbf{j} + 6\mathbf{k} \\ &= 2(20\mathbf{i} - 18\mathbf{j} + 3\mathbf{k})\end{aligned}$$

$$20(x - 3) - 18(y - 2) + 3(z - 1) = 0$$

$$20x - 18y + 3z = 27$$

58. Let  $\mathbf{u} = \mathbf{k}$  and let  $\mathbf{v}$  be the vector from  $(4, 2, 1)$  to  $(-3, 5, 7)$ :  $\mathbf{v} = -7\mathbf{i} + 3\mathbf{j} + 6\mathbf{k}$

Because  $\mathbf{u}$  and  $\mathbf{v}$  both lie in the plane  $P$ , the normal vector to  $P$  is:

$$\begin{aligned}\mathbf{u} \times \mathbf{v} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 1 \\ -7 & 3 & 6 \end{vmatrix} = -3\mathbf{i} - 7\mathbf{j} = -(3\mathbf{i} + 7\mathbf{j})\end{aligned}$$

$$3(x - 4) + 7(y - 2) = 0$$

$$3x + 7y = 26$$

59.  $xy$ -plane: Let  $z = 0$ .

$$\text{Then } 0 = 4 - t \Rightarrow t = 4 \Rightarrow x = 1 - 2(4) = -7 \text{ and}$$

$$y = -2 + 3(4) = 10. \text{ Intersection: } (-7, 10, 0)$$

$xz$ -plane: Let  $y = 0$ .

$$\text{Then } 0 = -2 + 3t \Rightarrow t = \frac{2}{3} \Rightarrow x = 1 - 2\left(\frac{2}{3}\right) = -\frac{1}{3} \text{ and}$$

$$z = -4 + \frac{2}{3} = -\frac{10}{3}. \text{ Intersection: } \left(-\frac{1}{3}, 0, -\frac{10}{3}\right)$$

$yz$ -plane: Let  $x = 0$ .

$$\text{Then } 0 = 1 - 2t \Rightarrow t = \frac{1}{2} \Rightarrow y = -2 + 3\left(\frac{1}{2}\right) = -\frac{1}{2} \text{ and}$$

$$z = -4 + \frac{1}{2} = -\frac{7}{2}. \text{ Intersection: } \left(0, -\frac{1}{2}, -\frac{7}{2}\right)$$

60. Parametric equations:  $x = 2 + 3t$ ,  $y = -1 + t$ ,  $z = 3 + 2t$

$xy$ -plane: Let  $z = 0$ .

$$\text{Then } 3 + 2t = 0 \Rightarrow t = -\frac{3}{2} \Rightarrow x = 2 + 3\left(-\frac{3}{2}\right) = -\frac{5}{2} \text{ and}$$

$$y = -1 + \left(-\frac{3}{2}\right) = -\frac{5}{2}. \text{ Intersection: } \left(-\frac{5}{2}, -\frac{5}{2}, 0\right)$$

$xz$ -plane: Let  $y = 0$ .

$$\text{Then } t = 1 \Rightarrow x = 2 + 3(1) = 5 \text{ and}$$

$$z = 3 + 2(1) = 5. \text{ Intersection: } (5, 0, 5)$$

$yz$ -plane: Let  $x = 0$ .

$$\text{Then } 2 + 3t = 0 \Rightarrow t = -\frac{2}{3} \Rightarrow y = -1 - \frac{2}{3} = -\frac{5}{3} \text{ and}$$

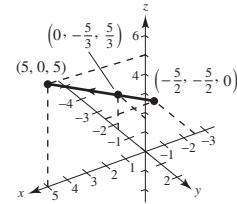
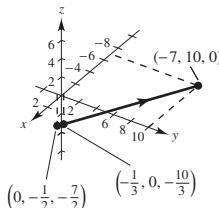
$$z = 3 + 2\left(-\frac{2}{3}\right) = \frac{5}{3}. \text{ Intersection: } \left(0, -\frac{5}{3}, \frac{5}{3}\right)$$

57. Let  $\mathbf{u} = \mathbf{i}$  and let  $\mathbf{v}$  be the vector from  $(1, -2, -1)$  to

$$(2, 5, 6): \mathbf{v} = \mathbf{i} + 7\mathbf{j} + 7\mathbf{k}$$

Because  $\mathbf{u}$  and  $\mathbf{v}$  both lie in the plane  $P$ , the normal vector to  $P$  is:

$$\begin{aligned}\mathbf{u} \times \mathbf{v} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 0 \\ 1 & 7 & 7 \end{vmatrix} = -7\mathbf{j} + 7\mathbf{k} = -7(\mathbf{j} - \mathbf{k}) \\ &= [y - (-2)] - [z - (-1)] = 0 \\ &\quad y - z = -1\end{aligned}$$



61. Let  $(x, y, z)$  be equidistant from  $(2, 2, 0)$  and  $(0, 2, 2)$ .

$$\begin{aligned}\sqrt{(x-2)^2 + (y-2)^2 + (z-0)^2} &= \sqrt{(x-0)^2 + (y-2)^2 + (z-2)^2} \\ x^2 - 4x + 4 + y^2 - 4y + 4 + z^2 &= x^2 + y^2 - 4y + 4 + z^2 - 4z + 4 \\ -4x + 8 &= -4z + 8 \\ x - z &= 0 \text{ Plane}\end{aligned}$$

62. Let  $(x, y, z)$  be equidistant from  $(1, 0, 2)$  and  $(2, 0, 1)$ .

$$\begin{aligned}\sqrt{(x-1)^2 + (y-0)^2 + (z-2)^2} &= \sqrt{(x-2)^2 + (y-0)^2 + (z-1)^2} \\ x^2 - 2x + 1 + y^2 + z^2 - 4z + 4 &= x^2 - 4x + 4 + y^2 + z^2 - 2z + 1 \\ -2x - 4z + 5 &= -4x - 2z + 5 \\ 2x - 2z &= 0 \\ x - z &= 0 \text{ Plane}\end{aligned}$$

63. Let  $(x, y, z)$  be equidistant from  $(-3, 1, 2)$  and  $(6, -2, 4)$ .

$$\begin{aligned}\sqrt{(x+3)^2 + (y-1)^2 + (z-2)^2} &= \sqrt{(x-6)^2 + (y+2)^2 + (z-4)^2} \\ x^2 + 6x + 9 + y^2 - 2y + 1 + z^2 - 4z + 4 &= x^2 - 12x + 36 + y^2 + 4y + 4 + z^2 - 8z + 16 \\ 6x - 2y - 4z + 14 &= -12x + 4y - 8z + 56 \\ 18x - 6y + 4z - 42 &= 0 \\ 9x - 3y + 2z - 21 &= 0 \text{ Plane}\end{aligned}$$

64. Let  $(x, y, z)$  be equidistant from  $(-5, 1, -3)$  and  $(2, -1, 6)$

$$\begin{aligned}\sqrt{(x+5)^2 + (y-1)^2 + (z+3)^2} &= \sqrt{(x-2)^2 + (y+1)^2 + (z-6)^2} \\ x^2 + 10x + 25 + y^2 - 2y + 1 + z^2 + 6z + 9 &= x^2 - 4x + 4 + y^2 + 2y + 1 + z^2 - 12z + 36 \\ 10x - 2y + 6z + 35 &= -4x + 2y - 12z + 41 \\ 14x - 4y + 18z - 6 &= 0 \\ 7x - 2y + 9z - 3 &= 0 \text{ Plane}\end{aligned}$$

65. The normal vectors to the planes are

$$\mathbf{n}_1 = \langle 5, -3, 1 \rangle, \mathbf{n}_2 = \langle 1, 4, 7 \rangle, \cos \theta = \frac{|\mathbf{n}_1 \cdot \mathbf{n}_2|}{\|\mathbf{n}_1\| \|\mathbf{n}_2\|} = 0.$$

So,  $\theta = \pi/2$  and the planes are orthogonal.

66. The normal vectors to the planes are

$$\mathbf{n}_1 = \langle 3, 1, -4 \rangle, \mathbf{n}_2 = \langle -9, -3, 12 \rangle.$$

Because  $\mathbf{n}_2 = -3\mathbf{n}_1$ , the planes are parallel, but not equal.

67. The normal vectors to the planes are

$$\mathbf{n}_1 = \mathbf{i} - 3\mathbf{j} + 6\mathbf{k}, \mathbf{n}_2 = 5\mathbf{i} + \mathbf{j} - \mathbf{k},$$

$$\cos \theta = \frac{|\mathbf{n}_1 \cdot \mathbf{n}_2|}{\|\mathbf{n}_1\| \|\mathbf{n}_2\|} = \frac{|5 - 3 - 6|}{\sqrt{46} \sqrt{27}} = \frac{4\sqrt{138}}{414} = \frac{2\sqrt{138}}{207}.$$

$$\text{So, } \theta = \arccos\left(\frac{2\sqrt{138}}{207}\right) \approx 83.5^\circ.$$

68. The normal vectors to the planes are

$$\mathbf{n}_1 = 3\mathbf{i} + 2\mathbf{j} - \mathbf{k}, \mathbf{n}_2 = \mathbf{i} - 4\mathbf{j} + 2\mathbf{k},$$

$$\cos \theta = \frac{|\mathbf{n}_1 \cdot \mathbf{n}_2|}{\|\mathbf{n}_1\| \|\mathbf{n}_2\|} = \frac{|3 - 8 - 2|}{\sqrt{14} \sqrt{21}} = \frac{7\sqrt{6}}{42} = \frac{\sqrt{6}}{6}.$$

$$\text{So, } \theta = \arccos\left(\frac{\sqrt{6}}{6}\right) \approx 65.9^\circ.$$

69. The normal vectors to the planes are  $\mathbf{n}_1 = \langle 1, -5, -1 \rangle$  and

$\mathbf{n}_2 = \langle 5, -25, -5 \rangle$ . Because  $\mathbf{n}_2 = 5\mathbf{n}_1$ , the planes are parallel, but not equal.

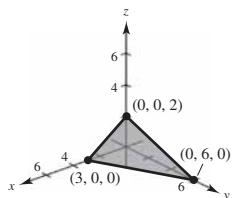
70. The normal vectors to the planes are

$$\mathbf{n}_1 = \langle 2, 0, -1 \rangle, \mathbf{n}_2 = \langle 4, 1, 8 \rangle,$$

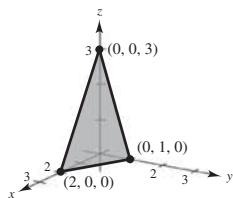
$$\cos \theta = \frac{|\mathbf{n}_1 \cdot \mathbf{n}_2|}{\|\mathbf{n}_1\| \|\mathbf{n}_2\|} = 0$$

So,  $\theta = \frac{\pi}{2}$  and the planes are orthogonal.

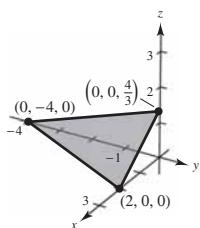
71.  $4x + 2y + 6z = 12$



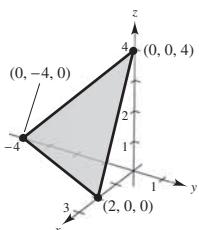
72.  $3x + 6y + 2z = 6$



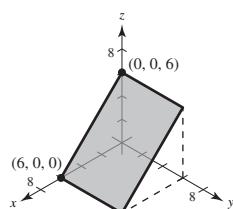
73.  $2x - y + 3z = 4$



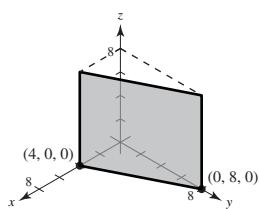
74.  $2x - y + z = 4$



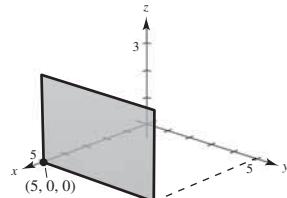
75.  $x + z = 6$



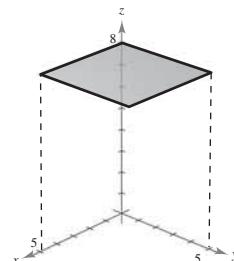
76.  $2x + y = 8$



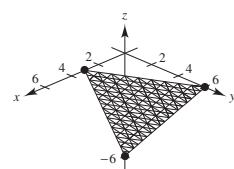
77.  $x = 5$



78.  $z = 8$

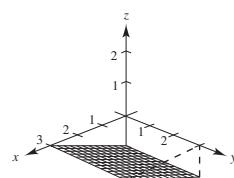


79.  $2x + y - z = 6$



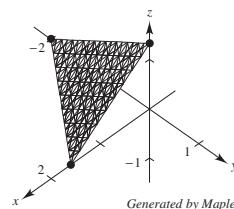
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80.  $x - 3z = 3$



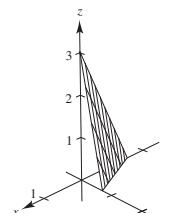
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81.  $-5x + 4y - 6z + 8 = 0$



Generated by Maple

82.  $2.1x - 4.7y - z + 3 = 0$



Generated by Mathematica

83.  $P_1: \mathbf{n} = \langle 15, -6, 24 \rangle$   $(0, -1, -1)$  not on plane  
 $P_2: \mathbf{n} = \langle -5, 2, -8 \rangle$   $(0, -1, -1)$  on plane  
 $P_3: \mathbf{n} = \langle 6, -4, 4 \rangle$   
 $P_4: \mathbf{n} = \langle 3, -2, -2 \rangle$

Planes  $P_1$  and  $P_2$  are parallel.

84.  $P_1: \mathbf{n} = \langle 2, -1, 3 \rangle$   $(4, 0, 0)$  on plane  
 $P_2: \mathbf{n} = \langle 3, -5, -2 \rangle$   
 $P_3: \mathbf{n} = \langle 8, -4, 12 \rangle$   $(4, 0, 0)$  not on plane  
 $P_4: \mathbf{n} = \langle -4, -2, 6 \rangle$

$P_1$  and  $P_3$  are parallel.

85.  $P_1: \mathbf{n} = \langle 3, -2, 5 \rangle$   $(1, -1, 1)$  on plane  
 $P_2: \mathbf{n} = \langle -6, 4, -10 \rangle$   $(1, -1, 1)$  not on plane  
 $P_3: \mathbf{n} = \langle -3, 2, 5 \rangle$   
 $P_4: \mathbf{n} = \langle 75, -50, 125 \rangle$   $(1, -1, 1)$  on plane

$P_1$  and  $P_4$  are identical.

$P_1 = P_4$  and is parallel to  $P_2$ .

86.  $P_1: \mathbf{n} = \langle -60, 90, 30 \rangle$  or  $\langle -2, 3, 1 \rangle$   $(0, 0, \frac{9}{10})$  on plane  
 $P_2: \mathbf{n} = \langle 6, -9, -3 \rangle$  or  $\langle -2, 3, 1 \rangle$   $(0, 0, -\frac{2}{3})$  on plane  
 $P_3: \mathbf{n} = \langle -20, 30, 10 \rangle$  or  $\langle -2, 3, 1 \rangle$   $(0, 0, \frac{5}{6})$  on plane  
 $P_4: \mathbf{n} = \langle 12, -18, 6 \rangle$  or  $\langle -2, 3, -1 \rangle$

$P_1, P_2$ , and  $P_3$  are parallel.

87. Each plane passes through the points  $(c, 0, 0), (0, c, 0)$ , and  $(0, 0, c)$ .

88.  $x = y = c$

Each plane is parallel to the  $z$ -axis.

89. If  $c = 0, z = 0$  is  $xy$ -plane.

If  $c \neq 0, cy + z = 0 \Rightarrow y = \frac{-1}{c}z$  is a plane parallel to  $x$ -axis and passing through the points  $(0, 0, 0)$  and  $(0, 1, -c)$ .

90.  $x + cz = 0$

If  $c = 0, z = 0$  is the  $yz$ -plane.

If  $c \neq 0, x + cz = 0$  is a plane parallel to the  $y$ -axis.

91. (a)  $\mathbf{n}_1 = 3\mathbf{i} + 2\mathbf{j} - \mathbf{k}$  and  $\mathbf{n}_2 = \mathbf{i} - 4\mathbf{j} + 2\mathbf{k}$

$$\cos \theta = \frac{|\mathbf{n}_1 \cdot \mathbf{n}_2|}{\|\mathbf{n}_1\| \|\mathbf{n}_2\|} = \frac{|-7|}{\sqrt{14}\sqrt{21}} = \frac{\sqrt{6}}{6}$$

$$\Rightarrow \theta \approx 1.1503 \approx 65.91^\circ$$

(b) The direction vector for the line is

$$\mathbf{n}_2 \times \mathbf{n}_1 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -4 & 2 \\ 3 & 2 & -1 \end{vmatrix} = 7(\mathbf{j} + 2\mathbf{k}).$$

Find a point of intersection of the planes.

$$6x + 4y - 2z = 14$$

$$x - 4y + 2z = 0$$

$$7x = 14$$

$$x = 2$$

Substituting 2 for  $x$  in the second equation, you have  $-4y + 2z = -2$  or  $z = 2y - 1$ . Letting  $y = 1$ , a point of intersection is  $(2, 1, 1)$ .

$$x = 2, y = 1 + t, z = 1 + 2t$$

92. (a)  $\mathbf{n}_1 = 6\mathbf{i} - 3\mathbf{j} + \mathbf{k}$ ,  $\mathbf{n}_2 = -\mathbf{i} + \mathbf{j} + 5\mathbf{k}$

$$\cos \theta = \frac{|\mathbf{n}_1 \cdot \mathbf{n}_2|}{\|\mathbf{n}_1\| \|\mathbf{n}_2\|} = \frac{-4}{\sqrt{46}\sqrt{27}} = \frac{-2\sqrt{138}}{207}$$

$$\theta \approx 1.6845 \approx 96.52^\circ$$

(b) The direction vector for the line is

$$\mathbf{n}_1 \times \mathbf{n}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 6 & -3 & 1 \\ -1 & 1 & 5 \end{vmatrix} = \langle -16, -31, 3 \rangle.$$

Find a point of intersection of the planes.

$$6x - 3y + z = 5 \Rightarrow 6x - 3y + z = 5$$

$$-x + y + 5z = 5 \Rightarrow \frac{-6x + 6y + 30z = 30}{3y + 31z = 35}$$

$$\text{Let } y = -9, z = 2 \Rightarrow x = -4 \Rightarrow (-4, -9, 2).$$

$$x = -4 - 16t, y = -9 - 31t, z = 2 + 3t$$

93. Writing the equation of the line in parametric form and substituting into the equation of the plane you have:

$$x = \frac{1}{2} + t, y = \frac{-3}{2} - t, z = -1 + 2t$$

$$2\left(\frac{1}{2} + t\right) - 2\left(\frac{-3}{2} - t\right) + (-1 + 2t) = 12, t = \frac{3}{2}$$

Substituting  $t = 3/2$  into the parametric equations for the line you have the point of intersection  $(2, -3, 2)$ .

The line does not lie in the plane.

94. Writing the equation of the line in parametric form and substituting into the equation of the plane you have:

$$x = 1 + 4t, y = 2t, z = 3 + 6t$$

$$2(1 + 4t) + 3(2t) = -5, t = \frac{-1}{2}$$

Substituting  $t = -\frac{1}{2}$  into the parametric equations for the line you have the point of intersection  $(-1, -1, 0)$ .

The line does not lie in the plane.

95. Writing the equation of the line in parametric form and substituting into the equation of the plane you have:

$$x = 1 + 3t, y = -1 - 2t, z = 3 + t$$

$$2(1 + 3t) + 3(-1 - 2t) = 10, -1 = 10, \text{ contradiction}$$

So, the line does not intersect the plane.

96. Writing the equation of the line in parametric form and substituting into the equation of the plane you have:

$$x = 4 + 2t, y = -1 - 3t, z = -2 + 5t$$

$$5(4 + 2t) + 3(-1 - 3t) = 17, t = 0$$

Substituting  $t = 0$  into the parametric equations for the line you have the point of intersection  $(4, -1, -2)$ .

The line does not lie in the plane.

97. Point:  $Q(0, 0, 0)$

$$\text{Plane: } 2x + 3y + z - 12 = 0$$

$$\text{Normal to plane: } \mathbf{n} = \langle 2, 3, 1 \rangle$$

$$\text{Point in plane: } P(6, 0, 0)$$

$$\text{Vector } \overline{PQ} = \langle -6, 0, 0 \rangle$$

$$D = \frac{|\overline{PQ} \cdot \mathbf{n}|}{\|\mathbf{n}\|} = \frac{|-12|}{\sqrt{14}} = \frac{6\sqrt{14}}{7}$$

98. Point:  $Q(0, 0, 0)$

$$\text{Plane: } 5x + y - z - 9 = 0$$

$$\text{Normal to plane: } \mathbf{n} = \langle 5, 1, -1 \rangle$$

$$\text{Point in plane: } P(0, 9, 0)$$

$$\text{Vector } \overline{PQ} = \langle 0, -9, 0 \rangle$$

$$D = \frac{|\overline{PQ} \cdot \mathbf{n}|}{\|\mathbf{n}\|} = \frac{|-9|}{\sqrt{27}} = \sqrt{3}$$

99. Point:  $Q(2, 8, 4)$

$$\text{Plane: } 2x + y + z = 5$$

$$\text{Normal to plane: } \mathbf{n} = \langle 2, 1, 1 \rangle$$

$$\text{Point in plane: } P(0, 0, 5)$$

$$\text{Vector: } \overline{PQ} = \langle 2, 8, -1 \rangle$$

$$D = \frac{|\overline{PQ} \cdot \mathbf{n}|}{\|\mathbf{n}\|} = \frac{11}{\sqrt{6}} = \frac{11\sqrt{6}}{6}$$

100. Point:  $Q(1, 3, -1)$

$$\text{Plane: } 3x - 4y + 5z - 6 = 0$$

$$\text{Normal to plane: } \mathbf{n} = \langle 3, -4, 5 \rangle$$

$$\text{Point in plane: } P(2, 0, 0)$$

$$\text{Vector } \overline{PQ} = \langle -1, 3, -1 \rangle$$

$$D = \frac{|\overline{PQ} \cdot \mathbf{n}|}{\|\mathbf{n}\|} = \frac{|-20|}{\sqrt{50}} = 2\sqrt{2}$$

101. The normal vectors to the planes are  $\mathbf{n}_1 = \langle 1, -3, 4 \rangle$  and

$$\mathbf{n}_2 = \langle 1, -3, 4 \rangle$$
. Because  $\mathbf{n}_1 = \mathbf{n}_2$ , the planes are

parallel. Choose a point in each plane.

$$P(10, 0, 0) \text{ is a point in } x - 3y + 4z = 10.$$

$$Q(6, 0, 0) \text{ is a point in } x - 3y + 4z = 6.$$

$$\overline{PQ} = \langle -4, 0, 0 \rangle, D = \frac{|\overline{PQ} \cdot \mathbf{n}_1|}{\|\mathbf{n}_1\|} = \frac{4}{\sqrt{26}} = \frac{2\sqrt{26}}{13}$$

102. The normal vectors to the planes are  $\mathbf{n}_1 = \langle 4, -4, 9 \rangle$  and

$$\mathbf{n}_2 = \langle 4, -4, 9 \rangle$$
. Because  $\mathbf{n}_1 = \mathbf{n}_2$ , the planes are

parallel. Choose a point in each plane.

$$P(-5, 0, 3) \text{ is a point in } 4x - 4y + 9z = 7.$$

$$Q(0, 0, 2) \text{ is a point in } 4x - 4y + 9z = 18.$$

$$\overline{PQ} = \langle 5, 0, -1 \rangle$$

$$D = \frac{|\overline{PQ} \cdot \mathbf{n}_1|}{\|\mathbf{n}_1\|} = \frac{11}{\sqrt{113}} = \frac{11\sqrt{113}}{113}$$

103. The normal vectors to the planes are  $\mathbf{n}_1 = \langle -3, 6, 7 \rangle$  and

$$\mathbf{n}_2 = \langle 6, -12, -14 \rangle$$
. Because  $\mathbf{n}_2 = -2\mathbf{n}_1$ , the planes are

parallel. Choose a point in each plane.

$$P(0, -1, 1) \text{ is a point in } -3x + 6y + 7z = 1.$$

$$Q\left(\frac{25}{6}, 0, 0\right) \text{ is a point in } 6x - 12y - 14z = 25.$$

$$\overline{PQ} = \left\langle \frac{25}{6}, 1, -1 \right\rangle$$

$$D = \frac{|\overline{PQ} \cdot \mathbf{n}_1|}{\|\mathbf{n}_1\|} = \frac{|-27/2|}{\sqrt{94}} = \frac{27}{2\sqrt{94}} = \frac{27\sqrt{94}}{188}$$

- 104.** The normal vectors to the planes are  $\mathbf{n}_1 = \langle 2, 0, -4 \rangle$  and  $\mathbf{n}_2 = \langle 2, 0, -4 \rangle$ . Because  $\mathbf{n}_1 = \mathbf{n}_2$ , the planes are parallel. Choose a point in each plane.

$P(2, 0, 0)$  is a point in  $2x - 4z = 4$ .

$Q(5, 0, 0)$  is a point in  $2x - 4z = 10$ .

$$\overline{PQ} = \langle 3, 0, 0 \rangle, D = \frac{|\overline{PQ} \cdot \mathbf{n}_1|}{\|\mathbf{n}_1\|} = \frac{6}{\sqrt{20}} = \frac{3\sqrt{5}}{5}$$

- 105.**  $\mathbf{u} = \langle 4, 0, -1 \rangle$  is the direction vector for the line.

$Q(1, 5, -2)$  is the given point, and  $P(-2, 3, 1)$  is on the line.

$$\overline{PQ} = \langle 3, 2, -3 \rangle$$

$$\overline{PQ} \times \mathbf{u} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 2 & -3 \\ 4 & 0 & -1 \end{vmatrix} = \langle -2, -9, -8 \rangle$$

$$D = \frac{\|\overline{PQ} \times \mathbf{u}\|}{\|\mathbf{u}\|} = \frac{\sqrt{149}}{\sqrt{17}} = \frac{\sqrt{2533}}{17}$$

- 106.**  $\mathbf{u} = \langle 2, 1, 2 \rangle$  is the direction vector for the line.

$Q(1, -2, 4)$  is the given point, and  $P(0, -3, 2)$  is a point on the line (let  $t = 0$ ).

$$\overline{PQ} = \langle 1, 1, 2 \rangle$$

$$\overline{PQ} \times \mathbf{u} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 2 \\ 2 & 1 & 2 \end{vmatrix} = \langle 0, 2, -1 \rangle$$

$$D = \frac{\|\overline{PQ} \times \mathbf{u}\|}{\|\mathbf{u}\|} = \frac{\sqrt{5}}{\sqrt{9}} = \frac{\sqrt{5}}{3}$$

- 107.**  $\mathbf{u} = \langle -1, 1, -2 \rangle$  is the direction vector for the line.

$Q(-2, 1, 3)$  is the given point, and  $P(1, 2, 0)$  is on the line (let  $t = 0$  in the parametric equations for the line).

$$\overline{PQ} = \langle -3, -1, 3 \rangle$$

$$\overline{PQ} \times \mathbf{u} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -3 & -1 & 3 \\ -1 & 1 & -2 \end{vmatrix} = \langle -1, -9, -4 \rangle$$

$$D = \frac{\|\overline{PQ} \times \mathbf{u}\|}{\|\mathbf{u}\|} = \frac{\sqrt{1+81+16}}{\sqrt{1+1+4}} = \frac{\sqrt{98}}{6} = \frac{7}{\sqrt{3}} = \frac{7\sqrt{3}}{3}$$

- 108.**  $\mathbf{u} = \langle 0, 3, 1 \rangle$  is the direction vector for the line.

$Q(4, -1, 5)$  is the given point, and  $P(3, 1, 1)$  is on the line.

$$\overline{PQ} = \langle 1, -2, 4 \rangle$$

$$\overline{PQ} \times \mathbf{u} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -2 & 4 \\ 0 & 3 & 1 \end{vmatrix} = \langle -14, -1, 3 \rangle$$

$$D = \frac{\|\overline{PQ} \times \mathbf{u}\|}{\|\mathbf{u}\|} = \frac{\sqrt{14^2 + 1 + 9}}{\sqrt{9+1}} = \sqrt{\frac{206}{10}} = \sqrt{\frac{103}{5}} = \frac{\sqrt{515}}{5}$$

- 109.** The direction vector for  $L_1$  is  $\mathbf{v}_1 = \langle -1, 2, 1 \rangle$ .

The direction vector for  $L_2$  is  $\mathbf{v}_2 = \langle 3, -6, -3 \rangle$ .

Because  $\mathbf{v}_2 = -3\mathbf{v}_1$ , the lines are parallel.

Let  $Q(2, 3, 4)$  to be a point on  $L_1$  and  $P(0, 1, 4)$  a point on  $L_2$ .  $\overline{PQ} = \langle 2, 0, 0 \rangle$ .

$\mathbf{u} = \mathbf{v}_2$  is the direction vector for  $L_2$ .

$$\overline{PQ} \times \mathbf{v}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 2 & 0 \\ 3 & -6 & -3 \end{vmatrix} = \langle -6, 6, -18 \rangle$$

$$D = \frac{\|\overline{PQ} \times \mathbf{v}_2\|}{\|\mathbf{v}_2\|} = \frac{\sqrt{36+36+324}}{\sqrt{9+36+9}} = \sqrt{\frac{396}{54}} = \sqrt{\frac{22}{3}} = \frac{\sqrt{66}}{3}$$

- 110.** The direction vector for  $L_1$  is  $\mathbf{v}_1 = \langle 6, 9, -12 \rangle$ .

The direction vector for  $L_2$  is  $\mathbf{v}_2 = \langle 4, 6, -8 \rangle$ .

Because  $\mathbf{v}_1 = \frac{3}{2}\mathbf{v}_2$ , the lines are parallel.

Let  $Q(3, -2, 1)$  to be a point on  $L_1$  and  $P(-1, 3, 0)$  a point on  $L_2$ .  $\overline{PQ} = \langle 4, -5, 1 \rangle$ .

$\mathbf{u} = \mathbf{v}_2$  is the direction vector for  $L_2$ .

$$\overline{PQ} \times \mathbf{v}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & -5 & 1 \\ 4 & 6 & -8 \end{vmatrix} = \langle 34, 36, 44 \rangle$$

$$D = \frac{\|\overline{PQ} \times \mathbf{v}_2\|}{\|\mathbf{v}_2\|} = \frac{\sqrt{34^2 + 36^2 + 44^2}}{\sqrt{16+36+64}} = \frac{\sqrt{4388}}{\sqrt{116}} = \sqrt{\frac{1097}{29}} = \frac{\sqrt{31813}}{29}$$

- 111.** The parametric equations of a line  $L$  parallel to  $\mathbf{v} = \langle a, b, c \rangle$  and passing through the point  $P(x_1, y_1, z_1)$  are

$$x = x_1 + at, y = y_1 + bt, z = z_1 + ct.$$

The symmetric equations are

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}.$$

- 112.** The equation of the plane containing  $P(x_1, y_1, z_1)$  and having normal vector  $\mathbf{n} = \langle a, b, c \rangle$  is

$$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0.$$

You need  $\mathbf{n}$  and  $P$  to find the equation.

- 113.** Simultaneously solve the two linear equations representing the planes and substitute the values back into one of the original equations. Then choose a value for  $t$  and form the corresponding parametric equations for the line of intersection.

- 114.**  $x = a$ : plane parallel to  $yz$ -plane containing  $(a, 0, 0)$

- $y = b$ : plane parallel to  $xz$ -plane containing  $(0, b, 0)$

- $z = c$ : plane parallel to  $xy$ -plane containing  $(0, 0, c)$

- 121.**  $0.92x - 1.03y + z = 0.02 \Rightarrow z = 0.02 - 0.92x + 1.03y$

(a)

Year	1999	2000	2001	2002	2003	2004	2005
$x$	1.4	1.4	1.4	1.6	1.6	1.7	1.7
$y$	7.3	7.1	7.0	7.0	6.9	6.9	6.9
$z$	6.2	6.1	5.9	5.8	5.6	5.5	5.6
Model $z$	6.25	6.05	5.94	5.76	5.66	5.56	5.56

The approximations are close to the actual values.

- (b) According to the model, if  $x$  and  $z$  decrease, then so will  $y$ . (Answers will vary.)

- 122.** On one side you have the points  $(0, 0, 0)$ ,  $(6, 0, 0)$ , and  $(-1, -1, 8)$ .

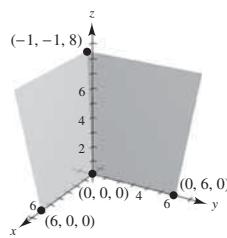
$$\mathbf{n}_1 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 6 & 0 & 0 \\ -1 & -1 & 8 \end{vmatrix} = -48\mathbf{j} - 6\mathbf{k}$$

On the adjacent side you have the points  $(0, 0, 0)$ ,  $(0, 6, 0)$ , and  $(-1, -1, 8)$ .

$$\mathbf{n}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 6 & 0 \\ -1 & -1 & 8 \end{vmatrix} = 48\mathbf{i} + 6\mathbf{k}$$

$$\cos \theta = \frac{|\mathbf{n}_1 \cdot \mathbf{n}_2|}{\|\mathbf{n}_1\| \|\mathbf{n}_2\|} = \frac{36}{2340} = \frac{1}{65}$$

$$\theta = \arccos \frac{1}{65} \approx 89.1^\circ$$



- 115.** (a) The planes are parallel if their normal vectors are parallel:

$$\langle a_1, b_1, c_1 \rangle = t \langle a_2, b_2, c_2 \rangle, t \neq 0$$

- (b) The planes are perpendicular if their normal vectors are perpendicular:

$$\langle a_1, b_1, c_1 \rangle \cdot \langle a_2, b_2, c_2 \rangle = 0$$

- 116.** Yes. If  $\mathbf{v}_1$  and  $\mathbf{v}_2$  are the direction vectors for the lines  $L_1$  and  $L_2$ , then  $\mathbf{v} = \mathbf{v}_1 \times \mathbf{v}_2$  is perpendicular to both  $L_1$  and  $L_2$ .

- 117.** An equation for the plane is

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1 \Rightarrow bcx + acy + abz = abc$$

For example, letting  $y = z = 0$ , the  $x$ -intercept is  $(a, 0, 0)$ .

- 118.** (a) Matches (iii)

- (b) Matches (i)

- (c) Matches (iv)

- (d) Matches (ii)

- 119.** Sphere

$$(x - 3)^2 + (y + 2)^2 + (z - 5)^2 = 16$$

- 120.** Parallel planes

$$4x - 3y + z = 10 \pm 4\|\mathbf{n}\| = 10 \pm 4\sqrt{26}$$

123.  $L_1: x_1 = 6 + t, y_1 = 8 - t, z_1 = 3 + t$

$L_2: x_2 = 1 + t, y_2 = 2 + t, z_2 = 2t$

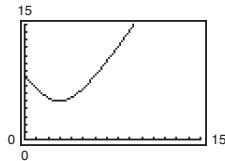
(a) At  $t = 0$ , the first insect is at  $P_1(6, 8, 3)$  and the second insect is at  $P_2(1, 2, 0)$ .

$$\text{Distance} = \sqrt{(6-1)^2 + (8-2)^2 + (3-0)^2} = \sqrt{70} \approx 8.37 \text{ inches}$$

$$(b) \text{Distance} = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2} = \sqrt{5^2 + (6-2t)^2 + (3-t)^2} = \sqrt{5t^2 - 30t + 70}, 0 \leq t \leq 10$$

(c) The distance is never zero.

(d) Using a graphing utility, the minimum distance is 5 inches when  $t = 3$  minutes.



124. First find the distance  $D$  from the point  $Q(-3, 2, 4)$  to the plane. Let  $P(4, 0, 0)$  be on the plane.

$\mathbf{n} = \langle 2, 4, -3 \rangle$  is the normal to the plane.

$$D = \frac{|\overline{PQ} \cdot \mathbf{n}|}{\|\mathbf{n}\|} = \frac{|\langle -7, 2, 4 \rangle \cdot \langle 2, 4, -3 \rangle|}{\sqrt{4+16+9}} = \frac{|-14+8-12|}{\sqrt{29}} = \frac{18}{\sqrt{29}} = \frac{18\sqrt{29}}{29}$$

The equation of the sphere with center  $(-3, 2, 4)$  and radius  $18\sqrt{29}/29$  is  $(x+3)^2 + (y-2)^2 + (z-4)^2 = \frac{324}{29}$ .

125. The direction vector  $\mathbf{v}$  of the line is the normal to the plane,  $\mathbf{v} = \langle 3, -1, 4 \rangle$ .

The parametric equations of the line are

$$x = 5 + 3t, y = 4 - t, z = -3 + 4t.$$

To find the point of intersection, solve for  $t$  in the following equation:

$$3(5 + 3t) - (4 - t) + 4(-3 + 4t) = 7$$

$$26t = 8$$

$$t = \frac{4}{13}$$

Point of intersection:

$$\left(5 + 3\left(\frac{4}{13}\right), 4 - \frac{4}{13}, -3 + 4\left(\frac{4}{13}\right)\right) = \left(\frac{77}{13}, \frac{48}{13}, -\frac{22}{13}\right)$$

126. The normal to the plane,  $\mathbf{n} = \langle 2, -1, -3 \rangle$  is perpendicular to the direction vector  $\mathbf{v} = \langle 2, 4, 0 \rangle$  of the line because  $\langle 2, -1, -3 \rangle \cdot \langle 2, 4, 0 \rangle = 0$ .

So, the plane is parallel to the line. To find the distance between them, let  $Q(-2, -1, 4)$  be on the line and

$$P(2, 0, 0)$$
 on the plane.  $\overline{PQ} = \langle -4, -1, 4 \rangle$ .

$$\begin{aligned} D &= \frac{|\overline{PQ} \cdot \mathbf{n}|}{\|\mathbf{n}\|} \\ &= \frac{|\langle -4, -1, 4 \rangle \cdot \langle 2, -1, -3 \rangle|}{\sqrt{4+1+9}} = \frac{19}{\sqrt{14}} = \frac{19\sqrt{14}}{14} \end{aligned}$$

127. The direction vector of the line  $L$  through  $(1, -3, 1)$  and  $(3, -4, 2)$  is  $\mathbf{v} = \langle 2, -1, 1 \rangle$ .

The parametric equations for  $L$  are

$$x = 1 + 2t, y = -3 - t, z = 1 + t.$$

Substituting these equations into the equation of the plane gives

$$(1 + 2t) - (-3 - t) + (1 + t) = 2$$

$$4t = -3$$

$$t = -\frac{3}{4}.$$

Point of intersection:

$$\left(1 + 2\left(-\frac{3}{4}\right), -3 + \frac{3}{4}, 1 - \frac{3}{4}\right) = \left(-\frac{1}{2}, -\frac{9}{4}, \frac{1}{4}\right)$$

128. The unknown line  $L$  is perpendicular to the normal vector  $\mathbf{n} = \langle 1, 1, 1 \rangle$  of the plane, and perpendicular to the direction vector  $\mathbf{u} = \langle 1, 1, -1 \rangle$ . So, the direction vector of  $L$  is

$$\mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 1 \\ 1 & 1 & -1 \end{vmatrix} = \langle -2, 2, 0 \rangle.$$

The parametric equations for  $L$  are  $x = 1 - 2t, y = 2t, z = 2$ .

129. True

130. False. They may be skew lines.

(See Section Project)

131. True

132. False. The lines  $x = t$ ,  $y = 0$ ,  $z = 1$  and  $x = 0$ ,  $y = t$ ,  $z = 1$  are both parallel to the plane  $z = 0$ , but the lines are not parallel.

133. False. Planes  $7x + y - 11z = 5$  and  $5x + 2y - 4z = 1$  are both perpendicular to plane  $2x - 3y + z = 3$ , but are not parallel.

134. True.

## Section 11.6 Surfaces in Space

1. Ellipsoid

Matches graph (c)

2. Hyperboloid of two sheets

Matches graph (e)

3. Hyperboloid of one sheet

Matches graph (f)

4. Elliptic cone

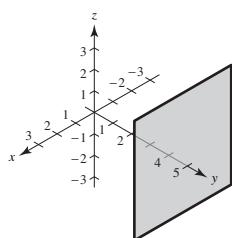
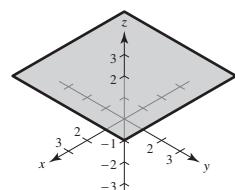
Matches graph (b)

5. Elliptic paraboloid

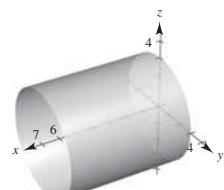
Matches graph (d)

6. Hyperbolic paraboloid

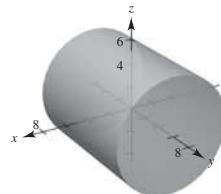
Matches graph (a)

7.  $y = 5$ Plane is parallel to the  $xz$ -plane.8.  $z = 2$ Plane is parallel to the  $xy$ -plane.9.  $y^2 + z^2 = 9$ 

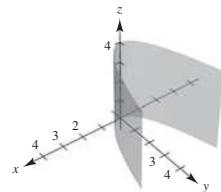
The  $x$ -coordinate is missing so you have a right circular cylinder with rulings parallel to the  $x$ -axis. The generating curve is a circle.

10.  $x^2 + z^2 = 25$ 

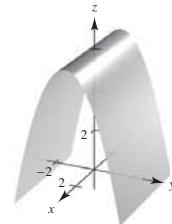
The  $y$ -coordinate is missing so you have a right circular cylinder with rulings parallel to the  $y$ -axis. The generating curve is a circle.

11.  $y = x^2$ 

The  $z$ -coordinate is missing so you have a parabolic cylinder with rulings parallel to the  $z$ -axis. The generating curve is a parabola.

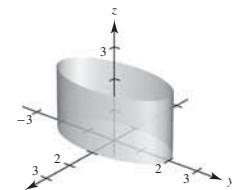
12.  $y^2 + z = 6$ 

The  $x$ -coordinate is missing so you have a parabolic cylinder with the rulings parallel to the  $x$ -axis. The generating curve is a parabola.

13.  $4x^2 + y^2 = 4$ 

$$\frac{x^2}{1} + \frac{y^2}{4} = 1$$

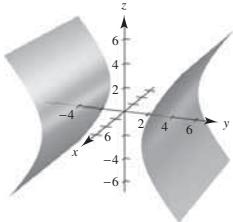
The  $z$ -coordinate is missing so you have an elliptic cylinder with rulings parallel to the  $z$ -axis. The generating curve is an ellipse.



14.  $y^2 - z^2 = 16$

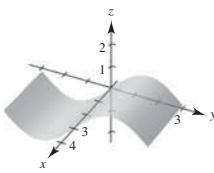
$$\frac{y^2}{16} - \frac{z^2}{16} = 1$$

The  $x$ -coordinate is missing so you have a hyperbolic cylinder with rulings parallel to the  $x$ -axis. The generating curve is a hyperbola.



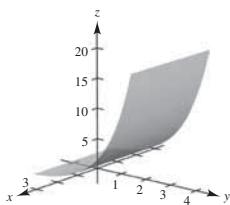
15.  $z = \sin y$

The  $x$ -coordinate is missing so you have a cylindrical surface with rulings parallel to the  $x$ -axis. The generating curve is the sine curve.



16.  $z = e^y$

The  $x$ -coordinate is missing so you have a cylindrical surface with rulings parallel to the  $x$ -axis. The generating curve is the exponential curve.



17.  $z = x^2 + y^2$

(a) You are viewing the paraboloid from the  $x$ -axis:

$$(20, 0, 0)$$

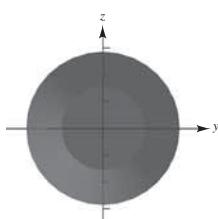
(b) You are viewing the paraboloid from above, but not on the  $z$ -axis:  $(10, 10, 20)$

(c) You are viewing the paraboloid from the  $z$ -axis:  $(0, 0, 20)$

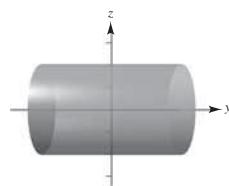
(d) You are viewing the paraboloid from the  $y$ -axis:  $(0, 20, 0)$

18.  $y^2 + z^2 = 4$

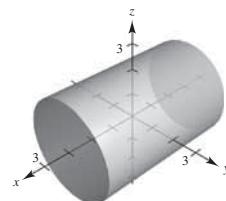
(a) From  $(10, 0, 0)$ :



(b) From  $(0, 10, 0)$ :



(c) From  $(10, 10, 10)$ :



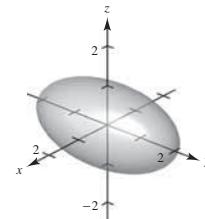
19.  $\frac{x^2}{1} + \frac{y^2}{4} + \frac{z^2}{1} = 1$

Ellipsoid

$$xy\text{-trace: } \frac{x^2}{1} + \frac{y^2}{4} = 1 \text{ ellipse}$$

$$xz\text{-trace: } x^2 + z^2 = 1 \text{ circle}$$

$$yz\text{-trace: } \frac{y^2}{4} + \frac{z^2}{1} = 1 \text{ ellipse}$$



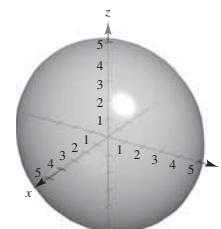
20.  $\frac{x^2}{16} + \frac{y^2}{25} + \frac{z^2}{25} = 1$

Ellipsoid

$$xy\text{-trace: } \frac{x^2}{16} + \frac{y^2}{25} = 1 \text{ ellipse}$$

$$xz\text{-trace: } \frac{x^2}{16} + \frac{z^2}{25} = 1 \text{ ellipse}$$

$$yz\text{-trace: } y^2 + z^2 = 25 \text{ circle}$$



21.  $16x^2 - y^2 + 16z^2 = 4$

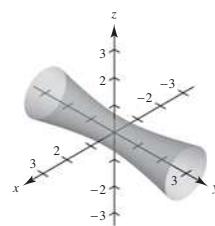
$$4x^2 - \frac{y^2}{4} + 4z^2 = 1$$

Hyperboloid of one sheet

$$xy\text{-trace: } 4x^2 - \frac{y^2}{4} = 1 \text{ hyperbola}$$

$$xz\text{-trace: } 4(x^2 + z^2) = 1 \text{ circle}$$

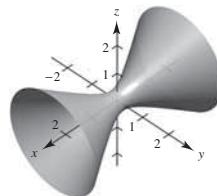
$$yz\text{-trace: } \frac{-y^2}{4} + 4z^2 = 1 \text{ hyperbola}$$



22.  $-8x^2 + 18y^2 + 18z^2 = 2$

$9y^2 + 9z^2 - 4x^2 = 1$

Hyperboloid of one sheet

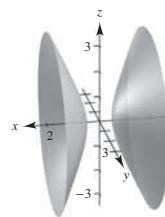
xy-trace:  $9y^2 - 4x^2 = 1$  hyperbolayz-trace:  $9y^2 + 9z^2 = 1$  circlexz-trace:  $9z^2 - 4x^2 = 1$  hyperbola

23.  $4x^2 - y^2 - z^2 = 1$

Hyperboloid of two sheets

xy-trace:  $4x^2 - y^2 = 1$  hyperbola

yz-trace: none

xz-trace:  $4x^2 - z^2 = 1$  hyperbola

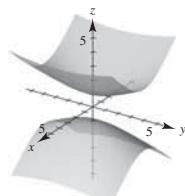
24.  $z^2 - x^2 - \frac{y^2}{4} = 1$

Hyperboloid of two sheets

xy-trace: none

xz-trace:  $z^2 - x^2 = 1$  hyperbolayz-trace:  $z^2 - \frac{y^2}{4} = 1$  hyperbola

$$z = \pm\sqrt{10}; \frac{x^2}{9} + \frac{y^2}{36} = 1 \text{ ellipse}$$

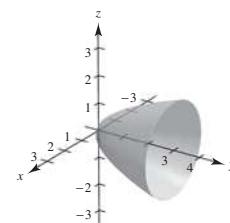


25.  $x^2 - y + z^2 = 0$

Elliptic paraboloid

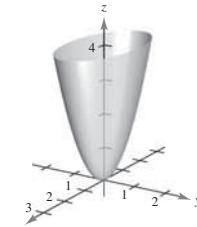
xy-trace:  $y = x^2$ xz-trace:  $x^2 + z^2 = 0$ ,  
point  $(0, 0, 0)$ yz-trace:  $y = z^2$ 

$y = 1: x^2 + z^2 = 1$



26.  $z = x^2 + 4y^2$

Elliptic paraboloid

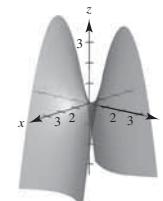
xy-trace: point  $(0, 0, 0)$ xz-trace:  $z = x^2$  parabolayz-trace:  $z = 4y^2$  parabola

27.  $x^2 - y^2 + z = 0$

Hyperbolic paraboloid

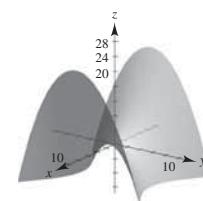
xy-trace:  $y = \pm x$ xz-trace:  $z = -x^2$ yz-trace:  $z = y^2$ 

$y = \pm 1: z = 1 - x^2$



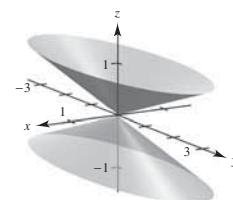
28.  $3z = -y^2 + x^2$

Hyperbolic paraboloid

xy-trace:  $y = \pm x$ xz-trace:  $z = \frac{1}{3}x^2$ yz-trace:  $z = -\frac{1}{3}y^2$ 

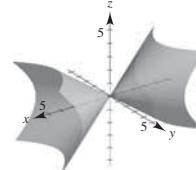
29.  $z^2 = x^2 + \frac{y^2}{9}$

Elliptic cone

xy-trace: point  $(0, 0, 0)$ xz-trace:  $z = \pm x$ yz-trace:  $z = \pm \frac{y}{3}$ When  $z = \pm 1$ ,  $x^2 + \frac{y^2}{9} = 1$  ellipse

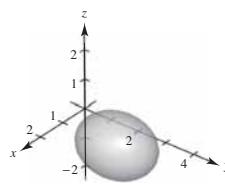
30.  $x^2 = 2y^2 + 2z^2$

Elliptic Cone

xy-trace:  $x = \pm\sqrt{2}y$ xz-trace:  $x = \pm\sqrt{2}z$ yz-trace: point:  $(0, 0, 0)$ 

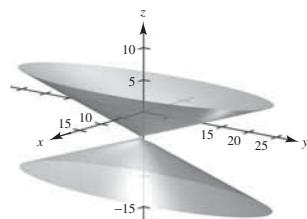
31.  $16x^2 + 9y^2 + 16z^2 - 32x - 36y + 36 = 0$   
 $16(x^2 - 2x + 1) + 9(y^2 - 4y + 4) + 16z^2 = -36 + 16 + 36$   
 $16(x - 1)^2 + 9(y - 2)^2 + 16z^2 = 16$   
 $\frac{(x - 1)^2}{1} + \frac{(y - 2)^2}{16/9} + \frac{z^2}{1} = 1$

Ellipsoid with center  $(1, 2, 0)$ .

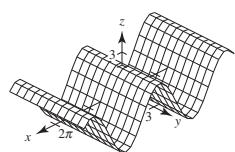


32.  $9x^2 + y^2 - 9z^2 - 54x - 4y - 54z + 4 = 0$   
 $9(x^2 - 6x + 9) + (y^2 - 4y + 4) - 9(z^2 + 6z + 9) = 81 - 81$   
 $9(x - 3)^2 + (y - 2)^2 - 9(z + 3)^2 = 0$

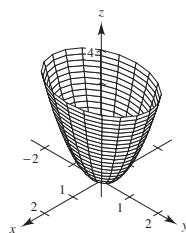
Elliptic cone with center  $(3, 2, -3)$ .



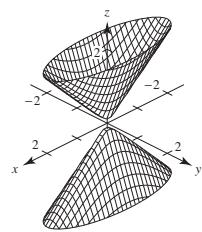
33.  $z = 2 \cos x$



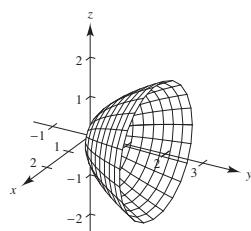
34.  $z = x^2 + 0.5y^2$



35.  $z^2 = x^2 + 7.5y^2$   
 $z = \pm\sqrt{x^2 + 7.5y^2}$

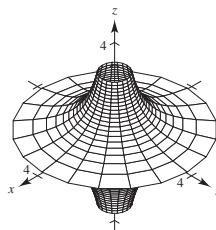


36.  $3.25y = x^2 + z^2$   
 $z^2 = 3.25y - x^2$   
 $z = \pm\sqrt{3.25y - x^2}$

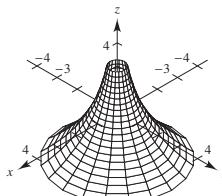


37.  $x^2 + y^2 = \left(\frac{2}{z}\right)^2$

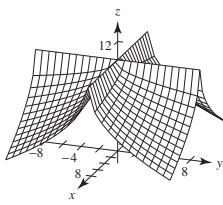
$$y = \pm\sqrt{\frac{4}{z^2} - x^2}$$



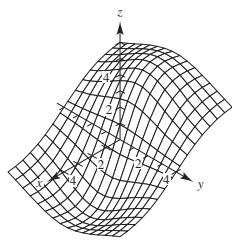
38.  $x^2 + y^2 = e^{-z}$   
 $-\ln(x^2 + y^2) = z$



39.  $z = 10 - \sqrt{|xy|}$



40.  $z = \frac{-x}{8 + x^2 + y^2}$

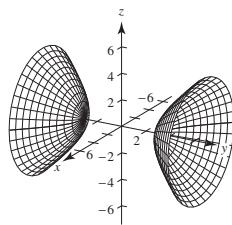


41.  $6x^2 - 4y^2 + 6z^2 = -36$

$$6z^2 = 4y^2 - 6x^2 - 36$$

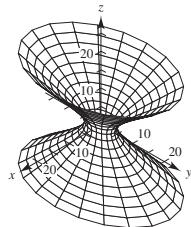
$$3z^2 = 2y^2 - 3x^2 - 18$$

$$z = \pm \frac{1}{\sqrt{3}} \sqrt{2y^2 - 3x^2 - 18}$$



42.  $9x^2 + 4y^2 - 8z^2 = 72$

$$z = \pm \sqrt{\frac{9}{8}x^2 + \frac{1}{2}y^2 - 9}$$

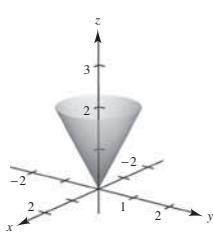


43.  $z = 2\sqrt{x^2 + y^2}$

$$z = 2$$

$$2\sqrt{x^2 + y^2} = 2$$

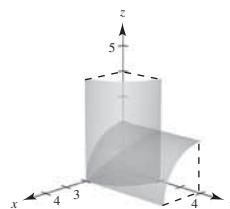
$$x^2 + y^2 = 1$$



44.  $z = \sqrt{4 - x^2}$

$$y = \sqrt{4 - x^2}$$

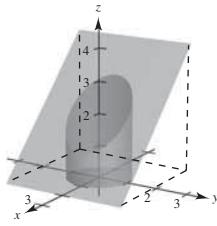
$$x = 0, y = 0, z = 0$$



45.  $x^2 + y^2 = 1$

$$x + z = 2$$

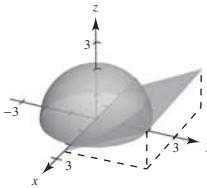
$$z = 0$$



46.  $z = \sqrt{4 - x^2 - y^2}$

$$y = 2z$$

$$z = 0$$



47.  $x^2 + z^2 = [r(y)]^2$  and  $z = r(y) = \pm 2\sqrt{y}$ ; so,  
 $x^2 + z^2 = 4y$ .

48.  $x^2 + z^2 = [r(y)]^2$  and  $z = r(y) = 3y$ ; so,

$$x^2 + z^2 = 9y^2.$$

49.  $x^2 + y^2 = [r(z)]^2$  and  $y = r(z) = \frac{z}{2}$ ; so,

$$x^2 + y^2 = \frac{z^2}{4}, 4x^2 + 4y^2 = z^2.$$

50.  $y^2 + z^2 = [r(x)]^2$  and  $z = r(x) = \frac{1}{2}\sqrt{4 - x^2}$ ; so,  
 $y^2 + z^2 = \frac{1}{4}(4 - x^2)$ ,  $x^2 + 4y^2 + 4z^2 = 4$ .

51.  $y^2 + z^2 = [r(x)]^2$  and  $y = r(x) = \frac{2}{x}$ ; so,

$$y^2 + z^2 = \left(\frac{2}{x}\right)^2, y^2 + z^2 = \frac{4}{x^2}.$$

52.  $x^2 + y^2 = [r(z)]^2$  and  $y = r(z) = e^z$ ; so,

$$x^2 + y^2 = e^{2z}.$$

53.  $x^2 + y^2 - 2z = 0$

$$x^2 + y^2 = (\sqrt{2z})^2$$

Equation of generating curve:  $y = \sqrt{2z}$  or  $x = \sqrt{2z}$

54.  $x^2 + z^2 = \cos^2 y$

Equation of generating curve:  $x = \cos y$  or  $z = \cos y$

55. Let  $C$  be a curve in a plane and let  $L$  be a line not in a parallel plane. The set of all lines parallel to  $L$  and intersecting  $C$  is called a cylinder.  $C$  is called the generating curve of the cylinder, and the parallel lines are called rulings.

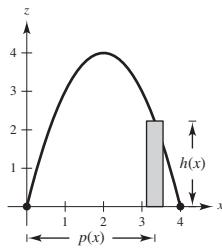
56. The trace of a surface is the intersection of the surface with a plane. You find a trace by setting one variable equal to a constant, such as  $x = 0$  or  $z = 2$ .

57. See pages 814 and 815.

58. In the  $xz$ -plane,  $z = x^2$  is a parabola.

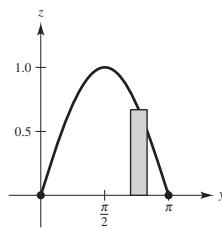
In three-space,  $z = x^2$  is a cylinder.

59.  $V = 2\pi \int_0^4 x(4x - x^2) dx = 2\pi \left[ \frac{4x^3}{3} - \frac{x^4}{4} \right]_0^4 = \frac{218\pi}{3}$



60.  $V = 2\pi \int_0^\pi y \sin y dy$

$$= 2\pi [\sin y - y \cos y]_0^\pi = 2\pi^2$$



61.  $z = \frac{x^2}{2} + \frac{y^2}{4}$

(a) When  $z = 2$  we have  $2 = \frac{x^2}{2} + \frac{y^2}{4}$ , or

$$1 = \frac{x^2}{4} + \frac{y^2}{8}$$

Major axis:  $2\sqrt{8} = 4\sqrt{2}$

Minor axis:  $2\sqrt{4} = 4$

$$c^2 = a^2 - b^2, c^2 = 4, c = 2$$

Foci:  $(0, \pm 2, 2)$

(b) When  $z = 8$  we have  $8 = \frac{x^2}{2} + \frac{y^2}{4}$ , or

$$1 = \frac{x^2}{16} + \frac{y^2}{32}$$

Major axis:  $2\sqrt{32} = 8\sqrt{2}$

Minor axis:  $2\sqrt{16} = 8$

$$c^2 = 32 - 16 = 16, c = 4$$

Foci:  $(0, \pm 4, 8)$

62.  $z = \frac{x^2}{2} + \frac{y^2}{4}$

(a) When  $y = 4$  you have  $z = \frac{x^2}{2} + 4$ ,

$$4\left(\frac{1}{2}\right)(z - 4) = x^2.$$

Focus:  $\left(0, 4, \frac{9}{2}\right)$

(b) When  $x = 2$  you have

$$z = 2 + \frac{y^2}{4}, 4(z - 2) = y^2.$$

Focus:  $(2, 0, 3)$

63. If  $(x, y, z)$  is on the surface, then

$$(y + 2)^2 = x^2 + (y - 2)^2 + z^2$$

$$y^2 + 4y + 4 = x^2 + y^2 - 4y + 4 + z^2$$

$$x^2 + z^2 = 8y$$

Elliptic paraboloid

Traces parallel to  $xz$ -plane are circles.

64. If  $(x, y, z)$  is on the surface, then

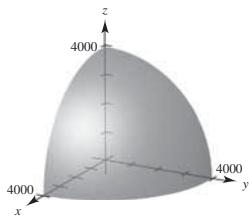
$$z^2 = x^2 + y^2 + (z - 4)^2$$

$$z^2 = x^2 + y^2 + z^2 - 8z + 16$$

$$8z = x^2 + y^2 + 16 \Rightarrow z = \frac{x^2}{8} + \frac{y^2}{8} + 2$$

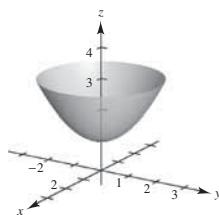
Elliptic paraboloid shifted up 2 units. Traces parallel to  $xy$ -plane are circles.

65.  $\frac{x^2}{3963^2} + \frac{y^2}{3963^2} + \frac{z^2}{3950^2} = 1$

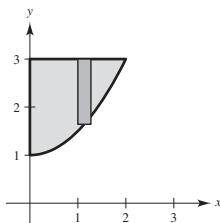


66. (a)  $x^2 + y^2 = [r(z)]^2$   
 $= [\sqrt{2(z-1)}]^2$

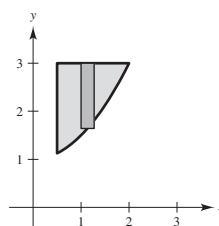
$$x^2 + y^2 - 2z + 2 = 0$$



(b)  $V = 2\pi \int_0^2 x \left[ 3 - \left( \frac{1}{2}x^2 + 1 \right) \right] dx$   
 $= 2\pi \int_0^2 \left( 2x - \frac{1}{2}x^3 \right) dx$   
 $= 2\pi \left[ x^2 - \frac{x^4}{8} \right]_0^2 = 4\pi \approx 12.6 \text{ cm}^3$



(c)  $V = 2\pi \int_{1/2}^2 x \left[ 3 - \left( \frac{1}{2}x^2 + 1 \right) \right] dx$   
 $= 2\pi \int_{1/2}^2 \left( 2x - \frac{1}{2}x^3 \right) dx$   
 $= 2\pi \left[ x^2 - \frac{x^4}{8} \right]_{1/2}^2$   
 $= 4 - \frac{31\pi}{64} = \frac{225\pi}{64} \approx 11.04 \text{ cm}^3$



67.  $z = \frac{y^2}{b^2} - \frac{x^2}{a^2}, z = bx + ay$

$$\begin{aligned} bx + ay &= \frac{y^2}{b^2} - \frac{x^2}{a^2} \\ \frac{1}{a^2} \left( x^2 + a^2bx + \frac{a^4b^2}{4} \right) &= \frac{1}{b^2} \left( y^2 - ab^2y + \frac{a^2b^4}{4} \right) \\ \frac{\left( x + \frac{a^2b}{2} \right)^2}{a^2} &= \frac{\left( y - \frac{ab^2}{2} \right)^2}{b^2} \\ y &= \pm \frac{b}{a} \left( x + \frac{a^2b}{2} \right) + \frac{ab^2}{2} \end{aligned}$$

Letting  $x = at$ , you obtain the two intersecting lines  
 $x = at, y = -bt, z = 0$  and  $x = at,$   
 $y = bt + ab^2, z = 2abt + a^2b^2.$

68. Equating twice the first equation with the second equation:

$$\begin{aligned} 2x^2 + 6y^2 - 4z^2 + 4y - 8 &= 2x^2 + 6y^2 - 4z^2 - 3x - 2 \\ 4y - 8 &= -3x - 2 \\ 3x + 4y &= 6, \text{ a plane} \end{aligned}$$

69. True. A sphere is a special case of an ellipsoid (centered at origin, for example)

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

having  $a = b = c.$

70. False. For example, the surface  $x^2 + z^2 = e^{-2y}$  can be formed by revolving the graph of  $x = e^{-y}$  about the  $y$ -axis, as the graph of  $z = e^{-y}$  about the  $y$ -axis.

71. False. The trace  $x = 2$  of the ellipsoid

$$\frac{x^2}{4} + \frac{y^2}{9} + z^2 = 1 \text{ is the point } (2, 0, 0).$$

72. False. Traces perpendicular to the axis are ellipses.

73. The Klein bottle *does not* have both an “inside” and an “outside.” It is formed by inserting the small open end through the side of the bottle and making it contiguous with the top of the bottle.

## Section 11.7 Cylindrical and Spherical Coordinates

1.  $(-7, 0, 5)$ , cylindrical

$$x = r \cos \theta = -7 \cos 0 = -7$$

$$y = r \sin \theta = -7 \sin 0 = 0$$

$$z = 5$$

$(-7, 0, 5)$ , rectangular

2.  $(2, -\pi, -4)$ , cylindrical

$$x = r \cos \theta = 2 \cos(-\pi) = -2$$

$$y = r \sin \theta = 2 \sin(-\pi) = 0$$

$$z = -4$$

$(-2, 0, -4)$ , rectangular

3.  $\left(3, \frac{\pi}{4}, 1\right)$ , cylindrical

$$x = 3 \cos \frac{\pi}{4} = \frac{3\sqrt{2}}{2}$$

$$y = 3 \sin \frac{\pi}{4} = \frac{3\sqrt{2}}{2}$$

$$z = 1$$

$\left(\frac{3\sqrt{2}}{2}, \frac{3\sqrt{2}}{2}, 1\right)$ , rectangular

4.  $\left(6, -\frac{\pi}{4}, 2\right)$ , cylindrical

$$x = 6 \cos\left(-\frac{\pi}{4}\right) = 3\sqrt{2}$$

$$y = 6 \sin\left(-\frac{\pi}{4}\right) = -3\sqrt{2}$$

$$z = 2$$

$(3\sqrt{2}, -3\sqrt{2}, 2)$ , rectangular

5.  $\left(4, \frac{7\pi}{6}, 3\right)$ , cylindrical

$$x = 4 \cos \frac{7\pi}{6} = -2\sqrt{3}$$

$$y = 4 \sin \frac{7\pi}{6} = -2$$

$$z = 3$$

$(-2\sqrt{3}, -2, 3)$ , rectangular

6.  $\left(-0.5, \frac{4\pi}{3}, 8\right)$ , cylindrical

$$x = -\frac{1}{2} \cos \frac{4\pi}{3} = \frac{1}{4}$$

$$y = -\frac{1}{2} \sin \frac{4\pi}{3} = \frac{\sqrt{3}}{4}$$

$$z = 8$$

$\left(\frac{1}{4}, \frac{\sqrt{3}}{4}, 8\right)$ , rectangular

7.  $(0, 5, 1)$ , rectangular

$$r = \sqrt{(0)^2 + (5)^2} = 5$$

$$\theta = \arctan \frac{5}{0} = \frac{\pi}{2}$$

$$z = 1$$

$\left(5, \frac{\pi}{2}, 1\right)$ , cylindrical

8.  $(2\sqrt{2}, -2\sqrt{2}, 4)$ , rectangular

$$r = \sqrt{(2\sqrt{2})^2 + (-2\sqrt{2})^2} = 4$$

$$\theta = \arctan(-1) = -\frac{\pi}{4}$$

$$z = -4$$

$\left(4, -\frac{\pi}{4}, 4\right)$ , cylindrical

9.  $(2, -2, -4)$ , rectangular

$$r = \sqrt{2^2 + (-2)^2} = 2\sqrt{2}$$

$$\theta = \arctan(-1) = -\frac{\pi}{4}$$

$$z = -4$$

$\left(2\sqrt{2}, -\frac{\pi}{4}, -4\right)$ , cylindrical

10.  $(3, -3, 7)$ , rectangular

$$r = \sqrt{3^2 + (-3)^2} = \sqrt{18} = 3\sqrt{2}$$

$$\theta = \arctan(-1) = -\frac{\pi}{4}$$

$$z = 7$$

$\left(3\sqrt{2}, -\frac{\pi}{4}, 7\right)$ , cylindrical

11.  $(1, \sqrt{3}, 4)$ , rectangular

$$r = \sqrt{1^2 + (\sqrt{3})^2} = 2$$

$$\theta = \arctan \sqrt{3} = \frac{\pi}{3}$$

$$z = 4$$

$$\left(2, \frac{\pi}{3}, 4\right), \text{ cylindrical}$$

12.  $(2\sqrt{3}, -2, 6)$ , rectangular

$$r = \sqrt{12 + 4} = 4$$

$$\theta = \arctan\left(-\frac{1}{\sqrt{3}}\right) = \frac{5\pi}{6}$$

$$z = 6$$

$$\left(4, -\frac{\pi}{6}, 6\right), \text{ cylindrical}$$

13.  $z = 4$  is the equation in cylindrical coordinates.

(plane)

14.  $x = 9$ , rectangular equation

$$r \cos \theta = 9$$

$$r = 9 \sec \theta, \text{ cylindrical equation}$$

15.  $x^2 + y^2 + z^2 = 17$ , rectangular equation

$$r^2 + z^2 = 17, \text{ cylindrical equation}$$

16.  $z = x^2 + y^2 - 11$ , rectangular equation

$$z = r^2 - 11, \text{ cylindrical equation}$$

17.  $y = x^2$ , rectangular equation

$$r \sin \theta = (r \cos \theta)^2$$

$$\sin \theta = r \cos^2 \theta$$

$$r = \sec \theta \cdot \tan \theta, \text{ cylindrical equation}$$

18.  $x^2 + y^2 = 8x$ , rectangular equation

$$r^2 = 8r \cos \theta$$

$$r = 8 \cos \theta, \text{ cylindrical equation}$$

19.  $y^2 = 10 - z^2$ , rectangular equation

$$(r \sin \theta)^2 = 10 - z^2$$

$$r^2 \sin^2 \theta + z^2 = 10, \text{ cylindrical equation}$$

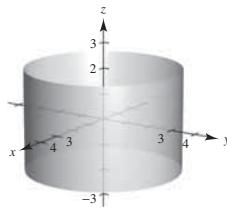
20.  $x^2 + y^2 + z^2 - 3z = 0$ , rectangular equation

$$r^2 + z^2 - 3z = 0, \text{ cylindrical equation}$$

21.  $r = 3$

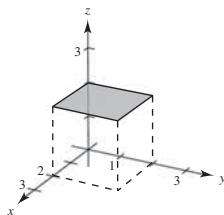
$$\sqrt{x^2 + y^2} = 3$$

$$x^2 + y^2 = 9$$



22.  $z = 2$

Same



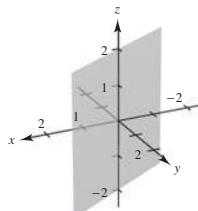
23.  $\theta = \frac{\pi}{6}$

$$\tan \frac{\pi}{6} = \frac{y}{x}$$

$$\frac{1}{\sqrt{3}} = \frac{y}{x}$$

$$x = \sqrt{3}y$$

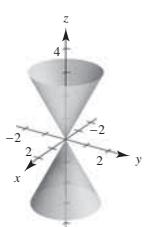
$$x - \sqrt{3}y = 0$$



24.  $r = \frac{z}{2}$

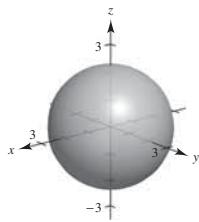
$$\sqrt{x^2 + y^2} = \frac{z}{2}$$

$$x^2 + y^2 - \frac{z^2}{4} = 0$$



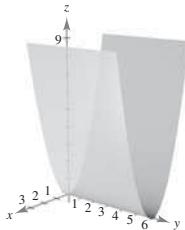
25.  $r^2 + z^2 = 5$

$x^2 + y^2 + z^2 = 5$



26.  $z = r^2 \cos^2 \theta$

$z = x^2$



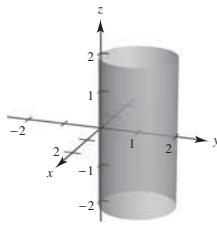
27.  $r = 2 \sin \theta$

$r^2 = 2r \sin \theta$

$x^2 + y^2 = 2y$

$x^2 + y^2 - 2y = 0$

$x^2 + (y - 1)^2 = 1$



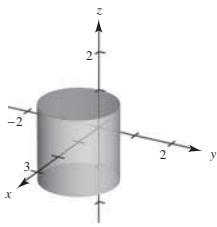
28.  $r = 2 \cos \theta$

$r^2 = 2r \cos \theta$

$x^2 + y^2 = 2x$

$x^2 + y^2 - 2x = 0$

$(x - 1)^2 + y^2 = 1$



29.  $(4, 0, 0)$ , rectangular

$\rho = \sqrt{4^2 + 0^2 + 0^2} = 4$

$\tan \theta = \frac{y}{x} = 0 \Rightarrow \theta = 0$

$\phi = \arccos 0 = \frac{\pi}{2}$

$\left(4, 0, \frac{\pi}{2}\right)$ , spherical

30.  $(-4, 0, 0)$ , rectangular

$\rho = \sqrt{(-4)^2 + 0^2 + 0^2} = 4$

$\tan \theta = \frac{y}{x} = 0 \Rightarrow \theta = 0$

$\theta = \arccos\left(\frac{z}{\rho}\right) = \arccos(0) = \frac{\pi}{2}$

$\left(4, 0, \frac{\pi}{2}\right)$ , spherical

31.  $(-2, 2\sqrt{3}, 4)$ , rectangular

$\rho = \sqrt{(-2)^2 + (2\sqrt{3})^2 + 4^2} = 4\sqrt{2}$

$\tan \theta = \frac{y}{x} = \frac{2\sqrt{3}}{-2} = -\sqrt{3}$

$\theta = \frac{2\pi}{3}$

$\phi = \arccos\frac{1}{\sqrt{2}} = \frac{\pi}{4}$

$\left(4\sqrt{2}, \frac{2\pi}{3}, \frac{\pi}{4}\right)$ , spherical

32.  $(2, 2, 4\sqrt{2})$ , rectangular

$\rho = \sqrt{2^2 + 2^2 + (4\sqrt{2})^2} = 2\sqrt{10}$

$\tan \theta = \frac{y}{x} = 1 \Rightarrow \theta = \frac{\pi}{4}$

$\phi = \arccos\frac{2}{\sqrt{5}}$

$\left(2\sqrt{10}, \frac{\pi}{4}, \arccos\frac{2}{\sqrt{5}}\right)$ , spherical

33.  $(\sqrt{3}, 1, 2\sqrt{3})$ , rectangular

$\rho = \sqrt{3 + 1 + 12} = 4$

$\tan \theta = \frac{y}{x} = \frac{1}{\sqrt{3}}$

$\theta = \frac{\pi}{6}$

$\phi = \arccos\frac{\sqrt{3}}{2} = \frac{\pi}{6}$

$\left(4, \frac{\pi}{6}, \frac{\pi}{6}\right)$ , spherical

34.  $(-1, 2, 1)$ , rectangular

$$\rho = \sqrt{(-1)^2 + 2^2 + 1^2} = \sqrt{6}$$

$$\tan \theta = \frac{y}{x} = -2 \Rightarrow \theta = \arctan(-2) + \pi$$

$$\phi = \arccos\left(\frac{1}{\sqrt{6}}\right)$$

$$\left(\sqrt{6}, \arctan(-2) + \pi, \arccos\frac{1}{\sqrt{6}}\right), \text{spherical}$$

35.  $\left(4, \frac{\pi}{6}, \frac{\pi}{4}\right)$ , spherical

$$x = 4 \sin \frac{\pi}{6} \cos \frac{\pi}{4} = \sqrt{6}$$

$$y = 4 \sin \frac{\pi}{6} \sin \frac{\pi}{4} = \sqrt{2}$$

$$z = 4 \cos \frac{\pi}{6} = 2\sqrt{2}$$

$$(\sqrt{6}, \sqrt{2}, 2\sqrt{2}), \text{rectangular}$$

36.  $\left(12, \frac{3\pi}{4}, \frac{\pi}{9}\right)$ , spherical

$$x = 12 \sin \frac{\pi}{9} \cos \frac{3\pi}{4} \approx -2.902$$

$$y = 12 \sin \frac{\pi}{9} \sin \frac{3\pi}{4} \approx 2.902$$

$$z = 12 \cos \frac{\pi}{9} \approx 11.276$$

$$(-2.902, 2.902, 11.276), \text{rectangular}$$

37.  $\left(12, -\frac{\pi}{4}, 0\right)$ , spherical

$$x = 12 \sin 0 \cos\left(-\frac{\pi}{4}\right) = 0$$

$$y = 12 \sin 0 \sin\left(-\frac{\pi}{4}\right) = 0$$

$$z = 12 \cos 0 = 12$$

$$(0, 0, 12), \text{rectangular}$$

38.  $\left(9, \frac{\pi}{4}, \pi\right)$ , spherical

$$x = 9 \sin \pi \cos \frac{\pi}{4} = 0$$

$$y = 9 \sin \pi \sin \frac{\pi}{4} = 0$$

$$z = 9 \cos \pi = -9$$

$$(0, 0, -9), \text{rectangular}$$

39.  $\left(5, \frac{\pi}{4}, \frac{3\pi}{4}\right)$ , spherical

$$x = 5 \sin \frac{3\pi}{4} \cos \frac{\pi}{4} = \frac{5}{2}$$

$$y = 5 \sin \frac{3\pi}{4} \sin \frac{\pi}{4} = \frac{5}{2}$$

$$z = 5 \cos \frac{3\pi}{4} = -\frac{5\sqrt{2}}{2}$$

$$\left(\frac{5}{2}, \frac{5}{2}, -\frac{5\sqrt{2}}{2}\right), \text{rectangular}$$

40.  $\left(6, \pi, \frac{\pi}{2}\right)$ , spherical

$$x = 6 \sin \frac{\pi}{2} \cos \pi = -6$$

$$y = 6 \sin \frac{\pi}{2} \sin \pi = 0$$

$$z = 6 \cos \frac{\pi}{2} = 0$$

$$(-6, 0, 0), \text{rectangular}$$

41.  $y = 2$ , rectangular equation

$$\rho \sin \phi \sin \theta = 2$$

$$\rho = 2 \csc \phi \csc \theta, \text{spherical equation}$$

42.  $z = 6$ , rectangular equation

$$\rho \cos \phi = 6$$

$$\rho = 6 \sec \phi, \text{spherical equation}$$

43.  $x^2 + y^2 + z^2 = 49$ , rectangular equation

$$\rho^2 = 49$$

$$\rho = 7, \text{spherical equation}$$

44.  $x^2 + y^2 - 3z^2 = 0$ , rectangular equation

$$x^2 + y^2 + z^2 = 4z^2$$

$$\rho^2 = 4\rho^2 \cos^2 \phi$$

$$1 = 4 \cos^2 \phi$$

$$\cos \phi = \frac{1}{2}$$

$$\phi = \frac{\pi}{3}, (\text{cone}) \text{spherical equation}$$

45.  $x^2 + y^2 = 16$ , rectangular equation

$$\rho^2 \sin^2 \phi \sin^2 \theta + \rho^2 \sin^2 \phi \cos^2 \theta = 16$$

$$\rho^2 \sin^2 \phi (\sin^2 \theta + \cos^2 \theta) = 16$$

$$\rho^2 \sin^2 \phi = 16$$

$$\rho \sin \phi = 4$$

$$\rho = 4 \csc \phi, \text{ spherical equation}$$

46.  $x = 13$ , rectangular equation

$$\rho \sin \phi \cos \theta = 13$$

$$\rho = 13 \csc \phi \sec \theta, \text{ spherical equation}$$

47.  $x^2 + y^2 = 2z^2$ , rectangular equation

$$\rho^2 \sin^2 \phi \cos^2 \theta + \rho^2 \sin^2 \phi \sin^2 \theta = 2\rho^2 \cos^2 \phi$$

$$\rho^2 \sin^2 \phi [\cos^2 \theta + \sin^2 \theta] = 2\rho^2 \cos^2 \phi$$

$$\rho^2 \sin^2 \phi = 2\rho^2 \cos^2 \theta$$

$$\frac{\sin^2 \phi}{\cos^2 \phi} = 2$$

$$\tan^2 \phi = 2$$

$$\tan \phi = \pm \sqrt{2}, \text{ spherical equation}$$

48.  $x^2 + y^2 + z^2 - 9z = 0$ , rectangular equation

$$\rho^2 - 9\rho \cos \phi = 0$$

$$\rho = 9 \cos \phi, \text{ spherical equation}$$

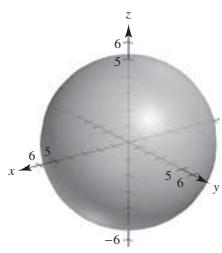
51.  $\phi = \frac{\pi}{6}$

$$\cos \phi = \frac{z}{\sqrt{x^2 + y^2 + z^2}}$$

$$\frac{\sqrt{3}}{2} = \frac{z}{\sqrt{x^2 + y^2 + z^2}}$$

$$\frac{3}{4} = \frac{z^2}{x^2 + y^2 + z^2}$$

$$3x^2 + 3y^2 - z^2 = 0, z \geq 0$$

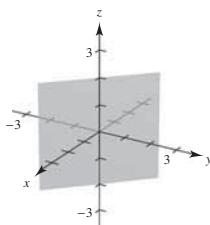
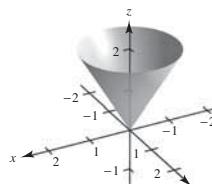


50.  $\theta = \frac{3\pi}{4}$

$$\tan \theta = \frac{y}{x}$$

$$-1 = \frac{y}{x}$$

$$x + y = 0$$



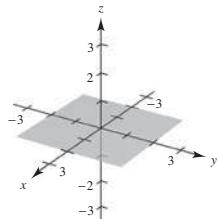
52.  $\phi = \frac{\pi}{2}$

$$\cos \phi = \frac{z}{\sqrt{x^2 + y^2 + z^2}}$$

$$0 = \frac{z}{\sqrt{x^2 + y^2 + z^2}}$$

$$z = 0$$

*xy*-plane

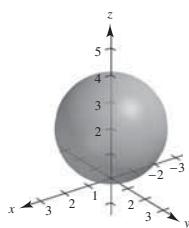


53.  $\rho = 4 \cos \phi$

$$\sqrt{x^2 + y^2 + z^2} = \frac{4z}{\sqrt{x^2 + y^2 + z^2}}$$

$$x^2 + y^2 + z^2 - 4z = 0$$

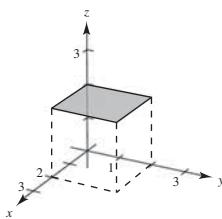
$$x^2 + y^2 + (z - 2)^2 = 4, z \geq 0$$



54.  $\rho = 2 \sec \phi$

$$\rho \cos \phi = 2$$

$$z = 2$$

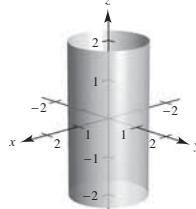


55.  $\rho = \csc \phi$

$$\rho \sin \phi = 1$$

$$\sqrt{x^2 + y^2} = 1$$

$$x^2 + y^2 = 1$$

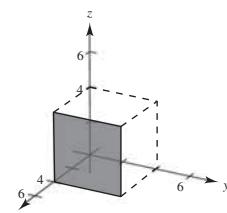


56.  $\rho = 4 \csc \phi \sec \phi$

$$= \frac{4}{\sin \phi \cos \theta}$$

$$\rho \sin \phi \cos \theta = 4$$

$$x = 4$$



57.  $\left(4, \frac{\pi}{4}, 0\right)$ , cylindrical

$$\rho = \sqrt{4^2 + 0^2} = 4$$

$$\theta = \frac{\pi}{4}$$

$$\phi = \arccos 0 = \frac{\pi}{2}$$

$\left(4, \frac{\pi}{4}, \frac{\pi}{2}\right)$ , spherical

58.  $\left(3, -\frac{\pi}{4}, 0\right)$ , cylindrical

$$\rho = \sqrt{3^2 + 0^2} = 3$$

$$\theta = -\frac{\pi}{4}$$

$$\phi = \arccos\left(\frac{0}{9}\right) = \frac{\pi}{2}$$

$\left(3, -\frac{\pi}{4}, \frac{\pi}{2}\right)$ , spherical

59.  $\left(4, \frac{\pi}{2}, 4\right)$ , cylindrical

$$\rho = \sqrt{4^2 + 4^2} = 4\sqrt{2}$$

$$\theta = \frac{\pi}{2}$$

$$\phi = \arccos\left(\frac{4}{4\sqrt{2}}\right) = \frac{\pi}{4}$$

$\left(4\sqrt{2}, \frac{\pi}{2}, \frac{\pi}{4}\right)$ , spherical

60.  $\left(2, \frac{2\pi}{3}, -2\right)$ , cylindrical

$$\rho = \sqrt{2^2 + (-2)^2} = 2\sqrt{2}$$

$$\theta = \frac{2\pi}{3}$$

$$\phi = \arccos\left(\frac{-1}{\sqrt{2}}\right) = \frac{3\pi}{4}$$

$$\left(2\sqrt{2}, \frac{2\pi}{3}, \frac{3\pi}{4}\right)$$
, spherical

65.  $\left(10, \frac{\pi}{6}, \frac{\pi}{2}\right)$ , spherical

$$r = 10 \sin \frac{\pi}{2} = 10$$

$$\theta = \frac{\pi}{6}$$

$$z = 10 \cos \frac{\pi}{2} = 0$$

$$\left(10, \frac{\pi}{6}, 0\right)$$
, cylindrical

61.  $\left(4, -\frac{\pi}{6}, -\frac{\pi}{6}\right)$ , cylindrical

$$\rho = \sqrt{4^2 + 6^2} = 2\sqrt{13}$$

$$\theta = -\frac{\pi}{6}$$

$$\phi = \arccos\frac{3}{\sqrt{13}}$$

$$\left(2\sqrt{13}, -\frac{\pi}{6}, \arccos\frac{3}{\sqrt{13}}\right)$$
, spherical

66.  $\left(4, \frac{\pi}{18}, \frac{\pi}{2}\right)$ , spherical

$$r = 4 \sin \frac{\pi}{2} = 4$$

$$\theta = \frac{\pi}{18}$$

$$z = 4 \cos \frac{\pi}{2} = 0$$

$$\left(4, \frac{\pi}{18}, 0\right)$$
, cylindrical

62.  $\left(-4, \frac{\pi}{3}, 4\right)$ , cylindrical

$$\rho = \sqrt{(-4)^2 + 4^2} = 4\sqrt{2}$$

$$\theta = \frac{\pi}{3}$$

$$\phi = \arccos\frac{1}{\sqrt{2}} = \frac{\pi}{4}$$

$$\left(4\sqrt{2}, \frac{\pi}{3}, \frac{\pi}{4}\right)$$
, spherical

67.  $\left(36, \pi, \frac{\pi}{2}\right)$ , spherical

$$r = \rho \sin \phi = 36 \sin \frac{\pi}{2} = 36$$

$$\theta = \pi$$

$$z = \rho \cos \phi = 36 \cos \frac{\pi}{2} = 0$$

$$(36, \pi, 0)$$
, cylindrical

63.  $(12, \pi, 5)$ , cylindrical

$$\rho = \sqrt{12^2 + 5^2} = 13$$

$$\theta = \pi$$

$$\phi = \arccos\frac{5}{13}$$

$$\left(13, \pi, \arccos\frac{5}{13}\right)$$
, spherical

68.  $\left(18, \frac{\pi}{3}, \frac{\pi}{3}\right)$ , spherical

$$r = \rho \sin \phi = 18 \sin \frac{\pi}{3} = 9$$

$$\theta = \frac{\pi}{3}$$

$$z = \rho \cos \phi = 18 \cos \frac{\pi}{3} = 9\sqrt{3}$$

$$\left(9, \frac{\pi}{3}, 9\sqrt{3}\right)$$
, cylindrical

64.  $\left(4, \frac{\pi}{2}, 3\right)$ , cylindrical

$$\rho = \sqrt{4^2 + 3^2} = 5$$

$$\theta = \frac{\pi}{2}$$

$$\phi = \arccos\frac{3}{5}$$

$$\left(5, \frac{\pi}{2}, \arccos\frac{3}{5}\right)$$
, spherical

69.  $\left(6, -\frac{\pi}{6}, \frac{\pi}{3}\right)$ , spherical

$$r = 6 \sin \frac{\pi}{3} = 3\sqrt{3}$$

$$\theta = -\frac{\pi}{6}$$

$$z = 6 \cos \frac{\pi}{3} = 3$$

$$\left(3\sqrt{3}, -\frac{\pi}{6}, 3\right)$$
, cylindrical

70.  $\left(5, -\frac{5\pi}{6}, \pi\right)$ , spherical

$$r = 5 \sin \pi = 0$$

$$\theta = -\frac{5\pi}{6}$$

$$z = 5 \cos \pi = -5$$

$$\left(0, -\frac{5\pi}{6}, -5\right)$$
, cylindrical

71.  $\left(8, \frac{7\pi}{6}, \frac{\pi}{6}\right)$ , spherical

$$r = 8 \sin \frac{\pi}{6} = 4$$

$$\theta = \frac{7\pi}{6}$$

$$z = 8 \cos \frac{\pi}{6} = \frac{8\sqrt{3}}{2}$$

$$\left(4, \frac{7\pi}{6}, 4\sqrt{3}\right)$$
, cylindrical

72.  $\left(7, \frac{\pi}{4}, \frac{3\pi}{4}\right)$ , spherical

$$r = 7 \sin \frac{3\pi}{4} = \frac{7\sqrt{2}}{2}$$

$$\theta = \frac{\pi}{4}$$

$$z = 7 \cos \frac{3\pi}{4} = -\frac{7\sqrt{2}}{2}$$

$$\left(\frac{7\sqrt{2}}{2}, \frac{\pi}{4}, -\frac{7\sqrt{2}}{2}\right)$$
, cylindrical

RectangularCylindricalSpherical

73.  $(4, 6, 3)$

$$(7.211, 0.983, 3)$$

$$(7.810, 0.983, 1.177)$$

74.  $(6, -2, -3)$

$$(6.325, -0.322, -3)$$

$$(7.000, -0.322, 2.014)$$

75.  $(4.698, 1.710, 8)$

$$\left(5, \frac{\pi}{9}, 8\right)$$

$$(9.434, 0.349, 0.559)$$

76.  $(7.317, -6.816, 6)$

$$(10, -0.75, 6)$$

$$(11.662, -0.750, 1.030)$$

77.  $(-7.071, 12.247, 14.142)$

$$(14.142, 2.094, 14.142)$$

$$\left(20, \frac{2\pi}{3}, \frac{\pi}{4}\right)$$

78.  $(6.115, 1.561, 4.052)$

$$(6.311, 0.25, 5.052)$$

$$(7.5, 0.25, 1)$$

79.  $(3, -2, 2)$

$$(3.606, -0.588, 2)$$

$$(4.123, -0.588, 1.064)$$

80.  $(3\sqrt{2}, 3\sqrt{2}, -3)$

$$(6, 0.785, -3)$$

$$(6.708, 0.785, 2.034)$$

81.  $\left(\frac{5}{2}, \frac{4}{3}, -\frac{3}{2}\right)$

$$(2.833, 0.490, -1.5)$$

$$(3.206, 0.490, 2.058)$$

82.  $(0, -5, 4)$

$$(5, -1.571, 4)$$

$$(6.403, -1.571, 0.896)$$

83.  $(-3.536, 3.536, -5)$

$$\left(5, \frac{3\pi}{4}, -5\right)$$

$$(7.071, 2.356, 2.356)$$

84.  $(-1.732, 1, 3)$

$$\left(-2, \frac{11\pi}{6}, 3\right)$$

$$(3.606, 2.618, 0.588)$$

**Note:** use the cylindrical coordinate  $\left(2, \frac{5\pi}{6}, 3\right)$

- | <u>Rectangular</u>   | <u>Cylindrical</u>  | <u>Spherical</u>                                |
|--|---|---|
| 85. $(2.804, -2.095, 6)$   | $(-3.5, 2.5, 6)$  | $(6.946, 5.642, 0.528)$                         |
| [Note: Use the cylindrical coordinates $(3.5, 5.642, 6)$ ]               |   |   |
| 86. $(2.207, 7.949, -4)$   | $(8.25, 1.3, -4)$   | $(9.169, 1.3, 2.022)$                           |
| 87. $(-1.837, 1.837, 1.5)$   | $(2.598, 2.356, 1.5)$   | $\left(3, \frac{3\pi}{4}, \frac{\pi}{3}\right)$ |
| 88. $(0, 0, -8)$   | $(0, -0.524, -8)$   | $\left(8, -\frac{\pi}{6}, \pi\right)$           |
| 89. $r = 5$  | 97. Rectangular to spherical: $\rho^2 = x^2 + y^2 + z^2$  |   |
| Cylinder   | $\tan \theta = \frac{y}{x}$   |   |
| Matches graph (d)  | $\phi = \arccos\left(\frac{z}{\sqrt{x^2 + y^2 + z^2}}\right)$                                     |   |
| 90. $\theta = \frac{\pi}{4}$   | Spherical to rectangular: $x = \rho \sin \phi \cos \theta$  |   |
| Plane  | $y = \rho \sin \phi \sin \theta$  |   |
| Matches graph (e)  | $z = \rho \cos \phi$  |   |
| 91. $\rho = 5$   | 98. (a) $r = a$ Cylinder with $z$ -axis symmetry  |   |
| Sphere   | $\theta = b$ Plane perpendicular to $xy$ -plane   |   |
| Matches graph (c)  | $z = c$ Plane parallel to $xy$ -plane   |   |
| 92. $\phi = \frac{\pi}{4}$   | (b) $\rho = a$ Sphere   |   |
| Cone   | $\theta = b$ Vertical half-plane  |   |
| Matches graph (a)  | $\phi = c$ Half-cone  |   |
| 93. $r^2 = z, x^2 + y^2 = z$   | 99. $x^2 + y^2 + z^2 = 25$  |   |
| Paraboloid   | (a) $r^2 + z^2 = 25$  |   |
| Matches graph (f)  | (b) $\rho^2 = 25 \Rightarrow \rho = 5$  |   |
| 94. $\rho = 4 \sec \phi, z = \rho \cos \phi = 4$                         | 100. $4(x^2 + y^2) = z^2$   |   |
| Plane  | (a) $4r^2 = z^2 \Rightarrow 2r = z$   |   |
| Matches graph (b)  | (b) $4(\rho^2 \sin^2 \phi \cos^2 \theta + \rho^2 \sin^2 \phi \sin^2 \theta) = \rho^2 \cos^2 \phi$ |   |
| 95. Rectangular to cylindrical: $r^2 = x^2 + y^2$                        | $4 \sin^2 \phi = \cos^2 \phi,$  |   |
|  | $\tan^2 \phi = \frac{1}{4},$  |   |
|  | $\tan \phi = \frac{1}{2} \Rightarrow \phi = \arctan \frac{1}{2}$                                  |   |
| Cylindrical to rectangular: $x = r \cos \theta$                          | 101. $x^2 + y^2 + z^2 - 2z = 0$   |   |
| $y = r \sin \theta$  | (a) $r^2 + z^2 - 2z = 0 \Rightarrow r^2 + (z - 1)^2 = 1$  |   |
| $z = z$  | (b) $\rho^2 - 2\rho \cos \phi = 0$  |   |
| 96. $\theta = c$ is a half-plane because of the restriction $r \geq 0$ . | $\rho(\rho - 2 \cos \phi) = 0$  |   |
|  | $\rho = 2 \cos \phi$  |   |

**102.**  $x^2 + y^2 = z$

- $r^2 = z$
- $\rho^2 \sin^2 \phi = \rho \cos \phi$   
 $\rho \sin^2 \phi = \cos \phi$   
 $\rho = \frac{\cos \phi}{\sin^2 \phi}$   
 $\rho = \csc \phi \cot \phi$

**103.**  $x^2 + y^2 = 4y$

- $r^2 = 4r \sin \theta, r = 4 \sin \theta$
- $\rho^2 \sin^2 \phi = 4\rho \sin \phi \sin \theta$   
 $\rho \sin \phi(\rho \sin \phi - 4 \sin \theta) = 0$   
 $\rho = \frac{4 \sin \theta}{\sin \phi}$   
 $\rho = 4 \sin \theta \csc \phi$

**104.**  $x^2 + y^2 = 36$

- $r^2 = 36 \Rightarrow r = 6$
- $\rho^2 \sin^2 \phi \cos^2 \theta + \rho^2 \sin^2 \phi \sin^2 \theta = 36$   
 $\rho^2 \sin^2 \phi = 36$   
 $\rho = 6 \csc \phi$

**105.**  $x^2 - y^2 = 9$

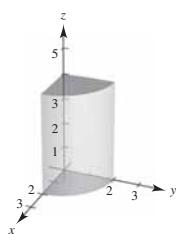
- $r^2 \cos^2 \theta - r^2 \sin^2 \theta = 9$   
 $r^2 = \frac{9}{\cos^2 \theta - \sin^2 \theta}$
- $\rho^2 \sin^2 \phi \cos^2 \theta - \rho^2 \sin^2 \phi \sin^2 \theta = 9$   
 $\rho^2 \sin^2 \phi = \frac{9}{\cos^2 \theta - \sin^2 \theta}$   
 $\rho^2 = \frac{9 \csc^2 \phi}{\cos^2 \theta - \sin^2 \theta}$

**106.**  $y = 4$

- $r \sin \theta = 4 \Rightarrow r = 4 \csc \theta$
- $\rho \sin \phi \sin \theta = 4,$   
 $\rho = 4 \csc \phi \csc \theta$

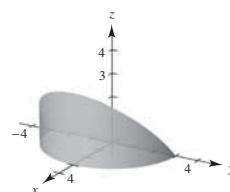
**107.**  $0 \leq \theta \leq \frac{\pi}{2}$

$$\begin{aligned}0 &\leq r \leq 2 \\0 &\leq z \leq 4\end{aligned}$$



**108.**  $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$

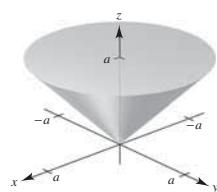
$$\begin{aligned}0 &\leq r \leq 3 \\0 &\leq z \leq r \cos \theta\end{aligned}$$



**109.**  $0 \leq \theta \leq 2\pi$

$$0 \leq r \leq a$$

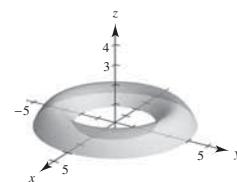
$$r \leq z \leq a$$



**110.**  $0 \leq \theta \leq 2\pi$

$$2 \leq r \leq 4$$

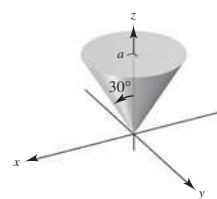
$$z^2 \leq -r^2 + 6r - 8$$



**111.**  $0 \leq \theta \leq 2\pi$

$$0 \leq \phi \leq \frac{\pi}{6}$$

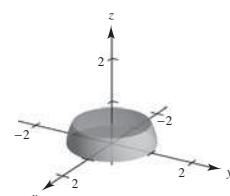
$$0 \leq \rho \leq a \sec \phi$$



**112.**  $0 \leq \theta \leq 2\pi$

$$\frac{\pi}{4} \leq \phi \leq \frac{\pi}{2}$$

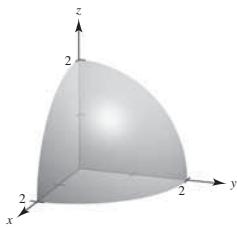
$$0 \leq \rho \leq 1$$



**113.**  $0 \leq \theta \leq \frac{\pi}{2}$

$$0 \leq \phi \leq \frac{\pi}{2}$$

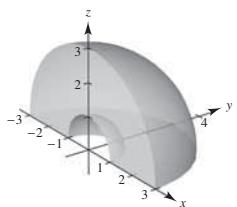
$$0 \leq \rho \leq 2$$



**114.**  $0 \leq \theta \leq \pi$

$$0 \leq \phi \leq \frac{\pi}{2}$$

$$1 \leq \rho \leq 3$$

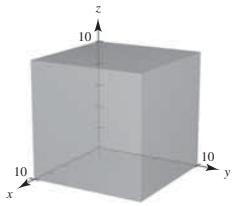


**115. Rectangular**

$$0 \leq x \leq 10$$

$$0 \leq y \leq 10$$

$$0 \leq z \leq 10$$



**116. Cylindrical:**

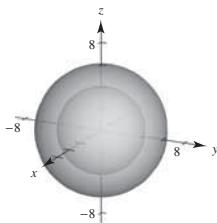
$$0.75 \leq r \leq 1.25$$

$$0 \leq z \leq 8$$



**117. Spherical**

$$4 \leq \rho \leq 6$$

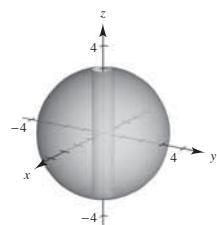


**118. Cylindrical**

$$\frac{1}{2} \leq r \leq 3$$

$$0 \leq \theta \leq 2\pi$$

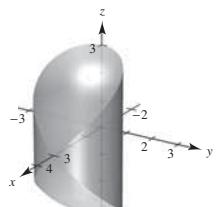
$$-\sqrt{9 - r^2} \leq z \leq \sqrt{9 - r^2}$$



**119. Cylindrical coordinates:**

$$r^2 + z^2 \leq 9,$$

$$r \leq 3 \cos \theta, 0 \leq \theta \leq \pi$$

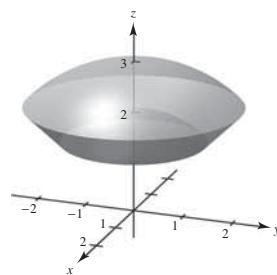


**120. Spherical coordinates:**

$$\rho \geq 2$$

$$\rho \leq 3$$

$$0 \leq \phi \leq \frac{\pi}{4}$$



**121. False.**  $r = z \Rightarrow x^2 + y^2 = z^2$  is a cone.

**122. True.** They both represent spheres of radius 2 centered at the origin.

123. False.  $(r, \theta, z) = (0, 0, 1)$  and  $(r, \theta, z) = (0, \pi, 1)$  represent the same point  $(x, y, z) = (0, 0, 1)$ .

124. True (except for the origin).

125.  $z = \sin \theta, r = 1$

$$z = \sin \theta = \frac{y}{r} = \frac{y}{1} = y$$

The curve of intersection is the ellipse formed by the intersection of the plane  $z = y$  and the cylinder  $r = 1$ .

126.  $\rho = 2 \sec \phi \Rightarrow \rho \cos \phi = 2 \Rightarrow z = 2$  plane  
 $\rho = 4$  sphere

The intersection of the plane and the sphere is a circle.

## Review Exercises for Chapter 11

1.  $P = (1, 2), Q = (4, 1), R = (5, 4)$

(a)  $\mathbf{u} = \overrightarrow{PQ} = \langle 4 - 1, 1 - 2 \rangle = \langle 3, -1 \rangle$   
 $\mathbf{v} = \overrightarrow{PR} = \langle 5 - 1, 4 - 2 \rangle = \langle 4, 2 \rangle$

(b)  $\mathbf{u} = 3\mathbf{i} - \mathbf{j}$

(c)  $\|\mathbf{v}\| = \sqrt{4^2 + 2^2} = \sqrt{20} = 2\sqrt{5}$

(d)  $2\mathbf{u} + \mathbf{v} = 2\langle 3, -1 \rangle + \langle 4, 2 \rangle = \langle 10, 0 \rangle$

2.  $P = (-2, -1), Q = (5, -1), R = (2, 4)$

(a)  $\mathbf{u} = \overrightarrow{PQ} = \langle 7, 0 \rangle$

$\mathbf{v} = \overrightarrow{PR} = \langle 4, 5 \rangle$

(b)  $\mathbf{u} = 7\mathbf{i}$

(c)  $\|\mathbf{v}\| = \sqrt{4^2 + 5^2} = \sqrt{41}$

(d)  $2\mathbf{u} + \mathbf{v} = 2\langle 7, 0 \rangle + \langle 4, 5 \rangle = \langle 18, 5 \rangle$

3.  $\mathbf{v} = \|\mathbf{v}\|(\cos \theta \mathbf{i} + \sin \theta \mathbf{j})$

$= 8(\cos 60^\circ \mathbf{i} + \sin 60^\circ \mathbf{j})$

$= 8\left(\frac{1}{2}\mathbf{i} + \frac{\sqrt{3}}{2}\mathbf{j}\right) = 4\mathbf{i} + 4\sqrt{3}\mathbf{j} = \langle 4, 4\sqrt{3} \rangle$

4.  $\mathbf{v} = \|\mathbf{v}\| \cos \theta \mathbf{i} + \|\mathbf{v}\| \sin \theta \mathbf{j}$

$= \frac{1}{2} \cos 225^\circ \mathbf{i} + \frac{1}{2} \sin 225^\circ \mathbf{j}$

$= -\frac{\sqrt{2}}{4}\mathbf{i} - \frac{\sqrt{2}}{4}\mathbf{j} = \left\langle -\frac{\sqrt{2}}{4}, -\frac{\sqrt{2}}{4} \right\rangle$

5.  $z = 0, y = 4, x = -5: (-5, 4, 0)$

6.  $x = z = 0, y = -7: (0, -7, 0)$

7. Looking down from the positive  $x$ -axis towards the  $yz$ -plane, the point is either in the first quadrant ( $y > 0, z > 0$ ) or in the third quadrant ( $y < 0, z < 0$ ). The  $x$ -coordinate can be any number.

8. Looking towards the  $xy$ -plane from the positive  $z$ -axis.

The point is either in the second quadrant ( $x < 0, y > 0$ ) or in the fourth quadrant ( $x > 0, y < 0$ ). The  $z$ -coordinate can be any number.

9.  $(x - 3)^2 + (y + 2)^2 + (z - 6)^2 = \left(\frac{15}{2}\right)^2$

10. Center:  $\left(\frac{0+4}{2}\right), \left(\frac{0+6}{2}\right), \left(\frac{4+0}{2}\right) = (2, 3, 2)$

Radius:

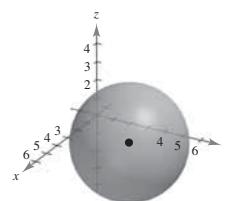
$$\sqrt{(2-0)^2 + (3-0)^2 + (2-4)^2} = \sqrt{4+9+4} = \sqrt{17}$$

$$(x-2)^2 + (y-3)^2 + (z-2)^2 = 17$$

11.  $(x^2 - 4x + 4) + (y^2 - 6y + 9) + z^2 = -4 + 4 + 9$   
 $(x-2)^2 + (y-3)^2 + z^2 = 9$

Center:  $(2, 3, 0)$

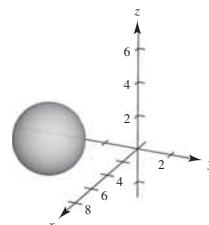
Radius: 3



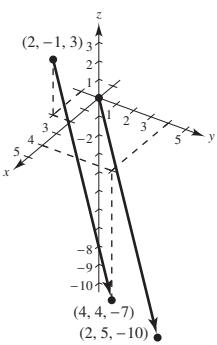
12.  $(x^2 - 10x + 25) + (y^2 + 6y + 9) + (z^2 - 4z + 4) = -34 + 25 + 9 + 4$   
 $(x-5)^2 + (y+3)^2 + (z-2)^2 = 4$

Center:  $(5, -3, 2)$

Radius: 2



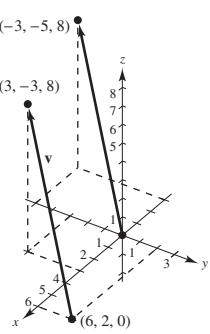
13. (a), (d)



(b)  $\mathbf{v} = \langle 4 - 2, 4 - (-1), -7 - 3 \rangle = \langle 2, 5, -10 \rangle$

(c)  $\mathbf{v} = 2\mathbf{i} + 5\mathbf{j} - 10\mathbf{k}$

14. (a), (d)



(b)  $\mathbf{v} = \langle 3 - 6, -3 - 2, 8 - 0 \rangle = \langle -3, -5, 8 \rangle$

(c)  $\mathbf{v} = -3\mathbf{i} - 5\mathbf{j} + 8\mathbf{k}$

15.  $\mathbf{v} = \langle -1 - 3, 6 - 4, 9 + 1 \rangle = \langle -4, 2, 10 \rangle$

$\mathbf{w} = \langle 5 - 3, 3 - 4, -6 + 1 \rangle = \langle 2, -1, -5 \rangle$

Because  $-2\mathbf{w} = \mathbf{v}$ , the points lie in a straight line.

16.  $\mathbf{v} = \langle 8 - 5, -5 + 4, 5 - 7 \rangle = \langle 3, -1, -2 \rangle$

$\mathbf{w} = \langle 11 - 5, 6 + 4, 3 - 7 \rangle = \langle 6, 10, -4 \rangle$

Because  $\mathbf{v}$  and  $\mathbf{w}$  are not parallel, the points do not lie in a straight line.

17. Unit vector:  $\frac{\mathbf{u}}{\|\mathbf{u}\|} = \left\langle \frac{2}{\sqrt{38}}, \frac{3}{\sqrt{38}}, \frac{5}{\sqrt{38}} \right\rangle$

18.  $8 \frac{\langle 6, -3, 2 \rangle}{\sqrt{49}} = \frac{8}{7} \langle 6, -3, 2 \rangle = \left\langle \frac{48}{7}, -\frac{24}{7}, \frac{16}{7} \right\rangle$

19.  $P = \langle 5, 0, 0 \rangle, Q = \langle 4, 4, 0 \rangle, R = \langle 2, 0, 6 \rangle$

(a)  $\mathbf{u} = \overrightarrow{PQ} = \langle -1, 4, 0 \rangle$

$\mathbf{v} = \overrightarrow{PR} = \langle -3, 0, 6 \rangle$

(b)  $\mathbf{u} \cdot \mathbf{v} = (-1)(-3) + 4(0) + 0(6) = 3$

(c)  $\mathbf{v} \cdot \mathbf{v} = 9 + 36 = 45$

20.  $P = \langle 2, -1, 3 \rangle, Q = \langle 0, 5, 1 \rangle, R = \langle 5, 5, 0 \rangle$

(a)  $\mathbf{u} = \overrightarrow{PQ} = \langle -2, 6, -2 \rangle$

$\mathbf{v} = \overrightarrow{PR} = \langle 3, 6, -3 \rangle$

(b)  $\mathbf{u} \cdot \mathbf{v} = (-2)(3) + (6)(6) + (-2)(-3) = 36$

(c)  $\mathbf{v} \cdot \mathbf{v} = 9 + 36 + 9 = 54$

21.  $\mathbf{u} = \langle 7, -2, 3 \rangle, \mathbf{v} = \langle -1, 4, 5 \rangle$

Because  $\mathbf{u} \cdot \mathbf{v} = 0$ , the vectors are orthogonal.

22.  $\mathbf{u} = \langle -4, 3, -6 \rangle, \mathbf{v} = \langle 16, -12, 24 \rangle$

Because  $\mathbf{v} = -4\mathbf{u}$ , the vectors are parallel.

23.  $\mathbf{u} = 5 \left( \cos \frac{3\pi}{4} \mathbf{i} + \sin \frac{3\pi}{4} \mathbf{j} \right) = \frac{5\sqrt{2}}{2} [-\mathbf{i} + \mathbf{j}]$

$\mathbf{v} = 2 \left( \cos \frac{2\pi}{3} \mathbf{i} + \sin \frac{2\pi}{3} \mathbf{j} \right) = -\mathbf{i} + \sqrt{3} \mathbf{j}$

$\mathbf{u} \cdot \mathbf{v} = \frac{5\sqrt{2}}{2} (1 + \sqrt{3})$

$\|\mathbf{u}\| = 5 \quad \|\mathbf{v}\| = 2$

$\cos \theta = \frac{|\mathbf{u} \cdot \mathbf{v}|}{\|\mathbf{u}\| \|\mathbf{v}\|} = \frac{(5\sqrt{2}/2)(1 + \sqrt{3})}{5(2)} = \frac{\sqrt{2} + \sqrt{6}}{4}$

$\theta = \arccos \frac{\sqrt{2} + \sqrt{6}}{4} = 15^\circ \left[ \text{or, } \frac{3\pi}{4} \cdot \frac{2\pi}{3} = \frac{\pi}{12} \text{ or } 15^\circ \right]$

24.  $\mathbf{u} = 6\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}, \mathbf{v} = -\mathbf{i} + 5\mathbf{j}$

$\cos \theta = \frac{|\mathbf{u} \cdot \mathbf{v}|}{\|\mathbf{u}\| \|\mathbf{v}\|} = \frac{|-6 + 10|}{\sqrt{49}\sqrt{26}} = \frac{4}{7\sqrt{26}}$

$\theta \approx 83.6^\circ$

25.  $\mathbf{u} = \langle 10, -5, 15 \rangle, \mathbf{v} = \langle -2, 1, -3 \rangle$

 $\mathbf{u} = -5\mathbf{v} \Rightarrow \mathbf{u}$  is parallel to  $\mathbf{v}$  and in the opposite direction.

$\theta = \pi$

26.  $\mathbf{u} = \langle 1, 0, -3 \rangle$

$\mathbf{v} = \langle 2, -2, 1 \rangle$

$\mathbf{u} \cdot \mathbf{v} = -1$

$\|\mathbf{u}\| = \sqrt{10}$

$\|\mathbf{v}\| = 3$

$\cos \theta = \frac{|\mathbf{u} \cdot \mathbf{v}|}{\|\mathbf{u}\| \|\mathbf{v}\|} = \frac{1}{3\sqrt{10}}$

$\theta \approx 83.9^\circ$

27. There are many correct answers.

For example:  $\mathbf{v} = \pm\langle 6, -5, 0 \rangle$ .

$$\begin{aligned} 28. W &= \mathbf{F} \cdot \overrightarrow{PQ} = \|\mathbf{F}\| \|\overrightarrow{PQ}\| \cos \theta = (75)(8) \cos 30^\circ \\ &= 300\sqrt{3} \text{ ft-lb} \end{aligned}$$

In Exercises 29–38,  $\mathbf{u} = \langle 3, -2, 1 \rangle$ ,  $\mathbf{v} = \langle 2, -4, -3 \rangle$ ,  
 $\mathbf{w} = \langle -1, 2, 2 \rangle$ .

$$\begin{aligned} 29. \mathbf{u} \cdot \mathbf{u} &= 3(3) + (-2)(-2) + (1)(1) \\ &= 14 = (\sqrt{14})^2 = \|\mathbf{u}\|^2 \end{aligned}$$

$$\begin{aligned} 30. \cos \theta &= \frac{|\mathbf{u} \cdot \mathbf{v}|}{\|\mathbf{u}\| \|\mathbf{v}\|} = \frac{11}{\sqrt{14}\sqrt{29}} \\ \theta &= \arccos\left(\frac{11}{\sqrt{14}\sqrt{29}}\right) \approx 56.9^\circ \end{aligned}$$

$$\begin{aligned} 31. \text{proj}_{\mathbf{u}} \mathbf{w} &= \left( \frac{\mathbf{u} \cdot \mathbf{w}}{\|\mathbf{u}\|^2} \right) \mathbf{u} \\ &= -\frac{5}{14} \langle 3, -2, 1 \rangle \\ &= \left\langle -\frac{15}{14}, \frac{10}{14}, -\frac{5}{14} \right\rangle \\ &= \left\langle -\frac{15}{14}, \frac{5}{7}, -\frac{5}{14} \right\rangle \end{aligned}$$

$$32. \text{Work} = \mathbf{u} \cdot \mathbf{w} = -3 - 4 + 2 = -5$$

$$33. \mathbf{n} = \mathbf{v} \times \mathbf{w} = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -4 & -3 \\ -1 & 2 & 2 \end{bmatrix} = -2\mathbf{i} - \mathbf{j}$$

$$\|\mathbf{n}\| = \sqrt{5}$$

$$\frac{\mathbf{n}}{\|\mathbf{n}\|} = \frac{1}{\sqrt{5}}(-2\mathbf{i} - \mathbf{j}), \text{ unit vector or } \frac{1}{\sqrt{5}}(2\mathbf{i} + \mathbf{j})$$

$$34. \mathbf{u} \times \mathbf{v} = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -2 & 1 \\ 2 & -4 & -3 \end{bmatrix} = 10\mathbf{i} + 11\mathbf{j} - 8\mathbf{k}$$

$$\mathbf{v} \times \mathbf{u} = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -4 & -3 \\ 3 & -2 & 1 \end{bmatrix} = -10\mathbf{i} - 11\mathbf{j} + 8\mathbf{k}$$

$$\text{So, } \mathbf{u} \times \mathbf{v} = -(\mathbf{v} \times \mathbf{u})$$

$$35. V = |\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})| = |\langle 3, -2, 1 \rangle \cdot \langle -2, -1, 0 \rangle| = |-4| = 4$$

$$36. \mathbf{u} \times (\mathbf{v} + \mathbf{w}) = \langle 3, -2, 1 \rangle \times \langle 1, -2, -1 \rangle$$

$$= \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -2 & 1 \\ 1 & -2 & -1 \end{bmatrix} = 4\mathbf{i} + 4\mathbf{j} - 4\mathbf{k}$$

$$\mathbf{u} \times \mathbf{v} = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -2 & 1 \\ 2 & -4 & -3 \end{bmatrix} = 10\mathbf{i} + 11\mathbf{j} - 8\mathbf{k}$$

$$\mathbf{u} \times \mathbf{w} = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -2 & 1 \\ -1 & 2 & 2 \end{bmatrix} = -6\mathbf{i} - 7\mathbf{j} + 4\mathbf{k}$$

$$(\mathbf{u} \times \mathbf{v}) + (\mathbf{u} \times \mathbf{w}) = 4\mathbf{i} + 4\mathbf{j} - 4\mathbf{k} = \mathbf{u} \times (\mathbf{v} + \mathbf{w})$$

$$37. \text{Area parallelogram} = \|\mathbf{u} \times \mathbf{v}\| = \|\langle 10, 11, -8 \rangle\| = \sqrt{10^2 + 11^2 + (-8)^2} \quad (\text{See Exercises 34, 36}) \\ = \sqrt{285}$$

$$38. \text{Area triangle} = \frac{1}{2} \|\mathbf{v} \times \mathbf{w}\| = \frac{1}{2} \sqrt{(-2)^2 + (-1)^2} = \frac{\sqrt{5}}{2} \quad (\text{See Exercise 33})$$

$$39. \mathbf{F} = c(\cos 20^\circ \mathbf{j} + \sin 20^\circ \mathbf{k})$$

$$\overrightarrow{PQ} = 2\mathbf{k}$$

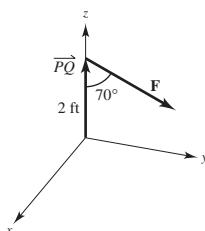
$$\overrightarrow{PQ} \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 2 \\ 0 & c \cos 20^\circ & c \sin 20^\circ \end{vmatrix} = -2c \cos 20^\circ \mathbf{i}$$

$$200 = \|\overrightarrow{PQ} \times \mathbf{F}\| = 2c \cos 20^\circ$$

$$c = \frac{100}{\cos 20^\circ}$$

$$\mathbf{F} = \frac{100}{\cos 20^\circ} (\cos 20^\circ \mathbf{j} + \sin 20^\circ \mathbf{k}) = 100(\mathbf{j} + \tan 20^\circ \mathbf{k})$$

$$\|\mathbf{F}\| = 100\sqrt{1 + \tan^2 20^\circ} = 100 \sec 20^\circ \approx 106.4 \text{ lb}$$



40.  $V = |\mathbf{u} \cdot (\mathbf{v} \cdot \mathbf{w})| = \begin{vmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & -1 & 2 \end{vmatrix} = 2(5) = 10$

41.  $\mathbf{v} = \langle 9 - 3, 11 - 0, 6 - 2 \rangle = \langle 6, 11, 4 \rangle$

(a) Parametric equations:

$$x = 3 + 6t, y = 11t, z = 2 + 4t$$

(b) Symmetric equations:  $\frac{x - 3}{6} = \frac{y}{11} = \frac{z - 2}{4}$

42.  $\mathbf{v} = \langle 8 + 1, 10 - 4, 5 - 3 \rangle = \langle 9, 6, 2 \rangle$

(a) Parametric equations:

$$x = -1 + 9t, y = 4 + 6t, z = 3 + 2t$$

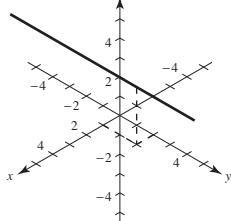
(b) Symmetric equations:  $\frac{x + 1}{9} = \frac{y - 4}{6} = \frac{z - 3}{2}$

43.  $\mathbf{v} = \mathbf{j}$

(a)  $x = 1, y = 2 + t, z = 3$

(b) None

(c)

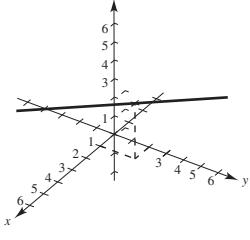


44. Direction numbers: 1, 1, 1

(a)  $x = 1 + t, y = 2 + t, z = 3 + t$

(b)  $x - 1 = y - 2 = z - 3$

(c)



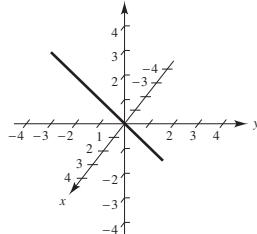
45.  $3x - 3y - 7z = -4, x - y + 2z = 3$

Solving simultaneously, you have  $z = 1$ . Substituting  $z = 1$  into the second equation, you have  $y = x - 1$ . Substituting for  $x$  in this equation you obtain two points on the line of intersection,  $(0, -1, 1), (1, 0, 1)$ . The direction vector of the line of intersection is  $\mathbf{v} = \mathbf{i} + \mathbf{j}$ .

(a)  $x = t, y = -1 + t, z = 1$

(b)  $x = y + 1, z = 1$

(c)



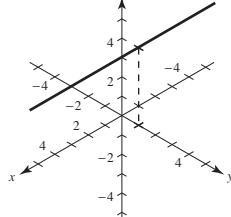
46.  $\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -5 & 1 \\ -3 & 1 & 4 \end{vmatrix} = -21\mathbf{i} - 11\mathbf{j} - 13\mathbf{k}$

Direction numbers: 21, 11, 13

(a)  $x = 21t, y = 1 + 11t, z = 4 + 13t$

(b)  $\frac{x}{21} = \frac{y - 1}{11} = \frac{z - 4}{13}$

(c)



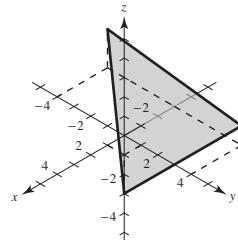
47.  $P = (-3, -4, 2), Q = (-3, 4, 1), R = (1, 1, -2)$

$\overrightarrow{PQ} = \langle 0, 8, -1 \rangle, \overrightarrow{PR} = [4, 5, -4]$

$\mathbf{n} = \overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 8 & -1 \\ 4 & 5 & -4 \end{vmatrix} = -27\mathbf{i} - 4\mathbf{j} - 32\mathbf{k}$

$-27(x + 3) - 4(y + 4) - 32(z - 2) = 0$

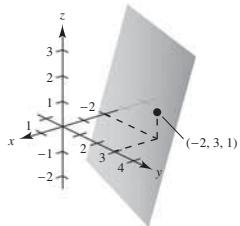
$27x + 4y + 32z = -33$



48.  $\mathbf{n} = 3\mathbf{i} - \mathbf{j} + \mathbf{k}$

$$3(x+2) - 1(y-3) + 1(z-1) = 0$$

$$3x - y + z + 8 = 0$$



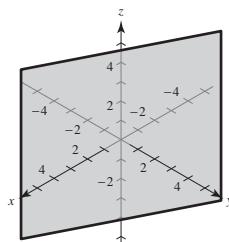
49. The two lines are parallel as they have the same direction numbers,  $-2, 1, 1$ . Therefore, a vector parallel to the plane is  $\mathbf{v} = -2\mathbf{i} + \mathbf{j} + \mathbf{k}$ . A point on the first line is  $(1, 0, -1)$  and a point on the second line is  $(-1, 1, 2)$ . The vector  $\mathbf{u} = 2\mathbf{i} - \mathbf{j} - 3\mathbf{k}$  connecting these two points is also parallel to the plane. Therefore, a normal to the plane is

$$\mathbf{v} \times \mathbf{u} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & 1 & 1 \\ 2 & -1 & -3 \end{vmatrix}$$

$$= -2\mathbf{i} - 4\mathbf{j} = -2(\mathbf{i} + 2\mathbf{j}).$$

Equation of the plane:  $(x-1) + 2y = 0$

$$x + 2y = 1$$



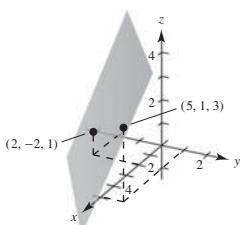
50. Let  $\mathbf{v} = \langle 5-2, 1+2, 3-1 \rangle = \langle 3, 3, 2 \rangle$  be the direction vector for the line through the two points. Let  $\mathbf{n} = \langle 2, 1, -1 \rangle$  be the normal vector to the plane. Then

$$\mathbf{v} \times \mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 3 & 2 \\ 2 & 1 & -1 \end{vmatrix} = \langle -5, 7, -3 \rangle$$

is the normal to the unknown plane.

$$-5(x-5) + 7(y-1) - 3(z-3) = 0$$

$$-5x + 7y - 3z + 27 = 0$$



51.  $Q(1, 0, 2)$  point

$$2x - 3y + 6z = 6$$

A point  $P$  on the plane is  $(3, 0, 0)$ .

$$\overrightarrow{PQ} = \langle -2, 0, 2 \rangle$$

$\mathbf{n} = \langle 2, -3, 6 \rangle$  normal to plane

$$D = \frac{|\overrightarrow{PQ} \cdot \mathbf{n}|}{\|\mathbf{n}\|} = \frac{8}{7}$$

52.  $Q(3, -2, 4)$  point

$$2x - 5y + z = 10$$

A point  $P$  on the plane is  $(5, 0, 0)$ .

$$\overrightarrow{PQ} = \langle -2, -2, 4 \rangle$$

$\mathbf{n} = \langle 2, -5, 1 \rangle$  normal to plane

$$D = \frac{|\overrightarrow{PQ} \cdot \mathbf{n}|}{\|\mathbf{n}\|} = \frac{10}{\sqrt{30}} = \frac{\sqrt{30}}{3}$$

53. The normal vectors to the planes are the same,  $\mathbf{n} = \langle 5, -3, 1 \rangle$ .

Choose a point in the first plane  $P(0, 0, 2)$ . Choose a point in the second plane,  $Q(0, 0, -3)$ .

$$\overrightarrow{PQ} = \langle 0, 0, -5 \rangle$$

$$D = \frac{|\overrightarrow{PQ} \cdot \mathbf{n}|}{\|\mathbf{n}\|} = \frac{|-5|}{\sqrt{35}} = \frac{5}{\sqrt{35}} = \frac{\sqrt{35}}{7}$$

54.  $Q(-5, 1, 3)$  point

$\mathbf{u} = \langle 1, -2, -1 \rangle$  direction vector

$P(1, 3, 5)$  point on line

$$\overrightarrow{PQ} = \langle -6, -2, -2 \rangle$$

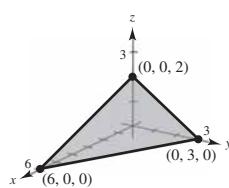
$$\overrightarrow{PQ} \times \mathbf{u} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -6 & -2 & -2 \\ 1 & -2 & -1 \end{vmatrix} = \langle -2, -8, 14 \rangle$$

$$D = \frac{\|\overrightarrow{PQ} \times \mathbf{u}\|}{\|\mathbf{u}\|} = \frac{\sqrt{264}}{\sqrt{6}} = 2\sqrt{11}$$

55.  $x + 2y + 3z = 6$

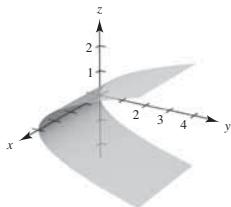
Plane

Intercepts:  $(6, 0, 0), (0, 3, 0), (0, 0, 2)$ ,



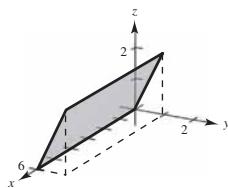
56.  $y = z^2$

Because the  $x$ -coordinate is missing, you have a cylindrical surface with rulings parallel to the  $x$ -axis. The generating curve is a parabola in the  $yz$ -coordinate plane.



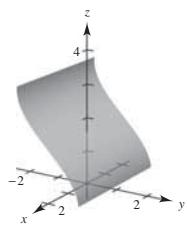
57.  $y = \frac{1}{2}z$

Plane with rulings parallel to the  $x$ -axis.



58.  $y = \cos z$

Because the  $x$ -coordinate is missing, you have a cylindrical surface with rulings parallel to the  $x$ -axis. The generating curve is  $y = \cos z$ .



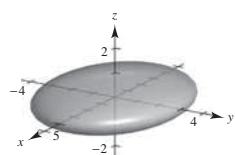
59.  $\frac{x^2}{16} + \frac{y^2}{9} + z^2 = 1$

Ellipsoid

xy-trace:  $\frac{x^2}{16} + \frac{y^2}{9} = 1$

xz-trace:  $\frac{x^2}{16} + z^2 = 1$

yz-trace:  $\frac{y^2}{9} + z^2 = 1$



60.  $16x^2 + 16y^2 - 9z^2 = 0$

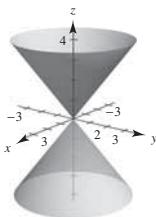
Cone

xy-trace: point  $(0, 0, 0)$

xz-trace:  $z = \pm \frac{4x}{3}$

yz-trace:  $z = \pm \frac{4y}{3}$

$z = 4, x^2 + y^2 = 9$



61.  $\frac{x^2}{16} - \frac{y^2}{9} + z^2 = -1$

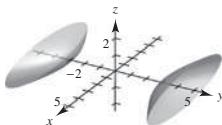
$\frac{y^2}{9} - \frac{x^2}{16} - z^2 = 1$

Hyperboloid of two sheets

xy-trace:  $\frac{y^2}{9} - \frac{x^2}{16} = 1$

xz-trace: None

yz-trace:  $\frac{y^2}{9} - z^2 = 1$



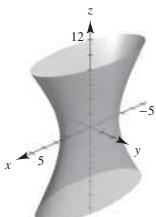
62.  $\frac{x^2}{25} + \frac{y^2}{4} - \frac{z^2}{100} = 1$

Hyperboloid of one sheet

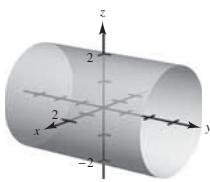
xy-trace:  $\frac{x^2}{25} + \frac{y^2}{4} = 1$

xz-trace:  $\frac{x^2}{25} - \frac{z^2}{100} = 1$

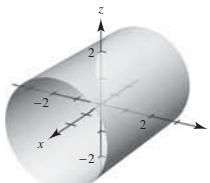
yz-trace:  $\frac{y^2}{4} - \frac{z^2}{100} = 1$



63.  $x^2 + z^2 = 4$ .

Cylinder of radius 2 about  $y$ -axis

64.  $y^2 + z^2 = 16$ .

Cylinder of radius 4 about  $x$ -axis

65. Let  $y = r(x) = 2\sqrt{x}$  and revolve the curve about the  $x$ -axis.

66.  $x^2 + 2y^2 + z^2 = 3y$

$x^2 + z^2 = 3y - 2y^2$

Let  $x^2 = 3y - 2y^2$  (Trace in  $xy$ -plane)

Then  $x = \sqrt{3y - 2y^2}$  is a generating curve. Revolve the curve about the  $y$ -axis.

67.  $z^2 = 2y$  revolved about  $y$ -axis

$z = \pm\sqrt{2y}$

$x^2 + z^2 = [r(y)]^2 = 2y$

$x^2 + z^2 = 2y$

68.  $2x + 3z = 1$  revolved about the  $x$ -axis

$z = \frac{1 - 2x}{3}$

$y^2 + z^2 = [r(x)]^2 = \left(\frac{1 - 2x}{3}\right)^2$ , Cone

69.  $(-2\sqrt{2}, 2\sqrt{2}, 2)$ , rectangular

(a)  $r = \sqrt{(-2\sqrt{2})^2 + (2\sqrt{2})^2} = 4$ ,

$\theta = \arctan(-1) = \frac{3\pi}{4}, z = 2,$

$\left(4, \frac{3\pi}{4}, 2\right)$ , cylindrical

(b)  $\rho = \sqrt{(-2\sqrt{2})^2 + (2\sqrt{2})^2 + (2)^2} = 2\sqrt{5}$ ,

$\theta = \frac{3\pi}{4}, \phi = \arccos \frac{2}{2\sqrt{5}} = \arccos \frac{1}{\sqrt{5}}$ ,

$\left(2\sqrt{5}, \frac{3\pi}{4}, \arccos \frac{\sqrt{5}}{5}\right)$ , spherical

70.  $\left(\frac{\sqrt{3}}{4}, \frac{3}{4}, \frac{3\sqrt{3}}{2}\right)$ , rectangular

(a)  $r = \sqrt{\left(\frac{\sqrt{3}}{4}\right)^2 + \left(\frac{3}{4}\right)^2} = \frac{\sqrt{3}}{2}$ ,

$\theta = \arctan \sqrt{3} = \frac{\pi}{3}$ ,

$z = \frac{3\sqrt{3}}{2}, \left(\frac{\sqrt{3}}{2}, \frac{\pi}{2}, \frac{3\sqrt{3}}{2}\right)$ , cylindrical

(b)  $\rho = \sqrt{\left(\frac{\sqrt{3}}{4}\right)^2 + \left(\frac{3}{4}\right)^2 + \left(\frac{3\sqrt{3}}{2}\right)^2} = \frac{\sqrt{30}}{2}, \theta = \frac{\pi}{3}$ ,

$\phi = \arccos \frac{3}{\sqrt{10}}, \left(\frac{\sqrt{30}}{2}, \frac{\pi}{3}, \arccos \frac{3}{\sqrt{10}}\right)$ , spherical

71.  $\left(100, -\frac{\pi}{6}, 50\right)$ , cylindrical

$\rho = \sqrt{100^2 + 50^2} = 50\sqrt{5}$

$\theta = -\frac{\pi}{6}$

$\phi = \arccos \left( \frac{50}{50\sqrt{5}} \right) = \arccos \left( \frac{1}{\sqrt{5}} \right) \approx 63.4^\circ \text{ or } 1.107$

$\left(50\sqrt{5}, -\frac{\pi}{6}, 63.4^\circ\right)$ , spherical or  $\left(50\sqrt{5}, -\frac{\pi}{6}, 1.1071\right)$

72.  $\left(81, -\frac{5\pi}{6}, 27\sqrt{3}\right)$ , cylindrical

$\rho = \sqrt{6561 + 2187} = 54\sqrt{3}$

$\theta = -\frac{5\pi}{6}$

$\phi = \arccos \left( \frac{27\sqrt{3}}{54\sqrt{3}} \right) = \arccos \frac{1}{2} = \frac{\pi}{3}$

$\left(54\sqrt{3}, -\frac{5\pi}{6}, \frac{\pi}{3}\right)$ , spherical

73.  $\left(25, -\frac{\pi}{4}, \frac{3\pi}{4}\right)$ , spherical

$$r^2 = \left(25 \sin\left(\frac{3\pi}{4}\right)\right)^2 \Rightarrow r = \frac{25\sqrt{2}}{2}$$

$$\theta = -\frac{\pi}{4}$$

$$z = \rho \cos \phi = 25 \cos \frac{3\pi}{4} = -\frac{25\sqrt{2}}{2}$$

$$\left(\frac{25\sqrt{2}}{2}, -\frac{\pi}{4}, -\frac{25\sqrt{2}}{2}\right)$$
, cylindrical

74.  $\left(12, -\frac{\pi}{2}, \frac{2\pi}{3}\right)$ , spherical

$$r^2 = \left(12 \sin\left(\frac{2\pi}{3}\right)\right)^2 \Rightarrow r = 6\sqrt{3}$$

$$\theta = -\frac{\pi}{2}$$

$$z = \rho \cos \phi = 12 \cos\left(\frac{2\pi}{3}\right) = -6$$

$$\left(6\sqrt{3}, -\frac{\pi}{2}, -6\right)$$
, cylindrical

75.  $x^2 - y^2 = 2z$

(a) Cylindrical:

$$r^2 \cos^2 \theta - r^2 \sin^2 \theta = 2z \Rightarrow r^2 \cos 2\theta = 2z$$

(b) Spherical:

$$\rho^2 \sin^2 \phi \cos^2 \theta - \rho^2 \sin^2 \phi \sin^2 \theta = 2\rho \cos \phi$$

$$\rho \sin^2 \phi \cos 2\theta - 2 \cos \phi = 0$$

$$\rho = 2 \sec 2\theta \cos \phi \csc^2 \phi$$

76.  $x^2 + y^2 + z^2 = 16$

(a) Cylindrical:  $r^2 + z^2 = 16$

(b) Spherical:  $\rho = 4$

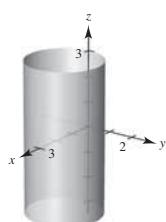
77.  $r = 5 \cos \theta$ , cylindrical equation

$$r^2 = 5r \cos \theta$$

$$x^2 + y^2 = 5x$$

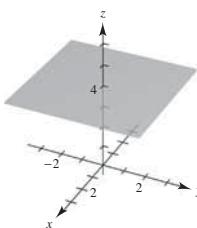
$$x^2 - 5x + \frac{25}{4} + y^2 = \frac{25}{4}$$

$$\left(x - \frac{5}{2}\right)^2 + y^2 = \left(\frac{5}{2}\right)^2$$
, rectangular equation



78.  $z = 4$ , cylindrical equation

$z = 4$ , rectangular equation

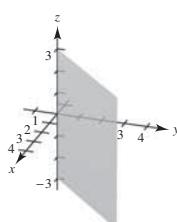


79.  $\theta = \frac{\pi}{4}$ , spherical coordinates

$$\tan \theta = \tan \frac{\pi}{4} = 1$$

$$\frac{y}{x} = 1$$

$y = x$ ,  $x \geq 0$ , rectangular coordinates, half-plane

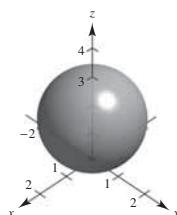


80.  $\rho = 3 \cos \theta$ , spherical coordinates

$$\sqrt{x^2 + y^2 + z^2} = \frac{3z}{\sqrt{x^2 + y^2 + z^2}}$$

$$x^2 + y^2 + z^2 - 3z = 0$$

$$x^2 + y^2 + \left(z - \frac{3}{2}\right)^2 = \left(\frac{3}{2}\right)^2$$
, rectangular coordinates, sphere



## Problem Solving for Chapter 11

1.  $\mathbf{a} + \mathbf{b} + \mathbf{c} = \mathbf{0}$

$$\mathbf{b} \times (\mathbf{a} + \mathbf{b} + \mathbf{c}) = \mathbf{0}$$

$$(\mathbf{b} \times \mathbf{a}) + (\mathbf{b} \times \mathbf{c}) = \mathbf{0}$$

$$\|\mathbf{a} \times \mathbf{b}\| = \|\mathbf{b} \times \mathbf{c}\|$$

$$\|\mathbf{b} \times \mathbf{c}\| = \|\mathbf{b}\| \|\mathbf{c}\| \sin A$$

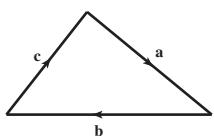
$$\|\mathbf{a} \times \mathbf{b}\| = \|\mathbf{a}\| \|\mathbf{b}\| \sin C$$

Then,

$$\frac{\sin A}{\|\mathbf{a}\|} = \frac{\|\mathbf{b} \times \mathbf{c}\|}{\|\mathbf{a}\| \|\mathbf{b}\| \|\mathbf{c}\|}$$

$$= \frac{\|\mathbf{a} \times \mathbf{b}\|}{\|\mathbf{a}\| \|\mathbf{b}\| \|\mathbf{c}\|}$$

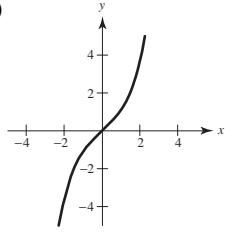
$$= \frac{\sin C}{\|\mathbf{c}\|}.$$



The other case,  $\frac{\sin A}{\|\mathbf{a}\|} = \frac{\sin B}{\|\mathbf{b}\|}$  is similar.

2.  $f(x) = \int_0^x \sqrt{t^4 + 1} dt$

(a)



(b)  $f'(x) = \sqrt{x^4 + 1}$

$$f'(0) = 1 = \tan \theta$$

$$\theta = \frac{\pi}{4}$$

$$\mathbf{u} = \frac{1}{\sqrt{2}}(\mathbf{i} + \mathbf{j}) = \left\langle \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \right\rangle$$

$$(c) \pm \left\langle \frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2} \right\rangle$$

(d) The line is  $y = x$ :  $x = t$ ,  $y = t$ .

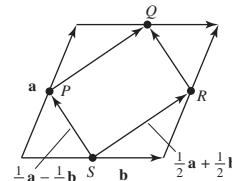
3. Label the figure as indicated.

From the figure, you see that

$$\overline{SP} = \frac{1}{2}\mathbf{a} - \frac{1}{2}\mathbf{b} = \overline{RQ} \text{ and } \overline{SR} = \frac{1}{2}\mathbf{a} + \frac{1}{2}\mathbf{b} = \overline{PQ}.$$

Because  $\overline{SP} = \overline{RQ}$  and  $\overline{SR} = \overline{PQ}$ ,

$PSRQ$  is a parallelogram.



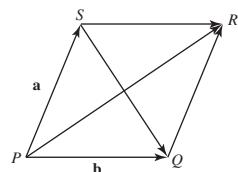
4. Label the figure as indicated.

$$\overline{PR} = \mathbf{a} + \mathbf{b}$$

$$\overline{SQ} = \mathbf{b} - \mathbf{a}$$

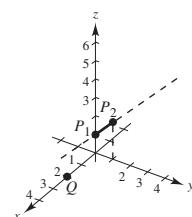
$$(\mathbf{a} + \mathbf{b}) \cdot (\mathbf{b} - \mathbf{a}) = \|\mathbf{b}\|^2 - \|\mathbf{a}\|^2 = 0, \text{ because}$$

$$\|\mathbf{a}\| = \|\mathbf{b}\| \text{ in a rhombus.}$$



5. (a)  $\mathbf{u} = \langle 0, 1, 1 \rangle$  is the direction vector of the line determined by  $P_1$  and  $P_2$ .

$$\begin{aligned} D &= \frac{\|\overline{P_1Q} \times \mathbf{u}\|}{\|\mathbf{u}\|} \\ &= \frac{\|\langle 2, 0, -1 \rangle \times \langle 0, 1, 1 \rangle\|}{\sqrt{2}} \\ &= \frac{\|\langle 1, -2, 2 \rangle\|}{\sqrt{2}} = \frac{3}{\sqrt{2}} = \frac{3\sqrt{2}}{2} \end{aligned}$$

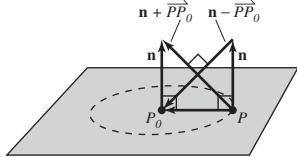


(b) The shortest distance to the line segment is  $\|P_1Q\| = \|\langle 2, 0, -1 \rangle\| = \sqrt{5}$ .

6.  $(\mathbf{n} + \overrightarrow{PP_0}) \perp (\mathbf{n} - \overrightarrow{PP_0})$

Figure is a square.

So,  $\|\overrightarrow{PP_0}\| = \|\mathbf{n}\|$  and the points  $P$  form a circle of radius  $\|\mathbf{n}\|$  in the plane with center at  $P_0$ .



7. (a)  $V = \pi \int_0^1 (\sqrt{z})^2 dz = \left[ \pi \frac{z^2}{2} \right]_0^1 = \frac{1}{2} \pi$

Note:  $\frac{1}{2}$  (base)(altitude) =  $\frac{1}{2} \pi(1) = \frac{1}{2} \pi$

(b)  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = z$ : (slice at  $z = c$ )

$$\frac{x^2}{(\sqrt{ca})^2} + \frac{y^2}{(\sqrt{cb})^2} = 1$$

At  $z = c$ , figure is ellipse of area

$$\pi(\sqrt{ca})(\sqrt{cb}) = \pi abc.$$

$$V = \int_0^k \pi abc \cdot dc = \left[ \frac{\pi abc^2}{2} \right]_0^k = \frac{\pi abk^2}{2}$$

(c)  $V = \frac{1}{2}(\pi abk)k = \frac{1}{2}(\text{area of base})(\text{height})$

8. (a)  $V = 2 \int_0^r \pi(r^2 - x^2) dx = 2\pi \left[ r^2x - \frac{x^3}{3} \right]_0^r = \frac{4}{3}\pi r^3$

(b) At height  $z = d > 0$ ,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{d^2}{c^2} = 1$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 - \frac{d^2}{c^2} = \frac{c^2 - d^2}{c^2}$$

$$\frac{x^2}{a^2(c^2 - d^2)} + \frac{y^2}{b^2(c^2 - d^2)} = 1.$$

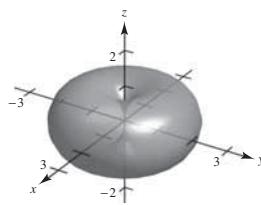
$$\text{Area} = \pi \sqrt{\left( \frac{a^2(c^2 - d^2)}{c^2} \right) \left( \frac{b^2(c^2 - d^2)}{c^2} \right)} = \frac{\pi ab}{c^2} (c^2 - d^2)$$

$$V = 2 \int_0^c \frac{\pi ab}{c^2} (c^2 - d^2) dd$$

$$= \frac{2\pi ab}{c^2} \left[ c^2d - \frac{d^3}{3} \right]_0^c = \frac{4}{3}\pi abc$$

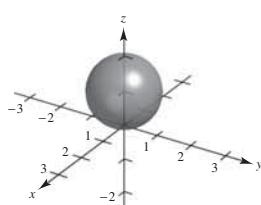
9. (a)  $\rho = 2 \sin \phi$

Torus



(b)  $\rho = 2 \cos \phi$

Sphere



10. (a)  $r = 2 \cos \theta$

Cylinder

(b)  $z = r^2 \cos 2\theta$   
 $z^2 = x^2 - y^2$

Hyperbolic paraboloid

11. From Exercise 66, Section 11.4,

$$(\mathbf{u} \times \mathbf{v}) \times (\mathbf{w} \times \mathbf{z}) = [(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{z}] \mathbf{w} - [(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}] \mathbf{z}.$$

12.  $x = -t + 3, y = \frac{1}{2}t + 1, z = 2t - 1; Q = (4, 3, s)$

(a)  $\mathbf{u} = \langle -2, 1, 4 \rangle$  direction vector for line

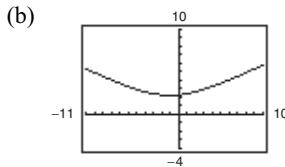
$P = (3, 1, -1)$  point on line

$$\overline{PQ} = \langle 1, 2, s+1 \rangle$$

$$\overline{PQ} \times \mathbf{u} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & s+1 \\ -2 & 1 & 4 \end{vmatrix}$$

$$= (7-s)\mathbf{i} + (-6-2s)\mathbf{j} + 5\mathbf{k}$$

$$D = \frac{\|\overline{PQ} \times \mathbf{u}\|}{\|\mathbf{u}\|} = \frac{\sqrt{(7-s)^2 + (-6-2s)^2 + 25}}{\sqrt{21}}$$



The minimum is  $D \approx 2.2361$  at  $s = -1$ .

(c) Yes, there are slant asymptotes. Using  $s = x$ , you have

$$D(s) = \frac{1}{\sqrt{21}}\sqrt{5x^2 + 10x + 110} = \frac{\sqrt{5}}{\sqrt{21}}\sqrt{x^2 + 2x + 22} = \frac{\sqrt{5}}{\sqrt{21}}\sqrt{(x+1)^2 + 21} \rightarrow \pm\sqrt{\frac{5}{21}}(x+1)$$

$$y = \pm\frac{\sqrt{105}}{21}(s+1) \text{ slant asymptotes.}$$

13. (a)  $\mathbf{u} = \|\mathbf{u}\|(\cos 0 \mathbf{i} + \sin 0 \mathbf{j}) = \|\mathbf{u}\|\mathbf{i}$

Downward force  $\mathbf{w} = -\mathbf{j}$

$$\begin{aligned} \mathbf{T} &= \|\mathbf{T}\|(\cos(90^\circ + \theta)\mathbf{i} + \sin(90^\circ + \theta)\mathbf{j}) \\ &= \|\mathbf{T}\|(-\sin \theta \mathbf{i} + \cos \theta \mathbf{j}) \\ \mathbf{0} &= \mathbf{u} + \mathbf{w} + \mathbf{T} = \|\mathbf{u}\|\mathbf{i} - \mathbf{j} + \|\mathbf{T}\|(-\sin \theta \mathbf{i} + \cos \theta \mathbf{j}) \\ \|\mathbf{u}\| &= \sin \theta \|\mathbf{T}\| \\ 1 &= \cos \theta \|\mathbf{T}\| \end{aligned}$$

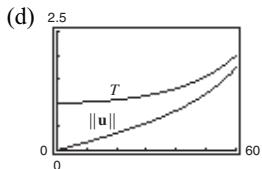
If  $\theta = 30^\circ$ ,  $\|\mathbf{u}\| = (1/2)\|\mathbf{T}\|$  and  $1 = (\sqrt{3}/2)\|\mathbf{T}\| \Rightarrow \|\mathbf{T}\| = \frac{2}{\sqrt{3}} \approx 1.1547$  lb and  $\|\mathbf{u}\| = \frac{1}{2} \left( \frac{2}{\sqrt{3}} \right) \approx 0.5774$  lb

(b) From part (a),  $\|\mathbf{u}\| = \tan \theta$  and  $\|\mathbf{T}\| = \sec \theta$ .

Domain:  $0 \leq \theta \leq 90^\circ$

(c)

$\theta$	$0^\circ$	$10^\circ$	$20^\circ$	$30^\circ$	$40^\circ$	$50^\circ$	$60^\circ$
$\mathbf{T}$	1	1.0154	1.0642	1.1547	1.3054	1.5557	2
$\ \mathbf{u}\ $	0	0.1763	0.3640	0.5774	0.8391	1.1918	1.7321



(e) Both are increasing functions.

(f)  $\lim_{\theta \rightarrow \pi/2^-} T = \infty$  and  $\lim_{\theta \rightarrow \pi/2^-} \|\mathbf{u}\| = \infty$ .

Yes. As  $\theta$  increases, both  $T$  and  $\|\mathbf{u}\|$  increase.

14. (a) The tension  $T$  is the same in each tow line.

$$\begin{aligned} 6000\mathbf{i} &= T(\cos 20^\circ + \cos(-20))\mathbf{i} + T(\sin 20^\circ + \sin(-20))\mathbf{j} \\ &= 2T\cos 20^\circ\mathbf{i} \\ \Rightarrow T &= \frac{6000}{2\cos 20^\circ} \approx 3192.5 \text{ lb} \end{aligned}$$

- (b) As in part (a),  $6000\mathbf{i} = 2T\cos\theta\mathbf{i}$

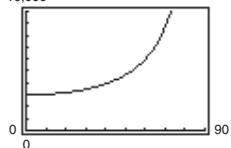
$$\Rightarrow T = \frac{3000}{\cos\theta}$$

Domain:  $0 < \theta < 90^\circ$

(c)

$\theta$	$10^\circ$	$20^\circ$	$30^\circ$	$40^\circ$	$50^\circ$	$60^\circ$
$T$	3046.3	3192.5	3464.1	3916.2	4667.2	6000.0

(d)



- (e) As  $\theta$  increases, there is less force applied in the direction of motion.

15. Let  $\theta = \alpha - \beta$ , the angle between  $\mathbf{u}$  and  $\mathbf{v}$ . Then

$$\sin(\alpha - \beta) = \frac{\|\mathbf{u} \times \mathbf{v}\|}{\|\mathbf{u}\| \|\mathbf{v}\|} = \frac{\|\mathbf{v} \times \mathbf{u}\|}{\|\mathbf{u}\| \|\mathbf{v}\|}.$$

For  $\mathbf{u} = \langle \cos \alpha, \sin \alpha, 0 \rangle$  and  $\mathbf{v} = \langle \cos \beta, \sin \beta, 0 \rangle$ ,  $\|\mathbf{u}\| = \|\mathbf{v}\| = 1$  and

$$\mathbf{v} \times \mathbf{u} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos \beta & \sin \beta & 0 \\ \cos \alpha & \sin \alpha & 0 \end{vmatrix} = (\sin \alpha \cos \beta - \cos \alpha \sin \beta)\mathbf{k}.$$

So,  $\sin(\alpha - \beta) = \|\mathbf{v} \times \mathbf{u}\| = \sin \alpha \cos \beta - \cos \alpha \sin \beta$ .

16. (a) Los Angeles:  $(4000, -118.24^\circ, 55.95^\circ)$

Rio de Janeiro:  $(4000, -43.23^\circ, 112.90^\circ)$

- (b) Los Angeles:  $x = 4000 \sin(55.95^\circ) \cos(-118.24^\circ)$

Rio de Janeiro:  $x = 4000 \sin(112.90^\circ) \cos(-43.23^\circ)$

$$y = 4000 \sin(55.95^\circ) \sin(-118.24^\circ)$$

$$y = 4000 \sin(112.90^\circ) \sin(-43.23^\circ)$$

$$z = 4000 \cos(55.95^\circ)$$

$$z = 4000 \cos(112.90^\circ)$$

$$(x, y, z) \approx (-1568.2, -2919.7, 2239.7)$$

$$(x, y, z) \approx (2684.7, -2523.8, -1556.5)$$

$$(c) \cos \theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \|\mathbf{v}\|} = \frac{(-1568.2)(2684.7) + (-2919.7)(-2523.8) + (2239.7)(-1556.5)}{(4000)(4000)} \approx -0.02047$$

$$\theta \approx 91.17^\circ \text{ or } 1.59 \text{ radians}$$

$$(d) s = r\theta = 4000(1.59) \approx 6360 \text{ miles}$$

(e) For Boston and Honolulu:

a. Boston:  $(4000, -71.06^\circ, 47.64^\circ)$       Honolulu:  $(4000, -157.86^\circ, 68.69^\circ)$

b. Boston:  $x = 4000 \sin 47.64^\circ \cos(-71.06^\circ)$       Honolulu:  $x = 4000 \sin 68.69^\circ \cos(-157.86^\circ)$

$$y = 4000 \sin 47.64^\circ \sin(-71.06^\circ)$$

$$y = 4000 \sin 68.69^\circ \sin(-157.86^\circ)$$

$$z = 4000 \cos 47.64^\circ$$

$$z = 4000 \cos 68.69^\circ$$

$$(959.4, -2795.7, 2695.1)$$

$$(-3451.7, -1404.4, 1453.7)$$

c.  $\cos \theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \|\mathbf{v}\|} = \frac{(959.4)(-3451.7) + (-2795.7)(-1404.4) + (2695.1)(1453.7)}{(4000)(4000)} \approx 0.28329$

$$\theta \approx 73.54^\circ \text{ or } 1.28 \text{ radians}$$

d.  $s = r\theta = 4000(1.28) \approx 5120 \text{ miles}$

17. From Theorem 11.13 and Theorem 11.7 (6) you have

$$\begin{aligned} D &= \frac{|\overrightarrow{PQ} \cdot \mathbf{n}|}{\|\mathbf{n}\|} \\ &= \frac{|\mathbf{w} \cdot (\mathbf{u} \times \mathbf{v})|}{\|\mathbf{u} \times \mathbf{v}\|} = \frac{|\mathbf{u} \times \mathbf{v} \cdot \mathbf{w}|}{\|\mathbf{u} \times \mathbf{v}\|} = \frac{|\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})|}{\|\mathbf{u} \times \mathbf{v}\|}. \end{aligned}$$

18. Assume one of  $a, b, c$ , is not zero, say  $a$ . Choose a point in the first plane such as  $(-d_1/a, 0, 0)$ . The distance between this point and the second plane is

$$\begin{aligned} D &= \frac{|a(-d_1/a) + b(0) + c(0) + d_2|}{\sqrt{a^2 + b^2 + c^2}} \\ &= \frac{|-d_1 + d_2|}{\sqrt{a^2 + b^2 + c^2}} = \frac{|d_1 - d_2|}{\sqrt{a^2 + b^2 + c^2}}. \end{aligned}$$

19.  $x^2 + y^2 = 1$  cylinder

$z = 2y$  plane

Introduce a coordinate system in the plane  $z = 2y$ .

The new  $u$ -axis is the original  $x$ -axis.

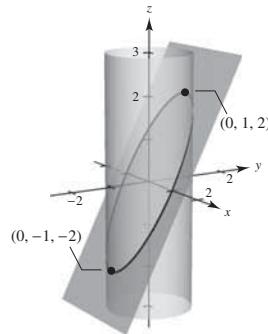
The new  $v$ -axis is the line  $z = 2y, x = 0$ .

Then the intersection of the cylinder and plane satisfies the equation of an ellipse:

$$x^2 + y^2 = 1$$

$$x^2 + \left(\frac{z}{2}\right)^2 = 1$$

$$x^2 + \frac{z^2}{4} = 1 \quad \text{ellipse}$$



20. Essay.

# C H A P T E R   1 2

## Vector-Valued Functions

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# C H A P T E R 12

## Vector-Valued Functions

### Section 12.1 Vector-Valued Functions

1.  $\mathbf{r}(t) = \frac{1}{t+1}\mathbf{i} + \frac{t}{2}\mathbf{j} - 3t\mathbf{k}$

Component functions:  $f(t) = \frac{1}{t+1}$

$$g(t) = \frac{t}{2}$$

$$h(t) = -3t$$

Domain:  $(-\infty, -1) \cup (-1, \infty)$

3.  $\mathbf{r}(t) = \ln t\mathbf{i} - e^t\mathbf{j} - t\mathbf{k}$

Component functions:  $f(t) = \ln t$

$$g(t) = -e^t$$

$$h(t) = -t$$

Domain:  $(0, \infty)$

2.  $\mathbf{r}(t) = \sqrt{4-t^2}\mathbf{i} + t^2\mathbf{j} - 6t\mathbf{k}$

Component functions:  $f(t) = \sqrt{4-t^2}$

$$g(t) = t^2$$

$$h(t) = -6t$$

Domain:  $[-2, 2]$

4.  $\mathbf{r}(t) = \sin t\mathbf{i} + 4 \cos t\mathbf{j} + t\mathbf{k}$

Component functions:  $f(t) = \sin t$

$$g(t) = 4 \cos t$$

$$h(t) = t$$

Domain:  $(-\infty, \infty)$

5.  $\mathbf{r}(t) = \mathbf{F}(t) + \mathbf{G}(t) = (\cos t\mathbf{i} - \sin t\mathbf{j} + \sqrt{t}\mathbf{k}) + (\cos t\mathbf{i} + \sin t\mathbf{j}) = 2 \cos t\mathbf{i} + \sqrt{t}\mathbf{k}$

Domain:  $[0, \infty)$

6.  $\mathbf{r}(t) = \mathbf{F}(t) - \mathbf{G}(t) = (\ln t\mathbf{i} + 5t\mathbf{j} - 3t^2\mathbf{k}) - (\mathbf{i} + 4t\mathbf{j} - 3t^2\mathbf{k})$

$$= (\ln t - 1)\mathbf{i} + (5t - 4t)\mathbf{j} + (-3t^2 + 3t^2)\mathbf{k} = (\ln t - 1)\mathbf{i} + t\mathbf{j}$$

Domain:  $(0, \infty)$

7.  $\mathbf{r}(t) = \mathbf{F}(t) \times \mathbf{G}(t) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \sin t & \cos t & 0 \\ 0 & \sin t & \cos t \end{vmatrix} = \cos^2 t\mathbf{i} - \sin t \cos t\mathbf{j} + \sin^2 t\mathbf{k}$

Domain:  $(-\infty, \infty)$

8.  $\mathbf{r}(t) = \mathbf{F}(t) \times \mathbf{G}(t) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ t^3 & -t & t \\ \sqrt[3]{t} & \frac{1}{t+1} & t+2 \end{vmatrix} = \left( -t(t+2) - \frac{t}{t+1} \right) \mathbf{i} - \left( t^3(t+2) - t\sqrt[3]{t} \right) \mathbf{j} + \left( \frac{t^3}{t+1} + t\sqrt[3]{t} \right) \mathbf{k}$

Domain:  $(-\infty, -1), (-1, \infty)$

9.  $\mathbf{r}(t) = \frac{1}{2}t^2\mathbf{i} - (t-1)\mathbf{j}$

(a)  $\mathbf{r}(1) = \frac{1}{2}\mathbf{i}$

(b)  $\mathbf{r}(0) = \mathbf{j}$

(c)  $\mathbf{r}(s+1) = \frac{1}{2}(s+1)^2\mathbf{i} - (s+1-1)\mathbf{j} = \frac{1}{2}(s+1)^2\mathbf{i} - s\mathbf{j}$

(d)  $\mathbf{r}(2+\Delta t) - \mathbf{r}(2) = \frac{1}{2}(2+\Delta t)^2\mathbf{i} - (2+\Delta t-1)\mathbf{j} - (2\mathbf{i} - \mathbf{j}) = \left( 2 + 2\Delta t + \frac{1}{2}(\Delta t)^2 \right) \mathbf{i} - (1 + \Delta t)\mathbf{j} - 2\mathbf{i} + \mathbf{j}$

$$= \left( 2\Delta t + \frac{1}{2}(\Delta t)^2 \right) \mathbf{i} - (\Delta t)\mathbf{j} = \frac{1}{2}\Delta t(\Delta t + 4)\mathbf{i} - \Delta t\mathbf{j}$$

10.  $\mathbf{r}(t) = \cos t\mathbf{i} + 2 \sin t\mathbf{j}$

(a)  $\mathbf{r}(0) = \mathbf{i}$

(b)  $\mathbf{r}\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}\mathbf{i} + \sqrt{2}\mathbf{j}$

(c)  $\mathbf{r}(\theta - \pi) = \cos(\theta - \pi)\mathbf{i} + 2 \sin(\theta - \pi)\mathbf{j} = -\cos \theta\mathbf{i} - 2 \sin \theta\mathbf{j}$

(d)  $\mathbf{r}\left(\frac{\pi}{6} + \Delta t\right) - \mathbf{r}\left(\frac{\pi}{6}\right) = \cos\left(\frac{\pi}{6} + \Delta t\right)\mathbf{i} + 2 \sin\left(\frac{\pi}{6} + \Delta t\right)\mathbf{j} - \left(\cos\left(\frac{\pi}{6}\right)\mathbf{i} + 2 \sin\frac{\pi}{6}\mathbf{j}\right)$

11.  $\mathbf{r}(t) = \ln t\mathbf{i} + \frac{1}{t}\mathbf{j} + 3t\mathbf{k}$

(a)  $\mathbf{r}(2) = \ln 2\mathbf{i} + \frac{1}{2}\mathbf{j} + 6\mathbf{k}$

(b)  $\mathbf{r}(-3)$  is not defined. ( $\ln(-3)$  does not exist.)

(c)  $\mathbf{r}(t - 4) = \ln(t - 4)\mathbf{i} + \frac{1}{t - 4}\mathbf{j} + 3(t - 4)\mathbf{k}$

(d)  $\mathbf{r}(1 + \Delta t) - \mathbf{r}(1) = \ln(1 + \Delta t)\mathbf{i} + \frac{1}{1 + \Delta t}\mathbf{j} + 3(1 + \Delta t)\mathbf{k} - (0\mathbf{i} + \mathbf{j} + 3\mathbf{k}) = \ln(1 + \Delta t)\mathbf{i} + \left(\frac{-\Delta t}{1 + \Delta t}\right)\mathbf{j} + (3\Delta t)\mathbf{k}$

12.  $\mathbf{r}(t) = \sqrt{t}\mathbf{i} + t^{3/2}\mathbf{j} + e^{-t/4}\mathbf{k}$

(a)  $\mathbf{r}(0) = \mathbf{k}$

(b)  $\mathbf{r}(4) = 2\mathbf{i} + 8\mathbf{j} + e^{-1}\mathbf{k}$

(c)  $\mathbf{r}(c + 2) = \sqrt{c + 2}\mathbf{i} + (c + 2)^{3/2}\mathbf{j} + e^{-[(c+2)/4]}\mathbf{k}$

(d)  $\mathbf{r}(9 + \Delta t) - \mathbf{r}(9) = (\sqrt{9 + \Delta t})\mathbf{i} + (9 + \Delta t)^{3/2}\mathbf{j} + e^{-[9 + \Delta t/4]}\mathbf{k} - (3\mathbf{i} + 27\mathbf{j} + e^{-9/4}\mathbf{k})$   
 $= (\sqrt{9 + \Delta t} - 3)\mathbf{i} + ((9 + \Delta t)^{3/2} - 27)\mathbf{j} + (e^{-[(9 + \Delta t)/4]} - e^{-9/4}\mathbf{k})$

13.  $\mathbf{r}(t) = \sqrt{t}\mathbf{i} + 3t\mathbf{j} - 4t\mathbf{k}$

$$\begin{aligned}\|\mathbf{r}(t)\| &= \sqrt{(\sqrt{t})^2 + (3t)^2 + (-4t)^2} \\ &= \sqrt{t + 9t^2 + 16t^2} = \sqrt{t(1 + 25t)}\end{aligned}$$

14.  $\mathbf{r}(t) = \sin 3t\mathbf{i} + \cos 3t\mathbf{j} + t\mathbf{k}$

$$\|\mathbf{r}(t)\| = \sqrt{(\sin 3t)^2 + (\cos 3t)^2 + t^2} = \sqrt{1 + t^2}$$

15.  $P(0, 0, 0), Q(3, 1, 2)$

$\mathbf{v} = \overrightarrow{PQ} = \langle 3, 1, 2 \rangle$

$\mathbf{r}(t) = 3t\mathbf{i} + t\mathbf{j} + 2t\mathbf{k}, 0 \leq t \leq 1$

$x = 3t, y = t, z = 2t, 0 \leq t \leq 1$ , Parametric equation

(Answers may vary)

16.  $P(0, 2, -1), Q(4, 7, 2)$

$\mathbf{v} = \overrightarrow{PQ} = \langle 4, 5, 3 \rangle$

$\mathbf{r}(t) = 4t\mathbf{i} + (2 + 5t)\mathbf{j} + (-1 + 3t)\mathbf{k}, 0 \leq t \leq 1$

$x = 4t, y = 2 + 5t, z = -1 + 3t,$

$0 \leq t \leq 1$ , Parametric equation

(Answers may vary)

17.  $P(-2, 5, -3), Q(-1, 4, 9)$

$\mathbf{v} = \overrightarrow{PQ} = \langle 1, -1, 12 \rangle$

$\mathbf{r}(t) = (-2 + t)\mathbf{i} + (5 - t)\mathbf{j} + (-3 + 12t)\mathbf{k}, 0 \leq t \leq 1$

$x = -2 + t, y = 5 - t, z = -3 + 12t,$

$0 \leq t \leq 1$ , Parametric equation

(Answers may vary)

18.  $P(1, -6, 8), Q(-3, -2, 5)$

$\mathbf{v} = \overrightarrow{PQ} = \langle -4, 4, -3 \rangle$

$\mathbf{r}(t) = (1 - 4t)\mathbf{i} + (-6 + 4t)\mathbf{j} + (8 - 3t)\mathbf{k}, 0 \leq t \leq 1$

$x = 1 - 4t, y = -6 + 4t, z = 8 - 3t,$

$0 \leq t \leq 1$ , Parametric equation

(Answers may vary)

19.  $\mathbf{r}(t) \cdot \mathbf{u}(t) = (3t - 1)(t^2) + \left(\frac{1}{4}t^3\right)(-8) + 4(t^3)$

$= 3t^3 - t^2 - 2t^3 + 4t^3 = 5t^3 - t^2$ , a scalar.

No, the dot product is a scalar-valued function.

20.  $\mathbf{r}(t) \cdot \mathbf{u}(t) = (3 \cos t)(4 \sin t) + (2 \sin t)(-6 \cos t) + (t - 2)(t^2) = t^3 - 2t^2$ , a scalar.

No, the dot product is a scalar-valued function.

21.  $\mathbf{r}(t) = t\mathbf{i} + 2t\mathbf{j} + t^2\mathbf{k}, -2 \leq t \leq 2$

$$x = t, y = 2t, z = t^2$$

So,  $z = x^2$ . Matches (b)

22.  $\mathbf{r}(t) = \cos(\pi t)\mathbf{i} + \sin(\pi t)\mathbf{j} + t^2\mathbf{k}, -1 \leq t \leq 1$

$$x = \cos(\pi t), y = \sin(\pi t), z = t^2$$

So,  $x^2 + y^2 = 1$ . Matches (c)

23.  $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + e^{0.75t}\mathbf{k}, -2 \leq t \leq 2$

$$x = t, y = t^2, z = e^{0.75t}$$

So,  $y = x^2$ . Matches (d)

24.  $\mathbf{r}(t) = t\mathbf{i} + \ln t\mathbf{j} + \frac{2t}{3}\mathbf{k}, 0.1 \leq t \leq 5$

$$x = t, y = \ln t, z = \frac{2t}{3}$$

So,  $z = \frac{2}{3}x$  and  $y = \ln x$ . Matches (a)

25. (a) View from the negative  $x$ -axis:  $(-20, 0, 0)$

(b) View from above the first octant:  $(10, 20, 10)$

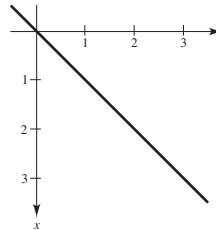
(c) View from the  $z$ -axis:  $(0, 0, 20)$

(d) View from the positive  $x$ -axis:  $(20, 0, 0)$

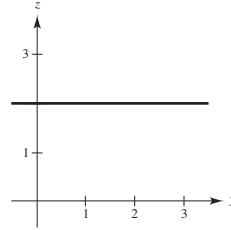
26.  $\mathbf{r}(t) = t\mathbf{i} + t\mathbf{j} + 2\mathbf{k}$

$$x = t, y = t, z = 2 \Rightarrow x = y$$

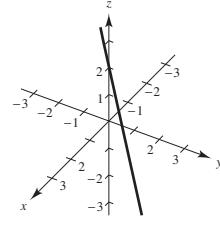
(a)  $(0, 0, 20)$



(b)  $(10, 0, 0)$



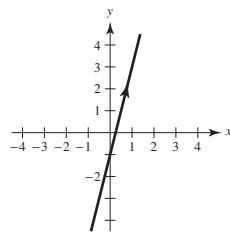
(c)  $(5, 5, 5)$



27.  $x = \frac{t}{4} \Rightarrow t = 4x$

$$y = t - 1$$

$$y = 4x - 1$$

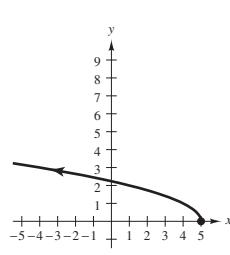


28.  $x = 5 - t \Rightarrow t = 5 - x$

$$y = \sqrt{t}$$

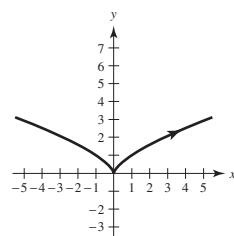
$$y = \sqrt{5 - x}$$

Domain:  $t \geq 0$

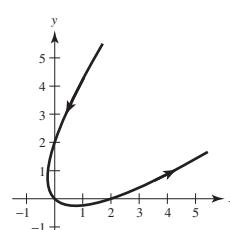


29.  $x = t^3, y = t^2$

$$y = x^{2/3}$$

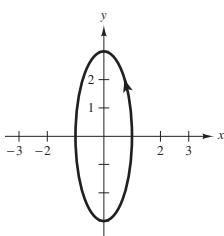


30.  $x = t^2 + t, y = t^2 - t$



31.  $x = \cos \theta, y = 3 \sin \theta$

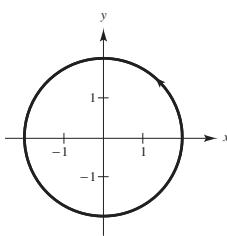
$$x^2 + \frac{y^2}{9} = 1, \text{ Ellipse}$$



32.  $x = 2 \cos t$

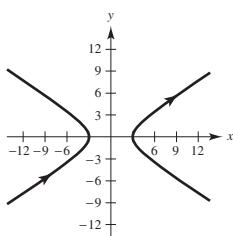
$$y = 2 \sin t$$

$$x^2 + y^2 = 4, \text{ circle}$$



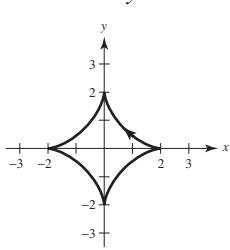
33.  $x = 3 \sec \theta, y = 2 \tan \theta$

$$\frac{x^2}{9} = \frac{y^2}{4} + 1, \text{ Hyperbola}$$



34.  $x = 2 \cos^3 t, y = 2 \sin^3 t$

$$\left(\frac{x}{2}\right)^{2/3} + \left(\frac{y}{2}\right)^{2/3} = \cos^2 t + \sin^2 t \\ = 1 \\ x^{2/3} + y^{2/3} = 2^{2/3}$$

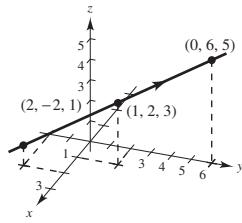


35.  $x = -t + 1$

$$y = 4t + 2$$

$$z = 2t + 3$$

Line passing through  
the points:  $(0, 6, 5), (1, 2, 3)$



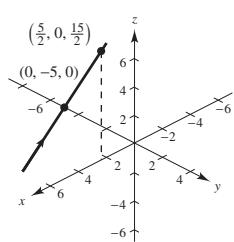
36.  $x = t$

$$y = 2t - 5$$

$$y = 3t$$

Line passing through

the points:  $(0, -5, 0), \left(\frac{5}{2}, 0, \frac{15}{2}\right)$

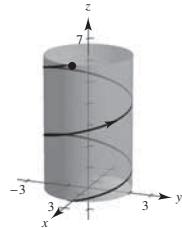


37.  $x = 2 \cos t, y = 2 \sin t, z = t$

$$\frac{x^2}{4} + \frac{y^2}{4} = 1$$

$$z = t$$

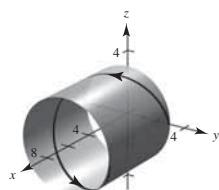
Circular helix



38.  $x = t, y = 3 \cos t, z = 3 \sin t$

$$y^2 + z^2 = (3 \cos t)^2 + (3 \sin t)^2 = 9$$

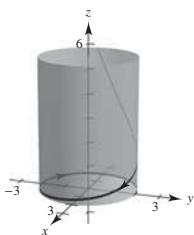
Circular helix along cylinder  $y^2 + z^2 = 9$



39.  $x = 2 \sin t, y = 2 \cos t, z = e^{-t}$

$$x^2 + y^2 = 4$$

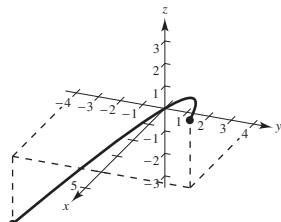
$$z = e^{-t}$$



40.  $x = t^2, y = 2t, z = \frac{3}{2}t$

$$x = \frac{y^2}{4}, z = \frac{3}{4}y$$

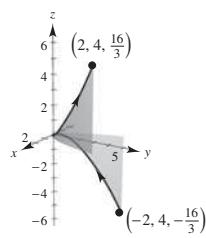
$t$	-2	-1	0	1	2
$x$	4	1	0	1	4
$y$	-4	-2	0	2	4
$z$	-3	− $\frac{3}{2}$	0	$\frac{3}{2}$	3



41.  $x = t, y = t^2, z = \frac{2}{3}t^3$

$$y = x^2, z = \frac{2}{3}x^3$$

$t$	-2	-1	0	1	2
$x$	-2	-1	0	1	2
$y$	4	1	0	1	4
$z$	$-\frac{16}{3}$	$-\frac{2}{3}$	0	$\frac{2}{3}$	$\frac{16}{3}$



42.  $x = \cos t + t \sin t$

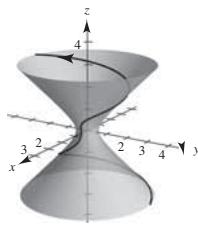
$$y = \sin t - t \cos t$$

$$z = t$$

$$x^2 + y^2 = 1 + t^2 = 1 + z^2$$

$$\text{or } x^2 + y^2 - z^2 = 1$$

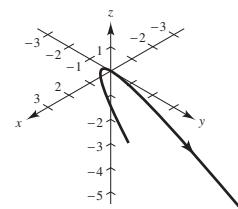
$$z = t$$



Helix along a hyperboloid of one sheet

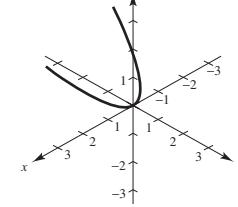
43.  $\mathbf{r}(t) = -\frac{1}{2}t^2\mathbf{i} + t\mathbf{j} - \frac{\sqrt{3}}{2}t^2\mathbf{k}$

Parabola



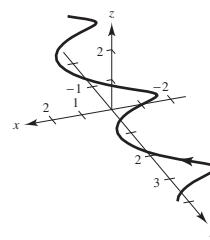
44.  $\mathbf{r}(t) = t\mathbf{i} - \frac{\sqrt{3}}{2}t^2\mathbf{j} + \frac{1}{2}t^2\mathbf{k}$

Parabola



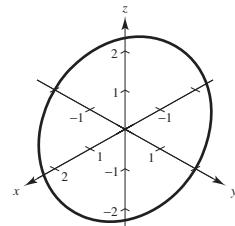
45.  $\mathbf{r}(t) = \sin t\mathbf{i} + \left(\frac{\sqrt{3}}{2} \cos t - \frac{1}{2}t\right)\mathbf{j} + \left(\frac{1}{2} \cos t + \frac{\sqrt{3}}{2}\right)\mathbf{k}$

Helix

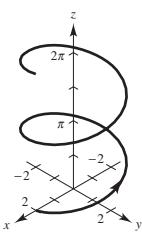


46.  $\mathbf{r}(t) = -\sqrt{2} \sin t\mathbf{i} + 2 \cos t\mathbf{j} + \sqrt{2} \sin t\mathbf{k}$

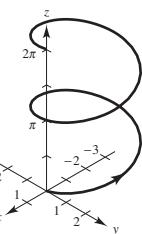
Ellipse



47.

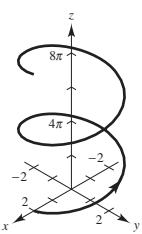


(a)



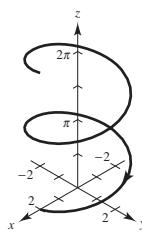
The helix is translated 2 units back on the x-axis.

(b)



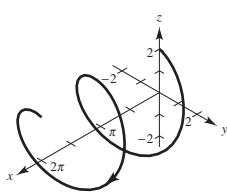
The height of the helix increases at a faster rate.

(c)



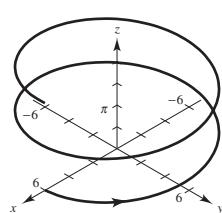
The orientation of the helix is reversed.

(d)



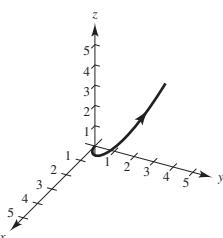
The axis of the helix is the x-axis.

(e)



The radius of the helix is increased from 2 to 6.

48.  $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + \frac{1}{2}t^3\mathbf{k}$



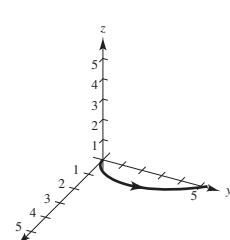
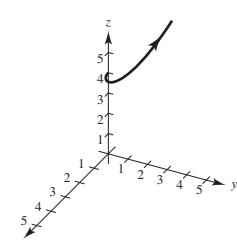
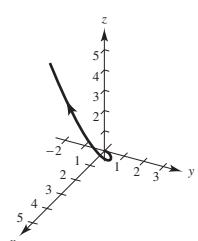
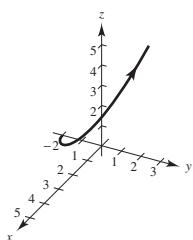
- (a)  $\mathbf{u}(t) = \mathbf{r}(t) - 2\mathbf{j}$  is a translation 2 units to the left along the  $y$ -axis.

- (b)  $\mathbf{u}(t) = t^2\mathbf{i} + t\mathbf{j} + \frac{1}{2}t^3\mathbf{k}$  has the roles of  $x$  and  $y$  interchanged. The graph is a reflection in the plane  $x = y$ .

- (c)  $\mathbf{u}(t) = \mathbf{r}(t) + 4\mathbf{k}$  is an upward shift 4 units.

- (d)  $\mathbf{u}(t) = t\mathbf{i} + t^2\mathbf{j} + \frac{1}{8}t^3\mathbf{k}$  shrinks the  $z$ -value by a factor of 4. The curve rises more slowly.

- (e)  $\mathbf{u}(t) = \mathbf{r}(-t)$  reverses the orientation.



49.  $y = x + 5$

Let  $x = t$ , then  $y = t + 5$

$$\mathbf{r}(t) = t\mathbf{i} + (t + 5)\mathbf{j}$$

50.  $2x - 3y + 5 = 0$

Let  $x = t$ , then  $y = \frac{1}{3}(2t + 5)$ .

$$\mathbf{r}(t) = t\mathbf{i} + \frac{1}{3}(2t + 5)\mathbf{j}$$

51.  $y = (x - 2)^2$

Let  $x = t$ , then  $y = (t - 2)^2$ .

$$\mathbf{r}(t) = t\mathbf{i} + (t - 2)^2\mathbf{j}$$

52.  $y = 4 - x^2$

Let  $x = t$ , then  $y = 4 - t^2$ .

$$\mathbf{r}(t) = t\mathbf{i} + (4 - t^2)\mathbf{j}$$

53.  $x^2 + y^2 = 25$

Let  $x = 5 \cos t$ , then  $y = 5 \sin t$ .

$$\mathbf{r}(t) = 5 \cos t\mathbf{i} + 5 \sin t\mathbf{j}$$

54.  $(x - 2)^2 + y^2 = 4$

Let  $x - 2 = 2 \cos t$ ,  $y = 2 \sin t$ .

$$\mathbf{r}(t) = (2 + 2 \cos t)\mathbf{i} + 2 \sin t\mathbf{j}$$

55.  $\frac{x^2}{16} - \frac{y^2}{4} = 1$

Let  $x = 4 \sec t$ ,  $y = 2 \tan t$ .

$$\mathbf{r}(t) = 4 \sec t\mathbf{i} + 2 \tan t\mathbf{j}$$

56.  $\frac{x^2}{9} + \frac{y^2}{16} = 1$

Let  $x = 3 \cos t$  and  $y = 4 \sin t$

Then  $\frac{x^2}{9} + \frac{y^2}{16} = \cos^2 t + \sin^2 t = 1$

$$\mathbf{r}(t) = 3 \cos t\mathbf{i} + 4 \sin t\mathbf{j}$$

57.  $\mathbf{r}_1(t) = t\mathbf{i} + t^2\mathbf{j}, 0 \leq t \leq 2$  ( $y = x^2$ )

$$\mathbf{r}_2(t) = (2 - t)\mathbf{i} + 4\mathbf{j}, 0 \leq t \leq 2$$

$$\mathbf{r}_3(t) = (4 - t)\mathbf{j}, 0 \leq t \leq 4$$

(Other answers possible)

58.  $\mathbf{r}_1(t) = t\mathbf{i}, 0 \leq t \leq 10$  ( $\mathbf{r}_1(0) = \mathbf{0}, \mathbf{r}_1(10) = 10\mathbf{i}$ )

$$\mathbf{r}_2(t) = 10(\cos t\mathbf{i} + \sin t\mathbf{j}),$$

$$0 \leq t \leq \frac{\pi}{4} \left( \mathbf{r}_2(0) = 10\mathbf{i}, \mathbf{r}_2\left(\frac{\pi}{4}\right) = 5\sqrt{2}\mathbf{i} + 5\sqrt{2}\mathbf{j} \right)$$

$$\mathbf{r}_3(t) = 5\sqrt{2}(1-t)\mathbf{i} + 5\sqrt{2}(1-t)\mathbf{j},$$

$$0 \leq t \leq 1 \left( \mathbf{r}_3(0) = 5\sqrt{2}\mathbf{i} + 5\sqrt{2}\mathbf{j}, \mathbf{r}_3(1) = \mathbf{0} \right)$$

(Other answers possible)

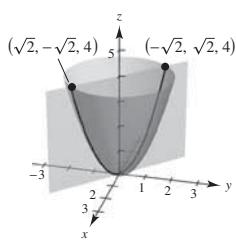
59.  $z = x^2 + y^2, x + y = 0$

Let  $x = t$ , then  $y = -x = -t$

$$\text{and } z = x^2 + y^2 = 2t^2.$$

$$\text{So, } x = t, y = -t, z = 2t^2.$$

$$\mathbf{r}(t) = t\mathbf{i} - t\mathbf{j} + 2t^2\mathbf{k}$$



62.  $4x^2 + 4y^2 + z^2 = 16, x = z^2$

If  $z = t$ , then  $x = t^2$  and  $y = \frac{1}{2}\sqrt{16 - 4t^4 - t^2}$ .

$t$	-1.3	-1.2	-1	0	1	1.2
$x$	1.69	1.44	1	0	1	1.44
$y$	0.85	1.25	1.66	2	1.66	1.25
$z$	-1.3	-1.2	-1	0	1	1.2

$$\mathbf{r}(t) = t^2\mathbf{i} + \frac{1}{2}\sqrt{16 - 4t^4 - t^2}\mathbf{j} + t\mathbf{k}$$

63.  $x^2 + y^2 + z^2 = 4, x + z = 2$

Let  $x = 1 + \sin t$ , then  $z = 2 - x = 1 - \sin t$  and  $x^2 + y^2 + z^2 = 4$ .

$$(1 + \sin t)^2 + y^2 + (1 - \sin t)^2 = 2 + 2 \sin^2 t + y^2 = 4$$

$$y^2 = 2 \cos^2 t, y = \pm\sqrt{2} \cos t$$

$$x = 1 + \sin t, y = \pm\sqrt{2} \cos t$$

$$z = 1 - \sin t$$

$$\mathbf{r}(t) = (1 + \sin t)\mathbf{i} + \sqrt{2} \cos t\mathbf{j} - (1 - \sin t)\mathbf{k} \text{ and}$$

$$\mathbf{r}(t) = (1 + \sin t)\mathbf{i} - \sqrt{2} \cos t\mathbf{j} + (1 - \sin t)\mathbf{k}$$

60.  $z = x^2 + y^2, z = 4$

$$\text{So, } x^2 + y^2 = 4 \text{ or}$$

$$x = 2 \cos t, y = 2 \sin t, z = 4.$$

$$\mathbf{r}(t) = 2 \cos t\mathbf{i} + 2 \sin t\mathbf{j} + 4\mathbf{k}$$

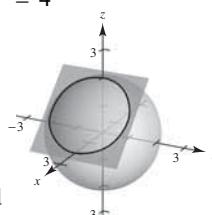
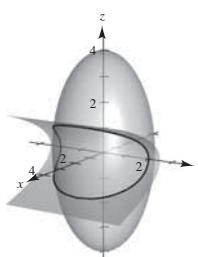
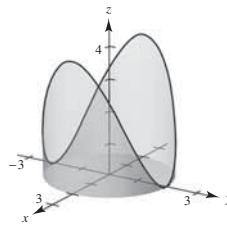
61.  $x^2 + y^2 = 4, z = x^2$

$$x = 2 \sin t, y = 2 \cos t$$

$$z = x^2 = 4 \sin^2 t$$

$t$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	$\pi$
$x$	0	1	$\sqrt{2}$	2	$\sqrt{2}$	0
$y$	2	$\sqrt{3}$	$\sqrt{2}$	0	$-\sqrt{2}$	-2
$z$	0	1	2	4	2	0

$$\mathbf{r}(t) = 2 \sin t\mathbf{i} + 2 \cos t\mathbf{j} + 4 \sin^2 t\mathbf{k}$$



$t$	$-\frac{\pi}{2}$	$-\frac{\pi}{6}$	0	$\frac{\pi}{6}$	$\frac{\pi}{2}$
$x$	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2
$y$	0	$\pm\frac{\sqrt{6}}{2}$	$\pm\sqrt{2}$	$\pm\frac{\sqrt{6}}{2}$	0
$z$	2	$\frac{3}{2}$	1	$\frac{1}{2}$	0

64.  $x^2 + y^2 + z^2 = 10, x + y = 4$

Let  $x = 2 + \sin t$ , then  $y = 2 - \sin t$  and  $z = \sqrt{2(1 - \sin^2 t)} = \sqrt{2} \cos t$ .

$t$	$-\frac{\pi}{2}$	$-\frac{\pi}{6}$	0	$\frac{\pi}{6}$	$\frac{\pi}{2}$	$\pi$
$x$	1	$\frac{3}{2}$	2	$\frac{5}{2}$	3	2
$y$	3	$\frac{5}{2}$	2	$\frac{3}{2}$	1	2
$z$	0	$\frac{\sqrt{6}}{2}$	$\sqrt{2}$	$\frac{\sqrt{6}}{2}$	0	$-\sqrt{2}$



$$\mathbf{r}(t) = (2 + \sin t)\mathbf{i} + (2 - \sin t)\mathbf{j} + \sqrt{2} \cos t\mathbf{k}$$

65.  $x^2 + z^2 = 4, y^2 + z^2 = 4$

Subtracting, you have  $x^2 - y^2 = 0$  or  $y = \pm x$ .

So, in the first octant, if you let  $x = t$ , then  $x = t, y = t, z = \sqrt{4 - t^2}$ .

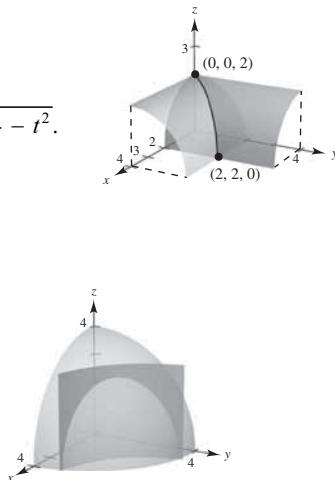
$$\mathbf{r}(t) = t\mathbf{i} + t\mathbf{j} + \sqrt{4 - t^2}\mathbf{k}$$

66.  $x^2 + y^2 + z^2 = 16, xy = 4$  (first octant)

Let  $x = t$ , then  $y = \frac{4}{t}$  and  $x^2 + y^2 + z^2 = t^2 + \frac{16}{t^2} + z^2 = 16$ .

$$z = \frac{1}{t}\sqrt{-t^4 + 16t^2 - 16}$$

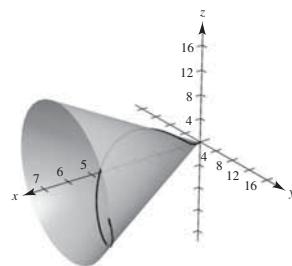
$$\left(\sqrt{8 - 4\sqrt{3}} \leq t \leq \sqrt{8 + 4\sqrt{3}}\right)$$



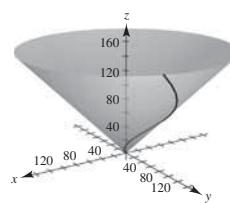
$t$	$\sqrt{8 + 4\sqrt{3}}$	1.5	2	2.5	3.0	3.5	$\sqrt{8 + 4\sqrt{3}}$
$x$	1.0	1.5	2	2.5	3.0	3.5	3.9
$y$	3.9	2.7	2	1.6	1.3	1.1	1.0
$z$	0	2.6	2.8	2.7	2.3	1.6	0

$$\mathbf{r}(t) = t\mathbf{i} + \frac{4}{t}\mathbf{j} + \frac{1}{t}\sqrt{-t^4 + 16t^2 - 16}\mathbf{k}$$

67.  $y^2 + z^2 = (2t \cos t)^2 + (2t \sin t)^2 = 4t^2 = 4x^2$



68.  $x^2 + y^2 = (e^{-t} \cos t)^2 + (e^{-t} \sin t)^2 = e^{-2t} = z^2$



69.  $\lim_{t \rightarrow \pi} (\mathbf{i} + \cos t\mathbf{j} + \sin t\mathbf{k}) = \pi\mathbf{i} - \mathbf{j}$

70.  $\lim_{t \rightarrow 2} \left( 3t\mathbf{i} + \frac{2}{t^2 - 1}\mathbf{j} + \frac{1}{t}\mathbf{k} \right) = 6\mathbf{i} + \frac{2}{3}\mathbf{j} + \frac{1}{2}\mathbf{k}$

71.  $\lim_{t \rightarrow 0} \left[ t^2 \mathbf{i} + 3t \mathbf{j} + \frac{1 - \cos t}{t} \mathbf{k} \right] = 0$

because

$$\lim_{t \rightarrow 0} \frac{1 - \cos t}{t} = \lim_{t \rightarrow 0} \frac{\sin t}{1} = 0. \text{ (L'Hôpital's Rule)}$$

72.  $\lim_{t \rightarrow 1} \left( \sqrt{t} \mathbf{i} + \frac{\ln t}{t^2 - 1} \mathbf{j} + \frac{1}{t-1} \mathbf{k} \right)$

does not exist because  $\lim_{t \rightarrow 1} \frac{1}{t-1}$  does not exist.

73.  $\lim_{t \rightarrow 0} \left[ e^t \mathbf{i} + \frac{\sin t}{t} \mathbf{j} + e^{-t} \mathbf{k} \right] = \mathbf{i} + \mathbf{j} + \mathbf{k}$

because

$$\lim_{t \rightarrow 0} \frac{\sin t}{t} = \lim_{t \rightarrow 0} \frac{\cos t}{1} = 1 \text{ (L'Hôpital's Rule)}$$

74.  $\lim_{t \rightarrow \infty} \left[ e^{-t} \mathbf{i} + \frac{1}{t} \mathbf{j} + \frac{t}{t^2 + 1} \mathbf{k} \right] = 0$

because

$$\lim_{t \rightarrow \infty} e^{-t} = 0, \lim_{t \rightarrow \infty} \frac{1}{t} = 0, \text{ and } \lim_{t \rightarrow \infty} \frac{t}{t^2 + 1} = 0.$$

75.  $\mathbf{r}(t) = t\mathbf{i} + \frac{1}{t}\mathbf{j}$

Continuous on  $(-\infty, 0), (0, \infty)$

76.  $\mathbf{r}(t) = \sqrt{t}\mathbf{i} + \sqrt{t-1}\mathbf{j}$

Continuous on  $[1, \infty)$

77.  $\mathbf{r}(t) = t\mathbf{i} + \arcsin t\mathbf{j} + (t-1)\mathbf{k}$

Continuous on  $[-1, 1]$

78.  $\mathbf{r}(t) = \langle 2e^{-t}, e^{-t}, \ln(t-1) \rangle$

Continuous on  $t-1 > 0$  or  $t > 1$ :  $(1, \infty)$ .

79.  $\mathbf{r}(t) = \langle e^{-t}, t^2, \tan t \rangle$

Discontinuous at  $t = \frac{\pi}{2} + n\pi$

Continuous on  $\left(-\frac{\pi}{2} + n\pi, \frac{\pi}{2} + n\pi\right)$

80.  $\mathbf{r}(t) = \langle 8, \sqrt{t}, \sqrt[3]{t} \rangle$

Continuous on  $[0, \infty)$

81.  $\mathbf{r}(t) = t^2 \mathbf{i} + (t-3)\mathbf{j} + t\mathbf{k}$

(a)  $\mathbf{s}(t) = \mathbf{r}(t) + 3\mathbf{k} = t^2 \mathbf{i} + (t-3)\mathbf{j} + (t+3)\mathbf{k}$

(b)  $\mathbf{s}(t) = \mathbf{r}(t) - 2\mathbf{i} = (t^2 - 2)\mathbf{i} + (t-3)\mathbf{j} + t\mathbf{k}$

(c)  $\mathbf{s}(t) = \mathbf{r}(t) + 5\mathbf{j} = t^2 \mathbf{i} + (t+2)\mathbf{j} + t\mathbf{k}$

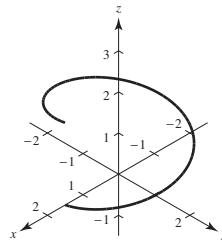
82. A vector-valued function  $\mathbf{r}$  is continuous at  $t = a$  if the limit of  $\mathbf{r}(t)$  exists as  $t \rightarrow a$  and  $\lim_{t \rightarrow a} \mathbf{r}(t) = \mathbf{r}(a)$ .

The function  $\mathbf{r}(t) = \begin{cases} \mathbf{i} + \mathbf{j} & t \geq 2 \\ -\mathbf{i} + \mathbf{j} & t < 2 \end{cases}$  is not continuous at  $t = 0$ .

83. One possible answer is

$$\mathbf{r}(t) = 1.5 \cos t\mathbf{i} + 1.5 \sin t\mathbf{j} + \frac{1}{\pi} t\mathbf{k}, 0 \leq t \leq 2\pi$$

Note that  $\mathbf{r}(2\pi) = 1.5\mathbf{i} + 2\mathbf{k}$ .



84. (a)  $x = -3 \cos t + 1, y = 5 \sin t + 2, z = 4$

$$\frac{(x-1)^2}{9} + \frac{(y-2)^2}{25} = 1, z = 4$$

(b)  $x = 4, y = -3 \cos t + 1, z = 5 \sin t + 2$

$$\frac{(y-1)^2}{9} + \frac{(z-2)^2}{25} = 1, x = 4$$

(c)  $x = 3 \cos t - 1, y = -5 \sin t - 2, z = 4$

$$\frac{(x+1)^2}{9} + \frac{(y+2)^2}{25} = 1, z = 4$$

(d)  $x = -3 \cos 2t + 1, y = 5 \sin 2t + 2, z = 4$

$$\frac{(x-1)^2}{9} + \frac{(y-2)^2}{25} = 1, z = 4$$

(a) and (d) represent the same graph

85. Let  $\mathbf{r}(t) = x_1(t)\mathbf{i} + y_1(t)\mathbf{j} + z_1(t)\mathbf{k}$  and  $\mathbf{u}(t) = x_2(t)\mathbf{i} + y_2(t)\mathbf{j} + z_2(t)\mathbf{k}$ . Then:

$$\begin{aligned}\lim_{t \rightarrow c} [\mathbf{r}(t) \times \mathbf{u}(t)] &= \lim_{t \rightarrow c} \{ [y_1(t)z_2(t) - y_2(t)z_1(t)]\mathbf{i} - [x_1(t)z_2(t) - x_2(t)z_1(t)]\mathbf{j} + [x_1(t)y_2(t) - x_2(t)y_1(t)]\mathbf{k} \} \\ &= \left[ \lim_{t \rightarrow c} y_1(t) \lim_{t \rightarrow c} z_2(t) - \lim_{t \rightarrow c} y_2(t) \lim_{t \rightarrow c} z_1(t) \right] \mathbf{i} - \left[ \lim_{t \rightarrow c} x_1(t) \lim_{t \rightarrow c} z_2(t) - \lim_{t \rightarrow c} x_2(t) \lim_{t \rightarrow c} z_1(t) \right] \mathbf{j} \\ &\quad + \left[ \lim_{t \rightarrow c} x_1(t) \lim_{t \rightarrow c} y_2(t) - \lim_{t \rightarrow c} x_2(t) \lim_{t \rightarrow c} y_1(t) \right] \mathbf{k} \\ &= \left[ \lim_{t \rightarrow c} x_1(t)\mathbf{i} + \lim_{t \rightarrow c} y_1(t)\mathbf{j} + \lim_{t \rightarrow c} z_1(t)\mathbf{k} \right] \times \left[ \lim_{t \rightarrow c} x_2(t)\mathbf{i} + \lim_{t \rightarrow c} y_2(t)\mathbf{j} + \lim_{t \rightarrow c} z_2(t)\mathbf{k} \right] \\ &= \lim_{t \rightarrow c} \mathbf{r}(t) \times \lim_{t \rightarrow c} \mathbf{u}(t)\end{aligned}$$

86. Let  $\mathbf{r}(t) = x_1(t)\mathbf{i} + y_1(t)\mathbf{j} + z_1(t)\mathbf{k}$  and  $\mathbf{u}(t) = x_2(t)\mathbf{i} + y_2(t)\mathbf{j} + z_2(t)\mathbf{k}$ . Then:

$$\begin{aligned}\lim_{t \rightarrow c} [\mathbf{r}(t) \cdot \mathbf{u}(t)] &= \lim_{t \rightarrow c} [x_1(t)x_2(t) + y_1(t)y_2(t) + z_1(t)z_2(t)] \\ &= \lim_{t \rightarrow c} x_1(t) \lim_{t \rightarrow c} x_2(t) + \lim_{t \rightarrow c} y_1(t) \lim_{t \rightarrow c} y_2(t) + \lim_{t \rightarrow c} z_1(t) \lim_{t \rightarrow c} z_2(t) \\ &= \left[ \lim_{t \rightarrow c} x_1(t)\mathbf{i} + \lim_{t \rightarrow c} y_1(t)\mathbf{j} + \lim_{t \rightarrow c} z_1(t)\mathbf{k} \right] \cdot \left[ \lim_{t \rightarrow c} x_2(t)\mathbf{i} + \lim_{t \rightarrow c} y_2(t)\mathbf{j} + \lim_{t \rightarrow c} z_2(t)\mathbf{k} \right] \\ &= \lim_{t \rightarrow c} \mathbf{r}(t) \cdot \lim_{t \rightarrow c} \mathbf{u}(t)\end{aligned}$$

87. Let  $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$ . Because  $\mathbf{r}$  is

continuous at  $t = c$ , then  $\lim_{t \rightarrow c} \mathbf{r}(t) = \mathbf{r}(c)$ .

$$\mathbf{r}(c) = x(c)\mathbf{i} + y(c)\mathbf{j} + z(c)\mathbf{k} \Rightarrow x(c), y(c), z(c)$$

are defined at  $c$ .

$$\begin{aligned}\|\mathbf{r}\| &= \sqrt{(x(t))^2 + (y(t))^2 + (z(t))^2} \\ \lim_{t \rightarrow c} \|\mathbf{r}\| &= \sqrt{(x(c))^2 + (y(c))^2 + (z(c))^2} = \|\mathbf{r}(c)\|\end{aligned}$$

So,  $\|\mathbf{r}\|$  is continuous at  $c$

88. Let

$$f(t) = \begin{cases} 1, & \text{if } t \geq 0 \\ -1, & \text{if } t < 0 \end{cases}$$

and  $\mathbf{r}(t) = f(t)\mathbf{i}$ . Then  $\mathbf{r}$  is not continuous at

$c = 0$ , whereas,  $\|\mathbf{r}\| = 1$  is continuous for all  $t$ .

89.  $\mathbf{r}(t) = t^2\mathbf{i} + (9t - 20)\mathbf{j} + t^2\mathbf{k}$

$$\mathbf{u}(s) = (3s + 4)\mathbf{i} + s^2\mathbf{j} + (5s - 4)\mathbf{k}.$$

Equating components:

$$t^2 = 3s + 4$$

$$9t - 20 = s^2$$

$$t^2 = 5s - 4$$

$$\text{So, } 3s + 4 = 5s - 4 \Rightarrow s = 4$$

$$9t - 20 = s^2 = 16 \Rightarrow t = 4.$$

The paths intersect at the same time  $t = 4$  at the point  $(16, 16, 16)$ . The particles collide.

90.  $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}$

$$\mathbf{u}(s) = (-2s + 3)\mathbf{i} + 8s\mathbf{j} + (12s + 2)\mathbf{k}$$

Equating components

$$t = -2s + 3$$

$$t^2 = 8s$$

$$t^3 = 12s + 2$$

$$(-2s + 3)^2 = 8s$$

$$4s^2 - 12s + 9 = 8s$$

$$4s^2 - 20s + 9 = 0$$

$$(2s - 9)(2s - 1) = 0$$

$$\text{For } s = \frac{1}{2}, t = -2\left(\frac{1}{2}\right) + 3 = 2.$$

$$\text{For } s = \frac{9}{2}, t = -2\left(\frac{9}{2}\right) + 3 = -6 \text{ and}$$

$$t^2 = 8\left(\frac{9}{2}\right) = 36 \text{ and } t^3 = 12\left(\frac{9}{2}\right) = 54. \text{ Impossible.}$$

The paths intersect at  $(2, 4, 8)$ , but at different times

$$\left(t = 2 \text{ and } s = \frac{1}{2}\right). \text{ No collision.}$$

91. No, not necessarily. See Exercise 90.

92. Yes. See Exercise 89.

93. True

94. False. The graph of  $x = y = z = t^3$  represents a line.

95. True. See Exercises 89 and 90.

96. True.  $y^2 + z^2 = t^2 \sin^2 t + t^2 \cos^2 t = t^2 = x$

## Section 12.2 Differentiation and Integration of Vector-Valued Functions

1.  $\mathbf{r}(t) = t^2\mathbf{i} + t\mathbf{j}, t_0 = 2$

$$x(t) = t^2, y(t) = t$$

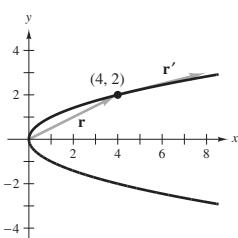
$$x = y^2$$

$$\mathbf{r}(2) = 4\mathbf{i} + 2\mathbf{j}$$

$$\mathbf{r}'(t) = 2t\mathbf{i} + \mathbf{j}$$

$$\mathbf{r}'(2) = 4\mathbf{i} + \mathbf{j}$$

$\mathbf{r}'(t_0)$  is tangent to the curve at  $t_0$ .



2.  $\mathbf{r}(t) = t\mathbf{i} + (t^2 - 1)\mathbf{j}, t_0 = 1$

$$x(t) = t, y(t) = t^2 - 1$$

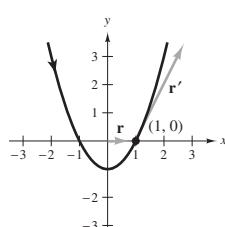
$$y = x^2 - 1$$

$$\mathbf{r}(1) = \mathbf{i}$$

$$\mathbf{r}'(t) = \mathbf{i} + 2t\mathbf{j}$$

$$\mathbf{r}'(1) = \mathbf{i} + 2\mathbf{j}$$

$\mathbf{r}'(t_0)$  is tangent to the curve at  $t_0$ .



3.  $\mathbf{r}(t) = t^2\mathbf{i} + \frac{1}{t}\mathbf{j}, t_0 = 2$

$$x(t) = t^2, y(t) = \frac{1}{t}$$

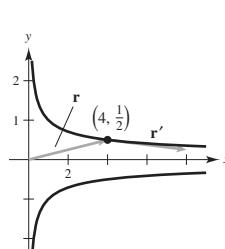
$$x = \frac{1}{y^2}$$

$$\mathbf{r}(2) = 4\mathbf{i} + \frac{1}{2}\mathbf{j}$$

$$\mathbf{r}'(t) = 2t\mathbf{i} - \frac{1}{t^2}\mathbf{j}$$

$$\mathbf{r}'(2) = 4\mathbf{i} - \frac{1}{4}\mathbf{j}$$

$\mathbf{r}'(t_0)$  is tangent to the curve at  $t_0$ .



4. (a)  $\mathbf{r}(t) = (1+t)\mathbf{i} + t^3\mathbf{j}, t_0 = 1$

$$x = 1+t$$

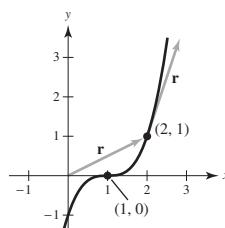
$$y = t^3 = (x-1)^3$$

(b)  $\mathbf{r}(1) = 2\mathbf{i} + \mathbf{j}$

$$\mathbf{r}'(t) = \mathbf{i} + 3t^2\mathbf{j}$$

$$\mathbf{r}'(1) = \mathbf{i} + 3\mathbf{j}$$

$\mathbf{r}'(t_0)$  is tangent to the curve at  $t_0$ .



5.  $\mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j}, t_0 = \frac{\pi}{2}$

$$x(t) = \cos t, y(t) = \sin t$$

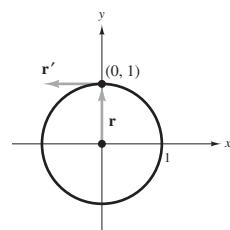
$$x^2 + y^2 = 1$$

$$\mathbf{r}\left(\frac{\pi}{2}\right) = \mathbf{j}$$

$$\mathbf{r}'(t) = -\sin t\mathbf{i} + \cos t\mathbf{j}$$

$$\mathbf{r}'\left(\frac{\pi}{2}\right) = -\mathbf{i}$$

$\mathbf{r}'(t_0)$  is tangent to the curve at  $t_0$ .



6.  $\mathbf{r}(t) = 3 \sin t\mathbf{i} + 4 \cos t\mathbf{j}, t_0 = \frac{\pi}{2}$

$$x(t) = 3 \sin t, y(t) = 4 \cos t$$

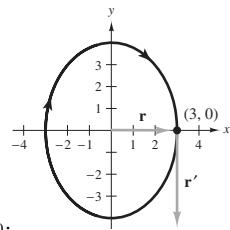
$$\left(\frac{x}{3}\right)^2 + \left(\frac{y}{4}\right)^2 = 1, \text{ ellipse}$$

$$\mathbf{r}\left(\frac{\pi}{2}\right) = 3\mathbf{i}$$

$$\mathbf{r}'(t) = 3 \cos t\mathbf{i} - 4 \sin t\mathbf{j}$$

$$\mathbf{r}'\left(\frac{\pi}{2}\right) = -4\mathbf{j}$$

$\mathbf{r}'(t_0)$  is tangent to the curve at  $t_0$ .



7.  $\mathbf{r}(t) = \langle e^t, e^{2t} \rangle, t_0 = 0$

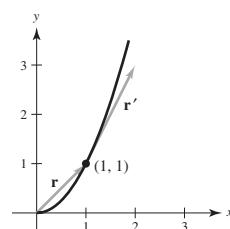
$$x(t) = e^t, y(t) = e^{2t} = (e^t)^2$$

$$y = x^2, x > 0$$

$$\mathbf{r}(0) = \langle 1, 1 \rangle$$

$$\mathbf{r}'(t) = \langle e^t, 2e^{2t} \rangle$$

$$\mathbf{r}'(0) = \langle 1, 2 \rangle$$



$\mathbf{r}'(t_0)$  is tangent to the curve at  $t_0$ .

8.  $\mathbf{r}(t) = \langle e^{-t}, e^t \rangle, t_0 = 0$

$$\mathbf{r}(t) = e^{-t} = \frac{1}{e^t}, y(t) = e^t$$

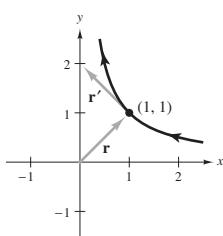
$$y = \frac{1}{x}, x > 0$$

$$\mathbf{r}(0) = \langle 1, 1 \rangle$$

$$\mathbf{r}'(t) = \langle -e^{-t}, e^t \rangle$$

$$\mathbf{r}'(0) = \langle -1, 1 \rangle$$

$\mathbf{r}'(t_0)$  is tangent to the curve at  $t_0$ .



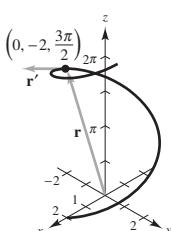
9. (a) and (b)  $\mathbf{r}(t) = 2 \cos t \mathbf{i} + 2 \sin t \mathbf{j} + t \mathbf{k}, t_0 = \frac{3\pi}{2}$

$$x^2 + y^2 = 4, z = t$$

$$\mathbf{r}'(t) = -2 \sin t \mathbf{i} + 2 \cos t \mathbf{j} + \mathbf{k}$$

$$\mathbf{r}\left(\frac{3\pi}{2}\right) = -2\mathbf{j} + \frac{3\pi}{2}\mathbf{k}$$

$$\mathbf{r}\left(\frac{3\pi}{2}\right) = 2\mathbf{i} + \mathbf{k}$$



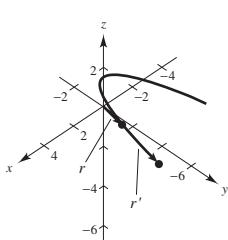
10.  $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + \frac{3}{2}\mathbf{k}, t_0 = 2$

$$y = x^2, z = \frac{3}{2}$$

$$\mathbf{r}'(t) = \mathbf{i} + 2t\mathbf{j}$$

$$\mathbf{r}(2) = 2\mathbf{i} + 4\mathbf{j} + \frac{3}{2}\mathbf{k}$$

$$\mathbf{r}'(2) = \mathbf{i} + 4\mathbf{j}$$



11.  $\mathbf{r}(t) = t^3\mathbf{i} - 3t\mathbf{j}$

$$\mathbf{r}'(t) = 3t^2\mathbf{i} - 3\mathbf{j}$$

12.  $\mathbf{r}(t) = \sqrt{t}\mathbf{i} + (1 - t^3)\mathbf{j}$

$$\mathbf{r}'(t) = \frac{1}{2\sqrt{t}}\mathbf{i} - 3t^2\mathbf{j}$$

13.  $\mathbf{r}(t) = \langle 2 \cos t, 5 \sin t \rangle$

$$\mathbf{r}'(t) = \langle -2 \sin t, 5 \cos t \rangle$$

14.  $\mathbf{r}(t) = \langle t \cos t, -2 \sin t \rangle$

$$\mathbf{r}'(t) = \langle -t \sin t + \cos t, -2 \cos t \rangle$$

25.  $\mathbf{r}(t) = 4 \cos t \mathbf{i} + 4 \sin t \mathbf{j}$

(a)  $\mathbf{r}'(t) = -4 \sin t \mathbf{i} + 4 \cos t \mathbf{j}$

(b)  $\mathbf{r}''(t) = -4 \cos t \mathbf{i} - 4 \sin t \mathbf{j}$

(c)  $\mathbf{r}'(t) \cdot \mathbf{r}''(t) = (-4 \sin t)(-4 \cos t) + 4 \cos t(-4 \sin t) = 0$

15.  $\mathbf{r}(t) = 6t\mathbf{i} - 7t^2\mathbf{j} + t^3\mathbf{k}$

$$\mathbf{r}'(t) = 6\mathbf{i} - 14t\mathbf{j} + 3t^2\mathbf{k}$$

16.  $\mathbf{r}(t) = \frac{1}{t}\mathbf{i} + 16t\mathbf{j} + \frac{t^2}{2}\mathbf{k}$

$$\mathbf{r}'(t) = -\frac{1}{t^2}\mathbf{i} + 16\mathbf{j} + t\mathbf{k}$$

17.  $\mathbf{r}(t) = a \cos^3 t \mathbf{i} + a \sin^3 t \mathbf{j} + \mathbf{k}$

$$\mathbf{r}'(t) = -3a \cos^2 t \sin t \mathbf{i} + 3a \sin^2 t \cos t \mathbf{j}$$

18.  $\mathbf{r}(t) = 4\sqrt{t}\mathbf{i} + t^2\sqrt{t}\mathbf{j} + \ln t^2\mathbf{k}$

$$\begin{aligned} \mathbf{r}'(t) &= \frac{2}{\sqrt{t}}\mathbf{i} + \left(2t\sqrt{t} + \frac{t^2}{2\sqrt{t}}\right)\mathbf{j} + \frac{2}{t}\mathbf{k} \\ &= \frac{2}{\sqrt{t}}\mathbf{i} + \frac{5t^{3/2}}{2}\mathbf{j} + \frac{2}{t}\mathbf{k} \end{aligned}$$

19.  $\mathbf{r}(t) = e^{-t}\mathbf{i} + 4\mathbf{j} + 5te^t\mathbf{k}$

$$\mathbf{r}'(t) = -e^{-t}\mathbf{i} + (5e^t + 5te^t)\mathbf{k}$$

20.  $\mathbf{r}(t) = \langle t^3, \cos 3t, \sin 3t \rangle$

$$\mathbf{r}'(t) = \langle 3t^2, -3 \sin 3t, 3 \cos 3t \rangle$$

21.  $\mathbf{r}(t) = \langle t \sin t, t \cos t, t \rangle$

$$\mathbf{r}'(t) = \langle \sin t + t \cos t, \cos t - t \sin t, 1 \rangle$$

22.  $\mathbf{r}(t) = \langle \arcsin t, \arccos t, 0 \rangle$

$$\mathbf{r}'(t) = \left\langle \frac{1}{\sqrt{1-t^2}}, -\frac{1}{\sqrt{1-t^2}}, 0 \right\rangle$$

23.  $\mathbf{r}(t) = t^3\mathbf{i} + \frac{1}{2}t^2\mathbf{j}$

(a)  $\mathbf{r}'(t) = 3t^2\mathbf{i} + t\mathbf{j}$

(b)  $\mathbf{r}''(t) = 6t\mathbf{i} + \mathbf{j}$

(c)  $\mathbf{r}'(t) \cdot \mathbf{r}''(t) = 3t^2(6t) + t = 18t^3 + t$

24.  $\mathbf{r}(t) = (t^2 + t)\mathbf{i} + (t^2 - t)\mathbf{j}$

(a)  $\mathbf{r}'(t) = (2t + 1)\mathbf{i} + (2t - 1)\mathbf{j}$

(b)  $\mathbf{r}''(t) = 2\mathbf{i} + 2\mathbf{j}$

(c)  $\mathbf{r}'(t) \cdot \mathbf{r}''(t) = (2t + 1)(2) + (2t - 1)(2) = 8t$

26.  $\mathbf{r}(t) = 8 \cos t \mathbf{i} + 3 \sin t \mathbf{j}$

(a)  $\mathbf{r}'(t) = -8 \sin t \mathbf{i} + 3 \cos t \mathbf{j}$

(b)  $\mathbf{r}''(t) = -8 \cos t \mathbf{i} - 3 \sin t \mathbf{j}$

(c)  $\mathbf{r}'(t) \cdot \mathbf{r}''(t) = (-8 \sin t)(-8 \cos t) + 3 \cos t(-3 \sin t) = 55 \sin t \cos t$

27.  $\mathbf{r}(t) = \frac{1}{2}t^2 \mathbf{i} - t \mathbf{j} + \frac{1}{6}t^3 \mathbf{k}$

(a)  $\mathbf{r}'(t) = t \mathbf{i} - \mathbf{j} + \frac{1}{2}t^2 \mathbf{k}$

(b)  $\mathbf{r}''(t) = \mathbf{i} + t \mathbf{k}$

(c)  $\mathbf{r}'(t) \cdot \mathbf{r}''(t) = t(1) - 1(0) + \frac{1}{2}t^2(t) = t + \frac{t^3}{2}$

28.  $\mathbf{r}(t) = t \mathbf{i} + (2t + 3) \mathbf{j} + (3t - 5) \mathbf{k}$

(a)  $\mathbf{r}'(t) = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$

(b)  $\mathbf{r}''(t) = 0$

(c)  $\mathbf{r}'(t) \cdot \mathbf{r}''(t) = 0$

29.  $\mathbf{r}(t) = \langle \cos t + t \sin t, \sin t - t \cos t, t \rangle$

(a)  $\mathbf{r}'(t) = \langle -\sin t + \sin t + t \cos t, \cos t - \cos t + t \sin t, 1 \rangle = \langle t \cos t, t \sin t, 1 \rangle$

(b)  $\mathbf{r}''(t) = \langle \cos t - t \sin t, \sin t + t \cos t, 0 \rangle$

(c)  $\mathbf{r}'(t) \cdot \mathbf{r}''(t) = (t \cos t)(\cos t - t \sin t) + (t \sin t)(\sin t + t \cos t) = t$

30.  $\mathbf{r}(t) = \langle e^{-t}, t^2, \tan(t) \rangle$

(a)  $\mathbf{r}'(t) = \langle -e^{-t}, 2t, \sec^2 t \rangle$

(b)  $\mathbf{r}''(t) = \langle e^{-t}, 2, 2 \sec^2 t \tan t \rangle$

(c)  $\mathbf{r}'(t) \cdot \mathbf{r}''(t) = -e^{-2t} + 4t + 2 \sec^4 t \tan t$

31.  $\mathbf{r}(t) = \cos(\pi t) \mathbf{i} + \sin(\pi t) \mathbf{j} + t^2 \mathbf{k}, t_0 = -\frac{1}{4}$

$\mathbf{r}'(t) = -\pi \sin(\pi t) \mathbf{i} + \pi \cos(\pi t) \mathbf{j} + 2t \mathbf{k}$

$\mathbf{r}'\left(-\frac{1}{4}\right) = \frac{\sqrt{2}\pi}{2} \mathbf{i} + \frac{\sqrt{2}\pi}{2} \mathbf{j} - \frac{1}{2} \mathbf{k}$

$$\left\| \mathbf{r}'\left(-\frac{1}{4}\right) \right\| = \sqrt{\left(\frac{\sqrt{2}\pi}{2}\right)^2 + \left(\frac{\sqrt{2}\pi}{2}\right)^2 + \left(-\frac{1}{2}\right)^2} = \sqrt{\pi^2 + \frac{1}{4}} = \frac{\sqrt{4\pi^2 + 1}}{2}$$

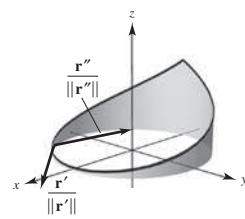
$\frac{\mathbf{r}'(-1/4)}{\|\mathbf{r}'(-1/4)\|} = \frac{1}{\sqrt{4\pi^2 + 1}} (\sqrt{2}\pi \mathbf{i} + \sqrt{2}\pi \mathbf{j} - \mathbf{k})$

$\mathbf{r}''(t) = -\pi^2 \cos(\pi t) \mathbf{i} - \pi^2 \sin(\pi t) \mathbf{j} + 2\mathbf{k}$

$\mathbf{r}''\left(-\frac{1}{4}\right) = -\frac{\sqrt{2}\pi^2}{2} \mathbf{i} + \frac{\sqrt{2}\pi^2}{2} \mathbf{j} + 2\mathbf{k}$

$$\left\| \mathbf{r}''\left(-\frac{1}{4}\right) \right\| = \sqrt{\left(-\frac{\sqrt{2}\pi^2}{2}\right)^2 + \left(\frac{\sqrt{2}\pi^2}{2}\right)^2 + (2)^2} = \sqrt{\pi^4 + 4}$$

$\frac{\mathbf{r}''(-1/4)}{\|\mathbf{r}''(-1/4)\|} = \frac{1}{2\sqrt{\pi^4 + 4}} (-\sqrt{2}\pi^2 \mathbf{i} + \sqrt{2}\pi^2 \mathbf{j} + 4\mathbf{k})$



32.  $\mathbf{r}(t) = \frac{3}{2}\mathbf{i} + t^2\mathbf{j} + e^{-t}\mathbf{k}, t_0 = \frac{1}{4}$

$$\mathbf{r}'(t) = \frac{3}{2}\mathbf{i} + 2t\mathbf{j} - e^{-t}\mathbf{k}, \mathbf{r}\left(\frac{1}{4}\right) = \frac{3}{8}\mathbf{i} + \frac{1}{16}\mathbf{j} + e^{-1/4}\mathbf{k}$$

$$\mathbf{r}'\left(\frac{1}{4}\right) = \frac{3}{2}\mathbf{i} + \frac{1}{2}\mathbf{j} - e^{-1/4}\mathbf{k}$$

$$\left\| \mathbf{r}'\left(\frac{1}{4}\right) \right\| = \sqrt{\frac{9}{4} + \frac{1}{4} + e^{-1/2}} = \frac{1}{2}\sqrt{10 + 4e^{-1/2}}$$

$$\left\| \mathbf{r}'(1/4) \right\| = \frac{3}{\sqrt{10 + 4e^{-1/2}}}\mathbf{i} + \frac{1}{\sqrt{10 + 4e^{-1/2}}}\mathbf{j} - \frac{2e^{-1/4}}{\sqrt{10 + 4e^{-1/2}}}\mathbf{k}$$

$$\mathbf{r}''(t) = 2\mathbf{j} + e^{-t}\mathbf{k}, \mathbf{r}''\left(\frac{1}{4}\right) = 2\mathbf{j} + e^{-1/4}\mathbf{k}$$

$$\left\| \mathbf{r}''\left(\frac{1}{4}\right) \right\| = \sqrt{4 + e^{-1/2}}$$

$$\left\| \mathbf{r}''\left(\frac{1}{4}\right) \right\| = \frac{2}{\sqrt{4 + e^{-1/2}}}\mathbf{j} + \frac{e^{-1/4}}{\sqrt{4 + e^{-1/2}}}\mathbf{k}$$

33.  $\mathbf{r}(t) = t^2\mathbf{i} + t^3\mathbf{j}$   
 $\mathbf{r}'(t) = 2t\mathbf{i} + 3t^2\mathbf{j}$   
 $\mathbf{r}'(0) = \mathbf{0}$   
 Smooth on  $(-\infty, 0), (0, \infty)$

34.  $\mathbf{r}(t) = \frac{1}{t-1}\mathbf{i} + 3t\mathbf{j}$   
 $\mathbf{r}'(t) = -\frac{1}{(t-1)^2}\mathbf{i} + 3\mathbf{j}$

Not continuous when  $t = 1$

Smooth on  $(-\infty, 1), (1, \infty)$

35.  $\mathbf{r}(\theta) = 2\cos^3 \theta\mathbf{i} + 3\sin^3 \theta\mathbf{j}$   
 $\mathbf{r}'(\theta) = -6\cos^2 \theta \sin \theta\mathbf{i} + 9\sin^2 \theta \cos \theta\mathbf{j}$

$$\mathbf{r}'\left(\frac{n\pi}{2}\right) = \mathbf{0}$$

Smooth on  $\left(\frac{n\pi}{2}, \frac{(n+1)\pi}{2}\right)$ ,  $n$  any integer.

36.  $\mathbf{r}(\theta) = (\theta + \sin \theta)\mathbf{i} + (1 - \cos \theta)\mathbf{j}$   
 $\mathbf{r}'(\theta) = (1 + \cos \theta)\mathbf{i} + \sin \theta\mathbf{j}$   
 $\mathbf{r}'((2n-1)\pi) = \mathbf{0}$ ,  $n$  any integer  
 Smooth on  $((2n-1)\pi, (2n+1)\pi)$

37.  $\mathbf{r}(\theta) = (\theta - 2\sin \theta)\mathbf{i} + (1 - 2\cos \theta)\mathbf{j}$   
 $\mathbf{r}'(\theta) = (1 - 2\cos \theta)\mathbf{i} + (2\sin \theta)\mathbf{j}$   
 $\mathbf{r}'(\theta) \neq \mathbf{0}$  for any value of  $\theta$   
 Smooth on  $(-\infty, \infty)$

38.  $\mathbf{r}(t) = \frac{2t}{8+t^3}\mathbf{i} + \frac{2t^2}{8+t^3}\mathbf{j}$   
 $\mathbf{r}'(t) = \frac{16-4t^3}{(t^3+8)^2}\mathbf{i} + \frac{32t-2t^4}{(t^3+8)^2}\mathbf{j}$   
 $\mathbf{r}'(t) \neq \mathbf{0}$  for any value of  $t$ .

$\mathbf{r}$  is not continuous when  $t = -2$ .  
 Smooth on  $(-\infty, -2), (-2, \infty)$

39.  $\mathbf{r}(t) = (t-1)\mathbf{i} + \frac{1}{t}\mathbf{j} - t^2\mathbf{k}$   
 $\mathbf{r}'(t) = \mathbf{i} - \frac{1}{t^2}\mathbf{j} - 2t\mathbf{k} \neq \mathbf{0}$   
 $\mathbf{r}$  is smooth or all  $t \neq 0$ :  $(-\infty, 0), (0, \infty)$

40.  $\mathbf{r}(t) = e'\mathbf{i} - e^{-t}\mathbf{j} + 3t\mathbf{k}$   
 $\mathbf{r}'(t) = e'\mathbf{i} + e^{-t}\mathbf{j} + 3\mathbf{k} \neq \mathbf{0}$   
 $\mathbf{r}$  is smooth for all  $t$ :  $(-\infty, \infty)$

41.  $\mathbf{r}(t) = t\mathbf{i} - 3t\mathbf{j} + \tan t\mathbf{k}$   
 $\mathbf{r}'(t) = \mathbf{i} - 3\mathbf{j} + \sec^2 t\mathbf{k} \neq \mathbf{0}$

$\mathbf{r}$  is smooth for all  $t \neq \frac{\pi}{2} + n\pi = \frac{2n+1}{2}\pi$ .

Smooth on intervals of form  $\left(-\frac{\pi}{2} + n\pi, \frac{\pi}{2} + n\pi\right)$ ,  
 $n$  is an integer.

42.  $\mathbf{r}(t) = \sqrt{t}\mathbf{i} + (t^2 - 1)\mathbf{j} + \frac{1}{4}t\mathbf{k}$   
 $\mathbf{r}'(t) = \frac{1}{2\sqrt{t}}\mathbf{i} + 2t\mathbf{j} + \frac{1}{4}\mathbf{k} \neq \mathbf{0}$   
 $\mathbf{r}$  is smooth for all  $t > 0$ :  $(0, \infty)$

43.  $\mathbf{r}(t) = t\mathbf{i} + 3t\mathbf{j} + t^2\mathbf{k}$ ,  $\mathbf{u}(t) = 4t\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}$

(a)  $\mathbf{r}'(t) = \mathbf{i} + 3\mathbf{j} + 2t\mathbf{k}$

(b)  $\mathbf{r}''(t) = 2\mathbf{k}$

(c)  $\mathbf{r}(t) \cdot \mathbf{u}(t) = 4t^2 + 3t^3 + t^5$

$$D_t[\mathbf{r}(t) \cdot \mathbf{u}(t)] = 8t + 9t^2 + 5t^4$$

(d)  $3\mathbf{r}(t) - \mathbf{u}(t) = -t\mathbf{i} + (9t - t^2)\mathbf{j} + (3t^2 - t^3)\mathbf{k}$

$$D_t[3\mathbf{r}(t) - \mathbf{u}(t)] = -\mathbf{i} + (9 - 2t)\mathbf{j} + (6t - 3t^2)\mathbf{k}$$

(e)  $\mathbf{r}(t) \times \mathbf{u}(t) = 2t^4\mathbf{i} - (t^4 - 4t^3)\mathbf{j} + (t^3 - 12t^2)\mathbf{k}$

$$D_t[\mathbf{r}(t) \times \mathbf{u}(t)] = 8t^3\mathbf{i} + (12t^2 - 4t^3)\mathbf{j} + (3t^2 - 24t)\mathbf{k}$$

(f)  $\|\mathbf{r}(t)\| = \sqrt{10t^2 + t^4} = t\sqrt{10 + t^2}$

$$D_t[\|\mathbf{r}(t)\|] = \frac{10 + 2t^2}{\sqrt{10 + t^2}}$$

44.  $\mathbf{r}(t) = t\mathbf{i} + 2 \sin t\mathbf{j} + 2 \cos t\mathbf{k}$

$\mathbf{u}(t) = \frac{1}{t}\mathbf{i} + 2 \sin t\mathbf{j} + 2 \cos t\mathbf{k}$

(a)  $\mathbf{r}'(t) = \mathbf{i} + 2 \cos t\mathbf{j} - 2 \sin t\mathbf{k}$

(b)  $\mathbf{r}''(t) = -2 \sin t\mathbf{j} - 2 \cos t\mathbf{k}$

(c)  $\mathbf{r}(t) \cdot \mathbf{u}(t) = 1 + 4 \sin^2 t + 4 \cos^2 t = 5$

$$D_t[\mathbf{r}(t) \cdot \mathbf{u}(t)] = 0, t \neq 0$$

(d)  $3\mathbf{r}(t) - \mathbf{u}(t) = \left(3t - \frac{1}{t}\right)\mathbf{i} + 4 \sin t\mathbf{j} + 4 \cos t\mathbf{k}$

$$D_t[3\mathbf{r}(t) - \mathbf{u}(t)] = \left(3 - \frac{1}{t^2}\right)\mathbf{i} + 4 \cos t\mathbf{j} - 4 \sin t\mathbf{k}$$

(e)  $\mathbf{r}(t) \times \mathbf{u}(t) = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ t & 2 \sin t & 2 \cos t \\ \frac{1}{t} & 2 \sin t & 2 \cos t \end{bmatrix} = 2 \cos t \left(\frac{1}{t} - t\right)\mathbf{j} + 2 \sin t \left(t - \frac{1}{t}\right)\mathbf{k}$

$$D_t[\mathbf{r}(t) \times \mathbf{u}(t)] = \left[-2 \sin t \left(\frac{1}{t} - t\right) + 2 \cos t \left(-\frac{1}{t^2} - 1\right)\right]\mathbf{j} + \left[2 \cos t \left(t - \frac{1}{t}\right) + 2 \sin t \left(1 + \frac{1}{t^2}\right)\right]\mathbf{k}$$

(f)  $\|\mathbf{r}(t)\| = \sqrt{t^2 + 4}$

$$D_t(\|\mathbf{r}(t)\|) = \frac{1}{2}(t^2 + 4)^{-1/2}(2t) = \frac{t}{\sqrt{t^2 + 4}}$$

45.  $\mathbf{r}(t) = t\mathbf{i} + 2t^2\mathbf{j} + t^3\mathbf{k}$ ,  $\mathbf{u}(t) = t^4\mathbf{k}$

(a)  $\mathbf{r}(t) \cdot \mathbf{u}(t) = t^7$

(i)  $D_t[\mathbf{r}(t) \cdot \mathbf{u}(t)] = 7t^6$

(ii) Alternate Solution:

$$D_t[\mathbf{r}(t) \cdot \mathbf{u}(t)] = \mathbf{r}(t) \cdot \mathbf{u}'(t) + \mathbf{r}'(t) \cdot \mathbf{u}(t) = (t\mathbf{i} + 2t^2\mathbf{j} + t^3\mathbf{k}) \cdot (4t^3\mathbf{k}) + (\mathbf{i} + 4t\mathbf{j} + 3t^2\mathbf{k}) \cdot (t^4\mathbf{k}) = 4t^6 + 3t^6 = 7t^6$$

$$(b) \quad \mathbf{r}(t) \times \mathbf{u}(t) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ t & 2t^2 & t^3 \\ 0 & 0 & t^4 \end{vmatrix} = 2t^6\mathbf{i} - t^5\mathbf{j}$$

$$(i) \quad D_t[\mathbf{r}(t) \times \mathbf{u}(t)] = 12t^5\mathbf{i} - 5t^4\mathbf{j}$$

$$(ii) \text{ Alternate Solution: } D_t[\mathbf{r}(t) \times \mathbf{u}(t)] = \mathbf{r}(t) \times \mathbf{u}'(t) + \mathbf{r}'(t) \times \mathbf{u}(t) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ t & 2t^2 & t^3 \\ 0 & 0 & 4t^3 \end{vmatrix} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 4t & 3t^2 \\ 0 & 0 & t^4 \end{vmatrix} = 12t^5\mathbf{i} - 5t^4\mathbf{j}$$

$$46. \quad \mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j} + t\mathbf{k}, \quad \mathbf{u}(t) = \mathbf{j} + t\mathbf{k}$$

$$(a) \quad \mathbf{r}(t) \cdot \mathbf{u}(t) = \sin t + t^2$$

$$(i) \quad D_t[\mathbf{r}(t) \cdot \mathbf{u}(t)] = \cos t + 2t$$

(ii) Alternate Solution:

$$\begin{aligned} D_t[\mathbf{r}(t) \cdot \mathbf{u}(t)] &= \mathbf{r}(t) \cdot \mathbf{u}'(t) + \mathbf{r}'(t) \cdot \mathbf{u}(t) \\ &= (\cos t\mathbf{i} + \sin t\mathbf{j} + t\mathbf{k}) \cdot \mathbf{k} + (-\sin t\mathbf{i} + \cos t\mathbf{j} + \mathbf{k}) \cdot (\mathbf{j} + t\mathbf{k}) = t + \cos t + t = 2t + \cos t \end{aligned}$$

$$(b) \quad \mathbf{r}(t) \times \mathbf{u}(t) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos t & \sin t & t \\ 0 & 1 & t \end{vmatrix} = (t \sin t - t)\mathbf{i} - (t \cos t)\mathbf{j} + \cos t\mathbf{k}$$

$$(i) \quad D_t[\mathbf{r}(t) \times \mathbf{u}(t)] = (t \cos t + \sin t - 1)\mathbf{i} - (\cos t - t \sin t)\mathbf{j} - \sin t\mathbf{k}$$

(ii) Alternate Solution:

$$\begin{aligned} D_t[\mathbf{r}(t) \times \mathbf{u}(t)] &= \mathbf{r}(t) \times \mathbf{u}'(t) + \mathbf{r}'(t) \times \mathbf{u}(t) \\ &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos t & \sin t & t \\ 0 & 0 & 1 \end{vmatrix} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\sin t & \cos t & 1 \\ 0 & 1 & t \end{vmatrix} = (\sin t + t \cos t - 1)\mathbf{i} + (t \sin t - \cos t)\mathbf{j} - \sin t\mathbf{k} \end{aligned}$$

$$47. \quad \mathbf{r}(t) = 3 \sin t\mathbf{i} + 4 \cos t\mathbf{j}$$

$$\mathbf{r}'(t) = 3 \cos t\mathbf{i} - 4 \sin t\mathbf{j}$$

$$\mathbf{r}(t) \cdot \mathbf{r}'(t) = 9 \sin t \cos t - 16 \cos t \sin t = -7 \sin t \cos t$$

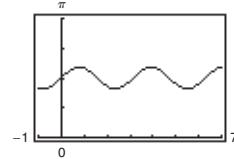
$$\cos \theta = \frac{\mathbf{r}(t) \cdot \mathbf{r}'(t)}{\|\mathbf{r}(t)\| \|\mathbf{r}'(t)\|} = \frac{-7 \sin t \cos t}{\sqrt{9 \sin^2 t + 16 \cos^2 t} \sqrt{9 \cos^2 t + 16 \sin^2 t}}$$

$$\theta = \arccos \left[ \frac{-7 \sin t \cos t}{\sqrt{(9 \sin^2 t + 16 \cos^2 t)} \sqrt{(9 \cos^2 t + 16 \sin^2 t)}} \right]$$

$$\theta = 1.855 \text{ maximum at } t = 3.927 = \left( \frac{5\pi}{4} \right) \text{ and } t = 0.785 = \left( \frac{\pi}{4} \right).$$

$$\theta = 1.287 \text{ minimum at } t = 2.356 = \left( \frac{3\pi}{4} \right) \text{ and } t = 5.498 = \left( \frac{7\pi}{4} \right).$$

$$\theta = \frac{\pi}{2} = (1.571) \text{ for } t = \frac{n\pi}{2}, n = 0, 1, 2, 3, \dots$$



48.  $\mathbf{r}(t) = t^2\mathbf{i} + t\mathbf{j}$

$$\mathbf{r}'(t) = 2t\mathbf{i} + \mathbf{j}$$

$$\mathbf{r}(t) \cdot \mathbf{r}'(t) = 2t^3 + t$$

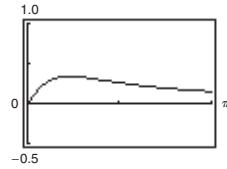
$$\|\mathbf{r}(t)\| = \sqrt{t^4 + t^2}, \|\mathbf{r}'(t)\| = \sqrt{4t^2 + 1}$$

$$\cos \theta = \frac{2t^3 + t}{\sqrt{t^4 + t^2}\sqrt{4t^2 + 1}}$$

$$\theta = \arccos \left[ \frac{2t^3 + t}{\sqrt{t^4 + t^2}\sqrt{4t^2 + 1}} \right]$$

$$\theta = 0.340 (\approx 19.47^\circ) \text{ maximum at } t = 0.707 = \left( \frac{\sqrt{2}}{2} \right).$$

$\theta \neq \frac{\pi}{2}$  for any  $t$ .



49.  $\mathbf{r}'(t) = \lim_{\Delta t \rightarrow 0} \frac{\mathbf{r}(t + \Delta t) - \mathbf{r}(t)}{\Delta t} = \lim_{\Delta t \rightarrow 0} \frac{[3(t + \Delta t) + 2]\mathbf{i} + [1 - (t + \Delta t)^2]\mathbf{j} - (3t + 2)\mathbf{i} - (1 - t^2)\mathbf{j}}{\Delta t}$

$$= \lim_{\Delta t \rightarrow 0} \frac{(3\Delta t)\mathbf{i} - (2t(\Delta t) + (\Delta t)^2)\mathbf{j}}{\Delta t} = \lim_{\Delta t \rightarrow 0} 3\mathbf{i} - (2t + \Delta t)\mathbf{j} = 3\mathbf{i} - 2\mathbf{j}$$

50.  $\mathbf{r}'(t) = \lim_{\Delta t \rightarrow 0} \frac{\mathbf{r}(t + \Delta t) - \mathbf{r}(t)}{\Delta t}$

$$= \lim_{\Delta t \rightarrow 0} \frac{\left[ \sqrt{t + \Delta t}\mathbf{i} + \frac{3}{t + \Delta t}\mathbf{j} - 2(t + \Delta t)\mathbf{k} \right] - \left[ \sqrt{t}\mathbf{i} + \frac{3}{t}\mathbf{j} - 2t\mathbf{k} \right]}{\Delta t} = \lim_{\Delta t \rightarrow 0} \left[ \frac{\sqrt{t + \Delta t} - \sqrt{t}}{\Delta t} \mathbf{i} + \frac{\frac{3}{t + \Delta t} - \frac{3}{t}}{\Delta t} \mathbf{j} - 2\mathbf{k} \right]$$

$$= \lim_{\Delta t \rightarrow 0} \left[ \frac{\Delta t}{\Delta t(\sqrt{t + \Delta t} + \sqrt{t})} \mathbf{i} + \frac{-3\Delta t}{(t + \Delta t)t(\Delta t)} \mathbf{j} - 2\mathbf{k} \right] = \lim_{\Delta t \rightarrow 0} \left[ \frac{1}{\sqrt{t + \Delta t} + \sqrt{t}} \mathbf{i} - \frac{3}{(t + \Delta t)t} \mathbf{j} - 2\mathbf{k} \right] = \frac{1}{2\sqrt{t}} \mathbf{i} - \frac{3}{t^2} \mathbf{j} - 2\mathbf{k}$$

51.  $\mathbf{r}'(t) = \lim_{\Delta t \rightarrow 0} \frac{\mathbf{r}(t + \Delta t) - \mathbf{r}(t)}{\Delta t}$

$$= \lim_{\Delta t \rightarrow 0} \frac{\langle (t + \Delta t)^2, 0, 2(t + \Delta t) \rangle - \langle t^2, 0, 2t \rangle}{\Delta t} = \lim_{\Delta t \rightarrow 0} \frac{\langle 2t\Delta t + (\Delta t)^2, 0, 2\Delta t \rangle}{\Delta t} = \lim_{\Delta t \rightarrow 0} \langle 2t + \Delta t, 0, 2 \rangle = \langle 2t, 0, 2 \rangle$$

52.  $\mathbf{r}'(t) = \lim_{\Delta t \rightarrow 0} \frac{\mathbf{r}(t + \Delta t) - \mathbf{r}(t)}{\Delta t} = \lim_{\Delta t \rightarrow 0} \frac{\langle 0, \sin(t + \Delta t), 4(t + \Delta t) \rangle - \langle 0, \sin t, 4t \rangle}{\Delta t}$

$$= \lim_{\Delta t \rightarrow 0} \frac{\langle 0, \sin t \cdot \cos(\Delta t) + \sin(\Delta t)\cos t - \sin t, 4\Delta t \rangle}{\Delta t} = \lim_{\Delta t \rightarrow 0} \left\langle 0, \frac{\sin t(\cos(\Delta t) - 1)}{\Delta t} + \cos t \left( \frac{\sin(\Delta t)}{\Delta t} \right), 4 \right\rangle$$

$$= \langle 0, 0 + \cos t, 4 \rangle = \langle 0, \cos t, 4 \rangle$$

53.  $\int (2t\mathbf{i} + \mathbf{j} + \mathbf{k})dt = t^2\mathbf{i} + t\mathbf{j} + t\mathbf{k} + \mathbf{C}$

54.  $\int (4t^3\mathbf{i} + 6t\mathbf{j} - 4\sqrt{t}\mathbf{k})dt = t^4\mathbf{i} + 3t^2\mathbf{j} - \frac{8}{3}t^{3/2}\mathbf{k} + \mathbf{C}$

55.  $\int \left( \frac{1}{t}\mathbf{i} + \mathbf{j} - t^{3/2}\mathbf{k} \right) dt = \ln t\mathbf{i} + t\mathbf{j} - \frac{2}{5}t^{5/2}\mathbf{k} + \mathbf{C}$

56.  $\int \left[ \ln t\mathbf{i} + \frac{1}{t}\mathbf{j} + \mathbf{k} \right] dt = (t \ln t - t)\mathbf{i} + \ln t\mathbf{j} + t\mathbf{k} + \mathbf{C}$

(Integration by parts)

57.  $\int [(2t - 1)\mathbf{i} + 4t^3\mathbf{j} + 3\sqrt{t}\mathbf{k}]dt = (t^2 - t)\mathbf{i} + t^4\mathbf{j} + 2t^{3/2}\mathbf{k} + \mathbf{C}$

58.  $\int [e^t\mathbf{i} + \sin t\mathbf{j} + \cos t\mathbf{k}]dt = e^t\mathbf{i} - \cos t\mathbf{j} + \sin t\mathbf{k} + \mathbf{C}$

59.  $\int [\sec^2 t\mathbf{i} + \frac{1}{1+t^2}\mathbf{j}]dt = \tan t\mathbf{i} + \arctan t\mathbf{j} + \mathbf{C}$

60.  $\int [e^{-t} \sin t\mathbf{i} + e^{-t} \cos t\mathbf{j}]dt = \frac{e^{-t}}{2}(-\sin t - \cos t)\mathbf{i} + \frac{e^{-t}}{2}(-\cos t + \sin t)\mathbf{j} + \mathbf{C}$

61.  $\int_0^1 (8t\mathbf{i} + t\mathbf{j} - \mathbf{k}) dt = [4t^2\mathbf{i}]_0^1 + \left[\frac{t^2}{2}\mathbf{j}\right]_0^1 - [\mathbf{k}]_0^1 = 4\mathbf{i} + \frac{1}{2}\mathbf{j} - \mathbf{k}$

62.  $\int_{-1}^1 (\mathbf{i} + t^3\mathbf{j} + \sqrt[3]{t}\mathbf{k}) dt = \left[\frac{t^2}{2}\mathbf{i}\right]_{-1}^1 + \left[\frac{t^4}{4}\mathbf{j}\right]_{-1}^1 + \left[\frac{3}{4}t^{4/3}\mathbf{k}\right]_{-1}^1 = \mathbf{0}$

63.  $\int_0^{\pi/2} [(a \cos t)\mathbf{i} + (a \sin t)\mathbf{j} + \mathbf{k}] dt = [a \sin t\mathbf{i}]_0^{\pi/2} - [a \cos t\mathbf{j}]_0^{\pi/2} + [\mathbf{k}]_0^{\pi/2} = a\mathbf{i} + a\mathbf{j} + \frac{\pi}{2}\mathbf{k}$

64.  $\int_0^{\pi/4} [(\sec t \tan t)\mathbf{i} + (\tan t)\mathbf{j} + (2 \sin t \cos t)\mathbf{k}] dt = [\sec t\mathbf{i} + \ln|\sec t|\mathbf{j} + \sin^2 t\mathbf{k}]_0^{\pi/4} = (\sqrt{2} - 1)\mathbf{i} + \ln\sqrt{2}\mathbf{j} + \frac{1}{2}\mathbf{k}$

65.  $\int_0^2 (t\mathbf{i} + e^t\mathbf{j} - te^t\mathbf{k}) dt = \left[\frac{t^2}{2}\mathbf{i}\right]_0^2 + [e^t\mathbf{j}]_0^2 - [(t-1)e^t\mathbf{k}]_0^2 = 2\mathbf{i} + (e^2 - 1)\mathbf{j} - (e^2 + 1)\mathbf{k}$

66.  $\|\mathbf{i} + t^2\mathbf{j}\| = \sqrt{t^2 + t^4} = t\sqrt{1 + t^2}$  for  $t \geq 0$

$$\int_0^3 \|\mathbf{i} + t^2\mathbf{j}\| dt = \int_0^3 t\sqrt{1 + t^2} dt = \left[\frac{1}{3}(1 + t^2)^{3/2}\right]_0^3 = \frac{1}{3}(10^{3/2} - 1)$$

67.  $\mathbf{r}(t) = \int (4e^{2t}\mathbf{i} + 3e^t\mathbf{j}) dt = 2e^{2t}\mathbf{i} + 3e^t\mathbf{j} + \mathbf{C}$

$\mathbf{r}(0) = 2\mathbf{i} + 3\mathbf{j} + \mathbf{C} = 2\mathbf{i} \Rightarrow \mathbf{C} = -3\mathbf{j}$

$\mathbf{r}(t) = 2e^{2t}\mathbf{i} + 3(e^t - 1)\mathbf{j}$

70.  $\mathbf{r}''(t) = -4 \cos t\mathbf{j} - 3 \sin t\mathbf{k}$

$\mathbf{r}'(t) = -4 \sin t\mathbf{j} + 3 \cos t\mathbf{k} + \mathbf{C}_1$

$\mathbf{r}'(0) = 3\mathbf{k} = 3\mathbf{k} + \mathbf{C}_1 \Rightarrow \mathbf{C}_1 = \mathbf{0}$

$\mathbf{r}(t) = 4 \cos t\mathbf{j} + 3 \sin t\mathbf{k} + \mathbf{C}_2$

$\mathbf{r}(0) = 4\mathbf{j} + \mathbf{C}_2 = 4\mathbf{j} \Rightarrow \mathbf{C}_2 = \mathbf{0}$

$\mathbf{r}(t) = 4 \cos t\mathbf{j} + 3 \sin t\mathbf{k}$

68.  $\mathbf{r}(t) = \int (3t^2\mathbf{j} + 6\sqrt{t}\mathbf{k}) dt = t^3\mathbf{j} + 4t^{3/2}\mathbf{k} + \mathbf{C}$

$\mathbf{r}(0) = \mathbf{C} = \mathbf{i} + 2\mathbf{j}$

$\mathbf{r}(t) = \mathbf{i} + (2 + t^3)\mathbf{j} + 4t^{3/2}\mathbf{k}$

71.  $\mathbf{r}(t) = \int (te^{-t^2}\mathbf{i} - e^{-t}\mathbf{j} + \mathbf{k}) dt = -\frac{1}{2}e^{-t^2}\mathbf{i} + e^{-t}\mathbf{j} + t\mathbf{k} + \mathbf{C}$

$\mathbf{r}(0) = -\frac{1}{2}\mathbf{i} + \mathbf{j} + \mathbf{C} = \frac{1}{2}\mathbf{i} - \mathbf{j} + \mathbf{k} \Rightarrow \mathbf{C} = \mathbf{i} - 2\mathbf{j} + \mathbf{k}$

69.  $\mathbf{r}'(t) = \int -32\mathbf{j} dt = -32t\mathbf{j} + \mathbf{C}_1$

$\mathbf{r}'(0) = \mathbf{C}_1 = 600\sqrt{3}\mathbf{i} + 600\mathbf{j}$

$\mathbf{r}'(t) = 600\sqrt{3}\mathbf{i} + (600 - 32t)\mathbf{j}$

$\mathbf{r}(t) = \int [600\sqrt{3}\mathbf{i} + (600 - 32t)\mathbf{j}] dt$

$= 600\sqrt{3}\mathbf{i} + (600t - 16t^2)\mathbf{j} + \mathbf{C}$

$\mathbf{r}(0) = \mathbf{C} = \mathbf{0}$

$\mathbf{r}(t) = 600\sqrt{3}\mathbf{i} + (600t - 16t^2)\mathbf{j}$

$$\begin{aligned} \mathbf{r}(t) &= \left(1 - \frac{1}{2}e^{-t^2}\right)\mathbf{i} + (e^{-t} - 2)\mathbf{j} + (t + 1)\mathbf{k} \\ &= \left(\frac{2 - e^{-t^2}}{2}\right)\mathbf{i} + (e^{-t} - 2)\mathbf{j} + (t + 1)\mathbf{k} \end{aligned}$$

72.  $\mathbf{r}(t) = \int \left[ \frac{1}{1+t^2} \mathbf{i} + \frac{1}{t^2} \mathbf{j} + \frac{1}{t} \mathbf{k} \right] dt$

$$= \arctan t \mathbf{i} - \frac{1}{t} \mathbf{j} + \ln t \mathbf{k} + \mathbf{C}$$

$$\mathbf{r}(1) = \frac{\pi}{4} \mathbf{i} - \mathbf{j} + \mathbf{C} = 2\mathbf{i} \Rightarrow \mathbf{C} = \left(2 - \frac{\pi}{4}\right) \mathbf{i} + \mathbf{j}$$

$$\mathbf{r}(t) = \left[2 - \frac{\pi}{4} + \arctan t\right] \mathbf{i} + \left(1 - \frac{1}{t}\right) \mathbf{j} + \ln t \mathbf{k}$$

73. See “Definition of the Derivative of a Vector-Valued Function” and Figure 12.8 on page 842.

74. To find the integral of a vector-valued function, you integrate each component function separately. The constant of integration  $\mathbf{C}$  is a constant vector.

75. At  $t = t_0$ , the graph of  $\mathbf{u}(t)$  is increasing in the  $x, y$ , and  $z$  directions simultaneously.

76. The graph of  $\mathbf{u}(t)$  does not change position relative to the  $xy$ -plane.

77. Let  $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$ . Then  
 $\mathbf{c}\mathbf{r}(t) = cx(t)\mathbf{i} + cy(t)\mathbf{j} + cz(t)\mathbf{k}$  and

$$D_t[\mathbf{c}\mathbf{r}(t)] = cx'(t)\mathbf{i} + cy'(t)\mathbf{j} + cz'(t)\mathbf{k} \\ = c[x'(t)\mathbf{i} + y'(t)\mathbf{j} + z'(t)\mathbf{k}] = \mathbf{c}\mathbf{r}'(t).$$

78. Let  $\mathbf{r}(t) = x_1(t)\mathbf{i} + y_1(t)\mathbf{j} + z_1(t)\mathbf{k}$  and  $\mathbf{u}(t) = x_2(t)\mathbf{i} + y_2(t)\mathbf{j} + z_2(t)\mathbf{k}$ .

$$\mathbf{r}(t) \pm \mathbf{u}(t) = [x_1(t) \pm x_2(t)]\mathbf{i} + [y_1(t) \pm y_2(t)]\mathbf{j} + [z_1(t) \pm z_2(t)]\mathbf{k} \\ D_t[\mathbf{r}(t) \pm \mathbf{u}(t)] = [x_1'(t) \pm x_2'(t)]\mathbf{i} + [y_1'(t) \pm y_2'(t)]\mathbf{j} + [z_1'(t) \pm z_2'(t)]\mathbf{k} \\ = [x_1'(t)\mathbf{i} + y_1'(t)\mathbf{j} + z_1'(t)\mathbf{k}] \pm [x_2'(t)\mathbf{i} + y_2'(t)\mathbf{j} + z_2'(t)\mathbf{k}] = \mathbf{r}'(t) \pm \mathbf{u}'(t)$$

79. Let  $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$ , then  $w(t)\mathbf{r}(t) = w(t)x(t)\mathbf{i} + w(t)y(t)\mathbf{j} + w(t)z(t)\mathbf{k}$ .

$$D_t[w(t)\mathbf{r}(t)] = [w(t)x'(t) + w'(t)x(t)]\mathbf{i} + [w(t)y'(t) + w'(t)y(t)]\mathbf{j} + [w(t)z'(t) + w'(t)z(t)]\mathbf{k} \\ = w(t)[x'(t)\mathbf{i} + y'(t)\mathbf{j} + z'(t)\mathbf{k}] + w'(t)[x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}] = w(t)\mathbf{r}'(t) + w'(t)\mathbf{r}(t)$$

80. Let  $\mathbf{r}(t) = x_1(t)\mathbf{i} + y_1(t)\mathbf{j} + z_1(t)\mathbf{k}$  and  $\mathbf{u}(t) = x_2(t)\mathbf{i} + y_2(t)\mathbf{j} + z_2(t)\mathbf{k}$ .

$$\mathbf{r}(t) \times \mathbf{u}(t) = [y_1(t)z_2(t) - z_1(t)y_2(t)]\mathbf{i} - [x_1(t)z_2(t) - z_1(t)x_2(t)]\mathbf{j} + [x_1(t)y_2(t) - y_1(t)x_2(t)]\mathbf{k} \\ D_t[\mathbf{r}(t) \times \mathbf{u}(t)] = [y_1(t)z_2'(t) + y_1'(t)z_2(t) - z_1(t)y_2'(t) - z_1'(t)y_2(t)]\mathbf{i} - [x_1(t)z_2'(t) + x_1'(t)z_2(t) - z_1(t)x_2'(t) - z_1'(t)x_2(t)]\mathbf{j} \\ + [x_1(t)y_2'(t) + x_1'(t)y_2(t) - y_1(t)x_2'(t) - y_1'(t)x_2(t)]\mathbf{k} \\ = \{[y_1(t)z_2'(t) - z_1(t)y_2'(t)]\mathbf{i} - [x_1(t)z_2'(t) - z_1(t)x_2'(t)]\mathbf{j} + [x_1(t)y_2'(t) - y_1(t)x_2'(t)]\mathbf{k}\} \\ + \{[y_1'(t)z_2(t) - z_1'(t)y_2(t)]\mathbf{i} - [x_1'(t)z_2(t) - z_1'(t)x_2(t)]\mathbf{j} + [x_1'(t)y_2(t) - y_1'(t)x_2(t)]\mathbf{k}\} \\ = \mathbf{r}(t) \times \mathbf{u}'(t) + \mathbf{r}'(t) \times \mathbf{u}(t)$$

81. Let  $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$ . Then  $\mathbf{r}(w(t)) = x(w(t))\mathbf{i} + y(w(t))\mathbf{j} + z(w(t))\mathbf{k}$  and

$$D_t[\mathbf{r}(w(t))] = x'(w(t))w'(t)\mathbf{i} + y'(w(t))w'(t)\mathbf{j} + z'(w(t))w'(t)\mathbf{k} \quad (\text{Chain Rule}) \\ = w'(t)[x'(w(t))\mathbf{i} + y'(w(t))\mathbf{j} + z'(w(t))\mathbf{k}] = w'(t)\mathbf{r}'(w(t)).$$

82. Let  $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$ . Then  $\mathbf{r}'(t) = x'(t)\mathbf{i} + y'(t)\mathbf{j} + z'(t)\mathbf{k}$ .

$$\mathbf{r}(t) \times \mathbf{r}'(t) = [y(t)z'(t) - z(t)y'(t)]\mathbf{i} - [x(t)z'(t) - z(t)x'(t)]\mathbf{j} + [x(t)y'(t) - y(t)x'(t)]\mathbf{k} \\ D_t[\mathbf{r}(t) \times \mathbf{r}'(t)] = [y(t)z''(t) + y'(t)z'(t) - z(t)y''(t) - z'(t)y'(t)]\mathbf{i} - [x(t)z''(t) + x'(t)z'(t) - z(t)x''(t) - z'(t)x'(t)]\mathbf{j} \\ + [x(t)y''(t) + x'(t)y'(t) - y(t)x''(t) - y'(t)x'(t)]\mathbf{k} \\ = [y(t)z''(t) - z(t)y''(t)]\mathbf{i} - [x(t)z''(t) - z(t)x''(t)]\mathbf{j} + [x(t)y''(t) - y(t)x''(t)]\mathbf{k} = \mathbf{r}(t) \times \mathbf{r}''(t)$$

83. Let  $\mathbf{r}(t) = x_1(t)\mathbf{i} + y_1(t)\mathbf{j} + z_1(t)\mathbf{k}$ ,  $\mathbf{u}(t) = x_2(t)\mathbf{i} + y_2(t)\mathbf{j} + z_2(t)\mathbf{k}$ , and  $\mathbf{v}(t) = x_3(t)\mathbf{i} + y_3(t)\mathbf{j} + z_3(t)\mathbf{k}$ . Then:

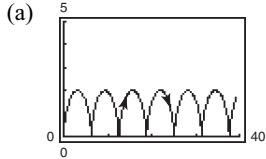
$$\begin{aligned}\mathbf{r}(t) \cdot [\mathbf{u}(t) \times \mathbf{v}(t)] &= x_1(t)[y_2(t)z_3(t) - z_2(t)y_3(t)] - y_1(t)[x_2(t)z_3(t) - z_2(t)x_3(t)] + z_1(t)[x_2(t)y_3(t) - y_2(t)x_3(t)] \\ D_t[\mathbf{r}(t) \cdot (\mathbf{u}(t) \times \mathbf{v}(t))] &= x_1(t)y_2(t)z_3'(t) + x_1(t)y_2'(t)z_3(t) + x_1'(t)y_2(t)z_3(t) - x_1(t)y_3(t)z_2'(t) \\ &\quad - x_1(t)y_3'(t)z_2(t) - x_1'(t)y_3(t)z_2(t) - y_1(t)x_2(t)z_3'(t) - y_1(t)x_2'(t)z_3(t) - y_1'(t)x_2(t)z_3(t) \\ &\quad + y_1(t)z_2(t)x_3'(t) + y_1(t)z_2'(t)x_3(t) + y_1'(t)z_2(t)x_3(t) + z_1(t)x_2(t)y_3'(t) + z_1(t)x_2'(t)y_3(t) \\ &\quad + z_1'(t)x_2(t)y_3(t) - z_1(t)y_2(t)x_3'(t) - z_1(t)y_2'(t)x_3(t) - z_1'(t)y_2(t)x_3(t) \\ &= \left\{ x_1'(t)[y_2(t)z_3(t) - y_3(t)z_2(t)] + y_1'(t)[-x_2(t)z_3(t) + z_2(t)x_3(t)] + z_1'(t)[x_2(t)y_3(t) - y_2(t)x_3(t)] \right\} \\ &\quad + \left\{ x_1(t)[y_2'(t)z_3(t) - y_3(t)z_2'(t)] + y_1(t)[-x_2'(t)z_3(t) + z_2'(t)x_3(t)] + z_1(t)[x_2'(t)y_3(t) - y_2'(t)x_3(t)] \right\} \\ &\quad + \left\{ x_1(t)[y_2(t)z_3'(t) - y_3'(t)z_2(t)] + y_1(t)[-x_2(t)z_3'(t) + z_2(t)x_3'(t)] + z_1(t)[x_2(t)y_3'(t) - y_2(t)x_3'(t)] \right\} \\ &= \mathbf{r}'(t) \cdot [\mathbf{u}(t) \times \mathbf{v}(t)] + \mathbf{r}(t) \cdot [\mathbf{u}'(t) \times \mathbf{v}(t)] + \mathbf{r}(t) \cdot [\mathbf{u}(t) \times \mathbf{v}'(t)]\end{aligned}$$

84. Let  $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$ . If  $\mathbf{r}(t) \cdot \mathbf{r}(t)$  is constant, then:

$$\begin{aligned}x^2(t) + y^2(t) + z^2(t) &= C \\ D_t[x^2(t) + y^2(t) + z^2(t)] &= D_t[C] \\ 2x(t)x'(t) + 2y(t)y'(t) + 2z(t)z'(t) &= 0 \\ 2[x(t)x'(t) + y(t)y'(t) + z(t)z'(t)] &= 0 \\ 2[\mathbf{r}(t) \cdot \mathbf{r}'(t)] &= 0.\end{aligned}$$

So,  $\mathbf{r}(t) \cdot \mathbf{r}'(t) = 0$ .

85.  $\mathbf{r}(t) = (t - \sin t)\mathbf{i} + (1 - \cos t)\mathbf{j}$



The curve is a cycloid.

$$(b) \mathbf{r}'(t) = (1 - \cos t)\mathbf{i} + \sin t\mathbf{j}$$

$$\mathbf{r}''(t) = \sin t\mathbf{i} + \cos t\mathbf{j}$$

$$\begin{aligned}\|\mathbf{r}'(t)\| &= \sqrt{1 - 2 \cos t + \cos^2 t + \sin^2 t} \\ &= \sqrt{2 - 2 \cos t}\end{aligned}$$

Minimum of  $\|\mathbf{r}'(t)\|$  is 0, ( $t = 0$ ).

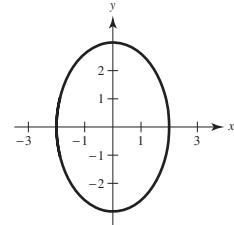
Maximum of  $\|\mathbf{r}'(t)\|$  is 2, ( $t = \pi$ ).

$$\|\mathbf{r}''(t)\| = \sqrt{\sin^2 t + \cos^2 t} = 1$$

Minimum and maximum of  $\|\mathbf{r}'(t)\|$  is 1.

86.  $\mathbf{r}(t) = 2 \cos t\mathbf{i} + 3 \sin t\mathbf{j}$

(a) Ellipse



$$(b) \mathbf{r}'(t) = -2 \sin t\mathbf{i} + 3 \cos t\mathbf{j}$$

$$\mathbf{r}''(t) = -2 \cos t\mathbf{i} - 3 \sin t\mathbf{j}$$

$$\|\mathbf{r}'(t)\| = \sqrt{4 \sin^2 t + 9 \cos^2 t}$$

Minimum of  $\|\mathbf{r}'(t)\|$  is 2, ( $t = \pi/2$ ).

Maximum of  $\|\mathbf{r}'(t)\|$  is 3, ( $t = 0$ ).

87.  $\mathbf{r}(t) = e^t \sin t \mathbf{i} + e^t \cos t \mathbf{j}$

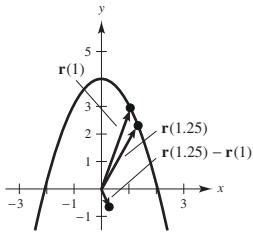
$$\mathbf{r}'(t) = (e^t \cos t + e^t \sin t) \mathbf{i} + (e^t \cos t - e^t \sin t) \mathbf{j}$$

$$\mathbf{r}''(t) = (-e^t \sin t + e^t \cos t + e^t \sin t + e^t \cos t) \mathbf{i} + (e^t \cos t - e^t \sin t - e^t \sin t - e^t \cos t) \mathbf{j} = 2e^t \cos t \mathbf{i} - 2e^t \sin t \mathbf{j}$$

$$\mathbf{r}(t) \cdot \mathbf{r}''(t) = 2e^{2t} \sin t \cos t - 2e^{2t} \sin t \cos t = 0$$

So,  $\mathbf{r}(t)$  is always perpendicular to  $\mathbf{r}''(t)$ .

88. (a)  $\mathbf{r}(t) = t\mathbf{i} + (4 - t^2)\mathbf{j}$



(b)  $\mathbf{r}(1) = \mathbf{i} + 3\mathbf{j}$

$$\mathbf{r}(1.25) = 1.25\mathbf{i} + 2.4375\mathbf{j}$$

$$\mathbf{r}(1.25) - \mathbf{r}(1) = 0.25\mathbf{i} - 0.5625\mathbf{j}$$

(c)  $\mathbf{r}'(t) = \mathbf{i} - 2t\mathbf{j}$

$$\mathbf{r}'(1) = \mathbf{i} - 2\mathbf{j}$$

$$\frac{\mathbf{r}(1.25) - \mathbf{r}(1)}{1.25 - 1} = \frac{0.25\mathbf{i} - 0.5625\mathbf{j}}{0.25} = \mathbf{i} - 2.25\mathbf{j}$$

This vector approximates  $\mathbf{r}'(1)$ .

## Section 12.3 Velocity and Acceleration

1.  $\mathbf{r}(t) = 3t\mathbf{i} + (t - 1)\mathbf{j}$

$$\mathbf{v}(t) = \mathbf{r}'(t) = 3\mathbf{i} + \mathbf{j}$$

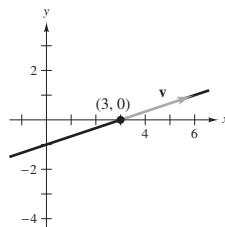
$$\mathbf{a}(t) = \mathbf{r}''(t) = \mathbf{0}$$

$$x = 3t, y = t - 1,$$

$$y = \frac{x}{3} - 1$$

At  $(3, 0)$ ,  $t = 1$ .

$$\mathbf{v}(1) = 3\mathbf{i} + \mathbf{j}, \mathbf{a}(1) = \mathbf{0}$$

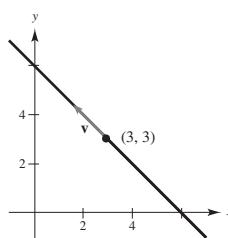


2.  $\mathbf{r}(t) = (6 - t)\mathbf{i} + t\mathbf{j}$

$$\mathbf{v}(t) = \mathbf{r}'(t) = -\mathbf{i} + \mathbf{j}$$

$$\mathbf{a}(t) = \mathbf{r}''(t) = \mathbf{0}$$

$$x = 6 - t, y = t, y = 6 - x$$



89. True

90. False. The definite integral is a vector, not a real number.

91. False. Let  $\mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j} + \mathbf{k}$ .

$$\|\mathbf{r}(t)\| = \sqrt{2}$$

$$\frac{d}{dt} [\|\mathbf{r}(t)\|] = 0$$

$$\mathbf{r}'(t) = -\sin t\mathbf{i} + \cos t\mathbf{j}$$

$$\|\mathbf{r}'(t)\| = 1$$

92. False.

$$D[\mathbf{r}(t) \cdot \mathbf{u}(t)] = \mathbf{r}(t) \cdot \mathbf{u}'(t) + \mathbf{r}'(t) \cdot \mathbf{u}(t)$$

(See Theorem 2.2, part 4)

3.  $\mathbf{r}(t) = t^2\mathbf{i} + t\mathbf{j}$

$$\mathbf{v}(t) = \mathbf{r}'(t) = 2t\mathbf{i} + \mathbf{j}$$

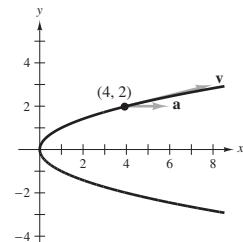
$$\mathbf{a}(t) = \mathbf{r}''(t) = 2\mathbf{i}$$

$$x = t^2, y = t, x = y^2$$

At  $(4, 2)$ ,  $t = 2$ .

$$\mathbf{v}(2) = 4\mathbf{i} + \mathbf{j}$$

$$\mathbf{a}(2) = 2\mathbf{i}$$



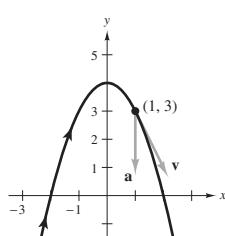
4.  $\mathbf{r}(t) = t\mathbf{i} + (-t^2 + 4)\mathbf{j}$

$$\mathbf{v}(t) = \mathbf{r}'(t) = \mathbf{i} - 2t\mathbf{j}$$

$$\mathbf{a}(t) = \mathbf{r}''(t) = -2\mathbf{j}$$

$$x = t, y = -t^2 + 4 = 4 - x^2$$

At  $(1, 3)$ ,  $t = 1$ ,  $\mathbf{v}(1) = \mathbf{i} - 2\mathbf{j}$ ,  $\mathbf{a}(1) = -2\mathbf{j}$



5.  $\mathbf{r}(t) = t^2\mathbf{i} + t^3\mathbf{j}$

$$\mathbf{v}(t) = \mathbf{r}'(t) = 2t\mathbf{i} + 3t^2\mathbf{j}$$

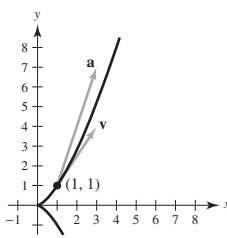
$$\mathbf{a}(t) = \mathbf{r}''(t) = 2\mathbf{i} + 6t\mathbf{j}$$

$$x = t^2, y = t^3, x = y^{2/3}$$

At  $(1, 1), t = 1$ .

$$\mathbf{v}(1) = 2\mathbf{i} + 3\mathbf{j}$$

$$\mathbf{a}(1) = 2\mathbf{i} + 6\mathbf{j}$$



6.  $\mathbf{r}(t) = \left(\frac{1}{4}t^3 + 1\right)\mathbf{i} + t\mathbf{j}$

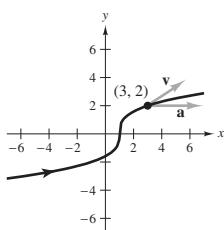
$$\mathbf{v}(t) = \mathbf{r}'(t) = \frac{3}{4}t^2\mathbf{i} + \mathbf{j}$$

$$\mathbf{a}(t) = \mathbf{r}''(t) = \frac{3}{2}\mathbf{i}$$

$$x = \frac{1}{4}t^3 + 1, y = t \Rightarrow x = \frac{1}{4}y^3 + 1$$

$$\Rightarrow y = \sqrt[3]{4(x-1)}$$

At  $(3, 2), t = 2, \mathbf{v}(2) = 3\mathbf{i} + \mathbf{j}, \mathbf{a}(2) = 3\mathbf{i}$



7.  $\mathbf{r}(t) = 2 \cos t\mathbf{i} + 2 \sin t\mathbf{j}$

$$\mathbf{v}(t) = \mathbf{r}'(t) = -2 \sin t\mathbf{i} + 2 \cos t\mathbf{j}$$

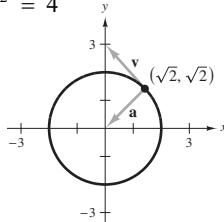
$$\mathbf{a}(t) = \mathbf{r}''(t) = -2 \cos t\mathbf{i} - 2 \sin t\mathbf{j}$$

$$x = 2 \cos t, y = 2 \sin t, x^2 + y^2 = 4$$

At  $(\sqrt{2}, \sqrt{2}), t = \frac{\pi}{4}$ .

$$\mathbf{v}\left(\frac{\pi}{4}\right) = -\sqrt{2}\mathbf{i} + \sqrt{2}\mathbf{j}$$

$$\mathbf{a}\left(\frac{\pi}{4}\right) = -\sqrt{2}\mathbf{i} - \sqrt{2}\mathbf{j}$$



8.  $\mathbf{r}(t) = 3 \cos t\mathbf{i} + 2 \sin t\mathbf{j}$

$$\mathbf{v}(t) = -3 \sin t\mathbf{i} + 2 \cos t\mathbf{j}$$

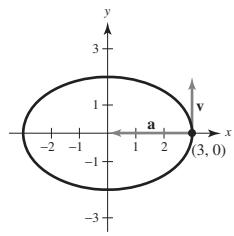
$$\mathbf{a}(t) = -3 \cos t\mathbf{i} - 2 \sin t\mathbf{j}$$

$$x = 3 \cos t, y = 2 \sin t, \frac{x^2}{9} + \frac{y^2}{4} = 1 \text{ Ellipse}$$

At  $(3, 0), t = 0$ .

$$\mathbf{v}(0) = 2\mathbf{j}$$

$$\mathbf{a}(0) = -3\mathbf{i}$$



9.  $\mathbf{r}(t) = \langle t - \sin t, 1 - \cos t \rangle$

$$\mathbf{v}(t) = \mathbf{r}'(t) = \langle 1 - \cos t, \sin t \rangle$$

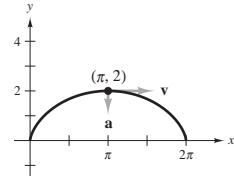
$$\mathbf{a}(t) = \mathbf{r}''(t) = \langle \sin t, \cos t \rangle$$

$x = t - \sin t, y = 1 - \cos t$  (cycloid)

At  $(\pi, 2), t = \pi$ .

$$\mathbf{v}(\pi) = \langle 2, 0 \rangle = 2\mathbf{i}$$

$$\mathbf{a}(\pi) = \langle 0, -1 \rangle = -\mathbf{j}$$



10.  $\mathbf{r}(t) = \langle e^{-t}, e^t \rangle$

$$\mathbf{v}(t) = \mathbf{r}'(t) = \langle -e^{-t}, e^t \rangle$$

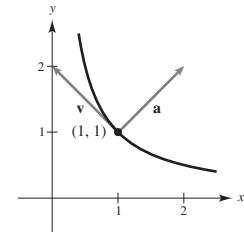
$$\mathbf{a}(t) = \mathbf{r}''(t) = \langle e^{-t}, e^t \rangle$$

$$x = e^{-t} = \frac{1}{e^t}, y = e^t, y = \frac{1}{x}$$

At  $(1, 1), t = 0$ .

$$\mathbf{v}(0) = \langle -1, 1 \rangle = -\mathbf{i} + \mathbf{j}$$

$$\mathbf{a}(0) = \langle 1, 1 \rangle = \mathbf{i} + \mathbf{j}$$



11.  $\mathbf{r}(t) = t\mathbf{i} + 5t\mathbf{j} + 3t\mathbf{k}$

$$\mathbf{v}(t) = \mathbf{i} + 5\mathbf{j} + 3\mathbf{k}$$

$$s(t) = \|\mathbf{v}(t)\| = \sqrt{1 + 5^2 + 3^2} = \sqrt{35}$$

$$\mathbf{a}(t) = \mathbf{0}$$

12.  $\mathbf{r}(t) = 4t\mathbf{i} + 4t\mathbf{j} + 2t\mathbf{k}$

$$\mathbf{v}(t) = 4\mathbf{i} + 4\mathbf{j} + 2\mathbf{k}$$

$$s(t) = \|\mathbf{v}(t)\| = \sqrt{16 + 16 + 4} = 6$$

$$\mathbf{a}(t) = \mathbf{0}$$

13.  $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + \frac{t^2}{2}\mathbf{k}$

$$\mathbf{v}(t) = \mathbf{i} + 2t\mathbf{j} + t\mathbf{k}$$

$$s(t) = \|\mathbf{v}(t)\| = \sqrt{1 + 4t^2 + t^2} = \sqrt{1 + 5t^2}$$

$$\mathbf{a}(t) = 2\mathbf{j} + \mathbf{k}$$

14.  $\mathbf{r}(t) = 3t\mathbf{i} + t\mathbf{j} + \frac{1}{4}t^2\mathbf{k}$

$$\mathbf{v}(t) = 3\mathbf{i} + \mathbf{j} + \frac{1}{2}t\mathbf{k}$$

$$s(t) = \|\mathbf{v}(t)\| = \sqrt{9 + 1 + \frac{1}{4}t^2} = \sqrt{10 + \frac{1}{4}t^2}$$

$$\mathbf{a}(t) = \frac{1}{2}\mathbf{k}$$

15.  $\mathbf{r}(t) = t\mathbf{i} + t\mathbf{j} + \sqrt{9 - t^2}\mathbf{k}$

$$\mathbf{v}(t) = \mathbf{i} + \mathbf{j} - \frac{t}{\sqrt{9 - t^2}}\mathbf{k}$$

$$s(t) = \|\mathbf{v}(t)\| = \sqrt{1 + 1 + \frac{t^2}{9 - t^2}} = \sqrt{\frac{18 - t^2}{9 - t^2}}$$

$$\mathbf{a}(t) = -\frac{9}{(9 - t^2)^{3/2}}\mathbf{-k}$$

16.  $\mathbf{r}(t) = t^2\mathbf{i} + t\mathbf{j} + 2t^{3/2}\mathbf{k}$

$$\mathbf{v}(t) = 2t\mathbf{i} + \mathbf{j} + 3\sqrt{t}\mathbf{k}$$

$$s(t) = \|\mathbf{v}(t)\| = \sqrt{4t^2 + 1 + 9t} = \sqrt{4t^2 + 9t + 1}$$

$$\mathbf{a}(t) = 2\mathbf{i} + \frac{3}{2\sqrt{t}}\mathbf{k}$$

17.  $\mathbf{r}(t) = \langle 4t, 3 \cos t, 3 \sin t \rangle$

$$\mathbf{v}(t) = \langle 4, -3 \sin t, 3 \cos t \rangle = 4\mathbf{i} - 3 \sin t\mathbf{j} + 3 \cos t\mathbf{k}$$

$$s(t) = \|\mathbf{v}(t)\| = \sqrt{16 + 9 \sin^2 t + 9 \cos^2 t} = 5$$

$$\mathbf{a}(t) = \langle 0, -3 \cos t, -3 \sin t \rangle = -3 \cos t\mathbf{j} - 3 \sin t\mathbf{k}$$

18.  $\mathbf{r}(t) = \langle 2 \cos t, 2 \sin t, t^2 \rangle$

$$\mathbf{v}(t) = \langle -2 \sin t, 2 \cos t, 2t \rangle$$

$$s(t) = \|\mathbf{v}(t)\| = \sqrt{4 \sin^2 t + 4 \cos^2 t + 4t^2} = 2\sqrt{1 + t^2}$$

$$\mathbf{a}(t) = \langle -2 \cos t, -2 \sin t, 2 \rangle$$

19.  $\mathbf{r}(t) = \langle e^t \cos t, e^t \sin t, e^t \rangle$

$$\mathbf{v}(t) = \langle e^t \cos t - e^t \sin t, e^t \sin t + e^t \cos t, e^t \rangle$$

$$s(t) = \|\mathbf{v}(t)\|$$

$$= \sqrt{e^{2t}(\cos t - \sin t)^2 + e^{2t}(\cos t + \sin t)^2 + e^{2t}} \\ = e^t \sqrt{3}$$

$$\mathbf{a}(t) = -2e^t \sin t\mathbf{i} + 2e^t \cos t\mathbf{j} + e^t \mathbf{k}$$

20.  $\mathbf{r}(t) = \left\langle \ln t, \frac{1}{t}, t^4 \right\rangle$

$$\mathbf{v}(t) = \left\langle \frac{1}{t}, -\frac{1}{t^2}, 4t^3 \right\rangle$$

$$s(t) = \|\mathbf{v}(t)\| = \sqrt{\frac{1}{t^2} + \frac{1}{t^4} + 16t^6} = \frac{1}{t^2} \sqrt{1 + t^2 + 16t^{10}}$$

$$\mathbf{a}(t) = \left\langle -\frac{1}{t^2}, \frac{2}{t^3}, 12t^2 \right\rangle$$

21. (a)  $\mathbf{r}(t) = \left\langle t, -t^2, \frac{t^3}{4} \right\rangle, t_0 = 1$

$$\mathbf{r}'(t) = \left\langle 1, -2t, \frac{3t^2}{4} \right\rangle$$

$$\mathbf{r}'(1) = \left\langle 1, -2, \frac{3}{4} \right\rangle$$

$$x = 1 + t, y = -1 - 2t, z = \frac{1}{4} + \frac{3}{4}t$$

(b)  $\mathbf{r}(1 + 0.1) \approx \left\langle 1 + 0.1, -1 - 2(0.1), \frac{1}{4} + \frac{3}{4}(0.1) \right\rangle \\ = \langle 1.100, -1.200, 0.325 \rangle$

22. (a)  $\mathbf{r}(t) = \left\langle t, \sqrt{25 - t^2}, \sqrt{25 - t^2} \right\rangle, t_0 = 3$

$$\mathbf{r}'(t) = \left\langle 1, \frac{-t}{\sqrt{25 - t^2}}, \frac{-t}{\sqrt{25 - t^2}} \right\rangle$$

$$\mathbf{r}'(3) = \left\langle 1, -\frac{3}{4}, -\frac{3}{4} \right\rangle$$

$$x = 3 + t, y = z = 4 - \frac{3}{4}t$$

(b)  $\mathbf{r}(3 + 0.1) \approx \left\langle 3 + 0.1, 4 - \frac{3}{4}(0.1), 4 - \frac{3}{4}(0.1) \right\rangle \\ = \langle 3.100, 3.925, 3.925 \rangle$

23.  $\mathbf{a}(t) = \mathbf{i} + \mathbf{j} + \mathbf{k}, \mathbf{v}(0) = \mathbf{0}, \mathbf{r}(0) = \mathbf{0}$

$$\mathbf{v}(t) = \int (\mathbf{i} + \mathbf{j} + \mathbf{k}) dt = t\mathbf{i} + t\mathbf{j} + t\mathbf{k} + \mathbf{C}$$

$$\mathbf{v}(0) = \mathbf{C} = \mathbf{0}, \mathbf{v}(t) = t\mathbf{i} + t\mathbf{j} + t\mathbf{k}, \mathbf{v}(t) = t(\mathbf{i} + \mathbf{j} + \mathbf{k})$$

$$\mathbf{r}(t) = \int (t\mathbf{i} + t\mathbf{j} + t\mathbf{k}) dt = \frac{t^2}{2}(\mathbf{i} + \mathbf{j} + \mathbf{k}) + \mathbf{C}$$

$$\mathbf{r}(0) = \mathbf{C} = \mathbf{0}, \mathbf{r}(t) = \frac{t^2}{2}(\mathbf{i} + \mathbf{j} + \mathbf{k}),$$

$$\mathbf{r}(2) = 2(\mathbf{i} + \mathbf{j} + \mathbf{k}) = 2\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$$

24.  $\mathbf{a}(t) = 2\mathbf{i} + 3\mathbf{k}, \mathbf{v}(0) = 4\mathbf{j}, \mathbf{r}(0) = \mathbf{0}$

$$\mathbf{v}(t) = \int (2\mathbf{i} + 3\mathbf{k}) dt = 2t\mathbf{i} + 3t\mathbf{k} + \mathbf{C}$$

$$\mathbf{v}(0) = \mathbf{C} = 4\mathbf{j} \Rightarrow \mathbf{v}(t) = 2t\mathbf{i} + 4\mathbf{j} + 3t\mathbf{k}$$

$$\mathbf{r}(t) = \int (2t\mathbf{i} + 4\mathbf{j} + 3t\mathbf{k}) dt = t^2\mathbf{i} + 4t\mathbf{j} + \frac{3}{2}t^2\mathbf{k} + \mathbf{C}$$

$$\mathbf{r}(0) = \mathbf{C} = \mathbf{0} \Rightarrow \mathbf{r}(t) = t^2\mathbf{i} + 4t\mathbf{j} + \frac{3}{2}t^2\mathbf{k}$$

$$\mathbf{r}(2) = 4\mathbf{i} + 8\mathbf{j} + 6\mathbf{k}$$

25.  $\mathbf{a}(t) = t\mathbf{j} + t\mathbf{k}$ ,  $\mathbf{v}(1) = 5\mathbf{j}$ ,  $\mathbf{r}(1) = \mathbf{0}$

$$\mathbf{v}(t) = \int(t\mathbf{j} + t\mathbf{k}) dt = \frac{t^2}{2}\mathbf{j} + \frac{t^2}{2}\mathbf{k} + \mathbf{C}$$

$$\mathbf{v}(1) = \frac{1}{2}\mathbf{j} + \frac{1}{2}\mathbf{k} + \mathbf{C} = 5\mathbf{j} \Rightarrow \mathbf{C} = \frac{9}{2}\mathbf{j} - \frac{1}{2}\mathbf{k}$$

$$\mathbf{v}(t) = \left(\frac{t^2}{2} + \frac{9}{2}\right)\mathbf{j} + \left(\frac{t^2}{2} - \frac{1}{2}\right)\mathbf{k}$$

$$\mathbf{r}(t) = \int \left[ \left(\frac{t^2}{2} + \frac{9}{2}\right)\mathbf{j} + \left(\frac{t^2}{2} - \frac{1}{2}\right)\mathbf{k} \right] dt = \left(\frac{t^3}{6} + \frac{9}{2}t\right)\mathbf{j} + \left(\frac{t^3}{6} - \frac{1}{2}t\right)\mathbf{k} + \mathbf{C}$$

$$\mathbf{r}(1) = \frac{14}{3}\mathbf{j} - \frac{1}{3}\mathbf{k} + \mathbf{C} = \mathbf{0} \Rightarrow \mathbf{C} = -\frac{14}{3}\mathbf{j} + \frac{1}{3}\mathbf{k}$$

$$\mathbf{r}(t) = \left(\frac{t^3}{6} + \frac{9}{2}t - \frac{14}{3}\right)\mathbf{j} + \left(\frac{t^3}{6} - \frac{1}{2}t + \frac{1}{3}\right)\mathbf{k}$$

$$\mathbf{r}(2) = \frac{17}{3}\mathbf{j} + \frac{2}{3}\mathbf{k}$$

26.  $\mathbf{a}(t) = -32\mathbf{k}$

$$\mathbf{v}(t) = \int -32\mathbf{k} dt = -32t\mathbf{k} + \mathbf{C}$$

$$\mathbf{v}(0) = \mathbf{C} = 3\mathbf{i} - 2\mathbf{j} + \mathbf{k} \Rightarrow \mathbf{v}(t) = 3\mathbf{i} - 2\mathbf{j} + (1 - 32t)\mathbf{k}$$

$$\mathbf{r}(t) = \int [3\mathbf{i} - 2\mathbf{j} + (1 - 32t)\mathbf{k}] dt = 3t\mathbf{i} - 2t\mathbf{j} + (t - 16t^2)\mathbf{k} + \mathbf{C}$$

$$\mathbf{r}(0) = \mathbf{C} = 5\mathbf{j} + 2\mathbf{k} \Rightarrow \mathbf{r}(t) = 3t\mathbf{i} + (5 - 2t)\mathbf{j} + (2 + t - 16t^2)\mathbf{k}$$

$$\mathbf{r}(2) = 6\mathbf{i} + \mathbf{j} - 60\mathbf{k}$$

27.  $\mathbf{a}(t) = -\cos t\mathbf{i} - \sin t\mathbf{j}$ ,  $\mathbf{v}(0) = \mathbf{j} + \mathbf{k}$ ,  $\mathbf{r}(0) = \mathbf{i}$

$$\mathbf{v}(t) = \int(-\cos t\mathbf{i} - \sin t\mathbf{j}) dt = -\sin t\mathbf{i} + \cos t\mathbf{j} + \mathbf{C}$$

$$\mathbf{v}(0) = \mathbf{j} + \mathbf{C} = \mathbf{j} + \mathbf{k} \Rightarrow \mathbf{C} = \mathbf{k}$$

$$\mathbf{v}(t) = -\sin t\mathbf{i} + \cos t\mathbf{j} + \mathbf{k}$$

$$\mathbf{r}(t) = \int(-\sin t\mathbf{i} + \cos t\mathbf{j} + \mathbf{k}) dt = \cos t\mathbf{i} + \sin t\mathbf{j} + t\mathbf{k} + \mathbf{C}$$

$$\mathbf{r}(0) = \mathbf{i} + \mathbf{C} = \mathbf{i} \Rightarrow \mathbf{C} = \mathbf{0}$$

$$\mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j} + t\mathbf{k}$$

$$\mathbf{r}(2) = (\cos 2)\mathbf{i} + (\sin 2)\mathbf{j} + 2\mathbf{k}$$

28.  $\mathbf{a}(t) = e^t\mathbf{i} - 8\mathbf{k}$

$$\mathbf{v}(t) = \int(e^t\mathbf{i} - 8\mathbf{k}) dt = e^t\mathbf{i} - 8t\mathbf{k} + \mathbf{C}$$

$$\mathbf{v}(0) = \mathbf{i} + \mathbf{C} = 2\mathbf{i} + 3\mathbf{j} + \mathbf{k} \Rightarrow \mathbf{C} = \mathbf{i} + 3\mathbf{j} + \mathbf{k}$$

$$\mathbf{v}(t) = (e^t + 1)\mathbf{i} + 3\mathbf{j} + (1 - 8t)\mathbf{k}$$

$$\mathbf{r}(t) = \int [(e^t + 1)\mathbf{i} + 3\mathbf{j} + (1 - 8t)\mathbf{k}] dt$$

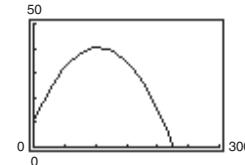
$$= (e^t + t)\mathbf{i} + 3t\mathbf{j} + (t - 4t^2)\mathbf{k} + \mathbf{C}$$

$$\mathbf{r}(0) = \mathbf{i} + \mathbf{C} = \mathbf{0} \Rightarrow \mathbf{C} = -\mathbf{i}$$

$$\mathbf{r}(t) = (e^t + t - 1)\mathbf{i} + 3t\mathbf{j} + (t - 4t^2)\mathbf{k}$$

29.  $\mathbf{r}(t) = (88 \cos 30^\circ)\mathbf{i} + [10 + (88 \sin 30^\circ)t - 16t^2]\mathbf{j}$

$$= 44\sqrt{3}\mathbf{i} + (10 + 44t - 16t^2)\mathbf{j}$$



30.  $\mathbf{r}(t) = (900 \cos 45^\circ)\mathbf{i} + [3 + (900 \sin 45^\circ)t - 16t^2]\mathbf{j} = 450\sqrt{2}\mathbf{i} + (3 + 450\sqrt{2}t - 16t^2)\mathbf{j}$

The maximum height occurs when  $y'(t) = 450\sqrt{2} - 32t = 0$ , which implies that  $t = (225\sqrt{2})/16$ .

The maximum height reached by the projectile is

$$y = 3 + 450\sqrt{2}\left(\frac{225\sqrt{2}}{16}\right) - 16\left(\frac{225\sqrt{2}}{16}\right)^2 = \frac{50,649}{8} = 6331.125 \text{ feet.}$$

The range is determined by setting  $y(t) = 3 + 450\sqrt{2}t - 16t^2 = 0$  which implies that

$$t = \frac{-450\sqrt{2} - \sqrt{405,192}}{-32} \approx 39.779 \text{ seconds}$$

$$\text{Range: } x = 450\sqrt{2}\left(\frac{-450\sqrt{2} - \sqrt{405,192}}{-32}\right) \approx 25,315.500 \text{ feet}$$

31.  $\mathbf{r}(t) = (v_0 \cos \theta)\mathbf{i} + \left[h + (v_0 \sin \theta)t - \frac{1}{2}gt^2\right]\mathbf{j} = \frac{v_0}{\sqrt{2}}\mathbf{i} + \left(3 + \frac{v_0}{\sqrt{2}}t - 16t^2\right)\mathbf{j}$

$$\frac{v_0}{\sqrt{2}}t = 300 \text{ when } 3 + \frac{v_0}{\sqrt{2}}t - 16t^2 = 3.$$

$$t = \frac{300\sqrt{2}}{v_0}, \frac{v_0}{\sqrt{2}}\left(\frac{300\sqrt{2}}{v_0}\right) - 16\left(\frac{300\sqrt{2}}{v_0}\right)^2 = 0, 300 - \frac{300^2(32)}{v_0^2} = 0$$

$$v_0^2 = 300(32), v_0 = \sqrt{9600} = 40\sqrt{6}, v_0 = 40\sqrt{6} \approx 97.98 \text{ ft/sec}$$

The maximum height is reached when the derivative of the vertical component is zero.

$$y(t) = 3 + \frac{tv_0}{\sqrt{2}} - 16t^2 = 3 + \frac{40\sqrt{6}}{\sqrt{2}}t - 16t^2 = 3 + 40\sqrt{3}t - 16t^2$$

$$y'(t) = 40\sqrt{3} - 32t = 0$$

$$t = \frac{40\sqrt{3}}{32} = \frac{5\sqrt{3}}{4}$$

$$\text{Maximum height: } y\left(\frac{5\sqrt{3}}{4}\right) = 3 + 40\sqrt{3}\left(\frac{5\sqrt{3}}{4}\right) - 16\left(\frac{5\sqrt{3}}{4}\right)^2 = 78 \text{ feet}$$

32.  $50 \text{ mi/h} = \frac{220}{3} \text{ ft/sec}$

$$\mathbf{r}(t) = \left(\frac{220}{3} \cos 15^\circ\right)\mathbf{i} + \left[5 + \left(\frac{220}{3} \sin 15^\circ\right)t - 16t^2\right]\mathbf{j}$$

The ball is 90 feet from where it is thrown when  $x = \frac{220}{3} \cos 15^\circ t = 90 \Rightarrow t = \frac{27}{22 \cos 15^\circ} \approx 1.2706 \text{ seconds.}$

The height of the ball at this time is  $y = 5 + \left(\frac{220}{3} \sin 15^\circ\right)\left(\frac{27}{22 \cos 15^\circ}\right) - 16\left(\frac{27}{22 \cos 15^\circ}\right)^2 \approx 3.286 \text{ feet.}$

33.  $x(t) = t(v_0 \cos \theta)$  or  $t = \frac{x}{v_0 \cos \theta}$

$$y(t) = t(v_0 \sin \theta) - 16t^2 + h$$

$$y = \frac{x}{v_0 \cos \theta}(v_0 \sin \theta) - 16\left(\frac{x^2}{v_0^2 \cos^2 \theta}\right) + h = (\tan \theta)x - \left(\frac{16}{v_0^2} \sec^2 \theta\right)x^2 + h$$

34.  $y = x - 0.005x^2$

From Exercise 33 we know that  $\tan \theta$  is the coefficient of  $x$ . So,  $\tan \theta = 1$ ,  $\theta = (\pi/4)$  rad =  $45^\circ$ . Also

$$\frac{16}{v_0^2} \sec^2 \theta = \text{negative of coefficient of } x^2$$

$$\frac{16}{v_0^2}(2) = 0.005 \text{ or } v_0 = 80 \text{ ft/sec}$$

$$\mathbf{r}(t) = (40\sqrt{2}t)\mathbf{i} + (40\sqrt{2}t - 16t^2)\mathbf{j}. \text{ Position function}$$

When  $40\sqrt{2}t = 60$ ,

$$t = \frac{60}{40\sqrt{2}} = \frac{3\sqrt{2}}{4}$$

$$\mathbf{v}(t) = 40\sqrt{2}\mathbf{i} + (40\sqrt{2} - 32t)\mathbf{j}$$

$$\mathbf{v}\left(\frac{3\sqrt{2}}{4}\right) = 40\sqrt{2}\mathbf{i} + (40\sqrt{2} - 24\sqrt{2})\mathbf{j}$$

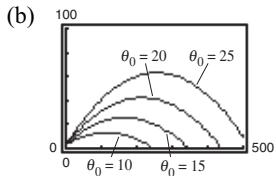
$$= 8\sqrt{2}(5\mathbf{i} + 2\mathbf{j}). \text{ Direction}$$

$$\text{Speed} = \left\| \mathbf{v}\left(\frac{3\sqrt{2}}{4}\right) \right\| = 8\sqrt{2}\sqrt{25+4} = 8\sqrt{58} \text{ ft/sec}$$

37.  $100 \text{ mi/h} = \left(100 \frac{\text{miles}}{\text{hr}}\right) \left(5280 \frac{\text{feet}}{\text{mile}}\right) / \left(3600 \frac{\text{sec}}{\text{hour}}\right) = \frac{440}{3} \text{ ft/sec}$

(a)  $\mathbf{r}(t) = \left(\frac{440}{3} \cos \theta_0\right)\mathbf{i} + \left[3 + \left(\frac{440}{3} \sin \theta_0\right)t - 16t^2\right]\mathbf{j}$

Graphing these curves together with  $y = 10$  shows that  $\theta_0 = 20^\circ$ .



(c) You want

$$x(t) = \left(\frac{440}{3} \cos \theta\right)t \geq 400 \text{ and } y(t) = 3 + \left(\frac{440}{3} \sin \theta\right)t - 16t^2 \geq 10.$$

From  $x(t)$ , the minimum angle occurs when  $t = 30/(11 \cos \theta)$ . Substituting this for  $t$  in  $y(t)$  yields:

$$3 + \left(\frac{440}{3} \sin \theta\right)\left(\frac{30}{11 \cos \theta}\right) - 16\left(\frac{30}{11 \cos \theta}\right)^2 = 10$$

$$400 \tan \theta - \frac{14,400}{121} \sec^2 \theta = 7$$

$$\frac{14,400}{121}(1 + \tan^2 \theta) - 400 \tan \theta + 7 = 0$$

$$14,400 \tan^2 \theta - 48,400 \tan \theta + 15,247 = 0$$

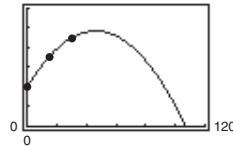
$$\tan \theta = \frac{48,400 \pm \sqrt{48,400^2 - 4(14,400)(15,247)}}{2(14,400)}$$

$$\theta = \tan^{-1}\left(\frac{48,400 - \sqrt{1,464,332,800}}{28,800}\right) \approx 19.38^\circ$$

35.  $\mathbf{r}(t) = t\mathbf{i} + (-0.004t^2 + 0.37t + 6)\mathbf{j}$

(a)  $y = -0.004x^2 + 0.37x + 6$

(b)



(c)  $y' = -0.008x + 0.37 = 0 \Rightarrow x = 45.8375$   
and  $y(45.8375) \approx 14.56 \text{ ft}$

(d) From Exercise 33,  $\tan \theta = 0.3667 \Rightarrow \theta \approx 20.14^\circ$

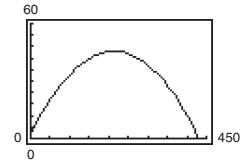
$$\frac{16 \sec^2 \theta}{v_0^2} = 0.004 \Rightarrow v_0^2 = \frac{16 \sec^2 \theta}{0.004} = \frac{4000}{\cos^2 \theta}$$

$$\Rightarrow v_0 \approx 67.4 \text{ ft/sec.}$$

36.  $\mathbf{r}(t) = 140(\cos 22^\circ)\mathbf{i} + (2.5 + 140(\sin 22^\circ)t - 16t^2)\mathbf{j}$

When  $x = 375$ ,  $t \approx 2.889$   
and  $y \approx 20.47 \text{ feet.}$

So, the ball clears  
the 10-foot fence.



38.  $h = 7$  feet,  $\theta = 35^\circ$ , 30 yards = 90 feet

$$\mathbf{r}(t) = (v_0 \cos 35^\circ)\mathbf{i} + [7 + (v_0 \sin 35^\circ)t - 16t^2]\mathbf{j}$$

$$(a) v_0 \cos 35^\circ t = 90 \text{ when } 7 + (v_0 \sin 35^\circ)t - 16t^2 = 4$$

$$t = \frac{90}{v_0 \cos 35^\circ}$$

$$7 + (v_0 \sin 35^\circ)\left(\frac{90}{v_0 \cos 35^\circ}\right) - 16\left(\frac{90}{v_0 \cos 35^\circ}\right)^2 = 4$$

$$90 \tan 35^\circ + 3 = \frac{129,600}{v_0^2 \cos^2 35^\circ}$$

$$v_0^2 = \frac{129,600}{\cos^2 35^\circ(90 \tan 35^\circ + 3)}$$

$$v_0 \approx 54.088 \text{ ft/sec}$$

- (b) The maximum height occurs when  $y'(t) = v_0 \sin 35^\circ - 32t = 0$ .

$$t = \frac{v_0 \sin 35^\circ}{32} \approx 0.969 \text{ sec}$$

At this time, the height is  $y(0.969) \approx 22.0$  ft.

$$(c) x(t) = 90 \Rightarrow (v_0 \cos 35^\circ)t = 90$$

$$t = \frac{90}{54.088 \cos 35^\circ} \approx 2.0 \text{ sec}$$

39.  $\mathbf{r}(t) = (v \cos \theta)\mathbf{i} + [(v \sin \theta)t - 16t^2]\mathbf{j}$

- (a) You want to find the minimum initial speed  $v$  as a function of the angle  $\theta$ . Because the bale must be thrown to the position  $(16, 8)$ , you have

$$16 = (v \cos \theta)t$$

$$8 = (v \sin \theta)t - 16t^2.$$

$t = 16/(v \cos \theta)$  from the first equation. Substituting into the second equation and solving for  $v$ , you obtain:

$$8 = (v \sin \theta)\left(\frac{16}{v \cos \theta}\right) - 16\left(\frac{16}{v \cos \theta}\right)^2$$

$$1 = 2\left(\frac{\sin \theta}{\cos \theta}\right) - 512\left(\frac{1}{v^2 \cos^2 \theta}\right)$$

$$512\left(\frac{1}{v^2 \cos^2 \theta}\right) = 2\left(\frac{\sin \theta}{\cos \theta}\right) - 1$$

$$\frac{1}{v^2} = \left(2 \frac{\sin \theta}{\cos \theta} - 1\right) \frac{\cos^2 \theta}{512} = \frac{2 \sin \theta \cos \theta - \cos^2 \theta}{512}$$

$$v^2 = \frac{512}{2 \sin \theta \cos \theta - \cos^2 \theta}$$

You minimize  $f(\theta) = \frac{512}{2 \sin \theta \cos \theta - \cos^2 \theta}$ .

$$f'(\theta) = -512 \left( \frac{2 \cos^2 \theta - 2 \sin^2 \theta + 2 \sin \theta \cos \theta}{(2 \sin \theta \cos \theta - \cos^2 \theta)^2} \right)$$

$$f'(\theta) = 0 \Rightarrow 2 \cos(2\theta) + \sin(2\theta) = 0$$

$$\tan(2\theta) = -2$$

$$\theta \approx 1.01722 \approx 58.28^\circ$$

Substituting into the equation for  $v$ ,  $v \approx 28.78$  ft/sec.

(b) If  $\theta = 45^\circ$ ,

$$16 = (v \cos \theta)t = v \frac{\sqrt{2}}{2}t$$

$$8 = (v \sin \theta)t - 16t^2 = v \frac{\sqrt{2}}{2}t - 16t^2$$

$$\text{From part (a), } v^2 = \frac{512}{2(\sqrt{2}/2)(\sqrt{2}/2) - (\sqrt{2}/2)^2} = \frac{512}{1/2} = 1024 \Rightarrow v = 32 \text{ ft/sec.}$$

40. Place the origin directly below the plane. Then  $\theta = 0$ ,  $v_0 = 792$  and

$$\mathbf{r}(t) = (v_0 \cos \theta)\mathbf{i} + (30,000 + (v_0 \sin \theta)t - 16t^2)\mathbf{j} = 792\mathbf{i} + (30,000 - 16t^2)\mathbf{j}$$

$$\mathbf{v}(t) = 792\mathbf{i} - 32t\mathbf{j}$$

At time of impact,  $30,000 - 16t^2 = 0 \Rightarrow t^2 = 1875 \Rightarrow t \approx 43.3$  seconds.

$$\mathbf{r}(43.3) = 34,294.6\mathbf{i}$$

$$\mathbf{v}(43.3) = 792\mathbf{i} - 1385.6\mathbf{j}$$

$$\|\mathbf{v}(43.3)\| = 1596 \text{ ft/sec} = 1088 \text{ mi/h}$$

$$\tan \alpha = \frac{30,000}{34,294.6} \approx 0.8748 \Rightarrow \alpha \approx 0.7187(41.18^\circ)$$

41.  $\mathbf{r}(t) = (v_0 \cos \theta)\mathbf{i} + [(v_0 \sin \theta)t - 16t^2]\mathbf{j}$

$$(v_0 \sin \theta)t - 16t^2 = 0 \text{ when } t = 0 \text{ and } t = \frac{v_0 \sin \theta}{16}.$$

The range is

$$x = (v_0 \cos \theta)t = (v_0 \cos \theta) \frac{v_0 \sin \theta}{16} = \frac{v_0^2}{32} \sin 2\theta.$$

So,

$$x = \frac{1200^2}{32} \sin(2\theta) = 3000 \Rightarrow \sin 2\theta = \frac{1}{15} \Rightarrow \theta \approx 1.91^\circ.$$

43. (a)  $\theta = 10^\circ$ ,  $v_0 = 66$  ft/sec

$$\mathbf{r}(t) = (66 \cos 10^\circ)\mathbf{i} + [0 + (66 \sin 10^\circ)t - 16t^2]\mathbf{j}$$

$$\mathbf{r}(t) \approx (65t)\mathbf{i} + (11.46t - 16t^2)\mathbf{j}$$

Maximum height: 2.052 feet

Range: 46.557 feet

(b)  $\theta = 10^\circ$ ,  $v_0 = 146$  ft/sec

$$\mathbf{r}(t) = (146 \cos 10^\circ)\mathbf{i} + [0 + (146 \sin 10^\circ)t - 16t^2]\mathbf{j}$$

$$\mathbf{r}(t) \approx (143.78t)\mathbf{i} + (25.35t - 16t^2)\mathbf{j}$$

Maximum height: 10.043 feet

Range: 227.828 feet

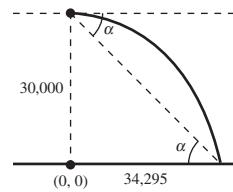
(c)  $\theta = 45^\circ$ ,  $v_0 = 66$  ft/sec

$$\mathbf{r}(t) = (66 \cos 45^\circ)\mathbf{i} + [0 + (66 \sin 45^\circ)t - 16t^2]\mathbf{j}$$

$$\mathbf{r}(t) \approx (46.67t)\mathbf{i} + (46.67t - 16t^2)\mathbf{j}$$

Maximum height: 34.031 feet

Range: 136.125 feet



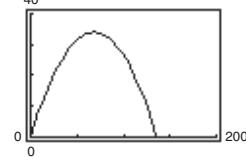
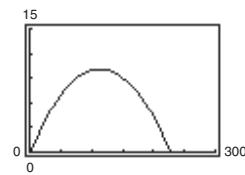
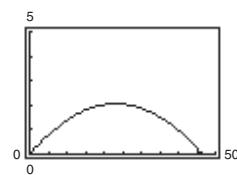
42. From Exercise 41, the range is

$$x = \frac{v_0^2}{32} \sin 2\theta$$

$$\text{So, } x = 200 = \frac{v_0^2}{32} \sin(24^\circ)$$

$$\Rightarrow v_0^2 = 6400/\sin(24^\circ)$$

$$\Rightarrow v_0 \approx 125.4 \text{ ft/sec}$$



(d)  $\theta = 45^\circ, v_0 = 146 \text{ ft/sec}$

$$\mathbf{r}(t) = (146 \cos 45^\circ)\mathbf{i} + [0 + (146 \sin 45^\circ)t - 16t^2]\mathbf{j}$$

$$\mathbf{r}(t) \approx (103.24t)\mathbf{i} + (103.24t - 16t^2)\mathbf{j}$$

Maximum height: 166.531 feet

Range: 666.125 feet

(e)  $\theta = 60^\circ, v_0 = 66 \text{ ft/sec}$

$$\mathbf{r}(t) = (66 \cos 60^\circ)\mathbf{i} + [0 + (66 \sin 60^\circ)t - 16t^2]\mathbf{j}$$

$$\mathbf{r}(t) \approx (33t)\mathbf{i} + (57.16t - 16t^2)\mathbf{j}$$

Maximum height: 51.047 feet

Range: 117.888 feet

(f)  $\theta = 60^\circ, v_0 = 146 \text{ ft/sec}$

$$\mathbf{r}(t) = (146 \cos 60^\circ)\mathbf{i} + [0 + (146 \sin 60^\circ)t - 16t^2]\mathbf{j}$$

$$\mathbf{r}(t) \approx (73t)\mathbf{i} + (126.44t - 16t^2)\mathbf{j}$$

Maximum height: 249.797 feet

Range: 576.881 feet

44. (a)  $\mathbf{r}(t) = t(v_0 \cos \theta)\mathbf{i} + (tv_0 \sin \theta - 16t^2)\mathbf{j}$

$$t(v_0 \sin \theta - 16t) = 0 \text{ when } t = \frac{v_0 \sin \theta}{16}.$$

$$\text{Range: } x = v_0 \cos \theta \left( \frac{v_0 \sin \theta}{32} \right) = \left( \frac{v_0^2}{32} \right) \sin 2\theta$$

The range will be maximum when

$$\frac{dx}{dt} = \left( \frac{v_0^2}{32} \right) 2 \cos 2\theta = 0$$

or

$$2\theta = \frac{\pi}{2}, \theta = \frac{\pi}{4} \text{ rad.}$$

(b)  $y(t) = tv_0 \sin \theta - 16t^2$

$$\frac{dy}{dt} = v_0 \sin \theta - 32t = 0 \text{ when } t = \frac{v_0 \sin \theta}{32}.$$

Maximum height:

$$y \left( \frac{v_0 \sin \theta}{32} \right) = \frac{v_0^2 \sin^2 \theta}{32} - 16 \frac{v_0^2 \sin^2 \theta}{32^2} = \frac{v_0^2 \sin^2 \theta}{64}$$

Minimum height when  $\sin \theta = 1$ , or  $\theta = \frac{\pi}{2}$ .

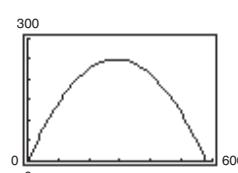
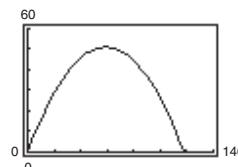
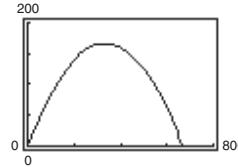
46.  $\mathbf{r}(t) = (v_0 \cos \theta)\mathbf{i} + [h + (v_0 \sin \theta)t - 4.9t^2]\mathbf{j}$

$$= (v_0 \cos 8^\circ)\mathbf{i} + [(v_0 \sin 8^\circ)t - 4.9t^2]\mathbf{j}$$

$$x = 50 \text{ when } (v_0 \cos 8^\circ)t = 50 \Rightarrow t = \frac{50}{v_0 \cos 8^\circ}. \text{ For this value of } t, y = 0:$$

$$(v_0 \sin 8^\circ) \left( \frac{50}{v_0 \cos 8^\circ} \right) - 4.9 \left( \frac{50}{v_0 \cos 8^\circ} \right)^2 = 0$$

$$50 \tan 8^\circ = \frac{(4.9)(2500)}{v_0^2 \cos^2 8^\circ} \Rightarrow v_0^2 = \frac{(4.9)50}{\tan 8^\circ \cos^2 8^\circ} \approx 1777.698 \Rightarrow v_0 \approx 42.2 \text{ m/sec}$$



45.  $\mathbf{r}(t) = (v_0 \cos \theta)\mathbf{i} + [h + (v_0 \sin \theta)t - 4.9t^2]\mathbf{j}$

$$= (100 \cos 30^\circ)\mathbf{i} + [1.5 + (100 \sin 30^\circ)t - 4.9t^2]\mathbf{j}$$

The projectile hits the ground when

$$-4.9t^2 + 100\left(\frac{1}{2}\right)t + 1.5 = 0 \Rightarrow t \approx 10.234 \text{ seconds.}$$

So the range is  $(100 \cos 30^\circ)(10.234) \approx 886.3$  meters.

The maximum height occurs when  $dy/dt = 0$ .

$$100 \sin 30 = 9.8t \Rightarrow t \approx 5.102 \text{ sec}$$

The maximum height is

$$y = 1.5 + (100 \sin 30^\circ)(5.102) - 4.9(5.102)^2 \approx 129.1 \text{ meters.}$$

47.  $\mathbf{r}(t) = b(\omega t - \sin \omega t)\mathbf{i} + b(1 - \cos \omega t)\mathbf{j}$

$$\mathbf{v}(t) = b(\omega - \omega \cos \omega t)\mathbf{i} + b\omega \sin \omega t\mathbf{j} = b\omega(1 - \cos \omega t)\mathbf{i} + b\omega \sin \omega t\mathbf{j}$$

$$\mathbf{a}(t) = (b\omega^2 \sin \omega t)\mathbf{i} + (b\omega^2 \cos \omega t)\mathbf{j} = b\omega^2[\sin(\omega t)\mathbf{i} + \cos(\omega t)\mathbf{j}]$$

$$\|\mathbf{v}(t)\| = \sqrt{2}b\omega\sqrt{1 - \cos(\omega t)}$$

$$\|\mathbf{a}(t)\| = b\omega^2$$

(a)  $\|\mathbf{v}(t)\| = 0$  when  $\omega t = 0, 2\pi, 4\pi, \dots$

(b)  $\|\mathbf{v}(t)\|$  is maximum when  $\omega t = \pi, 3\pi, \dots$ , then  $\|\mathbf{v}(t)\| = 2b\omega$ .

48.  $\mathbf{r}(t) = b(\omega t - \sin \omega t)\mathbf{i} + b(1 - \cos \omega t)\mathbf{j}$

$$\mathbf{v}(t) = b\omega[(1 - \cos \omega t)\mathbf{i} + (\sin \omega t)\mathbf{j}]$$

$$\text{Speed} = \|\mathbf{v}(t)\| = b\omega\sqrt{1 - 2\cos \omega t + \cos^2 \omega t + \sin^2 \omega t} = \sqrt{2}b\omega\sqrt{1 - \cos \omega t}.$$

The speed has a maximum value of  $2b\omega$  when  $\omega t = \pi, 3\pi, \dots$

$60 \text{ mi/h} = 88 \text{ ft/sec} = 88 \text{ rad/sec}$  (since  $b = 1$ ).

So, the maximum speed of a point on the tire is twice the speed of the car:

$$2(88) \text{ ft/sec} = 120 \text{ mi/h}$$

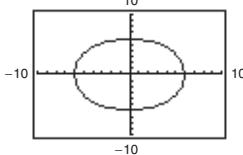
49.  $\mathbf{v}(t) = -b\omega \sin(\omega t)\mathbf{i} + b\omega \cos(\omega t)\mathbf{j}$

$$\mathbf{r}(t) \cdot \mathbf{v}(t) = -b^2\omega \sin(\omega t) \cos(\omega t) + b^2\omega \sin(\omega t) \cos(\omega t) = 0$$

So,  $\mathbf{r}(t)$  and  $\mathbf{v}(t)$  are orthogonal.

50. (a) Speed =  $\|\mathbf{v}\| = \sqrt{b^2\omega^2 \sin^2(\omega t) + b^2\omega^2 \cos^2(\omega t)} = \sqrt{b^2\omega^2[\sin^2(\omega t) + \cos^2(\omega t)]} = b\omega$

(b)



The graphing utility draws the circle faster for greater values of  $\omega$ .

51.  $\mathbf{a}(t) = -b\omega^2 \cos(\omega t)\mathbf{i} - b\omega^2 \sin(\omega t)\mathbf{j} = -b\omega^2[\cos(\omega t)\mathbf{i} + \sin(\omega t)\mathbf{j}] = -\omega^2\mathbf{r}(t)$

$\mathbf{a}(t)$  is a negative multiple of a unit vector from  $(0, 0)$  to  $(\cos \omega t, \sin \omega t)$  and so  $\mathbf{a}(t)$  is directed toward the origin.

52.  $\|\mathbf{a}(t)\| = b\omega^2 \|\cos(\omega t)\mathbf{i} + \sin(\omega t)\mathbf{j}\| = b\omega^2$

54.  $\|\mathbf{v}(t)\| = 30 \text{ mi/h} = 44 \text{ ft/sec}$

53.  $\|\mathbf{a}(t)\| = \omega^2 b, b = 2$

$$\omega = \frac{\|\mathbf{v}(t)\|}{b} = \frac{44}{300} \text{ rad/sec}$$

$$1 = m(32)$$

$$\mathbf{F} = m(\omega^2 b) = \frac{1}{32}(2\omega^2) = 10$$

$$\omega = 4\sqrt{10} \text{ rad/sec}$$

$$\|\mathbf{v}(t)\| = b\omega = 8\sqrt{10} \text{ ft/sec}$$

$$\|\mathbf{a}(t)\| = b\omega^2$$

$$\|\mathbf{F}\| = m(b\omega^2) = \frac{3400}{32}(300)\left(\frac{44}{300}\right)^2 = \frac{2057}{3} \text{ lb}$$

Let  $n$  be normal to the road.

$$\|\mathbf{n}\| \cos \theta = 3400$$

$$\|\mathbf{n}\| \sin \theta = \frac{2057}{3}$$

$$\text{Dividing, } \tan \theta = \frac{121}{600}$$

$$\theta \approx 11.4^\circ$$

55. To find the range, set

$$y(t) = h + (v_0 \sin \theta)t - \frac{1}{2}gt^2 = 0 \text{ then}$$

$0 = \left(\frac{1}{2}g\right)t^2 - (v_0 \sin \theta)t - h$ . By the Quadratic Formula, (discount the negative value)

$$\begin{aligned} t &= \frac{v_0 \sin \theta + \sqrt{(-v_0 \sin \theta)^2 - 4[(1/2)g](-h)}}{2[(1/2)g]} \\ &= \frac{v_0 \sin \theta + \sqrt{v_0^2 \sin^2 \theta + 2gh}}{g} \text{ second.} \end{aligned}$$

At this time,

$$\begin{aligned} x(t) &= v_0 \cos \theta \left( \frac{v_0 \sin \theta + \sqrt{v_0^2 \sin^2 \theta + 2gh}}{g} \right) \\ &= \frac{v_0 \cos \theta}{g} \left( v_0 \sin \theta + \sqrt{v_0^2 \left( \sin^2 \theta + \frac{2gh}{v_0^2} \right)} \right) \\ &= \frac{v_0^2 \cos \theta}{g} \left( \sin \theta + \sqrt{\sin^2 \theta + \frac{2gh}{v_0^2}} \right) \text{ feet.} \end{aligned}$$

56.  $h = 6$  feet,  $v_0 = 45$  feet per second,  $\theta = 42.5^\circ$ . From Exercise 55,

$$\begin{aligned} t &= \frac{45 \sin 42.5^\circ + \sqrt{(45)^2 \sin^2 42.5^\circ + 2(32)(6)}}{32} \\ &\approx 2.08 \text{ seconds.} \end{aligned}$$

At this time,  $x(t) \approx 69.02$  feet.

57.  $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$  Position vector

$\mathbf{v}(t) = x'(t)\mathbf{i} + y'(t)\mathbf{j} + z'(t)\mathbf{k}$  Velocity vector

$\mathbf{a}(t) = x''(t)\mathbf{i} + y''(t)\mathbf{j} + z''(t)\mathbf{k}$  Acceleration vector

$$\begin{aligned} \text{Speed} &= \|\mathbf{v}(t)\| = \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2} \\ &= C, C \text{ is a constant.} \end{aligned}$$

$$\frac{d}{dt} [x'(t)^2 + y'(t)^2 + z'(t)^2] = 0$$

$$2x'(t)x''(t) + 2y'(t)y''(t) + 2z'(t)z''(t) = 0$$

$$2[x'(t)x''(t) + y'(t)y''(t) + z'(t)z''(t)] = 0$$

$$\mathbf{v}(t) \cdot \mathbf{a}(t) = 0$$

Orthogonal

58.  $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$

$y(t) = m(x(t)) + b$ ,  $m$  and  $b$  are constants.

$$\mathbf{r}(t) = x(t)\mathbf{i} + [m(x(t)) + b]\mathbf{j}$$

$$\mathbf{v}(t) = x'(t)\mathbf{i} + mx'(t)\mathbf{j}$$

$$\mathbf{s}(t) = \sqrt{[x'(t)]^2 + [mx'(t)]^2} = C, C \text{ is a constant.}$$

$$\text{So, } x'(t) = \frac{C}{\sqrt{1+m^2}}$$

$$x''(t) = 0$$

$$\mathbf{a}(t) = x''(t)\mathbf{i} + mx''(t)\mathbf{j} = \mathbf{0}.$$

59.  $\mathbf{r}(t) = 6 \cos t\mathbf{i} + 3 \sin t\mathbf{j}$

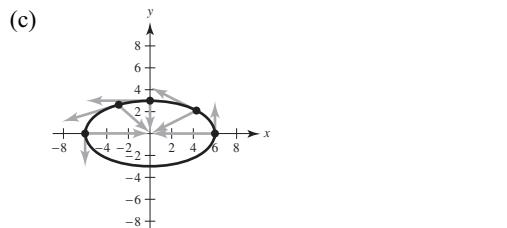
$$(a) \mathbf{v}(t) = \mathbf{r}'(t) = -6 \sin t\mathbf{i} + 3 \cos t\mathbf{j}$$

$$\|\mathbf{v}(t)\| = \sqrt{36 \sin^2 t + 9 \cos^2 t}$$

$$= 3\sqrt{4 \sin^2 t + \cos^2 t} = 3\sqrt{3 \sin^2 t + 1}$$

$$\mathbf{a}(t) = \mathbf{v}'(t) = -6 \cos t\mathbf{i} - 3 \sin t\mathbf{j}$$

(b)	<table border="1"> <tr> <td><math>t</math></td><td>0</td><td><math>\frac{\pi}{4}</math></td><td><math>\frac{\pi}{2}</math></td><td><math>\frac{2\pi}{3}</math></td><td><math>\pi</math></td></tr> <tr> <td>Speed</td><td>3</td><td><math>\frac{3}{2}\sqrt{10}</math></td><td>6</td><td><math>\frac{3}{2}\sqrt{13}</math></td><td>3</td></tr> </table>	$t$	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\pi$	Speed	3	$\frac{3}{2}\sqrt{10}$	6	$\frac{3}{2}\sqrt{13}$	3
$t$	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\pi$								
Speed	3	$\frac{3}{2}\sqrt{10}$	6	$\frac{3}{2}\sqrt{13}$	3								



(d) The speed is increasing when the angle between  $\mathbf{v}$  and  $\mathbf{a}$  is in the interval

$$\left[0, \frac{\pi}{2}\right].$$

The speed is decreasing when the angle is in the interval

$$\left[\frac{\pi}{2}, \pi\right].$$

60.  $\mathbf{r}(t) = a \cos \omega t \mathbf{i} + b \sin \omega t \mathbf{j}$

$$(a) \mathbf{r}'(t) = \mathbf{v}(t) = -a\omega \sin \omega t \mathbf{i} + b\omega \cos \omega t \mathbf{j}$$

$$\text{Speed} = \|\mathbf{v}(t)\| = \sqrt{a^2 \omega^2 \sin^2 \omega t + b^2 \omega^2 \cos^2 \omega t}$$

$$(b) \mathbf{a}(t) = \mathbf{v}'(t) = -a\omega^2 \cos \omega t \mathbf{i} - b\omega^2 \sin \omega t \mathbf{j}$$

$$= \omega^2(-a \cos \omega t \mathbf{i} - b \sin \omega t \mathbf{j})$$

$$= -\omega^2 \mathbf{r}(t)$$

61. The velocity of an object involves both magnitude and direction of motion, whereas speed involves only magnitude.

62. (a) The speed is increasing.

(b) The speed is decreasing.

63. (a)  $\mathbf{r}_1(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$

$$\mathbf{r}_2(t) = \mathbf{r}_1(2t)$$

Velocity:  $\mathbf{r}_2'(t) = 2\mathbf{r}_1'(2t)$

Acceleration:  $\mathbf{r}_2''(t) = 4\mathbf{r}_1''(2t)$

(b) In general, if  $\mathbf{r}_3(t) = \mathbf{r}_1(\omega t)$ , then:

Velocity:  $\mathbf{r}_3'(t) = \omega\mathbf{r}_1'(\omega t)$

Acceleration:  $\mathbf{r}_3''(t) = \omega^2\mathbf{r}_1''(\omega t)$

64.  $\mathbf{a}(t) = \sin t\mathbf{i} - \cos t\mathbf{j}$

$$\mathbf{v}(t) = \int \mathbf{a}(t) dt = -\cos t\mathbf{i} = \sin t\mathbf{j} + \mathbf{C}_1$$

$$\mathbf{v}(0) = -\mathbf{i} = -\mathbf{i} + \mathbf{C}_1 \Rightarrow \mathbf{C}_1 = \mathbf{0}$$

$$\mathbf{v}(t) = -\cos t\mathbf{i} - \sin t\mathbf{j}$$

$$\mathbf{r}(t) = \int \mathbf{v}(t) dt = -\sin t\mathbf{i} + \cos t\mathbf{j} + \mathbf{C}_2$$

$$\mathbf{r}(0) = \mathbf{j} = \mathbf{j} + \mathbf{C}_2 \Rightarrow \mathbf{C}_2 = \mathbf{0}$$

$$\mathbf{r}(t) = -\sin t\mathbf{i} + \cos t\mathbf{j}$$

The path is a circle.

65. False. The acceleration is the derivative of the velocity.

66. True

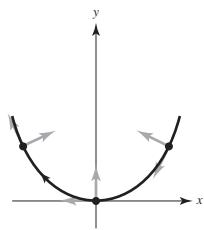
67. True

68. False. For example,  $6t\mathbf{r}(t) = t^3\mathbf{i}$ . Then  $\mathbf{v}(t) = 3t^2\mathbf{i}$  and

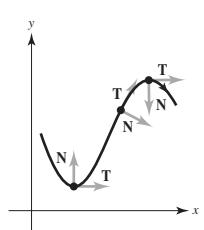
$\mathbf{a}(t) = 6t\mathbf{i}$ .  $\mathbf{v}(t)$  is not orthogonal to  $\mathbf{a}(t)$ .

## Section 12.4 Tangent Vectors and Normal Vectors

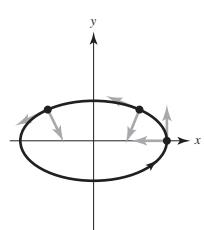
1.



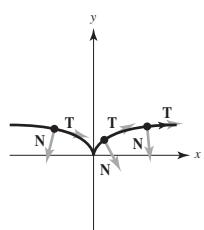
2.



3.



4.



5.  $\mathbf{r}(t) = t^2\mathbf{i} + 2t\mathbf{j}, t = 1$

$$\mathbf{r}'(t) = 2t\mathbf{i} + 2\mathbf{j}, \|\mathbf{r}'(t)\| = \sqrt{4t^2 + 4} = 2\sqrt{t^2 + 1}$$

$$\mathbf{T}(1) = \frac{\mathbf{r}'(1)}{\|\mathbf{r}'(1)\|} = \frac{1}{\sqrt{2}}(\mathbf{i} + \mathbf{j}) = \frac{\sqrt{2}}{2}\mathbf{i} + \frac{\sqrt{2}}{2}\mathbf{j}$$

6.  $\mathbf{r}(t) = t^3\mathbf{i} + 2t^2\mathbf{j}$

$$\mathbf{r}'(t) = 3t^2\mathbf{i} + 4t\mathbf{j}$$

$$\|\mathbf{r}'(t)\| = \sqrt{9t^4 + 16t^2}$$

$$\mathbf{T}(1) = \frac{\mathbf{r}'(1)}{\|\mathbf{r}'(1)\|} = \frac{1}{\sqrt{9+16}}(3\mathbf{i} + 4\mathbf{j}) = \frac{3}{5}\mathbf{i} + \frac{4}{5}\mathbf{j}$$

7.  $\mathbf{r}(t) = 4 \cos t\mathbf{i} + 4 \sin t\mathbf{j}, t = \frac{\pi}{4}$

$$\mathbf{r}'(t) = -4 \sin t\mathbf{i} + 4 \cos t\mathbf{j}$$

$$\|\mathbf{r}'(t)\| = \sqrt{16 \sin^2 t + 16 \cos^2 t} = 4$$

$$\mathbf{T}\left(\frac{\pi}{4}\right) = \frac{\mathbf{r}'\left(\frac{\pi}{4}\right)}{\|\mathbf{r}'\left(\frac{\pi}{4}\right)\|} = -\frac{\sqrt{2}}{2}\mathbf{i} + \frac{\sqrt{2}}{2}\mathbf{j}$$

8.  $\mathbf{r}(t) = 6 \cos t\mathbf{i} + 2 \sin t\mathbf{j}$

$$\mathbf{r}'(t) = -6 \sin t\mathbf{i} + 2 \cos t\mathbf{j}$$

$$\|\mathbf{r}'(t)\| = \sqrt{36 \sin^2 t + 4 \cos^2 t}$$

$$\mathbf{T}\left(\frac{\pi}{3}\right) = \frac{\mathbf{r}'\left(\frac{\pi}{3}\right)}{\|\mathbf{r}'\left(\frac{\pi}{3}\right)\|} = \frac{-3\sqrt{3}\mathbf{i} + \mathbf{j}}{\sqrt{36(3/4) + (1/4)}} = \frac{1}{\sqrt{28}}(-3\sqrt{3}\mathbf{i} + \mathbf{j})$$

9.  $\mathbf{r}(t) = 3t\mathbf{i} - \ln t\mathbf{j}, t = e$

$$\mathbf{r}'(t) = 3\mathbf{i} - \frac{1}{t}\mathbf{j}$$

$$\mathbf{r}'(e) = 3\mathbf{i} - \frac{1}{e}\mathbf{j}$$

$$\mathbf{T}(e) = \frac{\mathbf{r}'(e)}{\|\mathbf{r}'(e)\|}$$

$$= \frac{3\mathbf{i} - \frac{1}{e}\mathbf{j}}{\sqrt{9 + \frac{1}{e^2}}} = \frac{3e\mathbf{i} - \mathbf{j}}{\sqrt{9e^2 + 1}} \approx 0.9926\mathbf{i} - 0.1217\mathbf{j}$$

10.  $\mathbf{r}(t) = e^t \cos t\mathbf{i} + e^t \sin t\mathbf{j}, t = 0$

$$\mathbf{r}'(t) = (e^t \cos t - e^t \sin t)\mathbf{i} + e^t \mathbf{j}$$

$$\mathbf{r}'(0) = \mathbf{i} + \mathbf{j}$$

$$\mathbf{T}(0) = \frac{\mathbf{r}'(0)}{\|\mathbf{r}'(0)\|} = \frac{\mathbf{i} + \mathbf{j}}{\sqrt{2}} = \frac{\sqrt{2}}{2}\mathbf{i} + \frac{\sqrt{2}}{2}\mathbf{j}$$

11.  $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + t\mathbf{k}, P(0, 0, 0)$

$$\mathbf{r}'(t) = \mathbf{i} + 2t\mathbf{j} + \mathbf{k}$$

When  $t = 0$ ,  $\mathbf{r}'(0) = \mathbf{i} + \mathbf{k}$ ,  $[t = 0 \text{ at } (0, 0, 0)]$ .

$$\mathbf{T}(0) = \frac{\mathbf{r}'(0)}{\|\mathbf{r}'(0)\|} = \frac{\sqrt{2}}{2}(\mathbf{i} + \mathbf{k})$$

Direction numbers:  $a = 1, b = 0, c = 1$

Parametric equations:  $x = t, y = 0, z = t$

12.  $\mathbf{r}(t) = t^2\mathbf{i} + t\mathbf{j} + \frac{4}{3}\mathbf{k}, P\left(1, 1, \frac{4}{3}\right)$

$$\mathbf{r}'(t) = 2t\mathbf{i} + \mathbf{j}$$

When  $t = 1$ ,  $\mathbf{r}'(t) = \mathbf{r}'(1) = 2\mathbf{i} + \mathbf{j}$   $[t = 1 \text{ at } \left(1, 1, \frac{4}{3}\right)]$ .

$$\mathbf{T}(1) = \frac{\mathbf{r}'(1)}{\|\mathbf{r}'(1)\|} = \frac{2\mathbf{i} + \mathbf{j}}{\sqrt{5}} = \frac{\sqrt{5}}{5}(2\mathbf{i} + \mathbf{j})$$

Direction numbers:  $a = 2, b = 1, c = 0$

Parametric equations:  $x = 2t + 1, y = t + 1, z = \frac{4}{3}$

13.  $\mathbf{r}(t) = 3 \cos t\mathbf{i} + 3 \sin t\mathbf{j} + t\mathbf{k}, P(3, 0, 0)$

$$\mathbf{r}'(t) = -3 \sin t\mathbf{i} + 3 \cos t\mathbf{j} + \mathbf{k}$$

$$t = 0 \text{ at } P(3, 0, 0)$$

$$\mathbf{r}'(0) = 3\mathbf{j} + \mathbf{k}$$

$$\mathbf{T}(0) = \frac{\mathbf{r}'(0)}{\|\mathbf{r}'(0)\|} = \frac{3\mathbf{j} + \mathbf{k}}{\sqrt{10}}$$

Direction numbers:  $a = 0, b = 3, c = 1$

Parametric equations:  $x = 3, y = 3t, z = t$

14.  $\mathbf{r}(t) = \langle t, t, \sqrt{4 - t^2} \rangle, P(1, 1, \sqrt{3})$

$$\mathbf{r}'(t) = \left\langle 1, 1, -\frac{t}{4 - t^2} \right\rangle$$

When  $t = 1$ ,  $\mathbf{r}'(1) = \left\langle 1, 1, -\frac{1}{\sqrt{3}} \right\rangle, t = 1 \text{ at } (1, 1, \sqrt{3})$ .

$$\mathbf{T}(1) = \frac{\mathbf{r}'(1)}{\|\mathbf{r}'(1)\|} = \frac{\sqrt{21}}{7} \left\langle 1, 1, -\frac{1}{\sqrt{3}} \right\rangle$$

Direction numbers:  $a = 1, b = 1, c = -\frac{1}{\sqrt{3}}$

Parametric equations:  $x = t + 1, y = t + 1,$

$$z = -\frac{1}{\sqrt{3}}t + \sqrt{3}$$

15.  $\mathbf{r}(t) = \langle 2 \cos t, 2 \sin t, 4 \rangle, P(\sqrt{2}, \sqrt{2}, 4)$

$$\mathbf{r}'(t) = \langle -2 \sin t, 2 \cos t, 0 \rangle$$

When  $t = \frac{\pi}{4}$ ,  $\mathbf{r}'\left(\frac{\pi}{4}\right) = \langle -\sqrt{2}, \sqrt{2}, 0 \rangle,$

$$\left[ t = \frac{\pi}{4} \text{ at } (\sqrt{2}, \sqrt{2}, 4) \right].$$

$$\mathbf{T}\left(\frac{\pi}{4}\right) = \frac{\mathbf{r}'(\pi/4)}{\|\mathbf{r}'(\pi/4)\|} = \frac{1}{2} \langle -\sqrt{2}, \sqrt{2}, 0 \rangle$$

Direction numbers:  $a = -\sqrt{2}, b = \sqrt{2}, c = 0$

Parametric equations:  $x = -\sqrt{2}t + \sqrt{2}, y = \sqrt{2}t + \sqrt{2}, z = 4$

16.  $\mathbf{r}(t) = \langle 2 \sin t, 2 \cos t, 4 \sin^2 t \rangle, P(1, \sqrt{3}, 1)$

$$\mathbf{r}'(t) = \langle 2 \cos t, -2 \sin t, 8 \sin t \cos t \rangle$$

When  $t = \frac{\pi}{6}$ ,  $\mathbf{r}'\left(\frac{\pi}{6}\right) = \langle \sqrt{3}, -1, 2\sqrt{3} \rangle,$

$$\left[ t = \frac{\pi}{6} \text{ at } (1, \sqrt{3}, 1) \right].$$

$$\mathbf{T}\left(\frac{\pi}{6}\right) = \frac{\mathbf{r}'(\pi/6)}{\|\mathbf{r}'(\pi/6)\|} = \frac{1}{4} \langle \sqrt{3}, -1, 2\sqrt{3} \rangle$$

Direction numbers:  $a = \sqrt{3}, b = -1, c = 2\sqrt{3}$

Parametric equations:  $x = \sqrt{3}t + 1, y = -t + \sqrt{3}, z = 2\sqrt{3}t + 1$

17.  $\mathbf{r}(t) = \left\langle t, t^2, \frac{2}{3}t^3 \right\rangle$

$$\mathbf{r}'(t) = \langle 1, 2t, 2t^2 \rangle$$

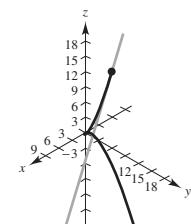
When  $t = 3$ ,  $\mathbf{r}'(3) = \langle 1, 6, 18 \rangle,$

$$\left[ t = 3 \text{ at } (3, 9, 18) \right].$$

$$\mathbf{T}(3) = \frac{\mathbf{r}'(3)}{\|\mathbf{r}'(3)\|} = \frac{1}{19} \langle 1, 6, 18 \rangle$$

Direction numbers:  $a = 1, b = 6, c = 18$

Parametric equations:  $x = t + 3, y = 6t + 9, z = 18t + 18$



18.  $\mathbf{r}(t) = 3 \cos t\mathbf{i} + 4 \sin t\mathbf{j} + \frac{1}{2}\mathbf{k}$

$$\mathbf{r}'(t) = -3 \sin t\mathbf{i} + 4 \cos t\mathbf{j} + \frac{1}{2}\mathbf{k}$$

When  $t = \frac{\pi}{2}$ ,  $\mathbf{r}\left(\frac{\pi}{2}\right) = -3\mathbf{i} + \frac{1}{2}\mathbf{k}$ ,

$$\left[ t = \frac{\pi}{2} \text{ at } \left(0, 4, \frac{\pi}{4}\right) \right].$$

$$\mathbf{T}\left(\frac{\pi}{2}\right) = \frac{\mathbf{r}'(\pi/2)}{\|\mathbf{r}'(\pi/2)\|} = \frac{2}{\sqrt{37}}\left(-3\mathbf{i} + \frac{1}{2}\mathbf{k}\right) = \frac{1}{\sqrt{37}}(-6\mathbf{i} + \mathbf{k})$$

Direction numbers:  $a = -6$ ,  $b = 0$ ,  $c = 1$

Parametric equations:  $x = -6t$ ,  $y = 4$ ,  $z = t + \frac{\pi}{4}$

19.  $\mathbf{r}(t) = t\mathbf{i} + \ln t\mathbf{j} + \sqrt{t}\mathbf{k}$ ,  $t_0 = 1$

$$\mathbf{r}'(t) = \mathbf{i} + \frac{1}{t}\mathbf{j} + \frac{1}{2\sqrt{t}}\mathbf{k}$$

$$\mathbf{T}(1) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = \frac{\mathbf{i} + \mathbf{j} + (1/2)\mathbf{k}}{\sqrt{1+1+(1/4)}} = \frac{2}{3}\mathbf{i} + \frac{2}{3}\mathbf{j} + \frac{1}{3}\mathbf{k}$$

Tangent line:  $x = 1 + t$ ,  $y = t$ ,  $z = 1 + \frac{1}{2}t$

$$\begin{aligned} \mathbf{r}(t_0 + 0.1) &= \mathbf{r}(1.1) \approx 1.1\mathbf{i} + 0.1\mathbf{j} + 1.05\mathbf{k} \\ &= \langle 1.1, 0.1, 1.05 \rangle \end{aligned}$$

20.  $\mathbf{r}(t) = e^{-t}\mathbf{i} + 2 \cos t\mathbf{j} + 2 \sin t\mathbf{k}$ ,  $t_0 = 0$

$$\mathbf{r}'(t) = -e^{-t}\mathbf{i} - 2 \sin t\mathbf{j} + 2 \cos t\mathbf{k}$$

$$\mathbf{r}(0) = \mathbf{i} + 2\mathbf{j}$$
,  $\mathbf{r}'(0) = -\mathbf{i} + 2\mathbf{k}$ ,  $\|\mathbf{r}'(0)\| = \sqrt{5}$

$$\mathbf{T}(0) = \frac{\mathbf{r}'(0)}{\|\mathbf{r}'(0)\|} = \frac{-\mathbf{i} + 2\mathbf{k}}{\sqrt{5}}$$

Parametric equations:

$$x(s) = 1 - s$$
,  $y(s) = 2$ ,  $z(s) = 2s$

$$\mathbf{r}(t_0 + 0.1) = \mathbf{r}(0 + 0.1)$$

$$\approx \langle 1 - 0.1, 2, 2(0.1) \rangle = \langle 0.9, 2, 0.2 \rangle$$

21.  $\mathbf{r}(4) = \langle 2, 16, 2 \rangle$

$$\mathbf{u}(8) = \langle 2, 16, 2 \rangle$$

So the curves intersect.

$$\mathbf{r}'(t) = \left\langle 1, 2t, \frac{1}{2} \right\rangle$$

$$\mathbf{u}'(s) = \left\langle \frac{1}{4}, 2, \frac{1}{3}s^{-2/3} \right\rangle$$

$$\cos \theta = \frac{\mathbf{r}'(4) \cdot \mathbf{u}'(8)}{\|\mathbf{r}'(4)\| \|\mathbf{u}'(8)\|} \approx \frac{16.29167}{16.29513} \Rightarrow \theta \approx 1.2^\circ$$

22.  $\mathbf{r}(0) = \langle 0, 1, 0 \rangle$

$$\mathbf{u}(0) = \langle 0, 1, 0 \rangle$$

So the curves intersect.

$$\mathbf{r}'(t) = \langle 1, -\sin t, \cos t \rangle$$

$$\mathbf{u}'(s) = \left\langle -\sin s \cos s - \cos s, -\sin s \cos s - \cos s, \frac{1}{2} \cos 2s + \frac{1}{2} \right\rangle$$

$$\mathbf{u}'(0) = \langle -1, 0, 1 \rangle$$

$$\cos \theta = \frac{\mathbf{r}'(0) \cdot \mathbf{u}'(0)}{\|\mathbf{r}'(0)\| \|\mathbf{u}'(0)\|} = 0 \Rightarrow \theta = \frac{\pi}{2}$$

23.  $\mathbf{r}(t) = t\mathbf{i} + \frac{1}{2}t^2\mathbf{j}$ ,  $t = 2$

$$\mathbf{r}'(t) = \mathbf{i} + t\mathbf{j}$$

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = \frac{\mathbf{i} + t\mathbf{j}}{\sqrt{1+t^2}}$$

$$\mathbf{T}'(t) = \frac{-t}{(t^2+1)^{3/2}}\mathbf{i} + \frac{1}{(t^2+1)^{3/2}}\mathbf{j}$$

$$\mathbf{T}'(2) = \frac{-2}{5^{3/2}}\mathbf{i} + \frac{1}{5^{3/2}}\mathbf{j}$$

$$\mathbf{N}(2) = \frac{\mathbf{T}'(2)}{\|\mathbf{T}'(2)\|} = \frac{1}{\sqrt{5}}(-2\mathbf{i} + \mathbf{j}) = \frac{-2\sqrt{5}}{5}\mathbf{i} + \frac{\sqrt{5}}{5}\mathbf{j}$$

24.  $\mathbf{r}(t) = t\mathbf{i} + \frac{6}{t}\mathbf{j}$ ,  $t = 3$

$$\mathbf{r}'(t) = \mathbf{i} - \frac{6}{t^2}\mathbf{j}$$

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = \frac{1}{\sqrt{1+(36/t^4)}}\left(\mathbf{i} - \frac{6}{t^2}\mathbf{j}\right)$$

$$= \frac{t^2}{\sqrt{t^4+36}}\left(\mathbf{i} - \frac{6}{t^2}\mathbf{j}\right)$$

$$\mathbf{T}'(t) = \frac{72t}{(t^4+36)^{3/2}}\mathbf{i} + \frac{12t^3}{(t^4+36)^{3/2}}\mathbf{j}$$

$$\mathbf{T}'(2) = \frac{144}{52^{3/2}}\mathbf{i} + \frac{96}{52^{3/2}}\mathbf{j}$$

$$\mathbf{N}(2) = \frac{\mathbf{T}'(2)}{\|\mathbf{T}'(2)\|} + \frac{1}{\sqrt{13}}(3\mathbf{i} + 2\mathbf{j})$$

**25.**  $\mathbf{r}(t) = \ln t\mathbf{i} + (t+1)\mathbf{j}, t = 2$

$$\mathbf{r}'(t) = \frac{1}{t}\mathbf{i} + \mathbf{j}$$

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = \frac{\frac{1}{t}\mathbf{i} + \mathbf{j}}{\sqrt{\frac{1}{t^2} + 1}} = \frac{\mathbf{i} + t\mathbf{j}}{\sqrt{1+t^2}}$$

$$\mathbf{T}'(t) = \frac{-t}{(1+t^2)^{3/2}}\mathbf{i} + \frac{1}{(1+t^2)^{3/2}}\mathbf{j}$$

$$\mathbf{T}'(2) = \frac{-2}{5^{3/2}}\mathbf{i} + \frac{1}{5^{3/2}}\mathbf{j}$$

$$\mathbf{N}(2) = \frac{\mathbf{T}'(2)}{\|\mathbf{T}'(2)\|} = \frac{-2\sqrt{5}}{5}\mathbf{i} + \frac{\sqrt{5}}{5}\mathbf{j}$$

**27.**  $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + \ln t\mathbf{k}, t = 1$

$$\mathbf{r}'(t) = \mathbf{i} + 2t\mathbf{j} + \frac{1}{t}\mathbf{k}$$

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = \frac{\mathbf{i} + 2t\mathbf{j} + \frac{1}{t}\mathbf{k}}{\sqrt{1+4t^2+\frac{1}{t^2}}} = \frac{t\mathbf{i} + 2t^2\mathbf{j} + \mathbf{k}}{\sqrt{4t^4+t^2+1}}$$

$$\mathbf{T}'(t) = \frac{1-4t^4}{(4t^4+t^2+1)^{3/2}}\mathbf{i} + \frac{2t^3+4t}{(4t^4+t^2+1)^{3/2}}\mathbf{j} + \frac{-8t^3-t}{(4t^4+t^2+1)^{3/2}}\mathbf{k}$$

$$\mathbf{T}'(1) = \frac{-3}{6^{3/2}}\mathbf{i} + \frac{6}{6^{3/2}}\mathbf{j} + \frac{-9}{6^{3/2}}\mathbf{k} = \frac{3}{6^{3/2}}[-\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}]$$

$$\mathbf{N}(1) = \frac{-\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}}{\sqrt{14}} = \frac{-\sqrt{14}}{14}\mathbf{i} + \frac{2\sqrt{14}}{14}\mathbf{j} - \frac{3\sqrt{14}}{14}\mathbf{k}$$

**28.**  $\mathbf{r}(t) = \sqrt{2}t\mathbf{i} + e^t\mathbf{j} + e^{-t}\mathbf{k}, t = 0$

$$\mathbf{r}'(t) = \sqrt{2}\mathbf{i} + e^t\mathbf{j} - e^{-t}\mathbf{k}$$

$$\|\mathbf{r}'(t)\| = \sqrt{2+e^{2t}+e^{-2t}} = \sqrt{(e^t+e^{-t})^2} = e^t + e^{-t}$$

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = \frac{\sqrt{2}\mathbf{i} + e^t\mathbf{j} - e^{-t}\mathbf{k}}{e^t + e^{-t}}$$

$$\mathbf{T}'(t) = \frac{\sqrt{2}(e^{-t}-e^t)}{(e^t+e^{-t})^2}\mathbf{i} + \frac{2}{(e^t+e^{-t})^2}\mathbf{j} + \frac{2}{(e^t+e^{-t})^2}\mathbf{k}$$

$$\mathbf{T}'(0) = \frac{1}{2}\mathbf{j} + \frac{1}{2}\mathbf{k}$$

$$\mathbf{N}(0) = \frac{\sqrt{2}}{2}\mathbf{j} + \frac{\sqrt{2}}{2}\mathbf{k}$$

**26.**  $\mathbf{r}(t) = \pi \cos t\mathbf{i} + \pi \sin t\mathbf{j}, t = \frac{\pi}{6}$

$$\mathbf{r}'(t) = -\pi \sin t\mathbf{i} + \pi \cos t\mathbf{j}$$

$$\|\mathbf{r}'(t)\| = \pi$$

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = -\sin t\mathbf{i} + \cos t\mathbf{j}$$

$$\mathbf{T}'(t) = -\cos t\mathbf{i} - \sin t\mathbf{j}, \|\mathbf{T}'(t)\| = 1$$

$$\mathbf{T}\left(\frac{\pi}{6}\right) = -\frac{\sqrt{3}}{2}\mathbf{i} - \frac{1}{2}\mathbf{j}$$

$$\mathbf{N}\left(\frac{\pi}{6}\right) = \frac{\mathbf{T}\left(\frac{\pi}{6}\right)}{\left\|\mathbf{T}\left(\frac{\pi}{6}\right)\right\|} = -\frac{\sqrt{3}}{2}\mathbf{i} - \frac{1}{2}\mathbf{j}$$

**27.**  $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + \ln t\mathbf{k}, t = 1$

$$\mathbf{r}'(t) = \mathbf{i} + 2t\mathbf{j} + \frac{1}{t}\mathbf{k}$$

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = \frac{\mathbf{i} + 2t\mathbf{j} + \frac{1}{t}\mathbf{k}}{\sqrt{1+4t^2+\frac{1}{t^2}}} = \frac{t\mathbf{i} + 2t^2\mathbf{j} + \mathbf{k}}{\sqrt{4t^4+t^2+1}}$$

$$\mathbf{T}'(t) = \frac{1-4t^4}{(4t^4+t^2+1)^{3/2}}\mathbf{i} + \frac{2t^3+4t}{(4t^4+t^2+1)^{3/2}}\mathbf{j} + \frac{-8t^3-t}{(4t^4+t^2+1)^{3/2}}\mathbf{k}$$

$$\mathbf{T}'(1) = \frac{-3}{6^{3/2}}\mathbf{i} + \frac{6}{6^{3/2}}\mathbf{j} + \frac{-9}{6^{3/2}}\mathbf{k} = \frac{3}{6^{3/2}}[-\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}]$$

$$\mathbf{N}(1) = \frac{-\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}}{\sqrt{14}} = \frac{-\sqrt{14}}{14}\mathbf{i} + \frac{2\sqrt{14}}{14}\mathbf{j} - \frac{3\sqrt{14}}{14}\mathbf{k}$$

**28.**  $\mathbf{r}(t) = \sqrt{2}t\mathbf{i} + e^t\mathbf{j} + e^{-t}\mathbf{k}, t = 0$

**29.**  $\mathbf{r}(t) = 6 \cos t\mathbf{i} + 6 \sin t\mathbf{j} + \mathbf{k}, t = \frac{3\pi}{4}$

$$\mathbf{r}'(t) = -6 \sin t\mathbf{i} + 6 \cos t\mathbf{j}$$

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = -\sin t\mathbf{i} + \cos t\mathbf{j}$$

$$\mathbf{T}'(t) = -\cos t\mathbf{i} - \sin t\mathbf{j}, \|\mathbf{T}'(t)\| = 1$$

$$\mathbf{N}\left(\frac{3\pi}{4}\right) = \frac{\mathbf{T}\left(\frac{3\pi}{4}\right)}{\left\|\mathbf{T}\left(\frac{3\pi}{4}\right)\right\|} = \frac{\sqrt{2}}{2}\mathbf{i} - \frac{\sqrt{2}}{2}\mathbf{j}$$

**30.**  $\mathbf{r}(t) = \cos 3t\mathbf{i} + 2 \sin 3t\mathbf{j} + \mathbf{k}, t = \pi$

$$\mathbf{r}'(t) = -3 \sin 3t\mathbf{i} + 6 \cos 3t\mathbf{j}$$

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = \frac{-3 \sin 3t\mathbf{i} + 6 \cos 3t\mathbf{j}}{\sqrt{9 \sin^2 3t + 36 \cos^2 3t}}$$

The normal vector is perpendicular to  $\mathbf{T}(t)$  and points toward the  $z$ -axis:

$$\mathbf{N}(t) = \frac{-6 \cos 3t\mathbf{i} - 3 \sin 3t\mathbf{j}}{\sqrt{9 \sin^2 3t + 36 \cos^2 3t}}$$

$$\mathbf{N}(\pi) = \frac{6\mathbf{i}}{\sqrt{36}} = \mathbf{i}$$

31.  $\mathbf{r}(t) = 4t\mathbf{i}$

$$\mathbf{v}(t) = 4\mathbf{i}$$

$$\mathbf{a}(t) = \mathbf{0}$$

$$\mathbf{T}(t) = \frac{\mathbf{v}(t)}{\|\mathbf{v}(t)\|} = \frac{4\mathbf{i}}{4} = \mathbf{i}$$

$$\mathbf{T}'(t) = \mathbf{0}$$

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|} \text{ is undefined.}$$

The path is a line and the speed is constant.

32.  $\mathbf{r}(t) = 4t\mathbf{i} - 2t\mathbf{j}$

$$\mathbf{v}(t) = 4\mathbf{i} - 2\mathbf{j}$$

$$\mathbf{a}(t) = \mathbf{0}$$

$$\mathbf{T}(t) = \frac{\mathbf{v}(t)}{\|\mathbf{v}(t)\|} = \frac{1}{\sqrt{5}}(2\mathbf{i} - \mathbf{j})$$

$$\mathbf{T}'(t) = \mathbf{0}$$

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|} \text{ is undefined.}$$

The path is a line and the speed is constant.

33.  $\mathbf{r}(t) = 4t^2\mathbf{i}$

$$\mathbf{v}(t) = 8t\mathbf{i}$$

$$\mathbf{a}(t) = 8\mathbf{i}$$

$$\mathbf{T}(t) = \frac{\mathbf{v}(t)}{\|\mathbf{v}(t)\|} = \frac{8t\mathbf{i}}{8t} = \mathbf{i}$$

$$\mathbf{T}'(t) = \mathbf{0}$$

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|} \text{ is undefined.}$$

The path is a line and the speed is variable.

34.  $\mathbf{r}(t) = t^2\mathbf{j} + \mathbf{k}$

$$\mathbf{v}(t) = 2t\mathbf{j}$$

$$\mathbf{a}(t) = 2\mathbf{j}$$

$$\mathbf{T}(t) = \frac{\mathbf{v}(t)}{\|\mathbf{v}(t)\|} = \frac{2t\mathbf{j}}{2t} = \mathbf{j}$$

$$\mathbf{T}'(t) = \mathbf{0}$$

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|} \text{ is undefined.}$$

The path is a line and the speed is variable.

35.  $\mathbf{r}(t) = t\mathbf{i} + \frac{1}{t}\mathbf{j}, \mathbf{v}(t) = \mathbf{i} - \frac{1}{t^2}\mathbf{j}, \mathbf{v}(1) = \mathbf{i} - \mathbf{j},$

$$\mathbf{a}(t) = \frac{2}{t^3}\mathbf{j}, \mathbf{a}(1) = 2\mathbf{j}$$

$$\mathbf{T}(t) = \frac{\mathbf{v}(t)}{\|\mathbf{v}(t)\|} = \frac{t^2}{\sqrt{t^4 + 1}}\left(\mathbf{i} - \frac{1}{t^2}\mathbf{j}\right) = \frac{1}{\sqrt{t^4 + 1}}(t^2\mathbf{i} - \mathbf{j})$$

$$\mathbf{T}(1) = \frac{1}{\sqrt{2}}(\mathbf{i} - \mathbf{j}) = \frac{\sqrt{2}}{2}(\mathbf{i} - \mathbf{j})$$

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|} = \frac{\frac{2t}{(t^4 + 1)^{3/2}}\mathbf{i} + \frac{2t^3}{(t^4 + 1)^{3/2}}\mathbf{j}}{\frac{2t}{(t^4 + 1)}} \\ = \frac{1}{\sqrt{t^4 + 1}}(\mathbf{i} + t^2\mathbf{j})$$

$$\mathbf{N}(1) = \frac{1}{\sqrt{2}}(\mathbf{i} + \mathbf{j}) = \frac{\sqrt{2}}{2}(\mathbf{i} + \mathbf{j})$$

$$a_T = \mathbf{a} \cdot \mathbf{T} = -\sqrt{2}$$

$$a_N = \mathbf{a} \cdot \mathbf{N} = \sqrt{2}$$

36.  $\mathbf{r}(t) = t^2\mathbf{i} + 2t\mathbf{j}, t = 1$

$$\mathbf{v}(t) = 2t\mathbf{i} + 2\mathbf{j}, \mathbf{v}(1) = 2\mathbf{i} + 2\mathbf{j}$$

$$\mathbf{a}(t) = 2\mathbf{i}, \mathbf{a}(1) = 2\mathbf{i}$$

$$\mathbf{T}(t) = \frac{\mathbf{v}(t)}{\|\mathbf{v}(t)\|} = \frac{1}{\sqrt{4t^2 + 4}}(2t\mathbf{i} + 2\mathbf{j}) = \frac{1}{\sqrt{t^2 + 1}}(\mathbf{i} + \mathbf{j})$$

$$\mathbf{T}(1) = \frac{1}{\sqrt{2}}(\mathbf{i} + \mathbf{j}) = \frac{\sqrt{2}}{2}\mathbf{i} + \frac{\sqrt{2}}{2}\mathbf{j}$$

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|} = \frac{\frac{1}{(t^2 + 1)^{3/2}}\mathbf{i} + \frac{-t}{(t^2 + 1)^{3/2}}\mathbf{j}}{\frac{1}{t^2 + 1}} \\ = \frac{1}{\sqrt{t^2 + 1}}(\mathbf{i} + t\mathbf{j})$$

$$\mathbf{N}(1) = \frac{\sqrt{2}}{2}\mathbf{i} - \frac{\sqrt{2}}{2}\mathbf{j}$$

$$a_T = \mathbf{a} \cdot \mathbf{T} = \sqrt{2}$$

$$a_N = \mathbf{a} \cdot \mathbf{N} = \sqrt{2}$$

37.  $\mathbf{r}(t) = (t - t^3)\mathbf{i} + 2t^2\mathbf{j}, t = 1$

$$\mathbf{v}(t) = (1 - 3t^2)\mathbf{i} + 4t\mathbf{j}, \mathbf{v}(1) = -2\mathbf{i} + 4\mathbf{j}$$

$$\mathbf{a}(t) = -6t\mathbf{i} + 4\mathbf{j}, \mathbf{a}(1) = -6\mathbf{i} + 4\mathbf{j}$$

$$\mathbf{T}(t) = \frac{\mathbf{v}(t)}{\|\mathbf{v}(t)\|} = \frac{(1 - 3t^2)\mathbf{i} + 4t\mathbf{j}}{\sqrt{9t^4 + 10t^2 + 1}}$$

$$\mathbf{T}(1) = \frac{-2\mathbf{i} + 4\mathbf{j}}{\sqrt{20}} = \frac{-\mathbf{i} + 2\mathbf{j}}{\sqrt{5}} = \frac{-\sqrt{5}}{5}(\mathbf{i} - 2\mathbf{j})$$

$$\mathbf{T}'(t) = \frac{-16t(3t^2 + 1)}{(9t^4 + 10t^2 + 1)^{3/2}}\mathbf{i} + \frac{4 - 36t^4}{(9t^4 + 10t^2 + 1)^{3/2}}\mathbf{j}$$

$$\mathbf{T}'(1) = \frac{-64}{20^{3/2}}\mathbf{i} + \frac{-32}{20^{3/2}}\mathbf{j}$$

$$\mathbf{N}(1) = \frac{-2\mathbf{i} - \mathbf{j}}{\sqrt{5}}$$

$$a_{\mathbf{T}} = \mathbf{a} \cdot \mathbf{T} = \frac{1}{\sqrt{5}}(6 + 8) = \frac{14\sqrt{5}}{5}$$

$$a_{\mathbf{N}} = \mathbf{a} \cdot \mathbf{N} = \frac{1}{\sqrt{5}}(12 - 4) = \frac{8\sqrt{5}}{5}$$

40.  $\mathbf{r}(t) = e^t\mathbf{i} + e^{-t}\mathbf{j} + t\mathbf{k}, t = 0$

$$\mathbf{v}(t) = e^t\mathbf{i} - e^{-t}\mathbf{j} + \mathbf{k}, \mathbf{v}(0) = \mathbf{i} - \mathbf{j} + \mathbf{k}$$

$$\mathbf{a}(t) = e^t\mathbf{i} + e^{-t}\mathbf{j}, \mathbf{a}(0) = \mathbf{i} + \mathbf{j}$$

$$\mathbf{T}(t) = \frac{\mathbf{v}(t)}{\|\mathbf{v}(t)\|} = \frac{e^t\mathbf{i} - e^{-t}\mathbf{j} + \mathbf{k}}{\sqrt{e^{2t} + e^{-2t} + 1}}$$

$$\mathbf{T}(0) = \frac{\mathbf{i} - \mathbf{j} + \mathbf{k}}{\sqrt{3}}$$

$$\mathbf{T}'(t) = \frac{e^{2t}(e^{2t} + 2)}{(e^{4t} + e^{2t} + 1)^{3/2}}\mathbf{i} + \frac{e^{2t}(2e^{2t} + 1)}{(e^{4t} + e^{2t} + 1)^{3/2}}\mathbf{j} + \frac{e^t(1 - e^{4t})}{(e^{4t} + e^{2t} + 1)^{3/2}}\mathbf{k}$$

$$\mathbf{T}'(0) = \frac{3}{3^{3/2}}\mathbf{i} + \frac{3}{3^{3/2}}\mathbf{j}$$

$$\mathbf{N}(0) = \frac{\sqrt{2}}{2}\mathbf{i} + \frac{\sqrt{2}}{2}\mathbf{j}$$

$$a_{\mathbf{T}} = \mathbf{a} \cdot \mathbf{T} = 0$$

$$a_{\mathbf{N}} = \mathbf{a} \cdot \mathbf{N} = \sqrt{2}$$

38.  $\mathbf{r}(t) = (t^3 - 4t)\mathbf{i} + (t^2 - 1)\mathbf{j}, t = 0$

$$\mathbf{v}(t) = (3t^2 - 4)\mathbf{i} + 2t\mathbf{j}, \mathbf{v}(0) = -4\mathbf{i}$$

$$\mathbf{a}(t) = 6\mathbf{i} + 2\mathbf{j}, \mathbf{a}(0) = 2\mathbf{j}$$

$$\mathbf{T}(t) = \frac{\mathbf{v}(t)}{\|\mathbf{v}(t)\|} = \frac{(3t^2 - 4)\mathbf{i} + 2t\mathbf{j}}{\sqrt{9t^4 - 20t^2 + 16}}$$

$$\mathbf{T}(0) = \frac{-4\mathbf{i}}{\sqrt{16}} = -\mathbf{i}$$

$$\mathbf{T}'(t) = \frac{4t(3t^2 + 4)}{(9t^4 - 20t^2 + 16)^{3/2}}\mathbf{i} + \frac{32 - 18t^4}{(9t^4 - 20t^2 + 16)^{3/2}}\mathbf{j}$$

$$\mathbf{T}'(1) = \frac{32}{16^{3/2}}\mathbf{j} = \frac{1}{2}\mathbf{j}$$

$$\mathbf{N}(1) = \mathbf{j}$$

$$a_{\mathbf{T}} = \mathbf{a} \cdot \mathbf{T} = 0$$

$$a_{\mathbf{N}} = \mathbf{a} \cdot \mathbf{N} = \sqrt{2}$$

39.  $\mathbf{r}(t) = e^t\mathbf{i} + e^{-2t}\mathbf{j}, t = 0$

$$\mathbf{v}(t) = e^t\mathbf{i} - 2e^{-2t}\mathbf{j}, \mathbf{v}(0) = \mathbf{i} - 2\mathbf{j}$$

$$\mathbf{a}(t) = e^t\mathbf{i} + 4e^{-2t}\mathbf{j}, \mathbf{a}(0) = \mathbf{i} + 4\mathbf{j}$$

$$\mathbf{T}(t) = \frac{\mathbf{v}(t)}{\|\mathbf{v}(t)\|} = \frac{e^t\mathbf{i} - 2e^{-2t}\mathbf{j}}{\sqrt{4e^{-4t} + e^{2t}}}$$

$$\mathbf{T}(0) = \frac{\mathbf{i} - 2\mathbf{j}}{\sqrt{5}}$$

$$\mathbf{N}(0) = \frac{2\mathbf{i} + \mathbf{j}}{\sqrt{5}}$$

$$a_{\mathbf{T}} = \mathbf{a} \cdot \mathbf{T} = \frac{1}{\sqrt{5}}(1 - 8) = \frac{-7\sqrt{5}}{5}$$

$$a_{\mathbf{N}} = \mathbf{a} \cdot \mathbf{N} = \frac{1}{\sqrt{5}}(2 + 4) = \frac{6\sqrt{5}}{5}$$

41.  $\mathbf{r}(t) = (e^t \cos t)\mathbf{i} + (e^t \sin t)\mathbf{j}$

$$\mathbf{v}(t) = e^t(\cos t - \sin t)\mathbf{i} + e^t(\cos t + \sin t)\mathbf{j}$$

$$\mathbf{a}(t) = e^t(-2 \sin t)\mathbf{i} + e^t(2 \cos t)\mathbf{j}$$

$$\text{At } t = \frac{\pi}{2}, \mathbf{T} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{1}{\sqrt{2}}(-\mathbf{i} + \mathbf{j}) = \frac{\sqrt{2}}{2}(-\mathbf{i} + \mathbf{j}).$$

Motion along  $\mathbf{r}$  is counterclockwise. So,

$$\mathbf{N} = \frac{1}{\sqrt{2}}(-\mathbf{i} - \mathbf{j}) = -\frac{\sqrt{2}}{2}(\mathbf{i} + \mathbf{j}).$$

$$a_T = \mathbf{a} \cdot \mathbf{T} = \sqrt{2}e^{\pi/2}$$

$$a_N = \mathbf{a} \cdot \mathbf{N} = \sqrt{2}e^{\pi/2}$$

43.  $\mathbf{r}(t_0) = (\cos \omega t_0 + \omega t_0 \sin \omega t_0)\mathbf{i} + (\sin \omega t_0 - \omega t_0 \cos \omega t_0)\mathbf{j}$

$$\mathbf{v}(t_0) = (\omega^2 t_0 \cos \omega t_0)\mathbf{i} + (\omega^2 t_0 \sin \omega t_0)\mathbf{j}$$

$$\mathbf{a}(t_0) = \omega^2[(\cos \omega t_0 - \omega t_0 \sin \omega t_0)\mathbf{i} + (\omega t_0 \cos \omega t_0 + \sin \omega t_0)\mathbf{j}]$$

$$\mathbf{T}(t_0) = \frac{\mathbf{v}}{\|\mathbf{v}\|} = (\cos \omega t_0)\mathbf{i} + (\sin \omega t_0)\mathbf{j}$$

Motion along  $\mathbf{r}$  is counterclockwise. So

$$\mathbf{N}(t_0) = (-\sin \omega t_0)\mathbf{i} + (\cos \omega t_0)\mathbf{j}.$$

$$a_T = \mathbf{a} \cdot \mathbf{T} = \omega^2$$

$$a_N = \mathbf{a} \cdot \mathbf{N} = \omega^2(\omega t_0) = \omega^3 t_0$$

44.  $\mathbf{r}(t_0) = (\omega t_0 - \sin \omega t_0)\mathbf{i} + (1 - \cos \omega t_0)\mathbf{j}$

$$\mathbf{v}(t_0) = \omega[(1 - \cos \omega t_0)\mathbf{i} + (\sin \omega t_0)\mathbf{j}]$$

$$\mathbf{a}(t_0) = \omega^2[(\sin \omega t_0)\mathbf{i} + (\cos \omega t_0)\mathbf{j}]$$

$$\mathbf{T} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{(1 - \cos \omega t_0)\mathbf{i} + (\sin \omega t_0)\mathbf{j}}{\sqrt{2}\sqrt{1 - \cos \omega t_0}}$$

Motion along  $\mathbf{r}$  is clockwise. So,

$$\mathbf{N} = \frac{(\sin \omega t_0)\mathbf{i} - (1 - \cos \omega t_0)\mathbf{j}}{\sqrt{2}\sqrt{1 - \cos \omega t_0}}$$

$$a_T = \mathbf{a} \cdot \mathbf{T} = \frac{\omega^2 \sin \omega t_0}{\sqrt{2}\sqrt{1 - \cos \omega t_0}} = \frac{\omega^2}{\sqrt{2}}\sqrt{1 + \cos \omega t_0}$$

$$a_N = \mathbf{a} \cdot \mathbf{N} = \frac{\omega^2}{\sqrt{2}}\sqrt{1 - \cos \omega t_0}$$

45.  $\mathbf{r}(t) = a \cos \omega t\mathbf{i} + a \sin \omega t\mathbf{j}$

$$\mathbf{v}(t) = -a\omega \sin \omega t\mathbf{i} + a\omega \cos \omega t\mathbf{j}$$

$$\mathbf{a}(t) = -a\omega^2 \cos \omega t\mathbf{i} - a\omega^2 \sin \omega t\mathbf{j}$$

$$\mathbf{T}(t) = \frac{\mathbf{v}(t)}{\|\mathbf{v}(t)\|} = -\sin \omega t\mathbf{i} + \cos \omega t\mathbf{j}$$

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|} = -\cos \omega t\mathbf{i} - \sin \omega t\mathbf{j}$$

$$a_T = \mathbf{a} \cdot \mathbf{T} = 0$$

$$a_N = \mathbf{a} \cdot \mathbf{N} = a\omega^2$$

42.  $\mathbf{r}(t) = a \cos(\omega t)\mathbf{i} + b \sin(\omega t)\mathbf{j}$

$$\mathbf{v}(t) = -a\omega \sin(\omega t)\mathbf{i} + b\omega \cos(\omega t)\mathbf{j}$$

$$\mathbf{a}(t) = -a\omega^2 \cos(\omega t)\mathbf{i} - b\omega^2 \sin(\omega t)\mathbf{j}$$

$$\mathbf{a}(0) = -a\omega^2\mathbf{i}$$

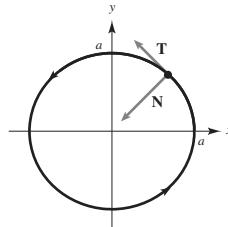
$$\mathbf{T}(0) = \frac{\mathbf{v}(0)}{\|\mathbf{v}(0)\|} = \mathbf{j}$$

Motion along  $\mathbf{r}(t)$  is counterclockwise. So,  $\mathbf{N}(0) = -\mathbf{i}$ .

$$a_T = \mathbf{a} \cdot \mathbf{T} = 0$$

$$a_N = \mathbf{a} \cdot \mathbf{N} = a\omega^2$$

46.  $\mathbf{T}(t)$  points in the direction that  $\mathbf{r}$  is moving.  $\mathbf{N}(t)$  points in the direction that  $\mathbf{r}$  is turning, toward the concave side of the curve.



47. Speed:  $\|\mathbf{v}(t)\| = a\omega$

The speed is constant because  $a_T = 0$ .

48. If the angular velocity  $\omega$  is halved,

$$a_N = a\left(\frac{\omega}{2}\right)^2 = \frac{a\omega^2}{4}.$$

$a_N$  is changed by a factor of  $\frac{1}{4}$ .

49.  $\mathbf{r}(t) = t\mathbf{i} + \frac{1}{t}\mathbf{j}, t_0 = 2$

$$x = t, y = \frac{1}{t} \Rightarrow xy = 1$$

$$\mathbf{r}'(t) = \mathbf{i} - \frac{1}{t^2}\mathbf{j}$$

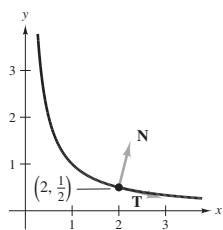
$$\mathbf{T}(t) = \frac{t^2\mathbf{i} - \mathbf{j}}{\sqrt{t^4 + 1}}$$

$$\mathbf{N}(t) = \frac{\mathbf{i} + t^2\mathbf{j}}{\sqrt{t^4 + 1}}$$

$$\mathbf{r}(2) = 2\mathbf{i} + \frac{1}{2}\mathbf{j}$$

$$\mathbf{T}(2) = \frac{\sqrt{17}}{17}(4\mathbf{i} - \mathbf{j})$$

$$\mathbf{N}(2) = \frac{\sqrt{17}}{17}(\mathbf{i} + 4\mathbf{j})$$



50.  $\mathbf{r}(t) = t^3\mathbf{i} + t\mathbf{j}, t_0 = 1$

$$x = t^3, y = t \Rightarrow x = y^3 \text{ or } y = x^{1/3}$$

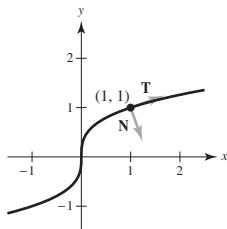
$$\mathbf{r}'(t) = 3t^2\mathbf{i} + \mathbf{j}$$

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = \frac{3t^2\mathbf{i} + \mathbf{j}}{\sqrt{9t^4 + 1}}$$

$$\mathbf{T}(1) = \frac{3\mathbf{i} + \mathbf{j}}{\sqrt{10}}$$

$$= \frac{3\sqrt{10}}{10}\mathbf{i} + \frac{\sqrt{10}}{10}\mathbf{j}$$

$$\mathbf{N}(1) = \frac{\sqrt{10}}{10}\mathbf{i} - \frac{3\sqrt{10}}{10}\mathbf{j}$$



51.  $\mathbf{r}(t) = 4t\mathbf{i} + 4t^2\mathbf{j}, t_0 = \frac{1}{4}$

$$x = 4t,$$

$$y = 4t^2 = 4\left(\frac{x}{4}\right)^2 = \frac{x^2}{4}$$

$$\mathbf{r}\left(\frac{1}{4}\right) = \mathbf{i} + \frac{1}{4}\mathbf{j}$$

$$\mathbf{r}'(t) = 4\mathbf{i} + 8t\mathbf{j}$$

$$\mathbf{T}(t) = \frac{4\mathbf{i} + 8t\mathbf{j}}{\sqrt{16 + 64t^2}} = \frac{\mathbf{i} + 2t\mathbf{j}}{\sqrt{1 + 4t^2}}$$

$$\mathbf{T}\left(\frac{1}{4}\right) = \frac{\mathbf{i} + \frac{1}{2}\mathbf{j}}{\sqrt{1 + \frac{1}{4}}} = \frac{2\mathbf{i} + \mathbf{j}}{\sqrt{5}} = \frac{2\sqrt{5}}{5}\mathbf{i} + \frac{\sqrt{5}}{5}\mathbf{j}$$

$\mathbf{N}\left(\frac{1}{4}\right)$  is perpendicular to  $\mathbf{T}\left(\frac{1}{4}\right)$ :

$$\mathbf{N}\left(\frac{1}{4}\right) = \frac{-\mathbf{i} + 2\mathbf{j}}{\sqrt{5}}$$

52.  $\mathbf{r}(t) = (2t + 1)\mathbf{i} - t^2\mathbf{j}, t_0 = 2$

$$x = 2t + 1,$$

$$y = -t^2 = -\left(\frac{x - 1}{2}\right)^2$$

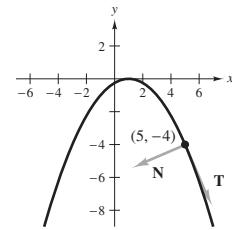
$$\mathbf{r}(2) = 5\mathbf{i} - 4\mathbf{j}$$

$$\mathbf{r}'(t) = 2\mathbf{i} - 2t\mathbf{j}$$

$$\mathbf{T}(t) = \frac{2\mathbf{i} - 2t\mathbf{j}}{\sqrt{4 + 4t^2}} = \frac{\mathbf{i} - t\mathbf{j}}{\sqrt{1 + t^2}}$$

$$\mathbf{T}(2) = \frac{\mathbf{i} - 2\mathbf{j}}{\sqrt{5}}$$

$$\mathbf{N}(2) = \frac{-2\mathbf{i} - \mathbf{j}}{\sqrt{5}}, \text{ perpendicular to } \mathbf{T}(2)$$



53.  $\mathbf{r}(t) = 2 \cos t\mathbf{i} + 2 \sin t\mathbf{j}, t_0 = \frac{\pi}{4}$

$$x = 2 \cos t, y = 2 \sin t \Rightarrow x^2 + y^2 = 4$$

$$\mathbf{r}'(t) = -2 \sin t\mathbf{i} + 2 \cos t\mathbf{j}$$

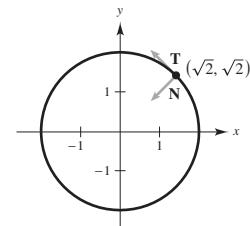
$$\mathbf{T}(t) = \frac{1}{2}(-2 \sin t\mathbf{i} + 2 \cos t\mathbf{j}) = -\sin t\mathbf{i} + \cos t\mathbf{j}$$

$$\mathbf{N}(t) = -\cos t\mathbf{i} - \sin t\mathbf{j}$$

$$\mathbf{r}\left(\frac{\pi}{4}\right) = \sqrt{2}\mathbf{i} + \sqrt{2}\mathbf{j}$$

$$\mathbf{T}\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}(-\mathbf{i} + \mathbf{j})$$

$$\mathbf{N}\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}(-\mathbf{i} - \mathbf{j})$$



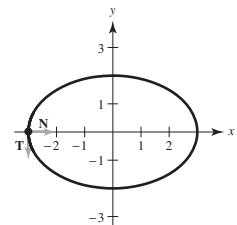
54.  $\mathbf{r}(t) = 3 \cos t\mathbf{i} + 2 \sin t\mathbf{j}, t_0 = \pi$

$$\frac{x}{3} = \cos t, \frac{y}{2} = \sin t \Rightarrow \frac{x^2}{9} + \frac{y^2}{4} = 1, \text{ Ellipse}$$

$$\mathbf{r}'(t) = -3 \sin t\mathbf{i} + 2 \cos t\mathbf{j}$$

$$\mathbf{r}'(\pi) = -2\mathbf{j} \Rightarrow \mathbf{T}(\pi) = -\mathbf{j}$$

$$\mathbf{N}(\pi) = \mathbf{i}$$



**55.**  $\mathbf{r}(t) = t\mathbf{i} + 2t\mathbf{j} - 3t\mathbf{k}, t = 1$

$$\mathbf{v}(t) = \mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$$

$$\mathbf{a}(t) = \mathbf{0}$$

$$\mathbf{T}(t) = \frac{\mathbf{v}}{\|\mathbf{v}\|}$$

$$= \frac{1}{\sqrt{14}}(\mathbf{i} + 2\mathbf{j} - 3\mathbf{k})$$

$$= \frac{\sqrt{14}}{14}(\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}) = \mathbf{T}(1)$$

$$\mathbf{N}(t) = \frac{\mathbf{T}'}{\|\mathbf{T}'\|} \text{ is undefined.}$$

$a_T, a_N$  are not defined.

**56.**  $\mathbf{r}(t) = 4t\mathbf{i} - 4t\mathbf{j} + 2t\mathbf{k}$

$$\mathbf{v}(t) = 4\mathbf{i} - 4\mathbf{j} + 2\mathbf{k}$$

$$\mathbf{a}(t) = \mathbf{0}$$

$$\mathbf{T}(t) = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{1}{3}(2\mathbf{i} - 2\mathbf{j} + \mathbf{k})$$

$$\mathbf{N}(t) = \frac{\mathbf{T}'}{\|\mathbf{T}'\|} \text{ is undefined.}$$

$a_T, a_N$  are not defined.

**58.**  $\mathbf{r}(t) = 3t\mathbf{i} - t\mathbf{j} + t^2\mathbf{k}, t = -1$

$$\mathbf{v}(t) = 3\mathbf{i} - \mathbf{j} + 2t\mathbf{k}, \|\mathbf{v}(t)\| = \sqrt{10 + 4t^2}$$

$$\mathbf{a}(t) = 2\mathbf{k}, \|\mathbf{a}\| = 2$$

$$\mathbf{T}(t) = \frac{\mathbf{v}(t)}{\|\mathbf{v}(t)\|} = \frac{3\mathbf{i} - \mathbf{j} + 2t\mathbf{k}}{\sqrt{10 + 4t^2}}$$

$$\mathbf{a}(-1) = 2\mathbf{k}, \mathbf{T}(-1) = \frac{1}{\sqrt{14}}(3\mathbf{i} - \mathbf{j} - 2\mathbf{k})$$

$$a_T = \mathbf{a}(-1) \cdot \mathbf{T}(-1) = \frac{-4}{\sqrt{14}}$$

$$a_N = \sqrt{\|\mathbf{a}\|^2 - a_T^2} = \sqrt{4 - \frac{16}{14}} = \sqrt{\frac{20}{7}} = \frac{2\sqrt{35}}{7}$$

$$\mathbf{a} = 2\mathbf{k} = a_T \mathbf{T} + a_N \mathbf{N} = \frac{-4}{\sqrt{14}} \left[ \frac{1}{\sqrt{14}}(3\mathbf{i} - \mathbf{j} - 2\mathbf{k}) \right] + \frac{2\sqrt{35}}{7} \mathbf{N}$$

$$\mathbf{N} = \frac{7}{2\sqrt{35}} \left[ 2\mathbf{k} + \frac{2}{7}(3\mathbf{i} - \mathbf{j} - 2\mathbf{k}) \right] = \frac{\sqrt{35}}{10} \left[ \frac{6}{7}\mathbf{i} - \frac{2}{7}\mathbf{j} + \frac{10}{7}\mathbf{k} \right]$$

**57.**  $\mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j} + 2t\mathbf{k}, t = \frac{\pi}{3}$

$$\mathbf{v}(t) = -\sin t\mathbf{i} + \cos t\mathbf{j} + 2\mathbf{k}$$

$$\mathbf{a}(t) = -\cos t\mathbf{i} - \sin t\mathbf{j}, \|\mathbf{a}\| = 1$$

$$\|\mathbf{v}(t)\| = \sqrt{\sin^2 t + \cos^2 t + 4} = \sqrt{5}$$

$$\mathbf{T}(t) = \frac{\mathbf{v}(t)}{\|\mathbf{v}(t)\|} = \frac{1}{\sqrt{5}}(-\sin t\mathbf{i} + \cos t\mathbf{j} + 2\mathbf{k})$$

$$a_T = \mathbf{a}(t) \cdot \mathbf{T}(t) = 0$$

$$a_N = \sqrt{\|\mathbf{a}\|^2 - a_T^2} = \sqrt{1 - 0} = 1$$

$$\mathbf{T}\left(\frac{\pi}{3}\right) = \frac{1}{\sqrt{5}} \left( -\frac{\sqrt{3}}{2}\mathbf{i} + \frac{1}{2}\mathbf{j} + 2\mathbf{k} \right)$$

$$\mathbf{a}\left(\frac{\pi}{3}\right) = -\frac{1}{2}\mathbf{i} - \frac{\sqrt{3}}{2}\mathbf{j} = a_T \mathbf{T} + a_N \mathbf{N} = \mathbf{N}$$

59.  $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + \frac{t^2}{2}\mathbf{k}, t = 1$

$$\mathbf{v}(t) = \mathbf{i} + 2t\mathbf{j} + t\mathbf{k}$$

$$\mathbf{v}(1) = \mathbf{i} + 2\mathbf{j} + \mathbf{k}$$

$$\mathbf{a}(t) = 2\mathbf{j} + \mathbf{k}$$

$$\mathbf{T}(t) = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{1}{\sqrt{1+5t^2}}(\mathbf{i} + 2t\mathbf{j} + t\mathbf{k})$$

$$\mathbf{T}(1) = \frac{\sqrt{6}}{6}(\mathbf{i} + 2\mathbf{j} + \mathbf{k})$$

$$\mathbf{N}(t) = \frac{\mathbf{T}'}{\|\mathbf{T}'\|} = \frac{\frac{-5\mathbf{i} + 2\mathbf{j} + \mathbf{k}}{(1+5t^2)^{3/2}}}{\frac{\sqrt{5}}{1+5t^2}} = \frac{-5\mathbf{i} + 2\mathbf{j} + \mathbf{k}}{\sqrt{5}\sqrt{1+5t^2}}$$

$$\mathbf{N}(1) = \frac{\sqrt{30}}{30}(-5\mathbf{i} + 2\mathbf{j} + \mathbf{k})$$

$$a_T = \mathbf{a} \cdot \mathbf{T} = \frac{5\sqrt{6}}{6}$$

$$a_N = \mathbf{a} \cdot \mathbf{N} = \frac{\sqrt{30}}{6}$$

60.  $\mathbf{r}(t) = (2t-1)\mathbf{i} + t^2\mathbf{j} - 4t\mathbf{k}, t = 2$

$$\mathbf{v}(t) = 2\mathbf{i} + 2t\mathbf{j} - 4\mathbf{k},$$

$$\|\mathbf{v}(t)\| = \sqrt{20 + 4t^2} = 2\sqrt{5 + t^2}$$

$$\mathbf{a}(t) = 2\mathbf{j}, \|\mathbf{a}\| = 2$$

$$\mathbf{T}(t) = \frac{2\mathbf{i} + 2t\mathbf{j} - 4\mathbf{k}}{2\sqrt{5+t^2}} = \frac{1}{\sqrt{5+t^2}}(\mathbf{i} + 2t\mathbf{j} - 2\mathbf{k})$$

$$\mathbf{a}(2) = 2\mathbf{j}, \mathbf{T}(2) = \frac{1}{3}(\mathbf{i} + 2\mathbf{j} - 2\mathbf{k})$$

$$a_T = \mathbf{a}(2) \cdot \mathbf{T}(2) = \frac{4}{3}$$

$$a_N = \sqrt{\|\mathbf{a}\|^2 - a_T^2} = \sqrt{4 - \frac{16}{9}} = \frac{2}{3}\sqrt{5}$$

$$\mathbf{a} = 2\mathbf{j} = a_T \mathbf{T} + a_N \mathbf{N} = \frac{4}{9}(\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}) + \frac{2}{3}\sqrt{5}\mathbf{N}$$

$$\mathbf{N} = \frac{3}{2\sqrt{5}} \left[ 2\mathbf{j} - \frac{4}{9}(\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}) \right]$$

$$= \frac{-2\sqrt{5}}{15}\mathbf{i} + \frac{\sqrt{5}}{3}\mathbf{j} + \frac{4\sqrt{5}}{15}\mathbf{k}$$

61.  $\mathbf{r}(t) = e^t \sin t\mathbf{i} + e^t \cos t\mathbf{j} + e^t\mathbf{k}$

$$\mathbf{v}(t) = (e^t \cos t + e^t \sin t)\mathbf{i} + (-e^t \sin t + e^t \cos t)\mathbf{j} + e^t\mathbf{k}$$

$$\mathbf{v}(0) = \mathbf{i} + \mathbf{j} + \mathbf{k}$$

$$\mathbf{a}(t) = 2e^t \cos t\mathbf{i} - 2e^t \sin t\mathbf{j} + e^t\mathbf{k}$$

$$\mathbf{a}(0) = 2\mathbf{i} + \mathbf{k}$$

$$\mathbf{T}(t) = \frac{\mathbf{v}}{\|\mathbf{v}\|}$$

$$= \frac{1}{\sqrt{3}}[(\cos t + \sin t)\mathbf{i} + (-\sin t + \cos t)\mathbf{j} + \mathbf{k}]$$

$$\mathbf{T}(0) = \frac{1}{\sqrt{3}}[\mathbf{i} + \mathbf{j} + \mathbf{k}]$$

$$\mathbf{N}(t) = \frac{1}{\sqrt{2}}[(-\sin t + \cos t)\mathbf{i} + (-\cos t - \sin t)\mathbf{j}]$$

$$\mathbf{N}(0) = \frac{\sqrt{2}}{2}\mathbf{i} - \frac{\sqrt{2}}{2}\mathbf{j}$$

$$a_T = \mathbf{a} \cdot \mathbf{T} = \sqrt{3}$$

$$a_N = \mathbf{a} \cdot \mathbf{N} = \sqrt{2}$$

62.  $\mathbf{r}(t) = e^t\mathbf{i} + 2t\mathbf{j} + e^{-t}\mathbf{k}, t = 0$

$$\mathbf{v}(t) = e^t\mathbf{i} + 2\mathbf{j} - e^{-t}\mathbf{k}, \|\mathbf{v}(t)\| = \sqrt{e^{2t} + 4 + e^{-2t}}$$

$$\mathbf{a}(t) = e^t\mathbf{i} + e^{-t}\mathbf{k}, \|\mathbf{a}(t)\| = \sqrt{e^{2t} + e^{-2t}}$$

$$\mathbf{v}(0) = \mathbf{i} + 2\mathbf{j} - \mathbf{k}, \mathbf{a}(0) = \mathbf{i} + \mathbf{k}$$

$$\mathbf{T}(0) = \frac{\mathbf{i} + 2\mathbf{j} - \mathbf{k}}{\sqrt{6}}$$

$$a_T = \mathbf{a} \cdot \mathbf{T} = 0$$

$$a_N = \sqrt{\|\mathbf{a}\|^2 - a_T^2} = \sqrt{2}$$

$$\mathbf{a} = \mathbf{i} + \mathbf{k} = a_T \mathbf{T} + a_N \mathbf{N} = \sqrt{2}\mathbf{N}$$

$$\Rightarrow \mathbf{N} = \frac{1}{\sqrt{2}}(\mathbf{i} + \mathbf{k})$$

63.  $\mathbf{r}(t) = 4\mathbf{i} + 3 \cos t\mathbf{j} + 3 \sin t\mathbf{k}, t = \frac{\pi}{2}$

$$\mathbf{v}(t) = 4\mathbf{i} - 3 \sin t\mathbf{j} + 3 \cos t\mathbf{k}$$

$$\mathbf{v}\left(\frac{\pi}{2}\right) = 4\mathbf{i} - 3\mathbf{j}$$

$$\mathbf{a}(t) = -3 \cos t\mathbf{j} - 3 \sin t\mathbf{k}$$

$$\mathbf{a}\left(\frac{\pi}{2}\right) = -3\mathbf{k}$$

$$\mathbf{T}(t) = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{1}{5}(4\mathbf{i} - 3 \sin t\mathbf{j} + 3 \cos t\mathbf{k})$$

$$\mathbf{T}\left(\frac{\pi}{2}\right) = \frac{1}{5}(4\mathbf{i} - 3\mathbf{j})$$

$$\mathbf{N}(t) = \frac{\mathbf{T}'}{\|\mathbf{T}'\|} = -\cos t\mathbf{j} - \sin t\mathbf{k}$$

$$\mathbf{N}\left(\frac{\pi}{2}\right) = -\mathbf{k}$$

$$a_T = \mathbf{a} \cdot \mathbf{T} = 0$$

$$a_N = \mathbf{a} \cdot \mathbf{N} = 3$$

64.  $\mathbf{r}(t) = (2 + \cos t)\mathbf{i} + (1 - \sin t)\mathbf{j} + \frac{t}{3}\mathbf{k}, t = \pi$

$$\mathbf{v}(t) = -\sin t\mathbf{i} - \cos t\mathbf{j} + \frac{1}{3}\mathbf{k}, \|\mathbf{v}(t)\| = \frac{\sqrt{10}}{3}$$

$$\mathbf{v}(\pi) = \mathbf{j} + \frac{1}{3}\mathbf{k}$$

$$\mathbf{a}(t) = -\cos t\mathbf{i} + \sin t\mathbf{j}$$

$$\mathbf{a}(\pi) = \mathbf{i}$$

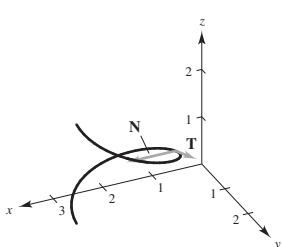
$$\mathbf{T}(t) = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{3\sqrt{10}}{10}\left(-\sin t\mathbf{i} - \cos t\mathbf{j} + \frac{1}{3}\mathbf{k}\right)$$

$$\mathbf{T}(\pi) = \frac{3\sqrt{10}}{10}\left(\mathbf{j} + \frac{1}{3}\mathbf{k}\right)$$

$$\mathbf{N}(t) = \frac{\mathbf{T}'}{\|\mathbf{T}'\|} = \frac{\frac{3\sqrt{10}}{10}(-\cos t\mathbf{i} + \sin t\mathbf{j})}{\frac{3\sqrt{10}}{10}} = -\cos t\mathbf{i} + \sin t\mathbf{j}$$

$$\mathbf{N}(\pi) = \mathbf{i}$$

$$a_T = \mathbf{a} \cdot \mathbf{T} = 0, a_N = \mathbf{a} \cdot \mathbf{N} = 1$$



65.  $\mathbf{r}(t) = t\mathbf{i} + 3t^2\mathbf{j} + \frac{t^2}{2}\mathbf{k}$

$$\mathbf{v}(t) = \mathbf{i} + 6t\mathbf{j} + t\mathbf{k}$$

$$\mathbf{v}(2) = \mathbf{i} + 12\mathbf{j} + 2\mathbf{k}$$

$$\mathbf{a}(t) = 6\mathbf{j} + \mathbf{k}$$

$$\mathbf{T}(t) = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{1}{\sqrt{1+37t^2}}(\mathbf{i} + 6t\mathbf{j} + t\mathbf{k})$$

$$\mathbf{T}(2) = \frac{1}{\sqrt{149}}(\mathbf{i} + 12\mathbf{j} + 2\mathbf{k})$$

$$\mathbf{N}(t) = \frac{\mathbf{T}'}{\|\mathbf{T}'\|}$$

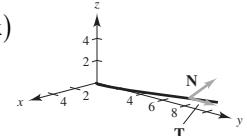
$$= \frac{1}{(1+37t^2)^{3/2}}[-37\mathbf{i} + 6\mathbf{j} + \mathbf{k}]$$

$$= \frac{\sqrt{37}}{1+37^2}[-37\mathbf{i} + 6\mathbf{j} + \mathbf{k}]$$

$$\mathbf{N}(2) = \frac{1}{\sqrt{37}\sqrt{149}}[-74\mathbf{i} + 6\mathbf{j} + \mathbf{k}]$$

$$= \frac{1}{\sqrt{5513}}(-74\mathbf{i} + 6\mathbf{j} + \mathbf{k})$$

$$a_T = \mathbf{a} \cdot \mathbf{T} = \frac{74}{\sqrt{149}}$$



66.  $\mathbf{r}(t) = t^2\mathbf{i} + \mathbf{j} + 2t\mathbf{k}, t = 1$

$$\mathbf{v}(t) = 2t\mathbf{i} + 2\mathbf{k}$$

$$\mathbf{v}(1) = 2\mathbf{i} + 2\mathbf{k}$$

$$\mathbf{a}(t) = 2\mathbf{i}$$

$$\mathbf{T}(t) = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{1}{\sqrt{4t^2+4}}(2t\mathbf{i} + 2\mathbf{k}) = \frac{t\mathbf{i} + \mathbf{k}}{\sqrt{t^2+1}}$$

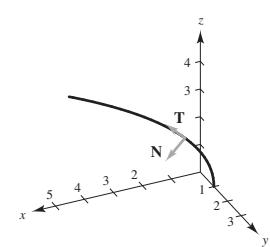
$$\mathbf{T}(1) = \frac{1}{\sqrt{2}}(\mathbf{i} + \mathbf{k})$$

$$\mathbf{N}(t) = \frac{\mathbf{T}'}{\|\mathbf{T}'\|} = \frac{(\mathbf{i} - t\mathbf{k})/(t^2+1)^{3/2}}{\sqrt{1+t^2}/(t^2+1)^{3/2}} = \frac{i - t\mathbf{k}}{\sqrt{t^2+1}}$$

$$\mathbf{N}(1) = \frac{1}{\sqrt{2}}(\mathbf{i} - \mathbf{k})$$

$$a_T = \mathbf{a} \cdot \mathbf{T} = \frac{2}{\sqrt{2}} = \sqrt{2}$$

$$a_N = \mathbf{a} \cdot \mathbf{N} = \frac{2}{\sqrt{2}} = \sqrt{2}$$



67. Let  $C$  be a smooth curve represented by  $\mathbf{r}$  on an open interval  $I$ . The unit tangent vector  $\mathbf{T}(t)$  at  $t$  is defined as

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|}, \mathbf{r}'(t) \neq 0.$$

The principal unit normal vector  $\mathbf{N}(t)$  at  $t$  is defined as

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|}, \mathbf{T}'(t) \neq 0.$$

The tangential and normal components of acceleration are defined as  $\mathbf{a}(t) = a_T \mathbf{T}(t) + a_N \mathbf{N}(t)$ .

71.  $\mathbf{r}(t) = \langle \pi t - \sin \pi t, 1 - \cos \pi t \rangle$

The graph is a cycloid.

$$(a) \mathbf{r}(t) = \langle \pi t - \sin \pi t, 1 - \cos \pi t \rangle$$

$$\mathbf{v}(t) = \langle \pi - \pi \cos \pi t, \pi \sin \pi t \rangle$$

$$\mathbf{a}(t) = \langle \pi^2 \sin \pi t, \pi^2 \cos \pi t \rangle$$

$$\mathbf{T}(t) = \frac{\mathbf{v}(t)}{\|\mathbf{v}(t)\|} = \frac{1}{\sqrt{2(1 - \cos \pi t)}} \langle 1 - \cos \pi t, \sin \pi t \rangle$$

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|} = \frac{1}{\sqrt{2(1 - \cos \pi t)}} \langle \sin \pi t, -1 + \cos \pi t \rangle$$

$$a_T = \mathbf{a} \cdot \mathbf{T} = \frac{1}{\sqrt{2(1 - \cos \pi t)}} [\pi^2 \sin \pi t(1 - \cos \pi t) + \pi^2 \cos \pi t \sin \pi t] = \frac{\pi^2 \sin \pi t}{\sqrt{2(1 - \cos \pi t)}}$$

$$a_N = \mathbf{a} \cdot \mathbf{N} = \frac{1}{\sqrt{2(1 - \cos \pi t)}} [\pi^2 \sin^2 \pi t + \pi^2 \cos \pi t(-1 + \cos \pi t)] = \frac{\pi^2(1 - \cos \pi t)}{\sqrt{2(1 - \cos \pi t)}} = \frac{\pi^2 \sqrt{2(1 - \cos \pi t)}}{2}$$

$$\text{When } t = \frac{1}{2}: a_T = \frac{\pi^2}{\sqrt{2}} = \frac{\sqrt{2}\pi^2}{2}, a_N = \frac{\sqrt{2}\pi^2}{2}$$

$$\text{When } t = 1: a_T = 0, a_N = \pi^2$$

$$\text{When } t = \frac{3}{2}: a_T = -\frac{\sqrt{2}\pi^2}{2}, a_N = \frac{\sqrt{2}\pi^2}{2}$$

$$(b) \text{ Speed: } s = \|\mathbf{v}(t)\| = \pi \sqrt{2(1 - \cos \pi t)}$$

$$\frac{ds}{dt} = \frac{\pi^2 \sin \pi t}{\sqrt{2(1 - \cos \pi t)}} = a_T$$

$$\text{When } t = \frac{1}{2}: a_T = \frac{\sqrt{2}\pi^2}{2} > 0 \Rightarrow \text{the speed is increasing.}$$

$$\text{When } t = 1: a_T = 0 \Rightarrow \text{the height is maximum.}$$

$$\text{When } t = \frac{3}{2}: a_T = -\frac{\sqrt{2}\pi^2}{2} < 0 \Rightarrow \text{the speed is decreasing.}$$

68. The unit tangent vector points in the direction of motion.

69. (a) If  $a_N = 0$ , then the motion is in a straight line.

- (b) If  $a_T = 0$ , then the speed is constant.

70.  $\mathbf{r}(t) = 3t\mathbf{i} + 4t\mathbf{j}$

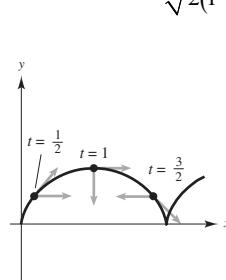
$$\mathbf{v}(t) = \mathbf{r}'(t) = 3\mathbf{i} + 4\mathbf{j}, \|\mathbf{v}(t)\| = \sqrt{9 + 16} = 5$$

$$\mathbf{a}(t) = \mathbf{v}'(t) = \mathbf{0}$$

$$\mathbf{T}(t) = \frac{\mathbf{v}(t)}{\|\mathbf{v}(t)\|} = \frac{3}{5}\mathbf{i} + \frac{4}{5}\mathbf{j}$$

$$\mathbf{T}'(t) = \mathbf{0} \Rightarrow \mathbf{N}(t) \text{ does not exist.}$$

The path is a line. The speed is constant (5).



72. (a)  $\mathbf{r}(t) = \langle \cos \pi t + \pi t \sin \pi t, \sin \pi t - \pi t \cos \pi t \rangle$

$$\mathbf{v}(t) = \langle -\pi \sin \pi t + \pi \sin \pi t + \pi^2 t \cos \pi t, \pi \cos \pi t - \pi \cos \pi t + \pi^2 t \sin \pi t \rangle = \langle \pi^2 t \cos \pi t, \pi^2 t \sin \pi t \rangle$$

$$\mathbf{a}(t) = \langle \pi^2 \cos \pi t - \pi^3 t \sin \pi t, \pi^2 \sin \pi t + \pi^3 t \cos \pi t \rangle$$

$$\mathbf{T}(t) = \frac{\mathbf{v}(t)}{\|\mathbf{v}(t)\|} = \langle \cos \pi t, \sin \pi t \rangle$$

$$a_T = \mathbf{a} \cdot \mathbf{T} = \cos \pi t (\pi^2 \cos \pi t - \pi^3 t \sin \pi t) + \sin \pi t (\pi^2 \sin \pi t + \pi^3 t \cos \pi t) = \pi^2$$

$$a_N = \sqrt{\|\mathbf{a}\|^2 - a_T^2} = \sqrt{\pi^4(1 + \pi^2 t^2) - \pi^4} = \pi^3 t$$

When  $t = 1$ ,  $a_T = \pi^2$ ,  $a_N = \pi^3$ . When  $t = 2$ ,  $a_T = \pi^2$ ,  $a_N = 2\pi^3$ .

(b) Because  $a_T = \pi^2 > 0$  for all values of  $t$ , the speed is increasing when  $t = 1$  and  $t = 2$ .

73.  $\mathbf{r}(t) = 2 \cos t \mathbf{i} + 2 \sin t \mathbf{j} + \frac{t}{2} \mathbf{k}$ ,  $t_0 = \frac{\pi}{2}$

$$\mathbf{r}'(t) = -2 \sin t \mathbf{i} + 2 \cos t \mathbf{j} + \frac{1}{2} \mathbf{k}$$

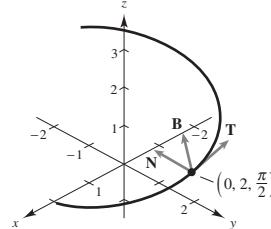
$$\mathbf{T}(t) = \frac{2\sqrt{17}}{17} \left( -2 \sin t \mathbf{i} + 2 \cos t \mathbf{j} + \frac{1}{2} \mathbf{k} \right)$$

$$\mathbf{N}(t) = -\cos t \mathbf{i} - \sin t \mathbf{j}$$

$$\mathbf{r}\left(\frac{\pi}{2}\right) = 2\mathbf{j} + \frac{\pi}{4}\mathbf{k}$$

$$\mathbf{T}\left(\frac{\pi}{2}\right) = \frac{2\sqrt{17}}{17} \left( -2\mathbf{i} + \frac{1}{2}\mathbf{k} \right) = \frac{\sqrt{17}}{17} (-4\mathbf{i} + \mathbf{k})$$

$$\mathbf{N}\left(\frac{\pi}{2}\right) = -\mathbf{j}$$



74.  $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + \frac{t^3}{3}\mathbf{k}$ ,  $t_0 = 1$

$$\mathbf{r}'(t) = \mathbf{i} + 2t\mathbf{j} + t^2\mathbf{k}$$

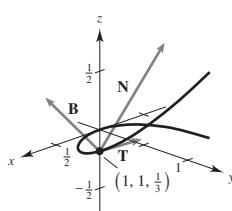
$$\mathbf{T}(t) = \frac{1}{\sqrt{1+4t^2+t^4}} (\mathbf{i} + 2t\mathbf{j} + t^2\mathbf{k})$$

$$\mathbf{N}(t) = \frac{1}{\sqrt{1+4t^2+t^4} \sqrt{1+t^2+t^4}} [(-2t-t^3)\mathbf{i} + (1-t^4)\mathbf{j} + (t+2t^3)\mathbf{k}]$$

$$\mathbf{r}(1) = \mathbf{i} + \mathbf{j} + \frac{1}{3}\mathbf{k}$$

$$\mathbf{T}(1) = \frac{1}{\sqrt{6}} (\mathbf{i} + 2\mathbf{j} + \mathbf{k})$$

$$\mathbf{N}(1) = \frac{1}{\sqrt{6}\sqrt{3}} (-3\mathbf{i} + 3\mathbf{k}) = \frac{\sqrt{2}}{2} (-\mathbf{i} + \mathbf{k})$$



$$\mathbf{B}(1) = \mathbf{T}(1) \times \mathbf{N}(1) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\sqrt{6}}{6} & \frac{\sqrt{6}}{3} & \frac{\sqrt{6}}{6} \\ -\frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \end{vmatrix} = \frac{\sqrt{3}}{3}\mathbf{i} - \frac{\sqrt{3}}{3}\mathbf{j} + \frac{\sqrt{3}}{3}\mathbf{k} = \frac{\sqrt{3}}{3}(\mathbf{i} - \mathbf{j} + \mathbf{k})$$

75.  $\mathbf{r}(t) = \mathbf{i} + \sin t\mathbf{j} + \cos t\mathbf{k}$ ,  $t_0 = \frac{\pi}{4}$

$$\mathbf{r}'(t) = \cos t\mathbf{j} - \sin t\mathbf{k},$$

$$\|\mathbf{r}'(t)\| = 1$$

$$\mathbf{r}\left(\frac{\pi}{4}\right) = \mathbf{T}\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}\mathbf{j} - \frac{\sqrt{2}}{2}\mathbf{k}$$

$$\mathbf{T}'(t) = -\sin t\mathbf{j} - \cos t\mathbf{k},$$

$$\mathbf{N}\left(\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}\mathbf{j} - \frac{\sqrt{2}}{2}\mathbf{k}$$

$$\mathbf{B}\left(\frac{\pi}{4}\right) = \mathbf{T}\left(\frac{\pi}{4}\right) \times \mathbf{N}\left(\frac{\pi}{4}\right) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ 0 & -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{vmatrix} = -\mathbf{i}$$

76.  $\mathbf{r}(t) = 2e^t\mathbf{i} + e^t \cos t\mathbf{j} + e^t \sin t\mathbf{k}$ ,  $t_0 = 0$

$$\mathbf{r}'(t) = 2e^t\mathbf{i} + (e^t \cos t - e^t \sin t)\mathbf{j} + (e^t \sin t + e^t \cos t)\mathbf{k}$$

$$\mathbf{r}'(0) = 2\mathbf{i} + \mathbf{j} + \mathbf{k} \Rightarrow \mathbf{T}(0) = \frac{1}{\sqrt{6}}(2\mathbf{i} + \mathbf{j} + \mathbf{k})$$

$$\begin{aligned} \|\mathbf{r}'(t)\|^2 &= 4e^{2t} + e^{2t} \cos^2 t + e^{2t} \sin^2 t - 2e^{2t} \cos t \sin t + e^{2t} \sin^2 t + e^{2t} \cos^2 t + 2e^{2t} \sin t \cos t \\ &= 4e^{2t} + 2e^{2t}(\cos^2 t + \sin^2 t) = 6e^{2t} \end{aligned}$$

$$\|\mathbf{r}'(t)\| = \sqrt{6}e^t$$

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = \frac{1}{\sqrt{6}}[2\mathbf{i} + (\cos t - \sin t)\mathbf{j} + (\sin t + \cos t)\mathbf{k}]$$

$$\mathbf{T}'(t) = \frac{1}{\sqrt{6}}[(-\sin t - \cos t)\mathbf{j} + (\cos t - \sin t)\mathbf{k}]$$

$$\mathbf{T}'(0) = \frac{1}{\sqrt{6}}[-\mathbf{j} + \mathbf{k}] \Rightarrow \mathbf{N}(0) = \frac{-\sqrt{2}}{2}\mathbf{j} + \frac{\sqrt{2}}{2}\mathbf{k}$$

$$\mathbf{B}(0) = \mathbf{T}(0) \times \mathbf{N}(0) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ 0 & -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{vmatrix} = \frac{\sqrt{3}}{3}\mathbf{i} - \frac{\sqrt{3}}{3}\mathbf{j} - \frac{\sqrt{3}}{3}\mathbf{k}$$

77.  $\mathbf{r}(t) = 4 \sin t\mathbf{i} + 4 \cos t\mathbf{j} + 2t\mathbf{k}, t_0 = \frac{\pi}{3}$

$$\mathbf{r}'(t) = 4 \cos t\mathbf{i} - 4 \sin t\mathbf{j} + 2\mathbf{k},$$

$$\|\mathbf{r}'(t)\| = \sqrt{16 \cos^2 t + 16 \sin^2 t + 4} = \sqrt{20} = 2\sqrt{5}$$

$$\mathbf{r}\left(\frac{\pi}{3}\right) = 2\mathbf{i} - 2\sqrt{3}\mathbf{j} + 2\mathbf{k}$$

$$\mathbf{T}\left(\frac{\pi}{3}\right) = \frac{1}{2\sqrt{5}}(2\mathbf{i} - 2\sqrt{3}\mathbf{j} + 2\mathbf{k}) = \frac{\sqrt{5}}{5}\mathbf{i} - \frac{\sqrt{15}}{5}\mathbf{j} + \frac{\sqrt{5}}{5}\mathbf{k} = \frac{\sqrt{5}}{5}(\mathbf{i} - \sqrt{3}\mathbf{j} + \mathbf{k})$$

$$\mathbf{T}'(t) = \frac{1}{2\sqrt{5}}(-4 \sin t\mathbf{i} - 4 \cos t\mathbf{j})$$

$$\mathbf{N}\left(\frac{\pi}{3}\right) = -\frac{\sqrt{3}}{2}\mathbf{i} - \frac{1}{2}\mathbf{j}$$

$$\mathbf{B}\left(\frac{\pi}{3}\right) = \mathbf{T}\left(\frac{\pi}{3}\right) \times \mathbf{N}\left(\frac{\pi}{3}\right) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\sqrt{5}}{5} & -\frac{\sqrt{15}}{5} & \frac{\sqrt{5}}{5} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \end{vmatrix} = \frac{\sqrt{5}}{10}\mathbf{i} - \frac{\sqrt{15}}{10}\mathbf{j} - \frac{4\sqrt{5}}{10}\mathbf{k} = \frac{\sqrt{5}}{10}(\mathbf{i} - \sqrt{3}\mathbf{j} - 4\mathbf{k})$$

78.  $\mathbf{r}(t) = 3 \cos 2t\mathbf{i} + 3 \sin 2t\mathbf{j} + t\mathbf{k}, t = \frac{\pi}{4}$

$$\mathbf{r}'(t) = -6 \sin 2t\mathbf{i} + 6 \cos 2t\mathbf{j} + \mathbf{k}$$

$$\|\mathbf{r}'(t)\| = \sqrt{37}$$

$$\mathbf{r}'\left(\frac{\pi}{4}\right) = -6\mathbf{i} + \mathbf{k}$$

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = \frac{1}{\sqrt{37}}(-6 \sin 2t\mathbf{i} + 6 \cos 2t\mathbf{j} + \mathbf{k})$$

$$\mathbf{T}'(t) = \frac{1}{\sqrt{37}}(-12 \cos 2t\mathbf{i} - 12 \sin 2t\mathbf{j})$$

$$\mathbf{T}\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{37}}(-6\mathbf{i} + \mathbf{k})$$

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|} = -\cos 2t\mathbf{i} - \sin 2t\mathbf{j}$$

$$\mathbf{N}\left(\frac{\pi}{4}\right) = -j$$

$$\mathbf{B}\left(\frac{\pi}{4}\right) = \mathbf{T}\left(\frac{\pi}{4}\right) \times \mathbf{N}\left(\frac{\pi}{4}\right) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{-6}{\sqrt{37}} & 0 & \frac{1}{\sqrt{37}} \\ 0 & -1 & 0 \end{vmatrix} = \frac{1}{\sqrt{37}}\mathbf{i} + \frac{6}{\sqrt{37}}\mathbf{k}$$

79. From Theorem 12.3 you have:

$$\mathbf{r}(t) = (v_0 t \cos \theta) \mathbf{i} + (h + v_0 t \sin \theta - 16t^2) \mathbf{j}$$

$$\mathbf{v}(t) = v_0 \cos \theta \mathbf{i} + (v_0 \sin \theta - 32t) \mathbf{j}$$

$$\mathbf{a}(t) = -32 \mathbf{j}$$

$$\mathbf{T}(t) = \frac{(v_0 \cos \theta) \mathbf{i} + (v_0 \sin \theta - 32t) \mathbf{j}}{\sqrt{v_0^2 \cos^2 \theta + (v_0 \sin \theta - 32t)^2}}$$

$$\mathbf{N}(t) = \frac{(v_0 \sin \theta - 32t) \mathbf{i} - v_0 \cos \theta \mathbf{j}}{\sqrt{v_0^2 \cos^2 \theta + (v_0 \sin \theta - 32t)^2}} \quad (\text{Motion is clockwise.})$$

$$a_T = \mathbf{a} \cdot \mathbf{T} = \frac{-32(v_0 \sin \theta - 32t)}{\sqrt{v_0^2 \cos^2 \theta + (v_0 \sin \theta - 32t)^2}}$$

$$a_N = \mathbf{a} \cdot \mathbf{N} = \frac{32v_0 \cos \theta}{\sqrt{v_0^2 \cos^2 \theta + (v_0 \sin \theta - 32t)^2}}$$

Maximum height when  $v_0 \sin \theta - 32t = 0$ ; (vertical component of velocity)

At maximum height,  $a_T = 0$  and  $a_N = 32$ .

80.  $\theta = 45^\circ$ ,  $v_0 = 150$

$$v_0 \cos \theta = 150 \cdot \frac{\sqrt{2}}{2} = 75\sqrt{2}$$

$$v_0 \sin \theta - 32t = 150 \cdot \frac{\sqrt{2}}{2} - 32t = 75\sqrt{2} - 32t$$

$$a_T = \frac{-32(75\sqrt{2} - 32t)}{\sqrt{11250 + (75\sqrt{2} - 32t)^2}} = \frac{16(32t - 75\sqrt{2})}{\sqrt{256t^2 - 1200\sqrt{2}t + 5625}}$$

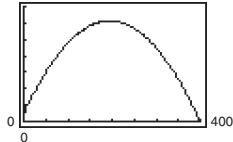
$$a_N = \frac{32(75\sqrt{2})}{\sqrt{11250 + (75\sqrt{2} - 32t)^2}} = \frac{1200\sqrt{2}}{\sqrt{256t^2 - 1200\sqrt{2}t + 5625}}$$

At the maximum height,  $a_T = 0$  and  $a_N = 32$ .

81. (a)  $\mathbf{r}(t) = (v_0 \cos \theta) \mathbf{i} + [h + (v_0 \sin \theta)t - \frac{1}{2}gt^2] \mathbf{j}$

$$= (120 \cos 30^\circ) \mathbf{i} + [5 + (120 \sin 30^\circ)t - 16t^2] \mathbf{j} = 60\sqrt{3} \mathbf{i} + [5 + 60t - 16t^2] \mathbf{j}$$

(b)



Maximum height  $\approx 61.25$  feet

range  $\approx 398.2$  feet

(c)  $\mathbf{v}(t) = 60\sqrt{3} \mathbf{i} + (60 - 32t) \mathbf{j}$

$$\text{Speed} = \|\mathbf{v}(t)\| = \sqrt{3600(3) + (60 - 32t)^2} = 8\sqrt{16t^2 - 60t + 225}$$

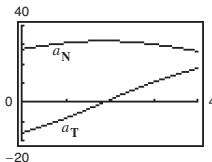
$$\mathbf{a}(t) = -32 \mathbf{j}$$

$T$	0.5	1.0	1.5	2.0	2.5	3.0
Speed	112.85	107.63	104.61	104.0	105.83	109.98

- (e) From Exercise 79, using  $\mathbf{v}_0 = 120$  and  $\theta = 30^\circ$ ,

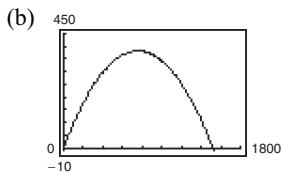
$$a_T = \frac{-32(60 - 32t)}{\sqrt{(60\sqrt{3})^2 + (60 - 32t)^2}}$$

$$a_N = \frac{32(60\sqrt{3})}{\sqrt{(60\sqrt{3})^2 + (60 - 32t)^2}}$$



At  $t = 1.875$ ,  $a_T = 0$  and the projectile is at its maximum height. When  $a_T$  and  $a_N$  have opposite signs, the speed is decreasing.

$$\begin{aligned} 82. (a) \quad \mathbf{r}(t) &= (v_0 \cos \theta)\mathbf{i} + \left[ h + (v_0 \sin \theta)t - \frac{1}{2}gt^2 \right] \mathbf{j} \\ &= (220 \cos 45^\circ)\mathbf{i} + \left[ 4 + (220 \sin 45^\circ)t - 16t^2 \right] \mathbf{j} \\ &= 110\sqrt{2}\mathbf{i} + \left[ 4 + 110\sqrt{2}t - 16t^2 \right] \mathbf{j} \end{aligned}$$



Maximum height  $\approx 382.125$  at  $t \approx 4.86$

Range  $\approx 1516.4$

$$\begin{aligned} (c) \quad \mathbf{v}(t) &= 110\sqrt{2}\mathbf{i} + \left[ 110\sqrt{2} - 32t \right] \mathbf{j} \\ \|\mathbf{v}(t)\| &= \sqrt{(110\sqrt{2})^2 + (110\sqrt{2} - 32t)^2} \\ \mathbf{a}(t) &= -32\mathbf{j} \end{aligned}$$

$t$	0.5	1.0	1.5	2.0	2.5	3.0
Speed	208.99	198.67	189.13	180.51	172.94	166.58

83.  $\mathbf{r}(t) = \langle 10 \cos 10\pi t, 10 \sin 10\pi t, 4 + 4t \rangle$ ,  $0 \leq t \leq \frac{1}{20}$

$$\begin{aligned} (a) \quad \mathbf{r}'(t) &= \langle -100\pi \sin(10\pi t), 100\pi \cos(10\pi t), 4 \rangle \\ \|\mathbf{r}'(t)\| &= \sqrt{(100\pi)^2 \sin^2(10\pi t) + (100\pi)^2 \cos^2(10\pi t) + 16} \\ &= \sqrt{(100\pi)^2 + 16} = 4\sqrt{625\pi^2 + 1} \approx 314 \text{ mi/h} \end{aligned}$$

(b)  $a_T = 0$  and  $a_N = 1000\pi^2$

$a_T = 0$  because the speed is constant.

84.  $600 \text{ mi/h} = 880 \text{ ft/sec}$ 

$$\mathbf{r}(t) = 880\mathbf{i} + (-16t^2 + 36,000)\mathbf{j}$$

$$\mathbf{v}(t) = 880\mathbf{i} - 32t\mathbf{j}$$

$$\mathbf{a}(t) = -32\mathbf{j}$$

$$\mathbf{T}(t) = \frac{880\mathbf{i} - 32t\mathbf{j}}{\sqrt{4t^2 + 3025}} = \frac{55\mathbf{i} - 2t\mathbf{j}}{\sqrt{4t^2 + 3025}}$$

Motion along  $\mathbf{r}$  is clockwise, therefore

$$\mathbf{N}(t) = \frac{-2t\mathbf{i} - 55\mathbf{j}}{\sqrt{4t^2 + 3025}}$$

$$a_T = \mathbf{a} \cdot \mathbf{T} = \frac{64t}{\sqrt{4t^2 + 3025}}$$

$$a_N = \mathbf{a} \cdot \mathbf{N} = \frac{1760}{\sqrt{4t^2 + 3025}}$$

85.  $\mathbf{r}(t) = (a \cos \omega t)\mathbf{i} + (a \sin \omega t)\mathbf{j}$ From Exercise 45, we know  $\mathbf{a} \cdot \mathbf{T} = 0$  and

$$\mathbf{a} \cdot \mathbf{N} = a\omega^2.$$

(a) Let  $\omega_0 = 2\omega$ . Then

$$\mathbf{a} \cdot \mathbf{N} = a\omega_0^2 = a(2\omega)^2 = 4a\omega^2$$

or the centripetal acceleration is increased by a factor of 4 when the velocity is doubled.

(b) Let  $a_0 = a/2$ . Then

$$\mathbf{a} \cdot \mathbf{N} = a_0\omega^2 = \left(\frac{a}{2}\right)\omega^2 = \left(\frac{1}{2}\right)a\omega^2$$

or the centripetal acceleration is halved when the radius is halved.

90. Let  $x$  = distance from the satellite to the center of the earth ( $x = r + 4000$ ). Then:

$$\begin{aligned} v &= \frac{2\pi x}{t} = \frac{2\pi x}{24(3600)} = \sqrt{\frac{9.56 \times 10^4}{x}} \\ \frac{4\pi^2 x^2}{(24)^2 (3600)^2} &= \frac{9.56 \times 10^4}{x} \\ x^3 &= \frac{(9.56 \times 10^4)(24)^2 (3600)^2}{4\pi^2} \Rightarrow x \approx 26,245 \text{ mi} \\ v &\approx \frac{2\pi(26,245)}{24(3600)} \approx 1.92 \text{ mi/sec} \approx 6871 \text{ mi/h} \end{aligned}$$

91. False. You could be turning.

92. True. All the motion is in the tangential direction.

93. (a)  $\mathbf{r}(t) = \cosh(bt)\mathbf{i} + \sinh(bt)\mathbf{j}, b > 0$ 

$$x = \cosh(bt), y = \sinh(bt)$$

$$x^2 - y^2 = \cosh^2(bt) - \sinh^2(bt) = 1, \text{ hyperbola}$$

(b)  $\mathbf{v}(t) = b \sinh(bt)\mathbf{i} + b \cosh(bt)\mathbf{j}$ 

$$\mathbf{a}(t) = b^2 \cosh(bt)\mathbf{i} + b^2 \sinh(bt)\mathbf{j} = b^2 \mathbf{r}(t)$$

$$86. \mathbf{r}(t) = (r \cos \omega t)\mathbf{i} + (r \sin \omega t)\mathbf{j}$$

$$\mathbf{v}(t) = (-r\omega \sin \omega t)\mathbf{i} + (r\omega \cos \omega t)\mathbf{j}$$

$$\|\mathbf{v}(t)\| = r\omega\sqrt{1} = r\omega = v$$

$$\mathbf{a}(t) = (-r\omega^2 \cos \omega t)\mathbf{i} + (r\omega \cos \omega t)\mathbf{j}$$

$$\|\mathbf{a}(t)\| = r\omega^2$$

$$(a) F = m\|\mathbf{a}(t)\| = m(r\omega^2) = \frac{m}{r}(r^2\omega^2) = \frac{mv^2}{r}$$

(b) By Newton's Law:

$$\frac{mv^2}{r} = \frac{GMm}{r^2}, v^2 = \frac{GM}{r}, v = \sqrt{\frac{GM}{r}}$$

$$87. v = \sqrt{\frac{GM}{r}} = \sqrt{\frac{9.56 \times 10^4}{4000 + 115}} \approx 4.82 \text{ mi/sec}$$

$$88. v = \sqrt{\frac{GM}{r}} = \sqrt{\frac{9.56 \times 10^4}{4000 + 245}} \approx 4.75 \text{ mi/sec}$$

$$89. v = \sqrt{\frac{9.56 \times 10^4}{4385}} \approx 4.67 \text{ mi/sec}$$

94. Let  $\mathbf{T}(t) = \cos \phi \mathbf{i} + \sin \phi \mathbf{j}$  be the unit tangent vector.

Then

$$\begin{aligned}\mathbf{T}'(t) &= \frac{d\mathbf{T}}{dt} \\ &= \frac{d}{d\phi} \frac{d\mathbf{T}}{dt} = -(\sin \phi \mathbf{i} - \cos \phi \mathbf{j}) \frac{d\phi}{dt} = \mathbf{M} \frac{d\phi}{dt}.\end{aligned}$$

$$\mathbf{M} = -\sin \phi \mathbf{i} + \cos \phi \mathbf{j}$$

$$= \cos[\phi + (\pi/2)] \mathbf{i} + \sin[\phi + (\pi/2)] \mathbf{j}$$

and is rotated counterclockwise through an angle of  $\pi/2$  from  $\mathbf{T}$ .

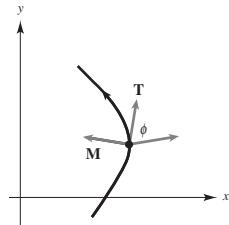
If  $d\phi/dt > 0$ , then the curve bends to the left and

$\mathbf{M}$  has the same direction as  $\mathbf{T}'$ .

So,  $\mathbf{M}$  has the same direction as

$$\mathbf{N} = \frac{\mathbf{T}'}{\|\mathbf{T}'\|},$$

which is toward the concave side of the curve.

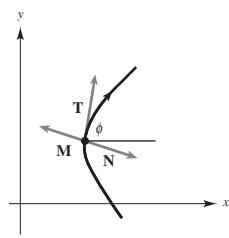


If  $d\phi/dt < 0$ , then the curve bends to the right and

$\mathbf{M}$  has the opposite direction as  $\mathbf{T}'$ . Thus,

$$\mathbf{N} = \frac{\mathbf{T}'}{\|\mathbf{T}'\|}$$

again points to the concave side of the curve.



95.  $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$

$$y(t) = m(x(t)) + b, m \text{ and } b \text{ are constants.}$$

$$\mathbf{r}(t) = x(t)\mathbf{i} + [m(x(t)) + b]\mathbf{j}$$

$$\mathbf{v}(t) = x'(t)\mathbf{i} + mx'(t)\mathbf{j}$$

$$\|\mathbf{v}(t)\| = \sqrt{[x'(t)]^2 + [mx'(t)]^2} = |x'(t)|\sqrt{1+m^2}$$

$$\mathbf{T}(t) = \frac{\mathbf{v}(t)}{\|\mathbf{v}(t)\|} = \frac{\pm(\mathbf{i} + m\mathbf{j})}{\sqrt{1+m^2}}, \text{ constant}$$

So,  $\mathbf{T}'(t) = \mathbf{0}$ .

## Section 12.5 Arc Length and Curvature

1.  $\mathbf{r}(t) = 3t\mathbf{i} - t\mathbf{j}, [0, 3]$

$$\frac{dx}{dt} = 3, \frac{dy}{dt} = -1, \frac{dz}{dt} = 0$$

$$s = \int_0^3 \sqrt{3^2 + (-1)^2} dt = \left[ \sqrt{10}t \right]_0^3 = 3\sqrt{10}$$

96. Using  $\mathbf{a} = a_T \mathbf{T} + a_N \mathbf{N}$ ,  $\mathbf{T} \times \mathbf{T} = \mathbf{0}$ , and  $\|\mathbf{T} \times \mathbf{N}\| = 1$ , you have:

$$\begin{aligned}\mathbf{v} \times \mathbf{a} &= \|\mathbf{v}\| \mathbf{T} \times (a_T \mathbf{T} + a_N \mathbf{N}) \\ &= \|\mathbf{v}\| a_T (\mathbf{T} \times \mathbf{T}) + \|\mathbf{v}\| a_N (\mathbf{T} \times \mathbf{N}) \\ &= \|\mathbf{v}\| a_N (\mathbf{T} \times \mathbf{N}) \\ \|\mathbf{v} \times \mathbf{a}\| &= \|\mathbf{v}\| a_N \|\mathbf{T} \times \mathbf{N}\| = \|\mathbf{v}\| a_N \\ \text{So, } a_N &= \frac{\|\mathbf{v} \times \mathbf{a}\|}{\|\mathbf{v}\|}.\end{aligned}$$

$$97. \|\mathbf{a}\|^2 = \mathbf{a} \cdot \mathbf{a}$$

$$\begin{aligned}&= (a_T \mathbf{T} + a_N \mathbf{N}) \cdot (a_T \mathbf{T} + a_N \mathbf{N}) \\ &= a_T^2 \|\mathbf{T}\|^2 + 2a_T a_N \mathbf{T} \cdot \mathbf{N} + a_N^2 \|\mathbf{N}\|^2 \\ &= a_T^2 + a_N^2 \\ a_N^2 &= \|\mathbf{a}\|^2 - a_T^2\end{aligned}$$

Because  $a_N > 0$ , we have  $a_N = \sqrt{\|\mathbf{a}\|^2 - a_T^2}$ .

$$98. F = ma = (1) \frac{dv}{dt} = \frac{dv}{dt} \text{ Force}$$

$$x = at + bt^2 + ct^3$$

$$v = \frac{dx}{dt} = a + 2bt + 3ct^2$$

$$\frac{dv}{dt} = 2b + 6ct$$

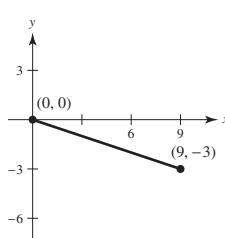
$$F^2 = 4b^2 + 24bct + 36c^2t^2$$

$$= 4b^2 + 12c + (2bt + 3ct^2)$$

$$= 4b^2 + 12c + (v - a)$$

$$F = f(v) = \pm \sqrt{4b^2 - 12ac + 12cv}$$

The sign of the radical is the sign of  $2b + 6ct$ , which cannot change.

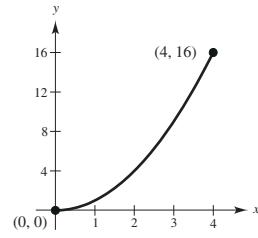


2.  $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j}$

$$\frac{dx}{dt} = 1, \frac{dy}{dt} = 2t, \frac{dz}{dt} = 0$$

$$s = \int_0^4 \sqrt{1+4t^2} dt$$

$$= \frac{1}{4} \left[ 2t\sqrt{1+4t^2} + \ln|2t + \sqrt{1+4t}| \right]_0^4 = \frac{1}{4} \left[ 8\sqrt{65} + \ln(8 + \sqrt{65}) \right] \approx 16.819$$

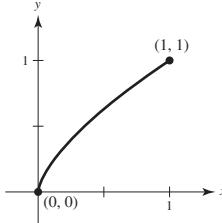


3.  $\mathbf{r}(t) = t^3\mathbf{i} + t^2\mathbf{j}, [0, 1]$

$$\frac{dx}{dt} = 3t^2, \frac{dy}{dt} = 2t, \frac{dz}{dt} = 0$$

$$s = \int_0^1 \sqrt{9t^4 + 4t^2} dt = \int_0^1 \sqrt{9t^2 + 4} t dt$$

$$= \frac{1}{18} \int_0^1 (9t^2 + 4)^{1/2} (18t) dt = \frac{1}{27} \left[ (9t^2 + 4)^{3/2} \right]_0^1 = \frac{1}{27} (13^{3/2} - 8) \approx 1.4397$$

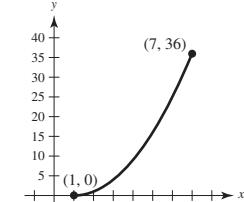


4.  $\mathbf{r}(t) = (t+1)\mathbf{i} + t^2\mathbf{j}, 0 \leq t \leq 6$

$$\frac{dx}{dt} = 1, \frac{dy}{dt} = 2t$$

$$s = \int_0^6 \sqrt{1+4t^2} dt$$

$$= \left[ \frac{1}{4} \ln(\sqrt{4t^2 + 1} + 2t) + \frac{1}{2} t \sqrt{4t^2 + 1} \right]_0^6 = \frac{1}{4} \ln(\sqrt{145} + 12) + 3\sqrt{145}$$

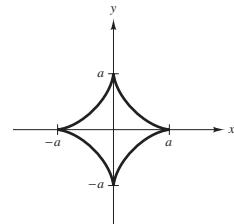


5.  $\mathbf{r}(t) = a \cos^3 t\mathbf{i} + a \sin^3 t\mathbf{j}$

$$\frac{dx}{dt} = -3a \cos^2 t \sin t, \frac{dy}{dt} = 3a \sin^2 t \cos t$$

$$s = 4 \int_0^{\pi/2} \sqrt{[-3a \cos^2 t \sin t]^2 + [3a \sin^2 t \cos t]^2} dt$$

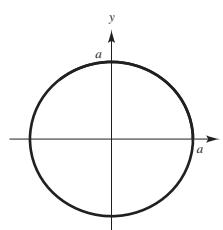
$$= 12a \int_0^{\pi/2} \sin t \cos t dt = 3a \int_0^{\pi/2} 2 \sin 2t dt = [-3a \cos 2t]_0^{\pi/2} = 6a$$



6.  $\mathbf{r}(t) = a \cos t\mathbf{i} + a \sin t\mathbf{j}$

$$\frac{dx}{dt} = -a \sin t, \frac{dy}{dt} = a \cos t$$

$$s = \int_0^{2\pi} \sqrt{a^2 \sin^2 t + a^2 \cos^2 t} dt = \int_0^{2\pi} a dt = [at]_0^{2\pi} = 2\pi a$$



7. (a)  $\mathbf{r}(t) = (v_0 \cos \theta)t\mathbf{i} + \left[ h + (v_0 \sin \theta)t - \frac{1}{2}gt^2 \right]\mathbf{j}$

$$= (100 \cos 45^\circ)t\mathbf{i} + \left[ 3 + (100 \sin 45^\circ)t - \frac{1}{2}(32)t^2 \right]\mathbf{j} = 50\sqrt{2}\mathbf{i} + \left[ 3 + 50\sqrt{2}t - 16t^2 \right]\mathbf{j}$$

(b)  $\mathbf{v}(t) = 50\sqrt{2}\mathbf{i} + (50\sqrt{2} - 32t)\mathbf{j}$

$$50\sqrt{2} - 32t = 0 \Rightarrow t = \frac{25\sqrt{2}}{16}$$

$$\text{Maximum height: } 3 + 50\sqrt{2} \left( \frac{25\sqrt{2}}{16} \right) - 16 \left( \frac{15\sqrt{2}}{16} \right)^2 = 81.125 \text{ ft}$$

(c)  $3 + 50\sqrt{2}t - 16t^2 = 0 \Rightarrow t \approx 4.4614$

Range:  $50\sqrt{2}(4.4614) \approx 315.5$  feet

(d)  $s = \int_0^{4.4614} \sqrt{(50\sqrt{2})^2 + (50\sqrt{2} - 32t)^2} dt \approx 362.9$  feet

8. (a)  $\mathbf{r}(t) = (v_0 \cos \theta)\mathbf{i} + \left[(v_0 \sin \theta)t - \frac{1}{2}gt^2\right]\mathbf{j}$

$$y(t) = (v_0 \sin \theta)t - \frac{1}{2}gt^2$$

$$y'(t) = v_0 \sin \theta - gt = 0 \text{ when } t = \frac{v_0 \sin \theta}{g}$$

Maximum height when  $\sin \theta = 1$ , or  $\theta = \frac{\pi}{2}$ .

(b)  $y(t) = (v_0 \sin \theta)t - \frac{1}{2}gt^2 = 0 \Rightarrow t = \frac{2v_0 \sin \theta}{g}$

$$\text{Range: } x(t) = (v_0 \cos \theta) \left( \frac{2v_0 \sin \theta}{g} \right) = \frac{v_0^2}{g} \sin^2 \theta$$

The range  $x(t)$  is a maximum for  $\sin 2\theta = 1$ , or  $\theta = \frac{\pi}{4}$ .

(c)  $x'(t) = v_0 \cos \theta$

$$y'(t) = v_0 \sin \theta - gt$$

$$x'(t)^2 + y'(t)^2 = v_0^2 \cos^2 \theta + (v_0 \sin \theta - gt)^2 = v_0^2 \cos^2 \theta + v_0^2 \sin^2 \theta - 2v_0^2 g \sin \theta t + g^2 t^2 = v_0^2 - 2v_0 g \sin \theta t + g^2 t^2$$

$$s(\theta) = \int_0^{2v_0 \sin \theta / g} [v_0^2 - 2v_0 g \sin \theta t + g^2 t^2]^{1/2} dt$$

Because  $v_0 = 96$  ft/sec, you have

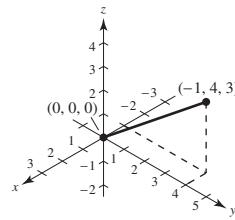
$$s(\theta) = \int_0^{6 \sin \theta} [96^2 - (6144 \sin \theta)t + 1024t^2]^{1/2} dt$$

Using a computer algebra system,  $s(\theta)$  is a maximum for  $\theta \approx 0.9855 \approx 56.5^\circ$ .

9.  $\mathbf{r}(t) = -t\mathbf{i} + 4t\mathbf{j} + 3t\mathbf{k}, [0, 1]$

$$\frac{dx}{dt} = -1, \frac{dy}{dt} = 4, \frac{dz}{dt} = 3$$

$$s = \int_0^1 \sqrt{1 + 16 + 9} dt = \left[ \sqrt{26}t \right]_0^1 = \sqrt{26}$$

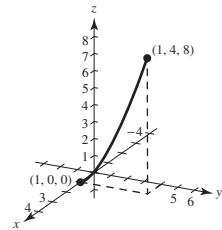


10.  $\mathbf{r}(t) = \mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}, [0, 2]$

$$\frac{dx}{dt} = 0, \frac{dy}{dt} = 2t, \frac{dz}{dt} = 3t^2$$

$$s = \int_0^2 \sqrt{4t^2 + 9t^4} dt$$

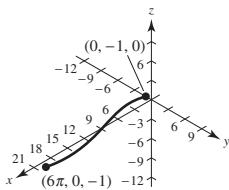
$$= \int_0^2 \sqrt{4 + 9t^2} t dt = \frac{1}{27} (4 + 9t^2)^{3/2} \Big|_0^2 = \frac{1}{27} (40^{3/2} - 4^{3/2}) = \frac{1}{27} [80\sqrt{10} - 8]$$



11.  $\mathbf{r}(t) = \langle 4t, -\cos t, \sin t \rangle, [0, \frac{3\pi}{2}]$

$$\frac{dx}{dt} = 4, \frac{dy}{dt} = \sin t, \frac{dz}{dt} = \cos t$$

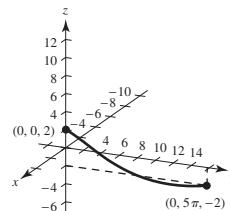
$$s = \int_0^{3\pi/2} \sqrt{16 + \sin^2 t + \cos^2 t} dt = \int_0^{3\pi/2} \sqrt{17} dt = \left[ \sqrt{17}t \right]_0^{3\pi/2} = \frac{3\pi}{2}\sqrt{17}$$



12.  $\mathbf{r}(t) = \langle 2 \sin t, 5t, 2 \cos t \rangle, [0, \pi]$

$$\frac{dx}{dt} = 2 \cos t, \frac{dy}{dt} = 5, \frac{dz}{dt} = -2 \sin t$$

$$s = \int_0^\pi \sqrt{4 \cos^2 t + 25 + 4 \sin^2 t} dt = \int_0^\pi \sqrt{29} dt = \sqrt{29}\pi$$

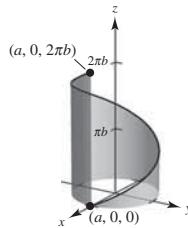


13.  $\mathbf{r}(t) = a \cos t \mathbf{i} + a \sin t \mathbf{j} + bt \mathbf{k}$

$$\frac{dx}{dt} = -a \sin t, \frac{dy}{dt} = a \cos t, \frac{dz}{dt} = b$$

$$s = \int_0^{2\pi} \sqrt{a^2 \sin^2 t + a^2 \cos^2 t + b^2} dt$$

$$= \int_0^{2\pi} \sqrt{a^2 + b^2} dt = \left[ \sqrt{a^2 + b^2} t \right]_0^{2\pi} = 2\pi\sqrt{a^2 + b^2}$$

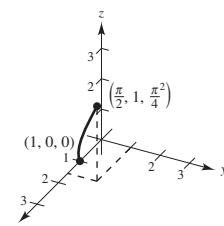


14.  $\mathbf{r}(t) = \langle \cos t + t \sin t, \sin t - t \cos t, t^2 \rangle$

$$\frac{dx}{dt} = t \cos t, \frac{dy}{dt} = t \sin t, \frac{dz}{dt} = 2t$$

$$s = \int_0^{\pi/2} \sqrt{(t \cos t)^2 + (t \sin t)^2 + (2t)^2} dt$$

$$= \int_0^{\pi/2} \sqrt{5t^2} dt = \sqrt{5} \frac{t^2}{2} \Big|_0^{\pi/2} = \frac{\sqrt{5}\pi^2}{8}$$



15.  $\mathbf{r}(t) = t^2 \mathbf{i} + t \mathbf{j} + \ln t \mathbf{k}$

$$\frac{dx}{dt} = 2t, \frac{dy}{dt} = 1, \frac{dz}{dt} = \frac{1}{t}$$

$$s = \int_1^3 \sqrt{(2t)^2 + (1)^2 + \left(\frac{1}{t}\right)^2} dt = \int_1^3 \sqrt{\frac{4t^4 + t^2 + 1}{t^2}} dt = \int_1^3 \frac{\sqrt{4t^4 + t^2 + 1}}{t} dt \approx 8.37$$

16.  $\mathbf{r}(t) = \sin \pi t \mathbf{i} + \cos \pi t \mathbf{j} + t^3 \mathbf{k}$

$$\frac{dx}{dt} = \pi \cos \pi t, \frac{dy}{dt} = -\pi \sin \pi t, \frac{dz}{dt} = 3t^2$$

$$s = \int_0^2 \sqrt{(\pi \cos \pi t)^2 + (-\pi \sin \pi t)^2 + (3t^2)^2} dt = \int_0^2 \sqrt{\pi^2 + 9t^4} dt \approx 11.15$$

**17.**  $\mathbf{r}(t) = t\mathbf{i} + (4 - t^2)\mathbf{j} + t^3\mathbf{k}$ ,  $0 \leq t \leq 2$

(a)  $\mathbf{r}(0) = \langle 0, 4, 0 \rangle$ ,  $\mathbf{r}(2) = \langle 2, 0, 8 \rangle$

$$\begin{aligned}\text{distance} &= \sqrt{2^2 + 4^2 + 8^2} = \sqrt{84} \\ &= 2\sqrt{21} \approx 9.165\end{aligned}$$

(b)  $\mathbf{r}(0) = \langle 0, 4, 0 \rangle$

$$\mathbf{r}(0.5) = \langle 0.5, 3.75, 0.125 \rangle$$

$$\mathbf{r}(1) = \langle 1, 3, 1 \rangle$$

$$\mathbf{r}(1.5) = \langle 1.5, 1.75, 3.375 \rangle$$

$$\mathbf{r}(2) = \langle 2, 0, 8 \rangle$$

$$\begin{aligned}\text{distance} &\approx \sqrt{(0.5)^2 + (0.25)^2 + (0.125)^2} \\ &\quad + \sqrt{(0.5)^2 + (0.75)^2 + (0.875)^2} \\ &\quad + \sqrt{(0.5)^2 + (1.25)^2 + (2.375)^2} \\ &\quad + \sqrt{(0.5)^2 + (1.75)^2 + (4.625)^2} \\ &\approx 0.5728 + 1.2562 + 2.7300 + 4.9702 \\ &\approx 9.529\end{aligned}$$

(c) Increase the number of line segments.

(d) Using a graphing utility, you obtain 9.57057.

**19.**  $\mathbf{r}(t) = \langle 2 \cos t, 2 \sin t, t \rangle$

(a)  $s = \int_0^t \sqrt{[x'(u)]^2 + [y'(u)]^2 + [z'(u)]^2} du = \int_0^t \sqrt{(-2 \sin u)^2 + (2 \cos u)^2 + (1)^2} du = \int_0^t \sqrt{5} du = [\sqrt{5}u]_0^t = \sqrt{5}t$

(b)  $\frac{s}{\sqrt{5}} = t$

$$x = 2 \cos\left(\frac{s}{\sqrt{5}}\right), y = 2 \sin\left(\frac{s}{\sqrt{5}}\right), z = \frac{s}{\sqrt{5}}$$

$$\mathbf{r}(s) = 2 \cos\left(\frac{s}{\sqrt{5}}\right)\mathbf{i} + 2 \sin\left(\frac{s}{\sqrt{5}}\right)\mathbf{j} + \frac{s}{\sqrt{5}}\mathbf{k}$$

(c) When  $s = \sqrt{5}$ :  $x = 2 \cos 1 \approx 1.081$

$$y = 2 \sin 1 \approx 1.683$$

$$z = 1$$

$$(1.081, 1.683, 1.000)$$

When  $s = 4$ :  $x = 2 \cos \frac{4}{\sqrt{5}} \approx -0.433$

$$y = 2 \sin \frac{4}{\sqrt{5}} \approx 1.953$$

$$z = \frac{4}{\sqrt{5}} \approx 1.789$$

$$(-0.433, 1.953, 1.789)$$

(d)  $\|\mathbf{r}'(s)\| = \sqrt{\left(-\frac{2}{\sqrt{5}} \sin\left(\frac{s}{\sqrt{5}}\right)\right)^2 + \left(\frac{2}{\sqrt{5}} \cos\left(\frac{s}{\sqrt{5}}\right)\right)^2 + \left(\frac{1}{\sqrt{5}}\right)^2} = \sqrt{\frac{4}{5} + \frac{1}{5}} = 1$

**20.**  $\mathbf{r}(t) = \left\langle 4(\sin t - t \cos t), 4(\cos t + t \sin t), \frac{3}{2}t^2 \right\rangle$

(a)  $s = \int_0^t \sqrt{[x'(u)]^2 + [y'(u)]^2 + [z'(u)]^2} du$

$$= \int_0^t \sqrt{(4u \sin u)^2 + (4u \cos u)^2 + (3u)^2} du = \int_0^t \sqrt{16u + 9u^2} du = \int_0^t 5u du = \frac{5}{2}t^2$$

**18.**  $\mathbf{r}(t) = 6 \cos\left(\frac{\pi t}{4}\right)\mathbf{i} + 2 \sin\left(\frac{\pi t}{4}\right)\mathbf{j} + t\mathbf{k}$ ,  $0 \leq t \leq 2$

(a)  $\mathbf{r}(0) = 6\mathbf{i} = \langle 6, 0, 0 \rangle$

$$\mathbf{r}(2) = 2\mathbf{j} + 2\mathbf{k} = \langle 0, 2, 2 \rangle$$

$$\text{distance} = \sqrt{6^2 + 2^2 + 2^2} = \sqrt{44}$$

$$= 2\sqrt{11} \approx 6.633$$

(b)  $\mathbf{r}(0) = \langle 6, 0, 0 \rangle$

$$\mathbf{r}(0.5) = \langle 5.543, 0.765, 0.5 \rangle$$

$$\mathbf{r}(1.0) = \langle 4.243, 1.414, 1.0 \rangle$$

$$\mathbf{r}(1.5) = \langle 2.296, 1.848, 1.5 \rangle$$

$$\mathbf{r}(2.0) = \langle 0, 2, 2 \rangle$$

$$\text{distance} \approx 6.9698$$

(c) Increase the number of line segments.

(d) Using a graphing utility, you obtain

$$s = \int_0^2 \|\mathbf{r}'(t)\| dt \approx 7.0105.$$

$$(b) t = \sqrt{\frac{2s}{5}}$$

$$x = 4\left(\sin\sqrt{\frac{2s}{5}} - \sqrt{\frac{2s}{5}} \cos\sqrt{\frac{2s}{5}}\right)$$

$$y = 4\left(\cos\sqrt{\frac{2s}{5}} + \sqrt{\frac{2s}{5}} \sin\sqrt{\frac{2s}{5}}\right)$$

$$z = \frac{3}{2}\left(\sqrt{\frac{2s}{5}}\right)^2 = \frac{3s}{5}$$

$$\mathbf{r}(s) = 4\left(\sin\sqrt{\frac{2s}{5}} - \sqrt{\frac{2s}{5}} \cos\sqrt{\frac{2s}{5}}\right)\mathbf{i} + 4\left(\cos\sqrt{\frac{2s}{5}} + \sqrt{\frac{2s}{5}} \sin\sqrt{\frac{2s}{5}}\right)\mathbf{j} + \frac{3s}{5}\mathbf{k}$$

(c) When  $s = \sqrt{5}$ :

$$x = 4\left(\sin\sqrt{\frac{2\sqrt{5}}{5}} - \sqrt{\frac{2\sqrt{5}}{5}} \cos\sqrt{\frac{2\sqrt{5}}{5}}\right) \approx -1.030$$

$$y = 4\left(\cos\sqrt{\frac{2\sqrt{5}}{5}} + \sqrt{\frac{2\sqrt{5}}{5}} \sin\sqrt{\frac{2\sqrt{5}}{5}}\right) \approx 5.408$$

$$z = \frac{3\sqrt{5}}{5} \approx 1.342$$

$$(-1.030, 5.408, 1.342)$$

When  $s = 4$ :

$$x = 4\left(\sin\sqrt{\frac{8}{5}} - \sqrt{\frac{8}{5}} \cos\sqrt{\frac{8}{5}}\right) \approx 2.291$$

$$y = 4\left(\cos\sqrt{\frac{8}{5}} + \sqrt{\frac{8}{5}} \sin\sqrt{\frac{8}{5}}\right) \approx 6.029$$

$$z = \frac{12}{5} = 2.4$$

$$(2.291, 6.029, 2.400)$$

$$(d) \|\mathbf{r}'(s)\| = \sqrt{\left(\frac{4}{5}\sin\sqrt{\frac{2s}{5}}\right)^2 + \left(\frac{4}{5}\cos\sqrt{\frac{2s}{5}}\right)^2 + \left(\frac{3}{5}\right)^2} = \sqrt{\frac{16}{25} + \frac{9}{25}} = 1$$

$$21. \mathbf{r}(s) = \left(1 + \frac{\sqrt{2}}{2}s\right)\mathbf{i} + \left(1 - \frac{\sqrt{2}}{2}s\right)\mathbf{j}$$

$$\mathbf{r}'(s) = \frac{\sqrt{2}}{2}\mathbf{i} - \frac{\sqrt{2}}{2}\mathbf{j} \text{ and } \|\mathbf{r}'(s)\| = \sqrt{\frac{1}{2} + \frac{1}{2}} = 1$$

$$\mathbf{T}(s) = \frac{\mathbf{r}'(s)}{\|\mathbf{r}'(s)\|} = \mathbf{r}'(s)$$

$$\mathbf{T}'(s) = \mathbf{0} \Rightarrow K = \|\mathbf{T}'(s)\| = 0 \text{ (The curve is a line.)}$$

$$22. \mathbf{r}(s) = (3 + s)\mathbf{i} + \mathbf{j}$$

$$\mathbf{r}'(s) = \mathbf{i} \text{ and } \|\mathbf{r}'(s)\| = 1$$

$$\mathbf{T}(s) = \mathbf{r}'(s)$$

$$\mathbf{T}'(s) = \mathbf{0} \Rightarrow K = \|\mathbf{T}'(s)\| = 0 \text{ (The curve is a line.)}$$

$$23. \mathbf{r}(s) = 2\cos\left(\frac{s}{\sqrt{5}}\right)\mathbf{i} + 2\sin\left(\frac{s}{\sqrt{5}}\right)\mathbf{j} + \frac{s}{\sqrt{5}}\mathbf{k}$$

$$\begin{aligned} \mathbf{T}(s) &= \mathbf{r}'(s) \\ &= -\frac{2}{\sqrt{5}}\sin\left(\frac{s}{\sqrt{5}}\right)\mathbf{i} + \frac{2}{\sqrt{5}}\cos\left(\frac{s}{\sqrt{5}}\right)\mathbf{j} + \frac{1}{\sqrt{5}}\mathbf{k} \end{aligned}$$

$$\mathbf{T}'(s) = -\frac{2}{5}\cos\left(\frac{s}{\sqrt{5}}\right)\mathbf{i} - \frac{2}{5}\sin\left(\frac{s}{\sqrt{5}}\right)\mathbf{j}$$

$$K = \|\mathbf{T}'(s)\| = \frac{2}{5}$$

24.  $\mathbf{r}(s) = 4\left(\sin\sqrt{\frac{2s}{5}} - \sqrt{\frac{2s}{5}} \cos\sqrt{\frac{2s}{5}}\right)\mathbf{i} + 4\left(\cos\sqrt{\frac{2s}{5}} + \sqrt{\frac{2s}{5}} \sin\sqrt{\frac{2s}{5}}\right)\mathbf{j} + \frac{3s}{5}\mathbf{k}$

$$\mathbf{T}(s) = \mathbf{r}'(s) = \frac{4}{5}\sin\sqrt{\frac{2s}{5}}\mathbf{i} + \frac{4}{5}\cos\sqrt{\frac{2s}{5}}\mathbf{j} + \frac{3}{5}\mathbf{k}$$

$$\mathbf{T}'(s) = \frac{4}{25}\sqrt{\frac{5}{2s}}\cos\sqrt{\frac{2s}{5}}\mathbf{i} - \frac{4}{25}\sqrt{\frac{5}{2s}}\sin\sqrt{\frac{2s}{5}}\mathbf{j}$$

$$K = \|\mathbf{T}'(s)\| = \frac{4}{25}\sqrt{\frac{5}{2s}} = \frac{2\sqrt{10s}}{25s}$$

25.  $\mathbf{r}(t) = 4t\mathbf{i} - 2t\mathbf{j}$

$$\mathbf{v}(t) = 4\mathbf{i} - 2\mathbf{j}$$

$$\mathbf{T}(t) = \frac{1}{\sqrt{5}}(2\mathbf{i} - \mathbf{j})$$

$$\mathbf{T}'(t) = \mathbf{0}$$

$$K = \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|} = 0$$

(The curve is a line.)

26.  $\mathbf{r}(t) = t^2\mathbf{i} + \mathbf{j}$

$$\mathbf{v}(t) = 2t\mathbf{i}$$

$$\mathbf{T}(t) = \mathbf{i}$$

$$\mathbf{T}'(t) = \mathbf{0}$$

$$K = \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|} = 0$$

27.  $\mathbf{r}(t) = t\mathbf{i} + \frac{1}{t}\mathbf{j}$

$$\mathbf{v}(t) = \mathbf{i} - \frac{1}{t^2}\mathbf{j}$$

$$\mathbf{v}(1) = \mathbf{i} - \mathbf{j}$$

$$\mathbf{a}(t) = \frac{2}{t^3}\mathbf{j}$$

$$\mathbf{a}(1) = 2\mathbf{j}$$

$$\mathbf{T}(t) = \frac{t^2\mathbf{i} - \mathbf{j}}{\sqrt{t^4 + 1}}$$

$$\mathbf{N}(t) = \frac{1}{(t^4 + 1)^{1/2}}(\mathbf{i} + t^2\mathbf{j})$$

$$\mathbf{N}(1) = \frac{1}{\sqrt{2}}(\mathbf{i} + \mathbf{j})$$

$$K = \frac{\mathbf{a} \cdot \mathbf{N}}{\|\mathbf{v}\|^2} = \frac{\sqrt{2}}{2}$$

28.  $\mathbf{r}(t) = t\mathbf{i} + \frac{1}{9}t^3\mathbf{j}$

$$\mathbf{v}(t) = \mathbf{i} + \frac{1}{3}t^2\mathbf{j}$$

$$\mathbf{v}(2) = \mathbf{i} + \frac{4}{2}\mathbf{j}, \|\mathbf{v}(2)\| = \sqrt{1 + \frac{16}{9}} = \frac{5}{3}$$

$$\mathbf{a}(t) = \frac{2}{3}t\mathbf{j}$$

$$\mathbf{a}(2) = \frac{4}{3}\mathbf{j}$$

$$\mathbf{T}(t) = \frac{\mathbf{i} + \frac{1}{3}t^2\mathbf{j}}{\sqrt{1 + \frac{t^4}{9}}} = \frac{3\mathbf{i} + t^2\mathbf{j}}{\sqrt{9 + t^4}}$$

$$\mathbf{T}(2) = \frac{3\mathbf{i} + 4\mathbf{j}}{5}$$

$$\mathbf{N}(2) = \frac{-4\mathbf{i} + 3\mathbf{j}}{5}$$

$$K = \frac{\mathbf{a} \cdot \mathbf{N}}{\|\mathbf{v}\|^2} = \frac{4/5}{25/9} = \frac{36}{125}$$

29.  $\mathbf{r}(t) = \langle t, \sin t \rangle$

$$\mathbf{r}'(t) = \langle 1, \cos t \rangle, \|\mathbf{r}'(t)\| = \sqrt{1 + \cos^2 t}$$

$$\mathbf{r}'\left(\frac{\pi}{2}\right) = \langle 1, 0 \rangle, \left\|\mathbf{r}'\left(\frac{\pi}{2}\right)\right\| = 1$$

$$\mathbf{a}(t) = \langle 0, -\sin t \rangle, \mathbf{a}\left(\frac{\pi}{2}\right) = \langle 0, -1 \rangle$$

$$\mathbf{T}(t) = \frac{1}{\sqrt{1 + \cos^2 t}} \langle 1, \cos t \rangle$$

$$\mathbf{T}\left(\frac{\pi}{2}\right) = \langle 1, 0 \rangle$$

$$\mathbf{N}\left(\frac{\pi}{2}\right) = \langle 0, -1 \rangle$$

$$K = \frac{\mathbf{a} \cdot \mathbf{N}}{\|\mathbf{v}\|^2} = \frac{1}{1} = 1$$

30.  $\mathbf{r}(t) = \langle 5 \cos t, 4 \sin t \rangle, t = \frac{\pi}{3}$

$$x(t) = 5 \cos t, y(t) = 4 \sin t$$

$$\begin{aligned} K &= \frac{|x'y'' - y'x''|}{\left[ (x')^2 + (y')^2 \right]^{3/2}} \\ &= \frac{|(-5 \sin t)(-4 \sin t) - (4 \cos t)(-5 \cos t)|}{[25 \sin^2 t + 16 \cos^2 t]^{3/2}} \\ &= \frac{20}{[25 \sin^2 t + 16 \cos^2 t]^{3/2}} \\ K\left(\frac{\pi}{3}\right) &= \frac{20}{[25(3/4) + 16(1/4)]^{3/2}} = \frac{160\sqrt{91}}{8281} \end{aligned}$$

31.  $\mathbf{r}(t) = 4 \cos 2\pi t \mathbf{i} + 4 \sin 2\pi t \mathbf{j}$

$$\mathbf{r}'(t) = -8\pi \sin 2\pi t \mathbf{i} + 8\pi \cos 2\pi t \mathbf{j}$$

$$\mathbf{T}(t) = -\sin 2\pi t \mathbf{i} + \cos 2\pi t \mathbf{j}$$

$$\mathbf{T}'(t) = -2\pi \cos 2\pi t \mathbf{i} - 2\pi \sin 2\pi t \mathbf{j}$$

$$K = \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|} = \frac{2\pi}{8\pi} = \frac{1}{4}$$

32.  $\mathbf{r}(t) = 2 \cos \pi t \mathbf{i} + \sin \pi t \mathbf{j}$

$$\mathbf{r}'(t) = -2\pi \sin \pi t \mathbf{i} + \pi \cos \pi t \mathbf{j}$$

$$\|\mathbf{r}'(t)\| = \pi \sqrt{4 \sin^2 \pi t + \cos^2 \pi t}$$

$$\mathbf{T}(t) = \frac{-2 \sin \pi t \mathbf{i} + \cos \pi t \mathbf{j}}{\sqrt{4 \sin^2 \pi t + \cos^2 \pi t}}$$

$$\mathbf{T}'(t) = \frac{-2\pi \cos \pi t \mathbf{i} - 4\pi \sin \pi t \mathbf{j}}{(4 \sin^2 \pi t + \cos^2 \pi t)^{3/2}}$$

$$\begin{aligned} K &= \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|} = \frac{\frac{2\pi}{\pi \sqrt{4 \sin^2 \pi t + \cos^2 \pi t}}}{\frac{2}{\sqrt{4 \sin^2 \pi t + \cos^2 \pi t}}} \\ &= \frac{2}{(4 \sin^2 \pi t + \cos^2 \pi t)^{3/2}} \end{aligned}$$

33.  $\mathbf{r}(t) = a \cos \omega t \mathbf{i} + a \sin \omega t \mathbf{j}$

$$\mathbf{r}'(t) = -a\omega \sin \omega t \mathbf{i} + a\omega \cos \omega t \mathbf{j}$$

$$\mathbf{T}(t) = -\sin \omega t \mathbf{i} + \cos \omega t \mathbf{j}$$

$$\mathbf{T}'(t) = -\omega \cos \omega t \mathbf{i} - \omega \sin \omega t \mathbf{j}$$

$$K = \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|} = \frac{\omega}{a\omega} = \frac{1}{a}$$

34.  $\mathbf{r}(t) = a \cos(\omega t) \mathbf{i} + b \sin(\omega t) \mathbf{j}$

$$\mathbf{r}'(t) = -a\omega \sin(\omega t) \mathbf{i} + b\omega \cos(\omega t) \mathbf{j}$$

$$\mathbf{T}(t) = \frac{-a \sin(\omega t) \mathbf{i} + b \cos(\omega t) \mathbf{j}}{\sqrt{a^2 \sin^2(\omega t) + b^2 \cos^2(\omega t)}}$$

$$\mathbf{T}'(t) = \frac{-ab^2 \omega \cos(\omega t) \mathbf{i} - a^2 b \omega \sin(\omega t) \mathbf{j}}{\left[ a^2 \sin^2(\omega t) + b^2 \cos^2(\omega t) \right]^{3/2}}$$

$$K = \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|} = \frac{\frac{ab\omega}{\sqrt{a^2 \sin^2(\omega t) + b^2 \cos^2(\omega t)}}}{\frac{\omega \sqrt{a^2 \sin^2(\omega t) + b^2 \cos^2(\omega t)}}{\sqrt{a^2 \sin^2(\omega t) + b^2 \cos^2(\omega t)}}}$$

$$= \frac{ab}{\left[ a^2 \sin^2(\omega t) + b^2 \cos^2(\omega t) \right]^{3/2}}$$

35.  $\mathbf{r}(t) = \langle a(\omega t - \sin \omega t), a(1 - \cos \omega t) \rangle$

From Exercise 44, Section 12.4, we have:

$$\mathbf{a} \cdot \mathbf{N} = \frac{a\omega^2}{\sqrt{2}} \cdot \sqrt{1 - \cos \omega t}$$

$$K = \frac{\mathbf{a}(t) \cdot \mathbf{N}(t)}{\|\mathbf{v}(t)\|^2}$$

$$= \frac{\left( \frac{a\omega^2}{\sqrt{2}} \right) \sqrt{1 - \cos \omega t}}{2a^2 \omega^2 (1 - \cos \omega t)} = \frac{\sqrt{2}}{4a\sqrt{1 - \cos \omega t}}$$

36.  $\mathbf{r}(t) = \langle \cos \omega t + \omega t \sin \omega t, \sin \omega t - \omega t \cos \omega t \rangle$

From Exercise 43, Section 12.4, we have:

$$\mathbf{a} \cdot \mathbf{N} = \omega^3 t$$

$$K = \frac{\mathbf{a}(t) \cdot \mathbf{N}(t)}{\|\mathbf{v}\|^2} = \frac{\omega^3 t}{\omega^4 t^2} = \frac{1}{\omega t}$$

37.  $\mathbf{r}(t) = t \mathbf{i} + t^2 \mathbf{j} + \frac{t^2}{2} \mathbf{k}$

$$\mathbf{r}'(t) = \mathbf{i} + 2t \mathbf{j} + t \mathbf{k}$$

$$\mathbf{T}(t) = \frac{\mathbf{i} + 2t \mathbf{j} + t \mathbf{k}}{\sqrt{1 + 5t^2}}$$

$$\mathbf{T}'(t) = \frac{-5t \mathbf{i} + 2 \mathbf{j} + \mathbf{k}}{(1 + 5t^2)^{3/2}}$$

$$K = \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|} = \frac{\frac{\sqrt{5}}{\sqrt{1 + 5t^2}}}{\frac{\sqrt{1 + 5t^2}}{\sqrt{1 + 5t^2}}} = \frac{\sqrt{5}}{(1 + 5t^2)^{3/2}}$$

38.  $\mathbf{r}(t) = 2t^2\mathbf{i} + t\mathbf{j} + \frac{1}{2}t^2\mathbf{k}$

$$\mathbf{r}'(t) = 4t\mathbf{i} + \mathbf{j} + t\mathbf{k}$$

$$\mathbf{T}(t) = \frac{4t\mathbf{i} + \mathbf{j} + t\mathbf{k}}{\sqrt{1+17t^2}}$$

$$\mathbf{T}'(t) = \frac{4\mathbf{i} - 17t\mathbf{j} + \mathbf{k}}{(1+17t^2)^{3/2}}$$

$$K = \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|} = \frac{\sqrt{289t^2 + 17}}{(1+17t^2)^{3/2}} / (1+17t^2)^{1/2}$$

$$= \frac{\sqrt{17}}{(1+17t^2)^{3/2}}$$

40.  $\mathbf{r}(t) = e^{2t}\mathbf{i} + e^{2t} \cos t\mathbf{j} + e^{2t} \sin t\mathbf{k}$

$$\mathbf{r}'(t) = 2e^{2t}\mathbf{i} + (2e^{2t} \cos t - e^{2t} \sin t)\mathbf{j} + (2e^{2t} \sin t + e^{2t} \cos t)\mathbf{k} = e^{2t}[2\mathbf{i} + (2 \cos t - \sin t)\mathbf{j} + (2 \sin t + \cos t)\mathbf{k}]$$

$$\|\mathbf{r}'(t)\| = e^{2t}[4 + (4 \cos^2 t - 4 \cos t \sin t + \sin^2 t) + (4 \sin^2 t + 4 \sin t \cos t + \cos^2 t)]^{1/2} = e^{2t}[9]^{1/2} = 3e^{2t}$$

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = \frac{2}{3}\mathbf{i} + \left(\frac{2}{3} \cos t - \frac{1}{3} \sin t\right)\mathbf{j} + \left(\frac{2}{3} \sin t + \frac{1}{3} \cos t\right)\mathbf{k}$$

$$\mathbf{T}'(t) = \left(-\frac{2}{3} \sin t - \frac{1}{3} \cos t\right)\mathbf{j} + \left(\frac{2}{3} \cos t - \frac{1}{3} \sin t\right)\mathbf{k}$$

$$\|\mathbf{T}'(t)\| = \left[\left(\frac{4}{9} \sin^2 t + \frac{1}{9} \cos^2 t + \frac{4}{9} \sin t \cos t\right) + \left(\frac{4}{9} \cos^2 t + \frac{1}{9} \sin^2 t - \frac{4}{9} \cos t \sin t\right)\right]^{1/2} = \frac{\sqrt{5}}{3}$$

$$K = \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|} = \frac{\frac{\sqrt{5}}{3}}{3e^{2t}} = \frac{\sqrt{5}}{9e^{2t}}$$

41.  $\mathbf{r}(t) = 3t\mathbf{i} + 2t^2\mathbf{j}, P(-3, 2) \Rightarrow t = -1$

$$x = 3t, x' = 3, x'' = 0$$

$$y = 2t^2, y' = 4t, y'' = 4$$

$$K = \frac{|x'y'' - y'x''|}{[(x')^2 + (y')^2]^{3/2}} = \frac{|3(4) - 0|}{[9 + (4t)^2]^{3/2}}$$

$$\text{At } t = -1, K = \frac{12}{(9+16)^{3/2}} = \frac{12}{125}$$

42.  $\mathbf{r}(t) = e^t\mathbf{i} + 4t\mathbf{j}, P(1, 0) \Rightarrow t = 0$

$$x = e^t, x' = e^t, x'' = e^t$$

$$y = 4t, y' = 4, y'' = 0$$

$$K = \frac{|x'y'' - y'x''|}{[(x')^2 + (y')^2]^{3/2}} = \frac{|0 - 4|}{(1+16)^{3/2}} = \frac{4}{17^{3/2}}$$

39.  $\mathbf{r}(t) = 4t\mathbf{i} + 3 \cos t\mathbf{j} + 3 \sin t\mathbf{k}$

$$\mathbf{r}'(t) = 4\mathbf{i} - 3 \sin t\mathbf{j} + 3 \cos t\mathbf{k}$$

$$\mathbf{T}(t) = \frac{1}{5}[4\mathbf{i} - 3 \sin t\mathbf{j} + 3 \cos t\mathbf{k}]$$

$$\mathbf{T}'(t) = \frac{1}{5}[-3 \cos t\mathbf{j} - 3 \sin t\mathbf{k}]$$

$$K = \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|} = \frac{3/5}{5} = \frac{3}{25}$$

43.  $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + \frac{t^3}{4}\mathbf{k}, P(2, 4, 2) \Rightarrow t = 2$

$$\mathbf{r}'(t) = \mathbf{i} + 2t\mathbf{j} + \frac{3}{4}t^2\mathbf{k}$$

$$\mathbf{r}'(2) = \mathbf{i} + 4\mathbf{j} + 3\mathbf{k}, \|\mathbf{r}'(2)\| = \sqrt{26}$$

$$\mathbf{r}''(t) = 2\mathbf{j} + \frac{3}{2}t\mathbf{k}$$

$$\mathbf{r}''(2) = 2\mathbf{j} + 3\mathbf{k}$$

$$\mathbf{r}'(2) \times \mathbf{r}''(2) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 4 & 3 \\ 0 & 2 & 3 \end{vmatrix} = 6\mathbf{i} - 3\mathbf{j} + 2\mathbf{k}$$

$$\|\mathbf{r}'(2) \times \mathbf{r}''(2)\| = \sqrt{49} = 7$$

$$K = \frac{\|\mathbf{r}' \times \mathbf{r}''\|}{\|\mathbf{r}'\|^3} = \frac{7}{26^{3/2}} = \frac{7\sqrt{26}}{676}$$

44.  $\mathbf{r}(t) = e^t \cos t \mathbf{i} + e^t \sin t \mathbf{j} + e^t \mathbf{k}$ ,  $P(1, 0, 1) \Rightarrow t = 0$

$$\mathbf{r}'(t) = (e^t \cos t - e^t \sin t) \mathbf{i} + (e^t \sin t + e^t \cos t) \mathbf{j} + e^t \mathbf{k}$$

$$\|\mathbf{r}'(t)\| = e^t \sqrt{(\cos^2 t - 2 \cos t \sin t + \sin^2 t) + (\sin^2 t + 2 \sin t \cos t + \cos^2 t) + 1} = \sqrt{3}e^t$$

$$\mathbf{T}(t) = \frac{1}{\sqrt{3}} [(\cos t - \sin t) \mathbf{i} + (\sin t + \cos t) \mathbf{j} + \mathbf{k}]$$

$$\mathbf{T}'(t) = \frac{1}{\sqrt{3}} [(-\sin t - \cos t) \mathbf{i} + (\cos t - \sin t) \mathbf{j}]$$

$$\mathbf{r}'(0) = \mathbf{i} + \mathbf{j} + \mathbf{k} \Rightarrow \|\mathbf{r}'(0)\| = \sqrt{3}$$

$$\mathbf{T}'(0) = \frac{1}{\sqrt{3}} (-\mathbf{i} + \mathbf{j}) \Rightarrow \|\mathbf{T}'(0)\| = \frac{\sqrt{2}}{\sqrt{3}}$$

$$K = \frac{\|\mathbf{T}'(0)\|}{\|\mathbf{r}'(0)\|} = \frac{\sqrt{2}/\sqrt{3}}{\sqrt{3}} = \frac{\sqrt{2}}{3}$$

45.  $y = 3x - 2$

Because  $y'' = 0$ ,  $K = 0$ , and the radius of curvature is undefined.

46.  $y = mx + b$

Because  $y'' = 0$ ,  $K = 0$ , and the radius of curvature is undefined.

47.  $y = 2x^2 + 3$ ,  $x = -1$

$$y' = 4x$$

$$y'' = 4$$

$$K = \frac{4}{[1 + (-4)^2]^{3/2}} = \frac{4}{17^{3/2}} \approx 0.057$$

$$\frac{1}{K} = \frac{17^{3/2}}{4} \approx 17.523 \text{ (radius of curvature)}$$

48.  $y = 2x + \frac{4}{x}$ ,  $x = 1$

$$y' = 2 - \frac{4}{x^2}, y'(1) = -2$$

$$y'' = \frac{8}{x^3}, y''(1) = 8$$

$$K = \frac{|y''|}{[1 + (y')^2]^{3/2}} = \frac{8}{(1 + 4)^{3/2}} = \frac{8}{5^{3/2}}$$

$$\frac{1}{K} = \frac{5^{3/2}}{8} \text{ (radius of curvature)}$$

49.  $y = \cos 2x$ ,  $x = 2\pi$

$$y' = -2 \sin 2x$$

$$y'' = -4 \cos 2x$$

At  $x = 2\pi$ ,  $y = 1$ ,  $y' = 0$ ,  $y'' = -4$

$$K = \frac{|-4|}{[1 + 0^2]^{3/2}} = 4$$

$$\frac{1}{K} = \frac{1}{4}$$

50.  $y = e^{3x}$ ,  $x = 0$

$$y' = 3e^{3x}, y'' = 9e^{3x}$$

At  $x = 0$ ,  $y = 1$ ,  $y' = 3$ ,  $y'' = 9$

$$K = \frac{9}{[1 + 3^2]^{3/2}} = \frac{9}{10^{3/2}}$$

$$\frac{1}{K} = \frac{10\sqrt{10}}{9}$$

51.  $y = \sqrt{a^2 - x^2}$ ,  $x = 0$

$$y' = \frac{-x}{\sqrt{a^2 - x^2}}$$

$$y'' = \frac{a^2}{(a^2 - x^2)^{3/2}}$$

At  $x = 0$ :  $y' = 0$

$$y'' = \frac{1}{a}$$

$$K = \frac{1/a}{(1 + 0^2)^{3/2}} = \frac{1}{a}$$

$$\frac{1}{K} = a \text{ (radius of curvature)}$$

52.  $y = \frac{3}{4}\sqrt{16 - x^2}$

$$y' = \frac{-9x}{16y}$$

$$y'' = \frac{-[9 + (16y')^2]}{16y}$$

At  $x = 0$ :  $y' = 0$

$$y'' = -\frac{3}{16}$$

$$K = \left| \frac{-3/16}{(1+0^2)^{3/2}} \right| = \frac{3}{16}$$

$$\frac{1}{K} = \frac{16}{3} \text{ (radius of curvature)}$$

53.  $y = x^3$ ,  $x = 2$

$$y' = 3x^2, y'' = 6x$$

At  $x = 2$ ,  $y = 8$ ,  $y' = 12$ ,  $y'' = 12$

$$K = \frac{12}{[1 + (12)^2]^{3/2}} = \frac{12}{(145)^{3/2}}$$

$$\frac{1}{K} = \frac{145\sqrt{145}}{12}$$

54.  $y = x^n$ ,  $x = 1$ ,  $n \geq 2$

$$y' = nx^{n-1}$$

$$y'' = n(n-1)x^{n-2}$$

At  $x = 1$ ,  $y = 1$ ,  $y' = n$ ,  $y'' = n(n-1)$

$$K = \frac{n(n-1)}{[1+n^2]^{3/2}}$$

55. (a) Point on circle:  $\left(\frac{\pi}{2}, 1\right)$

$$\text{Center: } \left(\frac{\pi}{2}, 0\right)$$

$$\text{Equation: } \left(x - \frac{\pi}{2}\right)^2 + y^2 = 1$$

(b) The circles have different radii because the curvature

$$\text{is different and } r = \frac{1}{K}.$$

56. (a)  $y = \frac{4x^2}{x^2 + 3}$

$$y' = \frac{24x}{(x^2 + 3)^2}$$

$$y'' = \frac{72(1-x^2)}{(x^2 + 3)^3}$$

At  $x = 0$ :  $y' = 0$

$$y'' = \frac{72}{27} = \frac{8}{3}$$

$$K = \frac{8/3}{(1+0^2)^{3/2}} = \frac{8}{3}$$

$$r = \frac{1}{K} = \frac{3}{8}$$

Center:  $\left(0, \frac{3}{8}\right)$

Equation:  $x^2 + \left(y - \frac{3}{8}\right)^2 = \frac{9}{64}$

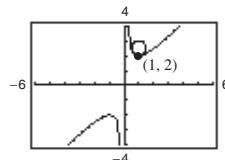
(b) The circles have different radii since the curvature is different and  $r = \frac{1}{K}$ .

57.  $y = x + \frac{1}{x}$ ,  $y' = \frac{1}{x^2}$ ,  $y'' = \frac{2}{x^3}$

$$K = \frac{2}{(1+0^2)^{3/2}} = 2 \text{ at } (1, 2)$$

Radius of curvature = 1/2. Because the tangent line is horizontal at  $(1, 2)$ , the normal line is vertical. The center of the circle is 1/2 unit above the point  $(1, 2)$  at  $(1, 5/2)$ .

Circle:  $(x-1)^2 + \left(y - \frac{5}{2}\right)^2 = \frac{1}{4}$



58.  $y = \ln x, \quad x = 1$

$$y' = \frac{1}{x}, \quad y'' = -\frac{1}{x^2}$$

$$y'(1) = 1, \quad y''(1) = -1$$

$$K = \frac{|-1|}{(1 + (1)^2)^{3/2}} = \frac{1}{2^{3/2}}, r = \frac{1}{K} = 2^{3/2} = 2\sqrt{2}$$

The slope of the tangent line at  $(1, 0)$  is  $y'(1) = 1$ .

The slope of the normal line is  $-1$ .

Equation of normal line:  $y = -(x - 1) = -x + 1$

The center of the circle is on the normal line  $2\sqrt{2}$  units away from the point  $(1, 0)$ .

$$\sqrt{(1-x)^2 + (0-y)^2} = 2\sqrt{2}$$

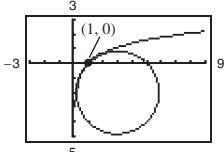
$$(1-x)^2 + (x-1)^2 = 8$$

$$2x^2 - 4x + 2 = 8$$

$$2(x^2 - 2x - 3) = 0$$

$$2(x-3)(x+1) = 0$$

$$x = 3 \text{ or } x = -1$$



Because the circle is below the curve,  $x = 3$  and  $y = -2$ .

Center of circle:  $(3, -2)$

Equation of circle:  $(x - 3)^2 + (y + 2)^2 = 8$

59.  $y = e^x, \quad y = 0$

$$y' = e^x, \quad y'' = e^x$$

$$y'(0) = 1, \quad y''(0) = 1$$

$$K = \frac{1}{(1 + 1^2)^{3/2}} = \frac{1}{2^{3/2}} = \frac{1}{2\sqrt{2}}, r = \frac{1}{K} = 2\sqrt{2}$$

The slope of the tangent line at  $(0, 1)$  is  $y'(0) = 1$ .

The slope of the normal line is  $-1$ .

Equation of normal line:  $y - 1 = -x$  or  $y = -x + 1$

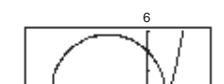
The center of the circle is on the normal line  $2\sqrt{2}$  units away from the point  $(0, 1)$ .

$$\sqrt{(0-x)^2 + (1-y)^2} = 2\sqrt{2}$$

$$x^2 + x^2 = 8$$

$$x^2 = 4$$

$$x = \pm 2$$



Because the circle is above the curve,  $x = -2$  and  $y = 3$ .

Center of circle:  $(-2, 3)$

Equation of circle:  $(x + 2)^2 + (y - 3)^2 = 8$

60.  $y = \frac{1}{3}x^3, \quad x = 1$

$$y' = x^2, \quad y'' = 2$$

$$y'(1) = 1, \quad y''(1) = 2$$

$$K = \frac{2}{(1 + 1^2)^{3/2}} = \frac{1}{\sqrt{2}}, r = \frac{1}{K} = \sqrt{2}$$

The slope of the tangent line at  $\left(1, \frac{1}{3}\right)$  is  $y'(1) = 1$ .

The slope of the normal line is  $-1$ .

Equation of normal line:  $y - \frac{1}{3} = -(x - 1)$  or  $y = -x + \frac{4}{3}$

The center of the circle is on the normal line  $\sqrt{2}$  units away from the point  $\left(1, \frac{1}{3}\right)$ .

$$\sqrt{(1-x)^2 + \left(\frac{1}{3} - y\right)^2} = \sqrt{2}$$

$$(1-x)^2 + (x-1)^2 = 2$$

$$(x-1)^2 = 1$$

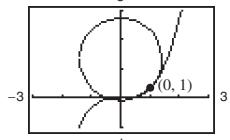
$$x = 0 \text{ or } x = 2$$

Because the circle is above the curve,  $x = 0$  and

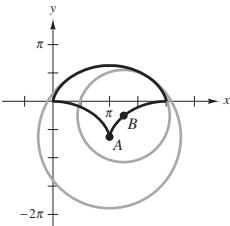
$$y = \frac{4}{3}$$

Center of circle:  $\left(0, \frac{4}{3}\right)$

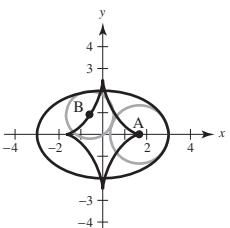
$$\text{Equation of circle: } x^2 + \left(y - \frac{4}{3}\right)^2 = 2$$



61.



62.



63.  $y = (x - 1)^2 + 3, \quad y' = 2(x - 1), \quad y'' = 2$

$$K = \frac{2}{(1 + [2(x - 1)]^2)^{3/2}} = \frac{2}{[1 + 4(x - 1)^2]^{3/2}}$$

(a)  $K$  is maximum when  $x = 1$  or at the vertex  $(1, 3)$ .

(b)  $\lim_{x \rightarrow \infty} K = 0$

64.  $y = x^3$ ,  $y' = 3x^2$ ,  $y'' = 6x$

$$K = \left| \frac{6x}{(1+9x^4)^{3/2}} \right|$$

(a)  $K$  is maximum at  $\left(\frac{1}{\sqrt[4]{45}}, \frac{1}{\sqrt[4]{45^3}}\right), \left(\frac{-1}{\sqrt[4]{45}}, \frac{-1}{\sqrt[4]{45^3}}\right)$ .

(b)  $\lim_{x \rightarrow \infty} K = 0$

65.  $y = x^{2/3}$ ,  $y' = \frac{2}{3}x^{-1/3}$ ,  $y'' = -\frac{2}{9}x^{-4/3}$

$$K = \left| \frac{(-2/9)x^{-4/3}}{\left[1 + (4/9)x^{-2/3}\right]^{3/2}} \right| = \left| \frac{6}{x^{1/3}(9x^{2/3} + 4)^{3/2}} \right|$$

(a)  $K \rightarrow \infty$  as  $x \rightarrow 0$ . No maximum

(b)  $\lim_{x \rightarrow \infty} K = 0$

66.  $y = \frac{1}{x}$ ,  $y' = -\frac{1}{x^2}$ ,  $y'' = \frac{2}{x^3}$ . Assume  $x > 0$ .

$$K = \frac{|y''|}{\left[1 + (y')^2\right]^{3/2}} = \frac{|2/x^3|}{\left(1 + 1/x^4\right)^{3/2}} = \frac{2x^3}{(x^4 + 1)^{3/2}}$$

$$\frac{dK}{dx} = \frac{6x^2(1-x^4)}{(x^4 + 1)^{5/2}}$$

(a)  $K$  has a maximum at  $x = 1$   
(and  $x = -1$  by symmetry).

(b)  $\lim_{x \rightarrow \infty} K = 0$

69.  $y = \sinh x$ ,  $y' = \cosh x$ ,  $y'' = \sinh x$

$$K = \frac{|\sinh x|}{\left[1 + (\cosh x)^2\right]^{3/2}}$$

(a) By symmetry, consider  $x \geq 0$ , so  $|\sinh x| = \sinh x$ :

$$K' = \frac{\left[1 + (\cosh x)^2\right]^{3/2} \cosh x - (\sinh x)^2 \left(1 + (\cosh x)^2\right)^{1/2} 2 \sinh x \cosh x}{\left[1 + (\cosh x)^2\right]^3}$$

Setting  $K' = 0$ :

$$\left[1 + (\cosh x)^2\right] \cosh x = (\sinh x)^2 3 \cosh x$$

$$1 + (\cosh x)^2 = 3(\sinh x)^2$$

$$1 + (1 + \sinh^2 x) = 3 \sinh^2 x$$

$$\sinh^2(x) = 1$$

$$x = \operatorname{arcsinh}(1) \approx 0.8814, \text{ Maximum}$$

Also,  $x = -\operatorname{arcsinh}(1)$

Maximum curvature at  $(\pm \operatorname{arcsinh}(1), 1)$

(b)  $\lim_{x \rightarrow \infty} K = 0$

67.  $y = \ln x$ ,  $y' = \frac{1}{x}$ ,  $y'' = -\frac{1}{x^2}$

$$K = \left| \frac{-1/x^2}{\left[1 + (1/x)^2\right]^{3/2}} \right| = \frac{x}{(x^2 + 1)^{3/2}}$$

$$\frac{dK}{dx} = \frac{-2x^2 + 1}{(x^2 + 1)^{5/2}}$$

(a)  $K$  has a maximum when  $x = \frac{1}{\sqrt{2}}$ .

(b)  $\lim_{x \rightarrow \infty} K = 0$

68.  $y = e^x$ ,  $y' = y'' = e^x$

$$K = \frac{|y''|}{\left[1 + (y')^2\right]^{3/2}} = \frac{e^x}{(1 + e^{2x})^{3/2}}$$

$$\frac{dK}{dx} = \frac{e^x(1 - 2e^{2x})}{(1 + e^{2x})^{5/2}}$$

(a)  $1 - 2e^{2x} = 0 \Rightarrow e^{2x} = \frac{1}{2} \Rightarrow x = \frac{1}{2} \ln\left(\frac{1}{2}\right) = -\frac{1}{2} \ln 2$

$K$  has maximum curvature at  $x = -\frac{1}{2} \ln 2$ .

(b)  $\lim_{x \rightarrow \infty} K = 0$

70.  $y = \cosh x$ ,  $y' = \sinh x$ ,  $y'' = \cosh x$

$$K = \frac{\cosh x}{[1 + \sinh^2 x]^{3/2}} = \frac{\cosh x}{[\cosh^2 x]^{3/2}} = \frac{1}{\cosh^2 x}$$

(a)  $K' = \frac{-2 \sinh x}{\cosh^3 x} = 0$  when  $x = 0$ , Maximum

(b)  $\lim_{x \rightarrow \infty} K = 0$

71.  $y = 1 - x^3$ ,  $y' = -3x^2$ ,  $y'' = -6x$

$$K = \frac{|-6x|}{[1 + 9x^4]^{3/2}}$$

Curvature is 0 at  $x = 0$ :  $(0, 1)$ .

72.  $y = (x - 1)^3 + 3$ ,  $y' = 3(x - 1)^2$ ,  $y'' = 6(x - 1)$

$$K = \frac{|y''|}{[1 + (y')^2]^{3/2}} = \frac{|6(x - 1)|}{[1 + 9(x - 1)^4]^{3/2}} = 0 \text{ at } x = 1.$$

Curvature is 0 at  $(1, 3)$ .

73.  $y = \cos x$ ,  $y' = -\sin x$ ,  $y'' = -\cos x$

$$K = \frac{|y''|}{[1 + (y')^2]^{3/2}} = \frac{|-\cos x|}{(1 + \sin^2 x)^{3/2}} = 0 \text{ for}$$

$$x = \frac{\pi}{2} + K\pi.$$

Curvature is 0 at  $\left(\frac{\pi}{2} + K\pi, 0\right)$ .

78.  $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j}$

(a)  $\frac{dx}{dt} = 1$ ,  $\frac{dy}{dt} = 2t$

$$\begin{aligned} s &= \int_0^2 \sqrt{1 + 4t^2} dt = \frac{1}{2} \int_0^2 \sqrt{1 + 4t^2} (2t) dt (u = 2t) = \frac{1}{2} \cdot \frac{1}{2} \left[ 2t\sqrt{1 + 4t^2} + \ln|2t + \sqrt{1 + 4t^2}| \right]_0^2 \quad (\text{Theorem 8.2}) \\ &= \frac{1}{4} \left[ 4\sqrt{17} + \ln|4 + \sqrt{17}| \right] \approx 4.647 \end{aligned}$$

(b) Let  $y = x^2$ ,  $y' = 2x$ ,  $y'' = 2$

At  $t = 0$ ,  $x = 0$ ,  $y = 0$ ,  $y' = 0$ ,  $y'' = 2$ ,  $K = 2$

$$[1 + 0]^{3/2} = 2$$

At  $t = 1$ ,  $x = 1$ ,  $y = 1$ ,  $y' = 2$ ,  $y'' = 2$

$$K = \frac{2}{[1 + (2)^2]^{3/2}} = \frac{2}{5^{3/2}} \approx 0.179$$

At  $t = 2$ ,  $x = 2$ ,  $y = 4$ ,  $y' = 4$ ,  $y'' = 2$

$$K = \frac{2}{[1 + 16]^{3/2}} = \frac{2}{17^{3/2}} \approx 0.0285$$

(c) As  $t$  changes from 0 to 2, the curvature decreases.

74.  $y = \sin x$ ,  $y' = \cos x$ ,  $y'' = -\sin x$

$$K = \frac{|\sin x|}{(1 + \cos^2 x)^{3/2}} = 0 \text{ for } x = n\pi.$$

Curvature is 0 for  $x = n\pi$ :  $(n\pi, 0)$

75. (a)  $s = \int_a^b \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2} dt = \int_a^b \|r'(t)\| dt$

(b) Plane:  $K = \left\| \frac{d\mathbf{T}}{dS} \right\| = \|\mathbf{T}'(s)\|$

Space:  $K = \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|} = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3}$

Answers will vary

76. The curve is a line.

77.  $K = \frac{|y''|}{[1 + (y')^2]^{3/2}}$

At the smooth relative extremum  $y' = 0$ , so  $K = |y''|$ .

Yes, for example,  $y = x^4$  has a curvature of 0 at its relative minimum  $(0, 0)$ . The curvature is positive at any other point on the curve.

79. Endpoints of the major axis:  $(\pm 2, 0)$

Endpoints of the minor axis:  $(0, \pm 1)$

$$x^2 + 4y^2 = 4$$

$$2x + 8yy' = 0$$

$$y' = -\frac{x}{4y}$$

$$y'' = \frac{(4y)(-1) - (-x)(4y')}{16y^2}$$

$$= \frac{-4y - (x^2/y)}{16y^2} = \frac{-(4y^2 + x^2)}{16y^3} = \frac{-1}{4y^3}$$

$$K = \frac{|-1/4y^3|}{\left[1 + (-x/4y)^2\right]^{3/2}} = \frac{|-16|}{(16y^2 + x^2)^{3/2}}$$

$$= \frac{16}{(12y^2 + 4)^{3/2}} = \frac{16}{(16 - 3x^2)^{3/2}}$$

So, because  $-2 \leq x \leq 2$ ,  $K$  is largest when  $x = \pm 2$  and smallest when  $x = 0$ .

80.  $y_1 = ax(b-x)$ ,  $y_2 = \frac{x}{x+2}$

You observe that  $(0, 0)$  is a solution point to both equations. So, the point  $P$  is origin.

$$y_1 = ax(b-x)$$
,  $y_1' = a(b-2x)$ ,  $y_1'' = -2a$

$$y_2 = \frac{x}{x+2}$$
,  $y_2' = \frac{2}{(x+2)^2}$ ,  $y_2'' = \frac{-4}{(x+2)^3}$

$$\text{At } P, y_1'(0) = ab \text{ and } y_2'(0) = \frac{2}{(0+2)^2} = \frac{1}{2}.$$

Because the curves have a common tangent at  $P$ ,

$$y_1'(0) = y_2'(0) \text{ or } ab = \frac{1}{2}. \text{ So, } y_1'(0) = \frac{1}{2}.$$
 Because the

curves have the same curvature at  $P$ ,  $K_1(0) = K_2(0)$ .

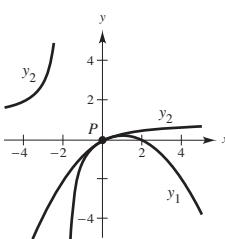
$$K_1(0) = \left| \frac{y_1''(0)}{\left[1 + (y_1(0))^2\right]^{3/2}} \right| = \left| \frac{-2a}{\left[1 + (1/2)^2\right]^{3/2}} \right|$$

$$K_2(0) = \left| \frac{y_2''(0)}{\left[1 + (y_2(0))^2\right]^{3/2}} \right| = \left| \frac{-1/2}{\left[1 + (1/2)^2\right]^{3/2}} \right|$$

So,  $2a = \pm \frac{1}{2}$  or  $a = \pm \frac{1}{4}$ . In order that the curves intersect at only one point, the parabola must be concave downward. So,

$$a = \frac{1}{4} \text{ and } b = \frac{1}{2a} = 2.$$

$$y_1 = \frac{1}{4}x(2-x) \text{ and } y_2 = \frac{x}{x+2}$$



81.  $f(x) = x^4 - x^2$

$$(a) K = \frac{2|6x^2 - 1|}{|16x^6 - 16x^4 + 4x^2 + 1|^{3/2}}$$

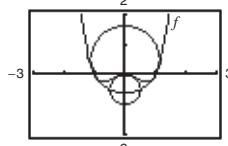
(b) For  $x = 0$ ,  $K = 2$ .  $f(0) = 0$ . At  $(0, 0)$ , the circle of curvature has radius  $\frac{1}{2}$ . Using the symmetry of

$$\text{the graph of } f, \text{ you obtain } x^2 + \left(y + \frac{1}{2}\right)^2 = \frac{1}{4}.$$

For  $x = 1$ ,  $K = (2\sqrt{5})/5$ .  $f(1) = 0$ . At  $(1, 0)$ , the circle of curvature has radius  $\frac{\sqrt{5}}{2} = \frac{1}{K}$ .

Using the graph of  $f$ , you see that the center of curvature is  $(0, \frac{1}{2})$ . So,

$$x^2 + \left(y - \frac{1}{2}\right)^2 = \frac{5}{4}.$$



To graph these circles, use

$$y = -\frac{1}{2} \pm \sqrt{\frac{1}{4} - x^2} \text{ and } y = \frac{1}{2} \pm \sqrt{\frac{5}{4} - x^2}.$$

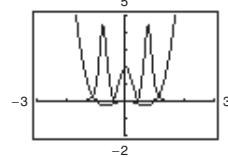
(c) The curvature tends to be greatest near the extrema of  $f$ , and  $K$  decreases as  $x \rightarrow \pm\infty$ .  $f$  and  $K$ , however, do not have the same critical numbers.

Critical numbers of  $f$ :

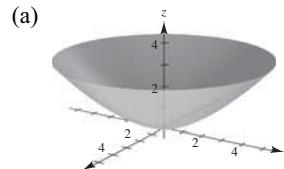
$$x = 0, \pm\frac{\sqrt{2}}{2} \approx \pm 0.7071$$

Critical numbers of  $K$ :

$$x = 0, \pm 0.7647, \pm 0.4082$$



82.  $y = \frac{1}{4}x^{8/5}, 0 \leq x \leq 5$



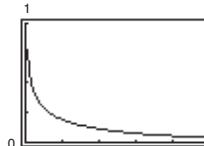
(rotated about  $y$ -axis)

$$(b) V = \int_0^5 2\pi x \left(\frac{5^{8/5}}{4} - \frac{x^{8/5}}{4}\right) dx = \frac{125\pi 5^{3/5}}{9} \approx 114.6 \text{ cm}^3$$

$$(c) y' = \frac{2}{5}x^{3/5}, y'' = \frac{6}{25}x^{-2/5} = \frac{6}{25x^{2/5}}$$

$$K = \frac{\frac{2}{5}x^{2/5}}{\left[1 + \frac{4}{25}x^{6/5}\right]^{3/2}}$$

$$= \frac{6}{25x^{2/5}\left[1 + \frac{4}{25}x^{6/5}\right]^{3/2}}$$



(d) No, the curvature approaches  $\infty$  as  $x \rightarrow 0^+$ . So, any spherical object will hit the sides of the goblet before touching the bottom  $(0, 0)$ .

83. (a) Imagine dropping the circle  $x^2 + (y - k)^2 = 16$

into the parabola  $y = x^2$ . The circle will drop to the point where the tangents to the circle and parabola are equal.

$$y = x^2 \text{ and } x^2 + (y - k)^2 = 16 \Rightarrow x^2 + (x^2 - k)^2 = 16$$

Taking derivatives,  $2x + 2(y - k)y' = 0$  and  $y' = 2x$ . So,

$$(y - k)y' = -x \Rightarrow y' = \frac{-x}{y - k}.$$

So,

$$\frac{-x}{y - k} = 2x \Rightarrow -x = 2x(y - k) \Rightarrow -1 = 2(x^2 - k) \Rightarrow x^2 - k = -\frac{1}{2}.$$

So,

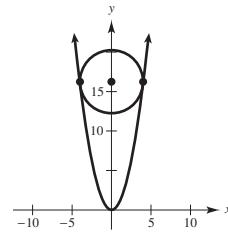
$$x^2 + (x^2 - k)^2 = x^2 + \left(-\frac{1}{2}\right)^2 = 16 \Rightarrow x^2 = 15.75.$$

Finally,  $k = x^2 + \frac{1}{2} = 16.25$ , and the center of the circle is 16.25 units from the vertex of

the parabola. Because the radius of the circle is 4, the circle is 12.25 units from the vertex.

(b) In 2-space, the parabola  $z = y^2$  (or  $z = x^2$ ) has a curvature of  $K = 2$  at  $(0, 0)$ . The radius of

the largest sphere that will touch the vertex has radius  $= 1/K = \frac{1}{2}$ .



84.  $s = \frac{c}{\sqrt{K}}$

$$y = \frac{1}{3}x^3$$

$$y' = x^2$$

$$y'' = 2x$$

$$K = \left| \frac{2x}{(1+x^4)^{3/2}} \right|$$

$$\text{When } x = 1: K = \frac{1}{\sqrt{2}}$$

$$s = \frac{c}{\sqrt{1/\sqrt{2}}} = \sqrt[4]{2}c$$

$$30 = \sqrt[4]{2}c \Rightarrow c = \frac{30}{\sqrt[4]{2}}$$

$$\text{At } x = \frac{3}{2}, K = \frac{3}{[1+(81/16)]^{3/2}} \approx 0.201$$

$$s = \left(\frac{3}{2}\right) = \frac{c}{\sqrt{K}} = \frac{30/\sqrt[4]{2}}{\sqrt{K}} \approx 56.27 \text{ mi/h}$$

85.  $P(x_0, y_0)$  point on curve  $y = f(x)$ . Let  $(\alpha, \beta)$  be the center of curvature. The radius of curvature is  $\frac{1}{K}$ .

$$y' = f'(x). \text{ Slope of normal line at } (x_0, y_0) \text{ is } \frac{-1}{f'(x_0)}.$$

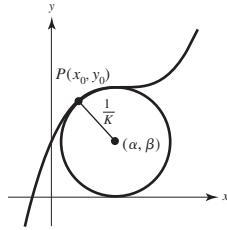
$$\text{Equation of normal line: } y - y_0 = \frac{-1}{f'(x_0)}(x - x_0)$$

$$(\alpha, \beta) \text{ is on the normal line: } -f'(x_0)(\beta - y_0) = \alpha - x_0 \quad \text{Equation 1}$$

$$(x_0, y_0) \text{ lies on the circle: } (x_0 - \alpha)^2 + (y_0 - \beta)^2 = \left(\frac{1}{K}\right)^2 = \left[\frac{\left(1 + f'(x_0)^2\right)^{3/2}}{|f''(x_0)|}\right]^2 \quad \text{Equation 2}$$

Substituting Equation 1 into Equation 2:

$$\begin{aligned} [f'(x_0)(\beta - y_0)]^2 + (y_0 - \beta)^2 &= \left(\frac{1}{K}\right)^2 \\ (\beta - y_0)^2 + \left[1 + f'(x_0)^2\right]^2 &= \frac{\left(1 + f'(x_0)^2\right)^3}{(f''(x_0))^2} \\ (\beta - y_0)^2 &= \frac{\left[1 + f'(x_0)^2\right]^2}{f''(x_0)^2} \end{aligned}$$



When  $f''(x_0) > 0$ ,  $\beta - y_0 > 0$ , and if  $f''(x_0) < 0$ , then  $\beta - y_0 < 0$ .

$$\begin{aligned} \text{So } \beta - y_0 &= \frac{1 + f'(x_0)^2}{f''(x_0)} \\ \beta &= y_0 + \frac{1 + f'(x_0)^2}{f''(x_0)} = y_0 + z \end{aligned}$$

Similarly,  $\alpha = x_0 - f'(x_0)z$ .

86. (a)  $y = f(x) = e^x$ ,  $f'(x) = f''(x) = e^x$ ,  $(0, 1)$

$$z = \frac{1 + f'(0)^2}{f''(0)} = 2$$

$$(\alpha, \beta) = (0 - 2, 1 + 2) = (-2, 3)$$

$$(b) \quad y = \frac{x^2}{2}, y' = x, y'' = 1, \left(1, \frac{1}{2}\right)$$

$$z = \frac{1 + f'(1)^2}{f''(1)} = 2$$

$$(\alpha, \beta) = \left(1 - 2, \frac{1}{2} + 2\right) = \left(-1, \frac{5}{2}\right)$$

$$(c) \quad y = x^2, y' = 2x, y'' = 2, (0, 0)$$

$$z = \frac{1 + f'(0)^2}{f''(0)} = \frac{1}{2}$$

$$(\alpha, \beta) = \left(0, 0 + \frac{1}{2}\right) = \left(0, \frac{1}{2}\right)$$

87.  $r(\theta) = r \cos \theta \mathbf{i} + r \sin \theta \mathbf{j} = f(\theta) \cos \theta \mathbf{i} + f(\theta) \sin \theta \mathbf{j}$

$$x(\theta) = f(\theta) \cos \theta$$

$$y(\theta) = f(\theta) \sin \theta$$

$$x'(\theta) = -f(\theta) \sin \theta + f'(\theta) \cos \theta$$

$$y'(\theta) = f(\theta) \cos \theta + f'(\theta) \sin \theta$$

$$x''(\theta) = -f(\theta) \cos \theta - f'(\theta) \sin \theta - f'(\theta) \sin \theta + f''(\theta) \cos \theta = -f(\theta) \cos \theta - 2f'(\theta) \sin \theta + f''(\theta) \cos \theta$$

$$y''(\theta) = -f(\theta) \sin \theta + f'(\theta) \cos \theta + f'(\theta) \cos \theta + f''(\theta) \sin \theta = -f(\theta) \sin \theta + 2f'(\theta) \cos \theta + f''(\theta) \sin \theta$$

$$K = \frac{|x'y'' - y'x''|}{[(x')^2 + (y')^2]^{3/2}} = \frac{|f^2(\theta) - f(\theta)f''(\theta) + 2(f'(\theta))^2|}{[f^2(\theta) + (f'(\theta))^2]^{3/2}} = \frac{|r^2 - rr'' + 2(r')^2|}{[r^2 + (r')^2]^{3/2}}$$

88. (a)  $r = 1 + \sin \theta$

$$r' = \cos \theta$$

$$r'' = -\sin \theta$$

$$K = \frac{|2(r')^2 - rr'' + r^2|}{[(r')^2 + r^2]^{3/2}} = \frac{|2 \cos^2 \theta - (1 + \sin \theta)(-\sin \theta) + (1 + \sin \theta)^2|}{\sqrt{[\cos^2 \theta + (1 + \sin \theta)^2]^3}} = \frac{3(1 + \sin \theta)}{\sqrt{8(1 + \sin \theta)^3}} = \frac{3}{2\sqrt{2(1 + \sin \theta)}}$$

(b)  $r = \theta$

$$r' = 1$$

$$r'' = 0$$

$$K = \frac{|2(r')^2 - rr'' + r^2|}{[(r')^2 + r^2]^{3/2}} = \frac{2 + \theta^2}{(1 + \theta^2)^{3/2}}$$

(c)  $r = a \sin \theta$

$$r' = a \cos \theta$$

$$r'' = -a \sin \theta$$

$$K = \frac{|2(r')^2 - rr'' + r^2|}{[(r')^2 + r^2]^{3/2}} = \frac{|2a^2 \cos^2 \theta + a^2 \sin^2 \theta + a^2 \sin^2 \theta|}{\sqrt{[a^2 \cos^2 \theta + a^2 \sin^2 \theta]^3}} = \frac{2a^2}{a^3} = \frac{2}{a}, a > 0$$

(d)  $r = e^\theta$

$$r' = e^\theta$$

$$r'' = e^\theta$$

$$K = \frac{|2(r')^2 - rr'' + r^2|}{[(r')^2 + r^2]^{3/2}} = \frac{2e^{2\theta}}{(2e^{2\theta})^{3/2}} = \frac{1}{\sqrt{2}e^\theta}$$

89.  $r = e^{a\theta}, a > 0$

$$r' = ae^{a\theta}$$

$$r'' = a^2 e^{a\theta}$$

$$K = \frac{|2(r')^2 - rr'' + r^2|}{[(r')^2 + r^2]^{3/2}} = \frac{|2a^2 e^{2a\theta} - a^2 e^{2a\theta} + e^{2a\theta}|}{[a^2 e^{2a\theta} + e^{2a\theta}]^{3/2}} = \frac{1}{e^{a\theta} \sqrt{a^2 + 1}}$$

(a) As  $\theta \rightarrow \infty, K \rightarrow 0$ .

(b) As  $a \rightarrow \infty, K \rightarrow 0$ .

90. At the pole,  $r = 0$ .

$$K = \frac{|2(r')^2 - rr'' + r^2|}{[(r')^2 + r^2]^{3/2}} = \frac{|2(r')^2|}{|r'|^3} = \frac{2}{|r'|}$$

91.  $r = 4 \sin 2\theta$

$$r' = 8 \cos 2\theta$$

$$\text{At the pole: } K = \frac{2}{|r'(0)|} = \frac{2}{8} = \frac{1}{4}$$

92.  $r = 6 \cos 3\theta$

$$r' = -18 \sin 3\theta$$

At the pole,

$$\theta = \frac{\pi}{6}, r'\left(\frac{\pi}{6}\right) = -18,$$

and

$$K = \frac{2}{|r'(\pi/6)|} = \frac{2}{|-18|} = \frac{1}{9}.$$

93.  $x = f(t), y = g(t)$

$$y' = \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{g'(t)}{f'(t)}$$

$$y'' = \frac{d}{dt} \left[ \frac{g'(t)}{f'(t)} \right]$$

$$= \frac{f'(t)g''(t) - g'(t)f''(t)}{[f'(t)]^2} = \frac{f'(t)g''(t) - g'(t)f''(t)}{[f'(t)]^3}$$

$$K = \frac{|y''|}{\left[1 + (y')^2\right]^{3/2}} = \frac{\left| \frac{f'(t)g''(t) - g'(t)f''(t)}{[f'(t)]^3} \right|}{\left[1 + \left(\frac{g'(t)}{f'(t)}\right)^2\right]^{3/2}}$$

$$= \frac{\left| f'(t)g''(t) - g'(t)f''(t) \right|}{\sqrt{\left\{ \frac{[f'(t)]^2 + [g'(t)]^2}{[f'(t)]^2} \right\}^3}} = \frac{\left| f'(t)g''(t) - g'(t)f''(t) \right|}{\left( [f'(t)]^2 + [g'(t)]^2 \right)^{3/2}}$$

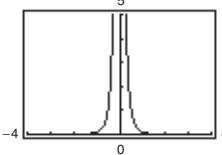
94.  $x(t) = t^3, x'(t) = 3t^2, x''(t) = 6t$

$$y(t) = \frac{1}{2}t^2, y'(t) = t, y''(t) = 1$$

$$K = \frac{|(3t^2)(1) - (t)(6t)|}{[(3t^2)^2 + (t)^2]^{3/2}}$$

$$= \frac{3t^2}{|t^3|(9t^2 + 1)^{3/2}} = \frac{3}{|t|(9t^2 + 1)^{3/2}}$$

$K \rightarrow 0$  as  $t \rightarrow \pm\infty$



$$95. x(\theta) = a(\theta - \sin \theta) \quad y(\theta) = a(1 - \cos \theta)$$

$$x'(\theta) = a(1 - \cos \theta) \quad y'(\theta) = a \sin \theta$$

$$x''(\theta) = a \sin \theta \quad y''(\theta) = a \cos \theta$$

$$\begin{aligned} K &= \frac{|x'(\theta)y''(\theta) - y'(\theta)x''(\theta)|}{\left[x'(\theta)^2 + y'(\theta)^2\right]^{3/2}} \\ &= \frac{|a^2(1 - \cos \theta)\cos \theta - a^2 \sin^2 \theta|}{\left[a^2(1 - \cos \theta)^2 + a^2 \sin^2 \theta\right]^{3/2}} \\ &= \frac{1}{a} \frac{|\cos \theta - 1|}{[2 - 2 \cos \theta]^{3/2}} \\ &= \frac{1}{a} \frac{1 - \cos \theta}{2\sqrt{2}[1 - \cos \theta]^{3/2}} (1 - \cos \theta \geq 0) \\ &= \frac{1}{2a\sqrt{2 - 2 \cos \theta}} = \frac{1}{4a} \csc\left(\frac{\theta}{2}\right) \end{aligned}$$

Minimum:  $\frac{1}{4a}$  ( $\theta = \pi$ )

Maximum: none ( $K \rightarrow \infty$  as  $\theta \rightarrow 0$ )

96. (a)  $\mathbf{r}(t) = 3t^2\mathbf{i} + (3t - t^3)\mathbf{j}$

$$\mathbf{v}(t) = 6t\mathbf{i} + (3 - 3t^2)\mathbf{j}$$

$$\frac{ds}{dt} = \|\mathbf{v}(t)\| = 3(1 + t^2), \frac{d^2s}{dt^2} = 6t$$

$$K = \frac{2}{3(1 + t^2)^2}$$

$$a_T = \frac{d^2s}{dt^2} = 6t$$

$$a_N = K \left( \frac{ds}{dt} \right)^2 = \frac{2}{3(1 + t^2)^2} \cdot 9(1 + t^2)^2 = 6$$

$$(b) \quad \mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + \frac{1}{2}t^2\mathbf{k}$$

$$\mathbf{v}(t) = \mathbf{i} + 2t\mathbf{j} + t\mathbf{k}$$

$$\frac{ds}{dt} = \|\mathbf{v}(t)\| = \sqrt{5t^2 + 1}$$

$$\frac{d^2s}{dt^2} = \frac{5t}{\sqrt{5t^2 + 1}}$$

$$\mathbf{a}(t) = 2\mathbf{j} + \mathbf{k}$$

$$\mathbf{r}'(t) \times \mathbf{r}''(t) = \mathbf{v}(t) \times \mathbf{a}(t)$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2t & t \\ 0 & 2 & 1 \end{vmatrix} = -\mathbf{j} + 2\mathbf{k}$$

$$K = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3} = \frac{\sqrt{5}}{(5t^2 + 1)^{3/2}}$$

$$a_T = \frac{d^2s}{dt^2} = \frac{5t}{\sqrt{5t^2 + 1}}$$

$$a_N = K \left( \frac{ds}{dt} \right)^2 = \frac{\sqrt{5}}{(5t^2 + 1)^{3/2}} (5t^2 + 1)$$

$$= \frac{\sqrt{5}}{\sqrt{5t^2 + 1}}$$

$$100. (a) \quad K = \|\mathbf{T}'(s)\| = \left\| \frac{d\mathbf{T}}{ds} \right\| = \left\| \frac{d\mathbf{T}}{dt} \cdot \frac{dt}{ds} \right\|, \text{ by the Chain Rule}$$

$$= \left\| \frac{d\mathbf{T}/dt}{ds/dt} \right\| = \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{v}(t)\|} = \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|}$$

$$(b) \quad \mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = \frac{\mathbf{r}'(t)}{ds/dt}$$

$$\mathbf{r}'(t) = \frac{ds}{dt} \mathbf{T}(t)$$

$$\mathbf{r}''(t) = \left( \frac{d^2s}{dt^2} \right) \mathbf{T}(t) + \frac{ds}{dt} \mathbf{T}'(t)$$

$$\mathbf{r}'(t) \times \mathbf{r}''(t) = \left( \frac{ds}{dt} \right) \left( \frac{d^2s}{dt^2} \right) [\mathbf{T}(t) \times \mathbf{T}(t)] + \left( \frac{ds}{dt} \right)^2 [\mathbf{T}(t) \times \mathbf{T}'(t)]$$

Because  $\mathbf{T}(t) \times \mathbf{T}(t) = \mathbf{0}$  and  $\frac{ds}{dt} = \|\mathbf{r}'(t)\|$ , you have:

$$\mathbf{r}'(t) \times \mathbf{r}''(t) = \|\mathbf{r}'(t)\|^2 [\mathbf{T}(t) \times \mathbf{T}'(t)]$$

$$\|\mathbf{r}'(t) \times \mathbf{r}''(t)\| = \|\mathbf{r}'(t)\|^2 \|\mathbf{T}(t) \times \mathbf{T}'(t)\| = \|\mathbf{r}'(t)\|^2 \|\mathbf{T}(t)\| \|\mathbf{T}'(t)\| = \|\mathbf{r}'(t)\|^2 (1) K \|\mathbf{r}'(t)\| \text{ from (a)}$$

$$\text{So, } \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3} = K.$$

$$(c) \quad K = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3} = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^2} = \frac{\|\mathbf{v}(t) \times \mathbf{a}(t)\|}{\|\mathbf{r}'(t)\|^2} = \frac{\mathbf{a}(t) \cdot \mathbf{N}(t)}{\|\mathbf{r}'(t)\|^2}$$

$$97. \quad F = ma_N = mK \left( \frac{ds}{dt} \right)^2$$

$$= \left( \frac{5500 \text{ lb}}{32 \text{ ft/sec}^2} \right) \left( \frac{1}{100 \text{ ft}} \right) \left( \frac{30(5280) \text{ ft}}{3600 \text{ sec}} \right)^2 = 3327.5 \text{ lb}$$

$$98. \quad F = ma_N = mK \left( \frac{ds}{dt} \right)^2$$

$$= \left( \frac{6400 \text{ lb}}{32 \text{ ft/sec}^2} \right) \left( \frac{1}{250 \text{ ft}} \right) \left( \frac{35(5280) \text{ ft}}{3600 \text{ sec}} \right)^2$$

$$= \frac{94864}{45} \approx 2108.1 \text{ lb}$$

$$99. \quad y = \cosh x = \frac{e^x + e^{-x}}{2}$$

$$y' = \frac{e^x - e^{-x}}{2} = \sinh x$$

$$y'' = \frac{e^x + e^{-x}}{2} = \cosh x$$

$$K = \frac{|\cosh x|}{\left[ 1 + (\sinh x)^2 \right]^{3/2}} = \frac{\cosh x}{(\cosh^2 x)^{3/2}} = \frac{1}{\cosh^2 x} = \frac{1}{y^2}$$

**101.** False

**102.** False

$$\text{Curvature} = \frac{1}{\text{radius}}$$

**103.** True

**104.** True

$$a_N = K \left( \frac{ds}{dt} \right)^2$$

**105.** Let  $\mathbf{r} = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$ . Then  $r = \|\mathbf{r}\| = \sqrt{x(t)^2 + y(t)^2 + z(t)^2}$  and  $r' = x'(t)\mathbf{i} + y'(t)\mathbf{j} + z'(t)\mathbf{k}$ . Then,

$$\begin{aligned} r \left( \frac{dr}{dt} \right) &= \sqrt{x(t)^2 + y(t)^2 + z(t)^2} \left[ \frac{1}{2} (x(t)^2 + y(t)^2 + z(t)^2)^{-1/2} \cdot (2x(t)x'(t) + 2y(t)y'(t) + 2z(t)z'(t)) \right] \\ &= x(t)x'(t) + y(t)y'(t) + z(t)z'(t) = \mathbf{r} \cdot \mathbf{r}'. \end{aligned}$$

**106.**  $\mathbf{F} = m\mathbf{a} \Rightarrow m\mathbf{a} = \frac{-GmM}{r^3}\mathbf{r}$

$$\mathbf{a} = -\frac{GM}{r^3}\mathbf{r}$$

Because  $\mathbf{r}$  is a constant multiple of  $\mathbf{a}$ , they are parallel. Because  $\mathbf{a} = \mathbf{r}''$  is parallel to  $\mathbf{r}$ ,  $\mathbf{r} \times \mathbf{r}'' = \mathbf{0}$ . Also,

$$\left( \frac{d}{dt} (\mathbf{r} \times \mathbf{r}') \right) = \mathbf{r}' \times \mathbf{r}' + \mathbf{r} \times \mathbf{r}'' = \mathbf{0} + \mathbf{0} = \mathbf{0}. \text{ So, } \mathbf{r} \times \mathbf{r}' \text{ is a constant vector which we will denote by } \mathbf{L}.$$

**107.** Let  $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$  where  $x, y$ , and  $z$  are function of  $t$ , and  $r = \|\mathbf{r}\|$ .

$$\begin{aligned} \frac{d}{dt} \left[ \frac{\mathbf{r}}{r} \right] &= \frac{r\mathbf{r}' - \mathbf{r}(dr/dt)}{r^2} = \frac{r\mathbf{r}' - \mathbf{r}[(\mathbf{r} \cdot \mathbf{r}')/r]}{r^2} \\ &= \frac{r^2\mathbf{r}' - (\mathbf{r} \cdot \mathbf{r}')\mathbf{r}}{r^3} \text{ (using Exercise 105)} \\ &= \frac{(x^2 + y^2 + z^2)(x'\mathbf{i} + y'\mathbf{j} + z'\mathbf{k}) - (xx' + yy' + zz')(x\mathbf{i} + y\mathbf{j} + z\mathbf{k})}{r^3} \\ &= \frac{1}{r^3} [(x'y^2 + x'z^2 - xy' - xz')\mathbf{i} + (x^2y' + z^2y' - xx'y - zz'y)\mathbf{j} + (x^2z' + y^2z' - xx'z - yy'z)\mathbf{k}] \\ &= \frac{1}{r^3} \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ yz' - y'z & -(xz' - x'z) & xy' - x'y \\ x & y & z \end{vmatrix} = \frac{1}{r^3} \{[\mathbf{r} \times \mathbf{r}'] \times \mathbf{r}\} \end{aligned}$$

$$\begin{aligned} \mathbf{108.} \frac{d}{dt} \left[ \frac{\mathbf{r}'}{GM} \times \mathbf{L} - \frac{\mathbf{r}}{r} \right] &= \frac{1}{GM} [\mathbf{r}' \times \mathbf{0} + \mathbf{r}'' \times \mathbf{L}] - \frac{1}{r^3} \{[\mathbf{r} \times \mathbf{r}'] \times \mathbf{r}\} \\ &= \frac{1}{GM} \left[ \mathbf{0} + \left( \frac{-GM\mathbf{r}}{r^3} \right) \times [\mathbf{r} \times \mathbf{r}'] \right] - \frac{1}{r^3} \{[\mathbf{r} \times \mathbf{r}'] \times \mathbf{r}\} \\ &= -\frac{\mathbf{r}}{r^3} \times [\mathbf{r} \times \mathbf{r}'] - \frac{1}{r^3} \{[\mathbf{r} \times \mathbf{r}'] \times \mathbf{r}\} \\ &= \frac{1}{r^3} \{[\mathbf{r} \times \mathbf{r}'] \times \mathbf{r} - [\mathbf{r} \times \mathbf{r}'] \times \mathbf{r}\} = \mathbf{0} \end{aligned}$$

So,  $\left( \frac{\mathbf{r}'}{GM} \right) \times \mathbf{L} - \left( \frac{\mathbf{r}}{r} \right)$  is a constant vector which we will denote by  $\mathbf{e}$ .

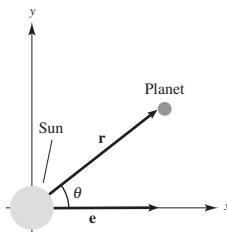
109. From Exercise 106, you have concluded that planetary motion is planar. Assume that the planet moves in the  $xy$ -plane with the sun at the origin. From Exercise 108, you have

$$\mathbf{r}' \times \mathbf{L} = GM\left(\frac{\mathbf{r}}{r} + \mathbf{e}\right).$$

Because  $\mathbf{r}' \times \mathbf{L}$  and  $\mathbf{r}$  are both perpendicular to  $\mathbf{L}$ , so

is  $\mathbf{e}$ . So,  $\mathbf{e}$  lies in the  $xy$ -plane. Situate the coordinate system so that  $\mathbf{e}$  lies along the positive  $x$ -axis and  $\theta$  is the angle between  $\mathbf{e}$  and  $\mathbf{r}$ . Let  $e = \|\mathbf{e}\|$ . Then  $\mathbf{r} \cdot \mathbf{e} = \|\mathbf{r}\| \|\mathbf{e}\| \cos \theta = re \cos \theta$ . Also,

$$\begin{aligned}\|\mathbf{L}\|^2 &= \mathbf{L} \cdot \mathbf{L} \\ &= (\mathbf{r} \times \mathbf{r}') \cdot \mathbf{L} \\ &= \mathbf{r} \cdot (\mathbf{r}' \times \mathbf{L}) \\ &= \mathbf{r} \cdot \left[ GM\left(\mathbf{e} + \frac{\mathbf{r}}{r}\right) \right] \\ &= GM\left[\mathbf{r} \cdot \mathbf{e} + \frac{\mathbf{r} \cdot \mathbf{r}}{r}\right] \\ &= GM[re \cos \theta + r].\end{aligned}$$



$$\text{So, } \frac{\|\mathbf{L}\|^2/GM}{1 + e \cos \theta} = r$$

and the planetary motion is a conic section. Because the planet returns to its initial position periodically, the conic is an ellipse.

110.  $\|\mathbf{L}\| = \|\mathbf{r} \times \mathbf{r}'\|$

Let:  $\mathbf{r} = r(\cos \theta \mathbf{i} + \sin \theta \mathbf{j})$

$$\mathbf{r}' = r(-\sin \theta \mathbf{i} + \cos \theta \mathbf{j}) \frac{d\theta}{dt} \left( \frac{d\mathbf{r}}{dt} = \frac{d\mathbf{r}}{d\theta} \cdot \frac{d\theta}{dt} \right)$$

$$\text{Then: } \mathbf{r} \times \mathbf{r}' = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ r \cos \theta & r \sin \theta & 0 \\ -r \sin \theta \frac{d\theta}{dt} & r \cos \theta \frac{d\theta}{dt} & 0 \end{vmatrix} = r^2 \frac{d\theta}{dt} \mathbf{k} \text{ and } \|\mathbf{L}\| = \|\mathbf{r} \times \mathbf{r}'\| = r^2 \frac{d\theta}{dt}.$$

111.  $A = \frac{1}{2} \int_{\alpha}^{\beta} r^2 d\theta$

So,

$$\frac{dA}{dt} = \frac{dA}{d\theta} \frac{d\theta}{dt} = \frac{1}{2} r^2 \frac{d\theta}{dt} = \frac{1}{2} \|\mathbf{L}\|$$

and  $\mathbf{r}$  sweeps out area at a constant rate.

112. Let  $P$  denote the period. Then

$$A = \int_0^P \frac{dA}{dt} dt = \frac{1}{2} \|\mathbf{L}\| P.$$

Also, the area of an ellipse is  $\pi ab$  where  $2a$  and  $2b$  are the lengths of the major and minor axes.

$$\pi ab = \frac{1}{2} \|\mathbf{L}\| P$$

$$P = \frac{2\pi ab}{\|\mathbf{L}\|}$$

$$P^2 = \frac{4\pi^2 a^2}{\|\mathbf{L}\|^2} (a^2 - c^2) = \frac{4\pi^2 a^2}{\|\mathbf{L}\|^2} a^2 (1 - e^2) = \frac{4\pi^2 a^4}{\|\mathbf{L}\|^2} \left(\frac{ed}{a}\right) = \frac{4\pi^2 ed}{\|\mathbf{L}\|^2} a^3 = \frac{4\pi^2 (\|\mathbf{L}\|^2/GM)}{\|\mathbf{L}\|^2} a^3 = \frac{4\pi^2}{GM} a^3 = K a^3$$

## Review Exercises for Chapter 12

1.  $\mathbf{r}(t) = \tan t\mathbf{i} + \mathbf{j} + t\mathbf{k}$

(a) Domain:  $t \neq \frac{\pi}{2} + n\pi, n$  an integer

(b) Continuous for all  $t \neq \frac{\pi}{2} + n\pi, n$  an integer

2.  $\mathbf{r}(t) = \sqrt{t}\mathbf{i} + \frac{1}{t-4}\mathbf{j} + \mathbf{k}$

(a) Domain:  $[0, 4)$  and  $(4, \infty)$

(b) Continuous except at  $t = 4$

5.  $\mathbf{r}(t) = (2t+1)\mathbf{i} + t^2\mathbf{j} - \sqrt{t+2}\mathbf{k}$

(a)  $\mathbf{r}(0) = \mathbf{i} - \sqrt{2}\mathbf{k}$

(b)  $\mathbf{r}(-2) = -3\mathbf{i} + 4\mathbf{j}$

(c)  $\mathbf{r}(c-1) = (2c-1)\mathbf{i} + (c-1)^2\mathbf{j} - \sqrt{c+1}\mathbf{k}$

$$\begin{aligned} \text{(d)} \quad & \mathbf{r}(1 + \Delta t) - \mathbf{r}(1) = [2(1 + \Delta t) + 1]\mathbf{i} + (1 + \Delta t)^2\mathbf{j} - \sqrt{3 + \Delta t}\mathbf{k} - (3\mathbf{i} + \mathbf{j} - \sqrt{3}\mathbf{k}) \\ &= 2\Delta t\mathbf{i} + (\Delta t^2 + 2\Delta t)\mathbf{j} - (\sqrt{3 + \Delta t} - \sqrt{3})\mathbf{k} \end{aligned}$$

6. (a)  $\mathbf{r}(0) = 3\mathbf{i} + \mathbf{j}$

(b)  $\mathbf{r}\left(\frac{\pi}{2}\right) = -\frac{\pi}{2}\mathbf{k}$

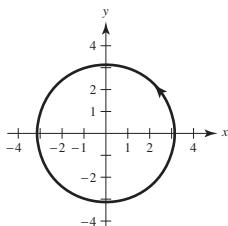
(c)  $\mathbf{r}(s - \pi) = 3 \cos(s - \pi)\mathbf{i} + (1 - \sin(s - \pi))\mathbf{j} - (s - \pi)\mathbf{k}$

$$\begin{aligned} \text{(d)} \quad & \mathbf{r}(\pi + \Delta t) - \mathbf{r}(\pi) = (3 \cos(\pi + \Delta t)\mathbf{i} + (1 - \sin(\pi + \Delta t))\mathbf{j} - (\pi + \Delta t)\mathbf{k}) - (-3\mathbf{i} + \mathbf{j} - \pi\mathbf{k}) \\ &= (-3 \cos \Delta t + 3)\mathbf{i} + \sin \Delta t - \Delta t\mathbf{k} \end{aligned}$$

7.  $\mathbf{r}(t) = \langle \pi \cos t, \pi \sin t \rangle$

$x = \pi \cos t, y = \pi \sin t$

$x^2 + y^2 = \pi^2$ , circle

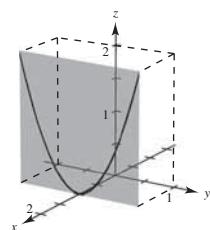


9.  $\mathbf{r}(t) = \mathbf{i} + t\mathbf{j} + t^2\mathbf{k}$

$x = 1$

$y = t$

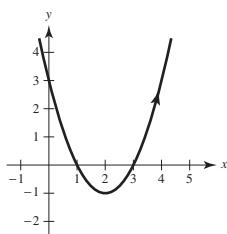
$z = t^2 \Rightarrow z = y^2$



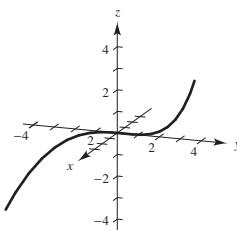
8.  $\mathbf{r}(t) = \langle t+2, t^2-1 \rangle$

$x = t+2 \Rightarrow t = x-2$

$y = t^2 - 1 = (x-2)^2 - 1$ , parabola

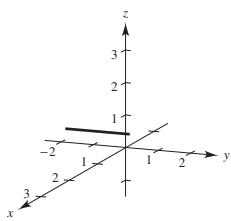


10.  $\mathbf{r}(t) = t^2\mathbf{i} + 3t\mathbf{j} + t^3\mathbf{k}$



11.  $\mathbf{r}(t) = \mathbf{i} + \sin t \mathbf{j} + \mathbf{k}$

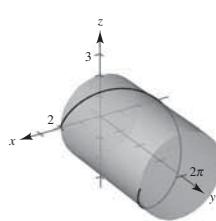
$$x = 1, y = \sin t, z = 1$$



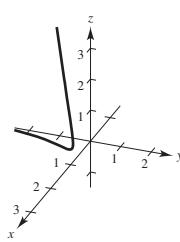
12.  $\mathbf{r}(t) = 2 \cos t \mathbf{i} + t \mathbf{j} + 2 \sin t \mathbf{k}$

$$x = 2 \cos t, y = t, z = 2 \sin t$$

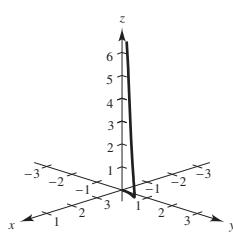
$$x^2 + z^2 = 4$$



13.  $\mathbf{r}(t) = t \mathbf{i} + \ln t \mathbf{j} + \frac{1}{2} t^2 \mathbf{k}$



14.  $\mathbf{r}(t) = \frac{1}{2} t \mathbf{i} + \sqrt{t} \mathbf{j} + \frac{1}{4} t^3 \mathbf{k}$



15. One possible answer is:

$$\mathbf{r}_1(t) = 3t \mathbf{i} + 4t \mathbf{j}, 0 \leq t \leq 1$$

$$\mathbf{r}_2(t) = 3 \mathbf{i} + (4-t) \mathbf{j}, 0 \leq t \leq 4$$

$$\mathbf{r}_3(t) = (3-t) \mathbf{i}, 0 \leq t \leq 3$$

23.  $\mathbf{r}(t) = 3t \mathbf{i} + (t-1) \mathbf{j}, \mathbf{u}(t) = t \mathbf{i} + t^2 \mathbf{j} + \frac{2}{3} t^3 \mathbf{k}$

(a)  $\mathbf{r}'(t) = 3 \mathbf{i} + \mathbf{j}$

(b)  $\mathbf{r}''(t) = 0$

(c)  $\mathbf{r}(t) \cdot \mathbf{u}(t) = 3t^2 + t^2(t-1) = t^3 + 2t^2$

$$D_t [\mathbf{r}(t) \cdot \mathbf{u}(t)] = 3t^2 + 4t$$

16. One possible answer is:

$$\mathbf{r}_1(t) = 4t \mathbf{i}, 0 \leq t \leq 1$$

$$\mathbf{r}_2(t) = 4 \cos t \mathbf{i} + 4 \sin t \mathbf{j}, 0 \leq t \leq \frac{\pi}{2}$$

$$\mathbf{r}_3(t) = (4-t) \mathbf{j}, 0 \leq t \leq 4$$

17. The vector joining the points is  $\langle 7, 4, -10 \rangle$ . One path is

$$\mathbf{r}(t) = \langle -2 + 7t, -3 + 4t, 8 - 10t \rangle.$$

18. The  $x$ - and  $y$ -components are  $2 \cos t$  and  $2 \sin t$ . At

$$t = \frac{3\pi}{2},$$

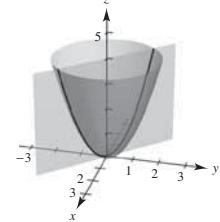
the staircase has made  $\frac{3}{4}$  of a revolution and is 2 meters high. So, one answer is

$$\mathbf{r}(t) = 2 \cos t \mathbf{i} + 2 \sin t \mathbf{j} + \frac{4}{3\pi} t \mathbf{k}.$$

19.  $z = x^2 + y^2, x + y = 0, t = x$

$$x = t, y = -t, z = 2t^2$$

$$\mathbf{r}(t) = t \mathbf{i} - t \mathbf{j} + 2t^2 \mathbf{k}$$

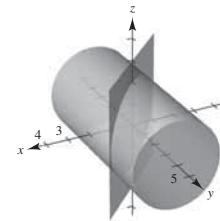


20.  $x^2 + z^2 = 4, x - y = 0, t = x$

$$x = t, y = t, z = \pm\sqrt{4-t^2}$$

$$\mathbf{r}(t) = t \mathbf{i} + t \mathbf{j} + \sqrt{4-t^2} \mathbf{k}$$

$$\mathbf{r}(t) = t \mathbf{i} + t \mathbf{j} - \sqrt{4-t^2} \mathbf{k}$$



21.  $\lim_{t \rightarrow 4^-} (t \mathbf{i} + \sqrt{4-t} \mathbf{j} + \mathbf{k}) = 4 \mathbf{i} + \mathbf{k}$

22.  $\lim_{t \rightarrow 0} \left( \frac{\sin 2t}{t} \mathbf{i} + e^{-t} \mathbf{j} + e^t \mathbf{k} \right) = \left( \lim_{t \rightarrow 0} \frac{2 \cos 2t}{1} \right) \mathbf{i} + \mathbf{j} + \mathbf{k}$   
 $= 2 \mathbf{i} + \mathbf{j} + \mathbf{k}$

$$(d) \mathbf{u}(t) - 2\mathbf{r}(t) = -5t\mathbf{i} + (t^2 - 2t + 2)\mathbf{j} + \frac{2}{3}t^3\mathbf{k}$$

$$D_t[\mathbf{u}(t) - 2\mathbf{r}(t)] = -5\mathbf{i} + (2t - 2)\mathbf{j} + 2t^2\mathbf{k}$$

$$(e) \|\mathbf{r}(t)\| = \sqrt{10t^2 - 2t + 1}$$

$$D_t[\|\mathbf{r}(t)\|] = \frac{10t - 1}{\sqrt{10t^2 - 2t + 1}}$$

$$(f) \mathbf{r}(t) \times \mathbf{u}(t) = \frac{2}{3}(t^4 - t^3)\mathbf{i} - 2t^4\mathbf{j} + (3t^3 - t^2 + t)\mathbf{k}$$

$$D_t[\mathbf{r}(t) \times \mathbf{u}(t)] = \left(\frac{8}{3}t^3 - 2t^2\right)\mathbf{i} - 8t^3\mathbf{j} + (9t^2 - 2t + 1)\mathbf{k}$$

24.  $\mathbf{r}(t) = \sin t\mathbf{i} + \cos t\mathbf{j} + t\mathbf{k}, \mathbf{u}(t) = \sin t\mathbf{i} + \cos t\mathbf{j} + \frac{1}{t}\mathbf{k}$

$$(a) \mathbf{r}'(t) = \cos t\mathbf{i} - \sin t\mathbf{j} + \mathbf{k}$$

$$(b) \mathbf{r}''(t) = -\sin t\mathbf{i} - \cos t\mathbf{j}$$

$$(c) \mathbf{r}(t) \cdot \mathbf{u}(t) = 2$$

$$D_t[\mathbf{r}(t) \cdot \mathbf{u}(t)] = 0$$

$$(d) \mathbf{u}(t) - 2\mathbf{r}(t) = -\sin t\mathbf{i} - \cos t\mathbf{j} + \left(\frac{1}{t} - 2t\right)\mathbf{k}$$

$$D_t[\mathbf{u}(t) - 2\mathbf{r}(t)] = -\cos t\mathbf{i} + \sin t\mathbf{j} + \left(-\frac{1}{t^2} - 2\right)\mathbf{k}$$

$$(e) \|\mathbf{r}(t)\| = \sqrt{1 + t^2}$$

$$D_t[\|\mathbf{r}(t)\|] = \frac{t}{\sqrt{1 + t^2}}$$

$$(f) \mathbf{r}(t) \times \mathbf{u}(t) = \left(\frac{1}{t}\cos t - t\cos t\right)\mathbf{i} - \left(\frac{1}{t}\sin t - t\sin t\right)\mathbf{j}$$

$$D_t[\mathbf{r}(t) \times \mathbf{u}(t)] = \left(-\frac{1}{t}\sin t - \frac{1}{t^2}\cos t + t\sin t - \cos t\right)\mathbf{i} - \left(\frac{1}{t}\cos t - \frac{1}{t^2}\sin t - t\cos t - \sin t\right)\mathbf{j}$$

25.  $x(t)$  and  $y(t)$  are increasing functions at  $t = t_0$ , and  $z(t)$  is a decreasing function at  $t = t_0$ .

26. The graph of  $\mathbf{u}$  is parallel to the  $yz$ -plane.

27.  $\int (\cos t\mathbf{i} + \cos t\mathbf{j}) dt = \sin t\mathbf{i} + (t\sin t + \cos t)\mathbf{j} + \mathbf{C}$

28.  $\int (\ln t\mathbf{i} + t\ln t\mathbf{j} + \mathbf{k}) dt = (t\ln t - t)\mathbf{i} + \frac{t^2}{4}(-1 + 2\ln t)\mathbf{j} + t\mathbf{k} + \mathbf{C}$

29.  $\int \|\cos t\mathbf{i} + \sin t\mathbf{j} + t\mathbf{k}\| dt = \int \sqrt{1 + t^2} dt = \frac{1}{2} \left[ t\sqrt{1 + t^2} + \ln \left| t + \sqrt{1 + t^2} \right| \right] + \mathbf{C}$

30.  $\int (t\mathbf{j} + t^2\mathbf{k}) \times (\mathbf{i} + t\mathbf{j} + t\mathbf{k}) dt = \int [(t^2 - t^3)\mathbf{i} + t^2\mathbf{j} - t\mathbf{k}] dt = \left(\frac{t^3}{3} - \frac{t^4}{4}\right)\mathbf{i} + \frac{t^3}{3}\mathbf{j} - \frac{t^2}{2}\mathbf{k} + \mathbf{C}$

31.  $\int_{-2}^2 (3t\mathbf{i} + 2t^2\mathbf{j} - t^3\mathbf{k}) dt = \left[ \frac{3t^2}{2}\mathbf{i} + \frac{2t^3}{3}\mathbf{j} - \frac{t^4}{4}\mathbf{k} \right]_{-2}^2 = \frac{32}{3}\mathbf{j}$

32.  $\int_0^1 (\sqrt{t} \mathbf{j} + t \sin t \mathbf{k}) dt = \left[ \frac{2}{3} t^{3/2} \mathbf{j} + (\sin t - t \cos t) \mathbf{k} \right]_0^1 = \frac{2}{3} \mathbf{j} + (\sin 1 - \cos 1) \mathbf{k}$

33.  $\int_0^2 (e^{t/2} \mathbf{i} - 3t^2 \mathbf{j} - \mathbf{k}) dt = \left[ 2e^{t/2} \mathbf{i} - t^3 \mathbf{j} - t \mathbf{k} \right]_0^2 = (2e - 2) \mathbf{i} - 8 \mathbf{j} - 2 \mathbf{k}$

34.  $\int_{-1}^1 (t^3 \mathbf{i} - \arcsin t \mathbf{j} - t^2 \mathbf{k}) dt = \left[ \frac{t^4}{4} \mathbf{i} - \left( t \arcsin t + \sqrt{1-t^2} \right) \mathbf{j} - \frac{t^3}{3} \mathbf{k} \right]_{-1}^1 = -\frac{2}{3} \mathbf{k}$

35.  $\mathbf{r}(t) = \int (2t \mathbf{i} + e^t \mathbf{j} + e^{-t} \mathbf{k}) dt = t^2 \mathbf{i} + e^t \mathbf{j} - e^{-t} \mathbf{k} + \mathbf{C}$   
 $\mathbf{r}(0) = \mathbf{j} - \mathbf{k} + \mathbf{C} = \mathbf{i} + 3 \mathbf{j} - 5 \mathbf{k} \Rightarrow \mathbf{C} = \mathbf{i} + 2 \mathbf{j} - 4 \mathbf{k}$   
 $\mathbf{r}(t) = (t^2 + 1) \mathbf{i} + (e^t + 2) \mathbf{j} - (e^{-t} + 4) \mathbf{k}$

36.  $\mathbf{r}(t) = \int (\sec t \mathbf{i} + \tan t \mathbf{j} + t^2 \mathbf{k}) dt$   
 $= \ln |\sec t + \tan t| \mathbf{i} - \ln |\cos t| \mathbf{j} + \frac{t^3}{3} \mathbf{k} + \mathbf{C}$   
 $\mathbf{r}(0) = \mathbf{C} = 3 \mathbf{k}$   
 $\mathbf{r}(t) = \ln |\sec t + \tan t| \mathbf{i} - \ln |\cos t| \mathbf{j} + \left( \frac{t^3}{3} + 3 \right) \mathbf{k}$

37.  $\mathbf{r}(t) = 4t \mathbf{i} + t^3 \mathbf{j} - t \mathbf{k}$   
 $\mathbf{r}'(t) = \mathbf{v}(t) = 4 \mathbf{i} + 3t^2 \mathbf{j} - \mathbf{k}$   
Speed  $= \|\mathbf{v}\| = \sqrt{16 + 9t^4 + 1} = \sqrt{17 + 9t^4}$   
 $\mathbf{a}(t) = 6t \mathbf{j}$

38.  $\mathbf{r}(t) = \sqrt{t} \mathbf{i} + 5t \mathbf{j} + 2t^2 \mathbf{k}$   
 $\mathbf{r}'(t) = \mathbf{v}(t) = \frac{1}{2\sqrt{t}} \mathbf{i} + 5 \mathbf{j} + 4t \mathbf{k}$   
Speed  $= \|\mathbf{v}\| = \sqrt{\frac{1}{4t} + 25 + 16t^2}$   
 $\mathbf{a}(t) = -\frac{1}{4t^{3/2}} \mathbf{i} + 4 \mathbf{k}$

39.  $\mathbf{r}(t) = \langle \cos^3 t, \sin^3 t, 3t \rangle$   
 $\mathbf{v}(t) = \mathbf{r}'(t) = \langle -3 \cos^2 t \sin t, 3 \sin^2 t \cos t, 3 \rangle$   
 $\|\mathbf{v}(t)\| = \sqrt{9 \cos^4 t \sin^2 t + 9 \sin^4 t \cos^2 t + 9} = 3\sqrt{\cos^2 t \sin^2 t (\cos^2 t + \sin^2 t) + 1} = 3\sqrt{\cos^2 t \sin^2 t + 1}$   
 $\mathbf{a}(t) = \mathbf{v}'(t) = \langle -6 \cos t (-\sin^2 t) + (-3 \cos^2 t) \cos t, 6 \sin t \cos^2 t + 3 \sin^2 t (-\sin t), 0 \rangle$   
 $= \langle 3 \cos t (2 \sin^2 t - \cos^2 t), 3 \sin t (2 \cos^2 t - \sin^2 t), 0 \rangle$

40.  $\mathbf{r}(t) = \langle t, -\tan t, e^t \rangle$   
 $\mathbf{r}'(t) = \mathbf{v}(t) = \langle 1, -\sec^2 t, e^t \rangle$   
 $\|\mathbf{v}(t)\| = \sqrt{1 + \sec^4 t + e^{2t}}$   
 $\mathbf{r}''(t) = \mathbf{a}(t) = \langle 0, -2 \sec^2 t \cdot \tan t, e^t \rangle$

41.  $\mathbf{r}(t) = \left\langle \ln(t-3), t^2, \frac{1}{2}t \right\rangle, t_0 = 4$   
 $\mathbf{r}'(t) = \left\langle \frac{1}{t-3}, 2t, \frac{1}{2} \right\rangle$   
 $\mathbf{r}'(4) = \left\langle 1, 8, \frac{1}{2} \right\rangle$  direction numbers  
Because  $\mathbf{r}(4) = \langle 0, 16, 2 \rangle$ , the parametric equations are  
 $x = t, y = 16 + 8t, z = 2 + \frac{1}{2}t.$   
 $\mathbf{r}(t_0 + 0.1) = \mathbf{r}(4.1) \approx \langle 0.1, 16.8, 2.05 \rangle$

42.  $\mathbf{r}(t) = \langle 3 \cosh t, \sinh t, -2t \rangle, t_0 = 0$   
 $\mathbf{r}'(t) = \langle 3 \sinh t, \cosh t, -2 \rangle$   
 $\mathbf{r}'(0) = \langle 0, 1, -2 \rangle$  direction numbers  
Because  $\mathbf{r}(0) = \langle 3, 0, 0 \rangle$ , the parametric equations are  
 $x = 3, y = t, z = -2t.$

$\mathbf{r}(t_0 + 0.1) = \mathbf{r}(0.1) \approx \langle 3, 0.1, -0.2 \rangle$

43.  $\mathbf{r}(t) = \left\langle (v_0 \cos \theta)t, (v_0 \sin \theta)t - \frac{1}{2}gt^2 \right\rangle$   
 $= \langle 42\sqrt{3}t, 42t - 16t^2 \rangle$

$42t = 16t^2 \Rightarrow t = 0, \frac{21}{8}$

Range  $= 42\sqrt{3} \left( \frac{21}{8} \right) = \frac{441\sqrt{3}}{4} \approx 190.96 \text{ ft}$

44.  $y = -16t^2 + 6 = 0 \Rightarrow t = \frac{\sqrt{6}}{4}$

$$x = v_0 t \Rightarrow 4 = v_0 \left( \frac{\sqrt{6}}{4} \right) \Rightarrow v_0 = \frac{16}{\sqrt{6}}$$

$$v_0 = \frac{8\sqrt{6}}{3} \approx 6.532 \text{ ft/sec}$$

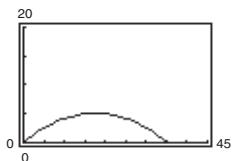
45. Range =  $x = \frac{v_0^2}{9.8} \sin 2\theta = 95$

$$v_0^2 = \frac{9.8(95)}{\sin(40^\circ)}$$

$$v_0 \approx 38.06 \text{ m/sec}$$

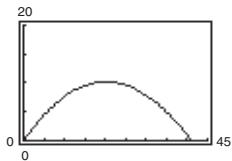
46.  $\mathbf{r}(t) = [(v_0 \cos \theta)t] \mathbf{i} + \left[ (v_0 \sin \theta)t - \frac{1}{2}(9.8)t^2 \right] \mathbf{j}$

(a)  $\mathbf{r}(t) = [(20 \cos 30^\circ)t] \mathbf{i} + [(20 \sin 30^\circ)t - 4.9t^2] \mathbf{j}$



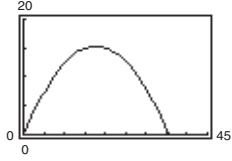
Maximum height  $\approx 5.1$  m; Range  $\approx 35.3$  m

(b)  $\mathbf{r}(t) = [(20 \cos 45^\circ)t] \mathbf{i} + [(20 \sin 45^\circ)t - 4.9t^2] \mathbf{j}$



Maximum height  $\approx 10.2$  m; Range  $\approx 40.8$  m

(c)  $\mathbf{r}(t) = [(20 \cos 60^\circ)t] \mathbf{i} + [(20 \sin 60^\circ)t - 4.9t^2] \mathbf{j}$



Maximum height  $\approx 15.3$  m; Range  $\approx 35.3$  m

(Note that  $45^\circ$  gives the longest range)

47.  $\mathbf{r}(t) = (2 - t)\mathbf{i} + 3t\mathbf{j}$

$$\mathbf{v}(t) = -\mathbf{i} + 3\mathbf{j}$$

$$\|\mathbf{v}\| = \sqrt{1 + 9} = \sqrt{10}$$

$$\mathbf{a}(t) = \mathbf{0}$$

$$\mathbf{T}(t) = \frac{1}{\sqrt{10}}(-\mathbf{i} + 3\mathbf{j})$$

$\mathbf{N}(t)$  is not defined

$$\mathbf{a} \cdot \mathbf{T} = 0$$

$\mathbf{a} \cdot \mathbf{N}$  does not exist

(The curve is a line)

48.  $\mathbf{r}(t) = (1 + 4t)\mathbf{i} + (2 - 3t)\mathbf{j}$

$$\mathbf{v}(t) = 4\mathbf{i} - 3\mathbf{j}$$

$$\|\mathbf{v}\| = 5$$

$$\mathbf{a}(t) = \mathbf{0}$$

$$\mathbf{T}(t) = \frac{1}{5}(4\mathbf{i} - 3\mathbf{j})$$

$\mathbf{N}(t)$  does not exist.

$$\mathbf{a} \cdot \mathbf{T} = 0$$

$\mathbf{a} \cdot \mathbf{N}$  does not exist.

49.  $\mathbf{r}(t) = t\mathbf{i} + \sqrt{t}\mathbf{j}$

$$\mathbf{v}(t) = \mathbf{i} + \frac{1}{2\sqrt{t}}\mathbf{j}$$

$$\|\mathbf{v}(t)\| = \frac{\sqrt{4t+1}}{2\sqrt{t}}$$

$$\mathbf{a}(t) = -\frac{1}{4t\sqrt{t}}\mathbf{j}$$

$$\mathbf{T}(t) = \frac{\mathbf{i} + (1/2\sqrt{t})\mathbf{j}}{(\sqrt{4t+1})/2\sqrt{t}} = \frac{2\sqrt{t}\mathbf{i} + \mathbf{j}}{\sqrt{4t+1}}$$

$$\mathbf{N}(t) = \frac{\mathbf{i} - 2\sqrt{t}\mathbf{j}}{\sqrt{4t+1}}$$

$$\mathbf{a} \cdot \mathbf{T} = \frac{-1}{4t\sqrt{t}\sqrt{4t+1}}$$

$$\mathbf{a} \cdot \mathbf{N} = \frac{1}{2t\sqrt{4t+1}}$$

50.  $\mathbf{r}(t) = 2(t+1)\mathbf{i} + \frac{2}{t+1}\mathbf{j}$

$$\mathbf{v}(t) = 2\mathbf{i} - \frac{2}{(t+1)^2}\mathbf{j}$$

$$\|\mathbf{v}(t)\| = \frac{2\sqrt{(t+1)^4 + 1}}{(t+1)^2}$$

$$\mathbf{a}(t) = \frac{4}{(t+1)^3}\mathbf{j}$$

$$\mathbf{T}(t) = \frac{(t+1)^2\mathbf{i} - \mathbf{j}}{\sqrt{(t+1)^4 + 1}}$$

$$\mathbf{N}(t) = \frac{\mathbf{i} + (t+1)^2\mathbf{j}}{\sqrt{(t+1)^4 + 1}}$$

$$\mathbf{a} \cdot \mathbf{T} = \frac{-4}{(t+1)^3\sqrt{(t+1)^4 + 1}}$$

$$\mathbf{a} \cdot \mathbf{N} = \frac{4(t+1)^2}{(t+1)^3\sqrt{(t+1)^4 + 1}} = \frac{4}{\left(t+1\sqrt{(t+1)^4 + 1}\right)}$$

51.  $\mathbf{r}(t) = e^t\mathbf{i} + e^{-t}\mathbf{j}$

$$\mathbf{v}(t) = e^t\mathbf{i} - e^{-t}\mathbf{j}$$

$$\|\mathbf{v}(t)\| = \sqrt{e^{2t} + e^{-2t}}$$

$$\mathbf{a}(t) = e^t\mathbf{i} + e^{-t}\mathbf{j}$$

$$\mathbf{T}(t) = \frac{e^t\mathbf{i} - e^{-t}\mathbf{j}}{\sqrt{e^{2t} + e^{-2t}}}$$

$$\mathbf{N}(t) = \frac{e^{-t}\mathbf{i} + e^t\mathbf{j}}{\sqrt{e^{2t} + e^{-2t}}}$$

$$\mathbf{a} \cdot \mathbf{T} = \frac{e^{2t} - e^{-2t}}{\sqrt{e^{2t} + e^{-2t}}}$$

$$\mathbf{a} \cdot \mathbf{N} = \frac{2}{\sqrt{e^{2t} - e^{-2t}}}$$

52.  $\mathbf{r}(t) = t \cos t\mathbf{i} + t \sin t\mathbf{j}$

$$\mathbf{v}(t) = \mathbf{r}'(t) = (-t \sin t + \cos t)\mathbf{i} + (t \cos t + \sin t)\mathbf{j}$$

$$\|\mathbf{v}(t)\| = \text{speed} = \sqrt{(-t \sin t + \cos t)^2 + (t \cos t + \sin t)^2} = \sqrt{t^2 + 1}$$

$$\mathbf{a}(t) = \mathbf{r}''(t) = (-t \cos t - 2 \sin t)\mathbf{i} + (-t \sin t + 2 \cos t)\mathbf{j}$$

$$\mathbf{T}(t) = \frac{\mathbf{v}(t)}{\|\mathbf{v}(t)\|} = \frac{(-t \sin t + \cos t)\mathbf{i} + (t \cos t + \sin t)\mathbf{j}}{\sqrt{t^2 + 1}}$$

$$\mathbf{N}(t) = \frac{-(t \cos t + \sin t)\mathbf{i} + (-t \sin t + \cos t)\mathbf{j}}{\sqrt{t^2 + 1}}$$

$$\mathbf{a}(t) \cdot \mathbf{T}(t) = \frac{t}{\sqrt{t^2 + 1}}$$

$$\mathbf{a}(t) \cdot \mathbf{N}(t) = \frac{t^2 + 2}{\sqrt{t^2 + 1}}$$

53.  $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + \frac{1}{2}t^2\mathbf{k}$

$$\mathbf{v}(t) = \mathbf{i} + 2t\mathbf{j} + t\mathbf{k}$$

$$\|\mathbf{v}\| = \sqrt{1 + 5t^2}$$

$$\mathbf{a}(t) = 2\mathbf{j} + \mathbf{k}$$

$$\mathbf{T}(t) = \frac{\mathbf{i} + 2t\mathbf{j} + t\mathbf{k}}{\sqrt{1 + 5t^2}}$$

$$\mathbf{N}(t) = \frac{-5t\mathbf{i} + 2\mathbf{j} + \mathbf{k}}{\sqrt{5}\sqrt{1 + 5t^2}}$$

$$\mathbf{a} \cdot \mathbf{T} = \frac{5t}{\sqrt{1 + 5t^2}}$$

$$\mathbf{a} \cdot \mathbf{N} = \frac{5}{\sqrt{5}\sqrt{1 + 5t^2}} = \frac{\sqrt{5}}{\sqrt{1 + 5t^2}}$$

54.  $\mathbf{r}(t) = (t-1)\mathbf{i} + t\mathbf{j} + \frac{1}{t}\mathbf{k}$

$$\mathbf{v}(t) = \mathbf{i} + \mathbf{j} - \frac{1}{t^2}\mathbf{k}$$

$$\|\mathbf{v}(t)\| = \frac{\sqrt{2t^4 + 1}}{t^2}$$

$$\mathbf{a}(t) = \frac{2}{t^3}\mathbf{k}$$

$$\mathbf{T}(t) = \frac{t^2\mathbf{i} + t^2\mathbf{j} - \mathbf{k}}{\sqrt{2t^4 + 1}}$$

$$\mathbf{N}(t) = \frac{\mathbf{i} + \mathbf{j} + 2t^2\mathbf{k}}{\sqrt{2}\sqrt{2t^4 + 1}}$$

$$\mathbf{a} \cdot \mathbf{T} = \frac{-2}{t^3\sqrt{2t^4 + 1}}$$

$$\mathbf{a} \cdot \mathbf{N} = \frac{4}{t\sqrt{2}\sqrt{2t^4 + 1}}$$

55.  $\mathbf{r}(t) = 2 \cos t\mathbf{i} + 2 \sin t\mathbf{j} + t\mathbf{k}, t = \pi/3$

$$\mathbf{r}(\pi/3) = \mathbf{i} + \sqrt{3}\mathbf{j} + \frac{\pi}{3}\mathbf{k}. \text{ Point: } (1, \sqrt{3}, \pi/3)$$

$$\mathbf{r}'(t) = -2 \sin t\mathbf{i} + 2 \cos t\mathbf{j} + \mathbf{k}$$

$$\mathbf{r}'(\pi/3) = -\sqrt{3}\mathbf{i} + \mathbf{j} + \mathbf{k}$$

Direction numbers:  $-\sqrt{3}, 1, 1$

$$x = 1 - \sqrt{3}t, y = \sqrt{3} + t, z = \pi/3 + t$$

56.  $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + \frac{2}{3}t^3\mathbf{k}, x = t, y = t^2, z = \frac{2}{3}t^3$

When  $t = 2, x = 2, y = 4, z = \frac{16}{3}$ .

$$\mathbf{r}'(t) = \mathbf{i} + 2t\mathbf{j} + 2t^2\mathbf{k}$$

Direction numbers when  $t = 2, a = 1, b = 4, c = 8$

$$x = t + 2, y = 4t + 4, z = 8t + \frac{16}{3}$$

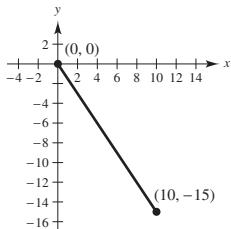
57.  $v = \sqrt{\frac{GM}{r}} = \sqrt{\frac{9.56 \times 10^4}{4000 + 550}} \approx 4.58 \text{ mi/sec}$

58. Factor of 4

59.  $\mathbf{r}(t) = 2t\mathbf{i} - 3t\mathbf{j}, 0 \leq t \leq 5$

$$\mathbf{r}'(t) = 2\mathbf{i} - 3\mathbf{j}$$

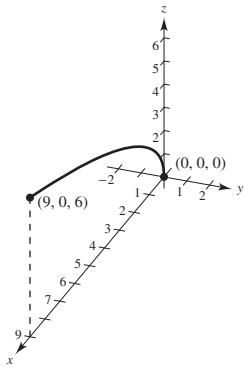
$$s = \int_a^b \|\mathbf{r}'(t)\| dt = \int_0^5 \sqrt{4+9} dt = \left[ \sqrt{13} t \right]_0^5 = 5\sqrt{13}$$



60.  $\mathbf{r}(t) = t^2\mathbf{i} + 2t\mathbf{k}, 0 \leq t \leq 3$

$$\mathbf{r}'(t) = 2t\mathbf{i} + 2\mathbf{k}$$

$$\begin{aligned} s &= \int_a^b \|\mathbf{r}'(t)\| dt = \int_0^3 \sqrt{4t^2 + 4} dt \\ &= \left[ \ln \sqrt{t^2 + 1} + t \right]_0^3 + t\sqrt{t^2 + 1} \\ &= \ln(\sqrt{10} + 3) + 3\sqrt{10} \approx 11.3053 \end{aligned}$$

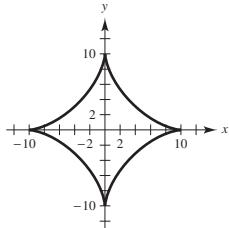


61.  $\mathbf{r}(t) = 10 \cos^3 t \mathbf{i} + 10 \sin^3 t \mathbf{j}$

$$\mathbf{r}'(t) = -30 \cos^2 t \sin t \mathbf{i} + 30 \sin^2 t \cos t \mathbf{j}$$

$$\begin{aligned} \|\mathbf{r}'(t)\| &= 30 \sqrt{\cos^4 t \sin^2 t + \sin^4 t \cos^2 t} \\ &= 30 |\cos t \sin t| \end{aligned}$$

$$s = 4 \int_0^{\pi/2} 30 \cos t \cdot \sin t dt = \left[ 120 \frac{\sin^2 t}{2} \right]_0^{\pi/2} = 60$$

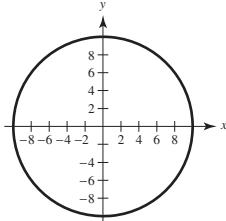


62.  $\mathbf{r}(t) = 10 \cos t \mathbf{i} + 10 \sin t \mathbf{j}$

$$\mathbf{r}'(t) = -10 \sin t \mathbf{i} + 10 \cos t \mathbf{j}$$

$$\|\mathbf{r}'(t)\| = 10$$

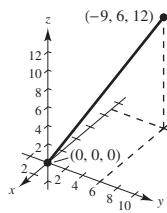
$$s = \int_0^{2\pi} 10 dt = 20\pi$$



63.  $\mathbf{r}(t) = -3t\mathbf{i} + 2t\mathbf{j} + 4t\mathbf{k}, 0 \leq t \leq 3$

$$\mathbf{r}'(t) = -3\mathbf{i} + 2\mathbf{j} + 4\mathbf{k}$$

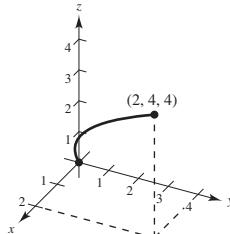
$$\begin{aligned} s &= \int_a^b \|\mathbf{r}'(t)\| dt = \int_0^3 \sqrt{9+4+16} dt \\ &= \int_0^3 \sqrt{29} dt = 3\sqrt{29} \end{aligned}$$



64.  $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + 2t\mathbf{k}, 0 \leq t \leq 2$

$$\mathbf{r}'(t) = \mathbf{i} + 2t\mathbf{j} + 2\mathbf{k}, \|\mathbf{r}'(t)\| = \sqrt{5 + 4t^2}$$

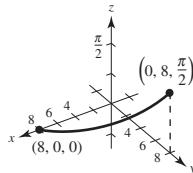
$$\begin{aligned} s &= \int_a^b \|\mathbf{r}'(t)\| dt = \int_0^2 \sqrt{5 + 4t^2} dt \\ &= \sqrt{21} + \frac{5}{4} \ln 5 - \frac{5}{4} \ln(\sqrt{105} - 4\sqrt{5}) \approx 6.2638 \end{aligned}$$



65.  $\mathbf{r}(t) = \langle 8 \cos t, 8 \sin t, t \rangle, 0 \leq t \leq \frac{\pi}{2}$

$$\mathbf{r}'(t) = \langle -8 \sin t, 8 \cos t, 1 \rangle, \|\mathbf{r}'(t)\| = \sqrt{65}$$

$$s = \int_a^b \|\mathbf{r}'(t)\| dt = \int_0^{\pi/2} \sqrt{65} dt = \frac{\pi\sqrt{65}}{2}$$



66.  $\mathbf{r}(t) = \langle 2(\sin t - t \cos t), 2(\cos t + t \sin t), t \rangle, 0 \leq t \leq \frac{\pi}{2}$

$$\mathbf{r}'(t) = \langle 2t \sin t, 2t \cos t, 1 \rangle, \|\mathbf{r}'(t)\| = \sqrt{4t^2 + 1}$$

$$s = \int_a^b \|\mathbf{r}'(t)\| dt = \int_0^{\pi/2} \sqrt{4t^2 + 1} dt$$

$$= \frac{1}{4} \ln(\sqrt{\pi^2 + 1} + \pi) + \frac{\pi}{4} \sqrt{\pi^2 + 1} \approx 3.055$$

67.  $\mathbf{r}(t) = 3t\mathbf{i} + 2t\mathbf{j}$

Line

$$K = 0$$

68.  $\mathbf{r}(t) = 2\sqrt{t}\mathbf{i} + 3t\mathbf{j}$

$$\mathbf{r}'(t) = \frac{1}{\sqrt{t}}\mathbf{i} + 3\mathbf{j}, \|\mathbf{r}'(t)\| = \sqrt{\frac{1}{t} + 9} = \sqrt{\frac{1+9t}{t}}$$

$$\mathbf{r}''(t) = -\frac{1}{2}t^{-3/2}\mathbf{i}$$

$$\mathbf{r}' \times \mathbf{r}'' = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{1}{\sqrt{t}} & 3 & 0 \\ -\frac{1}{2}t^{-3/2} & 0 & 0 \end{vmatrix} = \frac{3}{2}t^{-3/2}\mathbf{k}; \|\mathbf{r}' \times \mathbf{r}''\| = \frac{3}{2t^{3/2}}$$

$$K = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3} = \frac{3/2t^{3/2}}{(1+9t)^{3/2}/t^{3/2}} = \frac{3}{2(1+9t)^{3/2}}$$

69.  $\mathbf{r}(t) = 2t\mathbf{i} + \frac{1}{2}t^2\mathbf{j} + t^2\mathbf{k}$

$$\mathbf{r}'(t) = 2\mathbf{i} + t\mathbf{j} + 2t\mathbf{k}, \|\mathbf{r}'\| = \sqrt{5t^2 + 4}$$

$$\mathbf{r}''(t) = \mathbf{j} + 2\mathbf{k}$$

$$\mathbf{r}' \times \mathbf{r}'' = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & t & 2t \\ 0 & 1 & 2 \end{vmatrix} = -4\mathbf{j} + 2\mathbf{k}, \|\mathbf{r}' \times \mathbf{r}''\| = \sqrt{20}$$

$$K = \frac{\|\mathbf{r}' \times \mathbf{r}''\|}{\|\mathbf{r}'\|^3} = \frac{\sqrt{20}}{(5t^2 + 4)^{3/2}} = \frac{2\sqrt{5}}{(4 + 5t^2)^{3/2}}$$

70.  $\mathbf{r}(t) = 2t\mathbf{i} + 5 \cos t\mathbf{j} + 5 \sin t\mathbf{k}$

$$\mathbf{r}'(t) = 2\mathbf{i} - 5 \sin t\mathbf{j} + 5 \cos t\mathbf{k}, \|\mathbf{r}'(t)\| = \sqrt{29}$$

$$\mathbf{r}''(t) = 5 \cos t\mathbf{j} - 5 \sin t\mathbf{k}$$

$$\mathbf{r}' \times \mathbf{r}'' = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -5 \sin t & 5 \cos t \\ 0 & -5 \cos t & -5 \sin t \end{vmatrix}$$

$$= 25\mathbf{i} + 10 \sin t\mathbf{j} - 10 \cos t\mathbf{k}$$

$$\|\mathbf{r}' \times \mathbf{r}''\| = \sqrt{725}$$

$$K = \frac{\|\mathbf{r}' \times \mathbf{r}''\|}{\|\mathbf{r}'\|^3} = \frac{\sqrt{725}}{(29)^{3/2}} = \frac{\sqrt{25 \cdot 29}}{29\sqrt{29}} = \frac{5}{29}$$

71.  $\mathbf{r}(t) = \frac{1}{2}t^2\mathbf{i} + t\mathbf{j} + \frac{1}{3}t^3\mathbf{k}, P\left(\frac{1}{2}, 1, \frac{1}{3}\right) \Rightarrow t = 1$

$$\mathbf{r}'(t) = t\mathbf{i} + \mathbf{j} + t^2\mathbf{k}, \mathbf{r}'(1) = \mathbf{i} + \mathbf{j} + \mathbf{k}$$

$$\mathbf{r}''(t) = \mathbf{i} + 2t\mathbf{k}, \mathbf{r}''(1) = \mathbf{i} + 2\mathbf{k}$$

$$\mathbf{r}' \times \mathbf{r}'' = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 1 \\ 1 & 0 & 2 \end{vmatrix} = 2\mathbf{i} - \mathbf{j} - \mathbf{k}$$

$$K = \frac{\|\mathbf{r}' \times \mathbf{r}''\|}{\|\mathbf{r}'\|^3} = \frac{\sqrt{4+1+1}}{(\sqrt{3})^3} = \frac{\sqrt{6}}{3\sqrt{3}} = \frac{\sqrt{2}}{3}$$

72.  $\mathbf{r}(t) = 4 \cos t\mathbf{i} + 3 \sin t\mathbf{j} + t\mathbf{k}, P(-4, 0, \pi) \Rightarrow t = \pi$

$$\mathbf{r}'(t) = -4 \sin t\mathbf{i} + 3 \cos t\mathbf{j} + \mathbf{k}, \mathbf{r}'(\pi) = -3\mathbf{j} + \mathbf{k}$$

$$\mathbf{r}''(t) = -4 \cos t\mathbf{i} - 3 \sin t\mathbf{j}, \mathbf{r}''(\pi) = 4\mathbf{i}$$

$$\mathbf{r}' \times \mathbf{r}'' = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & -3 & 1 \\ 4 & 0 & 0 \end{vmatrix} = 4\mathbf{j} + 12\mathbf{k}$$

$$K = \frac{\|\mathbf{r}' \times \mathbf{r}''\|}{\|\mathbf{r}'\|^3} = \frac{\sqrt{16+144}}{(9+1)^{3/2}} = \frac{2}{5}$$

73.  $y = \frac{1}{2}x^2 + 2$

$$y' = x$$

$$y'' = 1$$

$$K = \frac{|y''|}{\left[1 + (y')^2\right]^{3/2}} = \frac{1}{(1+x^2)^{3/2}}$$

$$\text{At } x = 4, K = \frac{1}{17^{3/2}} \text{ and } r = 17^{3/2} = 17\sqrt{17}.$$

74.  $y = e^{-x/2}$

$$y' = -\frac{1}{2}e^{-x/2}, y'' = \frac{1}{4}e^{-x/2}$$

$$K = \frac{|y''|}{\left[1 + (y')^2\right]^{3/2}} = \frac{\frac{1}{4}e^{-x/2}}{\left[1 + \frac{1}{4}e^{-x}\right]^{3/2}}$$

$$\text{At } x = 0, K = \frac{1/4}{(5/4)^{3/2}} = \frac{2}{5^{3/2}} = \frac{2}{5\sqrt{5}} = \frac{2\sqrt{5}}{25},$$

$$r = \frac{5\sqrt{5}}{2}.$$

75.  $y = \ln x$

$$y' = \frac{1}{x}$$

$$y'' = -\frac{1}{x^2}$$

$$K = \frac{|y''|}{\left[1 + (y')^2\right]^{3/2}} = \frac{1/x^2}{\left[1 + (1/x)^2\right]^{3/2}}$$

$$\text{At } x = 1, K = \frac{1}{2^{3/2}} = \frac{1}{2\sqrt{2}} = \frac{\sqrt{2}}{4} \text{ and } r = 2\sqrt{2}.$$

76.  $y = \tan x$

$$y' = \sec^2 x$$

$$y'' = 2 \sec^2 x \tan x$$

$$K = \frac{|y''|}{\left[1 + (y')^2\right]^{3/2}} = \frac{|2 \sec^2 x \tan x|}{\left[1 + \sec^4 x\right]^{3/2}}$$

$$\text{At } x = \frac{\pi}{4}, K = \frac{4}{5^{3/2}} = \frac{4}{5\sqrt{5}} = \frac{4\sqrt{5}}{25} \text{ and } r = \frac{5\sqrt{5}}{4}.$$

77. The curvature changes abruptly from zero to a nonzero constant at the points  $B$  and  $C$ .

78.  $y = ax^5 + bx^3 + cx$

$$y' = 5ax^4 + 3bx^2 + c$$

$$y'' = 20ax^3 + 6bx$$

$$K = \frac{|20ax^4 + 6bx|}{\left[1 + (5ax^4 + 3bx^2 + c)^2\right]^{3/2}}$$

$$\text{At } x = 1: k = 0 \Rightarrow 20a + 6b = 0$$

$$y' = 0 \Rightarrow 5a + 3b + c = 0$$

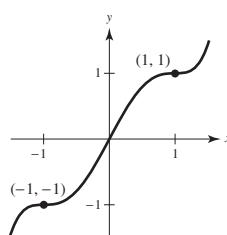
$$y(1) = 1 \Rightarrow a + b + c = 1$$

Solving these 3 equations for  $a, b, c$ , you obtain

$$a = \frac{3}{8}, b = -\frac{5}{4}, c = \frac{15}{8}. \text{ By symmetry, the same holds}$$

$$\text{at } x = -1.$$

$$y = \frac{3}{8}x^5 - \frac{5}{4}x^3 + \frac{15}{8}x$$



## Problem Solving for Chapter 12

1.  $x(t) = \int_0^t \cos\left(\frac{\pi u^2}{2}\right) du, y(t) = \int_0^t \sin\left(\frac{\pi u^2}{2}\right) du$

$$x'(t) = \cos\left(\frac{\pi t^2}{2}\right), y'(t) = \sin\left(\frac{\pi t^2}{2}\right)$$

$$(a) s = \int_0^a \sqrt{x'(t)^2 + y'(t)^2} dt = \int_0^a dt = a$$

$$(b) x''(t) = -\pi t \sin\left(\frac{\pi t^2}{2}\right), y''(t) = \pi t \cos\left(\frac{\pi t^2}{2}\right)$$

$$K = \frac{\left|\pi t \cos^2\left(\frac{\pi t^2}{2}\right) + \pi t \sin^2\left(\frac{\pi t^2}{2}\right)\right|}{1} = \pi t$$

At  $t = a, K = \pi a$ .

$$(c) K = \pi a = \pi \text{ (length)}$$

2.  $x^{2/3} + y^{2/3} = a^{2/3}$

$$\frac{2}{3}x^{-1/3} + \frac{2}{3}y^{-1/3}y' = 0$$

$$y' = \frac{-y^{1/3}}{x^{1/3}} \text{ Slope at } P(x,y).$$

$$\mathbf{r}(t) = \cos^3 t \mathbf{i} + \sin^3 t \mathbf{j}$$

$$\mathbf{r}'(t) = -3 \cos^2 t \sin t \mathbf{i} + 3 \sin^2 t \cos t \mathbf{j}$$

$$\|\mathbf{r}'(t)\| \mathbf{i} = |3 \cos t \sin t|$$

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = -\cos t \mathbf{i} + \sin t \mathbf{j}$$

$$\mathbf{T}'(t) = \sin t \mathbf{i} + \cos t \mathbf{j}$$

$$Q(0, 0, 0) \text{ origin}$$

$$P = (\cos^3 t, \sin^3 t, 0) \text{ on curve.}$$

$$\overline{PQ} \times \mathbf{T} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos^3 t & \sin^3 t & 0 \\ -\cos t & \sin t & 0 \end{vmatrix} = (\cos^3 t \sin t - \sin^3 t \cos t) \mathbf{k}$$

$$D = \frac{\|\overline{PQ} \times \mathbf{T}\|}{\|\mathbf{T}\|} = |\cos t \sin t|$$

$$K = \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|} = \frac{1}{|3 \cos t \sin t|}$$

So, the radius of curvature,  $\frac{1}{K}$ , is three times the distance from the origin to the tangent line.

3. Bomb:  $\mathbf{r}_1(t) = \langle 5000 - 400t, 3200 - 16t^2 \rangle$

Projectile:  $\mathbf{r}_2(t) = \langle (v_0 \cos \theta)t, (v_0 \sin \theta)t - 16t^2 \rangle$

At 1600 feet: Bomb:

$$3200 - 16t^2 = 1600 \Rightarrow t = 10 \text{ seconds.}$$

Projectile will travel 5 seconds:

$$5(v_0 \sin \theta) - 16(25) = 1600$$

$$v_0 \sin \theta = 400.$$

Horizontal position:

$$\text{At } t = 10, \text{ bomb is at } 5000 - 400(10) = 1000.$$

$$\text{At } t = 5, \text{ projectile is at } 5v_0 \cos \theta.$$

$$\text{So, } v_0 \cos \theta = 200.$$

Combining,

$$\frac{v_0 \sin \theta}{v_0 \cos \theta} = \frac{400}{200} \Rightarrow \tan \theta = 2 \Rightarrow \theta \approx 63.43^\circ.$$

$$v_0 = \frac{200}{\cos \theta} \approx 447.2 \text{ ft/sec}$$

4. Bomb:  $\mathbf{r}_1(t) = \langle 5000 + 400t, 3200 - 16t^2 \rangle$

Projectile:  $\mathbf{r}_2(t) = \langle (v_0 \cos \theta)t, (v_0 \sin \theta)t - 16t^2 \rangle$

At 1600 feet: Bomb:

$$3200 - 16t^2 = 1600 \Rightarrow t = 10$$

Projectile will travel 5 seconds:

$$5(v_0 \sin \theta) - 16(25) = 1600$$

$$v_0 \sin \theta = 400.$$

Horizontal position:

$$\text{At } t = 10, \text{ bomb is at } 5000 + 400(10) = 9000.$$

$$\text{At } t = 5, \text{ projectile is at } (v_0 \cos \theta)5.$$

So,

$$5v_0 \cos \theta = 9000$$

$$v_0 \cos \theta = 1800.$$

Combining,

$$\frac{v_0 \sin \theta}{v_0 \cos \theta} = \frac{400}{1800} \Rightarrow \tan \theta = \frac{2}{9} \Rightarrow \theta \approx 12.5^\circ.$$

$$v_0 = \frac{1800}{\cos \theta} \approx 1843.9 \text{ ft/sec}$$

5.  $x'(\theta) = 1 - \cos \theta, y'(\theta) = \sin \theta, 0 \leq \theta \leq 2\pi$

$$\sqrt{x'(\theta)^2 y'(\theta)^2} = \sqrt{(1 - \cos \theta)^2 + \sin^2 \theta}$$

$$= \sqrt{2 - 2 \cos \theta} = \sqrt{4 \sin^2 \frac{\theta}{2}}$$

$$s(t) = \int_{\pi}^t 2 \sin \frac{\theta}{2} d\theta = \left[ -4 \cos \frac{\theta}{2} \right]_{\pi}^t = -4 \cos \frac{t}{2}$$

$$x''(\theta) = \sin \theta, y''(\theta) = \cos \theta$$

$$K = \frac{|(1 - \cos \theta)\cos \theta - \sin \theta \sin \theta|}{\left( 2 \sin \frac{\theta}{2} \right)^3}$$

$$= \frac{|\cos \theta - 1|}{8 \sin^3 \frac{\theta}{2}}$$

$$= \frac{1}{4 \sin \frac{\theta}{2}}$$

So,  $\rho = \frac{1}{K} = 4 \sin \frac{t}{2}$  and

$$s^2 + \rho^2 = 16 \cos^2 \left( \frac{t}{2} \right) + 16 \sin^2 \left( \frac{t}{2} \right) = 16.$$

6.  $r = 1 - \cos \theta$

$$r' = \sin \theta$$

$$s(t) = \int_{\pi}^t \sqrt{(1 - \cos \theta)^2 + \sin^2 \theta} d\theta = \int_{\pi}^t \sqrt{2 - 2 \cos \theta} d\theta$$

$$= \int_{\pi}^t 2 \sin \frac{\theta}{2} d\theta = \left[ -4 \cos \frac{\theta}{2} \right]_{\pi}^t = -4 \cos \frac{t}{2}$$

$$K = \frac{|2(r')^2 - rr'' + r^2|}{[(r')^2 + r^2]^{3/2}}$$

$$= \frac{|2 \sin^2 \theta - (1 - \cos \theta)(\cos \theta) + (1 - \cos \theta)^2|}{8 \sin^3 \frac{\theta}{2}}$$

$$= \frac{|3 - 3 \cos \theta|}{8 \sin^3 \frac{\theta}{2}} = \frac{3}{4} \frac{\sin^2 \frac{\theta}{2}}{\sin^3 \frac{\theta}{2}} = \frac{3}{4 \sin \frac{\theta}{2}}$$

$$\rho = \frac{1}{K} = \frac{4 \sin \frac{\theta}{2}}{3}$$

$$s^2 + 9\rho^2 = 16 \cos^2 \frac{\theta}{2} + 16 \sin^2 \frac{\theta}{2} = 16$$

7.  $\|\mathbf{r}(t)\|^2 = \mathbf{r}(t) \cdot \mathbf{r}(t)$

$$\frac{d}{dt} (\|\mathbf{r}(t)\|^2) = 2\|\mathbf{r}(t)\| \frac{d}{dt} \|\mathbf{r}(t)\| = \mathbf{r}(t) \cdot \mathbf{r}'(t) + \mathbf{r}'(t) \cdot \mathbf{r}(t) \Rightarrow \frac{d}{dt} \|\mathbf{r}(t)\| = \frac{\mathbf{r}(t) \cdot \mathbf{r}'(t)}{\|\mathbf{r}(t)\|}$$

8. (a)  $\mathbf{r} = x\mathbf{i} + y\mathbf{j}$  position vector

$$\mathbf{r} = r \cos \theta \mathbf{i} + r \sin \theta \mathbf{j}$$

$$\frac{d\mathbf{r}}{dt} = \left[ \frac{dr}{dt} \cos \theta - r \sin \theta \frac{d\theta}{dt} \right] \mathbf{i} + \left[ \frac{dr}{dt} \sin \theta + r \cos \theta \frac{d\theta}{dt} \right] \mathbf{j}$$

$$\mathbf{a} = \frac{d^2\mathbf{r}}{dt^2} = \left[ \frac{d^2r}{dt^2} \cos \theta - \frac{dr}{dt} \sin \theta \frac{d\theta}{dt} - \frac{dr}{dt} \sin \theta \frac{d\theta}{dt} - r \cos \theta \left( \frac{d\theta}{dt} \right)^2 - r \sin \theta \frac{d^2\theta}{dt^2} \right] \mathbf{i}$$

$$+ \left[ \frac{d^2r}{dt^2} \sin \theta + \frac{dr}{dt} \cos \theta \frac{d\theta}{dt} + \frac{dr}{dt} \cos \theta \frac{d\theta}{dt} - r \sin \theta \left( \frac{d\theta}{dt} \right)^2 + r \cos \theta \frac{d^2\theta}{dt^2} \right]$$

$$a_r = \mathbf{a} \cdot \mathbf{u}_r = \mathbf{a} \cdot (\cos \theta \mathbf{i} + \sin \theta \mathbf{j})$$

$$= \left[ \frac{d^2r}{dt^2} \cos^2 \theta - 2 \frac{dr}{dt} \sin \theta \cos \theta \frac{d\theta}{dt} - r \cos^2 \theta \left( \frac{d\theta}{dt} \right)^2 - r \cos \theta \sin \theta \frac{d^2\theta}{dt^2} \right]$$

$$+ \left[ \frac{d^2r}{dt^2} \sin^2 \theta + 2 \frac{dr}{dt} \sin \theta \cos \theta \frac{d\theta}{dt} - r \sin^2 \theta \left( \frac{d\theta}{dt} \right)^2 + r \cos \theta \sin \theta \frac{d^2\theta}{dt^2} \right] = \frac{d^2r}{dt^2} - r \left( \frac{d\theta}{dt} \right)^2$$

$$a_\theta = \mathbf{a} \cdot \mathbf{u}_\theta = \mathbf{a} \cdot (-\sin \theta \mathbf{i} + \cos \theta \mathbf{j}) = 2 \frac{dr d\theta}{dr dt} + r \frac{d^2\theta}{dt^2}$$

$$\mathbf{a} = (\mathbf{a} \cdot \mathbf{u}_r)\mathbf{u}_r + (\mathbf{a} \cdot \mathbf{u}_\theta)\mathbf{u}_\theta = \left[ \frac{d^2r}{dt^2} - r \left( \frac{d\theta}{dt} \right)^2 \right] \mathbf{u}_r + \left[ 2 \frac{dr d\theta}{dt^2} + r \frac{d^2\theta}{dt^2} \right] \mathbf{u}_\theta$$

(b)  $\mathbf{r} = 42,000 \cos\left(\frac{\pi t}{12}\right) \mathbf{i} + 42,000 \sin\left(\frac{\pi t}{12}\right) \mathbf{j}$

$$\mathbf{r} = 42,000, \frac{dr}{dt} = 0, \frac{d^2r}{dt^2} = 0$$

$$\frac{d\theta}{dt} = \frac{\pi}{12}, \frac{d^2\theta}{dt^2} = 0$$

$$\text{So, } \mathbf{a} = -42,000 \left( \frac{\pi}{12} \right)^2 \mathbf{u}_r = -\frac{875}{3} \pi^2 \mathbf{u}_r.$$

Radial component:  $-\frac{875}{3} \pi^2$

Angular component: 0

9.  $\mathbf{r}(t) = 4 \cos t \mathbf{i} + 4 \sin t \mathbf{j} + 3t \mathbf{k}, t = \frac{\pi}{2}$

$$\mathbf{r}'(t) = -4 \sin t \mathbf{i} + 4 \cos t \mathbf{j} + 3 \mathbf{k}, \|\mathbf{r}'(t)\| = 5$$

$$\mathbf{r}''(t) = -4 \cos t \mathbf{i} - 4 \sin t \mathbf{j}$$

$$\mathbf{T} = -\frac{4}{5} \sin t \mathbf{i} + \frac{4}{5} \cos t \mathbf{j} + \frac{3}{5} \mathbf{k}$$

$$\mathbf{T}' = -\frac{4}{5} \cos t \mathbf{i} - \frac{4}{5} \sin t \mathbf{j}$$

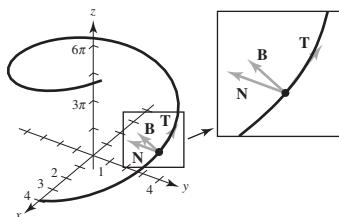
$$\mathbf{N} = -\cos t \mathbf{i} - \sin t \mathbf{j}$$

$$\mathbf{B} = \mathbf{T} \times \mathbf{N} = \frac{3}{5} \sin t \mathbf{i} - \frac{3}{5} \cos t \mathbf{j} + \frac{4}{5} \mathbf{k}$$

$$\text{At } t = \frac{\pi}{2}, \mathbf{T}\left(\frac{\pi}{2}\right) = -\frac{4}{5} \mathbf{i} + \frac{3}{5} \mathbf{k}$$

$$\mathbf{N}\left(\frac{\pi}{2}\right) = -\mathbf{j}$$

$$\mathbf{B}\left(\frac{\pi}{2}\right) = \frac{3}{5} \mathbf{i} + \frac{4}{5} \mathbf{k}$$



10.  $\mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j} - \mathbf{k}$ ,  $t = \frac{\pi}{4}$

$$\mathbf{r}'(t) = -\sin t\mathbf{i} + \cos t\mathbf{j}, \|\mathbf{r}'(t)\| = 1$$

$$\mathbf{T} = -\sin t\mathbf{i} + \cos t\mathbf{j}$$

$$\mathbf{T}' = -\cos t\mathbf{i} - \sin t\mathbf{j}$$

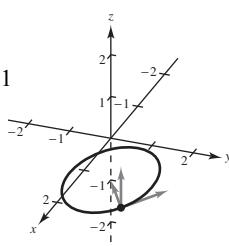
$$\mathbf{N} = -\cos t\mathbf{i} - \sin t\mathbf{j}$$

$$\mathbf{B} = \mathbf{T} \times \mathbf{N} = \mathbf{k}$$

$$\text{At } t = \frac{\pi}{4}, \mathbf{T}\left(\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}\mathbf{i} + \frac{\sqrt{2}}{2}\mathbf{j}$$

$$\mathbf{N}\left(\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}\mathbf{i} - \frac{\sqrt{2}}{2}\mathbf{j}$$

$$\mathbf{B}\left(\frac{\pi}{4}\right) = \mathbf{k}$$



12.  $y = \frac{1}{32}x^{5/2}$

$$y' = \frac{5}{64}x^{3/2}$$

$$y'' = \frac{15}{128}x^{1/2}$$

$$K = \left| \frac{\frac{15}{128}x^{1/2}}{\left(1 + \frac{25}{4096}x^3\right)^{3/2}} \right|$$

At the point  $(4, 1)$ ,  $K = \frac{120}{(89)^{3/2}} \Rightarrow r = \frac{1}{K} = \frac{(89)^{3/2}}{120} \approx 7$ .

13.  $\mathbf{r}(t) = \langle t \cos \pi t, t \sin \pi t \rangle, 0 \leq t \leq 2$

11. (a)  $\|\mathbf{B}\| = \|\mathbf{T} \times \mathbf{N}\| = 1$  constant length  $\Rightarrow \frac{d\mathbf{B}}{ds} \perp \mathbf{B}$

$$\frac{d\mathbf{B}}{ds} = \frac{d}{ds}(\mathbf{T} \times \mathbf{N}) = (\mathbf{T} \times \mathbf{N}') + (\mathbf{T}' \times \mathbf{N})$$

$$\mathbf{T} \cdot \frac{d\mathbf{B}}{ds} = \mathbf{T} \cdot (\mathbf{T} \times \mathbf{N}') + \mathbf{T} \cdot (\mathbf{T}' \times \mathbf{N})$$

$$= (\mathbf{T} \times \mathbf{T}) \cdot \mathbf{N}' + \mathbf{T} \cdot \left( \mathbf{T}' \times \frac{\mathbf{T}'}{\|\mathbf{T}'\|} \right) = 0$$

$$\text{So, } \frac{d\mathbf{B}}{ds} \perp \mathbf{B} \text{ and } \frac{d\mathbf{B}}{ds} \perp \mathbf{T} \Rightarrow \frac{d\mathbf{B}}{ds} = \tau \mathbf{N}$$

for some scalar  $\tau$ .

(b)  $\mathbf{B} = \mathbf{T} \times \mathbf{N}$ . Using Section 11.4, exercise 66,

$$\mathbf{B} \times \mathbf{N} = (\mathbf{T} \times \mathbf{N}) \times \mathbf{N} = -\mathbf{N} \times (\mathbf{T} \times \mathbf{N})$$

$$= -[(\mathbf{N} \cdot \mathbf{N})\mathbf{T} - (\mathbf{N} \cdot \mathbf{T})\mathbf{N}]$$

$$= -\mathbf{T}$$

$$\mathbf{B} \times \mathbf{T} = (\mathbf{T} \times \mathbf{N}) \times \mathbf{T} = -\mathbf{T} \times (\mathbf{T} \times \mathbf{N})$$

$$= -[(\mathbf{T} \cdot \mathbf{N})\mathbf{T} - (\mathbf{T} \cdot \mathbf{T})\mathbf{N}]$$

$$= \mathbf{N}.$$

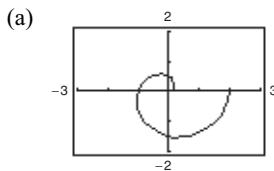
$$\text{Now, } K\mathbf{N} = \left\| \frac{d\mathbf{T}}{ds} \right\| \frac{\mathbf{T}'(s)}{\|\mathbf{T}'(s)\|} = \mathbf{T}'(s) = \frac{d\mathbf{T}}{ds}$$

Finally,

$$\mathbf{N}'(s) = \frac{d}{ds}(\mathbf{B} \times \mathbf{T}) = (\mathbf{B} \times \mathbf{T}') + (\mathbf{B}' \times \mathbf{T})$$

$$= (\mathbf{B} \times K\mathbf{N}) + (-\tau\mathbf{N} \times \mathbf{T})$$

$$= -K\mathbf{T} + \tau\mathbf{B}.$$



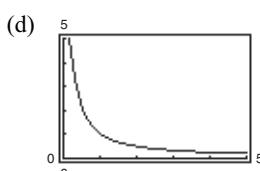
(b) Length  $= \int_0^2 \|\mathbf{r}'(t)\| dt$   
 $= \int_0^2 \sqrt{\pi^2 t^2 + 1} dt$   
 $\approx 6.766$  (graphing utility)

(c)  $K = \frac{\pi(\pi^2 t^2 + 2)}{[\pi^2 t^2 + 1]^{3/2}}$

$$K(0) = 2\pi$$

$$K(1) = \frac{\pi(\pi^2 + 2)}{(\pi^2 + 1)^{3/2}} \approx 1.04$$

$$K(2) \approx 0.51$$



(e)  $\lim_{t \rightarrow \infty} K = 0$

(f) As  $t \rightarrow \infty$ , the graph spirals outward and the curvature decreases.

14. (a) Eliminate the parameter to see that the Ferris wheel has a radius of 15 meters and is centered at  $16\mathbf{j}$ . At  $t = 0$ ,

the friend is located at  $\mathbf{r}_1(0) = \mathbf{j}$ , which is the low point on the Ferris wheel.

(b) If a revolution takes  $\Delta t$  seconds, then

$$\frac{\pi(t + \Delta t)}{10} = \frac{\pi t}{10} + 2\pi$$

and so  $\Delta t = 20$  seconds. The Ferris wheel makes three revolutions per minute.

- (c) The initial velocity is  $\mathbf{r}'_2(t_0) = -8.03\mathbf{i} + 11.47\mathbf{j}$ . The speed is  $\sqrt{8.03^2 + 11.47^2} \approx 14$  m/sec. The angle of inclination is  $\arctan\left(\frac{11.47}{8.03}\right) \approx 0.96$  radians or  $55^\circ$ .

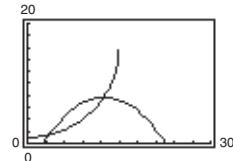
- (d) Although you may start with other values,  $t_0 = 0$  is a fine choice. The graph at the right shows two points of intersection. At  $t = 3.15$  sec the friend is near the vertex of the parabola, which the object reaches when

$$t - t_0 = -\frac{11.47}{2(-4.9)} \approx 1.17 \text{ sec.}$$

So, after the friend reaches the low point on the Ferris wheel, wait  $t_0 = 2$  sec before throwing the object in order to allow it to be within reach.

- (e) The approximate time is 3.15 seconds after starting to rise from the low point on the Ferris wheel. The friend has a constant speed of  $\|\mathbf{r}'_1(t)\| = 15$  m/sec. The speed of the object at that time is

$$\|\mathbf{r}'_2(3.15)\| = \sqrt{8.03^2 + [11.47 - 9.8(3.15 - 2)]^2} \approx 8.03 \text{ m/sec.}$$



# C H A P T E R 13

## Functions of Several Variables

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# C H A P T E R    1 3

## Functions of Several Variables

### Section 13.1 Introduction to Functions of Several Variables

1. No, it is not the graph of a function. For some values of  $x$  and  $y$  (for example,  $(x, y) = (0, 0)$ ), there are 2  $z$ -values.

2. Yes, it is the graph of a function.

3.  $x^2z + 3y^2 - xy = 10$

$$x^2z = 10 + xy - 3y^2$$

$$z = \frac{10 + xy - 3y^2}{x^2}$$

Yes,  $z$  is a function of  $x$  and  $y$ .

4.  $xz^2 + 2xy - y^2 = 4$

No,  $z$  is not a function of  $x$  and  $y$ . For example,  $(x, y) = (1, 0)$  corresponds to both  $z = \pm 2$ .

5.  $\frac{x^2}{4} + \frac{y^2}{9} + z^2 = 1$

No,  $z$  is not a function of  $x$  and  $y$ . For example,  $(x, y) = (0, 0)$  corresponds to both  $z = \pm 1$ .

6.  $z + x \ln y - 8yz = 0$

$$z(1 - 8y) = -x \ln y$$

$$z = \frac{x \ln y}{8y - 1}$$

Yes,  $z$  is a function of  $x$  and  $y$ .

7.  $f(x, y) = xy$

(a)  $f(3, 2) = 3(2) = 6$

(b)  $f(-1, 4) = -1(4) = -4$

(c)  $f(30, 5) = 30(5) = 150$

(d)  $f(5, y) = 5y$

(e)  $f(x, 2) = 2x$

(f)  $f(5, t) = 5t$

8.  $f(x, y) = 4 - x^2 - 4y^2$

(a)  $f(0, 0) = 4$

(b)  $f(0, 1) = 4 - 0 - 4 = 0$

(c)  $f(2, 3) = 4 - 4 - 36 = -36$

(d)  $f(1, y) = 4 - 1 - 4y^2 = 3 - 4y^2$

(e)  $f(x, 0) = 4 - x^2 - 0 = 4 - x^2$

(f)  $f(t, 1) = 4 - t^2 - 4 = -t^2$

9.  $f(x, y) = xe^y$

(a)  $f(5, 0) = 5e^0 = 5$

(b)  $f(3, 2) = 3e^2$

(c)  $f(2, -1) = 2e^{-1} = \frac{2}{e}$

(d)  $f(5, y) = 5e^y$

(e)  $f(x, 2) = xe^2$

(f)  $f(t, t) = te^t$

10.  $g(x, y) = \ln|x + y|$

(a)  $g(1, 0) = \ln|1 + 0| = 0$

(b)  $g(0, -1) = \ln|0 - 1| = \ln 1 = 0$

(c)  $g(0, e) = \ln|0 + e| = 1$

(d)  $g(1, 1) = \ln|1 + 1| = \ln 2$

$$(e) g\left(e, \frac{e}{2}\right) = \ln\left|e + \frac{e}{2}\right| = \ln\left(\frac{3e}{2}\right) = \ln 3 + \ln e - \ln 2 \\ = 1 + \ln 3 - \ln 2$$

(f)  $g(2, 5) = \ln|2 + 5| = \ln 7$

11.  $h(x, y, z) = \frac{xy}{z}$

(a)  $h(2, 3, 9) = \frac{2(3)}{9} = \frac{2}{3}$

(b)  $h(1, 0, 1) = \frac{1(0)}{1} = 0$

(c)  $h(-2, 3, 4) = \frac{(-2)(3)}{4} = -\frac{3}{2}$

(d)  $h(5, 4, -6) = \frac{5(4)}{-6} = -\frac{10}{3}$

12.  $f(x, y, z) = \sqrt{x + y + z}$

(a)  $f(0, 5, 4) = \sqrt{0 + 5 + 4} = 3$

(b)  $f(6, 8, -3) = \sqrt{6 + 8 - 3} = \sqrt{11}$

(c)  $f(4, 6, 2) = \sqrt{4 + 6 + 2} = \sqrt{12} = 2\sqrt{3}$

(d)  $f(10, -4, -3) = \sqrt{10 - 4 - 3} = \sqrt{3}$

13.  $f(x, y) = x \sin y$

(a)  $f\left(2, \frac{\pi}{4}\right) = 2 \sin \frac{\pi}{4} = \sqrt{2}$

(b)  $f(3, 1) = 3 \sin(1)$

(c)  $f\left(-3, \frac{\pi}{3}\right) = -3 \sin \frac{\pi}{3} = -3\left(\frac{\sqrt{3}}{2}\right) = \frac{-3\sqrt{3}}{2}$

(d)  $f\left(4, \frac{\pi}{2}\right) = 4 \sin \frac{\pi}{2} = 4$

14.  $V(r, h) = \pi r^2 h$

(a)  $V(3, 10) = \pi(3^2)10 = 90\pi$

(b)  $V(5, 2) = \pi(5^2)2 = 50\pi$

(c)  $V(4, 8) = \pi(4^2)8 = 128\pi$

(d)  $V(6, 4) = \pi(6^2)4 = 144\pi$

15.  $g(x, y) = \int_x^y (2t - 3) dt$

$= [t^2 - 3t]_x^y = y^2 - 3y - x^2 + 3x$

(a)  $g(4, 0) = 0 - 16 + 12 = -4$

(b)  $g(4, 1) = (1 - 3) - 16 + 12 = -6$

(c)  $g\left(4, \frac{3}{2}\right) = \left(\frac{9}{4} - \frac{9}{2}\right) - 16 + 12 = -\frac{25}{4}$

(d)  $g\left(\frac{3}{2}, 0\right) = 0 - \frac{9}{4} + \frac{9}{2} = \frac{9}{4}$

16.  $g(x, y) = \int_x^y \frac{1}{t} dt = \ln|t|_x^y = \ln|y| - \ln|x| = \ln\left|\frac{y}{x}\right|$

(a)  $g(4, 1) = \ln \frac{1}{4} = -\ln 4$

(b)  $g(6, 3) = \ln \frac{3}{6} = -\ln 2$

(c)  $g(2, 5) = \ln \frac{5}{2}$

(d)  $g\left(\frac{1}{2}, 7\right) = \ln \frac{7}{\left(\frac{1}{2}\right)} = \ln 14$

17.  $f(x, y) = 2x + y^2$

(a)  $\frac{f(x + \Delta x, y) - f(x, y)}{\Delta x} = \frac{2(x + \Delta x) + y^2 - (2x + y^2)}{\Delta x} = \frac{2\Delta x}{\Delta x} = 2, \Delta x \neq 0$

(b)  $\frac{f(x, y + \Delta y) - f(x, y)}{\Delta y} = \frac{2x + (y + \Delta y)^2 - 2x - y^2}{\Delta y} = \frac{2y\Delta y + (\Delta y^2)}{\Delta y} = 2y + \Delta y, \Delta y \neq 0$

18.  $f(x, y) = 3x^2 - 2y$

(a)  $\frac{f(x + \Delta x, y) - f(x, y)}{\Delta x} = \frac{3(x + \Delta x)^2 - 2y - (3x^2 - 2y)}{\Delta x} = \frac{6x\Delta x + 3(\Delta x)^2}{\Delta x} = 6x + 3\Delta x, \Delta x \neq 0$

(b)  $\frac{f(x, y + \Delta y) - f(x, y)}{\Delta y} = \frac{3x^2 - 2(y + \Delta y) - (3x^2 - 2y)}{\Delta y} = \frac{-2\Delta y}{\Delta y} = -2, \Delta y \neq 0$

19.  $f(x, y) = x^2 + y^2$

Domain:

{(x, y): x is any real number, y is any real number}

Range:  $z \geq 0$ 

22.  $f(x, y) = \frac{y}{\sqrt{x}}$

Domain:  $\{(x, y): x > 0\}$ 

Range: all real numbers

20.  $f(x, y) = e^{xy}$

Domain: Entire  $xy$ -planeRange:  $z > 0$ 

23.  $z = \frac{x + y}{xy}$

Domain:  $\{(x, y): x \neq 0 \text{ and } y \neq 0\}$ 

Range: all real numbers

21.  $g(x, y) = x\sqrt{y}$

Domain:  $\{(x, y): y \geq 0\}$ 

Range: all real numbers

24.  $z = \frac{xy}{x - y}$

Domain:  $\{(x, y): x \neq y\}$

Range: all real numbers

25.  $f(x, y) = \sqrt{4 - x^2 - y^2}$

Domain:  $4 - x^2 - y^2 \geq 0$

$$x^2 + y^2 \leq 4$$

$$\{(x, y): x^2 + y^2 \leq 4\}$$

Range:  $0 \leq z \leq 2$

26.  $f(x, y) = \sqrt{4 - x^2 - 4y^2}$

Domain:  $4 - x^2 - 4y^2 \geq 0$

$$x^2 + 4y^2 \leq 4$$

$$\frac{x^2}{4} + \frac{y^2}{1} \leq 1$$

$$\left\{ (x, y): \frac{x^2}{4} + \frac{y^2}{1} \leq 1 \right\}$$

Range:  $0 \leq z \leq 2$

27.  $f(x, y) = \arccos(x + y)$

Domain:  $\{(x, y): -1 \leq x + y \leq 1\}$

Range:  $0 \leq z \leq \pi$

28.  $f(x, y) = \arcsin\left(\frac{y}{x}\right)$

Domain:  $\left\{ (x, y): -1 \leq \frac{y}{x} \leq 1 \right\}$

Range:  $-\frac{\pi}{2} \leq z \leq \frac{\pi}{2}$

29.  $f(x, y) = \ln(4 - x - y)$

Domain:  $4 - x - y > 0$

$$x + y < 4$$

$$\{(x, y): y < -x + 4\}$$

Range: all real numbers

30.  $f(x, y) = \ln(xy - 6)$

Domain:  $xy - 6 > 0$

$$xy > 6$$

$$\{(x, y): xy > 6\}$$

Range: all real numbers

31.  $f(x, y) = \frac{-4x}{x^2 + y^2 + 1}$

(a) View from the positive  $x$ -axis:  $(20, 0, 0)$

(b) View where  $x$  is negative,  $y$  and  $z$  are positive:  $(-15, 10, 20)$

(c) View from the first octant:  $(20, 15, 25)$

(d) View from the line  $y = x$  in the  $xy$ -plane:  $(20, 20, 0)$

32. (a) Domain:

$$\{(x, y): x \text{ is any real number, } y \text{ is any real number}\}$$

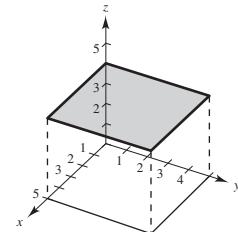
Range:  $-2 \leq z \leq 2$

(b)  $z = 0$  when  $x = 0$  which represents points on the  $y$ -axis.

(c) No. When  $x$  is positive,  $z$  is negative. When  $x$  is negative,  $z$  is positive. The surface does not pass through the first octant, the octant where  $y$  is negative and  $x$  and  $z$  are positive, the octant where  $y$  is positive and  $x$  and  $z$  are negative, and the octant where  $x$ ,  $y$  and  $z$  are all negative.

33.  $f(x, y) = 4$

Plane:  $z = 4$

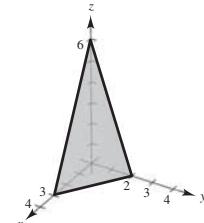


34.  $f(x, y) = 6 - 2x - 3y$

Plane

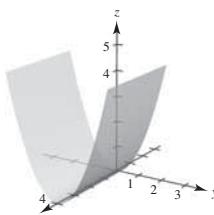
Domain: entire  $xy$ -plane

Range:  $-\infty < z < \infty$



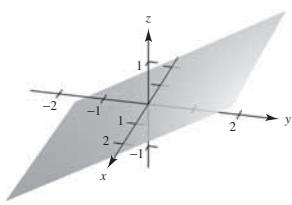
35.  $f(x, y) = y^2$

Because the variable  $x$  is missing, the surface is a cylinder with rulings parallel to the  $x$ -axis. The generating curve is  $z = y^2$ . The domain is the entire  $xy$ -plane and the range is  $z \geq 0$ .



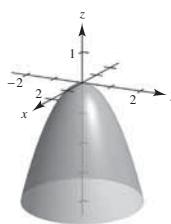
36.  $g(x, y) = \frac{1}{2}y$

Plane:  $z = \frac{1}{2}y$



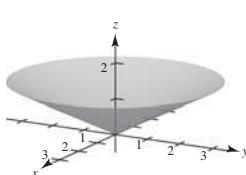
37.  $z = -x^2 - y^2$

Paraboloid

Domain: entire  $xy$ -planeRange:  $z \leq 0$ 

38.  $z = \frac{1}{2}\sqrt{x^2 + y^2}$

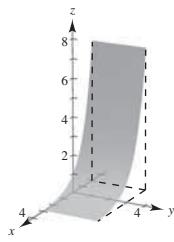
Cone

Domain of  $f$ : entire  $xy$ -planeRange:  $z \geq 0$ 

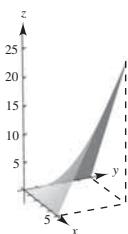
39.  $f(x, y) = e^{-x}$

Because the variable  $y$  is missing, the surface is a cylinder with rulings parallel to the  $y$ -axis. The generating curve is  $z = e^{-x}$ .

The domain is the entire  $xy$ -plane and the range is  $z > 0$ .

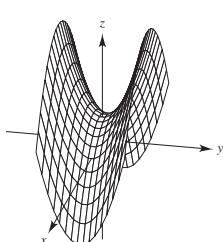


40.  $f(x, y) = \begin{cases} xy, & x \geq 0, y \geq 0 \\ 0, & \text{elsewhere} \end{cases}$

Domain of  $f$ : entire  $xy$ -planeRange:  $z \geq 0$ 

41.  $z = y^2 - x^2 + 1$

Hyperbolic paraboloid

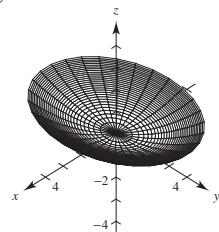
Domain: entire  $xy$ -planeRange:  $-\infty < z < \infty$ 

42.  $f(x, y) = \frac{1}{12}\sqrt{144 - 16x^2 - 9y^2}$

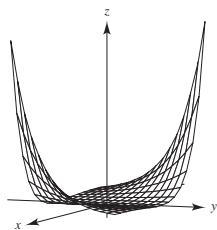
Semi-ellipsoid

Domain: set of all points lying on or inside the ellipse

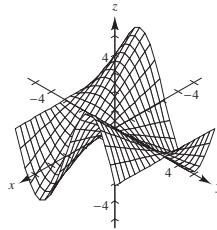
$$\left(\frac{x^2}{9}\right) + \left(\frac{y^2}{16}\right) = 1$$

Range:  $0 \leq z \leq 1$ 

43.  $f(x, y) = x^2 e^{(-xy)/2}$



44.  $f(x, y) = x \sin y$



45.  $z = e^{1-x^2-y^2}$

Level curves:

$$c = e^{1-x^2-y^2}$$

$$\ln c = 1 - x^2 - y^2$$

$$x^2 + y^2 = 1 - \ln c$$

Circles centered at  $(0, 0)$ 

Matches (c)

46.  $z = e^{1-x^2+y^2}$

Level curves:

$$c = e^{1-x^2+y^2}$$

$$\ln c = 1 - x^2 + y^2$$

$$x^2 - y^2 = 1 - \ln c$$

Hyperbolas centered at  $(0, 0)$ 

Matches (d)

47.  $z = \ln|y - x^2|$

Level curves:

$$c = \ln|y - x^2|$$

$$\pm e^c = y - x^2$$

$$y = x^2 \pm e^c$$

Parabolas

Matches (b)

48.  $z = \cos\left(\frac{x + 2y^2}{4}\right)$

Level curves:

$$c = \cos\left(\frac{x^2 + 2y^2}{4}\right)$$

$$\cos^{-1} c = \frac{x^2 + 2y^2}{4}$$

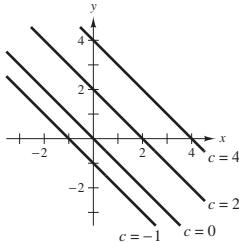
$$x^2 + 2y^2 = 4 \cos^{-1} c$$

Ellipses

Matches (a)

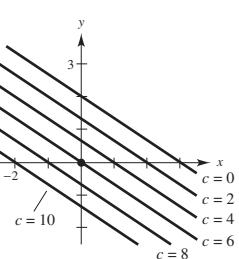
49.  $z = x + y$

Level curves are parallel lines of the form  $x + y = c$ .



50.  $f(x, y) = 6 - 2x - 3y$

The level curves are of the form  $6 - 2x - 3y = c$  or  $2x + 3y = 6 - c$ . So, the level curves are straight lines with a slope of  $-\frac{2}{3}$ .

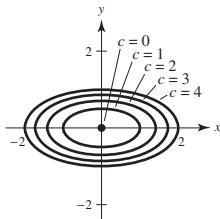


51.  $z = x^2 + 4y^2$

The level curves are ellipses of the form

$$x^2 + 4y^2 = c$$

(except  $x^2 + 4y^2 = 0$  is the point  $(0, 0)$ ).



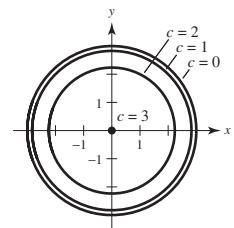
52.  $f(x, y) = \sqrt{9 - x^2 - y^2}$

The level curves are of the form

$$c = \sqrt{9 - x^2 - y^2}$$

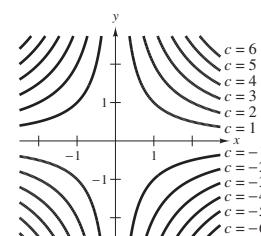
$$x^2 + y^2 = 9 - c^2, \text{ circles.}$$

$(x^2 + y^2 = 0$  is the point  $(0, 0)$ .)



53.  $f(x, y) = xy$

The level curves are hyperbolas of the form  $xy = c$ .

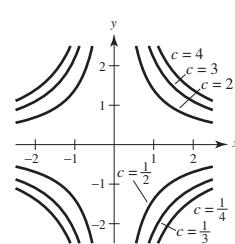


54.  $f(x, y) = e^{xy/2}$

The level curves are of the form

$$e^{xy/2} = c, \text{ or } \ln c = \frac{xy}{2}.$$

So, the level curves are hyperbolas.



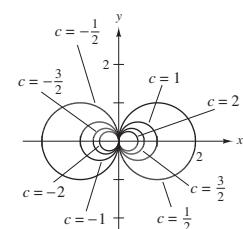
55.  $f(x, y) = \frac{x}{x^2 + y^2}$

The level curves are of the form

$$c = \frac{x}{x^2 + y^2}$$

$$x^2 - \frac{x}{c} + y^2 = 0$$

$$\left(x - \frac{1}{2c}\right)^2 + y^2 = \left(\frac{1}{2c}\right)^2.$$



So, the level curves are circles passing through the origin and centered at  $(\pm 1/(2c), 0)$ .

56.  $f(x, y) = \ln(x - y)$

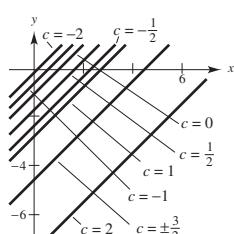
The level curves are of the form

$$c = \ln(x - y)$$

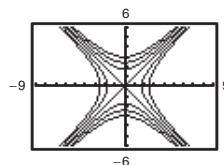
$$e^c = x - y$$

$$y = x - e^c.$$

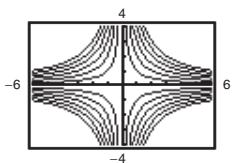
So, the level curves are parallel lines of slope 1 passing through the fourth quadrant.



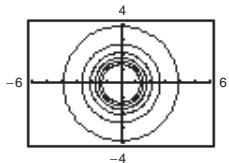
57.  $f(x, y) = x^2 - y^2 + 2$



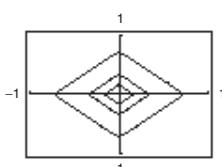
58.  $f(x, y) = |xy|$



59.  $g(x, y) = \frac{8}{1 + x^2 + y^2}$



60.  $h(x, y) = 3 \sin(|x| + |y|)$



61. The graph of a function of two variables is the set of all points  $(x, y, z)$  for which  $z = f(x, y)$  and  $(x, y)$  is in the domain of  $f$ . The graph can be interpreted as a surface in space. Level curves are the scalar fields  $f(x, y) = c$ , where  $c$  is a constant.

62. No, the following graphs are not hemispheres.

$$z = e^{-(x^2+y^2)}$$

$$z = x^2 + y^2$$

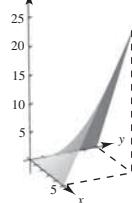
63.  $f(x, y) = \frac{x}{y}$

The level curves are the lines  $c = \frac{x}{y}$  or  $y = \frac{1}{c}x$ .

These lines all pass through the origin.

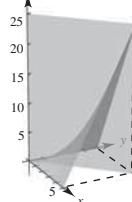
64.  $f(x, y) = xy, x \geq 0, y \geq 0$

(a)



- (b)  $g$  is a vertical translation of  $f$  three units downward.
- (c)  $g$  is a reflection of  $f$  in the  $xy$ -plane.
- (d) The graph of  $g$  is lower than the graph of  $f$ . If  $z = f(x, y)$  is on the graph of  $f$ , then  $\frac{1}{2}z$  is on the graph of  $g$ .

(e)



65. The surface is sloped like a saddle. The graph is not unique. Any vertical translation would have the same level curves.

One possible function is

$$f(x, y) = |xy|.$$

66. The surface could be an ellipsoid centered at  $(0, 1, 0)$ .

One possible function is

$$f(x, y) = x^2 + \frac{(y - 1)^2}{4} - 1.$$

67.  $V(I, R) = 1000 \left[ \frac{1 + 0.06(1 - R)}{1 + I} \right]^{10}$

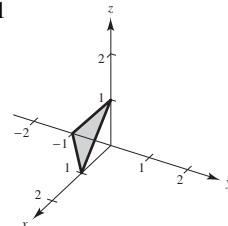
Inflation Rate			
Tax Rate	0	0.03	0.05
0	1790.85	1332.56	1099.43
0.28	1526.43	1135.80	937.09
0.35	1466.07	1090.90	900.04

68.  $A(r, t) = 5000e^{rt}$

Number of Year				
Rate	5	10	15	20
0.02	5525.85	6107.01	6749.29	7459.12
0.03	5809.17	6749.29	7841.56	9110.59
0.04	6107.01	7459.12	9110.59	11,127.70
0.05	6420.13	8243.61	10,585.00	13,591.41

69.  $f(x, y, z) = x - y + z, c = 1$

$$1 = x - y + z, \text{ Plane}$$

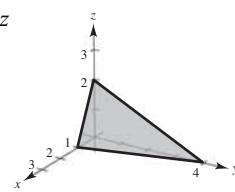


70.  $f(x, y, z) = 4x + y + 2z$

$$c = 4$$

$$4 = 4x + y + 2z$$

Plane

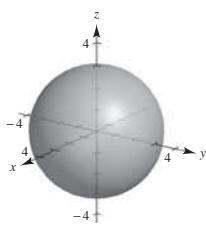


71.  $f(x, y, z) = x^2 + y^2 + z^2$

$$c = 9$$

$$9 = x^2 + y^2 + z^2$$

Sphere



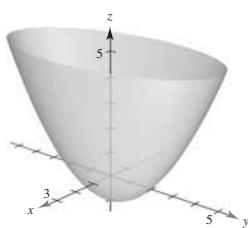
72.  $f(x, y, z) = x^2 + \frac{1}{4}y^2 - z$

$$c = 1$$

$$1 = x^2 + \frac{1}{4}y^2 - z$$

Elliptic paraboloid

Vertex:  $(0, 0, -1)$

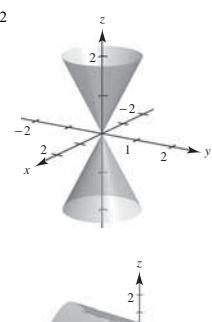


73.  $f(x, y, z) = 4x^2 + 4y^2 - z^2$

$$c = 0$$

$$0 = 4x^2 + 4y^2 - z^2$$

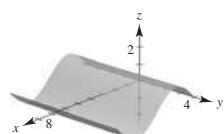
Elliptic cone



74.  $f(x, y, z) = \sin x - z$

$$c = 0$$

$$0 = \sin x - z \text{ or } z = \sin x$$



75.  $N(d, L) = \left(\frac{d-4}{4}\right)^2 L$

(a)  $N(22, 12) = \left(\frac{22-4}{4}\right)^2 (12) = 243$  board-feet

(b)  $N(30, 12) = \left(\frac{30-4}{4}\right)^2 (12) = 507$  board-feet

76.  $w = \frac{1}{x-y}, y < x$

(a)  $w(15, 9) = \frac{1}{15-9} = \frac{1}{6}$  h = 10 min

(b)  $w(15, 13) = \frac{1}{15-13} = \frac{1}{2}$  h = 30 min

(c)  $w(12, 7) = \frac{1}{12-7} = \frac{1}{5}$  h = 12 min

(d)  $w(5, 2) = \frac{1}{5-2} = \frac{1}{3}$  h = 20 min

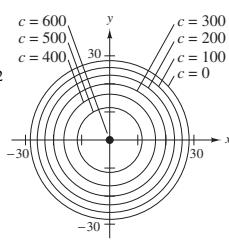
77.  $T = 600 - 0.75x^2 - 0.75y^2$

The level curves are of the form

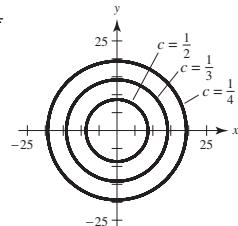
$$c = 600 - 0.75x^2 - 0.75y^2$$

$$x^2 + y^2 = \frac{600 - c}{0.75}$$

The level curves are circles centered at the origin.



78.  $V(x, y) = \frac{5}{\sqrt{25 + x^2 + y^2}}$



79.  $f(x, y) = 100x^{0.6}y^{0.4}$

$$f(2x, 2y) = 100(2x)^{0.6}(2y)^{0.4}$$

$$= 100(2)^{0.6}x^{0.6}(2)^{0.4}y^{0.4}$$

$$= 100(2)^{0.6}(2)^{0.4}x^{0.6}y^{0.4}$$

$$= 2[100x^{0.6}y^{0.4}] = 2f(x, y)$$

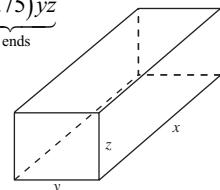
80.  $z = Cx^a y^{1-a}$

$$\ln z = \ln C + a \ln x + (1-a) \ln y$$

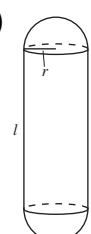
$$\ln z - \ln y = \ln C + a \ln x - a \ln y$$

$$\ln \frac{z}{y} = \ln C + a \ln \frac{x}{y}$$

81.  $C = \underbrace{1.20xy}_{\text{base}} + \underbrace{2(0.75)xz}_{\text{front and back}} + \underbrace{2(0.75)yz}_{\text{2 ends}}$   
 $= 1.20xy + 1.50(xz + yz)$



82.  $V = \pi r^2 l + \frac{4}{3}\pi r^3 = \frac{\pi r^2}{3}(3l + 4r)$



83.  $PV = kT$

(a)  $26(2000) = k(300) \Rightarrow k = \frac{520}{3}$

(b)  $P = \frac{kT}{V} = \frac{520}{3} \left(\frac{T}{V}\right)$

The level curves are of the form

$$c = \frac{520}{3} \left(\frac{T}{V}\right), \text{ or } V = \frac{520}{3c}T.$$

These are lines through the origin with slope  $\frac{520}{3c}$ .

84. (a)  $z = f(x, y) = 0.026x + 0.316y + 5.04$

Year	2002	2003	2004	2005	2006	2007
$z$	35.2	39.5	43.6	49.4	53.2	61.6
Model	35.3	39.4	44.0	49.4	53.3	61.8

(b)  $y$  has the greater influence because its coefficient (0.316) is greater than  $x$ 's coefficient (0.026).

(c)  $f(x, 95) = 0.026x + 0.316(95) + 5.04 = 0.026x + 35.06$

This gives the shareholder's equity in terms of net sales  $x$ , assuming total assets of  $y = 95$  (billion).

85. (a) Highest pressure at  $C$

(b) Lowest pressure at  $A$

(c) Highest wind velocity at  $B$

86. Southwest

87. (a) No; the level curves are uneven and sporadically spaced.

(b) Use more colors.

88. (a) The different colors represent various amplitudes.

(b) No, the level curves are uneven and sporadically spaced.

89. False. Let

$$f(x, y) = 2xy$$

$$f(1, 2) = f(2, 1), \text{ but } 1 \neq 2.$$

90. False. Let

$$f(x, y) = 5.$$

$$\text{Then, } f(2x, 2y) = 5 \neq 2^2 f(x, y).$$

91. True

92. False. If there were a point  $(x, y)$  on the level curves

$$f(x, y) = C_1 \text{ and } f(x, y) = C_2, \text{ then } C_1 = C_2.$$

## Section 13.2 Limits and Continuity

1.  $\lim_{(x, y) \rightarrow (1, 0)} x = 1$

$$f(x, y) = x, L = 1$$

We need to show that for all  $\varepsilon > 0$ , there exist a  $\delta$ -neighborhood about  $(1, 0)$  such that

$$|f(x, y) - L| = |x - 1| < \varepsilon$$

Whenever  $(x, y) \neq (1, 0)$  lies in the neighborhood.

From  $0 < \sqrt{(x-1)^2 + (y-0)^2} < \delta$ , it follows that

$$|x-1| = \sqrt{(x-1)^2} \leq \sqrt{(x-1)^2 + (y-0)^2} < \delta.$$

So, choose  $\delta = \varepsilon$  and the limit is verified.

2. Let  $\varepsilon > 0$  be given. We need to find  $\delta > 0$  such that

$$|f(x, y) - L| = |x - 4| < \varepsilon$$

whenever

$$0 < \sqrt{(x-a)^2 + (y-b)^2} = \sqrt{(x-4)^2 + (y+1)^2} < \delta.$$

Take  $\delta = \varepsilon$ .

Then if  $0 < \sqrt{(x-4)^2 + (y+1)^2} < \delta = \varepsilon$ , we have

$$\sqrt{(x-4)^2} < \varepsilon$$

$$|x-4| < \varepsilon.$$

3.  $\lim_{(x, y) \rightarrow (1, -3)} y = -3. f(x, y) = y, L = -3$

We need to show that for all  $\varepsilon > 0$ , there exists a  $\delta$ -neighborhood about  $(1, -3)$  such that

$$|f(x, y) - L| = |y + 3| < \varepsilon$$

whenever  $(x, y) \neq (1, -3)$  lies in the neighborhood.

From  $0 < \sqrt{(x-1)^2 + (y+3)^2} < \delta$  it follows that

$$|y+3| = \sqrt{(y+3)^2} \leq \sqrt{(x-1)^2 + (y+3)^2} < \delta.$$

So, choose  $\delta = \varepsilon$  and the limit is verified.

4. Let  $\varepsilon > 0$  be given. We need to find  $\delta > 0$  such that

$$|f(x, y) - L| = |y - b| < \varepsilon$$

whenever  $0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta$ . Take  $\delta = \varepsilon$ .

Then if  $0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta = \varepsilon$ , we have

$$\sqrt{(y-b)^2} < \varepsilon$$

$$|y-b| < \varepsilon.$$

5.  $\lim_{(x,y) \rightarrow (a,b)} [f(x,y) - g(x,y)] = \lim_{(x,y) \rightarrow (a,b)} f(x,y) - \lim_{(x,y) \rightarrow (a,b)} g(x,y) = 4 - 3 = 1$

6.  $\lim_{(x,y) \rightarrow (a,b)} \left[ \frac{5f(x,y)}{g(x,y)} \right] = \frac{5 \left[ \lim_{(x,y) \rightarrow (a,b)} f(x,y) \right]}{\lim_{(x,y) \rightarrow (a,b)} g(x,y)} = \frac{5(4)}{3} = \frac{20}{3}$

7.  $\lim_{(x,y) \rightarrow (a,b)} [f(x,y)g(x,y)] = \left[ \lim_{(x,y) \rightarrow (a,b)} f(x,y) \right] \left[ \lim_{(x,y) \rightarrow (a,b)} g(x,y) \right] = 4(3) = 12$

8.  $\lim_{(x,y) \rightarrow (a,b)} \left[ \frac{f(x,y) + g(x,y)}{f(x,y)} \right] = \frac{\lim_{(x,y) \rightarrow (a,b)} f(x,y) + \lim_{(x,y) \rightarrow (a,b)} g(x,y)}{\lim_{(x,y) \rightarrow (a,b)} f(x,y)} = \frac{4+3}{4} = \frac{7}{4}$

9.  $\lim_{(x,y) \rightarrow (2,1)} (2x^2 + y) = 8 + 1 = 9$

Continuous everywhere

10.  $\lim_{(x,y) \rightarrow (0,0)} (x + 4y + 1) = 0 + 4(0) + 1 = 1$

Continuous everywhere

11.  $\lim_{(x,y) \rightarrow (1,2)} e^{xy} = e^{1(2)} = e^2$

Continuous everywhere

12.  $\lim_{(x,y) \rightarrow (2,4)} \frac{x+y}{x^2+1} = \frac{2+4}{2^2+1} = \frac{6}{5}$

Continuous everywhere

13.  $\lim_{(x,y) \rightarrow (0,2)} \frac{x}{y} = \frac{0}{2} = 0$

Continuous for all  $y \neq 0$

14.  $\lim_{(x,y) \rightarrow (-1,2)} \frac{x+y}{x-y} = \frac{-1+2}{-1-2} = -\frac{1}{3}$

Continuous for all  $x \neq y$ .

15.  $\lim_{(x,y) \rightarrow (1,1)} \frac{xy}{x^2+y^2} = \frac{1}{2}$

Continuous except at  $(0,0)$

16.  $\lim_{(x,y) \rightarrow (1,1)} \frac{x}{\sqrt{x+y}} = \frac{1}{\sqrt{1+1}} = \frac{\sqrt{2}}{2}$

Continuous for  $x+y > 0$

17.  $\lim_{(x,y) \rightarrow (\pi/4,2)} y \cos(xy) = 2 \cos \frac{\pi}{2} = 0$

Continuous everywhere

18.  $\lim_{(x,y) \rightarrow (2\pi,4)} \sin \frac{x}{y} = \sin \frac{2\pi}{4} = 1$

Continuous for all  $y \neq 0$

19.  $\lim_{(x,y) \rightarrow (0,1)} \frac{\arcsin xy}{1-xy} = \frac{\arcsin 0}{1} = 0$

Continuous for  $xy \neq 1, |xy| \leq 1$

20.  $\lim_{(x,y) \rightarrow (0,1)} \frac{\arccos \left( \frac{x}{y} \right)}{1+xy} = \frac{\arccos 0}{1} = \frac{\pi}{2}$

Continuous for  $xy \neq -1, y \neq 0, 0 \leq \frac{x}{y} \leq \pi$

21.  $\lim_{(x,y,z) \rightarrow (1,3,4)} \sqrt{x+y+z} = \sqrt{1+3+4} = 2\sqrt{2}$

Continuous for  $x+y+z \geq 0$

22.  $\lim_{(x,y,z) \rightarrow (-2,1,0)} xe^{yz} = (-2)e^{1(0)} = -2$

Continuous everywhere

23.  $\lim_{(x,y) \rightarrow (1,1)} \frac{xy-1}{1+xy} = \frac{1-1}{1+1} = 0$

24.  $\lim_{(x,y) \rightarrow (1,-1)} \frac{x^2y}{1+xy^2} = \frac{-1}{1+1} = -\frac{1}{2}$

25.  $\lim_{(x,y) \rightarrow (0,0)} \frac{1}{x+y}$  does not exist

Because the denominator  $x+y$  approaches 0 as  $(x,y) \rightarrow (0,0)$ .

26.  $\lim_{(x,y) \rightarrow (0,0)} \frac{1}{x^2y^2}$  does not exist because the denominator  $xy$  approaches 0 as  $(x,y) \rightarrow (0,0)$ .

27.  $\lim_{(x,y) \rightarrow (2,2)} \frac{x^2-y^2}{x-y} = \lim_{(x,y) \rightarrow (2,2)} \frac{(x-y)(x+y)}{x-y} = \lim_{(x,y) \rightarrow (2,2)} (x+y) = 4$

28.  $\lim_{(x,y) \rightarrow (0,0)} \frac{x^4-4y^4}{x^2+2y^2} = \lim_{(x,y) \rightarrow (0,0)} \frac{(x^2-2y^2)(x^2+2y^2)}{x^2+2y^2} = \lim_{(x,y) \rightarrow (0,0)} (x^2-2y^2) = 0$

29.  $\lim_{(x,y) \rightarrow (0,0)} \frac{x-y}{\sqrt{x} - \sqrt{y}}$

does not exist because you can't approach  $(0,0)$  from negative values of  $x$  and  $y$ .

30. 
$$\begin{aligned} & \lim_{(x,y) \rightarrow (2,1)} \frac{x-y-1}{\sqrt{x-y}-1} \cdot \frac{\sqrt{x-y}+1}{\sqrt{x-y}+1} \\ &= \lim_{(x,y) \rightarrow (2,1)} \frac{(x-y-1)(\sqrt{x-y}+1)}{(x-y)-1} \\ &= \lim_{(x,y) \rightarrow (2,1)} (\sqrt{x-y}+1) = 2 \end{aligned}$$

31. The limit does not exist because along the line  $y = 0$  you have

$$\lim_{(x,y) \rightarrow (0,0)} \frac{x+y}{x^2+y} = \lim_{(x,0) \rightarrow (0,0)} \frac{x}{x^2} = \lim_{(x,0) \rightarrow (0,0)} \frac{1}{x}$$

which does not exist.

32. The limit does not exist because along the line  $x = y$  you have

$$\lim_{(x,y) \rightarrow (0,0)} \frac{x}{x^2-y^2} = \lim_{(x,x) \rightarrow (0,0)} \frac{x}{x^2-x^2} = \lim_{(x,x) \rightarrow (0,0)} \frac{x}{0}.$$

Because the denominator is 0, the limit does not exist.

33.  $\lim_{(x,y) \rightarrow (0,0)} \frac{x^2}{(x^2+1)(y^2+1)} = \frac{0}{(1)(1)} = 0$

34.  $\lim_{(x,y) \rightarrow (0,0)} \ln(x^2 + y^2)$  does not exist

because  $\ln(x^2 + y^2) \rightarrow -\infty$  as  $(x,y) \rightarrow (0,0)$ .

35. The limit does not exist because along the path  $x = 0, y = 0$ , you have

$$\lim_{(x,y,z) \rightarrow (0,0,0)} \frac{xy + yz + xz}{x^2 + y^2 + z^2} = \lim_{(0,0,z) \rightarrow (0,0,0)} \frac{0}{z^2} = 0$$

whereas along the path  $x = y = z$ , you have

$$\begin{aligned} \lim_{(x,y,z) \rightarrow (0,0,0)} \frac{xy + yz + xz}{x^2 + y^2 + z^2} &= \lim_{(x,x,x) \rightarrow (0,0,0)} \frac{x^2 + x^2 + x^2}{x^2 + x^2 + x^2} \\ &= 1 \end{aligned}$$

36. The limit does not exist because along the path  $y = z = 0$ , you have

$$\lim_{(x,y,z) \rightarrow (0,0,0)} \frac{xy + yz^2 + xz^2}{x^2 + y^2 + z^2} = \lim_{(x,0,0) \rightarrow (0,0,0)} \frac{0}{x^2} = 0$$

However, along the path  $z = 0, x = y$ , you have

$$\begin{aligned} \lim_{(x,y,z) \rightarrow (0,0,0)} \frac{xy + yz^2 + xz^2}{x^2 + y^2 + z^2} &= \lim_{(x,x,0) \rightarrow (0,0,0)} \frac{x^2}{x^2 + x^2} \\ &= \frac{1}{2} \end{aligned}$$

37.  $\lim_{(x,y) \rightarrow (0,0)} e^{xy} = 1$

Continuous everywhere

38.  $\lim_{(x,y) \rightarrow (0,0)} \left[ 1 - \frac{\cos(x^2 + y^2)}{x^2 + y^2} \right] = -\infty$

The limit does not exist.

Continuous except at  $(0,0)$

39.  $f(x, y) = \frac{xy}{x^2 + y^2}$

Continuous except at  $(0,0)$

Path:  $y = 0$

$(x, y)$	$(1, 0)$	$(0.5, 0)$	$(0.1, 0)$	$(0.01, 0)$	$(0.001, 0)$
$f(x, y)$	0	0	0	0	0

Path:  $y = x$

$(x, y)$	$(1, 1)$	$(0.5, 0.5)$	$(0.1, 0.1)$	$(0.01, 0.01)$	$(0.001, 0.001)$
$f(x, y)$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$

The limit does not exist because along the path  $y = 0$  the function equals 0, whereas

along the path  $y = x$  the function equals  $\frac{1}{2}$ .

40.  $f(x, y) = \frac{y}{x^2 + y^2}$

Continuous except at  $(0, 0)$

Path:  $y = x$

$(x, y)$	$(1, 1)$	$(0.5, 0.5)$	$(0.1, 0.1)$	$(0.01, 0.01)$	$(0.001, 0.001)$
$f(x, y)$	$\frac{1}{2}$	1	5	50	500

Path:  $y = 0$

$(x, y)$	$(1, 0)$	$(0.5, 0)$	$(0.1, 0)$	$(0.01, 0)$	$(0.001, 0)$
$f(x, y)$	0	0	0	0	0

The limit does not exist because along the path  $y = 0$  the function equals 0, whereas along the path  $y = x$  the function tends to infinity.

41.  $f(x, y) = -\frac{xy^2}{x^2 + y^4}$

Continuous except at  $(0, 0)$

Path:  $x = y^2$

$(x, y)$	$(1, 1)$	$(0.25, 0.5)$	$(0.01, 0.1)$	$(0.0001, 0.01)$	$(0.000001, 0.001)$
$f(x, y)$	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$

Path:  $x = -y^2$

$(x, y)$	$(-1, 1)$	$(-0.25, 0.5)$	$(-0.01, 0.1)$	$(-0.0001, 0.01)$	$(-0.000001, 0.001)$
$f(x, y)$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$

The limit does not exist because along the path  $x = y^2$  the function equals  $-\frac{1}{2}$ , whereas along the path  $x = -y^2$  the function equals  $\frac{1}{2}$ .

42.  $f(x, y) = \frac{2x - y^2}{2x^2 + y}$

Continuous except at  $(0, 0)$

Path:  $y = 0$

$(x, y)$	$(1, 0)$	$(0.25, 0)$	$(0.01, 0)$	$(0.001, 0)$	$(0.000001, 0)$
$f(x, y)$	1	4	100	1000	1,000,000

Path:  $y = x$

$(x, y)$	$(1, 1)$	$(0.25, 0.25)$	$(0.01, 0.01)$	$(0.001, 0.001)$	$(0.0001, 0.0001)$
$f(x, y)$	$\frac{1}{3}$	1.17	1.95	1.995	2.0

The limit does not exist because along the  $y = 0$  the function tends to infinity, whereas along the line  $y = x$  the function tends to 2.

43.  $\lim_{(x,y) \rightarrow (0,0)} \frac{x^4 - y^4}{x^2 + y^2} = \lim_{(x,y) \rightarrow (0,0)} \frac{(x^2 + y^2)(x^2 - y^2)}{x^2 + y^2} = \lim_{(x,y) \rightarrow (0,0)} (x^2 - y^2) = 0$

So,  $f$  is continuous everywhere, whereas  $g$  is continuous everywhere except at  $(0,0)$ .  $g$  has a removable discontinuity at  $(0,0)$ .

44.  $\lim_{(x,y) \rightarrow (0,0)} \frac{4x^4 - y^4}{2x^2 + y^2} = \lim_{(x,y) \rightarrow (0,0)} \frac{(2x^2 + y^2)(2x^2 - y^2)}{2x^2 + y^2} = \lim_{(x,y) \rightarrow (0,0)} (2x^2 - y^2) = 0$

So,  $g$  is continuous everywhere, whereas  $f$  is continuous everywhere except  $(0,0)$ .  $f$  has a removable discontinuity at  $(0,0)$ .

45.  $\lim_{(x,y) \rightarrow (0,0)} \frac{4x^2y^2}{x^2 + y^2} = 0$

So,  $\lim_{(x,y) \rightarrow (0,0)} f(x,y) = \lim_{(x,y) \rightarrow (0,0)} g(x,y) = 0$ .

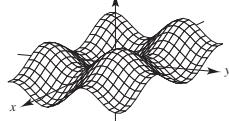
$f$  is continuous at  $(0,0)$ , whereas  $g$  is not continuous at  $(0,0)$ .

46.  $\lim_{(x,y) \rightarrow (0,0)} f(x,y) = \lim_{(x,y) \rightarrow (0,0)} \left( \frac{x^2 + 2xy^2 + y^2}{x^2 + y^2} \right)$   
 $= \lim_{(x,y) \rightarrow (0,0)} \left( 1 + \frac{2xy^2}{x^2 + y^2} \right) = 1$

(same limit for  $g$ )

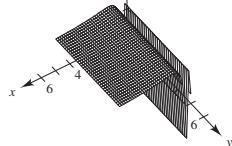
So,  $f$  is not continuous at  $(0,0)$ , whereas  $g$  is continuous at  $(0,0)$ .

47.  $\lim_{(x,y) \rightarrow (0,0)} \sin x + \sin y = 0$



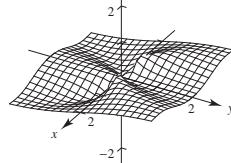
48.  $\lim_{(x,y) \rightarrow (0,0)} \sin \frac{1}{x} + \cos \frac{1}{x}$

Does not exist



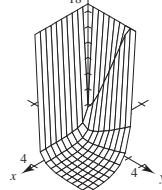
49.  $\lim_{(x,y) \rightarrow (0,0)} \frac{x^2y}{x^4 + 2y^2}$

Does not exist. Use the paths  $x = 0$  and  $y = x^2$ .



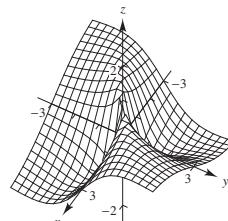
50.  $\lim_{(x,y) \rightarrow (0,0)} \frac{x^2 + y^2}{x^2y}$

Does not exist

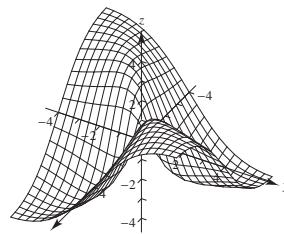


51.  $\lim_{(x,y) \rightarrow (0,0)} \frac{5xy}{x^2 + 2y^2}$

Does not exist. Use the paths  $x = 0$  and  $x = y$ .



52.  $\lim_{(x,y) \rightarrow (0,0)} \frac{6xy}{x^2 + y^2 + 1} = 0$



53.  $\lim_{(x,y) \rightarrow (0,0)} \frac{xy^2}{x^2 + y^2} = \lim_{r \rightarrow 0} \frac{(r \cos \theta)(r^2 \sin^2 \theta)}{r^2}$   
 $= \lim_{r \rightarrow 0} (r \cos \theta \sin^2 \theta) = 0$

54.  $\lim_{(x,y) \rightarrow (0,0)} \frac{x^3 + y^3}{x^2 + y^2} = \lim_{r \rightarrow 0} \frac{r^3(\cos^3 \theta + \sin^3 \theta)}{r^2}$   
 $= \lim_{r \rightarrow 0} r(\cos^3 \theta + \sin^3 \theta) = 0$

55.  $\lim_{(x,y) \rightarrow (0,0)} \frac{x^2y^2}{x^2 + y^2} = \lim_{r \rightarrow 0} \frac{r^4 \cos^2 \theta \sin^2 \theta}{r^2}$   
 $= \lim_{r \rightarrow 0} r^2 \cos^2 \theta \sin^2 \theta = 0$

**56.**  $x = r \cos \theta, y = r \sin \theta, \sqrt{x^2 + y^2} = r, x^2 - y^2 = r^2(\cos^2 \theta - \sin^2 \theta)$

$$\lim_{(x,y) \rightarrow (0,0)} \frac{x^2 - y^2}{\sqrt{x^2 + y^2}} = \lim_{r \rightarrow 0} \frac{r^2(\cos^2 \theta - \sin^2 \theta)}{r} = \lim_{r \rightarrow 0} r(\cos^2 \theta - \sin^2 \theta) = 0$$

**57.**  $\lim_{(x,y) \rightarrow (0,0)} \cos(x^2 + y^2) = \lim_{r \rightarrow 0} \cos(r^2) = \cos(0) = 1$

**58.**  $\lim_{(x,y) \rightarrow (0,0)} \sin \sqrt{x^2 + y^2} = \lim_{r \rightarrow 0} \sin(r) = \sin(0) = 0$

**59.**  $\sqrt{x^2 + y^2} = r$

$$\lim_{(x,y) \rightarrow (0,0)} \frac{\sin \sqrt{x^2 + y^2}}{\sqrt{x^2 + y^2}} = \lim_{r \rightarrow 0^+} \frac{\sin(r)}{r} = 1$$

**60.**  $\lim_{(x,y) \rightarrow (0,0)} \frac{\sin(x^2 + y^2)}{x^2 + y^2} = \lim_{r \rightarrow 0} \frac{\sin r^2}{r^2} = \lim_{r \rightarrow 0} \frac{2r \cos r^2}{2r} = \lim_{r \rightarrow 0} \cos r^2 = 1$

**61.**  $x^2 + y^2 = r^2$

$$\lim_{(x,y) \rightarrow (0,0)} \frac{1 - \cos(x^2 + y^2)}{x^2 + y^2} = \lim_{r \rightarrow 0} \frac{1 - \cos(r^2)}{r^2} = 0$$

**62.**  $x^2 + y^2 = r^2$

$$\lim_{(x,y) \rightarrow (0,0)} (x^2 + y^2) \ln(x^2 + y^2) = \lim_{r \rightarrow 0} r^2 \ln(r^2) = \lim_{r \rightarrow 0^+} 2r^2 \ln(r)$$

By L'Hôpital's Rule,  $\lim_{r \rightarrow 0^+} 2r^2 \ln(r) = \lim_{r \rightarrow 0^+} \frac{2 \ln(r)}{1/r^2} = \lim_{r \rightarrow 0^+} \frac{2/r}{-2/r^3} = \lim_{r \rightarrow 0^+} (-r^2) = 0$

**63.**  $f(x, y, z) = \frac{1}{\sqrt{x^2 + y^2 + z^2}}$

Continuous except at  $(0, 0, 0)$

**64.**  $f(x, y, z) = \frac{z}{x^2 + y^2 - 4}$

Continuous for  $x^2 + y^2 \neq 4$ .

**65.**  $f(x, y, z) = \frac{\sin z}{e^x + e^y}$

Continuous everywhere

**66.**  $f(x, y, z) = xy \sin z$

Continuous everywhere

**67.** For  $xy \neq 0$ , the function is clearly continuous.

For  $xy \neq 0$ , let  $z = xy$ . Then

$$\lim_{z \rightarrow 0} \frac{\sin z}{z} = 1$$

implies that  $f$  is continuous for all  $x, y$ .

**68.** For  $x^2 \neq y^2$ , the function is clearly continuous.

For  $x^2 \neq y^2$ , let  $z = x^2 - y^2$ . Then

$$\lim_{z \rightarrow 0} \frac{\sin(z)}{z} = 1$$

implies that  $f$  is continuous for all  $x, y$ .

**69.**  $f(t) = t^2, g(x, y) = 2x - 3y$

$$f(g(x, y)) = f(2x - 3y) = (2x - 3y)^2$$

Continuous everywhere

**70.**  $f(t) = \frac{1}{t}$

$$g(x, y) = x^2 + y^2$$

Continuous except at  $(0, 0)$

**71.**  $f(t) = \frac{1}{t}, g(x, y) = 2x - 3y$

$$f(g(x, y)) = f(2x - 3y) = \frac{1}{2x - 3y}$$

Continuous for all  $y \neq \frac{2}{3}x$

72.  $f(t) = \frac{1}{1-t}$ ,  $g(x, y) = x^2 + y^2$

$$f(g(x, y)) = f(x^2 + y^2) = \frac{1}{1 - x^2 - y^2}$$

Continuous for  $x^2 + y^2 \neq 1$

73.  $f(x, y) = x^2 - 4y$

$$(a) \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{[(x + \Delta x)^2 - 4y] - (x^2 - 4y)}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{2x\Delta x + (\Delta x)^2}{\Delta x} = \lim_{\Delta x \rightarrow 0} (2x + \Delta x) = 2x$$

$$(b) \lim_{\Delta y \rightarrow 0} \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y} = \lim_{\Delta y \rightarrow 0} \frac{[x^2 - 4(y + \Delta y)] - (x^2 - 4y)}{\Delta y} = \lim_{\Delta y \rightarrow 0} \frac{-4\Delta y}{\Delta y} = \lim_{\Delta y \rightarrow 0} (-4) = -4$$

74.  $f(x, y) = x^2 + y^2$

$$(a) \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{[(x + \Delta x)^2 + y^2] - (x^2 + y^2)}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{2x\Delta x + (\Delta x)^2}{\Delta x} = \lim_{\Delta x \rightarrow 0} (2x + \Delta x) = 2x$$

$$(b) \lim_{\Delta y \rightarrow 0} \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y} = \lim_{\Delta y \rightarrow 0} \frac{[x^2 + (y + \Delta y)^2] - (x^2 + y^2)}{\Delta y} = \lim_{\Delta y \rightarrow 0} \frac{2y\Delta y + (\Delta y)^2}{\Delta y} = \lim_{\Delta y \rightarrow 0} (2y + \Delta y) = 2y$$

75.  $f(x, y) = \frac{x}{y}$

$$(a) \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{\frac{x + \Delta x}{y} - \frac{x}{y}}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{y}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{1}{y} = \frac{1}{y}$$

$$(b) \lim_{\Delta y \rightarrow 0} \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y} = \lim_{\Delta y \rightarrow 0} \frac{\frac{x}{y + \Delta y} - \frac{x}{y}}{\Delta y} = \lim_{\Delta y \rightarrow 0} \frac{xy - (xy + x\Delta y)}{(y + \Delta y)y\Delta y} = \lim_{\Delta y \rightarrow 0} \frac{-x\Delta y}{(y + \Delta y)y\Delta y} = \lim_{\Delta y \rightarrow 0} \frac{-x}{(y + \Delta y)y} = \frac{-x}{y^2}$$

76.  $f(x, y) = \frac{1}{x + y}$

$$(a) \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{\frac{1}{x + \Delta x + y} - \frac{1}{x + y}}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{(x + y) - (x + \Delta x + y)}{(x + \Delta x + y)(x + y)\Delta x}$$

$$= \lim_{\Delta x \rightarrow 0} \frac{-\Delta x}{(x + \Delta x + y)(x + y)\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{-1}{(x + \Delta x + y)(x + y)} = \frac{-1}{(x + y)^2}$$

$$(b) \text{ By symmetry, } \lim_{\Delta y \rightarrow 0} \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y} = \frac{-1}{(x + y)^2}.$$

77.  $f(x, y) = 3x + xy - 2y$

$$(a) \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{3(x + \Delta x) + (x + \Delta x)y - 2y - (3x + xy - 2y)}{\Delta x}$$

$$= \lim_{\Delta x \rightarrow 0} \frac{3\Delta x + y\Delta x}{\Delta x} = \lim_{\Delta x \rightarrow 0} (3 + y) = 3 + y$$

$$(b) \lim_{\Delta y \rightarrow 0} \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y} = \lim_{\Delta y \rightarrow 0} \frac{3x + x(y + \Delta y) - 2(y + \Delta y) - (3x + xy - 2y)}{\Delta y}$$

$$= \lim_{\Delta y \rightarrow 0} \frac{x\Delta y - 2\Delta y}{\Delta y} = \lim_{\Delta y \rightarrow 0} (x - 2) = x - 2$$

78.  $f(x, y) = \sqrt{y}(y + 1)$

$$(a) \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{\sqrt{y}(y + 1) - \sqrt{y}(y + 1)}{\Delta x} = 0$$

$$\begin{aligned} (b) \lim_{\Delta y \rightarrow 0} \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y} &= \lim_{\Delta y \rightarrow 0} \frac{(y + \Delta y)^{3/2} + (y + \Delta y)^{1/2} - (y^{3/2} + y^{1/2})}{\Delta y} \\ &= \lim_{\Delta y \rightarrow 0} \frac{(y + \Delta y)^{3/2} - y^{3/2}}{\Delta y} + \lim_{\Delta y \rightarrow 0} \frac{(y + \Delta y)^{1/2} - y^{1/2}}{\Delta y} \\ &= \frac{3}{2}y^{1/2} + \frac{1}{2}y^{-1/2} \quad (\text{L'Hôpital's Rule}) \\ &= \frac{3y + 1}{2\sqrt{y}} \end{aligned}$$

79. True. Assuming  $f(x, 0)$  exists for  $x \neq 0$ .

83.  $\lim_{(x, y) \rightarrow (0, 0)} \frac{x^2 + y^2}{xy}$

80. False. Let  $f(x, y) = \frac{xy}{x^2 + y^2}$ .

See Exercise 39.

81. False. Let  $f(x, y) = \begin{cases} \ln(x^2 + y^2), & (x, y) \neq (0, 0) \\ 0, & x = 0, y = 0 \end{cases}$

82. True

(a) Along  $y = ax$ :

$$\begin{aligned} \lim_{(x, ax) \rightarrow (0, 0)} \frac{x^2 + (ax)^2}{x(ax)} &= \lim_{x \rightarrow 0} \frac{x^2(1 + a^2)}{ax^2} \\ &= \frac{1 + a^2}{a}, a \neq 0 \end{aligned}$$

If  $a = 0$ , then  $y = 0$  and the limit does not exist.

(b) Along

$$y = x^2: \lim_{(x, x^2) \rightarrow (0, 0)} \frac{x^2 + (x^2)^2}{x(x^2)} = \lim_{x \rightarrow 0} \frac{1 + x^2}{x}$$

Limit does not exist.

(c) No, the limit does not exist. Different paths result in different limits.

84.  $f(x, y) = \frac{x^2 y}{x^4 + y^2}$

$$(a) y = ax: f(x, ax) = \frac{x^2(ax)}{x^4 + (ax)^2} = \frac{ax}{x^2 + a^2}$$

$$\text{If } a \neq 0, \lim_{(x, ax) \rightarrow (0, 0)} \frac{ax}{x^2 + a^2} = 0.$$

$$(b) y = x^2: f(x, x^2) = \frac{x^2(x^2)}{x^4 + (x^2)^2} = \frac{x^4}{2x^4}$$

$$\lim_{(x, x^2) \rightarrow (0, 0)} \frac{x^4}{2x^4} = \frac{1}{2}$$

(c) No, the limit does not exist.  $f$  approaches different numbers along different paths.

$$\begin{aligned} 85. \lim_{(x, y, z) \rightarrow (0, 0, 0)} \frac{xyz}{x^2 + y^2 + z^2} &= \lim_{\rho \rightarrow 0^+} \frac{(\rho \sin \phi \cos \theta)(\rho \sin \phi \sin \theta)(\rho \cos \phi)}{\rho^2} \\ &= \lim_{\rho \rightarrow 0^+} \rho [\sin^2 \phi \cos \theta \sin \theta \cos \phi] = 0 \end{aligned}$$

86.  $\lim_{(x, y, z) \rightarrow (0, 0, 0)} \tan^{-1} \left[ \frac{1}{x^2 + y^2 + z^2} \right] = \lim_{\rho \rightarrow 0^+} \tan^{-1} \left[ \frac{1}{\rho^2} \right] = \frac{\pi}{2}$

87. As  $(x, y) \rightarrow (0, 1)$ ,  $x^2 + 1 \rightarrow 1$  and  $x^2 + (y - 1)^2 \rightarrow 0$ .

$$\text{So, } \lim_{(x, y) \rightarrow (0, 1)} \tan^{-1} \left[ \frac{x^2 + 1}{x^2 + (y - 1)^2} \right] = \frac{\pi}{2}.$$

88.  $\lim_{(x, y) \rightarrow (0, 0)} f(x, y) = \lim_{r \rightarrow 0} (r \cos \theta)(r \sin \theta) \frac{r^2 \cos^2 \theta - r^2 \sin^2 \theta}{r^2} = \lim_{r \rightarrow 0} r^2 [\cos \theta \sin \theta (\cos^2 \theta - \sin^2 \theta)] = 0$

So, define  $f(0, 0) = 0$ .

89. Because  $\lim_{(x, y) \rightarrow (a, b)} f(x, y) = L_1$ , then for  $\varepsilon/2 > 0$ , there corresponds  $\delta_1 > 0$  such that  $|f(x, y) - L_1| < \varepsilon/2$  whenever  $0 < \sqrt{(x - a)^2 + (y - b)^2} < \delta_1$ .

Because  $\lim_{(x, y) \rightarrow (a, b)} g(x, y) = L_2$ , then for  $\varepsilon/2 > 0$ , there corresponds  $\delta_2 > 0$  such that  $|g(x, y) - L_2| < \varepsilon/2$  whenever  $0 < \sqrt{(x - a)^2 + (y - b)^2} < \delta_2$ .

Let  $\delta$  be the smaller of  $\delta_1$  and  $\delta_2$ . By the triangle inequality, whenever  $\sqrt{(x - a)^2 + (y - b)^2} < \delta$ , we have

$$|f(x, y) + g(x, y) - (L_1 + L_2)| = |(f(x, y) - L_1) + (g(x, y) - L_2)| \leq |f(x, y) - L_1| + |g(x, y) - L_2| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

$$\text{So, } \lim_{(x, y) \rightarrow (a, b)} [f(x, y) + g(x, y)] = L_1 + L_2.$$

90. Given that  $f(x, y)$  is continuous, then  $\lim_{(x, y) \rightarrow (a, b)} f(x, y) = f(a, b) < 0$ , which means that for each  $\varepsilon > 0$ , there corresponds a  $\delta > 0$  such that  $|f(x, y) - f(a, b)| < \varepsilon$  whenever

$$0 < \sqrt{(x - a)^2 + (y - b)^2} < \delta.$$

Let  $\varepsilon = |f(a, b)|/2$ , then  $f(x, y) < 0$  for every point in the corresponding  $\delta$  neighborhood because

$$\begin{aligned} |f(x, y) - f(a, b)| &< \frac{|f(a, b)|}{2} \Rightarrow -\frac{|f(a, b)|}{2} < f(x, y) - f(a, b) < \frac{|f(a, b)|}{2} \\ &\Rightarrow \frac{3}{2}f(a, b) < f(x, y) < \frac{1}{2}f(a, b) < 0. \end{aligned}$$

91. See the definition on page 899. Show that the value of  $\lim_{(x, y) \rightarrow (x_0, y_0)} f(x, y)$  is not the same for two different paths to  $(x_0, y_0)$ .

92. See the definition on page 902.

93. (a) True

(b) False. The convergence along one path does not imply convergence along all paths.

$$(c) \text{ False. Let } f(x, y) = 4 \left[ \frac{(x - 2)^2 - (y - 3)^2}{(x - 2)^2 + (y - 3)^2} \right]^2.$$

(d) True

94. (a) No. The existence of  $f(2, 3)$  has no bearing on the existence of the limit as  $(x, y) \rightarrow (2, 3)$ .

(b) No,  $f(2, 3)$  can equal any number, or not even be defined.

### Section 13.3 Partial Derivatives

1.  $f_x(4, 1) < 0$

2.  $f_y(-1, -2) < 0$

3.  $f_y(4, 1) > 0$

4.  $f_x(-1, -1) = 0$

5. No,  $y$  only occurs in the numerator.

6. Yes,  $x$  occurs in both the numerator and denominator.

7. Yes,  $x$  occurs in both the numerator and denominator.

8. No,  $y$  only occurs in the numerator.

9.  $f(x, y) = 2x - 5y + 3$

$$f_x(x, y) = 2$$

$$f_y(x, y) = -5$$

10.  $f(x, y) = x^2 - 2y^2 + 4$

$$f_x(x, y) = 2x$$

$$f_y(x, y) = -4y$$

11.  $f(x, y) = x^2y^3$

$$f_x(x, y) = 2xy^3$$

$$f_y(x, y) = 3x^2y^2$$

12.  $f(x, y) = 4x^3y^{-2}$

$$f_x(x, y) = 12x^2y^{-2}$$

$$f_y(x, y) = -8x^3y^{-3}$$

13.  $z = x\sqrt{y}$

$$\frac{\partial z}{\partial x} = \sqrt{y}$$

$$\frac{\partial z}{\partial y} = \frac{x}{2\sqrt{y}}$$

14.  $z = 2y^2\sqrt{x}$

$$\frac{\partial z}{\partial x} = \frac{y^2}{\sqrt{x}}$$

$$\frac{\partial z}{\partial y} = 4y\sqrt{x}$$

15.  $z = x^2 - 4xy + 3y^2$

$$\frac{\partial z}{\partial x} = 2x - 4y$$

$$\frac{\partial z}{\partial y} = -4x + 6y$$

16.  $z = y^3 - 2xy^2 - 1$

$$\frac{\partial z}{\partial x} = -2y^2$$

$$\frac{\partial z}{\partial y} = 3y^2 - 4xy$$

17.  $z = e^{xy}$

$$\frac{\partial z}{\partial x} = ye^{xy}$$

$$\frac{\partial z}{\partial y} = xe^{xy}$$

18.  $z = e^{x/y} = e^{xy^{-1}}$

$$\frac{\partial z}{\partial x} = \frac{1}{y}e^{x/y}$$

$$\frac{\partial z}{\partial y} = \frac{-x}{y^2}e^{x/y}$$

19.  $z = x^2e^{2y}$

$$\frac{\partial z}{\partial x} = 2xe^{2y}$$

$$\frac{\partial z}{\partial y} = 2x^2e^{2y}$$

20.  $z = ye^{y/x} = ye^{yx^{-1}}$

$$\frac{\partial z}{\partial x} = ye^{yx^{-1}}[-yx^{-2}] = \frac{-y^2}{x^2}e^{y/x}$$

$$\frac{\partial z}{\partial y} = e^{y/x} + \frac{1}{x}ye^{y/x} = e^{y/x}\left(1 + \frac{y}{x}\right)$$

21.  $z = \ln \frac{x}{y} = \ln x - \ln y$

$$\frac{\partial z}{\partial x} = \frac{1}{x}$$

$$\frac{\partial z}{\partial y} = -\frac{1}{y}$$

22.  $z = \ln\sqrt{xy} = \frac{1}{2}\ln(xy)$

$$\frac{\partial z}{\partial x} = \frac{1}{2} \frac{y}{xy} = \frac{1}{2x}$$

$$\frac{\partial z}{\partial y} = \frac{1}{2} \frac{x}{xy} = \frac{1}{2y}$$

23.  $z = \ln(x^2 + y^2)$

$$\frac{\partial z}{\partial x} = \frac{2x}{x^2 + y^2}$$

$$\frac{\partial z}{\partial y} = \frac{2y}{x^2 + y^2}$$

24.  $z = \ln \frac{x+y}{x-y} = \ln(x+y) - \ln(x-y)$

$$\frac{\partial z}{\partial x} = \frac{1}{x+y} - \frac{1}{x-y} = \frac{-2y}{(x+y)(x-y)}$$

$$\frac{\partial z}{\partial y} = \frac{1}{x+y} + \frac{1}{x-y} = \frac{2x}{(x+y)(x-y)}$$

25.  $z = \frac{x^2}{2y} + \frac{3y^2}{x}$

$$\frac{\partial z}{\partial x} = \frac{2x}{2y} - \frac{3y^2}{x^2} = \frac{x^3 - 3y^3}{x^2 y}$$

$$\frac{\partial z}{\partial y} = \frac{-x^2}{2y^2} + \frac{6y}{x} = \frac{12y^3 - x^3}{2xy^2}$$

26.  $f(x, y) = \frac{xy}{x^2 + y^2}$

$$f_x(x, y) = \frac{(x^2 + y^2)(y) - (xy)(2x)}{(x^2 + y^2)^2} = \frac{y^3 - x^2 y}{(x^2 + y^2)^2}$$

$$f_y(x, y) = \frac{(x^2 + y^2)(x) - (xy)(2y)}{(x^2 + y^2)^2} = \frac{x^3 - xy^2}{(x^2 + y^2)^2}$$

27.  $h(x, y) = e^{-(x^2+y^2)}$

$$h_x(x, y) = -2xe^{-(x^2+y^2)}$$

$$h_y(x, y) = -2ye^{-(x^2+y^2)}$$

28.  $g(x, y) = \ln \sqrt{x^2 + y^2} = \frac{1}{2} \ln(x^2 + y^2)$

$$g_x(x, y) = \frac{1}{2} \frac{2x}{x^2 + y^2} = \frac{x}{x^2 + y^2}$$

$$g_y(x, y) = \frac{1}{2} \frac{2y}{x^2 + y^2} = \frac{y}{x^2 + y^2}$$

29.  $f(x, y) = \sqrt{x^2 + y^2}$

$$f_x(x, y) = \frac{1}{2} (x^2 + y^2)^{-1/2} (2x) = \frac{x}{\sqrt{x^2 + y^2}}$$

$$f_y(x, y) = \frac{1}{2} (x^2 + y^2)^{-1/2} (2y) = \frac{y}{\sqrt{x^2 + y^2}}$$

30.  $f(x, y) = \sqrt{2x + y^3}$

$$\frac{\partial f}{\partial x} = \frac{1}{2} (2x + y^3)^{-1/2} (2) = \frac{1}{\sqrt{2x + y^3}}$$

$$\frac{\partial f}{\partial y} = \frac{1}{2} (2x + y^3)^{-1/2} (3y^2) = \frac{3y^2}{2\sqrt{2x + y^3}}$$

31.  $z = \cos xy$

$$\frac{\partial z}{\partial x} = -y \sin xy$$

$$\frac{\partial z}{\partial y} = -x \sin xy$$

32.  $z = \sin(x + 2y)$

$$\frac{\partial z}{\partial x} = \cos(x + 2y)$$

$$\frac{\partial z}{\partial y} = 2 \cos(x + 2y)$$

33.  $z = \tan(2x - y)$

$$\frac{\partial z}{\partial x} = 2 \sec^2(2x - y)$$

$$\frac{\partial z}{\partial y} = -\sec^2(2x - y)$$

34.  $z = \sin 5x \cos 5y$

$$\frac{\partial z}{\partial x} = 5 \cos 5x \cos 5y$$

$$\frac{\partial z}{\partial y} = -5 \sin 5x \sin 5y$$

35.  $z = e^y \sin xy$

$$\frac{\partial z}{\partial x} = ye^y \cos xy$$

$$\frac{\partial z}{\partial y} = e^y \sin xy + xe^y \cos x$$

$$= e^y(x \cos xy + \sin xy)$$

36.  $z = \cos(x^2 + y^2)$

$$\frac{\partial z}{\partial x} = -2x \sin(x^2 + y^2)$$

$$\frac{\partial z}{\partial y} = -2y \sin(x^2 + y^2)$$

37.  $z = \sinh(2x + 3y)$

$$\frac{\partial z}{\partial x} = 2 \cosh(2x + 3y)$$

$$\frac{\partial z}{\partial y} = 3 \cosh(2x + 3y)$$

38.  $z = \cosh xy^2$

$$\frac{\partial z}{\partial x} = y^2 \sinh xy^2$$

$$\frac{\partial z}{\partial y} = 2xy \sinh xy^2$$

39.  $f(x, y) = \int_x^y (t^2 - 1) dt$

$$= \left[ \frac{t^3}{3} - t \right]_x^y = \left( \frac{y^3}{3} - y \right) - \left( \frac{x^3}{3} - x \right)$$

$$f_x(x, y) = -x^2 + 1 = 1 - x^2$$

$$f_y(x, y) = y^2 - 1$$

[You could also use the Second Fundamental Theorem of Calculus.]

40.  $f(x, y) = \int_x^y (2t + 1) dt + \int_y^x (2t - 1) dt$

$$= \int_x^y (2t + 1) dt - \int_x^y (2t - 1) dt$$

$$= \int_x^y 2 dt = [2t]_x^y = 2y - 2x$$

$$f_x(x, y) = -2$$

$$f_y(x, y) = 2$$

42.  $f(x, y) = x^2 - 2xy + y^2 = (x - y)^2$

$$\frac{\partial f}{\partial x} = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x}$$

$$= \lim_{\Delta x \rightarrow 0} \frac{(x + \Delta x)^2 - 2(x + \Delta x)y + y^2 - x^2 + 2xy - y^2}{\Delta x} = \lim_{\Delta x \rightarrow 0} (2x + \Delta x - 2y) = 2(x - y)$$

$$\frac{\partial f}{\partial y} = \lim_{\Delta y \rightarrow 0} \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y}$$

$$= \lim_{\Delta y \rightarrow 0} \frac{x^2 - 2x(y + \Delta y) + (y + \Delta y)^2 - x^2 + 2xy - y^2}{\Delta y} = \lim_{\Delta y \rightarrow 0} (-2x + 2y + \Delta y) = 2(y - x)$$

43.  $f(x, y) = \sqrt{x + y}$

$$\frac{\partial f}{\partial x} = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x}$$

$$= \lim_{\Delta x \rightarrow 0} \frac{\sqrt{x + \Delta x + y} - \sqrt{x + y}}{\Delta x}$$

$$= \lim_{\Delta x \rightarrow 0} \frac{(\sqrt{x + \Delta x + y} - \sqrt{x + y})(\sqrt{x + \Delta x + y} + \sqrt{x + y})}{\Delta x (\sqrt{x + \Delta x + y} + \sqrt{x + y})} = \lim_{\Delta x \rightarrow 0} \frac{1}{\sqrt{x + \Delta x + y} + \sqrt{x + y}} = \frac{1}{2\sqrt{x + y}}$$

$$\frac{\partial f}{\partial y} = \lim_{\Delta y \rightarrow 0} \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y} = \lim_{\Delta y \rightarrow 0} \frac{\sqrt{x + y + \Delta y} - \sqrt{x + y}}{\Delta y}$$

$$= \lim_{\Delta y \rightarrow 0} \frac{(\sqrt{x + y + \Delta y} - \sqrt{x + y})(\sqrt{x + y + \Delta y} + \sqrt{x + y})}{\Delta y (\sqrt{x + y + \Delta y} + \sqrt{x + y})}$$

$$= \lim_{\Delta y \rightarrow 0} \frac{1}{\sqrt{x + y + \Delta y} + \sqrt{x + y}} = \frac{1}{2\sqrt{x + y}}$$

41.  $f(x, y) = 3x + 2y$

$$\frac{\partial f}{\partial x} = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x}$$

$$= \lim_{\Delta x \rightarrow 0} \frac{3(x + \Delta x) + 2y - (3x + 2y)}{\Delta x}$$

$$= \lim_{\Delta x \rightarrow 0} \frac{3\Delta x}{\Delta x} = 3$$

$$\frac{\partial f}{\partial y} = \lim_{\Delta y \rightarrow 0} \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y}$$

$$= \lim_{\Delta y \rightarrow 0} \frac{3x + 2(y + \Delta y) - (3x + 2y)}{\Delta y}$$

$$= \lim_{\Delta y \rightarrow 0} \frac{2\Delta y}{\Delta y} = 2$$

44.  $f(x, y) = \frac{1}{x + y}$

$$\frac{\partial f}{\partial x} = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{\frac{1}{x + \Delta x + y} - \frac{1}{x + y}}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{-1}{(x + \Delta x + y)(x + y)} = \frac{-1}{(x + y)^2}$$

$$\frac{\partial f}{\partial y} = \lim_{\Delta y \rightarrow 0} \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y} = \lim_{\Delta y \rightarrow 0} \frac{\frac{1}{x + y + \Delta} - \frac{1}{x + y}}{\Delta y} = \lim_{\Delta y \rightarrow 0} \frac{-1}{(x + y + \Delta y)(x + y)} = \frac{-1}{(x + y)^2}$$

45.  $f(x, y) = e^y \sin x$

$$f_x(x, y) = e^y \cos x$$

At  $(\pi, 0)$ ,  $f_x(\pi, 0) = -1$ .

$$f_y(x, y) = e^y \sin x$$

At  $(\pi, 0)$ ,  $f_y(\pi, 0) = 0$ .

46.  $f(x, y) = e^{-x} \cos y$

$$f_x(x, y) = -e^{-x} \cos y$$

At  $(0, 0)$ ,  $f_x(0, 0) = -1$ .

$$f_y(x, y) = -e^{-x} \sin y$$

At  $(0, 0)$ ,  $f_y(0, 0) = 0$ .

47.  $f(x, y) = \cos(2x - y)$

$$f_x(x, y) = -2 \sin(2x - y)$$

At  $\left(\frac{\pi}{4}, \frac{\pi}{3}\right)$ ,  $f_x\left(\frac{\pi}{4}, \frac{\pi}{3}\right) = -2 \sin\left(\frac{\pi}{2} - \frac{\pi}{3}\right) = -1$ .

$$f_y(x, y) = \sin(2x - y)$$

At  $\left(\frac{\pi}{4}, \frac{\pi}{3}\right)$ ,  $f_y\left(\frac{\pi}{4}, \frac{\pi}{3}\right) = \sin\left(\frac{\pi}{2} - \frac{\pi}{3}\right) = \frac{1}{2}$ .

48.  $f(x, y) = \sin xy$

$$f_x(x, y) = y \cos xy$$

At  $\left(2, \frac{\pi}{4}\right)$ ,  $f_x\left(2, \frac{\pi}{4}\right) = \frac{\pi}{4} \cos \frac{\pi}{2} = 0$ .

$$f_y(x, y) = x \cos xy$$

At  $\left(2, \frac{\pi}{4}\right)$ ,  $f_y\left(2, \frac{\pi}{4}\right) = 2 \cos \frac{\pi}{2} = 0$ .

49.  $f(x, y) = \arctan \frac{y}{x}$

$$f_x(x, y) = \frac{1}{1 + (y^2/x^2)} \left(-\frac{y}{x^2}\right) = \frac{-y}{x^2 + y^2}$$

At  $(2, -2)$ :  $f_x(2, -2) = \frac{1}{4}$

$$f_y(x, y) = \frac{1}{1 + (y^2/x^2)} \left(\frac{1}{x}\right) = \frac{x}{x^2 + y^2}$$

At  $(2, -2)$ :  $f_y(2, -2) = \frac{1}{4}$

50.  $f(x, y) = \arccos(xy)$

$$f_x(x, y) = \frac{-y}{\sqrt{1 - x^2 y^2}}$$

At  $(1, 1)$ ,  $f_x$  is undefined.

$$f_y(x, y) = \frac{-x}{\sqrt{1 - x^2 y^2}}$$

At  $(1, 1)$ ,  $f_y$  is undefined.

51.  $f(x, y) = \frac{xy}{x - y}$

$$f_x(x, y) = \frac{y(x - y) - xy}{(x - y)^2} = \frac{-y^2}{(x - y)^2}$$

At  $(2, -2)$ :  $f_x(2, -2) = -\frac{1}{4}$

$$f_y(x, y) = \frac{x(x - y) + xy}{(x - y)^2} = \frac{x^2}{(x - y)^2}$$

At  $(2, -2)$ :  $f_y(2, -2) = \frac{1}{4}$

52.  $f(x, y) = \frac{2xy}{\sqrt{4x^2 + 5y^2}}$

$$f_x(x, y) = \frac{10y^3}{(4x^2 + 5y^2)^{3/2}}$$

At  $(1, 1)$ ,  $f_x(1, 1) = \frac{10}{9^{3/2}} = \frac{10}{27}$ .

$$f_y(x, y) = \frac{8x^3}{(4x^2 + 5y^2)^{3/2}}$$

At  $(1, 1)$ ,  $f_y(1, 1) = \frac{8}{9^{3/2}} = \frac{8}{27}$ .

53.  $g(x, y) = 4 - x^2 - y^2$

$$g_x(x, y) = -2x$$

At  $(1, 1)$ :  $g_x(1, 1) = -2$

$$g_y(x, y) = -2y$$

At  $(1, 1)$ :  $g_y(1, 1) = -2$

54.  $h(x, y) = x^2 - y^2$

$$h_x(x, y) = 2x$$

At  $(-2, 1)$ :  $h_x(-2, 1) = -4$

$$h_y(x, y) = -2y$$

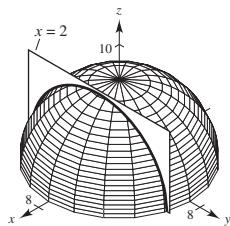
At  $(-2, 1)$ :  $h_y(-2, 1) = -2$

55.  $z = \sqrt{49 - x^2 - y^2}$ ,  $x = 2$ ,  $(2, 3, 6)$

Intersecting curve:  $z = \sqrt{45 - y^2}$

$$\frac{\partial z}{\partial y} = \frac{-y}{\sqrt{45 - y^2}}$$

At  $(2, 3, 6)$ :  $\frac{\partial z}{\partial y} = \frac{-3}{\sqrt{45 - 9}} = -\frac{1}{2}$

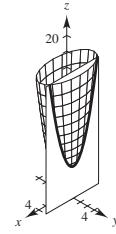


56.  $z = x^2 + 4y^2$ ,  $y = 1$ ,  $(2, 1, 8)$

Intersecting curve:  $z = x^2 + 4$

$$\frac{\partial z}{\partial x} = 2x$$

At  $(2, 1, 8)$ :  $\frac{\partial z}{\partial x} = 2(2) = 4$

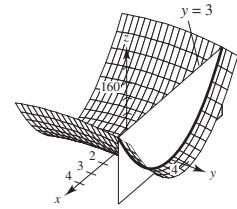


57.  $z = 9x^2 - y^2$ ,  $y = 3$ ,  $(1, 3, 0)$

Intersecting curve:  $z = 9x^2 - 9$

$$\frac{\partial z}{\partial x} = 18x$$

At  $(1, 3, 0)$ :  $\frac{\partial z}{\partial x} = 18(1) = 18$

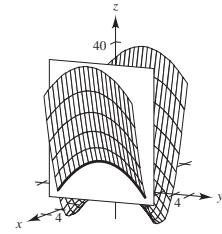


58.  $z = 9x^2 - y^2$ ,  $x = 1$ ,  $(1, 3, 0)$

Intersecting curve:  $z = 9 - y^2$

$$\frac{\partial z}{\partial y} = -2y$$

At  $(1, 3, 0)$ :  $\frac{\partial z}{\partial y} = -2(3) = -6$



59.  $H(x, y, z) = \sin(x + 2y + 3z)$

$$H_x(x, y, z) = \cos(x + 2y + 3z)$$

$$H_y(x, y, z) = 2 \cos(x + 2y + 3z)$$

$$H_z(x, y, z) = 3 \cos(x + 2y + 3z)$$

60.  $f(x, y, z) = 3x^2y - 5xyz + 10yz^2$

$$f_x(x, y, z) = 6xy - 5yz$$

$$f_y(x, y, z) = 3x^2 - 5xz + 10z^2$$

$$f_z(x, y, z) = -5xy + 20yz$$

61.  $w = \sqrt{x^2 + y^2 + z^2}$

$$\frac{\partial w}{\partial x} = \frac{x}{\sqrt{x^2 + y^2 + z^2}}$$

$$\frac{\partial w}{\partial y} = \frac{y}{\sqrt{x^2 + y^2 + z^2}}$$

$$\frac{\partial w}{\partial z} = \frac{z}{\sqrt{x^2 + y^2 + z^2}}$$

62.  $w = \frac{7xz}{x + y} = 7xz(x + y)^{-1}$

$$\frac{\partial w}{\partial x} = \frac{(x + y)(7z) - 7xz}{(x + y)^2} = \frac{7yz}{(x + y)^2}$$

$$\frac{\partial w}{\partial y} = \frac{-7xz}{(x + y)^2}$$

$$\frac{\partial w}{\partial z} = \frac{7x}{x + y}$$

63.  $F(x, y, z) = \ln\sqrt{x^2 + y^2 + z^2} = \frac{1}{2}\ln(x^2 + y^2 + z^2)$

$$F_x(x, y, z) = \frac{x}{x^2 + y^2 + z^2}$$

$$F_y(x, y, z) = \frac{y}{x^2 + y^2 + z^2}$$

$$F_z(x, y, z) = \frac{z}{x^2 + y^2 + z^2}$$

64.  $G(x, y, z) = \frac{1}{\sqrt{1 - x^2 - y^2 - z^2}}$

$$G_x(x, y, z) = \frac{x}{(1 - x^2 - y^2 - z^2)^{3/2}}$$

$$G_y(x, y, z) = \frac{y}{(1 - x^2 - y^2 - z^2)^{3/2}}$$

$$G_z(x, y, z) = \frac{z}{(1 - x^2 - y^2 - z^2)^{3/2}}$$

65.  $f(x, y, z) = x^3yz^2$

$$f_x(x, y, z) = 3x^2yz^2$$

$$f_x(1, 1, 1) = 3$$

$$f_y(x, y, z) = x^3z^2$$

$$f_y(1, 1, 1) = 1$$

$$f_z(x, y, z) = 2x^3yz$$

$$f_z(1, 1, 1) = 2$$

66.  $f(x, y, z) = x^2y^3 + 2xyz - 3yz$

$$f_x(x, y, z) = 2xy^3 + 2yz$$

$$f_x(-2, 1, 2) = -4 + 4 = 0$$

$$f_y(x, y, z) = 3x^2y^2 + 2xz - 3z$$

$$f_y(-2, 1, 2) = 12 - 8 - 6 = -2$$

$$f_z(x, y, z) = 2xy - 3y$$

$$f_z(-2, 1, 2) = -4 - 3 = -7$$

67.  $f(x, y, z) = \frac{x}{yz}$

$$f_x(x, y, z) = \frac{1}{yz}$$

$$f_x(1, -1, -1) = 1$$

$$f_y(x, y, z) = \frac{-x}{y^2z}$$

$$f_y(1, -1, -1) = 1$$

$$f_z(x, y, z) = \frac{-x}{yz^2}$$

$$f_z(1, -1, -1) = 1$$

68.  $f(x, y, z) = \frac{xy}{x + y + z}$

$$f_x(x, y, z) = \frac{(x + y + z)y - xy}{(x + y + z)^2} = \frac{y^2 + yz}{(x + y + z)^2}$$

$$f_x(3, 1, -1) = \frac{1 - 1}{3^2} = 0$$

$$f_y(x, y, z) = \frac{(x + y + z)x - xy}{(x + y + z)^2} = \frac{x^2 + xz}{(x + y + z)^2}$$

$$f_y(3, 1, -1) = \frac{9 - 3}{3^2} = \frac{2}{3}$$

$$f_z(x, y, z) = \frac{(x + y + z)(0) - xy}{(x + y + z)^2} = \frac{-xy}{(x + y + z)^2}$$

$$f_z(3, 1, -1) = \frac{-3}{9} = \frac{-1}{3}$$

69.  $f(x, y, z) = z \sin(x + y)$

$$f_x(x, y, z) = z \cos(x + y)$$

$$f_x\left(0, \frac{\pi}{2}, -4\right) = -4 \cos \frac{\pi}{2} = 0$$

$$f_y(x, y, z) = z \cos(x + y)$$

$$f_y\left(0, \frac{\pi}{2}, -4\right) = -4 \cos \frac{\pi}{2} = 0$$

$$f_z(x, y, z) = \sin(x + y)$$

$$f_z\left(0, \frac{\pi}{2}, -4\right) = \sin \frac{\pi}{2} = 1$$

70.  $\sqrt{3x^2 + y^2 - 2z^2}$

$$f_x(x, y, z) = \frac{6x}{2\sqrt{3x^2 + y^2 - 2z^2}} = \frac{3x}{\sqrt{3x^2 + y^2 - 2z^2}}$$

$$f_x(1, -2, 1) = \frac{6}{2\sqrt{3 + 4 - 2}} = \frac{3}{\sqrt{5}} = \frac{3\sqrt{5}}{5}$$

$$f_y(x, y, z) = \frac{2y}{2\sqrt{3x^2 + y^2 - 2z^2}} = \frac{y}{\sqrt{3x^2 + y^2 - 2z^2}}$$

$$f_y(1, -2, 1) = \frac{-2}{\sqrt{5}} = \frac{-2\sqrt{5}}{5}$$

$$f_z(x, y, z) = \frac{-4z}{2\sqrt{3x^2 + y^2 - 2z^2}} = \frac{-2z}{\sqrt{3x^2 + y^2 - 2z^2}}$$

$$f_z(1, -2, 1) = \frac{-2}{\sqrt{5}} = \frac{-2\sqrt{5}}{5}$$

71.  $z = 3xy^2$

$$\frac{\partial z}{\partial x} = 3y^2, \frac{\partial^2 z}{\partial x^2} = 0, \frac{\partial^2 z}{\partial y \partial x} = 6y$$

$$\frac{\partial z}{\partial y} = 6xy, \frac{\partial^2 z}{\partial y^2} = 6x, \frac{\partial^2 z}{\partial x \partial y} = 6y$$

72.  $z = x^2 + 3y^2$

$$\frac{\partial z}{\partial x} = 2x, \quad \frac{\partial^2 z}{\partial x^2} = 2, \quad \frac{\partial^2 z}{\partial y \partial x} = 0$$

$$\frac{\partial z}{\partial y} = 6y, \quad \frac{\partial^2 z}{\partial y^2} = 6, \quad \frac{\partial^2 z}{\partial x \partial y} = 0$$

73.  $z = x^2 - 2xy + 3y^2$

$$\frac{\partial z}{\partial x} = 2x - 2y$$

$$\frac{\partial^2 z}{\partial x^2} = 2$$

$$\frac{\partial^2 z}{\partial y \partial x} = -2$$

$$\frac{\partial z}{\partial y} = -2x + 6y$$

$$\frac{\partial^2 z}{\partial y^2} = 6$$

$$\frac{\partial^2 z}{\partial x \partial y} = -2$$

74.  $z = x^4 - 3x^2y^2 + y^4$

$$\frac{\partial z}{\partial x} = 4x^3 - 6xy^2$$

$$\frac{\partial^2 z}{\partial x^2} = 12x^2 - 6y^2$$

$$\frac{\partial^2 z}{\partial y \partial x} = -12xy$$

$$\frac{\partial z}{\partial y} = -6x^2y + 4y^3$$

$$\frac{\partial^2 z}{\partial y^2} = -6x^2 + 12y^2$$

$$\frac{\partial^2 z}{\partial x \partial y} = -12xy$$

75.  $z = \sqrt{x^2 + y^2}$

$$\frac{\partial z}{\partial x} = \frac{x}{\sqrt{x^2 + y^2}}$$

$$\frac{\partial^2 z}{\partial x^2} = \frac{y^2}{(x^2 + y^2)^{3/2}}$$

$$\frac{\partial^2 z}{\partial y \partial x} = \frac{-xy}{(x^2 + y^2)^{3/2}}$$

$$\frac{\partial z}{\partial y} = \frac{y}{\sqrt{x^2 + y^2}}$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{x^2}{(x^2 + y^2)^{3/2}}$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{-xy}{(x^2 + y^2)^{3/2}}$$

76.  $z = \ln(x - y)$

$$\frac{\partial z}{\partial x} = \frac{1}{x - y}$$

$$\frac{\partial^2 z}{\partial x^2} = -\frac{1}{(x - y)^2}$$

$$\frac{\partial^2 z}{\partial y \partial x} = \frac{1}{(x - y)^2}$$

$$\frac{\partial z}{\partial y} = \frac{-1}{x - y} = \frac{1}{y - x}$$

$$\frac{\partial^2 z}{\partial y^2} = -\frac{1}{(x - y)^2}$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{1}{(x - y)^2}$$

$$\text{So, } \frac{\partial^2 z}{\partial y \partial x} = \frac{\partial^2 z}{\partial x \partial y}.$$

77.  $z = e^x \tan y$

$$\frac{\partial z}{\partial x} = e^x \tan y$$

$$\frac{\partial^2 z}{\partial x^2} = e^x \tan y$$

$$\frac{\partial^2 z}{\partial y \partial x} = e^x \sec^2 y$$

$$\frac{\partial z}{\partial y} = e^x \sec^2 y$$

$$\frac{\partial^2 z}{\partial y^2} = 2e^x \sec^2 y \tan y$$

$$\frac{\partial^2 z}{\partial x \partial y} = e^x \sec^2 y$$

78.  $z = 2xe^y - 3ye^{-x}$

$$\frac{\partial z}{\partial x} = 2e^y + 3ye^{-x}$$

$$\frac{\partial^2 z}{\partial x^2} = -3ye^{-x}$$

$$\frac{\partial^2 z}{\partial y \partial x} = 2e^y + 3ye^{-x}$$

$$\frac{\partial z}{\partial y} = 2xe^y - 3e^{-x}$$

$$\frac{\partial^2 z}{\partial y^2} = 2xe^y$$

$$\frac{\partial^2 z}{\partial x \partial y} = 2e^y + 3e^{-x}$$

79.  $z = \cos xy$

$$\frac{\partial z}{\partial x} = -y \sin xy, \quad \frac{\partial^2 z}{\partial x^2} = -y^2 \cos xy$$

$$\frac{\partial^2 z}{\partial y \partial x} = -yx \cos xy - \sin xy$$

$$\frac{\partial z}{\partial y} = -x \sin xy, \quad \frac{\partial^2 z}{\partial y^2} = -x^2 \cos xy$$

$$\frac{\partial^2 z}{\partial x \partial y} = -xy \cos xy - \sin xy$$

80.  $z = \arctan \frac{y}{x}$

$$\frac{\partial z}{\partial x} = \frac{1}{1 + (y^2/x^2)} \left( -\frac{y}{x^2} \right) = \frac{-y}{x^2 + y^2}$$

$$\frac{\partial^2 z}{\partial x^2} = \frac{2xy}{(x^2 + y^2)^2}$$

$$\frac{\partial^2 z}{\partial y \partial x} = \frac{-(x^2 + y^2) + y(2y)}{(x^2 + y^2)^2} = \frac{y^2 - x^2}{(x^2 + y^2)^2}$$

$$\frac{\partial z}{\partial y} = \frac{1}{1 + (y^2/x^2)} \left( \frac{1}{x} \right) = \frac{x}{x^2 + y^2}$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{-2xy}{(x^2 + y^2)^2}$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{(x^2 + y^2) - x(2x)}{(x^2 + y^2)^2} = \frac{y^2 - x^2}{(x^2 + y^2)^2}$$

81.  $f_x(x, y) = 2x + y - 2 = 0$

$$f_y(x, y) = x + 2y + 2 = 0$$

$$2x + y - 2 = 0 \Rightarrow y = 2 - 2x$$

$$x + 2(2 - 2x) + 2 = 0 \Rightarrow -3x + 6 = 0 \Rightarrow x = 2,$$

$$y = -2$$

Point:  $(2, -2)$

82.  $f_x(x, y) = 2x - y - 5 = 0$

$$f_y(x, y) = -x + 2y + 1 = 0$$

$$2x - y - 5 = 0 \Rightarrow y = 2x - 5$$

$$-x + 2(2x - 5) + 1 = 0 \Rightarrow 3x - 9 = 0 \Rightarrow x = 3,$$

$$y = 1$$

Point:  $(3, 1)$

83.  $f_x(x, y) = 2x + 4y - 4, f_y(x, y) = 4x + 2y + 16$

$$f_x = f_y = 0: 2x + 4y = 4$$

$$4x + 2y = -16$$

Solving for  $x$  and  $y$ ,

$$x = -6 \text{ and } y = 4.$$

84.  $f_x(x, y) = 2x - y = 0$

$$f_y(x, y) = -x + 2y = 0$$

$$2x - y = 0 \Rightarrow y = 2x$$

$$-x + 2(2x) = 0 \Rightarrow x = 0, y = 0$$

Point:  $(0, 0)$

85.  $f_x(x, y) = -\frac{1}{x^2} + y, f_y(x, y) = -\frac{1}{y^2} + x$

$$f_x = f_y = 0: -\frac{1}{x^2} + y = 0 \text{ and } -\frac{1}{y^2} + x = 0$$

$$y = \frac{1}{x^2} \text{ and } x = \frac{1}{y^2}$$

$$y = y^4 \Rightarrow y = 1 = x$$

Points:  $(1, 1)$

86.  $f_x(x, y) = 9x^2 - 12y, f_y(x, y) = -12x + 3y^2$

$$f_x = f_y = 0: 9x^2 - 12y = 0 \Rightarrow 3x^2 = 4y$$

$$3y^2 - 12x = 0 \Rightarrow y^2 = 4x$$

Solving for  $x$  in the second equation,  $x = y^2/4$ , you obtain  $3(y^2/4)^2 = 4y$ .

$$3y^4 = 64y \Rightarrow y = 0 \text{ or } y = \frac{4}{3^{1/3}}$$

$$\Rightarrow x = 0 \text{ or } x = \frac{1}{4} \left( \frac{16}{3^{2/3}} \right)$$

Points:  $(0, 0), \left( \frac{4}{3^{2/3}}, \frac{4}{3^{1/3}} \right)$

87.  $f_x(x, y) = (2x + y)e^{x^2+xy+y^2} = 0$

$$f_y(x, y) = (x + 2y)e^{x^2+xy+y^2} = 0$$

$$2x + y = 0 \Rightarrow y = -2x$$

$$x + 2(-2x) = 0 \Rightarrow x = 0 \Rightarrow y = 0$$

Point:  $(0, 0)$

88.  $f_x(x, y) = \frac{2x}{x^2 + y^2 + 1} = 0 \Rightarrow x = 0$

$$f_y(x, y) = \frac{2y}{x^2 + y^2 + 1} = 0 \Rightarrow y = 0$$

Points:  $(0, 0)$

89.  $z = x \sec y$

$$\frac{\partial z}{\partial x} = \sec y$$

$$\frac{\partial^2 z}{\partial x^2} = 0$$

$$\frac{\partial^2 z}{\partial y \partial x} = \sec y \tan y$$

$$\frac{\partial z}{\partial y} = x \sec y \tan y$$

$$\frac{\partial^2 z}{\partial y^2} = x \sec y (\sec^2 y + \tan^2 y)$$

$$\frac{\partial^2 z}{\partial x \partial y} = \sec y \tan y$$

So,  $\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial^2 z}{\partial x \partial y}$

There are no points for which  $z_x = 0 = z_y$ , because

$$\frac{\partial z}{\partial x} = \sec y \neq 0.$$

90.  $z = \sqrt{25 - x^2 - y^2}$

$$\frac{\partial z}{\partial x} = \frac{-x}{\sqrt{25 - x^2 - y^2}}$$

$$\frac{\partial^2 z}{\partial x^2} = \frac{y^2 - 25}{(25 - x^2 - y^2)^{3/2}}$$

$$\frac{\partial^2 z}{\partial y \partial x} = \frac{-xy}{(25 - x^2 - y^2)^{3/2}}$$

$$\frac{\partial z}{\partial y} = \frac{-y}{\sqrt{25 - x^2 - y^2}}$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{x^2 - 25}{(25 - x^2 - y^2)^{3/2}}$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{-xy}{(25 - x^2 - y^2)^{3/2}}$$

$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial y} = 0$  if  $x = y = 0$

91.  $z = \ln\left(\frac{x}{x^2 + y^2}\right) = \ln x - \ln(x^2 + y^2)$

$$\frac{\partial z}{\partial x} = \frac{1}{x} - \frac{2x}{x^2 + y^2} = \frac{y^2 - x^2}{x(x^2 + y^2)}$$

$$\frac{\partial^2 z}{\partial x^2} = \frac{x^4 - 4x^2y^2 - y^4}{x^2(x^2 + y^2)^2}$$

$$\frac{\partial^2 z}{\partial y \partial x} = \frac{4xy}{(x^2 + y^2)^2}$$

$$\frac{\partial z}{\partial y} = -\frac{2y}{x^2 + y^2}$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{2(y^2 - x^2)}{(x^2 + y^2)^2}$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{4xy}{(x^2 + y^2)^2}$$

There are no points for which  $z_x = z_y = 0$ .

92.  $z = \frac{xy}{x - y}$

$$\frac{\partial z}{\partial x} = \frac{y(x - y) - xy}{(x - y)^2} = \frac{-y^2}{(x - y)^2}$$

$$\frac{\partial^2 z}{\partial x^2} = \frac{2y^2}{(x - y)^3}$$

$$\frac{\partial^2 z}{\partial y \partial x} = \frac{(x - y)^2(-2y) + y^2(2)(x - y)(-1)}{(x - y)^4} = \frac{-2xy}{(x - y)^3}$$

$$\frac{\partial z}{\partial y} = -\frac{x(x - y) + xy}{(x - y)^2} = \frac{x^2}{(x - y)^2}$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{2x^2}{(x - y)^3}$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{(x - y)^2(2x) - x^2(2)(x - y)}{(x - y)^4} = \frac{-2xy}{(x - y)^3}$$

There are no points for which  $z_x = z_y = 0$ .

93.  $f(x, y, z) = xyz$

$$f_x(x, y, z) = yz$$

$$f_y(x, y, z) = xz$$

$$f_{yy}(x, y, z) = 0$$

$$f_{xy}(x, y, z) = z$$

$$f_{yx}(x, y, z) = z$$

$$f_{yxy}(x, y, z) = 0$$

$$f_{xxy}(x, y, z) = 0$$

$$f_{yxy}(x, y, z) = 0$$

So,  $f_{xyy} = f_{yxy} = f_{yxx} = 0$ .

94.  $f(x, y, z) = x^2 - 3xy + 4yz + z^3$

$$f_x(x, y, z) = 2x - 3y$$

$$f_y(x, y, z) = -3x + 4z$$

$$f_{yy}(x, y, z) = 0$$

$$f_{xy}(x, y, z) = -3$$

$$f_{yx}(x, y, z) = -3$$

$$f_{yxx}(x, y, z) = 0$$

$$f_{yyy}(x, y, z) = 0$$

$$f_{yxy}(x, y, z) = 0$$

So,  $f_{xyy} = f_{yxy} = f_{yxx} = 0$ .

95.  $f(x, y, z) = e^{-x} \sin yz$

$$f_x(x, y, z) = -e^{-x} \sin yz$$

$$f_y(x, y, z) = ze^{-x} \cos yz$$

$$f_{yy}(x, y, z) = -z^2 e^{-x} \sin yz$$

$$f_{xy}(x, y, z) = -ze^{-x} \cos yz$$

$$f_{yx}(x, y, z) = -ze^{-x} \cos yz$$

$$f_{yxx}(x, y, z) = z^2 e^{-x} \sin yz$$

$$f_{xyy}(x, y, z) = z^2 e^{-x} \sin yz$$

$$f_{yxy}(x, y, z) = z^2 e^{-x} \sin yz$$

So,  $f_{xyy} = f_{yxy} = f_{yxx}$ .

96.  $f(x, y, z) = \frac{2z}{x+y}$

$$f_x(x, y, z) = \frac{-2z}{(x+y)^2}$$

$$f_y(x, y, z) = \frac{-2z}{(x+y)^2}$$

$$f_{yy}(x, y, z) = \frac{4z}{(x+y)^3}$$

$$f_{xy}(x, y, z) = \frac{4z}{(x+y)^3}$$

$$f_{yx}(x, y, z) = \frac{4z}{(x+y)^3}$$

$$f_{yxx}(x, y, z) = \frac{-12z}{(x+y)^4}$$

$$f_{xyy}(x, y, z) = \frac{-12z}{(x+y)^4}$$

$$f_{yxy}(x, y, z) = \frac{-12z}{(x+y)^4}$$

97.  $z = 5xy$

$$\frac{\partial z}{\partial x} = 5y$$

$$\frac{\partial^2 z}{\partial x^2} = 0$$

$$\frac{\partial z}{\partial y} = 5x$$

$$\frac{\partial^2 z}{\partial y^2} = 0$$

$$\text{So, } \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0 + 0 = 0.$$

98.  $z = \sin x \left( \frac{e^y - e^{-y}}{2} \right)$

$$\frac{\partial z}{\partial x} = \cos x \left( \frac{e^y - e^{-y}}{2} \right)$$

$$\frac{\partial^2 z}{\partial x^2} = -\sin x \left( \frac{e^y - e^{-y}}{2} \right)$$

$$\frac{\partial z}{\partial y} = \sin x \left( \frac{e^y + e^{-y}}{2} \right)$$

$$\frac{\partial^2 z}{\partial y^2} = \sin x \left( \frac{e^y - e^{-y}}{2} \right)$$

So,

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = -\sin x \left( \frac{e^y - e^{-y}}{2} \right) + \sin x \left( \frac{e^y - e^{-y}}{2} \right) = 0.$$

99.  $z = e^x \sin y$

$$\frac{\partial z}{\partial x} = e^x \sin y$$

$$\frac{\partial^2 z}{\partial x^2} = e^x \sin y$$

$$\frac{\partial z}{\partial y} = e^x \cos y$$

$$\frac{\partial^2 z}{\partial y^2} = -e^x \sin y$$

$$\text{So, } \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = e^x \sin y - e^x \sin y = 0.$$

100.  $z = \arctan \frac{y}{x}$

From Exercise 80, we have

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = \frac{2xy}{(x^2 + y^2)^2} + \frac{-2xy}{(x^2 + y^2)^2} = 0.$$

101.  $z = \sin(x - ct)$

$$\frac{\partial z}{\partial t} = -c \cos(x - ct)$$

$$\frac{\partial^2 z}{\partial t^2} = -c^2 \sin(x - ct)$$

$$\frac{\partial z}{\partial x} = \cos(x - ct)$$

$$\frac{\partial^2 z}{\partial x^2} = -\sin(x - ct)$$

So,  $\frac{\partial^2 z}{\partial t^2} = c^2 \left( \frac{\partial^2 z}{\partial x^2} \right)$ .

102.  $z = \cos(4x + 4ct)$

$$\frac{\partial z}{\partial t} = -4c \sin(4x + 4ct)$$

$$\frac{\partial^2 z}{\partial t^2} = -16c^2 \cos(4x + 4ct)$$

$$\frac{\partial z}{\partial x} = -4 \sin(4x + 4ct)$$

$$\frac{\partial^2 z}{\partial x^2} = -16 \cos(4x + 4ct)$$

$$\frac{\partial^2 z}{\partial t^2} = c^2 (-16 \cos(4x + 4ct)) = c^2 \left( \frac{\partial^2 z}{\partial x^2} \right)$$

103.  $z = \ln(x + ct)$

$$\frac{\partial z}{\partial t} = \frac{c}{x + ct}$$

$$\frac{\partial^2 z}{\partial t^2} = \frac{-c^2}{(x + ct)^2}$$

$$\frac{\partial z}{\partial x} = \frac{1}{x + ct}$$

$$\frac{\partial^2 z}{\partial x^2} = \frac{-1}{(x + ct)^2}$$

$$\frac{\partial^2 z}{\partial t^2} = \frac{-c^2}{(x + ct)^2} = c^2 \left( \frac{\partial^2 z}{\partial x^2} \right)$$

104.  $z = \sin(wct) \sin(wx)$

$$\frac{\partial z}{\partial t} = wc \cos(wct) \sin(wx)$$

$$\frac{\partial^2 z}{\partial t^2} = -w^2 c^2 \sin(wct) \sin(wx)$$

$$\frac{\partial z}{\partial x} = w \sin(wct) \cos(wx)$$

$$\frac{\partial^2 z}{\partial x^2} = -w^2 \sin(wct) \sin(wx)$$

So,  $\frac{\partial^2 z}{\partial t^2} = c^2 \left( \frac{\partial^2 z}{\partial x^2} \right)$ .

105.  $z = e^{-t} \cos \frac{x}{c}$

$$\frac{\partial z}{\partial t} = -e^{-t} \cos \frac{x}{c}$$

$$\frac{\partial z}{\partial x} = -\frac{1}{c} e^{-t} \sin \frac{x}{c}$$

$$\frac{\partial^2 z}{\partial x^2} = -\frac{1}{c^2} e^{-t} \cos \frac{x}{c}$$

So,  $\frac{\partial z}{\partial t} = c^2 \left( \frac{\partial^2 z}{\partial x^2} \right)$ .

106.  $z = e^{-t} \sin \frac{x}{c}$

$$\frac{\partial z}{\partial t} = -e^{-t} \sin \frac{x}{c}$$

$$\frac{\partial z}{\partial x} = \frac{1}{c} e^{-t} \cos \frac{x}{c}$$

$$\frac{\partial^2 z}{\partial x^2} = -\frac{1}{c^2} e^{-t} \sin \frac{x}{c}$$

So,  $\frac{\partial z}{\partial t} = c^2 \left( \frac{\partial^2 z}{\partial x^2} \right)$ .

107. Yes. The function  $f(x, y) = \cos(3x - 2y)$  satisfies both equations.

108. A function  $f(x, y)$  with the given partial derivatives does not exist.

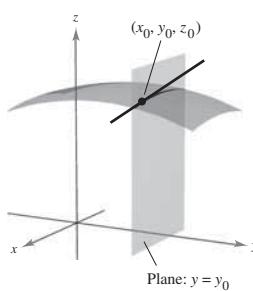
109.  $\frac{\partial}{\partial x}[f_x(x, y, z)] = 0$

There are no  $xs$  in the expression.

110.  $\frac{\partial}{\partial x}[f(x, y, z)] = \left( \sinh \frac{y}{z} \right)^{(y^2 - 2\sqrt{y-1})z}$

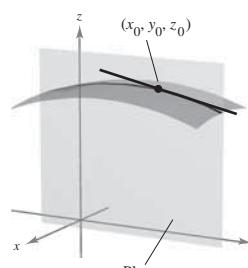
111. If  $z = f(x, y)$ , then to find  $f_x$  you consider  $y$  constant and differentiate with respect to  $x$ . Similarly, to find  $f_y$ , you consider  $x$  constant and differentiate with respect to  $y$ .

112.



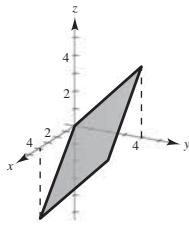
$\frac{\partial f}{\partial x}$  denotes the slope of surface in the  $x$ -direction.

$\frac{\partial f}{\partial y}$  denotes the slope of the surface in the  $y$ -direction.



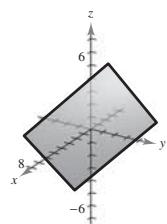
113. The plane  $z = -x + y = f(x, y)$  satisfies

$$\frac{\partial f}{\partial x} < 0 \text{ and } \frac{\partial f}{\partial y} > 0.$$



114. The plane  $z = x + y = f(x, y)$  satisfies

$$\frac{\partial f}{\partial x} > 0 \text{ and } \frac{\partial f}{\partial y} > 0.$$



115. In this case, the mixed partials are equal,  $f_{xy} = f_{yx}$ .

See Theorem 13.3.

116.  $f(x, y) = \sin(x - 2y)$

$$f_x(x, y) = \cos(x - 2y)$$

$$f_{xx}(x, y) = -\sin(x - 2y)$$

$$f_{xy}(x, y) = 2 \sin(x - 2y)$$

$$f_y(x, y) = -2 \cos(x - 2y)$$

$$f_{yy}(x, y) = -4 \sin(x - 2y)$$

$$f_{yx}(x, y) = 2 \sin(x - 2y)$$

So,  $f_{xy}(x, y) = f_{yx}(x, y)$ .

117.  $R = 200x_1 + 200x_2 - 4x_1^2 - 8x_1x_2 - 4x_2^2$

(a)  $\frac{\partial R}{\partial x_1} = 200 - 8x_1 - 8x_2$

$$\text{At } (x_1, x_2) = (4, 12), \frac{\partial R}{\partial x_1} = 200 - 32 - 96 = 72.$$

(b)  $\frac{\partial R}{\partial x_2} = 200 - 8x_1 - 8x_2$

$$\text{At } (x_1, x_2) = (4, 12), \frac{\partial R}{\partial x_2} = 72.$$

121. An increase in either price will cause a decrease in demand.

122.  $V(I, R) = 1000 \left[ \frac{1 + 0.06(1 - R)}{1 + I} \right]^{10}$

$$V_I(I, R) = 10,000 \left[ \frac{1 + 0.06(1 - R)}{1 + I} \right]^9 \left[ \frac{1 + 0.06(1 - R)}{(1 + I)^2} \right] = -10,000 \left[ \frac{(1 + 0.06(1 - R))^{10}}{(1 + I)^{11}} \right]$$

$$V_I(0.03, 0.28) = -11,027.20$$

$$V_R(I, R) = 10,000 \left[ \frac{1 + 0.06(1 - R)}{1 + I} \right]^9 \left[ \frac{-0.06}{1 + I} \right] = -600 \left[ \frac{(1 + 0.06(1 - R))^9}{(1 + I)^{10}} \right]$$

$$V_R(0.03, 0.28) = -653.26$$

The rate of inflation has the greater negative influence.

118. (a)  $C = 32\sqrt{xy} + 175x + 205y + 1050$

$$\frac{\partial C}{\partial x} = 16\sqrt{\frac{y}{x}} + 175$$

$$\left. \frac{\partial C}{\partial x} \right|_{(80, 20)} = 16\sqrt{\frac{1}{4}} + 175 = 183$$

$$\frac{\partial C}{\partial y} = 16\sqrt{\frac{x}{y}} + 205$$

$$\left. \frac{\partial C}{\partial y} \right|_{(80, 20)} = 16\sqrt{4} + 205 = 237$$

- (b) The fireplace-insert stove results in the cost increasing at a faster rate because  $\frac{\partial C}{\partial y} > \frac{\partial C}{\partial x}$ .

119.  $IQ(M, C) = 100 \frac{M}{C}$

$$IQ_M = \frac{100}{C}, IQ_M(12, 10) = 10$$

$$IQ_c = \frac{-100M}{C^2}, IQ_c(12, 10) = -12$$

When the chronological age is constant,  $IQ$  increases at a rate of 10 points per mental age year.

When the mental age is constant,  $IQ$  decreases at a rate of 12 points per chronological age year.

120.  $f(x, y) = 200x^{0.7}y^{0.3}$

(a)  $\frac{\partial f}{\partial x} = 140x^{-0.3}y^{0.3} = 140\left(\frac{y}{x}\right)^{0.3}$

At  $(x, y) = (1000, 500)$ ,

$$\frac{\partial f}{\partial x} = 140\left(\frac{500}{1000}\right)^{0.3} = 140\left(\frac{1}{2}\right)^{0.3} \approx 113.72.$$

(b)  $\frac{\partial f}{\partial x} = 60x^{0.7}y^{-0.7} = 60\left(\frac{x}{y}\right)^{0.7}$

At  $(x, y) = (1000, 500)$ ,

$$\frac{\partial f}{\partial x} = 60\left(\frac{1000}{500}\right)^{0.7} = 60(2)^{0.7} \approx 97.47.$$

123.  $T = 500 - 0.6x^2 - 1.5y^2$

$$\frac{\partial T}{\partial x} = -1.2x, \frac{\partial T}{\partial x}(2, 3) = -2.4^\circ/\text{m}$$

$$\frac{\partial T}{\partial y} = -3y = \frac{\partial T}{\partial y}(2, 3) = -9^\circ/\text{m}$$

124.  $A = 0.885t - 22.4h + 1.20th - 0.544$

$$(a) \frac{\partial A}{\partial t} = 0.885 + 1.20h$$

$$\frac{\partial A}{\partial t}(30^\circ, 0.80) = 0.885 + 1.20(0.80) = 1.845$$

$$\frac{\partial A}{\partial h} = -22.4 + 1.20t$$

$$\frac{\partial A}{\partial h}(30^\circ, 0.80) = -22.4 + 1.20(30^\circ) = 13.6$$

- (b) The humidity has a greater effect on  $A$  because its coefficient  $-22.4$  is larger than that of  $t$ .

125.  $PV = \frac{n}{xB}RT$

$$T = \frac{PV}{\frac{n}{xB}R} \Rightarrow \frac{\partial T}{\partial P} = \frac{V}{\frac{n}{xB}R}$$

$$P = \frac{\frac{n}{xB}RT}{V} \Rightarrow \frac{\partial P}{\partial V} = -\frac{\frac{n}{xB}RT}{V^2}$$

$$V = \frac{\frac{n}{xB}RT}{P} \Rightarrow \frac{\partial V}{\partial T} = \frac{\frac{n}{xB}R}{P}$$

$$\frac{\partial T}{\partial P} \cdot \frac{\partial P}{\partial V} \cdot \frac{\partial V}{\partial T} = \left( \frac{V}{\frac{n}{xB}R} \right) \left( -\frac{\frac{n}{xB}RT}{V^2} \right) \left( \frac{\frac{n}{xB}R}{P} \right)$$

$$= -\frac{\frac{n}{xB}RT}{VP} = -\frac{\frac{n}{xB}RT}{\frac{n}{xB}RT} = -1$$

128.  $z = -1.2225x^2 + 0.0096y^2 + 71.381x - 4.121y - 354.65$

$$(a) \frac{\partial z}{\partial x} = -2.445x + 71.381$$

$$\frac{\partial^2 z}{\partial x^2} = -2.445$$

$$\frac{\partial z}{\partial y} = 0.0192y - 4.121$$

$$\frac{\partial^2 z}{\partial y^2} = 0.0192$$

$$(b) \text{ Concave downward } \left( \frac{\partial^2 z}{\partial x^2} < 0 \right)$$

The rate of increase of Medicare expenses ( $z$ ) is declining with respect to worker's compensation expenses ( $x$ ).

$$(c) \text{ Concave upward } \left( \frac{\partial^2 z}{\partial y^2} > 0 \right)$$

The rate of increase of Medicare expenses ( $z$ ) is increasing with respect to public assistance expenses ( $y$ ).

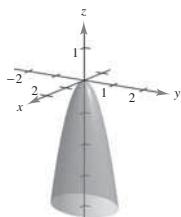
126.  $U = -5x^2 + xy - 3y^2$

$$(a) U_x = -10x + y$$

$$(b) U_y = x - 6y$$

- (c)  $U_x(2, 3) = -17$  and  $U_y(2, 3) = -16$ . The person should consume one more unit of  $y$  because the rate of decrease of satisfaction is less for  $y$ .

(d)



127.  $z = -0.92x + 1.03y + 0.02$

$$(a) \frac{\partial z}{\partial x} = -0.92$$

$$\frac{\partial z}{\partial y} = 1.03$$

- (b) As the consumption of flavored milk ( $x$ ) increases, the consumption of plain light and skim milk ( $z$ ) decreases. As the consumption of plain reduced-fat milk ( $y$ ) decreases, the consumption of plain light and skim milk decreases.

**129.** False

Let  $z = x + y + 1$ .

**130.** True

**131.** True

**132.** True

$$\text{133. } f(x, y) = \begin{cases} \frac{xy(x^2 - y^2)}{x^2 + y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$$

$$\text{(a) } f_x(x, y) = \frac{(x^2 + y^2)(3x^2y - y^3) - (x^3y - xy^3)(2x)}{(x^2 + y^2)^2} = \frac{y(x^4 + 4x^2y^2 - y^4)}{(x^2 + y^2)^2}$$

$$f_y(x, y) = \frac{(x^2 + y^2)(x^3 - 3xy^2) - (x^3y - xy^3)(2y)}{(x^2 + y^2)^2} = \frac{x(x^4 - 4x^2y^2 - y^4)}{(x^2 + y^2)^2}$$

$$\text{(b) } f_x(0, 0) = \lim_{\Delta x \rightarrow 0} \frac{f(\Delta x, 0) - f(0, 0)}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{0/\left[(\Delta x)^2\right] - 0}{\Delta x} = 0$$

$$f_y(0, 0) = \lim_{\Delta y \rightarrow 0} \frac{f(0, \Delta y) - f(0, 0)}{\Delta y} = \lim_{\Delta y \rightarrow 0} \frac{0/\left[(\Delta y)^2\right] - 0}{\Delta y} = 0$$

$$\text{(c) } f_{xy}(0, 0) = \left. \frac{\partial}{\partial y} \left( \frac{\partial f}{\partial x} \right) \right|_{(0, 0)} = \lim_{\Delta y \rightarrow 0} \frac{f_x(0, \Delta y) - f_x(0, 0)}{\Delta y} = \lim_{\Delta y \rightarrow 0} \frac{\Delta y(-(\Delta y)^4)}{\left((\Delta y)^2\right)^2 (\Delta y)} = \lim_{\Delta y \rightarrow 0} (-1) = -1$$

$$f_{yx}(0, 0) = \left. \frac{\partial}{\partial x} \left( \frac{\partial f}{\partial y} \right) \right|_{(0, 0)} = \lim_{\Delta x \rightarrow 0} \frac{f_y(\Delta x, 0) - f_y(0, 0)}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{\Delta x(\Delta x)^4}{\left((\Delta x)^2\right)^2 (\Delta x)} = \lim_{\Delta x \rightarrow 0} 1 = 1$$

(d)  $f_{yx}$  or  $f_{xy}$  or both are not continuous at  $(0, 0)$ .

$$\text{134. } f(x, y) = \int_x^y \sqrt{1 + t^3} dt$$

By the Second Fundamental Theorem of Calculus,

$$\frac{\partial f}{\partial x} = \frac{d}{dx} \int_x^y \sqrt{1 + t^3} dt = -\frac{d}{dx} \int_y^x \sqrt{1 + t^3} dt = -\sqrt{1 + x^3}$$

$$\frac{\partial f}{\partial y} = \frac{d}{dy} \int_x^y \sqrt{1 + t^3} dt = \sqrt{1 + y^3}.$$

$$\text{135. } f(x, y) = (x^3 + y^3)^{1/3}$$

$$\text{(a) } f_x(0, 0) = \lim_{\Delta x \rightarrow 0} \frac{f(0 + \Delta x, 0) - f(0, 0)}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{\Delta x}{\Delta x} = 1$$

$$f_y(0, 0) = \lim_{\Delta y \rightarrow 0} \frac{f(0, 0 + \Delta y) - f(0, 0)}{\Delta y} = \lim_{\Delta y \rightarrow 0} \frac{\Delta y}{\Delta y} = 1$$

(b)  $f_x(x, y)$  and  $f_y(x, y)$  fail to exist for  
 $y = -x, x \neq 0$ .

$$\text{136. } f(x, y) = (x^2 + y^2)^{2/3}$$

$$\text{For } (x, y) \neq (0, 0), f_x(x, y) = \frac{2}{3}(x^2 + y^2)^{-1/3}(2x) = \frac{4x}{3(x^2 + y^2)^{1/3}}.$$

For  $(x, y) = (0, 0)$ , use the definition of partial derivative.

$$f_x(0, 0) = \lim_{\Delta x \rightarrow 0} \frac{f(0 + \Delta x, 0) - f(0, 0)}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{(\Delta x)^{4/3}}{\Delta x} = \lim_{\Delta x \rightarrow 0} (\Delta x)^{1/3} = 0$$

## Section 13.4 Differentials

1.  $z = 2x^2y^3$

$$dz = 4xy^3 dx + 6x^2y^2 dy$$

2.  $z = \frac{x^2}{y}$

$$dz = \frac{2x}{y} dx - \frac{x^2}{y^2} dy$$

3.  $z = \frac{-1}{x^2 + y^2}$

$$dz = \frac{2x}{(x^2 + y^2)^2} dx + \frac{2y}{(x^2 + y^2)^2} dy$$

$$= \frac{2}{(x^2 + y^2)^2} (x dx + y dy)$$

4.  $w = \frac{x+y}{z-3y}$

$$dw = \frac{1}{z-3y} dx + \frac{3x+z}{(z-3y)^2} dy - \frac{x+y}{(z-3y)^2} dz$$

5.  $z = x \cos y - y \cos x$

$$\begin{aligned} dz &= (\cos y + y \sin x) dx + (-x \sin y - \cos x) dy \\ &= (\cos y + y \sin x) dx - (x \sin y + \cos x) dy \end{aligned}$$

6.  $z = \left(\frac{1}{2}\right)(e^{x^2+y^2} - e^{-x^2-y^2})$

$$\begin{aligned} dz &= 2x \left( \frac{e^{x^2+y^2} + e^{-x^2-y^2}}{2} \right) dx \\ &\quad + 2y \left( \frac{e^{x^2+y^2} + e^{-x^2-y^2}}{2} \right) dy \\ &= (e^{x^2+y^2} + e^{-x^2-y^2})(x dx + y dy) \end{aligned}$$

7.  $z = e^x \sin y$

$$dz = (e^x \sin y) dx + (e^x \cos y) dy$$

8.  $w = e^y \cos x + z^2$

$$dw = -e^y \sin x dx + e^y \cos x dy + 2z dz$$

9.  $w = 2z^3y \sin x$

$$dw = 2z^3y \cos x dx + 2z^3 \sin x dy + 6z^2y \sin x dz$$

10.  $w = x^2yz^2 + \sin yz$

$$\begin{aligned} dw &= 2xyz^2 dx + (x^2z^2 + z \cos yz) dy \\ &\quad + (2x^2yz + y \cos yz) dz \end{aligned}$$

11.  $f(x, y) = 2x - 3y$

(a)  $f(2, 1) = 1$

$$f(2.1, 1.05) = 1.05$$

$$\Delta z = f(2.1, 1.05) - f(2, 1) = 0.05$$

(b)  $dz = 2 dx - 3 dy = 2(0.1) - 3(0.05) = 0.05$

12.  $f(x, y) = x^2 + y^2$

(a)  $f(2, 1) = 5$

$$f(2.1, 1.05) = 5.5125$$

$$\Delta z = f(2.1, 1.05) - f(2, 1) = 0.5125$$

(b)  $dz = 2x dx + 2y dy$

$$= 2(2)(0.1) + 2(1)(0.05) = 0.5$$

13.  $f(x, y) = 16 - x^2 - y^2$

(a)  $f(2, 1) = 11$

$$f(2.1, 1.05) = 10.4875$$

$$\Delta z = f(2.1, 1.05) - f(2, 1) = -0.5125$$

(b)  $dz = -2x dx - 2y dy$

$$= -2(2)(0.1) - 2(1)(0.05) = -0.5$$

14.  $f(x, y) = \frac{y}{x}$

(a)  $f(2, 1) = 0.5$

$$f(2.1, 1.05) = 0.5$$

$$\Delta z = f(2.1, 1.05) - f(2, 1) = 0$$

(b)  $dz = \frac{-y}{x^2} dx + \frac{1}{x} dy = \frac{-1}{4}(0.1) + \frac{1}{2}(0.05) = 0$

15.  $f(x, y) = ye^x$

(a)  $f(2, 1) = e^2 \approx 7.3891$

$$f(2.1, 1.05) = 1.05e^{2.1} \approx 8.5745$$

$$\Delta z = f(2.1, 1.05) - f(2, 1) = 1.1854$$

(b)  $dz = ye^x dx + e^x dy$

$$= e^2(0.1) + e^2(0.05) \approx 1.1084$$

16.  $f(x, y) = x \cos y$

(a)  $f(2, 1) = 2 \cos 1 \approx 1.0806$

$$f(2.1, 1.05) = 2.1 \cos 1.05 \approx 1.0449$$

$$\Delta z = f(2.1, 1.05) - f(2, 1) = -0.0357$$

(b)  $dz = \cos y dx - x \sin y dy$

$$= \cos 1(0.1) - 2 \sin 1(0.05) \approx -0.0301$$

17. Let  $z = x^2y$ ,  $x = 2$ ,  $y = 9$ ,  $dx = 0.01$ ,  $dy = 0.02$ .

Then:  $dz = 2xy\,dx + x^2\,dy$

$$(2.01)^2(9.02) - 2^2 \cdot 9 \approx 2(2)(9)(0.01) + 2^2(0.02) = 0.44$$

18. Let  $z = \sqrt{x^2 + y^2}$ ,  $x = 5$ ,  $y = 3$ ,  $dx = 0.05$ ,  $dy = 0.1$ .

Then:

$$dz = \frac{x}{\sqrt{x^2 + y^2}}\,dx + \frac{y}{\sqrt{x^2 + y^2}}\,dy$$

$$\sqrt{(5.05)^2 + (3.1)^2} - \sqrt{5^2 + 3^2} \approx \frac{5}{\sqrt{5^2 + 3^2}}(0.05) + \frac{3}{\sqrt{5^2 + 3^2}}(0.1) = \frac{0.55}{\sqrt{34}} \approx 0.094$$

19. Let  $z = (1 - x^2)/y^2$ ,  $x = 3$ ,  $y = 6$ ,  $dx = 0.05$ ,  $dy = -0.05$ . Then:

$$dz = -\frac{2x}{y^2}\,dx + \frac{-2(1 - x^2)}{y^3}\,dy$$

$$\frac{1 - (3.05)^2}{(5.95)^2} - \frac{1 - 3^2}{6^2} \approx -\frac{2(3)}{6^2}(0.05) - \frac{2(1 - 3^2)}{6^3}(-0.05) \approx -0.012$$

20. Let  $z = \sin(x^2 + y^2)$ ,  $x = y = 1$ ,  $dx = 0.05$ ,  $dy = -0.05$ . Then:  $dz = 2x \cos(x^2 + y^2)\,dx + 2y \cos(x^2 + y^2)\,dy$

$$\sin[(1.05)^2 + (0.95)^2] - \sin 2 \approx 2(1) \cos(1^2 + 1^2)(0.05) + 2(1) \cos(1^2 + 1^2)(-0.05) = 0$$

21. See the definition on page 918.

22. In general, the accuracy worsens as  $\Delta x$  and  $\Delta y$  increases.

23. The tangent plane to the surface  $z = f(x, y)$  at the point  $P$  is a linear approximation of  $z$ .

24. If  $z = f(x, y)$ , then  $\Delta z \approx dz$  is the propagated error,

and  $\frac{\Delta z}{z} \approx \frac{dz}{z}$  is the relative error.

27.  $V = \frac{\pi r^2 h}{3}$ ,  $r = 4$ ,  $h = 8$

$$dV = \frac{2\pi rh}{3}dr + \frac{\pi r^2}{3}dh = \frac{\pi r}{3}(2h\,dr + r\,dh) = \frac{4\pi}{3}(16\,dr + 4\,dh)$$

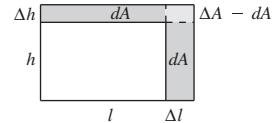
$$\Delta V = \frac{\pi}{3}[(r + \Delta r)^2(h + \Delta h) - r^2h] = \frac{\pi}{3}[(4 + \Delta r)^2(8 + \Delta h) - 128]$$

25.  $A = lh$

$$dA = l\,dh + h\,dl$$

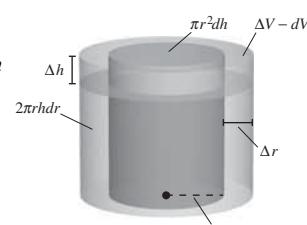
$$\begin{aligned} \Delta A &= (1 + dl)(h + dh) - lh \\ &= h\,dl + l\,dh + dl\,dh \end{aligned}$$

$$\Delta A - dA = dl\,dh$$



26.  $V = \pi r^2 h$

$$dV = 2\pi rh\,dr + \pi r^2\,dh$$



$\Delta r$	$\Delta h$	$dV$	$\Delta V$	$\Delta V - dV$
0.1	0.1	8.3776	8.5462	0.1686
0.1	-0.1	5.0265	5.0255	-0.0010
0.001	0.002	0.1005	0.1006	0.0001
-0.0001	0.0002	-0.0034	-0.0034	0.0000

28.  $S = \pi r \sqrt{r^2 + h^2}$ ,  $r = 6, h = 16$

$$\frac{dS}{dr} = \pi(r^2 + h^2)^{1/2} + \pi r^2(r^2 + h^2)^{-1/2} = \pi \frac{2r^2 + h^2}{\sqrt{r^2 + h^2}}$$

$$\frac{dS}{dh} = \pi \frac{rh}{\sqrt{r^2 + h^2}}$$

$$dS = \frac{\pi}{\sqrt{r^2 + h^2}} [(2r^2 + h^2)dr + (rh)dh] = \frac{\pi}{\sqrt{292}} [328 dr + 96 dh]$$

$$S(6, 16) = 322.101353$$

$$\Delta S = \pi(r + \Delta r)\sqrt{(r + \Delta r)^2 + (h + \Delta h)^2} = \pi(6 + \Delta r)\sqrt{(6 + \Delta r)^2 + (16 + \Delta h)^2} - 322.101353$$

$\Delta r$	$\Delta h$	$dS$	$\Delta S$	$\Delta S - dS$
0.1	0.1	7.7951	7.8375	0.0424
0.1	-0.1	4.2653	4.2562	-0.0091
0.001	0.002	0.0956	0.0956	0.0000
-0.0001	0.0002	-0.0025	-0.0025	-0.0000

29.  $z = -0.92x + 1.03y + 0.02$

(a)  $dz = -0.92 dx + 1.03 dy$

(b)  $dz = -0.92[\pm 0.25] + 1.03[\pm 0.25] = \mp 0.23 \pm 0.2575$

Maximum error:  $\pm 0.4875$

$$\text{Relative error: } \left| \frac{dz}{z} \right| = \frac{|\pm 0.4875|}{-0.92(1.9) + 1.03(7.5) + 0.02} \approx 0.081 = 8.1\%$$

30.  $(x, y) = (7.2, 2.5)$ ,  $|dx| \leq 0.05$ ,  $|dy| \leq 0.05$

$$\begin{aligned} r &= \sqrt{x^2 + y^2} \Rightarrow dr = \frac{x}{\sqrt{x^2 + y^2}} dx + \frac{y}{\sqrt{x^2 + y^2}} dy = \frac{7.2}{\sqrt{7.2^2 + 2.5^2}} dx + \frac{2.5}{\sqrt{7.2^2 + 2.5^2}} dy \\ &\approx 0.9447 dx + 0.3280 dy \end{aligned}$$

$$|dr| \leq 1.2727(0.05) \approx 0.064$$

$$\theta = \arctan\left(\frac{y}{x}\right) \Rightarrow d\theta = \frac{-y}{x^2 + y^2} dx + \frac{x}{x^2 + y^2} dy \approx -0.0430 dx + 0.1239 dy$$

Using the worst case scenario,  $dx = -0.05$  and  $dy = 0.05$ , you see that  $|d\theta| \leq 0.0083$ .

31.  $V = \pi r^2 h \Rightarrow dV = (2\pi rh)dr + (\pi r^2)dh$

$$\frac{dV}{V} = 2\frac{dr}{r} + \frac{dh}{h} = 2(0.04) + (0.02) = 0.10 = 10\%$$

32.  $A = \frac{1}{2}ab \sin C$

$$dA = \frac{1}{2}[(b \sin C)da + (a \sin C)db + (ab \cos C)dC]$$

$$= \frac{1}{2}[4(\sin 45^\circ)(\pm \frac{1}{16}) + 3(\sin 45^\circ)(\pm \frac{1}{16}) + 12(\cos 45^\circ)(\pm 0.02)] \approx \pm 0.24 \text{ in.}^2$$

33.  $C = 35.74 + 0.6215T - 35.75v^{0.16} + 0.4275Tv^{0.16}$

$$\frac{\partial C}{\partial T} = 0.6215 + 0.4275v^{0.16}$$

$$\frac{\partial C}{\partial v} = -5.72v^{-0.84} + 0.0684Tv^{-0.84}$$

$$dC = \frac{\partial C}{\partial T}dT + \frac{\partial C}{\partial v}dv = \left(0.6215 + 0.4275(23)^{0.16}\right)(\pm 1) + \left(-5.72(23)^{-0.84} + 0.0684(8)(23)^{-0.84}\right)(\pm 3)$$

$$= \pm 1.3275 \pm 1.1143 = \pm 2.4418 \text{ Maximum propagated error}$$

$$\frac{dC}{C} = \frac{\pm 2.4418}{-12.6807} \approx \pm 0.19 \text{ or } 19\%$$

34.  $a = \frac{v^2}{r}$

$$da = \frac{2v}{r}dv - \frac{v^2}{r^2}dr$$

$$\frac{da}{a} = 2\frac{dv}{v} - \frac{dr}{r} = 2(0.03) - (-0.02) = 0.08 = 8\%$$

**Note:** The maximum error will occur when  $dv$  and  $dr$  differ in signs.

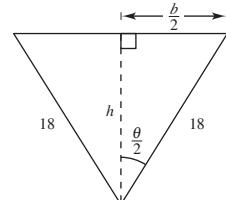
35. (a)  $V = \frac{1}{2}bhl = \left(18 \sin \frac{\theta}{2}\right)\left(18 \cos \frac{\theta}{2}\right)(16)(12) = 31,104 \sin \theta \text{ in.}^3 = 18 \sin \theta \text{ ft}^3$

$V$  is maximum when  $\sin \theta = 1$  or  $\theta = \pi/2$ .

(b)  $V = \frac{s^2}{2}(\sin \theta)l$

$$dV = s(\sin \theta)l ds + \frac{s^2}{2}l(\cos \theta)d\theta + \frac{s^2}{2}(\sin \theta)dl$$

$$= 18\left(\sin \frac{\pi}{2}\right)(16)(12)\left(\frac{1}{2}\right) + \frac{18^2}{2}(16)(12)\left(\cos \frac{\pi}{2}\right)\left(\frac{\pi}{90}\right) + \frac{18^2}{2}\left(\sin \frac{\pi}{2}\right)\left(\frac{1}{2}\right) = 1809 \text{ in.}^3 \approx 1.047 \text{ ft}^3$$



36. (a) Using the Law of Cosines:

$$a^2 = b^2 + c^2 - 2bc \cos A = 330^2 + 420^2 - 2(330)(420)\cos 90^\circ$$

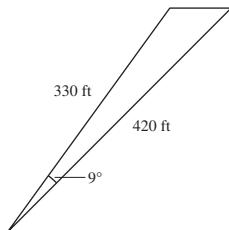
$$a \approx 107.3 \text{ ft.}$$

(b)  $a = \sqrt{b^2 + 420^2 - 2b(420)\cos \theta}$

$$da = \frac{1}{2}\left[b^2 + 420^2 - 840b \cos \theta\right]^{-1/2}\left[(2b - 840 \cos \theta)db + 840b \sin \theta d\theta\right]$$

$$= \frac{1}{2}\left[330^2 + 420^2 - 840(330)\left(\cos \frac{\pi}{20}\right)\right]^{-1/2}\left[\left(2(330) - 840 \cos \frac{\pi}{20}\right)(6) + 840(330)\left(\sin \frac{\pi}{20}\right)\left(\frac{\pi}{180}\right)\right]$$

$$\approx \frac{1}{2}[11512.79]^{-1/2}[\pm 1774.79] \approx \pm 8.27 \text{ ft}$$



37.  $P = \frac{E^2}{R}$ ,  $\left|\frac{dE}{E}\right| = 3\% = 0.03$ ,  $\left|\frac{dR}{R}\right| = 4\% = 0.04$

$$dP = \frac{2E}{R}dE - \frac{E^2}{R^2}dR$$

$$\frac{dP}{P} = \left[\frac{2E}{R}dE - \frac{E^2}{R^2}dR\right]/P = \left[\frac{2E}{R}dE - \frac{E^2}{R^2}dR\right]/(E^2/R) = \frac{2}{E}dE - \frac{1}{R}dR$$

Using the worst case scenario,  $\frac{dE}{E} = 0.03$  and  $\frac{dR}{R} = -0.04$ :  $\frac{dP}{P} \leq 2(0.03) - (-0.04) = 0.10 = 10\%$ .

38.  $\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$

$$R = \frac{R_1 R_2}{R_1 + R_2}$$

$$dR_1 = \Delta R_1 = 0.5$$

$$dR_2 = \Delta R_2 = -2$$

$$\Delta R \approx dR = \frac{\partial R}{\partial R_1} dR_1 + \frac{\partial R}{\partial R_2} dR_2 = \frac{R_2^2}{(R_1 + R_2)^2} \Delta R_1 + \frac{R_1^2}{(R_1 + R_2)^2} \Delta R_2$$

When  $R_1 = 10$  and  $R_2 = 15$ , we have  $\Delta R \approx \frac{15^2}{(10+15)^2}(0.5) + \frac{10^2}{(10+15)^2}(-2) = -0.14$  ohm.

39.  $L = 0.00021 \left( \ln \frac{2h}{r} - 0.75 \right)$

$$dL = 0.00021 \left[ \frac{dh}{h} - \frac{dr}{r} \right] = 0.00021 \left[ \frac{(\pm 1/100)}{100} - \frac{(\pm 1/16)}{2} \right] \approx (\pm 6.6) \times 10^{-6}$$

$$L = 0.00021(\ln 100 - 0.75) \pm dL \approx 8.096 \times 10^{-4} \pm 6.6 \times 10^{-6} \text{ micro henrys}$$

40.  $T = 2\pi \sqrt{\frac{L}{g}}$

$$dg = 32.23 - 32.09 = 0.14$$

$$dL = 2.48 - 2.50 = -0.02$$

$$\Delta T \approx dT = \frac{\partial T}{\partial g} dg + \frac{\partial T}{\partial L} dL = \frac{-\pi}{g} \sqrt{\frac{L}{g}} dg + \frac{\pi}{\sqrt{Lg}} dL$$

When  $g = 32.09$  and  $L = 2.50$ ,  $\Delta T \approx \frac{-\pi}{32.09} \sqrt{\frac{2.5}{32.09}}(0.14) + \frac{\pi}{\sqrt{(2.5)(32.09)}}(-0.02) \approx -0.0108$  seconds.

41.  $z = f(x, y) = x^2 - 2x + y$

$$\begin{aligned} \Delta z &= f(x + \Delta x, y + \Delta y) - f(x, y) = (x^2 + 2x(\Delta x) + (\Delta x)^2 - 2x - 2(\Delta x) + y + (\Delta y)) - (x^2 - 2x + y) \\ &= 2x(\Delta x) + (\Delta x)^2 - 2(\Delta x) + (\Delta y) = (2x - 2)\Delta x + \Delta y + \Delta x(\Delta x) + 0(\Delta y) \\ &= f_x(x, y)\Delta x + f_y(x, y)\Delta y + \varepsilon_1\Delta x + \varepsilon_2\Delta y \text{ where } \varepsilon_1 = \Delta x \text{ and } \varepsilon_2 = 0. \end{aligned}$$

As  $(\Delta x, \Delta y) \rightarrow (0, 0)$ ,  $\varepsilon_1 \rightarrow 0$  and  $\varepsilon_2 \rightarrow 0$ .

42.  $z = f(x, y) = x^2 + y^2$

$$\begin{aligned} \Delta z &= f(x + \Delta x, y + \Delta y) - f(x, y) = x^2 + 2x(\Delta x) + (\Delta x)^2 + y^2 + 2y(\Delta y) + (\Delta y)^2 - (x^2 + y^2) \\ &= 2x(\Delta x) + 2y(\Delta y) + \Delta x(\Delta x) + \Delta y(\Delta y) = f_x(x, y)\Delta x + f_y(x, y)\Delta y + \varepsilon_1\Delta x + \varepsilon_2\Delta y \text{ where } \varepsilon_1 = \Delta x \text{ and } \varepsilon_2 = \Delta y. \end{aligned}$$

As  $(\Delta x, \Delta y) \rightarrow (0, 0)$ ,  $\varepsilon_1 \rightarrow 0$  and  $\varepsilon_2 \rightarrow 0$ .

43.  $z = f(x, y) = x^2 y$

$$\begin{aligned} \Delta z &= f(x + \Delta x, y + \Delta y) - f(x, y) = (x^2 + 2x(\Delta x) + (\Delta x)^2)(y + \Delta y) - x^2 y \\ &= 2xy(\Delta x) + y(\Delta x)^2 + x^2\Delta y + 2x(\Delta x)(\Delta y) + (\Delta x)^2\Delta y = 2xy(\Delta x) + x^2\Delta y + (y\Delta x)\Delta x + [2x\Delta x + (\Delta x)^2]\Delta y \\ &= f_x(x, y)\Delta x + f_y(x, y)\Delta y + \varepsilon_1\Delta x + \varepsilon_2\Delta y \text{ where } \varepsilon_1 = y(\Delta x) \text{ and } \varepsilon_2 = 2x\Delta x + (\Delta x)^2. \end{aligned}$$

As  $(\Delta x, \Delta y) \rightarrow (0, 0)$ ,  $\varepsilon_1 \rightarrow 0$  and  $\varepsilon_2 \rightarrow 0$ .

44.  $z = f(x, y) = 5x - 10y + y^3$

$$\begin{aligned}\Delta z &= f(x + \Delta x, y + \Delta y) - f(x, y) \\&= 5x + 5\Delta x - 10y - 10\Delta y + y^3 + 3y^2(\Delta y) + 3y(\Delta y)^2 + (\Delta y)^3 - (5x - 10y + y^3) \\&= 5(\Delta x) + (3y^2 - 10)(\Delta y) + 0(\Delta x) + (3y(\Delta y) + (\Delta y)^2)\Delta y \\&= f_x(x, y)\Delta x + f_y(x, y)\Delta y + \varepsilon_1\Delta x + \varepsilon_2\Delta y \text{ where } \varepsilon_1 = 0 \text{ and } \varepsilon_2 = 3y(\Delta y) + (\Delta y)^2.\end{aligned}$$

As  $(\Delta x, \Delta y) \rightarrow (0, 0)$ ,  $\varepsilon_1 \rightarrow 0$  and  $\varepsilon_2 \rightarrow 0$ .

45.  $f(x, y) = \begin{cases} \frac{3x^2y}{x^4 + y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$

$$f_x(0, 0) = \lim_{\Delta x \rightarrow 0} \frac{f(\Delta x, 0) - f(0, 0)}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{\frac{0}{(\Delta x)^4} - 0}{\Delta x} = 0$$

$$f_y(0, 0) = \lim_{\Delta y \rightarrow 0} \frac{f(0, \Delta y) - f(0, 0)}{\Delta y} = \lim_{\Delta y \rightarrow 0} \frac{\frac{0}{(\Delta y)^2} - 0}{\Delta y} = 0$$

So, the partial derivatives exist at  $(0, 0)$ .

Along the line  $y = x$ :  $\lim_{(x, y) \rightarrow (0, 0)} f(x, y) = \lim_{x \rightarrow 0} \frac{3x^3}{x^4 + x^2} = \lim_{x \rightarrow 0} \frac{3x}{x^2 + 1} = 0$

Along the curve  $y = x^2$ :  $\lim_{(x, y) \rightarrow (0, 0)} f(x, y) = \frac{3x^4}{2x^4} = \frac{3}{2}$

$f$  is not continuous at  $(0, 0)$ . So,  $f$  is not differentiable at  $(0, 0)$ . (See Theorem 12.5)

46.  $f(x, y) = \begin{cases} \frac{5x^2y}{x^3 + y^3}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$

$$f_x(0, 0) = \lim_{\Delta x \rightarrow 0} \frac{f(\Delta x, 0) - f(0, 0)}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{0 - 0}{\Delta x} = 0$$

$$f_y(0, 0) = \lim_{\Delta y \rightarrow 0} \frac{f(0, \Delta y) - f(0, 0)}{\Delta y} = \lim_{\Delta y \rightarrow 0} \frac{0 - 0}{\Delta y} = 0$$

So, the partial derivatives exist at  $(0, 0)$ .

Along the line  $y = x$ :  $\lim_{(x, y) \rightarrow (0, 0)} f(x, y) = \lim_{x \rightarrow 0} \frac{5x^3}{2x^3} = \frac{5}{2}$

Along the line  $x = 0$ ,  $\lim_{(x, y) \rightarrow (0, 0)} f(x, y) = 0$ .

So,  $f$  is not continuous at  $(0, 0)$ . Therefore  $f$  is not differentiable at  $(0, 0)$ .

47. If  $f$  is differentiable at  $(x_0, y_0)$ , then the partial derivative

$f_x(x_0, y_0)$  exists, which implies that  $f(x, y_0)$  is differentiable at  $(x_0, y_0)$ . If  $f(x, y) = \sqrt{x^2 + y^2}$ , then  $f(x, 0) = \sqrt{x^2} = |x|$ . Because  $|x|$  is not differentiable at  $x = 0$ ,  $f(x, y) = \sqrt{x^2 + y^2}$  is not differentiable.

48.  $f(x, y) = \sqrt{x^2 + y^2}$

(a)  $f(1, 2) = \sqrt{1 + 4} = \sqrt{5} \approx 2.2361$

$$f(1.05, 2.1) = \sqrt{1.05^2 + 2.1^2} \approx 2.3479$$

(b)  $\Delta z = f(1.05, 2.1) - f(1, 2) = 0.1118$

(c)  $dz = \frac{x}{\sqrt{x^2 + y^2}}dx + \frac{y}{\sqrt{x^2 + y^2}}dy$

$$= \frac{1}{\sqrt{5}}(0.5) + \frac{2}{\sqrt{5}}(0.1) \approx 0.1118$$

The results are the same.

## Section 13.5 Chain Rules for Functions of Several Variables

1.  $w = x^2 + y^2$

$$x = 2t, y = 3t$$

$$\frac{dw}{dt} = \frac{\partial w}{\partial x} \frac{dx}{dt} + \frac{\partial w}{\partial y} \frac{dy}{dt} = (2x)(2) + (2y)(3)$$

$$= 4x + 6y = 8t + 18t = 26t$$

2.  $w = \sqrt{x^2 + y^2}$

$$x = \cos t, y = e^t$$

$$\begin{aligned}\frac{dw}{dt} &= \frac{\partial w}{\partial x} \frac{dx}{dt} + \frac{\partial w}{\partial y} \frac{dy}{dt} \\ &= \frac{x}{\sqrt{x^2 + y^2}}(-\sin t) + \frac{y}{\sqrt{x^2 + y^2}}e^t \\ &= \frac{-x \sin t + ye^t}{\sqrt{x^2 + y^2}} = \frac{-\cos t \sin t + e^{2t}}{\sqrt{\cos^2 t + e^{2t}}}\end{aligned}$$

3.  $w = x \sin y$

$$x = e^t, y = \pi - t$$

$$\begin{aligned}\frac{dw}{dt} &= \frac{\partial w}{\partial x} \frac{dx}{dt} + \frac{\partial w}{\partial y} \frac{dy}{dt} = \sin y(e^t) + x \cos y(-1) \\ &= \sin(\pi - t)e^t - e^t \cos(\pi - t) = e^t \sin t + e^t \cos t\end{aligned}$$

4.  $w = \ln \frac{y}{x}$

$$x = \cos t$$

$$y = \sin t$$

$$\begin{aligned}\frac{dw}{dt} &= \left(\frac{-1}{x}\right)(-\sin t) + \left(\frac{1}{y}\right)(\cos t) \\ &= \tan t + \cot t = \frac{1}{\sin t \cos t}\end{aligned}$$

8.  $w = xy \cos z$

$$x = t$$

$$y = t^2$$

$$z = \arccos t$$

$$\begin{aligned}(a) \quad \frac{dw}{dt} &= (y \cos z)(1) + (x \cos z)(2t) + (-xy \sin z)\left(-\frac{1}{\sqrt{1-t^2}}\right) = t^2(t) + t(t)(2t) - t(t^2)\sqrt{1-t^2}\left(\frac{-1}{\sqrt{1-t^2}}\right) \\ &= t^3 + 2t^3 + t^3 = 4t^3\end{aligned}$$

$$(b) \quad w = t^4, \frac{dw}{dt} = 4t^3$$

9.  $w = xy + xz + yz, x = t - 1, y = t^2 - 1, z = t$

$$\begin{aligned}(a) \quad \frac{dw}{dt} &= \frac{\partial w}{\partial x} \frac{dx}{dt} + \frac{\partial w}{\partial y} \frac{dy}{dt} + \frac{\partial w}{\partial z} \frac{dz}{dt} = (y + z) + (x + z)(2t) + (x + y) \\ &= (t^2 - 1 + t) + (t - 1 + t)(2t) + (t - 1 + t^2 - 1) = 3(2t^2 - 1)\end{aligned}$$

$$(b) \quad w = (t - 1)(t^2 - 1) + (t - 1)t + (t^2 - 1)t$$

$$\frac{dw}{dt} = 2t(t - 1) + (t^2 - 1) + 2t - 1 + 3t^2 - 1 = 3(2t^2 - 1)$$

5.  $w = xy, x = e^t, y = e^{-2t}$

$$\begin{aligned}(a) \quad \frac{dw}{dt} &= \frac{\partial w}{\partial x} \frac{dx}{dt} + \frac{\partial w}{\partial y} \frac{dy}{dt} \\ &= y(e^t) + x(-2e^{-2t}) = e^{-2t}e^t - e^t 2e^{-2t} = -e^{-t}\end{aligned}$$

$$(b) \quad w = e^t e^{-2t} = e^{-t}$$

$$\frac{dw}{dt} = -e^{-t}$$

6.  $w = \cos(x - y), x = t^2, y = 1$

$$\begin{aligned}(a) \quad \frac{dw}{dt} &= -\sin(x - y)(2t) + \sin(x - y)(0) \\ &= -2t \sin(x - y) = -2t \sin(t^2 - 1)\end{aligned}$$

$$(b) \quad w = \cos(t^2 - 1), \frac{dw}{dt} = -2t \sin(t^2 - 1)$$

7.  $w = x^2 + y^2 + z^2, x = \cos t, y = \sin t, z = e^t$

$$\begin{aligned}(a) \quad \frac{dw}{dt} &= \frac{\partial w}{\partial x} \frac{dx}{dt} + \frac{\partial w}{\partial y} \frac{dy}{dt} + \frac{\partial w}{\partial z} \frac{dz}{dt} \\ &= 2x(-\sin t) + 2y(\cos t) + 2z(e^t) \\ &= -2 \cos t \sin t + 2 \sin t \cos t + 2e^{2t} = 2e^{2t}\end{aligned}$$

$$(b) \quad w = \cos^2 t + \sin^2 t + e^{2t} = 1 + e^{2t}$$

$$\frac{dw}{dt} = 2e^{2t}$$

10.  $w = xy^2 + x^2z + yz^2$ ,  $x = t^2$ ,  $y = 2t$ ,  $z = 2$

$$\begin{aligned} \text{(a)} \quad \frac{dw}{dt} &= \frac{\partial w}{\partial x} \frac{dx}{dt} + \frac{\partial w}{\partial y} \frac{dy}{dt} + \frac{\partial w}{\partial z} \frac{dz}{dt} \\ &= (y^2 + 2xz)(2t) + (2xy + z^2)(2) + (x^2 + 2yz)(0) = (4t^2 + 4t^2)(2t) + (4t^3 + 4)(2) = 24t^3 + 8 \end{aligned}$$

$$\text{(b)} \quad w = t^2(4t^2) + t^4(2) + 2t(4) = 6t^4 + 8$$

$$\frac{dw}{dt} = 24t^3 + 8$$

11. Distance =  $f(t) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} = \sqrt{(10 \cos 2t - 7 \cos t)^2 + (6 \sin 2t - 4 \sin t)^2}$

$$\begin{aligned} f'(t) &= \frac{1}{2} \left[ (10 \cos 2t - 7 \cos t)^2 + (6 \sin 2t - 4 \sin t)^2 \right]^{-1/2} \\ &\quad \left[ [2(10 \cos 2t - 7 \cos t)(-20 \sin 2t + 7 \sin t)] + [2(6 \sin 2t - 4 \sin t)(12 \cos 2t - 4 \cos t)] \right] \\ f'\left(\frac{\pi}{2}\right) &= \frac{1}{2} \left[ (-10)^2 + 4^2 \right]^{-1/2} \left[ [2(-10)(7)] + [2(-4)(-12)] \right] = \frac{1}{2}(116)^{-1/2}(-44) = \frac{-22}{2\sqrt{29}} = \frac{-11\sqrt{29}}{29} \approx -2.04 \end{aligned}$$

12. Distance =  $f(t) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = \sqrt{[48 + (\sqrt{3} - \sqrt{2})]^2 + [48t(1 - \sqrt{2})]^2} = 48t\sqrt{8 - 2\sqrt{2} - 2\sqrt{6}}$

$$f'(t) = 48\sqrt{8 - 2\sqrt{2} - 2\sqrt{6}} = f'(1)$$

13.  $w = \ln(x + y)$ ,  $x = e^t$ ,  $y = e^{-t}$

$$\begin{aligned} \frac{dw}{dt} &= \frac{\partial w}{\partial x} \frac{dx}{dt} + \frac{\partial w}{\partial y} \frac{dy}{dt} \\ &= \frac{1}{x + y}(e^t) + \frac{1}{x + y}(-e^{-t}) = \frac{e^t - e^{-t}}{e^t + e^{-t}} \\ \frac{d^2w}{dt^2} &= \frac{(e^t + e^{-t})(e^t + e^{-t}) - (e^t - e^{-t})(e^t - e^{-t})}{(e^t + e^{-t})^2} \\ &= \frac{4}{(e^t + e^{-t})^2} \end{aligned}$$

$$\text{At } t = 0, \frac{d^2w}{dt^2} = \frac{4}{(1+1)^2} = 1.$$

14.  $w = \frac{x^2}{y}$

$$x = t^2$$

$$y = t + 1$$

$$t = 1$$

$$\begin{aligned} \frac{dw}{dt} &= \frac{\partial w}{\partial x} \frac{dx}{dt} + \frac{\partial w}{\partial y} \frac{dy}{dt} = \frac{2x}{y}(2t) + \frac{-x^2}{y^2}(1) \\ &= \frac{2t^2(2t)}{t+1} - \frac{t^4}{(t+1)^2} = \frac{(t+1)(4t^3) - t^4}{(t+1)^2} = \frac{3t^4 + 4t^3}{(t+1)^2} \end{aligned}$$

$$\frac{d^2w}{dt^2} = \frac{(t+1)^2(12t^3 + 12t^2) - (3t^4 + 4t^3)2(t+1)}{(t+1)^4}$$

$$\text{At } t = 1: \frac{d^2w}{dt^2} = \frac{4(24) - (7)(4)}{16} = \frac{68}{16} = 4.25$$

15.  $w = x^2 + y^2$

$$x = s + t, y = s - t$$

$$\frac{\partial w}{\partial s} = 2x(1) + 2y(1) = 2(s+t) + 2(s-t) = 4s$$

$$\frac{\partial w}{\partial t} = 2x(1) + 2y(-1) = 2(s+t) - 2(s-t) = 4t$$

When  $s = 1$  and  $t = 0$ ,  $\frac{\partial w}{\partial s} = 4$  and  $\frac{\partial w}{\partial t} = 0$ .

16.  $w = y^3 - 3x^2y$

$$x = e^s, y = e^t$$

$$\frac{\partial w}{\partial s} = -6xy(e^s) + (3y^2 - 3x^2)(0) = -6e^s e^t e^s = -6e^{2s+t}$$

$$\begin{aligned} \frac{\partial w}{\partial t} &= (-6xy)(0) + (3y^2 - 3x^2)e^t = (3e^{2t} - 3e^{2s})e^t \\ &= 3e^{3t} - 3e^{2s+t} \end{aligned}$$

When  $s = -1$  and  $t = 2$ ,  $\frac{\partial w}{\partial s} = -6$  and  $\frac{\partial w}{\partial t} = 3e^6 - 3$ .

17.  $w = \sin(2x + 3y)$

$$x = s + t$$

$$y = s - t$$

$$\frac{\partial w}{\partial s} = 2 \cos(2x + 3y) + 3 \cos(2x + 3y)$$

$$= 5 \cos(2x + 3y) = 5 \cos(5s - t)$$

$$\frac{\partial w}{\partial t} = 2 \cos(2x + 3y) - 3 \cos(2x + 3y)$$

$$= -\cos(2x + 3y) = -\cos(5s - t)$$

When  $s = 0$  and  $t = \frac{\pi}{2}$ ,  $\frac{\partial w}{\partial s} = 0$  and  $\frac{\partial w}{\partial t} = 0$ .

**18.**  $w = x^2 - y^2$

$$x = s \cos t$$

$$y = s \sin t$$

$$\frac{\partial w}{\partial s} = 2x \cos t - 2y \sin t$$

$$= 2s \cos^2 t - 2s \sin^2 t = 2s \cos 2t$$

$$\frac{\partial w}{\partial t} = 2x(-s \sin t) - 2y(s \cos t) = -2s^2 \sin 2t$$

When  $s = 3$  and  $t = \frac{\pi}{4}$ ,  $\frac{\partial w}{\partial s} = 0$  and  $\frac{\partial w}{\partial t} = -18$ .

**19.**  $w = \frac{yz}{x}$ ,  $x = \theta^2$ ,  $y = r + \theta$ ,  $z = r - \theta$

$$(a) \quad \frac{\partial w}{\partial r} = \frac{-yz}{x^2}(0) + \frac{z}{x}(1) + \frac{y}{x}(1) = \frac{z + y}{x} = \frac{2r}{\theta^2}$$

$$\frac{\partial w}{\partial \theta} = \frac{-yz}{x^2}(2\theta) + \frac{z}{x}(1) + \frac{y}{x}(-1)$$

$$= \frac{-(r + \theta)(r - \theta)}{\theta^4}(2\theta) + \frac{(r - \theta) - (r + \theta)}{\theta^2}$$

$$= \frac{2(\theta^2 - r^2)}{\theta^3} - \frac{2}{\theta} = \frac{-2r^2}{\theta^3}$$

$$(b) \quad w = \frac{yz}{x} = \frac{(r + \theta)(r - \theta)}{\theta^2} = \frac{r^2}{\theta^2} - 1$$

$$\frac{\partial w}{\partial r} = \frac{2r}{\theta^2}$$

$$\frac{\partial w}{\partial \theta} = \frac{-2r^2}{\theta^3}$$

**20.**  $w = x^2 - 2xy + y^2$ ,  $x = r + \theta$ ,  $y = r - \theta$

$$(a) \quad \frac{\partial w}{\partial r} = (2x - 2y)(1) + (-2x + 2y)(1) = 0$$

$$\frac{\partial w}{\partial \theta} = (2x - 2y)(1) + (-2x + 2y)(-1) = 4x - 4y = 4(x - y) = 4[(r + \theta) - (r - \theta)] = 8\theta$$

$$(b) \quad w = (r + \theta)^2 - 2(r + \theta)(r - \theta) + (r - \theta)^2 = (r^2 + 2r\theta + \theta^2) - 2(r^2 - \theta^2) + (r^2 - 2r\theta + \theta^2) = 4\theta^2$$

$$\frac{\partial w}{\partial r} = 0$$

$$\frac{\partial w}{\partial \theta} = 8\theta$$

**21.**  $w = \arctan \frac{y}{x}$ ,  $x = r \cos \theta$ ,  $y = r \sin \theta$

$$(a) \quad \frac{\partial w}{\partial r} = \frac{-y}{x^2 + y^2} \cos \theta + \frac{x}{x^2 + y^2} \sin \theta = \frac{-r \sin \theta \cos \theta}{r^2} + \frac{r \cos \theta \sin \theta}{r^2} = 0$$

$$\frac{\partial w}{\partial \theta} = \frac{-y}{x^2 + y^2}(-r \sin \theta) + \frac{x}{x^2 + y^2}(r \cos \theta) = \frac{-(r \sin \theta)(-r \sin \theta)}{r^2} + \frac{(r \cos \theta)(r \cos \theta)}{r^2} = 1$$

$$(b) \quad w = \arctan \frac{r \sin \theta}{r \cos \theta} = \arctan(\tan \theta) = \theta$$

$$\frac{\partial w}{\partial r} = 0$$

$$\frac{\partial w}{\partial \theta} = 1$$

**22.**  $w = \sqrt{25 - 5x^2 - 5y^2}$ ,  $x = r \cos \theta$ ,  $y = r \sin \theta$

$$(a) \quad \frac{\partial w}{\partial r} = \frac{-5x}{\sqrt{25 - 5x^2 - 5y^2}} \cos \theta + \frac{-5y}{\sqrt{25 - 5x^2 - 5y^2}} \sin \theta = \frac{-5r \cos^2 \theta - 5r \sin^2 \theta}{\sqrt{25 - 5x^2 - 5y^2}} = \frac{-5r}{\sqrt{25 - 5r^2}}$$

$$\frac{\partial w}{\partial \theta} = \frac{-5x}{\sqrt{25 - 5x^2 - 5y^2}}(-r \sin \theta) + \frac{-5y}{\sqrt{25 - 5x^2 - 5y^2}}(r \cos \theta) = \frac{-5r^2 \sin^2 \theta \cos \theta - 5r^2 \sin \theta \cos \theta}{\sqrt{25 - 5x^2 - 5y^2}} = 0$$

$$(b) w = \sqrt{25 - 5r^2}$$

$$\frac{\partial w}{\partial r} = \frac{-5r}{\sqrt{25 - 5r^2}}; \frac{\partial w}{\partial \theta} = 0$$

$$23. w = xyz, x = s + t, y = s - t, z = st^2$$

$$\begin{aligned}\frac{\partial w}{\partial s} &= yz(1) + xz(1) + xy(t^2) \\ &= (s - t)st^2 + (s + t)st^2 + (s + t)(s - t)t^2 = 2s^2t^2 + s^2t^2 - t^4 = 3s^2t^2 - t^4 = t^2(3s^2 - t^2)\end{aligned}$$

$$\begin{aligned}\frac{\partial w}{\partial t} &= yz(1) + xz(-1) + xy(2st) = (s - t)st^2 - (s + t)st^2 + (s + t)(s - t)(2st) = -2st^3 + 2s^3t - 2st^3 = 2s^3t - 4st^3 \\ &= 2st(s^2 - 2t^2)\end{aligned}$$

$$24. w = x^2 + y^2 + z^2, x = t \sin s, y = t \cos s, z = st^2$$

$$\begin{aligned}\frac{\partial w}{\partial s} &= 2x + \cos s + 2y(-t \sin s) + 2z(t^2) \\ &= 2t^2 \sin s \cos s - 2t^2 \sin s \cos s + 2st^4 = 2st^4 \\ \frac{\partial w}{\partial t} &= 2x \sin s + 2y \cos s + 2z(2st) \\ &= 2t \sin^2 s + 2t \cos^2 s + 4s^2t^3 = 2t + 4s^2t^3\end{aligned}$$

$$25. w = ze^{xy}, x = s - t, y = s + t, z = st$$

$$\begin{aligned}\frac{\partial w}{\partial s} &= yze^{xy}(1) + xze^{xy}(1) + e^{xy}(t) \\ &= e^{(s-t)(s+t)}[(s + t)st + (s - t)st + t] \\ &= e^{(s-t)(s+t)}[2s^2t + t] = te^{s^2-t^2}(2s^2 + 1) \\ \frac{\partial w}{\partial t} &= yze^{xy}(-1) + xze^{xy}(1) + e^{xy}(s) \\ &= e^{(s-t)(s+t)}[-(s + t)(st) + (s - t)st + s] \\ &= e^{(s-t)(s+t)}[-2st^2 + s] = se^{s^2-t^2}(1 - 2t^2)\end{aligned}$$

$$26. w = x \cos yz, x = s^2, y = t^2, z = s - 2t$$

$$\begin{aligned}\frac{\partial w}{\partial s} &= \cos(yz)(2s) - xz \sin(yz)(0) - xy \sin(yz)(1) \\ &= \cos(st^2 - 2t^3)2s - s^2t^2 \sin(st^2 - 2t^3) \\ \frac{\partial w}{\partial t} &= \cos(yz)(0) - xz \sin(yz)(2t) - xy \sin(yz)(-2) \\ &= -2s^2t(s - 2t) \sin(st^2 - 2t^3) + 2s^2t^2 \sin(st^2 - 2t^3) \\ &= (6s^2t^2 - 2s^3t) \sin(st^2 - 2t^3)\end{aligned}$$

$$27. x^2 - xy + y^2 - x + y = 0$$

$$\frac{dy}{dx} = -\frac{F_x(x, y)}{F_y(x, y)} = -\frac{2x - y - 1}{-x + 2y + 1} = \frac{y - 2x + 1}{2y - x + 1}$$

$$28. \sec xy + \tan xy + 5 = 0$$

$$\begin{aligned}\frac{dy}{dx} &= -\frac{F_x(x, y)}{F_y(x, y)} = -\frac{y \sec xy \tan xy + y \sec^2 xy}{x \sec xy \tan xy + x \sec^2 xy} \\ &= \frac{-y(\sec xy \tan xy + \sec^2 xy)}{x(\sec xy \tan xy + \sec^2 xy)} = -\frac{y}{x}\end{aligned}$$

$$29. \ln \sqrt{x^2 + y^2} + x + y = 4$$

$$\frac{1}{2} \ln(x^2 + y^2) + x + y - 4 = 0$$

$$\frac{dy}{dx} = -\frac{F_x(x, y)}{F_y(x, y)} = -\frac{\frac{x}{x^2 + y^2} + 1}{\frac{y}{x^2 + y^2} + 1} = -\frac{x + x^2 + y^2}{y + x^2 + y^2}$$

$$30. \frac{x}{x^2 + y^2} - y^2 - 6 = 0$$

$$\begin{aligned}\frac{dy}{dx} &= -\frac{F_x(x, y)}{F_y(x, y)} \\ &= -\frac{(y^2 - x^2)/(x^2 + y^2)^2}{(-2xy)/(x^2 + y^2)^2 - 2y} \\ &= \frac{y^2 - x^2}{2xy + 2y(x^2 + y^2)^2} \\ &= \frac{y^2 - x^2}{2xy + 2yx^4 + 4x^2y^3 + 2y^5}\end{aligned}$$

$$31. F(x, y, z) = x^2 + y^2 + z^2 - 1$$

$$F_x = 2x, F_y = 2y, F_z = 2z$$

$$\frac{\partial_z}{\partial_x} = -\frac{F_x}{F_z} = -\frac{x}{z}$$

$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{y}{z}$$

32.  $F(x, y, z) = xz + yz + xy$

$$F_x = z + y$$

$$F_y = z + x$$

$$F_z = x + y$$

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{y+z}{x+y}$$

$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{x+z}{x+y}$$

33.  $F(x, y, z) = x^2 + 2yz + z^2 - 1 = 0$

$$\frac{\partial z}{\partial x} = -\frac{F_x(x, y, z)}{F_z(x, y, z)} = \frac{-2x}{2y+2z} = \frac{-x}{y+z}$$

$$\frac{\partial z}{\partial y} = -\frac{F_y(x, y, z)}{F_z(x, y, z)} = \frac{-2z}{2y+2z} = \frac{-z}{y+z}$$

34.  $x + \sin(y + z) = 0$

(i)  $1 + \frac{\partial z}{\partial x} \cos(y + z) = 0$  implies

$$\frac{\partial z}{\partial x} = -\frac{1}{\cos(y + z)} = -\sec(y + z).$$

(ii)  $\left(1 + \frac{\partial z}{\partial y}\right) \cos(y + z) = 0$  implies  $\frac{\partial z}{\partial y} = -1$ .

35.  $F(x, y, z) = \tan(x + y) + \tan(y + z) - 1$

$$F_x = \sec^2(x + y)$$

$$F_y = \sec^2(x + y) + \sec^2(y + z)$$

$$F_z = \sec^2(y + z)$$

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{\sec^2(x + y)}{\sec^2(y + z)}$$

$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{\sec^2(x + y) + \sec^2(y + z)}{\sec^2(y + z)}$$

$$= -\left(\frac{\sec^2(x + y)}{\sec^2(y + z)} + 1\right)$$

36.  $F(x, y, z) = e^x \sin(y + z) - z$

$$F_x = e^x \sin(y + z)$$

$$F_y = e^x \cos(y + z)$$

$$F_z = e^x \cos(y + z) - 1$$

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = \frac{e^x \sin(y + z)}{1 - e^x \cos(y + z)}$$

$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = \frac{e^x \cos(y + z)}{1 - e^x \cos(y + z)}$$

37.  $F(x, y, z) = e^{xz} + xy = 0$

$$\frac{\partial z}{\partial x} = -\frac{F_x(x, y, z)}{F_z(x, y, z)} = -\frac{ze^{xz} + y}{xe^{xz}}$$

$$\frac{\partial z}{\partial y} = -\frac{F_y(x, y, z)}{F_z(x, y, z)} = \frac{-x}{xe^{xz}} = \frac{-1}{e^{xz}} = -e^{-xz}$$

38.  $x \ln y + y^2 z + z^2 - 8 = 0$

(i)  $\frac{\partial z}{\partial x} = \frac{-F_x(x, y, z)}{F_z(x, y, z)} = \frac{-\ln y}{y^2 + 2z}$

(ii)  $\frac{\partial z}{\partial y} = \frac{-F_y(x, y, z)}{F_z(x, y, z)} = -\frac{\frac{x}{y} + 2yz}{y^2 + 2z} = -\frac{x + 2y^2 z}{y^3 + 2yz}$

39.  $F(x, y, z, w) = xy + yz - wz + wx - s$

$$F_x = y + w$$

$$F_y = x + z$$

$$F_z = y - w$$

$$F_w = -z + x$$

$$\frac{\partial w}{\partial x} = -\frac{F_x}{F_w} = -\frac{y+w}{-z+x} = \frac{y+w}{z-x}$$

$$\frac{\partial w}{\partial y} = -\frac{F_y}{F_w} = -\frac{x+z}{-z+x} = \frac{x+z}{z-x}$$

$$\frac{\partial w}{\partial z} = -\frac{F_z}{F_w} = -\frac{y-w}{-z+x} = \frac{y-w}{z-x}$$

40.  $x^2 + y^2 - z^2 - 5yw + 10w^2 - 2 = F(x, y, z, w)$

$$F_x = 2x, F_y = 2y - 5w, F_z = 2z, F_w = -5y + 20w$$

$$\frac{\partial w}{\partial x} = -\frac{F_x}{F_w} = \frac{-2x}{-5y+20w} = \frac{2x}{5y-20w}$$

$$\frac{\partial w}{\partial y} = -\frac{F_y}{F_w} = \frac{5w-2y}{20w-5y}$$

$$\frac{\partial w}{\partial z} = -\frac{F_z}{F_w} = \frac{2z}{5y-20w}$$

41.  $F(x, y, z, w) = \cos xy + \sin yz + wz - 20$

$$\frac{\partial w}{\partial x} = \frac{-F_x}{F_w} = \frac{y \sin xy}{z}$$

$$\frac{\partial w}{\partial y} = \frac{-F_y}{F_w} = \frac{x \sin xy - z \cos yz}{z}$$

$$\frac{\partial w}{\partial z} = \frac{-F_z}{F_w} = \frac{-y \cos yz + w}{z}$$

42.  $F(x, y, z, w) = w - \sqrt{x-y} - \sqrt{y-z} = 0$

$$\begin{aligned}\frac{\partial w}{\partial x} &= \frac{-F_x}{F_w} = \frac{1}{2} \frac{(x-y)^{-1/2}}{1} = \frac{1}{2\sqrt{x-y}} \\ \frac{\partial w}{\partial y} &= \frac{-F_y}{F_w} = \frac{-1}{2} (x-y)^{-1/2} + \frac{1}{2} (y-z)^{-1/2} = \frac{-1}{2\sqrt{x-y}} + \frac{1}{2\sqrt{y-z}} \\ \frac{\partial w}{\partial z} &= \frac{-F_z}{F_w} = \frac{-1}{2\sqrt{y-z}}\end{aligned}$$

43. (a)  $f(x, y) = \frac{xy}{\sqrt{x^2 + y^2}}$

$$f(tx, ty) = \frac{(tx)(ty)}{\sqrt{(tx)^2 + (ty)^2}} = t \left( \frac{xy}{\sqrt{x^2 + y^2}} \right) = tf(x, y)$$

Degree: 1

(b)  $xf_x(x, y) + yf_y(x, y) = x \left( \frac{y^3}{(x^2 + y^2)^{3/2}} \right) + y \left( \frac{x^3}{(x^2 + y^2)^{3/2}} \right) = \frac{xy}{\sqrt{x^2 + y^2}} = f(x, y)$

44. (a)  $f(x, y) = x^3 - 3xy^2 + y^3$

$$f(tx, ty) = (tx)^3 - 3(tx)(ty)^2 + (ty)^3 = t^3(x^3 - 3xy^2 + y^3) = t^3 f(x, y)$$

Degree: 3

(b)  $xf_x(x, y) + yf_y(x, y) = x(3x^2 - 3y^2) + y(-6xy + 3y^2) = 3x^3 - 9xy^2 + 3y^3 = 3f(x, y)$

45. (a)  $f(x, y) = e^{x/y}$

$$f(tx, ty) = e^{tx/ty} = e^{x/y} = f(x, y)$$

Degree: 0

(b)  $xf_x(x, y) + yf_y(x, y) = x \left( \frac{1}{y} e^{x/y} \right) + y \left( -\frac{x}{y^2} e^{x/y} \right) = 0$

46. (a)  $f(x, y) = \frac{x^2}{\sqrt{x^2 + y^2}}$

$$f(tx, ty) = \frac{(tx)^2}{\sqrt{(tx)^2 + (ty)^2}} = t \left( \frac{x^2}{\sqrt{x^2 + y^2}} \right) = tf(x, y)$$

Degree: 1

(b)  $xf_x(x, y) + yf_y(x, y) = x \left[ \frac{x^3 + 2xy^2}{(x^2 + y^2)^{3/2}} \right] + y \left[ \frac{-x^2y}{(x^2 + y^2)^{3/2}} \right] = \frac{x^4 + x^2y^2}{(x^2 + y^2)^{3/2}} = \frac{x^2(x^2 + y^2)}{(x^2 + y^2)^{3/2}} = \frac{x^2}{\sqrt{x^2 + y^2}} = f(x, y)$

47.  $\frac{dw}{dt} = \frac{\partial w}{\partial x} \frac{dx}{dt} + \frac{\partial w}{\partial y} \frac{dy}{dt} = \frac{\partial f}{\partial x} \frac{dg}{dt} + \frac{\partial f}{\partial y} \frac{dh}{dt}$

At  $t = 2$ ,  $x = 4$ ,  $y = 3$ ,  $f_x(4, 3) = -5$  and  $f_y(4, 3) = 7$ .

So,  $\frac{dw}{dt} = (-5)(-1) + (7)(6) = 47$

48.  $\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial s}$

$$= \frac{\partial f}{\partial x} \frac{\partial g}{\partial s} + \frac{\partial f}{\partial y} \frac{\partial h}{\partial s} = (-5)(-3) + (7)(5) = 50$$

$$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial t}$$

$$= \frac{\partial f}{\partial x} \frac{\partial g}{\partial t} + \frac{\partial f}{\partial y} \frac{\partial h}{\partial t} = (-5)(-2) + (7)(8) = 66$$

49.  $\frac{dw}{dt} = \frac{\partial w}{\partial x} \frac{dx}{dt} + \frac{\partial w}{\partial y} \frac{dy}{dt}$  (Page 925)

50.  $\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial s}$

$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial t}$  (Page 925)

51.  $\frac{dy}{dx} = -\frac{f_x(x, y)}{f_y(x, y)}$

$\frac{\partial z}{\partial x} = -\frac{f_x(x, y, z)}{f_z(x, y, z)}$

$\frac{\partial z}{\partial y} = -\frac{f_y(x, y, z)}{f_z(x, y, z)}$

53.  $V = \pi r^2 h$

$$\frac{dV}{dt} = \pi \left( 2rh \frac{dr}{dt} + r^2 \frac{dh}{dt} \right) = \pi r \left( 2h \frac{dr}{dt} + r \frac{dh}{dt} \right) = \pi (12) [2(36)(6) + 12(-4)] = 4608\pi \text{ in.}^3/\text{min}$$

$S = 2\pi r(r + h)$

$$\frac{dS}{dt} = 2\pi \left[ (2r + h) \frac{dr}{dt} + r \frac{dh}{dt} \right] = 2\pi [(24 + 36)(6) + 12(-4)] = 624\pi \text{ in.}^2/\text{min}$$

54.  $V = \frac{1}{3}\pi r^2 h$

$$\frac{dV}{dt} = \frac{1}{3}\pi \left( 2rh \frac{dr}{dt} + r^2 \frac{dh}{dt} \right) = \frac{1}{3}\pi [2(12)(36)(6) + (12)^2(-4)] = 1536\pi \text{ in.}^3/\text{min}$$

$S = \pi r \sqrt{r^2 + h^2} + \pi r^2$  (Surface area includes base.)

$$\begin{aligned} \frac{dS}{dt} &= \pi \left[ \left( \sqrt{r^2 + h^2} + \frac{r^2}{\sqrt{r^2 + h^2}} + 2r \right) \frac{dr}{dt} + \frac{rh}{\sqrt{r^2 + h^2}} \frac{dh}{dt} \right] \\ &= \pi \left[ \left( \sqrt{12^2 + 36^2} + \frac{144}{\sqrt{12^2 + 36^2}} + 2(12) \right) (6) + \frac{36(12)}{\sqrt{12^2 + 36^2}} (-4) \right] \\ &= \pi \left[ \left( 12\sqrt{10} + \frac{12}{\sqrt{10}} \right) (6) + 144 + \frac{36}{\sqrt{10}} (-4) \right] = \frac{648\pi}{\sqrt{10}} + 144\pi \text{ in.}^2/\text{min} = \frac{36\pi}{5} (20 + 9\sqrt{10}) \text{ in.}^2/\text{min} \end{aligned}$$

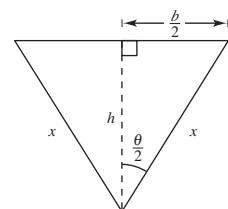
55.  $pV = mRT$

$$T = \frac{1}{mR}(pV)$$

$$\frac{dT}{dt} = \frac{1}{mR} \left[ V \frac{dp}{dt} + p \frac{dV}{dt} \right]$$

56.  $A = \frac{1}{2}bh = \left( x \sin \frac{\theta}{2} \right) \left( x \cos \frac{\theta}{2} \right) = \frac{x^2}{2} \sin \theta$

$$\begin{aligned} \frac{dA}{dt} &= \frac{\partial A}{\partial x} \frac{dx}{dt} + \frac{\partial A}{\partial \theta} \frac{d\theta}{dt} = x \sin \theta \frac{dx}{dt} + \frac{x^2}{2} \cos \theta \frac{d\theta}{dt} \\ &= 6 \left( \sin \frac{\pi}{4} \right) \left( \frac{1}{2} \right) + \frac{6^2}{2} \left( \cos \frac{\pi}{4} \right) \left( \frac{\pi}{90} \right) = \frac{3\sqrt{2}}{2} + \frac{\pi\sqrt{2}}{10} \text{ m}^2/\text{hr} \approx 2.566 \text{ m}^2/\text{h} \end{aligned}$$



57.  $I = \frac{1}{2}m(r_1^2 + r_2^2)$   
 $\frac{dI}{dt} = \frac{1}{2}m\left[2r_1\frac{dr_1}{dt} + 2r_2\frac{dr_2}{dt}\right] = m[(6)(2) + (8)(2)] = 28m \text{ cm}^2/\text{sec}$

58.  $V = \frac{\pi}{3}(r^2 + rR + R^2)h$   
 $\frac{dV}{dt} = \frac{\pi}{3}\left[(2r + R)h\frac{dr}{dt} + (r + 2R)h\frac{dR}{dt} + (r^2 + rR + R^2)\frac{dh}{dt}\right]$   
 $= \frac{\pi}{3}\left[2(15) + 25\right](10)(4) + \left[15 + 2(25)\right](10)(4) + \left[(15)^2 + (15)(25) + (25)^2\right](12)$

$$= \frac{\pi}{3}(19,500)$$

$$= 6,500\pi \text{ cm}^3/\text{min}$$

$$S = \pi(R + r)\sqrt{(R - r)^2 + h^2}$$

$$\frac{dS}{dt} = \pi\left\{\left[\sqrt{(R - r)^2 + h^2} - (R + r)\frac{(R - r)}{\sqrt{(R - r)^2 + h^2}}\right]\frac{dr}{dt} + \left[\sqrt{(R - r)^2 + h^2} + (R + r)\frac{(R - r)}{\sqrt{(R - r)^2 + h^2}}\right]\frac{dR}{dt}\right.$$

$$\left.+ (R + r)\frac{h}{\sqrt{(R - r)^2 + h^2}}\frac{dh}{dt}\right\}$$

$$= \pi\left\{\left[\sqrt{(25 - 15)^2 + 10^2} - (25 + 15)\frac{25 - 15}{\sqrt{(25 - 15)^2 + 10^2}}\right](4)\right.$$

$$\left.+\left[\sqrt{(25 - 15)^2 + 10^2} + (25 + 15)\frac{25 - 15}{\sqrt{(25 - 15)^2 + 10^2}}\right](4) + (25 + 15)\left[\frac{10}{\sqrt{(25 - 15)^2 + 10^2}}(12)\right]\right\}$$

$$= 320\sqrt{2}\pi \text{ cm}^2/\text{min}$$

59.  $w = f(x, y)$   
 $x = u - v$   
 $y = v - u$   
 $\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}\frac{dx}{du} + \frac{\partial w}{\partial y}\frac{dy}{du} = \frac{\partial w}{\partial x} - \frac{\partial w}{\partial y}$   
 $\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x}\frac{dx}{dv} + \frac{\partial w}{\partial y}\frac{dy}{dv} = -\frac{\partial w}{\partial x} + \frac{\partial w}{\partial y}$   
 $\frac{\partial w}{\partial u} + \frac{\partial w}{\partial v} = 0$

60.  $w = (x - y)\sin(y - x)$   
 $\frac{\partial w}{\partial x} = -(x - y)\cos(y - x) + \sin(y - x)$   
 $\frac{\partial w}{\partial y} = (x - y)\cos(y - x) - \sin(y - x)$   
 $\frac{\partial w}{\partial x} + \frac{\partial w}{\partial y} = 0$

61.  $w = f(x, y)$ ,  $x = r \cos \theta$ ,  $y = r \sin \theta$

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x} \cos \theta + \frac{\partial w}{\partial y} \sin \theta$$

$$\frac{\partial w}{\partial \theta} = \frac{\partial w}{\partial x}(-r \sin \theta) + \frac{\partial w}{\partial y}(r \cos \theta)$$

$$(a) \quad r \cos \theta \frac{\partial w}{\partial r} = \frac{\partial w}{\partial x} r \cos^2 \theta + \frac{\partial w}{\partial y} r \sin \theta \cos \theta$$

$$-\sin \theta \frac{\partial w}{\partial \theta} = \frac{\partial w}{\partial x}(r \sin^2 \theta) - \frac{\partial w}{\partial y} r \sin \theta \cos \theta$$

$$r \cos \theta \frac{\partial w}{\partial r} - \sin \theta \frac{\partial w}{\partial \theta} = \frac{\partial w}{\partial x}(r \cos^2 \theta + r \sin^2 \theta)$$

$$r \frac{\partial w}{\partial x} = \frac{\partial w}{\partial r}(r \cos \theta) - \frac{\partial w}{\partial \theta} \sin \theta$$

$$\frac{\partial w}{\partial x} = \frac{\partial w}{\partial r} \cos \theta - \frac{\partial w}{\partial \theta} \frac{\sin \theta}{r} \quad (\text{First Formula})$$

$$r \sin \theta \frac{\partial w}{\partial r} = \frac{\partial w}{\partial x} r \sin \theta \cos \theta + \frac{\partial w}{\partial y} r \sin^2 \theta$$

$$\cos \theta \frac{\partial w}{\partial \theta} = \frac{\partial w}{\partial x}(-r \sin \theta \cos \theta) + \frac{\partial w}{\partial y}(r \cos^2 \theta)$$

$$r \sin \theta \frac{\partial w}{\partial r} + \cos \theta \frac{\partial w}{\partial \theta} = \frac{\partial w}{\partial y}(r \sin^2 \theta + r \cos^2 \theta)$$

$$r \frac{\partial w}{\partial y} = \frac{\partial w}{\partial r} r \sin \theta + \frac{\partial w}{\partial \theta} \cos \theta$$

$$\frac{\partial w}{\partial y} = \frac{\partial w}{\partial r} \sin \theta + \frac{\partial w}{\partial \theta} \frac{\cos \theta}{r} \quad (\text{Second Formula})$$

$$(b) \quad \left( \frac{\partial w}{\partial r} \right)^2 + \frac{1}{r^2} \left( \frac{\partial w}{\partial \theta} \right)^2 = \left( \frac{\partial w}{\partial x} \right)^2 \cos^2 \theta + 2 \frac{\partial w}{\partial x} \frac{\partial w}{\partial y} \sin \theta \cos \theta + \left( \frac{\partial w}{\partial y} \right)^2 \sin^2 \theta + \left( \frac{\partial w}{\partial x} \right)^2 \sin^2 \theta$$

$$- 2 \frac{\partial w}{\partial x} \frac{\partial w}{\partial y} \sin \theta \cos \theta + \left( \frac{\partial w}{\partial y} \right)^2 \cos^2 \theta = \left( \frac{\partial w}{\partial x} \right)^2 + \left( \frac{\partial w}{\partial y} \right)^2$$

62.  $w = \arctan \frac{y}{x}$ ,  $x = r \cos \theta$ ,  $y = r \sin \theta$

$$= \arctan \left( \frac{r \sin \theta}{r \cos \theta} \right) = \arctan(\tan \theta) = \theta \text{ for } -\frac{\pi}{2} < \theta < \frac{\pi}{2}$$

$$\frac{\partial w}{\partial x} = \frac{-y}{x^2 + y^2}, \quad \frac{\partial w}{\partial y} = \frac{x}{x^2 + y^2}, \quad \frac{\partial w}{\partial r} = 0, \quad \frac{\partial w}{\partial \theta} = 1$$

$$\left( \frac{\partial w}{\partial x} \right)^2 + \left( \frac{\partial w}{\partial y} \right)^2 = \frac{y^2}{(x^2 + y^2)^2} + \frac{x^2}{(x^2 + y^2)^2} = \frac{1}{x^2 + y^2} = \frac{1}{r^2}$$

$$\left( \frac{\partial w}{\partial r} \right)^2 + \left( \frac{1}{r^2} \right) \left( \frac{\partial w}{\partial \theta} \right)^2 = 0 + \frac{1}{r^2}(1) = \frac{1}{r^2}$$

$$\text{So, } \left( \frac{\partial w}{\partial x} \right)^2 + \left( \frac{\partial w}{\partial y} \right)^2 = \left( \frac{\partial w}{\partial r} \right)^2 + \frac{1}{r^2} \left( \frac{\partial w}{\partial \theta} \right)^2.$$

63. Given  $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$  and  $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ ,  $x = r \cos \theta$  and  $y = r \sin \theta$ .

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x} \cos \theta + \frac{\partial u}{\partial y} \sin \theta = \frac{\partial v}{\partial y} \cos \theta - \frac{\partial v}{\partial x} \sin \theta$$

$$\frac{\partial v}{\partial \theta} = \frac{\partial v}{\partial x}(-r \sin \theta) + \frac{\partial v}{\partial y}(r \cos \theta) = r \left[ \frac{\partial v}{\partial y} \cos \theta - \frac{\partial v}{\partial x} \sin \theta \right]$$

$$\text{So, } \frac{\partial u}{\partial r} = \frac{1}{r} \frac{\partial v}{\partial \theta}.$$

$$\frac{\partial v}{\partial r} = \frac{\partial v}{\partial x} \cos \theta + \frac{\partial v}{\partial y} \sin \theta = -\frac{\partial u}{\partial y} \cos \theta + \frac{\partial u}{\partial x} \sin \theta$$

$$\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}(-r \sin \theta) + \frac{\partial u}{\partial y}(r \cos \theta) = -r \left[ -\frac{\partial u}{\partial y} \cos \theta + \frac{\partial u}{\partial x} \sin \theta \right]$$

$$\text{So, } \frac{\partial v}{\partial r} = -\frac{1}{r} \frac{\partial u}{\partial \theta}.$$

64. Note first that

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = \frac{x}{x^2 + y^2}$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} = \frac{y}{x^2 + y^2}.$$

$$\frac{\partial u}{\partial r} = \frac{x}{x^2 + y^2} \cos \theta + \frac{y}{x^2 + y^2} \sin \theta = \frac{r \cos^2 \theta + r \sin^2 \theta}{r^2} = \frac{1}{r}$$

$$\frac{\partial v}{\partial \theta} = \frac{-y}{x^2 + y^2}(-r \sin \theta) + \frac{x}{x^2 + y^2}(r \cos \theta) = \frac{r^2 \sin^2 \theta + r^2 \cos^2 \theta}{r^2} = 1$$

$$\text{So, } \frac{\partial u}{\partial r} = \frac{1}{r} \frac{\partial v}{\partial \theta}.$$

$$\frac{\partial v}{\partial r} = \frac{-y}{x^2 + y^2} \cos \theta + \frac{x}{x^2 + y^2} \sin \theta = \frac{-r \sin \theta \cos \theta + r \sin \theta \cos \theta}{r^2} = 0$$

$$\frac{\partial u}{\partial \theta} = \frac{x}{x^2 + y^2}(-r \sin \theta) + \frac{y}{x^2 + y^2}(r \cos \theta) = \frac{-r^2 \sin \theta \cos \theta + r^2 \sin \theta \cos \theta}{r^2} = 0$$

$$\text{So, } \frac{\partial v}{\partial r} = -\frac{1}{r} \frac{\partial u}{\partial \theta}.$$

## Section 13.6 Directional Derivatives and Gradients

1.  $f(x, y) = 3x - 4xy + 9y$

$$\mathbf{v} = \frac{3}{5}\mathbf{i} + \frac{4}{5}\mathbf{j}$$

$$\nabla f(x, y) = (3 - 4y)\mathbf{i} + (9 - 4x)\mathbf{j}$$

$$\nabla f(1, 2) = -5\mathbf{i} + 5\mathbf{j}$$

$$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{3}{5}\mathbf{i} + \frac{4}{5}\mathbf{j}$$

$$D_{\mathbf{u}} f(1, 2) = \nabla f(1, 2) \cdot \mathbf{u} = -3 + 4 = 1$$

2.  $f(x, y) = x^3 - y^3$ ,  $\mathbf{v} = \frac{\sqrt{2}}{2}(\mathbf{i} + \mathbf{j})$

$$\nabla f(x, y) = 3x^2\mathbf{i} - 3y^2\mathbf{j}$$

$$\nabla f(4, 3) = 48\mathbf{i} - 27\mathbf{j}$$

$$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{\sqrt{2}}{2}\mathbf{i} + \frac{\sqrt{2}}{2}\mathbf{j}$$

$$D_{\mathbf{u}} f(4, 3) = \nabla f(4, 3) \cdot \mathbf{u}$$

$$= 24\sqrt{2} - \frac{27}{2}\sqrt{2} = \frac{21}{2}\sqrt{2}$$

3.  $f(x, y) = xy$

$$\mathbf{v} = \frac{1}{2}(\mathbf{i} + \sqrt{3}\mathbf{j})$$

$$\nabla f(x, y) = y\mathbf{i} + x\mathbf{j}$$

$$\nabla f(0, -2) = -2\mathbf{i}$$

$$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{1}{2}\mathbf{i} + \frac{\sqrt{3}}{2}\mathbf{j}$$

$$D_{\mathbf{u}}f(0, -2) = \nabla f(0, -2) \cdot \mathbf{u} = -1$$

4.  $f(x, y) = \frac{x}{y}$

$$\mathbf{v} = -\mathbf{j}$$

$$\nabla f(x, y) = \frac{1}{y}\mathbf{i} - \frac{x}{y^2}\mathbf{j}$$

$$\nabla f(1, 1) = \mathbf{i} - \mathbf{j}$$

$$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = -\mathbf{j}$$

$$D_{\mathbf{u}}f(1, 1) = \nabla f(1, 1) \cdot \mathbf{u} = 1$$

5.  $h(x, y) = e^x \sin y$

$$\mathbf{v} = -\mathbf{i}$$

$$\nabla h = e^x \sin y\mathbf{i} + e^x \cos y\mathbf{j}$$

$$\nabla h\left(1, \frac{\pi}{2}\right) = e\mathbf{i}$$

$$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = -\mathbf{i}$$

$$D_{\mathbf{u}}h\left(1, \frac{\pi}{2}\right) = \nabla h\left(1, \frac{\pi}{2}\right) \cdot \mathbf{u} = -e$$

6.  $g(x, y) = \arccos xy$

$$\mathbf{v} = \mathbf{j}$$

$$\nabla g(x, y) = \frac{-y}{\sqrt{1 - (xy)^2}}\mathbf{i} + \frac{-x}{\sqrt{1 - (xy)^2}}\mathbf{j}$$

$$\nabla g(1, 0) = -\mathbf{j}$$

$$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \mathbf{j}$$

$$D_{\mathbf{u}}g(1, 0) = \nabla g(1, 0) \cdot \mathbf{u} = -1$$

7.  $g(x, y) = \sqrt{x^2 + y^2}$

$$\mathbf{v} = 3\mathbf{i} - 4\mathbf{j}$$

$$\nabla g = \frac{x}{\sqrt{x^2 + y^2}}\mathbf{i} + \frac{y}{\sqrt{x^2 + y^2}}\mathbf{j}$$

$$\nabla g(3, 4) = \frac{3}{5}\mathbf{i} + \frac{4}{5}\mathbf{j}$$

$$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{3}{5}\mathbf{i} - \frac{4}{5}\mathbf{j}$$

$$D_{\mathbf{u}}g(3, 4) = \nabla g(3, 4) \cdot \mathbf{u} = -\frac{7}{25}$$

8.  $h(x, y) = e^{-(x^2+y^2)}$

$$\mathbf{v} = \mathbf{i} + \mathbf{j}$$

$$\nabla h = -2xe^{-(x^2+y^2)}\mathbf{i} - 2ye^{-(x^2+y^2)}\mathbf{j}$$

$$\nabla h(0, 0) = \mathbf{0}$$

$$D_{\mathbf{u}}h(0, 0) = \nabla h(0, 0) \cdot \mathbf{u} = 0$$

9.  $f(x, y, z) = x^2 + y^2 + z^2$

$$\mathbf{v} = \frac{\sqrt{3}}{3}(\mathbf{i} - \mathbf{j} + \mathbf{k})$$

$$\nabla f(x, y, z) = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$$

$$\nabla f(1, 1, 1) = 2\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$$

$$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{\sqrt{3}}{3}\mathbf{i} - \frac{\sqrt{3}}{3}\mathbf{j} + \frac{\sqrt{3}}{3}\mathbf{k}$$

$$D_{\mathbf{u}}f(1, 1, 1) = \nabla f(1, 1, 1) \cdot \mathbf{u} = \frac{2}{3}\sqrt{3}$$

10.  $f(x, y, z) = xy + yz + xz$

$$\mathbf{v} = 2\mathbf{i} + \mathbf{j} - \mathbf{k}$$

$$\nabla f(x, y, z) = (y + z)\mathbf{i} + (x + z)\mathbf{j} + (y + x)\mathbf{k}$$

$$\nabla f(1, 2, -1) = \mathbf{i} + 3\mathbf{k}$$

$$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{1}{\sqrt{6}}(2\mathbf{i} + \mathbf{j} - \mathbf{k})$$

$$D_{\mathbf{u}}f(1, 2, -1) = \nabla f(1, 2, -1) \cdot \mathbf{u}$$

$$= \frac{2}{\sqrt{6}} - \frac{3}{\sqrt{6}} = \frac{-\sqrt{6}}{6}$$

11.  $h(x, y, z) = xyz$

$$\mathbf{v} = \langle 2, 1, 2 \rangle$$

$$\nabla h = yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}$$

$$\nabla h(2, 1, 1) = \mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$$

$$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{2}{3}\mathbf{i} + \frac{1}{3}\mathbf{j} + \frac{2}{3}\mathbf{k}$$

$$D_{\mathbf{u}}h(2, 1, 1) = \nabla h(2, 1, 1) \cdot \mathbf{u} = \frac{8}{3}$$

12.  $h(x, y, z) = x \arctan yz$

$$\mathbf{v} = \langle 1, 2, -1 \rangle$$

$$\nabla h(x, y, z) = \arctan yz\mathbf{i} + \frac{xz}{1 + (yz)^2}\mathbf{j} + \frac{xy}{1 + (yz)^2}\mathbf{k}$$

$$\nabla h(4, 1, 1) = \frac{\pi}{4}\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$$

$$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \left\langle \frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, -\frac{1}{\sqrt{6}} \right\rangle$$

$$D_{\mathbf{u}}h(4, 1, 1) = \nabla h(4, 1, 1) \cdot \mathbf{u} = \frac{\pi + 8}{4\sqrt{6}} = \frac{(\pi + 8)\sqrt{6}}{24}$$

13.  $f(x, y) = x^2 + y^2$

$$\mathbf{u} = \frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j}$$

$$\nabla f = 2x\mathbf{i} + 2y\mathbf{j}$$

$$D_{\mathbf{u}}f = \nabla f \cdot \mathbf{u} = \frac{2}{\sqrt{2}}x + \frac{2}{\sqrt{2}}y = \sqrt{2}(x + y)$$

14.  $f(x, y) = \frac{y}{x + y}$

$$\mathbf{u} = \frac{\sqrt{3}}{2}\mathbf{i} - \frac{1}{2}\mathbf{j}$$

$$\nabla f = -\frac{y}{(x + y)^2}\mathbf{i} + \frac{x}{(x + y)^2}\mathbf{j}$$

$$\begin{aligned} D_{\mathbf{u}}f &= \nabla f \cdot \mathbf{u} = -\frac{\sqrt{3}y}{2(x + y)^2} - \frac{x}{2(x + y)^2} \\ &= -\frac{1}{2(x + y)^2}(\sqrt{3}y + x) \end{aligned}$$

15.  $f(x, y) = \sin(2x + y)$

$$u = \frac{1}{2}\mathbf{i} + \frac{\sqrt{3}}{2}\mathbf{j}$$

$$\nabla f = 2 \cos(2x + y)\mathbf{i} + \cos(2x + y)\mathbf{j}$$

$$\begin{aligned} D_{\mathbf{u}}f &= \nabla f \cdot \mathbf{u} = \cos(2x + y) + \frac{\sqrt{3}}{2} \cos(2x + y) \\ &= \left(\frac{2 + \sqrt{3}}{2}\right) \cos(2x + y) \end{aligned}$$

16.  $g(x, y) = xe^y$

$$\mathbf{u} = -\frac{1}{2}\mathbf{i} + \frac{\sqrt{3}}{2}\mathbf{j}$$

$$\nabla g = e^y\mathbf{i} + xe^y\mathbf{j}$$

$$D_{\mathbf{u}}g = -\frac{1}{2}e^y + \frac{\sqrt{3}}{2}xe^y = \frac{e^y}{2}(\sqrt{3}x - 1)$$

17.  $f(x, y) = x^2 + 3y^2$

$$\mathbf{v} = 3\mathbf{i} + 4\mathbf{j}$$

$$\nabla f = 2x\mathbf{i} + 6y\mathbf{j}, \quad \nabla f(1, 1) = 2\mathbf{i} + 6\mathbf{j}$$

$$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{3}{5}\mathbf{i} + \frac{4}{5}\mathbf{j}$$

$$D_{\mathbf{u}}f(1, 1) = \nabla f(1, 1) \cdot \mathbf{u} = \frac{6}{5} + \frac{24}{5} = 6$$

18.  $f(x, y) = \cos(x + y)$

$$\mathbf{v} = \frac{\pi}{2}\mathbf{i} - \pi\mathbf{j}$$

$$\nabla f = -\sin(x + y)\mathbf{i} - \sin(x + y)\mathbf{j}$$

$$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{1}{\sqrt{5}}\mathbf{i} - \frac{2}{\sqrt{5}}\mathbf{j}$$

$$\begin{aligned} D_{\mathbf{u}}f &= -\frac{1}{\sqrt{5}}\sin(x + y) + \frac{2}{\sqrt{5}}\sin(x + y) \\ &= \frac{1}{\sqrt{5}}\sin(x + y) = \frac{\sqrt{5}}{5}\sin(x + y) \end{aligned}$$

At  $(0, \pi)$ ,  $D_{\mathbf{u}}f = 0$ .

19.  $g(x, y, z) = xye^z$

$$\mathbf{v} = -2\mathbf{i} - 4\mathbf{j}$$

$$\nabla g = ye^z\mathbf{i} + xe^z\mathbf{j} + xye^z\mathbf{k}$$

At  $(2, 4, 0)$ ,  $\nabla g = 4\mathbf{i} + 2\mathbf{j} + 8\mathbf{k}$ .

$$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = -\frac{1}{\sqrt{5}}\mathbf{i} - \frac{2}{\sqrt{5}}\mathbf{j}$$

$$D_{\mathbf{u}}g = \nabla g \cdot \mathbf{u} = -\frac{4}{\sqrt{5}} - \frac{4}{\sqrt{5}} = -\frac{8}{\sqrt{5}}$$

20.  $h(x, y, z) = \ln(x + y + z)$

$$\mathbf{v} = 3\mathbf{i} + 3\mathbf{j} + \mathbf{k}$$

$$\nabla h = \frac{1}{x + y + z}(\mathbf{i} + \mathbf{j} + \mathbf{k})$$

At  $(1, 0, 0)$ ,  $\nabla h = \mathbf{i} + \mathbf{j} + \mathbf{k}$ .

$$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{1}{\sqrt{19}}(3\mathbf{i} + 3\mathbf{j} + \mathbf{k})$$

$$D_{\mathbf{u}}h = \nabla h \cdot \mathbf{u} = \frac{7}{\sqrt{19}} = \frac{7\sqrt{19}}{19}$$

21.  $f(x, y) = 3x + 5y^2 + 1$

$$\nabla f(x, y) = 3\mathbf{i} + 10y\mathbf{j}$$

$$\nabla f(2, 1) = 3\mathbf{i} + 10\mathbf{j}$$

22.  $g(x, y) = 2xe^{y/x}$

$$\nabla g(x, y) = \left(-\frac{2y}{x}e^{y/x} + 2e^{y/x}\right)\mathbf{i} + 2e^{y/x}\mathbf{j}$$

$$\nabla g(2, 0) = 2\mathbf{i} + 2\mathbf{j}$$

23.  $z = \ln(x^2 - y)$

$$\nabla z(x, y) = \frac{2x}{x^2 - y}\mathbf{i} - \frac{1}{x^2 - y}\mathbf{j}$$

$$\nabla z(2, 3) = 4\mathbf{i} - \mathbf{j}$$

24.  $z = \cos(x^2 + y^2)$

$$\nabla z(x, y) = -2x \sin(x^2 + y^2)\mathbf{i} - 2y \sin(x^2 + y^2)\mathbf{j}$$

$$\nabla z(3, -4) = -6 \sin 25\mathbf{i} + 8 \sin 25\mathbf{j} \approx 0.7941\mathbf{i} - 1.0588\mathbf{j}$$

25.  $w = 3x^2 - 5y^2 + 2z^2$

$$\nabla w(x, y, z) = 6x\mathbf{i} - 10y\mathbf{j} + 4z\mathbf{k}$$

$$\nabla w(1, 1, -2) = 6\mathbf{i} - 10\mathbf{j} - 8\mathbf{k}$$

26.  $w = x \tan(y + z)$

$$\nabla w(x, y, z) = \tan(y + z)\mathbf{i} + x \sec^2(y + z)\mathbf{j}$$

$$+ x \sec^2(y + z)\mathbf{k}$$

$$\nabla w(4, 3, -1) = \tan 2\mathbf{i} + 4 \sec^2 2\mathbf{j} + 4 \sec^2 2\mathbf{k}$$

27.  $\overrightarrow{PQ} = \mathbf{i} + \mathbf{j}$ ,  $\mathbf{u} = \frac{\sqrt{2}}{2}\mathbf{i} + \frac{\sqrt{2}}{2}\mathbf{j}$

$$\nabla g(x, y) = 2x\mathbf{i} + 2y\mathbf{j}$$

$$\nabla g(1, 2) = 2\mathbf{i} + 4\mathbf{j}$$

$$D_{\mathbf{u}} g = \nabla g \cdot \mathbf{u} = \sqrt{2} + 2\sqrt{2} = 3\sqrt{2}$$

28.  $\overrightarrow{PQ} = 4\mathbf{i} + 2\mathbf{j}$ ,  $\mathbf{u} = \frac{2}{\sqrt{5}}\mathbf{i} + \frac{1}{\sqrt{5}}\mathbf{j}$

$$\nabla f = 6x\mathbf{i} - 2y\mathbf{j}$$

$$\nabla f(-1, 4) = -6\mathbf{i} - 8\mathbf{j}$$

$$D_{\mathbf{u}} f = \nabla f \cdot \mathbf{u} = -\frac{12}{\sqrt{5}} - \frac{8}{\sqrt{5}} = -4\sqrt{5}$$

29.  $\overrightarrow{PQ} = 2\mathbf{i} + \mathbf{j}$ ,  $\mathbf{u} = \frac{2}{\sqrt{5}}\mathbf{i} + \frac{1}{\sqrt{5}}\mathbf{j}$

$$\nabla f = e^y \cos x\mathbf{i} + e^y \sin x\mathbf{j}$$

$$\nabla f(0, 0) = \mathbf{i}$$

$$D_{\mathbf{u}} f = \nabla f \cdot \mathbf{u} = \frac{2}{\sqrt{5}} = \frac{2\sqrt{5}}{5}$$

30.  $\overrightarrow{PQ} = -\frac{\pi}{2}\mathbf{i} + \pi\mathbf{j}$ ,  $\mathbf{u} = -\frac{1}{\sqrt{5}}\mathbf{i} + \frac{2}{\sqrt{5}}\mathbf{j}$

$$\nabla f = 2 \cos 2x \cos y\mathbf{i} - \sin 2x \sin y\mathbf{j}$$

$$\nabla f(\pi, 0) = 2\mathbf{i}$$

$$D_{\mathbf{u}} f = \nabla f \cdot \mathbf{u} = -\frac{2}{\sqrt{5}} = -\frac{2\sqrt{5}}{5}$$

31.  $f(x, y) = x^2 + 2xy$

$$\nabla f(x, y) = (2x + 2y)\mathbf{i} + 2x\mathbf{j}$$

$$\nabla f(1, 0) = 2\mathbf{i} + 2\mathbf{j}$$

$$\|\nabla f(1, 0)\| = 2\sqrt{2}$$

32.  $f(x, y) = \frac{x+y}{y+1}$

$$\nabla f(x, y) = \frac{1}{y+1}\mathbf{i} + \frac{1-x}{(y+1)^2}\mathbf{j}$$

$$\nabla f(0, 1) = \frac{1}{2}\mathbf{i} + \frac{1}{4}\mathbf{j}$$

$$\|\nabla f(0, 1)\| = \sqrt{\frac{1}{4} + \frac{1}{16}} = \frac{1}{4}\sqrt{5}$$

33.  $h(x, y) = x \tan y$

$$\nabla h(x, y) = \tan y\mathbf{i} + x \sec^2 y\mathbf{j}$$

$$\nabla h\left(2, \frac{\pi}{4}\right) = \mathbf{i} + 4\mathbf{j}$$

$$\|\nabla h\left(2, \frac{\pi}{4}\right)\| = \sqrt{17}$$

34.  $h(x, y) = y \cos(x - y)$

$$\nabla h(x, y) = -y \sin(x - y)\mathbf{i}$$

$$+ [\cos(x - y) + y \sin(x - y)]\mathbf{j}$$

$$\nabla h\left(0, \frac{\pi}{3}\right) = \frac{\sqrt{3}\pi}{6}\mathbf{i} + \left(\frac{3 - \sqrt{3}\pi}{6}\right)\mathbf{j}$$

$$\|\nabla h\left(0, \frac{\pi}{3}\right)\| = \sqrt{\frac{3\pi^2}{36} + \frac{9 - 6\sqrt{3}\pi + 3\pi^2}{36}}$$

$$= \frac{\sqrt{3(2\pi^2 - 2\sqrt{3}\pi + 3)}}{6}$$

35.  $g(x, y) = ye^{-x}$

$$\nabla g(x, y) = -ye^{-x}\mathbf{i} + e^{-x}\mathbf{j}$$

$$\nabla g(0, 5) = -5\mathbf{i} + \mathbf{j}$$

$$\|\nabla g(0, 5)\| = \sqrt{26}$$

36.  $g(x, y) = \ln \sqrt[3]{x^2 + y^2} = \frac{1}{3} \ln(x^2 + y^2)$

$$\nabla g(x, y) = \frac{1}{3} \left[ \frac{2x}{x^2 + y^2}\mathbf{i} + \frac{2y}{x^2 + y^2}\mathbf{j} \right]$$

$$\nabla g(1, 2) = \frac{1}{3} \left( \frac{2}{5}\mathbf{i} + \frac{4}{5}\mathbf{j} \right) = \frac{2}{15}(\mathbf{i} + 2\mathbf{j})$$

$$\|\nabla g(1, 2)\| = \frac{2\sqrt{5}}{15}$$

37.  $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$

$$\nabla f(x, y, z) = \frac{1}{\sqrt{x^2 + y^2 + z^2}}(x\mathbf{i} + y\mathbf{j} + z\mathbf{k})$$

$$\nabla f(1, 4, 2) = \frac{1}{\sqrt{21}}(\mathbf{i} + 4\mathbf{j} + 2\mathbf{k})$$

$$\|\nabla f(1, 4, 2)\| = 1$$

38.  $w = \frac{1}{\sqrt{1-x^2-y^2-z^2}}$

$$\nabla w = \frac{1}{(\sqrt{1-x^2-y^2-z^2})^3} (x\mathbf{i} + y\mathbf{j} + z\mathbf{k})$$

$$\nabla w(0,0,0) = \mathbf{0}$$

$$\|\nabla w(0,0,0)\| = 0$$

39.  $w = xy^2z^2$

$$\nabla w = y^2z^2\mathbf{i} + 2xyz^2\mathbf{j} + 2xy^2z\mathbf{k}$$

$$\nabla w(2,1,1) = \mathbf{i} + 4\mathbf{j} + 4\mathbf{k}$$

$$\|\nabla w(2,1,1)\| = \sqrt{33}$$

40.  $f(x, y, z) = xe^{yz}$

$$\nabla f(x, y, z) = e^{yz}\mathbf{i} + xze^{yz}\mathbf{j} + xye^{yz}\mathbf{k}$$

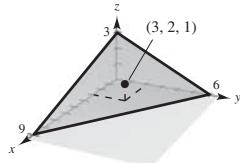
$$\nabla f(2, 0, -4) = \mathbf{i} - 8\mathbf{j}$$

$$\|\nabla f(2, 0, -4)\| = \sqrt{65}$$

For exercises 41–46,  $f(x, y) = 3 - \frac{x}{3} - \frac{y}{2}$  and

$$D_\theta f(x, y) = -\left(\frac{1}{3}\right)\cos\theta - \left(\frac{1}{2}\right)\sin\theta.$$

41.  $f(x, y) = 3 - \frac{x}{3} - \frac{y}{2}$



42. (a)  $D_{\pi/4}f(3, 2) = -\left(\frac{1}{3}\right)\frac{\sqrt{2}}{2} - \left(\frac{1}{2}\right)\frac{\sqrt{2}}{2} = -\frac{5\sqrt{2}}{12}$

(b)  $D_{2\pi/3}f(3, 2) = -\left(\frac{1}{3}\right)\left(-\frac{1}{2}\right) - \left(\frac{1}{2}\right)\frac{\sqrt{3}}{2} = \frac{2 - 3\sqrt{3}}{12}$

(c)  $D_{4\pi/3}f(3, 2) = -\left(\frac{1}{3}\right)\left(-\frac{1}{2}\right) - \left(\frac{1}{2}\right)\left(-\frac{\sqrt{3}}{2}\right)$

$$= \frac{2 + 3\sqrt{3}}{12}$$

(d)  $D_{-\pi/6}f(3, 2) = -\left(\frac{1}{3}\right)\left(\frac{\sqrt{3}}{2}\right) - \left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)$

$$= \frac{3 - 2\sqrt{3}}{12}$$

43. (a)  $\mathbf{u} = \left(\frac{1}{\sqrt{2}}\right)(\mathbf{i} + \mathbf{j})$

$$D_{\mathbf{u}}f = \nabla f \cdot \mathbf{u} = -\left(\frac{1}{3}\right)\frac{1}{\sqrt{2}} - \left(\frac{1}{2}\right)\frac{1}{\sqrt{2}} = -\frac{5\sqrt{2}}{12}$$

(b)  $\mathbf{v} = -3\mathbf{i} - 4\mathbf{j}$

$$\|\mathbf{v}\| = \sqrt{9 + 16} = 5$$

$$\mathbf{u} = -\frac{3}{5}\mathbf{i} - \frac{4}{5}\mathbf{j}$$

$$D_{\mathbf{u}}f = \nabla f \cdot \mathbf{u} = \frac{1}{5} + \frac{2}{5} = \frac{3}{5}$$

(c)  $\mathbf{v} = 3\mathbf{i} + 4\mathbf{j}$

$$\|\mathbf{v}\| = \sqrt{9 + 16} = 5$$

$$\mathbf{u} = -\frac{3}{5}\mathbf{i} + \frac{4}{5}\mathbf{j}$$

$$D_{\mathbf{u}}f = \nabla f \cdot \mathbf{u} = \frac{1}{5} - \frac{2}{5} = -\frac{1}{5}$$

(d)  $\mathbf{v} = \mathbf{i} + 3\mathbf{j}$

$$\|\mathbf{v}\| = \sqrt{10}$$

$$\mathbf{u} = \frac{1}{\sqrt{10}}\mathbf{i} + \frac{3}{\sqrt{10}}\mathbf{j}$$

$$D_{\mathbf{u}}f = \nabla f \cdot \mathbf{u} = \frac{-11}{6\sqrt{10}} = -\frac{11\sqrt{10}}{60}$$

44.  $\nabla f = -\left(\frac{1}{3}\right)\mathbf{i} - \left(\frac{1}{2}\right)\mathbf{j}$

45.  $\|\nabla f\| = \sqrt{\frac{1}{9} + \frac{1}{4}} = \frac{1}{6}\sqrt{13}$

46.  $\nabla f = -\frac{1}{3}\mathbf{i} - \frac{1}{2}\mathbf{j}$

$$\frac{\nabla f}{\|\nabla f\|} = \frac{1}{\sqrt{13}}(-2\mathbf{i} - 3\mathbf{j})$$

So,  $\mathbf{u} = (1/\sqrt{13})(3\mathbf{i} - 2\mathbf{j})$  and

$D_{\mathbf{u}}f(3, 2) = \nabla f \cdot \mathbf{u} = 0$ .  $\nabla f$  is the direction of greatest rate of change of  $f$ . So, in a direction orthogonal to  $\nabla f$ , the rate of change of  $f$  is 0.

47. (a) In the direction of the vector  $-4\mathbf{i} + \mathbf{j}$

(b)  $\nabla f = \frac{1}{10}(2x - 3y)\mathbf{i} + \frac{1}{10}(-3x + 2y)\mathbf{j}$

$$\nabla f(1, 2) = \frac{1}{10}(-4)\mathbf{i} + \frac{1}{10}(1)\mathbf{j} = -\frac{2}{5}\mathbf{i} + \frac{1}{10}\mathbf{j}$$

(Same direction as in part (a))

(c)  $-\nabla f = \frac{2}{5}\mathbf{i} - \frac{1}{10}\mathbf{j}$ , the direction opposite that of the gradient

48. (a) In the direction of the vector  $\mathbf{i} + \mathbf{j}$

$$(b) \nabla f = \frac{1}{2}y \frac{1}{2\sqrt{x}}\mathbf{i} + \frac{1}{2}\sqrt{x}\mathbf{j} = \frac{y}{4\sqrt{x}}\mathbf{i} + \frac{1}{2}\sqrt{x}\mathbf{j}$$

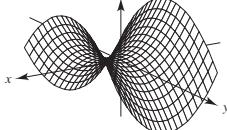
$$\nabla f(1, 2) = \frac{1}{2}\mathbf{i} + \frac{1}{2}\mathbf{j}$$

(Same direction as in part (a))

- (c)  $-\nabla f = -\frac{1}{2}\mathbf{i} - \frac{1}{2}\mathbf{j}$ , the direction opposite that of the gradient

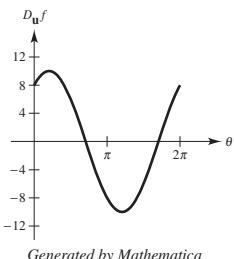
49.  $f(x, y) = x^2 - y^2, (4, -3, 7)$

(a)



(b)  $D_{\mathbf{u}} f(x, y) = \nabla f(x, y) \cdot \mathbf{u} = 2x \cos \theta - 2y \sin \theta$

$$D_{\mathbf{u}} f(4, -3) = 8 \cos \theta + 6 \sin \theta$$



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- (c) Zeros:  $\theta \approx 2.21, 5.36$

These are the angles  $\theta$  for which  $D_{\mathbf{u}} f(4, 3)$  equals zero.

(d)  $g(\theta) = D_{\mathbf{u}} f(4, -3) = 8 \cos \theta + 6 \sin \theta$

$$g'(\theta) = -8 \sin \theta + 6 \cos \theta$$

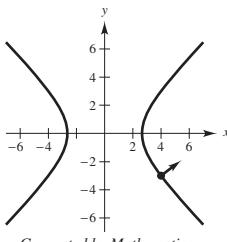
Critical numbers:  $\theta \approx 0.64, 3.79$

These are the angles for which  $D_{\mathbf{u}} f(4, -3)$  is a maximum (0.64) and minimum (3.79).

(e)  $\|\nabla f(4, -3)\| = \|2(4)\mathbf{i} - 2(-3)\mathbf{j}\| = \sqrt{64 + 36} = 10$ , the maximum value of  $D_{\mathbf{u}} f(4, -3)$ , at  $\theta \approx 0.64$ .

(f)  $f(x, y) = x^2 - y^2 = 7$

$\nabla f(4, -3) = 8\mathbf{i} + 6\mathbf{j}$  is perpendicular to the level curve at  $(4, -3)$ .



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50. (a)  $f(x, y) = \frac{8y}{1 + x^2 + y^2} = 2$

$$\Rightarrow 4y = 1 + x^2 + y^2$$

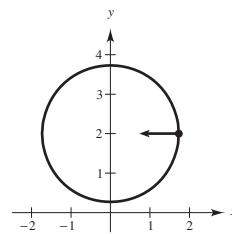
$$4 = y^2 - 4y + 4 + x^2 + 1$$

$$(y - 2)^2 + x^2 = 3$$

Circle: center:  $(0, 2)$ , radius:  $\sqrt{3}$

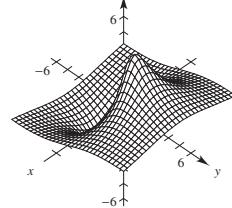
(b)  $\nabla f = \frac{-16xy}{(1 + x^2 + y^2)^2}\mathbf{i} + \frac{8 + 8x^2 - 8y^2}{(1 + x^2 + y^2)^2}\mathbf{j}$

$$\nabla f(\sqrt{3}, 2) = \frac{-\sqrt{3}}{2}\mathbf{i}$$



(c) The directional derivative of  $f$  is 0 in the direction  $\pm \mathbf{j}$ .

(d)



51.  $f(x, y) = 6 - 2x - 3y$

$$c = 6, P = (0, 0)$$

$$\nabla f(x, y) = -2\mathbf{i} - 3\mathbf{j}$$

$$6 - 2x - 3y = 6$$

$$0 = 2x + 3y$$

$$\nabla f(0, 0) = -2\mathbf{i} - 3\mathbf{j}$$

52.  $f(x, y) = x^2 + y^2$

$$c = 25, P = (3, 4)$$

$$\nabla f(x, y) = 2x\mathbf{i} + 2y\mathbf{j}$$

$$x^2 + y^2 = 25$$

$$\nabla f(3, 4) = 6\mathbf{i} + 8\mathbf{j}$$

53.  $f(x, y) = xy$

$$c = -3, P = (-1, 3)$$

$$\nabla f(x, y) = y\mathbf{i} + x\mathbf{j}$$

$$xy = -3$$

$$\nabla f(-1, 3) = 3\mathbf{i} - \mathbf{j}$$

54.  $f(x, y) = \frac{x}{x^2 + y^2}$

$$c = \frac{1}{2}, P = (1, 1)$$

$$\nabla f(x, y) = \frac{y^2 - x^2}{(x^2 + y^2)^2} \mathbf{i} - \frac{2xy}{(x^2 + y^2)^2} \mathbf{j}$$

$$\frac{x}{x^2 + y^2} = \frac{1}{2}$$

$$x^2 + y^2 - 2x = 0$$

$$\nabla f(1, 1) = -\frac{1}{2} \mathbf{j}$$

55.  $f(x, y) = 4x^2 - y$

(a)  $\nabla f(x, y) = 8x\mathbf{i} - \mathbf{j}$

$$\nabla f(2, 10) = 16\mathbf{i} - \mathbf{j}$$

(b)  $\|16\mathbf{i} - \mathbf{j}\| = \sqrt{257}$

$\frac{1}{\sqrt{257}}(16\mathbf{i} - \mathbf{j})$  is a unit vector normal to the level curve  $4x^2 - y = 6$  at  $(2, 10)$ .

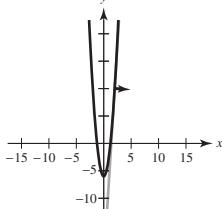
(c) The vector  $\mathbf{i} + 16\mathbf{j}$  is tangent to the level curve.

$$\text{Slope} = \frac{16}{1} = 16$$

$$y - 10 = 16(x - 2)$$

$y = 16x - 22$  Tangent line

(d)



56.  $f(x, y) = x - y^2$

(a)  $\nabla f(x, y) = \mathbf{i} - 2y\mathbf{j}$

$$\nabla f(4, -1) = \mathbf{i} + 2\mathbf{j}$$

(b)  $\|\nabla f(4, -1)\| = \sqrt{5}$

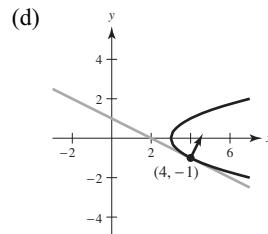
$\frac{1}{\sqrt{5}}(\mathbf{i} + 2\mathbf{j})$  is a unit vector normal to the level curve  $x - y^2 = 3$  at  $(4, -1)$ .

(c) The vector  $2\mathbf{i} - \mathbf{j}$  is tangent to the level curve.

$$\text{Slope} = -\frac{1}{2}$$

$$y + 1 = -\frac{1}{2}(x - 4)$$

$y = -\frac{1}{2}x + 1$  Tangent line



57.  $f(x, y) = 3x^2 - 2y^2$

(a)  $\nabla f = 6x\mathbf{i} - 4y\mathbf{j}$

$$\nabla f(1, 1) = 6\mathbf{i} - 4\mathbf{j}$$

(b)  $\|\nabla f(1, 1)\| = \sqrt{36 + 16} = 2\sqrt{13}$

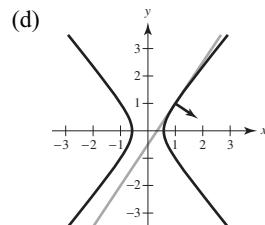
$\frac{1}{\sqrt{13}}(6\mathbf{i} - 4\mathbf{j})$  is a unit vector normal to the level curve  $3x^2 - 2y^2 = 1$  at  $(1, 1)$ .

(c) The vector  $2\mathbf{i} + 3\mathbf{j}$  is tangent to the level curve.

$$\text{Slope} = \frac{3}{2}$$

$$y - 1 = \frac{3}{2}(x - 1)$$

$$y = \frac{3}{2}x - \frac{1}{2} \quad \text{Tangent line}$$



58.  $f(x, y) = 9x^2 + 4y^2$

(a)  $\nabla f = 18x\mathbf{i} + 8y\mathbf{j}$

$$\nabla f(2, -1) = 36\mathbf{i} - 8\mathbf{j}$$

(b)  $\|\nabla f(2, -1)\| = \sqrt{1360} = 4\sqrt{85}$

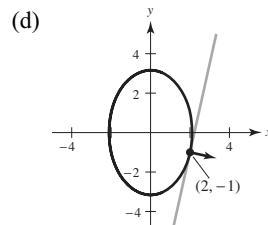
$\frac{1}{\sqrt{85}}(36\mathbf{i} - 8\mathbf{j})$  is a unit vector normal to the level curve  $9x^2 + 4y^2 = 40$  at  $(2, -1)$ .

(c) The vector  $2\mathbf{i} + 9\mathbf{j}$  is tangent to the level curve.

$$\text{Slope} = \frac{9}{2}$$

$$y + 1 = \frac{9}{2}(x - 2)$$

$$y = \frac{9}{2}x - 10 \quad \text{Tangent line}$$



59. See the definition, page 934.

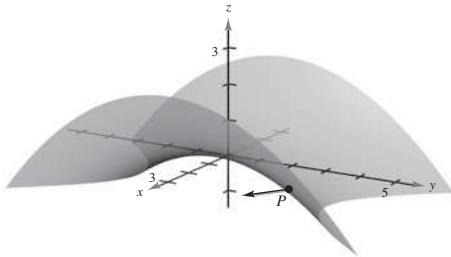
60. Let  $f(x, y)$  be a function of two variables and  $\mathbf{u} = \cos \theta \mathbf{i} + \sin \theta \mathbf{j}$  a unit vector.

(a) If  $\theta = 0^\circ$ , then  $D_{\mathbf{u}} f = \frac{\partial f}{\partial x}$ .

(b) If  $\theta = 90^\circ$ , then  $D_{\mathbf{u}} f = \frac{\partial f}{\partial y}$ .

61. See the definition, pages 936 and 937.

62.

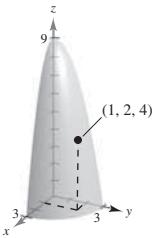


63. The gradient vector is normal to the level curves. See Theorem 13.12.

64.  $f(x, y) = 9 - x^2 - y^2$  and

$$\begin{aligned} D_\theta f(x, y) &= -2x \cos \theta - 2y \sin \theta \\ &= -2(x \cos \theta + y \sin \theta) \end{aligned}$$

(a)  $f(x, y) = 9 - x^2 - y^2$



(b)  $D_{-\pi/4} f(1, 2) = -2 \left( \frac{\sqrt{2}}{2} - \sqrt{2} \right) = \sqrt{2}$

(c)  $D_{\pi/3} f(1, 2) = -2 \left( \frac{1}{2} + \sqrt{3} \right) = -(1 + 2\sqrt{3})$

(d)  $\nabla f(1, 2) = -2\mathbf{i} - 4\mathbf{j}$

$\|\nabla f(1, 2)\| = \sqrt{4 + 16} = \sqrt{20} = 2\sqrt{5}$

(e)  $\nabla f(1, 2) = -2\mathbf{i} - 4\mathbf{j}$

$$\frac{\nabla f(1, 2)}{\|\nabla f(1, 2)\|} = \frac{1}{\sqrt{5}}(-2\mathbf{i} - 2\mathbf{j})$$

Therefore,  $\mathbf{u} = (1/\sqrt{5})(-2\mathbf{i} + \mathbf{j})$  and

$D_{\mathbf{u}} f(1, 2) = \nabla f(1, 2) \cdot \mathbf{u} = 0$ .

65.  $T = \frac{x}{x^2 + y^2}$

$$\nabla T = \frac{y^2 - x^2}{(x^2 + y^2)^2} \mathbf{i} - \frac{2xy}{(x^2 + y^2)^2} \mathbf{j}$$

$$\nabla T(3, 4) = \frac{7}{625} \mathbf{i} - \frac{24}{625} \mathbf{j} = \frac{1}{625}(7\mathbf{i} - 24\mathbf{j})$$

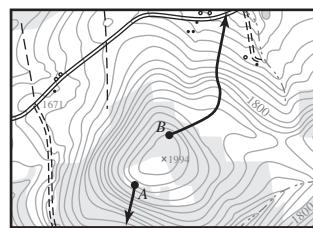
66.  $h(x, y) = 5000 - 0.001x^2 - 0.004y^2$

$\nabla h = -0.002x\mathbf{i} - 0.008y\mathbf{j}$

$\nabla h(500, 300) = -\mathbf{i} - 2.4\mathbf{j}$  or

$5\nabla h = -(5\mathbf{i} + 12\mathbf{j})$

67.



68. The wind speed is greatest at B.

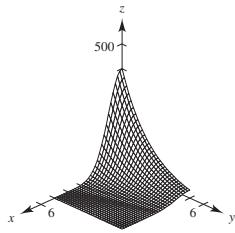
69.  $T(x, y) = 400 - 2x^2 - y^2$ ,  $P = (10, 10)$

$$\begin{aligned} \frac{dx}{dt} &= -4x & \frac{dy}{dt} &= -2y \\ x(t) &= C_1 e^{-4t} & y(t) &= C_2 e^{-2t} \\ 10 &= x(0) = C_1 & 10 &= y(0) = C_2 \\ x(t) &= 10e^{-4t} & y(t) &= 10e^{-2t} \\ x &= \frac{y^2}{10} & y^2(t) &= 100e^{-4t} \\ y^2 &= 10x & \end{aligned}$$

70.  $T(x, y) = 100 - x^2 - 2y^2$ ,  $P = (4, 3)$

$$\begin{aligned} \frac{dx}{dt} &= -2x & \frac{dy}{dt} &= -4y \\ x(t) &= C_1 e^{-2t} & y(t) &= C_2 e^{-4t} \\ 4 &= x(0) = C_1 & 3 &= y(0) = C_2 \\ x(t) &= 4e^{-2t} & y(t) &= 3e^{-4t} \\ \frac{3x^2}{16} &= e^{-4t} = y \Rightarrow u = \frac{3}{16}x^2 & \end{aligned}$$

71. (a)



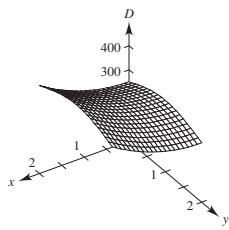
$$(b) \nabla T(x, y) = 400e^{-(x^2+y)/2} \left[ (-x)\mathbf{i} - \frac{1}{2}\mathbf{j} \right]$$

$$\nabla T(3, 5) = 400e^{-7} \left[ -3\mathbf{i} - \frac{1}{2}\mathbf{j} \right]$$

There will be no change in directions perpendicular to the gradient:  $\pm(\mathbf{i} - 6\mathbf{j})$

- (c) The greatest increase is in the direction of the gradient:  $-3\mathbf{i} - \frac{1}{2}\mathbf{j}$

72. (a)



- (b) The graph of  $-D = -250 - 30x^2 - 50 \sin(\pi y/2)$  would model the ocean floor.

$$(c) D(1, 0.5) = 250 + 30(1) + 50 \sin \frac{\pi}{4} \approx 315.4 \text{ ft}$$

$$(d) \frac{\partial D}{\partial x} = 60x \text{ and } \frac{\partial D}{\partial x}(1, 0.5) = 60$$

$$(e) \frac{\partial D}{\partial y} = 25\pi \cos \frac{\pi y}{2} \text{ and}$$

$$\frac{\partial D}{\partial y}(1, 0.5) = 25\pi \cos \frac{\pi}{4} \approx 55.5$$

$$(f) \nabla D = 60x\mathbf{i} + 25\pi \cos \left( \frac{\pi y}{2} \right) \mathbf{j}$$

$$\nabla D(1, 0.5) = 60\mathbf{i} + 55.5\mathbf{j}$$

73. True

74. False

$$D_{\mathbf{u}} f(x, y) = \sqrt{2} > 1 \text{ when } \mathbf{u} = \left( \cos \frac{\pi}{4} \right) \mathbf{i} + \left( \sin \frac{\pi}{4} \right) \mathbf{j}$$

75. True

76. True

77. Let  $f(x, y, z) = e^x \cos y + \frac{z^2}{2} + C$ . Then  $\nabla f(x, y, z) = e^x \cos y \mathbf{i} - e^x \sin y \mathbf{j} + z \mathbf{k}$ .

78. We cannot use Theorem 13.9 because  $f$  is not a differentiable function of  $x$  and  $y$ . So, we use the definition of directional derivatives.

$$D_{\mathbf{u}} f(x, y) = \lim_{t \rightarrow 0} \frac{f(x + t \cos \theta, y + t \sin \theta) - f(x, y)}{t}$$

$$D_{\mathbf{u}} f(0, 0) = \lim_{t \rightarrow 0} \frac{f\left[0 + \left(\frac{t}{\sqrt{2}}\right), 0 + \left(\frac{t}{\sqrt{2}}\right)\right] - f(0, 0)}{t} = \lim_{t \rightarrow 0} \frac{1}{t} \left[ \frac{4\left(\frac{t}{\sqrt{2}}\right)\left(\frac{t}{\sqrt{2}}\right)}{\left(\frac{t^2}{2}\right) + \left(\frac{t^2}{2}\right)} \right] = \lim_{t \rightarrow 0} \frac{1}{t} \left[ \frac{2t^2}{t^2} \right] = \lim_{t \rightarrow 0} \frac{2}{t} \text{ which does not exist.}$$

$$\text{If } f(0, 0) = 2, \text{ then } D_{\mathbf{u}} f(0, 0) = \lim_{t \rightarrow 0} \frac{f\left(0 + \frac{t}{\sqrt{2}}, 0 + \frac{t}{\sqrt{2}}\right) - 2}{t} = \lim_{t \rightarrow 0} \frac{1}{t} \left[ \frac{2t^2}{t^2} - 2 \right] = 0$$

which implies that the directional derivative exists.

79. (a)  $f(x, y) = \sqrt[3]{xy}$  is the composition of two continuous functions,  $h(x, y) = xy$  and  $g(z) = z^{1/3}$ , and therefore continuous by Theorem 13.2.

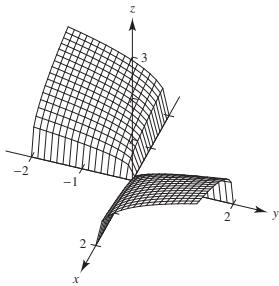
$$(b) f_x(0, 0) = \lim_{\Delta x \rightarrow 0} \frac{f(0 + \Delta x, 0) - f(0, 0)}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{(0 + \Delta x)^{1/3} - 0}{\Delta x} = 0$$

$$f_y(0, 0) = \lim_{\Delta y \rightarrow 0} \frac{f(0, 0 + \Delta y) - f(0, 0)}{\Delta y} = \lim_{\Delta y \rightarrow 0} \frac{(0 + \Delta y)^{1/3} - 0}{\Delta y} = 0$$

Let  $\mathbf{u} = \cos \theta \mathbf{i} + \sin \theta \mathbf{j}$ ,  $\theta \neq 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$ . Then

$$D_{\mathbf{u}} f(0, 0) = \lim_{t \rightarrow 0} \frac{f(0 + t \cos \theta, 0 + t \sin \theta) - f(0, 0)}{t} = \lim_{t \rightarrow 0} \frac{\sqrt[3]{t^2 \cos \theta \sin \theta}}{t} = \lim_{t \rightarrow 0} \frac{\sqrt[3]{\cos \theta \sin \theta}}{t^{1/3}}, \text{ does not exist.}$$

(c)



## Section 13.7 Tangent Planes and Normal Lines

$$1. F(x, y, z) = 3x - 5y + 3z - 15 = 0$$

$$3x - 5y + 3z = 15 \text{ Plane}$$

$$2. F(x, y, z) = x^2 + y^2 + z^2 - 25 = 0$$

$$x^2 + y^2 + z^2 = 25$$

Sphere, radius 5, centered at origin.

$$3. F(x, y, z) = 4x^2 + 9y^2 - 4z^2 = 0$$

$$4x^2 + 9y^2 = 4z^2 \text{ Elliptic cone}$$

$$4. F(x, y, z) = 16x^2 - 9y^2 + 36z = 0$$

$$16x^2 - 9y^2 + 36z = 0 \text{ Hyperbolic paraboloid}$$

$$5. F(x, y, z) = 3x + 4y + 12z = 0$$

$$\nabla F = 3\mathbf{i} + 4\mathbf{j} + 12\mathbf{k}, \|\nabla F\| = \sqrt{9 + 16 + 144} = 13$$

$$\mathbf{n} = \frac{\nabla F}{\|\nabla F\|} = \frac{3}{13}\mathbf{i} + \frac{4}{13}\mathbf{j} + \frac{12}{13}\mathbf{k}$$

$$6. F(x, y, z) = x + y + z - 4$$

$$\nabla F = \mathbf{i} + \mathbf{j} + \mathbf{k}$$

$$\mathbf{n} = \frac{\nabla F}{\|\nabla F\|} = \frac{1}{\sqrt{3}}(\mathbf{i} + \mathbf{j} + \mathbf{k}) = \frac{\sqrt{3}}{3}(\mathbf{i} + \mathbf{j} + \mathbf{k})$$

$$7. F(x, y, z) = x^2 + y^2 + z^2 - 6$$

$$\nabla F = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$$

$$\nabla F(1, 1, 2) = 2\mathbf{i} + 2\mathbf{j} + 4\mathbf{k}$$

$$\|\nabla F(1, 1, 2)\| = \sqrt{4 + 4 + 16} = 2\sqrt{6}$$

$$\mathbf{n} = \frac{\nabla F}{\|\nabla F\|} = \frac{1}{\sqrt{6}}\mathbf{i} + \frac{1}{\sqrt{6}}\mathbf{j} + \frac{2}{\sqrt{6}}\mathbf{k}$$

$$8. F(x, y, z) = \sqrt{x^2 + y^2} - z$$

$$\nabla F(x, y, z) = \frac{x}{\sqrt{x^2 + y^2}}\mathbf{i} + \frac{y}{\sqrt{x^2 + y^2}}\mathbf{j} - \mathbf{k}$$

$$\nabla F(3, 4, 5) = \frac{3}{5}\mathbf{i} + \frac{4}{5}\mathbf{j} - \mathbf{k}$$

$$\mathbf{n} = \frac{\nabla F}{\|\nabla F\|} = \frac{5}{5\sqrt{2}}\left(\frac{3}{5}\mathbf{i} + \frac{4}{5}\mathbf{j} - \mathbf{k}\right)$$

$$= \frac{1}{5\sqrt{2}}(3\mathbf{i} + 4\mathbf{j} - 5\mathbf{k})$$

$$= \frac{\sqrt{2}}{10}(3\mathbf{i} + 4\mathbf{j} - 5\mathbf{k})$$

$$9. F(x, y, z) = x^3 - z$$

$$\nabla F = 3x^2\mathbf{i} - \mathbf{k}$$

$$\nabla F(2, -1, 8) = 12\mathbf{i} - \mathbf{k}$$

$$\|\nabla F(2, -1, 8)\| = \sqrt{145}$$

$$\mathbf{n} = \frac{\nabla F}{\|\nabla F\|} = \frac{1}{\sqrt{145}}(12\mathbf{i} - \mathbf{k})$$

10.  $F(x, y, z) = x^2y^4 - z$

$$\nabla F(x, y, z) = 2xy^4\mathbf{i} + 4x^2y^3\mathbf{j} - \mathbf{k}$$

$$\nabla F(1, 2, 16) = 32\mathbf{i} + 32\mathbf{j} - \mathbf{k}$$

$$\begin{aligned}\mathbf{n} &= \frac{\nabla F}{\|\nabla F\|} = \frac{1}{\sqrt{2049}}(32\mathbf{i} + 32\mathbf{j} - \mathbf{k}) \\ &= \frac{\sqrt{2049}}{2049}(32\mathbf{i} + 32\mathbf{j} - \mathbf{k})\end{aligned}$$

11.  $F(x, y, z) = x^2 + 3y + z^3 - 9$

$$\nabla F(x, y, z) = 2x\mathbf{i} + 3\mathbf{j} + 3z^2\mathbf{k}$$

$$\nabla F(2, -1, 2) = 4\mathbf{i} + 3\mathbf{j} + 12\mathbf{k}$$

$$\mathbf{n} = \frac{\nabla F}{\|\nabla F\|} = \frac{1}{13}(4\mathbf{i} + 3\mathbf{j} + 12\mathbf{k})$$

12.  $F(x, y, z) = x^2y^3 - y^2z + 2xz^3 - 4$

$$\nabla F = (2xy^3 + 2z^3)\mathbf{i} + (3x^2y^2 - 2yz)\mathbf{j} + (6xz^2 - y^2)\mathbf{k}$$

$$\nabla F(-1, 1, -1) = -4\mathbf{i} + 5\mathbf{j} - 7\mathbf{k}$$

$$\|\nabla F(-1, 1, -1)\| = 3\sqrt{10}$$

$$\mathbf{n} = \frac{\nabla F}{\|\nabla F\|} = \frac{1}{3\sqrt{10}}(-4\mathbf{i} + 5\mathbf{j} - 7\mathbf{k})$$

13.  $F(x, y, z) = \ln\left(\frac{x}{y-z}\right) = \ln x - \ln(y-z)$

$$\nabla F(x, y, z) = \frac{1}{x}\mathbf{i} - \frac{1}{y-z}\mathbf{j} + \frac{1}{y-z}\mathbf{k}$$

$$\nabla F(1, 4, 3) = \mathbf{i} - \mathbf{j} + \mathbf{k}$$

$$\mathbf{n} = \frac{\nabla F}{\|\nabla F\|} = \frac{1}{\sqrt{3}}(\mathbf{i} - \mathbf{j} + \mathbf{k}) = \frac{\sqrt{3}}{3}(\mathbf{i} - \mathbf{j} + \mathbf{k})$$

14.  $F(x, y, z) = ze^{x^2-y^2} - 3$

$$\nabla F(x, y, z) = 2xze^{x^2-y^2}\mathbf{i} - 2yze^{x^2-y^2}\mathbf{j} + e^{x^2-y^2}\mathbf{k}$$

$$\nabla F(2, 2, 3) = 12\mathbf{i} - 12\mathbf{j} + \mathbf{k}$$

$$\mathbf{n} = \frac{\nabla F}{\|\nabla F\|} = \frac{1}{17}(12\mathbf{i} - 12\mathbf{j} + \mathbf{k})$$

15.  $F(x, y, z) = -x \sin y + z - 4$

$$\nabla F(x, y, z) = -\sin y\mathbf{i} - x \cos y\mathbf{j} + \mathbf{k}$$

$$\nabla F\left(6, \frac{\pi}{6}, 7\right) = -\frac{1}{2}\mathbf{i} - 3\sqrt{3}\mathbf{j} + \mathbf{k}$$

$$\begin{aligned}\mathbf{n} &= \frac{\nabla F}{\|\nabla F\|} = \frac{2}{\sqrt{113}}\left(-\frac{1}{2}\mathbf{i} - 3\sqrt{3}\mathbf{j} + \mathbf{k}\right) \\ &= \frac{1}{\sqrt{113}}(-\mathbf{i} - 6\sqrt{3}\mathbf{j} + 2\mathbf{k}) \\ &= \frac{\sqrt{113}}{113}(-\mathbf{i} - 6\sqrt{3}\mathbf{j} + 2\mathbf{k})\end{aligned}$$

16.  $F(x, y, z) = \sin(x - y) - z - 2$

$$\nabla F(x, y, z) = \cos(x - y)\mathbf{i} - \cos(x - y)\mathbf{j} - \mathbf{k}$$

$$\nabla F\left(\frac{\pi}{3}, \frac{\pi}{6}, -\frac{3}{2}\right) = \frac{\sqrt{3}}{2}\mathbf{i} - \frac{\sqrt{3}}{2}\mathbf{j} - \mathbf{k}$$

$$\begin{aligned}\mathbf{n} &= \frac{\nabla F}{\|\nabla F\|} = \frac{2}{\sqrt{10}}\left(\frac{\sqrt{3}}{2}\mathbf{i} - \frac{\sqrt{3}}{2}\mathbf{j} - \mathbf{k}\right) \\ &= \frac{1}{\sqrt{10}}(\sqrt{3}\mathbf{i} - \sqrt{3}\mathbf{j} - 2\mathbf{k}) \\ &= \frac{\sqrt{10}}{10}(\sqrt{3}\mathbf{i} - \sqrt{3}\mathbf{j} - 2\mathbf{k})\end{aligned}$$

17.  $f(x, y) = x^2 + y^2 + 3, (2, 1, 8)$

$$F(x, y, z) = x^2 + y^2 + 3 - z$$

$$F_x(x, y, z) = 2x \quad F_y(x, y, z) = 2y \quad F_z(x, y, z) = -1$$

$$F_x(2, 1, 8) = 4 \quad F_y(2, 1, 8) = 2 \quad F_z(2, 1, 8) = -1$$

$$4(x - 2) + 2(y - 1) - 1(z - 8) = 0$$

$$4x + 2y - z = 2$$

18.  $f(x, y) = \frac{y}{x}, (1, 2, 2)$

$$F(x, y, z) = \frac{y}{x} - z$$

$$F_x(x, y, z) = -\frac{y}{x^2} \quad F_y(x, y, z) = \frac{1}{x} \quad F_z(x, y, z) = -1$$

$$F_x(1, 2, 2) = -2 \quad F_y(1, 2, 2) = 1 \quad F_z(1, 2, 2) = -1$$

$$-2(x - 1) + (y - 2) - (z - 2) = 0$$

$$-2x + y - z + 2 = 0$$

$$2x - y + z = 2$$

19.  $f(x, y) = \sqrt{x^2 + y^2}, (3, 4, 5)$

$$F(x, y, z) = \sqrt{x^2 + y^2} - z$$

$$F_x(x, y, z) = \frac{x}{\sqrt{x^2 + y^2}} \quad F_y(x, y, z) = \frac{y}{\sqrt{x^2 + y^2}} \quad F_z(x, y, z) = -1$$

$$F_x(3, 4, 5) = \frac{3}{5} \quad F_y(3, 4, 5) = \frac{4}{5} \quad F_z(3, 4, 5) = -1$$

$$\frac{3}{5}(x - 3) + \frac{4}{5}(y - 4) - (z - 5) = 0$$

$$3(x - 3) + 4(y - 4) - 5(z - 5) = 0$$

$$3x + 4y - 5z = 0$$

20.  $g(x, y) = \arctan \frac{y}{x}, (1, 0, 0)$

$$G(x, y, z) = \arctan \frac{y}{x} - z$$

$$G_x(x, y, z) = \frac{-\left(y/x^2\right)}{1 + \left(y^2/x^2\right)} = \frac{-y}{x^2 + y^2} \quad G_y(x, y, z) = \frac{1/x}{1 + \left(y^2/x^2\right)} = \frac{x}{x^2 + y^2} \quad G_z(x, y, z) = -1$$

$$G_x(1, 0, 0) = 0$$

$$G_y(1, 0, 0) = 1$$

$$G_z(1, 0, 0) = -1$$

$$y - z = 0$$

21.  $g(x, y) = x^2 + y^2, (1, -1, 2)$

$$G(x, y, z) = x^2 + y^2 - z$$

$$G_x(x, y, z) = 2x \quad G_y(x, y, z) = 2y \quad G_z(x, y, z) = -1$$

$$G_x(1, -1, 2) = 2 \quad G_y(1, -1, 2) = -2 \quad G_z(1, -1, 2) = -1$$

$$2(x - 1) - 2(y + 1) - 1(z - 2) = 0$$

$$2x - 2y - z = 2$$

22.  $z = x^2 - 2xy + y^2, (1, 2, 1)$

$$F(x, y, z) = x^2 - 2xy + y^2 - z$$

$$F_x(x, y, z) = 2x - 2y \quad F_y(x, y, z) = -2x + 2y \quad F_z(x, y, z) = -1$$

$$F_x(1, 2, 1) = -2 \quad F_y(1, 2, 1) = 2 \quad F_z(1, 2, 1) = -1$$

$$-2(x - 1) + 2(y - 2) - (z - 1) = 0$$

$$-2x + 2y - z - 1 = 0$$

$$2x - 2y + z = -1$$

23.  $f(x, y) = 2 - \frac{2}{3}x - y, (3, -1, 1)$

$$F(x, y, z) = 2 - \frac{2}{3}x - y - z$$

$$F_x(x, y, z) = -\frac{2}{3}, \quad F_y(x, y, z) = -1, \quad F_z(x, y, z) = -1$$

$$-\frac{2}{3}(x - 3) - (y + 1) - (z - 1) = 0$$

$$-\frac{2}{3}x - y - z + 2 = 0$$

$$2x + 3y + 3z = 6$$

24.  $z = e^x(\sin y + 1), \left(0, \frac{\pi}{2}, 2\right)$

$$F(x, y, z) = e^x(\sin y + 1) - z$$

$$F_x(x, y, z) = e^x(\sin y + 1) \quad F_y(x, y, z) = e^x \cos y \quad F_z(x, y, z) = -1$$

$$F_x\left(0, \frac{\pi}{2}, 2\right) = 2 \quad F_y\left(0, \frac{\pi}{2}, 2\right) = 0 \quad F_z\left(0, \frac{\pi}{2}, 2\right) = -1$$

$$2x - z = -2$$

25.  $h(x, y) = \ln\sqrt{x^2 + y^2}, (3, 4, \ln 5)$

$$H(x, y, z) = \ln\sqrt{x^2 + y^2} - z = \frac{1}{2}\ln(x^2 + y^2) - z$$

$$H_x(x, y, z) = \frac{x}{x^2 + y^2} \quad H_y(x, y, z) = \frac{y}{x^2 + y^2} \quad H_z(x, y, z) = -1$$

$$H_x(3, 4, \ln 5) = \frac{3}{25} \quad H_y(3, 4, \ln 5) = \frac{4}{25} \quad H_z(3, 4, \ln 5) = -1$$

$$\frac{3}{25}(x - 3) + \frac{4}{25}(y - 4) - (z - \ln 5) = 0$$

$$3(x - 3) + 4(y - 4) - 25(z - \ln 5) = 0$$

$$3x + 4y - 25z = 25(1 - \ln 5)$$

26.  $h(x, y) = \cos y, \left(5, \frac{\pi}{4}, \frac{\sqrt{2}}{2}\right)$

$$H(x, y, z) = \cos y - z$$

$$H_x(x, y, z) = 0 \quad H_y(x, y, z) = -\sin y \quad H_z(x, y, z) = -1$$

$$H_x\left(5, \frac{\pi}{4}, \frac{\sqrt{2}}{2}\right) = 0 \quad H_y\left(5, \frac{\pi}{4}, \frac{\sqrt{2}}{2}\right) = -\frac{\sqrt{2}}{2} \quad H_z\left(5, \frac{\pi}{4}, \frac{\sqrt{2}}{2}\right) = -1$$

$$-\frac{\sqrt{2}}{2}\left(y - \frac{\pi}{4}\right) - \left(z - \frac{\sqrt{2}}{2}\right) = 0$$

$$-\frac{\sqrt{2}}{2}y - z + \frac{\sqrt{2}\pi}{8} + \frac{\sqrt{2}}{2} = 0$$

$$4\sqrt{2}y + 8z = \sqrt{2}(\pi + 4)$$

27.  $x^2 + 4y^2 + z^2 = 36, (2, -2, 4)$

$$F(x, y, z) = x^2 + 4y^2 + z^2 - 36$$

$$F_x(x, y, z) = 2x \quad F_y(x, y, z) = 8y \quad F_z(x, y, z) = 2z$$

$$F_x(2, -2, 4) = 4 \quad F_y(2, -2, 4) = -16 \quad F_z(2, -2, 4) = 8$$

$$4(x - 2) - 16(y + 2) + 8(z - 4) = 0$$

$$(x - 2) - 4(y + 2) + 2(z - 4) = 0$$

$$x - 4y + 2z = 18$$

**28.**  $x^2 + 2z^2 = y^2, (1, 3, -2)$

$$F(x, y, z) = x^2 - y^2 + 2z^2$$

$$F_x(x, y, z) = 2x \quad F_y(x, y, z) = -2y \quad F_z(x, y, z) = 4z$$

$$F_x(1, 3, -2) = 2 \quad F_y(1, 3, -2) = -6 \quad F_z(1, 3, -2) = -8$$

$$2(x - 1) - 6(y - 3) - 8(z + 2) = 0$$

$$(x - 1) - 3(y - 3) - 4(z + 2) = 0$$

$$x - 3y - 4z = 0$$

**29.**  $xy^2 + 3x - z^2 = 8, (1, -3, 2)$

$$F(x, y, z) = xy^2 + 3x - z^2 - 8$$

$$F_x(x, y, z) = y^2 + 3 \quad F_y(x, y, z) = 2xy \quad F_z(x, y, z) = -2z$$

$$F_x(1, -3, 2) = 12 \quad F_y(1, -3, 2) = -6 \quad F_z(1, -3, 2) = -4$$

$$12(x - 1) - 6(y + 3) - 4(z - 2) = 0$$

$$12x - 6y - 4z = 22$$

$$6x - 3y - 2z = 11$$

**30.**  $x = y(2z - 3), (4, 4, 2)$

$$F(x, y, z) = x - 2yz + 3y$$

$$F_x(x, y, z) = 1 \quad F_y(x, y, z) = -2z + 3 \quad F_z(x, y, z) = -2y$$

$$F_x(4, 4, 2) = 1 \quad F_y(4, 4, 2) = -1 \quad F_z(4, 4, 2) = -8$$

$$(x - 4) - 1(y - 4) - 8(z - 2) = 0$$

$$x - y - 8z = -16$$

$$-x + y + 8z = 16$$

**31.**  $x + y + z = 9, (3, 3, 3)$

$$F(x, y, z) = x + y + z - 9$$

$$F_x(x, y, z) = 1 \quad F_y(x, y, z) = 1 \quad F_z(x, y, z) = 1$$

$$F_x(3, 3, 3) = 1 \quad F_y(3, 3, 3) = 1 \quad F_z(3, 3, 3) = 1$$

$$(x - 3) + (y - 3) + (z - 3) = 0$$

$$x + y + z = 9 \text{ (same plane!)}$$

Direction numbers: 1, 1, 1

Line:  $x - 3 = y - 3 = z - 3$

**32.**  $x^2 + y^2 + z^2 + 9, (1, 2, 2)$

$$F(x, y, z) = x^2 + y^2 + z^2 - 9$$

$$F_x(x, y, z) = 2x \quad F_y(x, y, z) = 2y \quad F_z(x, y, z) = 2z$$

$$F_x(1, 2, 2) = 2 \quad F_y(1, 2, 2) = 4 \quad F_z(1, 2, 2) = 4$$

Direction numbers: 1, 2, 2

Plane:  $(x - 1) + 2(y - 2) + 2(z - 2) = 0, x + 2y + 2z = 9$

$$\text{Line: } \frac{x - 1}{1} = \frac{y - 2}{2} = \frac{z - 2}{2}$$

33.  $x^2 + y^2 + z = 9, (1, 2, 4)$

$$F(x, y, z) = x^2 + y^2 + z - 9$$

$$F_x(x, y, z) = 2x \quad F_y(x, y, z) = 2y \quad F_z(x, y, z) = 1$$

$$F_x(1, 2, 4) = 2 \quad F_y(1, 2, 4) = 4 \quad F_z(1, 2, 4) = 1$$

Direction numbers: 2, 4, 1

$$\text{Plane: } 2(x - 1) + 4(y - 2) + (z - 4) = 0, 2x + 4y + z = 14$$

$$\text{Line: } \frac{x - 1}{2} = \frac{y - 2}{4} = \frac{z - 4}{1}$$

34.  $z = 16 - x^2 - y^2, (2, 2, 8)$

$$F(x, y, z) = 16 - x^2 - y^2 - z$$

$$F_x(x, y, z) = -2x \quad F_y(x, y, z) = -2y \quad F_z(x, y, z) = -1$$

$$F_x(2, 2, 8) = -4 \quad F_y(2, 2, 8) = -4 \quad F_z(2, 2, 8) = -1$$

$$-4(x - 2) - 4(y - 2) - (z - 8) = 0$$

$$-4x - 4y - z = -24$$

$$4x + 4y + z = 24$$

Direction numbers: 4, 4, 1

$$\text{Line: } \frac{x - 2}{4} = \frac{y - 2}{4} = z - 8$$

35.  $z = x^2 - y^2, (3, 2, 5)$

$$F(x, y, z) = x^2 - y^2 - z$$

$$F_x(x, y, z) = 2x \quad F_y(x, y, z) = -2y \quad F_z(x, y, z) = -1$$

$$F_x(3, 2, 5) = 6 \quad F_y(3, 2, 5) = -4 \quad F_z(3, 2, 5) = -1$$

$$6(x - 3) - 4(y - 2) - (z - 5) = 0$$

$$6x - 4y - z = 5$$

Direction numbers: 6, -4, -1

$$\text{Line: } \frac{x - 3}{6} = \frac{y - 2}{-4} = \frac{z - 5}{-1}$$

36.  $xy - z = 0, (-2, -3, 6)$

$$F(x, y, z) = xy - z$$

$$F_x(x, y, z) = y \quad F_y(x, y, z) = x \quad F_z(x, y, z) = -1$$

$$F_x(-2, -3, 6) = -3 \quad F_y(-2, -3, 6) = -2 \quad F_z(-2, -3, 6) = -1$$

Direction numbers: 3, 2, 1

$$\text{Plane: } 3(x + 2) + 2(y + 3) + (z - 6) = 0, 3x + 2y + z = -6$$

$$\text{Line: } \frac{x + 2}{3} = \frac{y + 3}{2} = \frac{z - 6}{1}$$

37.  $xyz = 10, (1, 2, 5)$

$$F(x, y, z) = xyz - 10$$

$$F_x(x, y, z) = yz \quad F_y(x, y, z) = xz \quad F_z(x, y, z) = xy$$

$$F_x(1, 2, 5) = 10 \quad F_y(1, 2, 5) = 5 \quad F_z(1, 2, 5) = 2$$

Direction numbers: 10, 5, 2

$$\text{Plane: } 10(x - 1) + 5(y - 2) + 2(z - 5) = 0, 10x + 5y + 2z = 30$$

$$\text{Line: } \frac{x - 1}{10} = \frac{y - 2}{5} = \frac{z - 5}{2}$$

38.  $z = ye^{2xy}, (0, 2, 2)$

$$F(x, y, z) = ye^{2xy} - z$$

$$F_x(x, y, z) = 2y^2e^{2xy} \quad F_y(x, y, z) = (1 + 2xy)e^{2xy} \quad F_z(x, y, z) = -1$$

$$F_x(0, 2, 2) = 8 \quad F_y(0, 2, 2) = 1 \quad F_z(0, 2, 2) = -1$$

$$8(x - 0) + (y - 2) - (z - 2) = 0$$

$$8x + y - z = 0$$

Direction number: 8, 1, -1

$$\text{Line: } \frac{x}{8} = \frac{y - 2}{1} = \frac{z - 2}{-1}$$

39.  $z = \arctan \frac{y}{x}, \left(1, 1, \frac{\pi}{4}\right)$

$$F(x, y, z) = \arctan \frac{y}{x} - z$$

$$F_x(x, y, z) = \frac{-y}{x^2 + y^2} \quad F_y(x, y, z) = \frac{x}{x^2 + y^2} \quad F_z(x, y, z) = -1$$

$$F_x\left(1, 1, \frac{\pi}{4}\right) = -\frac{1}{2} \quad F_y\left(1, 1, \frac{\pi}{4}\right) = \frac{1}{2} \quad F_z\left(1, 1, \frac{\pi}{4}\right) = -1$$

Direction numbers: 1, -1, 2

$$\text{Plane: } (x - 1) - (y - 1) + 2\left(z - \frac{\pi}{4}\right) = 0, x - y + 2z = \frac{\pi}{2}$$

$$\text{Line: } \frac{x - 1}{1} = \frac{y - 1}{-1} = \frac{z - (\pi/4)}{2}$$

40.  $y \ln(xz^2) = 2, (e, 2, 1)$

$$F(x, y, z) = y[\ln x + 2 \ln z] - 2$$

$$F_x(x, y, z) = \frac{y}{x} \quad F_y(x, y, z) = \ln x + 2 \ln z \quad F_z(x, y, z) = \frac{2y}{z}$$

$$F_x(e, 2, 1) = \frac{2}{e} \quad F_y(e, 2, 1) = 1 \quad F_z(e, 2, 1) = 4$$

$$\frac{2}{e}(x - e) + (y - 2) + 4(z - 1) = 0$$

$$\frac{2}{e}x + y + 4z = 8$$

Direction numbers:  $\frac{2}{e}, 1, 4$

$$\frac{x - e}{(2/e)} = \frac{y - 2}{1} = \frac{z - 1}{4}$$

41.  $F(x, y, z) = x^2 + y^2 - 5 \quad G(x, y, z) = x - z$

$$\nabla F(x, y, z) = 2x\mathbf{i} + 2y\mathbf{j} \quad \nabla G(x, y, z) = \mathbf{i} - \mathbf{k}$$

$$\nabla F(1, 1, 1) = 2\mathbf{i} + 2\mathbf{j} \quad \nabla G(1, 1, 1) = \mathbf{i} - \mathbf{k}$$

$$(a) \quad \nabla F \times \nabla G = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 2 & 0 \\ 1 & 0 & -1 \end{vmatrix} = -2\mathbf{i} + 2\mathbf{j} - 2\mathbf{k} = -2(\mathbf{i} - \mathbf{j} + \mathbf{k})$$

Direction numbers: 1, -1, 1

$$\text{Line: } x - 1 = \frac{y - 1}{-1} = z - 1$$

$$(b) \cos \theta = \frac{|\nabla F \cdot \nabla G|}{\|\nabla F\| \|\nabla G\|} = \frac{2}{(2\sqrt{2})\sqrt{2}} = \frac{1}{2}$$

Not orthogonal

$$42. F(x, y, z) = x^2 + y^2 - z \quad G(x, y, z) = 4 - y - z$$

$$\nabla F(x, y, z) = 2x\mathbf{i} + 2y\mathbf{j} - \mathbf{k} \quad \nabla G(x, y, z) = -\mathbf{j} - \mathbf{k}$$

$$\nabla F(2, -1, 5) = 4\mathbf{i} - 2\mathbf{j} - \mathbf{k} \quad \nabla G(2, -1, 5) = -\mathbf{j} - \mathbf{k}$$

$$(a) \nabla F \times \nabla G = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & -2 & -1 \\ 0 & -1 & -1 \end{vmatrix} = \mathbf{i} + 4\mathbf{j} - 4\mathbf{k}$$

$$\text{Direction numbers: } 1, 4, -4. \frac{x-2}{1} = \frac{y+1}{4} = \frac{z-5}{-4}$$

$$(b) \cos \theta = \frac{|\nabla F \cdot \nabla G|}{\|\nabla F\| \|\nabla G\|} = \frac{3}{\sqrt{21}\sqrt{2}} = \frac{3}{\sqrt{42}} = \frac{\sqrt{42}}{14}; \text{ not orthogonal}$$

$$43. F(x, y, z) = x^2 + z^2 - 25 \quad G(x, y, z) = y^2 + z^2 - 25$$

$$\nabla F = 2x\mathbf{i} + 2z\mathbf{k} \quad \nabla G = 2y\mathbf{j} + 2z\mathbf{k}$$

$$\nabla F(3, 3, 4) = 6\mathbf{i} + 8\mathbf{k} \quad \nabla G(3, 3, 4) = 6\mathbf{j} + 8\mathbf{k}$$

$$(a) \nabla F \times \nabla G = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 6 & 0 & 8 \\ 0 & 6 & 8 \end{vmatrix} = -48\mathbf{i} - 48\mathbf{j} + 36\mathbf{k} = -12(4\mathbf{i} + 4\mathbf{j} - 3\mathbf{k})$$

$$\text{Direction numbers: } 4, 4, -3. \frac{x-3}{4} = \frac{y-3}{4} = \frac{z-4}{-3}$$

$$(b) \cos \theta = \frac{|\nabla F \cdot \nabla G|}{\|\nabla F\| \|\nabla G\|} = \frac{64}{(10)(10)} = \frac{16}{25}; \text{ not orthogonal}$$

$$44. F(x, y, z) = \sqrt{x^2 + y^2} - z \quad G(x, y, z) = 5x - 2y + 3z = 22$$

$$\nabla F(x, y, z) = \frac{x}{\sqrt{x^2 + y^2}}\mathbf{i} + \frac{y}{\sqrt{x^2 + y^2}}\mathbf{j} - \mathbf{k} \quad \nabla G(x, y, z) = 5\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$$

$$\nabla F(3, 4, 5) = \frac{3}{5}\mathbf{i} + \frac{4}{5}\mathbf{j} - \mathbf{k} \quad \nabla G(3, 4, 5) = 5\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$$

$$(a) \nabla F \times \nabla G = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3/5 & 4/5 & -1 \\ 5 & -2 & 3 \end{vmatrix} = \frac{2}{5}\mathbf{i} - \frac{34}{5}\mathbf{j} - \frac{26}{5}\mathbf{k}$$

$$\text{Direction numbers: } 1, -17, -13$$

$$\frac{x-3}{1} = \frac{y-4}{-17} = \frac{z-5}{-13} \text{ Tangent line}$$

$$(b) \cos \theta = \frac{|\nabla F \cdot \nabla G|}{\|\nabla F\| \|\nabla G\|} = \frac{-8/5}{\sqrt{2}\sqrt{38}} = \frac{-8}{5\sqrt{76}} \text{ Not orthogonal}$$

$$45. F(x, y, z) = x^2 + y^2 + z^2 - 14 \quad G(x, y, z) = x - y - z$$

$$\nabla F(x, y, z) = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k} \quad \nabla G(x, y, z) = \mathbf{i} - \mathbf{j} - \mathbf{k}$$

$$\nabla F(3, 1, 2) = 6\mathbf{i} + 2\mathbf{j} + 4\mathbf{k} \quad \nabla G(3, 1, 2) = \mathbf{i} - \mathbf{j} - \mathbf{k}$$

$$(a) \nabla F \times \nabla G = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 6 & 2 & 4 \\ 1 & -1 & -1 \end{vmatrix} = 2\mathbf{i} + 10\mathbf{j} - 8\mathbf{k} = 2[\mathbf{i} + 5\mathbf{j} - 4\mathbf{k}]$$

Direction numbers: 1, 5, -4

$$\text{Line: } \frac{x-3}{1} = \frac{y-1}{5} = \frac{z-2}{-4}$$

$$(b) \cos \theta = \frac{|\nabla F \cdot \nabla G|}{\|\nabla F\| \|\nabla G\|} = 0 \Rightarrow \text{orthogonal}$$

$$46. F(x, y, z) = x^2 + y^2 - z \quad G(x, y, z) = x + y + 6z - 33$$

$$\nabla F(x, y, z) = 2x\mathbf{i} + 2y\mathbf{j} - \mathbf{k} \quad \nabla G(x, y, z) = \mathbf{i} + \mathbf{j} + 6\mathbf{k}$$

$$\nabla F(1, 2, 5) = 2\mathbf{i} + 4\mathbf{j} - \mathbf{k} \quad \nabla G(1, 2, 5) = \mathbf{i} + \mathbf{j} + 6\mathbf{k}$$

$$(a) \nabla F \times \nabla G = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 4 & -1 \\ 1 & 1 & 6 \end{vmatrix} = 25\mathbf{i} - 13\mathbf{j} - 2\mathbf{k}$$

$$\text{Direction numbers: } 25, -13, -2. \frac{x-1}{25} = \frac{y-2}{-13} = \frac{z-5}{-2}$$

$$(b) \cos \theta = \frac{|\nabla F \cdot \nabla G|}{\|\nabla F\| \|\nabla G\|} = 0; \text{ orthogonal}$$

$$47. F(x, y, z) = 3x^2 + 2y^2 - z - 15, (2, 2, 5)$$

$$\nabla F(x, y, z) = 6x\mathbf{i} + 4y\mathbf{j} - \mathbf{k}$$

$$\nabla F(2, 2, 5) = 12\mathbf{i} + 8\mathbf{j} - \mathbf{k}$$

$$\cos \theta = \frac{|\nabla F(2, 2, 5) \cdot \mathbf{k}|}{\|\nabla F(2, 2, 5)\|} = \frac{1}{\sqrt{209}}$$

$$\theta = \arccos\left(\frac{1}{\sqrt{209}}\right) \approx 86.03^\circ$$

$$48. F(x, y, z) = 2xy - z^3, (2, 2, 2)$$

$$\nabla F = 2y\mathbf{i} + 2x\mathbf{j} - 3z^2\mathbf{k}$$

$$\nabla F(2, 2, 2) = 4\mathbf{i} + 4\mathbf{j} - 12\mathbf{k}$$

$$\cos \theta = \frac{|\nabla F(2, 2, 2) \cdot \mathbf{k}|}{\|\nabla F(2, 2, 2)\|} = \frac{|-12|}{\sqrt{176}} = \frac{3\sqrt{11}}{11}$$

$$\theta = \arccos\left(\frac{3\sqrt{11}}{11}\right) \approx 25.24^\circ$$

$$49. F(x, y, z) = x^2 - y^2 + z, (1, 2, 3)$$

$$\nabla F(x, y, z) = 2x\mathbf{i} - 2y\mathbf{j} + \mathbf{k}$$

$$\nabla F(1, 2, 3) = 2\mathbf{i} - 4\mathbf{j} + \mathbf{k}$$

$$\cos \theta = \frac{|\nabla F(1, 2, 3) \cdot \mathbf{k}|}{\|\nabla F(1, 2, 3)\|} = \frac{1}{\sqrt{21}}$$

$$\theta = \arccos \frac{1}{\sqrt{21}} \approx 77.40^\circ$$

$$50. F(x, y, z) = x^2 + y^2 - 5, (2, 1, 3)$$

$$\nabla F(x, y, z) = 2x\mathbf{i} + 2y\mathbf{j}$$

$$\nabla F(2, 1, 3) = 4\mathbf{i} + 2\mathbf{j}$$

$$\cos \theta = \frac{|\nabla F(2, 1, 3) \cdot \mathbf{k}|}{\|\nabla F(2, 1, 3)\|} = 0$$

$$\theta = \arccos 0 = 90^\circ$$

$$51. F(x, y, z) = 3 - x^2 - y^2 + 6y - z$$

$$\nabla F(x, y, z) = -2x\mathbf{i} + (-2y + 6)\mathbf{j} - \mathbf{k}$$

$$-2x = 0, x = 0$$

$$-2y + 6 = 0, y = 3$$

$$z = 3 - 0^2 - 3^2 + 6(3) = 12$$

$$(0, 3, 12) \text{ (vertex of paraboloid)}$$

$$52. F(x, y, z) = 3x^2 + 2y^2 - 3x + 4y - z - 5$$

$$\nabla F(x, y, z) = (6x - 3)\mathbf{i} + (4y + 4)\mathbf{j} - \mathbf{k}$$

$$6x - 3 = 0, x = \frac{1}{2}$$

$$4y + 4 = 0, y = -1$$

$$z = 3\left(\frac{1}{2}\right)^2 + 2(-1)^2 - 3\left(\frac{1}{2}\right) + 4(-1) - 5 = -\frac{31}{4}$$

$$\left(\frac{1}{2}, -1, -\frac{31}{4}\right)$$

53.  $F(x, y, z) = x^2 - xy + y^2 - 2x - 2y - z$   
 $\nabla F(x, y, z) = (2x - y - 2)\mathbf{i} + (-x + 2y - 2)\mathbf{j} - \mathbf{k}$   
 $2x - y - 2 = 0$   
 $-x + 2y - 2 = 0$   
 $y = 2x - 2 \Rightarrow -x + 2(2x - 2) - 2$   
 $= 3x - 6 = 0 \Rightarrow x = 2$   
 $y = 2, z = -4$   
Point:  $(2, 2, -4)$

54.  $F(x, y, z) = 4x^2 + 4xy - 2y^2 + 8x - 5y - 4 - z$   
 $\nabla F(x, y, z) = (8x + 4y + 8)\mathbf{i} + (4x - 4y - 5)\mathbf{j} - \mathbf{k}$   
 $8x + 4y + 8 = 0$   
 $4x - 4y - 5 = 0$   
Adding,  $12x + 3 = 0 \Rightarrow x = -\frac{1}{4} \Rightarrow y = -\frac{3}{2}$ , and  
 $z = -\frac{5}{4}$   
Point:  $(-\frac{1}{4}, -\frac{3}{2}, -\frac{5}{4})$

57.  $F(x, y, z) = x^2 + 2y^2 + 3z^2 - 3, (-1, 1, 0)$   
 $F_x(x, y, z) = 2x \quad F_y(x, y, z) = 4y \quad F_z(x, y, z) = 6z$   
 $F_x(-1, 1, 0) = -2 \quad F_y(-1, 1, 0) = 4 \quad F_z(-1, 1, 0) = 0$   
 $-2(x + 1) + 4(y - 1) + 0(z - 0) = 0$   
 $-2x + 4y = 6$   
 $-x + 2y = 3$   
 $G(x, y, z) = x^2 + y^2 + z^2 + 6x - 10y + 14, (-1, 1, 0)$   
 $G_x(x, y, z) = 2x + 6 \quad G_y(x, y, z) = 2y - 10 \quad G_z(x, y, z) = 2z$   
 $G_x(-1, 1, 0) = 4 \quad G_y(-1, 1, 0) = -8 \quad G_z(-1, 1, 0) = 0$   
 $4(x + 1) - 8(y - 1) + 0(z - 0) = 0$   
 $4x - 8y + 12 = 0$   
 $-x + 2y = 3$

The tangent planes are the same.

58.  $F(x, y, z) = x^2 + y^2 + z^2 - 8x - 12y + 4z + 42, (2, 3, -3)$   
 $F_x(x, y, z) = 2x - 8 \quad F_y(x, y, z) = 2y - 12 \quad F_z(x, y, z) = 2z + 4$   
 $F_x(2, 3, -3) = -4 \quad F_y(2, 3, -3) = -6 \quad F_z(2, 3, -3) = -2$   
 $-4(x - 2) - 6(y - 3) - 2(z + 3) = 0$   
 $-4x - 6y - 2z + 20 = 0$   
 $2x + 3y + z = 10$   
 $G(x, y, z) = x^2 + y^2 + 2z - 7, (2, 3, -3)$   
 $G_x(x, y, z) = 2x \quad G_y(x, y, z) = 2y \quad G_z(x, y, z) = 2$   
 $G_x(2, 3, -3) = 4 \quad G_y(2, 3, -3) = 6 \quad G_z(2, 3, -3) = 2$   
 $4(x - 2) + 6(y - 3) + 2(z + 3) = 0$   
 $4x + 6y + 2z - 20 = 0$   
 $2x + 3y + z = 10$

The tangent planes are the same.

55.  $F(x, y, z) = 5xy - z$   
 $\nabla F(x, y, z) = 5y\mathbf{i} + 5x\mathbf{j} - \mathbf{k}$   
 $5y = 0$   
 $5x = 0$   
 $x = y = z = 0$   
Point:  $(0, 0, 0)$

56.  $F(x, y, z) = xy + \frac{1}{x} + \frac{1}{y} - z$   
 $\nabla F(x, y, z) = \left(y - \frac{1}{x^2}\right)\mathbf{i} + \left(x - \frac{1}{y^2}\right)\mathbf{j} - \mathbf{k}$   
 $y = \frac{1}{x^2}$   
 $x = \frac{1}{y^2} = x^4 \Rightarrow x = 1, y = 1, z = 3$   
Point:  $(1, 1, 3)$

59. (a)  $F(x, y, z) = 2xy^2 - z$ ,  $F(1, 1, 2) = 2 - 2 = 0$

$$G(x, y, z) = 8x^2 - 5y^2 - 8z + 13, G(1, 1, 2) = 8 - 5 - 16 + 13 = 0$$

So,  $(1, 1, 2)$  lies on both surfaces.

(b)  $\nabla F = 2y^2\mathbf{i} + 4xy\mathbf{j} - \mathbf{k}$ ,  $\nabla F(1, 1, 2) = 2\mathbf{i} + 4\mathbf{j} - \mathbf{k}$

$$\nabla G = 16x\mathbf{i} - 10y\mathbf{j} - 8\mathbf{k}$$

$$\nabla F \cdot \nabla G = 2(16) + 4(-10) + (-1)(-8) = 0$$

The tangent planes are perpendicular at  $(1, 1, 2)$ .

60. (a)  $F(x, y, z) = x^2 + y^2 + z^2 + 2x - 4y - 4z - 12$

$$F(1, -2, 1) = 0$$

$$G(x, y, z) = 4x^2 + y^2 + 16z^2 - 24$$

$$G(1, -2, 1) = 0$$

So,  $(1, -2, 1)$  lies on both surfaces.

(b)  $\nabla F = (2x + 2)\mathbf{i} + (2y - 4)\mathbf{j} + (2z - 4)\mathbf{k}$

$$\nabla F(1, -2, 1) = 4\mathbf{i} - 8\mathbf{j} - 2\mathbf{k}$$

$$\nabla G = 8x\mathbf{i} + 2y\mathbf{j} + 32z\mathbf{k}$$

$$\nabla G(1, -2, 1) = 8\mathbf{i} - 4\mathbf{j} + 32\mathbf{k}$$

$$\nabla F \cdot \nabla G = 32 + 32 - 64 = 0$$

The planes are perpendicular at  $(1, -2, 1)$ .

61.  $F(x, y, z) = x^2 + 4y^2 + z^2 - 9$

$$\nabla F = 2x\mathbf{i} + 8y\mathbf{j} + 2z\mathbf{k}$$

This normal vector is parallel to the line with direction number  $-4, 8, -2$ .

So,  $2x = -4t \Rightarrow x = -2t$

$$8y = 8t \Rightarrow y = t$$

$$2z = -2t \Rightarrow z = -t$$

$$x^2 + 4y^2 + z^2 - 9 = 4t^2 + 4t^2 + t^2 - 9 = 0 \Rightarrow t = \pm 1$$

There are two points on the ellipse where the tangent plane is perpendicular to the line:

$$(-2, 1, -1) \quad (t = 1)$$

$$(2, -1, 1) \quad (t = -1)$$

62.  $F(x, y, z) = x^2 + 4y^2 - z^2 - 1$

$$\nabla F = 2x\mathbf{i} + 8y\mathbf{j} - 2z\mathbf{k}$$

The normal to the plane,  $\mathbf{n} = \mathbf{i} + 4\mathbf{j} - \mathbf{k}$

must be parallel to  $\nabla F$ .

So,  $2x = t \Rightarrow x = \frac{t}{2}$

$$8y = 4t \Rightarrow y = \frac{t}{2}$$

$$-2z = -t \Rightarrow z = \frac{t}{2}$$

$$x^2 + 4y^2 - z^2 = \frac{t^2}{4} + t^2 - \frac{t^2}{4} = t^2 = 1 \Rightarrow t = \pm 1.$$

Two points:  $\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$  ( $t = 1$ ) and  $\left(-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right)$  ( $t = -1$ )

63.  $F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0) = 0$

(Theorem 13.13)

64. For a sphere, the common object is the center of the sphere. For a right circular cylinder, the common object is the axis of the cylinder.

65. Answers will vary.

66. (a)  $x^2 - y^2 + z^2 = 0, (5, 13, -12)$

$$F(x, y, z) = x^2 - y^2 + z^2$$

$$F_x(x, y, z) = 2x \quad F_y(x, y, z) = -2y \quad F_z(x, y, z) = 2z$$

$$F_x(5, 13, -12) = 10 \quad F_y(5, 13, -12) = -26 \quad F_z(x, y, z) = -24$$

Direction numbers: 5, -13, -12

Plane:  $5(x - 5) - 13(y - 13) - 12(z + 12) = 0$

$$5x - 13y - 12z = 0$$

(b) Line:  $\frac{x - 5}{5} = \frac{y - 13}{-13} = \frac{z + 12}{-12}$

67.  $z = f(x, y) = \frac{4xy}{(x^2 + 1)(y^2 + 1)}, -2 \leq x \leq 2, 0 \leq y \leq 3$

(a) Let  $F(x, y, z) = \frac{4xy}{(x^2 + 1)(y^2 + 1)} - z$

$$\nabla F(x, y, z) = \frac{4y}{y^2 + 1} \left( \frac{x^2 + 1 - 2x^2}{(x^2 + 1)^2} \right) \mathbf{i} + \frac{4x}{x^2 + 1} \left( \frac{y^2 + 1 - 2y^2}{(y^2 + 1)^2} \right) \mathbf{j} - \mathbf{k} = \frac{4y(1 - x^2)}{(y^2 + 1)(x^2 + 1)^2} \mathbf{i} + \frac{4x(1 - y^2)}{(x^2 + 1)(y^2 + 1)^2} \mathbf{j} - \mathbf{k}$$

$$\nabla F(1, 1, 1) = -\mathbf{k}$$

Direction numbers: 0, 0, -1

Line:  $x = 1, y = 1, z = 1 - t$

Tangent plane:  $0(x - 1) + 0(y - 1) - 1(z - 1) = 0 \Rightarrow z = 1$

(b)  $\nabla F(-1, 2, -\frac{4}{5}) = 0\mathbf{i} + \frac{-4(-3)}{(2)(5)^2}\mathbf{j} - \mathbf{k} = \frac{6}{25}\mathbf{j} - \mathbf{k}$

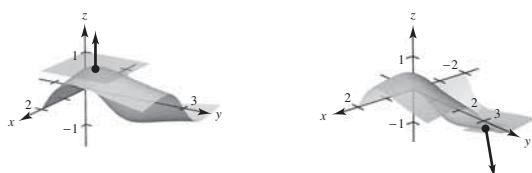
Line:  $x = -1, y = 2 + \frac{6}{25}t, z = -\frac{4}{5} - t$

Plane:  $0(x + 1) + \frac{6}{25}(y - 2) - 1\left(z + \frac{4}{5}\right) = 0$

$$6y - 12 - 25z - 20 = 0$$

$$6y - 25z - 32 = 0$$

(c)



68. (a)  $f(x, y) = \frac{\sin y}{x}, -3 \leq x \leq 3, 0 \leq y \leq 2\pi$

Let  $F(x, y, z) = \frac{\sin y}{x} - z$

$$\nabla F(x, y, z) = \frac{-\sin y}{x^2} \mathbf{i} + \frac{\cos y}{x} \mathbf{j} - \mathbf{k}$$

$$\nabla F\left(2, \frac{\pi}{2}, \frac{1}{2}\right) = -\frac{1}{4} \mathbf{i} - \mathbf{k}$$

Direction numbers:  $-\frac{1}{4}, 0, -1$  or  $1, 0, 4$

Line:  $x = 2 + t, y = \frac{\pi}{2}, z = \frac{1}{2} + 4t$

Tangent plane:  $1(x - 2) + 0\left(y - \frac{\pi}{2}\right) + 4\left(z - \frac{1}{2}\right) = 0 \Rightarrow x + 4z - 4 = 0$

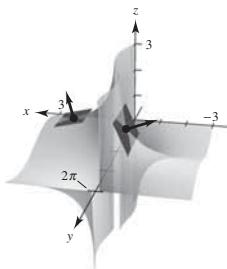
(b)  $\nabla F\left(-\frac{2}{3}, \frac{3\pi}{2}, \frac{3}{2}\right) = \frac{9}{4} \mathbf{i} - \mathbf{k}$

Direction numbers:  $\frac{9}{4}, 0, -1$  or  $9, 0, -4$

Line:  $x = -\frac{2}{3} + 9t, y = \frac{3\pi}{2}, z = \frac{3}{2} - 4t$

Tangent plane:  $9\left(x + \frac{2}{3}\right) + 0\left(y - \frac{3\pi}{2}\right) - 4\left(z - \frac{3}{2}\right) = 0 \Rightarrow 9x - 4z + 12 = 0$

(c)



69.  $f(x, y) = 6 - x^2 - \frac{y^2}{4}, g(x, y) = 2x + y$

(a)  $F(x, y, z) = z + x^2 + \frac{y^2}{4} - 6 \quad G(x, y, z) = z - 2x - y$

$$\nabla F(x, y, z) = 2x \mathbf{i} + \frac{1}{2}y \mathbf{j} + \mathbf{k} \quad \nabla G(x, y, z) = -2 \mathbf{i} - \mathbf{j} + \mathbf{k}$$

$$\nabla F(1, 2, 4) = 2 \mathbf{i} + \mathbf{j} + \mathbf{k} \quad \nabla G(1, 2, 4) = -2 \mathbf{i} - \mathbf{j} + \mathbf{k}$$

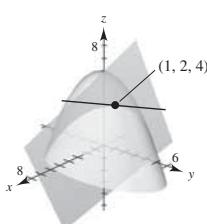
The cross product of these gradients is parallel to the curve of intersection.

$$\nabla F(1, 2, 4) \times \nabla G(1, 2, 4) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & 1 \\ -2 & -1 & 1 \end{vmatrix} = 2 \mathbf{i} - 4 \mathbf{j}$$

Using direction numbers  $1, -2, 0$ , you get  $x = 1 + t, y = 2 - 2t, z = 4$ .

$$\cos \theta = \frac{\nabla F \cdot \nabla G}{\|\nabla F\| \|\nabla G\|} = \frac{-4 - 1 + 1}{\sqrt{6} \sqrt{6}} = \frac{-4}{6} \Rightarrow \theta \approx 48.2^\circ$$

(b)



70. (a)  $f(x, y) = \sqrt{16 - x^2 - y^2 + 2x - 4y}$

$$g(x, y) = \frac{\sqrt{2}}{2} \sqrt{1 - 3x^2 + y^2 + 6x + 4y}$$

(b)  $f(x, y) = g(x, y)$

$$16 - x^2 - y^2 + 2x - 4y = \frac{1}{2}(1 - 3x^2 + y^2 + 6x + 4y)$$

$$32 - 2x^2 - 2y^2 + 4x - 8y = 1 - 3x^2 + y^2 + 6x + 4y$$

$$x^2 - 2x + 31 = 3y^2 + 12y$$

$$(x^2 - 2x + 1) + 42 = 3(y^2 + 4y + 4)$$

$$(x - 1)^2 + 42 = 3(y + 2)^2$$

To find points of intersection, let  $x = 1$ . Then

$$3(y + 2)^2 = 42$$

$$(y + 2)^2 = 14$$

$$y = -2 \pm \sqrt{14}$$

$\nabla f(1, -2 + \sqrt{14}) = -\sqrt{2}\mathbf{j}$ ,  $\nabla g(1, -2 + \sqrt{14}) = (1/\sqrt{2})\mathbf{j}$ . The normals to  $f$  and  $g$  at this point are  $-\sqrt{2}\mathbf{j} - \mathbf{k}$  and  $(-1/\sqrt{2})\mathbf{j} - \mathbf{k}$ , which are orthogonal.

Similarly,  $\nabla f(1, -2 - \sqrt{14}) = \sqrt{2}\mathbf{j}$  and  $\nabla g(1, -2 - \sqrt{14}) = (-1/\sqrt{2})\mathbf{j}$  and the normals are  $\sqrt{2}\mathbf{j} - \mathbf{k}$  and  $(-1/\sqrt{2})\mathbf{j} - \mathbf{k}$ , which are also orthogonal.

(c) No, showing that the surfaces are orthogonal at 2 points does not imply that they are orthogonal at every point of intersection.

71.  $F(x, y, z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1$

$$F_x(x, y, z) = \frac{2x}{a^2}$$

$$F_y(x, y, z) = \frac{2y}{b^2}$$

$$F_z(x, y, z) = \frac{2z}{c^2}$$

$$\text{Plane: } \frac{2x_0}{a^2}(x - x_0) + \frac{2y_0}{b^2}(y - y_0) + \frac{2z_0}{c^2}(z - z_0) = 0$$

$$\frac{x_0x}{a^2} + \frac{y_0y}{b^2} + \frac{z_0z}{c^2} = \frac{x_0^2}{a^2} + \frac{y_0^2}{b^2} + \frac{z_0^2}{c^2} = 1$$

72.  $F(x, y, z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} - 1$

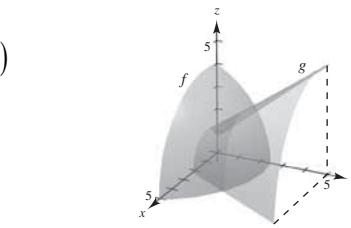
$$F_x(x, y, z) = \frac{2x}{a^2}$$

$$F_y(x, y, z) = \frac{2y}{b^2}$$

$$F_z(x, y, z) = \frac{-2z}{c^2}$$

$$\text{Plane: } \frac{2x_0}{a^2}(x - x_0) + \frac{2y_0}{b^2}(y - y_0) - \frac{2z_0}{c^2}(z - z_0) = 0$$

$$\frac{x_0x}{a^2} + \frac{y_0y}{b^2} - \frac{z_0z}{c^2} = \frac{x_0^2}{a^2} + \frac{y_0^2}{b^2} - \frac{z_0^2}{c^2} = 1$$



73.  $F(x, y, z) = a^2x^2 + b^2y^2 - z^2$

$$F_x(x, y, z) = 2a^2x$$

$$F_y(x, y, z) = 2b^2y$$

$$F_z(x, y, z) = -2z$$

Plane:

$$2a^2x_0(x - x_0) + 2b^2y_0(y - y_0) - 2z_0(z - z_0) = 0$$

$$a^2x_0x + b^2y_0y - z_0z = a^2x_0^2 + b^2y_0^2 - z_0^2 = 0$$

So, the plane passes through the origin.

74.  $z = xf\left(\frac{y}{x}\right)$

$$F(x, y, z) = xf\left(\frac{y}{x}\right) - z$$

$$F_x(x, y, z) = f\left(\frac{y}{x}\right) + xf'\left(\frac{y}{x}\right)\left(-\frac{y}{x^2}\right) = f\left(\frac{y}{x}\right) - \frac{y}{x}f'\left(\frac{y}{x}\right)$$

$$F_y(x, y, z) = xf'\left(\frac{y}{x}\right)\left(\frac{1}{x}\right) = f'\left(\frac{y}{x}\right)$$

$$F_z(x, y, z) = -1$$

Tangent plane at  $(x_0, y_0, z_0)$ :

$$\begin{aligned} & \left[ f\left(\frac{y_0}{x_0}\right) - \frac{y_0}{x_0}f'\left(\frac{y_0}{x_0}\right) \right] (x - x_0) + f'\left(\frac{y_0}{x_0}\right)(y - y_0) - (z - z_0) = 0 \\ & \left[ f\left(\frac{y_0}{x_0}\right) - \frac{y_0}{x_0}f'\left(\frac{y_0}{x_0}\right) \right] x - x_0f\left(\frac{y_0}{x_0}\right) + y_0f'\left(\frac{y_0}{x_0}\right) + y_0f'\left(\frac{y_0}{x_0}\right) - y_0f'\left(\frac{y_0}{x_0}\right) - z + x_0f\left(\frac{y_0}{x_0}\right) = 0 \\ & \left[ f\left(\frac{y_0}{x_0}\right) - \frac{y_0}{x_0}f'\left(\frac{y_0}{x_0}\right) \right] x + f'\left(\frac{y_0}{x_0}\right)y - z = 0 \end{aligned}$$

So, the plane passes through the origin  $(x, y, z) = (0, 0, 0)$ .

75.  $f(x, y) = e^{x-y}$

$$f_x(x, y) = e^{x-y}, f_y(x, y) = -e^{x-y}$$

$$f_{xx}(x, y) = e^{x-y}, f_{yy}(x, y) = e^{x-y}, f_{xy}(x, y) = -e^{x-y}$$

(a)  $P_1(x, y) \approx f(0, 0) + f_x(0, 0)x + f_y(0, 0)y = 1 + x - y$

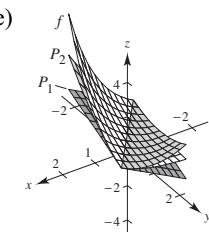
(b)  $P_2(x, y) \approx f(0, 0) + f_x(0, 0)x + f_y(0, 0)y + \frac{1}{2}f_{xx}(0, 0)x^2 + f_{xy}(0, 0)xy + \frac{1}{2}f_{yy}(0, 0)y^2 = 1 + x - y + \frac{1}{2}x^2 - xy + \frac{1}{2}y^2$

(c) If  $x = 0$ ,  $P_2(0, y) = 1 - y + \frac{1}{2}y^2$ . This is the second-degree Taylor polynomial for  $e^{-y}$ .

If  $y = 0$ ,  $P_2(x, 0) = 1 + x + \frac{1}{2}x^2$ . This is the second-degree Taylor polynomial for  $e^x$ .

(d)

$x$	$y$	$f(x, y)$	$P_1(x, y)$	$P_2(x, y)$
0	0	1	1	1
0	0.1	0.9048	0.9000	0.9050
0.2	0.1	1.1052	1.1000	1.1050
0.2	0.5	0.7408	0.7000	0.7450
1	0.5	1.6487	1.5000	1.6250



76.  $f(x, y) = \cos(x + y)$

$$f_x(x, y) = -\sin(x + y), f_y(x, y) = -\sin(x + y)$$

$$f_{xx}(x, y) = -\cos(x + y), f_{yy}(x, y) = -\cos(x + y), f_{xy}(x, y) = -\cos(x + y)$$

(a)  $P_1(x, y) \approx f(0, 0) + f_x(0, 0)x + f_y(0, 0)y = 1$

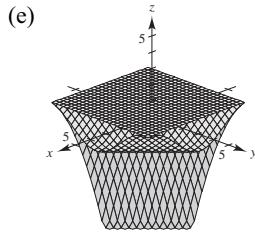
(b)  $P_2(x, y) \approx f(0, 0) + f_x(0, 0)x + f_y(0, 0)y + \frac{1}{2}f_{xx}(0, 0)x^2 + f_{xy}(0, 0)xy + \frac{1}{2}f_{yy}(0, 0)y^2$   
 $= 1 - \frac{1}{2}x^2 - xy - \frac{1}{2}y^2$

(c) If  $x = 0$ ,  $P_2(0, y) = 1 - \frac{1}{2}y^2$ . This is the second-degree Taylor polynomial for  $\cos y$ .

If  $y = 0$ ,  $P_2(x, 0) = 1 - \frac{1}{2}x^2$ . This is the second-degree Taylor polynomial for  $\cos x$ .

(d)

$x$	$y$	$f(x, y)$	$P_1(x, y)$	$P_2(x, y)$
0	0	1	1	1
0	0.1	0.9950	1	0.9950
0.2	0.1	0.9553	1	0.9950
0.2	0.5	0.7648	1	0.7550
1	0.5	0.0707	1	-0.1250



77. Given  $z = f(x, y)$ , then:

$$\begin{aligned} F(x, y, z) &= f(x, y) - z = 0 \\ \nabla F(x_0, y_0, z_0) &= f_x(x_0, y_0)\mathbf{i} + f_y(x_0, y_0)\mathbf{j} - \mathbf{k} \\ \cos \theta &= \frac{|\nabla F(x_0, y_0, z_0) \cdot \mathbf{k}|}{\|\nabla F(x_0, y_0, z_0)\| \|\mathbf{k}\|} \\ &= \frac{|-1|}{\sqrt{[f_x(x_0, y_0)]^2 + [f_y(x_0, y_0)]^2 + (-1)^2}} \\ &= \frac{1}{\sqrt{[f_x(x_0, y_0)]^2 + [f_y(x_0, y_0)]^2 + 1}} \end{aligned}$$

78. Given  $w = F(x, y, z)$  where  $F$  is differentiable at

$$(x_0, y_0, z_0) \text{ and } \nabla F(x_0, y_0, z_0) \neq \mathbf{0},$$

the level surface of  $F$  at  $(x_0, y_0, z_0)$  is of the form  $F(x, y, z) = C$  for some constant  $C$ . Let

$$G(x, y, z) = F(x, y, z) - C = 0.$$

Then  $\nabla G(x_0, y_0, z_0) = \nabla F(x_0, y_0, z_0)$  where  $\nabla G(x_0, y_0, z_0)$  is normal to  $F(x, y, z) - C = 0$  at  $(x_0, y_0, z_0)$ . So,  $\nabla F(x_0, y_0, z_0)$  is normal to the level surface through  $(x_0, y_0, z_0)$ .

## Section 13.8 Extrema of Functions of Two Variables

1.  $g(x, y) = (x - 1)^2 + (y - 3)^2 \geq 0$

Relative minimum:  $(1, 3, 0)$

**Check:**  $g_x = 2(x - 1) = 0 \Rightarrow x = 1$

$$g_y = 2(y - 3) = 0 \Rightarrow y = 3$$

$$g_{xx} = 2, g_{yy} = 2, g_{xy} = 0, d = (2)(2) - 0 = 4 > 0$$

At critical point  $(1, 3)$ ,  $d > 0$  and  $g_{xx} > 0 \Rightarrow$  relative minimum at  $(1, 3, 0)$ .

2.  $g(x, y) = s - (x - 3)^2 - (y + 2)^2 \leq 5$

Relative maximum:  $(3, -2, 5)$

**Check:**  $g_x = -2(x - 3) = 0 \Rightarrow x = 3$

$$g_y = -2(y + 2) = 0 \Rightarrow y = -2$$

$$g_{xx} = -2, g_{yy} = -2, g_{xy} = 0$$

$$d = (-2)(-2) - 0 = 4 > 0$$

At critical point  $(3, -2)$ ,  $d > 0$  and  $g_{xx} < 0 \Rightarrow$  relative maximum at  $(3, -2, 5)$ .

3.  $f(x, y) = \sqrt{x^2 + y^2 + 1} \geq 1$

Relative minimum:  $(0, 0, 1)$

**Check:**  $f_x = \frac{x}{\sqrt{x^2 + y^2 + 1}} = 0 \Rightarrow x = 0$

$$f_y = \frac{y}{\sqrt{x^2 + y^2 + 1}} = 0 \Rightarrow y = 0$$

$$f_{xx} = \frac{y^2 + 1}{(x^2 + y^2 + 1)^{3/2}}$$

$$f_{yy} = \frac{x^2 + 1}{(x^2 + y^2 + 1)^{3/2}}$$

$$f_{xy} = \frac{-xy}{(x^2 + y^2 + 1)^{3/2}}$$

At the critical point  $(0, 0)$ ,  $f_{xx} > 0$  and

$$f_{xx}f_{yy} - (f_{xy})^2 > 0.$$

So,  $(0, 0, 1)$  is a relative minimum.

4.  $f(x, y) = \sqrt{25 - (x - 2)^2 - y^2} \leq 5$

Relative maximum:  $(2, 0, 5)$

**Check:**  $f_x = -\frac{x - 2}{\sqrt{25 - (x - 2)^2 - y^2}} = 0 \Rightarrow x = 2$

$$f_y = -\frac{y}{\sqrt{25 - (x - 2)^2 - y^2}} = 0 \Rightarrow y = 0$$

$$f_{xx} = -\frac{25 - y^2}{[25 - (x - 2)^2 - y^2]^{3/2}}$$

$$f_{yy} = -\frac{25 - (x - 2)^2}{[25 - (x - 2)^2 - y^2]^{3/2}}$$

$$f_{xy} = -\frac{y(x - 2)}{[25 - (x - 2)^2 - y^2]^{3/2}}$$

At the critical point  $(2, 0)$ ,  $f_{xx} < 0$

$$\text{and } f_{xx}f_{yy} - (f_{xy})^2 > 0.$$

So,  $(2, 0, 5)$  is a relative maximum.

5.  $f(x, y) = x^2 + y^2 + 2x - 6y + 6 = (x + 1)^2 + (y - 3)^2 - 4 \geq -4$

Relative minimum:  $(-1, 3, -4)$

**Check:**  $f_x = 2x + 2 = 0 \Rightarrow x = -1$

$$f_y = 2y - 6 = 0 \Rightarrow y = 3$$

$$f_{xx} = 2, f_{yy} = 2, f_{xy} = 0$$

At the critical point  $(-1, 3)$ ,  $f_{xx} > 0$  and  $f_{xx}f_{yy} - (f_{xy})^2 > 0$ . So,  $(-1, 3, -4)$  is a relative minimum.

6.  $f(x, y) = -x^2 - y^2 + 10x + 12y - 64$

$$= -(x^2 - 10x + 25) - (y^2 - 12y + 36) + 25 + 36 - 64 = -(x - 5)^2 - (y - 6)^2 - 3 \leq -3$$

Relative maximum:  $(5, 6, -3)$

**Check:**  $f_x = -2x + 10 = 0 \Rightarrow x = 5$

$$f_y = -2y + 12 = 0 \Rightarrow y = 6$$

$$f_{xx} = -2, f_{yy} = -2, f_{xy} = 0, d = (-2)(-2) - 0 = 4 > 0$$

At critical point  $(5, 6)$ ,  $d > 0$  and  $f_{xx} < 0 \Rightarrow$  relative maximum at  $(5, 6, -3)$ .

7.  $f(x, y) = 3x^2 + 2y^2 - 6x - 4y + 16$

$$\begin{aligned} f_x &= 6x - 6 = 0 \\ f_y &= 4y - 4 = 0 \end{aligned} \left. \begin{aligned} x &= 1, y = 1 \end{aligned} \right.$$

$$f_{xx} = 6, f_{yy} = 4, f_{xy} = 0, d = 6(4) - 0 = 24 > 0.$$

At the critical point  $(1, 1)$ ,  $d > 0$

and  $f_{xx} > 0 \Rightarrow (1, 1, 11)$  is a relative minimum.

8.  $f(x, y) = -3x^2 - 2y^2 + 3x - 4y + 5$

$$f_x = -6x + 3 = 0 \text{ when } x = \frac{1}{2}.$$

$$f_y = -4y - 4 = 0 \text{ when } y = -1.$$

$$f_{xx} = -6, f_{yy} = -4, f_{xy} = 0$$

At the critical point  $(\frac{1}{2}, -1)$ ,  $f_{xx} < 0$

$$\text{and } f_{xx}f_{yy} - (f_{xy})^2 > 0.$$

So,  $(\frac{1}{2}, -1, \frac{31}{4})$  is relative maximum.

9.  $f(x, y) = -x^2 - 5y^2 + 10x - 10y - 28$

$$\begin{cases} f_x = -2x + 10 = 0 \\ f_y = -10y - 10 = 0 \end{cases} \quad \left. \begin{array}{l} x = 5, y = -1 \\ \text{Solving simultaneously} \end{array} \right\}$$

$$f_{xx} = -2, f_{yy} = -10, f_{xy} = 0, d = (-2)(-10) > 0.$$

At the critical number  $(5, -1)$ ,  $d > 0$

and  $f_{xx} < 0 \Rightarrow (5, -1, 2)$  is a relative maximum.

10.  $f(x, y) = 2x^2 + 2xy + y^2 + 2x - 3$

$$\begin{cases} f_x = 4x + 2y + 2 = 0 \\ f_y = 2x + 2y = 0 \end{cases} \quad \left. \begin{array}{l} \text{Solving simultaneously} \\ \text{yields } x = -1 \text{ and } y = 1. \end{array} \right\}$$

$$f_{xx} = 4, f_{yy} = 2, f_{xy} = 2$$

At the critical point  $(-1, 1)$ ,  $f_{xx} > 0$

$$\text{and } f_{xx}f_{yy} - (f_{xy})^2 > 0.$$

So,  $(-1, 1, -4)$  is a relative minimum.

11.  $f(x, y) = z = x^2 + xy + \frac{1}{2}y^2 - 2x + y$

$$\begin{cases} f_x = 2x + y - 2 = 0 \\ f_y = x + y + 1 = 0 \end{cases} \quad \left. \begin{array}{l} \text{Solving simultaneously} \\ \text{yields } x = 3, y = -4 \end{array} \right\}$$

$$f_{xx} = 2, f_{yy} = 1, f_{xy} = 1, d = 2(1) - 1 = 1 > 0.$$

At the critical point  $(3, -4)$ ,  $d > 0$

and  $f_{xx} > 0 \Rightarrow (3, -4, -5)$  is a relative minimum.

12.  $f(x, y) = -5x^2 + 4xy - y^2 + 16x + 10$

$$\begin{cases} f_x = -10x + 4y + 16 = 0 \\ f_y = 4x - 2y = 0 \end{cases} \quad \left. \begin{array}{l} \text{Solving simultaneously} \\ \text{yields } x = 8 \text{ and } y = 16. \end{array} \right\}$$

$$f_{xx} = -10, f_{yy} = -2, f_{xy} = 4$$

At the critical point  $(8, 16)$ ,  $f_{xx} < 0$

$$\text{and } f_{xx}f_{yy} - (f_{xy})^2 > 0.$$

So,  $(8, 16, 74)$  is a relative maximum.

13.  $f(x, y) = \sqrt{x^2 + y^2}$

$$\begin{cases} f_x = \frac{x}{\sqrt{x^2 + y^2}} = 0 \\ f_y = \frac{y}{\sqrt{x^2 + y^2}} = 0 \end{cases} \quad \left. \begin{array}{l} x = y = 0 \\ \text{Solving simultaneously} \end{array} \right\}$$

Because  $f(x, y) \geq 0$  for all  $(x, y)$  and

$f(0, 0) = 0, (0, 0, 0)$  is a relative minimum.

14.  $h(x, y) = (x^2 + y^2)^{1/3} + 2$

$$\begin{cases} h_x = \frac{2x}{3(x^2 + y^2)^{2/3}} = 0 \\ h_y = \frac{2y}{3(x^2 + y^2)^{2/3}} = 0 \end{cases} \quad \left. \begin{array}{l} x = 0, y = 0 \\ \text{Solving simultaneously} \end{array} \right\}$$

Because  $h(x, y) \geq 2$  for all  $(x, y)$ ,  $(0, 0, 2)$  is a relative minimum.

15.  $g(x, y) = 4 - |x| - |y|$

$(0, 0)$  is the only critical point. Because  $g(x, y) \leq 4$  for all  $(x, y)$ ,  $(0, 0, 4)$  is a relative maximum.

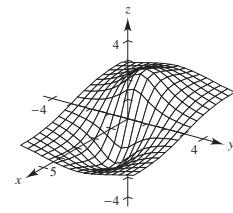
16.  $f(x, y) = |x + y| - 2$

Since  $f(x, y) \geq -2$  for all  $(x, y)$ , the relative minima of  $f$  consist of all points  $(x, y)$  satisfying  $x + y = 0$ .

17.  $z = \frac{-4x}{x^2 + y^2 + 1}$

Relative minimum:  $(1, 0, -2)$

Relative maximum:  $(-1, 0, 2)$

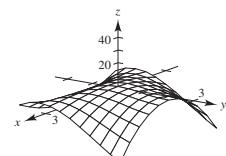


18.  $f(x, y) = y^3 - 3yx^2 - 3y^2 - 3x^2 + 1$

Relative maximum:  $(0, 0, 1)$

Saddle points:

$$(0, 2, -3), (\pm\sqrt{3}, -1, -3)$$

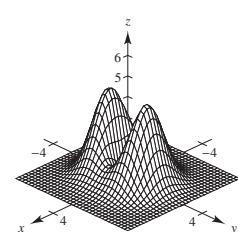


19.  $z = (x^2 + 4y^2)e^{1-x^2-y^2}$

Relative minimum:  $(0, 0, 0)$

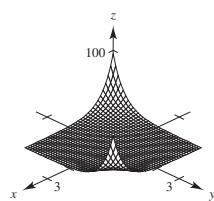
Relative maxima:  $(0, \pm 1, 4)$

Saddle points:  $(\pm 1, 0, 1)$



20.  $z = e^{xy}$

Saddle point:  $(0, 0, 1)$



21.  $h(x, y) = 80x + 80y - x^2 - y^2$

$$\begin{cases} h_x = 80 - 2x = 0 \\ h_y = 80 - 2y = 0 \end{cases} \quad \left. \begin{array}{l} x = y = 40 \\ \end{array} \right.$$

$$h_{xx} = -2, h_{yy} = -2, h_{xy} = 0,$$

$$d = (-2)(-2) - 0 = 4 > 0$$

At the critical point  $(40, 40)$ ,  $d > 0$  and

$h_{xx} < 0 \Rightarrow (40, 40, 3200)$  is a relative maximum.

22.  $g(x, y) = x^2 - y^2 - x - y$

$$\begin{cases} g_x = 2x - 1 = 0 \\ g_y = -2y - 1 = 0 \end{cases} \quad \left. \begin{array}{l} x = 1/2 \\ y = -1/2 \end{array} \right.$$

$$g_{xx} = 2, g_{yy} = -2, g_{xy} = 0, d = 2(-2) - 0 = -4 < 0$$

At the critical point  $(1/2, -1/2)$ ,  $d < 0$

$\Rightarrow (1/2, -1/2, 0)$  is a saddle point.

25.  $f(x, y) = x^2 - xy - y^2 - 3x - y$

$$f_x = 2x - y - 3 = 0$$

$$f_y = -x - 2y - 1 = 0$$

Solving simultaneously yields  $x = 1, y = -1$ .

$$f_{xx} = 2, f_{yy} = -2, f_{xy} = -1$$

$$d = (2)(-2) - (-1)^2 = -5 < 0$$

At the critical point  $(1, -1)$ ,  $d < 0 \Rightarrow (1, -1, -1)$  is a saddle point.

26.  $f(x, y) = 2xy - \frac{1}{2}(x^4 + y^2) + 1$

$$\begin{cases} f_x = 2y - 2x^3 \\ f_y = 2x - 2y^3 \end{cases} \quad \left. \begin{array}{l} \text{Solving by substitution yields 3 critical points:} \\ (0, 0), (1, 1), (-1, -1) \end{array} \right.$$

$$f_{xx} = -6x^2, f_{yy} = -6y^2, f_{xy} = 2$$

At  $(0, 0)$ ,  $f_{xx}f_{yy} - (f_{xy})^2 < 0 \Rightarrow (0, 0, 1)$  saddle point.

At  $(1, 1)$ ,  $f_{xx}f_{yy} - (f_{xy})^2 > 0$  and  $f_{xx} < 0 \Rightarrow (1, 1, 2)$  relative maximum.

At  $(-1, -1)$ ,  $f_{xx}f_{yy} - (f_{xy})^2 > 0$  and  $f_{xx} < 0 \Rightarrow (-1, -1, 2)$  relative maximum.

27.  $f(x, y) = e^{-x} \sin y$

$$\begin{cases} f_x = -e^{-x} \sin y = 0 \\ f_y = e^{-x} \cos y = 0 \end{cases} \quad \left. \begin{array}{l} \text{Because } e^{-x} > 0 \text{ for all } x \text{ and } \sin y \text{ and } \cos y \text{ are never} \\ \text{both zero for a given value of } y, \text{ there are no critical points.} \end{array} \right.$$

23.  $g(x, y) = xy$

$$\begin{cases} g_x = y \\ g_y = x \end{cases} \quad \left. \begin{array}{l} x = 0 \text{ and } y = 0 \\ \end{array} \right.$$

$$g_{xx} = 0, g_{yy} = 0, g_{xy} = 1$$

At the critical point  $(0, 0)$ ,  $g_{xx}g_{yy} - (g_{xy})^2 < 0$ .

So,  $(0, 0, 0)$  is a saddle point.

24.  $h(x, y) = x^2 - 3xy - y^2$

$$\begin{cases} h_x = 2x - 3y = 0 \\ h_y = -3x - 2y = 0 \end{cases} \quad \left. \begin{array}{l} \text{Solving simultaneously} \\ \text{yields } x = 0 \text{ and } y = 0. \end{array} \right.$$

$$h_{xx} = 2, h_{yy} = -2, h_{xy} = -3$$

At the critical point  $(0, 0)$ ,  $h_{xx}h_{yy} - (h_{xy})^2 < 0$ .

So,  $(0, 0, 0)$  is a saddle point.

28.  $f(x, y) = \left( \frac{1}{2} - x^2 + y^2 \right) e^{1-x^2-y^2}$

$$\begin{cases} f_x = (2x^3 - 2xy^2 - 3x)e^{1-x^2-y^2} = 0 \\ f_y = (2x^2y - 2y^3 + y)e^{1-x^2-y^2} = 0 \end{cases}$$

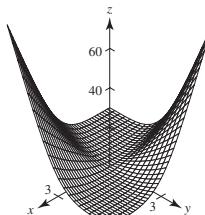
Solving yields the critical points  $(0, 0), \left(0, \pm \frac{\sqrt{2}}{2}\right), \left(\pm \frac{\sqrt{6}}{2}, 0\right)$ .

$$\begin{aligned} f_{xx} &= (-4x^4 + 4x^2y^2 + 12x^2 - 2y^2 - 3)e^{1-x^2-y^2} \\ f_{yy} &= (4y^4 - 4x^2y^2 + 2x^2 - 8y^2 + 1)e^{1-x^2-y^2} \\ f_{xy} &= (-4x^3y + 4xy^3 + 2xy)e^{1-x^2-y^2} \end{aligned}$$

At the critical point  $(0, 0)$ ,  $f_{xx}f_{yy} - (f_{xy})^2 < 0$ . So,  $(0, 0, e/2)$  is a saddle point. At the critical points  $\left(0, \pm \sqrt{2}/2\right)$ ,  $f_{xx} < 0$  and  $f_{xx}f_{yy} - (f_{xy})^2 > 0$ . So,  $\left(0, \pm \sqrt{2}/2, \sqrt{e}\right)$  are relative maxima. At the critical points  $\left(\pm \sqrt{6}/2, 0\right)$ ,  $f_{xx} > 0$  and  $f_{xx}f_{yy} - (f_{xy})^2 > 0$ . So,  $\left(\pm \sqrt{6}/2, 0, -\sqrt{e}/e\right)$  are relative minima.

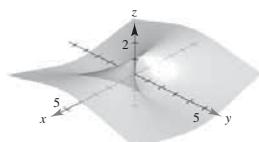
29.  $z = \frac{(x-y)^4}{x^2+y^2} \geq 0$ .  $z = 0$  if  $x = y \neq 0$ .

Relative minimum at all points  $(x, x), x \neq 0$ .



30.  $z = \frac{(x^2 - y^2)^2}{x^2 + y^2} \geq 0$ .  $z = 0$  if  $x^2 = y^2 \neq 0$ .

Relative minima at all points  $(x, x)$  and  $(x, -x), x \neq 0$ .



31.  $f_{xx}f_{yy} - (f_{xy})^2 = (9)(4) - 6^2 = 0$

Insufficient information.

32.  $f_{xx} < 0$  and  $f_{xx}f_{yy} - (f_{xy})^2 = (-3)(-8) - 2^2 > 0$

$f$  has a relative maximum at  $(x_0, y_0)$

33.  $f_{xx}f_{yy} - (f_{xy})^2 = (-9)(6) - 10^2 < 0$

$f$  has a saddle point at  $(x_0, y_0)$ .

34.  $f_{xx} > 0$  and  $f_{xx}f_{yy} - (f_{xy})^2 = (25)(8) - 10^2 > 0$

$f$  has a relative minimum at  $(x_0, y_0)$

35.  $d = f_{xx}f_{yy} - (f_{xy})^2 = (2)(8) - f_{xy}^2 = 16 - f_{xy}^2 > 0$   
 $\Rightarrow f_{xy}^2 < 16 \Rightarrow -4 < f_{xy} < 4$

36.  $d = f_{xx}f_{yy} - (f_{xy})^2 < 0$  if  $f_{xx}$  and  $f_{yy}$  have opposite signs. So,  $(a, b, f(a, b))$  is a saddle point. For example, consider  $f(x, y) = x^2 - y^2$  and  $(a, b) = (0, 0)$ .

37.  $f(x, y) = x^3 + y^3$

$$\begin{cases} f_x = 3x^2 = 0 \\ f_y = 3y^2 = 0 \end{cases} \Rightarrow x = y = 0$$

Critical point:  $(0, 0)$

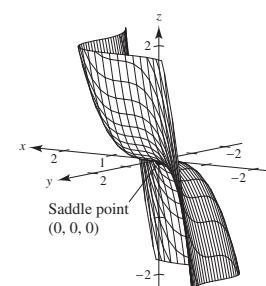
(b)  $f_{xx} = 6x, f_{yy} = 6y, f_{xy} = 0$

At  $(0, 0)$ ,  $f_{xx}f_{yy} - (f_{xy})^2 = 0$ .

$(0, 0, 0)$  is a saddle point.

(c) Test fails at  $(0, 0)$ .

(d)



38.  $f(x, y) = x^3 + y^3 - 6x^2 + 9y^2 + 12x + 27y + 19$

(a)  $f_x = 3x^2 - 12x + 12 = 0$   
 $f_y = 3y^2 + 18y + 27 = 0$

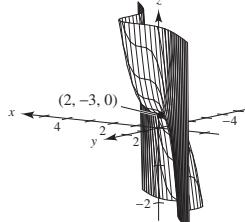
(b)  $f_{xx} = 6x - 12, f_{yy} = 6y + 18, f_{xy} = 0$

At  $(2, -3)$ ,  $f_{xx}f_{yy} - (f_{xy})^2 = 0$ .

$(2, -3, 0)$  is a saddle point.

(c) Test fails at  $(2, -3)$ .

(d)



39.  $f(x, y) = (x - 1)^2(y + 4)^2 \geq 0$

(a)  $f_x = 2(x - 1)(y + 4)^2 = 0$   
 $f_y = 2(x - 1)^2(y + 4) = 0$

(b)  $f_{xx} = 2(y + 4)^2$

$f_{yy} = 2(x - 1)^2$

$f_{xy} = 4(x - 1)(y + 4)$

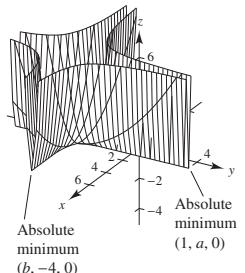
At both  $(1, a)$  and  $(b, -4)$ ,  $f_{xx}f_{yy} - (f_{xy})^2 = 0$ .

Because  $f(x, y) \geq 0$ , there are absolute minima

at  $(1, a, 0)$  and  $(b, -4, 0)$ .

(c) Test fails at  $(1, a)$  and  $(b, -4)$ .

(d)



40.  $f(x, y) = \sqrt{(x - 1)^2 + (y + 2)^2} \geq 0$

(a)  $f_x = \frac{x - 1}{\sqrt{(x - 1)^2 + (y + 2)^2}} = 0$   
 $f_y = \frac{y + 2}{\sqrt{(x - 1)^2 + (y + 2)^2}} = 0$

(b)  $f_{xx} = \frac{(y + 2)^2}{[(x - 1)^2 + (y + 2)^2]^{3/2}}$

$f_{yy} = \frac{(x - 1)^2}{[(x - 1)^2 + (y + 2)^2]^{3/2}}$

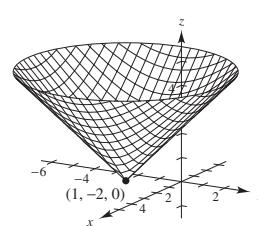
$f_{xy} = \frac{(x - 1)(y + 2)}{[(x - 1)^2 + (y + 2)^2]^{3/2}}$

At  $(1, -2)$ ,  $f_{xx}f_{yy} - (f_{xy})^2$  is undefined.

$(1, -2, 0)$  is an absolute minimum.

(c) Test fails at  $(1, -2)$ .

(d)



41.  $f(x, y) = x^{2/3} + y^{2/3} \geq 0$

(a)  $f_x = \frac{2}{3x^{1/3}}$   
 $f_y = \frac{2}{3y^{1/3}}$

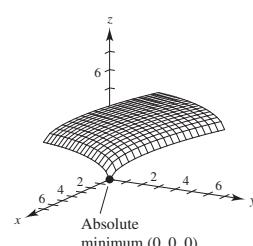
(b)  $f_{xx} = \frac{-2}{9x^{4/3}}, f_{yy} = \frac{-2}{9y^{4/3}}, f_{xy} = 0$

At  $(0, 0)$ ,  $f_{xx}f_{yy} - (f_{xy})^2$  is undefined.

$(0, 0, 0)$  is an absolute minimum.

(c) Test fails at  $(0, 0)$ .

(d)



42.  $f(x, y) = (x^2 + y^2)^{2/3} \geq 0$

(a) 
$$\begin{cases} f_x = \frac{4x}{3(x^2 + y^2)^{1/3}} \\ f_y = \frac{4y}{3(x^2 + y^2)^{1/3}} \end{cases}$$
  $f_x$  and  $f_y$  are undefined at  $x = 0, y = 0$ .  
Critical Point:  $(0, 0)$

(b) 
$$f_{xx} = \frac{4(x^2 + 3y^2)}{9(x^2 + y^2)^{4/3}}$$

$$f_{yy} = \frac{4(3x^2 + y^2)}{9(x^2 + y^2)^{4/3}}$$

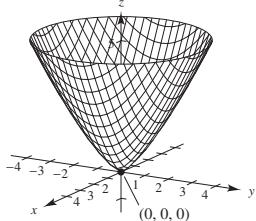
$$f_{xy} = \frac{-8xy}{9(x^2 + y^2)^{4/3}}$$

At  $(0, 0)$ ,  $f_{xx}f_{yy} - (f_{xy})^2$  is undefined.

$(0, 0, 0)$  is an absolute minimum.

(c) Test fails at  $(0, 0)$ .

(d)



43.  $f(x, y, z) = x^2 + (y - 3)^2 + (z + 1)^2 \geq 0$

$$\begin{cases} f_x = 2x = 0 \\ f_y = 2(y - 3) = 0 \\ f_z = 2(z + 1) = 0 \end{cases}$$
 Solving yields the critical point  $(0, 3, -1)$ .

Absolute minimum: 0 at  $(0, 3, -1)$

44.  $f(x, y, z) = 9 - [x(y - 1)(z + 2)]^2 \leq 9$

The absolute maximum value of  $f$  is 9, and realized at all points where  $x(y - 1)(z + 2) = 0$ .

So, the critical points are of the form  $(0, a, b), (c, 1, d), (e, f, -z)$

where  $a, b, c, d, e, f$  are real numbers.

45.  $f(x, y) = x^2 - 4xy + 5, R = \{(x, y); 1 \leq x \leq 4, 0 \leq y \leq 2\}$

$$\begin{cases} f_x = 2x - 4y = 0 \\ f_y = -4x = 0 \end{cases}$$
  $x = y = 0$  (not in region R)

Along  $y = 0, 1 \leq x \leq 4; f = x^2 + 5, f(1, 0) = 6, f(4, 0) = 21$ .

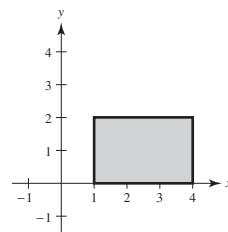
Along  $y = 2, 1 \leq x \leq 4; f = x^2 - 8x + 5, f' = 2x - 8 = 0$

$f(1, 2) = -2, f(4, 2) = -11$ .

Along  $x = 1, 0 \leq y \leq 2; f = -4y + 6, f(1, 0) = 6, f(1, 2) = -2$ .

Along  $x = 4, 0 \leq y \leq 2; f = 21 - 16y, f(4, 0) = 21, f(4, 2) = -11$ .

So, the maximum is  $(4, 0, 21)$  and the minimum is  $(4, 2, -11)$ .



46.  $f(x, y) = x^2 + xy, R = \{(x, y) : |x| \leq 2, |y| \leq 1\}$

$$\begin{cases} f_x = 2x + y = 0 \\ f_y = x = 0 \end{cases} \Rightarrow x = y = 0$$

$$f(0, 0) = 0$$

Along  $y = 1, -2 \leq x \leq 2, f = x^2 + x, f' = 2x + 1 = 0 \Rightarrow x = -\frac{1}{2}$ .

Thus,  $f(-2, 1) = 2, f(-\frac{1}{2}, 1) = -\frac{1}{4}$  and  $f(2, 1) = 6$ .

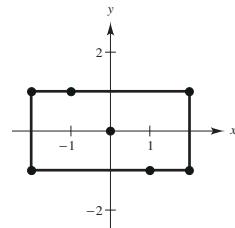
Along  $y = -1, -2 \leq x \leq 2, f = x^2 - x, f' = 2x - 1 = 0 \Rightarrow x = \frac{1}{2}$ .

Thus,  $f(-2, -1) = 6, f(\frac{1}{2}, -1) = -\frac{1}{4}, f(2, -1) = 2$ .

Along  $x = 2, -1 \leq y \leq 1, f = 4 + 2y \Rightarrow f' = 2 \neq 0$ .

Along  $x = -2, -1 \leq y \leq 1, f = 4 - 2y \Rightarrow f' = -2 \neq 0$ .

So, the maxima are  $f(2, 1) = 6$  and  $f(-2, -1) = 6$  and the minima are  $f(-\frac{1}{2}, 1) = -\frac{1}{4}$  and  $f(\frac{1}{2}, -1) = -\frac{1}{4}$ .



47.  $f(x, y) = 12 - 3x - 2y$  has no critical points. On the line  $y = x + 1, 0 \leq x \leq 1$ ,

$$f(x, y) = f(x) = 12 - 3x - 2(x + 1) = -5x + 10$$

and the maximum is 10, the minimum is 5. On the line  $y = -2x + 4, 1 \leq x \leq 2$ ,

$$f(x, y) = f(x) = 12 - 3x - 2(-2x + 4) = x + 4$$

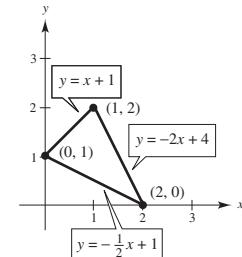
and the maximum is 6, the minimum is 5. On the line  $y = -\frac{1}{2}x + 1, 0 \leq x \leq 2$ ,

$$f(x, y) = f(x) = 12 - 3x - 2(-\frac{1}{2}x + 1) = -2x + 10$$

and the maximum is 10, the minimum is 6.

Absolute maximum: 10 at  $(0, 1)$

Absolute minimum: 5 at  $(1, 2)$



48.  $f(x, y) = (2x - y)^2$

$$f_x = 4(2x - y) = 0 \Rightarrow 2x = y$$

$$f_y = -2(2x - y) = 0 \Rightarrow 2x = y$$

On the line  $y = x + 1, 0 \leq x \leq 1$ ,

$$f(x, y) = f(x) = (2x - (x + 1))^2 = (x - 1)^2$$

and the maximum is 1, the minimum is 0. On the line  $y = -\frac{1}{2}x + 1, 0 \leq x \leq 2$ ,

$$f(x, y) = f(x) = (2x - (-\frac{1}{2}x + 1))^2 = (\frac{5}{2}x - 1)^2$$

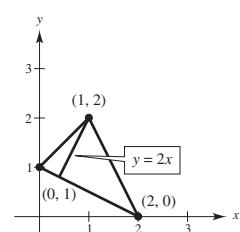
and the maximum is 16, the minimum is 0. On the line  $y = -2x + 4, 1 \leq x \leq 2$ ,

$$f(x, y) = f(x) = (2x - (-2x + 4))^2 = (4x - 4)^2$$

and the maximum is 16, the minimum is 0.

Absolute maximum: 16 at  $(2, 0)$

Absolute Minimum: 0 at  $(1, 2)$  and along the line  $y = 2x$ .



49.  $f(x, y) = 3x^2 + 2y^2 - 4y$

$$\begin{cases} f_x = 6x = 0 \Rightarrow x = 0 \\ f_y = 4y - 4 = 0 \Rightarrow y = 1 \end{cases} \quad f(0, 1) = -2$$

On the line  $y = 4, -2 \leq x \leq 2$ ,

$$f(x, y) = f(x) = 3x^2 + 32 - 16 = 3x^2 + 16$$

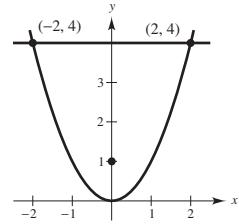
and the maximum is 28, the minimum is 16. On the curve  $y = x^2, -2 \leq x \leq 2$ ,

$$f(x, y) = f(x) = 3x^2 + 2(x^2)^2 - 4x^2 = 2x^4 - x^2 = x^2(2x^2 - 1)$$

and the maximum is 28, the minimum is  $-\frac{1}{8}$ .

Absolute maximum: 28 at  $(\pm 2, 4)$

Absolute minimum: -2 at  $(0, 1)$



50.  $f(x, y) = 2x - 2xy + y^2$

$$\begin{cases} f_x = 2 - 2y = 0 \Rightarrow y = 1 \\ f_y = 2y - 2x = 0 \Rightarrow y = x \Rightarrow x = 1 \end{cases} \quad f(1, 1) = 1$$

On the line  $y = 1, -1 \leq x \leq 1$ ,

$$f(x, y) = f(x) = 2x - 2x + 1 = 1.$$

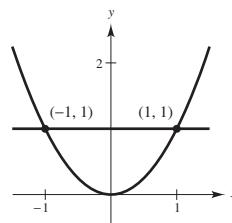
On the curve  $y = x^2, -1 \leq x \leq 1$

$$f(x, y) = f(x) = 2x - 2x(x^2) + (x^2)^2 = x^4 - 2x^3 + 2x$$

and the maximum is 1, the minimum is  $-\frac{11}{16}$ .

Absolute maximum: 1 at  $(1, 1)$  and on  $y = 1$

Absolute minimum:  $-\frac{11}{16} = -0.6875$  at  $(-\frac{1}{2}, \frac{1}{4})$



51.  $f(x, y) = x^2 + 2xy + y^2, R = \{(x, y) : |x| \leq 2, |y| \leq 1\}$

$$\begin{cases} f_x = 2x + 2y = 0 \\ f_y = 2x + 2y = 0 \end{cases} \quad y = -x$$

$$f(x, -x) = x^2 - 2x^2 + x^2 = 0$$

Along  $y = 1, -2 \leq x \leq 2$ ,

$$f = x^2 + 2x + 1, f' = 2x + 2 = 0 \Rightarrow x = -1, f(-2, 1) = 1, f(-1, 1) = 0, f(2, 1) = 9.$$

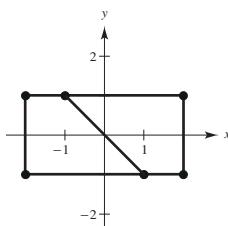
Along  $y = -1, -2 \leq x \leq 2$ ,

$$f = x^2 - 2x + 1, f' = 2x - 2 = 0 \Rightarrow x = 1, f(-2, -1) = 9, f(1, -1) = 0, f(2, -1) = 1.$$

Along  $x = 2, -1 \leq y \leq 1, f = 4 + 4y + y^2, f' = 2y + 4 \neq 0$ .

Along  $x = -2, -1 \leq y \leq 1, f = 4 - 4y + y^2, f' = 2y - 4 \neq 0$ .

So, the maxima are  $f(-2, -1) = 9$  and  $f(2, 1) = 9$ , and the minima are  $f(x, -x) = 0, -1 \leq x \leq 1$ .



52.  $f(x, y) = x^2 + 2xy + y^2, R = \{(x, y) : x^2 + y^2 \leq 8\}$

$$\begin{cases} f_x = 2x + 2y = 0 \\ f_y = 2x + 2y = 0 \end{cases} \Rightarrow y = -x$$

$$f(x, -x) = x^2 - 2x^2 + x^2 = 0$$

On the boundary  $x^2 + y^2 = 8$ , we have  $y^2 = 8 - x^2$  and  $y = \pm\sqrt{8 - x^2}$ . Thus,

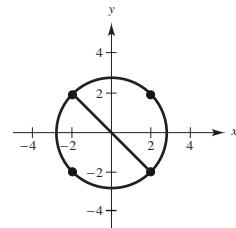
$$f = x^2 \pm 2x\sqrt{8 - x^2} + (8 - x^2) = 8 \pm 2x\sqrt{8 - x^2}$$

$$f' = \pm \left[ (8 - x^2)^{-1/2}(-2x^2) + 2(8 - x^2)^{1/2} \right] = \pm \frac{16 - 4x^2}{\sqrt{8 - x^2}}.$$

Then,  $f' = 0$  implies  $16 = 4x^2$  or  $x = \pm 2$ .

$$f(2, 2) = f(-2, -2) = 16 \text{ and } f(2, -2) = f(-2, 2) = 0$$

So, the maxima are  $f(2, 2) = 16$  and  $f(-2, -2) = 16$ , and the minima are  $f(x, -x) = 0, |x| \leq 2$ .

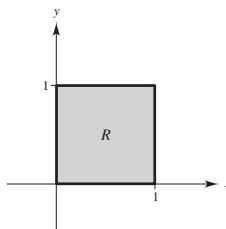


53.  $f(x, y) = \frac{4xy}{(x^2 + 1)(y^2 + 1)}, R = \{(x, y) : 0 \leq x \leq 1, 0 \leq y \leq 1\}$

$$f_x = \frac{4(1 - x^2)y}{(y^2 + 1)(x^2 + 1)^2} = 0 \Rightarrow x = 1 \text{ or } y = 0$$

$$f_y = \frac{4(1 - y^2)x}{(x^2 + 1)(y^2 + 1)^2} = 0 \Rightarrow x = 0 \text{ or } y = 1$$

For  $x = 0, y = 0$ , also, and  $f(0, 0) = 0$ .



For  $x = 1, y = 1, f(1, 1) = 1$ .

The absolute maximum is  $1 = f(1, 1)$ .

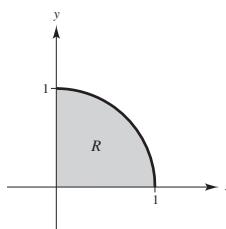
The absolute minimum is  $0 = f(0, 0)$ . (In fact,  $f(0, y) = f(x, 0) = 0$ .)

54.  $f(x, y) = \frac{4xy}{(x^2 + 1)(y^2 + 1)}, R = \{(x, y) : x \geq 0, y \geq 0, x^2 + y^2 \leq 1\}$

$$f_x = \frac{4(1 - x^2)y}{(y^2 + 1)(x^2 + 1)^2} = 0 \Rightarrow x = 1 \text{ or } y = 0$$

$$f_y = \frac{4(1 - y^2)x}{(x^2 + 1)(y^2 + 1)^2} = 0 \Rightarrow y = 1 \text{ or } x = 0$$

For  $x = 0, y = 0$ , also, and  $f(0, 0) = 0$ .



For  $x = 1$  and  $y = 1$ , the point  $(1, 1)$  is outside  $R$ .

For  $x^2 + y^2 = 1, f(x, y) = f(x, \sqrt{1 - x^2}) = \frac{4x\sqrt{1 - x^2}}{2 + x^2 - x^4}$ , and the maximum occurs at  $x = \frac{\sqrt{2}}{2}, y = \frac{\sqrt{2}}{2}$ .

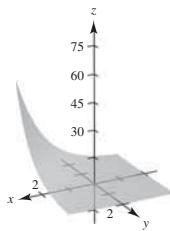
$$\text{Absolute maximum is } \frac{8}{9} = f\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right).$$

The absolute minimum is  $0 = f(0, 0)$ . (In fact,  $f(0, y) = f(x, 0) = 0$ )

55. In this case, the point  $A$  will be a saddle point. The function could be  $f(x, y) = xy$ .

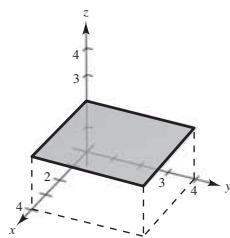
56.  $A$  and  $B$  are relative extrema.  $C$  and  $D$  are saddle points.

57.



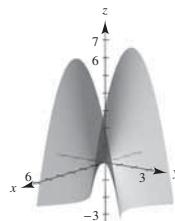
No extrema

58.



Extrema at all  $(x, y)$

59.



Saddle point

## Section 13.9 Applications of Extrema of Functions of Two Variables

1. A point on the plane is given by

$(x, y, z) = (x, y, 3 - x + y)$ . The square of the distance from  $(0, 0, 0)$  to this point is

$$S = x^2 + y^2 + (3 - x + y)^2.$$

$$S_x = 2x - 2(3 - x + y)$$

$$S_y = 2y + 2(3 - x + y)$$

From the equations  $S_x = 0$  and  $S_y = 0$  we obtain

$$4x - 2y = 6$$

$$-2x + 4y = -6.$$

Solving simultaneously, we have  $x = 1$ ,  $y = -1$ ,  $z = 1$ .

So, the distance is  $\sqrt{1^2 + (-1)^2 + 1^2} = \sqrt{3}$ .

60.  $f(x, y) = x^2 - y^2$ ,  $g(x, y) = x^2 + y^2$

- (a)  $f_x = 2x = 0$ ,  $f_y = -2y = 0 \Rightarrow (0, 0)$  is a critical point.

- $g_x = 2x = 0$ ,  $g_y = 2y = 0 \Rightarrow (0, 0)$  is a critical point.

- (b)  $f_{xx} = 2$ ,  $f_{yy} = -2$ ,  $f_{xy} = 0$

- $d = 2(-2) - 0 < 0 \Rightarrow (0, 0)$  is a saddle point.

- $g_{xx} = 2$ ,  $g_{yy} = 2$ ,  $g_{xy} = 0$

- $d = 2(2) - 0 > 0 \Rightarrow (0, 0)$  is a relative minimum.

61. False.

Let  $f(x, y) = 1 - |x| - |y|$ .

$(0, 0, 1)$  is a relative maximum, but  $f_x(0, 0)$  and  $f_y(0, 0)$  do not exist.

62. False. Consider  $f(x, y) = x^2 - y^2$ .

Then  $f_x(0, 0) = f_y(0, 0) = 0$ , but  $(0, 0, 0)$  is a saddle point.

63. False. Let  $f(x, y) = x^2 y^2$  (See Example 4 on page 958).

64. False.

Let  $f(x, y) = x^4 - 2x^2 + y^2$ .

Relative minima:  $(\pm 1, 0, -1)$

Saddle point:  $(0, 0, 0)$

2. A point on the plane is given by

$(x, y, z) = (x, y, 3 - x + y)$ . The square of the distance from  $(1, 2, 3)$  to this point is

$$\begin{aligned} S &= (x - 1)^2 + (y - 2)^2 + (3 - x + y - 3)^2 \\ &= (x - 1)^2 + (y - 2)^2 + (y - x)^2. \end{aligned}$$

$$S_x = 2(x - 1) - 2(y - x)$$

$$S_y = 2(y - 2) + 2(y - x)$$

From the equation  $S_x = 0$  and  $S_y = 0$  we obtain

$$4x - 2y = 2$$

$$-2x + 4y = 4.$$

Solving simultaneously, we have

$$x = 4/3, y = 5/3, z = 10/3.$$

So, the distance is

$$\sqrt{\left(\frac{4}{3} - 1\right)^2 + \left(\frac{5}{3} - 2\right)^2 + \left(\frac{5}{3} - \frac{4}{3}\right)^2} = \frac{\sqrt{13}}{3}.$$

3. A point on the surface is given by  $(x, y, z) = (x, y, \sqrt{1 - 2x - 2y})$ . The square of the distance from  $(-2, -2, 0)$  to a point on the surface is given by

$$S = (x + 2)^2 + (y + 2)^2 + (\sqrt{1 - 2x - 2y} - 0)^2 = (x + 2)^2 + (y + 2)^2 + 1 - 2x - 2y.$$

$$S_x = 2(x + 2) - 2$$

$$S_y = 2(y + 2) - 2$$

From the equations  $S_x = 0$  and  $S_y = 0$ , we obtain  $\begin{cases} 2x + 2 = 0 \\ 2y + 2 = 0 \end{cases} \Rightarrow x = y = -1, z = \sqrt{5}$ .

So, the distance is  $\sqrt{(-1+2)^2 + (-1+2)^2 + (\sqrt{5})^2} = \sqrt{7}$ .

4. A point on the surface is given by  $(x, y, z) = (x, y, \sqrt{1 - 2x - 2y})$ . The square of the distance from  $(0, 0, 2)$  to a point on the surface is given by

$$S = x^2 + y^2 + (2 - \sqrt{1 - 2x - 2y})^2.$$

$$S_x = 2x + \frac{2(2 - \sqrt{1 - 2x - 2y})}{\sqrt{1 - 2x - 2y}} = 2x + \frac{4}{\sqrt{1 - 2x - 2y}} - 2$$

$$S_y = 2y + \frac{2(2 - \sqrt{1 - 2x - 2y})}{\sqrt{1 - 2x - 2y}} = 2y + \frac{4}{\sqrt{1 - 2x - 2y}} - 2$$

From the equation  $S_x = 0$  and  $S_y = 0$ , we obtain  $x = y$ .

$$\text{So, } 2x + \frac{4}{\sqrt{1 - 4x}} - 2 = 0.$$

Using a graphing utility,  $x = y \approx -0.3221$ .

The minimum distance is  $\sqrt{S} = \left[ x^2 + y^2 + (2 - \sqrt{1 - 2x - 2y})^2 \right]^{1/2} \approx 0.667$ .

5. Let  $x, y$ , and  $z$  be the numbers. Because  $xyz = 27$ ,

$$Z = \frac{27}{xy}.$$

$$S = x + y + z = x + y + \frac{27}{xy}.$$

$$S_x = 1 - \frac{27}{x^2y} = 0, S_y = 1 - \frac{27}{xy^2} = 0.$$

$$\begin{cases} x^2y = 27 \\ xy^2 = 27 \end{cases} \Rightarrow x = y = 3$$

So,  $x = y = z = 3$ .

6. Because  $x + y + z = 32$ ,  $z = 32 - x - y$ . So,

$$P = xy^2z = 32xy^2 - x^2y^2 - xy^3$$

$$P_x = 32y^2 - 2xy^2 - y^3 = y^2(32 - 2x - y) = 0$$

$$P_y = 64xy - 2x^2y - 3xy^2 = y(64x - 2x^2 - 3xy) = 0.$$

Ignoring the solution  $y = 0$  and substituting

$y = 32 - 2x$  into  $P_y = 0$ , we have

$$64x - 2x^2 - 3x(32 - 2x) = 0$$

$$4x(x - 8) = 0.$$

So,  $x = 8, y = 16$ , and  $z = 8$ .

7. Let  $x, y$ , and  $z$  be the numbers and let

$$S = x^2 + y^2 + z^2. \text{ Because}$$

$$x + y + z = 30, \text{ we have}$$

$$S = x^2 + y^2 + (30 - x - y)^2$$

$$S_x = 2x + 2(30 - x - y)(-1) = 0 \quad \begin{cases} 2x + y = 30 \\ x + 2y = 30 \end{cases}$$

$$S_y = 2y + 2(30 - x - y)(-1) = 0 \quad \begin{cases} x + 2y = 30 \\ x + y = 30 \end{cases}$$

Solving simultaneously yields  $x = 10$ ,

$y = 10$ , and  $z = 10$ .

8. Let  $x, y$ , and  $z$  be the numbers. Because  $xyz = 1, z = 1/xy$ .

$$S = x^2 + y^2 + z^2 = x^2 + y^2 + \frac{1}{x^2y^2}$$

$$S_x = 2x - \frac{2}{x^3y^2} = 0, S_y = 2y - \frac{2}{x^2y^3} = 0$$

$$\begin{cases} x(x^3y^2) = 1 \\ y(x^2y^3) = 1 \end{cases} \Rightarrow x^4y^2 = x^2y^4 \Rightarrow x = y$$

So,  $x = y = z = 1$ .

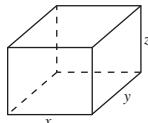
9. The volume is  $668.25 = xyz \Rightarrow z = \frac{668.25}{xy}$ .

$$C = 0.06(2yz + 2xz) + 0.11(xy) = 0.12\left(\frac{668.25}{x} + \frac{668.25}{y}\right) + 0.11(xy)$$

$$C = \frac{80.19}{x} + \frac{80.19}{y} + 0.11(xy)$$

$$C_x = \frac{-80.19}{x^2} + 0.11y = 0$$

$$C_y = \frac{-8.19}{y^2} + 0.11x = 0$$



Solving simultaneously,  $x = y = 9$  and  $z = 8.25$ .

$$\text{Minimum cost: } \frac{80.19}{9} + \frac{80.19}{9} + 0.11(xy) = \$26.73$$

10. Let  $x$ ,  $y$ , and  $z$  be the length, width, and height, respectively. Then  $C_0 = 1.5xy + 2yz + 2xz$  and  $z = \frac{C_0 - 1.5xy}{2(x + y)}$ .

The volume is given by

$$V = xyz = \frac{C_0xy - 1.5x^2y^2}{2(x + y)}$$

$$V_x = \frac{y^2(2C_0 - 3x^2 - 6xy)}{4(x + y)^2}$$

$$V_y = \frac{x^2(2C_0 - 3y^2 - 6xy)}{4(x + y)^2}.$$

In solving the system  $V_x = 0$  and  $V_y = 0$ , we note by the symmetry of the equations that  $y = x$ .

Substituting  $y = x$  into  $V_x = 0$  yields

$$\frac{x^2(2C_0 - 9x^2)}{16x^2} = 0, 2C_0 = 9x^2, x = \frac{1}{3}\sqrt{2C_0}, y = \frac{1}{3}\sqrt{2C_0}, \text{ and } z = \frac{1}{4}\sqrt{2C_0}.$$

11. Let  $a + b + c = k$ . Then  $V = \frac{4\pi abc}{3} = \frac{4}{3}\pi ab(k - a - b) = \frac{4}{3}\pi(kab - a^2b - ab^2)$

$$V_a = \frac{4\pi}{3}(kb - 2ab - b^2) = 0 \quad \left. \begin{aligned} kb - 2ab - b^2 &= 0 \\ ka - a^2 - 2ab &= 0 \end{aligned} \right\}$$

$$V_b = \frac{4\pi}{3}(ka - a^2 - 2ab) = 0 \quad \left. \begin{aligned} ka - a^2 - 2ab &= 0 \end{aligned} \right\}$$

Solving this system simultaneously yields  $a = b$  and substitution yields  $b = k/3$ . So, the solution is  $a = b = c = k/3$ .

12. Consider the sphere given by  $x^2 + y^2 + z^2 = r^2$  and let a vertex of the rectangular box be  $(x, y, \sqrt{r^2 - x^2 - y^2})$ .

Then the volume is given by

$$V = (2x)(2y)\left(2\sqrt{r^2 - x^2 - y^2}\right) = 8xy\sqrt{r^2 - x^2 - y^2}$$

$$V_x = 8\left(xy\frac{-x}{\sqrt{r^2 - x^2 - y^2}} + y\sqrt{r^2 - x^2 - y^2}\right) = \frac{8y}{\sqrt{r^2 - x^2 - y^2}}(r^2 - 2x^2 - y^2) = 0$$

$$V_y = 8\left(xy\frac{-y}{\sqrt{r^2 - x^2 - y^2}} + x\sqrt{r^2 - x^2 - y^2}\right) = \frac{8x}{\sqrt{r^2 - x^2 - y^2}}(r^2 - x^2 - 2y^2) = 0.$$

Solving the system

$$2x^2 + y^2 = r^2$$

$$x^2 + 2y^2 = r^2$$

yields the solution  $x = y = z = r/\sqrt{3}$ .

13. Let  $x$ ,  $y$ , and  $z$  be the length, width, and height, respectively and let  $V_0$  be the given volume.

Then  $V_0 = xyz$  and  $z = V_0/xy$ . The surface area is

$$S = 2xy + 2yz + 2xz = 2\left(xy + \frac{V_0}{x} + \frac{V_0}{y}\right)$$

$$S_x = 2\left(y - \frac{V_0}{x^2}\right) = 0 \quad x^2y - V_0 = 0$$

$$S_y = 2\left(x - \frac{V_0}{y^2}\right) = 0 \quad xy^2 - V_0 = 0.$$

Solving simultaneously yields  $x = \sqrt[3]{V_0}$ ,  $y = \sqrt[3]{V_0}$ , and  $z = \sqrt[3]{V_0}$ .

14.  $A = \frac{1}{2}[(30 - 2x) + (30 - 2x) + 2x \cos \theta]x \sin \theta = 30x \sin \theta - 2x^2 \sin \theta + x^2 \sin \theta \cos \theta$

$$\frac{\partial A}{\partial x} = 30 \sin \theta - 4x \sin \theta + 2x \sin \theta \cos \theta = 0$$

$$\frac{\partial A}{\partial \theta} = 30 \cos \theta - 2x^2 \cos \theta + x^2(2 \cos^2 \theta - 1) = 0$$

From  $\frac{\partial A}{\partial x} = 0$  we have  $15 - 2x + x \cos \theta = 0 \Rightarrow \cos \theta = \frac{2x - 15}{x}$ .

$$\text{From } \frac{\partial A}{\partial \theta} = 0 \text{ we obtain } 30x\left(\frac{2x - 15}{x}\right) - 2x^2\left(\frac{2x - 15}{x}\right) + x^2\left(2\left(\frac{2x - 15}{x}\right)^2 - 1\right) = 0$$

$$30(2x - 15) - 2x(2x - 15) + 2(2x - 15)^2 - x^2 = 0$$

$$3x^2 - 30x = 0$$

$$x = 10.$$

Then  $\cos \theta = \frac{1}{2} \Rightarrow \theta = 60^\circ$ .

15.  $R(x_1, x_2) = -5x_1^2 - 8x_2^2 - 2x_1x_2 + 42x_1 + 102x_2$

$$R_{x_1} = -10x_1 - 2x_2 + 42 = 0, 5x_1 + x_2 = 21$$

$$R_{x_2} = -16x_2 - 2x_1 + 102 = 0, x_1 + 8x_2 = 51$$

Solving this system yields  $x_1 = 3$  and  $x_2 = 6$ .

$$R_{x_1x_1} = -10$$

$$R_{x_1x_2} = -2$$

$$R_{x_2x_2} = -16$$

$$R_{x_1x_1} < 0 \text{ and } R_{x_1x_1}R_{x_2x_2} - (R_{x_1x_2})^2 > 0$$

So, revenue is maximized when  $x_1 = 3$  and  $x_2 = 6$ .

16.  $P(x_1, x_2) = 15(x_1 + x_2) - C_1 - C_2$

$$= 15x_1 + 15x_2 - (0.02x_1^2 + 4x_1 + 500) - (0.05x_2^2 + 4x_2 + 275) = -0.02x_1^2 - 0.05x_2^2 + 11x_1 + 11x_2 - 775$$

$$P_{x_1} = -0.04x_1 + 11 = 0, x_1 = 275$$

$$P_{x_2} = -0.10x_2 + 11 = 0, x_2 = 110$$

$$P_{x_1x_1} = -0.04$$

$$P_{x_1x_2} = 0$$

$$P_{x_2x_2} = -0.10$$

$$P_{x_1x_1} < 0 \text{ and } P_{x_1x_1}P_{x_2x_2} - (P_{x_1x_2})^2 > 0$$

So, profit is maximized when  $x_1 = 275$  and  $x_2 = 110$ .

17.  $P(p, q, r) = 2pq + 2pr + 2qr.$

$p + q + r = 1$  implies that  $r = 1 - p - q.$

$$P(p, q) = 2pq + 2p(1 - p - q) + 2q(1 - p - q)$$

$$= 2pq + 2p - 2p^2 - 2pq + 2q - 2pq - 2q^2 = -2pq + 2p + 2q - 2p^2 - 2q^2$$

$$\frac{\partial P}{\partial p} = -2q + 2 - 4p; \frac{\partial P}{\partial q} = -2p + 2 - 4q$$

Solving  $\frac{\partial P}{\partial p} = \frac{\partial P}{\partial q} = 0$  gives  $q + 2p = 1$   
 $p + 2q = 1$

$$\text{and so } p = q = \frac{1}{3} \text{ and } P\left(\frac{1}{3}, \frac{1}{3}\right) = -2\left(\frac{1}{9}\right) + 2\left(\frac{1}{3}\right) + 2\left(\frac{1}{3}\right) - 2\left(\frac{1}{9}\right) = \frac{6}{9} = \frac{2}{3}.$$

18. (a)  $H = -x \ln x - y \ln y - z \ln z, x + y + z = 1 = -x \ln x - y \ln y - (1 - x - y)\ln(1 - x - y)$

$$H_x = -1 - \ln x + 1 + \ln(1 - x - y) = 0$$

$$H_y = -1 - \ln y + 1 + \ln(1 - x - y) = 0$$

$$\ln(1 - x - y) = \ln x = \ln y \Rightarrow x = y.$$

$$\text{So, } \ln(1 - 2x) = \ln x \Rightarrow 1 - 2x = x \Rightarrow x = y = z = \frac{1}{3}.$$

(b)  $H = -\frac{1}{3} \ln\left(\frac{1}{3}\right) - \frac{1}{3} \ln\left(\frac{1}{3}\right) - \frac{1}{3} \ln\left(\frac{1}{3}\right) = -\ln\left(\frac{1}{3}\right) = \ln 3$

19. The distance from  $P$  to  $Q$  is  $\sqrt{x^2 + 4}$ . The distance from  $Q$  to  $R$  is  $\sqrt{(y - x)^2 + 1}$ . The distance from  $R$  to  $S$  is  $10 - y$ .

$$C = 3k\sqrt{x^2 + 4} + 2k\sqrt{(y - x)^2 + 1} + k(10 - y)$$

$$C_x = 3k\left(\frac{x}{\sqrt{x^2 + 4}}\right) + 2k\left(\frac{-(y - x)}{\sqrt{(y - x)^2 + 1}}\right) = 0$$

$$C_y = 2k\left(\frac{y - x}{\sqrt{(y - x)^2 + 1}}\right) - k = 0 \Rightarrow \frac{y - x}{\sqrt{(y - x)^2 + 1}} = \frac{1}{2}$$

$$3k\left(\frac{x}{\sqrt{x^2 + 4}}\right) + 2k\left(-\frac{1}{2}\right) = 0$$

$$\frac{x}{\sqrt{x^2 + 4}} = \frac{1}{3}$$

$$3x = \sqrt{x^2 + 4}$$

$$9x^2 = x^2 + 4$$

$$x^2 = \frac{1}{2}$$

$$x = \frac{\sqrt{2}}{2}$$

$$2(y - x) = \sqrt{(y - x)^2 + 1}$$

$$4(y - x)^2 = (y - x)^2 + 1$$

$$(y - x)^2 = \frac{1}{3}$$

$$y = \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{2}} = \frac{2\sqrt{3} + 3\sqrt{2}}{6}$$

So,  $x = \frac{\sqrt{2}}{2} \approx 0.707$  km and  $y = \frac{2\sqrt{3} + 3\sqrt{2}}{6} \approx 1.284$  km.

$$\begin{aligned} \mathbf{20. } S &= d_1 + d_2 + d_3 = \sqrt{(0-0)^2 + (y-0)^2} + \sqrt{(0-2)^2 + (y-2)^2} + \sqrt{(0+2)^2 + (y-2)^2} \\ &= y + 2\sqrt{4 + (y-2)^2} \end{aligned}$$

$$\frac{dS}{dy} = 1 + \frac{2(y-2)}{\sqrt{4 + (y-2)^2}} = 0 \text{ when } y = 2 - \frac{2\sqrt{3}}{3} = \frac{6-2\sqrt{3}}{3}.$$

The sum of the distance is minimized when  $y = \frac{2(3-\sqrt{3})}{3} \approx 0.845$ .

$$\begin{aligned} \mathbf{21. (a) } S(x, y) &= d_1 + d_2 + d_3 \\ &= \sqrt{(x-0)^2 + (y-0)^2} + \sqrt{(x+2)^2 + (y-2)^2} + \sqrt{(x-4)^2 + (y-2)^2} \\ &= \sqrt{x^2 + y^2} + \sqrt{(x+2)^2 + (y-2)^2} + \sqrt{(x-4)^2 + (y-2)^2} \end{aligned}$$

From the graph we see that the surface has a minimum.

$$\mathbf{(b) } S_x(x, y) = \frac{x}{\sqrt{x^2 + y^2}} + \frac{x+2}{\sqrt{(x+2)^2 + (y-2)^2}} + \frac{x-4}{\sqrt{(x-4)^2 + (y-2)^2}}$$

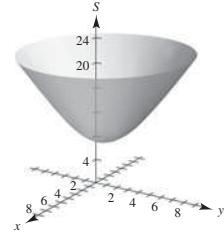
$$S_y(x, y) = \frac{y}{\sqrt{x^2 + y^2}} + \frac{y-2}{\sqrt{(x+2)^2 + (y-2)^2}} + \frac{y-2}{\sqrt{(x-4)^2 + (y-2)^2}}$$

$$\mathbf{(c) } -\nabla S(1, 1) = -S_x(1, 1)\mathbf{i} - S_y(1, 1)\mathbf{j} = -\frac{1}{\sqrt{2}}\mathbf{i} - \left(\frac{1}{\sqrt{2}} - \frac{2}{\sqrt{10}}\right)\mathbf{j}$$

$$\tan \theta = \frac{(2/\sqrt{10}) - (1/\sqrt{2})}{-1/\sqrt{2}} = 1 - \frac{2}{\sqrt{5}} \Rightarrow \theta \approx 186.027^\circ$$

$$\mathbf{(d) } (x_2, y_2) = (x_1 - S_x(x_1, y_1)t, y_1 - S_y(x_1, y_1)t) = \left(1 - \frac{1}{\sqrt{2}}t, 1 + \left(\frac{2}{\sqrt{10}} - \frac{1}{\sqrt{2}}\right)t\right)$$

$$\begin{aligned} S\left(1 - \frac{1}{\sqrt{2}}t, 1 + \left(\frac{2}{\sqrt{10}} - \frac{1}{\sqrt{2}}\right)t\right) &= \sqrt{2 + \left(\frac{2\sqrt{10}}{5} - 2\sqrt{2}\right)t + \left(1 - \frac{2\sqrt{5}}{5} + \frac{2}{5}\right)t^2} \\ &\quad + \sqrt{10 - \left(\frac{2\sqrt{10}}{5} + 2\sqrt{2}\right)t + \left(1 - \frac{2\sqrt{5}}{5} + \frac{2}{5}\right)t^2} \\ &\quad + \sqrt{10 - \left(\frac{2\sqrt{10}}{5} - 4\sqrt{2}\right)t + \left(1 - \frac{2\sqrt{5}}{5} + \frac{2}{5}\right)t^2} \end{aligned}$$



Using a computer algebra system, we find that the minimum occurs when  $t \approx 1.344$ . So  $(x_2, y_2) \approx (0.05, 0.90)$ .

$$\mathbf{(e) } (x_3, y_3) = (x_2 - S_x(x_2, y_2)t, y_2 - S_y(x_2, y_2)t) \approx (0.05 + 0.03t, 0.90 - 0.26t)$$

$$\begin{aligned} S(0.05 + 0.03t, 0.90 - 0.26t) &= \sqrt{(0.05 + 0.03t)^2 + (0.90 - 0.26t)^2} + \sqrt{(2.05 + 0.03t)^2 + (-1.10 - 0.26t)^2} \\ &\quad + \sqrt{(-3.95 + 0.03t)^2 + (-1.10 - 0.26t)^2} \end{aligned}$$

Using a computer algebra system, we find that the minimum occurs when  $t \approx 1.78$ . So  $(x_3, y_3) \approx (0.10, 0.44)$ .

$$(x_4, y_4) = (x_3 - S_x(x_3, y_3)t, y_3 - S_y(x_3, y_3)t) \approx (0.10 - 0.09t, 0.44 - 0.01t)$$

$$\begin{aligned} S(0.10 - 0.09t, 0.45 - 0.01t) &= \sqrt{(0.10 - 0.09t)^2 + (0.45 - 0.01t)^2} + \sqrt{(2.10 - 0.09t)^2 + (-1.55 - 0.01t)^2} \\ &\quad + \sqrt{(-3.90 - 0.09t)^2 + (-1.55 - 0.01t)^2} \end{aligned}$$

Using a computer algebra system, we find that the minimum occurs when  $t \approx 0.44$ . So  $(x_4, y_4) \approx (0.06, 0.44)$ .

**Note:** The minimum occurs at  $(x, y) = (0.0555, 0.3992)$

**(f)**   $-\nabla S(x, y)$  points in the direction that  $S$  decreases most rapidly. You would use  $\nabla S(x, y)$  for maximization problems.

22. (a)  $S = \sqrt{(x+4)^2 + y^2} + \sqrt{(x-1)^2 + (y-6)^2} + \sqrt{(x-12)^2 + (y-2)^2}$

The surface appears to have a minimum near  $(x, y) = (1, 5)$ .

(b)  $S_x = \frac{x+4}{\sqrt{(x+4)^2 + y^2}} + \frac{x-1}{\sqrt{(x-1)^2 + (y-6)^2}} + \frac{x-12}{\sqrt{(x-12)^2 + (y-2)^2}}$

$$S_y = \frac{y}{\sqrt{(x+4)^2 + y^2}} + \frac{y-6}{\sqrt{(x-1)^2 + (y-6)^2}} + \frac{y-2}{\sqrt{(x-12)^2 + (y-2)^2}}$$

(c) Let  $(x_1, y_1) = (1, 5)$ . Then  $-\nabla S(1, 5) = 0.258\mathbf{i} + 0.03\mathbf{j}$ . Direction  $\approx 6.6^\circ$

(d)  $t \approx 0.94$ ,  $x_2 \approx 1.24$ ,  $y_2 \approx 5.03$

(e)  $t \approx 3.56$ ,  $x_3 \approx 1.24$ ,  $y_3 \approx 5.06$ ,  $t \approx 1.04$ ,  $x_4 \approx 1.23$ ,  $y_4 \approx 5.06$

**Note:** Minimum occurs at  $(x, y) = (1.2335, 5.0694)$

(f)  $-\nabla S(x, y)$  points in the direction that  $S$  decreases most rapidly.

23. Write the equation to be maximized or minimized as a function of two variables. Set the partial derivatives equal to zero (or undefined) to obtain the critical points. Use the Second Partial Test to test for relative extrema using the critical points. Check the boundary points, too.

24. See pages 964 and 965.

25. (a)

$x$	$y$	$xy$	$x^2$
-2	0	0	4
0	1	0	0
2	3	6	4
$\sum x_i = 0$	$\sum y_i = 4$	$\sum x_i y_i = 6$	$\sum x_i^2 = 8$

$$a = \frac{3(6) - 0(4)}{3(8) - 0^2} = \frac{3}{4}, b = \frac{1}{3} \left[ 4 - \frac{3}{4}(0) \right] = \frac{4}{3}, y = \frac{3}{4}x + \frac{4}{3}$$

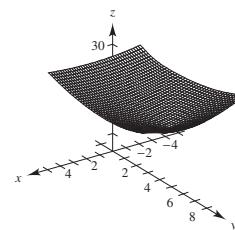
$$(b) S = \left( -\frac{3}{2} + \frac{4}{3} - 0 \right)^2 + \left( \frac{4}{3} - 1 \right)^2 + \left( \frac{3}{2} + \frac{4}{3} - 3 \right)^2 = \frac{1}{6}$$

26. (a)

$x$	$y$	$xy$	$x^2$
-3	0	0	9
-1	1	-1	1
1	1	1	1
3	2	6	9
$\sum x_i = 0$	$\sum y_i = 4$	$\sum x_i y_i = 6$	$\sum x_i^2 = 20$

$$a = \frac{4(6) - 0(4)}{4(20) - (0)^2} = \frac{3}{10}, b = \frac{1}{4} \left[ 4 - \frac{3}{10}(0) \right] = 1, y = \frac{3}{10}x + 1$$

$$(b) S = \left( \frac{1}{10} - 0 \right)^2 + \left( \frac{7}{10} - 1 \right)^2 + \left( \frac{13}{10} - 1 \right)^2 + \left( \frac{19}{10} - 2 \right)^2 = \frac{1}{5}$$



27. (a)

$x$	$y$	$xy$	$x^2$
0	4	0	0
1	3	3	1
1	1	1	1
2	0	0	4
$\sum x_i = 4$	$\sum y_i = 8$	$\sum x_i y_i = 4$	$\sum x_i^2 = 6$

$$a = \frac{4(4) - 4(8)}{4(6) - 4^2} = -2, b = \frac{1}{4}[8 + 2(4)] = 4, y = -2x + 4$$

$$(b) S = (4 - 4)^2 + (2 - 3)^2 + (2 - 1)^2 + (0 - 0)^2 = 2$$

28. (a)

$x$	$y$	$xy$	$x^2$
3	0	0	9
1	0	0	1
2	0	0	4
3	1	3	9
4	1	4	16
4	2	8	16
5	2	10	25
6	2	12	36
$\sum x_i = 28$	$\sum y_i = 8$	$\sum x_i y_i = 37$	$\sum x_i^2 = 116$

$$a = \frac{8(37) - (28)(8)}{8(116) - (28)^2} = \frac{72}{144} = \frac{1}{2}, b = \frac{1}{8}\left[8 - \frac{1}{2}(28)\right] = -\frac{3}{4}, y = \frac{1}{2}x - \frac{3}{4}$$

$$(b) S = \left(\frac{3}{4} - 0\right)^2 + \left(-\frac{1}{4} - 0\right)^2 + \left(\frac{1}{4} - 0\right)^2 + \left(\frac{3}{4} - 1\right)^2 + \left(\frac{5}{4} - 1\right)^2 + \left(\frac{5}{4} - 2\right)^2 + \left(\frac{7}{4} - 2\right)^2 + \left(\frac{9}{4} - 2\right)^2 = \frac{3}{2}$$

29.  $(0, 0), (1, 1), (3, 4), (4, 2), (5, 5)$ 

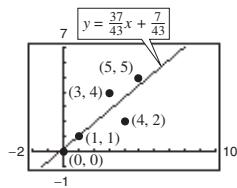
$$\sum x_i = 13, \quad \sum y_i = 12,$$

$$\sum x_i y_i = 46, \quad \sum x_i^2 = 51$$

$$a = \frac{5(46) - 13(12)}{5(51) - (13)^2} = \frac{74}{86} = \frac{37}{43}$$

$$b = \frac{1}{5}\left[12 - \frac{37}{43}(13)\right] = \frac{7}{43}$$

$$y = \frac{37}{43}x + \frac{7}{43}$$

30.  $(1, 0), (3, 3), (5, 6)$ 

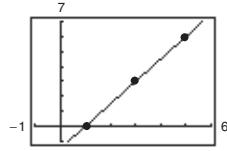
$$\sum x_i = 9, \quad \sum y_i = 9,$$

$$\sum x_i y_i = 39, \quad \sum x_i^2 = 35$$

$$a = \frac{3(39) - 9(9)}{3(35) - (9)^2} = \frac{36}{24} = \frac{3}{2}$$

$$b = \frac{1}{3}\left[9 - \frac{3}{2}(9)\right] = -\frac{9}{6} = -\frac{3}{2}$$

$$y = \frac{3}{2}x - \frac{3}{2}$$

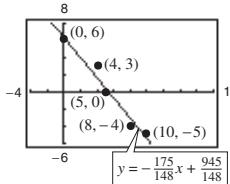


31.  $(0, 6), (4, 3), (5, 0), (8, -4), (10, -5)$

$$\begin{aligned}\sum x_i &= 27, \quad \sum y_i = 0, \\ \sum x_i y_i &= -70, \quad \sum x_i^2 = 205 \\ a &= \frac{5(-70) - (27)(0)}{5(205) - (27)^2} = \frac{-350}{296} = -\frac{175}{148}\end{aligned}$$

$$b = \frac{1}{5} \left[ 0 - \left( -\frac{175}{148} \right) (27) \right] = \frac{945}{148}$$

$$y = -\frac{175}{148}x + \frac{945}{148}$$



32.  $(6, 4), (1, 2), (3, 3), (8, 6), (11, 8), (13, 8); n = 6$

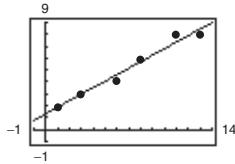
$$\sum x_i = 42 \quad \sum y_i = 31$$

$$\sum x_i y_i = 275 \quad \sum x_i^2 = 400$$

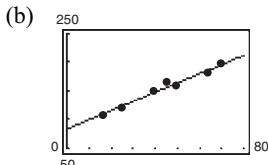
$$a = \frac{6(275) - (42)(31)}{6(400) - (42)^2} = \frac{29}{53} \approx 0.5472$$

$$b = \frac{1}{6} \left( 31 - \frac{29}{53} \cdot 42 \right) = \frac{425}{318} \approx 1.3365$$

$$y = \frac{29}{53}x + \frac{425}{318}$$



33. (a)  $y = 1.6x + 84$



(c) For each one-year increase in age, the pressure changes by approximately 1.6, the slope of the line.

34. (a) Using a graphing utility,  $y = -300x + 832$ .

(b) When  $x = 1.59$ ,  $y = -300(1.59) + 832 = 355$ .

35.  $(1.0, 32), (1.5, 41), (2.0, 48), (2.5, 53)$

$$\sum x_i = 7, \sum y_i = 174, \sum x_i y_i = 322, \sum x_i^2 = 13.5$$

$$a = 14, b = 19, y = 14x + 19$$

When  $x = 1.6$ ,  $y = 41.4$  bushels per acre.

36. (a)  $y = 2.09x - 60.2$

(b) For each one-point increase in the percent  $x$ ,  $y$  increases by about 2.09 million.

$$37. S(a, b, c) = \sum_{i=1}^n (y_i - ax_i^2 - bx_i - c)^2$$

$$\frac{\partial S}{\partial a} = \sum_{i=1}^n -2x_i^2(y_i - ax_i^2 - bx_i - c) = 0$$

$$\frac{\partial S}{\partial b} = \sum_{i=1}^n -2x_i(y_i - ax_i^2 - bx_i - c) = 0$$

$$\frac{\partial S}{\partial c} = -2 \sum_{i=1}^n (y_i - ax_i^2 - bx_i - c) = 0$$

$$a \sum_{i=1}^n x_i^4 + b \sum_{i=1}^n x_i^3 + c \sum_{i=1}^n x_i^2 = \sum_{i=1}^n x_i^2 y_i$$

$$a \sum_{i=1}^n x_i^3 + b \sum_{i=1}^n x_i^2 + c \sum_{i=1}^n x_i = \sum_{i=1}^n x_i y_i$$

$$a \sum_{i=1}^n x_i^2 + b \sum_{i=1}^n x_i + cn = \sum_{i=1}^n y_i$$

38. Let  $x$ ,  $y$ , and  $z$  be the length, width, and height, respectively. Then the sum of the length and girth is given by  $x + (2y + 2z) = 108$

or  $x = 108 - 2y - 2z$ . The volume is given by

$$V = xyz = 108yz - 2zy^2 - 2yz^2$$

$$V_y = 108z - 4yz - 2z^2 = z(108 - 4y - 2z) = 0$$

$$V_z = 108y - 2y^2 - 4yz = y(108 - 2y - 4z) = 0.$$

Solving the system  $4y + 2z = 108$

and  $2y + 4z = 108$ , we obtain the solution

$x = 36$  inches,  $y = 18$  inches, and  $z = 18$  inches.

39.  $(-2, 0), (-1, 0), (0, 1), (1, 2), (2, 5)$

$$\sum x_i = 0$$

$$\sum y_i = 8$$

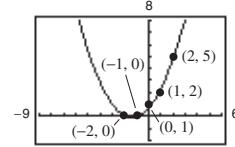
$$\sum x_i^2 = 10$$

$$\sum x_i^3 = 0$$

$$\sum x_i^4 = 34$$

$$\sum x_i y_i = 12$$

$$\sum x_i^2 y_i = 22$$



$$34a + 10c = 22, 10b = 12, 10a + 5c = 8$$

$$a = \frac{3}{7}, b = \frac{6}{5}, c = \frac{26}{35}, y = \frac{3}{7}x^2 + \frac{6}{5}x + \frac{26}{35}$$

40.  $(-4, 5), (-2, 6), (2, 6), (4, 2)$

$$\sum x_i = 0$$

$$\sum y_i = 19$$

$$\sum x_i^2 = 40$$

$$\sum x_i^3 = 0$$

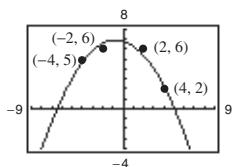
$$\sum x_i^4 = 544$$

$$\sum x_i y_i = -12$$

$$\sum x_i^2 y_i = 160$$

$$544a + 40c = 160, 40b = -12, 40a + 4c = 19$$

$$a = -\frac{5}{24}, b = -\frac{3}{10}, c = \frac{41}{6}, y = -\frac{5}{24}x^2 - \frac{3}{10}x + \frac{41}{6}$$



41.  $(0, 0), (2, 2), (3, 6), (4, 12)$

$$\sum x_i = 9$$

$$\sum y_i = 20$$

$$\sum x_i^2 = 29$$

$$\sum x_i^3 = 99$$

$$\sum x_i^4 = 353$$

$$\sum x_i y_i = 70$$

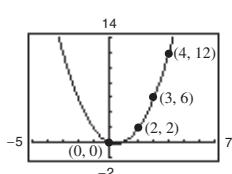
$$\sum x_i^2 y_i = 254$$

$$353a + 99b + 29c = 254$$

$$99a + 29b + 9c = 70$$

$$29a + 9b + 4c = 20$$

$$a = 1, b = -1, c = 0, y = x^2 - x$$



42.  $(0, 10), (1, 9), (2, 6), (3, 0)$

$$\sum x_i = 6$$

$$\sum y_i = 25$$

$$\sum x_i^2 = 14$$

$$\sum x_i^3 = 36$$

$$\sum x_i^4 = 98$$

$$\sum x_i y_i = 21$$

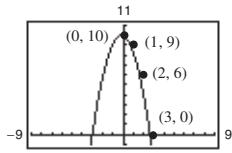
$$\sum x_i^2 y_i = 33$$

$$98a + 36b + 14c = 33$$

$$36a + 14b + 6c = 21$$

$$14a + 6b + 4c = 25$$

$$a = -\frac{5}{4}, b = \frac{9}{20}, c = \frac{199}{20}, y = -\frac{5}{4}x^2 + \frac{9}{20}x + \frac{199}{20}$$



43. (a)  $(0, 0), (2, 15), (4, 30),$

$$(6, 50), (8, 65), (10, 70)$$

$$\sum x_i = 30$$

$$\sum y_i = 230$$

$$\sum x_i^2 = 220$$

$$\sum x_i^3 = 1800$$

$$\sum x_i^4 = 15,664$$

$$\sum x_i y_i = 1670$$

$$\sum x_i^2 y_i = 13,500$$

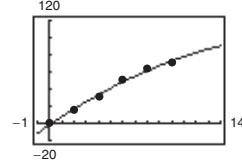
$$15,664a + 1800b + 220c = 13,500$$

$$1800a + 220b + 30c = 1670$$

$$220a + 30b + 6c = 230$$

$$y = -\frac{25}{112}x^2 + \frac{541}{56}x - \frac{25}{14} \approx -0.22x^2 + 9.66x - 1.79$$

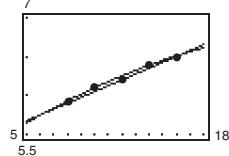
(b)



44. (a)  $y = 0.075x + 5.3$

(b)  $y = -0.002x^2 + 0.12x + 5.1$

(c)



(d) For 2014,  $x = 24$  and

$$y = 0.075(24) + 5.3 = 7.1 \text{ (billion)}$$

$$y = -0.002(24)^2 + 0.12(24) + 5.1 \approx 6.8 \text{ (billion).}$$

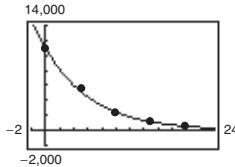
The quadratic model is less accurate because of the negative  $x^2$  coefficient.

45. (a)  $\ln P = -0.1499h + 9.3018$

(b)  $\ln P = -0.1499h + 9.3018$

$P = e^{-0.1499h+9.3018} = 10,957.7e^{-0.1499h}$

(c)

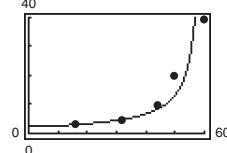


(d) Same answers

46. (a)  $\frac{1}{y} = ax + b = -0.0074x + 0.445$

$y = \frac{1}{-0.0074x + 0.445}$

(b)

(c) No. For  $x = 70$ ,  $y \approx -14$ , which is nonsense. $y = 1000$  which seems inaccurate.

47.  $S(a, b) = \sum_{i=1}^n (ax_i + b - y_i)^2$

$S_a(a, b) = 2a \sum_{i=1}^n x_i^2 + 2b \sum_{i=1}^n x_i - 2 \sum_{i=1}^n x_i y_i$

$S_b(a, b) = 2a \sum_{i=1}^n x_i + 2nb - 2 \sum_{i=1}^n y_i$

$S_{aa}(a, b) = 2 \sum_{i=1}^n x_i^2$

$S_{bb}(a, b) = 2n$

$S_{ab}(a, b) = 2 \sum_{i=1}^n x_i$

 $S_{aa}(a, b) > 0$  as long as  $x_i \neq 0$  for all  $i$ . (Note: If  $x_i = 0$  for all  $i$ , then  $x = 0$  is the least squares regression line.)

$d = S_{aa}S_{bb} - S_{ab}^2 = 4n \sum_{i=1}^n x_i^2 - 4 \left( \sum_{i=1}^n x_i \right)^2 = 4 \left[ n \sum_{i=1}^n x_i^2 - \left( \sum_{i=1}^n x_i \right)^2 \right] \geq 0$  since  $n \sum_{i=1}^n x_i^2 \geq \left( \sum_{i=1}^n x_i \right)^2$ .

As long as  $d \neq 0$ , the given values for  $a$  and  $b$  yield a minimum.

## Section 13.10 Lagrange Multipliers

1. Maximize
- $f(x, y) = xy$
- .

Constraint:  $x + y = 10$ 

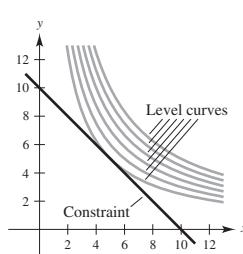
$\nabla f = \lambda \nabla g$

$y\mathbf{i} + x\mathbf{j} = \lambda(\mathbf{i} + \mathbf{j})$

$$\begin{cases} y = \lambda \\ x = \lambda \end{cases} \Rightarrow \begin{cases} x = y \\ x = \lambda \end{cases}$$

$x + y = 10 \Rightarrow x = y = 5$

$f(5, 5) = 25$



2. Maximize
- $f(x, y) = xy$
- .

Constraint:  $2x + y = 4$ 

$\nabla f = \lambda \nabla g$

$y\mathbf{i} + x\mathbf{j} = 2\lambda\mathbf{i} + \lambda\mathbf{j}$

$y = 2\lambda$

$x = \lambda$

$2x + y = 4 \Rightarrow 4\lambda = 4$

$\lambda = 1, x = 1, y = 2$

$f(1, 2) = 2$

3. Minimize  $f(x, y) = x^2 + y^2$ .

Constraint:  $x + y = 4$

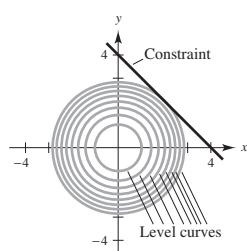
$$\nabla f = \lambda \nabla g$$

$$2x\mathbf{i} + 2y\mathbf{j} = \lambda\mathbf{i} + \lambda\mathbf{j}$$

$$\begin{cases} 2x = \lambda \\ 2y = \lambda \end{cases} \Rightarrow \begin{cases} x = y \\ x = y \end{cases}$$

$$x + y = 4 \Rightarrow x = y = 2$$

$$f(2, 2) = 8$$



4. Minimize  $f(x, y) = x^2 + y^2$ .

Constraint:  $2x + 4y = 5$

$$\nabla f = \lambda \nabla g$$

$$2x\mathbf{i} + 2y\mathbf{j} = 2\lambda\mathbf{i} + 4\lambda\mathbf{j}$$

$$2x = 2\lambda \Rightarrow x = \lambda$$

$$2y = 4\lambda \Rightarrow y = 2\lambda$$

$$2x + 4y = 5 \Rightarrow 10\lambda = 5$$

$$\lambda = \frac{1}{2}, x = \frac{1}{2}, y = 1$$

$$f\left(\frac{1}{2}, 1\right) = \frac{5}{4}$$

5. Minimize  $f(x, y) = x^2 + y^2$ .

Constraint:  $x + 2y - 5 = 0$

$$\nabla f = \lambda \nabla g$$

$$2x\mathbf{i} + 2y\mathbf{j} = \lambda(\mathbf{i} + 2\mathbf{j})$$

$$2x = \lambda \quad \left\{ \begin{array}{l} x = \lambda/2 \\ y = \lambda \end{array} \right.$$

$$2y = 2\lambda \quad \left\{ \begin{array}{l} y = \lambda \\ y = \lambda \end{array} \right.$$

$$x + 2y - 5 = 0$$

$$\frac{\lambda}{2} + 2\lambda = 5 \Rightarrow \lambda = 2, x = 1, y = 2$$

$$f(1, 2) = 5$$

6. Maximize  $f(x, y) = x^2 - y^2$ .

Constraint:  $2y - x^2 = 0$

$$\nabla f = \lambda \nabla g$$

$$2x\mathbf{i} - 2y\mathbf{j} = -2x\lambda\mathbf{i} + 2\lambda\mathbf{j}$$

$$2x = -2x\lambda \Rightarrow x = 0 \text{ or } \lambda = -1$$

If  $x = 0$ , then  $y = 0$  and  $f(0, 0) = 0$ .

If  $\lambda = -1$ ,

$$-2y = 2\lambda = -2 \Rightarrow y = 1 \Rightarrow x^2 = 2 \Rightarrow x = \sqrt{2}.$$

$$f(\sqrt{2}, 1) = 2 - 1 = 1, \text{ Maximum}$$

7. Maximize  $f(x, y) = 2x + 2xy + y$ .

Constraint:  $2x + y = 100$

$$\nabla f = \lambda \nabla g$$

$$(2 + 2y)\mathbf{i} + (2x + 1)\mathbf{j} = 2\lambda\mathbf{i} + \lambda\mathbf{j}$$

$$\begin{cases} 2 + 2y = 2\lambda \Rightarrow y = \lambda - 1 \\ 2x + 1 = \lambda \Rightarrow x = \frac{\lambda - 1}{2} \end{cases} \quad \left\{ \begin{array}{l} y = 2x \\ y = 2x \end{array} \right.$$

$$2x + y = 100 \Rightarrow 4x = 100$$

$$x = 25, y = 50$$

$$f(25, 50) = 2600$$

8. Minimize  $f(x, y) = 3x + y + 10$ .

Constraint:  $x^2y = 6$

$$\nabla f = \lambda \nabla g$$

$$3\mathbf{i} + \mathbf{j} = 2xy\lambda\mathbf{i} + x^2\lambda\mathbf{j}$$

$$\begin{cases} 3 = 2xy\lambda \Rightarrow \lambda = \frac{3}{2xy} \\ 1 = x^2\lambda \Rightarrow \lambda = \frac{1}{x^2} \end{cases} \quad \left\{ \begin{array}{l} 3x^2 = 2xy \Rightarrow y = \frac{3x}{2} \\ (x \neq 0) \end{array} \right.$$

$$x^2y = 6 \Rightarrow x^2\left(\frac{3x}{2}\right) = 6$$

$$x^3 = 4$$

$$x = \sqrt[3]{4}, y = \frac{\sqrt[3]{4}}{2}$$

$$f\left(\sqrt[3]{4}, \frac{\sqrt[3]{4}}{2}\right) = \frac{9\sqrt[3]{4} + 20}{2}$$

9. Note:  $f(x, y) = \sqrt{6 - x^2 - y^2}$  is maximum when  $g(x, y)$  is maximum.

Maximize  $g(x, y) = 6 - x^2 - y^2$ .

Constraint:  $x + y = 2$

$$\begin{cases} -2x = \lambda \\ -2y = \lambda \end{cases} \Rightarrow \begin{cases} x = y \\ x = y \end{cases}$$

$$x + y = 2 \Rightarrow x = y = 1$$

$$f(1, 1) = \sqrt{g(1, 1)} = 2$$

10. Note:  $f(x, y) = \sqrt{x^2 + y^2}$  is minimum when  $g(x, y)$  is minimum.

Minimize  $g(x, y) = x^2 + y^2$ .

Constraint:  $2x + 4y = 15$

$$\begin{cases} 2x = 2\lambda \\ 2y = 4\lambda \end{cases} \Rightarrow \begin{cases} y = 2x \\ y = 2x \end{cases}$$

$$2x + 4y = 15 \Rightarrow 10x = 15$$

$$x = \frac{3}{2}, y = 3$$

$$f\left(\frac{3}{2}, 3\right) = \sqrt{g\left(\frac{3}{2}, 3\right)} = \frac{3\sqrt{5}}{2}$$

11. Minimize  $f(x, y, z) = x^2 + y^2 + z^2$ .

Constraint:  $x + y + z = 9$

$$\begin{cases} 2x = \lambda \\ 2y = \lambda \\ 2z = \lambda \end{cases} \Rightarrow \begin{cases} x = y = z \\ x = y = z \end{cases}$$

$$x + y + z = 9 \Rightarrow x = y = z = 3$$

$$f(3, 3, 3) = 27$$

12. Maximize  $f(x, y, z) = xyz$ .

Constraint:  $x + y + z = 3$

$$\begin{cases} yz = \lambda \\ xz = \lambda \\ xy = \lambda \end{cases} \Rightarrow \begin{cases} yz = xz = xy \Rightarrow x = y = z \\ yz = xz = xy \Rightarrow x = y = z \end{cases}$$

$$x + y + z = 3 \Rightarrow x = y = z = 1$$

$$f(1, 1, 1) = 1$$

13. Minimize  $f(x, y, z) = x^2 + y^2 + z^2$ .

Constraint:  $x + y + z = 1$

$$\begin{cases} 2x = \lambda \\ 2y = \lambda \\ 2z = \lambda \end{cases} \Rightarrow \begin{cases} x = y = z \\ x = y = z \end{cases}$$

$$x + y + z = 1 \Rightarrow x = y = z = \frac{1}{3}$$

$$f\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right) = \frac{1}{3}$$

14. Minimize  $f(x, y) = x^2 - 10x + y^2 - 14y + 28$ .

Constraint:  $x + y = 10$

$$2x - 10 = \lambda \Rightarrow x = \frac{1}{2}(10 + \lambda)$$

$$2y - 14 = \lambda \Rightarrow y = \frac{1}{2}(14 + \lambda)$$

$$x + y = 10$$

$$\frac{1}{2}(10 + \lambda) + \frac{1}{2}(14 + \lambda) = 10$$

$$\lambda = -2$$

$$x = 4, y = 6$$

$$f(4, 6) = 16 - 40 + 36 - 84 + 28 = -44$$

15. Maximize or minimize  $f(x, y) = x^2 + 3xy + y^2$ .

Constraint:  $x^2 + y^2 \leq 1$

Case 1: On the circle  $x^2 + y^2 = 1$

$$\begin{cases} 2x + 3y = 2x\lambda \\ 3x + 2y = 2y\lambda \end{cases} \Rightarrow \begin{cases} x^2 = y^2 \\ x^2 = y^2 \end{cases}$$

$$x^2 + y^2 = 1 \Rightarrow x = \pm\frac{\sqrt{2}}{2}, y = \pm\frac{\sqrt{2}}{2}$$

$$\text{Maxima: } f\left(\pm\frac{\sqrt{2}}{2}, \pm\frac{\sqrt{2}}{2}\right) = \frac{5}{2}$$

$$\text{Minima: } f\left(\pm\frac{\sqrt{2}}{2}, \mp\frac{\sqrt{2}}{2}\right) = -\frac{1}{2}$$

Case 2: Inside the circle

$$\begin{cases} f_x = 2x + 3y = 0 \\ f_y = 3x + 2y = 0 \end{cases} \Rightarrow \begin{cases} x = y = 0 \\ x = y = 0 \end{cases}$$

$$f_{xx} = 2, f_{yy} = 2, f_{xy} = 3, f_{xx}f_{yy} - (f_{xy})^2 \leq 0$$

$$\text{Saddle point: } f(0, 0) = 0$$

By combining these two cases, we have a maximum

of  $\frac{5}{2}$  at  $\left(\pm\frac{\sqrt{2}}{2}, \pm\frac{\sqrt{2}}{2}\right)$  and a minimum of

$$-\frac{1}{2} \text{ at } \left(\pm\frac{\sqrt{2}}{2}, \mp\frac{\sqrt{2}}{2}\right).$$

**16.** Maximize or minimize  $f(x, y) = e^{-xy/4}$ .

Constraint:  $x^2 + y^2 \leq 1$

Case 1: On the circle  $x^2 + y^2 = 1$

$$\begin{cases} -(y/4)e^{-xy/4} = 2x\lambda \\ -(x/4)e^{-xy/4} = 2y\lambda \end{cases} \Rightarrow x^2 = y^2$$

$$x^2 + y^2 = 1 \Rightarrow x = \pm \frac{\sqrt{2}}{2}$$

$$\text{Maxima: } f\left(\pm \frac{\sqrt{2}}{2}, \mp \frac{\sqrt{2}}{2}\right) = e^{1/8} \approx 1.1331$$

$$\text{Minima: } f\left(\pm \frac{\sqrt{2}}{2}, \pm \frac{\sqrt{2}}{2}\right) = e^{-1/8} \approx 0.8825$$

Case 2: Inside the circle

$$\begin{cases} f_x = -(y/4)e^{-xy/4} = 0 \\ f_y = -(x/4)e^{-xy/4} = 0 \end{cases} \Rightarrow x = y = 0$$

$$f_{xx} = \frac{y^2}{16}e^{-xy/4}, f_{yy} = \frac{x^2}{16}e^{-xy/4}, f_{xy} = e^{-xy}\left(\frac{1}{16}xy - \frac{1}{4}\right)$$

$$\text{At } (0, 0), f_{xx}f_{yy} - (f_{xy})^2 < 0.$$

$$\text{Saddle point: } f(0, 0) = 1$$

Combining the two cases, we have a maximum

$$\text{of } e^{1/8} \text{ at } \left(\pm \frac{\sqrt{2}}{2}, \mp \frac{\sqrt{2}}{2}\right) \text{ and a minimum}$$

$$\text{of } e^{-1/8} \text{ at } \left(\pm \frac{\sqrt{2}}{2}, \pm \frac{\sqrt{2}}{2}\right).$$

**17.** Maximize  $f(x, y, z) = xyz$ .

Constraints:  $x + y + z = 32$

$$x - y + z = 0$$

$$\nabla f = \lambda \nabla g + \mu \nabla h$$

$$yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k} = \lambda(\mathbf{i} + \mathbf{j} + \mathbf{k}) + \mu(\mathbf{i} - \mathbf{j} + \mathbf{k})$$

$$yz = \lambda + \mu$$

$$xz = \lambda - \mu \quad \begin{cases} yz = xy \Rightarrow x = z \\ yz = xy \end{cases}$$

$$xy = \lambda + \mu$$

$$\begin{cases} x + y + z = 32 \\ x - y + z = 0 \end{cases} \quad \begin{cases} 2x + 2z = 32 \Rightarrow x = z = 8 \\ y = 16 \end{cases}$$

$$f(8, 16, 8) = 1024$$

**18.** Minimize  $f(x, y, z) = x^2 + y^2 + z^2$ .

Constraints:  $x + 2z = 6$

$$x + y = 12$$

$$\nabla f = \lambda \nabla g + \mu \nabla h$$

$$2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k} = \lambda(\mathbf{i} + 2\mathbf{k}) + \mu(\mathbf{i} + \mathbf{j})$$

$$\begin{cases} 2x = \lambda + \mu \\ 2y = \mu \\ 2z = 2\lambda \end{cases} \quad \begin{cases} 2x = 2y + z \\ 2x = 2y + z \end{cases}$$

$$x + 2z = 6 \Rightarrow z = \frac{6 - x}{2} = 3 - \frac{x}{2}$$

$$x + y = 12 \Rightarrow y = 12 - x$$

$$2x = 2(12 - x) + \left(3 - \frac{x}{2}\right) \Rightarrow \frac{9}{2}x = 27 \Rightarrow x = 6$$

$$x = 6, z = 0$$

$$f(6, 6, 0) = 72$$

**19.**  $f(x, y) = (x - 0)^2 + (y - 0)^2 = x^2 + y^2$

Constraint:  $x + y = 1$

$$\begin{cases} 2x = \lambda \\ 2y = \lambda \end{cases} \quad \begin{cases} x = \lambda/2 \\ y = \lambda/2 \end{cases} \Rightarrow x = y$$

$$x + y = 1$$

$$x = y = \frac{1}{2}$$

$$\text{Minimum distance: } \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2} = \frac{\sqrt{2}}{2}$$

**20.** Minimize the square of the distance  $f(x, y) = x^2 + y^2$

subject to the constraint  $2x + 3y = -1$ .

$$\begin{cases} 2x = 2\lambda \\ 2y = 3\lambda \end{cases} \quad \begin{cases} y = \frac{3x}{2} \\ y = \frac{3\lambda}{2} \end{cases}$$

$$2x + 3y = -1 \Rightarrow x = -\frac{2}{13}, y = -\frac{3}{13}$$

The point on the line is  $\left(-\frac{2}{13}, -\frac{3}{13}\right)$  and the desired

$$\text{distance is } d = \sqrt{\left(-\frac{2}{13}\right)^2 + \left(-\frac{3}{13}\right)^2} = \frac{\sqrt{13}}{13}.$$

21.  $f(x, y) = x^2 + (y - 2)^2$

Constraint:  $x - y = 4$

$$\begin{cases} 2x = \lambda \\ 2(y - 2) = -\lambda \end{cases} \quad \begin{cases} x = \lambda/2 \\ y = \frac{4 - \lambda}{2} \end{cases}$$

$$x - y = 4$$

$$\frac{\lambda}{2} - \left( \frac{4 - \lambda}{2} \right) = 4$$

$$\lambda = 6$$

$$x = 3, y = -1$$

$$\text{Minimum distance: } \sqrt{3^2 + (-1 - 2)^2} = 3\sqrt{2}$$

22.  $f(x, y) = (x - 1)^2 + y^2$

Constraint:  $x + 4y = 3$

$$\begin{cases} 2(x - 1) = \lambda \\ 2y = 4\lambda \end{cases} \quad \begin{cases} x = \frac{\lambda + 2}{2} \\ y = 2\lambda \end{cases}$$

$$x + 4y = 3$$

$$\frac{\lambda + 2}{2} + 4(2\lambda) = 3$$

$$\lambda + 2 + 16\lambda = 6$$

$$17\lambda = 4$$

$$\lambda = \frac{4}{17}$$

$$x = \frac{19}{17}, y = \frac{8}{17}$$

$$\text{Minimum distance: } \sqrt{\left(\frac{19}{17}\right)^2 + \left(\frac{8}{17}\right)^2} = \frac{5\sqrt{17}}{17}$$

23.  $f(x, y) = x^2 + (y - 3)^2$

Constraint:  $y - x^2 = 0$

$$2x = -2x\lambda$$

$$2(y - 3) = \lambda$$

$$y = x^2$$

If  $x = 0, y = 0$ , and  $f(0, 0) = 9 \Rightarrow \text{distance} = 3$ .

If  $x \neq 0, \lambda = -1, y = 5/2, x = \pm\sqrt{5/2}$

$$f\left(\pm\sqrt{5/2}, 5/2\right) = 5/2 + \left(\frac{1}{2}\right)^2 = \frac{11}{4} < 3$$

$$\text{Minimum distance: } \frac{\sqrt{11}}{2}$$

24.  $f(x, y) = (x + 3)^2 + y^2$

Constraint:  $y - x^2 = 0$

$$2(x + 3) = -2\lambda x$$

$$2y = \lambda$$

$$y = x^2$$

$$\lambda = 2y = 2x^2$$

$$2(x + 3) = -2(2x^3)$$

$$4x^3 + 2x + 6 = 0$$

$$2(x + 1)(2x^2 - 2x + 3) = 0 \Rightarrow x = -1, y = 1,$$

$$\text{Minimum distance: } \sqrt{(-1)^2 + (1^2)} = \sqrt{2}$$

25.  $f(x, y) = (x - 1/2)^2 + (y - 1)^2$

Constraint:  $y - x^2 - 1 = 0$

$$2\left(x - \frac{1}{2}\right) = -2x\lambda$$

$$2(y - 1) = \lambda$$

$$y = x^2 + 1$$

$$\lambda = 2y - 2 = 2(x^2 + 1) - 2 = 2x^2$$

$$2\left(x - \frac{1}{2}\right) = -2x(2x^2) = -4x^3$$

$$4x^3 + 2x - 1 = 0$$

$$x \approx 0.3855$$

$$y \approx 1.1486$$

$$\text{Minimum distance: } \sqrt{f(0.3855, 1.1486)} \approx 0.188$$

26. Minimize the square of the distance  $f(x, y) = x^2 + (y - 10)^2$  subject to the constraint  $(x - 4)^2 + y^2 = 4$ .

$$\begin{aligned} 2x &= 2(x - 4)\lambda \quad \left\{ \frac{x}{x - 4} = \frac{y - 10}{y} \Rightarrow y = -\frac{5}{2}x + 10 \right. \\ 2(y - 10) &= 2y\lambda \quad \left. \right\} \\ (x - 4)^2 + y^2 &= 4 \Rightarrow (x^2 - 8x + 16) + \left( \frac{25}{4}x^2 - 50x + 100 \right) = 4 \\ \frac{29}{4}x^2 - 58x + 112 &= 0 \end{aligned}$$

Using a graphing utility, we obtain  $x \approx 3.2572$  and  $x \approx 4.7428$  or by the Quadratic Formula,

$$x = \frac{58 \pm \sqrt{58^2 - 4(29/4)(112)}}{2(29/4)} = \frac{58 \pm 2\sqrt{29}}{29/2} = 4 \pm \frac{4\sqrt{29}}{29}.$$

Using the smaller value, we have  $x = 4\left(1 - \frac{\sqrt{29}}{29}\right)$  and  $y = \frac{10\sqrt{29}}{29} \approx 1.8570$ .

The point on the circle is  $\left[4\left(1 - \frac{\sqrt{29}}{29}\right), \frac{10\sqrt{29}}{29}\right]$  and the desired distance is  $d = \sqrt{16\left(1 - \frac{\sqrt{29}}{29}\right)^2 + \left(\frac{10\sqrt{29}}{29} - 10\right)^2} \approx 8.77$ .

The larger  $x$ -value does not yield a minimum.

27. Minimize the square of the distance

$$f(x, y, z) = (x - 2)^2 + (y - 1)^2 + (z - 1)^2$$

subject to the constraint  $x + y + z = 1$ .

$$\begin{aligned} 2(x - 2) &= \lambda \\ 2(y - 1) &= \lambda \quad \left\{ y = z \text{ and } y = x - 1 \right. \\ 2(z - 1) &= \lambda \quad \left. \right\} \end{aligned}$$

$$x + y + z = 1 \Rightarrow x + 2(x - 1) = 1$$

$$x = 1, y = z = 0$$

The point on the plane is  $(1, 0, 0)$  and the desired distance

$$\text{is } d = \sqrt{(1 - 2)^2 + (0 - 1)^2 + (0 - 1)^2} = \sqrt{3}.$$

28. Minimize the square of the distance

$$f(x, y, z) = (x - 4)^2 + y^2 + z^2$$

subject to the constraint  $\sqrt{x^2 + y^2} - z = 0$ .

$$\begin{aligned} 2(x - 4) &= \frac{x}{\sqrt{x^2 + y^2}}\lambda = \frac{x}{z}\lambda \\ 2y &= \frac{y}{\sqrt{x^2 + y^2}}\lambda = \frac{y}{z}\lambda \\ 2z &= -\lambda \end{aligned} \quad \left\{ \begin{array}{l} 2(x - 4) = -2x \\ 2y = -2y \end{array} \right.$$

$$\sqrt{x^2 + y^2} - z = 0, x = 2, y = 0, z = 2$$

The point on the plane is  $(2, 0, 2)$  and the desired distance

$$\text{is } d = \sqrt{(2 - 4)^2 + 0^2 + 2^2} = 2\sqrt{2}.$$

29. Maximize  $f(x, y, z) = z$  subject to the constraints

$$x^2 + y^2 - z^2 = 0 \text{ and } x + 2z = 4.$$

$$0 = 2x\lambda + \mu$$

$$0 = 2y\lambda \Rightarrow y = 0$$

$$1 = -2z\lambda + 2\mu$$

$$x^2 + y^2 - z^2 = 0$$

$$x + 2z = 4 \Rightarrow x = 4 - 2z$$

$$(4 - 2z)^2 + 0^2 - z^2 = 0$$

$$3z^2 - 16z + 16 = 0$$

$$(3z - 4)(z - 4) = 0$$

$$z = \frac{4}{3} \text{ or } z = 4$$

The maximum value of  $f$  occurs when  $z = 4$  at the point  $(-4, 0, 4)$ .

30. Maximize  $f(x, y, z) = z$  subject to the constraints

$$x^2 + y^2 + z^2 = 36 \text{ and } 2x + y - z = 2.$$

$$0 = 2x\lambda + 2\mu$$

$$0 = 2y\lambda + \mu \quad \left\{ \begin{array}{l} x = 2y \\ 1 = 2z\lambda - \mu \end{array} \right.$$

$$x^2 + y^2 + z^2 = 36$$

$$2x + y - z = 2 \Rightarrow z = 2x + y - 2 = 5y - 2$$

$$(2y)^2 + y^2 + (5y - 2)^2 = 36$$

$$30y^2 - 20y - 32 = 0$$

$$15y^2 - 10y - 16 = 0$$

$$y = \frac{5 \pm \sqrt{265}}{15}$$

Choosing the positive value for  $y$  we have the point

$$\left( \frac{10 + 2\sqrt{265}}{15}, \frac{5 + \sqrt{265}}{15}, \frac{-1 + \sqrt{265}}{3} \right).$$

31. Optimization problems that have restrictions or constraints on the values that can be used to produce the optimal solution are called constrained optimization problems.

32. See explanation at the bottom of page 971.

33. Minimize  $f(x, y, z) = x^2 + y^2 + z^2$ .

Constraint:  $g(x, y, z) = x - y + z = 3$

$$2x = \lambda \Rightarrow x = \lambda/2$$

$$2y = -\lambda \Rightarrow y = -\lambda/2$$

$$2z = \lambda \Rightarrow z = \lambda/2$$

$$x - y + z = 3$$

$$\frac{\lambda}{2} - \left(-\frac{\lambda}{2}\right) + \frac{\lambda}{2} = 3$$

$$\frac{3\lambda}{2} = 3$$

$$\lambda = 2$$

$$x = 1, y = -1, z = 1$$

$$\text{Minimum distance} = \sqrt{1^2 + (-1)^2 + 1^2} = \sqrt{3}$$

34. Minimize  $f(x, y, z) = (x - 1)^2 + (y - 2)^2 + (z - 3)^2$ .

Constraint:  $g(x, y, z) = x - y + z = 3$

$$2(x - 1) = \lambda \Rightarrow x = \frac{2 + \lambda}{2}$$

$$2(y - 2) = -\lambda \Rightarrow y = \frac{4 - \lambda}{2}$$

$$2(z - 3) = \lambda \Rightarrow z = \frac{6 + \lambda}{2}$$

$$x - y + z = 3$$

$$\frac{2 + \lambda}{2} - \frac{4 - \lambda}{2} + \frac{6 + \lambda}{2} = 3$$

$$3\lambda + 4 = 6$$

$$\lambda = \frac{2}{3}$$

$$x = \frac{4}{3}, y = \frac{5}{3}, z = \frac{10}{3}$$

$$\text{Minimum distance} = \left(\frac{1}{3}\right)^2 + \left(-\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^2 = \frac{\sqrt{3}}{3}$$

35. Minimize  $f(x, y, z) = x + y + z$ .

Constraint:  $g(x, y, z) = xyz = 27$

$$\begin{aligned} 1 &= \lambda yz \Rightarrow x = \lambda xyz \\ 1 &= \lambda xz \Rightarrow y = \lambda xyz \\ 1 &= \lambda xy \Rightarrow z = \lambda xyz \end{aligned} \Rightarrow x = y = z$$

$$xyz = 27$$

$$x^3 = 27 \Rightarrow x = y = z = 3$$

36. Maximize  $P(x, y, z) = xy^2z$ .

Constraint:  $g(x, y, z) = x + y + z = 32$

$$y^2z = \lambda$$

$$2xyz = \lambda$$

$$xy^2 = \lambda$$

$$x + y + z = 32$$

$$xy^2 = y^2z \Rightarrow x = z \quad (y \neq 0)$$

$$2xyz = xy^2 \Rightarrow 2x^2y = xy^2 \Rightarrow 2x = y$$

$$x + 2x + x = 32$$

$$x = 8$$

$$y = 16$$

$$z = 8$$

37. Minimize  $f(x, y, z) = 0.06(2yz + 2xz) + 0.11(xy)$ .

Constraint:  $g(x, y, z) = xyz = 668.25$

$$0.12z + 0.11y = yz\lambda$$

$$0.12z + 0.11x = xz\lambda$$

$$0.12(y + x) = xy\lambda$$

$$xyz = 668.25$$

$$0.12xz + 0.11yx = xyz\lambda = 0.12yz + 0.11xy \Rightarrow x = y$$

$$0.12(2x) = x^2\lambda \Rightarrow \lambda = \frac{0.24}{x}$$

$$0.12z + 0.11x = xz\left(\frac{0.24}{x}\right) = 0.24z \Rightarrow z = \frac{0.11x}{0.12} = \frac{11x}{12}$$

$$xyz = x^2\left(\frac{11}{12}x\right) = 668.25 \Rightarrow x = y = 9, z = \frac{33}{4}$$

$$f\left(9, 9, \frac{33}{4}\right) = \$26.73$$

**38.** Maximize  $f(x, y, z) = xyz$  (volume).

$$\text{Constraint: } g(x, y, z) = 1.5xy + 2xz + 2yz = C$$

$$yz = 1.5y\lambda + 2z\lambda$$

$$xz = 1.5x\lambda + 2z\lambda$$

$$xy = 2x\lambda + 2y\lambda$$

$$1.5xy + 2xz + 2yz = C$$

$$xyz = x[1.5y\lambda + 2z\lambda] = y[1.5x\lambda + 2z\lambda]$$

$$2xz\lambda = 2yz\lambda$$

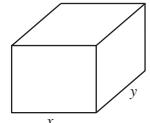
$$x = y \quad (\text{also by symmetry})$$

$$x^2 = 2x\lambda + 2x\lambda \Rightarrow \lambda = x/4.$$

$$xz = 1.5x\left(\frac{x}{4}\right) + 2z\left(\frac{x}{4}\right) \Rightarrow z = \frac{3}{4}x$$

$$1.5x^2 + 2x\left(\frac{3}{4}x\right) + 2x\left(\frac{3}{4}x\right) = C \Rightarrow x^2 = \frac{2}{9}C \Rightarrow x = \frac{\sqrt{2C}}{3},$$

$$y = \frac{\sqrt{2C}}{3}, z = \frac{\sqrt{2C}}{4}$$



**39.** Maximize  $f(x, y, z) = \frac{4\pi abc}{3}$ .

$$\text{Constraint: } a + b + c = K, \text{ constant}$$

$$\begin{aligned} \frac{4\pi bc}{3} &= \lambda \\ \frac{4\pi ac}{3} &= \lambda \\ \frac{4\pi ab}{3} &= \lambda \end{aligned} \left\{ \begin{aligned} \frac{4\pi bc}{3} &= \frac{4\pi ac}{3} \Rightarrow a = b \Rightarrow a = b = c = \frac{K}{3} \\ p + q + r &= 3p = 1 \Rightarrow p = \frac{1}{3} \text{ and} \end{aligned} \right.$$

$$\frac{4\pi ab}{3} = \lambda$$

So, ellipse is a sphere:

$$x^2 + y^2 + z^2 = a^2$$

**40.** Consider the sphere given by  $x^2 + y^2 + z^2 = r^2$

and let a vertex of the rectangular box be  $(x, y, z)$ .

$$\text{Maximize } V(x, y, z) = (2x)(2y)(2z) = 8xyz.$$

$$\text{Constraint: } g(x, y, z) = x^2 + y^2 + z^2 = r^2$$

$$\begin{aligned} 8yz &= 2x\lambda \Rightarrow 4xyz = \lambda x^2 \\ 8xz &= 2y\lambda \Rightarrow 4xyz = \lambda y^2 \\ 8xy &= 2z\lambda \Rightarrow 4xzy = \lambda z^2 \end{aligned} \left\{ \begin{aligned} x &= y = z \\ x + y + z &= 3x = 1 \Rightarrow x = y = z = \frac{1}{3} \end{aligned} \right.$$

$$x^2 + y^2 + z^2 = r^2$$

So, the rectangular box is a cube.

**41.** Maximize  $P(p, q, r) = 2pq + 2pr + 2qr$ .

$$\text{Constraint: } g(p, q, r) = p + q + r = 1$$

$$\begin{aligned} 2q + 2r &= \lambda \\ 2p + 2r &= \lambda \\ 2p + 2q &= \lambda \end{aligned} \left\{ \begin{aligned} p &= q = r \\ p + q + r &= 3p = 1 \Rightarrow p = \frac{1}{3} \text{ and} \end{aligned} \right.$$

$$P\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right) = 3\left(\frac{2}{9}\right) = \frac{2}{3}.$$

**42.** Maximize  $H(x, y, z) = -x \ln x - y \ln y - z \ln z$ .

$$\text{Constraint: } g(x, y, z) = x + y + z = 1$$

$$\begin{aligned} -\ln x - 1 &= \lambda \\ -\ln y - 1 &= \lambda \\ -\ln z - 1 &= \lambda \end{aligned} \left\{ \begin{aligned} x &= y = z \\ x + y + z &= 3x = 1 \Rightarrow x = y = z = \frac{1}{3} \end{aligned} \right.$$

$$(b) H\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right) = 3\left[-\frac{1}{3} \ln\left(\frac{1}{3}\right)\right] = \ln 3$$

43. Maximize  $V(x, y, z) = (2x)(2y)(2z) = 8xyz$  subject to the constraint  $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ .

$$\begin{aligned} 8yz &= \frac{2x}{a^2}\lambda \\ 8xz &= \frac{2y}{b^2}\lambda \\ 8xy &= \frac{2z}{c^2}\lambda \end{aligned}$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \Rightarrow \frac{3x^2}{a^2} = 1, \frac{3y^2}{b^2} = 1, \frac{3z^2}{c^2} = 1$$

$$x = \frac{a}{\sqrt{3}}, y = \frac{b}{\sqrt{3}}, z = \frac{c}{\sqrt{3}}$$

So, the dimensions of the box are  $\frac{2\sqrt{3}a}{3} \times \frac{2\sqrt{3}b}{3} \times \frac{2\sqrt{3}c}{3}$ .

44. (a) Yes. Lagrange multipliers can be used.

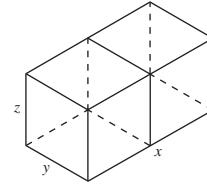
(b) Maximize  $V(x, y, z) = xyz$  subject to the constraint  $x + 2y + 2z = 108$ .

$$\begin{aligned} yz &= \lambda \\ xz &= 2\lambda \\ xy &= 2\lambda \end{aligned}$$

$$y = z \text{ and } x = 2y$$

$$x + 2y + 2z = 108 \Rightarrow 6y = 108, y = 18$$

$$x = 36, y = z = 18$$



Volume is maximum when the dimensions are  $36 \times 18 \times 18$  inches.

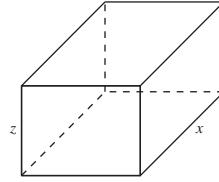
45. Minimize  $C(x, y, z) = 5xy + 3(2xz + 2yz + xy)$  subject to the constraint  $xyz = 480$ .

$$\begin{aligned} 8y + 6z &= yz\lambda \\ 8x + 6z &= xz\lambda \\ 6x + 6y &= xy\lambda \end{aligned}$$

$$x = y, 4y = 3z$$

$$xyz = 480 \Rightarrow \frac{4}{3}y^3 = 480$$

$$x = y = \sqrt[3]{360}, z = \frac{4}{3}\sqrt[3]{360}$$



Dimensions:  $\sqrt[3]{360} \times \sqrt[3]{360} \times \frac{4}{3}\sqrt[3]{360}$  feet.

46. (a) Maximize  $P(x, y, z) = xyz$  subject to the

constraint  $x + y + z = S$ .

$$\begin{aligned} yz &= \lambda \\ xz &= \lambda \\ xy &= \lambda \end{aligned}$$

$$x = y = z$$

$$x + y + z = S \Rightarrow x = y = z = \frac{S}{3}$$

$$\text{So, } xyz \leq \left(\frac{S}{3}\right)\left(\frac{S}{3}\right)\left(\frac{S}{3}\right), x, y, z > 0$$

$$xyz \leq \frac{S^3}{27}$$

$$\sqrt[3]{xyz} \leq \frac{S}{3}$$

$$\sqrt[3]{xyz} \leq \frac{x+y+z}{3}$$

(b) Maximize  $P = x_1x_2x_3 \cdots x_n$  subject to the constraint  $\sum_{i=1}^n x_i = S$ .

$$\left. \begin{array}{l} x_2x_3 \cdots x_n = \lambda \\ x_1x_3 \cdots x_n = \lambda \\ x_1x_2 \cdots x_n = \lambda \\ \vdots \\ x_1x_2x_3 \cdots x_{n-1} = \lambda \end{array} \right\} x_1 = x_2 = x_3 = \cdots = x_n$$

$$\sum_{i=1}^n x_i = S \Rightarrow x_1 = x_2 = x_3 = \cdots = x_n = \frac{S}{n}$$

So,

$$x_1x_2x_3 \cdots x_n \leq \left(\frac{S}{n}\right)\left(\frac{S}{n}\right)\left(\frac{S}{n}\right) \cdots \left(\frac{S}{n}\right), x_i \geq 0$$

$$x_1x_2x_3 \cdots x_n \leq \left(\frac{S}{n}\right)^n$$

$$\sqrt[n]{x_1x_2x_3 \cdots x_n} \leq \frac{S}{n}$$

$$\sqrt[n]{x_1x_2x_3 \cdots x_n} \leq \frac{x_1 + x_2 + x_3 + \cdots + x_n}{n}.$$

47. Minimize  $A(\pi, r) = 2\pi rh + 2\pi r^2$  subject to the constraint  $\pi r^2 h = V_0$ .

$$\left. \begin{array}{l} 2\pi h + 4\pi r = 2\pi rh\lambda \\ 2\pi r = \pi r^2\lambda \end{array} \right\} h = 2r$$

$$\pi r^2 h = V_0 \Rightarrow 2\pi r^3 = V_0$$

$$\text{Dimensions: } r = \sqrt[3]{\frac{V_0}{2\pi}} \text{ and } h = 2\sqrt[3]{\frac{V_0}{2\pi}}$$

48. Maximize  $T(x, y, z) = 100 + x^2 + y^2$  subject to the constraints  $x^2 + y^2 + z^2 = 50$  and  $x - z = 0$ .

$$\left. \begin{array}{l} 2x = 2x\lambda + \mu \\ 2y = 2y\lambda \\ 0 = 2z\lambda - \mu \end{array} \right\}$$

If  $y \neq 0$ , then  $\lambda = 1$  and  $\mu = 0, z = 0$ .

So,  $x = z = 0$  and  $y = \sqrt{50}$ .

$$T(0, \sqrt{50}, 0) = 100 + 50 = 150$$

If  $y = 0$  then  $x^2 + z^2 = 2x^2 = 50$  and  $x = z = \sqrt{50}/2$ .

$$T\left(\frac{\sqrt{50}}{2}, 0, \frac{\sqrt{50}}{2}\right) = 100 + \frac{50}{4} = 112.5$$

So, the maximum temperature is 150.

49. Using the formula Time =  $\frac{\text{Distance}}{\text{Rate}}$ , minimize  $T(x, y) = \frac{\sqrt{d_1^2 + x^2}}{v_1} + \frac{\sqrt{d_2^2 + y^2}}{v_2}$  subject to the constraint  $x + y = a$ .

$$\begin{cases} \frac{x}{v_1\sqrt{d_1^2 + x^2}} = \lambda \\ \frac{y}{v_2\sqrt{d_2^2 + y^2}} = \lambda \end{cases} \quad \begin{cases} \frac{x}{v_1\sqrt{d_1^2 + x^2}} = \frac{y}{v_2\sqrt{d_2^2 + y^2}} \\ x + y = a \end{cases}$$

Because  $\sin \theta_1 = \frac{x}{\sqrt{d_1^2 + x^2}}$

and  $\sin \theta_2 = \frac{y}{\sqrt{d_2^2 + y^2}}$ ,

we have  $\frac{x/\sqrt{d_1^2 + x^2}}{v_1} = \frac{y/\sqrt{d_2^2 + y^2}}{v_2}$  or

$$\frac{\sin \theta_1}{v_1} = \frac{\sin \theta_2}{v_2}.$$

50. Case 1: Minimize  $P(l, h) = 2h + l + \left(\frac{\pi l}{2}\right)$  subject to

the constraint  $lh + \left(\frac{\pi l^2}{8}\right) = A$ .

$$1 + \frac{\pi}{2} = \left(h + \frac{\pi l}{4}\right)\lambda$$

$$2 = l\lambda \Rightarrow \lambda = \frac{2}{l}, 1 + \frac{\pi}{2} = \frac{2h}{l} + \frac{\pi}{2}$$

$$l = 2h$$

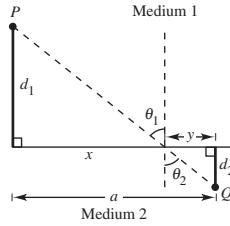
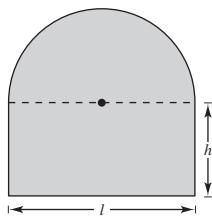
- Case 2: Minimize  $A(l, h) = lh + \left(\frac{\pi l^2}{8}\right)$  subject to

the constraint  $2h + l + \left(\frac{\pi l}{2}\right) = P$ .

$$h + \frac{\pi l}{4} = \left(1 + \frac{\pi}{2}\right)\lambda$$

$$l = 2\lambda \Rightarrow \lambda = \frac{l}{2}, h + \frac{\pi l}{4} = \frac{l}{2} + \frac{\pi l}{4}$$

$$h = \frac{l}{2} \text{ or } l = 2h$$



51. Maximize  $P(x, y) = 100x^{0.25}y^{0.75}$  subject to the constraint  $72x + 60y = 250,000$ .

$$25x^{-0.75}y^{0.75} = 72\lambda \Rightarrow \left(\frac{y}{x}\right)^{0.75} = \frac{72\lambda}{25}$$

$$75x^{0.25}y^{-0.25} = 60\lambda \Rightarrow \left(\frac{x}{y}\right)^{0.25} = \frac{60\lambda}{75}$$

$$\left(\frac{y}{x}\right)^{0.75}\left(\frac{y}{x}\right)^{0.25} = \left(\frac{72\lambda}{25}\right)\left(\frac{75}{60\lambda}\right)$$

$$\frac{y}{x} = \frac{18}{5}$$

$$y = \frac{18}{5}x$$

$$72x + 60\left(\frac{18}{5}x\right) = 288x = 250,000 \Rightarrow x = \frac{15,625}{18}$$

$$y = 3125$$

$$P\left(\frac{15625}{18}, 3125\right) \approx 226,869$$

52. Maximize  $P(x, y) = 100x^{0.4}y^{0.6}$  subject to the constraint  $72x + 60y = 250,000$ .

$$40x^{-0.6}y^{0.6} = 72\lambda \Rightarrow \left(\frac{y}{x}\right)^{0.6} = \frac{72\lambda}{40}$$

$$60x^{0.4}y^{-0.4} = 60\lambda \Rightarrow \left(\frac{x}{y}\right)^{0.4} = \frac{60\lambda}{60} = \lambda$$

$$\left(\frac{y}{x}\right)^{0.6} \left(\frac{y}{x}\right)^{0.4} = \frac{72\lambda}{40} \cdot \frac{1}{\lambda}$$

$$\frac{y}{x} = \frac{9}{5} \Rightarrow y = \frac{9}{5}x$$

$$72x + 60\left(\frac{9}{5}x\right) = 180x = 250,000 \Rightarrow x = \frac{125,000}{9}$$

$$y = 2500$$

$$P\left(\frac{125,000}{9}, 2500\right) \approx 496,399$$

54. Minimize  $C(x, y) = 72x + 60y$  subject to the constraint  $100x^{0.6}y^{0.4} = 50,000$ .

$$72 = 60x^{-0.4}y^{0.4}\lambda \Rightarrow \left(\frac{y}{x}\right)^{0.4} = \frac{72}{60\lambda}$$

$$60 = 40x^{0.6}y^{-0.6}\lambda \Rightarrow \left(\frac{x}{y}\right)^{0.6} = \frac{60}{40\lambda} = \frac{3}{2\lambda}$$

$$\left(\frac{y}{x}\right)^{0.4} \left(\frac{y}{x}\right)^{0.6} = \frac{72}{60\lambda} \cdot \frac{2\lambda}{3}$$

$$\frac{y}{x} = \frac{4}{5} \Rightarrow y = \frac{4}{5}x$$

$$100x^{0.6}\left(\frac{4}{5}x\right)^{0.4} = 50,000$$

$$x = \frac{500}{(4/5)^{0.4}}$$

$$y = \frac{400}{(4/5)^{0.4}}$$

$$C\left(\frac{500}{(4/5)^{0.4}}, \frac{400}{(4/5)^{0.4}}\right) \approx \$65,601.72$$

55. (a) Maximize  $g(\alpha, \beta, \gamma) = \cos \alpha \cos \beta \cos \gamma$  subject to the constraint  $\alpha + \beta + \gamma = \pi$ .

$$\begin{aligned} -\sin \alpha \cos \beta \cos \gamma &= \lambda \\ -\cos \alpha \sin \beta \cos \gamma &= \lambda \\ -\cos \alpha \cos \beta \sin \gamma &= \lambda \end{aligned} \quad \left| \begin{array}{l} \tan \alpha = \tan \beta = \tan \gamma \Rightarrow \alpha = \beta = \gamma \end{array} \right.$$

$$\alpha + \beta + \gamma = \pi \Rightarrow \alpha = \beta = \gamma = \frac{\pi}{3}$$

$$g\left(\frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3}\right) = \frac{1}{8}$$

53. Minimize  $C(x, y) = 72x + 60y$  subject to the constraint  $100x^{0.25}y^{0.75} = 50,000$ .

$$72 = 25x^{-0.75}y^{0.75}\lambda \Rightarrow \left(\frac{y}{x}\right)^{0.75} = \frac{72}{25\lambda}$$

$$60 = 75x^{0.25}y^{-0.25}\lambda \Rightarrow \left(\frac{x}{y}\right)^{0.25} = \frac{60}{75\lambda}$$

$$\left(\frac{y}{x}\right)^{0.75} \left(\frac{y}{x}\right)^{0.25} = \frac{72}{25\lambda} \cdot \frac{75\lambda}{60}$$

$$\frac{y}{x} = \frac{18}{5} \Rightarrow y = \frac{18}{5}x = 3.6x$$

$$100x^{0.25}(3.6x)^{0.75} = 50,000$$

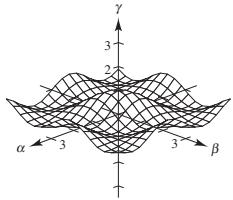
$$x = \frac{500}{3.6^{0.75}} \approx 191.3124$$

$$y = 3.6x \approx 688.7247$$

$$C(191.3124, 688.7247) \approx 55,097.97$$

(b)  $\alpha + \beta + \gamma = \pi \Rightarrow \gamma = \pi - (\alpha + \beta)$

$$\begin{aligned} g(\alpha + \beta) &= \cos \alpha \cos \beta \cos(\pi - (\alpha + \beta)) \\ &= \cos \alpha \cos \beta [\cos \pi \cos(\alpha + \beta) + \sin \pi \sin(\alpha + \beta)] \\ &= -\cos \alpha \cos \beta \cos(\alpha + \beta) \end{aligned}$$



56. Let  $r$  = radius of cylinder, and  $h$  = height of cylinder = height of cone.

$$S = 2\pi rh + 2\pi r\sqrt{h^2 + r^2} = \text{constant surface area}$$

$$V = \pi r^2 h + \frac{2\pi r^2 h}{3} = \frac{5\pi r^2 h}{3} \text{ volume}$$

We maximize  $f(r, h) = r^2 h$  subject to  $g(r, h) = rh + r\sqrt{h^2 + r^2} = C$ .

$$(C - rh)^2 = r^2(h^2 + r^2)$$

$$C^2 - 2Crh = r^4$$

$$h = \frac{C^2 - r^4}{2Cr}$$

$$f(r, h) = F(r) = r^2 \left[ \frac{C^2 - r^4}{2Cr} \right] = \frac{Cr}{2} - \frac{r^5}{2C}$$

$$F'(r) = \frac{C}{2} - \frac{5r^4}{2C} = 0$$

$$C^2 = 5r^4$$

$$r^2 = \frac{C}{\sqrt{5}}$$

$$F''(r) = \frac{-10r^3}{C}$$

$$h = \frac{C^2 - r^4}{2Cr} = \frac{C^2 - C^2/5}{2C(C^2/5)^{1/4}}$$

$$= \frac{(4/5)C}{2(C^2/5)^{1/4}}$$

$$= \frac{2C}{5r}$$

$$= \frac{2}{5r}(\sqrt{5}r^2)$$

$$= \frac{2\sqrt{5}}{5}r$$

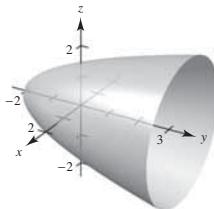
$$\text{So, } \frac{h}{r} = \frac{2\sqrt{5}}{5}.$$

By the Second Derivative Test, this is a maximum.

## Review Exercises for Chapter 13

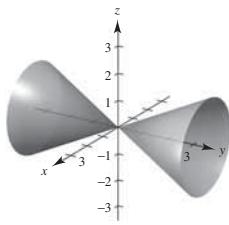
1.  $f(x, y, z) = x^2 - y + z^2 = 2$   
 $y = x^2 + z^2 - 2$

Elliptic paraboloid

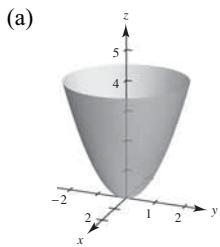


2.  $f(x, y, z) = 4x^2 - y^2 + 4z^2 = 0$

Elliptic cone.

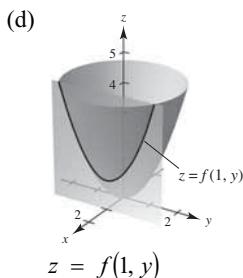


3.  $f(x, y) = x^2 + y^2$

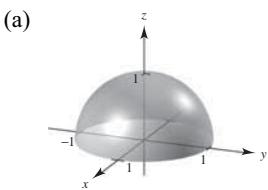


(b)  $g(x, y) = f(x, y) + 2$  is a vertical translation of  $f$  two units upward.

(c)  $g(x, y) = f(x, y - z)$  is a horizontal translation of  $f$  two units to the right. The vertex moves from  $(0, 0, 0)$  to  $(0, 2, 0)$ .



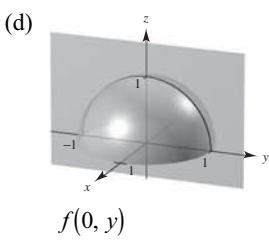
4.  $f(x, y) = \sqrt{1 - x^2 - y^2}$



(b)  $g(x, y) = f(x + 2, y)$  is the graph of  $f$  shifted 2 units back on the  $x$ -axis. The hemisphere has equation

$$(x + 2)^2 + y^2 + z^2 = 1, z \geq 0.$$

(c)  $g(x, y) = 4 - f(x, y)$  is the graph of  $f$  reflected in the  $xy$ -plane, and shifted 4 units upward.



$$z = \sqrt{1 - x^2 - y^2}$$

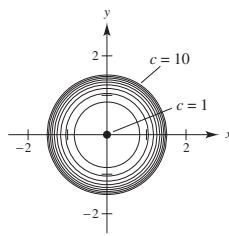
5.  $f(x, y) = e^{x^2+y^2}$

The level curves are of the form

$$c = e^{x^2+y^2}$$

$$\ln c = x^2 + y^2.$$

The level curves are circles centered at the origin.



Generated by Mathematica

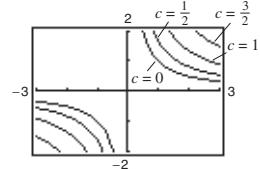
6.  $f(x, y) = \ln xy$

The level curves are of the form

$$c = \ln xy$$

$$e^c = xy.$$

The level curves are hyperbolas



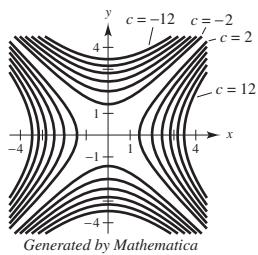
7.  $f(x, y) = x^2 - y^2$

The level curves are of the form

$$c = x^2 - y^2$$

$$1 = \frac{x^2}{c} - \frac{y^2}{c}.$$

The level curves are hyperbolas.



8.  $f(x, y) = \frac{x}{x + y}$

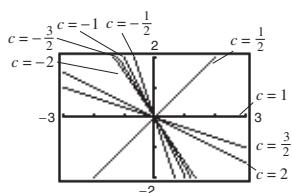
The level curves are of the form

$$c = \frac{x}{x + y}$$

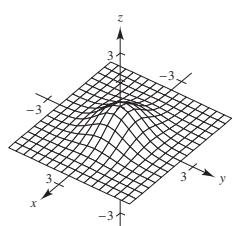
$$y = \left(\frac{1 - c}{c}\right)x.$$

The level curves are passing through the origin with

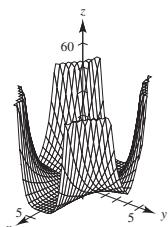
slope  $\frac{1 - c}{c}$ .



9.  $f(x, y) = e^{-(x^2 + y^2)}$



10.  $g(x, y) = |y|^{1+|x|}$



11.  $\lim_{(x, y) \rightarrow (1, 1)} \frac{xy}{x^2 + y^2} = \frac{1}{2}$

Continuous except at  $(0, 0)$ .

12.  $\lim_{(x, y) \rightarrow (1, 1)} \frac{xy}{x^2 - y^2}$

Does not exist.

Continuous except when  $y = \pm x$ .

13.  $\lim_{(x, y) \rightarrow (0, 0)} \frac{y + xe^{-y^2}}{1 + x^2} = \frac{0 + 0}{1 + 0} = 0$

Continuous everywhere.

14.  $\lim_{(x, y) \rightarrow (0, 0)} \frac{x^2 y}{x^4 + y^2}$

For  $y = x^2$ ,  $\frac{x^2 y}{x^4 + y^2} = \frac{x^4}{x^4 + x^4} \rightarrow \frac{1}{2}$ .

For  $y = 0$ ,  $\frac{x^2 y}{x^4 + y^2} = 0$  for  $x \neq 0$ .

The limit does not exist.

Continuous to all  $(x, y) \neq (0, 0)$

15.  $f(x, y) = e^x \cos y$

$$f_x = e^x \cos y$$

$$f_y = -e^x \sin y$$

16.  $f(x, y) = \frac{xy}{x + y}$

$$f_x = \frac{y(x + y) - xy}{(x + y)^2} = \frac{y^2}{(x + y)^2}$$

$$f_y = \frac{x^2}{(x + y)^2}$$

17.  $z = e^{-y} + e^{-x}$

$$\frac{\partial z}{\partial x} = -e^{-x}, \frac{\partial z}{\partial y} = -e^{-y}$$

18.  $z = \ln(x^2 + y^2 + 1)$

$$\frac{\partial z}{\partial x} = \frac{2x}{x^2 + y^2 + 1}$$

$$\frac{\partial z}{\partial y} = \frac{2y}{x^2 + y^2 + 1}$$

19.  $g(x, y) = \frac{xy}{x^2 + y^2}$

$$g_x = \frac{y(x^2 + y^2) - xy(2x)}{(x^2 + y^2)^2} = \frac{y(y^2 - x^2)}{(x^2 + y^2)^2}$$

$$g_y = \frac{x(x^2 - y^2)}{(x^2 + y^2)^2}$$

20.  $w = \sqrt{x^2 - y^2 - z^2}$

$$\frac{\partial w}{\partial x} = \frac{1}{2}(x^2 - y^2 - z^2)^{-1/2}(2x) = \frac{x}{\sqrt{x^2 - y^2 - z^2}}$$

$$\frac{\partial w}{\partial y} = \frac{-y}{\sqrt{x^2 - y^2 - z^2}}$$

$$\frac{\partial w}{\partial z} = \frac{-z}{\sqrt{x^2 - y^2 - z^2}}$$

21.  $f(x, y, z) = z \arctan \frac{y}{x}$

$$f_x = \frac{z}{1 + (y^2/x^2)} \left( -\frac{y}{x^2} \right) = \frac{-yz}{x^2 + y^2}$$

$$f_y = \frac{z}{1 + (y^2/x^2)} \left( \frac{1}{x} \right) = \frac{xz}{x^2 + y^2}$$

$$f_z = \arctan \frac{y}{x}$$

22.  $f(x, y, z) = \frac{1}{\sqrt{1 + x^2 + y^2 + z^2}}$

$$= (1 + x^2 + y^2 + z^2)^{-1/2}$$

$$\frac{\partial f}{\partial x} = -\frac{1}{2}(1 + x^2 + y^2 + z^2)^{-3/2}(2x)$$

$$= \frac{-x}{(1 + x^2 + y^2 + z^2)^{3/2}}$$

$$\frac{\partial f}{\partial y} = \frac{-y}{(1 + x^2 + y^2 + z^2)^{3/2}}$$

$$\frac{\partial f}{\partial z} = \frac{-z}{(1 + x^2 + y^2 + z^2)^{3/2}}$$

23.  $u(x, t) = ce^{-n^2 t} \sin(nx)$

$$\frac{\partial u}{\partial x} = cne^{-n^2 t} \cos(nx)$$

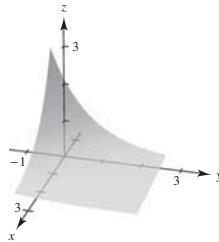
$$\frac{\partial u}{\partial t} = -cn^2 e^{-n^2 t} \sin(nx)$$

24.  $u(x, t) = c(\sin akx)\cos kt$

$$\frac{\partial u}{\partial x} = akc(\cos akx)\cos kt$$

$$\frac{\partial u}{\partial t} = -kc(\sin akx)\sin kt$$

25.



26.  $z = x^2 \ln(y + 1)$

$$\frac{\partial z}{\partial x} = 2x \ln(y + 1). \text{ At } (2, 0, 0), \frac{\partial z}{\partial x} = 0.$$

Slope in x-direction.

$$\frac{\partial z}{\partial y} = \frac{x^2}{1 + y}. \text{ At } (2, 0, 0), \frac{\partial z}{\partial y} = 4.$$

Slope in y-direction.

27.  $f(x, y) = 3x^2 - xy + 2y^3$

$$f_x = 6x - y$$

$$f_y = -x + 6y^2$$

$$f_{xx} = 6$$

$$f_{yy} = 12y$$

$$f_{xy} = -1$$

$$f_{yx} = -1$$

28.  $h(x, y) = \frac{x}{x + y}$

$$h_x = \frac{y}{(x + y)^2}$$

$$h_y = \frac{-x}{(x + y)^2}$$

$$h_{xx} = \frac{-2y}{(x + y)^3}$$

$$h_{yy} = \frac{2x}{(x + y)^3}$$

$$h_{xy} = \frac{(x + y)^2 - 2y(x + y)}{(x + y)^4} = \frac{x - y}{(x + y)^3}$$

$$h_{yx} = \frac{-(x + y)^2 + 2y(x + y)}{(x + y)^4} = \frac{x - y}{(x + y)^3}$$

29.  $h(x, y) = x \sin y + y \cos x$

$$h_x = \sin y - y \sin x$$

$$h_y = x \cos y + \cos x$$

$$h_{xx} = -y \cos x$$

$$h_{yy} = -x \sin y$$

$$h_{xy} = \cos y - \sin x$$

$$h_{yx} = \cos y - \sin x$$

30.  $g(x, y) = \cos(x - 2y)$

$$g_x = -\sin(x - 2y)$$

$$g_y = 2 \sin(x - 2y)$$

$$g_{xx} = -\cos(x - 2y)$$

$$g_{yy} = -4 \cos(x - 2y)$$

$$g_{xy} = 2 \cos(x - 2y)$$

$$g_{yx} = 2 \cos(x - 2y)$$

31.  $z = x^2 - y^2$

$$\frac{\partial z}{\partial x} = 2x$$

$$\frac{\partial^2 z}{\partial x^2} = 2$$

$$\frac{\partial z}{\partial y} = -2y$$

$$\frac{\partial^2 z}{\partial y^2} = -2$$

So,  $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$ .

32.  $z = x^3 - 3xy^2$

$$\frac{\partial z}{\partial x} = 3x^2 - 3y^2$$

$$\frac{\partial^2 z}{\partial x^2} = 6x$$

$$\frac{\partial z}{\partial y} = -6xy$$

$$\frac{\partial^2 z}{\partial y^2} = -6x$$

So,  $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$ .

33.  $z = \frac{y}{x^2 + y^2}$

$$\frac{\partial z}{\partial x} = \frac{-2xy}{(x^2 + y^2)^2}$$

$$\frac{\partial^2 z}{\partial x^2} = -2y \left[ \frac{-4x^2}{(x^2 + y^2)^3} + \frac{1}{(x^2 + y^2)^2} \right] = 2y \frac{3x^2 - y^2}{(x^2 + y^2)^3}$$

$$\frac{\partial z}{\partial y} = \frac{(x^2 + y^2) - 2y}{(x^2 + y^2)^2} = \frac{x^2 - y^2}{(x^2 + y^2)^2}$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{(x^2 + y^2)^2(-2y) - 2(x^2 - y^2)(x^2 + y^2)(2y)}{(x^2 + y^2)^4} = -2y \frac{3x^2 - y^2}{(x^2 + y^2)^3}$$

So,  $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$ .

34.  $z = e^y \sin x$

$$\frac{\partial z}{\partial x} = e^y \cos x$$

$$\frac{\partial^2 z}{\partial x^2} = -e^y \sin x$$

$$\frac{\partial z}{\partial y} = e^y \sin x$$

$$\frac{\partial^2 z}{\partial y^2} = e^y \sin x$$

So,  $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = -e^y \sin x + e^y \sin x = 0$ .

35.  $z = x \sin xy$

$$dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy = (xy \cos xy + \sin xy)dx + (x^2 \cos xy)dy$$

36.  $z = \frac{xy}{\sqrt{x^2 + y^2}}$

$$\begin{aligned} dz &= \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy \\ &= \left[ \frac{\sqrt{x^2 + y^2} y - xy \left( x/\sqrt{x^2 + y^2} \right)}{x^2 + y^2} \right] dx + \left[ \frac{\sqrt{x^2 + y^2} x - xy \left( y/\sqrt{x^2 + y^2} \right)}{x^2 + y^2} \right] dy = \frac{y^3}{(x^2 + y^2)^{3/2}} dx + \frac{x^3}{(x^2 + y^2)^{3/2}} dy \end{aligned}$$

37.  $z^2 = x^2 + y^2$

$$2z dz = 2x dx + 2y dy$$

$$dz = \frac{x}{z} dx + \frac{y}{z} dy = \frac{5}{13} \left( \frac{1}{2} \right) + \frac{12}{13} \left( \frac{1}{2} \right) = \frac{17}{26} \approx 0.654 \text{ cm}$$

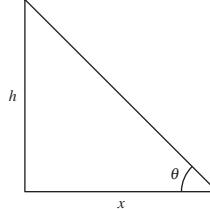
$$\text{Percentage error: } \frac{dz}{z} = \frac{17/26}{13} \approx 0.0503 \approx 5\%$$

38. From the accompanying figure we observe

$$\tan \theta = \frac{h}{x} \text{ or } h = x \tan \theta$$

$$dh = \frac{\partial h}{\partial x} dx + \frac{\partial h}{\partial \theta} d\theta = \tan \theta dx + x \sec^2 \theta d\theta.$$

$$\text{Letting } x = 100, dx = \pm \frac{1}{2}, \theta = \frac{11\pi}{60}, \text{ and } d\theta = \pm \frac{\pi}{180}.$$



(Note that we express the measurement of the angle in radians.) The maximum error is approximately

$$dh = \tan \left( \frac{11\pi}{60} \right) \left( \pm \frac{1}{2} \right) + 100 \sec^2 \left( \frac{11\pi}{60} \right) \left( \pm \frac{\pi}{180} \right) \approx \pm 0.3247 \pm 2.4814 \approx \pm 2.81 \text{ feet.}$$

39.  $V = \frac{1}{3}\pi r^2 h$

$$dV = \frac{2}{3}\pi r h dr + \frac{1}{3}\pi r^2 dh = \frac{2}{3}\pi(2)(5)\left(\pm \frac{1}{8}\right) + \frac{1}{3}\pi(2)^2\left(\pm \frac{1}{8}\right) = \pm \frac{5}{6}\pi \pm \frac{1}{6}\pi = \pm \pi \text{ in.}^3$$

40.  $A = \pi r \sqrt{r^2 + h^2}$

$$\begin{aligned} dA &= \left( \pi \sqrt{r^2 + h^2} + \frac{\pi r^2}{\sqrt{r^2 + h^2}} \right) dr + \frac{\pi r h}{\sqrt{r^2 + h^2}} dh \\ &= \frac{\pi(2r^2 + h^2)}{\sqrt{r^2 + h^2}} dr + \frac{\pi r h}{\sqrt{r^2 + h^2}} dh = \frac{\pi(8 + 25)}{\sqrt{29}} \left( \pm \frac{1}{8} \right) + \frac{10\pi}{\sqrt{29}} \left( \pm \frac{1}{8} \right) = \pm \frac{43\pi}{8\sqrt{29}} \end{aligned}$$

41.  $w = \ln(x^2 + y), x = 2t, y = 4 - t$

$$\begin{aligned} \text{(a) Chain Rule: } \frac{dw}{dt} &= \frac{\partial w}{\partial x} \frac{dx}{dt} + \frac{\partial w}{\partial y} \frac{dy}{dt} \\ &= \frac{2x}{x^2 + y}(2) + \frac{1}{x^2 + y}(-1) \\ &= \frac{8t - 1}{4t^2 + 4 - t} \end{aligned}$$

$\text{(b) Substitution: } w = \ln(x^2 + y) = \ln(4t^2 + 4 - t)$

$$\frac{dw}{dt} = \frac{1}{4t^2 + 4 - t}(8t - 1)$$

42.  $u = y^2 - x, x = \cos t, y = \sin t$

$$\begin{aligned} \text{(a) Chain Rule: } \frac{du}{dt} &= \frac{\partial u}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial t} \\ &= -1(-\sin t) + 2y(\cos t) \\ &= \sin t + 2(\sin t) \cos t \\ &= \sin t(1 + 2 \cos t) \end{aligned}$$

$\text{(b) Substitution: } u = \sin^2 t - \cos t$

$$\begin{aligned} \frac{du}{dt} &= 2 \sin t \cos t + \sin t \\ &= \sin t(1 + 2 \cos t) \end{aligned}$$

43.  $w = \frac{xy}{z}, x = 2r + t, y = rt, z = 2r - t$

(a) Chain Rule:  $\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial r} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial r} + \frac{\partial w}{\partial z} \frac{\partial z}{\partial r}$

$$\begin{aligned} &= \frac{y}{z}(2) + \frac{x}{z}(1) - \frac{xy}{z^2}(2) \\ &= \frac{2rt}{2r-t} + \frac{(2r+t)t}{2r-t} - \frac{2(2r+t)(rt)}{(2r-t)^2} \\ &= \frac{4r^2t - 4rt^2 - t^3}{(2r-t)^2} \\ \frac{\partial w}{\partial t} &= \frac{\partial w}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial t} + \frac{\partial w}{\partial z} \frac{\partial z}{\partial t} \\ &= \frac{y}{z}(1) + \frac{x}{z}(r) = \frac{xy}{z^2}(-1) \\ &= \frac{4r^2t - rt^2 + 4r^3}{(2r-t)^2} \end{aligned}$$

(b) Substitution:  $w = \frac{xy}{z} = \frac{(2r+t)(rt)}{2r-t} = \frac{2r^2t + rt^2}{2r-t}$

$$\begin{aligned} \frac{\partial w}{\partial r} &= \frac{4r^2t - 4rt^2 - t^3}{(2r-t)^2} \\ \frac{\partial w}{\partial t} &= \frac{4r^2t - rt^2 + 4r^3}{(2r-t)^2} \end{aligned}$$

44.  $u = x^2 + y^2 + z^2, x = r \cos t, y = r \sin t, z = t$

(a) Chain Rule:  $\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x} \frac{\partial x}{\partial r} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial r} + \frac{\partial u}{\partial z} \frac{\partial z}{\partial r}$

$$\begin{aligned} &= 2x \cos t + 2y \sin t + 2z(0) \\ &= 2(r \cos^2 t + r \sin^2 t) = 2r \\ \frac{\partial u}{\partial t} &= \frac{\partial u}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial t} + \frac{\partial u}{\partial z} \frac{\partial z}{\partial t} \\ &= 2x(-r \sin t) + 2y(r \cos t) + 2z = 2(-r^2 \sin t \cos t + r^2 \sin t \cos t) + 2t = 2t \end{aligned}$$

(b) Substitution:  $u(r, t) = r^2 \cos^2 t + r^2 \sin^2 t + t^2 = r^2 + t^2$

$$\begin{aligned} \frac{\partial u}{\partial r} &= 2r \\ \frac{\partial u}{\partial t} &= 2t \end{aligned}$$

45.  $x^2 + xy + y^2 + yz + z^2 = 0$

$$2x + y + y \frac{\partial z}{\partial x} + 2z \frac{\partial z}{\partial x} = 0$$

$$\frac{\partial z}{\partial x} = \frac{-2x - y}{y + 2z}$$

$$x + 2y + y \frac{\partial z}{\partial y} + z + 2z \frac{\partial z}{\partial y} = 0$$

$$\frac{\partial z}{\partial y} = \frac{-x - 2y - z}{y + 2z}$$

46.

$$xz^2 - y \sin z = 0$$

$$2xz \frac{\partial z}{\partial x} + z^2 - y \cos z \frac{\partial z}{\partial x} = 0$$

$$\frac{\partial z}{\partial x} = \frac{z^2}{y \cos z - 2xz}$$

$$2xz \frac{\partial z}{\partial y} - y \cos z \frac{\partial z}{\partial y} - \sin z = 0$$

$$\frac{\partial z}{\partial y} = \frac{\sin z}{2xz - y \cos z}$$

47.  $f(x, y) = x^2y$

$$\nabla f = 2xy\mathbf{i} + x^2\mathbf{j}$$

$$\nabla f(-5, 5) = -50\mathbf{i} + 25\mathbf{j}$$

$$\mathbf{u} = \frac{3}{5}\mathbf{i} - \frac{4}{5}\mathbf{j} \text{ unit vector}$$

$$\begin{aligned} D_{\mathbf{u}}f(-5, 5) &= \nabla f(-5, 5) \cdot \mathbf{u} \\ &= -30 - 20 = -50 \end{aligned}$$

48.  $f(x, y) = \frac{1}{4}y^2 - x^2$

$$\nabla f = -2x\mathbf{i} + \frac{1}{2}y\mathbf{j}$$

$$\nabla f(1, 4) = -2\mathbf{i} + 2\mathbf{j}$$

$$\mathbf{u} = \frac{1}{\sqrt{5}}\mathbf{v} = \frac{2\sqrt{5}}{5}\mathbf{i} + \frac{\sqrt{5}}{5}\mathbf{j}$$

$$D_{\mathbf{u}}f(1, 4) = \nabla f(1, 4) \cdot \mathbf{u}$$

$$= -\frac{4\sqrt{5}}{5} + \frac{2\sqrt{5}}{5} = -\frac{2\sqrt{5}}{5}$$

49.  $w = y^2 + xz$

$$\nabla w = z\mathbf{i} + 2y\mathbf{j} + x\mathbf{k}$$

$$\nabla w(1, 2, 2) = 2\mathbf{i} + 4\mathbf{j} + \mathbf{k}$$

$$\mathbf{u} = \frac{1}{3}\mathbf{v} = \frac{2}{3}\mathbf{i} - \frac{1}{3}\mathbf{j} + \frac{2}{3}\mathbf{k}$$

$$D_{\mathbf{u}}w(1, 2, 2) = \nabla w(1, 2, 2) \cdot \mathbf{u} = \frac{4}{3} - \frac{4}{3} + \frac{2}{3} = \frac{2}{3}$$

50.  $w = 5x^2 + 2xy - 3y^2z$

$$\nabla w = (10x + 2y)\mathbf{i} + (2x - 6yz)\mathbf{j} - 3y^2\mathbf{k}$$

$$\nabla w(1, 0, 1) = 10\mathbf{i} + 2\mathbf{j}$$

$$\mathbf{u} = \frac{1}{\sqrt{3}}(\mathbf{i} + \mathbf{j} - \mathbf{k})$$

$$D_{\mathbf{u}}w(1, 0, 1) = \nabla w(1, 0, 1) \cdot \mathbf{u}$$

$$= \frac{10}{\sqrt{3}} + \frac{2}{\sqrt{3}} = \frac{12}{\sqrt{3}} = 4\sqrt{3}$$

51.  $z = x^2y$

$$\nabla z = 2xy\mathbf{i} + x^2\mathbf{j}$$

$$\nabla_z(2, 1) = 4\mathbf{i} + 4\mathbf{j}$$

$$\|\nabla z(2, 1)\| = 4\sqrt{2}$$

52.  $z = e^{-x} \cos y$

$$\nabla z = -e^{-x} \cos y\mathbf{i} - e^{-x} \sin y\mathbf{j}$$

$$\nabla z\left(0, \frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}\mathbf{i} - \frac{\sqrt{2}}{2}\mathbf{j} = \left\langle -\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2} \right\rangle$$

$$\|\nabla z\left(0, \frac{\pi}{4}\right)\| = 1$$

53.  $z = \frac{y}{x^2 + y^2}$

$$\nabla z = -\frac{2xy}{(x^2 + y^2)^2}\mathbf{i} + \frac{x^2 - y^2}{(x^2 + y^2)^2}\mathbf{j}$$

$$\nabla z(1, 1) = -\frac{1}{2}\mathbf{i} = \left\langle -\frac{1}{2}, 0 \right\rangle$$

$$\|\nabla z(1, 1)\| = \frac{1}{2}$$

54.  $z = \frac{x^2}{x - y}$

$$\nabla z = \frac{x^2 - 2xy}{(x - y)^2}\mathbf{i} + \frac{x^2}{(x - y)^2}\mathbf{j}$$

$$\nabla z(2, 1) = 4\mathbf{j}$$

$$\|\nabla z(2, 1)\| = 4$$

55.  $f(x, y) = 9x^2 - 4y^2, c = 65, P(3, 2)$

(a)  $\nabla f(x, y) = 18x\mathbf{i} - 8y\mathbf{j}$

$$\nabla f(3, 2) = 54\mathbf{i} - 16\mathbf{j}$$

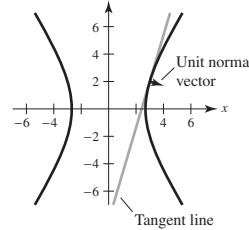
(b) Unit normal:  $\frac{54\mathbf{i} - 16\mathbf{j}}{\|54\mathbf{i} - 16\mathbf{j}\|} = \frac{1}{\sqrt{793}}(27\mathbf{i} - 8\mathbf{j})$

(c) Slope =  $\frac{27}{8}$ .

$$y - z = \frac{27}{8}(x - 3)$$

$$y = \frac{27}{8}x - \frac{65}{8} \text{ Tangent line}$$

(d)



56.  $f(x, y) = 4y \sin x - y, c = 3, P\left(\frac{\pi}{2}, 1\right)$

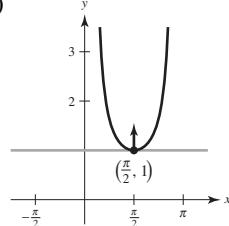
(a)  $\nabla f(x, y) = 4y \cos x\mathbf{i} + (4 \sin x - 1)\mathbf{j}$

$$\nabla f\left(\frac{\pi}{2}, 1\right) = 3\mathbf{j}$$

(b) Unit normal vector:  $\mathbf{j}$

(c) Tangent line horizontal:  $y = 1$

(d)



57.  $F(x, y, z) = x^2y - z = 0$

$$\nabla F = 2xy\mathbf{i} + x^2\mathbf{j} - \mathbf{k}$$

$$\nabla F(2, 1, 4) = 4\mathbf{i} + 4\mathbf{j} - \mathbf{k}$$

So, the equation of the tangent plane is

$$4(x - 2) + 4(y - 1) - (z - 4) = 0 \text{ or}$$

$$4x + 4y - z = 8,$$

and the equation of the normal line is

$$x = 4t + 2, y = 4t + 1, z = -t + 4.$$

58.  $F(x, y, z) = y^2 + z^2 - 25 = 0$

$$\nabla F = 2y\mathbf{j} + 2z\mathbf{k}$$

$$\nabla F(2, 3, 4) = 6\mathbf{j} + 8\mathbf{k} = 2(3\mathbf{j} + 4\mathbf{k})$$

So, the equation of the tangent plane is

$$3(y - 3) + 4(z - 4) = 0 \text{ or } 3y + 4z = 25,$$

and the equation of the normal line is

$$x = 2, \frac{y - 3}{3} = \frac{z - 4}{4}.$$

59.  $F(x, y, z) = x^2 + y^2 - 4x + 6y + z + 9 = 0$

$$\nabla F = (2x - 4)\mathbf{i} + (2y + 6)\mathbf{j} + \mathbf{k}$$

$$\nabla F(2, -3, 4) = \mathbf{k}$$

So, the equation of the tangent plane is

$$z - 4 = 0 \text{ or } z = 4,$$

and the equation of the normal line is

$$x = 2, y = -3, z = 4 + t.$$

60.  $F(x, y, z) = x^2 + y^2 + z^2 - 9 = 0$

$$\nabla F = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$$

$$\nabla F(1, 2, 2) = 2\mathbf{i} + 4\mathbf{j} + 4\mathbf{k} = 2(\mathbf{i} + 2\mathbf{j} + 2\mathbf{k})$$

So, the equation of the tangent plane is

$$(x - 1) + 2(y - 2) + 2(z - 2) = 0 \text{ or}$$

$$x + 2y + 2z = 9,$$

and the equation of the normal line is

$$\frac{x - 1}{1} = \frac{y - 2}{2} = \frac{z - 2}{2}.$$

64. (a)  $f(x, y) = \cos x + \sin y, f(0, 0) = 1$

$$f_x = -\sin x, f_x(0, 0) = 0$$

$$f_y = \cos y, f_y(0, 0) = 1$$

$$P_1(x, y) = 1 + y$$

(b)  $f_{xx} = -\cos x, f_{xx}(0, 0) = -1$

$$f_{yy} = -\sin y, f_{yy}(0, 0) = 0$$

$$f_{xy} = 0, f_{xy}(0, 0) = 0$$

$$P_2(x, y) = 1 + y - \frac{1}{2}x^2$$

61.  $F(x, y, z) = y^2 + z - 9 = 0, (2, 2, 5)$

$$G(x, y, z) = x - y$$

$$\nabla F = 2y\mathbf{j} + \mathbf{k}$$

$$\nabla G = \mathbf{i} - \mathbf{j}$$

$$\nabla F(2, 2, 5) = 4\mathbf{j} + \mathbf{k}$$

$$\nabla F \times \nabla G = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 4 & 1 \\ 1 & -1 & 0 \end{vmatrix} = \mathbf{i} + \mathbf{j} - 4\mathbf{k}$$

$$\text{Tangent line: } \frac{x - 2}{1} = \frac{y - 2}{1} = \frac{z - 5}{-4}$$

62.  $F(x, y, z) = x^2 - y^2 - z = 0$

$$G(x, y, z) = 3 - z = 0$$

$$\nabla F = 2x\mathbf{i} - 2y\mathbf{j} - \mathbf{k}$$

$$\nabla G = -\mathbf{k}$$

$$\nabla F(2, 1, 3) = 4\mathbf{i} - 2\mathbf{j} - \mathbf{k}$$

$$\nabla F \times \nabla G = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & -2 & -1 \\ 0 & 0 & -1 \end{vmatrix} = 2(\mathbf{i} + 2\mathbf{j})$$

So, the equation of the tangent line is

$$\frac{x - 2}{1} = \frac{y - 1}{2}, z = 3.$$

63.  $f(x, y, z) = x^2 + y^2 + z^2 - 14$

$$\nabla f(x, y, z) = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$$

$$\nabla f(2, 1, 3) = 4\mathbf{i} + 2\mathbf{j} + 6\mathbf{k}$$
 Normal vector to plane.

$$\cos \theta = \frac{|\mathbf{n} \cdot \mathbf{k}|}{\|\mathbf{n}\|} = \frac{6}{\sqrt{56}} = \frac{3\sqrt{14}}{14}$$

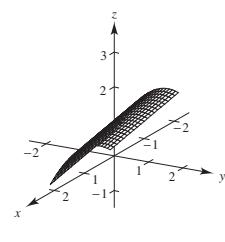
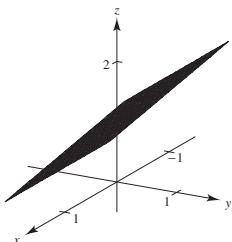
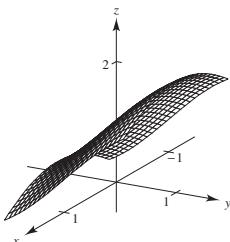
$$\theta = 36.7^\circ$$

(c) If  $y = 0$ , you obtain the 2<sup>nd</sup> degree Taylor polynomial for  $\cos x$ .

(d)

$x$	$y$	$f(x, y)$	$P_1(x, y)$	$P_2(x, y)$
0	0	1.0	1.0	1.0
0	0.1	1.0998	1.1	1.1
0.2	0.1	1.0799	1.1	1.095
0.5	0.3	1.1731	1.3	1.175
1	0.5	1.0197	1.5	1.0

(e)



The accuracy lessens as the distance from  $(0, 0)$  increases.

65.  $f(x, y) = 2x^2 + 6xy + 9y^2 + 8x + 14$

$$f_x = 4x + 6y + 8 = 0$$

$$f_y = 6x + 18y = 0, x = -3y$$

$$4(-3y) + 6y = -8 \Rightarrow y = \frac{4}{3}, x = -4$$

$$f_{xx} = 4$$

$$f_{yy} = 18$$

$$f_{xy} = 6$$

$$f_{xx}f_{yy} - (f_{xy})^2 = 4(18) - (6)^2 = 36 > 0.$$

So,  $(-4, \frac{4}{3}, -2)$  is a relative minimum.

66.  $f(x, y) = x^2 + 3xy + y^2 - 5x$

$$f_x = 2x + 3y - 5 = 0$$

$$f_y = 3x + 2y = 0 \quad \Rightarrow \quad y = -\frac{3}{2}x$$

$$2x + 3\left(-\frac{3}{2}x\right) = 5$$

$$4x - 9x = 10$$

$$x = -2, y = 3$$

$$f_{xx} = 2, f_{yy} = 2, f_{xy} = 3, d = 4 - 9 < 0$$

$\Rightarrow (-2, 3)$  is a saddle point.

67.  $f(x, y) = xy + \frac{1}{x} + \frac{1}{y}$

$$f_x = y - \frac{1}{x^2} = 0, x^2y = 1$$

$$f_y = x - \frac{1}{y^2} = 0, xy^2 = 1$$

So,  $x^2y = xy^2$  or  $x = y$  and substitution yields the critical point  $(1, 1)$ .

$$f_{xx} = \frac{2}{x^3}$$

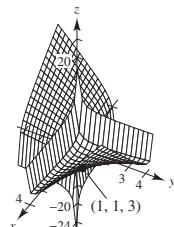
$$f_{xy} = 1$$

$$f_{yy} = \frac{2}{y^3}$$

At the critical point  $(1, 1)$ ,  $f_{xx} = 2 > 0$  and

$$f_{xx}f_{yy} - (f_{xy})^2 = 3 > 0.$$

So,  $(1, 1, 3)$  is a relative minimum.



68.  $z = 50(x + y) - (0.1x^3 + 20x + 150) - (0.05y^3 + 20.6y + 125)$   
 $z_x = 50 - 0.3x^2 - 20 = 0, x = \pm 10$   
 $z_y = 50 - 0.15y^2 - 20.6 = 0, y = \pm 14$

Critical Points:  $(10, 14), (10, -14), (-10, 14), (-10, -14)$

$$z_{xx} = -0.6x, z_{yy} = -0.3y, z_{xy} = 0$$

$$\text{At } (10, 14), z_{xx}z_{yy} - (z_{xy})^2 = (-6)(-4.2) - 0^2 > 0, z_{xx} < 0.$$

$(10, 14, 199.4)$  is a relative maximum.

$$\text{At } (10, -14), z_{xx}z_{yy} - (z_{xy})^2 = (-6)(4.2) - 0^2 < 0.$$

$(10, -14, -349.4)$  is a saddle point.

$$\text{At } (-10, 14), z_{xx}z_{yy} - (z_{xy})^2 = (6)(-4.2) - 0^2 < 0.$$

$(-10, 14, -200.6)$  is a saddle point.

$$\text{At } (-10, -14), z_{xx}z_{yy} - (z_{xy})^2 = (6)(4.2) - 0^2 > 0, z_{xx} < 0.$$

$(-10, -14, -749.4)$  is a relative minimum.

69. The level curves are hyperbolas. There is a critical point at  $(0, 0)$ , but there are no relative extrema. The gradient is normal to the level curve at any given point  $(x_0, y_0)$ .

70. The level curves indicate that there is a relative extremum at  $A$ , the center of the ellipse in the second quadrant, and that there is a saddle point at  $B$ , the origin.

71.  $P(x_1, x_2) = R - C_1 - C_2 = [225 - 0.4(x_1 + x_2)](x_1 + x_2) - (0.05x_1^2 + 15x_1 + 5400) - (0.03x_2^2 + 15x_2 + 6100)$   
 $= -0.45x_1^2 - 0.43x_2^2 - 0.8x_1x_2 + 210x_1 + 210x_2 - 11,500$

$$P_{x_1} = -0.9x_1 - 0.8x_2 + 210 = 0$$

$$0.9x_1 + 0.8x_2 = 210$$

$$P_{x_2} = -0.86x_2 - 0.8x_1 + 210 = 0$$

$$0.8x_1 + 0.86x_2 = 210$$

Solving this system yields  $x_1 \approx 94$  and  $x_2 \approx 157$ .

$$P_{x_1 x_1} = -0.9$$

$$P_{x_1 x_2} = -0.8$$

$$P_{x_2 x_2} = -0.86$$

$$P_{x_1 x_1} < 0$$

$$P_{x_1 x_1} P_{x_2 x_2} - (P_{x_1 x_2})^2 > 0$$

So, profit is maximum when  $x_1 \approx 94$  and  $x_2 \approx 157$ .

72. Minimize  $C(x_1, x_2) = 0.25x_1^2 + 10x_1 + 0.15x_2^2 + 12x_2$   
subject to the constraint  $x_1 + x_2 = 1000$ .

$$\begin{aligned} 0.50x_1 + 10 &= \lambda \\ 0.30x_2 + 12 &= \lambda \\ x_1 + x_2 &= 1000 \Rightarrow 3x_1 + 3x_2 = 3000 \\ 5x_1 - 3x_2 &= 20 \\ 8x_1 &= 3020 \\ x_1 &= 377.5 \\ x_2 &= 622.5 \end{aligned}$$

$$C(377.5, 622.5) = 104,997.50$$

74. Minimize the square of the distance:

$$\begin{aligned} f(x, y, z) &= (x - 2)^2 + (y - 2)^2 + (z^2 + y^2 - 0)^2 \\ f_x &= 2(x - 2) + 2(x^2 + y^2)2x = 0 \quad | \quad x - 2 + 2x^3 + 2xy^2 = 0 \\ f_y &= 2(y - 2) + 2(x^2 + y^2)2y = 0 \quad | \quad y - 2 + 2y^3 + 2x^2y = 0 \end{aligned}$$

Clearly  $x = y$  and hence:  $4x^3 + x - 2 = 0$ . Using a computer algebra system,  $x \approx 0.6894$ .

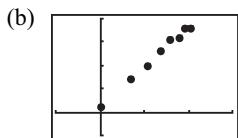
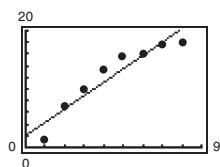
$$\text{So, } (\text{distance})^2 = (0.6894 - 2)^2 + (0.6894 - 2)^2 + [2(0.6894)]^2 \approx 4.3389.$$

Distance  $\approx 2.08$

75. (a)  $y = 0.004x^2 + 0.07x + 19.4$

(b) When  $x = 80$ ,  $y = 0.004(80)^2 + 0.07(80) + 19.4 \approx 50.6$  Kg.

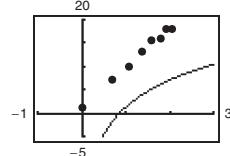
76. (a)  $y = 2.29t + 2.0$



Yes, the data appear linear.

- (c)  $y = 1.24 + 8.37 \ln t$

- (d)



73. Maximize  $f(x, y) = 4x + xy + 2y$  subject to the constraint  $20x + 4y = 2000$ .

$$\begin{aligned} 4 + y &= 20\lambda \\ x + 2 &= 4\lambda \\ 20x + 4y &= 2000 \Rightarrow 5x + y = 500 \\ 5x - y &= -6 \\ 10x &= 494 \\ x &= 49.4 \\ y &= 253 \end{aligned}$$

$$f(49.4, 253) = 13,201.8$$

77. Optimize  $f(x, y, z) = xy + yz + xz$  subject to the constraint  $x + y + z = 1$ .

$$\begin{aligned} y + z &= \lambda \\ x + z &= \lambda \\ x + y &= \lambda \end{aligned} \left\| \begin{array}{l} x = y = z \\ x + y + z = 1 \Rightarrow x = y = z = \frac{1}{3} \end{array} \right.$$

Maximum:  $f\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right) = \frac{1}{3}$

78. Optimize  $f(x, y) = x^2y$  subject to the constraint  $x + 2y = 2$ .

$$\begin{aligned} 2xy &= \lambda \\ x^2 &= 2\lambda \end{aligned} \left\| \begin{array}{l} x^2 = 4xy \Rightarrow x = 0 \text{ or } x = 4y \\ x + 2y = 2 \end{array} \right.$$

$$x + 2y = 2$$

If  $x = 0, y = 1$ . If  $x = 4y$ , then  $y = \frac{1}{3}, x = \frac{4}{3}$ .

Maximum:  $f\left(\frac{4}{3}, \frac{1}{3}\right) = \frac{16}{27}$

Minimum:  $f(0, 1) = 0$

$$79. PQ = \sqrt{x^2 + 4},$$

$$QR = \sqrt{y^2 + 1},$$

$$RS = z; x + y + z = 10$$

$$C = 3\sqrt{x^2 + 4} + 2\sqrt{y^2 + 1} + z$$

$$\text{Constraint: } x + y + z = 10$$

$$\nabla C = \lambda \nabla g$$

$$\frac{3x}{\sqrt{x^2 + 4}} \mathbf{i} + \frac{2y}{\sqrt{y^2 + 1}} \mathbf{j} + \mathbf{k} = \lambda [\mathbf{i} + \mathbf{j} + \mathbf{k}]$$

$$3x = \lambda \sqrt{x^2 + 4}$$

$$2y = \lambda \sqrt{y^2 + 1}$$

$$1 = \lambda$$

$$9x^2 = x^2 + 4 \Rightarrow x^2 = \frac{1}{2}$$

$$4y^2 = y^2 + 1 \Rightarrow y^2 = \frac{1}{3}$$

$$\text{So, } x = \frac{\sqrt{2}}{2} \approx 0.707 \text{ km,}$$

$$y = \frac{\sqrt{3}}{3} \approx 0.577 \text{ km,}$$

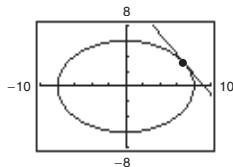
$$z = 10 - \frac{\sqrt{2}}{2} - \frac{\sqrt{3}}{3} \approx 8.716 \text{ km.}$$

80.  $f(x, y) = ax + by, x, y > 0$

$$\text{Constraint: } \frac{x^2}{64} + \frac{y^2}{36} = 1$$

- (a) Level curves of  $f(x, y) = 4x + 3y$  are lines of form  $y = -\frac{4}{3}x + C$ .

Using  $y = -\frac{4}{3}x + 12.3$ , you obtain  $x \approx 7, y \approx 3$ , and  $f(7, 3) = 28 + 9 = 37$ .



Constraint is an ellipse.

- (b) Level curves of  $f(x, y) = 4x + 9y$  are lines of form  $y = -\frac{4}{9}x + C$ .

Using  $y = -\frac{4}{9}x + 7$ , you obtain  $x \approx 4, y \approx 5.2$ , and  $f(4, 5.2) = 62.8$ .

## Problem Solving for Chapter 13

1. (a) The three sides have lengths 5, 6, and 5.

Thus,  $s = \frac{16}{2} = 8$  and  $A = \sqrt{8(3)(2)(3)} = 12$ .

- (b) Let  $f(a, b, c) = (\text{area})^2 = s(s - a)(s - b)(s - c)$ , subject to the constraint  $a + b + c = \text{constant}$  (perimeter).

Using Lagrange multipliers,

$$-s(s - b)(s - c) = \lambda$$

$$-s(s - a)(s - c) = \lambda$$

$$-s(s - a)(s - b) = \lambda.$$

From the first 2 equations

$$s - b = s - a \Rightarrow a = b.$$

Similarly,  $b = c$  and hence  $a = b = c$  which is an equilateral triangle.

- (c) Let  $f(a, b, c) = a + b + c$ , subject

to  $(\text{Area})^2 = s(s - a)(s - b)(s - c)$  constant.

Using Lagrange multipliers,

$$1 = -\lambda s(s - b)(s - c)$$

$$1 = -\lambda s(s - a)(s - c)$$

$$1 = -\lambda s(s - a)(s - b)$$

So,  $s - a = s - b \Rightarrow a = b$  and  $a = b = c$ .

$$2. V = \frac{4}{3}\pi r^3 + \pi r^2 h$$

Material =  $M = 4\pi r^2 + 2\pi r h$

$$V = 1000 \Rightarrow h = \frac{1000 - (4/3)\pi r^3}{\pi r^2}$$

$$\text{So, } M = 4\pi r^2 + 2\pi r \left( \frac{1000 - (4/3)\pi r^3}{\pi r^2} \right)$$

$$= 4\pi r^2 + \frac{2000}{r} - \frac{8}{3}\pi r^2$$

$$\frac{dM}{dr} = 8\pi r - \frac{2000}{r^2} - \frac{16}{3}\pi r = 0$$

$$8\pi r - \frac{16}{3}\pi r = \frac{2000}{r^2}$$

$$r^3 \left( \frac{8}{3}\pi \right) = 2000$$

$$r^3 = \frac{750}{\pi} \Rightarrow r = 5 \left( \frac{6}{\pi} \right)^{1/3}$$

$$\text{Then, } h = \frac{1000 - (4/3)\pi(750/\pi)}{\pi r^2} = 0.$$

The tank is a sphere of radius  $r = 5 \left( \frac{6}{\pi} \right)^{1/3}$ .

3. (a)  $F(x, y, z) = xyz - 1 = 0$

$$F_x = yz, F_y = xz, F_z = xy$$

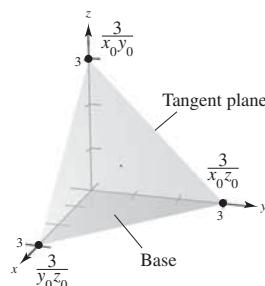
Tangent plane:

$$y_0 z_0(x - x_0) + x_0 z_0(y - y_0) + x_0 y_0(z - z_0) = 0$$

$$y_0 z_0 x + x_0 z_0 y + x_0 y_0 z = 3x_0 y_0 z_0 = 3$$

$$(b) V = \frac{1}{3}(\text{base})(\text{height})$$

$$= \frac{1}{3} \left( \frac{1}{2} \frac{3}{y_0 z_0} \frac{3}{x_0 z_0} \right) \left( \frac{3}{x_0 y_0} \right) = \frac{9}{2}$$



4. (a) As  $x \rightarrow \pm\infty$ ,  $f(x) = (x^3 - 1)^{1/3} \rightarrow x$  and  
hence  $\lim_{x \rightarrow \infty} [f(x) - g(x)] = \lim_{x \rightarrow \infty} [f(x) - g(x)] = 0$ .

- (b) Let  $(x_0, (x_0^3 - 1)^{1/3})$  be a point on the graph of  $f$ .

The line through this point perpendicular to  $g$  is  $y = -x + x_0 + \sqrt[3]{x_0^3 - 1}$ .

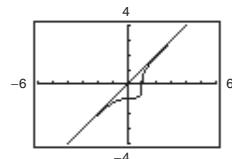
This line intersects  $g$  at the point

$$\left( \frac{1}{2} \left[ x_0 + \sqrt[3]{x_0^3 - 1} \right], \frac{1}{2} \left[ x_0 + \sqrt[3]{x_0^3 - 1} \right] \right).$$

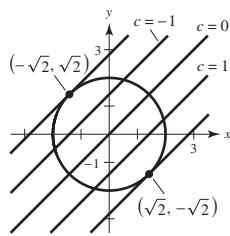
The square of the distance between these two points is  $h(x_0) = \frac{1}{2} \left( x_0 - \sqrt[3]{x_0^3 - 1} \right)^2$ .

$h$  is a maximum for  $x_0 = \frac{1}{\sqrt[3]{2}}$ . So, the point

on  $f$  farthest from  $g$  is  $\left( \frac{1}{\sqrt[3]{2}}, -\frac{1}{\sqrt[3]{2}} \right)$ .



5. (a)



Maximum value of  $f$  is  $f(\sqrt{2}, -\sqrt{2}) = 2\sqrt{2}$ .

Maximize  $f(x, y) = x - y$ .

Constraint:  $g(x, y) = x^2 + y^2 = 4$

$$\nabla f = \lambda \nabla g: \quad 1 = 2\lambda x$$

$$-1 = 2\lambda y$$

$$x^2 + y^2 = 4$$

$$2\lambda x = -2\lambda y \Rightarrow x = -y$$

$$2x^2 = 4 \Rightarrow x = \pm\sqrt{2}, y = \mp\sqrt{2}$$

$$f(\sqrt{2}, -\sqrt{2}) = 2\sqrt{2}, f(-\sqrt{2}, \sqrt{2}) = -2\sqrt{2}$$

(b)  $f(x, y) = x - y$

Constraint:  $x^2 + y^2 = 0 \Rightarrow (x, y) = (0, 0)$

Maximum and minimum values are 0.

Lagrange multipliers does not work:

$$\begin{cases} 1 = 2\lambda x \\ -1 = 2\lambda y \end{cases} \quad x = -y = 0, \text{ a contradiction.}$$

Note that  $\nabla g(0, 0) = \mathbf{0}$ .

8. (a)  $T(x, y) = 2x^2 + y^2 - y + 10 = 10$

$$2x^2 + y^2 - y + \frac{1}{4} = \frac{1}{4}$$

$$2x^2 + \left(y - \frac{1}{2}\right)^2 = \frac{1}{4}$$

$$\frac{x^2}{1/8} + \frac{\left(y - (1/2)\right)^2}{1/4} = 1 \quad \text{ellipse}$$

6. Heat Loss =  $H = k(5xy + xy + 3xz + 3xz + 3yz + 3yz)$

$$= k(6xy + 6xz + 6yz)$$

$$V = xyz = 1000 \Rightarrow z = \frac{1000}{xy}.$$

$$\text{Then } H = 6k\left(xy + \frac{1000}{y} + \frac{1000}{x}\right).$$

Setting  $H_x = H_y = 0$ , you obtain  $x = y = z = 10$ .

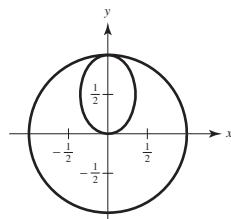
7.  $H = k(5xy + 6xz + 6yz)$

$$z = \frac{1000}{xy} \Rightarrow H = k\left(5xy + \frac{6000}{y} + \frac{6000}{x}\right).$$

$$H_x = 5y - \frac{6000}{x^2} = 0 \Rightarrow 5yx^2 = 6000$$

By symmetry,  $x = y \Rightarrow x^3 = y^3 = 1200$ .

$$\text{So, } x = y = 2\sqrt[3]{150} \text{ and } z = \frac{5}{3}\sqrt[3]{150}.$$



(b) On  $x^2 + y^2 = 1$ ,  $T(x, y) = T(y) = 2(1 - y^2) + y^2 - y + 10 = 12 - y^2 - y$

$$T'(y) = -2y - 1 = 0 \Rightarrow y = -\frac{1}{2}, x = \pm\frac{\sqrt{3}}{2}.$$

$$\text{Inside: } T_x = 4x - 0, T_y = 2y - 1 = 0 \Rightarrow \left(0, \frac{1}{2}\right)$$

$$T\left(0, \frac{1}{2}\right) = \frac{39}{4} \text{ minimum}$$

$$T\left(\pm\frac{\sqrt{3}}{2}, -\frac{1}{2}\right) = \frac{49}{4} \text{ maximum}$$

$$\begin{aligned}
 9. (a) \quad & \frac{\partial f}{\partial x} = Cax^{a-1}y^{1-a}, \frac{\partial f}{\partial y} = C(1-a)x^ay^{-a} \\
 & x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = Cax^ay^{1-a} + C(1-a)x^ay^{1-a} \\
 & = [Ca + C(1-a)]x^ay^{1-a} \\
 & = Cx^ay^{1-a} = f
 \end{aligned}$$

$$(b) \quad f(tx, ty) = C(tx)^a(ty)^{1-a} = Ct^ax^at^{1-a}y^{1-a} = Cx^ay^{1-a}(t) = tf(x, y)$$

10.  $x = r \cos \theta, y = r \sin \theta, z = z$

$$\begin{aligned}
 \frac{\partial u}{\partial \theta} &= \frac{\partial u}{\partial x} \frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial \theta} + \frac{\partial u}{\partial z} \frac{\partial z}{\partial \theta} = \frac{\partial u}{\partial x}(-r \sin \theta) + \frac{\partial u}{\partial y}r \cos \theta \text{ Similarly,} \\
 \frac{\partial u}{\partial r} &= \frac{\partial u}{\partial x} \cos \theta + \frac{\partial u}{\partial y} \sin \theta. \\
 \frac{\partial^2 u}{\partial \theta^2} &= (-r \sin \theta) \left[ \frac{\partial^2 u}{\partial x^2} \frac{\partial x}{\partial \theta} + \frac{\partial^2 u}{\partial x \partial y} \frac{\partial y}{\partial \theta} + \frac{\partial^2 u}{\partial x \partial z} \frac{\partial z}{\partial \theta} \right] - r \frac{\partial u}{\partial x} \cos \theta + (r \cos \theta) \left[ \frac{\partial^2 u}{\partial y \partial x} \frac{\partial x}{\partial \theta} + \frac{\partial^2 u}{\partial y^2} \frac{\partial y}{\partial \theta} + \frac{\partial^2 u}{\partial y \partial z} \frac{\partial z}{\partial \theta} \right] - r \frac{\partial u}{\partial y} \sin \theta \\
 &= \frac{\partial^2 u}{\partial x^2} r^2 \sin^2 \theta + \frac{\partial^2 u}{\partial y^2} r^2 \cos^2 \theta - 2 \frac{\partial^2 u}{\partial x \partial y} r^2 \sin \theta \cos \theta - \frac{\partial u}{\partial x} r \cos \theta - \frac{\partial u}{\partial y} r \sin \theta \\
 \text{Similarly, } \frac{\partial^2 u}{\partial r^2} &= \frac{\partial^2 u}{\partial x^2} \cos^2 \theta + \frac{\partial^2 u}{\partial y^2} \sin^2 \theta + 2 \frac{\partial^2 u}{\partial x \partial y} \cos \theta \sin \theta.
 \end{aligned}$$

Now observe that

$$\begin{aligned}
 \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2} &= \left[ \frac{\partial^2 u}{\partial x^2} \cos^2 \theta + \frac{\partial^2 u}{\partial y^2} \sin^2 \theta + 2 \frac{\partial^2 u}{\partial x \partial y} \cos \theta \sin \theta \right] + \frac{1}{r} \left[ \frac{\partial u}{\partial x} \cos \theta + \frac{\partial u}{\partial y} \sin \theta \right] \\
 &\quad + \left[ \frac{\partial^2 u}{\partial x^2} \sin^2 \theta + \frac{\partial^2 u}{\partial y^2} \cos^2 \theta - 2 \frac{\partial^2 u}{\partial x \partial y} \sin \theta \cos \theta - \frac{1}{r} \frac{\partial u}{\partial x} \cos \theta - \frac{1}{r} \frac{\partial u}{\partial y} \sin \theta \right] + \frac{\partial^2 u}{\partial z^2} \\
 &= \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}.
 \end{aligned}$$

So, Laplace's equation in cylindrical coordinates, is  $\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2} = 0$ .

$$11. (a) \quad x = 64(\cos 45^\circ)t = 32\sqrt{2}t$$

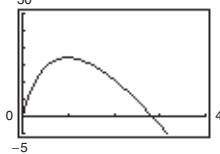
$$y = 64(\sin 45^\circ)t - 16t^2 = 32\sqrt{2}t - 16t^2$$

$$(b) \quad \tan \alpha = \frac{y}{x + 50}$$

$$\alpha = \arctan\left(\frac{y}{x + 50}\right) = \arctan\left(\frac{32\sqrt{2}t - 16t^2}{32\sqrt{2}t + 50}\right)$$

$$(c) \quad \frac{d\alpha}{dt} = \frac{1}{1 + \left(\frac{32\sqrt{2}t - 16t^2}{32\sqrt{2}t + 50}\right)^2} \cdot \frac{-64(8\sqrt{2}t^2 + 25t - 25\sqrt{2})}{(32\sqrt{2}t + 50)^2} = \frac{-16(8\sqrt{2}t^2 + 25t - 25\sqrt{2})}{64t^4 - 256\sqrt{2}t^3 + 1024t^2 + 800\sqrt{2}t + 625}$$

(d)



No. The rate of change of  $\alpha$  is greatest when the projectile is closest to the camera.

(e)  $\frac{d\alpha}{dt} = 0$  when

$$8\sqrt{2}t^2 + 25t - 25\sqrt{2} = 0$$

$$t = \frac{-25 + \sqrt{25^2 - 4(8\sqrt{2})(-25\sqrt{2})}}{2(8\sqrt{2})} \approx 0.98 \text{ second.}$$

No, the projectile is at its maximum height when  $dy/dt = 32\sqrt{2} - 32t = 0$  or  $t = \sqrt{2} \approx 1.41$  seconds.

12. (a)  $d = \sqrt{x^2 + y^2} = \sqrt{(32\sqrt{2}t)^2 + (32\sqrt{2}t - 16t^2)^2} = \sqrt{4096t^2 - 1024\sqrt{2}t^3 + 256t^4} = 16t\sqrt{t^2 - 4\sqrt{2}t + 16}$

(b)  $\frac{dd}{dt} = \frac{32(t^2 - 3\sqrt{2}t + 8)}{\sqrt{t^2 - 4\sqrt{2}t + 16}}$

(c) When  $t = 2$ :

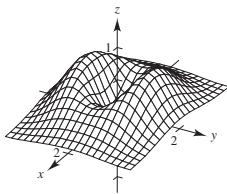
$$\frac{dd}{dt} = \frac{32(12 - 6\sqrt{2})}{\sqrt{20 - 8\sqrt{2}}} \approx 38.16 \text{ ft/sec}$$

(d)  $\frac{d^2d}{dt^2} = \frac{32(t^3 - 6\sqrt{2}t^2 + 36t - 32\sqrt{12})}{(t^2 - 4\sqrt{2}t + 16)^{3/2}} = 0$  when  $t \approx 1.943$  seconds. No. The projectile is at its maximum height when  $t = \sqrt{2}$ .

13. (a) There is a minimum at  $(0, 0, 0)$ , maxima at  $(0, \pm 1, 2/e)$  and saddle point at  $(\pm 1, 0, 1/e)$ :

$$\begin{aligned} f_x &= (x^2 + 2y^2)e^{-(x^2+y^2)}(-2x) + (2x)e^{-(x^2+y^2)} \\ &= e^{-(x^2+y^2)}[(x^2 + 2y^2)(-2x) + 2x] = e^{-(x^2+y^2)}[-2x^3 + 4xy^2 + 2x] = 0 \Rightarrow x^3 + 2xy^2 - x = 0 \\ f_y &= (x^2 + 2y^2)e^{-(x^2+y^2)}(-2y) + (4y)e^{-(x^2+y^2)} \\ &= e^{-(x^2+y^2)}[(x^2 + 2y^2)(-2y) + 4y] = e^{-(x^2+y^2)}[-4y^3 - 2x^2y + 4y] = 0 \Rightarrow 2y^3 + x^2y - 2y = 0 \end{aligned}$$

Solving the two equations  $x^3 + 2xy^2 - x = 0$  and  $2y^3 + x^2y - 2y = 0$ , you obtain the following critical points:  $(0, \pm 1), (\pm 1, 0), (0, 0)$ . Using the second derivative test, you obtain the results above.



(b) As in part (a), you obtain

$$f_x = e^{-(x^2+y^2)}[2x(x^2 - 1 - 2y^2)]$$

$$f_y = e^{-(x^2+y^2)}[2y(2 + x^2 - 2y^2)]$$

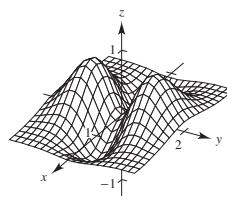
The critical numbers are  $(0, 0), (0, \pm 1), (\pm 1, 0)$ .

These yield

$(\pm 1, 0, -1/e)$  minima

$(0, \pm 1, 2/e)$  maxima

$(0, 0, 0)$  saddle



(c) In general, for  $\alpha > 0$  you obtain

$(0, 0, 0)$  minimum

$(0, \pm 1, \beta/e)$  maxima

$(\pm 1, 0, \alpha/e)$  saddle

For  $\alpha < 0$ , you obtain

$(\pm 1, 0, \alpha/e)$  minima

$(0, \pm 1, \beta/e)$  maxima

$(0, 0, 0)$  saddle

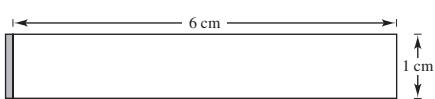
14. Given that  $f$  is a differentiable function such that

$$\nabla f(x_0, y_0) = \mathbf{0}, \text{ then } f_x(x_0, y_0) = 0 \text{ and } f_y(x_0, y_0) = 0.$$

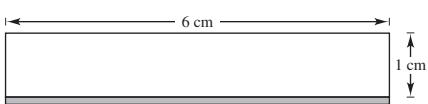
Therefore, the tangent plane is  $-(z - z_0) = 0$  or

$$z = z_0 = f(x_0, y_0) \text{ which is horizontal.}$$

15. (a)



- (b)



(c) The height has more effect since the shaded region in (b) is larger than the shaded region in (a).

$$(d) A = hl \Rightarrow dA = l dh + h dl$$

$$\text{If } dl = 0.01 \text{ and } dh = 0, \text{ then } dA = l(0.01) = 0.01.$$

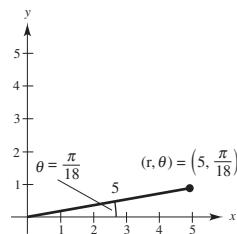
$$\text{If } dh = 0.01 \text{ and } dl = 0, \text{ then } dA = 6(0.01) = 0.06.$$

$$16. (r, \theta) = \left(5, \frac{\pi}{18}\right)$$

$$dr = \pm 0.05, d\theta = \pm 0.05$$

$$x = r \cos \theta = 5 \cos \frac{\pi}{18} \approx 4.924$$

$$y = r \sin \theta = 5 \sin \frac{\pi}{18} \approx 0.868$$



(a)  $dx$  should be more effected by changes in  $r$ .

$$dx = (\cos \theta)dr + (-r \sin \theta)d\theta \\ \approx (0.985)dr - 0.868d\theta$$

$dx$  is more effected by changes in  $r$  because  $0.985 > 0.868$ .

(b)  $dy$  should be more effected by changes in  $\theta$ .

$$dy = \sin \theta dr + r \cos \theta d\theta \\ \approx 0.174dr + 4.924d\theta$$

$dy$  is more effected by  $\theta$  because  $4.924 > 0.174$ .

17. Let  $g(x, y) = yf\left(\frac{x}{y}\right)$ .

$$g_y(x, y) = f\left(\frac{x}{y}\right) + yf'\left(\frac{x}{y}\right)\left(\frac{-x}{y^2}\right) = f\left(\frac{x}{y}\right) - \frac{x}{y}f'\left(\frac{x}{y}\right)$$

$$g_x(x, y) = yf'\left(\frac{x}{y}\right)\left(\frac{1}{y}\right) = f'\left(\frac{x}{y}\right)$$

$$\text{Tangent plane at } (x_0, y_0, z_0) \text{ is } f'\left(\frac{x_0}{y_0}\right)(x - x_0) + \left[ f\left(\frac{x_0}{y_0}\right) - \frac{x_0}{y_0}f'\left(\frac{x_0}{y_0}\right) \right](y - y_0) - 1\left(z - y_0f\left(\frac{x_0}{y_0}\right)\right) = 0$$

$$\Rightarrow f'\left(\frac{x_0}{y_0}\right)x + \left[ f\left(\frac{x_0}{y_0}\right) - \frac{x_0}{y_0}f'\left(\frac{x_0}{y_0}\right) \right]y - z = 0.$$

This plane passes through the origin, the common point of intersection.

18.  $x^2 + y^2 = 2x$

$$(x - 1)^2 + y^2 = 1 \text{ Circle}$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \text{Ellipse}$$

The circle and ellipse intersect at  $(x, y)$  and  $(x, -y)$  for a unique value of  $x$ .

$$y^2 = \frac{b^2}{a^2}(a^2 - x^2) \quad \text{Ellipse}$$

$$x^2 + \frac{b^2}{a^2}(a^2 - x^2) = 2x \quad \text{Circle}$$

$$\left(1 - \frac{b^2}{a^2}\right)x^2 - 2x + b^2 = 0 \quad \text{Quadratic}$$

For these to be a unique  $x$ -value, the discriminant must be 0.

$$4 - 4\left(1 - \frac{b^2}{a^2}\right)b^2 = 0$$

$$a^2 - a^2b^2 + b^4 = 0$$

We use lagrange multipliers to minimize the area  $f(a, b) = \pi ab$  of the ellipse subject to the constraint

$$g(a, b) = a^2 - a^2b^2 + b^4 = 0.$$

$$\nabla f = \lambda \nabla g$$

$$\langle \pi b, \pi a \rangle = \lambda \langle 2a - 2ab^2, -2a^2b + 4b^3 \rangle$$

$$\pi b = \lambda(2a - 2ab^2)$$

$$\pi a = \lambda(-2a^2b + 4b^3)$$

$$\lambda = \frac{\pi b}{2a - 2ab^2} = \frac{\pi a}{4b^3 - 2a^2b} \Rightarrow 4b^4 - 2a^2b^2 = 2a^2 - 2a^2b^2 \Rightarrow 2b^4 = a^2 \Rightarrow b^2 = \frac{a^2}{\sqrt{2}}$$

$$\text{Using the constraint, } a^2 - a^2b^2 + b^4 = 0, \quad a^2 - a^2 \frac{a^2}{\sqrt{2}} + \frac{a^2}{2} = 0$$

$$\frac{3}{2} = \frac{a^2}{\sqrt{2}}$$

$$a = \frac{3}{2}\sqrt{2}, b = \frac{\sqrt{6}}{2}.$$

$$\text{Ellipse: } \frac{x^2}{(9/2)} + \frac{y^2}{(3/2)} = 1$$

19.  $\frac{\partial u}{\partial t} = \frac{1}{2}[-\cos(x-t) + \cos(x+t)]$

$$\frac{\partial^2 u}{\partial t^2} = \frac{1}{2}[-\sin(x-t) - \sin(x+t)]$$

$$\frac{\partial u}{\partial x} = \frac{1}{2}[\cos(x-t) + \cos(x+t)]$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{2}[-\sin(x-t) - \sin(x+t)]$$

$$\text{Then, } \frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}.$$

20.  $u(x, t) = \frac{1}{2}[f(x - ct) + f(x + ct)]$

Let  $r = x - ct$  and  $s = x + ct$ .

$$\text{Then } u(r, s) = \frac{1}{2}[f(r) + f(s)].$$

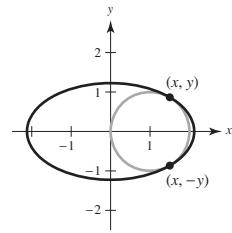
$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial r} \frac{\partial r}{\partial t} + \frac{\partial u}{\partial s} \frac{\partial s}{\partial t} = \frac{1}{2} \frac{df}{dr}(-c) + \frac{1}{2} \frac{df}{ds}(c)$$

$$\frac{\partial^2 u}{\partial t^2} = \frac{1}{2} \frac{d^2 f}{dr^2}(-c)^2 + \frac{1}{2} \frac{d^2 f}{ds^2}(c)^2 = \frac{c^2}{2} \left[ \frac{d^2 f}{dr^2} + \frac{d^2 f}{ds^2} \right]$$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r} \frac{\partial r}{\partial x} + \frac{\partial u}{\partial s} \frac{\partial s}{\partial x} = \frac{1}{2} \frac{df}{dr}(1) + \frac{1}{2} \frac{df}{ds}(1)$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{2} \frac{d^2 f}{dr^2}(1)^2 + \frac{1}{2} \frac{d^2 f}{ds^2}(1)^2 = \frac{1}{2} \left[ \frac{d^2 f}{dr^2} + \frac{d^2 f}{ds^2} \right]$$

$$\text{So, } \frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}.$$



# C H A P T E R 1 4

## Multiple Integration

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# C H A P T E R 14

## Multiple Integration

### Section 14.1 Iterated Integrals and Area in the Plane

1.  $\int_0^x (x + 2y) dy = [xy + y^2]_0^x = x^2 + x^2 = 2x^2$

3.  $\int_1^{2y} \frac{y}{x} dx = [y \ln x]_1^{2y}$   
 $= y \ln 2y - 0 = y \ln 2y, (y > 0)$

2.  $\int_x^{x^2} \frac{y}{x} dy = \left[ \frac{1}{2} \frac{y^2}{x} \right]_x^{x^2} = \frac{1}{2} \left( \frac{x^4}{x} - \frac{x^2}{x} \right) = \frac{x}{2} (x^2 - 1)$

4.  $\int_0^{\cos y} y dx = [yx]_0^{\cos y} = y \cos y$

5.  $\int_0^{\sqrt{4-x^2}} x^2 y dy = \left[ \frac{1}{2} x^2 y^2 \right]_0^{\sqrt{4-x^2}} = \frac{4x^2 - x^4}{2}$

6.  $\int_{x^3}^{\sqrt{x}} (x^2 + 3y^2) dy = [x^2 y + y^3]_{x^3}^{\sqrt{x}} = \left( x^2 \sqrt{x} + (\sqrt{x})^3 \right) - \left( x^2 x^3 + (x^3)^3 \right) = x^{5/2} + x^{3/2} - x^5 - x^9$

7.  $\int_{e^y}^y \frac{y \ln x}{x} dx = \left[ \frac{1}{2} y \ln^2 x \right]_{e^y}^y = \frac{1}{2} y [\ln^2 y - \ln^2 e^y] = \frac{y}{2} [\ln y^2 - y^2], (y > 0)$

8.  $\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} (x^2 + y^2) dx = \left[ \frac{1}{3} x^3 + y^2 x \right]_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} = 2 \left[ \frac{1}{3} (1 - y^2)^{3/2} + y^2 (1 - y^2)^{1/2} \right] = \frac{2\sqrt{1-y^2}}{3} (1 + 2y^2)$

9.  $\int_0^{x^3} y e^{-y/x} dy = [-xye^{-y/x}]_0^{x^3} + x \int_0^{x^3} e^{-y/x} dy = -x^4 e^{-x^2} - [x^2 e^{-y/x}]_0^{x^3} = x^2 (1 - e^{-x^2} - x^2 e^{-x^2})$

$u = y, du = dy, dv = e^{-y/x} dy, v = -xe^{-y/x}$

10.  $\int_y^{\pi/2} \sin^3 x \cos y dx = \int_y^{\pi/2} (1 - \cos^2 x) \sin x \cos y dx$   
 $= \left[ (-\cos x + \frac{1}{3} \cos^3 x) \cos y \right]_y^{\pi/2} = \left( \cos y - \frac{1}{3} \cos^3 y \right) \cos y$

11.  $\int_0^1 \int_0^2 (x + y) dy dx = \int_0^1 \left[ xy + \frac{1}{2} y^2 \right]_0^2 dx = \int_0^1 (2x + 2) dx = [x^2 + 2x]_0^1 = 3$

12.  $\int_{-1}^1 \int_{-2}^2 (x^2 - y^2) dy dx = \int_{-1}^1 \left[ x^2 y - \frac{y^3}{3} \right]_{-2}^2 dx = \int_{-1}^1 \left[ 2x^2 - \frac{8}{3} + 2x^2 - \frac{8}{3} \right] dx$   
 $= \int_{-1}^1 \left( 4x^2 - \frac{16}{3} \right) dx = \left[ \frac{4x^3}{3} - \frac{16}{3} x \right]_{-1}^1 = \left( \frac{4}{3} - \frac{16}{3} \right) - \left( -\frac{4}{3} + \frac{16}{3} \right) = -8$

13.  $\int_1^2 \int_0^4 (x^2 - 2y^2) dx dy = \int_1^2 \left[ \frac{x^3}{3} - 2xy^2 \right]_0^4 dy = \int_1^2 \left[ \frac{64}{3} - 8y^2 \right] dy = \left[ \frac{64}{3} y - \frac{8}{3} y^3 \right]_1^2 = \left( \frac{128}{3} - \frac{64}{3} \right) - \left( \frac{64}{3} - \frac{8}{3} \right) = \frac{8}{3}$

14.  $\int_{-1}^2 \int_1^3 (x + y^2) dx dy = \int_{-1}^2 \left[ \frac{x^2}{2} + xy^2 \right]_1^3 dy = \int_{-1}^2 \left[ \left( \frac{9}{2} + 3y^2 \right) - \left( \frac{1}{2} + y^2 \right) \right] dy$   
 $= \int_{-1}^2 (4 + 2y^2) dy = \left[ 4y + \frac{2}{3} y^3 \right]_{-1}^2 = \left( 8 + \frac{16}{3} \right) - \left( -4 - \frac{2}{3} \right) = 18$

$$15. \int_0^{\pi/2} \int_0^1 y \cos x \, dy \, dx = \int_0^{\pi/2} \left[ \frac{y^2}{2} \cos x \right]_0^1 \, dx = \int_0^{\pi/2} \frac{1}{2} \cos x \, dx = \left[ \frac{1}{2} \sin x \right]_0^{\pi/2} = \frac{1}{2}$$

$$16. \int_0^{\ln 4} \int_0^{\ln 3} e^{x+y} \, dy \, dx = \int_0^{\ln 4} \left[ e^{x+y} \right]_0^{\ln 3} \, dx \\ = \int_0^{\ln 4} \left[ e^{x+\ln 3} - e^x \right] \, dx = \left[ e^{x+\ln 3} - e^x \right]_0^{\ln 4} = (e^{\ln 4 + \ln 3} - e^{\ln 4}) - (e^{\ln 3} - 1) = (12 - 4) - (3 - 1) = 6$$

$$17. \int_0^\pi \int_0^{\sin x} (1 + \cos x) \, dy \, dx = \int_0^\pi \left[ (y + y \cos x) \right]_0^{\sin x} \, dx = \int_0^\pi [\sin x + \sin x \cos x] \, dx = \left[ -\cos x + \frac{1}{2} \sin^2 x \right]_0^\pi = 1 + 1 = 2$$

$$18. \int_1^4 \int_1^{\sqrt{x}} 2ye^{-x} \, dy \, dx = \int_1^4 \left[ y^2 e^{-x} \right]_1^{\sqrt{x}} \, dx = \int_1^4 (xe^{-x} - e^{-x}) \, dx = \left[ -xe^{-x} \right]_1^4 = -4e^{-4} + e^{-1} = \frac{1}{e} - \frac{4}{e^4}$$

$$19. \int_0^1 \int_0^x \sqrt{1-x^2} \, dy \, dx = \int_0^1 \left[ y \sqrt{1-x^2} \right]_0^x \, dx = \int_0^1 x \sqrt{1-x^2} \, dx = \left[ -\frac{1}{2} \left( \frac{2}{3} \right) (1-x^2)^{3/2} \right]_0^1 = \frac{1}{3}$$

$$20. \int_{-4}^4 \int_0^{x^2} \sqrt{64-x^3} \, dy \, dx = \int_{-4}^4 \left[ y \sqrt{64-x^3} \right]_0^{x^2} \, dx = \int_{-4}^4 x^2 \sqrt{64-x^3} \, dx = \left[ -\frac{2}{9} (64-x^3)^{3/2} \right]_{-4}^4 = 0 + \frac{2}{9} (128)^{3/2} = \frac{2048}{9} \sqrt{2}$$

$$21. \int_{-1}^5 \int_0^{3y} \left( 3 + x^2 + \frac{1}{4} y^2 \right) \, dx \, dy = \int_{-1}^5 \left[ 3x + \frac{x^3}{3} + \frac{1}{4} xy^2 \right]_0^{3y} \, dy \\ = \int_{-1}^5 \left[ 9y + 9y^3 + \frac{3}{4} y^5 \right] \, dy = \int_{-1}^5 \left[ 9y + \frac{39}{4} y^3 \right] \, dy = \left[ \frac{9}{2} y^2 + \frac{39}{16} y^4 \right]_{-1}^5 = \left( \frac{9}{2}(25) + \frac{39}{16}(625) \right) - \left( \frac{9}{2} + \frac{39}{16} \right) = 1629$$

$$22. \int_0^2 \int_y^{2y} (10 + 2x^2 + 2y^2) \, dx \, dy = \int_0^2 \left[ 10x + \frac{2x^3}{3} + 2y^2 x \right]_y^{2y} \, dy = \int_0^2 \left[ \left( 20y + \frac{16}{3} y^3 + 4y^3 \right) - \left( 10y + \frac{2}{3} y^3 + 2y^3 \right) \right] \, dy \\ = \int_0^2 \left[ 10y + \frac{20}{3} y^3 \right] \, dy = \left[ 5y^2 + \frac{5y^4}{3} \right]_0^2 = 20 + \frac{80}{3} = \frac{140}{3}$$

$$23. \int_0^1 \int_0^{\sqrt{1-y^2}} (x+y) \, dx \, dy = \int_0^1 \left[ \frac{1}{2} x^2 + xy \right]_0^{\sqrt{1-y^2}} \, dy = \int_0^1 \left[ \frac{1}{2} (1-y^2) + y \sqrt{1-y^2} \right] \, dy = \left[ \frac{1}{2} y - \frac{1}{6} y^3 - \frac{1}{2} \left( \frac{2}{3} \right) (1-y^2)^{3/2} \right]_0^1 = \frac{2}{3}$$

$$24. \int_0^2 \int_{3y^2-6y}^{2y-y^2} 3y \, dx \, dy = \int_0^2 [3xy]_{3y^2-6y}^{2y-y^2} \, dy = 3 \int_0^2 (8y^2 - 4y^3) \, dy = \left[ 3 \left( \frac{8}{3} y^3 - y^4 \right) \right]_0^2 = 16$$

$$25. \int_0^2 \int_0^{\sqrt{4-y^2}} \frac{2}{\sqrt{4-y^2}} \, dx \, dy = \int_0^2 \left[ \frac{2x}{\sqrt{4-y^2}} \right]_0^{\sqrt{4-y^2}} \, dy = \int_0^2 2 \, dy = [2y]_0^2 = 4$$

$$26. \int_1^3 \int_0^y \frac{4}{x^2 + y^2} \, dx \, dy = \int_1^3 \left[ \frac{4}{y} \arctan \left( \frac{x}{y} \right) \right]_0^y \, dy = \int_1^3 \frac{4}{y} \left( \frac{\pi}{4} \right) \, dy = [\pi \ln y]_1^3 = \pi \ln 3$$

$$27. \int_0^{\pi/2} \int_0^{2 \cos \theta} r \, dr \, d\theta = \int_0^{\pi/2} \left[ \frac{r^2}{2} \right]_0^{2 \cos \theta} \, d\theta = \int_0^{\pi/2} 2 \cos^2 \theta \, d\theta = \left[ \theta - \frac{1}{2} \sin 2\theta \right]_0^{\pi/2} = \frac{\pi}{2}$$

$$28. \int_0^{\pi/4} \int_{-\sqrt{3}}^{\sqrt{3} \cos \theta} r \, dr \, d\theta = \int_0^{\pi/4} \left[ \frac{r^2}{2} \right]_{-\sqrt{3}}^{\sqrt{3} \cos \theta} \, d\theta = \int_0^{\pi/4} \left( \frac{3 \cos^2 \theta}{2} - \frac{3}{2} \right) \, d\theta = \int_0^{\pi/4} \left( \frac{3}{4} (1 + \cos 2\theta) - \frac{3}{2} \right) \, d\theta \\ = \int_0^{\pi/4} \left( \frac{3}{4} \cos 2\theta - \frac{3}{4} \right) \, d\theta = \left[ \frac{3}{8} \sin 2\theta - \frac{3}{4} \theta \right]_0^{\pi/4} = \frac{3}{8} - \frac{3\pi}{16}$$

$$\begin{aligned}
 29. \int_0^{\pi/2} \int_0^{\sin \theta} \theta r dr d\theta &= \int_0^{\pi/2} \left[ \theta \frac{r^2}{2} \right]_0^{\sin \theta} d\theta = \int_0^{\pi/2} \frac{1}{2} \theta \sin^2 \theta d\theta \\
 &= \frac{1}{4} \int_0^{\pi/2} (\theta - \theta \cos 2\theta) d\theta = \frac{1}{4} \left[ \frac{\theta^2}{2} - \left( \frac{1}{4} \cos 2\theta + \frac{\theta}{2} \sin 2\theta \right) \right]_0^{\pi/2} = \frac{\pi^2}{32} + \frac{1}{8}
 \end{aligned}$$

$$30. \int_0^{\pi/4} \int_0^{\cos \theta} 3r^2 \sin \theta dr d\theta = \int_0^{\pi/4} \left[ r^3 \sin \theta \right]_0^{\cos \theta} d\theta = \int_0^{\pi/4} \cos^3 \sin \theta d\theta = \left[ -\frac{\cos^4 \theta}{4} \right]_0^{\pi/4} = -\frac{1}{4} \left[ \left( \frac{1}{\sqrt{2}} \right)^4 - 1 \right] = \frac{3}{16}$$

$$31. \int_1^\infty \int_0^{1/x} y dy dx = \int_1^\infty \left[ \frac{y^2}{2} \right]_0^{1/x} dx = \frac{1}{2} \int_1^\infty \frac{1}{x^2} dx = \left[ -\frac{1}{2x} \right]_1^\infty = 0 + \frac{1}{2} = \frac{1}{2}$$

$$32. \int_0^3 \int_0^\infty \frac{x^2}{1+y^2} dy dx = \int_0^3 \left[ x^2 \arctan y \right]_0^\infty dx = \int_0^3 x^2 \left( \frac{\pi}{2} \right) dx = \left[ \frac{\pi}{2} \cdot \frac{x^3}{3} \right]_0^3 = \frac{9\pi}{2}$$

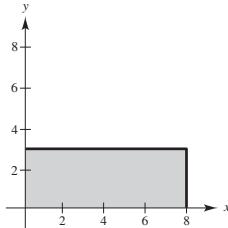
$$33. \int_1^\infty \int_1^\infty \frac{1}{xy} dx dy = \int_1^\infty \left[ \frac{1}{y} \ln x \right]_1^\infty dy = \int_1^\infty \left[ \frac{1}{y}(\infty) - \frac{1}{y}(0) \right] dy$$

Diverges

$$34. \int_0^\infty \int_0^\infty xy e^{-(x^2+y^2)} dx dy = \int_0^\infty \left[ -\frac{1}{2} ye^{-(x^2+y^2)} \right]_0^\infty dy = \int_0^\infty \frac{1}{2} ye^{-y^2} dy = \left[ -\frac{1}{4} e^{-y^2} \right]_0^\infty = \frac{1}{4}$$

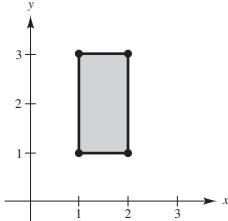
$$35. A = \int_0^8 \int_0^3 dy dx = \int_0^8 [y]_0^3 dx = \int_0^8 3 dx = [3x]_0^8 = 24$$

$$A = \int_0^3 \int_0^8 dx dy = \int_0^3 [x]_0^8 dy = \int_0^3 8 dy = [8y]_0^3 = 24$$



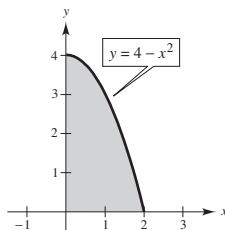
$$36. A = \int_1^2 \int_1^3 dy dx = \int_1^2 [y]_1^3 dx = \int_1^2 2 dx = [2x]_1^2 = 2$$

$$A = \int_1^3 \int_1^2 dx dy = \int_1^3 [x]_1^2 dy = \int_1^3 dy = [y]_1^3 = 2$$



$$37. A = \int_0^2 \int_0^{4-x^2} dy dx = \int_0^2 [y]_0^{4-x^2} dx = \int_0^2 (4-x^2) dx = \left[ 4x - \frac{x^3}{3} \right]_0^2 = \frac{16}{3}$$

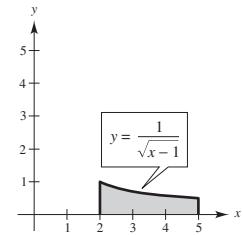
$$A = \int_0^4 \int_0^{\sqrt{4-y}} dx dy = \int_0^4 [x]_0^{\sqrt{4-y}} dy = \int_0^4 \sqrt{4-y} dy = -\int_0^4 (4-y)^{1/2} (-1) dy = \left[ -\frac{2}{3}(4-y)^{3/2} \right]_0^4 = \frac{2}{3}(8) = \frac{16}{3}$$



$$38. A = \int_2^5 \int_0^{1/\sqrt{x-1}} dy dx = \int_2^5 [y]_0^{1/\sqrt{x-1}} dx = \int_2^5 \frac{1}{\sqrt{x-1}} dx = \left[ 2\sqrt{x-1} \right]_2^5 = 2$$

$$A = \int_0^{1/2} \int_2^5 dx dy + \int_{1/2}^1 \int_2^{1+(1/y^2)} dx dy$$

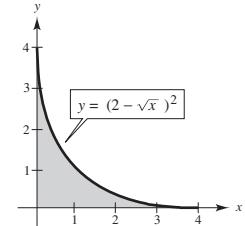
$$= \int_0^{1/2} [x]_2^5 dy + \int_{1/2}^1 [x]_2^{1+(1/y^2)} dy = \int_0^{1/2} 3 dy + \int_{1/2}^1 \left( \frac{1}{y^2} - 1 \right) dy = [3y]_0^{1/2} + \left[ -\frac{1}{y} - y \right]_{1/2}^1 = 2$$



$$39. A = \int_0^4 \int_0^{(2-\sqrt{x})^2} dy dx = \int_0^4 [y]_0^{(2-\sqrt{x})^2} dx = \int_0^4 (4 - 4\sqrt{x} + x) dx = \left[ 4x - \frac{8}{3}x\sqrt{x} + \frac{x^2}{2} \right]_0^4 = \frac{8}{3}$$

$$A = \int_0^4 \int_0^{(2-\sqrt{y})^2} dx dy = \frac{8}{3}$$

Integration steps are similar to those above.



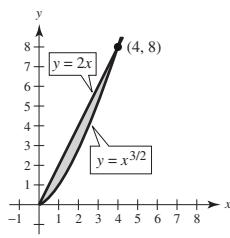
$$40. A = \int_0^4 \int_{x^{3/2}}^{2x} dy dx$$

$$= \int_0^4 [y]_{x^{3/2}}^{2x} dx = \int_0^4 (2x - x^{3/2}) dx$$

$$= \left[ x^2 - \frac{2}{5}x^{5/2} \right]_0^4 = 16 - \frac{2}{5}(32) = \frac{16}{5}$$

$$A = \int_0^8 \int_{y/2}^{y^{2/3}} dx dy = \int_0^8 \left( y^{2/3} - \frac{y}{2} \right) dy$$

$$= \left[ \frac{3}{5}y^{5/3} - \frac{y^2}{4} \right]_0^8 = \frac{3}{5}(32) - 16 = \frac{16}{5}$$



$$41. A = \int_0^3 \int_0^{2x/3} dy dx + \int_3^5 \int_0^{5-x} dy dx$$

$$= \int_0^3 [y]_0^{2x/3} dx + \int_3^5 [y]_0^{5-x} dx$$

$$= \int_0^3 \frac{2x}{3} dx + \int_3^5 (5-x) dx$$

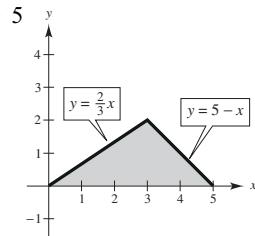
$$= \left[ \frac{1}{3}x^2 \right]_0^3 + \left[ 5x - \frac{1}{2}x^2 \right]_3^5 = 5$$

$$A = \int_0^2 \int_{3y/2}^{5-y} dx dy$$

$$= \int_0^2 [x]_{3y/2}^{5-y} dy$$

$$= \int_0^2 \left( 5 - y - \frac{3y}{2} \right) dy$$

$$= \int_0^2 \left( 5 - \frac{5y}{2} \right) dy = \left[ 5y - \frac{5}{4}y^2 \right]_0^2 = 5$$



$$42. A = \int_0^3 \int_0^x dy dx + \int_3^9 \int_0^{9/x} dy dx$$

$$= \int_0^3 [y]_0^x dx + \int_3^9 [y]_0^{9/x} dx = \int_0^3 x dx + \int_3^9 \frac{9}{x} dx$$

$$= \left[ \frac{1}{2}x^2 \right]_0^3 + [9 \ln x]_3^9$$

$$= \frac{9}{2} + 9(\ln 9 - \ln 3)$$

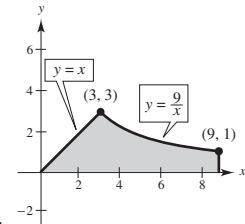
$$= \frac{9}{2}(1 + \ln 9)$$

$$A = \int_0^1 \int_y^9 dx dy + \int_1^3 \int_y^{9/y} dx dy$$

$$= \int_0^1 [x]_y^9 dy + \int_1^3 [x]_y^{9/y} dy$$

$$= \int_0^1 (9 - y) dy + \int_1^3 \left( \frac{9}{y} - y \right) dy$$

$$= \left[ 9y - \frac{1}{2}y^2 \right]_0^1 + \left[ 9 \ln y - \frac{1}{2}y^2 \right]_1^3 = \frac{9}{2}(1 + \ln 9)$$



$$43. \frac{A}{4} = \int_0^a \int_0^{(b/a)\sqrt{a^2-x^2}} dy dx = \int_0^a [y]_0^{(b/a)\sqrt{a^2-x^2}} dx$$

$$= \frac{b}{a} \int_0^a \sqrt{a^2 - x^2} dx = ab \int_0^{\pi/2} \cos^2 \theta d\theta$$

$$(x = a \sin \theta, dx = a \cos \theta d\theta)$$

$$= \frac{ab}{2} \int_0^{\pi/2} (1 + \cos 2\theta) d\theta$$

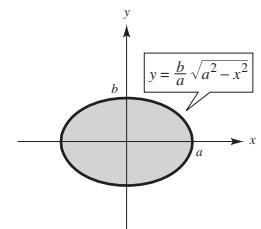
$$= \left[ \frac{ab}{2} \left( \theta + \frac{1}{2} \sin 2\theta \right) \right]_0^{\pi/2}$$

$$= \frac{\pi ab}{4}$$

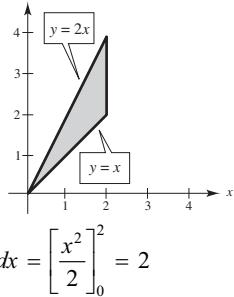
$$\text{So, } A = \pi ab.$$

$$\frac{A}{4} = \int_0^b \int_0^{(a/b)\sqrt{b^2-y^2}} dx dy = \frac{\pi ab}{4}$$

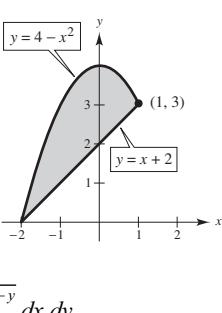
$$\text{So, } A = \pi ab. \text{ Integration steps are similar to those above.}$$



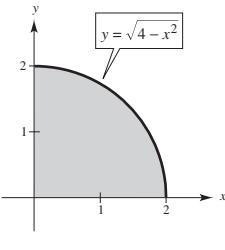
$$\begin{aligned}
 44. \quad A &= \int_0^2 \int_{y/2}^y dx dy + \int_2^4 \int_{y/2}^2 dx dy \\
 &= \int_0^2 \frac{y}{2} dy + \int_2^4 \left(2 - \frac{y}{2}\right) dy \\
 &= \left[\frac{y^2}{4}\right]_0^2 + \left[2y - \frac{y^2}{4}\right]_2^4 \\
 &= 1 + (4 - 3) = 2 \\
 A &= \int_0^2 \int_x^{2x} dy dx = \int_0^2 (2x - x) dx = \left[\frac{x^2}{2}\right]_0^2 = 2
 \end{aligned}$$



$$\begin{aligned}
 45. \quad A &= \int_{-2}^1 \int_{x+2}^{4-x^2} dy dx \\
 &= \int_{-2}^1 [y]_{x+2}^{4-x^2} dx \\
 &= \int_{-2}^1 (4 - x^2 - x - 2) dx \\
 &= \int_{-2}^1 (2 - x - x^2) dx \\
 &= \left[2x - \frac{1}{2}x^2 - \frac{1}{3}x^3\right]_{-2}^1 = \frac{9}{2} \\
 A &= \int_0^3 \int_{-\sqrt{4-y}}^{\sqrt{4-y}} dx dy + 2 \int_3^4 \int_0^{\sqrt{4-y}} dx dy \\
 &= \int_0^3 [x]_{-\sqrt{4-y}}^{\sqrt{4-y}} dy + 2 \int_3^4 [x]_0^{\sqrt{4-y}} dy \\
 &= \int_0^3 (y - 2 + \sqrt{4 - y}) dy + 2 \int_3^4 \sqrt{4 - y} dy \\
 &= \left[\frac{1}{2}y^2 - 2y - \frac{2}{3}(4 - y)^{3/2}\right]_0^3 - \left[\frac{4}{3}(4 - y)^{3/2}\right]_3^4 = \frac{9}{2}
 \end{aligned}$$

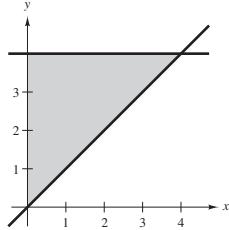


$$\begin{aligned}
 46. \quad A &= \int_0^2 \int_0^{\sqrt{4-x^2}} dy dx \\
 &= \int_0^2 \sqrt{4 - x^2} dx \\
 &= 4 \int_0^{\pi/2} \cos^2 \theta d\theta \\
 &= 2 \int_0^{\pi/2} (1 + \cos 2\theta) d\theta \\
 &= \left[2\left(\theta + \frac{1}{2}\sin 2\theta\right)\right]_0^{\pi/2} = \pi
 \end{aligned}$$

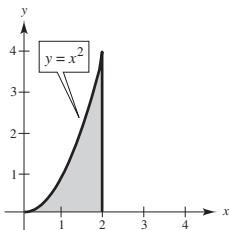


$$\begin{aligned}
 A &= \int_0^2 \int_0^{\sqrt{4-y^2}} dx dy = \int_0^2 \sqrt{4 - y^2} dy \\
 &= 4 \int_0^{\pi/2} \cos^2 \theta d\theta = 2 \int_0^{\pi/2} (1 + \cos 2\theta) d\theta \\
 &= \left[2\left(\theta + \frac{1}{2}\sin 2\theta\right)\right]_0^{\pi/2} = \pi \\
 (y &= 2 \sin \theta, dy = 2 \cos \theta d\theta, \sqrt{4 - y^2} = 2 \cos \theta)
 \end{aligned}$$

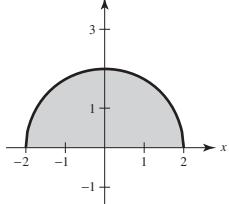
$$\begin{aligned}
 47. \quad \int_0^4 \int_0^y f(x, y) dx dy, 0 \leq x \leq y, 0 \leq y \leq 4 \\
 &= \int_0^4 \int_x^4 f(x, y) dy dx
 \end{aligned}$$



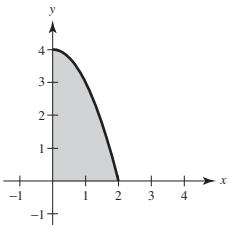
$$\begin{aligned}
 48. \quad \int_0^4 \int_{\sqrt{y}}^2 f(x, y) dx dy, \sqrt{y} \leq x \leq 2, 0 \leq y \leq 4 \\
 &= \int_0^2 \int_0^{x^2} f(x, y) dy dx
 \end{aligned}$$



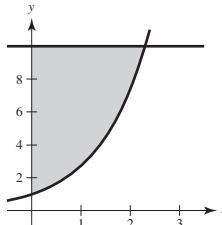
$$\begin{aligned}
 49. \quad \int_{-2}^2 \int_0^{\sqrt{4-x^2}} f(x, y) dy dx, 0 \leq y \leq \sqrt{4 - x^2}, -2 \leq x \leq 2 \\
 &= \int_0^2 \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} dx dy
 \end{aligned}$$



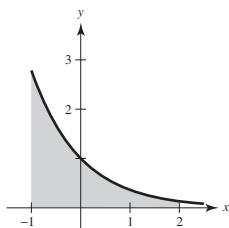
$$\begin{aligned}
 50. \quad \int_0^2 \int_0^{4-x^2} f(x, y) dy dx, 0 \leq y \leq 4 - x^2, 0 \leq x \leq 2 \\
 &= \int_0^4 \int_0^{\sqrt{4-y^2}} f(x, y) dx dy
 \end{aligned}$$



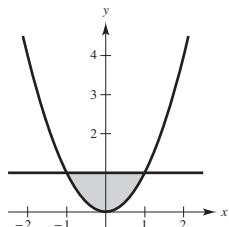
51.  $\int_1^{10} \int_0^{\ln y} f(x, y) dx dy, 0 \leq x \leq \ln y, 1 \leq y \leq 10$   
 $= \int_0^{\ln 10} \int_{e^x}^{10} f(x, y) dy dx$



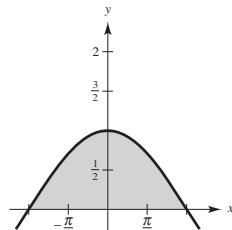
52.  $\int_{-1}^2 \int_0^{e^{-x}} f(x, y) dy dx, 0 \leq y \leq e^{-x}, -1 \leq x \leq 2$   
 $= \int_0^{e^{-2}} \int_{-1}^2 f(x, y) dx dy + \int_{e^{-2}}^e \int_{-1}^{-\ln y} f(x, y) dx dy$



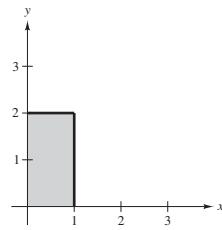
53.  $\int_{-1}^1 \int_{x^2}^1 f(x, y) dy dx, x^2 \leq y \leq 1, -1 \leq x \leq 1$   
 $= \int_0^1 \int_{-\sqrt{y}}^{\sqrt{y}} f(x, y) dx dy$



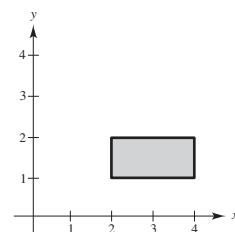
54.  $\int_{-\pi/2}^{\pi/2} \int_0^{\cos x} f(x, y) dy dx, 0 \leq y \leq \cos x, -\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$   
 $= \int_0^1 \int_{-\arccos y}^{\arccos y} f(x, y) dx dy$



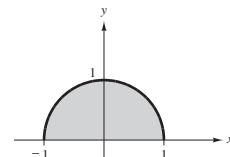
55.  $\int_0^1 \int_0^2 dy dx = \int_0^2 \int_0^1 dx dy = 2$



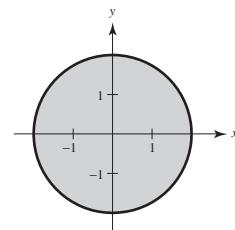
56.  $\int_1^2 \int_2^4 dx dy = \int_2^4 \int_1^2 dy dx = 2$



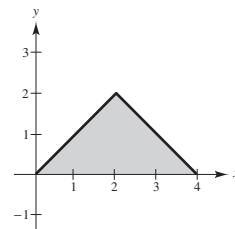
57.  $\int_0^1 \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} dx dy = \int_{-1}^1 \int_0^{\sqrt{1-x^2}} dy dx = \frac{\pi}{2}$



58.  $\int_{-2}^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} dy dx = \int_{-2}^2 \left( \sqrt{4-x^2} + \sqrt{4-x^2} \right) dx = 4\pi$   
 $\int_{-2}^2 \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} dx dy = 4\pi$

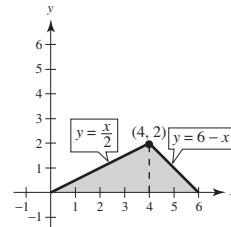


59.  $\int_0^2 \int_0^x dy dx + \int_2^4 \int_0^{4-x} dy dx = \int_0^2 \int_y^{4-y} dx dy = 4$

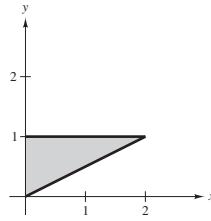


60.  $\int_0^4 \int_0^{x/2} dy dx + \int_4^6 \int_0^{6-x} dy dx = \int_0^4 \frac{x}{2} dx + \int_4^6 (6-x) dx = 4 + 2 = 6$

$$\int_0^2 \int_{2y}^{6-y} dx dy = \int_0^2 (6-3y) dy = \left[ 6y - \frac{3y^2}{2} \right]_0^2 = 6$$

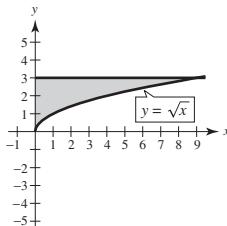


61.  $\int_0^2 \int_{x/2}^1 dy dx = \int_0^1 \int_0^{2y} dx dy = 1$

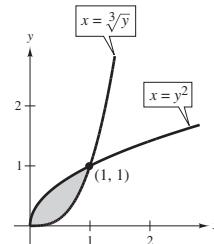


62.  $\int_0^9 \int_{\sqrt{x}}^3 dy dx = \int_0^9 (3 - \sqrt{x}) dx$   
 $= \left[ 3x - \frac{2}{3}x^{3/2} \right]_0^9 = 27 - 18 = 9$

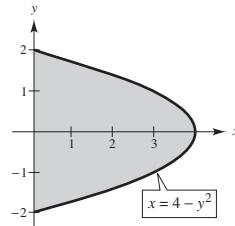
$$\int_0^3 \int_0^{y^2} dx dy = \int_0^3 y^2 dy = \left[ \frac{y^3}{3} \right]_0^3 = 9$$



63.  $\int_0^1 \int_{y^2}^{\sqrt{y}} dx dy = \int_0^1 \int_{x^3}^{\sqrt{x}} dy dx = \frac{5}{12}$

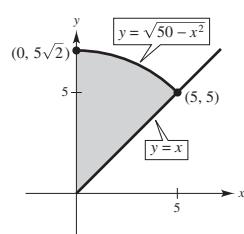


64.  $\int_{-2}^2 \int_0^{4-y^2} dx dy = \int_0^4 \int_{-\sqrt{4-x}}^{\sqrt{4-x}} dy dx = \frac{32}{3}$



65. The first integral arises using vertical representative rectangles. The second two integrals arise using horizontal representative rectangles.

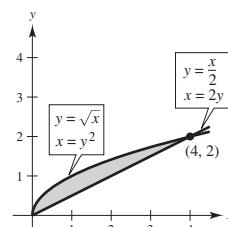
$$\begin{aligned} \int_0^5 \int_x^{\sqrt{50-x^2}} x^2 y^2 dy dx &= \int_0^5 \left[ \frac{1}{3}x^2(50-x^2)^{3/2} - \frac{1}{3}x^5 \right] dx = \frac{15,625}{24}\pi \\ \int_0^5 \int_0^y x^2 y^2 dx dy + \int_5^{\sqrt{50}} \int_0^{50-y^2} x^2 y^2 dx dy &= \int_0^5 \frac{1}{3}y^5 dy + \int_5^{\sqrt{50}} \frac{1}{3}(50-y^2)^{3/2} y^2 dy \\ &= \frac{15,625}{18} + \left( \frac{15,625}{18}\pi - \frac{15,625}{18} \right) = \frac{15,625}{24}\pi \end{aligned}$$



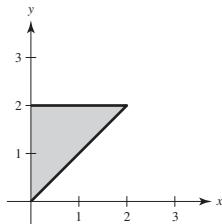
66. (a)  $A = \int_0^2 \int_{y^2}^{2y} dx dy = \int_0^2 [x]_{y^2}^{2y} dy = \int_0^2 (2y - y^2) dy = \left[ y^2 - \frac{y^3}{3} \right]_0^2 = 4 - \frac{8}{3} = \frac{4}{3}$

$$\begin{aligned} \text{(b)} \quad A &= \int_0^4 \int_{x/2}^{\sqrt{x}} dy dx = \int_0^4 [y]_{x/2}^{\sqrt{x}} dx = \int_0^4 \left( \sqrt{x} - \frac{x}{2} \right) dx \\ &= \left[ \frac{2}{3}x^{3/2} - \frac{x^2}{4} \right]_0^4 = \frac{16}{3} - 4 = \frac{4}{3} \end{aligned}$$

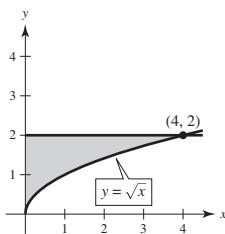
Integrals (a) and (b) are the same.



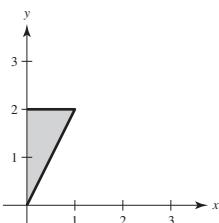
$$\begin{aligned}
 67. \int_0^2 \int_x^2 x\sqrt{1+y^3} dy dx &= \int_0^2 \int_0^y x\sqrt{1+y^3} dx dy \\
 &= \int_0^2 \left[ \sqrt{1+y^3} \cdot \frac{x^2}{2} \right]_0^y dy \\
 &= \frac{1}{2} \int_0^2 \sqrt{1+y^3} y^2 dy \\
 &= \left[ \frac{1}{2} \cdot \frac{1}{3} \cdot \frac{2}{3} (1+y^3)^{3/2} \right]_0^2 \\
 &= \frac{1}{9}(27) - \frac{1}{9}(1) = \frac{26}{9}
 \end{aligned}$$



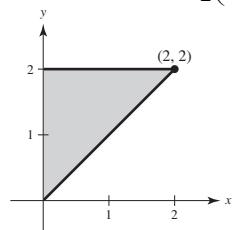
$$\begin{aligned}
 68. \int_0^4 \int_{\sqrt{x}}^2 \frac{3}{2+y^3} dy dx &= \int_0^2 \int_0^{y^2} \frac{3}{2+y^3} dx dy \\
 &= \int_0^2 \left[ \frac{3x}{2+y^3} \right]_0^{y^2} dy \\
 &= \int_0^2 \frac{3y^2}{2+y^3} dy = \left[ \ln|2+y^3| \right]_0^2 \\
 &= \ln 10 - \ln 2 = \ln 5
 \end{aligned}$$



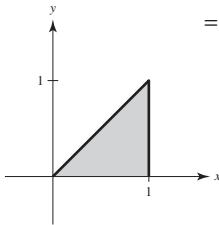
$$\begin{aligned}
 69. \int_0^1 \int_{2x}^2 4e^{y^2} dy dx &= \int_0^2 \int_0^{y/2} 4e^{y^2} dx dy \\
 &= \int_0^2 \left[ 4xe^{y^2} \right]_0^{y/2} dy = \int_0^2 2ye^{y^2} dy \\
 &= \left[ e^{y^2} \right]_0^2 = e^4 - 1
 \end{aligned}$$



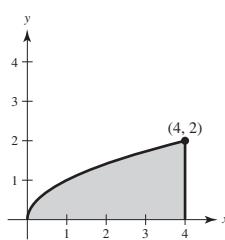
$$\begin{aligned}
 70. \int_0^2 \int_x^2 e^{-y^2} dy dx &= \int_0^2 \int_0^y e^{-y^2} dx dy \\
 &= \int_0^2 \left[ xe^{-y^2} \right]_0^y dy \\
 &= \int_0^2 ye^{-y^2} dy \\
 &= \left[ -\frac{1}{2}e^{-y^2} \right]_0^2 \\
 &= -\frac{1}{2}(e^{-4}) + \frac{1}{2}e^0 \\
 &= \frac{1}{2}\left(1 - \frac{1}{e^4}\right) \approx 0.4908
 \end{aligned}$$



$$\begin{aligned}
 71. \int_0^1 \int_y^1 \sin(x^2) dx dy &= \int_0^1 \int_0^x \sin(x^2) dy dx \\
 &= \int_0^1 \left[ y \sin(x^2) \right]_0^x dx \\
 &= \int_0^1 x \sin(x^2) dx \\
 &= \left[ -\frac{1}{2} \cos(x^2) \right]_0^1 \\
 &= -\frac{1}{2} \cos 1 + \frac{1}{2}(1) \\
 &= \frac{1}{2}(1 - \cos 1) \approx 0.2298
 \end{aligned}$$



$$\begin{aligned}
 72. \int_0^2 \int_{y^2}^4 \sqrt{x} \sin x dx dy &= \int_0^4 \int_0^{\sqrt{x}} \sqrt{x} \sin x dy dx \\
 &= \int_0^4 \left[ y \sqrt{x} \sin x \right]_0^{\sqrt{x}} dx \\
 &= \int_0^4 x \sin x dx \\
 &= \left[ \sin x - x \cos x \right]_0^4 \\
 &= \sin 4 - 4 \cos 4 \approx 1.858
 \end{aligned}$$



73.  $\int_0^2 \int_{x^2}^{2x} (x^3 + 3y^2) dy dx = \frac{1664}{105} \approx 15.848$

74.  $\int_0^1 \int_y^{2y} \sin(x+y) dx dy = \frac{\sin 2}{2} - \frac{\sin 3}{3} \approx 0.408$

75.  $\int_0^4 \int_0^y \frac{2}{(x+1)(y+1)} dx dy = (\ln 5)^2 \approx 2.590$

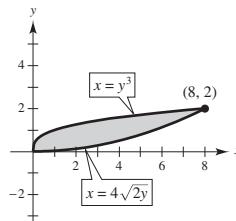
76.  $\int_0^a \int_0^{a-x} (x^2 + y^2) dy dx = \frac{a^4}{6}$

77. (a)  $x = y^3 \Leftrightarrow y = x^{1/3}$

$$x = 4\sqrt{2y} \Leftrightarrow x^2 = 32y \Leftrightarrow y = \frac{x^2}{32}$$

(b)  $\int_0^8 \int_{x^2/32}^{x^{1/3}} (x^2 y - xy^2) dy dx$

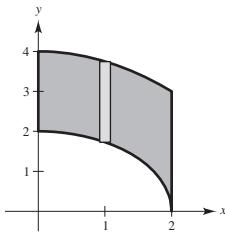
(c) Both integrals equal  $\frac{67,520}{693} \approx 97.43$ .



78. (a)  $y = \sqrt{4 - x^2} \Leftrightarrow x = \sqrt{4 - y^2}$

$$y = 4 - \frac{x^2}{4} \Leftrightarrow x = \sqrt{16 - 4y}$$

(b)  $\int_0^2 \int_{\sqrt{4-y^2}}^2 \frac{xy}{x^2 + y^2 + 1} dx dy + \int_2^3 \int_0^2 \frac{xy}{x^2 + y^2 + 1} dx dy + \int_3^4 \int_0^{\sqrt{16-4y}} \frac{xy}{x^2 + y^2 + 1} dx dy$



(c) Both orders of integration yield 1.11899.

79.  $\int_0^2 \int_0^{4-x^2} e^{xy} dy dx \approx 20.5648$

80.  $\int_0^2 \int_x^2 \sqrt{16 - x^2 - y^2} dy dx \approx 6.8520$

81.  $\int_0^{2\pi} \int_0^{1+\cos\theta} 6r^2 \cos\theta dr d\theta = \frac{15\pi}{2}$

82.  $\int_0^{\pi/2} \int_0^{1+\sin\theta} 15\theta r dr d\theta = \frac{45\pi^2}{32} + \frac{135}{8} \approx 30.7541$

83. An iterated integral is integration of a function of several variables. Integrate with respect to one variable while holding the other variables constant.

84. A region is vertically simple if it is bounded on the left and right by vertical lines, and bounded on the top and bottom by functions of  $x$ . A region is horizontally simple if it is bounded on the top and bottom by horizontal lines, and bounded on the left and right by functions of  $y$ .

85. The region is a rectangle.

86. The integrations might be easier. See Exercises 59–62.

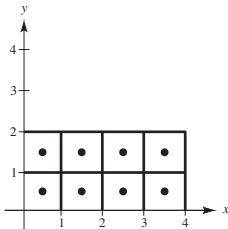
87. True

88. False, let  $f(x, y) = x$ .

## Section 14.2 Double Integrals and Volume

For Exercises 1–4,  $\Delta x_i = \Delta y_i = 1$  and the midpoints of the squares are

$$\left(\frac{1}{2}, \frac{1}{2}\right), \left(\frac{3}{2}, \frac{1}{2}\right), \left(\frac{5}{2}, \frac{1}{2}\right), \left(\frac{7}{2}, \frac{1}{2}\right), \left(\frac{1}{2}, \frac{3}{2}\right), \left(\frac{3}{2}, \frac{3}{2}\right), \left(\frac{5}{2}, \frac{3}{2}\right), \left(\frac{7}{2}, \frac{3}{2}\right).$$



1.  $f(x, y) = x + y$

$$\sum_{i=1}^8 f(x_i, y_i) \Delta x_i \Delta y_i = 1 + 2 + 3 + 4 + 2 + 3 + 4 + 5 = 24$$

$$\int_0^4 \int_0^2 (x + y) dy dx = \int_0^4 \left[ xy + \frac{y^2}{2} \right]_0^2 dx = \int_0^4 (2x + 2) dx = [x^2 + 2x]_0^4 = 24$$

2.  $f(x, y) = \frac{1}{2}x^2y$

$$\sum_{i=1}^{\infty} f(x_i, y_i) \Delta x_i \Delta y_i = \frac{1}{16} + \frac{9}{16} + \frac{25}{16} + \frac{49}{16} + \frac{3}{16} + \frac{27}{16} + \frac{75}{16} + \frac{147}{16} = 21$$

$$\int_0^4 \int_0^2 \frac{1}{2}x^2y dy dx = \int_0^4 \left[ \frac{x^2y^2}{4} \right]_0^2 dx = \int_0^4 x^2 dx = \left[ \frac{x^3}{3} \right]_0^4 = \frac{64}{3} \approx 21.3$$

3.  $f(x, y) = x^2 + y^2$

$$\sum_{i=1}^8 f(x_i, y_i) \Delta x_i \Delta y_i = \frac{2}{4} + \frac{10}{4} + \frac{26}{4} + \frac{50}{4} + \frac{10}{4} + \frac{18}{4} + \frac{34}{4} + \frac{58}{4} = 52$$

$$\int_0^4 \int_0^2 (x^2 + y^2) dy dx = \int_0^4 \left[ x^2y + \frac{y^3}{3} \right]_0^2 dx = \int_0^4 \left( 2x^2 + \frac{8}{3} \right) dx = \left[ \frac{2x^3}{3} + \frac{8x}{3} \right]_0^4 = \frac{160}{3}$$

4.  $f(x, y) = \frac{1}{(x+1)(y+1)}$

$$\sum_{i=1}^8 f(x_i, y_i) \Delta x_i \Delta y_i = \frac{4}{9} + \frac{4}{15} + \frac{4}{21} + \frac{4}{27} + \frac{4}{15} + \frac{4}{25} + \frac{4}{35} + \frac{4}{45} = \frac{7936}{4725} \approx 1.680$$

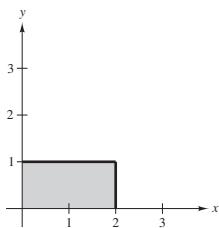
$$\begin{aligned} \int_0^4 \int_0^2 \frac{1}{(x+1)(y+1)} dy dx &= \int_0^4 \left[ \frac{1}{x+1} \ln(y+1) \right]_0^2 dx \\ &= \int_0^4 \frac{\ln 3}{x+1} dx = [\ln 3 \cdot \ln(x+1)]_0^4 = (\ln 3)(\ln 5) \approx 1.768 \end{aligned}$$

5.  $\int_0^4 \int_0^4 f(x, y) dy dx \approx (32 + 31 + 28 + 23) + (31 + 30 + 27 + 22) + (28 + 27 + 24 + 19) + (23 + 22 + 19 + 14)$   
 $= 400$

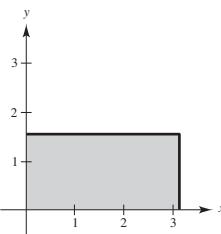
Using the corner of the  $i$ th square furthest from the origin, you obtain 272.

6.  $\int_0^2 \int_0^2 f(x, y) dy dx \approx 4 + 2 + 8 + 6 = 20$

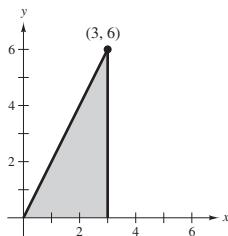
7.  $\int_0^2 \int_0^1 (1 + 2x + 2y) dy dx = \int_0^2 [y + 2xy + y^2]_0^1 dx = \int_0^2 (2 + 2x) dx = [2x + x^2]_0^2 = 8$



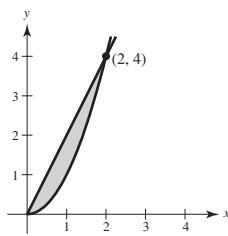
8.  $\int_0^\pi \int_0^{\pi/2} \sin^2 x \cos^2 y dy dx = \int_0^\pi \left[ \frac{1}{2} \sin^2 x \left( y + \frac{1}{2} \sin 2y \right) \right]_0^{\pi/2} dx$   
 $= \int_0^\pi \frac{1}{2} \sin^2 x \left( \frac{\pi}{2} \right) dx = \frac{\pi}{8} \int_0^\pi (1 - \cos 2x) dx = \left[ \frac{\pi}{8} \left( x - \frac{1}{2} \sin 2x \right) \right]_0^\pi = \frac{\pi^2}{8}$



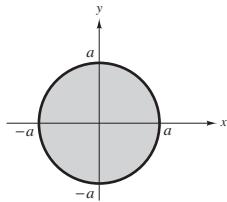
9.  $\int_0^6 \int_{y/2}^3 (x + y) dx dy = \int_0^6 \left[ \frac{1}{2} x^2 + xy \right]_{y/2}^3 dy = \int_0^6 \left( \frac{9}{2} + 3y - \frac{5}{8} y^2 \right) dy = \left[ \frac{9}{2}y + \frac{3}{2}y^2 - \frac{5}{24}y^3 \right]_0^6 = 36$



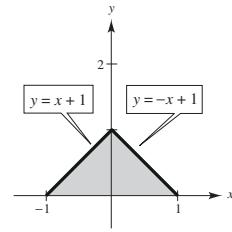
10.  $\int_0^4 \int_{(1/2)y}^{\sqrt{y}} x^2 y^2 dx dy = \int_0^4 \left[ \frac{x^3 y^2}{3} \right]_{(1/2)y}^{\sqrt{y}} dy = \int_0^4 \left( \frac{y^{7/2}}{3} - \frac{y^5}{24} \right) dy = \left[ \frac{2y^{9/2}}{27} - \frac{y^6}{144} \right]_0^4 = \frac{1024}{27} - \frac{256}{9} = \frac{256}{27}$



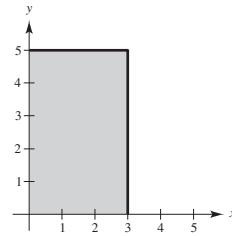
11.  $\int_{-a}^a \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} (x + y) dy dx = \int_{-a}^a \left[ xy + \frac{1}{2} y^2 \right]_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} dx = \int_{-a}^a 2x\sqrt{a^2 - x^2} dx = \left[ -\frac{2}{3}(a^2 - x^2)^{3/2} \right]_{-a}^a = 0$



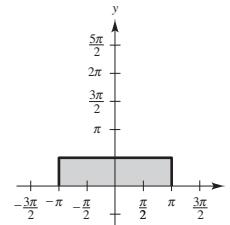
$$\begin{aligned}
 12. \int_0^1 \int_{y-1}^0 e^{x+y} dx dy + \int_0^1 \int_0^{1-y} e^{x+y} dx dy &= \int_0^1 [e^{x+y}]_{y-1}^0 dy + \int_0^1 [e^{x+y}]_0^{1-y} dy \\
 &= \int_0^1 (e - e^{2y-1}) dy = [ey - \frac{1}{2}e^{2y-1}]_0^1 = \frac{1}{2}(e + e^{-1})
 \end{aligned}$$



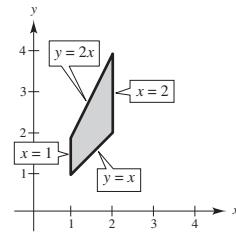
$$\begin{aligned}
 13. \int_0^5 \int_0^3 xy dx dy &= \int_0^3 \int_0^5 xy dy dx \\
 &= \int_0^3 [\frac{1}{2}xy^2]_0^5 dx = \frac{25}{2} \int_0^3 x dx = [\frac{25}{4}x^2]_0^3 = \frac{225}{4}
 \end{aligned}$$



$$\begin{aligned}
 14. \int_0^{\pi/2} \int_{-\pi}^{\pi} \sin x \sin y dx dy &= \int_{-\pi}^{\pi} \int_0^{\pi/2} \sin x \sin y dy dx \\
 &= \int_{-\pi}^{\pi} [-\sin x \cos y]_0^{\pi/2} dx = \int_{-\pi}^{\pi} \sin x dx = 0
 \end{aligned}$$



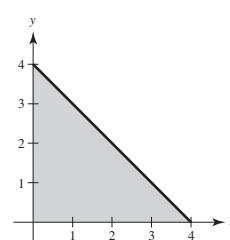
$$\begin{aligned}
 15. \int_1^2 \int_1^y \frac{y}{x^2 + y^2} dx dy + \int_2^4 \int_{y/2}^2 \frac{y}{x^2 + y^2} dx dy &= \int_1^2 \int_x^{2x} \frac{y}{x^2 + y^2} dy dx \\
 &= \frac{1}{2} \int_1^2 \left[ \ln(x^2 + y^2) \right]_x^{2x} dx \\
 &= \frac{1}{2} \int_1^2 (\ln 5x^2 - \ln 2x^2) dx \\
 &= \frac{1}{2} \ln \frac{5}{2} \int_1^2 dx = \left[ \frac{1}{2} \left( \ln \frac{5}{2} \right) x \right]_1^2 = \frac{1}{2} \ln \frac{5}{2}
 \end{aligned}$$



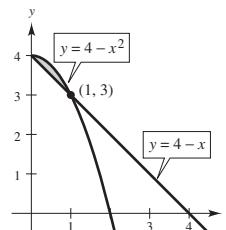
$$16. \int_0^4 \int_0^{4-x} xe^y dy dx = \int_0^4 \int_0^{4-y} xe^y dy dx$$

For the first integral, you obtain:

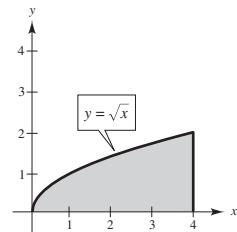
$$\begin{aligned}
 \int_0^4 [xe^y]_0^{4-x} dx &= \int_0^4 (xe^{4-x} - x) dx \\
 &= \left[ -e^{4-x}(1+x) - \frac{x^2}{2} \right]_0^4 = (-5 - 8) + e^4 = e^4 - 13.
 \end{aligned}$$



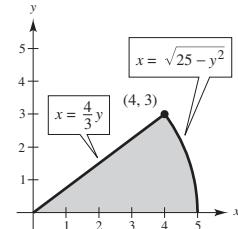
$$\begin{aligned}
 17. \int_3^4 \int_{4-y}^{\sqrt{4-y}} -2y dx dy &= \int_0^1 \int_{4-x}^{4-x^2} -2y dy dx \\
 &= \int_0^1 [-y^2]_{4-x}^{4-x^2} dx \\
 &= -\int_0^1 [(4-x^2)^2 - (4-x)^2] dx \\
 &= -\int_0^1 [16 - 8x^2 + x^4 - (16 - 8x + x^2)] dx \\
 &= -\left[ -3x^3 + \frac{x^5}{5} + 4x^2 \right]_0^1 = -\frac{6}{5}
 \end{aligned}$$



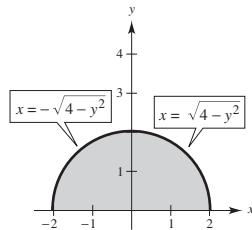
$$\begin{aligned}
 18. \int_0^2 \int_{y^2}^4 \frac{y}{1+x^2} dx dy &= \int_0^4 \int_0^{\sqrt{x}} \frac{y}{1+x^2} dy dx \\
 &= \frac{1}{2} \int_0^4 \left[ \frac{y^2}{1+x^2} \right]_0^{\sqrt{x}} dx = \frac{1}{2} \int_0^4 \frac{x}{1+x^2} dx = \left[ \frac{1}{4} \ln(1+x^2) \right]_0^4 = \frac{1}{4} \ln(17)
 \end{aligned}$$



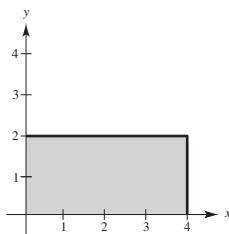
$$\begin{aligned}
 19. \int_0^4 \int_0^{3x/4} x dy dx + \int_4^5 \int_0^{\sqrt{25-x^2}} x dy dx &= \int_0^3 \int_{4y/3}^{\sqrt{25-y^2}} x dy dx \\
 &= \int_0^3 \left[ \frac{1}{2}x^2 \right]_{4y/3}^{\sqrt{25-y^2}} dy \\
 &= \frac{25}{18} \int_0^3 (9 - y^2) dy = \left[ \frac{25}{18} (9y - \frac{1}{3}y^3) \right]_0^3 = 25
 \end{aligned}$$



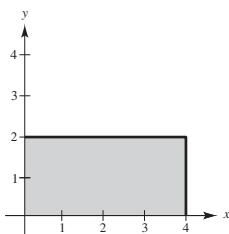
$$\begin{aligned}
 20. \int_0^2 \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} (x^2 + y^2) dx dy &= \int_{-2}^2 \int_0^{\sqrt{4-x^2}} (x^2 + y^2) dy dx \\
 &= \int_{-2}^2 \left[ x^2 y + \frac{1}{3}y^3 \right]_0^{\sqrt{4-x^2}} dx \\
 &= \int_{-2}^2 \left[ x^2 \sqrt{4-x^2} + \frac{1}{3}(4-x^2)^{3/2} \right] dx \\
 &= \left[ -\frac{x}{4}(4-x^2)^{3/2} + \frac{1}{2}\left(x\sqrt{4-x^2} + 4\arcsin\frac{x}{2}\right) + \frac{1}{12}\left[x(4-x^2)^{3/2} + 6x\sqrt{4-x^2} + 24\arctan\frac{x}{2}\right] \right]_{-2}^2 = 4\pi
 \end{aligned}$$



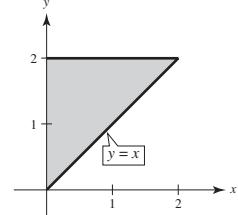
$$21. V = \int_0^4 \int_0^2 \frac{y}{2} dy dx = \int_0^4 \left[ \frac{y^2}{4} \right]_0^2 dx = \int_0^4 dx = 4$$



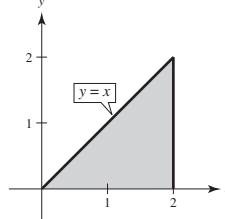
$$\begin{aligned}
 22. V &= \int_0^4 \int_0^2 (6-2y) dy dx \\
 &= \int_0^4 [6y - y^2]_0^2 dx = \int_0^4 8 dx = 32
 \end{aligned}$$



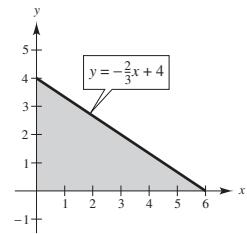
$$\begin{aligned}
 23. V &= \int_0^2 \int_0^y (4-x-y) dx dy \\
 &= \int_0^2 \left[ 4x - \frac{x^2}{2} - xy \right]_0^y dy \\
 &= \int_0^2 \left( 4y - \frac{y^2}{2} - y^2 \right) dy \\
 &= \left[ 2y^2 - \frac{y^3}{6} - \frac{y^3}{3} \right]_0^2 \\
 &= 8 - \frac{8}{6} - \frac{8}{3} = 4
 \end{aligned}$$



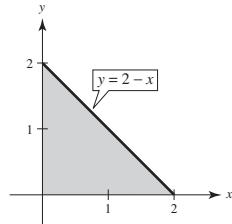
$$24. V = \int_0^2 \int_0^x 4 dy dx = \int_0^2 4x dx = 2x^2 \Big|_0^2 = 8$$



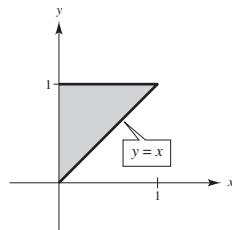
$$\begin{aligned}
 25. V &= \int_0^6 \int_0^{(-2/3)x+4} \left( \frac{12 - 2x - 3y}{4} \right) dy dx = \int_0^6 \left[ \frac{1}{4} \left( 12y - 2xy - \frac{3}{2}y^2 \right) \right]_0^{(-2/3)x+4} dx \\
 &= \int_0^6 \left( \frac{1}{6}x^2 - 2x + 6 \right) dx = \left[ \frac{1}{18}x^3 - x^2 + 6x \right]_0^6 = 12
 \end{aligned}$$



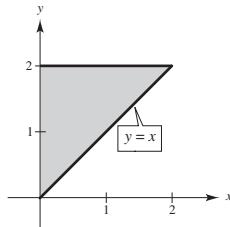
$$\begin{aligned}
 26. V &= \int_0^2 \int_0^{2-x} (2 - x - y) dy dx = \int_0^2 \left[ 2y - xy - \frac{y^2}{2} \right]_0^{2-x} dx \\
 &= \int_0^2 \frac{1}{2}(2 - x)^2 dx = -\frac{1}{6}(x - 2)^3 \Big|_0^2 = \frac{4}{3}
 \end{aligned}$$



$$\begin{aligned}
 27. V &= \int_0^1 \int_0^y (1 - xy) dx dy \\
 &= \int_0^1 \left[ x - \frac{x^2 y}{2} \right]_0^y dy = \int_0^1 \left( y - \frac{y^3}{2} \right) dy = \left[ \frac{y^2}{2} - \frac{y^4}{8} \right]_0^1 = \frac{3}{8}
 \end{aligned}$$



$$\begin{aligned}
 28. V &= \int_0^2 \int_0^y (4 - y^2) dx dy \\
 &= \int_0^2 (4y - y^3) dy = \left[ 2y^2 - \frac{y^4}{4} \right]_0^2 = 4
 \end{aligned}$$



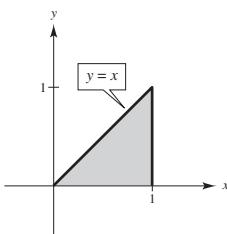
$$29. V = \int_0^\infty \int_0^\infty \frac{1}{(x+1)^2(y+1)^2} dy dx = \int_0^\infty \left[ -\frac{1}{(x+1)^2(y+1)} \right]_0^\infty dx = \int_0^\infty \frac{1}{(x+1)^2} dx = \left[ -\frac{1}{(x+1)} \right]_0^\infty = 1$$

$$30. V = \int_0^\infty \int_0^\infty e^{-(x+y)/2} dy dx = \int_0^\infty \left[ -2e^{-(x+y)/2} \right]_0^\infty dx = \int_0^\infty 2e^{-x/2} dx = \left[ -4e^{-x/2} \right]_0^\infty = 4$$

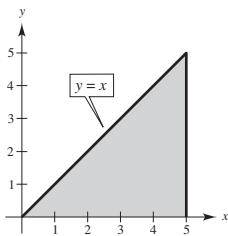
$$31. V = 4 \int_0^{2\sqrt{2}} \int_0^{\sqrt{8-x^2}} \sqrt{8 - x^2 - y^2} dy dx = \frac{32\sqrt{2}\pi}{3}$$

$$32. V = \int_0^1 \int_0^x \sqrt{1 - x^2} dy dx = \frac{1}{3}$$

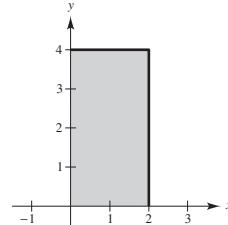
$$33. V = \int_0^1 \int_0^x xy dy dx = \int_0^1 \left[ \frac{1}{2}xy^2 \right]_0^x dx = \frac{1}{2} \int_0^1 x^3 dx = \left[ \frac{1}{8}x^4 \right]_0^1 = \frac{1}{8}$$



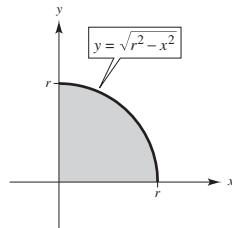
$$\begin{aligned}
 34. V &= \int_0^5 \int_0^x x \, dy \, dx \\
 &= \int_0^5 [xy]_0^x \, dx = \int_0^5 x^2 \, dx \\
 &= \left[ \frac{1}{3}x^3 \right]_0^5 = \frac{125}{3}
 \end{aligned}$$



$$\begin{aligned}
 35. V &= \int_0^2 \int_0^4 x^2 \, dy \, dx \\
 &= \int_0^2 [x^2 y]_0^4 \, dx = \int_0^2 4x^2 \, dx \\
 &= \left[ \frac{4x^3}{3} \right]_0^2 = \frac{32}{3}
 \end{aligned}$$

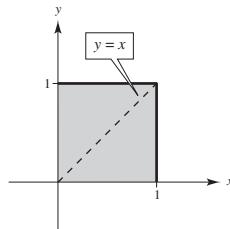


$$\begin{aligned}
 36. V &= 8 \int_0^r \int_0^{\sqrt{r^2 - x^2}} \sqrt{r^2 - x^2 - y^2} \, dy \, dx \\
 &= 4 \int_0^r \left[ y \sqrt{r^2 - x^2 - y^2} + (r^2 - x^2) \arcsin \frac{y}{\sqrt{r^2 - x^2}} \right]_0^{\sqrt{r^2 - x^2}} \, dx \\
 &= 4 \left( \frac{\pi}{2} \right) \int_0^r (r^2 - x^2) \, dx = \left[ 2\pi \left( r^2 x - \frac{1}{3} x^3 \right) \right]_0^r = \frac{4\pi r^3}{3}
 \end{aligned}$$

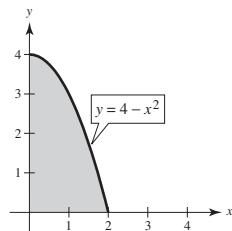


37. Divide the solid into two equal parts.

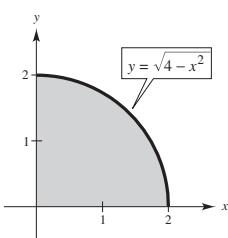
$$\begin{aligned}
 V &= 2 \int_0^1 \int_0^x \sqrt{1 - x^2} \, dy \, dx = 2 \int_0^1 \left[ y \sqrt{1 - x^2} \right]_0^x \, dx \\
 &= 2 \int_0^1 x \sqrt{1 - x^2} \, dx = \left[ -\frac{2}{3}(1 - x^2)^{3/2} \right]_0^1 = \frac{2}{3}
 \end{aligned}$$



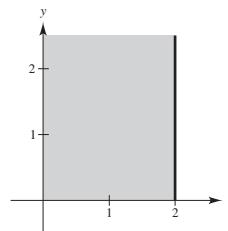
$$\begin{aligned}
 38. V &= \int_0^2 \int_0^{4-x^2} (4 - x^2) \, dy \, dx \\
 &= \int_0^2 (4 - x^2)(4 - x^2) \, dx \\
 &= \int_0^2 (16 - 8x^2 + x^4) \, dx = \left[ 16x - \frac{8x^3}{3} + \frac{x^5}{5} \right]_0^2 = 32 - \frac{64}{3} + \frac{32}{5} = \frac{256}{15}
 \end{aligned}$$



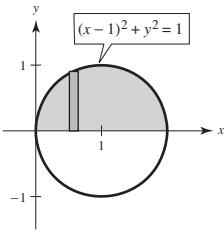
$$\begin{aligned}
 39. V &= \int_0^2 \int_0^{\sqrt{4-x^2}} (x + y) \, dy \, dx = \int_0^2 \left[ xy + \frac{1}{2}y^2 \right]_0^{\sqrt{4-x^2}} \, dx \\
 &= \int_0^2 \left( x\sqrt{4 - x^2} + 2 - \frac{1}{2}x^2 \right) \, dx \\
 &= \left[ -\frac{1}{3}(4 - x^2)^{3/2} + 2x - \frac{1}{6}x^3 \right]_0^2 = \frac{16}{3}
 \end{aligned}$$



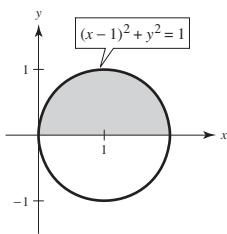
$$\begin{aligned}
 40. V &= \int_0^2 \int_0^\infty \frac{1}{1 + y^2} \, dy \, dx = \int_0^2 [\arctan y]_0^\infty \, dx \\
 &= \int_0^2 \frac{\pi}{2} \, dx = \left[ \frac{\pi x}{2} \right]_0^2 = \pi
 \end{aligned}$$



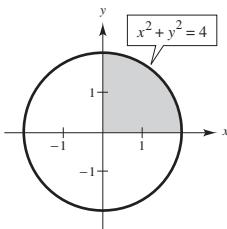
$$41. V = 2 \int_0^2 \int_0^{\sqrt{1-(x-1)^2}} ([4 - x^2 - y^2] - [4 - 2x]) dy dx \\ = 2 \int_0^2 \int_0^{\sqrt{1-(x-1)^2}} (2x - x^2 - y^2) dy dx$$



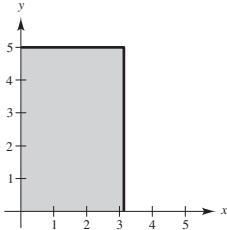
$$42. V = 2 \int_0^2 \int_0^{\sqrt{1-(x-1)^2}} [2x - (x^2 + y^2)] dy dx$$



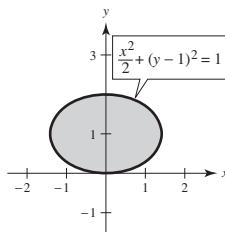
$$43. V = 4 \int_0^2 \int_0^{\sqrt{4-x^2}} (x^2 + y^2) dy dx$$



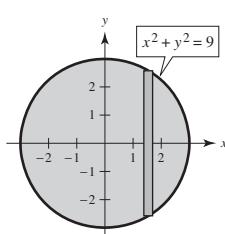
$$44. V = \int_0^5 \int_0^\pi \sin^2 x dx dy$$



$$45. V = \int_0^2 \int_{-\sqrt{2(y-1)^2}}^{\sqrt{2(y-1)^2}} [4y - (x^2 + 2y^2)] dx dy$$



$$46. V = \int_{-3}^3 \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} ([18 - x^2 - y^2] - [x^2 + y^2]) dy dx \\ = 4 \int_0^3 \int_0^{\sqrt{9-x^2}} (18 - 2x^2 - 2y^2) dy dx$$



$$47. z = 9 - x^2 - y^2, z = 0$$

$$V = 4 \int_0^3 \int_0^{\sqrt{9-x^2}} (9 - x^2 - y^2) dy dx = \frac{81\pi}{2}$$

$$48. V = \int_0^9 \int_0^{\sqrt{9-y}} \sqrt{9-y} dx dy = \frac{81}{2}$$

$$49. V = \int_0^2 \int_0^{-0.5x+1} \frac{2}{1+x^2+y^2} dy dx \approx 1.2315$$

$$50. V = \int_0^{16} \int_0^{4-\sqrt{y}} \ln(1+x+y) dx dy \approx 38.25$$

51.  $f$  is a continuous function such that

$0 \leq f(x, y) \leq 1$  over a region  $R$  of area 1. Let

$f(m, n)$  = the minimum value of  $f$  over  $R$  and

$f(M, N)$  = the maximum value of  $f$  over  $R$ . Then

$$f(m, n) \int_R \int dA \leq \int_R \int f(x, y) dA \leq f(M, N) \int_R \int dA.$$

Because  $\int_R \int dA = 1$  and

$0 \leq f(m, n) \leq f(M, N) \leq 1$ , you have

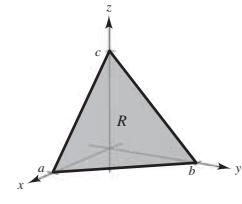
$$0 \leq f(m, n)(1) \leq \int_R \int f(x, y) dA \leq f(M, N)(1) \leq 1.$$

So,  $0 \leq \int_R \int f(x, y) dA \leq 1$ .

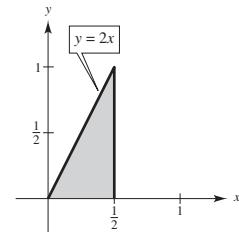
52.  $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$

$$z = c\left(1 - \frac{x}{a} - \frac{y}{b}\right)$$

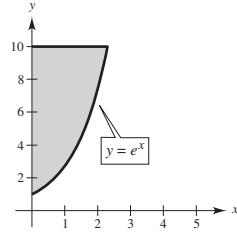
$$\begin{aligned} V &= \int_R \int f(x, y) dA = \int_0^a \int_0^{b[1-(x/a)]} c\left(1 - \frac{x}{a} - \frac{y}{b}\right) dy dx = c \int_0^a \left[y - \frac{xy}{a} - \frac{y^2}{2b}\right]_0^{b[1-(x/a)]} dx \\ &= c \int_0^a \left[b\left(1 - \frac{x}{a}\right) - \frac{xb}{a}\left(1 - \frac{x}{a}\right) - \frac{b^2}{2b}\left(1 - \frac{x}{a}\right)^2\right] dx \\ &= c \left[-\frac{ab}{2}\left(1 - \frac{x}{a}\right)^2 - \frac{x^2b}{2a} + \frac{x^3b}{3a^2} + \frac{ab}{6}\left(1 - \frac{x}{a}\right)^3\right]_0^a = c \left[\left(-\frac{ab}{2} + \frac{ab}{3}\right) - \left(-\frac{ab}{2} + \frac{ab}{6}\right)\right] = \frac{abc}{6} \end{aligned}$$



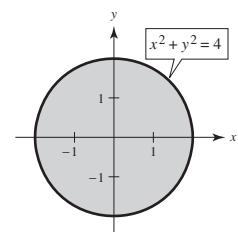
$$\begin{aligned} 53. \int_0^1 \int_{y/2}^{1/2} e^{-x^2} dx dy &= \int_0^{1/2} \int_0^{2x} e^{-x^2} dy dx \\ &= \int_0^{1/2} 2xe^{-x^2} dx = \left[-e^{-x^2}\right]_0^{1/2} = -e^{-1/4} + 1 = 1 - e^{-1/4} \approx 0.221 \end{aligned}$$



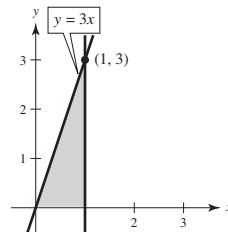
$$\begin{aligned} 54. \int_0^{\ln 10} \int_{e^x}^{10} \frac{1}{\ln y} dy dx &= \int_1^{10} \int_0^{\ln y} \frac{1}{\ln y} dx dy \\ &= \int_1^{10} \left[ \frac{x}{\ln y} \right]_0^{\ln y} dy \\ &= \int_1^{10} dy = [y]_1^{10} = 9 \end{aligned}$$



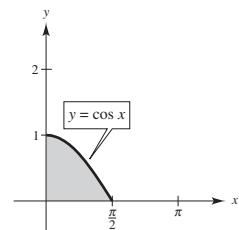
$$\begin{aligned} 55. \int_{-2}^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \sqrt{4-y^2} dy dx &= \int_{-2}^2 \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} \sqrt{4-y^2} dx dy = \int_{-2}^2 \left[ x\sqrt{4-y^2} \right]_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} dy \\ &= \int_{-2}^2 2(4-y^2) dy = \left[ 8y - \frac{2y^3}{3} \right]_{-2}^2 = \left( 16 - \frac{16}{3} \right) - \left( -16 + \frac{16}{3} \right) \\ &= \frac{64}{3} \end{aligned}$$



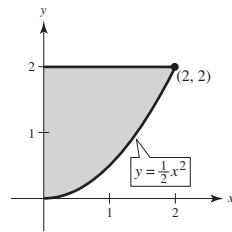
$$\begin{aligned} 56. \int_0^3 \int_{y/3}^1 \frac{1}{1+x^4} dx dy &= \int_0^1 \int_0^{3x} \frac{1}{1+x^4} dy dx = \int_0^1 \left[ \frac{y}{1+x^4} \right]_0^{3x} dx \\ &= \int_0^1 \frac{3x}{1+x^4} dx = \frac{3}{2} \arctan(x^2) \Big|_0^1 \\ &= \frac{3}{2} \left(\frac{\pi}{4}\right) = \frac{3\pi}{8} \end{aligned}$$



$$\begin{aligned} 57. \int_0^1 \int_0^{\arccos y} \sin x \sqrt{1+\sin^2 x} dx dy \\ &= \int_0^{\pi/2} \int_0^{\cos x} \sin x \sqrt{1+\sin^2 x} dy dx \\ &= \int_0^{\pi/2} (1+\sin^2 x)^{1/2} \sin x \cos x dx = \left[ \frac{1}{2} \cdot \frac{2}{3} (1+\sin^2 x)^{3/2} \right]_0^{\pi/2} = \frac{1}{3} [2\sqrt{2} - 1] \end{aligned}$$



$$\begin{aligned}
 58. \int_0^2 \int_{(1/2)x^2}^2 \sqrt{y} \cos y \, dy \, dx &= \int_0^2 \int_0^{\sqrt{2y}} \sqrt{y} \cos y \, dx \, dy \\
 &= \int_0^2 \sqrt{2y} \sqrt{y} \cos y \, dy = \sqrt{2} \int_0^2 y \cos y \, dy \\
 &= \sqrt{2} [\cos y + y \sin y]_0^2 = \sqrt{2} [\cos 2 + 2 \sin 2 - 1]
 \end{aligned}$$

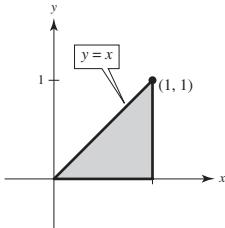


$$59. \text{Average} = \frac{1}{8} \int_0^4 \int_0^2 x \, dy \, dx = \frac{1}{8} \int_0^4 2x \, dx = \left[ \frac{x^2}{8} \right]_0^4 = 2$$

$$\begin{aligned}
 60. \text{Average} &= \frac{1}{15} \int_0^5 \int_0^3 2xy \, dy \, dx = \frac{1}{15} \int_0^5 [xy^2]_0^3 \, dx \\
 &= \frac{1}{15} \int_0^5 9x \, dx = \frac{1}{15} \left[ \frac{9x^2}{2} \right]_0^5 = \frac{15}{2}
 \end{aligned}$$

$$\begin{aligned}
 61. \text{Average} &= \frac{1}{4} \int_0^2 \int_0^2 (x^2 + y^2) \, dx \, dy \\
 &= \frac{1}{4} \int_0^2 \left[ \frac{x^3}{3} + xy^2 \right]_0^2 \, dy = \frac{1}{4} \int_0^2 \left( \frac{8}{3} + 2y^2 \right) \, dy \\
 &= \left[ \frac{1}{4} \left( \frac{8}{3}y + \frac{2}{3}y^3 \right) \right]_0^2 = \frac{8}{3}
 \end{aligned}$$

$$\begin{aligned}
 62. \text{Average} &= \frac{1}{(1/2)} \int_0^1 \int_0^x \frac{1}{x+y} \, dy \, dx \\
 &= 2 \int_0^1 [\ln|x+y|]_0^x \, dx = 2 \int_0^1 [\ln 2x - \ln x] \, dx \\
 &= 2 \int_0^1 \ln 2 \, dx = 2[x \ln 2]_0^1 = 2 \ln 2
 \end{aligned}$$



$$\begin{aligned}
 63. \text{Average} &= \frac{1}{1/2} \int_0^1 \int_x^1 e^{x+y} \, dy \, dx = 2 \int_0^1 e^{x+1} - e^{2x} \, dx \\
 &= 2 \left[ e^{x+1} - \frac{1}{2}e^{2x} \right]_0^1 = 2 \left[ e^2 - \frac{1}{2}e^2 - e + \frac{1}{2} \right] \\
 &= e^2 - 2e + 1 = (e-1)^2
 \end{aligned}$$

$$\begin{aligned}
 64. \text{Average} &= \frac{1}{\pi^2} \int_0^\pi \int_0^\pi \sin(x+y) \, dy \, dx \\
 &= \frac{1}{\pi^2} \int_0^\pi [-\cos(x+y)]_0^\pi \, dx \\
 &= \frac{1}{\pi^2} \int_0^\pi (-\cos(x+\pi) + \cos x) \, dx \\
 &= \frac{1}{\pi^2} \int_0^\pi 2 \cos x \, dx = \frac{1}{\pi^2} [2 \sin x]_0^\pi = 0
 \end{aligned}$$

$$\begin{aligned}
 65. \text{Average} &= \frac{1}{1250} \int_{300}^{325} \int_{200}^{250} 100x^{0.6} y^{0.4} \, dx \, dy \\
 &= \frac{1}{1250} \int_{300}^{325} \left[ (100y^{0.4}) \frac{x^{1.6}}{1.6} \right]_{200}^{250} \, dy \\
 &= \frac{128,844.1}{1250} \int_{300}^{325} y^{0.4} \, dy \\
 &= 103.0753 \left[ \frac{y^{1.4}}{1.4} \right]_{300}^{325} \approx 25,645.24
 \end{aligned}$$

$$\begin{aligned}
 66. \text{Average} &= \frac{1}{8} \int_0^2 \int_0^4 (20 - 4x^2 - y^2) \, dy \, dx \\
 &= \frac{1}{8} \left( \frac{224}{3} \right) = \frac{28}{3}^\circ \text{C}
 \end{aligned}$$

67. See the definition on page 994.

68. The value of  $\int_R \int f(x, y) \, dA$  would be  $kB$ .

69. (a) The total snowfall in the county  $R$ .  
(b) The average snowfall in  $R$ .

70. Part (b) is invalid. You cannot have the variable of integration  $y$  as a limit of integration.

71. No, the maximum possible value is  $(\text{Area})(6) = 6\pi$ .

72. The second is integrable. The first contains

$\int \sin y^2 \, dy$  which does not have an elementary antiderivation.

73.  $f(x, y) \geq 0$  for all  $(x, y)$  and

$$\begin{aligned}
 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) \, dA &= \int_0^5 \int_0^2 \frac{1}{10} \, dy \, dx \\
 &= \int_0^5 \frac{1}{5} \, dx = 1
 \end{aligned}$$

$$\begin{aligned}
 P(0 \leq x \leq 2, 1 \leq y \leq 2) &= \int_0^2 \int_1^2 \frac{1}{10} \, dy \, dx \\
 &= \int_0^2 \frac{1}{10} \, dx = \frac{1}{5}.
 \end{aligned}$$

74.  $f(x, y) \geq 0$  for all  $(x, y)$  and

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) dA = \int_0^2 \int_0^2 \frac{1}{4}xy dy dx = \int_0^2 \frac{x}{2} dx = 1$$

$$P(0 \leq x \leq 1, 1 \leq y \leq 2) = \int_0^1 \int_1^2 \frac{1}{4}xy dy dx = \int_0^1 \frac{3x}{8} dx = \frac{3}{16}.$$

75.  $f(x, y) \geq 0$  for all  $(x, y)$  and

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) dA = \int_0^3 \int_3^6 \frac{1}{27}(9 - x - y) dy dx = \int_0^3 \frac{1}{27} \left[ 9y - xy - \frac{y^2}{2} \right]_3^6 dx = \int_0^3 \left( \frac{1}{2} - \frac{1}{9}x \right) dx = \left[ \frac{x}{2} - \frac{x^2}{18} \right]_0^3 = 1$$

$$P(0 \leq x \leq 1, 4 \leq y \leq 6) = \int_0^1 \int_4^6 \frac{1}{27}(9 - x - y) dy dx = \int_0^1 \frac{2}{27}(4 - x) dx = \frac{7}{27}.$$

76.  $f(x, y) \geq 0$  for all  $(x, y)$  and

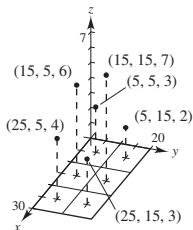
$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) dA = \int_0^{\infty} \int_0^{\infty} e^{-x-y} dy dx = \int_0^{\infty} \lim_{b \rightarrow \infty} \left[ -e^{-x-y} \right]_0^b dx = \int_0^{\infty} e^{-x} dx = \lim_{b \rightarrow \infty} \left[ -e^{-x} \right]_0^b = 1$$

$$P(0 \leq x \leq 1, x \leq y \leq 1) = \int_0^1 \int_x^1 e^{-x-y} dy dx = \int_0^1 \left[ -e^{-x-y} \right]_x^1 dx = \int_0^1 (e^{-2x} - e^{-x-1}) dx$$

$$= \left[ -\frac{1}{2}e^{-2x} + e^{-x-1} \right]_0^1 = \frac{1}{2}e^{-2} - e^{-1} + \frac{1}{2} = \frac{1}{2}(e^{-1} - 1)^2 \approx 0.1998.$$

77. Divide the base into six squares, and assume the height at the center of each square is the height of the entire square. So,

$$V \approx (4 + 3 + 6 + 7 + 3 + 2)(100) = 2500 \text{ m}^3.$$



78. Sample Program for T1-82:

Program: DOUBLE

```

: Input A
: Input B
: Input M
: Input C
: Input D
: Input N
: 0 → V
: (B - A)/M → G
: (D - C)/N → H
: For (I, 1, M, 1)
: For (J, 1, N, 1)
: A + 0.5G(2I - 1) → X
: C + 0.5H(2J - 1) → Y
: V + sin(√X + Y) × G × H → V
: End
: End
: Disp V

```

79.  $\int_0^1 \int_0^2 \sin \sqrt{x+y} dy dx \quad m = 4, n = 8$

(a) 1.78435

(b) 1.7879

80.  $\int_0^2 \int_0^4 20e^{-x^3/8} dy dx \quad m = 10, n = 20$

(a) 129.2018

(b) 129.2756

81.  $\int_4^6 \int_0^2 y \cos \sqrt{x} dx dy \quad m = 4, n = 8$

(a) 11.0571

(b) 11.0414

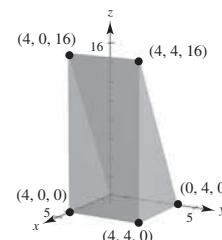
82.  $\int_1^4 \int_1^2 \sqrt{x^3 + y^3} dx dy \quad m = 6, n = 4$

(a) 13.956

(b) 13.9022

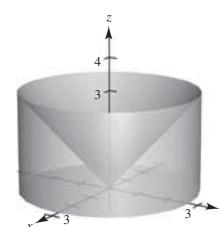
83.  $V \approx 125$

Matches d.



84.  $V \approx 50$

Matches a.

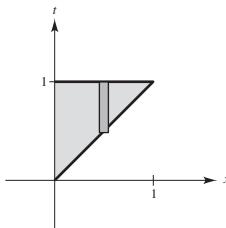


85. False

$$V = 8 \int_0^1 \int_0^{\sqrt{1-y^2}} \sqrt{1-x^2-y^2} dx dy$$

86. True

$$\begin{aligned} 87. \text{ Average} &= \int_0^1 f(x) dx = \int_0^1 \int_1^x e^{t^2} dt dx = - \int_0^1 \int_x^1 e^{t^2} dt dx \\ &= - \int_0^1 \int_0^t e^{t^2} dx dt = - \int_0^1 t e^{t^2} dt \\ &= \left[ -\frac{1}{2} e^{t^2} \right]_0^1 = -\frac{1}{2}(e-1) = \frac{1}{2}(1-e) \end{aligned}$$



90.  $z = x^2 + y^2 - 4$  is a paraboloid opening upward with vertex  $(0, 0, -4)$ . The double integral is minimized if  $z \leq 0$ . That is,

$$R = \{(x, y) : x^2 + y^2 \leq 4\}.$$

[The minimum value is  $-8\pi$ .]

$$\begin{aligned} 91. \int_0^2 (\tan^{-1} \pi x - \tan^{-1} x) dx &= \int_0^2 \int_x^{\pi x} \frac{1}{1+y^2} dy dx = \int_0^2 \int_{y/\pi}^y \frac{1}{1+y^2} dx dy + \int_2^{2\pi} \int_{y/\pi}^2 \frac{1}{1+y^2} dx dy \\ &= \int_0^2 \left[ \frac{x}{1+y^2} \right]_{y/\pi}^y dy + \int_2^{2\pi} \left[ \frac{x}{1+y^2} \right]_{y/\pi}^2 dy = \int_0^2 \left[ \frac{y}{1+y^2} - \frac{y/\pi}{1+y^2} \right] dy + \int_2^{2\pi} \left[ \frac{2}{1+y^2} - \frac{y/\pi}{1+y^2} \right] dy \\ &= \left[ \frac{1}{2} \left( 1 - \frac{1}{\pi} \right) \ln(1+y^2) \right]_0^2 + \left[ 2 \tan^{-1} y - \frac{1}{2\pi} \ln(1+y^2) \right]_0^{2\pi} \\ &= \frac{1}{2} \left( 1 - \frac{1}{\pi} \right) \ln 5 + 2 \tan^{-1}(2\pi) - \frac{1}{2\pi} \ln(1+4\pi^2) - 2 \tan^{-1}(2) + \frac{1}{2\pi} \ln(5) \\ &= \frac{1}{2} \ln 5 + 2 \tan^{-1}(2\pi) - 2 \tan^{-1}(2) - \frac{1}{2\pi} \ln(1+4\pi^2) \approx 0.8274 \end{aligned}$$

$$92. \int_0^3 \int_0^{\sqrt{9-y^2}} \sqrt{9-x^2-y^2} dx dy = \frac{9\pi}{2}$$

because this double integral represents the portion of the sphere  $x^2 + y^2 + z^2 = 9$  in the first octant.

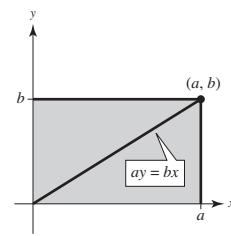
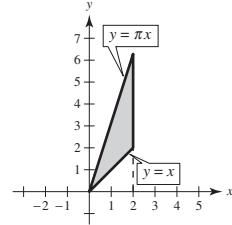
$$V = \frac{1}{8} \cdot \frac{4}{3}\pi(3)^3 = \frac{9\pi}{2}$$

$$93. \text{ Let } I = \int_0^a \int_0^b e^{\max\{b^2x^2, a^2y^2\}} dy dx.$$

Divide the rectangle into two parts by the diagonal line  $ay = bx$ . On lower triangle,

$$b^2x^2 \geq a^2y^2 \text{ because } y \leq \frac{b}{a}x.$$

$$\begin{aligned} I &= \int_0^a \int_0^{bx/a} e^{b^2x^2} dy dx + \int_0^b \int_0^{ay/b} e^{a^2y^2} dx dy = \int_0^a \frac{bx}{a} e^{b^2x^2} dx + \int_0^b \frac{ay}{b} e^{a^2y^2} dy \\ &= \frac{1}{2ab} \left[ e^{b^2x^2} \right]_0^a + \frac{1}{2ab} \left[ e^{a^2y^2} \right]_0^b = \frac{1}{2ab} \left[ e^{b^2a^2} - 1 + e^{a^2b^2} - 1 \right] = \frac{e^{a^2b^2} - 1}{ab} \end{aligned}$$



**94.** Assume such a function exists.

$$u(x) = 1 + \lambda \int_x^1 u(y)u(y-x) dy; \lambda > \frac{1}{2}, 0 \leq x \leq 1$$

$$\alpha = \int_0^1 u(x) dx = \int_0^1 dx + \lambda \int_0^1 \int_x^1 u(y)u(y-x) dy dx$$

Change the order of integration.

$$\alpha = \int_0^1 u(x) dx = 1 + \lambda \int_0^1 \int_0^y u(y)u(y-x) dx dy = 1 + \lambda \int_0^1 u(y) \left[ \int_0^y u(y-x) dx \right] dy$$

Hold  $y$  fixed and let  $z = y - x, dz = -dx$ .

$$\alpha = 1 + \lambda \int_0^1 u(y) \left[ \int_y^0 u(z)(-dz) \right] dy = 1 + \lambda \int_0^1 u(y) \left[ \int_0^y u(z) dz \right] dy$$

Let  $f(y) = \int_0^y u(z) dz$ . Then  $f'(y) = u(y), f(0) = 0, f(1) = \alpha$ .

$$\alpha = 1 + \lambda \int_0^1 f'(y)f(y) dy = 1 + \lambda \left[ \frac{f(y)^2}{2} \right]_0^1 = 1 + \lambda \left[ \frac{1}{2}f(1)^2 - \frac{1}{2}f(0)^2 \right] = 1 + \lambda \frac{1}{2}\alpha^2$$

$$\lambda\alpha^2 - 2\alpha + 2 = 0.$$

For  $\alpha$  to exist, the discriminant of this quadratic must be nonnegative.

$$b^2 - 4ac = 4 - 8\lambda \geq 0 \Rightarrow \lambda \leq \frac{1}{2}$$

But,  $\lambda > \frac{1}{2}$ , a contradiction.

### Section 14.3 Change of Variables: Polar Coordinates

1. Rectangular coordinates

2. Polar coordinates

3. Polar coordinates

4. Rectangular coordinates

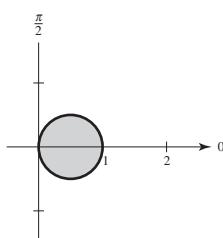
$$5. R = \{(r, \theta) : 0 \leq r \leq 8, 0 \leq \theta \leq \pi\}$$

$$6. R = \{(r, \theta) : 0 \leq r \leq 4 \sin \theta, 0 \leq \theta \leq \pi\}$$

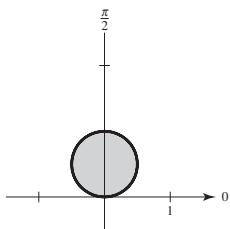
$$7. R = \{(r, \theta) : 0 \leq r \leq 3 + 3 \sin \theta, 0 \leq \theta \leq 2\pi\} \text{ Cardioid}$$

$$8. R = \{(r, \theta) : 0 \leq r \leq 4 \cos 3\theta, 0 \leq \theta \leq \pi\}$$

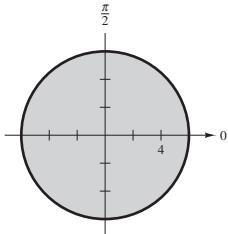
$$\begin{aligned} 9. \int_0^\pi \int_0^{\cos \theta} r dr d\theta \\ &= \int_0^\pi \left[ \frac{r^2}{2} \right]_0^{\cos \theta} d\theta \\ &= \int_0^\pi \frac{1}{2} \cos^2 \theta d\theta \\ &= \int_0^\pi \frac{1}{4} (1 + \cos 2\theta) d\theta \\ &= \frac{1}{4} \left[ \theta + \frac{1}{2} \sin 2\theta \right]_0^\pi = \frac{\pi}{4} \end{aligned}$$



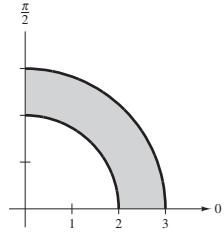
$$\begin{aligned} 10. \int_0^\pi \int_0^{\sin \theta} r^2 dr d\theta &= \int_0^\pi \left[ \frac{r^3}{3} \right]_0^{\sin \theta} d\theta \\ &= \frac{1}{3} \int_0^\pi \sin^3 \theta d\theta = \frac{1}{3} \int_0^\pi (1 - \cos^2 \theta) \sin \theta d\theta \\ &= \frac{1}{3} \left[ -\cos \theta + \frac{\cos^3 \theta}{3} \right]_0^\pi \\ &= \frac{1}{3} \left[ \left( 1 - \frac{1}{3} \right) - \left( -1 + \frac{1}{3} \right) \right] \\ &= \frac{4}{9} \end{aligned}$$



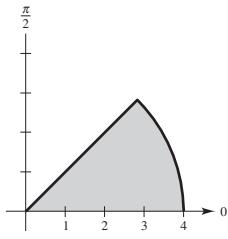
$$\begin{aligned}
 11. \int_0^{2\pi} \int_0^6 3r^2 \sin \theta \, dr \, d\theta &= \int_0^{2\pi} \left[ r^3 \sin \theta \right]_0^6 \, d\theta \\
 &= \int_0^{2\pi} 216 \sin \theta \, d\theta \\
 &= [-216 \cos \theta]_0^{2\pi} = 0
 \end{aligned}$$



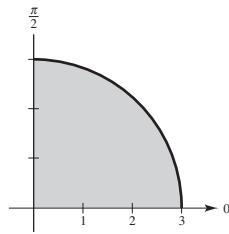
$$\begin{aligned}
 13. \int_0^{\pi/2} \int_2^3 \sqrt{9 - r^2} \, r \, dr \, d\theta &= \int_0^{\pi/2} \left[ -\frac{1}{3}(9 - r^2)^{3/2} \right]_2^3 \, d\theta \\
 &= \left[ \frac{5\sqrt{5}}{3}\theta \right]_0^{\pi/2} = \frac{5\sqrt{5}\pi}{6}
 \end{aligned}$$



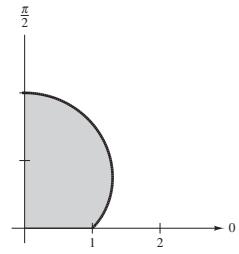
$$\begin{aligned}
 12. \int_0^{\pi/4} \int_0^4 r^2 \sin \theta \cos \theta \, dr \, d\theta &= \int_0^{\pi/4} \left[ \frac{r^3}{3} \sin \theta \cos \theta \right]_0^4 \, d\theta \\
 &= \left[ \left( \frac{64}{3} \right) \frac{\sin^2 \theta}{2} \right]_0^{\pi/4} = \frac{16}{3}
 \end{aligned}$$



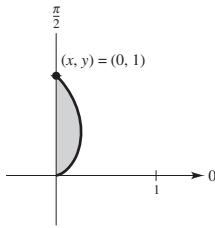
$$\begin{aligned}
 14. \int_0^{\pi/2} \int_0^3 r e^{-r^2} \, dr \, d\theta &= \int_0^{\pi/2} \left[ -\frac{1}{2} e^{-r^2} \right]_0^3 \, d\theta \\
 &= \left[ -\frac{1}{2} (e^{-9} - 1)\theta \right]_0^{\pi/2} \\
 &= \frac{\pi}{4} \left( 1 - \frac{1}{e^9} \right)
 \end{aligned}$$



$$\begin{aligned}
 15. \int_0^{\pi/2} \int_0^{1+\sin \theta} \theta r \, dr \, d\theta &= \int_0^{\pi/2} \left[ \frac{\theta r^2}{2} \right]_0^{1+\sin \theta} \, d\theta \\
 &= \int_0^{\pi/2} \frac{1}{2} \theta (1 + \sin \theta)^2 \, d\theta \\
 &= \left[ \frac{1}{8} \theta^2 + \sin \theta - \theta \cos \theta + \frac{1}{2} \theta \left( -\frac{1}{2} \cos \theta \cdot \sin \theta + \frac{1}{2} \theta \right) + \frac{1}{8} \sin^2 \theta \right]_0^{\pi/2} \\
 &= \frac{3}{32} \pi^2 + \frac{9}{8}
 \end{aligned}$$



$$16. \int_0^{\pi/2} \int_0^{1-\cos \theta} (\sin \theta) r \, dr \, d\theta = \int_0^{\pi/2} \left[ \left( \sin \theta \right) \frac{r^2}{2} \right]_0^{1-\cos \theta} \, d\theta = \int_0^{\pi/2} \frac{\sin \theta}{2} (1 - \cos \theta)^2 \, d\theta = \left[ \frac{1}{6} (1 - \cos(\theta))^3 \right]_0^{\pi/2} = \frac{1}{6}$$



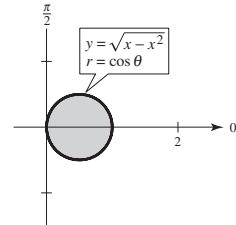
$$17. \int_0^a \int_0^{\sqrt{a^2 - y^2}} y \, dx \, dy = \int_0^{\pi/2} \int_0^a r^2 \sin \theta \, dr \, d\theta = \frac{a^3}{3} \int_0^{\pi/2} \sin \theta \, d\theta = \left[ \frac{a^3}{3} (-\cos \theta) \right]_0^{\pi/2} = \frac{a^3}{3}$$

18.  $\int_0^a \int_0^{\sqrt{a^2-x^2}} x dy dx = \int_0^{\pi/2} \int_0^a r^2 \cos \theta dr d\theta = \frac{a^3}{3} \int_0^{\pi/2} \cos \theta d\theta = \left[ \frac{a^3}{3} \sin \theta \right]_0^{\pi/2} = \frac{a^3}{3}$

19.  $\int_{-2}^2 \int_0^{\sqrt{4-x^2}} (x^2 + y^2) dy dx = \int_0^{\pi} \int_0^2 r^2 r dr d\theta = \int_0^{\pi} \left[ \frac{r^4}{4} \right]_0^2 d\theta = \int_0^{\pi} 4 d\theta = 4\pi$

20. Note that  $x - x^2 = -\left(x^2 - x + \frac{1}{4}\right) + \frac{1}{4} = \frac{1}{4} - \left(x - \frac{1}{2}\right)^2$ . So  $y = \sqrt{x - x^2} \Rightarrow y^2 + \left(x - \frac{1}{2}\right)^2 = \frac{1}{4}$ .

$$\begin{aligned} \int_0^1 \int_{-\sqrt{x-x^2}}^{\sqrt{x-x^2}} (x^2 + y^2) dy dx &= \int_{-\pi/2}^{\pi/2} \int_0^{\cos \theta} r^2 r dr d\theta = \int_{-\pi/2}^{\pi/2} \left[ \frac{r^4}{4} \right]_0^{\cos \theta} d\theta \\ &= \frac{1}{4} \int_{-\pi/2}^{\pi/2} \cos^4 \theta d\theta = \frac{1}{2} \int_0^{\pi} \cos^4 \theta d\theta \\ &= \frac{1}{2} \left( \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{\pi}{2} \right) = \frac{3\pi}{32} \quad (\text{Wallis's Formula}) \end{aligned}$$



21.  $\int_0^3 \int_0^{\sqrt{9-x^2}} (x^2 + y^2)^{3/2} dy dx = \int_0^{\pi/2} \int_0^3 r^4 dr d\theta = \frac{243}{5} \int_0^{\pi/2} d\theta = \frac{243\pi}{10}$

22.  $\int_0^2 \int_y^{\sqrt{8-y^2}} \sqrt{x^2 + y^2} dx dy = \int_0^{\pi/4} \int_0^{2\sqrt{2}} r^2 dr d\theta = \int_0^{\pi/4} \frac{(2\sqrt{2})^3}{3} d\theta = \left[ \frac{(2\sqrt{2})^3}{3} \theta \right]_0^{\pi/4} = \frac{(2\sqrt{2})^3}{3} \cdot \frac{\pi}{4} = \frac{4\sqrt{2}\pi}{3}$

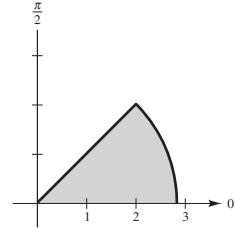
23.  $\int_0^2 \int_0^{\sqrt{2x-x^2}} xy dy dx = \int_0^{\pi/2} \int_0^{2\cos \theta} r^3 \cos \theta \sin \theta dr d\theta = 4 \int_0^{\pi/2} \cos^5 \theta \sin \theta d\theta = \left[ -\frac{4 \cos^6 \theta}{6} \right]_0^{\pi/2} = \frac{2}{3}$

$$\begin{aligned} 24. \int_0^4 \int_0^{\sqrt{4y-y^2}} x^2 dx dy &= \int_0^{\pi/2} \int_0^{4\sin \theta} r^3 \cos^2 \theta dr d\theta = \int_0^{\pi/2} 64 \sin^4 \theta \cos^2 \theta d\theta \\ &= 64 \int_0^{\pi/2} (\sin^4 \theta - \sin^6 \theta) d\theta = \frac{64}{6} \left[ \sin^5 \theta \cos \theta - \frac{\sin^3 \theta \cos \theta}{4} + \frac{3}{8}(\theta - \sin \theta \cos \theta) \right]_0^{\pi/2} = 2\pi \end{aligned}$$

25.  $\int_{-1}^1 \int_0^{\sqrt{1-x^2}} \cos(x^2 + y^2) dy dx = \int_0^{\pi} \int_0^1 \cos(r^2) r dr d\theta = \int_0^{\pi} \left[ \frac{1}{2} \sin(r^2) \right]_0^1 d\theta = \int_0^{\pi} \frac{1}{2} \sin(1) d\theta = \frac{\pi}{2} \sin(1) \approx 1.3218$

$$\begin{aligned} 26. \int_0^2 \int_0^{\sqrt{4-x^2}} \sin \sqrt{x^2 + y^2} dy dx &= \int_0^{\pi/2} \int_0^2 \sin(r) r dr d\theta = \int_0^{\pi/2} [\sin r - r \cos r]_0^2 d\theta \quad [\text{Integration by parts}] \\ &= \int_0^{\pi/2} (\sin 2 - 2 \cos 2) d\theta = \frac{\pi}{2} (\sin 2 - 2 \cos 2) \approx 2.7357 \end{aligned}$$

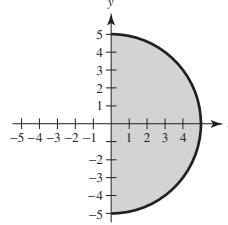
$$\begin{aligned} 27. \int_0^2 \int_0^x \sqrt{x^2 + y^2} dy dx + \int_2^{2\sqrt{2}} \int_0^{\sqrt{8-x^2}} \sqrt{x^2 + y^2} dy dx &= \int_0^{\pi/4} \int_0^{2\sqrt{2}} r^2 dr d\theta \\ &= \int_0^{\pi/4} \frac{16\sqrt{2}}{3} d\theta = \frac{4\sqrt{2}\pi}{3} \end{aligned}$$



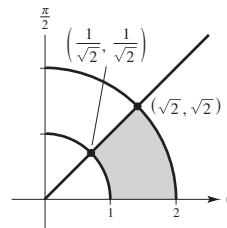
$$\begin{aligned} 28. \int_0^{(5\sqrt{2})/2} \int_0^x xy dy dx + \int_{(5\sqrt{2})/2}^5 \int_0^{\sqrt{25-x^2}} xy dy dx &= \int_0^{\pi/4} \int_0^5 r^3 \sin \theta \cos \theta dr d\theta \\ &= \int_0^{\pi/4} \frac{625}{4} \sin \theta \cos \theta d\theta = \left[ \frac{625}{8} \sin^2 \theta \right]_0^{\pi/4} = \frac{625}{16} \end{aligned}$$

$$\begin{aligned}
 29. \int_0^2 \int_0^{\sqrt{4-x^2}} (x+y) dy dx &= \int_0^{\pi/2} \int_0^2 (r \cos \theta + r \sin \theta) r dr d\theta = \int_0^{\pi/2} \int_0^2 (\cos \theta + \sin \theta) r^2 dr d\theta \\
 &= \frac{8}{3} \int_0^{\pi/2} (\cos \theta + \sin \theta) d\theta = \left[ \frac{8}{3}(\sin \theta - \cos \theta) \right]_0^{\pi/2} = \frac{16}{3}
 \end{aligned}$$

$$\begin{aligned}
 30. \int_{-\pi/2}^{\pi/2} \int_0^5 e^{-r^2/2} r dr d\theta &= \int_{-\pi/2}^{\pi/2} \left[ -e^{-r^2/2} \right]_0^5 d\theta \\
 &= \int_{-\pi/2}^{\pi/2} (1 - e^{-25/2}) d\theta \\
 &= \left[ (1 - e^{-25/2}) \theta \right]_{-\pi/2}^{\pi/2} = \pi(1 - e^{-25/2})
 \end{aligned}$$



$$\begin{aligned}
 31. \int_0^{1/\sqrt{2}} \int_{\sqrt{1-y^2}}^{\sqrt{4-y^2}} \arctan \frac{y}{x} dx dy + \int_{1/\sqrt{2}}^{\sqrt{2}} \int_y^{\sqrt{4-y^2}} \arctan \frac{y}{x} dx dy \\
 &= \int_0^{\pi/4} \int_1^2 \theta r dr d\theta \\
 &= \int_0^{\pi/4} \frac{3}{2} \theta d\theta = \left[ \frac{3\theta^2}{4} \right]_0^{\pi/4} = \frac{3\pi^2}{64}
 \end{aligned}$$



$$\begin{aligned}
 32. \int_0^3 \int_0^{\sqrt{9-x^2}} (9 - x^2 - y^2) dy dx &= \int_0^{\pi/2} \int_0^3 (9 - r^2) r dr d\theta \\
 &= \int_0^{\pi/2} \int_0^3 (9r - r^3) dr d\theta = \int_0^{\pi/2} \left[ \frac{9}{2}r^2 - \frac{1}{4}r^4 \right]_0^3 d\theta = \frac{81}{4} \int_0^{\pi/2} d\theta = \frac{81\pi}{8}
 \end{aligned}$$

$$\begin{aligned}
 33. V &= \int_0^{\pi/2} \int_0^1 (r \cos \theta)(r \sin \theta) r dr d\theta \\
 &= \frac{1}{2} \int_0^{\pi/2} \int_0^1 r^3 \sin 2\theta dr d\theta = \frac{1}{8} \int_0^{\pi/2} \sin 2\theta d\theta = \left[ -\frac{1}{16} \cos 2\theta \right]_0^{\pi/2} = \frac{1}{8}
 \end{aligned}$$

$$34. V = 4 \int_0^{\pi/2} \int_0^1 (r^2 + 3)r dr d\theta = 4 \int_0^{\pi/2} \left[ \frac{r^4}{4} + \frac{3r^2}{2} \right]_0^1 d\theta = 4 \int_0^{\pi/2} \frac{7}{4} d\theta = \frac{7\pi}{2}$$

$$35. V = \int_0^{2\pi} \int_0^5 r^2 dr d\theta = \int_0^{2\pi} \frac{125}{3} d\theta = \frac{250\pi}{3}$$

$$\begin{aligned}
 36. V &= \int_R \int \ln(x^2 + y^2) dA = \int_0^{2\pi} \int_1^2 (\ln r^2) r dr d\theta = 2 \int_0^{2\pi} \int_1^2 r \ln r dr d\theta \\
 &= 2 \int_0^{2\pi} \left[ \frac{r^2}{4} (-1 + 2 \ln r) \right]_1^2 d\theta = 2 \int_0^{2\pi} \left( \ln 4 - \frac{3}{4} \right) d\theta = 4\pi \left( \ln 4 - \frac{3}{4} \right)
 \end{aligned}$$

$$\begin{aligned}
 37. V &= 2 \int_0^{\pi/2} \int_0^{4 \cos \theta} \sqrt{16 - r^2} r dr d\theta = 2 \int_0^{\pi/2} \left[ -\frac{1}{3} (\sqrt{16 - r^2})^3 \right]_0^{4 \cos \theta} d\theta = -\frac{2}{3} \int_0^{\pi/2} (64 \sin^3 \theta - 64) d\theta \\
 &= \frac{128}{3} \int_0^{\pi/2} [1 - \sin \theta (1 - \cos^2 \theta)] d\theta = \frac{128}{3} \left[ \theta + \cos \theta - \frac{\cos^3 \theta}{3} \right]_0^{\pi/2} = \frac{64}{9} (3\pi - 4)
 \end{aligned}$$

$$38. V = \int_0^{2\pi} \int_1^4 \sqrt{16 - r^2} r dr d\theta = \int_0^{2\pi} \left[ -\frac{1}{3} (\sqrt{16 - r^2})^3 \right]_1^4 d\theta = \int_0^{2\pi} 5\sqrt{15} d\theta = 10\sqrt{15}\pi$$

$$39. V = \int_0^{2\pi} \int_a^4 \sqrt{16 - r^2} r dr d\theta = \int_0^{2\pi} \left[ -\frac{1}{3} (\sqrt{16 - r^2})^3 \right]_a^4 d\theta = \frac{1}{3} (\sqrt{16 - a^2})^3 (2\pi)$$

One-half the volume of the hemisphere is  $(64\pi)/3$ .

$$\frac{2\pi}{3} (16 - a^2)^{3/2} = \frac{64\pi}{3}$$

$$(16 - a^2)^{3/2} = 32$$

$$16 - a^2 = 32^{2/3}$$

$$a^2 = 16 - 32^{2/3} = 16 - 8\sqrt[3]{2}$$

$$a = \sqrt{4(4 - 2\sqrt[3]{2})} = 2\sqrt{4 - 2\sqrt[3]{2}} \approx 2.4332$$

$$40. x^2 + y^2 + z^2 = a^2 \Rightarrow z = \sqrt{a^2 - (x^2 + y^2)} = \sqrt{a^2 - r^2}$$

$$V = 8 \int_0^{\pi/2} \int_0^a \sqrt{a^2 - r^2} r dr d\theta \quad (8 \text{ times the volume in the first octant})$$

$$= 8 \int_0^{\pi/2} \left[ -\frac{1}{2} \cdot \frac{2}{3} (a^2 - r^2)^{3/2} \right]_0^a d\theta = 8 \int_0^{\pi/2} \frac{a^3}{3} d\theta = \left[ \frac{8a^3}{3}\theta \right]_0^{\pi/2} = \frac{4\pi a^3}{3}$$

$$41. \text{ Total volume } = V = \int_0^{2\pi} \int_0^4 25e^{-r^2/4} r dr d\theta = \int_0^{2\pi} \left[ -50e^{-r^2/4} \right]_0^4 d\theta = \int_0^{2\pi} -50(e^{-4} - 1) d\theta = (1 - e^{-4})100\pi \approx 308.40524$$

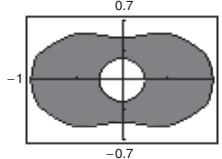
Let  $c$  be the radius of the hole that is removed.

$$\begin{aligned} \frac{1}{10}V &= \int_0^{2\pi} \int_0^c 25e^{-r^2/4} r dr d\theta = \int_0^{2\pi} \left[ -50e^{-r^2/4} \right]_0^c d\theta \\ &= \int_0^{2\pi} -50(e^{-c^2/4} - 1) d\theta \Rightarrow 30.84052 = 100\pi(1 - e^{-c^2/4}) \\ &\Rightarrow e^{-c^2/4} = 0.90183 \\ &\Rightarrow -\frac{c^2}{4} = -0.10333 \\ &\Rightarrow c^2 = 0.41331 \\ &\Rightarrow c = 0.6429 \end{aligned}$$

$$\Rightarrow \text{diameter} = 2c = 1.2858$$

$$42. \frac{-9}{4(x^2 + y^2 + 9)} \leq z \leq \frac{9}{4(x^2 + y^2 + 9)}, \frac{1}{4} \leq r \leq \frac{1}{2}(1 + \cos^2 \theta)$$

$$(a) \frac{-9}{4r^2 + 36} \leq z \leq \frac{9}{4r^2 + 36}$$

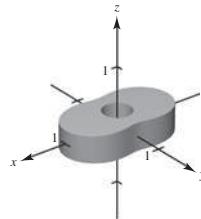


$$(b) \text{ Perimeter} = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta.$$

$$r = \frac{1}{2}(1 + \cos^2 \theta) = \frac{1}{2} + \frac{1}{2} \cos^2 \theta$$

$$\frac{dr}{d\theta} = -\cos \theta \sin \theta$$

$$\text{Perimeter} = 2 \int_0^{\pi} \sqrt{\frac{1}{4}(1 + \cos^2 \theta)^2 + \cos^2 \theta \sin^2 \theta} d\theta \approx 5.21$$



$$(c) V = 2 \int_0^{2\pi} \int_{1/4}^{1/2(1+\cos^2 \theta)} \frac{9}{4r^2 + 36} r dr d\theta \approx 0.8000$$

43.  $A = \int_0^\pi \int_0^{6\cos\theta} r dr d\theta = \int_0^\pi 18 \cos^2 \theta d\theta = 9 \int_0^\pi (1 + \cos 2\theta) d\theta = \left[ 9\left(\theta + \frac{1}{2}\sin 2\theta\right) \right]_0^\pi = 9\pi$

44.  $A = \int_0^{2\pi} \int_2^4 r dr d\theta = \int_0^{2\pi} 6 d\theta = 12\pi$

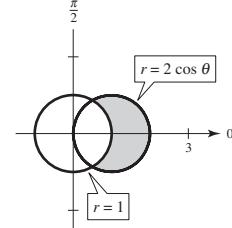
45.  $A = \int_0^{2\pi} \int_0^{1+\cos\theta} r dr d\theta = \frac{1}{2} \int_0^{2\pi} (1 + 2 \cos \theta + \cos^2 \theta) d\theta$   
 $= \frac{1}{2} \int_0^{2\pi} \left( 1 + 2 \cos \theta + \frac{1 + \cos 2\theta}{2} \right) d\theta = \frac{1}{2} \left[ \theta + 2 \sin \theta + \frac{1}{2} \left( \theta + \frac{1}{2} \sin 2\theta \right) \right]_0^{2\pi} = \frac{3\pi}{2}$

46.  $A = \int_0^{2\pi} \int_0^{2+\sin\theta} r dr d\theta = \frac{1}{2} \int_0^{2\pi} (2 + \sin \theta)^2 d\theta = \frac{1}{2} \int_0^{2\pi} (4 + 4 \sin \theta + \sin^2 \theta) d\theta$   
 $= \frac{1}{2} \int_0^{2\pi} \left( 4 + 4 \sin \theta + \frac{1 - \cos 2\theta}{2} \right) d\theta$   
 $= \frac{1}{2} \left[ 4\theta - 4 \cos \theta + \frac{1}{2} \theta - \frac{1}{4} \sin 2\theta \right]_0^{2\pi} = \frac{1}{2} [8\pi - 4 + \pi + 4] = \frac{9\pi}{2}$

47.  $A = 3 \int_0^{\pi/3} \int_0^{2\sin 3\theta} r dr d\theta = \frac{3}{2} \int_0^{\pi/3} 4 \sin^2 3\theta d\theta = 3 \int_0^{\pi/3} (1 - \cos 6\theta) d\theta = 3 \left[ \theta - \frac{1}{6} \sin 6\theta \right]_0^{\pi/3} = \pi$

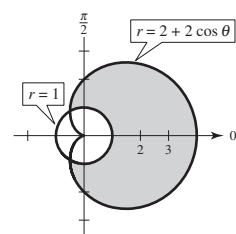
48.  $A = 8 \int_0^{\pi/4} \int_0^{3\cos 2\theta} r dr d\theta = 4 \int_0^{\pi/4} 9 \cos^2 2\theta d\theta = 18 \int_0^{\pi/4} (1 + \cos 4\theta) d\theta = 18 \left[ \theta + \frac{1}{4} \sin 4\theta \right]_0^{\pi/4} = \frac{9\pi}{2}$

49.  $r = 1 = 2 \cos \theta \Rightarrow \theta = \pm \frac{\pi}{3}$   
 $A = 2 \int_0^{\pi/3} \int_1^{2\cos\theta} r dr d\theta = 2 \int_0^{\pi/3} \left[ \frac{r^2}{2} \right]_1^{2\cos\theta} d\theta = 2 \int_0^{\pi/3} \left( 2 \cos^2 \theta - \frac{1}{2} \right) d\theta$   
 $= 2 \int_0^{\pi/3} \left( 1 + \cos 2\theta - \frac{1}{2} \right) d\theta = 2 \left[ \frac{1}{2}\theta + \frac{\sin 2\theta}{2} \right]_0^{\pi/3} = 2 \left[ \frac{\pi}{6} + \frac{\sqrt{3}}{4} \right] = \frac{\pi}{3} + \frac{\sqrt{3}}{2}$



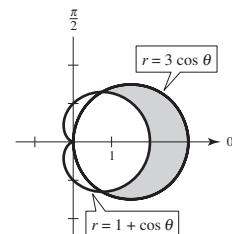
50.  $r = 2 + 2 \cos \theta = 1 \Rightarrow \cos \theta = -\frac{1}{2} \Rightarrow \theta = \frac{2\pi}{3}, \frac{4\pi}{3}$

$$\begin{aligned} A &= 2 \int_0^{2\pi/3} \int_1^{2+2\cos\theta} r dr d\theta = 2 \int_0^{2\pi/3} \left[ \frac{r^2}{2} \right]_1^{2+2\cos\theta} d\theta \\ &= \int_0^{2\pi/3} \left[ (2 + 2 \cos \theta)^2 - 1 \right] d\theta \\ &= \int_0^{2\pi/3} \left[ 3 + 8 \cos \theta + 4 \cos^2 \theta \right] d\theta = \int_0^{2\pi/3} \left[ 3 + 8 \cos \theta + 2(1 + \cos 2\theta) \right] d\theta \\ &= [5\theta + 8 \sin \theta + \sin 2\theta]_0^{2\pi/3} = \frac{10\pi}{3} + 4\sqrt{3} - \frac{\sqrt{3}}{2} = \frac{10\pi}{3} + \frac{7\sqrt{3}}{2} \end{aligned}$$



51.  $r = 3 \cos \theta = 1 + \cos \theta \Rightarrow \cos \theta = \frac{1}{2} \Rightarrow \theta = \pm \frac{\pi}{3}$

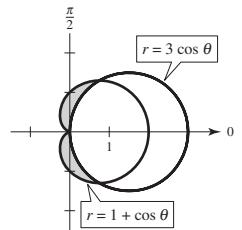
$$\begin{aligned} A &= 2 \int_0^{\pi/3} \int_{1+\cos\theta}^{3\cos\theta} r dr d\theta = 2 \int_0^{\pi/3} \left[ \frac{r^2}{2} \right]_{1+\cos\theta}^{3\cos\theta} d\theta = \int_0^{\pi/3} \left[ 9 \cos^2 \theta - (1 + \cos \theta)^2 \right] d\theta \\ &= \int_0^{\pi/3} (8 \cos^2 \theta - 2 \cos \theta - 1) d\theta = \int_0^{\pi/3} [4(1 + \cos 2\theta) - 2 \cos \theta - 1] d\theta \\ &= [3\theta + 2 \sin 2\theta - 2 \sin \theta]_0^{\pi/3} = 3\left(\frac{\pi}{3}\right) + \sqrt{3} - \sqrt{3} = \pi \end{aligned}$$



52.  $r = 1 + \cos \theta = 3 \cos \theta \Rightarrow \cos \theta = \frac{1}{2} \Rightarrow \theta = \pm \frac{\pi}{3}$

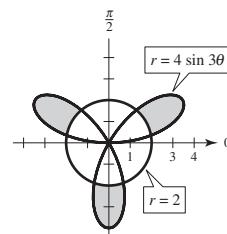
$$\begin{aligned}\frac{1}{2}A &= \int_{\pi/3}^{\pi/2} \int_{3 \cos \theta}^{1+\cos \theta} r dr d\theta + \int_{\pi/2}^{\pi} \int_0^{1+\cos \theta} r dr d\theta \\ &= \int_{\pi/3}^{\pi/2} \left[ \frac{r^2}{2} \right]_{3 \cos \theta}^{1+\cos \theta} d\theta + \int_{\pi/2}^{\pi} \left[ \frac{r^2}{2} \right]_0^{1+\cos \theta} d\theta = \int_{\pi/3}^{\pi/2} \frac{(1+\cos \theta)^2 - 9 \cos^2 \theta}{2} d\theta + \int_{\pi/2}^{\pi} \frac{(1+\cos \theta)^2}{2} d\theta \\ &= \int_{\pi/3}^{\pi/2} \frac{1+2\cos \theta - 4(1+\cos 2\theta)}{2} d\theta + \int_{\pi/2}^{\pi} \left( \frac{1}{2} + \cos \theta + \frac{1+\cos 2\theta}{4} \right) d\theta \\ &= \left[ -\frac{3}{2}\theta + \sin \theta - \sin 2\theta \right]_{\pi/3}^{\pi/2} + \left[ \frac{3}{4}\theta + \sin \theta + \frac{\sin 2\theta}{8} \right]_{\pi/2}^{\pi} = \left( -\frac{3\pi}{4} + 1 \right) - \left( -\frac{\pi}{2} + \frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2} \right) + \left( \frac{3\pi}{4} - \frac{3\pi}{8} - 1 \right) = \frac{\pi}{8}\end{aligned}$$

So,  $A = \frac{\pi}{4}$ .



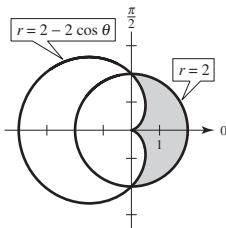
53.  $r = 4 \sin 3\theta = 2 \Rightarrow \sin 3\theta = \frac{1}{2} \Rightarrow 3\theta = \frac{\pi}{6}, \frac{5\pi}{6} \Rightarrow \theta = \frac{\pi}{18}, \frac{5\pi}{18}$

$$\begin{aligned}A &= 3 \int_{\pi/18}^{5\pi/18} \int_2^{4 \sin 3\theta} r dr d\theta = 3 \int_{\pi/18}^{5\pi/18} \left[ \frac{r^2}{2} \right]_2^{4 \sin 3\theta} d\theta = \frac{3}{2} \int_{\pi/18}^{5\pi/18} \left[ (4 \sin 3\theta)^2 - 4 \right] d\theta \\ &= \frac{3}{2} \int_{\pi/18}^{5\pi/18} \left[ 8(1 - \cos 6\theta) - 4 \right] d\theta = \frac{3}{2} \left[ 4\theta - \frac{4}{3} \sin 6\theta \right]_{\pi/18}^{5\pi/18} \\ &= \frac{3}{2} \left[ \left( \frac{10}{9}\pi - \frac{4}{3} \left( \frac{-\sqrt{3}}{2} \right) \right) - \left( \frac{2\pi}{9} - \frac{4}{3} \left( \frac{\sqrt{3}}{2} \right) \right) \right] = \frac{4}{3}\pi + 2\sqrt{3}\end{aligned}$$



54.  $r = 2 = 2 - 2 \cos \theta \Rightarrow \cos \theta = 0 \Rightarrow \theta = \pm \frac{\pi}{2}$

$$\begin{aligned}A &= 2 \int_0^{\pi/2} \int_{2-2 \cos \theta}^2 r dr d\theta \\ &= 2 \int_0^{\pi/2} \left[ \frac{r^2}{2} \right]_{2-2 \cos \theta}^2 d\theta \\ &= \int_0^{\pi/2} \left[ 4 - (2 - 2 \cos \theta)^2 \right] d\theta \\ &= \int_0^{\pi/2} (8 \cos \theta - 4 \cos^2 \theta) d\theta \\ &= \int_0^{\pi/2} (8 \cos \theta - 2(1 + \cos 2\theta)) d\theta \\ &= [8 \sin \theta - 2\theta - \sin 2\theta]_0^{\pi/2} = 8 - \pi\end{aligned}$$



55. Let  $R$  be a region bounded by the graphs of

$r = g_1(\theta)$  and  $r = g_2(\theta)$ , and the lines  $\theta = a$  and  $\theta = b$ .

When using polar coordinates to evaluate a double integral over  $R$ ,  $R$  can be partitioned into small polar sectors.

56. See Theorem 14.3.

57.  $r$ -simple regions have fixed bounds for  $\theta$ .  
 $\theta$ -simple regions have fixed bounds for  $r$ .

58. (a) Horizontal or polar representative elements  
(b) Polar representative element  
(c) Vertical or polar

59. (a)  $\int_{-3}^3 \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} f(x, y) dy dx$   
(b)  $\int_0^{2\pi} \int_0^3 f(r \cos \theta, r \sin \theta) r dr d\theta$   
(c) In general, the integral in part (b) is easier to evaluate. The endpoints of the region of integration are constants.

60.  $0 \leq r \leq 4, 0 \leq \theta \leq 2\pi, x^2 + y^2 = r^2$ .

Answer (c)

61. You would need to insert a factor of  $r$  because of the  $r dr d\theta$  nature of polar coordinate integrals. The plane regions would be sectors of circles.

62. (a) The volume of the subregion determined by the point  $(5, \pi/16, 7)$  is base  $\times$  height  $= (5 \cdot 10 \cdot \pi/8)(7)$ .

Adding up the 20 volumes, ending with  $(45 \cdot 10 \cdot \pi/8)(12)$ , you obtain

$$\begin{aligned} V &\approx 10 \cdot \frac{\pi}{8} [5(7 + 9 + 9 + 5) + 15(8 + 10 + 11 + 8) + 25(10 + 14 + 15 + 11) \\ &\quad + 35(12 + 15 + 18 + 16) + 45(9 + 10 + 14 + 12)] \\ &= \frac{5\pi}{4} [150 + 555 + 1250 + 2135 + 2025] \approx \frac{5\pi}{4} [6115] \approx 24,013.5 \text{ ft}^3 \end{aligned}$$

(b)  $(57)(24,013.5) = 1,368,769.5$  pounds

(c)  $(7.48)(24,013.5) \approx 179,621$  gallons

63.  $\int_{\pi/4}^{\pi/2} \int_0^5 r \sqrt{1+r^3} \sin \sqrt{\theta} dr d\theta \approx 56.051$

[Note: This integral equals  $\left(\int_{\pi/4}^{\pi/2} \sin \sqrt{\theta} d\theta\right) \left(\int_0^5 r \sqrt{1+r^3} dr\right)$ .]

64.  $\int_0^{\pi/4} \int_0^4 5e^{\sqrt{r\theta}} r dr d\theta \approx 87.130$

65. Volume = base  $\times$  height

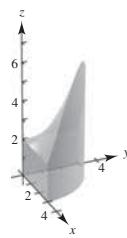
$\approx 8\pi \times 12 \approx 300$

Answer (c)



66. Volume = base  $\times$  height  $\approx \frac{9}{4}\pi \times 3 \approx 21$

Answer (a)



67. False

Let  $f(r, \theta) = r - 1$  where  $R$  is the circular sector  $0 \leq r \leq 6$  and  $0 \leq \theta \leq \pi$ . Then,

$$\int_R \int (r - 1) dA > 0 \text{ but } r - 1 > 0 \text{ for all } r.$$

68. True

69. (a)  $I^2 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-(x^2+y^2)/2} dA = 4 \int_0^{\pi/2} \int_0^{\infty} e^{-r^2/2} r dr d\theta = 4 \int_0^{\pi/2} \left[-e^{-r^2/2}\right]_0^{\infty} d\theta = 4 \int_0^{\pi/2} d\theta = 2\pi$

(b) So,  $I = \sqrt{2\pi}$ .

70. (a) Let  $u = \sqrt{2}x$ , then  $\int_{-\infty}^{\infty} e^{-x^2} dx = \int_{-\infty}^{\infty} e^{-u^2/2} \frac{1}{\sqrt{2}} du = \frac{1}{\sqrt{2}} (\sqrt{2\pi}) = \sqrt{\pi}$ .

(b) Let  $u = 2x$ , then  $\int_{-\infty}^{\infty} e^{-4x^2} dx = \int_{-\infty}^{\infty} e^{-u^2/4} \frac{1}{2} du = \frac{1}{2} \sqrt{\pi}$ .

71.  $\int_{-7}^7 \int_{-\sqrt{49-x^2}}^{\sqrt{49-x^2}} 4000e^{-0.01(x^2+y^2)} dy dx = \int_0^{2\pi} \int_0^7 4000e^{-0.01r^2} r dr d\theta = \int_0^{2\pi} \left[-200,000e^{-0.01r^2}\right]_0^7 d\theta$   
 $= 2\pi(-200,000)(e^{-0.49} - 1) = 400,000\pi(1 - e^{-0.49}) \approx 486,788$

$$\begin{aligned}
 72. \int_0^\infty \int_0^\infty k e^{-\left(x^2+y^2\right)} dy dx &= \int_0^{\pi/2} \int_0^\infty k e^{-r^2} r dr d\theta \\
 &= \int_0^{\pi/2} \left[ -\frac{k}{2} e^{-r^2} \right]_0^\infty d\theta \\
 &= \int_0^{\pi/2} \frac{k}{2} d\theta = \frac{k\pi}{4}
 \end{aligned}$$

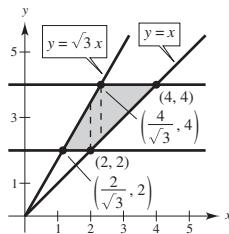
For  $f(x, y)$  to be a probability density function,

$$\begin{aligned}
 \frac{k\pi}{4} &= 1 \\
 k &= \frac{4}{\pi}.
 \end{aligned}$$

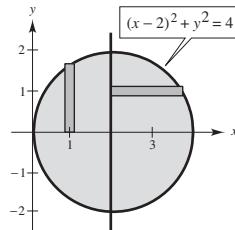
$$73. (a) \int_2^4 \int_{y/\sqrt{3}}^y f dx dy$$

$$\begin{aligned}
 (b) \quad &\int_{2/\sqrt{3}}^2 \int_2^{\sqrt{3x}} f dy dx \\
 &+ \int_2^{4/\sqrt{3}} \int_x^{\sqrt{3x}} f dy dx + \int_{4/\sqrt{3}}^4 \int_x^4 f dy dx
 \end{aligned}$$

$$(c) \int_{\pi/4}^{\pi/3} \int_{2 \csc \theta}^{4 \csc \theta} fr dr d\theta$$



$$\begin{aligned}
 74. (a) \quad &4 \int_0^2 \int_2^{2+\sqrt{4-y^2}} f dx dy \\
 (b) \quad &4 \int_0^2 \int_0^{\sqrt{4-(x-2)^2}} f dy dx \\
 (c) \quad &2 \int_0^{\pi/2} \int_0^{4 \cos \theta} fr dr d\theta
 \end{aligned}$$



$$75. A = \frac{\Delta \theta r_2^2}{2} - \frac{\Delta \theta r_1^2}{2} = \Delta \theta \left( \frac{r_1 + r_2}{2} \right) (r_2 - r_1) = r \Delta r \Delta \theta$$

## Section 14.4 Center of Mass and Moments of Inertia

$$1. m = \int_0^2 \int_0^2 xy dy dx = \int_0^2 \left[ \frac{xy^2}{2} \right]_0^2 dx = \int_0^2 2x dx = \left[ x^2 \right]_0^2 = 4$$

$$2. m = \int_0^3 \int_0^{9-x^2} xy dy dx = \int_0^3 \left[ \frac{xy^2}{2} \right]_0^{9-x^2} dx = \int_0^3 \frac{x(9-x^2)^2}{2} dx = \left[ -\frac{1}{4} \frac{(9-x^2)^3}{3} \right]_0^3 = 0 + \frac{1}{4}(243) = \frac{243}{4}$$

$$3. m = \int_0^{\pi/2} \int_0^1 (r \cos \theta)(r \sin \theta) r dr d\theta = \int_0^{\pi/2} \left[ (\cos \theta \sin \theta) \frac{r^4}{4} \right]_0^1 d\theta = \int_0^{\pi/2} \frac{1}{4} \cos \theta \sin \theta d\theta = \left[ \frac{1}{4} \cdot \frac{\sin^2 \theta}{2} \right]_0^{\pi/2} = \frac{1}{8}$$

$$\begin{aligned}
 4. m = \int_0^3 \int_3^{3+\sqrt{9-x^2}} xy dy dx &= \int_0^3 \left[ x \frac{y^2}{2} \right]_3^{3+\sqrt{9-x^2}} dx = \int_0^3 \frac{x}{2} \left( (3 + \sqrt{9-x^2})^2 - 9 \right) dx = \frac{1}{2} \int_0^3 [6x\sqrt{9-x^2} + 9x - x^3] dx \\
 &= \frac{1}{2} \left[ -2(9-x^2)^{3/2} + \frac{9x^2}{2} - \frac{x^4}{4} \right]_0^3 = \frac{1}{2} \left[ \frac{81}{2} - \frac{81}{4} + 54 \right] = \frac{297}{8}
 \end{aligned}$$

5. (a)  $m = \int_0^a \int_0^a k \, dy \, dx = ka^2$   
 $M_x = \int_0^a \int_0^a ky \, dy \, dx = \int_0^a \frac{ka^2}{2} \, dx = \frac{ka^3}{2}$   
 $M_y = \int_0^a \int_0^a kx \, dy \, dx = \frac{ka^3}{2}$   
 $\bar{x} = \frac{M_y}{m} = \frac{a}{2}, \bar{y} = \frac{M_x}{m} = \frac{a}{2}$   
 $(\bar{x}, \bar{y}) = \left( \frac{a}{2}, \frac{a}{2} \right)$  (center of square)

(b)  $m = \int_0^a \int_0^a ky \, dy \, dx = \frac{1}{2} ka^3$   
 $M_x = \int_0^a \int_0^a ky^2 \, dy \, dx = \frac{1}{3} ka^4$   
 $M_y = \int_0^a \int_0^a kyx \, dy \, dx = \frac{1}{4} ka^4$   
 $\bar{x} = \frac{M_y}{m} = \frac{a}{2}, \bar{y} = \frac{M_x}{m} = \frac{2a}{3}$   
 $(\bar{x}, \bar{y}) = \left( \frac{a}{2}, \frac{2a}{3} \right)$

(c)  $m = \int_0^a \int_0^a kx \, dy \, dx = \frac{1}{2} ka^3$   
 $M_x = \int_0^a \int_0^a kxy \, dy \, dx = \frac{1}{4} ka^4$   
 $M_y = \int_0^a \int_0^a kx^2 \, dy \, dx = \frac{1}{3} ka^3$   
 $\bar{x} = \frac{M_y}{m} = \frac{2a}{3}, \bar{y} = \frac{M_x}{m} = \frac{a}{2}$   
 $(\bar{x}, \bar{y}) = \left( \frac{2a}{3}, \frac{a}{2} \right)$

7. (a)  $m = \int_0^a \int_0^y k \, dx \, dy = \frac{1}{2} ka^2$   
 $M_x = \int_0^a \int_0^y ky \, dx \, dy = \frac{1}{3} ka^3$   
 $M_y = \int_0^a \int_0^y kx \, dx \, dy = \frac{1}{6} ka^3$   
 $\bar{x} = \frac{M_y}{m} = \frac{a}{3}, \bar{y} = \frac{M_x}{m} = \frac{2a}{3}$   
 $(\bar{x}, \bar{y}) = \left( \frac{a}{3}, \frac{2a}{3} \right)$

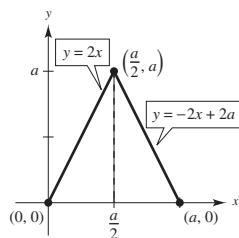
(c)  $m = \int_0^a \int_0^y kx \, dx \, dy = \frac{1}{6} ka^3$   
 $M_x = \int_0^a \int_0^y kxy \, dx \, dy = \frac{1}{8} ka^4$   
 $M_y = \int_0^a \int_0^y kx^2 \, dx \, dy = \frac{1}{12} ka^4$   
 $\bar{x} = \frac{M_y}{m} = \frac{a}{2}, \bar{y} = \frac{M_x}{m} = \frac{3a}{4}$   
 $(\bar{x}, \bar{y}) = \left( \frac{a}{2}, \frac{3a}{4} \right)$

6. (a)  $m = \int_0^a \int_0^b kxy \, dy \, dx = \frac{ka^2 b^2}{4}$   
 $M_x = \int_0^a \int_0^b kxy^2 \, dy \, dx = \frac{ka^2 b^3}{6}$   
 $M_y = \int_0^a \int_0^b kx^2 y \, dy \, dx = \frac{ka^3 b^2}{6}$   
 $\bar{x} = \frac{M_y}{m} = \frac{ka^3 b^2 / 6}{ka^2 b^2 / 4} = \frac{2a}{3},$   
 $\bar{y} = \frac{M_x}{m} = \frac{ka^2 b^3 / 6}{ka^2 b^2 / 4} = \frac{2}{3}b$   
 $(\bar{x}, \bar{y}) = \left( \frac{2a}{3}, \frac{2b}{3} \right)$

(b)  $m = \int_0^a \int_0^b k(x^2 + y^2) \, dy \, dx = \frac{kab}{3}(a^2 + b^2)$   
 $M_x = \int_0^a \int_0^b k(x^2 y + y^3) \, dy \, dx = \frac{kab^2}{12}(2a^2 + 3b^2)$   
 $M_y = \int_0^a \int_0^b k(x^3 + xy^2) \, dy \, dx = \frac{ka^2 b}{12}(3a^2 + 2b^2)$   
 $\bar{x} = \frac{M_y}{m} = \frac{(ka^2 b / 12)(3a^2 + 2b^2)}{(kab / 3)(a^2 + b^2)} = \frac{a(3a^2 + 2b^2)}{4(a^2 + b^2)}$   
 $\bar{y} = \frac{M_x}{m} = \frac{(kab^2 / 12)(2a^2 + 3b^2)}{(kab / 3)(a^2 + b^2)} = \frac{b(2a^2 + 3b^2)}{4(a^2 + b^2)}$   
 $(\bar{x}, \bar{y}) = \left( \frac{a(3a^2 + 2b^2)}{4(a^2 + b^2)}, \frac{b(2a^2 + 3b^2)}{4(a^2 + b^2)} \right)$

(b)  $m = \int_0^a \int_0^y ky \, dx \, dy = \frac{1}{3} ka^3$   
 $M_x = \int_0^a \int_0^y ky^2 \, dx \, dy = \frac{1}{4} ka^4$   
 $M_y = \int_0^a \int_0^y kxy \, dx \, dy = \frac{1}{8} ka^4$   
 $x = \frac{M_y}{m} = \frac{3a}{8}, \bar{y} = \frac{M_x}{m} = \frac{3a}{4}$   
 $(\bar{x}, \bar{y}) = \left( \frac{3a}{8}, \frac{3a}{4} \right)$

8. (a)  $m = \int_0^a \int_{y/2}^{a-y/2} k \, dx \, dy = \frac{1}{2} ka^2$   
 $M_x = \int_0^a \int_{y/2}^{a-y/2} ky \, dx \, dy = \frac{1}{6} ka^3$   
 $M_y = \int_0^a \int_{y/2}^{a-y/2} kx \, dx \, dy = \frac{1}{4} ka^3$   
 $\bar{x} = \frac{M_y}{m} = \frac{a}{2}, \bar{y} = \frac{M_x}{m} = \frac{a}{3}, (\bar{x}, \bar{y}) = \left(\frac{a}{2}, \frac{a}{3}\right)$



(b)  $m = \int_0^a \int_{y/2}^{a-y/2} kxy \, dx \, dy = \frac{1}{12} ka^4$   
 $M_x = \int_0^a \int_{y/2}^{a-y/2} kxy^2 \, dx \, dy = \frac{1}{24} ka^5$   
 $M_y = \int_0^a \int_{y/2}^{a-y/2} kx^2y \, dx \, dy = \frac{11}{240} ka^5$   
 $\bar{x} = \frac{M_y}{m} = \frac{11a}{20}, \bar{y} = \frac{M_x}{m} = \frac{a}{2}, (\bar{x}, \bar{y}) = \left(\frac{11a}{20}, \frac{a}{2}\right)$

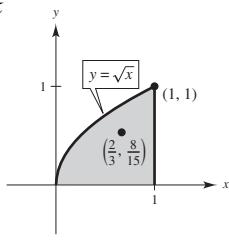
9. (a) The  $x$ -coordinate changes by 5:  $(\bar{x}, \bar{y}) = \left(\frac{a}{2} + 5, \frac{a}{2}\right)$

(b) The  $x$ -coordinate changes by 5:  $(\bar{x}, \bar{y}) = \left(\frac{a}{2} + 5, \frac{2a}{3}\right)$

(c)  $m = \int_5^{a+5} \int_0^a kx \, dy \, dx = \frac{1}{2} ka \left((a+5)^2 - 25\right)$   
 $M_x = \int_5^{a+5} \int_0^a kxy \, dy \, dx = \frac{1}{4} ka^2 \left((a+5)^2 - 25\right)$   
 $M_y = \int_5^{a+5} \int_0^a kx^2 \, dy \, dx = \frac{1}{3} ka \left((a+5)^3 - 125\right)$   
 $\bar{x} = \frac{M_y}{m} = \frac{2 \left[(a+5)^3 - 125\right]}{3 \left[(a+5)^2 - 25\right]} = \frac{2(a^2 + 15a + 75)}{3(a+10)}$   
 $y = \frac{M_x}{m} = \frac{a}{2}$   
 $(\bar{x}, \bar{y}) = \left(\frac{2(a^2 + 15a + 75)}{3(a+10)}, \frac{a}{2}\right)$

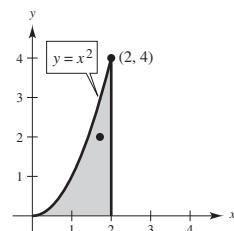
10. The  $x$ -coordinate changes by  $c$  units horizontally and  $d$  units vertically. This is not necessarily true for variable densities. See Exercise 9.

11.  $m = \int_0^1 \int_0^{\sqrt{x}} ky \, dy \, dx = \frac{1}{4} k$   
 $M_x = \int_0^1 \int_0^{\sqrt{x}} ky^2 \, dy \, dx = \frac{2}{15} k$   
 $M_y = \int_0^1 \int_0^{\sqrt{x}} kxy \, dy \, dx = \frac{1}{6} k$   
 $\bar{x} = \frac{M_y}{m} = \frac{2}{3}$   
 $\bar{y} = \frac{M_x}{m} = \frac{8}{15}$   
 $(\bar{x}, \bar{y}) = \left(\frac{2}{3}, \frac{8}{15}\right)$



12.  $m = \int_0^2 \int_0^{x^2} kxy \, dy \, dx = \frac{16}{3} k$

$M_x = \int_0^2 \int_0^{x^2} kxy^2 \, dy \, dx = \frac{32}{3} k$   
 $M_y = \int_0^2 \int_0^{x^2} kx^2y \, dy \, dx = \frac{64}{7} k$   
 $\bar{x} = \frac{M_y}{m} = \frac{12}{7}$   
 $\bar{y} = \frac{M_x}{m} = 2$   
 $(\bar{x}, \bar{y}) = \left(\frac{12}{7}, 2\right)$



13.  $m = \int_1^4 \int_0^{4/x} kx^2 dy dx = 30k$

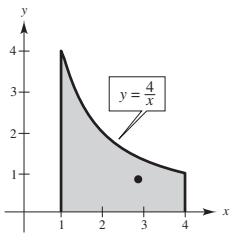
$$M_x = \int_1^4 \int_0^{4/x} kx^2 y dy dx = 24k$$

$$M_y = \int_1^4 \int_0^{4/x} kx^3 dy dx = 84k$$

$$\bar{x} = \frac{M_y}{m} = \frac{84k}{30k} = \frac{14}{5}$$

$$\bar{y} = \frac{M_x}{m} = \frac{24k}{30k} = \frac{4}{5}$$

$$(\bar{x}, \bar{y}) = \left( \frac{14}{5}, \frac{4}{5} \right)$$



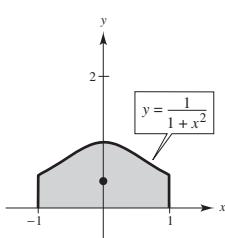
14.  $\bar{x} = 0$  by symmetry

$$m = \int_{-1}^1 \int_0^{\sqrt{1+x^2}} k dy dx = \frac{k\pi}{2}$$

$$M_x = \int_{-1}^1 \int_0^{\sqrt{1+x^2}} ky dy dx = \frac{k}{8}(2 + \pi)$$

$$\bar{y} = \frac{M_x}{m} = \frac{k}{8}(2 + \pi) \cdot \frac{2}{k\pi} = \frac{2 + \pi}{4\pi}$$

$$(\bar{x}, \bar{y}) = \left( 0, \frac{2 + \pi}{4\pi} \right)$$



15. (a)  $m = \int_0^1 \int_0^{e^x} k dy dx = k(e - 1)$

$$M_x = \int_0^1 \int_0^{e^x} ky dy dx = \frac{1}{4}k(e^2 - 1)$$

$$M_y = \int_0^1 \int_0^{e^x} kx dy dx = k$$

$$\bar{x} = \frac{M_y}{m} = \frac{1}{e - 1},$$

$$\bar{y} = \frac{M_x}{m} = \frac{e^2 - 1}{4(e - 1)} = \frac{e + 1}{4},$$

$$(\bar{x}, \bar{y}) = \left( \frac{1}{e - 1}, \frac{e + 1}{4} \right)$$

(b)  $m = \int_0^1 \int_0^{e^x} ky dy dx = \frac{e^2 - 1}{4}k$

$$M_x = \int_0^1 \int_0^{e^x} ky^2 dy dx = \frac{e^3 - 1}{9}k$$

$$M_y = \int_0^1 \int_0^{e^x} kxy dy dx = \frac{e^2 + 1}{8}k$$

$$\bar{x} = \frac{M_y}{m} = \frac{e^2 + 1}{2(e^2 - 1)}, \bar{y} = \frac{M_x}{m} = \frac{4(e^3 - 1)}{9(e^2 - 1)}$$

$$(\bar{x}, \bar{y}) = \left( \frac{e^2 + 1}{2(e^2 - 1)}, \frac{4(e^3 - 1)}{9(e^2 - 1)} \right)$$

16. (a)  $m = \int_0^1 \int_0^{e^{-x}} ky dy dx = \frac{1}{4}(1 - e^{-2})k$

$$M_x = \int_0^1 \int_0^{e^{-x}} ky^2 dy dx = \frac{1}{9}(1 - e^{-3})k$$

$$M_y = \int_0^1 \int_0^{e^{-x}} kxy dy dx = \frac{1}{8}(1 - 3e^{-2})k$$

$$\bar{x} = \frac{M_y}{m} = \frac{1 - 3e^{-2}}{2(1 - e^{-2})}$$

$$\bar{y} = \frac{M_x}{m} = \frac{4(1 - e^{-3})}{9(1 - e^{-2})}$$

$$(\bar{x}, \bar{y}) = \left( \frac{1 - 3e^{-2}}{2(1 - e^{-2})}, \frac{4(1 - e^{-3})}{9(1 - e^{-2})} \right)$$

(b)  $m = \int_0^1 \int_0^{e^{-x}} ky^2 dy dx = \frac{1}{9}(1 - e^{-3})k$

$$M_x = \int_0^1 \int_0^{e^{-x}} ky^3 dy dx = \frac{1}{16}(1 - e^{-4})k$$

$$M_y = \int_0^1 \int_0^{e^{-x}} kxy^2 dy dx = \frac{1}{27}(1 - 4e^{-3})k$$

$$\bar{x} = \frac{M_y}{m} = \frac{1 - 4e^{-3}}{3(1 - e^{-3})}$$

$$\bar{y} = \frac{M_x}{m} = \frac{9(1 - e^{-4})}{16(1 - e^{-3})}$$

$$(\bar{x}, \bar{y}) = \left( \frac{1 - 4e^{-3}}{3(1 - e^{-3})}, \frac{9(1 - e^{-4})}{16(1 - e^{-3})} \right)$$

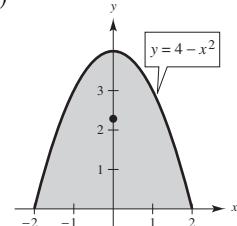
17.  $m = \int_{-2}^2 \int_0^{4-x^2} kx dy dx = \frac{256}{15}k$

$$M_x = \int_{-2}^2 \int_0^{4-x^2} kxy dy dx = \frac{4096}{105}k$$

$\bar{x} = 0$  (by symmetry)

$$\bar{y} = \frac{M_x}{m} = \frac{16}{7}$$

$$(\bar{x}, \bar{y}) = \left( 0, \frac{16}{7} \right)$$



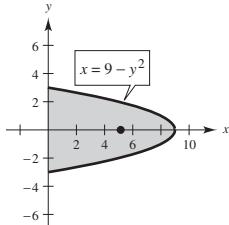
18.  $m = \int_{-3}^3 \int_0^{9-y^2} kx \, dx \, dy = \frac{648k}{5}$

$$M_x = \int_{-3}^3 \int_0^{9-y^2} kxy \, dx \, dy = 0 \quad (\text{by symmetry})$$

$$M_y = \int_{-3}^3 \int_0^{9-y^2} kx^2 \, dx \, dy = \frac{23,328k}{35}$$

$$\bar{x} = \frac{M_y}{m} = \frac{36}{7}, \quad \bar{y} = 0$$

$$(\bar{x}, \bar{y}) = \left( \frac{36}{7}, 0 \right)$$



21.  $m = \frac{\pi a^2 k}{8}$

$$M_x = \int_R \int ky \, dA = \int_0^{\pi/4} \int_0^a kr^2 \sin \theta \, dr \, d\theta = \frac{ka^3(2 - \sqrt{2})}{6}$$

$$M_y = \int_R \int kx \, dA = \int_0^{\pi/4} \int_0^a kr^2 \cos \theta \, dr \, d\theta = \frac{ka^3\sqrt{2}}{6}$$

$$\bar{x} = \frac{M_y}{m} = \frac{ka^3\sqrt{2}}{6} \cdot \frac{8}{\pi a^2 k} = \frac{4a\sqrt{2}}{3\pi}$$

$$\bar{y} = \frac{M_x}{m} = \frac{ka^3(2 - \sqrt{2})}{6} \cdot \frac{8}{\pi a^2 k} = \frac{4a(2 - \sqrt{2})}{3\pi}$$

$$(\bar{x}, \bar{y}) = \left( \frac{4a\sqrt{2}}{3\pi}, \frac{4a(2 - \sqrt{2})}{3\pi} \right)$$

22.  $m = \int_0^a \int_0^{\sqrt{a^2-x^2}} k(x^2 + y^2) \, dy \, dx = \int_0^{\pi/2} \int_0^a kr^3 \, dr \, d\theta = \frac{ka^4\pi}{8}$

$$M_x = \int_0^a \int_0^{\sqrt{a^2-x^2}} k(x^2 + y^2)y \, dy \, dx = \int_0^{\pi/2} \int_0^a kr^4 \sin \theta \, dr \, d\theta = \frac{ka^5}{5}$$

$M_y = M_x$  by symmetry

$$\bar{x} = \bar{y} = \frac{M_y}{m} = \frac{ka^5}{5} \cdot \frac{8}{ka^4\pi} = \frac{8a}{5\pi}$$

$$(\bar{x}, \bar{y}) = \left( \frac{8a}{5\pi}, \frac{8a}{5\pi} \right)$$

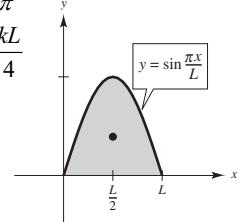
19.  $\bar{x} = \frac{L}{2}$  (by symmetry)

$$m = \int_0^L \int_0^{\sin(\pi x/L)} k \, dy \, dx = \frac{2kL}{\pi}$$

$$M_x = \int_0^L \int_0^{\sin(\pi x/L)} ky \, dy \, dx = \frac{kL}{4}$$

$$\bar{y} = \frac{M_x}{m} = \frac{\pi}{8}$$

$$(\bar{x}, \bar{y}) = \left( \frac{L}{2}, \frac{\pi}{8} \right)$$



20.  $m = \int_0^{L/2} \int_0^{\cos(\pi x/L)} ky \, dy \, dx = \frac{kL}{8}$

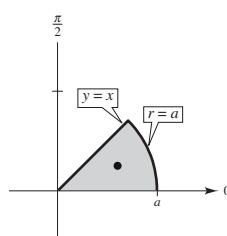
$$M_x = \int_0^{L/2} \int_0^{\cos(\pi y/L)} ky^2 \, dy \, dx = \frac{2kL}{9\pi}$$

$$M_y = \int_0^{L/2} \int_0^{\cos(\pi x/L)} kxy \, dy \, dx = \frac{L^2 k (\pi^2 - 4)}{32\pi^2}$$

$$\bar{x} = \frac{M_y}{m} = \frac{L(\pi^2 - 4)}{4\pi^2}$$

$$\bar{y} = \frac{M_x}{m} = \frac{16}{9\pi}$$

$$(\bar{x}, \bar{y}) = \left( \frac{L(\pi^2 - 4)}{4\pi^2}, \frac{16}{9\pi} \right)$$



23.  $m = \int_0^2 \int_0^{e^{-x}} kxy \, dy \, dx = \frac{1 - 5e^{-4}}{8} k$

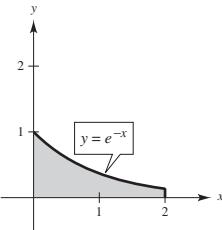
$$M_x = \int_0^2 \int_0^{e^{-x}} kxy^2 \, dy \, dx = \frac{1 - 7e^{-6}}{27} k$$

$$M_y = \int_0^2 \int_0^{e^{-x}} kx^2y \, dy \, dx = \frac{1 - 13e^{-4}}{8} k$$

$$\bar{x} = \frac{M_y}{m} = \frac{e^4 - 13}{e^4 - 5}$$

$$\bar{y} = \frac{M_x}{m} = \frac{8(e^6 - 7)}{27(e^6 - 5e^2)}$$

$$(\bar{x}, \bar{y}) = \left( \frac{e^4 - 13}{e^4 - 5}, \frac{8(e^6 - 7)}{27(e^6 - 5e^2)} \right)$$



24.  $m = \int_1^e \int_0^{\ln x} \frac{k}{x} \, dy \, dx = \frac{k}{2}$

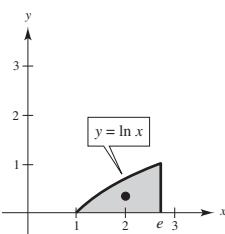
$$M_x = \int_1^e \int_0^{\ln x} \frac{k}{x} y \, dy \, dx = \frac{k}{6}$$

$$M_y = \int_1^e \int_0^{\ln x} \frac{k}{x} x \, dy \, dx = k$$

$$\bar{x} = \frac{M_y}{m} = \frac{k}{1} \cdot \frac{2}{k} = 2$$

$$\bar{y} = \frac{M_x}{m} = \frac{k}{6} \cdot \frac{2}{k} = \frac{1}{3}$$

$$(\bar{x}, \bar{y}) = \left( 2, \frac{1}{3} \right)$$



26.  $\bar{y} = 0$  by symmetry

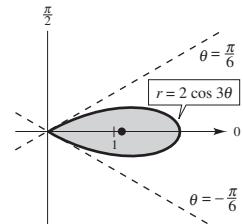
$$m = \int_R \int k \, dA = \int_0^{2\pi} \int_0^{1+\cos\theta} kr \, dr \, d\theta = \frac{3\pi k}{2}$$

$$M_y = \int_R \int kx \, dA = \int_0^{2\pi} \int_0^{1+\cos\theta} kr^2 \cos\theta \, dr \, d\theta = \frac{k}{3} \int_0^{2\pi} \cos\theta (1 + 3\cos\theta + 3\cos^2\theta + \cos^3\theta) \, d\theta$$

$$= \frac{k}{3} \int_0^{2\pi} \left[ \cos\theta + \frac{3}{2}(1 + \cos^2\theta) + 3\cos\theta(1 - \sin^2\theta) + \frac{1}{4}(1 + \cos 2\theta)^2 \right] \, d\theta = \frac{5k\pi}{4}$$

$$\bar{x} = \frac{M_y}{m} = \frac{5k\pi}{4} \cdot \frac{2}{3k\pi} = \frac{5}{6}$$

$$(\bar{x}, \bar{y}) = \left( \frac{5}{6}, 0 \right)$$



27.  $m = bh$

$$I_x = \int_0^b \int_0^h y^2 \, dy \, dx = \frac{bh^3}{3}$$

$$I_y = \int_0^b \int_0^h x^2 \, dy \, dx = \frac{b^3 h}{3}$$

$$\bar{x} = \sqrt{\frac{I_y}{m}} = \sqrt{\frac{b^3 h}{3} \cdot \frac{1}{bh}} = \sqrt{\frac{b^2}{3}} = \frac{b}{\sqrt{3}} = \frac{\sqrt{3}}{3} b$$

$$\bar{y} = \sqrt{\frac{I_x}{m}} = \sqrt{\frac{bh^3}{3} \cdot \frac{1}{bh}} = \sqrt{\frac{h^2}{3}} = \frac{h}{\sqrt{3}} = \frac{\sqrt{3}}{3} h$$

25.  $\bar{y} = 0$  by symmetry

$$m = \int_R \int k \, dA = \int_{-\pi/6}^{\pi/6} \int_0^{2\cos 3\theta} kr \, dr \, d\theta = \frac{k\pi}{3}$$

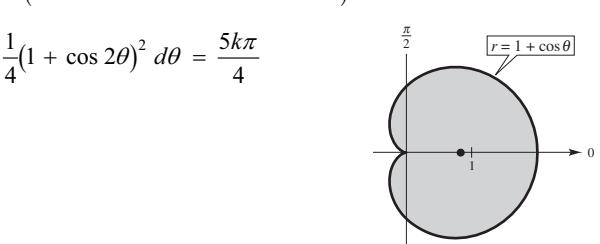
$$M_y = \int_R \int kx \, dA$$

$$= \int_{-\pi/6}^{\pi/6} \int_0^{2\cos 3\theta} kr^2 \cos\theta \, dr \, d\theta$$

$$= \frac{27\sqrt{3}}{40} k \approx 1.17k$$

$$\bar{x} = \frac{M_y}{m} = \frac{81\sqrt{3}}{40\pi} \approx 1.12$$

$$(\bar{x}, \bar{y}) \approx (1.12, 0)$$



28.  $m = \int_0^b \int_0^{h-(hx/b)} dy \, dx = \frac{bh}{2}$

$$I_x = \int_0^b \int_0^{h-(hx/b)} y^2 \, dy \, dx = \frac{bh^3}{12}$$

$$I_y = \int_0^b \int_0^{h-(hx/b)} x^2 \, dy \, dx = \frac{b^3 h}{12}$$

$$\bar{x} = \sqrt{\frac{I_y}{m}} = \sqrt{\frac{b^3 h / 12}{bh/2}} = \frac{b}{\sqrt{6}} = \frac{\sqrt{6}}{6} b$$

$$\bar{y} = \sqrt{\frac{I_x}{m}} = \sqrt{\frac{bh^3 / 12}{bh/2}} = \frac{h}{\sqrt{6}} = \frac{\sqrt{6}}{6} h$$

**29.**  $m = \pi a^2$

$$I_x = \int_R \int y^2 dA = \int_0^{2\pi} \int_0^a r^3 \sin^2 \theta dr d\theta = \frac{a^4 \pi}{4}$$

$$I_y = \int_R \int x^2 dA = \int_0^{2\pi} \int_0^a r^3 \cos^2 \theta dr d\theta = \frac{a^4 \pi}{4}$$

$$I_0 = I_x + I_y = \frac{a^4 \pi}{4} + \frac{a^4 \pi}{4} = \frac{a^4 \pi}{2}$$

$$\bar{x} = \bar{y} = \sqrt{\frac{I_x}{m}} = \sqrt{\frac{a^4 \pi}{4} \cdot \frac{1}{\pi a^2}} = \frac{a}{2}$$

**30.**  $m = \frac{\pi a^2}{2}$

$$I_x = \int_R \int y^2 dA = \int_0^\pi \int_0^a r^3 \sin^2 \theta dr d\theta = \frac{a^4 \pi}{8}$$

$$I_y = \int_R \int x^2 dA = \int_0^\pi \int_0^a r^3 \cos^2 \theta dr d\theta = \frac{a^4 \pi}{8}$$

$$I_0 = I_x + I_y = \frac{a^4 \pi}{8} + \frac{a^4 \pi}{8} = \frac{a^4 \pi}{4}$$

$$\bar{x} = \bar{y} = \sqrt{\frac{I_x}{m}} = \sqrt{\frac{a^4 \pi}{8} \cdot \frac{2}{\pi a^2}} = \frac{a}{2}$$

**32.**  $m = \pi ab$

$$I_x = 4 \int_0^a \int_0^{(b/a)\sqrt{a^2-x^2}} y^2 dy dx = 4 \int_0^a \frac{b^3}{3a^3} (a^2 - x^2)^{3/2} dx = \frac{4b^3}{3a^3} \int_0^a \left[ a^2 \sqrt{a^2 - x^2} - x^2 \sqrt{a^2 - x^2} \right] dx \\ = \frac{4b^3}{3a^3} \left[ \frac{a^2}{2} \left( x \sqrt{a^2 - x^2} + a^2 \arcsin \frac{x}{a} \right) - \frac{1}{8} \left[ x(2x^2 - a^2) \sqrt{a^2 - x^2} + a^4 \arcsin \frac{x}{a} \right] \right]_0^a = \frac{ab^3 \pi}{4}$$

$$I_y = 4 \int_0^b \int_0^{(a/b)\sqrt{b^2-y^2}} x^2 dy dx = \frac{a^3 b \pi}{4}$$

$$I_0 = I_y + I_x = \frac{a^3 b \pi}{4} + \frac{ab^3 \pi}{4} = \frac{ab \pi}{4} (a^2 + b^2)$$

$$\bar{x} = \sqrt{\frac{I_y}{m}} = \sqrt{\frac{a^3 b \pi}{4} \cdot \frac{1}{\pi ab}} = \frac{a}{2}$$

$$\bar{y} = \sqrt{\frac{I_x}{m}} = \sqrt{\frac{ab^3 \pi}{4} \cdot \frac{1}{\pi ab}} = \frac{b}{2}$$

**33.**  $\rho = ky$

$$m = k \int_0^a \int_0^b y dy dx = \frac{kab^2}{2}$$

$$I_x = k \int_0^a \int_0^b y^3 dy dx = \frac{kab^4}{4}$$

$$I_y = k \int_0^a \int_0^b x^2 yy dy dx = \frac{ka^3 b^2}{6}$$

$$I_0 = I_x + I_y = \frac{3kab^4 + 2kb^2 a^3}{12}$$

$$\bar{x} = \sqrt{\frac{I_y}{m}} = \sqrt{\frac{ka^3 b^2 / 6}{kab^2 / 2}} = \sqrt{\frac{a^2}{3}} = \frac{a}{\sqrt{3}} = \frac{\sqrt{3}}{3} a$$

$$\bar{y} = \sqrt{\frac{I_x}{m}} = \sqrt{\frac{kab^4 / 4}{kab^2 / 2}} = \sqrt{\frac{b^2}{2}} = \frac{b}{\sqrt{2}} = \frac{\sqrt{2}}{2} b$$

**31.**  $m = \frac{\pi a^2}{4}$

$$I_x = \int_R \int y^2 dA = \int_0^{\pi/2} \int_0^a r^3 \sin^2 \theta dr d\theta = \frac{\pi a^4}{16}$$

$$I_y = \int_R \int x^2 dA = \int_0^{\pi/2} \int_0^a r^3 \cos^2 \theta dr d\theta = \frac{\pi a^4}{16}$$

$$I_0 = I_x + I_y = \frac{\pi a^4}{16} + \frac{\pi a^4}{16} = \frac{\pi a^4}{8}$$

$$\bar{x} = \bar{y} = \sqrt{\frac{I_x}{m}} = \sqrt{\frac{\pi a^4}{16} \cdot \frac{4}{\pi a^2}} = \frac{a}{2}$$

**34.**  $\rho = ky$

$$m = 2k \int_0^a \int_0^{\sqrt{a^2-x^2}} y dy dx = k \int_0^a (a^2 - x^2) dx = \frac{2ka^3}{3}$$

$$I_x = k \int_{-a}^a \int_0^{\sqrt{a^2-x^2}} y^3 dy dx = \frac{4ka^5}{15}$$

$$I_y = k \int_{-a}^a \int_0^{\sqrt{a^2-x^2}} x^2 y dy dx = \frac{2ka^5}{15}$$

$$I_0 = I_x + I_y = \frac{2ka^5}{5}$$

$$\bar{x} = \sqrt{\frac{I_y}{m}} = \sqrt{\frac{2ka^5 / 15}{2ka^3 / 3}} = \sqrt{\frac{a^2}{5}} = \frac{a\sqrt{5}}{5}$$

$$\bar{y} = \sqrt{\frac{I_x}{m}} = \sqrt{\frac{4ka^5 / 15}{2ka^3 / 3}} = \sqrt{\frac{2a^2}{5}} = \frac{a\sqrt{10}}{5}$$

35.  $\rho = kx$

$$m = k \int_0^2 \int_0^{4-x^2} x \, dy \, dx = 4k$$

$$I_x = k \int_0^2 \int_0^{4-x^2} xy^2 \, dy \, dx = \frac{32k}{3}$$

$$I_y = k \int_0^2 \int_0^{4-x^2} x^3 \, dy \, dx = \frac{16k}{3}$$

$$I_0 = I_x + I_y = 16k$$

$$\bar{x} = \sqrt{\frac{I_y}{m}} = \sqrt{\frac{16k/3}{4k}} = \sqrt{\frac{4}{3}} = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$$

$$\bar{y} = \sqrt{\frac{I_x}{m}} = \sqrt{\frac{32k/3}{4k}} = \sqrt{\frac{8}{3}} = \frac{4}{\sqrt{6}} = \frac{2\sqrt{6}}{3}$$

36.  $\rho = kxy$

$$m = k \int_0^1 \int_{x^2}^x xy \, dy \, dx = \frac{k}{2} \int_0^1 (x^3 - x^5) \, dx = \frac{k}{24}$$

$$I_x = k \int_0^1 \int_{x^2}^x xy^3 \, dy \, dx = \frac{k}{4} \int_0^1 (x^5 - x^9) \, dx = \frac{k}{60}$$

$$I_y = k \int_0^1 \int_{x^2}^x x^3 y \, dy \, dx = \frac{k}{2} \int_0^1 (x^5 - x^7) \, dx = \frac{k}{48}$$

$$I_0 = I_x + I_y = \frac{9k}{240} = \frac{3k}{80}$$

$$\bar{x} = \sqrt{\frac{I_y}{m}} = \sqrt{\frac{k/48}{k/24}} = \frac{\sqrt{2}}{2}$$

$$\bar{y} = \sqrt{\frac{I_x}{m}} \sqrt{\frac{k/60}{k/24}} = \frac{\sqrt{10}}{5}$$

37.  $\rho = kxy$

$$m = \int_0^4 \int_0^{\sqrt{x}} kxy \, dy \, dx = \frac{32k}{3}$$

$$I_x = \int_0^4 \int_0^{\sqrt{x}} kxy^3 \, dy \, dx = 16k$$

$$I_y = \int_0^4 \int_0^{\sqrt{x}} kx^3 y \, dy \, dx = \frac{512k}{5}$$

$$I_0 = I_x + I_y = \frac{592k}{5}$$

$$\bar{x} = \sqrt{\frac{I_y}{m}} = \sqrt{\frac{512k}{5} \cdot \frac{3}{32k}} = \sqrt{\frac{48}{5}} = \frac{4\sqrt{15}}{5}$$

$$\bar{y} = \sqrt{\frac{I_x}{m}} = \sqrt{\frac{16k}{1} \cdot \frac{3}{32k}} = \sqrt{\frac{3}{2}} = \frac{\sqrt{6}}{2}$$

41.  $I = 2k \int_{-b}^b \int_0^{\sqrt{b^2-x^2}} (x-a)^2 \, dy \, dx = 2k \int_{-b}^b (x-a)^2 \sqrt{b^2-x^2} \, dx$

$$= 2k \left[ \int_{-b}^b x^2 \sqrt{b^2-x^2} \, dx - 2a \int_{-b}^b x \sqrt{b^2-x^2} \, dx + a^2 \int_{-b}^b \sqrt{b^2-x^2} \, dx \right] = 2k \left[ \frac{\pi b^4}{8} + 0 + \frac{\pi a^2 b^2}{2} \right] = \frac{k\pi b^2}{4} (b^2 + 4a^2)$$

42.  $I = \int_0^4 \int_0^2 k(x-6)^2 \, dy \, dx = \int_0^4 2k(x-6)^2 \, dx = \left[ \frac{2k}{3}(x-6)^3 \right]_0^4 = \frac{416k}{3}$

38.  $\rho = x^2 + y^2$

$$m = \int_0^1 \int_{x^2}^{\sqrt{x}} (x^2 + y^2) \, dy \, dx = \frac{6}{35}$$

$$I_x = \int_0^1 \int_{x^2}^{\sqrt{x}} (x^2 + y^2)y^2 \, dy \, dx = \frac{158}{2079}$$

$$I_y = \int_0^1 \int_{x^2}^{\sqrt{x}} (x^2 + y^2)x^2 \, dy \, dx = \frac{158}{2079}$$

$$I_0 = I_x + I_y = \frac{316}{2079}$$

$$\bar{x} = \sqrt{\frac{I_y}{m}} = \sqrt{\frac{158}{2079} \cdot \frac{35}{6}} = \sqrt{\frac{395}{891}} = \frac{\sqrt{351,945}}{891}$$

$$\bar{y} = \sqrt{\frac{I_x}{m}} = \bar{x} = \frac{\sqrt{351,945}}{891}$$

39.  $\rho = kx$

$$m = \int_0^1 \int_{x^2}^{\sqrt{x}} kx \, dy \, dx = \frac{3k}{20}$$

$$I_x = \int_0^1 \int_{x^2}^{\sqrt{x}} kxy^2 \, dy \, dx = \frac{3k}{56}$$

$$I_y = \int_0^1 \int_{x^2}^{\sqrt{x}} kx^3 \, dy \, dx = \frac{k}{18}$$

$$I_0 = I_x + I_y = \frac{55k}{504}$$

$$\bar{x} = \sqrt{\frac{I_y}{m}} = \sqrt{\frac{k}{18} \cdot \frac{20}{3k}} = \frac{\sqrt{30}}{9}$$

$$\bar{y} = \sqrt{\frac{I_x}{m}} = \sqrt{\frac{3k}{56} \cdot \frac{20}{3k}} = \frac{\sqrt{70}}{14}$$

40.  $\rho = ky$

$$m = 2 \int_0^2 \int_{x^3}^{4x} ky \, dy \, dx = \frac{512k}{21}$$

$$I_x = 2 \int_0^2 \int_{x^3}^{4x} ky^3 \, dy \, dx = \frac{32,768k}{65}$$

$$I_y = 2 \int_0^2 \int_{x^3}^{4x} kx^2 y \, dy \, dx = \frac{2048k}{45}$$

$$I_0 = I_x + I_y = \frac{321,536k}{585}$$

$$\bar{x} = \sqrt{\frac{I_y}{m}} = \sqrt{\frac{2048k}{45} \cdot \frac{21}{512k}} = \sqrt{\frac{28}{15}} = \frac{2\sqrt{105}}{15}$$

$$\bar{y} = \sqrt{\frac{I_x}{m}} = \sqrt{\frac{32,768k}{65} \cdot \frac{21}{512k}} = \frac{8\sqrt{1365}}{65}$$

$$43. I = \int_0^4 \int_0^{\sqrt{x}} kx(x-6)^2 dy dx = \int_0^4 kx\sqrt{x}(x^2 - 12x + 36) dx = k \left[ \frac{2}{9}x^{9/2} - \frac{24}{7}x^{7/2} + \frac{72}{5}x^{5/2} \right]_0^4 = \frac{42,752k}{315}$$

$$\begin{aligned} 44. I &= \int_{-a}^a \int_0^{\sqrt{a^2-x^2}} ky(y-a)^2 dy dx \\ &= \int_{-a}^a k \left[ \frac{y^4}{4} - \frac{2ay^3}{3} + \frac{a^2y^2}{2} \right]_0^{\sqrt{a^2-x^2}} dx \\ &= \int_{-a}^a k \left[ \frac{1}{4}(a^4 - 2a^2x^2 + x^4) - \frac{2a}{3}(a^2\sqrt{a^2-x^2} - x^2\sqrt{a^2-x^2}) + \frac{a^2}{2}(a^2 - x^2) \right] dx \\ &= k \left[ \frac{1}{4} \left( a^4x - \frac{2a^2x^3}{3} + \frac{x^5}{5} \right) - \frac{2a}{3} \left( \frac{a^2}{2} \left( x\sqrt{a^2-x^2} + a^2 \arcsin \frac{x}{a} \right) \right. \right. \\ &\quad \left. \left. - \frac{1}{8} \left( x(2x^2 - a^2)\sqrt{a^2-x^2} + a^4 \arcsin \frac{x}{a} \right) \right) + \frac{a^2}{2} \left( a^2x - \frac{x^3}{3} \right) \right]_{-a}^a \\ &= 2k \left[ \frac{1}{4} \left( a^5 - \frac{2}{3}a^5 + \frac{1}{5}a^5 \right) - \frac{2a}{3} \left( \frac{a^4\pi}{4} - \frac{a^4\pi}{16} \right) + \frac{a^2}{2} \left( a^3 - \frac{a^3}{3} \right) \right] = 2k \left( \frac{7a^5}{15} - \frac{a^5\pi}{8} \right) = ka^5 \left( \frac{56 - 15\pi}{60} \right) \end{aligned}$$

$$\begin{aligned} 45. I &= \int_0^a \int_0^{\sqrt{a^2-x^2}} k(a-y)(y-a)^2 dy dx = \int_0^a \int_0^{\sqrt{a^2-x^2}} k(a-y)^3 dy dx = \int_0^a \left[ -\frac{k}{4}(a-y)^4 \right]_0^{\sqrt{a^2-x^2}} dx \\ &= -\frac{k}{4} \int_0^a \left[ a^4 - 4a^3y + 6a^2y^2 - 4ay^3 + y^4 \right]_0^{\sqrt{a^2-x^2}} dx \\ &= -\frac{k}{4} \int_0^a \left[ a^4 - 4a^3\sqrt{a^2-x^2} + 6a^2(a^2-x^2) - 4a(a^2-x^2)\sqrt{a^2-x^2} + (a^4 - 2a^2x^2 + x^4) - a^4 \right] dx \\ &= -\frac{k}{4} \int_0^a \left[ 7a^4 - 8a^2x^2 + x^4 - 8a^3\sqrt{a^2-x^2} + 4ax^2\sqrt{a^2-x^2} \right] dx \\ &= -\frac{k}{4} \left[ 7a^4x - \frac{8a^2}{3}x^3 + \frac{x^5}{5} - 4a^3 \left( x\sqrt{a^2-x^2} + a^2 \arcsin \frac{x}{a} \right) + \frac{a}{2} \left( x(2x^2 - a^2)\sqrt{a^2-x^2} + a^4 \arcsin \frac{x}{a} \right) \right]_0^a \\ &= -\frac{k}{4} \left( 7a^5 - \frac{8}{3}a^5 + \frac{1}{5}a^5 - 2a^5\pi + \frac{1}{4}a^5\pi \right) = a^5k \left( \frac{7\pi}{16} - \frac{17}{15} \right) \end{aligned}$$

$$\begin{aligned} 46. I &= \int_{-2}^2 \int_0^{4-x^2} k(y-2)^2 dy dx = \int_{-2}^2 \left[ \frac{k}{3}(y-1)^3 \right]_0^{4-x^2} dx = \int_{-2}^2 \frac{k}{3}[(2-x^2) + 8] dx \\ &= \frac{k}{3} \int_{-2}^2 (16 - 12x^2 + 6x^4 - x^6) dx = \left[ \frac{k}{3} \left( 16x - 4x^3 + \frac{6}{5}x^5 - \frac{1}{7}x^7 \right) \right]_{-2}^2 = \frac{2k}{3} \left( 32 - 32 + \frac{192}{5} - \frac{128}{7} \right) = \frac{1408k}{105} \end{aligned}$$

47. Let  $\rho(x, y)$  be a continuous density function on the planar lamina  $R$ . The movements of mass with respect to the  $x$ - and  $y$ -axes are

$$M_x = \int_R \int y \rho(x, y) dA \text{ and } M_y = \int_R \int x \rho(x, y) dA.$$

If  $m$  is the mass of the lamina, then the center of mass is

$$(\bar{x}, \bar{y}) = \left( \frac{M_y}{m}, \frac{M_x}{m} \right).$$

48.  $I_x = \int_R \int y^2 \rho(x, y) dA$ , Moment of inertia about  $x$ -axis

$$I_y = \int_R \int x^2 \rho(x, y) dA$$
, Moment of inertia about  $y$ -axis

49. See the definition on page 1017.

50. (a)  $\rho(x, y) = ky$

$\bar{y}$  will increase.

(b)  $\rho(x, y) = k|2-x|$

$(\bar{x}, \bar{y})$  will be the same.

(c)  $\rho(x, y) = kxy$

Both  $\bar{x}$  and  $\bar{y}$  will increase.

(d)  $\rho(x, y) = k(4-x)(4-y)$

Both  $\bar{x}$  and  $\bar{y}$  will decrease.

51.  $\bar{y} = \frac{L}{2}$ ,  $A = bL$ ,  $h = \frac{L}{2}$

$$\begin{aligned} I_{\bar{y}} &= \int_0^b \int_0^L \left( y - \frac{L}{2} \right)^2 dy dx \\ &= \int_0^b \left[ \frac{[y - (L/2)]^3}{3} \right]_0^L dx = \frac{L^3 b}{12} \\ y_a &= \bar{y} - \frac{I_{\bar{y}}}{hA} = \frac{L}{2} - \frac{L^3 b / 12}{(L/2)(bL)} = \frac{L}{3} \end{aligned}$$

52.  $\bar{y} = \frac{a}{2}$ ,  $A = ab$ ,  $h = L - \frac{a}{2}$

$$\begin{aligned} I_{\bar{y}} &= \int_0^b \int_0^a \left( y - \frac{a}{2} \right)^2 dy dx = \frac{a^3 b}{12} \\ y_a &= \frac{a}{2} - \frac{a^3 b / 12}{[L - (a/2)]ab} = \frac{a(3L - 2a)}{3(2L - a)} \end{aligned}$$

53.  $\bar{y} = \frac{2L}{3}$ ,  $A = \frac{bL}{2}$ ,  $h = \frac{L}{3}$

$$\begin{aligned} I_{\bar{y}} &= 2 \int_0^{b/2} \int_{2Lx/b}^L \left( y - \frac{2L}{3} \right)^2 dy dx \\ &= \frac{2}{3} \int_0^{b/2} \left[ \left( y - \frac{2L}{3} \right)^3 \right]_{2Lx/b}^L dx \\ &= \frac{2}{3} \int_0^{b/2} \left[ \frac{L}{27} - \left( \frac{2Lx}{b} - \frac{2L}{3} \right)^3 \right] dx \\ &= \frac{2}{3} \left[ \frac{L^3 x}{27} - \frac{b}{8L} \left( \frac{2Lx}{b} - \frac{2L}{3} \right)^4 \right]_0^{b/2} = \frac{L^3 b}{36} \end{aligned}$$

$$y_a = \frac{2L}{3} - \frac{L^3 b / 36}{L^2 b / 6} = \frac{L}{2}$$

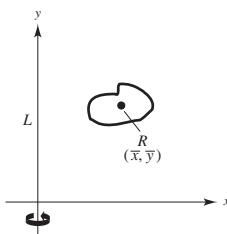
54.  $\bar{y} = 0$ ,  $A = \pi a^2$ ,  $h = L$

$$\begin{aligned} I_{\bar{y}} &= \int_{-a}^a \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} y^2 dy dx = \int_0^{2\pi} \int_0^a r^3 \sin^2 \theta dr d\theta \\ &= \int_0^{2\pi} \frac{a^4}{4} \sin^2 \theta d\theta = \frac{a^4 \pi}{4} \\ y_a &= -\frac{(a^4 \pi / 4)}{L \pi a^2} = -\frac{a^2}{4L} \end{aligned}$$

55. Orient the  $xy$ -coordinate system so that  $L$  is along the  $y$ -axis and  $R$  is the first quadrant. Then the volume of the solid is

$$\begin{aligned} V &= \int_R \int 2\pi x \, dA \\ &= 2\pi \int_R \int x \, dA \\ &= 2\pi \left( \int_R \int x \, dA \right) \int_R \int dA \\ &= 2\pi \bar{x} A. \end{aligned}$$

By our positioning,  $\bar{x} = r$ . So,  $V = 2\pi rA$ .



## Section 14.5 Surface Area

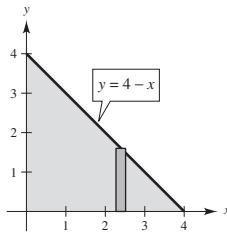
1.  $f(x, y) = 2x + 2y$

$$f_x = f_y = 2$$

$$\sqrt{1 + (f_x)^2 + (f_y)^2} = \sqrt{1 + 4 + 4} = 3$$

$$S = \int_0^4 \int_0^{4-x} 3 \, dy \, dx = 3 \int_0^4 (4 - x) \, dx$$

$$= 3 \left[ 4x - \frac{x^2}{2} \right]_0^4 = 24$$

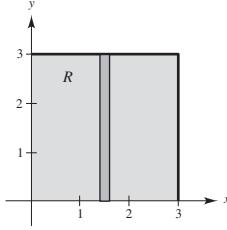


2.  $f(x, y) = 15 + 2x - 3y$

$$f_x = 2, f_y = -3$$

$$\sqrt{1 + (f_x)^2 + (f_y)^2} = \sqrt{14}$$

$$S = \int_0^3 \int_0^3 \sqrt{14} \, dy \, dx = \int_0^3 3\sqrt{14} \, dx = 9\sqrt{14}$$

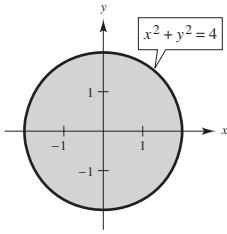


3.  $f(x, y) = 7 + 2x + 2y$

$$f_x = f_y = 2$$

$$\sqrt{1 + (f_x)^2 + (f_y)^2} = \sqrt{1 + 4 + 4} = 3$$

$$S = \int_0^{2\pi} \int_0^2 3r dr d\theta = \int_0^{2\pi} 6 d\theta = 12\pi$$

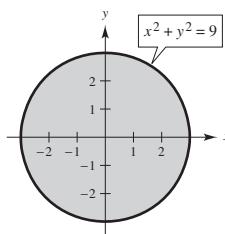


4.  $f(x, y) = 12 + 2x - 3y$

$$f_x = 2, f_y = -3$$

$$\sqrt{1 + (f_x)^2 + (f_y)^2} = \sqrt{1 + 4 + 9} = \sqrt{14}$$

$$S = \int_0^{2\pi} \int_0^3 \sqrt{14} r dr d\theta = \int_0^{2\pi} \frac{9\sqrt{14}}{2} d\theta = 9\sqrt{14} \pi$$



5.  $f(x, y) = 9 - x^2$

$$f_x = -2x, f_y = 0$$

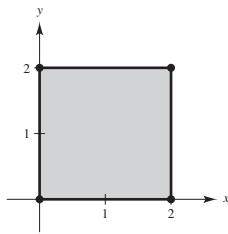
$$\sqrt{1 + (f_x)^2 + (f_y)^2} = \sqrt{1 + 4x^2}$$

$$S = \int_0^2 \int_0^2 \sqrt{1 + 4x^2} dy dx = 2 \int_0^2 \sqrt{1 + 4x^2} dx$$

$$= 2 \left[ \frac{1}{4} \ln(\sqrt{1 + 4x^2} + 2x) + \frac{x}{2} \sqrt{1 + 4x^2} \right]_0^2$$

$$= 2 \left[ \frac{1}{4} \ln(\sqrt{17} + 4) + \sqrt{17} \right]$$

$$= 2\sqrt{17} + \frac{1}{2} \ln(4 + \sqrt{17})$$



6.  $f(x, y) = y^2$

$R$  = square with vertices  $(0, 0), (3, 0), (0, 3), (3, 3)$

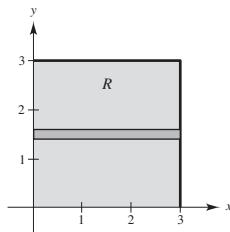
$$f_x = 0, f_y = 2y$$

$$\sqrt{1 + (f_x)^2 + (f_y)^2} = \sqrt{1 + 4y^2}$$

$$S = \int_0^3 \int_0^3 \sqrt{1 + 4y^2} dx dy = \int_0^3 3\sqrt{1 + 4y^2} dy$$

$$= \left[ \frac{3}{4} \left( 2y\sqrt{1 + 4y^2} + \ln|2y + \sqrt{1 + 4y^2}| \right) \right]_0^3$$

$$= \frac{3}{4} \left( 6\sqrt{37} + \ln|6 + \sqrt{37}| \right)$$



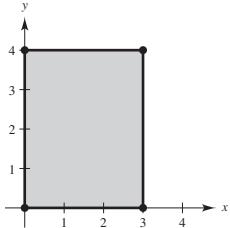
7.  $f(x, y) = 3 + x^{3/2}$

$$f_x = \frac{3}{2}x^{1/2}, f_y = 0$$

$$\sqrt{1 + (f_x)^2 + (f_y)^2} = \sqrt{1 + \frac{9}{4}x} = \frac{\sqrt{4 + 9x}}{2}$$

$$S = \int_0^3 \int_0^4 \frac{\sqrt{4 + 9x}}{2} dy dx = 4 \int_0^3 \frac{\sqrt{4 + 9x}}{2} dx$$

$$= \left[ \frac{4}{27} (4 + 9x)^{3/2} \right]_0^3 = \frac{4}{27} (31\sqrt{31} - 8)$$



8.  $f(x, y) = 2 + \frac{2}{3}y^{3/2}$

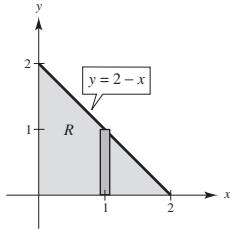
$$f_x = 0, f_y = y^{1/2}$$

$$\sqrt{1 + (f_x)^2 + (f_y)^2} = \sqrt{1 + y}$$

$$S = \int_0^2 \int_0^{2-y} \sqrt{1 + y} dx dy = \int_0^2 \sqrt{1 + y}(2 - y) dy$$

$$= \left[ 2(1 + y)^{3/2} - \frac{2}{5}(1 + y)^{5/2} \right]_0^2$$

$$= 2 \cdot 3^{3/2} - \frac{2}{5} \cdot 3^{5/2} - 2 + \frac{2}{5} = \frac{12}{5}\sqrt{3} - \frac{8}{5}$$



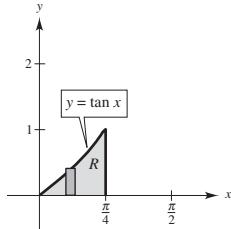
9.  $f(x, y) = \ln|\sec x|$

$$R = \left\{ (x, y) : 0 \leq x \leq \frac{\pi}{4}, 0 \leq y \leq \tan x \right\}$$

$$f_x = \tan x, f_y = 0$$

$$\sqrt{1 + (f_x)^2 + (f_y)^2} = \sqrt{1 + \tan^2 x} = \sec x$$

$$\begin{aligned} S &= \int_0^{\pi/4} \int_0^{\tan x} \sec x \, dy \, dx \\ &= \int_0^{\pi/4} \sec x \tan x \, dx \\ &= [\sec x]_0^{\pi/4} = \sqrt{2} - 1 \end{aligned}$$

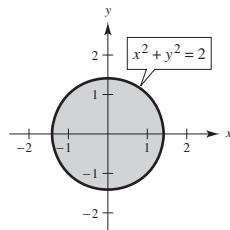


10.  $f(x, y) = 13 + x^2 - y^2$

$$f_x = 2x, f_y = -2y$$

$$\sqrt{1 + (f_x)^2 + (f_y)^2} = \sqrt{1 + 4x^2 + 4y^2}$$

$$\begin{aligned} S &= \int_0^{2\pi} \int_0^2 \sqrt{1 + 4r^2} r \, dr \, d\theta \\ &= \int_0^{2\pi} \left[ \frac{1}{12}(1 + 4r^2)^{3/2} \right]_0^2 d\theta \\ &= \int_0^{2\pi} \frac{1}{12}(17^{3/2} - 1) d\theta \\ &= \frac{\pi}{6}(17\sqrt{17} - 1) \end{aligned}$$



13.  $f(x, y) = \sqrt{a^2 - x^2 - y^2}$

$$R = \left\{ (x, y) : x^2 + y^2 \leq b^2, 0 < b < a \right\}$$

$$f_x = \frac{-x}{\sqrt{a^2 - x^2 - y^2}}, f_y = \frac{-y}{\sqrt{a^2 - x^2 - y^2}}$$

$$\sqrt{1 + (f_x)^2 + (f_y)^2} = \sqrt{1 + \frac{x^2}{a^2 - x^2 - y^2} + \frac{y^2}{a^2 - x^2 - y^2}} = \frac{a}{\sqrt{a^2 - x^2 - y^2}}$$

$$S = \int_{-b}^b \int_{-\sqrt{b^2-x^2}}^{\sqrt{b^2-x^2}} \frac{a}{\sqrt{a^2 - x^2 - y^2}} \, dy \, dx = \int_0^{2\pi} \int_0^b \frac{a}{\sqrt{a^2 - r^2}} r \, dr \, d\theta = 2\pi a(a - \sqrt{a^2 - b^2})$$

11.  $f(x, y) = \sqrt{x^2 + y^2}$

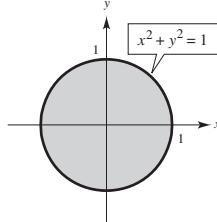
$$R = \left\{ (x, y) : 0 \leq f(x, y) \leq 1 \right\}$$

$$0 \leq \sqrt{x^2 + y^2} \leq 1, x^2 + y^2 \leq 1$$

$$f_x = \frac{x}{\sqrt{x^2 + y^2}}, f_y = \frac{y}{\sqrt{x^2 + y^2}}$$

$$\sqrt{1 + (f_x)^2 + (f_y)^2} = \sqrt{1 + \frac{x^2}{x^2 + y^2} + \frac{y^2}{x^2 + y^2}} = \sqrt{2}$$

$$S = \int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \sqrt{2} \, dy \, dx = \int_0^{2\pi} \int_0^1 \sqrt{2} r \, dr \, d\theta = \sqrt{2}\pi$$



12.  $f(x, y) = xy$

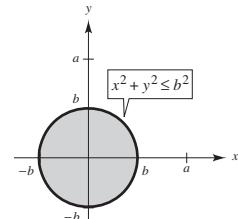
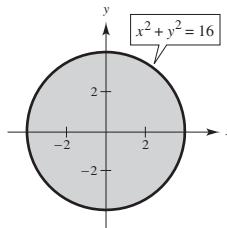
$$R = \left\{ (x, y) : x^2 + y^2 \leq 16 \right\}$$

$$f_x = y, f_y = x$$

$$\sqrt{1 + (f_x)^2 + (f_y)^2} = \sqrt{1 + y^2 + x^2}$$

$$S = \int_{-4}^4 \int_{-\sqrt{16-x^2}}^{\sqrt{16-x^2}} \sqrt{1 + y^2 + x^2} \, dy \, dx$$

$$= \int_0^{2\pi} \int_0^4 \sqrt{1 + r^2} r \, dr \, d\theta = \frac{2\pi}{3}(17\sqrt{17} - 1)$$

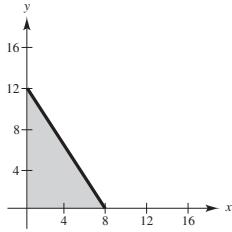


14. See Exercise 13.

$$\begin{aligned} S &= \int_{-a}^a \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \frac{a}{\sqrt{a^2-x^2-y^2}} dy dx \\ &= \int_0^{2\pi} \int_0^a \frac{a}{\sqrt{a^2-r^2}} r dr d\theta = 2\pi a^2 \end{aligned}$$

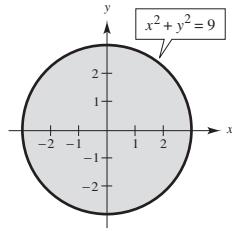
15.  $z = 24 - 3x - 2y$

$$\begin{aligned} \sqrt{1 + (f_x)^2 + (f_y)^2} &= \sqrt{14} \\ S &= \int_0^8 \int_0^{-(3/2)x+12} \sqrt{14} dy dx = 48\sqrt{14} \end{aligned}$$



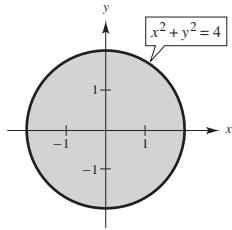
17.  $z = \sqrt{25 - x^2 - y^2}$

$$\begin{aligned} \sqrt{1 + (f_x)^2 + (f_y)^2} &= \sqrt{1 + \frac{x^2}{25 - x^2 - y^2} + \frac{y^2}{25 - x^2 - y^2}} = \frac{5}{\sqrt{25 - x^2 - y^2}} \\ S &= 2 \int_{-3}^3 \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \frac{5}{\sqrt{25 - (x^2 + y^2)}} dy dx = 2 \int_0^{2\pi} \int_0^3 \frac{5}{\sqrt{25 - r^2}} r dr d\theta = 20\pi \end{aligned}$$



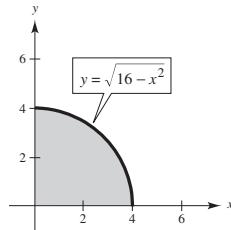
18.  $z = 2\sqrt{x^2 + y^2}$

$$\begin{aligned} \sqrt{1 + (f_x)^2 + (f_y)^2} &= \sqrt{1 + \frac{4x^2}{x^2 + y^2} + \frac{4y^2}{x^2 + y^2}} = \sqrt{5} \\ S &= \int_0^{2\pi} \int_0^2 \sqrt{5}r dr d\theta = 4\pi\sqrt{5} \end{aligned}$$



16.  $z = 16 - x^2 - y^2$

$$\begin{aligned} \sqrt{1 + (f_x)^2 + (f_y)^2} &= \sqrt{1 + 4x^2 + 4y^2} \\ S &= \int_0^4 \int_0^{\sqrt{16-x^2}} \sqrt{1 + 4(x^2 + y^2)} dy dx \\ &= \int_0^{\pi/2} \int_0^4 \sqrt{1 + 4r^2} r dr d\theta = \frac{\pi}{24}(65\sqrt{65} - 1) \end{aligned}$$

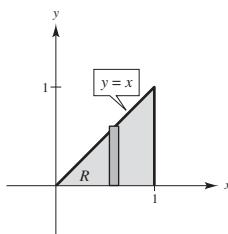


19.  $f(x, y) = 2y + x^2$

$R$  = triangle with vertices  $(0, 0), (1, 0), (1, 1)$

$$\sqrt{1 + (f_x)^2 + (f_y)^2} = \sqrt{5 + 4x^2}$$

$$S = \int_0^1 \int_0^x \sqrt{5 + 4x^2} dy dx = \frac{1}{12}(27 - 5\sqrt{5})$$

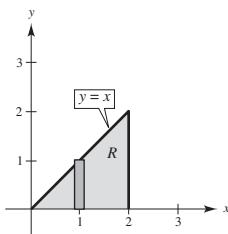


20.  $f(x, y) = 2x + y^2$

$R$  = triangle with vertices  $(0, 0), (2, 0), (2, 2)$

$$\sqrt{1 + (f_x)^2 + (f_y)^2} = \sqrt{5 + 4y^2}$$

$$\begin{aligned} S &= \int_0^2 \int_0^x \sqrt{5 + 4y^2} dy dx \\ &= \frac{5}{4} \ln\left(\frac{8\sqrt{21} + 37}{5}\right) + \frac{\sqrt{21}}{4} + \frac{5\sqrt{5}}{12} \end{aligned}$$



21.  $f(x, y) = 9 - x^2 - y^2$

$R = \{(x, y): 0 \leq f(x, y)\}$

$$0 \leq 9 - x^2 - y^2 \Rightarrow x^2 + y^2 \leq 9$$

$$fx = -2x, fy = -2y$$

$$\sqrt{1 + (f_x)^2 + (f_y)^2} = \sqrt{1 + 4x^2 + 4y^2}$$

$$S = \int_{-3}^3 \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \sqrt{1 + 4x^2 + 4y^2} dy dx$$

$$= \int_0^{2\pi} \int_0^3 \sqrt{1 + 4r^2} r dr d\theta$$

$$= \frac{\pi}{6}(37\sqrt{37} - 1) \approx 117.3187$$

22.  $f(x, y) = x^2 + y^2$

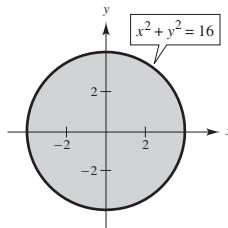
$R = \{(x, y): 0 \leq f(x, y) \leq 16\}$

$$0 \leq x^2 + y^2 \leq 16$$

$$fx = 2x, fy = 2y$$

$$\sqrt{1 + (f_x)^2 + (f_y)^2} = \sqrt{1 + 4x^2 + 4y^2}$$

$$\begin{aligned} S &= \int_{-4}^4 \int_{-\sqrt{16-x^2}}^{\sqrt{16-x^2}} \sqrt{1 + 4x^2 + 4y^2} dy dx \\ &= \int_0^{2\pi} \int_0^4 \sqrt{1 + 4r^2} dr d\theta = \frac{(65\sqrt{65} - 1)\pi}{6} \end{aligned}$$



23.  $f(x, y) = 4 - x^2 - y^2$

$R = \{(x, y): 0 \leq x \leq 1, 0 \leq y \leq 1\}$

$$fx = -2x, fy = -2y$$

$$\sqrt{1 + (f_x)^2 + (f_y)^2} = \sqrt{1 + 4x^2 + 4y^2}$$

$$S = \int_0^1 \int_0^1 \sqrt{1 + 4x^2 + 4y^2} dy dx \approx 1.8616$$

24.  $f(x, y) = \frac{2}{3}x^{3/2} + \cos x$

$R = \{(x, y): 0 \leq x \leq 1, 0 \leq y \leq 1\}$

$$fx = x^{1/2} - \sin x, fy = 0$$

$$\sqrt{1 + (f_x)^2 + (f_y)^2} = \sqrt{1 + (\sqrt{x} - \sin x)^2}$$

$$S = \int_0^1 \int_0^1 \sqrt{1 + (\sqrt{x} - \sin x)^2} dy dx \approx 1.02185$$

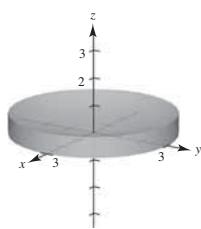
25. Surface area  $> (4) \cdot (6) = 24$

Matches (e)



26. Surface area  $\approx 9\pi$

Matches (c)



27.  $f(x, y) = e^x$

$R = \{(x, y): 0 \leq x \leq 1, 0 \leq y \leq 1\}$

$f_x = e^x, f_y = 0$

$\sqrt{1 + (f_x)^2 + (f_y)^2} = \sqrt{1 + e^{2x}}$

$S = \int_0^1 \int_0^1 \sqrt{1 + e^{2x}} dy dx$

$= \int_0^1 \sqrt{1 + e^{2x}} \approx 2.0035$

31.  $f(x, y) = e^{-x} \sin y$

$f_x = -e^{-x} \sin y, f_y = e^{-x} \cos y$

$\sqrt{1 + (f_x)^2 + (f_y)^2} = \sqrt{1 + e^{-2x} \sin^2 y + e^{-2x} \cos^2 y} = \sqrt{1 + e^{-2x}}$

$S = \int_{-2}^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \sqrt{1 + e^{-2x}} dy dx$

32.  $f(x, y) = \cos(x^2 + y^2)$

$R = \left\{(x, y): x^2 + y^2 \leq \frac{\pi}{2}\right\}$

$f_x = -2x \sin(x^2 + y^2), f_y = -2y \sin(x^2 + y^2)$

$\sqrt{1 + (f_x)^2 + (f_y)^2} = \sqrt{1 + 4x^2 \sin^2(x^2 + y^2) + 4y^2 \sin^2(x^2 + y^2)} = \sqrt{1 + 4[\sin^2(x^2 + y^2)](x^2 + y^2)}$

$S = \int_{-\sqrt{\pi/2}}^{\sqrt{\pi/2}} \int_{-\sqrt{(\pi/2)-x^2}}^{\sqrt{(\pi/2)-x^2}} \sqrt{1 + 4(x^2 + y^2) \sin^2(x^2 + y^2)} dy dx$

33.  $f(x, y) = e^{xy}$

$R = \{(x, y): 0 \leq x \leq 4, 0 \leq y \leq 10\}$

$f_x = ye^{xy}, f_y = xe^{xy}$

$\sqrt{1 + (f_x)^2 + (f_y)^2} = \sqrt{1 + y^2 e^{2xy} + x^2 e^{2xy}} = \sqrt{1 + e^{2xy}(x^2 + y^2)}$

$S = \int_0^4 \int_0^{10} \sqrt{1 + e^{2xy}(x^2 + y^2)} dy dx$

28.  $f(x, y) = \frac{2}{5}y^{5/2}$

$R = \{(x, y): 0 \leq x \leq 1, 0 \leq y \leq 1\}$

$f_x = 0, f_y = y^{3/2}$

$\sqrt{1 + (f_x)^2 + (f_y)^2} = \sqrt{1 + y^3}$

$S = \int_0^1 \int_0^1 \sqrt{1 + y^3} dx dy = \int_0^1 \sqrt{1 + y^3} dy \approx 1.1114$

29.  $f(x, y) = x^3 - 3xy + y^3$

$R = \text{square with vertices } (1, 1), (-1, 1), (-1, -1), (1, -1)$

$f_x = 3x^2 - 3y = 3(x^2 - y),$

$f_y = -3x + 3y^2 = 3(y^2 - x)$

$\sqrt{1 + (f_x)^2 + (f_y)^2} = \int_{-1}^1 \int_{-1}^1 \sqrt{1 + 9(x^2 - y)^2 + 9(y^2 - x)^2} dy dx$

30.  $f(x, y) = x^2 - 3xy - y^2$

$R = \{(x, y): 0 \leq x \leq 4, 0 \leq y \leq x\}$

$f_x = 2x - 3y, f_y = -3x - 2y = -(3x + 2y)$

$\sqrt{1 + (f_x)^2 + (f_y)^2} = \sqrt{1 + (2x - 3y)^2 + (3x + 2y)^2}$   
 $= \sqrt{1 + 13(x^2 + y^2)}$

$S = \int_0^4 \int_0^x \sqrt{1 + 13(x^2 + y^2)} dy dx$

34.  $f(x, y) = e^{-x} \sin y$

$$R = \{(x, y) : 0 \leq x \leq 4, 0 \leq y \leq x\}$$

$$f_x = -e^{-x} \sin y, f_y = e^{-x} \cos y$$

$$\sqrt{1 + (f_x)^2 + (f_y)^2} = \sqrt{1 + e^{-2x} \sin^2 y + e^{-2x} \cos^2 y} = \sqrt{1 + e^{-2x}}$$

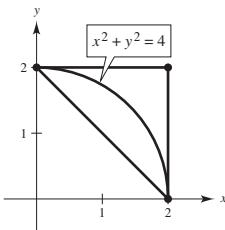
$$S = \int_0^4 \int_0^x \sqrt{1 + e^{-2x}} dy dx$$

35. See the definition on page 1021.

36.  $f(x, y) = x^2 + y^2$  is a paraboloid opening upward.

Using the figure below, you see that the surface areas satisfy:

(b) < (c) < (a)



37. No, the surface area is the same.

$$z = f(x, y) \quad \text{and} \quad z = f(x, y) + k$$

have the same partial derivatives.

40.  $f(x, y) = k\sqrt{x^2 + y^2}$

$$\sqrt{1 + (f_x)^2 + (f_y)^2} = \sqrt{1 + \frac{k^2 x^2}{x^2 + y^2} + \frac{k^2 y^2}{x^2 + y^2}} = \sqrt{k^2 + 1}$$

$$S = \int_R \int \sqrt{1 + (f_x)^2 + (f_y)^2} dA = \int_R \int \sqrt{k^2 + 1} dA = \sqrt{k^2 + 1} \int_R \int dA = A\sqrt{k^2 + 1} = \pi r^2 \sqrt{k^2 + 1}$$

41. (a)  $V = \int_R \int f(x, y)$

$$= 8 \int_R \int \sqrt{625 - x^2 - y^2} dA \quad \text{where } R \text{ is the region in the first quadrant}$$

$$= 8 \int_0^{\pi/2} \int_4^{25} \sqrt{625 - r^2} r dr d\theta = -4 \int_0^{\pi/2} \left[ \frac{2}{3} (625 - r^2)^{3/2} \right]_4^{25} d\theta$$

$$= -\frac{8}{3} [0 - 609\sqrt{609}] \cdot \frac{\pi}{2} = 812\pi\sqrt{609} \text{ cm}^3$$

(b)  $A = \int_R \int \sqrt{1 + (f_x)^2 + (f_y)^2} dA = 8 \int_R \int \sqrt{1 + \frac{x^2}{625 - x^2 - y^2} + \frac{y^2}{625 - x^2 - y^2}} dA$

$$= 8 \int_R \int \frac{25}{\sqrt{625 - x^2 - y^2}} dA = 8 \int_0^{\pi/2} \int_4^{25} \frac{25}{\sqrt{625 - r^2}} r dr d\theta$$

$$= \lim_{b \rightarrow 25^-} \left[ -200\sqrt{625 - r^2} \right]_4^b \cdot \frac{\pi}{2} = 100\pi\sqrt{609} \text{ cm}^2$$

38. (a) Yes. For example, let  $R$  be the square given by

$$0 \leq x \leq 1, 0 \leq y \leq 1,$$

and  $S$  the square parallel to  $R$  given by

$$0 \leq x \leq 1, 0 \leq y \leq 1, z = 1.$$

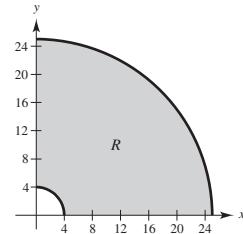
(b) Yes. Let  $R$  be the region in part (a) and  $S$  the surface given by  $f(x, y) = xy$ .

(c) No.

39.  $f(x, y) = \sqrt{1 - x^2}; f_x = \frac{-x}{\sqrt{1 - x^2}}, f_y = 0$

$$S = \int_R \int \sqrt{1 + (f_x)^2 + (f_y)^2} dA \\ = 16 \int_0^1 \int_0^x \frac{1}{\sqrt{1 - x^2}} dy dx$$

$$= 16 \int_0^1 \frac{x}{\sqrt{1 - x^2}} dx = \left[ -16(1 - x^2)^{1/2} \right]_0^1 = 16$$



42. (a)  $z = -\frac{1}{75}y^3 + \frac{4}{25}y^2 - \frac{16}{15}y + 25$

(b)  $V \approx 2(50) \int_0^{15} \left( -\frac{1}{75}y^3 + \frac{4}{25}y^2 - \frac{16}{15}y + 25 \right) dy = 100(266.25) = 26,625 \text{ cubic feet}$

(c)  $f(x, y) = -\frac{1}{75}y^3 + \frac{4}{25}y^2 - \frac{16}{15}y + 25$

$$f_x = 0, f_y = -\frac{1}{25}y^2 + \frac{8}{25}y - \frac{16}{15}$$

$$S = 2 \int_0^{50} \int_0^{15} \sqrt{1 + f_y^2 + f_x^2} dy dx \approx 3087.58 \text{ sq ft}$$

(d) Arc length  $\approx 30.8758$

$$\text{Surface area of roof } \approx 2(50)(30.8758) = 3087.58 \text{ sq ft}$$

## Section 14.6 Triple Integrals and Applications

$$\begin{aligned} 1. \int_0^3 \int_0^2 \int_0^1 (x + y + z) dx dz dy &= \int_0^3 \int_0^2 \left[ \frac{x^2}{2} + xy + xz \right]_0^1 dz dy = \int_0^3 \int_0^2 \left( \frac{1}{2} + y + z \right) dz dy = \int_0^3 \left[ \frac{1}{2}z + yz + \frac{z^2}{2} \right]_0^2 dy \\ &= \int_0^3 (1 + 2y + 2) dy = \left[ 3y + y^2 \right]_0^3 = 18 \end{aligned}$$

$$\begin{aligned} 2. \int_{-1}^1 \int_{-1}^1 \int_{-1}^1 x^2 y^2 z^2 dx dy dz &= \frac{1}{3} \int_{-1}^1 \int_{-1}^1 \left[ x^3 y^2 z^2 \right]_{-1}^1 dy dz \\ &= \frac{2}{3} \int_{-1}^1 \int_{-1}^1 y^2 z^2 dy dz = \frac{2}{9} \int_{-1}^1 \left[ y^3 z^2 \right]_{-1}^1 dz = \frac{4}{9} \int_{-1}^1 z^2 dz = \left[ \frac{4}{27} z^3 \right]_{-1}^1 = \frac{8}{27} \end{aligned}$$

$$3. \int_0^1 \int_0^x \int_0^{xy} x dz dy dx = \int_0^1 \int_0^x [xz]_0^{xy} dy dx = \int_0^1 \int_0^x x^2 y dy dx = \int_0^1 \left[ \frac{x^2 y^2}{2} \right]_0^x dx = \int_0^1 \frac{x^4}{2} dx = \left[ \frac{x^5}{10} \right]_0^1 = \frac{1}{10}$$

$$4. \int_0^9 \int_0^{y/3} \int_0^{\sqrt{y^2 - 9x^2}} z dz dx dy = \frac{1}{2} \int_0^9 \int_0^{y/3} (y^2 - 9x^2) dx dy = \frac{1}{2} \int_0^9 \left[ xy^2 - 3x^3 \right]_0^{y/3} dy = \frac{2}{18} \int_0^9 y^3 dy = \left[ \frac{1}{36} y^4 \right]_0^9 = \frac{729}{4}$$

$$\begin{aligned} 5. \int_1^4 \int_0^1 \int_0^x 2ze^{-x^2} dy dx dz &= \int_1^4 \int_0^1 \left[ (2ze^{-x^2})y \right]_0^x dx dz = \int_1^4 \int_0^1 2zxe^{-x^2} dx dz \\ &= \int_1^4 \left[ -ze^{-x^2} \right]_0^1 dz = \int_1^4 z(1 - e^{-1}) dz = \left[ (1 - e^{-1}) \frac{z^2}{2} \right]_1^4 = \frac{15}{2} \left( 1 - \frac{1}{e} \right) \end{aligned}$$

$$6. \int_1^4 \int_1^{e^2} \int_0^{1/xz} \ln z dy dz dx = \int_1^4 \int_1^{e^2} [(\ln z)y]_0^{1/xz} dz dx = \int_1^4 \int_1^{e^2} \frac{\ln z}{xz} dz dx = \int_1^4 \left[ \frac{(\ln z)^2}{2x} \right]_1^{e^2} dx = \int_1^4 \frac{2}{x} dx = \left[ 2 \ln |x| \right]_1^4 = 2 \ln 4$$

$$\begin{aligned} 7. \int_0^4 \int_0^{\pi/2} \int_0^{1-x} x \cos y dz dy dx &= \int_0^4 \int_0^{\pi/2} [(x \cos y)z]_0^{1-x} dy dx = \int_0^4 \int_0^{\pi/2} x(1-x) \cos y dy dx \\ &= \int_0^4 [x(1-x) \sin y]_0^{\pi/2} dx = \int_0^4 x(1-x) dx = \left[ \frac{x^2}{2} - \frac{x^3}{3} \right]_0^4 = 8 - \frac{64}{3} = -\frac{40}{3} \end{aligned}$$

$$8. \int_0^{\pi/2} \int_0^{y/2} \int_0^{1/y} \sin y dz dx dy = \int_0^{\pi/2} \int_0^{y/2} \frac{\sin y}{y} dx dy = \frac{1}{2} \int_0^{\pi/2} \sin y dy = \left[ -\frac{1}{2} \cos y \right]_0^{\pi/2} = \frac{1}{2}$$

$$9. \int_0^3 \int_{-\sqrt{9-y^2}}^{\sqrt{9-y^2}} \int_0^{y^2} y dz dx dy = \frac{324}{5}$$

10.  $\int_0^{\sqrt{2}} \int_0^{\sqrt{2-x^2}} \int_{2x^2+y^2}^{4-y^2} y \, dz \, dy \, dx = \int_0^{\sqrt{2}} \int_0^{\sqrt{2-x^2}} (4y - 2x^2y - 2y^3) \, dy \, dx = \frac{16\sqrt{2}}{15}$

11.  $\int_0^2 \int_0^{\sqrt{4-x^2}} \int_1^4 \frac{x^2 \sin y}{z} \, dz \, dy \, dx = \int_0^2 \int_0^{\sqrt{4-x^2}} \left[ x^2 \sin y \ln|z| \right]_1^4 \, dy \, dx$   
 $= \int_0^2 \left[ x^2 \ln 4(-\cos y) \right]_0^{\sqrt{4-x^2}} \, dx = \int_0^2 x^2 \ln 4 \left[ 1 - \cos \sqrt{4-x^2} \right] \, dx \approx 2.44167$

12.  $\int_0^3 \int_0^{2-(2y/3)} \int_0^{6-2y-3z} ze^{-x^2y^2} \, dx \, dz \, dy = \int_0^6 \int_0^{(6-x)/2} \int_0^{(6-x-2y)/3} ze^{-x^2y^2} \, dz \, dy \, dx$   
 $= \int_0^6 \int_0^{3-(x/2)} \frac{1}{2} \left( \frac{6-x-2y}{3} \right)^2 e^{-x^2y^2} \, dy \, dx \approx 2.118$

13.  $V = \int_0^5 \int_0^{5-x} \int_0^{5-x-y} dz \, dy \, dx$

14.  $V = \int_0^3 \int_0^{2x} \int_0^{9-x^2} dz \, dy \, dx$

15.  $V = \int_{-\sqrt{6}}^{\sqrt{6}} \int_{-\sqrt{6-x^2}}^{\sqrt{6-x^2}} \int_0^{6-x^2-y^2} dz \, dy \, dx = \int_{-\sqrt{6}}^{\sqrt{6}} \int_{-\sqrt{6-y^2}}^{\sqrt{6-y^2}} \int_0^{6-x^2-y^2} dz \, dx \, dy$

16.  $V = \int_{-4}^4 \int_{-\sqrt{16-x^2}}^{\sqrt{16-x^2}} \int_0^{\sqrt{16-x^2-y^2}} dz \, dy \, dx$

17.  $z = \frac{1}{2}(x^2 + y^2) \Rightarrow 2z = x^2 + y^2$

$x^2 + y^2 + z^2 = 2z + z^2 = 80 \Rightarrow z^2 + 2z - 80 = 0 \Rightarrow (z-8)(z+10) = 0 \Rightarrow z = 8 \Rightarrow x^2 + y^2 = 2z = 16$

$V = \int_{-4}^4 \int_{-\sqrt{16-x^2}}^{\sqrt{16-x^2}} \int_{1/2(x^2+y^2)}^{\sqrt{80-x^2-y^2}} dz \, dy \, dx$

18.  $V = \int_{-\sqrt{2}}^{\sqrt{2}} \int_{-\sqrt{(4-2x^2)/3}}^{\sqrt{(4-2x^2)/3}} \int_{x^2+3y^2}^{4-x^2} dz \, dy \, dx$

$z = 4 - x^2 = x^2 + 3y^2$

$4 = 2x^2 + 3y^2$

$1 = \frac{x^2}{2} + \frac{y^2}{(4/3)}$  ellipse

19.  $V = \int_{-2}^2 \int_0^{4-y^2} \int_0^x dz \, dx \, dy = \int_{-2}^2 \int_0^{4-y^2} x \, dx \, dy$   
 $= \frac{1}{2} \int_{-2}^2 (4 - y^2)^2 \, dy = \int_0^2 (16 - 8y^2 + y^4) \, dy = \left[ 16y - \frac{8}{3}y^3 + \frac{1}{5}y^5 \right]_0^2 = \frac{256}{15}$

20.  $V = \int_0^2 \int_0^2 \int_0^{2xy} dz \, dy \, dx = \int_0^2 \int_0^2 2xy \, dy \, dx = \int_0^2 \left[ xy^2 \right]_0^2 \, dx = \int_0^2 4x \, dx = 8$

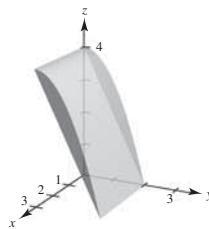
21.  $V = 8 \int_0^a \int_0^{\sqrt{a^2-x^2}} \int_0^{\sqrt{a^2-x^2-y^2}} dz \, dy \, dx = 8 \int_0^a \int_0^{\sqrt{a^2-x^2}} \sqrt{a^2 - x^2 - y^2} \, dy \, dx$   
 $= 4 \int_0^a \left[ y \sqrt{a^2 - x^2 - y^2} + (a^2 - x^2) \arcsin \left( \frac{y}{\sqrt{a^2 - x^2}} \right) \right]_{a^2-x^2}^{\sqrt{a^2-x^2}} \, dx$   
 $= 4 \left( \frac{\pi}{2} \right) \int_0^a (a^2 - x^2) \, dx = \left[ 2\pi \left( a^2 x - \frac{1}{3}x^3 \right) \right]_0^a = \frac{4}{3}\pi a^3$

$$\begin{aligned}
 22. V &= 4 \int_0^6 \int_0^{\sqrt{36-x^2}} \int_0^{36-x^2-y^2} dz dy dx = 4 \int_0^6 \int_0^{\sqrt{36-x^2}} (36 - x^2 - y^2) dy dx = 4 \int_0^6 \left[ 36y - x^2y - \frac{y^3}{3} \right]_0^{\sqrt{36-x^2}} dx \\
 &= 4 \int_0^6 \left[ 36\sqrt{36-x^2} - x^2\sqrt{36-x^2} - \frac{1}{3}(36-x^2)^{3/2} \right] dx \\
 &= 4 \left[ 9x\sqrt{36-x^2} + 324 \arcsin\left(\frac{x}{6}\right) + \frac{1}{6}x(36-x^2)^{3/2} \right]_0^6 = 4(162\pi) = 648\pi
 \end{aligned}$$

$$23. V = \int_0^2 \int_0^{4-x^2} \int_0^{4-x^2} dz dy dx = \int_0^2 (4 - x^2)^2 dx = \int_0^2 (16 - 8x^2 + x^4) dx = \left[ 16x - \frac{8}{3}x^3 + \frac{1}{5}x^5 \right]_0^2 = \frac{256}{15}$$

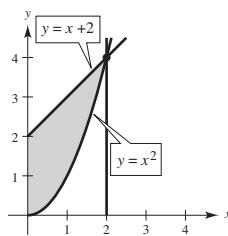
$$\begin{aligned}
 24. V &= \int_0^{\sqrt{2}} \int_0^{2-x^2} \int_0^{9-x^3} dz dy dx \\
 &= \int_0^{\sqrt{2}} \int_0^{2-x^2} (9 - x^3) dy dx \\
 &= \int_0^{\sqrt{2}} (9 - x^3)(2 - x^2) dx \\
 &= \int_0^{\sqrt{2}} (18 - 9x^2 - 2x^3 + x^5) dx \\
 &= \left[ 18x - 3x^3 - \frac{1}{2}x^4 + \frac{x^6}{6} \right]_0^{\sqrt{2}} \\
 &= 18\sqrt{2} - 6\sqrt{2} - 2 + \frac{4}{3} = 12\sqrt{2} - \frac{2}{3}
 \end{aligned}$$

$$\begin{aligned}
 25. V &= \int_0^3 \int_0^2 \int_{2-y}^{4-y^2} dz dy dx \\
 &= \int_0^3 \int_0^2 [4 - y^2 - 2 + y] dy dx \\
 &= \int_0^3 \left[ 2y - \frac{y^3}{3} + \frac{y^2}{2} \right]_0^2 dx \\
 &= \int_0^3 \left( 4 - \frac{8}{3} + 2 \right) dx \\
 &= \left[ \frac{10}{3}x \right]_0^3 = 10
 \end{aligned}$$

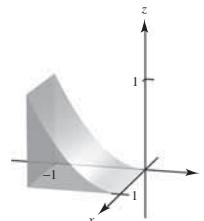


26. The region in the  $xy$ -plane is:

$$\begin{aligned}
 V &= \int_0^2 \int_{x^2}^{x+2} \int_0^x dz dy dx = \int_0^2 \int_{x^2}^{x+2} x dy dx \\
 &= \int_0^2 [xy]_{x^2}^{x+2} dx = \int_0^2 (x(x+2) - x^3) dx \\
 &= \left[ \frac{x^3}{3} + x^2 - \frac{x^4}{4} \right]_0^2 = \frac{8}{3} + 4 - 4 = \frac{8}{3}
 \end{aligned}$$

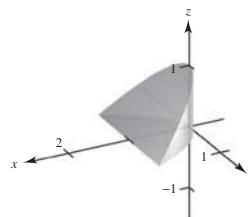


27.



$$\int_0^1 \int_0^1 \int_{-1}^{-\sqrt{z}} dy dz dx$$

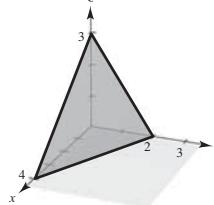
28.



$$\int_{-1}^1 \int_0^{1-y^2} \int_{y^2}^{1-z} dx dz dy$$

29. Plane:  $3x + 6y + 4z = 12$

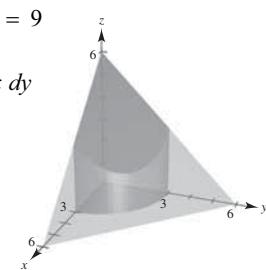
$$\int_0^3 \int_0^{(12-4z)/3} \int_0^{(12-4z-3x)/6} dy dx dz$$



30. Top plane:  $x + y + z = 6$

Side cylinder:  $x^2 + y^2 = 9$

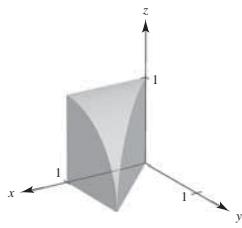
$$\int_0^3 \int_0^{\sqrt{9-y^2}} \int_0^{6-x-y} dz dx dy$$



31. Top cylinder:  $y^2 + z^2 = 1$

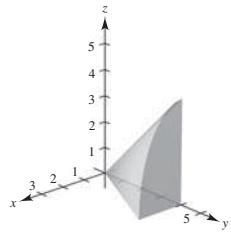
Side plane:  $x = y$

$$\int_0^1 \int_0^x \int_0^{\sqrt{1-y^2}} dz dy dx$$



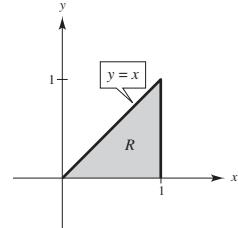
32. Elliptic cone:  $4x^2 + z^2 = y^2$

$$\int_0^4 \int_z^4 \int_0^{\sqrt{y^2-z^2}/2} dx dy dz$$



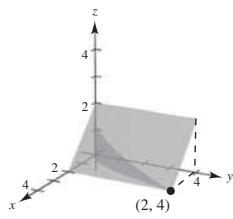
33.  $Q = \{(x, y, z): 0 \leq x \leq 1, 0 \leq y \leq x, 0 \leq z \leq 3\}$

$$\begin{aligned} \iiint_Q xyz \, dV &= \int_0^3 \int_0^1 \int_y^1 xyz \, dx \, dy \, dz = \int_0^3 \int_0^1 \int_0^x xyz \, dy \, dx \, dz \\ &= \int_0^1 \int_0^3 \int_y^1 xyz \, dx \, dz \, dy = \int_0^1 \int_0^3 \int_0^x xyz \, dy \, dz \, dx \\ &= \int_0^1 \int_y^1 \int_0^3 xyz \, dz \, dx \, dy = \int_0^1 \int_0^x \int_0^3 xyz \, dz \, dy \, dx \left( = \frac{9}{16} \right) \end{aligned}$$



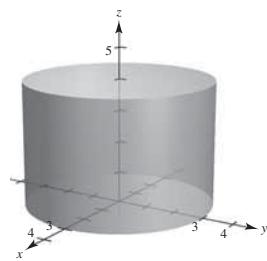
34.  $Q = \{(x, y, z): 0 \leq x \leq 2, x^2 \leq y \leq 4, 0 \leq z \leq 2 - x\}$

$$\begin{aligned} \iiint_Q xyz \, dV &= \int_0^2 \int_{x^2}^4 \int_0^{2-x} xyz \, dz \, dy \, dx \\ &= \int_0^4 \int_0^{\sqrt{y}} \int_0^{2-x} xyz \, dz \, dx \, dy \\ &= \int_0^2 \int_0^{2-x} \int_{x^2}^4 xyz \, dy \, dz \, dx \\ &= \int_0^2 \int_0^{2-x} \int_{x^2}^4 xyz \, dy \, dx \, dz \\ &= \int_0^2 \int_0^{(2-z)^2} \int_0^{\sqrt{y}} xyz \, dx \, dy \, dz + \int_0^2 \int_{(2-z)^2}^4 \int_0^{2-z} xyz \, dx \, dy \, dz \\ &= \int_0^4 \int_0^{2-\sqrt{y}} \int_0^{\sqrt{y}} xyz \, dx \, dz \, dy + \int_0^4 \int_{2-\sqrt{y}}^2 \int_0^{2-z} dx \, dz \, dy \left( = \frac{104}{21} \right) \end{aligned}$$



35.  $Q = \{(x, y, z): x^2 + y^2 \leq 9, 0 \leq z \leq 4\}$

$$\begin{aligned} \iiint_Q xyz \, dV &= \int_0^4 \int_{-3}^3 \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} xyz \, dy \, dx \, dz = \int_0^4 \int_{-3}^3 \int_{-\sqrt{9-y^2}}^{\sqrt{9-y^2}} xyz \, dx \, dy \, dz \\ &= \int_{-3}^3 \int_0^4 \int_{-\sqrt{9-y^2}}^{\sqrt{9-y^2}} xyz \, dx \, dz \, dy = \int_{-3}^3 \int_{-\sqrt{9-y^2}}^{\sqrt{9-y^2}} \int_0^4 xyz \, dz \, dx \, dy \\ &= \int_{-3}^3 \int_0^4 \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} xyz \, dy \, dz \, dx = \int_{-3}^3 \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \int_0^4 xyz \, dy \, dx \, dz \left( = 0 \right) \end{aligned}$$



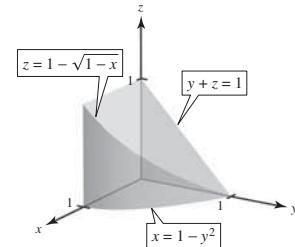
36.  $Q = \{(x, y, z): 0 \leq x \leq 1, y \leq 1 - x^2, 0 \leq z \leq 6\}$

$$\begin{aligned} \iiint_Q xyz \, dV &= \int_0^1 \int_0^{1-x^2} \int_0^6 xyz \, dz \, dy \, dx = \int_0^1 \int_0^{\sqrt{1-y}} \int_0^6 xyz \, dz \, dx \, dy \\ &= \int_0^1 \int_0^6 \int_0^{\sqrt{1-y}} xyz \, dx \, dz \, dy = \int_0^6 \int_0^1 \int_0^{\sqrt{1-y}} xyz \, dy \, dx \, dz \\ &= \int_0^1 \int_0^6 \int_0^{1-x^2} xyz \, dy \, dz \, dx = \int_0^6 \int_0^1 \int_0^{1-x^2} xyz \, dy \, dx \, dz = \frac{3}{2} \end{aligned}$$



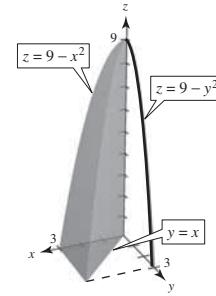
37.  $\mathcal{Q} = \{(x, y, z) : 0 \leq y \leq 1, 0 \leq x \leq 1 - y^2, 0 \leq z \leq 1 - y\}$

$$\begin{aligned}\int_0^1 \int_0^{1-y^2} \int_0^{1-y} dz dx dy &= \int_0^1 \int_0^{\sqrt{1-x}} \int_0^{1-y} dz dy dx \\&= \int_0^1 \int_0^{2z-z^2} \int_0^{1-z} dy dx dz + \int_0^1 \int_{2z-z^2}^1 \int_0^{\sqrt{1-x}} dy dx dz \\&= \int_0^1 \int_{1-\sqrt{1-x}}^1 \int_0^{1-z} dy dz dx + \int_0^1 \int_0^{1-\sqrt{1-x}} \int_0^{\sqrt{1-x}} dy dz dx \\&= \int_0^1 \int_0^{1-y} \int_0^{1-y^2} dx dz dy = \int_0^1 \int_0^{1-z} \int_0^{1-y^2} dx dy dz = \frac{5}{12}\end{aligned}$$



38.  $\mathcal{Q} = \{(x, y, z) : 0 \leq x \leq 3, 0 \leq y \leq x, 0 \leq z \leq 9 - x^2\}$

$$\begin{aligned}\int_0^3 \int_0^x \int_0^{9-x^2} dz dy dx &= \int_0^3 \int_y^3 \int_0^{9-x^2} dz dx dy \\&= \int_0^3 \int_0^{9-x^2} \int_0^x dy dz dx = \int_0^9 \int_0^{\sqrt{9-z}} \int_0^x dy dx dz \\&= \int_0^9 \int_0^{\sqrt{9-z}} \int_y^{\sqrt{9-z}} dx dy dz = \int_0^3 \int_0^{9-y^2} \int_y^{\sqrt{9-z}} dx dz dy = \frac{81}{4}\end{aligned}$$



39.  $m = k \int_0^6 \int_0^{4-(2x/3)} \int_0^{2-(y/2)-(x/3)} dz dy dx = 8k$

$$M_{yz} = k \int_0^6 \int_0^{4-(2x/3)} \int_0^{2-(y/2)-(x/3)} x dz dy dx = 12k$$

$$\bar{x} = \frac{M_{yz}}{m} = \frac{12k}{8k} = \frac{3}{2}$$

40.  $m = k \int_0^5 \int_0^{5-x} \int_0^{1/5(15-3x-3y)} y dz dy dx = \frac{125}{8}k$

$$M_{xz} = k \int_0^5 \int_0^{5-x} \int_0^{1/5(15-3x-3y)} y^2 dz dy dx = \frac{125}{4}k$$

$$\bar{y} = \frac{M_{xz}}{m} = 2$$

41.  $m = k \int_0^4 \int_0^4 \int_0^{4-x} x dz dy dx = k \int_0^4 \int_0^4 x(4-x) dy dx$

$$= 4k \int_0^4 (4x - x^2) dx = \frac{128k}{3}$$

$$M_{xy} = k \int_0^4 \int_0^4 \int_0^{4-x} xz dz dy dx = k \int_0^4 \int_0^4 x \frac{(4-x)^2}{2} dy dx$$

$$= 2k \int_0^4 (16x - 8x^2 + x^3) dx = \frac{128k}{3}$$

$$\bar{z} = \frac{M_{xy}}{m} = 1$$

42.  $m = k \int_0^b \int_0^{a[1-(y/b)]} \int_0^{c[1-(y/b)-(x/a)]} dz dx dy = \frac{kabc}{6}$

$$M_{xz} = k \int_0^b \int_0^{a[1-(y/b)]} \int_0^{c[1-(y/b)-(x/a)]} y dz dx dy = \frac{kab^2 c}{24}$$

$$\bar{y} = \frac{M_{xz}}{m} = \frac{kab^2 c / 24}{kabc / 6} = \frac{b}{4}$$

43.  $m = k \int_0^b \int_0^b \int_0^b xy dz dy dx = \frac{kb^5}{4}$

$$M_{yz} = k \int_0^b \int_0^b \int_0^b x^2 y dz dy dx = \frac{kb^6}{6}$$

$$M_{xz} = k \int_0^b \int_0^b \int_0^b xy^2 dz dy dx = \frac{kb^6}{6}$$

$$M_{xy} = k \int_0^b \int_0^b \int_0^b xyz dz dy dx = \frac{kb^6}{8}$$

$$\bar{x} = \frac{M_{yz}}{m} = \frac{kb^6 / 6}{kb^5 / 4} = \frac{2b}{3}$$

$$\bar{y} = \frac{M_{xz}}{m} = \frac{kb^6 / 6}{kb^5 / 4} = \frac{2b}{3}$$

$$\bar{z} = \frac{M_{xy}}{m} = \frac{kb^6 / 8}{kb^5 / 4} = \frac{b}{2}$$

44.  $m = k \int_0^a \int_0^b \int_0^c z dz dy dx = \frac{kabc^2}{2}$

$$M_{xy} = k \int_0^a \int_0^b \int_0^c z^2 dz dy dx = \frac{kabc^3}{3}$$

$$M_{yz} = k \int_0^a \int_0^b \int_0^c xz dz dy dx = \frac{ka^2 bc^2}{4}$$

$$M_{xz} = k \int_0^a \int_0^b \int_0^c yz dz dy dx = \frac{kab^2 c^2}{4}$$

$$\bar{x} = \frac{M_{yz}}{m} = \frac{ka^2 bc^2 / 4}{kabc^2 / 2} = \frac{a}{2}$$

$$\bar{y} = \frac{M_{xz}}{m} = \frac{kab^2 c^2 / 4}{kabc^2 / 2} = \frac{b}{2}$$

$$\bar{z} = \frac{M_{xy}}{m} = \frac{kabc^3 / 3}{kabc^2 / 2} = \frac{2c}{3}$$

45.  $\bar{x}$  will be greater than 2, whereas  $\bar{y}$  and  $\bar{z}$  will be unchanged.

46.  $\bar{z}$  will be greater than  $8/5$ , whereas  $\bar{x}$  and  $\bar{y}$  will be unchanged.

47.  $\bar{y}$  will be greater than 0, whereas  $\bar{x}$  and  $\bar{z}$  will be unchanged.

48.  $\bar{x}$ ,  $\bar{y}$  and  $\bar{z}$  will all be greater than their original values.

$$49. \quad m = \frac{1}{3}k\pi r^2 h$$

$$\bar{x} = \bar{y} = 0 \text{ by symmetry}$$

$$\begin{aligned} M_{xy} &= 4k \int_0^r \int_0^{\sqrt{r^2-x^2}} \int_{h\sqrt{x^2+y^2}/r}^h z \, dz \, dy \, dx \\ &= \frac{2kh^2}{r^2} \int_0^r \int_0^{\sqrt{r^2-x^2}} (r^2 - x^2 - y^2) \, dy \, dx \\ &= \frac{4kh^2}{3r^2} \int_0^r (r^2 - x^2)^{3/2} \, dx = \frac{k\pi r^2 h^2}{4} \\ \bar{z} &= \frac{M_{xy}}{m} = \frac{k\pi r^2 h^2 / 4}{k\pi r^2 h / 3} = \frac{3h}{4} \\ (\bar{x}, \bar{y}, \bar{z}) &= \left(0, 0, \frac{3h}{4}\right) \end{aligned}$$

$$51. \quad m = \frac{128k\pi}{3}$$

$$\bar{x} = \bar{y} = 0 \text{ by symmetry}$$

$$z = \sqrt{4^2 - x^2 - y^2}$$

$$\begin{aligned} M_{xy} &= 4k \int_0^4 \int_0^{\sqrt{4^2-x^2}} \int_0^{\sqrt{4^2-x^2-y^2}} z \, dz \, dy \, dx \\ &= 2k \int_0^4 \int_0^{\sqrt{4^2-x^2}} (4^2 - x^2 - y^2) \, dy \, dx = 2k \int_0^4 \left[ 16y - x^2y - \frac{1}{3}y^3 \right]_0^{\sqrt{4^2-x^2}} \, dx \\ &= \frac{4k}{3} \int_0^4 (4^2 - x^2)^{3/2} \, dx \\ &= \frac{1024k}{3} \int_0^{\pi/2} \cos^4 \theta \, d\theta \quad (\text{let } x = 4 \sin \theta) \\ &= 64\pi k \quad \text{by Wallis's Formula} \\ \bar{z} &= \frac{M_{xy}}{m} = \frac{64k\pi}{1} \cdot \frac{3}{128k\pi} = \frac{3}{2} \\ (\bar{x}, \bar{y}, \bar{z}) &= \left(0, 0, \frac{3}{2}\right) \end{aligned}$$

$$50. \quad m = 2k \int_0^3 \int_0^{\sqrt{9-x^2}} \int_0^y dz \, dy \, dx = 18k$$

$$M_{yz} = k \int_{-3}^3 \int_0^{\sqrt{9-x^2}} \int_0^y x \, dz \, dy \, dx = 0$$

$$M_{xz} = k \int_{-3}^3 \int_0^{\sqrt{9-x^2}} \int_0^y y \, dz \, dy \, dx = \frac{81\pi}{8}k$$

$$M_{xy} = k \int_{-3}^3 \int_0^{\sqrt{9-x^2}} \int_0^y z \, dz \, dy \, dx = \frac{81\pi}{16}k$$

$$\bar{x} = \frac{M_{yz}}{m} = 0$$

$$\bar{y} = \frac{M_{xz}}{m} = \frac{9\pi}{16}$$

$$\bar{z} = \frac{M_{xy}}{m} = \frac{9\pi}{32}$$

$$(\bar{x}, \bar{y}, \bar{z}) = \left(0, \frac{9\pi}{16}, \frac{9\pi}{32}\right)$$

52.  $\bar{x} = 0$

$$m = 2k \int_0^2 \int_0^1 \int_0^{1/(y^2+1)} dz dy dx = 2k \int_0^2 \int_0^1 \frac{1}{y^2+1} dy dx = 2k \left(\frac{\pi}{4}\right) \int_0^2 dx = k\pi$$

$$M_{xz} = 2k \int_0^2 \int_0^1 \int_0^{1/(y^2+1)} y dz dy dx = 2k \int_0^2 \int_0^1 \frac{y}{y^2+1} dy dx = k \int_0^2 (\ln 2) dx = k \ln 4$$

$$M_{xy} = 2k \int_0^2 \int_0^1 \int_0^{1/(y^2+1)} z dz dy dx$$

$$= k \int_0^2 \int_0^1 \frac{1}{(y^2+1)^2} dy dx = k \int_0^2 \left[ \frac{y}{2(y^2+1)} + \frac{1}{2} \arctan y \right]_0^1 dx = k \left( \frac{1}{4} + \frac{\pi}{8} \right) \int_0^2 dx = k \left( \frac{1}{2} + \frac{\pi}{4} \right)$$

$$\bar{y} = \frac{M_{xz}}{m} = \frac{k \ln 4}{k\pi} = \frac{\ln 4}{\pi}$$

$$\bar{z} = \frac{M_{xy}}{m} = k \left( \frac{1}{2} + \frac{\pi}{4} \right) / k\pi = \frac{2 + \pi}{4\pi}$$

$$(\bar{x}, \bar{y}, \bar{z}) = \left( 0, \frac{\ln 4}{\pi}, \frac{2 + \pi}{4\pi} \right)$$

53.  $f(x, y) = \frac{5}{12}y$

$$m = k \int_0^{20} \int_0^{-(3/5)x+12} \int_0^{(5/12)y} dz dy dx = 200k$$

$$M_{yz} = k \int_0^{20} \int_0^{-(3/5)x+12} \int_0^{(5/12)y} x dz dy dx = 1000k$$

$$M_{xz} = k \int_0^{20} \int_0^{-(3/5)x+12} \int_0^{(5/12)y} y dz dy dx = 1200k$$

$$M_{xy} = k \int_0^{20} \int_0^{-(3/5)x+12} \int_0^{(5/12)y} z dz dy dx = 250k$$

$$\bar{x} = \frac{M_{yz}}{m} = \frac{1000k}{200k} = 5$$

$$\bar{y} = \frac{M_{xz}}{m} = \frac{1200k}{200k} = 6$$

$$\bar{z} = \frac{M_{xy}}{m} = \frac{250k}{200k} = \frac{5}{4}$$

$$(\bar{x}, \bar{y}, \bar{z}) = \left( 5, 6, \frac{5}{4} \right)$$

54.  $f(x, y) = \frac{1}{15}(60 - 12x - 20y)$

$$m = k \int_0^5 \int_0^{-(3/5)x+3} \int_0^{(1/15)(60-12x-20y)} dz dy dx = 10k$$

$$M_{yz} = k \int_0^5 \int_0^{-(3/5)x+3} \int_0^{(1/15)(60-12x-20y)} x dz dy dx = \frac{25k}{2}$$

$$M_{xz} = k \int_0^5 \int_0^{-(3/5)x+3} \int_0^{-(1/15)(60-12x-20y)} y dz dy dx = \frac{15k}{2}$$

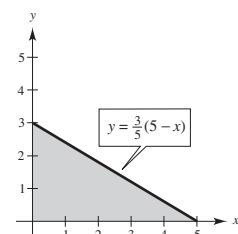
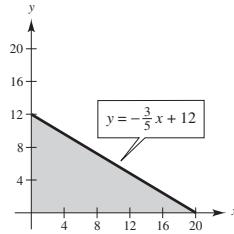
$$M_{xy} = k \int_0^5 \int_0^{-(3/5)x+3} \int_0^{(1/15)(60-12x-20y)} z dz dy dx = 10k$$

$$\bar{x} = \frac{M_{yz}}{m} = \frac{25k/2}{10k} = \frac{5}{4}$$

$$\bar{y} = \frac{M_{xz}}{m} = \frac{15k/2}{10k} = \frac{3}{4}$$

$$\bar{z} = \frac{M_{xy}}{m} = \frac{10k}{10k} = 1$$

$$(\bar{x}, \bar{y}, \bar{z}) = \left( \frac{5}{4}, \frac{3}{4}, 1 \right)$$



$$\begin{aligned}
 55. \text{ (a)} \quad I_x &= k \int_0^a \int_0^a \int_0^a (y^2 + z^2) dx dy dz = ka \int_0^a \int_0^a (y^2 + z^2) dy dz \\
 &= ka \int_0^a \left[ \frac{1}{3}y^3 + z^2y \right]_0^a dz = ka \int_0^a \left( \frac{1}{3}a^3 + az^2 \right) dz = \left[ ka \left( \frac{1}{3}a^3z + \frac{1}{3}az^3 \right) \right]_0^a = \frac{2ka^5}{3} \\
 I_x &= I_y = I_z = \frac{2ka^5}{3} \text{ by symmetry}
 \end{aligned}$$

$$\begin{aligned}
 \text{(b)} \quad I_x &= k \int_0^a \int_0^a \int_0^a (y^2 + z^2) xyz dx dy dz = \frac{ka^2}{2} \int_0^a \int_0^a (y^3z + yz^3) dy dz \\
 &= \frac{ka^2}{2} \int_0^a \left[ \frac{y^4z}{4} + \frac{y^2z^3}{2} \right]_0^a dz = \frac{ka^4}{8} \int_0^a (a^2z + 2z^3) dz = \left[ \frac{ka^4}{8} \left( \frac{a^2z^2}{2} + \frac{2z^4}{4} \right) \right]_0^a = \frac{ka^8}{8} \\
 I_x &= I_y = I_z = \frac{ka^8}{8} \text{ by symmetry}
 \end{aligned}$$

$$56. \text{ (a)} \quad I_{xy} = k \int_{-a/2}^{a/2} \int_{-a/2}^{a/2} \int_{-a/2}^{a/2} z^2 dz dy dx = \frac{ka^5}{12}$$

$$I_{xz} = I_{yz} = \frac{ka^5}{12} \text{ by symmetry}$$

$$I_x = I_y = I_z = \frac{ka^5}{12} + \frac{ka^5}{12} = \frac{ka^5}{6}$$

$$\text{(b)} \quad I_{xy} = k \int_{-a/2}^{a/2} \int_{-a/2}^{a/2} \int_{-a/2}^{a/2} z^2 (x^2 + y^2) dz dy dx = \frac{a^3 k}{12} \int_{-a/2}^{a/2} \int_{-a/2}^{a/2} (x^2 + y^2) dy dx = \frac{a^7 k}{72}$$

$$I_{xy} = k \int_{-a/2}^{a/2} \int_{-a/2}^{a/2} \int_{-a/2}^{a/2} y^2 (x^2 + y^2) dz dy dx = ka \int_{-a/2}^{a/2} \int_{-a/2}^{a/2} (x^2 y^2 + y^4) dy dx = \frac{7ka^7}{360}$$

$I_{yz} = I_{xz}$  by symmetry

$$I_x = I_{xy} + I_{xz} = \frac{a^7 k}{30}$$

$$I_y = I_{xy} + I_{yz} = \frac{a^7 k}{30}$$

$$I_z = I_{yz} + I_{xz} = \frac{7ka^7}{180}$$

$$\begin{aligned}
 57. \text{ (a)} \quad I_x &= k \int_0^4 \int_0^4 \int_0^{4-x} (y^2 + z^2) dz dy dx = k \int_0^4 \int_0^4 \left[ y^2(4-x) + \frac{1}{3}(4-x)^3 \right] dy dx \\
 &= k \int_0^4 \left[ \frac{y^3}{3}(4-x) + \frac{y}{3}(4-x)^3 \right]_0^4 dx = k \int_0^4 \left[ \frac{64}{3}(4-x) + \frac{4}{3}(4-x)^3 \right] dx = k \left[ -\frac{32}{3}(4-x)^2 - \frac{1}{3}(4-x)^4 \right]_0^4 = 256k \\
 I_y &= k \int_0^4 \int_0^4 \int_0^{4-x} (x^2 + z^2) dz dy dx = k \int_0^4 \int_0^4 \left[ x^2(4-x) + \frac{1}{3}(4-x)^3 \right] dy dx \\
 &= 4k \int_0^4 \left[ 4x^2 - x^3 + \frac{1}{3}(4-x)^3 \right] dx = 4k \left[ \frac{4}{3}x^3 - \frac{1}{4}x^4 - \frac{1}{12}(4-x)^4 \right]_0^4 = \frac{512k}{3} \\
 I_z &= k \int_0^4 \int_0^4 \int_0^{4-x} (x^2 + y^2) dz dy dx = k \int_0^4 \int_0^4 (x^2 + y^2)(4-x) dy dx \\
 &= k \int_0^4 \left[ \left( x^2 y + \frac{y^3}{3} \right)(4-x) \right]_0^4 dx = k \int_0^4 \left( 4x^2 + \frac{64}{3} \right)(4-x) dx = 256k
 \end{aligned}$$

$$\begin{aligned}
(b) \quad I_x &= k \int_0^4 \int_0^4 \int_0^{4-x} y(y^2 + z^2) dz dy dx = k \int_0^4 \int_0^4 \left[ y^3(4-x) + \frac{1}{3}y(4-x)^3 \right] dy dx \\
&= k \int_0^4 \left[ \frac{y^4}{4}(4-x) + \frac{y^2}{6}(4-x)^3 \right]_0^4 dx = k \int_0^4 \left[ 64(4-x) + \frac{8}{3}(4-x)^3 \right] dx = k \left[ -32(4-x)^2 - \frac{2}{3}(4-x)^4 \right]_0^4 = \frac{2048k}{3} \\
I_y &= k \int_0^4 \int_0^4 \int_0^{4-x} y(x^2 + z^2) dz dy dx = k \int_0^4 \int_0^4 \left[ x^2y(4-x) + \frac{1}{3}y(4-x)^3 \right] dy dx \\
&= 8k \int_0^4 \left[ 4x^2 - x^3 + \frac{1}{3}(4-x)^3 \right] dx = 8k \left[ \frac{4}{3}x^3 - \frac{1}{4}x^4 - \frac{1}{12}(4-x)^4 \right]_0^4 = \frac{1024k}{3} \\
I_z &= k \int_0^4 \int_0^4 \int_0^{4-x} y(x^2 + y^2) dz dy dx = k \int_0^4 \int_0^4 (x^2y + y^3)(4-x) dx \\
&= k \int_0^4 \left[ \left( \frac{x^2y^2}{2} + \frac{y^4}{4} \right)(4-x) \right]_0^4 dx = k \int_0^4 (8x^2 + 64)(4-x) dx \\
&= 8k \int_0^4 (32 - 8x + 4x^2 - x^3) dx = \left[ 8k \left( 32x - 4x^2 + \frac{4}{3}x^3 - \frac{1}{4}x^4 \right) \right]_0^4 = \frac{2048k}{3}
\end{aligned}$$

$$\begin{aligned}
58. (a) \quad I_{xy} &= k \int_0^4 \int_0^2 \int_0^{4-y^2} z^3 dz dy dx = k \int_0^4 \int_0^2 \frac{1}{4}(4-y^2)^4 dy dx = \frac{k}{4} \int_0^4 \int_0^2 (256 - 256y^2 + 96y^4 - 16y^6 + y^8) dy dx \\
&= \frac{k}{4} \int_0^4 \left[ 256y - \frac{256y^3}{3} + \frac{96y^5}{5} - \frac{16y^7}{7} + \frac{y^9}{9} \right]_0^2 dx = k \int_0^4 \frac{16,384}{945} dx = \frac{65,536k}{315} \\
I_{xz} &= k \int_0^4 \int_0^2 \int_0^{4-y^2} y^2 z dz dy dx = k \int_0^4 \int_0^2 \frac{1}{2}y^2(4-y^2)^2 dy dx \\
&= k \int_0^4 \int_0^2 \frac{1}{2}(16y^2 - 8y^4 + y^6) dy dx = \frac{k}{2} \int_0^4 \left[ \frac{16y^3}{3} - \frac{8y^5}{5} + \frac{y^7}{7} \right]_0^2 dx = \frac{k}{2} \int_0^4 \frac{1024}{105} dx = \frac{2048k}{105} \\
I_{yz} &= k \int_0^4 \int_0^2 \int_0^{4-y^2} x^2 z dz dy dx = k \int_0^4 \int_0^2 \frac{1}{2}x^2(4-y^2)^2 dy dx \\
&= k \int_0^4 \int_0^2 \frac{1}{2}x^2(16 - 8y^2 + y^4) dy dx = \frac{k}{2} \int_0^4 \left[ x^2 \left( 16y - \frac{8y^3}{3} + \frac{y^5}{5} \right) \right]_0^4 dx = \frac{k}{2} \int_0^4 \frac{256}{15} x^2 dx = \frac{8192k}{45} \\
I_x &= I_{xz} + I_{xy} = \frac{2048k}{9}, I_y = I_{yz} + I_{xy} = \frac{8192k}{21}, I_z = I_{yz} + I_{xz} = \frac{63,488k}{315}
\end{aligned}$$

$$\begin{aligned}
(b) \quad I_{xy} &= \int_0^4 \int_0^2 \int_0^{4-y^2} z^2(4-z) dz dy dx \\
&= k \int_0^4 \int_0^2 \int_0^{4-y^2} 4z^2 dz dy dx - k \int_0^4 \int_0^2 \int_0^{4-y^2} z^3 dz dy dx = \frac{32,768k}{105} - \frac{65,536k}{315} = \frac{32,768k}{315} \\
I_{xz} &= \int_0^4 \int_0^2 \int_0^{4-y^2} y^2(4-z) dz dy dx \\
&= k \int_0^4 \int_0^2 \int_0^{4-y^2} 4y^2 dz dy dx - k \int_0^4 \int_0^2 \int_0^{4-y^2} y^2 z dz dy dx = \frac{1024k}{15} - \frac{2048k}{105} = \frac{1024k}{21} \\
I_{yz} &= k \int_0^4 \int_0^2 \int_0^{4-y^2} x^2(4-z) dz dy dx \\
&= k \int_0^4 \int_0^2 \int_0^{4-y^2} 4x^2 dz dy dx - k \int_0^4 \int_0^2 \int_0^{4-y^2} x^2 z dz dy dx = \frac{4096k}{9} - \frac{8192k}{45} = \frac{4096k}{15} \\
I_x &= I_{xz} + I_{xy} = \frac{48,128k}{315}, I_y = I_{yz} + I_{xy} = \frac{118,784k}{315}, I_z = I_{xz} + I_{yz} = \frac{11,264k}{35}
\end{aligned}$$

$$\begin{aligned}
59. \quad I_{xy} &= k \int_{-L/2}^{L/2} \int_{-a}^a \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} z^2 \, dz \, dx \, dy = k \int_{-L/2}^{L/2} \int_{-a}^a \frac{2}{3}(a^2 - x^2) \sqrt{a^2 - x^2} \, dx \, dy \\
&= \frac{2}{3} \int_{-L/2}^{L/2} k \left[ \frac{a^2}{2} \left( x \sqrt{a^2 - x^2} + a^2 \arcsin \frac{x}{a} \right) - \frac{1}{8} x (2x^2 - a^2) \sqrt{a^2 - x^2} + a^4 \arcsin \frac{x}{a} \right]_a^a \, dy \\
&= \frac{2k}{3} \int_{-L/2}^{L/2} 2 \left( \frac{a^4 \pi}{4} - \frac{a^4 \pi}{16} \right) dy = \frac{a^4 \pi L k}{4}
\end{aligned}$$

Because  $m = \pi a^2 L k$ ,  $I_{xy} = ma^2/4$ .

$$\begin{aligned}
I_{xz} &= k \int_{-L/2}^{L/2} \int_{-a}^a \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} y^2 \, dz \, dx \, dy = 2k \int_{-L/2}^{L/2} \int_{-a}^a y^2 \sqrt{a^2 - x^2} \, dx \, dy \\
&= 2k \int_{-L/2}^{L/2} \left[ \frac{y^2}{2} \left( x \sqrt{a^2 - x^2} + a^2 \arcsin \frac{x}{a} \right) \right]_a^a \, dy = k \pi a^2 \int_{-L/2}^{L/2} y^2 \, dy = \frac{2k \pi a^2}{3} \left( \frac{L^3}{8} \right) = \frac{1}{12} m L^2 \\
I_{yz} &= k \int_{-L/2}^{L/2} \int_{-a}^a \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} x^2 \, dz \, dx \, dy = 2k \int_{-L/2}^{L/2} \int_{-a}^a x^2 \sqrt{a^2 - x^2} \, dx \, dy \\
&= 2k \int_{-L/2}^{L/2} \frac{1}{8} \left[ x (2x^2 - a^2) \sqrt{a^2 - x^2} + a^4 \arcsin \frac{x}{a} \right]_a^a \, dy = \frac{ka^4 \pi}{4} \int_{-L/2}^{L/2} dy = \frac{ka^4 \pi L}{4} = \frac{ma^2}{4} \\
I_x &= I_{xy} + I_{xz} = \frac{ma^2}{4} + \frac{mL^2}{12} = \frac{m}{12} (3a^2 + L^2) \\
I_y &= I_{xy} + I_{yz} = \frac{ma^2}{4} + \frac{ma^2}{4} = \frac{ma^2}{2} \\
I_z &= I_{xz} + I_{yz} = \frac{mL^2}{12} + \frac{ma^2}{4} = \frac{m}{12} (3a^2 + L^2)
\end{aligned}$$

$$\begin{aligned}
60. \quad I_{xy} &= \int_{-c/2}^{c/2} \int_{-a/2}^{a/2} \int_{-b/2}^{b/2} z^2 \, dz \, dy \, dx = \frac{b^3}{12} \int_{-c/2}^{c/2} \int_{-a/2}^{a/2} dy \, dx = \frac{1}{12} b^2 (abc) = \frac{1}{12} mb^2 \\
I_{xz} &= \int_{-c/2}^{c/2} \int_{-a/2}^{a/2} \int_{-b/2}^{b/2} y^2 \, dz \, dy \, dx = b \int_{-c/2}^{c/2} \int_{-a/2}^{a/2} y^2 \, dy \, dx = \frac{ba^3}{12} \int_{-c/2}^{c/2} dx = \frac{ba^3 c}{12} = \frac{1}{12} a^2 (abc) = \frac{1}{12} ma^2 \\
I_{yz} &= \int_{-c/2}^{c/2} \int_{-a/2}^{a/2} \int_{-b/2}^{b/2} x^2 \, dz \, dy \, dx = ab \int_{-c/2}^{c/2} x^2 \, dx = \frac{abc^3}{12} = \frac{1}{12} c^2 (abc) = \frac{1}{12} mc^2 \\
I_x &= I_{xy} + I_{xz} = \frac{1}{12} m (a^2 + b^2) \\
I_y &= I_{xy} + I_{yz} = \frac{1}{12} m (b^2 + c^2) \\
I_z &= I_{xz} + I_{yz} = \frac{1}{12} m (a^2 + c^2)
\end{aligned}$$

$$61. \int_{-1}^1 \int_{-1}^1 \int_0^{1-x} (x^2 + y^2) \sqrt{x^2 + y^2 + z^2} \, dz \, dy \, dx$$

$$62. \int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_0^{4-x^2-y^2} kx^2 (x^2 + y^2) \, dz \, dy \, dx$$

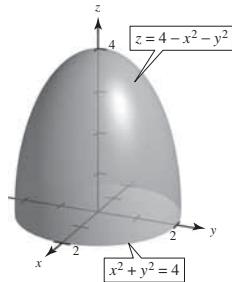
$$63. \rho = kz$$

$$(a) \quad m = \int_{-2}^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_0^{4-x^2-y^2} (kz) \, dz \, dy \, dx \left( = \frac{32k\pi}{3} \right)$$

$$(b) \quad \bar{x} = \bar{y} = 0 \text{ by symmetry}$$

$$\bar{z} = \frac{M_{xy}}{m} = \frac{1}{m} \int_{-2}^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_0^{4-x^2-y^2} kz^2 \, dz \, dy \, dx (= 2)$$

$$(c) \quad I_z = \int_{-2}^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_0^{4-x^2-y^2} (x^2 + y^2) kz \, dz \, dy \, dx \left( = \frac{32k\pi}{3} \right)$$



64.  $\rho = kxy$

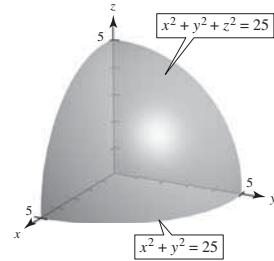
$$(a) m = \int_0^5 \int_0^{\sqrt{25-x^2}} \int_0^{\sqrt{25-x^2-y^2}} kxy \, dz \, dy \, dx \left( = \frac{625}{3}k \right)$$

$$(b) \bar{x} = \frac{M_{yz}}{m} = \frac{1}{m} \int_0^5 \int_0^{\sqrt{25-x^2}} \int_0^{\sqrt{25-x^2-y^2}} x(kxy) \, dz \, dy \, dx \left( = \frac{25\pi}{32} \right)$$

$\bar{y} = \bar{x}$  by symmetry

$$\bar{z} = \frac{M_{xy}}{m} = \frac{1}{m} \int_0^5 \int_0^{\sqrt{25-x^2}} \int_0^{\sqrt{25-x^2-y^2}} z(kxy) \, dz \, dy \, dx \left( = \frac{25}{16} \right)$$

$$(c) I_z = \int_0^5 \int_0^{\sqrt{25-x^2}} \int_0^{\sqrt{25-x^2-y^2}} (x^2 + y^2)kxy \, dz \, dy \, dx \left( = \frac{62500}{21}k \right)$$



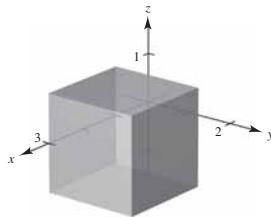
65. See the definition, page 1027.

See Theorem 14.4, page 1028.

66. Because the density increases as you move away from the axis of symmetry, the moment of inertia will increase.

67. (a) The annular solid on the right has the greater density.  
 (b) The annular solid on the right has the greater moment of inertia.  
 (c) The solid on the left will reach the bottom first. The solid on the right has a greater resistance to rotational motion.

68. The region of integration is a cube:



Answer: (a)

$$71. V = \frac{1}{3} \text{base} \times \text{height}$$

$$= \frac{1}{3} \left( \frac{1}{2}(2)(2) \right)(2) = \frac{4}{3}$$

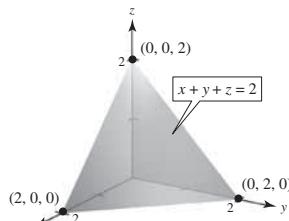
$$f(x, y, z) = x + y + z$$

$$\text{Plane: } x + y + z = 2$$

$$\text{Average value} = \frac{1}{V} \iiint_Q f(x, y, z) \, dV$$

$$= \frac{3}{4} \int_0^2 \int_0^{2-x} \int_0^{2-x-y} (x + y + z) \, dz \, dy \, dx = \frac{3}{4} \int_0^2 \int_0^{2-x} \frac{1}{2}(2-x-y)(x+y+2) \, dy \, dx$$

$$= \frac{3}{4} \int_0^2 \frac{1}{6}(x+4)(x-2)^2 \, dx = \frac{3}{4}(2) = \frac{3}{2}$$



69.  $V = 1$  (unit cube)

$$\begin{aligned} \text{Average value} &= \frac{1}{V} \iiint_Q f(x, y, z) \, dV \\ &= \int_0^1 \int_0^1 \int_0^1 (z^2 + 4) \, dx \, dy \, dz \\ &= \int_0^1 \int_0^1 (z^2 + 4) \, dy \, dz = \int_0^1 (z^2 + 4) \, dz \\ &= \left[ \frac{z^3}{3} + 4z \right]_0^1 = \frac{1}{3} + 4 = \frac{13}{3} \end{aligned}$$

70.  $V = 64$  (cube with sides of length 4)

$$\begin{aligned} \text{Average value} &= \frac{1}{V} \iiint_Q f(x, y, z) \, dV \\ &= \frac{1}{64} \int_0^4 \int_0^4 \int_0^4 xyz \, dx \, dy \, dz \\ &= \frac{1}{64} \int_0^4 \int_0^4 8yz \, dy \, dz \\ &= \frac{1}{8} \int_0^4 8z \, dz = \int_0^4 z \, dz = 8 \end{aligned}$$

72.  $V = \frac{4}{3}\pi(\sqrt{3})^3 = 4\sqrt{3}\pi$

Average value  $= \frac{1}{V} \iiint_Q f(x, y, z) dV = \frac{1}{4\sqrt{3}\pi} \int_{-\sqrt{3}}^{\sqrt{3}} \int_{-\sqrt{3-x^2}}^{\sqrt{3-x^2}} \int_{-\sqrt{3-x^2-y^2}}^{\sqrt{3-x^2-y^2}} (x + y) dz dy dx = 0$ , by symmetry

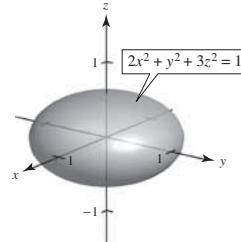
73.  $1 - 2x^2 - y^2 - 3z^2 \geq 0$

$2x^2 + y^2 + 3z^2 \leq 1$

$Q = \{(x, y, z) : 2x^2 + y^2 + 3z^2 \leq 1\}$  ellipsoid

$$\int_{-1/\sqrt{2}}^{1/\sqrt{2}} \int_{-\sqrt{1-2x^2}}^{\sqrt{1-2x^2}} \int_{-\sqrt{(1-2x^2-y^2)/3}}^{\sqrt{(1-2x^2-y^2)/3}} (1 - 2x^2 - y^2 - 3z^2) dz dy dx \approx 0.684$$

Exact value:  $\frac{4\sqrt{6}\pi}{45}$



74.  $1 - x^2 - y^2 - z^2 \geq 0$

$x^2 + y^2 + z^2 \leq 1$

$Q = \{(x, y, z) : x^2 + y^2 + z^2 \leq 1\}$  sphere

$$\int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{-\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}} (1 - x^2 - y^2 - z^2) dz dy dx \approx 1.6755$$

Exact value:  $\frac{8\pi}{15}$

75.  $\frac{14}{15} = \int_0^1 \int_0^{3-a-y^2} \int_a^{4-x-y^2} dz dx dy = \int_0^1 \int_0^{3-a-y^2} (4 - x - y^2 - a) dx dy$

$$= \int_0^1 \left[ (4 - y^2 - a)x - \frac{x^2}{2} \right]_0^{3-a-y^2} dy = \int_0^1 \left[ (4 - y^2 - a)(3 - a - y^2) - \frac{(3 - a - y^2)^2}{2} \right] dy = \frac{94}{15} - \frac{11a}{3} + \frac{1}{2}a^2$$

So,  $3a^2 - 22a + 32 = 0$

$(a - 2)(3a - 16) = 0$

$$a = 2, \frac{16}{3}.$$

76.  $x^2 + \frac{y^2}{b^2} + \frac{z^2}{9} = 1$

By symmetry, the volume in the first octant is

$$\frac{1}{8}(16\pi) = 2\pi.$$

$$2\pi = \int_0^1 \int_0^{b\sqrt{1-x^2}} \int_{-\sqrt{1-x^2-y^2/b^2}}^{\sqrt{1-x^2-y^2/b^2}} 1 dz dy dx$$

By trial and error,  $b = 4$ .

**Note:** Volume at ellipsoid  $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$  is  $\frac{4}{3}\pi abc$ .

77. Let  $y_k = 1 - x_k$ .

$$\frac{\pi}{2n}(x_1 + \dots + x_n) = \frac{\pi}{2n}(n - y_1 - y_2 - \dots - y_n) = \frac{\pi}{2} - \frac{\pi}{2n}(y_1 + \dots + y_n)$$

So,

$$\begin{aligned} I_1 &= \int_0^1 \int_0^1 \cdots \int_0^1 \cos^2 \left\{ \frac{\pi}{2n}(x_1 + \dots + x_n) \right\} dx_1 dx_2 \cdots dx_n \\ &= \int_1^0 \int_1^0 \cdots \int_1^0 \sin^2 \left\{ \frac{\pi}{2n}(y_1 + \dots + y_n) \right\} (-dy_1)(-dy_2) \cdots (-dy_n) = \int_0^1 \int_0^1 \cdots \int_0^1 \sin^2 \left\{ \frac{\pi}{2n}(x_1 + \dots + x_n) \right\} dx_1 dx_2 \cdots dx_n = I_2 \end{aligned}$$

$$I_1 + I_2 = 1 \Rightarrow I_1 = \frac{1}{2}.$$

$$\text{Finally, } \lim_{n \rightarrow \infty} I_1 = \frac{1}{2}.$$

## Section 14.7 Triple Integrals in Cylindrical and Spherical Coordinates

$$1. \int_{-1}^5 \int_0^{\pi/2} \int_0^3 r \cos \theta \, dr \, d\theta \, dz = \int_{-1}^5 \int_0^{\pi/2} \frac{9}{2} \cos \theta \, d\theta \, dz = \int_{-1}^5 \left[ \frac{9}{2} \sin \theta \right]_0^{\pi/2} dz = \int_{-1}^5 \frac{9}{2} \, dz = \left[ \frac{9}{2} z \right]_{-1}^5 = \frac{9}{2}(5 - (-1)) = 27$$

$$\begin{aligned} 2. \int_0^{\pi/4} \int_0^6 \int_0^{6-r} rz \, dz \, dr \, d\theta &= \int_0^{\pi/4} \int_0^6 \left[ \frac{rz^2}{2} \right]_0^{6-r} dr \, d\theta = \int_0^{\pi/4} \int_0^6 \frac{1}{2} (r^3 - 12r^2 + 36r) dr \, d\theta \\ &= \int_0^{\pi/4} \frac{1}{2} \left[ \frac{r^4}{4} - 4r^3 + 18r^2 \right]_0^6 d\theta = \int_0^{\pi/4} \frac{1}{2} (108) d\theta = 54 \left( \frac{\pi}{4} \right) = \frac{27\pi}{2} \end{aligned}$$

$$\begin{aligned} 3. \int_0^{\pi/2} \int_0^{2\cos^2 \theta} \int_0^{4-r^2} r \sin \theta \, dz \, dr \, d\theta &= \int_0^{\pi/2} \int_0^{2\cos^2 \theta} r(4 - r^2) \sin \theta \, dr \, d\theta = \int_0^{\pi/2} \left[ \left( 2r^2 - \frac{r^4}{4} \right) \sin \theta \right]_0^{2\cos^2 \theta} d\theta \\ &= \int_0^{\pi/2} [8 \cos^4 \theta - 4 \cos^8 \theta] \sin \theta \, d\theta = \left[ -\frac{8 \cos^5 \theta}{5} + \frac{4 \cos^9 \theta}{9} \right]_0^{\pi/2} = \frac{52}{45} \end{aligned}$$

$$4. \int_0^{\pi/2} \int_0^\pi \int_0^2 e^{-\rho^3} \rho^2 \, d\rho \, d\theta \, d\phi = \int_0^{\pi/2} \int_0^\pi \left[ -\frac{1}{3} e^{-\rho^3} \right]_0^2 d\theta \, d\phi = \int_0^{\pi/2} \int_0^\pi \frac{1}{3} (1 - e^{-8}) d\theta \, d\phi = \frac{\pi^2}{6} (1 - e^{-8})$$

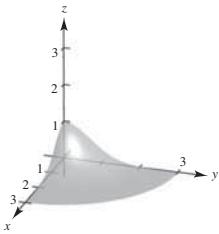
$$5. \int_0^{2\pi} \int_0^{\pi/4} \int_0^{\cos \phi} \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta = \frac{1}{3} \int_0^{2\pi} \int_0^{\pi/4} \cos^3 \phi \sin \phi \, d\phi \, d\theta = -\frac{1}{12} \int_0^{2\pi} \left[ \cos^4 \phi \right]_0^{\pi/4} d\theta = \frac{\pi}{8}$$

$$\begin{aligned} 6. \int_0^{\pi/4} \int_0^{\pi/4} \int_0^{\cos \theta} \rho^2 \sin \phi \cos \phi \, d\rho \, d\theta \, d\phi &= \frac{1}{3} \int_0^{\pi/4} \int_0^{\pi/4} \cos^3 \theta \sin \phi \cos \phi \, d\theta \, d\phi \\ &= \frac{1}{3} \int_0^{\pi/4} \int_0^{\pi/4} \sin \phi \cos \phi [\cos \theta (1 - \sin^2 \theta)] \, d\theta \, d\phi \\ &= \frac{1}{3} \int_0^{\pi/4} \sin \phi \cos \phi \left[ \sin \theta - \frac{\sin^3 \theta}{3} \right]_0^{\pi/4} \, d\phi \\ &= \frac{5\sqrt{2}}{36} \int_0^{\pi/4} \sin \phi \cos \phi \, d\phi = \left[ \frac{5\sqrt{2}}{36} \frac{\sin^2 \phi}{2} \right]_0^{\pi/4} = \frac{5\sqrt{2}}{144} \end{aligned}$$

$$7. \int_0^4 \int_0^z \int_0^{\pi/2} re^r \, d\theta \, dr \, dz = \pi(e^4 + 3)$$

$$8. \int_0^{\pi/2} \int_0^\pi \int_0^{\sin \theta} (2 \cos \phi) \rho^2 \, d\rho \, d\theta \, d\phi = \frac{8}{9}$$

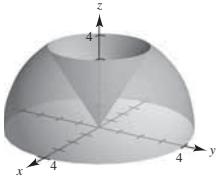
9.  $\int_0^{\pi/2} \int_0^3 \int_0^{e^{-r^2}} r \, dz \, dr \, d\theta = \int_0^{\pi/2} \int_0^3 re^{-r^2} \, dr \, d\theta = \int_0^{\pi/2} \left[ -\frac{1}{2}e^{-r^2} \right]_0^3 d\theta = \int_0^{\pi/2} \frac{1}{2}(1 - e^{-9}) \, d\theta = \frac{\pi}{4}(1 - e^{-9})$



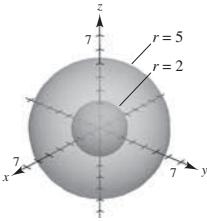
10.  $\int_0^{2\pi} \int_0^{\sqrt{5}} \int_2^{5-r^2} r \, dz \, dr \, d\theta = \int_0^{2\pi} \int_0^{\sqrt{5}} (5r - r^3) \, dr \, d\theta = \int_0^{2\pi} \left[ \frac{5r^2}{2} - \frac{r^4}{4} \right]_0^{\sqrt{5}} d\theta = \int_0^{2\pi} \left( \frac{25}{2} - \frac{25}{4} \right) d\theta = \frac{25}{4} \cdot 2\pi = \frac{25\pi}{2}$



11.  $\int_0^{2\pi} \int_{\pi/6}^{\pi/2} \int_0^4 \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta = \frac{64}{3} \int_0^{2\pi} \int_{\pi/6}^{\pi/2} \sin \phi \, d\phi \, d\theta = \frac{64}{3} \int_0^{2\pi} [-\cos \phi]_{\pi/6}^{\pi/2} d\theta = \frac{32\sqrt{3}}{3} \int_0^{2\pi} d\theta = \frac{64\sqrt{3}\pi}{3}$



12.  $\int_0^{2\pi} \int_0^{\pi} \int_0^5 \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta = \frac{117}{3} \int_0^{2\pi} \int_0^{\pi} \sin \phi \, d\phi \, d\theta = \frac{117}{3} \int_0^{2\pi} [-\cos \phi]_0^{\pi} d\theta = \frac{468\pi}{3} = 156\pi$



13.  $\int_0^{2\pi} \int_0^2 \int_{r^2}^4 r^2 \cos \theta \, dz \, dr \, d\theta = 0$

$$\int_0^{2\pi} \int_0^{\arctan(1/2)} \int_0^{4 \sec \phi} \rho^3 \sin^2 \phi \cos \theta \, d\rho \, d\phi \, d\theta + \int_0^{2\pi} \int_{\arctan(1/2)}^{\cot \phi \csc \phi} \rho^3 \sin^2 \phi \cos \theta \, d\rho \, d\phi \, d\theta = 0$$

14.  $\int_0^{\pi/2} \int_0^2 \int_0^{\sqrt{16-r^2}} r^2 \, dz \, dr \, d\theta = \frac{8\pi^2}{3} - 2\pi\sqrt{3}$

$$\int_0^{\pi/2} \int_0^{\pi/6} \int_0^4 \rho^3 \sin^2 \phi \, d\rho \, d\phi \, d\theta + \int_0^{\pi/2} \int_{\pi/6}^{\pi/2} \int_4^{2 \csc \phi} \rho^3 \sin^2 \phi \, d\rho \, d\phi \, d\theta = \frac{8\pi^2}{3} - 2\pi\sqrt{3}$$

15.  $\int_0^{2\pi} \int_0^a \int_a^{a+\sqrt{a^2-r^2}} r^2 \cos \theta \, dz \, dr \, d\theta = 0$

$$\int_0^{\pi/4} \int_0^{2\pi} \int_{a \sec \phi}^{2a \cos \phi} \rho^3 \sin^2 \phi \cos \theta \, d\rho \, d\theta \, d\phi = 0$$

16.  $\int_0^{\pi/2} \int_0^3 \int_0^{\sqrt{9-r^2}} \sqrt{r^2 + z^2} r dz dr d\theta = \frac{81\pi}{8}$

$$\int_0^{\pi/2} \int_0^{\pi/2} \int_0^3 \rho^3 \sin \phi d\rho d\phi d\theta = \frac{81\pi}{8}$$

17.  $V = 4 \int_0^{\pi/2} \int_0^{a \cos \theta} \int_0^{\sqrt{a^2 - r^2}} r dz dr d\theta = 4 \int_0^{\pi/2} \int_0^{a \cos \theta} r \sqrt{a^2 - r^2} dr d\theta$

$$= \frac{4}{3} a^3 \int_0^{\pi/2} (1 - \sin^3 \theta) d\theta = \frac{4}{3} a^3 \left[ \theta + \frac{1}{3} \cos \theta (\sin^2 \theta + 2) \right]_0^{\pi/2} = \frac{4}{3} a^3 \left( \frac{\pi}{2} - \frac{2}{3} \right) = \frac{2a^3}{9} (3\pi - 4)$$

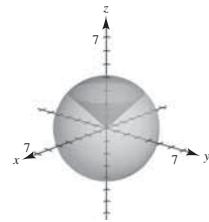
18.  $V = \frac{2}{3}\pi(4)^3 + 4 \left[ \int_0^{\pi/2} \int_0^{2\sqrt{2}} \int_0^r r dz dr d\theta + \int_0^{\pi/2} \int_{2\sqrt{2}}^4 \int_0^{\sqrt{16-r^2}} r dz dr d\theta \right]$

(Volume of lower hemisphere) + 4(Volume in the first octant)

$$V = \frac{128\pi}{3} + 4 \left[ \int_0^{\pi/2} \int_0^{2\sqrt{2}} r^2 dr d\theta + \int_0^{\pi/2} \int_{2\sqrt{2}}^4 r \sqrt{16 - r^2} dr d\theta \right]$$

$$= \frac{128\pi}{3} + 4 \left[ \frac{8\sqrt{2}\pi}{3} + \int_0^{\pi/2} \left[ -\frac{1}{3}(16 - r^2)^{3/2} \right]_{2\sqrt{2}}^4 d\theta \right]$$

$$= \frac{128\pi}{3} + 4 \left[ \frac{8\sqrt{2}\pi}{3} + \frac{8\sqrt{2}\pi}{3} \right] = \frac{128\pi}{3} + \frac{64\sqrt{2}\pi}{3} = \frac{64\pi}{3}(2 + \sqrt{2})$$



19. In the  $xy$ -plane,  $2x = 2x^2 + 2y^2 \Rightarrow$

$$0 = x^2 - x + y^2 \Rightarrow (x^2 - x + 1/4) + y^2 = 1/4 \\ \Rightarrow (x - 1/2)^2 + y^2 = (1/2)^2$$

In polar coordinates, use  $r = \cos \theta$  for this circle.

$$V = \int_0^\pi \int_0^{\cos \theta} \int_{2r^2}^{2r \cos \theta} r dz dr d\theta \\ = \int_0^\pi \int_0^{\cos \theta} (2r^2 \cos \theta - 2r^3) dr d\theta \\ = \int_0^\pi \left[ \frac{2r^3}{3} \cos \theta - \frac{r^4}{2} \right]_0^{\cos \theta} d\theta \\ = \int_0^\pi \left( \frac{2}{3} \cos^4 \theta - \frac{\cos^4 \theta}{2} \right) d\theta \\ = \frac{1}{6} \int_0^\pi \cos^4 \theta d\theta = \frac{\pi}{16}$$

20.  $2 - x^2 - y^2 = x^2 + y^2 \Rightarrow x^2 + y^2 = 1$

$$V = \int_0^{2\pi} \int_0^1 \int_{r^2}^{2-r^2} r dz dr d\theta \\ = \int_0^{2\pi} \int_0^1 r(2 - 2r^2) dr d\theta \\ = \int_0^{2\pi} \left[ r^2 - \frac{r^4}{2} \right]_0^1 d\theta = \frac{1}{2}(2\pi) = \pi$$

21.  $V = 2 \int_0^\pi \int_0^{a \cos \theta} \int_0^{\sqrt{a^2 - r^2}} r dz dr d\theta$

$$= 2 \int_0^\pi \int_0^{a \cos \theta} r \sqrt{a^2 - r^2} dr d\theta$$

$$= 2 \int_0^\pi \left[ -\frac{1}{3}(a^2 - r^2)^{3/2} \right]_0^{a \cos \theta} d\theta$$

$$= \frac{2a^3}{3} \int_0^\pi (1 - \sin^3 \theta) d\theta$$

$$= \frac{2a^3}{3} \left[ \theta + \cos \theta - \frac{\cos^3 \theta}{3} \right]_0^\pi = \frac{2a^3}{9}(3\pi - 4)$$

22.  $V = \int_0^{2\pi} \int_0^{\sqrt{2}} \int_r^{\sqrt{4-r^2}} r dz dr d\theta$

$$= \int_0^{2\pi} \int_0^{\sqrt{2}} (r \sqrt{4 - r^2} - r^2) dr d\theta$$

$$= \int_0^{2\pi} \left[ -\frac{1}{3}(4 - r^2)^{3/2} - \frac{r^3}{3} \right]_0^{\sqrt{2}} d\theta = \frac{8\pi}{3}(2 - \sqrt{2})$$

23.  $m = \int_0^{2\pi} \int_0^2 \int_{9-r \cos \theta - 2r \sin \theta}^{9-r \cos \theta + 2r \sin \theta} (kr) r dz dr d\theta$

$$= \int_0^{2\pi} \int_0^2 kr^2 (9 - r \cos \theta - 2r \sin \theta) dr d\theta$$

$$= \int_0^{2\pi} k \left[ 3r^3 - \frac{r^4}{4} \cos \theta - \frac{r^4}{2} \sin \theta \right]_0^2 d\theta$$

$$= \int_0^{2\pi} k [24 - 4 \cos \theta - 8 \sin \theta] d\theta$$

$$= k [24\theta - 4 \sin \theta + 8 \cos \theta]_0^{2\pi}$$

$$= k [48\pi + 8 - 8] = 48k\pi$$

$$\begin{aligned}
 24. \int_0^{\pi/2} \int_0^2 \int_0^{12e^{-r^2}} k r dz dr d\theta &= \int_0^{\pi/2} \int_0^2 12ke^{-r^2} r dr d\theta \\
 &= \int_0^{\pi/2} \left[ -6ke^{-r^2} \right]_0^2 \\
 &= \int_0^{\pi/2} (-6ke^{-4} + 6k) d\theta \\
 &= 3k\pi(1 - e^{-4})
 \end{aligned}$$

$$\begin{aligned}
 25. z &= h - \frac{h}{r_0} \sqrt{x^2 + y^2} = \frac{h}{r_0}(r_0 - r) \\
 V &= 4 \int_0^{\pi/2} \int_0^{r_0} \int_0^{h(r_0-r)/r_0} r dz dr d\theta \\
 &= \frac{4h}{r_0} \int_0^{\pi/2} \int_0^{r_0} (r_0 r - r^2) dr d\theta \\
 &= \frac{4h}{r_0} \int_0^{\pi/2} \frac{r_0^3}{6} d\theta \\
 &= \frac{4h}{r_0} \left( \frac{r_0^3}{6} \right) \left( \frac{\pi}{2} \right) = \frac{1}{3}\pi r_0^2 h
 \end{aligned}$$

26.  $\bar{x} = \bar{y} = 0$  by symmetry

$$m = \frac{1}{3}\pi r_0^2 h k \text{ from Exercise 25}$$

$$\begin{aligned}
 M_{xy} &= 4k \int_0^{\pi/2} \int_0^{r_0} \int_0^{h(r_0-r)/r_0} zr dz dr d\theta \\
 &= \frac{2kh^2}{r_0^2} \int_0^{\pi/2} \int_0^{r_0} (r_0^2 r - 2r_0 r^2 + r^3) dr d\theta \\
 &= \frac{2kh^2}{r_0^2} \left( \frac{r_0^4}{12} \right) \left( \frac{\pi}{2} \right) = \frac{k r_0^2 h^2 \pi}{12} \\
 \bar{z} &= \frac{M_{xy}}{m} = \frac{k r_0^2 h^2 \pi}{12} \left( \frac{3}{\pi r_0^2 h k} \right) = \frac{h}{4}
 \end{aligned}$$

$$27. \rho = k\sqrt{x^2 + y^2} = kr$$

$\bar{x} = \bar{y} = 0$  by symmetry

$$m = 4k \int_0^{\pi/2} \int_0^{r_0} \int_0^{h(r_0-r)/r_0} r^2 dz dr d\theta = \frac{1}{6}k\pi r_0^3 h$$

$$\begin{aligned}
 M_{xy} &= 4k \int_0^{\pi/2} \int_0^{r_0} \int_0^{h(r_0-r)/r_0} r^2 z dz dr d\theta \\
 &= \frac{1}{30}k\pi r_0^3 h^2 \\
 \bar{z} &= \frac{M_{xy}}{m} = \frac{k\pi r_0^3 h^2 / 30}{k\pi r_0^3 h / 6} = \frac{h}{5} \\
 (\bar{x}, \bar{y}, \bar{z}) &= \left( 0, 0, \frac{h}{5} \right)
 \end{aligned}$$

$$\begin{aligned}
 28. \rho &= kz \\
 \bar{x} &= \bar{y} = 0 \text{ by symmetry} \\
 m &= 4k \int_0^{\pi/2} \int_0^{r_0} \int_0^{h(r_0-r)/r_0} zr dz dr d\theta = \frac{1}{12}k\pi r_0^2 h^2
 \end{aligned}$$

$$\begin{aligned}
 M_{xy} &= 4k \int_0^{\pi/2} \int_0^{r_0} \int_0^{h(r_0-r)/r_0} z^2 r dz dr d\theta \\
 &= \frac{1}{30}k\pi r_0^2 h^3 \\
 \bar{z} &= \frac{M_{xy}}{m} = \frac{k\pi r_0^2 h^3 / 30}{k\pi r_0^2 h^2 / 12} = \frac{2h}{5} \\
 (\bar{x}, \bar{y}, \bar{z}) &= \left( 0, 0, \frac{2h}{5} \right)
 \end{aligned}$$

$$\begin{aligned}
 29. I_z &= 4k \int_0^{\pi/2} \int_0^{r_0} \int_0^{h(r_0-r)/r_0} r^3 dz dr d\theta \\
 &= \frac{4kh}{r_0} \int_0^{\pi/2} \int_0^{r_0} (r_0 r^3 - r^4) dr d\theta \\
 &= \frac{4kh}{r_0} \left( \frac{r_0^5}{20} \right) \left( \frac{\pi}{2} \right) = \frac{1}{10}k\pi r_0^4 h
 \end{aligned}$$

Because the mass of the core is  $m = kV = k\left(\frac{1}{3}\pi r_0^2 h\right)$  from Exercise 25, we have  $k = 3m/\pi r_0^2 h$ . So,

$$I_z = \frac{1}{10}k\pi r_0^4 h = \frac{1}{10}\left(\frac{3m}{\pi r_0^2 h}\right)\pi r_0^4 h = \frac{3}{10}mr_0^2.$$

$$\begin{aligned}
 30. I_z &= \iiint_Q (x^2 + y^2) \rho(x, y, z) dV \\
 &= 4k \int_0^{\pi/2} \int_0^{r_0} \int_0^{h(r_0-r)/r_0} r^4 dz dr d\theta \\
 &= 4kh \int_0^{\pi/2} \int_0^{r_0} \frac{r_0 - r}{r_0} r^4 dr d\theta \\
 &= 4kh \int_0^{\pi/2} \left[ \frac{r^5}{5} - \frac{r^6}{6r_0} \right]_0^{r_0} d\theta = 4kh \int_0^{\pi/2} \left[ \frac{r_0^5}{5} - \frac{r_0^5}{6} \right] d\theta \\
 &= 4kh \int_0^{\pi/2} \frac{1}{30}r_0^5 d\theta = 4kh \frac{1}{30}r_0^5 \frac{\pi}{2} = \frac{1}{15}r_0^5 \pi kh
 \end{aligned}$$

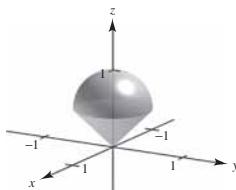
$$\begin{aligned}
 31. m &= k(\pi b^2 h - \pi a^2 h) = k\pi h(b^2 - a^2) \\
 I_z &= 4k \int_0^{\pi/2} \int_a^b \int_0^h r^3 dz dr d\theta \\
 &= 4kh \int_0^{\pi/2} \int_a^b r^3 dr d\theta = kh \int_0^{\pi/2} (b^4 - a^4) d\theta \\
 &= \frac{k\pi(b^4 - a^4)h}{2} = \frac{k\pi(b^2 - a^2)(b^2 + a^2)h}{2} \\
 &= \frac{1}{2}m(a^2 + b^2)
 \end{aligned}$$

$$\begin{aligned}
 32. m &= k\pi a^2 h \\
 I_z &= 2k \int_0^{\pi/2} \int_0^{2a \sin \theta} \int_0^h r^3 dz dr d\theta = \frac{3}{2}k\pi a^4 h = \frac{3}{2}ma^2
 \end{aligned}$$

$$\begin{aligned}
 33. \ V &= \int_0^{2\pi} \int_{\pi/4}^{\pi/2} \int_0^3 \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta \\
 &= \int_0^{2\pi} \int_{\pi/4}^{\pi/2} 9 \sin \phi \, d\phi \, d\theta \\
 &= \int_0^{2\pi} [-9 \cos \phi]_{\pi/4}^{\pi/2} \, d\theta \\
 &= \int_0^{2\pi} 9 \left( \frac{\sqrt{2}}{2} \right) \, d\theta = 18\pi \left( \frac{\sqrt{2}}{2} \right) = 9\pi\sqrt{2}
 \end{aligned}$$



$$\begin{aligned}
 34. \ x^2 + y^2 + z^2 &= z \\
 x^2 + y^2 + \left( z^2 - z + \frac{1}{4} \right) &= \frac{1}{4} \\
 x^2 + y^2 + \left( z - \frac{1}{2} \right)^2 &= \frac{1}{4}
 \end{aligned}$$

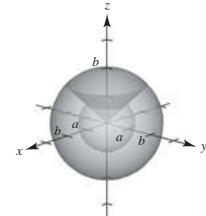


Sphere with center  $\left(0, 0, \frac{1}{2}\right)$ :  $\rho = \cos \phi$

$$V = \int_0^{2\pi} \int_0^{\pi/4} \int_0^{\cos \phi} \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta = \int_0^{2\pi} \int_0^{\pi/4} \frac{\cos^3 \phi}{3} \sin \phi \, d\phi \, d\theta = \int_0^{2\pi} \left[ \frac{-\cos^4 \phi}{12} \right]_0^{\pi/4} \, d\theta = \int_0^{2\pi} \frac{1}{12} \left( 1 - \frac{1}{4} \right) \, d\theta = \frac{\pi}{8}$$

$$35. \ V = \int_0^{2\pi} \int_0^{\pi} \int_0^{4 \sin \phi} \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta = 16\pi^2$$

$$\begin{aligned}
 36. \ V &= 8 \int_0^{\pi/4} \int_0^{\pi/2} \int_a^b \rho^2 \sin \phi \, d\rho \, d\theta \, d\phi \quad (\text{includes upper and lower cones}) \\
 &= \frac{8}{3} (b^3 - a^3) \int_0^{\pi/4} \int_0^{\pi/2} \sin \phi \, d\theta \, d\phi \\
 &= \frac{4\pi}{3} (b^3 - a^3) \int_0^{\pi/4} \sin \phi \, d\phi \\
 &= \left[ \frac{4\pi}{3} (b^3 - a^3) (-\cos \phi) \right]_0^{\pi/4} = \left( 1 - \frac{\sqrt{2}}{2} \right) \frac{4\pi}{3} (b^3 - a^3) = \frac{2\pi}{3} (2 - \sqrt{2}) (b^3 - a^3)
 \end{aligned}$$



$$\begin{aligned}
 37. \ m &= 8k \int_0^{\pi/2} \int_0^{\pi/2} \int_0^a \rho^3 \sin \phi \, d\rho \, d\theta \, d\phi \\
 &= 2ka^4 \int_0^{\pi/2} \int_0^{\pi/2} \sin \phi \, d\theta \, d\phi \\
 &= k\pi a^4 \int_0^{\pi/2} \sin \phi \, d\phi = \left[ k\pi a^4 (-\cos \phi) \right]_0^{\pi/2} = k\pi a^4
 \end{aligned}$$

$$39. \ m = \frac{2}{3} k\pi r^3$$

$\bar{x} = \bar{y} = 0$  by symmetry

$$\begin{aligned}
 M_{xy} &= 4k \int_0^{\pi/2} \int_0^{\pi/2} \int_0^r \rho^3 \cos \phi \sin \phi \, d\rho \, d\theta \, d\phi \\
 &= \frac{1}{2} kr^4 \int_0^{\pi/2} \int_0^{\pi/2} \sin 2\phi \, d\theta \, d\phi \\
 &= \frac{kr^4 \pi}{4} \int_0^{\pi/2} \sin 2\phi \, d\phi \\
 &= \left[ -\frac{1}{8} k\pi r^4 \cos 2\phi \right]_0^{\pi/2} = \frac{1}{4} k\pi r^4 \\
 \bar{z} &= \frac{M_{xy}}{m} = \frac{k\pi r^4 / 4}{2k\pi r^3 / 3} = \frac{3r}{8} \\
 (\bar{x}, \bar{y}, \bar{z}) &= \left( 0, 0, \frac{3r}{8} \right)
 \end{aligned}$$

$$\begin{aligned}
 38. \ m &= 8k \int_0^{\pi/2} \int_0^{\pi/2} \int_0^a \rho^3 \sin^2 \phi \, d\rho \, d\theta \, d\phi \\
 &= 2ka^4 \int_0^{\pi/2} \int_0^{\pi/2} \sin^2 \phi \, d\theta \, d\phi \\
 &= k\pi a^4 \int_0^{\pi/2} \sin^2 \phi \, d\phi \\
 &= \left[ k\pi a^4 \left( \frac{1}{2}\phi - \frac{1}{4}\sin 2\phi \right) \right]_0^{\pi/2} = k\pi a^4 \frac{\pi}{4} = \frac{1}{4} k\pi^2 a^4
 \end{aligned}$$

40.  $\bar{x} = \bar{y} = 0$  by symmetry

$$m = k\left(\frac{2}{3}\pi R^3 - \frac{2}{3}\pi r^3\right) = \frac{2}{3}k\pi(R^3 - r^3)$$

$$\begin{aligned} M_{xy} &= 4k \int_0^{\pi/2} \int_0^{\pi/2} \int_r^R \rho^3 \cos \phi \sin \phi d\rho d\theta d\phi \\ &= \frac{1}{2}k(R^4 - r^4) \int_0^{\pi/2} \int_0^{\pi/2} \sin 2\phi d\theta d\phi \\ &= \frac{1}{4}k\pi(R^4 - r^4) \int_0^{\pi/2} \sin 2\phi d\phi \\ &= \left[ -\frac{1}{8}k\pi(R^4 - r^4) \cos 2\phi \right]_0^{\pi/2} = \frac{1}{4}k\pi(R^4 - r^4) \\ \bar{z} &= \frac{M_{xy}}{m} = \frac{k\pi(R^4 - r^4)/4}{2k\pi(R^3 - r^3)/3} = \frac{3(R^4 - r^4)}{8(R^3 - r^3)} \\ (\bar{x}, \bar{y}, \bar{z}) &= \left( 0, 0, \frac{3(R^4 - r^4)}{8(R^3 - r^3)} \right) \end{aligned}$$

$$\begin{aligned} 41. I_z &= 4k \int_{\pi/4}^{\pi/2} \int_0^{\pi/2} \int_0^{\cos \phi} \rho^4 \sin^3 \phi d\rho d\theta d\phi \\ &= \frac{4}{5}k \int_{\pi/4}^{\pi/2} \int_0^{\pi/2} \cos^5 \phi \sin^3 \phi d\theta d\phi \\ &= \frac{2}{5}k\pi \int_{\pi/4}^{\pi/2} \cos^5 \phi (1 - \cos^2 \phi) \sin \phi d\phi \\ &= \left[ \frac{2}{5}k\pi \left( -\frac{1}{6}\cos^6 \phi + \frac{1}{8}\cos^8 \phi \right) \right]_{\pi/4}^{\pi/2} = \frac{k\pi}{192} \end{aligned}$$

$$46. \int_{\theta_1}^{\theta_2} \int_{\phi_1}^{\phi_2} \int_{\rho_1}^{\rho_2} f(\rho \sin \phi \cos \theta, \rho \sin \phi \sin \theta, \rho \cos \phi) \rho^2 \sin \phi d\rho d\phi d\theta$$

47. (a)  $r = r_0$ : right circular cylinder about  $z$ -axis

$\theta = \theta_0$ : plane parallel to  $z$ -axis

$z = z_0$ : plane parallel to  $xy$ -plane

(b)  $\rho = \rho_0$ : sphere of radius  $\rho_0$

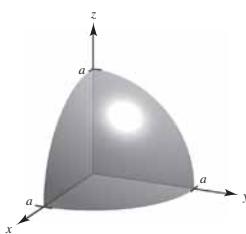
$\theta = \theta_0$ : plane parallel to  $z$ -axis

$\phi = \phi_0$ : cone

$$48. (a) \int_0^{\pi/2} \int_0^a \int_0^{\sqrt{a^2 - r^2}} \sqrt{r^2 + z^2} r dz dr d\theta$$

$$(b) \int_0^{\pi/2} \int_0^{\pi/2} \int_0^a \rho^3 \sin \phi d\rho d\phi d\theta$$

Integral (b) appears easier. The limits of integration are all constants.



$$\begin{aligned} 42. I_z &= 4k \int_0^{\pi/2} \int_0^{\pi/2} \int_r^R \rho^4 \sin^3 \phi d\rho d\theta d\phi \\ &= \frac{4k}{5}(R^5 - r^5) \int_0^{\pi/2} \int_0^{\pi/2} \sin^3 \phi d\theta d\phi \\ &= \frac{2k\pi}{5}(R^5 - r^5) \int_0^{\pi/2} \sin \phi (1 - \cos^2 \phi) d\phi \\ &= \left[ \frac{2k\pi}{5}(R^5 - r^5) \left( -\cos \phi + \frac{\cos^3 \phi}{3} \right) \right]_0^{\pi/2} \\ &= \frac{4k\pi}{15}(R^5 - r^5) \end{aligned}$$

$$\begin{aligned} 43. x &= r \cos \theta & x^2 + y^2 &= r^2 \\ y &= r \sin \theta & \tan \theta &= \frac{y}{x} \\ z &= z & z &= z \end{aligned}$$

$$\begin{aligned} 44. x &= \rho \sin \phi \cos \theta & \rho^2 &= x^2 + y^2 + z^2 \\ y &= \rho \sin \phi \sin \theta & \tan \theta &= \frac{y}{x} \\ z &= \rho \cos \phi & \cos \phi &= \frac{z}{\sqrt{x^2 + y^2 + z^2}} \end{aligned}$$

$$45. \int_{\theta_1}^{\theta_2} \int_{g_1(\theta)}^{g_2(\theta)} \int_{h_1(r \cos \theta, r \sin \theta)}^{h_2(r \cos \theta, r \sin \theta)} f(r \cos \theta, r \sin \theta, z) r dz dr d\theta$$

$$\begin{aligned}
49. \quad & 16 \int_0^a \int_0^{\sqrt{a^2-x^2}} \int_0^{\sqrt{a^2-x^2-y^2}} \int_0^{\sqrt{a^2-x^2-y^2-z^2}} dw dz dy dx = 16 \int_0^a \int_0^{\sqrt{a^2-x^2}} \int_0^{\sqrt{a^2-x^2-y^2}} \sqrt{a^2-x^2-y^2} dz dy dx \\
& = 16 \int_0^{\pi/2} \int_0^a \int_0^{\sqrt{a^2-r^2}} \sqrt{(a^2-r^2)-z^2} dz (r dr d\theta) \\
& = 16 \int_0^{\pi/2} \int_0^a \frac{1}{2} \left[ z \sqrt{(a^2-r^2)-z^2} + (a^2-r^2) \arcsin \frac{z}{\sqrt{a^2-r^2}} \right]_0^{\sqrt{a^2-r^2}} r dr d\theta \\
& = 8 \int_0^{\pi/2} \int_0^a \frac{\pi}{2} (a^2-r^2) r dr d\theta = 4\pi \int_0^{\pi/2} \left[ \frac{a^2 r^2}{2} - \frac{r^4}{4} \right]_0^a d\theta = a^4 \pi \int_0^{\pi/2} d\theta = \frac{a^4 \pi^2}{2}
\end{aligned}$$

$$\begin{aligned}
50. \quad & \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sqrt{x^2+y^2+z^2} e^{-(x^2+y^2+z^2)} dx dy dz \\
& = \int_0^{2\pi} \int_0^{\pi} \int_0^{\infty} \rho e^{-\rho^2} \rho^2 \sin \phi d\rho d\phi d\theta \\
& = \lim_{k \rightarrow \infty} \int_0^{2\pi} \int_0^{\pi} \int_0^k \rho^3 e^{-\rho^2} \sin \phi d\rho d\phi d\theta \\
& = \lim_{k \rightarrow \infty} \left( \int_0^{\pi} \sin \phi d\phi \right) \left( \int_0^{2\pi} d\theta \right) \left( \int_0^k \rho^3 e^{-\rho^2} d\rho \right) \\
& = (2)(2\pi) \lim_{k \rightarrow \infty} \left[ \frac{-(\rho^2+1)e^{-\rho^2}}{2} \right]_0^k = 4\pi \left[ \frac{1}{2} \right] = 2\pi
\end{aligned}$$

**51.**  $(x^2 + y^2 + z^2 + 8)^2 \leq 36(x^2 + y^2)$   
 In cylindrical coordinates,  
 $(r^2 + z^2 + 8)^2 \leq 36r^2$   
 $r^2 + z^2 + 8 \leq 6r$   
 $r^2 - 6r + 9 + z^2 - 1 \leq 0$   
 $(r - 3)^2 + z^2 \leq 1$ .  
 This is a torus: rotate  $(x - 3)^2 + z^2 = 1$  about the  $z$ -axis. By Pappus' Theorem,  
 $V = 2\pi(3)\pi = 6\pi^2$ .

## Section 14.8 Change of Variables: Jacobians

$$\begin{aligned}
1. \quad & x = -\frac{1}{2}(u - v) \\
& y = \frac{1}{2}(u + v) \\
& \frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial y}{\partial u} \frac{\partial x}{\partial v} = \left(-\frac{1}{2}\right)\left(\frac{1}{2}\right) - \left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = -\frac{1}{2}
\end{aligned}$$

$$\begin{aligned}
2. \quad & x = au + bv \\
& y = cu + dv \\
& \frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial y}{\partial u} \frac{\partial x}{\partial v} = ad - cb
\end{aligned}$$

$$\begin{aligned}
3. \quad & x = u - v^2 \\
& y = u + v \\
& \frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial y}{\partial u} \frac{\partial x}{\partial v} = (1)(1) - (1)(-2v) = 1 + 2v
\end{aligned}$$

$$\begin{aligned}
7. \quad & x = e^u \sin v \\
& y = e^u \cos v \\
& \frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial y}{\partial u} \frac{\partial x}{\partial v} = (e^u \sin v)(-e^u \sin v) - (e^u \cos v)(e^u \cos v) = -e^{2u}
\end{aligned}$$

$$\begin{aligned}
8. \quad & x = \frac{u}{v} \\
& y = u + v \\
& \frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial y}{\partial u} \frac{\partial x}{\partial v} = \left(\frac{1}{v}\right)(1) - (1)\left(-\frac{u}{v^2}\right) = \frac{1}{v} + \frac{u}{v^2} = \frac{u+v}{v^2}
\end{aligned}$$

$$\begin{aligned}
4. \quad & x = uv - 2u \\
& y = uv \\
& \frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial y}{\partial u} \frac{\partial x}{\partial v} = (v - 2)u - vu = -2u
\end{aligned}$$

$$\begin{aligned}
5. \quad & x = u \cos \theta - v \sin \theta \\
& y = u \sin \theta + v \cos \theta \\
& \frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial y}{\partial u} \frac{\partial x}{\partial v} = \cos^2 \theta + \sin^2 \theta = 1
\end{aligned}$$

$$\begin{aligned}
6. \quad & x = u + a \\
& y = v + a \\
& \frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial y}{\partial u} \frac{\partial x}{\partial v} = (1)(1) - (0)(0) = 1
\end{aligned}$$

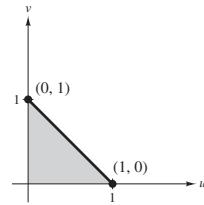
9.  $x = 3u + 2v$

$$y = 3v$$

$$v = \frac{y}{3}$$

$$u = \frac{x - 2v}{3} = \frac{x - 2(y/3)}{3} = \frac{x}{3} - \frac{2y}{9}$$

$(x, y)$	$(u, v)$
(0, 0)	(0, 0)
(3, 0)	(1, 0)
(2, 3)	(0, 1)



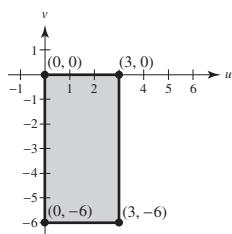
10.  $x = \frac{1}{3}(4u - v)$

$$y = \frac{1}{3}(u - v)$$

$$u = x - y$$

$$v = x - 4y$$

$(x, y)$	$(u, v)$
(0, 0)	(0, 0)
(4, 1)	(3, 0)
(2, 2)	(0, -6)
(6, 3)	(3, -6)



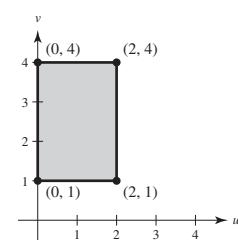
12.  $x = \frac{1}{3}(v - u)$

$$y = \frac{1}{3}(2v + u)$$

$$u = y - 2x$$

$$v = x + y$$

$(x, y)$	$(u, v)$
$\left(-\frac{1}{3}, \frac{4}{3}\right)$	(2, 1)
$\left(\frac{1}{3}, \frac{2}{3}\right)$	(0, 1)
$\left(\frac{4}{3}, \frac{8}{3}\right)$	(0, 4)
$\left(\frac{2}{3}, \frac{10}{3}\right)$	(2, 4)



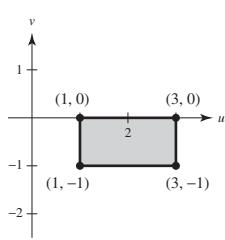
11.  $x = \frac{1}{2}(u + v)$

$$y = \frac{1}{2}(u - v)$$

$$u = x + y$$

$$v = x - y$$

$(x, y)$	$(u, v)$
$\left(\frac{1}{2}, \frac{1}{2}\right)$	(1, 0)
(0, 1)	(1, -1)
(1, 2)	(3, -1)
$\left(\frac{3}{2}, \frac{3}{2}\right)$	(3, 0)



13.  $\begin{cases} x - 2y = 0 \\ x + y = 4 \end{cases} \Rightarrow \begin{cases} 3y = 4 \\ y = \frac{4}{3}, \quad x = \frac{8}{3} \end{cases}$

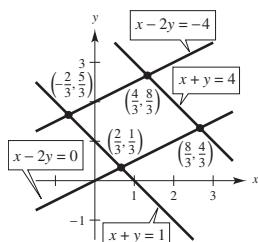
$$\begin{cases} x - 2y = -4 \\ x + y = 4 \end{cases} \Rightarrow \begin{cases} 3y = 8 \\ y = \frac{8}{3}, \quad x = \frac{4}{3} \end{cases}$$

$$\begin{cases} x - 2y = -4 \\ x + y = 1 \end{cases} \Rightarrow \begin{cases} 3y = 5 \\ y = \frac{5}{3}, \quad x = -\frac{2}{3} \end{cases}$$

$$\begin{cases} x - 2y = 0 \\ x + y = 1 \end{cases} \Rightarrow \begin{cases} 3y = 1 \\ y = \frac{1}{2}, \quad x = \frac{2}{3} \end{cases}$$

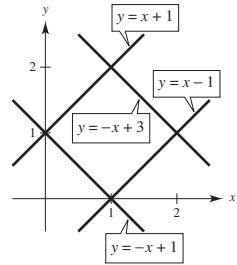
$$u = x + y \quad u - v = 3y \Rightarrow y = \frac{1}{3}(u - v)$$

$$v = x - 2y \quad 2u + v = 3x \Rightarrow x = \frac{1}{3}(2u + v)$$

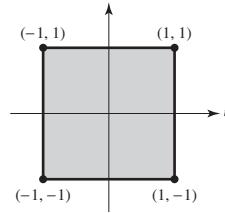


$$\iint_R 3xy \, dA = \int_{-2/3}^{2/3} \int_{1-x}^{(x+4)/2} 3xy \, dy \, dx + \int_{2/3}^{4/3} \int_{x/2}^{(x+4)/2} 3xy \, dy \, dx + \int_{4/3}^{8/3} \int_{x/2}^{4-x} 3xy \, dy \, dx = \frac{32}{27} + \frac{164}{27} + \frac{296}{27} = \frac{164}{9}$$

$$\begin{aligned}
 14. \iint_R (x+y)^2 \sin^2(x-y) dA &= \int_0^1 \int_{1-x}^{x+1} f(x) dy dx + \int_1^2 \int_{x-1}^{3-x} f(x) dy dx \\
 &= \left( \frac{-\cos^2(1)}{16} - \frac{5}{3} \sin(1) \cos(1) + \frac{15}{16} \sin^2(1) + \frac{17}{16} \right) + \left[ \cos^2 1 - \frac{8}{3} \sin(1) \cos(1) + 7/3 \right] \\
 &= \frac{13}{3} - \frac{13}{3} \sin(1) \cos(1) = \frac{13}{3} - \frac{13}{6} \sin(2) \approx 2.363
 \end{aligned}$$

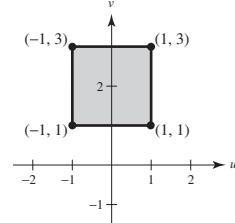


$$\begin{aligned}
 15. \quad x &= \frac{1}{2}(u+v) \\
 y &= \frac{1}{2}(u-v) \\
 \frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial y}{\partial u} \frac{\partial x}{\partial v} &= \left( \frac{1}{2} \right) \left( -\frac{1}{2} \right) - \left( \frac{1}{2} \right) \left( \frac{1}{2} \right) = -\frac{1}{2} \\
 \iint_R 4(x^2 + y^2) dA &= \int_{-1}^1 \int_{-1}^1 4 \left[ \frac{1}{4}(u+v)^2 + \frac{1}{4}(u-v)^2 \right] \left( \frac{1}{2} \right) dv du \\
 &= \int_{-1}^1 \int_{-1}^1 (u^2 + v^2) dv du = \int_{-1}^1 2 \left( u^2 + \frac{1}{3} \right) du = \left[ 2 \left( \frac{u^3}{3} + \frac{u}{3} \right) \right]_{-1}^1 = \frac{8}{3}
 \end{aligned}$$

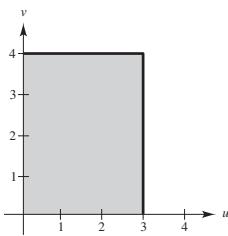


$$\begin{aligned}
 16. \quad x &= \frac{1}{2}(u+v), \quad u = x - y \\
 y &= -\frac{1}{2}(u-v), \quad v = x + y \\
 \frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial y}{\partial u} \frac{\partial x}{\partial v} &= \frac{1}{2} \left( \frac{1}{2} \right) - \left( -\frac{1}{2} \right) \left( \frac{1}{2} \right) = \frac{1}{2} \\
 \iint_R 60xy dA &= \int_{-1}^1 \int_1^3 60 \left( \frac{1}{2}(u+v) \right) \left( -\frac{1}{2}(u-v) \right) \left( \frac{1}{2} \right) dv du \\
 &= \int_{-1}^1 \int_1^3 -\frac{15}{2} (v^2 - u^2) dv du \\
 &= \int_{-1}^1 \left[ -\frac{15}{2} \left( \frac{v^3}{3} - u^2 v \right) \right]_1^3 du = \int_{-1}^1 \frac{15}{2} \left( 2u^2 - \frac{26}{3} \right) du = \left[ \frac{15}{2} \left( \frac{2}{3}u^3 - \frac{26}{3}u \right) \right]_{-1}^1 = 15 \left( \frac{2}{3} - \frac{26}{3} \right) = -120
 \end{aligned}$$

$(x, y)$	$(u, v)$
(0, 1)	(-1, 1)
(2, 1)	(1, 3)
(1, 2)	(-1, 3)
(1, 0)	(1, 1)



$$\begin{aligned}
 17. \quad x &= u + v \\
 y &= u \\
 \frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial y}{\partial u} \frac{\partial x}{\partial v} &= (1)(0) - (1)(1) = -1 \\
 \iint_R y(x-y) dA &= \int_0^3 \int_0^4 uv(1) dv du \\
 &= \int_0^3 8u du = 36
 \end{aligned}$$



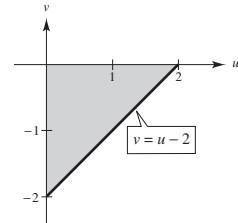
$$\begin{aligned}
 18. \quad x &= \frac{1}{2}(u+v) \\
 y &= \frac{1}{2}(u-v) \\
 \frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial y}{\partial u} \frac{\partial x}{\partial v} &= -\frac{1}{2}
 \end{aligned}$$

$$\iint_R 4(x+y)e^{x-y} dA = \int_0^2 \int_{u-2}^0 4ue^v \left( \frac{1}{2} \right) dv du$$

$$= \int_0^2 2u(1 - e^{u-2}) du$$

$$= 2 \left[ \frac{u^2}{2} - ue^{u-2} + e^{u-2} \right]_0^2$$

$$= 2(1 - e^{-2})$$

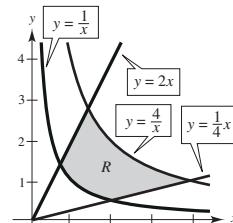


19.  $\int_R \int e^{-xy/2} dA$

$$R: y = \frac{x}{4}, y = 2x, y = \frac{1}{x}, y = \frac{4}{x}$$

$$x = \sqrt{v/u}, y = \sqrt{uv} \Rightarrow u = \frac{y}{x}, v = xy$$

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} -\frac{1}{2} \frac{v^{1/2}}{u^{3/2}} & \frac{1}{2} & \frac{1}{u^{1/2} v^{1/2}} \\ \frac{1}{2} \frac{v^{1/2}}{u^{1/2}} & \frac{1}{2} & \frac{u^{1/2}}{v^{1/2}} \end{vmatrix} = -\frac{1}{4} \left( \frac{1}{u} + \frac{1}{u} \right) = -\frac{1}{2u}$$



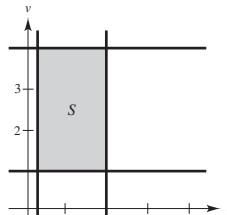
Transformed Region:

$$y = \frac{1}{x} \Rightarrow yx = 1 \Rightarrow v = 1$$

$$y = \frac{4}{x} \Rightarrow ux = 4 \Rightarrow v = 4$$

$$y = 2x \Rightarrow \frac{y}{x} = 2 \Rightarrow u = 2$$

$$y = \frac{x}{4} \Rightarrow \frac{y}{x} = \frac{1}{4} \Rightarrow u = \frac{1}{4}$$



$$\int_R \int e^{-xy/2} dA = \int_{1/4}^2 \int_1^4 e^{-v/2} \left( \frac{1}{2u} \right) dv du = - \int_{1/4}^2 \left[ \frac{e^{-v/2}}{u} \right]_1^4 du = - \int_{1/4}^2 (e^{-2} - e^{-1/2}) \frac{1}{u} du$$

$$= - \left[ (e^{-2} - e^{-1/2}) \ln u \right]_{1/4}^2 = -(e^{-2} - e^{-1/2}) \left( \ln 2 - \ln \frac{1}{4} \right) = (e^{-1/2} - e^{-2}) \ln 8 \approx 0.9798$$

20.  $x = \frac{u}{v}$

$$y = v$$

$$\frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial y}{\partial u} \frac{\partial x}{\partial v} = \frac{1}{v}$$

$$\int_R \int y \sin xy dA = \int_1^4 \int_1^4 v (\sin u) \frac{1}{v} dv du = \int_1^4 3 \sin u du = [-3 \cos u]_1^4 = 3(\cos 1 - \cos 4) \approx 3.5818$$

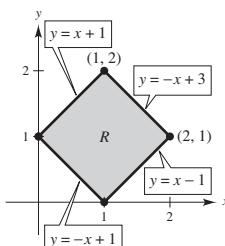
21.  $u = x - y = 1, \quad v = x + y = 1$

$$u = x - y = -1, \quad v = x + y = 3$$

$$x = \frac{1}{2}(u + v)$$

$$y = \frac{1}{2}(v - u)$$

$$\frac{\partial(x, y)}{\partial(u, v)} = \frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial y}{\partial u} \frac{\partial x}{\partial v} = \frac{1}{2} \left( \frac{1}{2} \right) - \left( -\frac{1}{2} \right) \left( \frac{1}{2} \right) = \frac{1}{2}$$



$$\begin{aligned} \int_R \int 48xy dA &= \int_1^3 \int_{-1}^1 48 \left( \frac{1}{2} \right) (u + v) \left( \frac{1}{2} \right) (v - u) \left( \frac{1}{2} \right) du dv = \int_1^3 \int_{-1}^1 6(v^2 - u^2) du dv = 6 \int_1^3 \left[ uv^2 - \frac{u^3}{3} \right]_{-1}^1 dv \\ &= 6 \int_1^3 \left( 2v^2 - \frac{2}{3} \right) dv = 6 \left[ \frac{2v^3}{3} - \frac{2}{3}v \right]_1^3 = 6 \left[ 18 - 2 - \frac{2}{3} + \frac{2}{3} \right] = 96 \end{aligned}$$

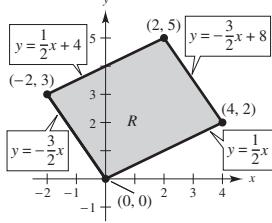
22.  $u = 2y - x = 0, v = 3x + 2y = 0$

$u = 2y - x = 8, v = 3x + 2y = 16$

$$x = \frac{1}{4}(v - u)$$

$$y = \frac{1}{8}(v + 3u)$$

$$\frac{\partial(x, y)}{\partial(u, v)} = \left(-\frac{1}{4}\right)\left(\frac{1}{8}\right) - \left(\frac{3}{8}\right)\left(\frac{1}{4}\right) = -\frac{1}{8}$$



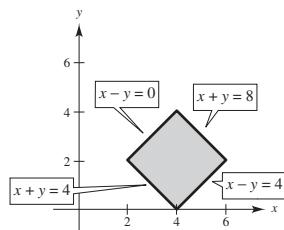
$$\int_R \int (3x + 2y)^2 \sqrt{2y - x} \, dA = \int_0^{16} \int_0^8 v u^{1/2} \, du \, dv = \int_0^{16} \left[ \frac{2}{3} v u^{3/2} \right]_0^8 \, dv = \int_0^{16} \frac{32}{3} \sqrt{2} v \, dv = \frac{16^2}{2} \left( \frac{32}{3} \sqrt{2} \right) = \frac{4096\sqrt{2}}{3}$$

23.  $u = x + y = 4, v = x - y = 0$

$u = x + y = 8, v = x - y = 4$

$$x = \frac{1}{2}(u + v), y = \frac{1}{2}(u - v)$$

$$\frac{\partial(x, y)}{\partial(u, v)} = -\frac{1}{2}$$



$$\int_R \int (x + y)e^{x-y} \, dA = \int_4^8 \int_0^4 ue^v \left( \frac{1}{2} \right) \, dv \, du = \frac{1}{2} \int_4^8 u(e^4 - 1) \, du = \left[ \frac{1}{4} u^2 (e^4 - 1) \right]_4^8 = 12(e^4 - 1)$$

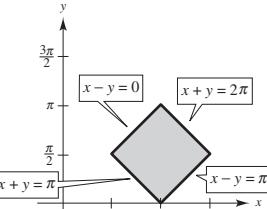
24.  $u = x + y = \pi, v = x - y = 0$

$u = x + y = 2\pi, v = x - y = \pi$

$$x = \frac{1}{2}(u + v), y = \frac{1}{2}(u - v)$$

$$\frac{\partial(x, y)}{\partial(u, v)} = -\frac{1}{2}$$

$$\int_R \int (x + y)^2 \sin^2(x - y) \, dA = \int_0^\pi \int_\pi^{2\pi} u^2 \sin^2 v \left( \frac{1}{2} \right) \, du \, dv = \int_0^\pi \left[ \frac{1}{2} \left( \frac{u^3}{3} \right) \frac{1 - \cos 2v}{2} \right]_{\pi}^{2\pi} \, dy = \left[ \frac{7\pi^3}{12} \left( v - \frac{1}{2} \sin 2v \right) \right]_0^\pi = \frac{7\pi^4}{12}$$



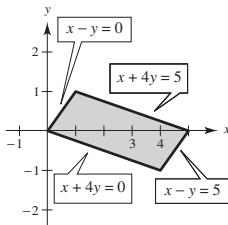
25.  $u = x + 4y = 0, v = x - y = 0$

$u = x + 4y = 5, v = x - y = 5$

$$x = \frac{1}{5}(u + 4v), y = \frac{1}{5}(u - v)$$

$$\frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial y}{\partial u} \frac{\partial x}{\partial v} = \left( \frac{1}{5} \right) \left( -\frac{1}{5} \right) - \left( \frac{1}{5} \right) \left( \frac{4}{5} \right) = -\frac{1}{5}$$

$$\int_R \int \sqrt{(x - y)(x + 4y)} \, dA = \int_0^5 \int_0^5 \sqrt{uv} \left( \frac{1}{5} \right) \, du \, dv = \int_0^5 \left[ \frac{1}{5} \left( \frac{2}{3} \right) u^{3/2} \sqrt{v} \right]_0^5 \, dv = \left[ \frac{2\sqrt{5}}{3} \left( \frac{2}{3} \right) v^{3/2} \right]_0^5 = \frac{100}{9}$$



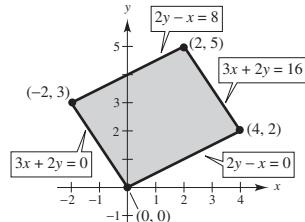
26.  $u = 3x + 2y = 0, v = 2y - x = 0$

$u = 3x + 2y = 16, v = 2y - x = 8$

$$x = \frac{1}{4}(u - v), y = \frac{1}{8}(u + 3v)$$

$$\frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial y}{\partial u} \frac{\partial x}{\partial v} = \frac{1}{4} \left( \frac{3}{8} \right) - \frac{1}{8} \left( -\frac{1}{4} \right) = \frac{1}{8}$$

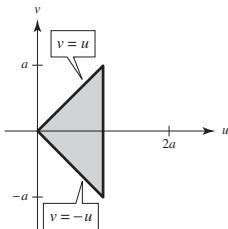
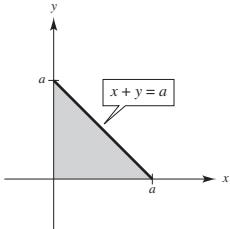
$$\int_R \int (3x + 2y)(2y - x)^{3/2} \, dA = \int_0^8 \int_0^{16} u v^{3/2} \left( \frac{1}{8} \right) \, du \, dv = \int_0^8 16v^{3/2} \, dv = \left( \frac{2}{5} \right) 16v^{5/2} \Big|_0^8 = \frac{4096}{5} \sqrt{2}$$



27.  $u = x + y, v = x - y, x = \frac{1}{2}(u + v), y = \frac{1}{2}(u - v)$

$$\frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial y}{\partial u} \frac{\partial x}{\partial v} = -\frac{1}{2}$$

$$\int_R \int \sqrt{x+y} \, dA = \int_0^a \int_{-u}^u \sqrt{u} \left( \frac{1}{2} \right) dv \, du = \int_0^a u \sqrt{u} \, du = \left[ \frac{2}{5} u^{5/2} \right]_0^a = \frac{2}{5} a^{5/2}$$

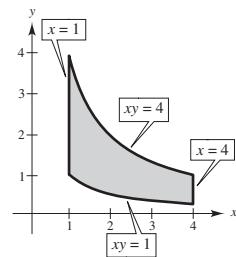


28.  $u = x = 1, v = xy = 1$

$$u = x = 4, v = xy = 4$$

$$x = u, \quad y = \frac{u}{v}$$

$$\frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial y}{\partial u} \frac{\partial x}{\partial v} = \frac{1}{u}$$



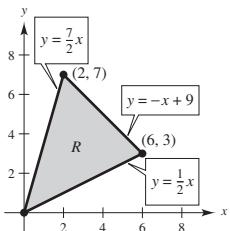
$$\int_R \int \frac{xy}{1+x^2y^2} \, dA = \int_1^4 \int_1^4 \frac{v}{1+v^2} \left( \frac{1}{u} \right) dv \, du = \int_1^4 \left[ \frac{1}{2} \ln(1+v^2) \right]_1^4 \frac{1}{u} du = \left[ \frac{1}{2} [\ln 17 - \ln 2] \ln u \right]_1^4 = \frac{1}{2} \left( \ln \frac{17}{2} \right) (\ln 4)$$

29.  $u = 2x - y$

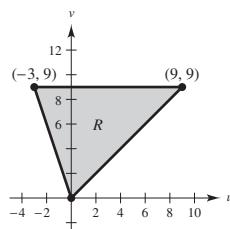
$$v = x + y$$

$$3x = u + v \Rightarrow x = \frac{1}{3}(u + v)$$

Then  $y = v - x = v - \frac{1}{3}(u + v) = \frac{1}{3}(2v - u)$ .

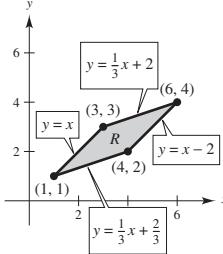


$(x, y)$	$(u, v)$
(0, 0)	(0, 0)
(6, 3)	(9, 9)
(2, 7)	(-3, 9)



One side is parallel to the  $u$ -axis.

30.



$$y - x = 0, \quad 3y - x = 6$$

$$y - x = -2, \quad 3y - x = 2$$

Let  $u = y - x$  and  $v = 3y - x$ .

Then  $x = \frac{1}{2}(v - 3u)$  and  $y = \frac{1}{2}(v + u)$ .

$$\text{So, } T(u, v) = (x, y) = \left( \frac{1}{2}(v - 3u), \frac{1}{2}(v + u) \right).$$

Note that:

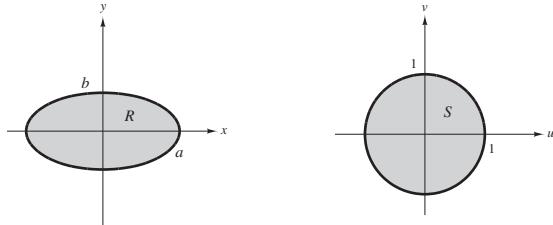
$(x, y)$	$(u, v)$
(1, 1)	(0, 2)
(4, 2)	(-2, 2)
(6, 4)	(-2, 6)
(3, 3)	(0, 6)

31.  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, x = au, y = bv$

$$\frac{(au)^2}{a^2} + \frac{(bv)^2}{b^2} = 1$$

$$u^2 + v^2 = 1$$

(a)  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad u^2 + v^2 = 1$



(b)  $\frac{\partial(x, y)}{\partial(u, v)} = \frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial y}{\partial u} \frac{\partial x}{\partial v} = (a)(b) - (0)(0) = ab$

(c)  $A = \int_S ab \, dS = ab(\pi(1)^2) = \pi ab$

32. (a)  $f(x, y) = 16 - x^2 - y^2$

$$R: \frac{x^2}{16} + \frac{y^2}{9} \leq 1$$

$$V = \int_R \int f(x, y) \, dA$$

Let  $x = 4u$  and  $y = 3v$ .

$$\begin{aligned} \int_R \int (16 - x^2 - y^2) \, dA &= \int_{-1}^1 \int_{-\sqrt{1-u^2}}^{\sqrt{1-u^2}} (16 - 16u^2 - 9v^2) 12 \, dv \, du \quad (\text{Let } u = r \cos \theta, v = r \sin \theta.) \\ &= \int_0^{2\pi} \int_0^1 (16 - 16r^2 \cos^2 \theta - 9r^2 \sin^2 \theta) 12r \, dr \, d\theta \\ &= 12 \int_0^{2\pi} \left[ 8r^2 - 4r^4 \cos^2 \theta - \frac{9}{4}r^4 \sin^2 \theta \right]_0^1 d\theta = 12 \int_0^{2\pi} \left[ 8 - 4 \cos^2 \theta - \frac{9}{4} \sin^2 \theta \right] d\theta \\ &= 12 \int_0^{2\pi} \left[ 8 - 4 \left( \frac{1 + \cos 2\theta}{2} \right) - \frac{9}{4} \left( \frac{1 - \cos 2\theta}{2} \right) \right] d\theta = 12 \int_0^{2\pi} \left[ \frac{39}{8} - \frac{7}{8} \cos 2\theta \right] d\theta \\ &= 12 \left[ \frac{39}{8}\theta - \frac{7}{16} \sin 2\theta \right]_0^{2\pi} = 12 \left[ \frac{39\pi}{4} \right] = 117\pi \end{aligned}$$

(b)  $f(x, y) = A \cos \left[ \frac{\pi}{2} \sqrt{\frac{x^2}{a^2} + \frac{y^2}{b^2}} \right]$

$$R: \frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1$$

Let  $x = au$  and  $y = bv$ .

$$\int_R \int f(x, y) \, dA = \int_{-1}^1 \int_{-\sqrt{1-u^2}}^{\sqrt{1-u^2}} A \cos \left[ \frac{\pi}{2} \sqrt{u^2 + v^2} \right] ab \, dv \, du$$

Let  $u = r \cos \theta, v = r \sin \theta$ .

$$\begin{aligned} Aab \int_0^{2\pi} \int_0^1 \cos \left[ \frac{\pi}{2} r \right] r \, dr \, d\theta &= Aab \left[ \frac{2r}{\pi} \sin \left( \frac{\pi r}{2} \right) + \frac{4}{\pi^2} \cos \left( \frac{\pi r}{2} \right) \right]_0^1 (2\pi) \\ &= 2\pi Aab \left[ \left( \frac{2}{\pi} + 0 \right) - \left( 0 + \frac{4}{\pi^2} \right) \right] = \frac{4(\pi - 2)Aab}{\pi} \end{aligned}$$

33. Jacobian  $= \frac{\partial(x, y)}{\partial(u, v)} = \frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial y}{\partial u} \frac{\partial x}{\partial v}$

34. See Theorem 14.5.

35.  $x = u(1-v)$ ,  $y = uv(1-w)$ ,  $z = uvw$

$$\frac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix} 1-v & -u & 0 \\ v(1-w) & u(1-w) & -uv \\ vw & uw & uv \end{vmatrix} = (1-v)[u^2v(1-w) + u^2vw] + u[uv^2(1-w) + uv^2w] = (1-v)(u^2v) + u(uv^2) = u^2v$$

36.  $x = 4u - v$ ,  $y = 4v - w$ ,  $z = u + w$

$$\frac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix} 4 & -1 & 0 \\ 0 & 4 & -1 \\ 1 & 0 & 1 \end{vmatrix} = 17$$

37.  $x = \frac{1}{2}(u+v)$ ,  $y = \frac{1}{2}(u-v)$ ,  $z = 2uvw$

$$\frac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix} 1/2 & 1/2 & 0 \\ 1/2 & -1/2 & 0 \\ 2vw & 2uw & 2uv \end{vmatrix} = 2uv[-1/4 - 1/4] = -uv$$

38.  $x = u - v + w$ ,  $y = 2uv$ ,  $z = u + v + w$

$$\frac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix} 1 & -1 & 1 \\ 2v & 2u & 0 \\ 1 & 1 & 1 \end{vmatrix} = 1(2u) + 1(2v) + 1(2v - 2u) = 4v$$

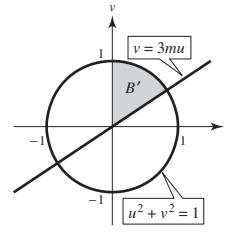
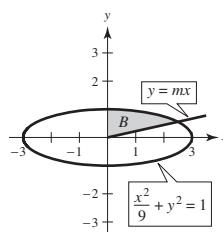
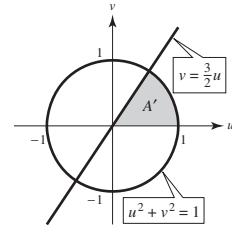
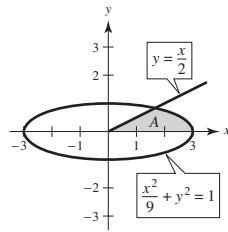
39.  $x = \rho \sin \phi \cos \theta$ ,  $y = \rho \sin \phi \sin \theta$ ,  $z = \rho \cos \phi$

$$\begin{aligned} \frac{\partial(x, y, z)}{\partial(\rho, \theta, \phi)} &= \begin{vmatrix} \sin \phi \cos \theta & -\rho \sin \phi \sin \theta & \rho \cos \phi \cos \theta \\ \sin \phi \sin \theta & \rho \sin \phi \cos \theta & \rho \cos \phi \sin \theta \\ \cos \phi & 0 & -\rho \sin \phi \end{vmatrix} \\ &= \cos \phi [-\rho^2 \sin \phi \cos \phi \sin^2 \theta - \rho^2 \sin \phi \cos \phi \cos^2 \theta] - \rho \sin \phi [\rho \sin^2 \phi \cos^2 \theta + \rho \sin^2 \phi \sin^2 \theta] \\ &= \cos \phi [-\rho^2 \sin \phi \cos \phi (\sin^2 \theta + \cos^2 \theta)] - \rho \sin \phi [\rho \sin^2 \phi (\cos^2 \theta + \sin^2 \theta)] \\ &= -\rho^2 \sin \phi \cos^2 \phi - \rho^2 \sin^3 \phi = -\rho^2 \sin \phi (\cos^2 \phi + \sin^2 \phi) = -\rho^2 \sin \phi \end{aligned}$$

40.  $x = r \cos \theta$ ,  $y = r \sin \theta$ ,  $z = z$

$$\frac{\partial(x, y, z)}{\partial(r, \theta, z)} = \begin{vmatrix} \cos \theta & -r \sin \theta & 0 \\ \sin \theta & r \cos \theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1[r \cos^2 \theta + r \sin^2 \theta] = r$$

41. Let  $u = \frac{x}{3}$ ,  $v = y \Rightarrow \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} 3 & 0 \\ 0 & 1 \end{vmatrix} = 3$ ,  $y = \frac{x}{2} \Rightarrow v = \frac{3u}{2}$ .



Region  $A$  is transformed to region  $A'$ , and region  $B$  is transformed to region  $B'$ .

$$A' = B' \Rightarrow \frac{2}{3} = 3m \Rightarrow m = \frac{2}{9}$$

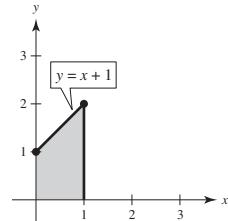
**Note:** You could also calculate the integrals directly.

## Review Exercises for Chapter 14

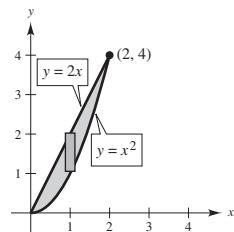
1.  $\int_1^{x^2} x \ln y dy = [xy(-1 + \ln y)]_1^{x^2} = x^3(-1 + \ln x^2) + x = x - x^3 + x^3 \ln x^2$

2.  $\int_y^{2y} (x^2 + y^2) dx = \left[ \frac{x^3}{3} + xy^2 \right]_y^{2y} = \frac{10y^3}{3}$

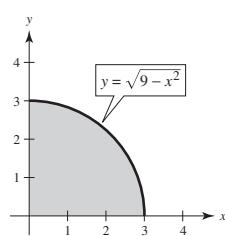
3.  $\int_0^1 \int_0^{1+x} (3x + 2y) dy dx = \int_0^1 [3xy + y^2]_0^{1+x} dx$   
 $= \int_0^1 (4x^2 + 5x + 1) dx = \left[ \frac{4}{3}x^3 + \frac{5}{2}x^2 + x \right]_0^1 = \frac{29}{6}$



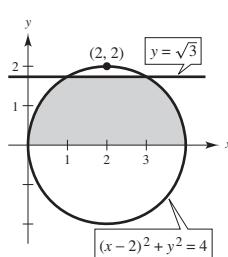
4.  $\int_0^2 \int_{x^2}^{2x} (x^2 + 2y) dy dx = \int_0^2 [x^2y + y^2]_{x^2}^{2x} dx$   
 $= \int_0^2 (4x^2 + 2x^3 - 2x^4) dx$   
 $= \left[ \frac{4}{3}x^3 + \frac{1}{2}x^4 - \frac{2}{5}x^5 \right]_0^2 = \frac{88}{15}$



5.  $\int_0^3 \int_0^{\sqrt{9-x^2}} 4x dy dx = \int_0^3 4x \sqrt{9-x^2} dx$   
 $= \left[ -\frac{4}{3}(9-x^2)^{3/2} \right]_0^3 = 36$



6.  $\int_0^{\sqrt{3}} \int_{2-\sqrt{4-y^2}}^{2+\sqrt{4-y^2}} dx dy = 2 \int_0^{\sqrt{3}} \sqrt{4-y^2} dy$   
 $= \left[ y\sqrt{4-y^2} + 4 \arcsin \frac{y}{2} \right]_0^{\sqrt{3}}$   
 $= \sqrt{3} + \frac{4\pi}{3}$



7.  $\int_0^3 \int_0^{(3-x)/3} dy dx = \int_0^1 \int_0^{3-3y} dx dy$   
 $A = \int_0^1 \int_0^{3-3y} dx dy = \int_0^1 (3-3y) dy = \left[ 3y - \frac{3}{2}y^2 \right]_0^1 = \frac{3}{2}$

8.  $\int_0^2 \int_0^x dy dx + \int_2^3 \int_0^{6-2x} dy dx = \int_0^2 \int_y^{(6-y)/2} dx dy$   
 $A = \int_0^2 \int_y^{(6-y)/2} dx dy = \frac{1}{2} \int_0^2 (6-3y) dy = \left[ \frac{1}{2}(6y - \frac{3}{2}y^2) \right]_0^2 = 3$

9.  $\int_{-5}^3 \int_{-\sqrt{25-x^2}}^{\sqrt{25-x^2}} dy dx = \int_{-4}^{-2} \int_{-\sqrt{25-y^2}}^{\sqrt{25-y^2}} dx dy + \int_{-4}^4 \int_{-\sqrt{25-y^2}}^{\sqrt{25-y^2}} dx dy + \int_4^5 \int_{-\sqrt{25-y^2}}^{\sqrt{25-y^2}} dx dy$   
 $A = 2 \int_{-5}^3 \int_0^{\sqrt{25-x^2}} dy dx = 2 \int_{-5}^3 \sqrt{25-x^2} dx = \left[ x\sqrt{25-x^2} + 25 \arcsin \frac{x}{5} \right]_{-5}^3 = \frac{25\pi}{2} + 12 + 25 \arcsin \frac{3}{5} \approx 67.36$

10.  $\int_0^4 \int_{x^2-2x}^{6x-x^2} dy dx = \int_{-1}^0 \int_{1-\sqrt{1+y}}^{1+\sqrt{1+y}} dy dx + \int_0^8 \int_{3-\sqrt{9-y}}^{1+\sqrt{1+y}} dx dy + \int_8^9 \int_{3-\sqrt{9-y}}^{3+\sqrt{9-y}} dx dy$

$$A = \int_0^4 \int_{x^2-2x}^{6x-x^2} dy dx = \int_0^4 (8x - 2x^2) dx = \left[ 4x^2 - \frac{2}{3}x^3 \right]_0^4 = \frac{64}{3}$$

11.  $A = 4 \int_0^1 \int_0^{x\sqrt{1-x^2}} dy dx = 4 \int_0^1 x\sqrt{1-x^2} dx = \left[ -\frac{4}{3}(1-x^2)^{3/2} \right]_0^1 = \frac{4}{3}$

$$A = 4 \int_0^{1/2} \int_{\sqrt{(1-\sqrt{1-4y^2})/2}}^{\sqrt{(1+\sqrt{1-4y^2})/2}} dx dy$$

12.  $A = \int_0^2 \int_0^{y^2+1} dx dy = \int_0^1 \int_0^2 dy dx + \int_1^2 \int_{\sqrt{x-1}}^2 dy dx = \frac{14}{3}$

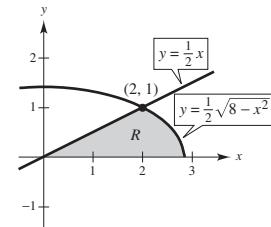
13.  $A = \int_2^5 \int_{x-3}^{\sqrt{x-1}} dy dx + 2 \int_1^2 \int_0^{\sqrt{x-1}} dy dx = \int_{-1}^2 \int_{y^2+1}^{y+3} dx dy = \frac{9}{2}$

14.  $A = \int_0^3 \int_{-y}^{2y-y^2} dx dy = \int_{-3}^0 \int_{-x}^{1+\sqrt{1-x}} dy dx + \int_0^1 \int_{1-\sqrt{1-x}}^{1+\sqrt{1-x}} dy dx = \frac{9}{2}$

15. Both integrations are over the common region  $R$  shown in the figure. Analytically,

$$\int_0^1 \int_{2y}^{2\sqrt{2-y^2}} (x+y) dx dy = \frac{4}{3} + \frac{4}{3}\sqrt{2}$$

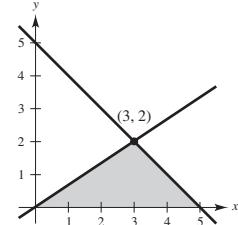
$$\int_0^2 \int_0^{x/2} (x+y) dy dx + \int_2^{2\sqrt{2}} \int_0^{\sqrt{8-x^2}/2} (x+y) dy dx = \frac{5}{3} + \left( \frac{4}{3}\sqrt{2} - \frac{1}{3} \right) = \frac{4}{3} + \frac{4}{3}\sqrt{2}$$



16. Both integrations are over the common region  $R$  shown in the figure. Analytically,

$$\int_0^2 \int_{3y/2}^{5-y} e^{x+y} dx dy = \frac{2}{5} + \frac{8}{5}e^5$$

$$\int_0^3 \int_0^{2x/3} e^{x+y} dy dx + \int_3^5 \int_0^{5-x} e^{x+y} dy dx = \left( \frac{3}{5}e^5 - e^3 + \frac{2}{5} \right) + (e^5 + e^3) = \frac{8}{5}e^5 + \frac{2}{5}$$



17.  $V = \int_0^4 \int_0^{x^2+4} (x^2 - y + 4) dy dx$

$$= \int_0^4 \left[ x^2y - \frac{1}{2}y^2 + 4y \right]_0^{x^2+4} dx$$

$$= \int_0^4 \left( \frac{1}{2}x^4 + 4x^2 + 8 \right) dx$$

$$= \left[ \frac{1}{10}x^5 + \frac{4}{3}x^3 + 8x \right]_0^4 = \frac{3296}{15}$$

18.  $V = \int_0^3 \int_0^x (x+y) dy dx$

$$= \int_0^3 \left[ xy + \frac{1}{2}y^2 \right]_0^x dx$$

$$= \frac{3}{2} \int_0^3 x^2 dx$$

$$= \left[ \frac{1}{2}x^3 \right]_0^3 = \frac{27}{2}$$

19. Area  $R = 16$

$$\text{Average Value} = \frac{1}{16} \int_{-2}^2 \int_{-2}^2 (16 - x^2 - y^2) dy dx$$

$$= \frac{1}{16} \int_{-2}^2 \left[ 16y - x^2y - \frac{y^3}{3} \right]_{-2}^2 dx$$

$$= \frac{1}{16} \int_{-2}^2 \left[ 64 - 4x^2 - \frac{16}{3} \right] dx$$

$$= \frac{1}{16} \left[ 64x - \frac{4x^3}{3} - \frac{16}{3}x \right]_{-2}^2$$

$$= \frac{1}{16} \left[ 256 - \frac{64}{3} - \frac{64}{3} \right] = \frac{40}{3}$$

**20.** Area  $R = 9$

$$\text{Average Value} = \frac{1}{9} \int_0^3 \int_0^3 (2x^2 + y^2) dy dx = \frac{1}{9} \int_0^3 \left[ 2x^2 y + \frac{y^3}{3} \right]_0^3 dx = \frac{1}{9} \int_0^3 (6x^2 + 9) dx = \frac{1}{9} [2x^3 + 9x]_0^3 = 9$$

**21.** Area  $R = 3(5) = 15$

$$\begin{aligned} \text{Average temperature} &= \frac{1}{15} \int_0^3 \int_0^5 (40 - 6x^2 - y^2) dy dx = \frac{1}{15} \int_0^3 \left[ 40y - 6x^2 y - \frac{y^3}{3} \right]_0^5 dx \\ &= \frac{1}{15} \int_0^3 \left[ 200 - 30x^2 - \frac{125}{3} \right] dx = \frac{1}{15} \left[ 200x - 10x^3 - \frac{125x}{3} \right]_0^3 = \frac{1}{15} [600 - 270 - 125] = 13\frac{2}{3}^\circ C \end{aligned}$$

**22.** Average  $= \frac{1}{150} \int_{45}^{60} \int_{40}^{50} [192x + 576y - x^2 - 5y^2 - 2xy - 5000] dx dy \approx 13,246.67$

$$\begin{aligned} \text{23. } \int_0^\infty \int_0^\infty kxye^{-(x+y)} dy dx &= \int_0^\infty \left[ -kxe^{-(x+y)}(y+1) \right]_0^\infty dx \\ &= \int_0^\infty kxe^{-x} dx \\ &= \left[ -k(x+1)e^{-x} \right]_0^\infty = k \end{aligned}$$

So,  $k = 1$ .

$$P = \int_0^1 \int_0^1 xye^{-(x+y)} dy dx \approx 0.070$$

$$\begin{aligned} \text{24. } \int_0^1 \int_0^x kxy dy dx &= \int_0^1 \left[ \frac{kxy^2}{2} \right]_0^x dx \\ &= \int_0^1 \frac{kx^3}{2} dx = \left[ \frac{kx^4}{8} \right]_0^1 = \frac{k}{8} \end{aligned}$$

Because  $\frac{k}{8} = 1$ , you have  $k = 8$ .

$$\begin{aligned} P &= \int_0^{0.25} \int_0^{0.5} 8xy dy dx \\ &= \int_0^{0.25} \left[ 4xy^2 \right]_0^{0.5} dx \\ &= \int_0^{0.25} x dx \\ &= \left[ \frac{x^2}{2} \right]_0^{0.25} = 0.125 \end{aligned}$$

$$\begin{aligned} \text{31. } \int_0^h \int_0^x \sqrt{x^2 + y^2} dy dx &= \int_0^{\pi/4} \int_0^{h \sec \theta} r^2 dr d\theta \\ &= \frac{h^3}{3} \int_0^{\pi/4} \sec^3 \theta d\theta = \frac{h^3}{6} [\sec \theta \tan \theta + \ln |\sec \theta + \tan \theta|]_0^{\pi/4} = \frac{h^3}{6} [\sqrt{2} + \ln(\sqrt{2} + 1)] \end{aligned}$$

$$\text{32. } \int_0^4 \int_0^{\sqrt{16-y^2}} (x^2 + y^2) dx dy = \int_0^{\pi/2} \int_0^4 r^3 dr d\theta = \int_0^{\pi/2} \left[ \frac{r^4}{4} \right]_0^4 d\theta = \int_0^{\pi/2} 64 d\theta = 32\pi$$

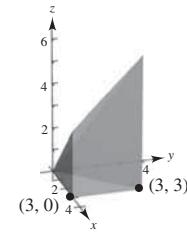
$$\text{33. } A = 2 \int_0^\pi \int_0^{2+\cos \theta} r dr d\theta = \int_0^\pi (2 + \cos \theta)^2 d\theta = \int_0^\pi \left[ 4 + 4 \cos \theta + \frac{1 + \cos 2\theta}{2} \right] d\theta = \left[ 4\theta + 4 \sin \theta + \frac{1}{2} \theta + \frac{\sin 2\theta}{4} \right]_0^\pi = \frac{9\pi}{2}$$

$$\text{34. } A = 4 \int_0^{\pi/2} \int_0^{2 \sin 2\theta} r dr d\theta = 2 \int_0^{\pi/2} (2 \sin 2\theta)^2 d\theta = 8 \int_0^{\pi/2} \frac{1 - \cos 4\theta}{2} d\theta = 4 \left[ \theta - \frac{\sin 4\theta}{4} \right]_0^{\pi/2} = 2\pi$$

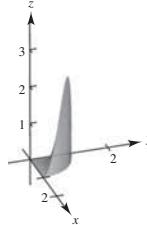
**25.** Volume  $\approx (\text{base})(\text{height})$

$$\approx \frac{9}{2}(3) = \frac{27}{2}$$

Matches (c)



**26.** Matches (c)



**27.** True

**28.** False,  $\int_0^1 \int_0^1 x dy dx \neq \int_1^2 \int_1^2 x dy dx$

**29.** True

**30.** True,

$$\int_0^1 \int_0^1 \frac{1}{1+x^2+y^2} dx dy < \int_0^1 \int_0^1 \frac{1}{1+x^2} dx dy = \frac{\pi}{4}$$

35.  $V = 4 \int_0^h \int_0^{\pi/2} \int_1^{\sqrt{1+z^2}} r dr d\theta dz = 2 \int_0^h \int_0^{\pi/2} (1 + z^2 - 1) d\theta dz = \pi \int_0^h z^2 dz = \left[ \pi \left( \frac{1}{3} z^3 \right) \right]_0^h = \frac{\pi h^3}{3}$

36.  $V = 8 \int_0^{\pi/2} \int_b^R \sqrt{R^2 - r^2} r dr d\theta = -\frac{8}{3} \int_0^{\pi/2} \left[ (R^2 - r^2)^{3/2} \right]_b^R d\theta = \frac{8}{3} (R^2 - b^2)^{3/2} \int_0^{\pi/2} d\theta = \frac{4}{3} \pi (R^2 - b^2)^{3/2}$

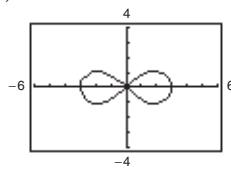
37. (a)  $(x^2 + y^2)^2 = 9(x^2 - y^2)$

$$(r^2)^2 = 9(r^2 \cos^2 \theta - r^2 \sin^2 \theta)$$

$$r^2 = 9(\cos^2 \theta - \sin^2 \theta)$$

$$= 9 \cos 2\theta$$

$$r = 3\sqrt{\cos 2\theta}$$



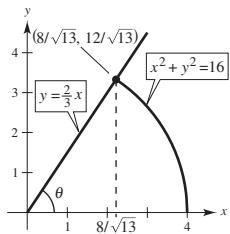
(b)  $A = 4 \int_0^{\pi/4} \int_0^{3\sqrt{\cos 2\theta}} r dr d\theta = 9$

(c)  $V = 4 \int_0^{\pi/4} \int_0^{3\sqrt{\cos 2\theta}} \sqrt{9 - r^2} r dr d\theta \approx 20.392$

38.  $\tan \theta = \frac{12\sqrt{13}}{8\sqrt{13}} = \frac{3}{2} \Rightarrow \theta \approx 0.9828$

The polar region is given by  $0 \leq r \leq 4$  and  $0 \leq \theta \leq 0.9828$ . So,

$$\int_0^{\arctan(3/2)} \int_0^4 (r \cos \theta)(r \sin \theta) r dr d\theta = \frac{288}{13}.$$



40.  $m = k \int_0^L \int_0^{(h/2)[2-(x/L)-(x^2/L^2)]} dy dx = \frac{kh}{2} \int_0^L \left( 2 - \frac{x}{L} - \frac{x^2}{L^2} \right) dx = \frac{7khL}{12}$

$$M_x = k \int_0^L \int_0^{(h/2)[2-(x/L)-(x^2/L^2)]} y dy dx = \frac{kh^2}{8} \int_0^L \left( 2 - \frac{x}{L} - \frac{x^2}{L^2} \right)^2 dx \\ = \frac{kh^2}{8} \int_0^L \left[ 4 - \frac{4x}{L} - \frac{3x^2}{L^2} + \frac{2x^3}{L^3} + \frac{x^4}{L^4} \right] dx = \frac{kh^2}{8} \left[ 4x - \frac{2x^2}{L} - \frac{x^3}{L^2} + \frac{x^4}{2L^3} + \frac{x^5}{5L^4} \right]_0^L = \frac{kh^2}{8} \cdot \frac{17L}{10} = \frac{17kh^2L}{80}$$

$$M_y = k \int_0^L \int_0^{(h/2)[2-(x/L)-(x^2/L^2)]} x dy dx$$

$$= \frac{kh}{2} \int_0^L \left( 2x - \frac{x^2}{L} - \frac{x^3}{L^2} \right) dx = \frac{kh}{2} \left[ x^2 - \frac{x^3}{3L} - \frac{x^4}{4L^2} \right]_0^L = \frac{kh}{2} \cdot \frac{5L^2}{12} = \frac{5khL^2}{24}$$

$$\bar{x} = \frac{M_y}{m} = \frac{5khL^2}{24} \cdot \frac{12}{7khL} = \frac{5L}{14}$$

$$\bar{y} = \frac{M_x}{m} = \frac{17kh^2L}{80} \cdot \frac{12}{7khL} = \frac{51h}{140}$$

39. (a)  $m = k \int_0^1 \int_{2x^3}^{2x} xy dy dx = \frac{k}{4}$

$$M_x = k \int_0^1 \int_{2x^3}^{2x} xy^2 dy dx = \frac{16k}{55}$$

$$M_y = k \int_0^1 \int_{2x^3}^{2x} x^2 y dy dx = \frac{8k}{45}$$

$$\bar{x} = \frac{M_y}{m} = \frac{32}{45}$$

$$\bar{y} = \frac{M_x}{m} = \frac{64}{55}$$

$$(\bar{x}, \bar{y}) = \left( \frac{32}{45}, \frac{64}{55} \right)$$

(b)  $m = k \int_0^1 \int_{2x^3}^{2x} (x^2 + y^2) dy dx = \frac{17k}{30}$

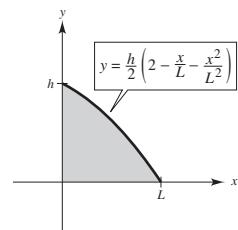
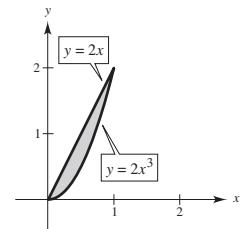
$$M_x = k \int_0^1 \int_{2x^3}^{2x} y(x^2 + y^2) dy dx = \frac{392k}{585}$$

$$M_y = k \int_0^1 \int_{2x^3}^{2x} x(x^2 + y^2) dy dx = \frac{156k}{385}$$

$$\bar{x} = \frac{M_y}{m} = \frac{936}{1309}$$

$$\bar{y} = \frac{M_x}{m} = \frac{784}{663}$$

$$(\bar{x}, \bar{y}) = \left( \frac{936}{1309}, \frac{784}{663} \right)$$



41.  $I_x = \int_R \int y^2 \rho(x, y) dA = \int_0^a \int_0^b kxy^2 dy dx = \frac{1}{6}kb^3a^2$   
 $I_y = \int_R \int x^2 \rho(x, y) dA = \int_0^a \int_0^b kx^3 dy dx = \frac{1}{4}kba^4$   
 $I_0 = I_x + I_y = \frac{1}{6}kb^3a^2 + \frac{1}{4}kba^4 = \frac{ka^2b}{12}(2b^2 + 3a^2)$   
 $m = \int_R \int \rho(x, y) dA = \int_0^a \int_0^b kx dy dx = \frac{1}{2}kba^2$   
 $\bar{x} = \sqrt{\frac{I_y}{m}} = \sqrt{\frac{(1/4)kba^4}{(1/2)kba^2}} = \sqrt{\frac{a^2}{2}} = \frac{a\sqrt{2}}{2}$   
 $\bar{y} = \sqrt{\frac{I_x}{m}} = \sqrt{\frac{(1/6)kb^3a^2}{(1/2)kba^2}} = \sqrt{\frac{b^2}{3}} = \frac{b\sqrt{3}}{3}$

42.  $I_x = \int_R \int y^2 \rho(x, y) dA = \int_0^2 \int_0^{4-x^2} ky^3 dy dx = \frac{16,384}{315}k$   
 $I_y = \int_R \int x^2 \rho(x, y) dA = \int_0^2 \int_0^{4-x^2} kx^2 y dy dx = \frac{512}{105}k$   
 $I_0 = I_x + I_y = \frac{16,384k}{315} + \frac{512k}{105} = \frac{17,920}{315}k = \frac{512}{9}k$   
 $m = \int_R \int \rho(x, y) dA = \int_0^2 \int_0^{4-x^2} ky dy dx = \frac{128}{15}k$   
 $\bar{x} = \sqrt{\frac{I_y}{m}} = \sqrt{\frac{512k/105}{128k/15}} = \sqrt{\frac{4}{7}} = \frac{2\sqrt{7}}{7}$   
 $\bar{y} = \sqrt{\frac{I_x}{m}} = \sqrt{\frac{16,384k/315}{128k/15}} = \sqrt{\frac{128}{21}} = \frac{8\sqrt{42}}{21}$

44.  $f(x, y) = 16 - x - y^2$

$R = \{(x, y): 0 \leq x \leq 2, 0 \leq y \leq x\}$

$f_x = -1, f_y = -2y$

$\sqrt{1 + (f_x)^2 + (f_y)^2} = \sqrt{2 + 4y^2}$

$S = \int_0^2 \int_y^2 \sqrt{2 + 4y^2} dx dy = \int_0^2 \left[ 2\sqrt{2 + 4y^2} - y\sqrt{2 + 4y^2} \right] dy$ 
 $= \left[ \frac{1}{2} \left( 2y\sqrt{2 + 4y^2} + 2 \ln|2y + \sqrt{2 + 4y^2}| \right) - \frac{1}{12} (2 + 4y^2)^{3/2} \right]_0^2$ 
 $= \left[ \frac{1}{2} (4\sqrt{18} + 2 \ln|4 + \sqrt{18}|) - \frac{1}{12} (18\sqrt{18}) \right] - \left[ \ln\sqrt{2} - \frac{2\sqrt{2}}{12} \right]$ 
 $= 6\sqrt{2} + \ln|4 + 3\sqrt{2}| - \frac{9\sqrt{2}}{2} - \ln\sqrt{2} + \frac{\sqrt{2}}{6} = \frac{5\sqrt{2}}{3} + \ln|2\sqrt{2} + 3|$

45.  $f(x, y) = 9 - y^2$

$f_x = 0, f_y = -2y$

$S = \int_R \int \sqrt{1 + (f_x)^2 + (f_y)^2} dA$ 
 $= \int_0^3 \int_{-y}^y \sqrt{1 + 4y^2} dx dy = \int_0^3 \left[ \sqrt{1 + 4y^2} x \right]_{-y}^y dy = \int_0^3 2y\sqrt{1 + 4y^2} dy = \frac{1}{4} \frac{2}{3} (1 + 4y^2)^{3/2} \Big|_0^3 = \frac{1}{6} [(37)^{3/2} - 1]$

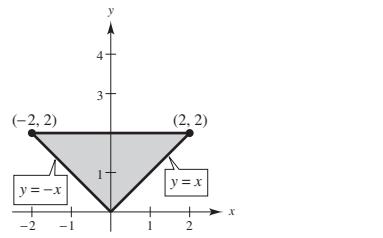
43.  $f(x, y) = 25 - x^2 - y^2$   
 $f_x = -2x, f_y = -2y$   
 $S = \int_R \int \sqrt{1 + (f_x)^2 + (f_y)^2} dA$ 
 $= \int_R \int \sqrt{1 + 4x^2 + 4y^2} dA$ 
 $= 4 \int_0^{\pi/2} \int_0^5 \sqrt{1 + 4r^2} r dr d\theta$ 
 $= \frac{1}{3} \int_0^{\pi/2} \left[ (1 + 4r^2)^{3/2} \right]_0^5 d\theta$ 
 $= \frac{1}{3} \int_0^{\pi/2} [(101)^{3/2} - 1] d\theta$ 
 $= \frac{\pi}{6} [101\sqrt{101} - 1]$

46.  $f(x, y) = 4 - x^2$ ,  $f_x = -2x$ ,  $f_y = 0$

$$S = \int_R \int \sqrt{1 + 4x^2} \, dA = \int_{-2}^0 \int_{-x}^2 \sqrt{1 + 4x^2} \, dy \, dx + \int_0^2 \int_x^2 \sqrt{1 + 4x^2} \, dy \, dx$$

These integrals are equal by symmetry.

$$\begin{aligned} S &= 2 \int_0^2 \int_x^2 \sqrt{1 + 4x^2} \, dy \, dx = \int_0^2 \left[ 2\sqrt{1 + 4x^2} - x\sqrt{1 + 4x^2} \right] dx \\ &= 2 \left[ \frac{1}{2} \ln(\sqrt{1 + 4x^2} + 2x) + x\sqrt{1 + 4x^2} - \frac{1}{12}(1 + 4x^2)^{3/2} \right]_0^2 = 2 \left[ \frac{1}{2} \ln(\sqrt{17} + 4) + 2\sqrt{17} - \frac{17}{12}\sqrt{17} + \frac{1}{12} \right] \approx 7.0717 \end{aligned}$$



47. (a)  $V = \int_0^{50} \int_0^{\sqrt{50^2 - x^2}} \left( 20 + \frac{xy}{100} - \frac{x+y}{5} \right) dy \, dx = \int_0^{50} \left[ 20\sqrt{50^2 - x^2} + \frac{x}{200}(50^2 - x^2) - \frac{x}{5}\sqrt{50^2 - x^2} - \frac{50^2 - x^2}{10} \right] dy$   
 $= \left[ 10 \left( x\sqrt{50^2 - x^2} + 50^2 \arcsin \frac{x}{50} \right) + \frac{25}{4}x^2 - \frac{x^4}{800} + \frac{1}{15}(50^2 - x^2)^{3/2} - 250x + \frac{x^3}{30} \right]_0^{50} \approx 30,415.74 \text{ ft}^3$

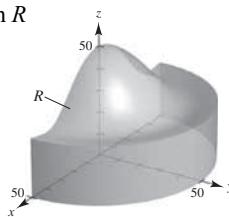
(b)  $z = 20 + \frac{xy}{100}$

$$\sqrt{1 + (f_x)^2 + (f_y)^2} = \sqrt{1 + \frac{y^2}{100^2} + \frac{x^2}{100^2}} = \frac{\sqrt{100^2 + x^2 + y^2}}{100}$$

$$S = \frac{1}{100} \int_0^{50} \int_0^{\sqrt{50^2 - x^2}} \sqrt{100^2 + x^2 + y^2} \, dy \, dx = \frac{1}{100} \int_0^{\pi/2} \int_0^{50} \sqrt{100^2 + r^2} r \, dr \, d\theta \approx 2081.53 \text{ ft}^2$$

48. (a) Graph of  $f(x, y) = z = 25 \left[ 1 + e^{-(x^2+y^2)/1000} \cos^2 \left( \frac{x^2+y^2}{1000} \right) \right]$

over region  $R$



(b) Surface area =  $\int_R \int \sqrt{1 + f_x(x, y)^2 + f_y(x, y)^2} \, dA$

Using a symbolic computer program, you obtain surface area  $\approx 4540$  sq. ft.

$$\begin{aligned} 49. \int_{-3}^3 \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \int_{x^2+y^2}^9 \sqrt{x^2 + y^2} \, dz \, dy \, dx &= \int_0^{2\pi} \int_0^3 \int_{r^2}^9 r^2 \, dz \, dr \, d\theta \\ &= \int_0^{2\pi} \int_0^3 (9r^2 - r^4) \, dr \, d\theta = \int_0^{2\pi} \left[ 3r^3 - \frac{r^5}{5} \right]_0^3 \, d\theta = \frac{162}{5} \int_0^{2\pi} \, d\theta = \frac{324\pi}{5} \end{aligned}$$

$$50. \int_{-2}^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_0^{(x^2+y^2)/2} (x^2 + y^2) \, dz \, dy \, dx = \int_0^{2\pi} \int_0^2 \int_0^{r^2/2} r^3 \, dz \, dr \, d\theta = \frac{1}{2} \int_0^{2\pi} \int_0^2 r^5 \, dr \, d\theta = \frac{16}{3} \int_0^{2\pi} \, d\theta = \frac{32\pi}{3}$$

$$\begin{aligned} 51. \int_0^a \int_0^b \int_0^c (x^2 + y^2 + z^2) \, dx \, dy \, dz &= \int_0^a \int_0^b \left( \frac{1}{3}c^3 + cy^2 + cz^2 \right) dy \, dz \\ &= \int_0^a \left( \frac{1}{3}bc^3 + \frac{1}{3}b^3c + bcz^2 \right) dz = \frac{1}{3}abc^3 + \frac{1}{3}ab^3c + \frac{1}{3}a^3bc = \frac{1}{3}abc(a^2 + b^2 + c^2) \end{aligned}$$

$$\begin{aligned} 52. \int_0^5 \int_0^{\sqrt{25-x^2}} \int_0^{\sqrt{25-x^2-y^2}} \frac{1}{1+x^2+y^2+z^2} \, dz \, dy \, dx &= \int_0^{\pi/2} \int_0^{\pi/2} \int_0^5 \frac{\rho^2}{1+\rho^2} \sin \phi \, d\rho \, d\phi \, d\theta \\ &= \int_0^{\pi/2} \int_0^{\pi/2} [\rho - \arctan \rho]_0^5 \sin \phi \, d\phi \, d\theta \\ &= \int_0^{\pi/2} [(5 - \arctan 5)(-\cos \phi)]_0^{\pi/2} \, d\theta = \frac{\pi}{2}(5 - \arctan 5) \end{aligned}$$

53.  $\int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{-\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}} (x^2 + y^2) dz dy dx = \int_0^{2\pi} \int_0^1 \int_{-\sqrt{1-r^2}}^{\sqrt{1-r^2}} r^3 dz dr d\theta = \frac{8\pi}{15}$

54.  $\int_0^2 \int_0^{\sqrt{4-x^2}} \int_0^{\sqrt{4-x^2-y^2}} xyz dz dy dx = \frac{4}{3}$

55.  $V = 4 \int_0^{\pi/2} \int_0^{2\cos\theta} \int_0^{\sqrt{4-r^2}} r dz dr d\theta = 4 \int_0^{\pi/2} \int_0^{2\cos\theta} r \sqrt{4-r^2} dr d\theta$   
 $= - \int_0^{\pi/2} \left[ \frac{4}{3} (4-r^2)^{3/2} \right]_0^{2\cos\theta} d\theta = \frac{32}{3} \int_0^{\pi/2} (1 - \sin^3 \theta) d\theta = \frac{32}{3} \left[ \theta + \cos \theta - \frac{1}{3} \cos^3 \theta \right]_0^{\pi/2} = \frac{32}{3} \left( \frac{\pi}{2} - \frac{2}{3} \right)$

56.  $V = 2 \int_0^{\pi/2} \int_0^{2\sin\theta} \int_0^{16-r^2} r dz dr d\theta = 2 \int_0^{\pi/2} \int_0^{2\sin\theta} r (16 - r^2) dr d\theta$   
 $= 2 \int_0^{\pi/2} (32 \sin^2 \theta - 4 \sin^4 \theta) d\theta = 8 \int_0^{\pi/2} (8 \sin^2 \theta - \sin^4 \theta) d\theta$   
 $= 8 \left[ 4\theta - 2 \sin 2\theta + \frac{1}{4} \sin^3 \theta \cos \theta - \frac{3}{4} \left( \frac{1}{2}\theta - \frac{1}{4} \sin 2\theta \right) \right]_0^{\pi/2} = \frac{29\pi}{2}$

57.  $m = 4k \int_{\pi/4}^{\pi/2} \int_0^{\pi/2} \int_0^{\cos\phi} \rho^2 \sin \phi d\rho d\theta d\phi$   
 $= \frac{4}{3} k \int_{\pi/4}^{\pi/2} \int_0^{\pi/2} \cos^3 \phi \sin \phi d\theta d\phi = \frac{2}{3} k\pi \int_{\pi/4}^{\pi/2} \cos^3 \phi \sin \phi d\phi = \left[ -\frac{2}{3} k\pi \left( \frac{1}{4} \cos^4 \phi \right) \right]_{\pi/4}^{\pi/2} = \frac{k\pi}{24}$

$M_{xy} = 4k \int_{\pi/4}^{\pi/2} \int_0^{\pi/2} \int_0^{\cos\phi} \rho^3 \cos \phi \sin \phi d\rho d\theta d\phi$   
 $= k \int_{\pi/4}^{\pi/2} \int_0^{\pi/2} \cos^5 \phi \sin \phi d\theta d\phi = \frac{1}{2} k\pi \int_{\pi/4}^{\pi/2} \cos^5 \phi \sin \phi d\phi = \left[ -\frac{1}{12} k\pi \cos^6 \phi \right]_{\pi/4}^{\pi/2} = \frac{k\pi}{96}$

$$\bar{z} = \frac{M_{xy}}{m} = \frac{k\pi/96}{k\pi/24} = \frac{1}{4}$$

$\bar{x} = \bar{y} = 0$  by symmetry

$$(\bar{x}, \bar{y}, \bar{z}) = \left( 0, 0, \frac{1}{4} \right)$$

58.  $m = 2k \int_0^{\pi/2} \int_0^a \int_0^{cr \sin \theta} r dz dr d\theta = 2kc \int_0^{\pi/2} \int_0^a r^2 \sin \theta dr d\theta = \frac{2}{3} kca^3 \int_0^{\pi/2} \sin \theta d\theta = \frac{2}{3} kca^3$

$$M_{xz} = 2k \int_0^{\pi/2} \int_0^a \int_0^{cr \sin \theta} r^2 \sin \theta dz dr d\theta = 2kc \int_0^{\pi/2} \int_0^a r^3 \sin^2 \theta dr d\theta = \frac{1}{2} kca^4 \int_0^{\pi/2} \sin^2 \theta d\theta = \frac{1}{8} \pi kca^4$$

$$M_{xy} = 2k \int_0^{\pi/2} \int_0^a \int_0^{cr \sin \theta} rz dz dr d\theta = kc^2 \int_0^{\pi/2} \int_0^a r^3 \sin^2 \theta dr d\theta = \frac{1}{4} kc^2 a^4 \int_0^{\pi/2} \sin^2 \theta d\theta = \frac{1}{16} \pi kc^2 a^4$$

$$\bar{x} = 0$$

$$\bar{y} = \frac{M_{xz}}{m} = \frac{\pi kca^4/8}{2kca^3/3} = \frac{3\pi a}{16}$$

$$\bar{z} = \frac{M_{xy}}{m} = \frac{\pi kc^2 a^4/16}{2kca^3/3} = \frac{3\pi ca}{32}$$

$$(\bar{x}, \bar{y}, \bar{z}) = \left( 0, \frac{3\pi a}{16}, \frac{3\pi ca}{32} \right)$$

59.  $m = k \int_0^{\pi/2} \int_0^{\pi/2} \int_0^a \rho^2 \sin \phi \, d\rho \, d\theta \, d\phi = \frac{k\pi a^3}{6}$

$$M_{xy} = k \int_0^{\pi/2} \int_0^{\pi/2} \int_0^a (\rho \cos \phi) \rho^2 \sin \phi \, d\rho \, d\theta \, d\phi = \frac{k\pi a^4}{16}$$

$$\bar{x} = \bar{y} = \bar{z} = \frac{M_{xy}}{m} = \frac{k\pi a^4}{16} \left( \frac{6}{k\pi a^3} \right) = \frac{3a}{8}$$

$$(\bar{x}, \bar{y}, \bar{z}) = \left( \frac{3a}{8}, \frac{3a}{8}, \frac{3a}{8} \right)$$

60.  $m = \frac{500\pi}{3} - \int_0^3 \int_0^{2\pi} \int_{-\sqrt{25-r^2}}^{\sqrt{25-r^2}} r \, dz \, d\theta \, dr = \frac{500\pi}{3} - \int_0^3 \int_0^{2\pi} (r\sqrt{25-r^2} - 4r) \, d\theta \, dr$

$$= \frac{500\pi}{3} - 2\pi \left[ -\frac{1}{3}(25-r^2)^{3/2} - 2r^2 \right]_0^3 = \frac{500\pi}{3} - 2\pi \left[ -\frac{64}{3} - 18 + \frac{125}{3} \right] = \frac{500\pi}{3} - \frac{14\pi}{3} = 162\pi$$

$\bar{x} = \bar{y} = 0$  by symmetry

$$M_{xy} = \int_0^{2\pi} \int_0^3 \int_{-\sqrt{25-r^2}}^4 zr \, dz \, dr \, d\theta + \int_0^{2\pi} \int_3^5 \int_{-\sqrt{25-r^2}}^{\sqrt{25-r^2}} zr \, dz \, dr \, d\theta = \int_0^{2\pi} \int_0^3 \left[ 8 - \frac{1}{2}(25-r^2) \right] r \, dr \, d\theta + 0$$

$$= \int_0^{2\pi} \int_0^3 \left[ \frac{1}{2}r^3 - \frac{9}{2}r \right] dr \, d\theta = \int_0^{2\pi} \left[ \frac{1}{8}r^4 - \frac{9}{4}r^2 \right]_0^3 d\theta = \left[ -\frac{81}{8}\theta \right]_0^{2\pi} = -\frac{81}{4}\pi$$

$$\bar{z} = \frac{M_{xy}}{m} = -\frac{81\pi}{4} \frac{1}{162\pi} = -\frac{1}{8}$$

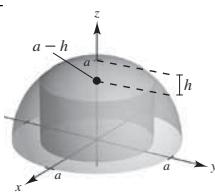
$$(\bar{x}, \bar{y}, \bar{z}) = \left( 0, 0, -\frac{1}{8} \right)$$

61.  $I_z = 4k \int_0^{\pi/2} \int_3^4 \int_0^{16-r^2} r^3 \, dz \, dr \, d\theta = 4k \int_0^{\pi/2} \int_3^4 (16r^3 - r^5) \, dr \, d\theta = \frac{833\pi k}{3}$

62.  $I_z = k \int_0^\pi \int_0^{2\pi} \int_0^a \rho^2 \sin^2 \phi (\rho) \rho^2 \sin \phi \, d\rho \, d\theta \, d\phi = \frac{4k\pi a^6}{9}$

63.  $z = f(x, y) = \sqrt{a^2 - x^2 - y^2} = \sqrt{a^2 - r^2}$

$$0 \leq r \leq \sqrt{2ah - h^2}$$



(a) Disc Method

$$\begin{aligned} V &= \pi \int_{a-h}^a (a^2 - y^2) \, dy = \pi \left[ a^2y - \frac{y^3}{3} \right]_{a-h}^a = \pi \left[ \left( a^3 - \frac{a^3}{3} \right) - \left( a^2(a-h) - \frac{(a-h)^3}{3} \right) \right] \\ &= \pi \left[ a^3 - \frac{a^3}{3} - a^3 + a^2h + \frac{a^3}{3} - a^2h + ah^2 - \frac{h^3}{3} \right] = \pi \left[ ah^2 - \frac{h^3}{3} \right] = \frac{1}{3}\pi h^2 [3a - h] \end{aligned}$$

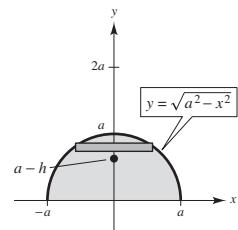
Equivalently, use spherical coordinates.

$$V = \int_0^{2\pi} \int_0^{\cos^{-1}(a-h/a)} \int_{(a-h)\sec\phi}^a \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta$$

(b)  $M_{xy} = \int_0^{2\pi} \int_0^{\cos^{-1}(a-h/a)} \int_{(a-h)\sec\phi}^a (\rho \cos \phi) \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta = \frac{1}{4}h^2\pi(2a-h)^2$

$$\bar{z} = \frac{M_{xy}}{V} = \frac{\frac{1}{4}h^2\pi(2a-h)^2}{\frac{1}{3}h^2\pi(3a-h)} = \frac{3(2a-h)^2}{4(3a-h)}$$

$$\text{Centroid: } \left( 0, 0, \frac{3(2a-h)^2}{4(3a-h)} \right)$$



(c) If  $h = a$ ,  $\bar{z} = \frac{3(a)^2}{4(2a)} = \frac{3}{8}a$ .

Centroid of hemisphere:  $\left(0, 0, \frac{3}{8}a\right)$

(d)  $\lim_{h \rightarrow 0} \bar{z} = \lim_{h \rightarrow 0} \frac{3(2a - h)^2}{4(3a - h)} = \frac{3(4a^2)}{12a} = a$

(e)  $x^2 + y^2 = \rho^2 \sin^2 \phi$

$$I_z = \int_0^{2\pi} \int_0^{\cos^{-1}(a-h/a)} \int_{(a-h)\sec\phi}^a (\rho^2 \sin^2 \phi) \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta = \frac{h^3}{30} (20a^2 - 15ah + 3h^2) \pi$$

(f) If  $h = a$ ,  $I_z = \frac{a^3 \pi}{30} (20a^2 - 15a^2 + 3a^2) = \frac{4}{15}a^5 \pi$ .

64.  $x^2 + y^2 + \frac{z^2}{a^2} = 1$

$$I_z = \iiint_Q (x^2 + y^2) \, dV = \int_{-a}^a \int_{-\sqrt{1-z^2-a^2}}^{\sqrt{1-z^2-a^2}} \int_{-\sqrt{1-y^2-z^2-a^2}}^{\sqrt{1-y^2-z^2-a^2}} (x^2 + y^2) \, dx \, dy \, dz = \frac{8}{15}\pi a$$

65.  $\int_0^{2\pi} \int_0^\pi \int_0^{6 \sin \phi} \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta$

Because  $\rho = 6 \sin \phi$  represents (in the  $yz$ -plane) a circle of radius 3 centered at  $(0, 3, 0)$ , the integral represents the volume of the torus formed by revolving  $(0 < \theta < 2\pi)$  this circle about the  $z$ -axis.

66.  $\int_0^\pi \int_0^2 \int_0^{1+r^2} r \, dz \, dr \, d\theta$

Because  $z = 1 + r^2$  represents a paraboloid with vertex  $(0, 0, 1)$ , this integral represents the volume of the solid below the paraboloid and above the semi-circle  $y = \sqrt{4 - x^2}$  in the  $xy$ -plane.

67.  $\frac{\partial(x, y)}{\partial(u, v)} = \frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial y}{\partial u} \frac{\partial x}{\partial v} = 1(-3) - 2(3) = -9$

68.  $\frac{\partial(x, y)}{\partial(u, v)} = \frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial y}{\partial u} \frac{\partial x}{\partial v} = (2u)(-2v) - (2u)(2v) = -8uv$

69.  $\frac{\partial(x, y)}{\partial(u, v)} = \frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial x}{\partial v} \frac{\partial y}{\partial u} = \frac{1}{2} \left( -\frac{1}{2} \right) - \frac{1}{2} \left( \frac{1}{2} \right) = -\frac{1}{2}$

$$x = \frac{1}{2}(u + v), y = \frac{1}{2}(u - v) \Rightarrow u = x + y, v = x - y$$

Boundaries in  $xy$ -plane

$$x + y = 3$$

$$x + y = 5$$

$$x - y = -1$$

$$x - y = 1$$

Boundaries in  $uv$ -plane

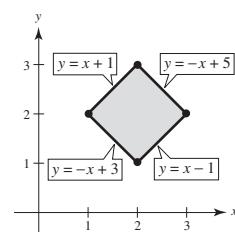
$$u = 3$$

$$u = 5$$

$$v = -1$$

$$v = 1$$

$$\begin{aligned} \int_R \int \ln(x + y) \, dA &= \int_3^5 \int_{-1}^1 \ln\left(\frac{1}{2}(u + v) + \frac{1}{2}(u - v)\right) \left(\frac{1}{2}\right) \, dv \, du = \int_3^5 \int_{-1}^1 \frac{1}{2} \ln u \, dv \, du = \int_3^5 \ln u \, du = [u \ln u - u]_3^5 \\ &= (5 \ln 5 - 5) - (3 \ln 3 - 3) = 5 \ln 5 - 3 \ln 3 - 2 \approx 2.751 \end{aligned}$$



70.  $\frac{\partial(x, y)}{\partial(u, v)} = \frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial x}{\partial v} \frac{\partial y}{\partial u} = 1\left(\frac{1}{u}\right) - 0 = \frac{1}{u}$

$$x = u, y = \frac{v}{u} \Rightarrow u = x, v = xy$$

Boundary in  $xy$ -plane

$$x = 1$$

$$x = 5$$

$$xy = 1$$

$$xy = 5$$

Boundary in  $uv$ -plane

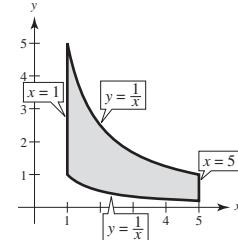
$$u = 1$$

$$u = 5$$

$$v = 1$$

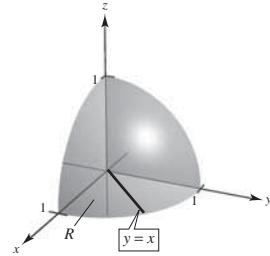
$$v = 5$$

$$\begin{aligned} \int_R \int \frac{x}{1+x^2 y^2} dA &= \int_1^5 \int_1^5 \frac{u}{1+u^2(v/u)^2} \left(\frac{1}{u}\right) du dv = \int_1^5 \int_1^5 \frac{1}{1+v^2} du dv = \int_1^5 \frac{4}{1+v^2} dv \\ &= 4 \arctan v \Big|_1^5 = 4 \arctan 5 - \pi \end{aligned}$$



## Problem Solving for Chapter 14

$$\begin{aligned} 1. V &= 16 \int_R \int \sqrt{1-x^2} dA \\ &= 16 \int_0^{\pi/4} \int_0^1 \sqrt{1-r^2 \cos^2 \theta} r dr d\theta = -\frac{16}{3} \int_0^{\pi/4} \frac{1}{\cos^2 \theta} \left[ (1-\cos^2 \theta)^{3/2} - 1 \right] d\theta \\ &= -\frac{16}{3} [\sec \theta + \cos \theta - \tan \theta]_0^{\pi/4} = 8(2 - \sqrt{2}) \approx 4.6863 \end{aligned}$$



$$2. z = \frac{1}{c}(d - ax - by) \text{ Plane}$$

$$f_x = -\frac{a}{c}, f_y = -\frac{b}{c}$$

$$\sqrt{1 + (f_x)^2 + (f_y)^2} = \sqrt{1 + \frac{a^2}{c^2} + \frac{b^2}{c^2}}$$

$$S = \int_R \int \sqrt{1 + \frac{a^2}{c^2} + \frac{b^2}{c^2}} dA = \frac{\sqrt{a^2 + b^2 + c^2}}{c} \int_R \int dA = \frac{\sqrt{a^2 + b^2 + c^2}}{c} A(R)$$

$$3. (a) \int \frac{du}{a^2 + u^2} = \frac{1}{a} \arctan \frac{u}{a} + C. \text{ Let } a^2 = 2 - u^2, u = v.$$

$$\text{Then } \int \frac{1}{(2-u^2)+v^2} dv = \frac{1}{\sqrt{2-u^2}} \arctan \frac{v}{\sqrt{2-u^2}} + C.$$

$$\begin{aligned} (b) I_1 &= \int_0^{\sqrt{2}/2} \left[ \frac{2}{\sqrt{2-u^2}} \arctan \frac{v}{\sqrt{2-u^2}} \right]_u^u du \\ &= \int_0^{\sqrt{2}/2} \frac{2}{\sqrt{2-u^2}} \left( \arctan \frac{u}{\sqrt{2-u^2}} - \arctan \frac{-u}{\sqrt{2-u^2}} \right) du = \int_0^{\sqrt{2}/2} \frac{4}{\sqrt{2-u^2}} \arctan \frac{u}{\sqrt{2-u^2}} du \end{aligned}$$

Let  $u = \sqrt{2} \sin \theta, du = \sqrt{2} \cos \theta d\theta, 2 - u^2 = 2 - 2 \sin^2 \theta = 2 \cos^2 \theta$ .

$$I_1 = 4 \int_0^{\pi/6} \frac{1}{\sqrt{2} \cos \theta} \arctan \left( \frac{\sqrt{2} \sin \theta}{\sqrt{2} \cos \theta} \right) \cdot \sqrt{2} \cos \theta d\theta = 4 \int_0^{\pi/6} \arctan(\tan \theta) d\theta = \frac{4\theta^2}{2} \Big|_0^{\pi/6} = 2 \left( \frac{\pi}{6} \right)^2 = \frac{\pi^2}{18}$$

$$\begin{aligned}
 (c) \quad I_2 &= \int_{-\sqrt{2}/2}^{\sqrt{2}} \left[ \frac{2}{\sqrt{2-u^2}} \arctan \frac{v}{\sqrt{2-u^2}} \right]_{u=-\sqrt{2}}^{-u+\sqrt{2}} du \\
 &= \int_{-\sqrt{2}/2}^{\sqrt{2}} \frac{2}{\sqrt{2-u^2}} \left[ \arctan \left( \frac{-u+\sqrt{2}}{\sqrt{2-u^2}} \right) - \arctan \left( \frac{u-\sqrt{2}}{\sqrt{2-u^2}} \right) \right] du = \int_{-\sqrt{2}/2}^{\sqrt{2}} \frac{4}{\sqrt{2-u^2}} \arctan \left( \frac{\sqrt{2}-u}{\sqrt{2-u^2}} \right) du
 \end{aligned}$$

Let  $u = \sqrt{2} \sin \theta$ .

$$I_2 = 4 \int_{\pi/6}^{\pi/2} \frac{1}{\sqrt{2} \cos \theta} \arctan \left( \frac{\sqrt{2}-\sqrt{2} \sin \theta}{\sqrt{2} \cos \theta} \right) \cdot \sqrt{2} \cos \theta d\theta = 4 \int_{\pi/6}^{\pi/2} \arctan \left( \frac{1-\sin \theta}{\cos \theta} \right) d\theta$$

$$(d) \quad \tan \left( \frac{1}{2} \left( \frac{\pi}{2} - \theta \right) \right) = \sqrt{\frac{1 - \cos((\pi/2) - \theta)}{1 + \cos((\pi/2) - \theta)}} = \sqrt{\frac{1 - \sin \theta}{1 + \sin \theta}} = \sqrt{\frac{(1 - \sin \theta)^2}{(1 + \sin \theta)(1 - \sin \theta)}} = \sqrt{\frac{(1 - \sin \theta)^2}{\cos^2 \theta}} = \frac{1 - \sin \theta}{\cos \theta}$$

$$\begin{aligned}
 (e) \quad I_2 &= 4 \int_{\pi/6}^{\pi/2} \arctan \left( \frac{1 - \sin \theta}{\cos \theta} \right) d\theta = 4 \int_{\pi/6}^{\pi/2} \arctan \left( \tan \left( \frac{1}{2} \left( \frac{\pi}{2} - \theta \right) \right) \right) d\theta = 4 \int_{\pi/6}^{\pi/2} \frac{1}{2} \left( \frac{\pi}{2} - \theta \right) d\theta = 2 \int_{\pi/6}^{\pi/2} \left( \frac{\pi}{2} - \theta \right) d\theta \\
 &= 2 \left[ \frac{\pi}{2} \theta - \frac{\theta^2}{2} \right]_{\pi/6}^{\pi/2} = 2 \left[ \left( \frac{\pi^2}{4} - \frac{\pi^2}{8} \right) - \left( \frac{\pi^2}{12} - \frac{\pi^2}{72} \right) \right] = 2 \left[ \frac{18 - 9 - 6 + 1}{72} \pi^2 \right] = \frac{4}{36} \pi^2 = \frac{\pi^2}{9}
 \end{aligned}$$

$$(f) \quad \frac{1}{1-xy} = 1 + (xy) + (xy)^2 + \dots \quad |xy| < 1$$

$$\begin{aligned}
 \int_0^1 \int_0^1 \frac{1}{1-xy} dx dy &= \int_0^1 \int_0^1 \left[ 1 + (xy) + (xy)^2 + \dots \right] dx dy = \int_0^1 \int_0^1 \sum_{K=0}^{\infty} (xy)^K dx dy = \sum_{K=0}^{\infty} \int_0^1 \frac{x^{K+1} y^K}{K+1} \Big|_0^1 dy \\
 &= \sum_{K=0}^{\infty} \int_0^1 \frac{y^K}{K+1} dy = \sum_{K=0}^{\infty} \frac{y^{K+1}}{(K+1)^2} \Big|_0^1 = \sum_{K=0}^{\infty} \frac{1}{(K+1)^2} = \sum_{n=1}^{\infty} \frac{1}{n^2}
 \end{aligned}$$

$$(g) \quad u = \frac{x+y}{\sqrt{2}}, v = \frac{y-x}{\sqrt{2}}$$

$$u - v = \frac{2x}{\sqrt{2}} \Rightarrow x = \frac{u-v}{\sqrt{2}}$$

$$u + v = \frac{2y}{\sqrt{2}} \Rightarrow y = \frac{u+v}{\sqrt{2}}$$

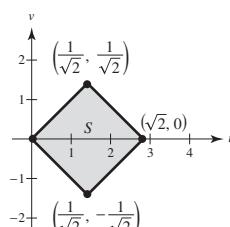
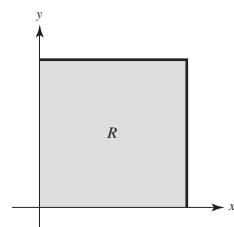
$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{vmatrix} = 1$$

$$\begin{array}{ccc} R & & S \\ (0, 0) & \leftrightarrow & (0, 0) \end{array}$$

$$(1, 0) \leftrightarrow \left( \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \right)$$

$$(0, 1) \leftrightarrow \left( \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right)$$

$$(1, 1) \leftrightarrow (\sqrt{2}, 0)$$



$$\int_0^1 \int_0^1 \frac{1}{1-xy} dx dy = \int_0^{\sqrt{2}/2} \int_{-u}^u \frac{1}{1 - \frac{u^2}{2} + \frac{v^2}{2}} dv du + \int_{\sqrt{2}/2}^{\sqrt{2}} \int_{u-\sqrt{2}}^{-u+\sqrt{2}} \frac{1}{1 - \frac{u^2}{2} + \frac{v^2}{2}} dv du = I_1 + I_2 = \frac{\pi^2}{18} + \frac{\pi^2}{9} = \frac{\pi^2}{6}$$

4.  $A: \int_0^{2\pi} \int_4^5 \left( \frac{r}{16} - \frac{r^2}{160} \right) r dr d\theta = \frac{1333\pi}{960} \approx 4.36 \text{ ft}^3$

$$B = \int_0^{2\pi} \int_9^{10} \left( \frac{r}{16} - \frac{r^2}{160} \right) r dr d\theta = \frac{523\pi}{960} \approx 1.71 \text{ ft}^3$$

The distribution is not uniform. Less water in region of greater area.

In one hour, the entire lawn receives

$$\int_0^{2\pi} \int_0^{10} \left( \frac{r}{16} - \frac{r^2}{160} \right) r dr d\theta = \frac{125\pi}{12} \approx 32.72 \text{ ft}^3.$$

5. Boundary in  $xy$ -plane

$$y = \sqrt{x}$$

$$y = \sqrt{2x}$$

$$y = \frac{1}{3}x^2$$

$$y = \frac{1}{4}x^2$$

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{1}{3}\left(\frac{v}{u}\right)^{2/3} & \frac{2}{3}\left(\frac{u}{v}\right)^{1/3} \\ \frac{2}{3}\left(\frac{v}{u}\right)^{1/3} & \frac{1}{3}\left(\frac{u}{v}\right)^{2/3} \end{vmatrix} = -\frac{1}{3}$$

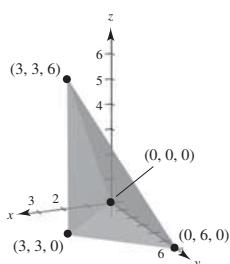
$$A = \int_R \int 1 dA = \int_S \int 1 \left| \frac{\partial(x, y)}{\partial(u, v)} \right| dA = \frac{1}{3}$$

6. (a)  $V = \int_0^{2\pi} \int_0^2 \int_2^{\sqrt{8-r^2}} r dz dr d\theta = \frac{8\pi}{3}(4\sqrt{2} - 5)$

$$(b) V = \int_0^{2\pi} \int_0^{\pi/4} \int_{2\sec\phi}^{2\sqrt{2}} \rho^2 \sin\phi d\rho d\phi d\theta$$

$$= \frac{8\pi}{3}(4\sqrt{2} - 5)$$

7.  $V = \int_0^3 \int_0^{2x} \int_x^{6-x} dy dz dx = 18$



8.  $\int_0^1 \int_0^1 x^n y^n dx dy = \int_0^1 \left[ \frac{x^{n+1}}{n+1} y^n \right]_0^1 dy$

$$= \int_0^1 \frac{1}{n+1} y^n dy$$

$$= \left[ \frac{y^{n+1}}{(n+1)^2} \right]_0^1 = \frac{1}{(n+1)^2}$$

$$\lim_{n \rightarrow \infty} \int_0^1 \int_0^1 x^n y^n dx dy = \lim_{n \rightarrow \infty} \frac{1}{(n+1)^2} = 0$$

9. From Exercise 69, Section 14.3,

$$\int_{-\infty}^{\infty} e^{-x^2/2} dx = \sqrt{2\pi}.$$

$$\text{So, } \int_0^{\infty} e^{-x^2/2} dx = \frac{\sqrt{2\pi}}{2} \text{ and } \int_0^{\infty} e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$$

$$\begin{aligned} \int_0^{\infty} x^2 e^{-x^2} dx &= \left[ -\frac{1}{2} x e^{-x^2} \right]_0^{\infty} + \frac{1}{2} \int_0^{\infty} e^{-x^2} dx \\ &= \frac{1}{2} \cdot \frac{\sqrt{\pi}}{2} = \frac{\sqrt{\pi}}{4} \end{aligned}$$

10. Let  $v = \ln\left(\frac{1}{x}\right)$ ,  $dv = -\frac{dx}{x}$ .

$$e^v = \frac{1}{x}, x = e^{-v}, dx = -e^{-v} dv$$

$$\int_0^1 \sqrt{\ln(1/x)} dx = \int_{\infty}^0 \sqrt{v}(-e^{-v}) dv = \int_0^{\infty} \sqrt{v} e^{-v} dv$$

Let  $u = \sqrt{v}$ ,  $u^2 = v$ ,  $2u du = dv$ .

$$\begin{aligned} \int_0^1 \sqrt{\ln(1/x)} dx &= \int_0^{\infty} u e^{-u^2} (2u du) \\ &= 2 \int_0^{\infty} u^2 e^{-u^2} du = 2 \left( \frac{\sqrt{\pi}}{4} \right) = \frac{\sqrt{\pi}}{2} \end{aligned}$$

(See Problem Solving #9.)

11.  $f(x, y) = \begin{cases} ke^{-(x+y)/a} & x \geq 0, y \geq 0 \\ 0 & \text{elsewhere} \end{cases}$

$$\begin{aligned} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) dA &= \int_0^{\infty} \int_0^{\infty} k e^{-(x+y)/a} dx dy \\ &= k \int_0^{\infty} e^{-x/a} dx \cdot \int_0^{\infty} e^{-y/a} dy \end{aligned}$$

These two integrals are equal to

$$\int_0^{\infty} e^{-x/a} dx = \lim_{b \rightarrow \infty} \left[ (-a)e^{-x/a} \right]_0^b = a.$$

So, assuming  $a, k > 0$ , you obtain

$$1 = ka^2 \text{ or } a = \frac{1}{\sqrt{k}}$$

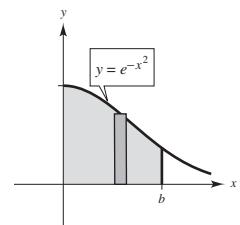
12. By the shell method,

$$V = \lim_{b \rightarrow \infty} \int_0^b 2\pi x e^{-x^2} dx = \lim_{b \rightarrow \infty} \left[ -\pi e^{-x^2} \right]_0^b = \pi.$$

This same volume is given by

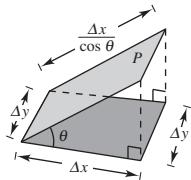
$$\begin{aligned} \pi &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-(x^2+y^2)} dy dx \\ &= 4 \int_0^{\infty} \int_0^{\infty} e^{-(x^2+y^2)} dy dx \\ &= 4 \int_0^{\infty} e^{-x^2} dx \int_0^{\infty} e^{-y^2} dy \\ &= 4 \left[ \int_0^{\infty} e^{-x^2} dx \right]^2. \end{aligned}$$

So,  $\int_0^{\infty} e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$ .



## 13. Essay

14.  $A = l \cdot w = \left( \frac{\Delta x}{\cos \theta} \right) \Delta y = \sec \theta \Delta x \Delta y$



Area in  $xy$ -plane:  $\Delta x \Delta y$

15. The greater the angle between the given plane and the  $xy$ -plane, the greater the surface area. So:  $z_2 < z_1 < z_4 < z_3$

16. Converting to polar coordinates,

$$\begin{aligned} \int_0^\infty \int_0^\infty \frac{1}{(1+x^2+y^2)^2} dx dy &= \int_0^\infty \int_0^{\pi/2} \frac{1}{(1+r^2)^2} r d\theta dr \\ &= \int_0^\infty \frac{r}{(1+r^2)^2} \left( \frac{\pi}{2} \right) dr \\ &= \lim_{t \rightarrow \infty} \int_0^t \frac{\pi}{4} (1+r^2)^{-2} (2r dr) \\ &= \lim_{t \rightarrow \infty} \left[ \frac{\pi}{4} \cdot \frac{-1}{1+r^2} \right]_0^t = \frac{\pi}{4} \end{aligned}$$

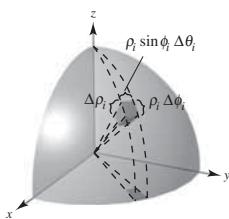
17.  $\int_0^1 \int_0^1 \frac{x-y}{(x+y)^3} dx dy = -\frac{1}{2}$

$$\int_0^1 \int_0^1 \frac{(x-y)}{(x+y)^3} dy dx = \frac{1}{2}$$

The results are not the same. Fubini's Theorem is not valid because  $f$  is not continuous on the region  $0 \leq x \leq 1, 0 \leq y \leq 1$ .

18. The volume of this spherical block can be determined as follows. One side is length  $\Delta\rho$ . Another side is  $\rho \Delta\phi$ . Finally, the third side is given by the length of an arc of angle  $\Delta\theta$  in a circle of radius  $\rho \sin \phi$ . Thus:

$$\begin{aligned} \Delta V &\approx (\Delta\rho)(\rho \Delta\phi)(\Delta\theta \rho \sin \phi) \\ &= \rho^2 \sin \phi \Delta\rho \Delta\phi \Delta\theta \end{aligned}$$



# C H A P T E R 1 5

## Vector Analysis

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# C H A P T E R 15

## Vector Analysis

### Section 15.1 Vector Fields

1. All vectors are parallel to  $x$ -axis.

Matches (d)

2. All vectors are parallel to  $y$ -axis.

Matches (c)

3. Vectors are in rotational pattern.

Matches (e)

4. All vectors point outward.

Matches (b)

5. Vectors are parallel to  $x$ -axis for  $y = n\pi$ .

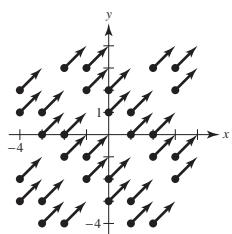
Matches (a)

6. Vectors along  $x$ -axis have no  $x$ -component.

Matches (f)

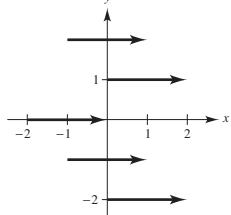
7.  $\mathbf{F}(x, y) = \mathbf{i} + \mathbf{j}$

$$\|\mathbf{F}\| = \sqrt{2}$$



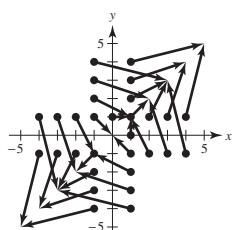
8.  $\mathbf{F}(x, y) = 2\mathbf{i}$

$$\|\mathbf{F}\| = 2$$



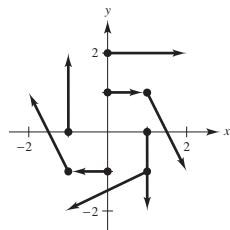
9.  $\mathbf{F}(x, y) = y\mathbf{i} + x\mathbf{j}$

$$\|\mathbf{F}\| = \sqrt{y^2 + x^2}$$



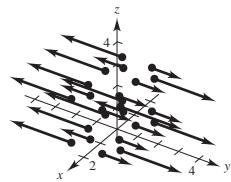
10.  $\mathbf{F}(x, y) = y\mathbf{i} - 2x\mathbf{j}$

$$\|\mathbf{F}\| = \sqrt{y^2 + 4x^2}$$



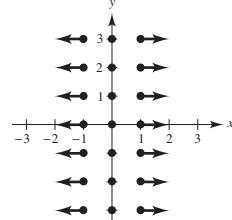
11.  $\mathbf{F}(x, y, z) = 3y\mathbf{j}$

$$\|\mathbf{F}\| = 3|y| = c$$



12.  $\mathbf{F}(x, y) = x\mathbf{i}$

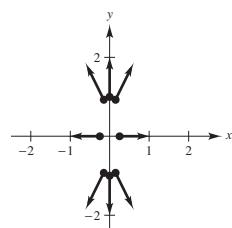
$$\|\mathbf{F}\| = |x| = c$$



13.  $\mathbf{F}(x, y) = 4x\mathbf{i} + y\mathbf{j}$

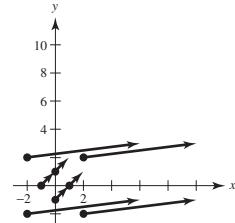
$$\|\mathbf{F}\| = \sqrt{16x^2 + y^2} = c$$

$$\frac{x^2}{c^2/16} + \frac{y^2}{c^2} = 1$$



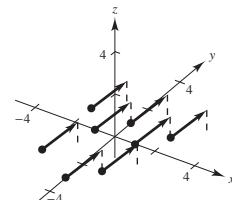
14.  $\mathbf{F}(x, y) = (x^2 + y^2)\mathbf{i} + \mathbf{j}$

$$\|\mathbf{F}\| = \sqrt{1 + (x^2 + y^2)^2}$$



15.  $\mathbf{F}(x, y, z) = \mathbf{i} + \mathbf{j} + \mathbf{k}$

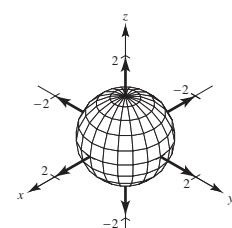
$$\|\mathbf{F}\| = \sqrt{3}$$



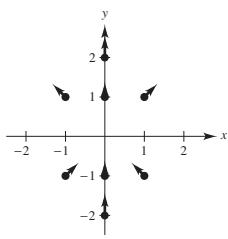
16.  $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$

$$\|\mathbf{F}\| = \sqrt{x^2 + y^2 + z^2} = c$$

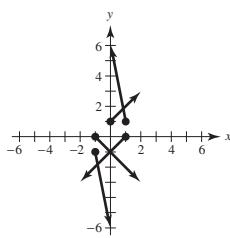
$$x^2 + y^2 + z^2 = c^2$$



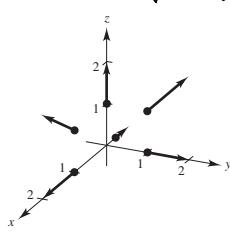
17.  $F(x, y) = \frac{1}{8}(2xy\mathbf{i} + y^2\mathbf{j})$



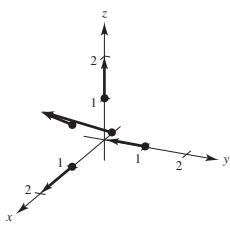
18.  $\mathbf{F}(x, y) = (2y - 3x)\mathbf{i} + (2y + 3x)\mathbf{j}$



19.  $F(x, y, z) = \frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{\sqrt{x^2 + y^2 + z^2}}$



20.  $\mathbf{F}(x, y, z) = x\mathbf{i} - y\mathbf{j} + z\mathbf{k}$



26.  $f(x, y, z) = \sqrt{x^2 + 4y^2 + z^2}$

$$f_x = \frac{x}{\sqrt{x^2 + 4y^2 + z^2}}$$

$$f_y = \frac{4y}{\sqrt{x^2 + 4y^2 + z^2}}$$

$$f_z = \frac{z}{\sqrt{x^2 + 4y^2 + z^2}}$$

$$\mathbf{F}(x, y, z) = \frac{x}{\sqrt{x^2 + 4y^2 + z^2}}\mathbf{i} + \frac{4y}{\sqrt{x^2 + 4y^2 + z^2}}\mathbf{j} + \frac{z}{\sqrt{x^2 + 4y^2 + z^2}}\mathbf{k}$$

21.  $f(x, y) = x^2 + 2y^2$

$$f_x(x, y) = 2x$$

$$f_y(x, y) = 4y$$

$$\mathbf{F}(x, y) = 2x\mathbf{i} + 4y\mathbf{j}$$

Note that  $\nabla f = \mathbf{F}$ .

22.  $f(x, y) = x^2 - \frac{1}{4}y^2$

$$f_x(x, y) = 2x$$

$$f_y(x, y) = -\frac{1}{2}y$$

$$\mathbf{F}(x, y) = 2x\mathbf{i} - \frac{1}{2}y\mathbf{j}$$

23.  $g(x, y) = 5x^2 + 3xy + y^2$

$$g_x(x, y) = 10x + 3y$$

$$g_y(x, y) = 3x + 2y$$

$$\mathbf{G}(x, y) = (10x + 3y)\mathbf{i} + (3x + 2y)\mathbf{j}$$

24.  $g(x, y) = \sin 3x \cos 4y$

$$g_x(x, y) = 3 \cos 3x \cos 4y$$

$$g_y(x, y) = -4 \sin 3x \sin 4y$$

$$\mathbf{G}(x, y) = 3 \cos 3x \cos 4y\mathbf{i} - 4 \sin 3x \sin 4y\mathbf{j}$$

25.  $f(x, y, z) = 6xyz$

$$f_x(x, y, z) = 6yz$$

$$f_y(x, y, z) = 6xz$$

$$f_z(x, y, z) = 6xy$$

$$\mathbf{F}(x, y, z) = 6yz\mathbf{i} + 6xz\mathbf{j} + 6xy\mathbf{k}$$

27.  $g(x, y, z) = z + ye^{x^2}$

$$g_x(x, y, z) = 2xye^{x^2}$$

$$g_y(x, y, z) = e^{x^2}$$

$$g_z(x, y, z) = 1$$

$$\mathbf{G}(x, y, z) = 2xye^{x^2}\mathbf{i} + e^{x^2}\mathbf{j} + \mathbf{k}$$

28.  $g(x, y, z) = \frac{y}{z} + \frac{z}{x} - \frac{xz}{y}$

$$g_x(x, y, z) = -\frac{z}{x^2} - \frac{z}{y}$$

$$g_y(x, y, z) = \frac{1}{z} + \frac{xz}{y^2}$$

$$g_z(x, y, z) = -\frac{y}{z^2} + \frac{1}{x} - \frac{x}{y}$$

$$\mathbf{G}(x, y, z) = \left(-\frac{z}{x^2} - \frac{z}{y}\right)\mathbf{i} + \left(\frac{1}{z} + \frac{xz}{y^2}\right)\mathbf{j} + \left(-\frac{y}{z^2} + \frac{1}{x} - \frac{x}{y}\right)\mathbf{k}$$

29.  $h(x, y, z) = xy \ln(x + y)$

$$h_x(x, y, z) = y \ln(x + y) + \frac{xy}{x + y}$$

$$h_y(x, y, z) = x \ln(x + y) + \frac{xy}{x + y}$$

$$h_z(x, y, z) = 0$$

$$\mathbf{H}(x, y, z) = \left[\frac{xy}{x + y} + y \ln(x + y)\right]\mathbf{i} + \left[\frac{xy}{x + y} + x \ln(x + y)\right]\mathbf{j}$$

30.  $h(x, y, z) = x \arcsin yz$

$$h_x(x, y, z) = \arcsin yz$$

$$h_y(x, y, z) = \frac{xz}{\sqrt{1 - y^2 z^2}}$$

$$h_z(x, y, z) = \frac{xy}{\sqrt{1 - y^2 z^2}}$$

$$\mathbf{H}(x, y, z) = (\arcsin yz)\mathbf{i} + \frac{xz}{\sqrt{1 - y^2 z^2}}\mathbf{j} + \frac{xy}{\sqrt{1 - y^2 z^2}}\mathbf{k}$$

31.  $\mathbf{F}(x, y) = xy^2\mathbf{i} + x^2y\mathbf{j}$

$M = xy^2$  and  $N = x^2y$  have continuous first partial derivatives.

$$\frac{\partial N}{\partial x} = 2xy = \frac{\partial M}{\partial y} \Rightarrow \mathbf{F} \text{ is conservative}$$

32.  $\mathbf{F}(x, y) = \frac{1}{x^2}(y\mathbf{i} - x\mathbf{j}) = \frac{y}{x^2}\mathbf{i} - \frac{1}{x}\mathbf{j}$

$M = y/x^2$  and  $N = -(1/x)$  have continuous first partial derivatives for all  $x \neq 0$ .

$$\frac{\partial N}{\partial x} = \frac{1}{x^2} = \frac{\partial M}{\partial y} \Rightarrow \mathbf{F} \text{ is conservative.}$$

33.  $\mathbf{F}(x, y) = \sin y\mathbf{i} + x \cos y\mathbf{j}$

$M = \sin y$  and  $N = x \cos y$  have continuous first partial derivatives.

$$\frac{\partial N}{\partial x} = \cos y = \frac{\partial M}{\partial y} \Rightarrow \mathbf{F} \text{ is conservative.}$$

34.  $\mathbf{F}(x, y) = \frac{1}{xy}(y\mathbf{i} - x\mathbf{j}) = \frac{1}{x}\mathbf{i} - \frac{1}{y}\mathbf{j}$

$M = 1/x$  and  $N = -1/y$  have continuous first partial derivatives for all  $x, y \neq 0$ .

$$\frac{\partial N}{\partial x} = 0 = \frac{\partial M}{\partial y} \Rightarrow \mathbf{F} \text{ is conservative.}$$

35.  $\mathbf{F}(x, y) = 5y^2(y\mathbf{i} + 3x\mathbf{j})$

$$M = 5y^3, N = 15xy^2$$

$$\frac{\partial N}{\partial x} = 15y^2 = \frac{\partial M}{\partial y} \Rightarrow \text{Conservative}$$

36.  $M = \frac{2}{y}e^{2x/y}, N = \frac{-2x}{y^2}e^{2x/y}$

$$\frac{\partial N}{\partial x} = \frac{-2(y+2x)}{y^3}e^{2x/y} = \frac{\partial M}{\partial y} \Rightarrow \text{Conservative}$$

37.  $M = \frac{1}{\sqrt{x^2 + y^2}}, N = \frac{1}{\sqrt{x^2 + y^2}}$

$$\frac{\partial N}{\partial x} = \frac{-x}{(x^2 + y^2)^{3/2}} \neq \frac{\partial M}{\partial y} = \frac{-y}{(x^2 + y^2)^{3/2}}$$

$\Rightarrow$  Not conservative

38.  $M = \frac{y}{\sqrt{1+xy}}, N = \frac{x}{\sqrt{1+xy}}$

$$\frac{\partial N}{\partial x} = \frac{xy+2}{2(xy+1)^{3/2}} = \frac{\partial M}{\partial y} \Rightarrow \text{Conservative}$$

39.  $\mathbf{F}(x, y) = y\mathbf{i} + x\mathbf{j}$

$$\frac{\partial}{\partial y}[y] = 1 = \frac{\partial}{\partial x}[x] \Rightarrow \text{Conservative}$$

$$f_x(x, y) = y, f_y(x, y) = x \Rightarrow f(x, y) = xy + k$$

40.  $\mathbf{F}(x, y) = 3x^2y^2\mathbf{i} + 2x^3y\mathbf{j}$

$$\frac{\partial}{\partial y}[3x^2y^2] = 6x^2y$$

$$\frac{\partial}{\partial x}[2x^3y] = 6x^2y$$

Conservative

$$f_x(x, y) = 3x^2y^2$$

$$f_y(x, y) = 2x^3y$$

$$f(x, y) = x^3y^2 + K$$

41.  $\mathbf{F}(x, y) = 2xy\mathbf{i} + x^2\mathbf{j}$

$$\frac{\partial}{\partial y}[2xy] = 2x$$

$$\frac{\partial}{\partial x}[x^2] = 2x$$

Conservative

$$f_x(x, y) = 2xy, f_y(x, y) = x^2, f(x, y) = x^2y + K$$

42.  $\mathbf{F}(x, y) = xe^{x^2y}(2y\mathbf{i} + x\mathbf{j})$

$$\frac{\partial}{\partial y}\left[2xye^{x^2y}\right] = 2xe^{x^2y} + 2x^3ye^{x^2y}$$

$$\frac{\partial}{\partial x}\left[x^2e^{x^2y}\right] = 2xe^{x^2y} + 2x^3ye^{x^2y}$$

Conservative

$$f_x(x, y) = 2xye^{x^2y}$$

$$f_y(x, y) = x^2e^{x^2y}$$

$$f(x, y) = e^{x^2y} + K$$

43.  $\mathbf{F}(x, y) = 15y^3\mathbf{i} - 5xy^2\mathbf{j}$

$$\frac{\partial}{\partial y}[15y^3] = 45y^2 \neq \frac{\partial}{\partial x}[-5xy^2] = -5y^2$$

Not conservative

44.  $\mathbf{F}(x, y) = \frac{1}{y^2}(y\mathbf{i} - 2x\mathbf{j})$

$$= \frac{1}{y}\mathbf{i} - \frac{2x}{y^2}\mathbf{j}$$

$$\frac{\partial}{\partial y}\left[\frac{1}{y}\right] = -\frac{1}{y^2}$$

$$\frac{\partial}{\partial x}\left[-\frac{2x}{y^2}\right] = -\frac{2}{y^2}$$

Not conservative

45.  $\mathbf{F}(x, y) = \frac{2y}{x}\mathbf{i} - \frac{x^2}{y^2}\mathbf{j}$

$$\frac{\partial}{\partial y}\left[\frac{2y}{x}\right] = \frac{2}{x}$$

$$\frac{\partial}{\partial x}\left[-\frac{x^2}{y^2}\right] = -\frac{2x}{y^2}$$

Not conservative

46.  $\mathbf{F}(x, y) = \frac{x}{x^2 + y^2}\mathbf{i} + \frac{y}{x^2 + y^2}\mathbf{j}$

$$\frac{\partial}{\partial y}\left[\frac{x}{x^2 + y^2}\right] = -\frac{2xy}{(x^2 + y^2)^2}$$

$$\frac{\partial}{\partial x}\left[\frac{y}{x^2 + y^2}\right] = -\frac{2xy}{(x^2 + y^2)^2}$$

Conservative

$$f_x(x, y) = \frac{x}{x^2 + y^2}$$

$$f_y(x, y) = \frac{y}{x^2 + y^2}$$

$$f(x, y) = \frac{1}{2}\ln(x^2 + y^2) + K$$

47.  $\mathbf{F}(x, y) = e^x(\cos y\mathbf{i} - \sin y\mathbf{j})$

$$\frac{\partial}{\partial y} [e^x \cos y] = -e^x \sin y$$

$$\frac{\partial}{\partial x} [-e^x \sin y] = -e^x \sin y$$

Conservative

$$f_x(x, y) = e^x \cos y$$

$$f_y(x, y) = -e^x \sin y$$

$$f(x, y) = e^x \cos y + K$$

48.  $\mathbf{F}(x, y) = \frac{2x}{(x^2 + y^2)^2}\mathbf{i} + \frac{2y}{(x^2 + y^2)^2}\mathbf{j}$

$$\frac{\partial}{\partial y} \left[ \frac{2x}{(x^2 + y^2)^2} \right] = -\frac{8xy}{(x^2 + y^2)^3}$$

$$\frac{\partial}{\partial x} \left[ \frac{2y}{(x^2 + y^2)^2} \right] = -\frac{8xy}{(x^2 + y^2)^3}$$

Conservative

$$f_x(x, y) = \frac{2x}{(x^2 + y^2)^2}$$

$$f_y(x, y) = \frac{2y}{(x^2 + y^2)^2}$$

$$f(x, y) = -\frac{1}{x^2 + y^2} + K$$

51.  $\mathbf{F}(x, y, z) = e^x \sin y \mathbf{i} - e^x \cos y \mathbf{j}, (0, 0, 1)$

$$\mathbf{curl F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ e^x \sin y & -e^x \cos y & 0 \end{vmatrix} = (-e^x \cos y - e^x \cos y)\mathbf{k} = -2e^x \cos y \mathbf{k}$$

$$\mathbf{curl F}(0, 0, 1) = -2\mathbf{k}$$

52.  $\mathbf{F}(x, y, z) = e^{-xyz}(\mathbf{i} + \mathbf{j} + \mathbf{k}), (3, 2, 0)$

$$\mathbf{curl F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ e^{-xyz} & e^{-xyz} & e^{-xyz} \end{vmatrix} = (-xz + xy)e^{-xyz}\mathbf{i} - (-yz + xy)e^{-xyz}\mathbf{j} + (-yz + xz)e^{-xyz}\mathbf{k}$$

$$\mathbf{curl F}(3, 2, 0) = 6\mathbf{i} - 6\mathbf{j}$$

53.  $\mathbf{F}(x, y, z) = \arctan\left(\frac{x}{y}\right)\mathbf{i} + \ln\sqrt{x^2 + y^2}\mathbf{j} + \mathbf{k}$

$$\mathbf{curl F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \arctan\left(\frac{x}{y}\right) & \frac{1}{2} \ln(x^2 + y^2) & 1 \end{vmatrix} = \left[ \frac{x}{x^2 + y^2} - \frac{(-x/y^2)}{1 + (x/y)^2} \right] \mathbf{k} = \frac{2x}{x^2 + y^2} \mathbf{k}$$

49.  $\mathbf{F}(x, y, z) = xyz\mathbf{i} + xyz\mathbf{j} + xyz\mathbf{k}, (2, 1, 3)$

$$\mathbf{curl F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xyz & xyz & xyz \end{vmatrix} = (xz - xy)\mathbf{i} - (yz - xy)\mathbf{j} + (yz - xz)\mathbf{k}$$

$$\begin{aligned} \mathbf{curl F}(2, 1, 3) &= (6 - 2)\mathbf{i} - (3 - 2)\mathbf{j} + (3 - 6)\mathbf{k} \\ &= 4\mathbf{i} - \mathbf{j} - 3\mathbf{k} \end{aligned}$$

50.  $\mathbf{F}(x, y, z) = x^2z\mathbf{i} - 2xz\mathbf{j} + yz\mathbf{k}, (2, -1, 3)$

$$\mathbf{curl F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2z & -2xz & yz \end{vmatrix} = (z + 2x)\mathbf{i} - (0 - x^2)\mathbf{j} + (-2z - 0)\mathbf{k} = (z + 2x)\mathbf{i} + x^2\mathbf{j} - 2z\mathbf{k}$$

$$\mathbf{curl F}(2, -1, 3) = 7\mathbf{i} + 4\mathbf{j} - 6\mathbf{k}$$

54.  $\mathbf{F}(x, y, z) = \frac{yz}{y-z}\mathbf{i} + \frac{xz}{x-z}\mathbf{j} + \frac{xy}{x-y}\mathbf{k}$

$$\begin{aligned}\operatorname{curl} \mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{yz}{y-z} & \frac{xz}{x-z} & \frac{xy}{x-y} \end{vmatrix} = \left[ \frac{x^2}{(x-y)^2} - \frac{x^2}{(x-z)^2} \right] \mathbf{i} - \left[ \frac{-y^2}{(x-y)^2} - \frac{y^2}{(y-z)^2} \right] \mathbf{j} + \left[ \frac{-z^2}{(x-z)^2} - \frac{-z^2}{(y-z)^2} \right] \mathbf{k} \\ &= x^2 \left[ \frac{1}{(x-y)^2} - \frac{1}{(x-z)^2} \right] \mathbf{i} + y^2 \left[ \frac{1}{(x-y)^2} + \frac{1}{(y-z)^2} \right] \mathbf{j} + z^2 \left[ \frac{1}{(y-z)^2} - \frac{1}{(x-z)^2} \right] \mathbf{k}\end{aligned}$$

55.  $\mathbf{F}(x, y, z) = \sin(x-y)\mathbf{i} + \sin(y-z)\mathbf{j} + \sin(z-x)\mathbf{k}$

$$\operatorname{curl} \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \sin(x-y) & \sin(y-z) & \sin(z-x) \end{vmatrix} = \cos(y-z)\mathbf{i} + \cos(z-x)\mathbf{j} + \cos(x-y)\mathbf{k}$$

56.  $\mathbf{F}(x, y, z) = \sqrt{x^2 + y^2 + z^2}(\mathbf{i} + \mathbf{j} + \mathbf{k})$

$$\operatorname{curl} \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \sqrt{x^2 + y^2 + z^2} & \sqrt{x^2 + y^2 + z^2} & \sqrt{x^2 + y^2 + z^2} \end{vmatrix} = \frac{(y-z)\mathbf{i} + (z-x)\mathbf{j} + (x-y)\mathbf{k}}{\sqrt{x^2 + y^2 + z^2}}$$

57.  $\mathbf{F}(x, y, z) = xy^2z^2\mathbf{i} + x^2yz^2\mathbf{j} + x^2y^2z\mathbf{k}$

$$\operatorname{curl} \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy^2z^2 & x^2yz^2 & x^2y^2z \end{vmatrix} = \mathbf{0}$$

Conservative

$$f_x(x, y, z) = xy^2z^2$$

$$f_y(x, y, z) = x^2yz^2$$

$$f_z(x, y, z) = x^2y^2z$$

$$f(x, y, z) = \frac{1}{2}x^2y^2z^2 + K$$

58.  $\mathbf{F}(x, y, z) = y^2z^3\mathbf{i} + 2xyz^3\mathbf{j} + 3xy^2z^2\mathbf{k}$

$$\operatorname{curl} \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2z^3 & 2xyz^3 & 3xy^2z^2 \end{vmatrix} = \mathbf{0}$$

Conservative

$$f(x, y, z) = xy^2z^3 + K$$

59.  $\mathbf{F}(x, y, z) = \sin z\mathbf{i} + \sin x\mathbf{j} + \sin y\mathbf{k}$

$$\begin{aligned}\operatorname{curl} \mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \sin z & \sin x & \sin y \end{vmatrix} \\ &= \cos y\mathbf{i} + \cos z\mathbf{j} + \cos x\mathbf{k} \neq \mathbf{0}\end{aligned}$$

Not conservative

60.  $\mathbf{F}(x, y, z) = ye^z\mathbf{i} + ze^x\mathbf{j} + xe^y\mathbf{k}$

$$\operatorname{curl} \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ ye^z & ze^x & xe^y \end{vmatrix} = (xe^y - e^x)\mathbf{i} - (e^y - ye^z)\mathbf{j} + (ze^x - e^z)\mathbf{k} \neq \mathbf{0}$$

Not conservative

61.  $\mathbf{F}(x, y, z) = \frac{z}{y}\mathbf{i} - \frac{xz}{y^2}\mathbf{j} + \frac{x}{y}\mathbf{k}$

$$\operatorname{curl} \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{z}{y} & -\frac{xz}{y^2} & \frac{x}{y} \end{vmatrix} = \left( -\frac{x}{y^2} + \frac{x}{y^2} \right) \mathbf{i} - \left( \frac{1}{y} - \frac{1}{y} \right) \mathbf{j} + \left( -\frac{z}{y^2} + \frac{z}{y^2} \right) \mathbf{k} = \mathbf{0}$$

Conservative

$$f_x(x, y, z) = \frac{z}{y}$$

$$f_y(x, y, z) = -\frac{xz}{y^2}$$

$$f_z(x, y, z) = \frac{x}{y}$$

$$f(x, y, z) = \frac{xz}{y} + K$$

62.  $\mathbf{F}(x, y, z) = \frac{x}{x^2 + y^2}\mathbf{i} + \frac{y}{x^2 + y^2}\mathbf{j} + \mathbf{k}$

$$\operatorname{curl} \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{x}{x^2 + y^2} & \frac{y}{x^2 + y^2} & 1 \end{vmatrix} = \mathbf{0}$$

Conservative

$$f_x(x, y, z) = \frac{x}{x^2 + y^2}$$

$$f_y(x, y, z) = \frac{y}{x^2 + y^2}$$

$$f_z(x, y, z) = 1$$

$$\begin{aligned} f(x, y, z) &= \int \frac{x}{x^2 + y^2} dx \\ &= \frac{1}{2} \ln(x^2 + y^2) + g(y, z) + K_1 \end{aligned}$$

$$\begin{aligned} f(x, y, z) &= \int \frac{y}{x^2 + y^2} dy \\ &= \frac{1}{2} \ln(x^2 + y^2) + h(x, z) + K_2 \end{aligned}$$

$$f(x, y, z) = \int dz = z + p(x, y) + K_3$$

$$f(x, y, z) = \frac{1}{2} \ln(x^2 + y^2) + z + K$$

66.  $\mathbf{F}(x, y, z) = \ln(x^2 + y^2)\mathbf{i} + xy\mathbf{j} + \ln(y^2 + z^2)\mathbf{k}$

$$\operatorname{div} \mathbf{F}(x, y, z) = \frac{\partial}{\partial x} [\ln(x^2 + y^2)] + \frac{\partial}{\partial y} [xy] + \frac{\partial}{\partial z} [\ln(y^2 + z^2)] = \frac{2x}{x^2 + y^2} + x + \frac{2z}{y^2 + z^2}$$

63.  $\mathbf{F}(x, y) = x^2\mathbf{i} + 2y^2\mathbf{j}$

$$\operatorname{div} \mathbf{F}(x, y) = \frac{\partial}{\partial x}(x^2) + \frac{\partial}{\partial y}(2y^2) = 2x + 4y$$

64.  $\mathbf{F}(x, y) = xe^x\mathbf{i} + ye^y\mathbf{i}$

$$\begin{aligned} \operatorname{div} \mathbf{F}(x, y) &= \frac{\partial}{\partial x}(xe^x) + \frac{\partial}{\partial y}(ye^y) \\ &= xe^x + e^x + ye^y + e^y \end{aligned}$$

65.  $\mathbf{F}(x, y, z) = \sin x\mathbf{i} + \cos y\mathbf{j} + z^2\mathbf{k}$

$$\begin{aligned} \operatorname{div} \mathbf{F}(x, y, z) &= \frac{\partial}{\partial x}[\sin x] + \frac{\partial}{\partial y}[\cos y] + \frac{\partial}{\partial z}[z^2] \\ &= \cos x - \sin y + 2z \end{aligned}$$

67.  $\mathbf{F}(x, y, z) = xyz\mathbf{i} + xy\mathbf{j} + z\mathbf{k}$

$$\operatorname{div} \mathbf{F}(x, y, z) = yz + x + 1$$

$$\operatorname{div} \mathbf{F}(2, 1, 1) = 1 + 2 + 1 = 4$$

68.  $\mathbf{F}(x, y, z) = x^2z\mathbf{i} - 2xz\mathbf{j} + yz\mathbf{k}$

$$\operatorname{div} \mathbf{F}(x, y, z) = 2xz + y$$

$$\operatorname{div} \mathbf{F}(2, -1, 3) = 11$$

69.  $\mathbf{F}(x, y, z) = e^x \sin y\mathbf{i} - e^x \cos y\mathbf{j} + z^2\mathbf{k}$

$$\operatorname{div} \mathbf{F}(x, y, z) = e^x \sin y + e^x \cos y + 2z$$

$$\operatorname{div} \mathbf{F}(3, 0, 0) = 0$$

70.  $\mathbf{F}(x, y, z) = \ln(xyz)(\mathbf{i} + \mathbf{j} + \mathbf{k})$

$$\operatorname{div} \mathbf{F}(x, y, z) = \frac{1}{x} + \frac{1}{y} + \frac{1}{z}$$

$$\operatorname{div} \mathbf{F}(3, 2, 1) = \frac{1}{3} + \frac{1}{2} + 1 = \frac{11}{6}$$

71. See the definition, page 1058. Examples include velocity fields, gravitational fields, and magnetic fields.

72. See the definition of Conservative Vector Field on page 1061. To test for a conservative vector field, see Theorem 15.1 and 15.2.

76.  $\mathbf{F}(x, y, z) = x\mathbf{i} - z\mathbf{k}$

$$\mathbf{G}(x, y, z) = x^2\mathbf{i} + y\mathbf{j} + z^2\mathbf{k}$$

$$\mathbf{F} \times \mathbf{G} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x & 0 & -z \\ x^2 & y & z^2 \end{vmatrix} = yz\mathbf{i} - (xz^2 + x^2z)\mathbf{j} + xy\mathbf{k}$$

$$\operatorname{curl}(\mathbf{F} \times \mathbf{G}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & -xz^2 - x^2z & xy \end{vmatrix} = (x + 2xz + x^2)\mathbf{i} - (y - y)\mathbf{j} + (-z^2 - 2xz - z)\mathbf{k} = x(x + 2z + 1)\mathbf{i} - z(z + 2x + 1)\mathbf{k}$$

77.  $\mathbf{F}(x, y, z) = xyz\mathbf{i} + y\mathbf{j} + z\mathbf{k}$

$$\operatorname{curl} \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xyz & y & z \end{vmatrix} = xy\mathbf{j} - xz\mathbf{k}$$

$$\operatorname{curl}(\operatorname{curl} \mathbf{F}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & xy & -xz \end{vmatrix} = z\mathbf{j} + y\mathbf{k}$$

73. See the definition on page 1064.

74. See the definition on page 1066.

75.  $\mathbf{F}(x, y, z) = \mathbf{i} + 3x\mathbf{j} + 2y\mathbf{k}$

$$\mathbf{G}(x, y, z) = x\mathbf{i} - y\mathbf{j} + z\mathbf{k}$$

$$\mathbf{F} \times \mathbf{G} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 3x & 2y \\ x & -y & z \end{vmatrix} = (3xz + 2y^2)\mathbf{i} - (z - 2xy)\mathbf{j} + (-y - 3x^2)\mathbf{k}$$

$$\operatorname{curl}(\mathbf{F} \times \mathbf{G}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3xz + 2y^2 & -z + 2xy & -y - 3x^2 \end{vmatrix} = (-1 + 1)\mathbf{i} - (-6x - 3x)\mathbf{j} + (2y - 4y)\mathbf{k} = 9x\mathbf{j} - 2y\mathbf{k}$$

78.  $\mathbf{F}(x, y, z) = x^2z\mathbf{i} - 2xz\mathbf{j} + yz\mathbf{k}$

$$\mathbf{curl}\ \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2z & -2xz & yz \end{vmatrix} = (z + 2x)\mathbf{i} + x^2\mathbf{j} - 2z\mathbf{k}$$

$$\mathbf{curl}(\mathbf{curl}\ \mathbf{F}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z + 2x & x^2 & -2z \end{vmatrix} = \mathbf{j} + 2x\mathbf{k}$$

79.  $\mathbf{F}(x, y, z) = \mathbf{i} + 3x\mathbf{j} + 2y\mathbf{k}$

$$\mathbf{G}(x, y, z) = x\mathbf{i} - y\mathbf{j} + z\mathbf{k}$$

$$\begin{aligned} \mathbf{F} \times \mathbf{G} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 3x & 2y \\ x & -y & z \end{vmatrix} \\ &= (3xz + 2y^2)\mathbf{i} - (z - 2xy)\mathbf{j} + (-y - 3x^2)\mathbf{k} \end{aligned}$$

$$\operatorname{div}(\mathbf{F} \times \mathbf{G}) = 3z + 2x$$

80.  $\mathbf{F}(x, y, z) = x\mathbf{i} - z\mathbf{k}$

$$\mathbf{G}(x, y, z) = x^2\mathbf{i} + y\mathbf{j} + z^2\mathbf{k}$$

$$\mathbf{F} \times \mathbf{G} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x & 0 & -z \\ x^2 & y & z^2 \end{vmatrix} = yz\mathbf{i} - (xz^2 + x^2z)\mathbf{j} + xy\mathbf{k}$$

$$\operatorname{div}(\mathbf{F} \times \mathbf{G}) = 0$$

81.  $\mathbf{F}(x, y, z) = xyz\mathbf{i} + y\mathbf{j} + z\mathbf{k}$

$$\mathbf{curl}\ \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xyz & y & z \end{vmatrix} = xy\mathbf{j} - xz\mathbf{k}$$

$$\operatorname{div}(\mathbf{curl}\ \mathbf{F}) = x - x = 0$$

82.  $\mathbf{F}(x, y, z) = x^2z\mathbf{i} - 2xz\mathbf{j} + yz\mathbf{k}$

$$\mathbf{curl}\ \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2z & -2xz & yz \end{vmatrix} = (z + 2x)\mathbf{i} + x^2\mathbf{j} - 2z\mathbf{k}$$

$$\operatorname{div}(\mathbf{curl}\ \mathbf{F}) = 2 - 2 = 0$$

83. Let  $\mathbf{F} = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$  and  $\mathbf{G} = Q\mathbf{i} + R\mathbf{j} + S\mathbf{k}$  where  $M, N, P, Q, R$ , and  $S$  have continuous partial derivatives.

$$\mathbf{F} + \mathbf{G} = (M + Q)\mathbf{i} + (N + R)\mathbf{j} + (P + S)\mathbf{k}$$

$$\begin{aligned} \mathbf{curl}(\mathbf{F} + \mathbf{G}) &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ M + Q & N + R & P + S \end{vmatrix} \\ &= \left[ \frac{\partial}{\partial y}(P + S) - \frac{\partial}{\partial z}(N + R) \right] \mathbf{i} - \left[ \frac{\partial}{\partial x}(P + S) - \frac{\partial}{\partial z}(M + Q) \right] \mathbf{j} + \left[ \frac{\partial}{\partial x}(N + R) - \frac{\partial}{\partial y}(M + Q) \right] \mathbf{k} \\ &= \left( \frac{\partial P}{\partial y} - \frac{\partial N}{\partial z} \right) \mathbf{i} - \left( \frac{\partial P}{\partial x} - \frac{\partial M}{\partial z} \right) \mathbf{j} + \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) \mathbf{k} + \left( \frac{\partial S}{\partial y} - \frac{\partial Q}{\partial z} \right) \mathbf{i} - \left( \frac{\partial S}{\partial x} - \frac{\partial Q}{\partial z} \right) \mathbf{j} + \left( \frac{\partial R}{\partial x} - \frac{\partial Q}{\partial y} \right) \mathbf{k} \\ &= \mathbf{curl}\ \mathbf{F} + \mathbf{curl}\ \mathbf{G} \end{aligned}$$

84. Let  $f(x, y, z)$  be a scalar function whose second partial derivatives are continuous.

$$\nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}$$

$$\mathbf{curl}(\nabla f) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{vmatrix} = \left( \frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial z \partial y} \right) \mathbf{i} - \left( \frac{\partial^2 f}{\partial x \partial z} - \frac{\partial^2 f}{\partial z \partial x} \right) \mathbf{j} + \left( \frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial y \partial x} \right) \mathbf{k} = \mathbf{0}$$

85. Let  $\mathbf{F} = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$  and  $\mathbf{G} = R\mathbf{i} + S\mathbf{j} + T\mathbf{k}$ .

$$\begin{aligned}\operatorname{div}(\mathbf{F} + \mathbf{G}) &= \frac{\partial}{\partial x}(M + R) + \frac{\partial}{\partial y}(N + S) + \frac{\partial}{\partial z}(P + T) = \frac{\partial M}{\partial x} + \frac{\partial R}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial S}{\partial y} + \frac{\partial P}{\partial z} + \frac{\partial T}{\partial z} \\ &= \left[ \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z} \right] + \left[ \frac{\partial R}{\partial x} + \frac{\partial S}{\partial y} + \frac{\partial T}{\partial z} \right] \\ &= \operatorname{div} \mathbf{F} + \operatorname{div} \mathbf{G}\end{aligned}$$

86. Let  $\mathbf{F} = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$  and  $\mathbf{G} = R\mathbf{i} + S\mathbf{j} + T\mathbf{k}$ .

$$\begin{aligned}\mathbf{F} \times \mathbf{G} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ M & N & P \\ R & S & T \end{vmatrix} = (NT - PS)\mathbf{i} - (MT - PR)\mathbf{j} + (MS - NR)\mathbf{k} \\ \operatorname{div}(\mathbf{F} \times \mathbf{G}) &= \frac{\partial}{\partial x}(NT - PS) + \frac{\partial}{\partial y}(PR - MT) + \frac{\partial}{\partial z}(MS - NR) \\ &= N \frac{\partial T}{\partial x} + T \frac{\partial N}{\partial x} - P \frac{\partial S}{\partial x} - S \frac{\partial P}{\partial x} + P \frac{\partial R}{\partial y} + R \frac{\partial P}{\partial y} - M \frac{\partial T}{\partial y} - T \frac{\partial M}{\partial y} + M \frac{\partial S}{\partial z} + S \frac{\partial M}{\partial z} - N \frac{\partial R}{\partial z} - R \frac{\partial N}{\partial z} \\ &= \left[ \left( \frac{\partial P}{\partial y} - \frac{\partial N}{\partial z} \right) R + \left( \frac{\partial M}{\partial z} - \frac{\partial P}{\partial x} \right) S + \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) T \right] - \left[ M \left( \frac{\partial T}{\partial y} - \frac{\partial S}{\partial z} \right) + N \left( \frac{\partial R}{\partial z} - \frac{\partial T}{\partial x} \right) + P \left( \frac{\partial S}{\partial x} - \frac{\partial R}{\partial y} \right) \right] \\ &= (\operatorname{curl} \mathbf{F}) \cdot \mathbf{G} - \mathbf{F} \cdot (\operatorname{curl} \mathbf{G})\end{aligned}$$

87.  $\mathbf{F} = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$

$$\begin{aligned}\nabla \times [\nabla f + (\nabla \times \mathbf{F})] &= \operatorname{curl}(\nabla f + (\nabla \times \mathbf{F})) \\ &= \operatorname{curl}(\nabla f) + \operatorname{curl}(\nabla \times \mathbf{F}) \quad (\text{Exercise 83}) \\ &= \operatorname{curl}(\nabla \times \mathbf{F}) \quad (\text{Exercise 84}) \\ &= \nabla \times (\nabla \times \mathbf{F})\end{aligned}$$

88. Let  $\mathbf{F} = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ .

$$\begin{aligned}\nabla \times (f\mathbf{F}) &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ fM & fN & fP \end{vmatrix} \\ &= \left( \frac{\partial f}{\partial y}P + f \frac{\partial P}{\partial y} - \frac{\partial f}{\partial z}N - f \frac{\partial N}{\partial z} \right) \mathbf{i} - \left( \frac{\partial f}{\partial x}P + f \frac{\partial P}{\partial x} - \frac{\partial f}{\partial z}M - f \frac{\partial M}{\partial z} \right) \mathbf{j} + \left( \frac{\partial f}{\partial x}N + f \frac{\partial N}{\partial x} - \frac{\partial f}{\partial y}M - f \frac{\partial M}{\partial y} \right) \mathbf{k} \\ &= f \left[ \left( \frac{\partial P}{\partial y} - \frac{\partial N}{\partial z} \right) \mathbf{i} - \left( \frac{\partial P}{\partial x} - \frac{\partial M}{\partial z} \right) \mathbf{j} + \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) \mathbf{k} \right] + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \\ M & N & P \end{vmatrix} = f[\nabla \times \mathbf{F}] + (\nabla f) \times \mathbf{F}\end{aligned}$$

89. Let  $\mathbf{F} = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ , then  $f\mathbf{F} = fM\mathbf{i} + fN\mathbf{j} + fP\mathbf{k}$ .

$$\begin{aligned}\operatorname{div}(f\mathbf{F}) &= \frac{\partial}{\partial x}(fM) + \frac{\partial}{\partial y}(fN) + \frac{\partial}{\partial z}(fP) = f \frac{\partial M}{\partial x} + M \frac{\partial f}{\partial x} + f \frac{\partial N}{\partial y} + N \frac{\partial f}{\partial y} + f \frac{\partial P}{\partial z} + P \frac{\partial f}{\partial z} \\ &= f \left( \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z} \right) + \left( \frac{\partial f}{\partial x}M + \frac{\partial f}{\partial y}N + \frac{\partial f}{\partial z}P \right) = f \operatorname{div} \mathbf{F} + \nabla f \cdot \mathbf{F}\end{aligned}$$

90. Let  $\mathbf{F} = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ .

$$\begin{aligned}\operatorname{curl} \mathbf{F} &= \left( \frac{\partial P}{\partial y} - \frac{\partial N}{\partial z} \right) \mathbf{i} - \left( \frac{\partial P}{\partial x} - \frac{\partial M}{\partial z} \right) \mathbf{j} + \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) \mathbf{k} \\ \operatorname{div}(\operatorname{curl} \mathbf{F}) &= \frac{\partial}{\partial x} \left[ \frac{\partial P}{\partial y} - \frac{\partial N}{\partial z} \right] - \frac{\partial}{\partial y} \left[ \frac{\partial P}{\partial x} - \frac{\partial M}{\partial z} \right] + \frac{\partial}{\partial z} \left[ \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right] \\ &= \frac{\partial^2 P}{\partial x \partial y} - \frac{\partial^2 N}{\partial x \partial z} - \frac{\partial^2 P}{\partial y \partial x} + \frac{\partial^2 M}{\partial y \partial z} + \frac{\partial^2 N}{\partial z \partial x} - \frac{\partial^2 M}{\partial z \partial y} = 0 \quad (\text{because the mixed partials are equal})\end{aligned}$$

In Exercises 91-93,  $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$  and  $f(x, y, z) = \|\mathbf{F}(x, y, z)\| = \sqrt{x^2 + y^2 + z^2}$ .

91.  $\ln f = \frac{1}{2} \ln(x^2 + y^2 + z^2)$

$$\nabla(\ln f) = \frac{x}{x^2 + y^2 + z^2} \mathbf{i} + \frac{y}{x^2 + y^2 + z^2} \mathbf{j} + \frac{z}{x^2 + y^2 + z^2} \mathbf{k} = \frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{x^2 + y^2 + z^2} = \frac{\mathbf{F}}{f^2}$$

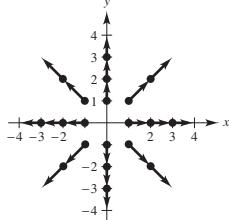
92.  $\frac{1}{f} = \frac{1}{\sqrt{x^2 + y^2 + z^2}}$

$$\nabla\left(\frac{1}{f}\right) = \frac{-x}{(x^2 + y^2 + z^2)^{3/2}} \mathbf{i} + \frac{-y}{(x^2 + y^2 + z^2)^{3/2}} \mathbf{j} + \frac{-z}{(x^2 + y^2 + z^2)^{3/2}} \mathbf{k} = \frac{-(x\mathbf{i} + y\mathbf{j} + z\mathbf{k})}{(\sqrt{x^2 + y^2 + z^2})^3} = \frac{\mathbf{F}}{f^3}$$

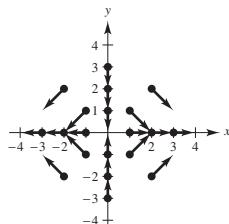
93.  $f^n = \left(\sqrt{x^2 + y^2 + z^2}\right)^n$

$$\begin{aligned}\nabla f^n &= n \left(\sqrt{x^2 + y^2 + z^2}\right)^{n-1} \frac{x}{\sqrt{x^2 + y^2 + z^2}} \mathbf{i} + n \left(\sqrt{x^2 + y^2 + z^2}\right)^{n-1} \frac{y}{\sqrt{x^2 + y^2 + z^2}} \mathbf{j} + n \left(\sqrt{x^2 + y^2 + z^2}\right)^{n-1} \frac{z}{\sqrt{x^2 + y^2 + z^2}} \mathbf{k} \\ &= n \left(\sqrt{x^2 + y^2 + z^2}\right)^{n-2} (x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) = nf^{n-2}\mathbf{F}\end{aligned}$$

94. (a)  $\mathbf{F}(x, y) = \frac{x\mathbf{i} + y\mathbf{j}}{\sqrt{x^2 + y^2}}$



(b)  $\mathbf{G}(x, y) = \frac{x\mathbf{i} - y\mathbf{j}}{\sqrt{x^2 + y^2}}$



(c) All the vectors are unit vectors. Those of  $\mathbf{F}$  point away from origin. Answers will vary.

95. True.  $\|\mathbf{F}(x, y)\| = \sqrt{16x^2 + y^4} \rightarrow 0$  as  $(x, y) \rightarrow (0, 0)$ .

96. True. If  $(x, y)$  is on the positive  $y$ -axis, then

$x = 0$  and  $y > 0$ . So,

$$\mathbf{F}(x, y) = \mathbf{F}(0, y) = -y^2 \mathbf{j}.$$

97. False. Curl is defined on vector fields, not scalar fields.

98. False. See Example 7.

## Section 15.2 Line Integrals

1.  $\mathbf{r}(t) = \begin{cases} t\mathbf{i} + t\mathbf{j}, & 0 \leq t \leq 1 \\ (2-t)\mathbf{i} + \sqrt{2-t}\mathbf{j}, & 1 \leq t \leq 2 \end{cases}$

2.  $\mathbf{r}(t) = \begin{cases} t\mathbf{i} + t^2\mathbf{j}, & 0 \leq t \leq 2 \\ (4-t)\mathbf{i} + 4\mathbf{j}, & 2 \leq t \leq 4 \\ (8-t)\mathbf{j}, & 4 \leq t \leq 8 \end{cases}$

3.  $\mathbf{r}(t) = \begin{cases} t\mathbf{i}, & 0 \leq t \leq 3 \\ 3\mathbf{i} + (t-3)\mathbf{j}, & 3 \leq t \leq 6 \\ (9-t)\mathbf{i} + 3\mathbf{j}, & 6 \leq t \leq 9 \\ (12-t)\mathbf{j}, & 9 \leq t \leq 12 \end{cases}$

4.  $\mathbf{r}(t) = \begin{cases} t\mathbf{i} + \frac{4}{5}t\mathbf{j}, & 0 \leq t \leq 5 \\ 5\mathbf{i} + (9-t)\mathbf{j}, & 5 \leq t \leq 9 \\ (14-t)\mathbf{i}, & 9 \leq t \leq 14 \end{cases}$

5.  $x^2 + y^2 = 9$

$$\frac{x^2}{9} + \frac{y^2}{9} = 1$$

$$\cos^2 t + \sin^2 t = 1$$

$$\cos^2 t = \frac{x^2}{9}$$

$$\sin^2 t = \frac{y^2}{9}$$

$$x = 3 \cos t$$

$$y = 3 \sin t$$

$$\mathbf{r}(t) = 3 \cos t\mathbf{i} + 3 \sin t\mathbf{j}$$

$$0 \leq t \leq 2\pi$$

9.  $\mathbf{r}(t) = \sin t\mathbf{i} + \cos t\mathbf{j} + 2\mathbf{k}, \quad 0 \leq t \leq \frac{\pi}{2}$

$$\mathbf{r}'(t) = \cos t\mathbf{i} - \sin t\mathbf{j}$$

$$\int_C (x^2 + y^2 + z^2) ds = \int_0^{\pi/2} (\sin^2 t + \cos^2 t + 4) \sqrt{\cos^2 t + \sin^2 t} dt = \int_0^{\pi/2} 5 dt = \frac{5\pi}{2}$$

10.  $\mathbf{r}(t) = 12t\mathbf{i} + 5t\mathbf{j} + 84t\mathbf{k}, \quad 0 \leq t \leq 1$

$$\mathbf{r}'(t) = 12\mathbf{i} + 5\mathbf{j} + 84\mathbf{k}$$

$$\int_C 2xyz ds = \int_0^1 2(12t)(5t)(84t) \sqrt{(12)^2 + 5^2 + (84)^2} dt = \int_0^1 10,080 t^3 (85) dt = 856,800 \left[ \frac{t^4}{4} \right]_0^1 = 214,200$$

11. (a)  $\mathbf{r}(t) = t\mathbf{i} + t\mathbf{j}, \quad 0 \leq t \leq 1$

(b)  $\mathbf{r}'(t) = \mathbf{i} + \mathbf{j}, \|\mathbf{r}'(t)\| = \sqrt{2}$

$$\begin{aligned} \int_C (x^2 + y^2) ds &= \int_0^1 (t^2 + t^2) \sqrt{2} dt \\ &= 2\sqrt{2} \left[ \frac{t^3}{3} \right]_0^1 = \frac{2\sqrt{2}}{3} \end{aligned}$$

6.  $\frac{x^2}{16} + \frac{y^2}{9} = 1$

$$\cos^2 t + \sin^2 t = 1$$

$$\cos^2 t = \frac{x^2}{16}$$

$$\sin^2 t = \frac{y^2}{9}$$

$$x = 4 \cos t$$

$$y = 3 \sin t$$

$$\mathbf{r}(t) = 4 \cos t\mathbf{i} + 3 \sin t\mathbf{j}$$

$$0 \leq t \leq 2\pi$$

7.  $\mathbf{r}(t) = 4\mathbf{i} + 3\mathbf{j}, \quad 0 \leq t \leq 1$

$$\mathbf{r}'(t) = 4\mathbf{i} + 3\mathbf{j}$$

$$\begin{aligned} \int_C xy ds &= \int_0^1 (4t)(3t) \sqrt{4^2 + 3^2} dt \\ &= \int_0^1 60t^2 dt = \left[ 20t^3 \right]_0^1 = 20 \end{aligned}$$

8.  $\mathbf{r}(t) = t\mathbf{i} + (2-t)\mathbf{j}, \quad 0 \leq t \leq 2$

$$\mathbf{r}'(t) = \mathbf{i} - \mathbf{j}$$

$$\begin{aligned} \int_C 3(x-y) ds &= \int_0^2 3(t - (2-t)) \sqrt{1^2 + (-1)^2} dt \\ &= 3\sqrt{2} \int_0^2 (2t-2) dt \\ &= 3\sqrt{2} \left[ t^2 - 2t \right]_0^2 = 0 \end{aligned}$$

12. (a)  $\mathbf{r}(t) = t\mathbf{i} + 2t\mathbf{j}, \quad 0 \leq t \leq 2$

(b)  $\mathbf{r}'(t) = \mathbf{i} + 2\mathbf{j}, \|\mathbf{r}'(t)\| = \sqrt{5}$

$$\int_C (x^2 + y^2) ds = \int_0^2 (t^2 + 4t^2) \sqrt{5} dt$$

$$= \left[ \sqrt{5} \frac{5t^3}{3} \right]_0^2 = \frac{40\sqrt{5}}{3}$$

13. (a)  $\mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j}, \quad 0 \leq t \leq \frac{\pi}{2}$

(b)  $\int_C (x^2 + y^2) ds = \int_0^{\pi/2} [\cos^2 t + \sin^2 t] \sqrt{(-\sin t)^2 + (\cos t)^2} dt = \int_0^{\pi/2} dt = \frac{\pi}{2}$

14. (a)  $\mathbf{r}(t) = 2 \cos t\mathbf{i} + 2 \sin t\mathbf{j}, \quad 0 \leq t \leq \frac{\pi}{2}$

(b)  $\int_C (x^2 + y^2) ds = \int_0^{\pi/2} [4 \cos^2 t + 4 \sin^2 t] \sqrt{(-2 \sin t)^2 + (2 \cos t)^2} dt = \int_0^{\pi/2} 8 dt = 4\pi$

15. (a)  $\mathbf{r}(t) = t\mathbf{i}, \quad 0 \leq t \leq 1$

(b)  $\mathbf{r}'(t) = \mathbf{i}, \|\mathbf{r}'(t)\| = 1$

$$\int_C (x + 4\sqrt{y}) ds = \int_0^1 t dt = \left[ \frac{t^2}{2} \right]_0^1 = \frac{1}{2}$$

16. (a)  $\mathbf{r}(t) = t\mathbf{j}, \quad 1 \leq t \leq 9$

(b)  $\mathbf{r}'(t) = \mathbf{j}, \|\mathbf{r}'(t)\| = 1$

$$\int_C (x + 4\sqrt{y}) ds = \int_1^9 4\sqrt{t} dt = \left[ \frac{8}{3}t^{3/2} \right]_1^9 = \frac{8}{3}(27 - 1) = \frac{208}{3}$$

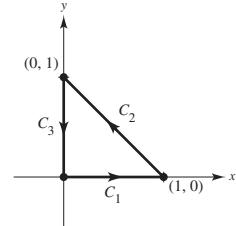
17. (a)  $\mathbf{r}(t) = \begin{cases} t\mathbf{i}, & 0 \leq t \leq 1 \\ (2-t)\mathbf{i} + (t-1)\mathbf{j}, & 1 \leq t \leq 2 \\ (3-t)\mathbf{j}, & 2 \leq t \leq 3 \end{cases}$

(b)  $\int_{C_1} (x + 4\sqrt{y}) ds = \int_0^1 t dt = \frac{1}{2}$

$$\int_{C_2} (x + 4\sqrt{y}) ds = \int_1^2 [(2-t) + 4\sqrt{t-1}] \sqrt{1+1} dt = \sqrt{2} \left[ 2t - \frac{t^2}{2} + \frac{8}{3}(t-1)^{3/2} \right]_1^2 = \frac{19\sqrt{2}}{6}$$

$$\int_{C_3} (x + 4\sqrt{y}) ds = \int_2^3 4\sqrt{3-t} dt = \left[ -\frac{8}{3}(3-t)^{3/2} \right]_2^3 = \frac{8}{3}$$

$$\int_C (x + 4\sqrt{y}) ds = \frac{1}{2} + \frac{19\sqrt{2}}{6} + \frac{8}{3} = \frac{19 + 19\sqrt{2}}{6} = \frac{19(1 + \sqrt{2})}{6}$$



18. (a)  $\mathbf{r}(t) = \begin{cases} t\mathbf{i}, & 0 \leq t \leq 2 \\ 2\mathbf{i} + (t-2)\mathbf{j}, & 2 \leq t \leq 4 \\ (6-t)\mathbf{i} + 2\mathbf{j}, & 4 \leq t \leq 6 \\ (8-t)\mathbf{j}, & 6 \leq t \leq 8 \end{cases}$

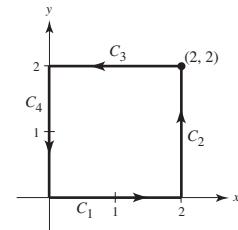
(b)  $\int_{C_1} (x + 4\sqrt{y}) ds = \int_0^2 t dt = 2$

$$\int_{C_2} (x + 4\sqrt{y}) ds = \int_2^4 (2 + 4\sqrt{t-2}) dt = 4 + \frac{16\sqrt{2}}{3}$$

$$\int_{C_3} (x + 4\sqrt{y}) ds = \int_4^6 ((6-t) + 4\sqrt{2}) dt = 2 + 8\sqrt{2}$$

$$\int_{C_4} (x + 4\sqrt{y}) ds = \int_6^8 4\sqrt{8-t} dt = \frac{16\sqrt{2}}{3}$$

$$\int_C (x + 4\sqrt{y}) ds = 2 + 4 + \frac{16\sqrt{2}}{3} + 2 + 8\sqrt{2} + \frac{16\sqrt{2}}{3} = 8 + \frac{56}{3}\sqrt{2}$$



19. (a)  $C_1: (0, 0, 0)$  to  $(1, 0, 0)$ :  $\mathbf{r}(t) = t\mathbf{i}, 0 \leq t \leq 1, \mathbf{r}'(t) = \mathbf{i}, \|\mathbf{r}'(t)\| = 1$

$$\int_{C_1} (2x + y^2 - z) ds = \int_0^1 2t dt = t^2 \Big|_0^1 = 1$$

$C_2: (1, 0, 0)$  to  $(1, 0, 1)$ :  $\mathbf{r}(t) = \mathbf{i} + t\mathbf{k}, 0 \leq t \leq 1, \mathbf{r}'(t) = \mathbf{k}, \|\mathbf{r}'(t)\| = 1$

$$\int_{C_2} (2x + y^2 - z) ds = \int_0^1 (2 - t) dt = \left[ 2t - \frac{t^2}{2} \right]_0^1 = \frac{3}{2}$$

$C_3: (1, 0, 1)$  to  $(1, 1, 1)$ :  $\mathbf{r}(t) = \mathbf{i} + t\mathbf{j} + \mathbf{k}, 0 \leq t \leq 1, \mathbf{r}'(t) = \mathbf{j}, \|\mathbf{r}'(t)\| = 1$

$$\int_{C_3} (2x + y^2 - z) ds = \int_0^1 (2 + t^2 - 1) dt = \left[ t + \frac{t^3}{3} \right]_0^1 = \frac{4}{3}$$

$$(b) \text{ Combining, } \int_C (2x + y^2 - z) ds = 1 + \frac{3}{2} + \frac{4}{3} = \frac{23}{6}.$$

20. (a)  $C_1: (0, 0, 0)$  to  $(0, 1, 0)$ :  $\mathbf{r}(t) = t\mathbf{j}, 0 \leq t \leq 1, \mathbf{r}'(t) = \mathbf{j}, \|\mathbf{r}'(t)\| = 1$

$$\int_{C_1} (2x + y^2 - z) ds = \int_0^1 t^2 dt = \frac{t^3}{3} \Big|_0^1 = \frac{1}{3}$$

$C_2: (0, 1, 0)$  to  $(0, 1, 1)$ :  $\mathbf{r}(t) = \mathbf{j} + t\mathbf{k}, 0 \leq t \leq 1, \mathbf{r}'(t) = \mathbf{k}, \|\mathbf{r}'(t)\| = 1$

$$\int_{C_2} (2x + y^2 - z) ds = \int_0^1 (1 - t) dt = \left[ t - \frac{t^2}{2} \right]_0^1 = \frac{1}{2}$$

$C_3: (0, 1, 1)$  to  $(0, 0, 0)$ :  $\mathbf{r}(t) = (1 - t)\mathbf{j} + (1 - t)\mathbf{k}, 0 \leq t \leq 1, \mathbf{r}'(t) = -\mathbf{j} - \mathbf{k}, \|\mathbf{r}'(t)\| = \sqrt{2}$

$$\int_{C_3} (2x + y^2 - z) ds = \int_0^1 [(1 - t)^2 - (1 - t)] \sqrt{2} dt = \int_0^1 (t^2 - t) \sqrt{2} dt = \sqrt{2} \left[ \frac{t^3}{3} - \frac{t^2}{2} \right]_0^1 = -\frac{\sqrt{2}}{6}$$

$$(b) \text{ Combining, } \int_C (2x + y^2 - z) ds = \frac{1}{3} + \frac{1}{2} - \frac{\sqrt{2}}{6} = \frac{5 - \sqrt{2}}{6}.$$

$$21. \rho(x, y, z) = \frac{1}{2}(x^2 + y^2 + z^2)$$

$$\mathbf{r}(t) = 2 \cos t\mathbf{i} + 2 \sin t\mathbf{j} + t\mathbf{k}, 0 \leq t \leq 4\pi$$

$$\mathbf{r}'(t) = -2 \sin t\mathbf{i} + 2 \cos t\mathbf{j} + \mathbf{k}$$

$$\|\mathbf{r}'(t)\| = \sqrt{4 \sin^2 t + 4 \cos^2 t + 1} = \sqrt{5}$$

$$\text{Mass} = \int_C \rho(x, y, z) ds$$

$$= \int_0^{4\pi} \frac{1}{2}(4 \cos^2 t + 4 \sin^2 t + t^2) \sqrt{5} dt$$

$$= \frac{\sqrt{5}}{2} \int_0^{4\pi} (4 + t^2) dt = \frac{\sqrt{5}}{2} \left[ 4t + \frac{t^3}{3} \right]_0^{4\pi}$$

$$= \frac{\sqrt{5}}{2} \left[ 16\pi + \frac{64\pi^3}{3} \right] = \frac{8\pi\sqrt{5}}{3} (4\pi^2 + 3) \approx 795.7$$

$$22. \rho(x, y, z) = z$$

$$\mathbf{r}(t) = 2 \cos t\mathbf{i} + 2 \sin t\mathbf{j} + t\mathbf{k}, 0 \leq t \leq 4\pi$$

$$\mathbf{r}'(t) = -2 \sin t\mathbf{i} + 2 \cos t\mathbf{j} + \mathbf{k}$$

$$\|\mathbf{r}'(t)\| = \sqrt{4 \sin^2 t + 4 \cos^2 t + 1} = \sqrt{5}$$

$$\text{Mass} = \int_C \rho(x, y, z) ds$$

$$= \int_0^{4\pi} t \sqrt{5} dt = \left[ \frac{t^2}{2} \sqrt{5} \right]_0^{4\pi} = 8\pi^2 \sqrt{5}$$

$$23. \mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j}, 0 \leq t \leq \pi$$

$$\mathbf{r}'(t) = -\sin t\mathbf{i} + \cos t\mathbf{j}, \|\mathbf{r}'(t)\| = 1$$

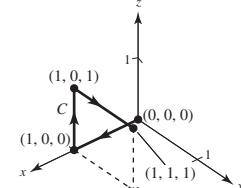
$$\text{Mass} = \int_C \rho(x, y) ds = \int_C (x + y) ds$$

$$= \int_0^\pi (\cos t + \sin t) dt$$

$$= [\sin t - \cos t]_0^\pi$$

$$= 1 + 1 = 2$$

$C_2: (1, 0, 0)$  to  $(1, 0, 1)$ :  $\mathbf{r}(t) = \mathbf{i} + t\mathbf{k}, 0 \leq t \leq 1, \mathbf{r}'(t) = \mathbf{k}, \|\mathbf{r}'(t)\| = 1$

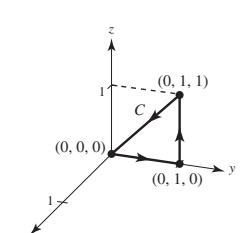


$C_3: (1, 0, 1)$  to  $(1, 1, 1)$ :  $\mathbf{r}(t) = \mathbf{i} + t\mathbf{j} + \mathbf{k}, 0 \leq t \leq 1, \mathbf{r}'(t) = \mathbf{j}, \|\mathbf{r}'(t)\| = 1$

$$\int_{C_3} (2x + y^2 - z) ds = \int_0^1 (2 + t^2 - 1) dt = \left[ t + \frac{t^3}{3} \right]_0^1 = \frac{4}{3}$$

$$(b) \text{ Combining, } \int_C (2x + y^2 - z) ds = 1 + \frac{3}{2} + \frac{4}{3} = \frac{23}{6}.$$

20. (a)  $C_1: (0, 0, 0)$  to  $(0, 1, 0)$ :  $\mathbf{r}(t) = t\mathbf{j}, 0 \leq t \leq 1, \mathbf{r}'(t) = \mathbf{j}, \|\mathbf{r}'(t)\| = 1$



$C_2: (0, 1, 0)$  to  $(0, 1, 1)$ :  $\mathbf{r}(t) = \mathbf{j} + t\mathbf{k}, 0 \leq t \leq 1, \mathbf{r}'(t) = \mathbf{k}, \|\mathbf{r}'(t)\| = 1$

$$\int_{C_2} (2x + y^2 - z) ds = \int_0^1 (1 - t) dt = \left[ t - \frac{t^2}{2} \right]_0^1 = \frac{1}{2}$$

$C_3: (0, 1, 1)$  to  $(0, 0, 0)$ :  $\mathbf{r}(t) = (1 - t)\mathbf{j} + (1 - t)\mathbf{k}, 0 \leq t \leq 1, \mathbf{r}'(t) = -\mathbf{j} - \mathbf{k}, \|\mathbf{r}'(t)\| = \sqrt{2}$

$$\int_{C_3} (2x + y^2 - z) ds = \int_0^1 [(1 - t)^2 - (1 - t)] \sqrt{2} dt = \int_0^1 (t^2 - t) \sqrt{2} dt = \sqrt{2} \left[ \frac{t^3}{3} - \frac{t^2}{2} \right]_0^1 = -\frac{\sqrt{2}}{6}$$

$$(b) \text{ Combining, } \int_C (2x + y^2 - z) ds = \frac{1}{3} + \frac{1}{2} - \frac{\sqrt{2}}{6} = \frac{5 - \sqrt{2}}{6}.$$

24.  $\mathbf{r}(t) = t^2\mathbf{i} + 2t\mathbf{j}, \quad 0 \leq t \leq 1$

$$\mathbf{r}'(t) = 2t\mathbf{i} + 2\mathbf{j}, \quad \|\mathbf{r}'(t)\| = \sqrt{4t^2 + 4}$$

$$\begin{aligned}\text{Mass} &= \int_C \rho(x, y) ds = \int_C \frac{3}{4}y ds \\ &= \int_0^1 \frac{3}{4}(2t)\sqrt{4t^2 + 4} dt \\ &= \int_0^1 3t(t^2 + 1)^{1/2} dt \\ &= \left[ (t^2 + 1)^{3/2} \right]_0^1 = 2\sqrt{2} - 1\end{aligned}$$

25.  $\mathbf{r}(t) = t^2\mathbf{i} + 2t\mathbf{j} + t\mathbf{k}, \quad 1 \leq t \leq 3$

$$\mathbf{r}'(t) = 2t\mathbf{i} + 2\mathbf{j} + \mathbf{k}, \quad \|\mathbf{r}'(t)\| = \sqrt{4t^2 + 5}$$

$$\begin{aligned}\text{Mass} &= \int_C \rho(x, y, z) ds = \int_C kz ds \\ &= \int_1^3 kt\sqrt{4t^2 + 5} dt \\ &= \frac{k(4t^2 + 5)^{3/2}}{12} \Big|_1^3 \\ &= \frac{k}{12} [41\sqrt{41} - 27]\end{aligned}$$

26.  $\mathbf{r}(t) = 2 \cos t\mathbf{i} + 2 \sin t\mathbf{j} + 3t\mathbf{k}, \quad 0 \leq t \leq 2\pi$

$$\mathbf{r}'(t) = -2 \sin t\mathbf{i} + 2 \cos t\mathbf{j} + 3\mathbf{k}$$

$$\|\mathbf{r}'(t)\| = \sqrt{4 + 9} = \sqrt{13}$$

$$\begin{aligned}\text{Mass} &= \int_C \rho(x, y, z) ds = \int_C (k + z) ds \\ &= \int_0^{2\pi} (k + 3t)\sqrt{13} dt \\ &= \sqrt{13} \left( kt + \frac{3t^2}{2} \right) \Big|_0^{2\pi} \\ &= \sqrt{13}(2\pi k + 6\pi^2)\end{aligned}$$

27.  $\mathbf{F}(x, y) = x\mathbf{i} + y\mathbf{j}$

$$C: \mathbf{r}(t) = t\mathbf{i} + t\mathbf{j}, \quad 0 \leq t \leq 1$$

$$\mathbf{F}(t) = t\mathbf{i} + t\mathbf{j}$$

$$\mathbf{r}'(t) = \mathbf{i} + \mathbf{j}$$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 (t + t) dt = \left[ t^2 \right]_0^1 = 1$$

28.  $\mathbf{F}(x, y) = xy\mathbf{i} + y\mathbf{j}$

$$C: \mathbf{r}(t) = 4 \cos t\mathbf{i} + 4 \sin t\mathbf{j}, \quad 0 \leq t \leq \frac{\pi}{2}$$

$$\mathbf{F}(t) = 16 \sin t \cos t\mathbf{i} + 4 \sin t\mathbf{j}$$

$$\mathbf{r}'(t) = -4 \sin t\mathbf{i} + 4 \cos t\mathbf{j}$$

$$\begin{aligned}\int_C \mathbf{F} \cdot d\mathbf{r} &= \int_0^{\pi/2} (-64 \sin^2 t \cos t + 16 \sin t \cos t) dt \\ &= \left[ -\frac{64}{3} \sin^3 t + 8 \sin^2 t \right]_0^{\pi/2} = -\frac{40}{3}\end{aligned}$$

29.  $\mathbf{F}(x, y) = 3x\mathbf{i} + 4y\mathbf{j}$

$$C: \mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j}, \quad 0 \leq t \leq \pi/2$$

$$\mathbf{F}(t) = 3 \cos t\mathbf{i} + 4 \sin t\mathbf{j}$$

$$\mathbf{r}'(t) = -\sin t\mathbf{i} + \cos t\mathbf{j}$$

$$\begin{aligned}\int_C \mathbf{F} \cdot d\mathbf{r} &= \int_0^{\pi/2} (-3 \cos t \sin t + 4 \sin t \cos t) dt \\ &= \left[ \frac{\sin^2 t}{2} \right]_0^{\pi/2} = 1/2\end{aligned}$$

30.  $\mathbf{F}(x, y) = 3x\mathbf{i} + 4y\mathbf{j}$

$$C: \mathbf{r}(t) = t\mathbf{i} + \sqrt{4 - t^2}\mathbf{j}, \quad -2 \leq t \leq 2$$

$$\mathbf{F}(t) = 3t\mathbf{i} + 4\sqrt{4 - t^2}\mathbf{j}$$

$$\mathbf{r}'(t) = \mathbf{i} - \frac{t}{\sqrt{4 - t^2}}\mathbf{j}$$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_{-2}^2 (3t - 4t) dt = \left[ -\frac{t^2}{2} \right]_{-2}^2 = 0$$

31.  $\mathbf{F}(x, y, z) = xy\mathbf{i} + xz\mathbf{j} + yz\mathbf{k}$

$$C: \mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + 2t\mathbf{k}, \quad 0 \leq t \leq 1$$

$$\mathbf{F}(t) = t^3\mathbf{i} + 2t^2\mathbf{j} + 2t^3\mathbf{k}$$

$$\mathbf{r}'(t) = \mathbf{i} + 2t\mathbf{j} + 2\mathbf{k}$$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 (t^3 + 4t^3 + 4t^3) dt = \left[ \frac{9t^4}{4} \right]_0^1 = \frac{9}{4}$$

32.  $\mathbf{F}(x, y, z) = x^2\mathbf{i} + y^2\mathbf{j} + z^2\mathbf{k}$

$$C: \mathbf{r}(t) = 2 \sin t\mathbf{i} + 2 \cos t\mathbf{j} + \frac{1}{2}t^2\mathbf{k}, \quad 0 \leq t \leq \pi$$

$$\mathbf{F}(t) = 4 \sin^2 t\mathbf{i} + 4 \cos^2 t\mathbf{j} + \frac{1}{4}t^4\mathbf{k}$$

$$\mathbf{r}'(t) = 2 \cos t\mathbf{i} - 2 \sin t\mathbf{j} + \frac{1}{2}t^3\mathbf{k}$$

$$\begin{aligned}\int_C \mathbf{F} \cdot d\mathbf{r} &= \int_0^\pi \left( 8 \sin^2 t \cos t - 8 \cos^2 t \sin t + \frac{1}{4}t^5 \right) dt \\ &= \left[ \frac{8}{3} \sin^3 t + \frac{8}{3} \cos^3 t + \frac{t^6}{24} \right]_0^\pi \\ &= -\frac{8}{3} + \frac{\pi^6}{24} - \frac{8}{3} = \frac{\pi^6}{24} - \frac{16}{3}\end{aligned}$$

33.  $\mathbf{F}(x, y, z) = x^2z\mathbf{i} + 6y\mathbf{j} + yz^2\mathbf{k}$

$$\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + \ln t\mathbf{k}, \quad 1 \leq t \leq 3$$

$$\mathbf{F}(t) = t^2 \ln t\mathbf{i} + 6t^2\mathbf{j} + t^2 \ln^2 t\mathbf{k}$$

$$d\mathbf{r} = \left( \mathbf{i} + 2t\mathbf{j} + \frac{1}{t}\mathbf{k} \right) dt$$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_1^3 \left[ t^2 \ln t + 12t^3 + t(\ln t)^2 \right] dt \approx 249.49$$

34.  $\mathbf{F}(x, y, z) = \frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{\sqrt{x^2 + y^2 + z^2}}$

$$\mathbf{r}(t) = t\mathbf{i} + t\mathbf{j} + e^t\mathbf{k}, 0 \leq t \leq 2$$

$$\mathbf{F}(t) = \frac{t\mathbf{i} + t\mathbf{j} + e^t\mathbf{k}}{\sqrt{2t^2 + e^{2t}}}$$

$$d\mathbf{r} = (\mathbf{i} + \mathbf{j} + e^t\mathbf{k}) dt$$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^2 \frac{1}{\sqrt{2t^2 + e^{2t}}} (2t + e^{2t}) dt \approx 6.91$$

36.  $\mathbf{F}(x, y) = x^2\mathbf{i} - xy\mathbf{j}$

C:  $x = \cos^3 t, y = \sin^3 t$  from  $(1, 0)$  to  $(0, 1)$

$$\mathbf{r}(t) = \cos^3 t\mathbf{i} + \sin^3 t\mathbf{j}, 0 \leq t \leq \frac{\pi}{2}$$

$$\mathbf{r}'(t) = -3\cos^2 t \sin t\mathbf{i} + 3\sin^2 t \cos t\mathbf{j}$$

$$\mathbf{F}(t) = \cos^6 t\mathbf{i} - \cos^3 t \sin^3 t\mathbf{j}$$

$$\begin{aligned} \mathbf{F} \cdot \mathbf{r}' &= -3\cos^8 t \sin t - 3\cos^4 t \sin^5 t = -3\cos^4 t \sin t (\cos^4 t + \sin^4 t) = -3\cos^4 t \sin t [\cos^4 t + (1 - \cos^2 t)^2] \\ &= -3\cos^4 t \sin t (2\cos^4 t - 2\cos^2 t + 1) = -6\cos^8 t \sin t + 6\cos^6 t \sin t - 3\cos^4 t \sin t \end{aligned}$$

$$\text{Work} = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{\pi/2} [-6\cos^8 t \sin t + 6\cos^6 t \sin t - 3\cos^4 t \sin t] dt$$

$$= \left[ \frac{2\cos^9 t}{3} - \frac{6\cos^7 t}{7} + \frac{3\cos^5 t}{5} \right]_0^{\pi/2} = -\frac{43}{105}$$

37.  $\mathbf{F}(x, y) = x\mathbf{i} + y\mathbf{j}$

C:  $\mathbf{r}(t) = \begin{cases} t\mathbf{i} & 0 \leq t \leq 1 \\ (2-t)\mathbf{i} + (t-1)\mathbf{j}, & 1 \leq t \leq 2 \\ (3-t)\mathbf{j} & 2 \leq t \leq 3 \end{cases}$

On  $C_1, \mathbf{F}(t) = t\mathbf{i}, \mathbf{r}'(t) = \mathbf{i}$

$$\text{Work} = \int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_0^1 t dt = \frac{1}{2}$$

On  $C_2, \mathbf{F}(t) = (2-t)\mathbf{i} + (t-1)\mathbf{j}, \mathbf{r}'(t) = -\mathbf{i} + \mathbf{j}$

$$\begin{aligned} \text{Work} &= \int_{C_2} \mathbf{F} \cdot d\mathbf{r} = \int_1^2 [(t-2) + (t-1)] dt \\ &= [t^2 - 3t]_1^2 \\ &= (4-6) - (1-3) = 0 \end{aligned}$$

On  $C_3, \mathbf{F}(t) = (3-t)\mathbf{j}, \mathbf{r}'(t) = -\mathbf{j}$

$$\begin{aligned} \text{Work} &= \int_{C_3} \mathbf{F} \cdot d\mathbf{r} = \int_2^3 (t-3) dt = \left[ \frac{t^2}{2} - 3t \right]_2^3 \\ &= \left( \frac{9}{2} - 9 \right) - (2-6) = -\frac{1}{2} \end{aligned}$$

$$\text{Total work} = \frac{1}{2} + 0 - \frac{1}{2} = 0$$

35.  $\mathbf{F}(x, y) = x\mathbf{i} + 2y\mathbf{j}$

C:  $\mathbf{r}(t) = t\mathbf{i} + t^3\mathbf{j}, 0 \leq t \leq 2$

$$\mathbf{r}'(t) = \mathbf{i} + 3t^2\mathbf{j}$$

$$\mathbf{F}(t) = t\mathbf{i} + 2t^3\mathbf{j}$$

$$\text{Work} = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^2 (t + 6t^5) dt = \left[ \frac{t^2}{2} + t^6 \right]_0^2 = 66$$

38.  $\mathbf{F}(x, y) = -y\mathbf{i} - x\mathbf{j}$

C: counterclockwise along the semicircle

$$y = \sqrt{4 - x^2} \text{ from } (2, 0) \text{ to } (-2, 0)$$

$$\mathbf{r}(t) = 2 \cos t\mathbf{i} + 2 \sin t\mathbf{j}, 0 \leq t \leq \pi$$

$$\mathbf{r}'(t) = -2 \sin t\mathbf{i} + 2 \cos t\mathbf{j}$$

$$\mathbf{F}(t) = -2 \sin t\mathbf{i} - 2 \cos t\mathbf{j}$$

$$\mathbf{F} \cdot \mathbf{r}' = 4 \sin^2 t - 4 \cos^2 t = -4 \cos 2t$$

$$\text{Work} = \int_C \mathbf{F} \cdot d\mathbf{r}$$

$$= -4 \int_0^\pi \cos 2t dt$$

$$= [-2 \sin 2t]_0^\pi = 0$$

39.  $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} - 5z\mathbf{k}$

C:  $\mathbf{r}(t) = 2 \cos t\mathbf{i} + 2 \sin t\mathbf{j} + tk, 0 \leq t \leq 2\pi$

$$\mathbf{r}'(t) = -2 \sin t\mathbf{i} + 2 \cos t\mathbf{j} + \mathbf{k}$$

$$\mathbf{F}(t) = 2 \cos t\mathbf{i} + 2 \sin t\mathbf{j} - 5t\mathbf{k}$$

$$\mathbf{F} \cdot \mathbf{r}' = -5t$$

$$\text{Work} = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} -5t dt = -10\pi^2$$

40.  $\mathbf{F}(x, y, z) = yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}$

C: line from  $(0, 0, 0)$  to  $(5, 3, 2)$

$$\mathbf{r}(t) = 5t\mathbf{i} + 3t\mathbf{j} + 2t\mathbf{k}, \quad 0 \leq t \leq 1$$

$$\mathbf{r}'(t) = 5\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$$

$$\mathbf{F}(t) = 6t^2\mathbf{i} + 10t^2\mathbf{j} + 15t^2\mathbf{k}$$

$$\mathbf{F} \cdot \mathbf{r}' = 90t^2$$

$$\text{Work} = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 90t^2 dt = 30$$

41. Because the vector field determined by  $\mathbf{F}$  points in the general direction of the path  $C$ ,  $\mathbf{F} \cdot \mathbf{T} > 0$  and work will be positive.

42. Because the vector field determined by  $\mathbf{F}$  points for the most part in the opposite direction of the path  $C$ ,  $\mathbf{F} \cdot \mathbf{T} < 0$  and work will be negative.

43. Because the vector field determined by  $\mathbf{F}$  is perpendicular to the path, work will be 0.

44. Because the vector field is perpendicular to the path, work will be 0.

46.  $\mathbf{F}(x, y) = x^2y\mathbf{i} + xy^{3/2}\mathbf{j}$

$$(a) \quad \mathbf{r}_1(t) = (t+1)\mathbf{i} + t^2\mathbf{j}, \quad 0 \leq t \leq 2$$

$$\mathbf{r}'_1(t) = \mathbf{i} + 2t\mathbf{j}$$

$$\mathbf{F}(t) = (t+1)^2 t^2 \mathbf{i} + (t+1)t^3 \mathbf{j}$$

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_0^2 \left[ (t+1)^2 t^2 + 2t^4(t+1) \right] dt = \frac{256}{5}$$

$$(b) \quad \mathbf{r}_2(t) = (1+2\cos t)\mathbf{i} + 4\cos^2 t\mathbf{j}, \quad 0 \leq t \leq \frac{\pi}{2}$$

$$\mathbf{r}'_2(t) = -2\sin t\mathbf{i} - 8\cos t \sin t\mathbf{j}$$

$$\mathbf{F}(t) = (1+2\cos t)^2 (4\cos^2 t)\mathbf{i} + (1+2\cos t)(8\cos^3 t)\mathbf{j}$$

$$\int_{C_2} \mathbf{F} \cdot d\mathbf{r} = \int_0^{\pi/2} \left[ (1+2\cos t)^2 (4\cos^2 t)(-2\sin t) - 8\cos t \sin t (1+2\cos t)(8\cos^3 t) \right] dt = -\frac{256}{5}$$

Both paths join  $(1, 0)$  and  $(3, 4)$ . The integrals are negatives of each other because the orientations are different.

47.  $\mathbf{F}(x, y) = y\mathbf{i} - x\mathbf{j}$

C:  $\mathbf{r}(t) = t\mathbf{i} - 2t\mathbf{j}$

$$\mathbf{r}'(t) = \mathbf{i} - 2\mathbf{j}$$

$$\mathbf{F}(t) = -2t\mathbf{i} - t\mathbf{j}$$

$$\mathbf{F} \cdot \mathbf{r}' = -2t + 2t = 0$$

$$\text{So, } \int_C \mathbf{F} \cdot d\mathbf{r} = 0.$$

45.  $\mathbf{F}(x, y) = x^2\mathbf{i} + xy\mathbf{j}$

$$(a) \quad \mathbf{r}_1(t) = 2t\mathbf{i} + (t-1)\mathbf{j}, \quad 1 \leq t \leq 3$$

$$\mathbf{r}'_1(t) = 2\mathbf{i} + \mathbf{j}$$

$$\mathbf{F}(t) = 4t^2\mathbf{i} + 2t(t-1)\mathbf{j}$$

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_1^3 (8t^2 + 2t(t-1)) dt = \frac{236}{3}$$

Both paths join  $(2, 0)$  and  $(6, 2)$ . The integrals are negatives of each other because the orientations are different.

$$(b) \quad \mathbf{r}_2(t) = 2(3-t)\mathbf{i} + (2-t)\mathbf{j}, \quad 0 \leq t \leq 2$$

$$\mathbf{r}'_2(t) = -2\mathbf{i} - \mathbf{j}$$

$$\mathbf{F}(t) = 4(3-t)^2\mathbf{i} + 2(3-t)(2-t)\mathbf{j}$$

$$\begin{aligned} \int_{C_2} \mathbf{F} \cdot d\mathbf{r} &= \int_0^2 \left[ -8(3-t)^2 - 2(3-t)(2-t) \right] dt \\ &= -\frac{236}{3} \end{aligned}$$

48.  $\mathbf{F}(x, y) = -3y\mathbf{i} + x\mathbf{j}$

C:  $\mathbf{r}(t) = t\mathbf{i} - t^3\mathbf{j}$

$$\mathbf{r}'(t) = \mathbf{i} - 3t^2\mathbf{j}$$

$$\mathbf{F}(t) = 3t^3\mathbf{i} + t\mathbf{j}$$

$$\mathbf{F} \cdot \mathbf{r}' = 3t^3 - 3t^3 = 0$$

$$\text{So, } \int_C \mathbf{F} \cdot d\mathbf{r} = 0.$$

**49.**  $\mathbf{F}(x, y) = (x^3 - 2x^2)\mathbf{i} + \left(x - \frac{y}{2}\right)\mathbf{j}$

$$C: \mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j}$$

$$\mathbf{r}'(t) = \mathbf{i} + 2t\mathbf{j}$$

$$\mathbf{F}(t) = \left(t^3 - 2t^2\right)\mathbf{i} + \left(t - \frac{t^2}{2}\right)\mathbf{j}$$

$$\mathbf{F} \cdot \mathbf{r}' = \left(t^3 - 2t^2\right) + 2t\left(t - \frac{t^2}{2}\right) = 0$$

So,  $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$ .

**51.**  $x = 2t, y = 10t, 0 \leq t \leq 1 \Rightarrow y = 5x$  or  $x = \frac{y}{5}, 0 \leq y \leq 10$

$$\int_C (x + 3y^2) dy = \int_0^{10} \left(\frac{y}{5} + 3y^2\right) dy = \left[\frac{y^2}{10} + y^3\right]_0^{10} = 1010$$

**52.**  $x = 2t, y = 10t, 0 \leq t \leq 1 \Rightarrow y = 5x, 0 \leq x \leq 2$

$$\int_C (x + 3y^2) dx = \int_0^2 (x + 75x^2) dx = \left[\frac{x^2}{2} + 25x^3\right]_0^2 = 202$$

**53.**  $x = 2t, y = 10t, 0 \leq t \leq 1 \Rightarrow x = \frac{y}{5}, 0 \leq y \leq 10, dx = \frac{1}{5} dy$

$$\int_C xy dx + y dy = \int_0^{10} \left(\frac{y^2}{25} + y\right) dy = \left[\frac{y^3}{75} + \frac{y^2}{2}\right]_0^{10} = \frac{190}{3} \text{ or}$$

$$y = 5x, dy = 5 dx, 0 \leq x \leq 2$$

$$\int_C xy dx + y dy = \int_0^2 (5x^2 + 25x) dx = \left[\frac{5x^3}{3} + \frac{25x^2}{2}\right]_0^2 = \frac{190}{3}$$

**54.**  $x = 2t, y = 10t, 0 \leq t \leq 1 \Rightarrow y = 5x, dy = 5 dx, 0 \leq x \leq 2$

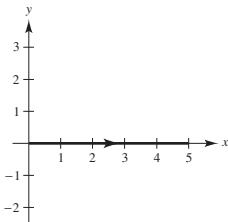
$$\begin{aligned} \int_C (3y - x) dx + y^2 dy &= \int_0^2 (3(5x) - x) dx + (5x)^2 5 dx = \int_0^2 (14x + 125x^2) dx \\ &= \left[7x^2 + \frac{125}{3}x^3\right]_0^2 = 28 + \frac{125}{3}(8) = \frac{1084}{3} \end{aligned}$$

**55.**  $\mathbf{r}(t) = t\mathbf{i}, 0 \leq t \leq 5$

$$x(t) = t, y(t) = 0$$

$$dx = dt, dy = 0$$

$$\int_C (2x - y) dx + (x + 3y) dy = \int_0^5 2t dt = 25$$



**50.**  $\mathbf{F}(x, y) = x\mathbf{i} + y\mathbf{j}$

$$C: \mathbf{r}(t) = 3 \sin t\mathbf{i} + 3 \cos t\mathbf{j}$$

$$\mathbf{r}'(t) = 3 \cos t\mathbf{i} - 3 \sin t\mathbf{j}$$

$$\mathbf{F}(t) = 3 \sin t\mathbf{i} + 3 \cos t\mathbf{j}$$

$$\mathbf{F} \cdot \mathbf{r}' = 9 \sin t \cos t - 9 \sin t \cos t = 0$$

So,  $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$ .

$$\int_C \mathbf{F} \cdot d\mathbf{r} = 0.$$

So,  $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$ .

$$\int_C (x + 3y^2) dy = \int_0^{10} \left(\frac{y}{5} + 3y^2\right) dy = \left[\frac{y^2}{10} + y^3\right]_0^{10} = 1010$$

**52.**  $x = 2t, y = 10t, 0 \leq t \leq 1 \Rightarrow y = 5x, 0 \leq x \leq 2$

$$\int_C (x + 3y^2) dx = \int_0^2 (x + 75x^2) dx = \left[\frac{x^2}{2} + 25x^3\right]_0^2 = 202$$

**53.**  $x = 2t, y = 10t, 0 \leq t \leq 1 \Rightarrow x = \frac{y}{5}, 0 \leq y \leq 10, dx = \frac{1}{5} dy$

$$\int_C xy dx + y dy = \int_0^{10} \left(\frac{y^2}{25} + y\right) dy = \left[\frac{y^3}{75} + \frac{y^2}{2}\right]_0^{10} = \frac{190}{3} \text{ or}$$

$$y = 5x, dy = 5 dx, 0 \leq x \leq 2$$

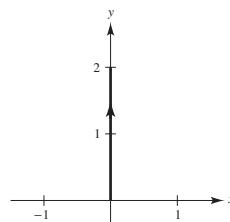
$$\int_C xy dx + y dy = \int_0^2 (5x^2 + 25x) dx = \left[\frac{5x^3}{3} + \frac{25x^2}{2}\right]_0^2 = \frac{190}{3}$$

**56.**  $\mathbf{r}(t) = t\mathbf{j}, 0 \leq t \leq 2$

$$x(t) = 0, y(t) = t$$

$$dx = 0, dy = dt$$

$$\int_C (2x - y) dx + (x + 3y) dy = \int_0^2 3t dt = \left[\frac{3}{2}t^2\right]_0^2 = 6$$



57.  $\mathbf{r}(t) = \begin{cases} t\mathbf{i}, & 0 \leq t \leq 3 \\ 3\mathbf{i} + (t-3)\mathbf{j}, & 3 \leq t \leq 6 \end{cases}$

$C_1: x(t) = t, y(t) = 0,$

$dx = dt, dy = 0$

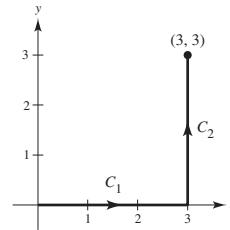
$$\int_{C_1} (2x - y) dx + (x + 3y) dy = \int_0^3 2t dt = 9$$

$C_2: x(t) = 3, y(t) = t - 3$

$dx = 0, dy = dt$

$$\int_{C_2} (2x - y) dx + (x + 3y) dy = \int_3^6 [3 + 3(t-3)] dt = \left[ \frac{3t^2}{2} - 6t \right]_3^6 = \frac{45}{2}$$

$$\int_C (2x - y) dx + (x + 3y) dy = 9 + \frac{45}{2} = \frac{63}{2}$$



58.  $\mathbf{r}(t) = \begin{cases} -t\mathbf{j}, & 0 \leq t \leq 3 \\ (t-3)\mathbf{i} - 3\mathbf{j}, & 3 \leq t \leq 5 \end{cases}$

$C_1: x(t) = 0, y(t) = -t$

$dx = 0, dy = -dt$

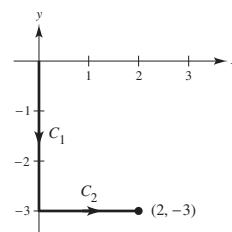
$$\int_{C_1} (2x - y) dx + (x + 3y) dy = \int_0^3 3t dt = \frac{27}{2}$$

$C_2: x(t) = t - 3, y(t) = -3$

$dx = dt, dy = 0$

$$\int_{C_2} (2x - y) dx + (x + 3y) dy = \int_3^5 [2(t-3) + 3] dt = \left[ (t-3)^2 + 3t \right]_3^5 = 10$$

$$\int_C (2x - y) dx + (x + 3y) dy = \frac{27}{2} + 10 = \frac{47}{2}$$



59.  $x(t) = t, y(t) = 1 - t^2, 0 \leq t \leq 1, dx = dt, dy = -2t dt$

$$\int_C (2x - y) dx + (x + 3y) dy = \int_0^1 [(2t - 1 + t^2) + (t + 3 - 3t^2)(-2t)] dt$$

$$= \int_0^1 (6t^3 - t^2 - 4t - 1) dt = \left[ \frac{3t^4}{2} - \frac{t^3}{3} - 2t^2 - t \right]_0^1 = -\frac{11}{6}$$

60.  $x(t) = t, y(t) = t^{3/2}, 0 \leq t \leq 4, dx = dt, dy = \frac{3}{2}t^{1/2} dt$

$$\int_C (2x - y) dx + (x + 3y) dy = \int_0^4 [(2t - t^{3/2}) + (t + 3t^{3/2})(\frac{3}{2}t^{1/2})] dt$$

$$= \int_0^4 (\frac{3}{2}t^2 + \frac{1}{2}t^{3/2} + 2t) dt = \left[ \frac{3}{2}t^3 + \frac{1}{5}t^{5/2} + t^2 \right]_0^4 = 96 + \frac{1}{5}(32) + 16 = \frac{592}{5}$$

61.  $x(t) = t, y(t) = 2t^2, 0 \leq t \leq 2$

$dx = dt, dy = 4t dt$

$$\int_C (2x - y) dx + (x + 3y) dy = \int_0^2 (2t - 2t^2) dt + (t + 6t^2)4t dt = \int_0^2 (24t^3 + 2t^2 + 2t) dt = \left[ 6t^4 + \frac{2}{3}t^3 + t^2 \right]_0^2 = \frac{316}{3}$$

62.  $x(t) = 4 \sin t, y(t) = 3 \cos t, 0 \leq t \leq \frac{\pi}{2}$

$dx = 4 \cos t dt, dy = -3 \sin t dt$

$$\int_C (2x - y) dx + (x + 3y) dy = \int_0^{\pi/2} (8 \sin t - 3 \cos t)(4 \cos t) dt + (4 \sin t + 9 \cos t)(-3 \sin t) dt$$

$$= \int_0^{\pi/2} (5 \sin t \cos t - 12 \cos^2 t - 12 \sin^2 t) dt = \left[ \frac{5}{2} \sin^2 t - 12t \right]_0^{\pi/2} = \frac{5}{2} - 6\pi$$

63.  $f(x, y) = h$

$C$ : line from  $(0, 0)$  to  $(3, 4)$

$$\mathbf{r} = 3t\mathbf{i} + 4t\mathbf{j}, \quad 0 \leq t \leq 1$$

$$\mathbf{r}'(t) = 3\mathbf{i} + 4\mathbf{j}$$

$$\|\mathbf{r}'(t)\| = 5$$

Lateral surface area:

$$\int_C f(x, y) ds = \int_0^1 5h dt = 5h$$

64.  $f(x, y) = y$

$C$ : line from  $(0, 0)$  to  $(4, 4)$

$$\mathbf{r}(t) = t\mathbf{i} + t\mathbf{j}, \quad 0 \leq t \leq 4$$

$$\mathbf{r}'(t) = \mathbf{i} + \mathbf{j}$$

$$\|\mathbf{r}'(t)\| = \sqrt{2}$$

Lateral surface area:

$$\int_C f(x, y) ds = \int_0^4 t(\sqrt{2}) dt = 8\sqrt{2}$$

67.  $f(x, y) = h$

$C$ :  $y = 1 - x^2$  from  $(1, 0)$  to  $(0, 1)$

$$\mathbf{r}(t) = (1-t)\mathbf{i} + \left[1 - (1-t)^2\right]\mathbf{j}, \quad 0 \leq t \leq 1$$

$$\mathbf{r}'(t) = -\mathbf{i} + 2(1-t)\mathbf{j}$$

$$\|\mathbf{r}'(t)\| = \sqrt{1 + 4(1-t)^2}$$

Lateral surface area:

$$\begin{aligned} \int_C f(x, y) ds &= \int_0^1 h\sqrt{1 + 4(1-t)^2} dt = -\frac{h}{4}\left[2(1-t)\sqrt{1 + 4(1-t)^2} + \ln\left|2(1-t) + \sqrt{1 + 4(1-t)^2}\right|\right]_0^1 \\ &= \frac{h}{4}\left[2\sqrt{5} + \ln(2 + \sqrt{5})\right] \approx 1.4789h \end{aligned}$$

68.  $f(x, y) = y + 1$

$C$ :  $y = 1 - x^2$  from  $(1, 0)$  to  $(0, 1)$

$$\mathbf{r}(t) = (1-t)\mathbf{i} + \left[1 - (1-t)^2\right]\mathbf{j}, \quad 0 \leq t \leq 1$$

$$\mathbf{r}'(t) = -\mathbf{i} + 2(1-t)\mathbf{j}$$

$$\|\mathbf{r}'(t)\| = \sqrt{1 + 4(1-t)^2}$$

Lateral surface area:

$$\begin{aligned} \int_C f(x, y) ds &= \int_0^1 \left[2 - (1-t)^2\right]\sqrt{1 + 4(1-t)^2} dt = 2\int_0^1 \sqrt{1 + 4(1-t)^2} dt - \int_0^1 (1-t)^2 \sqrt{1 + 4(1-t)^2} dt \\ &= -\frac{1}{2}\left[2(1-t)\sqrt{1 + 4(1-t)^2} + \ln\left|2(1-t) + \sqrt{1 + 4(1-t)^2}\right|\right]_0^1 \\ &\quad + \frac{1}{64}\left[2(1-t)\left[2(4)(1-t)^2 + 1\right]\sqrt{1 + 4(1-t)^2} - \ln\left|2(1-t) + \sqrt{1 + 4(1-t)^2}\right|\right]_0^1 \\ &= \frac{1}{2}\left[2\sqrt{5} + \ln(2 + \sqrt{5})\right] - \frac{1}{64}\left[18\sqrt{5} - \ln(2 + \sqrt{5})\right] \\ &= \frac{23}{32}\sqrt{5} + \frac{33}{64}\ln(2 + \sqrt{5}) = \frac{1}{64}\left[46\sqrt{5} + 33\ln(2 + \sqrt{5})\right] \approx 2.3515 \end{aligned}$$

65.  $f(x, y) = xy$

$C$ :  $x^2 + y^2 = 1$  from  $(1, 0)$  to  $(0, 1)$

$$\mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j}, \quad 0 \leq t \leq \frac{\pi}{2}$$

$$\mathbf{r}'(t) = -\sin t\mathbf{i} + \cos t\mathbf{j}$$

$$\|\mathbf{r}'(t)\| = 1$$

Lateral surface area:

$$\int_C f(x, y) ds = \int_0^{\pi/2} \cos t \sin t dt = \left[\frac{\sin^2 t}{2}\right]_0^{\pi/2} = \frac{1}{2}$$

66.  $f(x, y) = x + y$

$C$ :  $x^2 + y^2 = 1$  from  $(1, 0)$  to  $(0, 1)$

$$\mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j}, \quad 0 \leq t \leq \frac{\pi}{2}$$

$$\mathbf{r}'(t) = -\sin t\mathbf{i} + \cos t\mathbf{j}$$

$$\|\mathbf{r}'(t)\| = 1$$

Lateral surface area:

$$\begin{aligned} \int_C f(x, y) ds &= \int_0^{\pi/2} (\cos t + \sin t) dt \\ &= [\sin t - \cos t]_0^{\pi/2} = 2 \end{aligned}$$

68.  $f(x, y) = y + 1$

$C$ :  $y = 1 - x^2$  from  $(1, 0)$  to  $(0, 1)$

$$\mathbf{r}(t) = (1-t)\mathbf{i} + \left[1 - (1-t)^2\right]\mathbf{j}, \quad 0 \leq t \leq 1$$

$$\mathbf{r}'(t) = -\mathbf{i} + 2(1-t)\mathbf{j}$$

$$\|\mathbf{r}'(t)\| = \sqrt{1 + 4(1-t)^2}$$

Lateral surface area:

$$\begin{aligned} \int_C f(x, y) ds &= \int_0^1 \left[2 - (1-t)^2\right]\sqrt{1 + 4(1-t)^2} dt = 2\int_0^1 \sqrt{1 + 4(1-t)^2} dt - \int_0^1 (1-t)^2 \sqrt{1 + 4(1-t)^2} dt \\ &= -\frac{1}{2}\left[2(1-t)\sqrt{1 + 4(1-t)^2} + \ln\left|2(1-t) + \sqrt{1 + 4(1-t)^2}\right|\right]_0^1 \\ &\quad + \frac{1}{64}\left[2(1-t)\left[2(4)(1-t)^2 + 1\right]\sqrt{1 + 4(1-t)^2} - \ln\left|2(1-t) + \sqrt{1 + 4(1-t)^2}\right|\right]_0^1 \\ &= \frac{1}{2}\left[2\sqrt{5} + \ln(2 + \sqrt{5})\right] - \frac{1}{64}\left[18\sqrt{5} - \ln(2 + \sqrt{5})\right] \\ &= \frac{23}{32}\sqrt{5} + \frac{33}{64}\ln(2 + \sqrt{5}) = \frac{1}{64}\left[46\sqrt{5} + 33\ln(2 + \sqrt{5})\right] \approx 2.3515 \end{aligned}$$

69.  $f(x, y) = xy$

$C: y = 1 - x^2$  from  $(1, 0)$  to  $(0, 1)$

You could parameterize the curve  $C$  as in Exercises 67 and 68. Alternatively, let  $x = \cos t$ , then:

$$y = 1 - \cos^2 t = \sin^2 t$$

$$\mathbf{r}(t) = \cos t \mathbf{i} + \sin^2 t \mathbf{j}, \quad 0 \leq t \leq \frac{\pi}{2}$$

$$\mathbf{r}'(t) = -\sin t \mathbf{i} + 2 \sin t \cos t \mathbf{j}$$

$$\|\mathbf{r}'(t)\| = \sqrt{\sin^2 t + 4 \sin^2 t \cos^2 t} = \sin t \sqrt{1 + 4 \cos^2 t}$$

Lateral surface area:

$$\int_C f(x, y) ds = \int_0^{\pi/2} \cos t \sin^2 t \left( \sin t \sqrt{1 + 4 \cos^2 t} \right) dt = \int_0^{\pi/2} \sin^2 t \left[ (1 + 4 \cos^2 t)^{1/2} \sin t \cos t \right] dt$$

Let  $u = \sin^2 t$  and  $dv = (1 + 4 \cos^2 t)^{1/2} \sin t \cos t$ , then  $du = 2 \sin t \cos t dt$  and  $v = -\frac{1}{12}(1 + 4 \cos^2 t)^{3/2}$ .

$$\begin{aligned} \int_C f(x, y) ds &= \left[ -\frac{1}{12} \sin^2 t (1 + 4 \cos^2 t)^{3/2} \right]_0^{\pi/2} + \frac{1}{6} \int_0^{\pi/2} (1 + 4 \cos^2 t)^{3/2} \sin t \cos t dt \\ &= \left[ -\frac{1}{12} \sin^2 t (1 + 4 \cos^2 t)^{3/2} - \frac{1}{120} (1 + 4 \cos^2 t)^{5/2} \right]_0^{\pi/2} = \left( -\frac{1}{12} - \frac{1}{120} \right) + \frac{1}{120} (5)^{5/2} = \frac{1}{120} (25\sqrt{5} - 11) \approx 0.3742 \end{aligned}$$

70.  $f(x, y) = x^2 - y^2 + 4$

$C: x^2 + y^2 = 4$

$$\mathbf{r}(t) = 2 \cos t \mathbf{i} + 2 \sin t \mathbf{j}, \quad 0 \leq t \leq 2\pi$$

$$\mathbf{r}'(t) = -2 \sin t \mathbf{i} + 2 \cos t \mathbf{j}$$

$$\|\mathbf{r}'(t)\| = 2$$

Lateral surface area:

$$\int_C f(x, y) ds = \int_0^{2\pi} (4 \cos^2 t - 4 \sin^2 t + 4)(2) dt = 8 \int_0^{2\pi} (1 + \cos 2t) dt = \left[ 8(t + \frac{1}{2} \sin 2t) \right]_0^{2\pi} = 16\pi$$

71. (a)  $f(x, y) = 1 + y^2$

$$\mathbf{r}(t) = 2 \cos t \mathbf{i} + 2 \sin t \mathbf{j}, \quad 0 \leq t \leq 2\pi$$

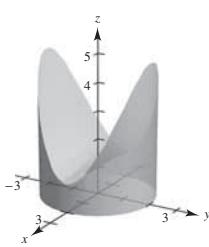
$$\mathbf{r}'(t) = -2 \sin t \mathbf{i} + 2 \cos t \mathbf{j}$$

$$\|\mathbf{r}'(t)\| = 2$$

$$S = \int_C f(x, y) ds = \int_0^{2\pi} (1 + 4 \sin^2 t)(2) dt = \left[ 2t + 4(t - \sin t \cos t) \right]_0^{2\pi} = 12\pi \approx 37.70 \text{ cm}^2$$

(b)  $0.2(12\pi) = \frac{12\pi}{5} \approx 7.54 \text{ cm}^3$

(c)



72.  $f(x, y) = 20 + \frac{1}{4}x$

$C: y = x^{3/2}, 0 \leq x \leq 40$

$$\mathbf{r}(t) = t\mathbf{i} + t^{3/2}\mathbf{j}, 0 \leq t \leq 40$$

$$\mathbf{r}'(t) = \mathbf{i} + \frac{3}{2}t^{1/2}\mathbf{j}$$

$$\|\mathbf{r}'(t)\| = \sqrt{1 + \left(\frac{9}{4}\right)t}$$

Lateral surface area:  $\int_C f(x, y) ds = \int_0^{40} \left(20 + \frac{1}{4}t\right) \sqrt{1 + \left(\frac{9}{4}\right)t} dt$

Let  $u = \sqrt{1 + \left(\frac{9}{4}\right)t}$ , then  $t = \frac{4}{9}(u^2 - 1)$  and  $dt = \frac{8}{9}u du$ .

$$\begin{aligned} \int_0^{40} \left(20 + \frac{1}{4}t\right) \sqrt{1 + \left(\frac{9}{4}\right)t} dt &= \int_1^{\sqrt{91}} \left[20 + \frac{1}{9}(u^2 - 1)\right] \left(u\right) \left(\frac{8}{9}u\right) du = \frac{8}{81} \int_1^{\sqrt{91}} (u^4 + 179u^2) du \\ &= \frac{8}{81} \left[ \frac{u^5}{5} + \frac{179u^3}{3} \right]_1^{\sqrt{91}} = \frac{850,304\sqrt{91} - 7184}{1215} \approx 6670.12 \end{aligned}$$

73.  $\mathbf{r}(t) = a \cos t\mathbf{i} + a \sin t\mathbf{j}, 0 \leq t \leq 2\pi$

$$\mathbf{r}'(t) = -a \sin t\mathbf{i} + a \cos t\mathbf{j}, \|\mathbf{r}'(t)\| = a$$

$$\begin{aligned} I_x &= \int_C y^2 \rho(x, y) ds = \int_0^{2\pi} (a^2 \sin^2 t)(1)a dt \\ &= a^3 \int_0^{2\pi} \sin^2 t dt = a^3 \pi \end{aligned}$$

$$\begin{aligned} I_y &= \int_C x^2 \rho(x, y) ds = \int_0^{2\pi} (a^2 \cos^2 t)(1)a dt \\ &= a^3 \int_0^{2\pi} \cos^2 t dt = a^3 \pi \end{aligned}$$

74.  $\mathbf{r}(t) = a \cos t\mathbf{i} + a \sin t\mathbf{j}, 0 \leq t \leq 2\pi$

$$\mathbf{r}'(t) = -a \sin t\mathbf{i} + a \cos t\mathbf{j}, \|\mathbf{r}'(t)\| = a$$

$$\begin{aligned} I_x &= \int_C y^2 \rho(x, y) ds = \int_0^{2\pi} (a^2 \sin^2 t)(\sin t)a^2 dt \\ &= a^4 \int_0^{2\pi} \sin^3 t dt = 0 \end{aligned}$$

$$\begin{aligned} I_y &= \int_C x^2 \rho(x, y) ds = \int_0^{2\pi} (a^2 \cos^2 t)(\sin t)a^2 dt \\ &= a^4 \int_0^{2\pi} \cos^2 t \sin t dt = 0 \end{aligned}$$

75. (a) Graph of:  $\mathbf{r}(t) = 3 \cos t\mathbf{i} + 3 \sin t\mathbf{j} + (1 + \sin^2 2t)\mathbf{k}, 0 \leq t \leq 2\pi$

For  $y = b$  constant,  $3 \sin t = b \Rightarrow \sin t = \frac{b}{3}$  and

$$\begin{aligned} 1 + \sin^2 2t &= 1 + (2 \sin t \cos t)^2 \\ &= 1 + 4 \sin^2 t \cos^2 t \\ &= 1 + 4 \sin^2 t (1 - \sin^2 t) = 1 + \frac{4}{9}b^2 \left(1 - \frac{b^2}{9}\right). \end{aligned}$$



(b) Consider the portion of the surface in the first quadrant. The curve  $z = 1 + \sin^2 2t$  is over the curve

$\mathbf{r}_1(t) = 3 \cos t\mathbf{i} + 3 \sin t\mathbf{j}, 0 \leq t \leq \pi/2$ . So, the total lateral surface area is

$$4 \int_C f(x, y) ds = 4 \int_0^{\pi/2} (1 + \sin^2 2t) 3 dt = 12 \left( \frac{3\pi}{4} \right) = 9\pi \text{ cm}^2.$$

(c) The cross sections parallel to the  $xz$ -plane are rectangles of height  $1 + 4(y/3)^2(1 - y^2/9)$  and base  $2\sqrt{9 - y^2}$ . So,

$$\text{Volume} = 2 \int_0^3 2\sqrt{9 - y^2} \left(1 + 4\frac{y^2}{9} \left(1 - \frac{y^2}{9}\right)\right) dy = \frac{27\pi}{2} \approx 42.412 \text{ cm}^3.$$

76.  $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j}, \quad 0 \leq t \leq 1$

$$\mathbf{r}'(t) = \mathbf{i} + 2t\mathbf{j}$$

$$\begin{aligned}\int_C \mathbf{F} \cdot d\mathbf{r} &\approx \frac{1-0}{3(4)} [5 + 4(4) + 2(4) + 4(6) + 11] \\ &= \frac{16}{3}\end{aligned}$$

$(x, y)$	$(0, 0)$	$\left(\frac{1}{4}, \frac{1}{16}\right)$	$\left(\frac{1}{2}, \frac{1}{4}\right)$	$\left(\frac{3}{4}, \frac{9}{16}\right)$	$(1, 1)$
$\mathbf{F}(x, y)$	$5\mathbf{i}$	$3.5\mathbf{i} + \mathbf{j}$	$2\mathbf{i} + 2\mathbf{j}$	$1.5\mathbf{i} + 3\mathbf{j}$	$\mathbf{i} + 5\mathbf{j}$
$\mathbf{r}'(t)$	$\mathbf{i}$	$\mathbf{i} + 0.5\mathbf{j}$	$\mathbf{i} + \mathbf{j}$	$\mathbf{i} + 1.5\mathbf{j}$	$\mathbf{i} + 2\mathbf{j}$
$\mathbf{F} \cdot \mathbf{r}'$	5	4	4	6	11

77.  $\mathbf{r}(t) = 3 \sin t\mathbf{i} + 3 \cos t\mathbf{j} + \frac{10}{2\pi} t\mathbf{k}, \quad 0 \leq t \leq 2\pi$

$$\mathbf{F} = 175\mathbf{k}$$

$$d\mathbf{r} = \left(3 \cos t\mathbf{i} - 3 \sin t\mathbf{j} + \frac{10}{2\pi}\mathbf{k}\right)dt$$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} \frac{1750}{2\pi} dt = \left[\frac{1750}{2\pi}t\right]_0^{2\pi} = 1750 \text{ ft} \cdot \text{lb}$$

78.  $W = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_C M dx + N dy$

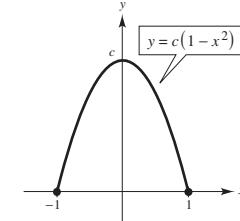
$$M = 15(4 - x^2)y = 60 - 15x^2(c - cx^2)$$

$$N = -15xy = -15x(c - cx^2)$$

$$dx = dx, dy = -2cx dx$$

$$W = \int_{-1}^1 \left[60 - 15x^2(c - cx^2) + (-15x(c - cx^2))(-2cx)\right] dx = 120 - 4c + 8c^2 \quad (\text{parabola})$$

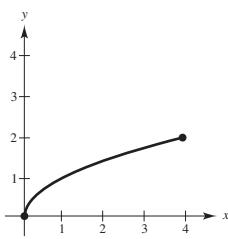
$W' = 16c - 4 = 0 \Rightarrow c = \frac{1}{4}$  yields the minimum work, 119.5. Along the straight line path,  $y = 0$ , the work is 120.



79. See the definition of Line Integral, page 1070. See Theorem 15.4.

80. See the definition, page 1074.

81. The greater the height of the surface over the curve, the greater the lateral surface area. So,  $z_3 < z_1 < z_2 < z_4$ .



82. (a) Work = 0

(b) Work is negative, because against force field.

(c) Work is positive, because with force field.

83. False

$$\int_C xy ds = \sqrt{2} \int_0^1 t^2 dt$$

84. False, the orientation of  $C$  does not affect the form.

$$\int_C f(x, y) ds.$$

85. False, the orientations are different.

86. False. For example, see Exercise 32.

87.  $\mathbf{F}(x, y) = (y - x)\mathbf{i} + xy\mathbf{j}$

$$\mathbf{r}(t) = kt(1-t)\mathbf{i} + t\mathbf{j}, \quad 0 \leq t \leq 1$$

$$\mathbf{r}'(t) = k(1-2t)\mathbf{i} + \mathbf{j}$$

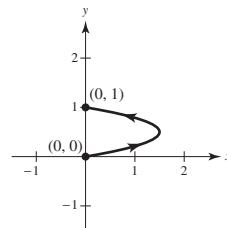
$$\text{Work} = 1 = \int_C \mathbf{F} \cdot d\mathbf{r}$$

$$= \int_0^1 [(t - kt(1-t))\mathbf{i} + kt^2(1-t)\mathbf{j}] \cdot [k(1-2t)\mathbf{i} + \mathbf{j}] dt$$

$$= \int_0^1 [(t - kt(1-t))k(1-2t) + kt^2(1-t)] dt$$

$$= \int_0^1 (-2k^2t^3 - kt^3 - kt^2 + 3k^2t^2 - k^2t + kt) dt = \frac{-k}{12}$$

$$k = -12$$



### Section 15.3 Conservative Vector Fields and Independence of Path

1.  $\mathbf{F}(x, y) = x^2\mathbf{i} + xy\mathbf{j}$

(a)  $\mathbf{r}_1(t) = t\mathbf{i} + t^2\mathbf{j}, \quad 0 \leq t \leq 1$

$$\mathbf{r}'_1(t) = \mathbf{i} + 2t\mathbf{j}$$

$$\mathbf{F}(t) = t^2\mathbf{i} + t^3\mathbf{j}$$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 (t^2 + 2t^4) dt = \frac{11}{15}$$

(b)  $\mathbf{r}_2(\theta) = \sin \theta \mathbf{i} + \sin^2 \theta \mathbf{j}, \quad 0 \leq \theta \leq \frac{\pi}{2}$

$$\mathbf{r}'_2(\theta) = \cos \theta \mathbf{i} + 2 \sin \theta \cos \theta \mathbf{j}$$

$$\mathbf{F}(t) = \sin^2 \theta \mathbf{i} + \sin^3 \theta \mathbf{j}$$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{\pi/2} (\sin^2 \theta \cos \theta + 2 \sin^4 \theta \cos \theta) d\theta = \left[ \frac{\sin^3 \theta}{3} + \frac{2 \sin^5 \theta}{5} \right]_0^{\pi/2} = \frac{11}{15}$$

2.  $\mathbf{F}(x, y) = (x^2 + y^2)\mathbf{i} - x\mathbf{j}$

(a)  $\mathbf{r}_1(t) = t\mathbf{i} + \sqrt{t}\mathbf{j}, \quad 0 \leq t \leq 4$

$$\mathbf{r}'_1(t) = \mathbf{i} + \frac{1}{2\sqrt{t}}\mathbf{j}$$

$$\mathbf{F}(t) = (t^2 + t)\mathbf{i} - t\mathbf{j}$$

$$\begin{aligned} \int_C \mathbf{F} \cdot d\mathbf{r} &= \int_0^4 \left( t^2 + t - \frac{1}{2}\sqrt{t} \right) dt \\ &= \left[ \frac{t^3}{3} + \frac{t^2}{2} - \frac{t^{3/2}}{3} \right]_0^4 = \frac{80}{3} \end{aligned}$$

(b)  $\mathbf{r}_2(w) = w^2\mathbf{i} + w\mathbf{j}, \quad 0 \leq w \leq 2$

$$\mathbf{r}'_2(w) = 2w\mathbf{i} + \mathbf{j}$$

$$\mathbf{F}(w) = (w^4 + w^2)\mathbf{i} - w^2\mathbf{j}$$

$$\begin{aligned} \int_C \mathbf{F} \cdot d\mathbf{r} &= \int_0^2 \left[ 2w(w^4 + w^2) - w^2 \right] dw \\ &= \left[ \frac{w^6}{3} + \frac{w^4}{2} - \frac{w^3}{3} \right]_0^2 = \frac{80}{3} \end{aligned}$$

3.  $\mathbf{F}(x, y) = y\mathbf{i} - x\mathbf{j}$

(a)  $\mathbf{r}_1(\theta) = \sec \theta \mathbf{i} + \tan \theta \mathbf{j}, \quad 0 \leq \theta \leq \frac{\pi}{3}$

$$\mathbf{r}'_1(\theta) = \sec \theta \tan \theta \mathbf{i} + \sec^2 \theta \mathbf{j}$$

$$\mathbf{F}(\theta) = \tan \theta \mathbf{i} - \sec \theta \mathbf{j}$$

$$\begin{aligned} \int_C \mathbf{F} \cdot d\mathbf{r} &= \int_0^{\pi/3} (\sec \theta \tan^2 \theta - \sec^3 \theta) d\theta \\ &= \int_0^{\pi/3} [\sec \theta (\sec^2 \theta - 1) - \sec^3 \theta] d\theta \\ &= - \int_0^{\pi/3} \sec \theta d\theta \\ &= [-\ln |\sec \theta + \tan \theta|]_0^{\pi/3} \\ &= -\ln(2 + \sqrt{3}) \approx -1.317 \end{aligned}$$

(b)  $\mathbf{r}_2(t) = \sqrt{t+1}\mathbf{i} + \sqrt{t}\mathbf{j}, \quad 0 \leq t \leq 3$

$$\mathbf{r}'_2(t) = \frac{1}{2\sqrt{t+1}}\mathbf{i} + \frac{1}{2\sqrt{t}}\mathbf{j}$$

$$\mathbf{F}(t) = \sqrt{t}\mathbf{i} - \sqrt{t+1}\mathbf{j}$$

$$\begin{aligned} \int_C \mathbf{F} \cdot d\mathbf{r} &= \int_0^3 \left[ \frac{\sqrt{t}}{2\sqrt{t+1}} - \frac{\sqrt{t+1}}{2\sqrt{t}} \right] dt \\ &= -\frac{1}{2} \int_0^3 \frac{1}{\sqrt{t}\sqrt{t+1}} dt \\ &= -\frac{1}{2} \int_0^3 \frac{1}{\sqrt{t^2 + t + (1/4) - (1/4)}} dt \\ &= -\frac{1}{2} \int_0^3 \frac{1}{\sqrt{[t + (1/2)]^2 - (1/4)}} dt \\ &= \left[ -\frac{1}{2} \ln \left| \left( t + \frac{1}{2} \right) + \sqrt{t^2 + t} \right| \right]_0^3 \\ &= -\frac{1}{2} \left[ \ln \left( \frac{7}{2} + 2\sqrt{3} \right) - \ln \left( \frac{1}{2} \right) \right] \\ &= -\frac{1}{2} \ln(7 + 4\sqrt{3}) \approx -1.317 \end{aligned}$$

4.  $\mathbf{F}(x, y) = y\mathbf{i} + x^2\mathbf{j}$

(a)  $\mathbf{r}_1(t) = (2+t)\mathbf{i} + (3-t)\mathbf{j}, \quad 0 \leq t \leq 3$

$$\mathbf{r}'_1(t) = \mathbf{i} - \mathbf{j}$$

$$\mathbf{F}(t) = (3-t)\mathbf{i} + (2+t)^2\mathbf{j}$$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^3 [(3-t) - (2+t)^2] dt = \left[ -\frac{(3-t)^2}{2} - \frac{(2+t)^3}{3} \right]_0^3 = -\frac{69}{2}$$

(b)  $\mathbf{r}_2(w) = (2 + \ln w)\mathbf{i} + (3 - \ln w)\mathbf{j}, \quad 1 \leq w \leq e^3$

$$\mathbf{r}'_2(w) = \frac{1}{w}\mathbf{i} - \frac{1}{w}\mathbf{j}$$

$$\mathbf{F}(w) = (3 - \ln w)\mathbf{i} + (2 + \ln w)^2\mathbf{j}$$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_1^{e^3} \left[ (3 - \ln w) \left( \frac{1}{w} \right) - (2 + \ln w)^2 \left( \frac{1}{w} \right) \right] dw = \left[ -\frac{(3 - \ln w)^2}{2} - \frac{(2 + \ln w)^3}{3} \right]_1^{e^3} = -\frac{69}{2}$$

5.  $\mathbf{F}(x, y) = e^x \sin y\mathbf{i} + e^x \cos y\mathbf{j}$

$$\frac{\partial N}{\partial x} = e^x \cos y \quad \frac{\partial M}{\partial y} = e^x \cos y$$

Because  $\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}$ ,  $\mathbf{F}$  is conservative.

6.  $\mathbf{F}(x, y) = 15x^2y^2\mathbf{i} + 10x^3y\mathbf{j}$

$$\frac{\partial N}{\partial x} = 30x^2y \quad \frac{\partial M}{\partial y} = 30x^2y$$

Because  $\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}$ ,  $\mathbf{F}$  is conservative.

7.  $\mathbf{F}(x, y) = \frac{1}{y}\mathbf{i} + \frac{x}{y^2}\mathbf{j}$

$$\frac{\partial N}{\partial x} = \frac{1}{y^2} \quad \frac{\partial M}{\partial y} = -\frac{1}{y^2}$$

Because  $\frac{\partial N}{\partial x} \neq \frac{\partial M}{\partial y}$ ,  $\mathbf{F}$  is not conservative.

8.  $\mathbf{F}(x, y, z) = y \ln z\mathbf{i} - x \ln z\mathbf{j} + \frac{xy}{z}\mathbf{k}$

$\operatorname{curl} \mathbf{F} \neq \mathbf{0}$  so  $\mathbf{F}$  is not conservative.

$$\left( \frac{\partial P}{\partial y} = \frac{x}{z} \neq -\frac{x}{z} = \frac{\partial N}{\partial z} \right)$$

9.  $\mathbf{F}(x, y, z) = y^2z\mathbf{i} + 2xyz\mathbf{j} + xy^2\mathbf{k}$

$\operatorname{curl} \mathbf{F} = \mathbf{0} \Rightarrow \mathbf{F}$  is conservative.

10.  $\mathbf{F}(x, y, z) = \sin(yz)\mathbf{i} + xz \cos(yz)\mathbf{j} + xy \sin(yz)\mathbf{k}$

$\operatorname{curl} \mathbf{F} \neq \mathbf{0}$ , so  $\mathbf{F}$  is not conservative.

11.  $\mathbf{F}(x, y) = 2xy\mathbf{i} + x^2\mathbf{j}$

(a)  $\mathbf{r}_1(t) = t\mathbf{i} + t^2\mathbf{j}, \quad 0 \leq t \leq 1$

$$\mathbf{r}'_1(t) = \mathbf{i} + 2t\mathbf{j}$$

$$\mathbf{F}(t) = 2t^3\mathbf{i} + t^2\mathbf{j}$$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 4t^3 dt = 1$$

(b)  $\mathbf{r}_2(t) = t\mathbf{i} + t^3\mathbf{j}, \quad 0 \leq t \leq 1$

$$\mathbf{r}'_2(t) = \mathbf{i} + 3t^2\mathbf{j}$$

$$\mathbf{F}(t) = 2t^4\mathbf{i} + t^2\mathbf{j}$$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 5t^4 dt = 1$$

12.  $\mathbf{F}(x, y) = ye^{xy}\mathbf{i} + xe^{xy}\mathbf{j}$

(a)  $\mathbf{r}_1(t) = t\mathbf{i} - (t-3)\mathbf{j}, \quad 0 \leq t \leq 3$

$$\mathbf{r}'_1(t) = \mathbf{i} - \mathbf{j}$$

$$\mathbf{F}(t) = -(t-3)e^{3t-t^2}\mathbf{i} + te^{3t-t^2}\mathbf{j}$$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^3 \left[ -(t-3)e^{3t-t^2} - te^{3t-t^2} \right] dt$$

$$= \int_0^3 e^{3t-t^2} (3-2t) dt$$

$$= \left[ e^{3t-t^2} \right]_0^3 = e^0 - e^0 = 0$$

(b)  $\mathbf{F}(x, y)$  is conservative because

$$\frac{\partial M}{\partial y} = xye^{xy} + e^{xy} = \frac{\partial N}{\partial x}$$

The potential function is  $f(x, y) = e^{xy} + K$ .

By Theorem 15.7,  $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$ .

13.  $\mathbf{F}(x, y) = y\mathbf{i} - x\mathbf{j}$

(a)  $\mathbf{r}_1(t) = t\mathbf{i} + t\mathbf{j}, \quad 0 \leq t \leq 1$

$$\mathbf{r}'_1(t) = \mathbf{i} + \mathbf{j}$$

$$\mathbf{F}(t) = t\mathbf{i} - t\mathbf{j}$$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = 0$$

(b)  $\mathbf{r}_2(t) = t\mathbf{i} + t^2\mathbf{j}, \quad 0 \leq t \leq 1$

$$\mathbf{r}'_2(t) = \mathbf{i} + 2t\mathbf{j}$$

$$\mathbf{F}(t) = t^2\mathbf{i} - t\mathbf{j}$$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 -t^2 dt = -\frac{1}{3}$$

(c)  $\mathbf{r}_3(t) = t\mathbf{i} + t^3\mathbf{j}, \quad 0 \leq t \leq 1$

$$\mathbf{r}'_3(t) = \mathbf{i} + 3t^2\mathbf{j}$$

$$\mathbf{F}(t) = t^3\mathbf{i} - t\mathbf{j}$$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 -2t^3 dt = -\frac{1}{2}$$

14.  $\mathbf{F}(x, y) = xy^2\mathbf{i} + 2x^2y\mathbf{j}$

(a)  $\mathbf{r}_1(t) = t\mathbf{i} + \frac{1}{t}\mathbf{j}, \quad 1 \leq t \leq 3$

$$\mathbf{r}'_1(t) = \mathbf{i} - \frac{1}{t^2}\mathbf{j}$$

$$\mathbf{F}(t) = \frac{1}{t}\mathbf{i} + 2t\mathbf{j}$$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_1^3 -\frac{1}{t} dt = [-\ln|t|]_1^3 = -\ln 3$$

(b)  $\mathbf{r}_2(t) = (t+1)\mathbf{i} - \frac{1}{3}(t-3)\mathbf{j}, \quad 0 \leq t \leq 2$

$$\mathbf{r}'_2(t) = \mathbf{i} - \frac{1}{3}\mathbf{j}$$

$$\mathbf{F}(t) = \frac{1}{9}(t+1)(t-3)^2\mathbf{i} - \frac{2}{3}(t+1)^2(t-3)\mathbf{j}$$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^2 \left[ \frac{1}{9}(t+1)(t-3)^2 + \frac{2}{9}(t+1)^2(t-3) \right] dt$$

$$= \frac{1}{9} \int_0^2 (3t^3 - 7t^2 - 7t + 3) dt \\ = \frac{1}{9} \left[ \frac{3t^4}{4} - \frac{7t^3}{3} - \frac{7t^2}{2} + 3t \right]_0^2 = -\frac{44}{27}$$

15.  $\int_C y^2 dx + 2xy dy$

Because  $\partial M/\partial y = \partial N/\partial x = 2y$ ,  $\mathbf{F}(x, y) = y^2\mathbf{i} + 2xy\mathbf{j}$  is conservative. The potential function is  $f(x, y) = xy^2 + k$ . So, you can use the Fundamental Theorem of Line Integrals.

(a)  $\int_C y^2 dx + 2xy dy = [x^2 y]_{(0,0)}^{(4,4)} = 64$

(b)  $\int_C y^2 dx + 2xy dy = [x^2 y]_{(-1,0)}^{(1,0)} = 0$

(c) and (d) Because  $C$  is a closed curve,  $\int_C y^2 dx + 2xy dy = 0$ .

16.  $\int_C (2x - 3y + 1) dx - (3x + y - 5) dy$

Because  $\partial M/\partial y = \partial N/\partial x = -3$ ,  $\mathbf{F}(x, y) = (2x - 3y + 1)\mathbf{i} - (3x + y - 5)\mathbf{j}$  is conservative. The potential function is  $f(x, y) = x^2 - 3xy - (y^2/2) + x + 5y + k$ .

(a) and (d) Because  $C$  is a closed curve,  $\int_C (2x - 3y + 1) dx - (3x + y - 5) dy = 0$ .

(b)  $\int_C (2x - 3y + 1) dx - (3x + y - 5) dy = \left[ x^2 - 3xy - \frac{y^2}{2} + x + 5y \right]_{(0,-1)}^{(0,1)} = 10$

(c)  $\int_C (2x - 3y + 1) dx - (3x + y - 5) dy = \left[ x^2 - 3xy - \frac{y^2}{2} + x + 5y \right]_{(0,1)}^{(2,e^2)} = \frac{1}{2}(3 - 2e^2 - e^4)$

17.  $\int_C 2xy \, dx + (x^2 + y^2) \, dy$

Because  $\partial M/\partial y = \partial N/\partial x = 2x$ ,  $\mathbf{F}(x, y) = 2xy\mathbf{i} + (x^2 + y^2)\mathbf{j}$  is conservative.

The potential function is  $f(x, y) = x^2y + \frac{y^3}{3} + k$ .

$$(a) \int_C 2xy \, dx + (x^2 + y^2) \, dy = \left[ x^2y + \frac{y^3}{3} \right]_{(5,0)}^{(0,4)} = \frac{64}{3}$$

$$(b) \int_C 2xy \, dx + (x^2 + y^2) \, dy = \left[ x^2y + \frac{y^3}{3} \right]_{(2,0)}^{(0,4)} = \frac{64}{3}$$

18.  $\int_C (x^2 + y^2) \, dx + 2xy \, dy$

Because  $\partial M/\partial y = \partial N/\partial x = 2y$ ,  $\mathbf{F}(x, y) = (x^2 + y^2)\mathbf{i} + 2xy\mathbf{j}$  is conservative. The potential function is  $f(x, y) = (x^3/3) + xy^2 + k$ .

$$(a) \int_C (x^2 + y^2) \, dx + 2xy \, dy = \left[ \frac{x^3}{3} + xy^2 \right]_{(0,0)}^{(8,4)} = \frac{896}{3}$$

$$(b) \int_C (x^2 + y^2) \, dx + 2xy \, dy = \left[ \frac{x^3}{3} + xy^2 \right]_{(2,0)}^{(0,2)} = -\frac{8}{3}$$

19.  $\mathbf{F}(x, y, z) = xyz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}$

Because  $\text{curl } \mathbf{F} = \mathbf{0}$ ,  $\mathbf{F}(x, y, z)$  is conservative. The potential function is  $f(x, y, z) = xyz + k$ .

$$(a) \mathbf{r}_1(t) = t\mathbf{i} + 2\mathbf{j} + t\mathbf{k}, \quad 0 \leq t \leq 4$$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = [xyz]_{(0,2,0)}^{(4,2,4)} = 32$$

$$(b) \mathbf{r}_2(t) = t^2\mathbf{i} + t\mathbf{j} + t^2\mathbf{k}, \quad 0 \leq t \leq 2$$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = [xyz]_{(0,0,0)}^{(4,2,4)} = 32$$

21.  $\mathbf{F}(x, y, z) = (2y + x)\mathbf{i} + (x^2 - z)\mathbf{j} + (2y - 4z)\mathbf{k}$

$\mathbf{F}(x, y, z)$  is not conservative.

$$(a) \mathbf{r}_1(t) = t\mathbf{i} + t^2\mathbf{j} + \mathbf{k}, \quad 0 \leq t \leq 1$$

$$\mathbf{r}_1'(t) = \mathbf{i} + 2t\mathbf{j}$$

$$\mathbf{F}(t) = (2t^2 + t)\mathbf{i} + (t^2 - 1)\mathbf{j} + (2t^2 - 4)\mathbf{k}$$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 (2t^3 + 2t^2 - t) \, dt = \frac{2}{3}$$

$$(b) \mathbf{r}_2(t) = t\mathbf{i} + t\mathbf{j} + (2t - 1)^2\mathbf{k}, \quad 0 \leq t \leq 1$$

$$\mathbf{r}_2'(t) = \mathbf{i} + \mathbf{j} + 4(2t - 1)\mathbf{k}$$

$$\mathbf{F}(t) = 3t\mathbf{i} + \left[ t^2 - (2t - 1)^2 \right] \mathbf{j} + \left[ 2t - 4(2t - 1)^2 \right] \mathbf{k}$$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 \left[ 3t + t^2 - (2t - 1)^2 + 8t(2t - 1) - 16(2t - 1)^3 \right] dt$$

$$= \int_0^1 \left[ 17t^2 - 5t - (2t - 1)^2 - 16(2t - 1)^3 \right] dt = \left[ \frac{17t^3}{3} - \frac{5t^2}{2} - \frac{(2t - 1)^3}{6} - 2(2t - 1)^4 \right]_0^1 = \frac{17}{6}$$

20.  $\mathbf{F}(x, y, z) = \mathbf{i} + z\mathbf{j} + y\mathbf{k}$

Because  $\text{curl } \mathbf{F} = \mathbf{0}$ ,  $\mathbf{F}(x, y, z)$  is conservative. The potential function is  $f(x, y, z) = x + yz + k$ .

$$(a) \mathbf{r}_1(t) = \cos t\mathbf{i} + \sin t\mathbf{j} + t^2\mathbf{k}, \quad 0 \leq t \leq \pi$$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = [x + yz]_{(1,0,0)}^{(-1,0,\pi^2)} = -2$$

$$(b) \mathbf{r}_2(t) = (1 - 2t)\mathbf{i} + \pi^2 t\mathbf{k}, \quad 0 \leq t \leq 1$$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = [x + yz]_{(1,0,0)}^{(-1,0,\pi^2)} = -2$$

22.  $\mathbf{F}(x, y, z) = -y\mathbf{i} + x\mathbf{j} + 3xz^2\mathbf{k}$

$\mathbf{F}(x, y, z)$  is not conservative.

(a)  $\mathbf{r}_1(t) = \cos t\mathbf{i} + \sin t\mathbf{j} + t\mathbf{k}, \quad 0 \leq t \leq \pi$

$$\mathbf{r}'_1(t) = -\sin t\mathbf{i} + \cos t\mathbf{j} + \mathbf{k}$$

$$\mathbf{F}(t) = -\sin t\mathbf{i} + \cos t\mathbf{j} + 3t^2 \cos t\mathbf{k}$$

$$\begin{aligned}\int_C \mathbf{F} \cdot d\mathbf{r} &= \int_0^\pi [\sin^2 t + \cos^2 t + 3t^2 \cos t] dt \\ &= \int_0^\pi [1 + 3t^2 \cos t] dt \\ &= [t]_0^\pi + 3[t^2 \sin t]_0^\pi - 6 \int_0^\pi t \sin t dt \\ &= [t + 3t^2 \sin t - 6(\sin t - t \cos t)]_0^\pi \\ &= -5\pi\end{aligned}$$

(b)  $\mathbf{r}_2(t) = (1 - 2t)\mathbf{i} + \pi t\mathbf{k}, \quad 0 \leq t \leq 1$

$$\mathbf{r}'_2(t) = -2\mathbf{i} + \pi\mathbf{k}$$

$$\mathbf{F}(t) = (1 - 2t)\mathbf{j} + 3\pi^2 t^2 (1 - 2t)\mathbf{k}$$

$$\begin{aligned}\int_C \mathbf{F} \cdot d\mathbf{r} &= \int_0^1 3\pi^3 t^2 (1 - 2t) dt \\ &= 3\pi^3 \int_0^1 (t^2 - 2t^3) dt \\ &= 3\pi^3 \left[ \frac{t^3}{3} - \frac{t^4}{2} \right]_0^1 = -\frac{\pi^3}{2}\end{aligned}$$

23.  $\mathbf{F}(x, y, z) = e^z(y\mathbf{i} + x\mathbf{j} + xy\mathbf{k})$

$\mathbf{F}(x, y, z)$  is conservative. The potential function is

$$f(x, y, z) = xye^z + k.$$

(a)  $\mathbf{r}_1(t) = 4 \cos t\mathbf{i} + 4 \sin t\mathbf{j} + 3\mathbf{k}, \quad 0 \leq t \leq \pi$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = [xye^z]_{(4, 0, 3)}^{(-4, 0, 3)} = 0$$

(b)  $\mathbf{r}_2(t) = (4 - 8t)\mathbf{i} + 3\mathbf{k}, \quad 0 \leq t \leq 1$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = [xye^z]_{(4, 0, 3)}^{(-4, 0, 3)} = 0$$

24.  $\mathbf{F}(x, y, z) = y \sin z\mathbf{i} + x \sin z\mathbf{j} + xy \cos x\mathbf{k}$

(a)  $\mathbf{r}_1(t) = t^2\mathbf{i} + t^2\mathbf{j}, \quad 0 \leq t \leq 2$

$$\mathbf{r}'_1(t) = 2t\mathbf{i} + 2t\mathbf{j}$$

$$\mathbf{F}(t) = t^4 \cos t^2\mathbf{k}$$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^2 0 dt = 0$$

(b)  $\mathbf{r}_2(t) = 4t\mathbf{i} + 4t\mathbf{j}, \quad 0 \leq t \leq 1$

$$\mathbf{r}'_2(t) = 4\mathbf{i} + 4\mathbf{j}$$

$$\mathbf{F}(t) = 16t^2 \cos(4t)\mathbf{k}$$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 0 dt = 0$$

25.  $\int_C (3y\mathbf{i} + 3x\mathbf{j}) \cdot d\mathbf{r} = [3xy]_{(0,0)}^{(3,8)} = 72$

26.  $\int_C [2(x+y)\mathbf{i} + 2(x+y)\mathbf{j}] \cdot d\mathbf{r} = [(x+y)^2]_{(-1,1)}^{(3,2)} = 5^2 - 0 = 25$

27.  $\int_C \cos x \sin y dx + \sin x \cos y dy = [\sin x \sin y]_{(0,-\pi)}^{(3\pi/2, \pi/2)} = -1$

28.  $\int_C \frac{y dx - x dy}{x^2 + y^2} = \left[ \arctan\left(\frac{x}{y}\right) \right]_{(1,1)}^{(2\sqrt{3}, 2)} = \frac{\pi}{3} - \frac{\pi}{4} = \frac{\pi}{12}$

29.  $\int_C e^x \sin y dx + e^x \cos y dy = [e^x \sin y]_{(0,0)}^{(2\pi, 0)} = 0$

30.  $\int_C \frac{2x}{(x^2 + y^2)^2} dx + \frac{2y}{(x^2 + y^2)^2} dy = \left[ -\frac{1}{x^2 + y^2} \right]_{(7,5)}^{(1,5)} = -\frac{1}{26} + \frac{1}{74} = \frac{-12}{481}$

31.  $\int_C (z + 2y) dx + (2x - z) dy + (x - y) dz$

$\mathbf{F}(x, y, z)$  is conservative and the potential function is  $f(x, y, z) = xz + 2xy - yz$

(a)  $[xz + 2xy - yz]_{(0,0,0)}^{(1,1,1)} = 2 - 0 = 2$

(b)  $[xz + 2xy - yz]_{(0,0,0)}^{(0,0,1)} + [xz + 2xy - yz]_{(0,0,1)}^{(1,1,1)} = 0 + 2 = 2$

(c)  $[xz + 2xy - yz]_{(0,0,0)}^{(1,0,0)} + [xz + 2xy - yz]_{(1,0,0)}^{(1,1,0)} + [xz + 2xy - yz]_{(1,1,0)}^{(1,1,1)} = 0 + 2 + (2 - 2) = 2$

32.  $\int_C zy \, dx + xz \, dy + xy \, dz$

**Note:** Because  $\mathbf{F}(x, y, z) = yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}$  is conservative and the potential function is  $f(x, y, z) = xyz + k$ , the integral is independent of path as illustrated below.

(a)  $[xyz]_{(0,0,0)}^{(1,1,1)} = 1$

(b)  $[xyz]_{(0,0,0)}^{(0,0,1)} + [xyz]_{(0,0,1)}^{(1,1,1)} = 0 + 1 = 1$

(c)  $[xyz]_{(0,0,0)}^{(1,0,0)} + [xyz]_{(1,0,0)}^{(1,1,0)} + [xyz]_{(1,1,0)}^{(1,1,1)} = 0 + 0 + 1 = 1$

33.  $\int_C -\sin x \, dx + z \, dy + y \, dz = [\cos x + yz]_{(0,0,0)}^{(\pi/2,3,4)} = 12 - 1 = 11$

34.  $\mathbf{F}(x, y, z)$  is conservative:  $f(x, y, z) = 3x^2 - 4yz + 10z^2$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = [3x^2 - 4yz + 10z^2]_{(0,0,0)}^{(3,4,0)} = 27$$

35.  $\mathbf{F}(x, y) = 9x^2y^2\mathbf{i} + (6x^3y - 1)\mathbf{j}$  is conservative.

$$\text{Work} = [3x^3y^2 - y]_{(0,0)}^{(5,9)} = 30,366$$

36.  $\mathbf{F}(x, y)$  is conservative.  $f(x, y) = \frac{x^2}{y}$

$$\text{Work} = \left[ \frac{x^2}{y} \right]_{(-1,1)}^{(3,2)} = \frac{9}{2} - 1 = \frac{7}{2}$$

37.  $\mathbf{r}(t) = 2 \cos 2\pi t \mathbf{i} + 2 \sin 2\pi t \mathbf{j}$

$$\mathbf{r}'(t) = -4\pi \sin 2\pi t \mathbf{i} + 4\pi \cos 2\pi t \mathbf{j}$$

$$\mathbf{a}(t) = -8\pi^2 \cos 2\pi t \mathbf{i} - 8\pi^2 \sin 2\pi t \mathbf{j}$$

$$\mathbf{F}(t) = m\mathbf{a}(t) = \frac{1}{32}\mathbf{a}(t) = -\frac{\pi^2}{4}(\cos 2\pi t \mathbf{i} + \sin 2\pi t \mathbf{j})$$

$$W = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_C -\frac{\pi^2}{4}(\cos 2\pi t \mathbf{i} + \sin 2\pi t \mathbf{j}) \cdot 4\pi(-\sin 2\pi t \mathbf{i} + \cos 2\pi t \mathbf{j}) dt = -\pi^3 \int_C 0 \, dt = 0$$

38.  $\mathbf{F}(x, y, z) = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$

Because  $\mathbf{F}(x, y, z)$  is conservative, the work done in moving a particle along any path from  $P$  to  $Q$  is

$$f(x, y, z) = [a_1x + a_2y + a_3z]_{P=(p_1, p_2, p_3)}^{Q=(q_1, q_2, q_3)} = a_1(q_1 - p_1) + a_2(q_2 - p_2) + a_3(q_3 - p_3) = \mathbf{F} \cdot \overrightarrow{PQ}.$$

39.  $\mathbf{F} = -175\mathbf{j}$

40. No. The force field is conservative.

(a)  $\mathbf{r}(t) = t\mathbf{i} + (50 - t)\mathbf{j}, \quad 0 \leq t \leq 50$

41. See Theorem 15.5.

$$d\mathbf{r} = (\mathbf{i} - \mathbf{j}) dt$$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{50} 175 \, dt = 8750 \text{ ft} \cdot \text{lbs}$$

42. A line integral is independent of path if  $\int_C \mathbf{F} \cdot d\mathbf{r}$  does not depend on the curve joining  $P$  and  $Q$ . See Theorem 15.6.

(b)  $\mathbf{r}(t) = t\mathbf{i} + \frac{1}{50}(50 - t)^2 \mathbf{j}, \quad 0 \leq t \leq 50$

$$d\mathbf{r} = \mathbf{i} - \frac{1}{25}(50 - t)\mathbf{j}$$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{50} (175) \frac{1}{25} (50 - t) \, dt$$

$$= 7 \left[ 50t - \frac{t^2}{2} \right]_0^{50} = 8750 \text{ ft} \cdot \text{lbs}$$

43. (a) For the circle  $\mathbf{r}(t) = a \cos t\mathbf{i} - a \sin t\mathbf{j}$ ,  $0 \leq t \leq 2\pi$ , you have  $x^2 + y^2 = a^2$ , and

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} \left( \frac{-a \sin t}{a^2} \mathbf{i} - \frac{a \cos t}{a^2} \mathbf{j} \right) \cdot (-a \sin t\mathbf{i} - a \cos t\mathbf{j}) dt = \int_0^{2\pi} (\sin^2 t + \cos^2 t) dt = 2\pi.$$

- (b) For this curve, the answer is the same,  $2\pi$ .  
 (c) For the opposite orientation, the answer is  $-2\pi$ .  
 (d) For the curve away from the origin, the answer is 0.

44. (a) The direct path along the line segment joining  $(-4, 0)$  to  $(3, 4)$  requires less work than the path going from  $(-4, 0)$  to  $(-4, 4)$  and then to  $(3, 4)$ .

- (b) The closed curve given by the line segments joining  $(-4, 0)$ ,  $(-4, 4)$ ,  $(3, 4)$ , and  $(-4, 0)$  satisfies  
 $\int_C \mathbf{F} \cdot d\mathbf{r} \neq 0$ .

45. Conservative.  $\int_C \mathbf{F} \cdot d\mathbf{r}$  is independent of path.

51. Let

$$\mathbf{F} = M\mathbf{i} + N\mathbf{j} = \frac{\partial f}{\partial y}\mathbf{i} - \frac{\partial f}{\partial x}\mathbf{j}.$$

Then  $\frac{\partial M}{\partial y} = \frac{\partial}{\partial y} \left( \frac{\partial f}{\partial y} \right) = \frac{\partial^2 f}{\partial y^2}$  and  $\frac{\partial N}{\partial x} = \frac{\partial}{\partial x} \left( -\frac{\partial f}{\partial x} \right) = -\frac{\partial^2 f}{\partial x^2}$ . Because  $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$  you have  $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ .

So,  $\mathbf{F}$  is conservative. Therefore, by Theorem 15.7, you have  $\int_C \left( \frac{\partial f}{\partial y} dx - \frac{\partial f}{\partial x} dy \right) = \int_C (M dx + N dy) = \int_C \mathbf{F} \cdot d\mathbf{r} = 0$  for every closed curve in the plane.

52. Because the sum of the potential and kinetic energies remains constant from point to point, if the kinetic energy is decreasing at a rate of 15 units per minute, then the potential energy is increasing at a rate of 15 units per minute.

53.  $\mathbf{F}(x, y) = \frac{y}{x^2 + y^2}\mathbf{i} - \frac{x}{x^2 + y^2}\mathbf{j}$

$$(a) M = \frac{y}{x^2 + y^2}$$

$$\frac{\partial M}{\partial y} = \frac{(x^2 + y^2)(1) - y(2y)}{(x^2 + y^2)^2} = \frac{x^2 - y^2}{(x^2 + y^2)^2}$$

$$N = -\frac{x}{x^2 + y^2}$$

$$\frac{\partial N}{\partial x} = \frac{(x^2 + y^2)(-1) + x(2x)}{(x^2 + y^2)^2} = \frac{x^2 - y^2}{(x^2 + y^2)^2}$$

$$\text{So, } \frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}.$$

- (b)  $\mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j}$ ,  $0 \leq t \leq \pi$

$$\mathbf{F} = \sin t\mathbf{i} - \cos t\mathbf{j}$$

$$d\mathbf{r} = (-\sin t\mathbf{i} + \cos t\mathbf{j}) dt$$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^\pi (-\sin^2 t - \cos^2 t) dt = [-t]_0^\pi = -\pi$$

46. Not conservative. The value of  $\int_C \mathbf{F} \cdot d\mathbf{r}$  from  $(-1, 0)$  to  $(1, 0)$  is positive if the path is above the  $x$ -axis, and negative if the path is below the  $x$ -axis.

47. False, it would be true if  $\mathbf{F}$  were conservative.

48. True

49. True

50. False, the requirement is  $\partial M / \partial y = \partial N / \partial x$ .

(c)  $\mathbf{r}(t) = \cos t\mathbf{i} - \sin t\mathbf{j}, \quad 0 \leq t \leq \pi$

$$\mathbf{F} = -\sin t\mathbf{i} - \cos t\mathbf{j}$$

$$d\mathbf{r} = (-\sin t\mathbf{i} - \cos t\mathbf{j}) dt$$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^\pi (\sin^2 t + \cos^2 t) dt = [t]_0^\pi = \pi$$

(d)  $\mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j}, \quad 0 \leq t \leq 2\pi$

$$\mathbf{F} = \sin t\mathbf{i} - \cos t\mathbf{j}$$

$$d\mathbf{r} = (\sin t\mathbf{i} + \cos t\mathbf{j}) dt$$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} (-\sin^2 t - \cos^2 t) dt = [-t]_0^{2\pi} = -2\pi$$

This does not contradict Theorem 15.7 because  $\mathbf{F}$  is not continuous at  $(0, 0)$  in  $R$  enclosed by curve  $C$ .

(e)  $\nabla \left( \arctan \frac{x}{y} \right) = \frac{1/y}{1 + (x/y)^2} \mathbf{i} + \frac{-x/y^2}{1 + (x/y)^2} \mathbf{j} = \frac{y}{x^2 + y^2} \mathbf{i} - \frac{x}{x^2 + y^2} \mathbf{j} = \mathbf{F}$

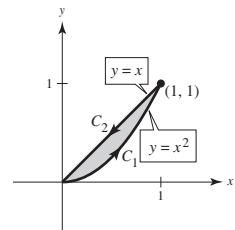
## Section 15.4 Green's Theorem

1.  $\mathbf{r}(t) = \begin{cases} t\mathbf{i} + t^2\mathbf{j}, & 0 \leq t \leq 1 \\ (2-t)\mathbf{i} + (2-t)\mathbf{j}, & 1 \leq t \leq 2 \end{cases}$

$$\begin{aligned} \int_C y^2 dx + x^2 dy &= \int_0^1 [t^4(dt) + t^2(2t dt)] + \int_1^2 [(2-t)^2(-dt) + (2-t)^2(-dt)] \\ &= \int_0^1 (t^4 + 2t^3) dt + \int_1^2 2(2-t)^2(-dt) = \left[ \frac{t^5}{5} + \frac{t^4}{2} \right]_0^1 + \left[ \frac{2(2-t)^3}{3} \right]_1^2 = \frac{7}{10} - \frac{2}{3} = \frac{1}{30} \end{aligned}$$

By Green's Theorem,

$$\begin{aligned} \int_R \int \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dA &= \int_0^1 \int_{x^2}^x (2x - 2y) dy dx = \int_0^1 [2xy - y^2]_x^x dx \\ &= \int_0^1 (x^2 - 2x^3 + x^4) dx = \left[ \frac{x^3}{3} - \frac{x^4}{2} + \frac{x^5}{5} \right]_0^1 = \frac{1}{30} \end{aligned}$$

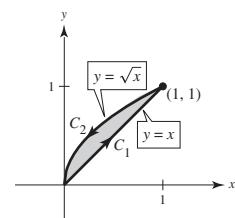


2.  $\mathbf{r}(t) = \begin{cases} t\mathbf{i} + t\mathbf{j}, & 0 \leq t \leq 1 \\ (2-t)\mathbf{i} + \sqrt{2-t}\mathbf{j}, & 1 \leq t \leq 2 \end{cases}$

$$\begin{aligned} \int_C y^2 dx + x^2 dy &= \int_0^1 [t^2(dt) + t^2(dt)] + \int_1^2 [(2-t)(-dt) + (2-t)^2 \left( -\frac{1}{2\sqrt{2-t}} dt \right)] \\ &= \int_0^1 2t^2 dt + \int_1^2 \left[ (t-2) - \frac{1}{2}(2-t)^{3/2} \right] dt \\ &= \left[ \frac{2t^3}{3} \right]_0^1 + \left[ \frac{(t-2)^2}{2} + \frac{(2-t)^{5/2}}{5} \right]_1^2 \\ &= \frac{2}{3} - \frac{1}{2} - \frac{1}{5} = -\frac{1}{30} \end{aligned}$$

By Green's Theorem,

$$\begin{aligned} \int_R \int \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dA &= \int_0^1 \int_x^{\sqrt{x}} (2x - 2y) dy dx = \int_0^1 [2xy - y^2]_x^{\sqrt{x}} dx \\ &= \int_0^1 (2x^{3/2} - x - x^2) dx = \left[ \frac{4}{5}x^{5/2} - \frac{x^2}{2} - \frac{x^3}{3} \right]_0^1 = -\frac{1}{30} \end{aligned}$$

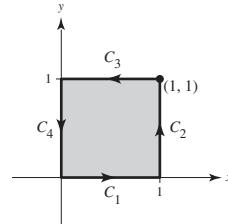


$$3. \mathbf{r}(t) = \begin{cases} t\mathbf{i} & 0 \leq t \leq 1 \\ \mathbf{i} + (t-1)\mathbf{j} & 1 \leq t \leq 2 \\ (3-t)\mathbf{i} + \mathbf{j} & 2 \leq t \leq 3 \\ (4-t)\mathbf{j} & 3 \leq t \leq 4 \end{cases}$$

$$\begin{aligned} \int_C y^2 dx + x^2 dy &= \int_0^1 [0 dt + t^2(0)] + \int_1^2 [(t-1)^2(0) + 1 dt] + \int_2^3 [1(-dt) + (3-t)^2(0)] + \int_3^4 [(4-t)^2(0) + 0(-dt)] \\ &= \int_1^2 dt + \int_2^3 -dt = 1 - 1 = 0 \end{aligned}$$

By Green's Theorem,

$$\begin{aligned} \int_R \int \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dA &= \int_0^1 \int_0^1 (2x - 2y) dy dx \\ &= \int_0^1 [2xy - y^2]_0^1 dx = \int_0^1 (2x - 1) dx = [x^2 - x]_0^1 = 0 \end{aligned}$$

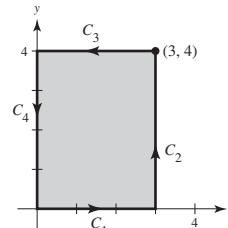


$$4. \mathbf{r}(t) = \begin{cases} t\mathbf{i} & 0 \leq t \leq 3 \\ 3\mathbf{i} + (t-3)\mathbf{j} & 3 \leq t \leq 7 \\ (10-t)\mathbf{i} + 4\mathbf{j} & 7 \leq t \leq 10 \\ (14-t)\mathbf{j} & 10 \leq t \leq 14 \end{cases}$$

$$\begin{aligned} \int_C y^2 dx + x^2 dy &= \int_0^3 [0(dt) + t^2(0)] + \int_3^7 [(t-3)^2(0) + 9 dt] + \int_7^{10} [16(-dt) + (10-t)^2(0)] + \int_{10}^{14} [0(-dt) + (14-t)^2(0)] \\ &= \int_3^7 9 dt + \int_7^{10} -16 dt = 9(7-3) + (-16)(10-7) = -12 \end{aligned}$$

By Green's Theorem,

$$\begin{aligned} \int_R \int \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dA &= \int_0^3 \int_0^4 (2x - 2y) dy dx = \int_0^3 [2xy - y^2]_0^4 dx \\ &= \int_0^3 [8x - 16] dx = [4x^2 - 16x]_0^3 = 36 - 48 = -12 \end{aligned}$$



5.  $C: x^2 + y^2 = 4$

Let  $x = 2 \cos t$  and  $y = 2 \sin t$ ,  $0 \leq t \leq 2\pi$ .

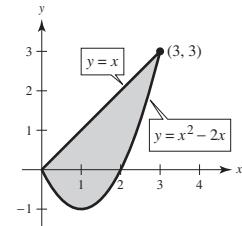
$$\begin{aligned} \int_C xe^y dx + e^x dy &= \int_0^{2\pi} [2 \cos t e^{2 \sin t} (-2 \sin t) + e^{2 \cos t} (2 \cos t)] dt \approx 19.99 \\ \int_R \int \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dA &= \int_{-2}^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (e^x - xe^y) dy dx = \int_{-2}^2 [2\sqrt{4-x^2}e^x - xe^{\sqrt{4-x^2}} + xe^{-\sqrt{4-x^2}}] dx \approx 19.99 \end{aligned}$$

6.  $C$ : boundary of the region lying between the graphs of  $y = x$  and  $y = x^3$

$$\begin{aligned} \int_C xe^y dx + e^x dy &= \int_0^1 (xe^{x^3} + 3x^2 e^x) dx + \int_1^0 (xe^x + e^x) dx \approx 2.936 - 2.718 \approx 0.22 \\ \int_R \int \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dA &= \int_0^1 \int_{x^3}^x (e^x - xe^y) dy dx = \int_0^1 (xe^{x^3} - x^3 e^x) dx \approx 0.22 \end{aligned}$$

In Exercises 7–10,  $\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 1$ .

$$\begin{aligned} 7. \int_C (y - x) dx + (2x - y) dy &= \int_0^3 \int_{x^2-2x}^x dy dx \\ &= \int_0^3 [x - (x^2 - 2x)] dx = \left[ -\frac{x^3}{3} + \frac{3x^2}{2} \right]_0^3 = -9 + \frac{27}{2} = \frac{9}{2} \end{aligned}$$

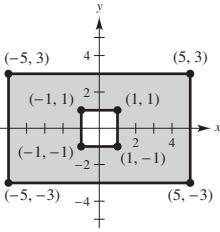


8. Because  $C$  is an ellipse with  $a = 2$  and  $b = 1$ , then  $R$  is an ellipse of area  $\pi ab = 2\pi$ . So, Green's Theorem yields

$$\int_C (y - x) dx + (2x - y) dy = \int_R \int 1 dA = \text{Area of ellipse} = 2\pi.$$

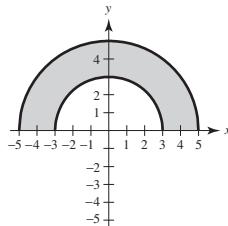
9. From the accompanying figure, we see that  $R$  is the shaded region. So, Green's Theorem yields

$$\int_C (y - x) dx + (2x - y) dy = \int_R \int 1 dA = \text{Area of } R = 6(10) - 2(2) = 56.$$



10.  $R$  is the shaded region of the accompanying figure.

$$\begin{aligned} \int_C (y - x) dx + (2x - y) dy &= \int_R \int 1 dA \\ &= \text{Area of shaded region} \\ &= \frac{1}{2}\pi[25 - 9] = 8\pi \end{aligned}$$



$$\begin{aligned} 11. \int_C 2xy dx + (x + y) dy &= \int_R \int \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dA \\ &= \int_{-1}^1 \int_0^{1-x^2} (1 - 2x) dy dx = \int_{-1}^1 [y - 2xy]_0^{1-x^2} dx = \int_{-1}^1 [(1 - x^2) - 2x(1 - x^2)] dx \\ &= \int_{-1}^1 [1 - x^2 - 2x + 2x^3] dx = \left[ x - \frac{x^3}{3} - x^2 + \frac{x^4}{2} \right]_{-1}^1 = \frac{1}{6} + \frac{7}{6} = \frac{4}{3} \end{aligned}$$

12. The given curves intersect at  $(0, 0)$  and  $(9, 3)$ . So, Green's Theorem yields

$$\begin{aligned} \int_C y^2 dx + xy dy &= \int_R \int (y - 2y) dA \\ &= \int_0^9 \int_0^{\sqrt{x}} -y dy dx = \int_0^9 \left[ \frac{-y^2}{2} \right]_0^{\sqrt{x}} dx = \int_0^9 \frac{-x}{2} dx = \left[ \frac{-x^2}{4} \right]_0^9 = -\frac{81}{4}. \end{aligned}$$

$$13. \int_C (x^2 - y^2) dx + 2xy dy = \int_R \int \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dA = \int_{-4}^4 \int_{-\sqrt{16-x^2}}^{\sqrt{16-x^2}} (2y + 2y) dy dx = \int_{-4}^4 [2y^2]_{-\sqrt{16-x^2}}^{\sqrt{16-x^2}} dx = 0$$

14. In this case, let  $y = r \sin \theta$ ,  $x = r \cos \theta$ . Then  $dA = r dr d\theta$  and Green's Theorem yields

$$\begin{aligned} \int_C (x^2 - y^2) dx + 2xy dy &= \int_R \int 4y dA = 4 \int_0^{2\pi} \int_0^{1+\cos \theta} r \sin \theta r dr d\theta \\ &= 4 \int_0^{2\pi} \int_0^{1+\cos \theta} r^2 \sin \theta dr d\theta = \frac{4}{3} \int_0^{2\pi} \sin \theta (1 + \cos \theta)^3 d\theta = \left[ -\frac{(1 + \cos \theta)^4}{3} \right]_0^{2\pi} = 0. \end{aligned}$$

15. Because  $\frac{\partial M}{\partial y} = -2e^x \sin 2y = \frac{\partial N}{\partial x}$  you have

$$\int_R \int \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dA = 0.$$

16. Because  $\frac{\partial M}{\partial y} = \frac{2x}{x^2 + y^2} = \frac{\partial N}{\partial x}$ ,

you have path independence and  

$$\int_R \int \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dA = 0.$$

17. By Green's Theorem,

$$\begin{aligned}\int_C \cos y \, dx + (xy - x \sin y) \, dy &= \int_R \int (y - \sin y + \sin y) \, dA = \int_0^1 \int_x^{\sqrt{x}} y \, dy \, dx = \int_0^1 \left[ \frac{y^2}{2} \right]_x^{\sqrt{x}} \, dx \\ &= \int_0^1 \left( \frac{x}{2} - \frac{x^2}{2} \right) \, dx = \left[ \frac{x^2}{4} - \frac{x^3}{6} \right]_0^1 = \frac{1}{4} - \frac{1}{6} = \frac{1}{12}\end{aligned}$$

18. By Green's Theorem,

$$\int_C (e^{-x^2/2} - y) \, dx + (e^{-y^2/2} + x) \, dy = \int_R \int 2 \, dA = 2(\text{Area of } R) = 2[\pi(6)^2 - \pi(2)(3)] = 60\pi.$$

19. By Green's Theorem,

$$\int_C (x - 3y) \, dx + (x + y) \, dy = \int_R \int [1 + 3] \, dA = 4[\text{Area Large Circle} - \text{Area Small Circle}] = 4[9\pi - \pi] = 32\pi$$

20. By Green's Theorem,

$$\begin{aligned}\int_C 3x^2 e^y \, dx + e^y \, dy &= \int_R \int -3x^2 e^y \, dA \\ &= \int_1^2 \int_{-2}^2 -3x^2 e^y \, dy \, dx + \int_{-1}^1 \int_1^2 -3x^2 e^y \, dy \, dx \\ &\quad + \int_{-2}^{-1} \int_{-2}^2 -3x^2 e^y \, dy \, dx + \int_{-1}^1 \int_{-2}^{-1} -3x^2 e^y \, dy \, dx \\ &= -7(e^2 - e^{-2}) - 2(e^2 - e) - 7(e^2 - e^{-2}) - 2(e^{-1} - e^{-2}) \\ &= -16e^2 + 16e^{-2} + 2e - 2e^{-1}.\end{aligned}$$

21.  $\mathbf{F}(x, y) = xy\mathbf{i} + (x + y)\mathbf{j}$

$$C: x^2 + y^2 = 1$$

$$\begin{aligned}\text{Work} &= \int_C xy \, dx + (x + y) \, dy = \int_R \int (1 - x) \, dA = \int_0^{2\pi} \int_0^1 (1 - r \cos \theta) r \, dr \, d\theta \\ &= \int_0^{2\pi} \left[ \frac{r^2}{2} - \frac{r^2}{2} \cos \theta \right]_0^1 \, d\theta = \int_0^{2\pi} \frac{1}{2}(1 - \cos \theta) \, d\theta = \left[ \frac{1}{2}\theta - \frac{1}{2}\sin \theta \right]_0^{2\pi} = \pi\end{aligned}$$

22.  $\mathbf{F}(x, y) = (e^x - 3y)\mathbf{i} + (e^y + 6x)\mathbf{j}$

$$C: r = 2 \cos \theta$$

$$\text{Work} = \int_C (e^x - 3y) \, dx + (e^y + 6x) \, dy = \int_R \int 9 \, dA = 9\pi \text{ because } r = 2 \cos \theta \text{ is a circle with a radius of one.}$$

23.  $\mathbf{F}(x, y) = (x^{3/2} - 3y)\mathbf{i} + (6x + 5\sqrt{y})\mathbf{j}$

$C$ : boundary of the triangle with vertices  $(0, 0), (5, 0), (0, 5)$

$$\text{Work} = \int_C (x^{3/2} - 3y) \, dx + (6x + 5\sqrt{y}) \, dy = \int_R \int 9 \, dA = 9\left(\frac{1}{2}\right)(5)(5) = \frac{225}{2}$$

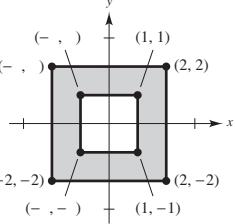
24.  $\mathbf{F}(x, y) = (3x^2 + y)\mathbf{i} + 4xy^2\mathbf{j}$

$C$ : boundary of the region bounded by the graphs of  $y = \sqrt{x}, y = 0, x = 9$

$$\text{Work} = \int_C (3x^2 + y) \, dx + 4xy^2 \, dy = \int_0^9 \int_0^{\sqrt{x}} (4y^2 - 1) \, dy \, dx = \int_0^9 \left( \frac{4}{3}x^{3/2} - x^{1/2} \right) \, dx = \frac{558}{5}$$

25.  $C$ : let  $x = a \cos t, y = a \sin t, 0 \leq t \leq 2\pi$ . By Theorem 15.9, you have

$$A = \frac{1}{2} \int_C x \, dy - y \, dx = \frac{1}{2} \int_0^{2\pi} [a \cos t(a \cos t) - a \sin t(-a \sin t)] \, dt = \frac{1}{2} \int_0^{2\pi} a^2 \, dt = \left[ \frac{a^2}{2}t \right]_0^{2\pi} = \pi a^2.$$



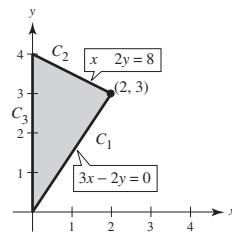
**26.** From the figure you see that

$$C_1: y = \frac{3}{2}x, dy = \frac{3}{2}dx, 0 \leq x \leq 2$$

$$C_2: y = -\frac{x}{2} + 4, dy = -\frac{1}{2}dx$$

$$C_3: x = 0, dx = 0.$$

$$\begin{aligned} A &= \frac{1}{2} \int_0^2 \left( \frac{3}{2}x - \frac{3}{2}x \right) dx + \frac{1}{2} \int_2^0 \left( -\frac{1}{2}x + \frac{x}{2} - 4 \right) dx + \frac{1}{2}(0) \\ &= \frac{1}{2} \int_2^0 (-4) dx = 2 \int_0^2 dx = 4 \end{aligned}$$

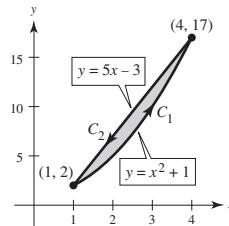


**27.**  $C_1: y = x^2 + 1, dy = 2x dx$

$$C_2: y = 5x - 3, dy = 5 dx$$

So, by Theorem 15.9 you have

$$\begin{aligned} A &= \frac{1}{2} \int_1^4 \left( x(2x) - (x^2 + 1) \right) dx + \frac{1}{2} \int_4^1 \left( x(5) - (5x - 3) \right) dx \\ &= \frac{1}{2} \left[ \frac{x^3}{3} - x \right]_1^4 + \frac{1}{2} [3x]_4^1 = \frac{1}{2}[18] + \frac{1}{2}[-9] = \frac{9}{2}. \end{aligned}$$



**28.** Because the loop of the folium is formed on the interval  $0 \leq t \leq \infty$ ,

$$dx = \frac{3(1 - 2t^3)}{(t^3 + 1)^2} dt \text{ and } dy = \frac{3(2t - t^4)}{(t^3 + 1)^2} dt,$$

you have

$$\begin{aligned} A &= \frac{1}{2} \int_0^\infty \left[ \left( \frac{3t}{t^3 + 1} \right) \frac{3(2t - t^4)}{(t^3 + 1)^2} - \left( \frac{3t^2}{t^3 + 1} \right) \frac{3(1 - 2t^3)}{(t^3 + 1)^2} \right] dt \\ &= \frac{9}{2} \int_0^\infty \frac{t^5 + t^2}{(t^3 + 1)^3} dt = \frac{9}{2} \int_0^\infty \frac{t^2(t^3 + 1)}{(t^3 + 1)^3} dt = \frac{3}{2} \int_0^\infty 3t^2(t^3 + 1)^{-2} dt = \left[ \frac{-3}{2(t^3 + 1)} \right]_0^\infty = \frac{3}{2}. \end{aligned}$$

**29.** See Theorem 15.8, page 1093.

**30.** See Theorem 15.9:  $A = \frac{1}{2} \int_C x dy - y dx$ .

**31.** For the moment about the  $x$ -axis,  $M_x = \int_R \int y dA$ . Let  $N = 0$  and  $M = -y^2/2$ . By Green's Theorem,

$$M_x = \int_C -\frac{y^2}{2} dx = -\frac{1}{2} \int_C y^2 dx \text{ and } \bar{y} = \frac{M_x}{2A} = -\frac{1}{2A} \int_C y^2 dx.$$

For the moment about the  $y$ -axis,  $M_y = \int_R \int x dA$ . Let  $N = x^2/2$  and  $M = 0$ . By Green's Theorem,

$$M_y = \int_C \frac{x^2}{2} dy = \frac{1}{2} \int_C x^2 dy \text{ and } \bar{x} = \frac{M_y}{2A} = \frac{1}{2A} \int_C x^2 dy.$$

**32.** By Theorem 15.9 and the fact that  $x = r \cos \theta, y = r \sin \theta$ , you have

$$A = \frac{1}{2} \int x dy - y dx = \frac{1}{2} \int (r \cos \theta)(r \cos \theta) d\theta - (r \sin \theta)(-r \sin \theta) d\theta = \frac{1}{2} \int_C r^2 d\theta.$$

33.  $A = \int_{-2}^2 (4 - x^2) dx = \left[ 4x - \frac{x^3}{3} \right]_{-2}^2 = \frac{32}{3}$

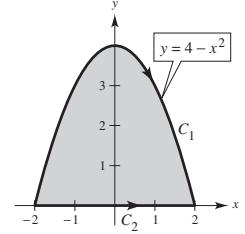
$$\bar{x} = \frac{1}{2A} \int_{C_1} x^2 dy + \frac{1}{2A} \int_{C_2} x^2 dy$$

For  $C_1$ ,  $dy = -2x dx$  and for  $C_2$ ,  $dy = 0$ . So,  $\bar{x} = \frac{1}{2(32/3)} \int_{-2}^2 x^2 (-2x) dx = \left[ \frac{3}{64} \left( -\frac{x^4}{2} \right) \right]_2^{-2} = 0$ .

To calculate  $\bar{y}$ , note that  $y = 0$  along  $C_2$ . So,

$$\bar{y} = \frac{-1}{2(32/3)} \int_{-2}^2 (4 - x^2)^2 dx = \frac{3}{64} \int_{-2}^2 (16 - 8x^2 + x^4) dx = \frac{3}{64} \left[ 16x - \frac{8x^3}{3} + \frac{x^5}{5} \right]_{-2}^2 = \frac{8}{5}.$$

$$(\bar{x}, \bar{y}) = \left( 0, \frac{8}{5} \right)$$



34. Because  $A = \text{area of semicircle} = \frac{\pi a^2}{2}$ , you have  $\frac{1}{2A} = \frac{1}{\pi a^2}$ . Note that  $y = 0$  and  $dy = 0$  along the boundary  $y = 0$ .

Let  $x = a \cos t$ ,  $y = a \sin t$ ,  $0 \leq t \leq \pi$ , then

$$\bar{x} = \frac{1}{\pi a^2} \int_0^\pi a^2 \cos^2 t (a \cos t) dt = \frac{a}{\pi} \int_0^\pi \cos^3 t dt = \frac{a}{\pi} \int_0^\pi (1 - \sin^2 t) \cos t dt = \frac{a}{\pi} \left[ \sin t - \frac{\sin^3 t}{3} \right]_0^\pi = 0$$

$$\bar{y} = \frac{-1}{\pi a^2} \int_0^\pi a^2 \sin^2 t (-a \sin t) dt = \frac{a}{\pi} \int_0^\pi \sin^3 t dt = \frac{a}{\pi} \left[ -\cos t + \frac{\cos^3 t}{3} \right]_0^\pi = \frac{4a}{3\pi}.$$

$$(\bar{x}, \bar{y}) = \left( 0, \frac{4a}{3\pi} \right)$$

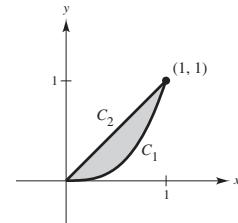
35. Because  $A = \int_0^1 (x - x^3) dx = \left[ \frac{x^2}{2} - \frac{x^4}{4} \right]_0^1 = \frac{1}{4}$ , you have  $\frac{1}{2A} = 2$ . On  $C_1$  you have  $y = x^3$ ,  $dy = 3x^2 dx$  and on  $C_2$  you

have  $y = x$ ,  $dy = dx$ . So,

$$\bar{x} = 2 \int_C x^2 dy = 2 \int_{C_1} x^2 (3x^2 dx) + 2 \int_{C_2} x^2 dx = 6 \int_0^1 x^4 dx + 2 \int_1^0 x^2 dx = \frac{6}{5} - \frac{2}{3} = \frac{8}{15}$$

$$\bar{y} = -2 \int_C y^2 dx = -2 \int_0^1 x^6 dx - 2 \int_1^0 x^2 dx = -\frac{2}{7} + \frac{2}{3} = \frac{8}{21}.$$

$$(\bar{x}, \bar{y}) = \left( \frac{8}{15}, \frac{8}{21} \right)$$



36. Because  $A = \frac{1}{2}(2a)(c) = ac$ , you have  $\frac{1}{2A} = \frac{1}{2ac}$ ,

$C_1$ :  $y = 0$ ,  $dy = 0$

$$C_2: y = \frac{c}{b-a}(x-a), dy = \frac{c}{b-a} dx$$

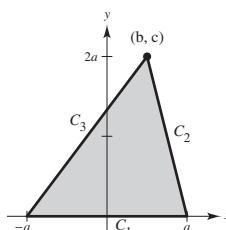
$$C_3: y = \frac{c}{b+a}(x+a), dy = \frac{c}{b+a} dx.$$

So,

$$\bar{x} = \frac{1}{2ac} \int_C x^2 dy = \frac{1}{2ac} \left[ \int_{-a}^a 0 + \int_a^b x^2 \frac{c}{b-a} dx + \int_b^a x^2 \frac{c}{b+a} dx \right] = \frac{1}{2ac} \left[ 0 + \frac{2abc}{3} \right] = \frac{b}{3}$$

$$\bar{y} = \frac{-1}{2ac} \int_C y^2 dx = \frac{-1}{2ac} \left[ 0 + \int_a^b \left( \frac{c}{b-a} \right)^2 (x-a)^2 dx + \int_b^a \left( \frac{c}{b+a} \right)^2 (x+a)^2 dx \right] = \frac{-1}{2ac} \left[ \frac{c^2(b-a)}{3} - \frac{c^2(b+a)}{3} \right] = \frac{c}{3}.$$

$$(\bar{x}, \bar{y}) = \left( \frac{b}{3}, \frac{c}{3} \right)$$



37.  $A = \frac{1}{2} \int_0^{2\pi} a^2 (1 - \cos \theta)^2 d\theta = \frac{a^2}{2} \int_0^{2\pi} \left(1 - 2 \cos \theta + \frac{1}{2} + \frac{\cos 2\theta}{2}\right) d\theta = \frac{a^2}{2} \left[ \frac{3\theta}{2} - 2 \sin \theta + \frac{1}{4} \sin 2\theta \right]_0^{2\pi} = \frac{a^2}{2} (3\pi) = \frac{3\pi a^2}{2}$

38.  $A = \frac{1}{2} \int_0^\pi a^2 \cos^2 3\theta d\theta = \frac{a^2}{2} \int_0^\pi \frac{1 + \cos 6\theta}{2} d\theta = \frac{a^2}{4} \left[ \theta + \frac{\sin 6\theta}{6} \right]_0^\pi = \frac{\pi a^2}{4}$

Note: In this case  $R$  is enclosed by  $r = a \cos 3\theta$  where  $0 \leq \theta \leq \pi$ .

39. In this case the inner loop has domain  $\frac{2\pi}{3} \leq \theta \leq \frac{4\pi}{3}$ . So,

$$A = \frac{1}{2} \int_{2\pi/3}^{4\pi/3} (1 + 4 \cos \theta + 4 \cos^2 \theta) d\theta = \frac{1}{2} \int_{2\pi/3}^{4\pi/3} (3 + 4 \cos \theta + 2 \cos 2\theta) d\theta = \frac{1}{2} [3\theta + 4 \sin \theta + \sin 2\theta]_{2\pi/3}^{4\pi/3} = \pi - \frac{3\sqrt{3}}{2}.$$

40. In this case,  $0 \leq \theta \leq 2\pi$  and you let  $u = \frac{\sin \theta}{1 + \cos \theta}$ ,  $\cos \theta = \frac{1 - u^2}{1 + u^2}$ ,  $d\theta = \frac{2 du}{1 + u^2}$ .

Now  $u \Rightarrow \infty$  as  $\theta \Rightarrow \pi$  and you have

$$\begin{aligned} A &= 2 \left( \frac{1}{2} \right) \int_0^\pi \frac{9}{(2 - \cos \theta)^2} d\theta = 9 \int_0^\infty \frac{\frac{2du}{1+u^2}}{4 - 4\left(\frac{1-u^2}{1+u^2}\right) + \frac{(1-u^2)^2}{(1+u^2)^2}} = 18 \int_0^\infty \frac{1+u^2}{(1+3u^2)^2} du \\ &= 18 \int_0^\infty \frac{1/3}{1+3u^2} du + 18 \int_0^\infty \frac{2/3}{(1+3u^2)^2} du = \left[ \frac{6}{\sqrt{3}} \arctan \sqrt{3} u \right]_0^\infty + \frac{12}{\sqrt{3}} \left( \frac{1}{2} \right) \left[ \frac{u}{1+3u^2} + \int \frac{\sqrt{3}}{1+3u^2} du \right]_0^\infty \\ &= \frac{6}{\sqrt{3}} \left( \frac{\pi}{2} \right) + \frac{6}{\sqrt{3}} \left[ \frac{u}{1+3u^2} \right]_0^\infty + \left[ \frac{6}{\sqrt{3}} \arctan \sqrt{3} u \right]_0^\infty = \frac{3\pi}{\sqrt{3}} + 0 + \frac{3\pi}{\sqrt{3}} = 2\sqrt{3}\pi. \end{aligned}$$

41. (a)  $\int_{C_1} y^3 dx + (27x - x^3) dy = \int_R \int [(27 - 3x^2) - 3y^2] dA$   
 $= \int_0^{2\pi} \int_0^1 (27 - 3r^2) r dr d\theta = \int_0^{2\pi} \left[ \frac{27r^2}{2} - \frac{3r^4}{4} \right]_0^1 d\theta = \int_0^{2\pi} \frac{51}{4} d\theta = \frac{51}{2}\pi$

(b) You want to find  $c$  such that  $\int_0^c (27 - 3r^2)r dr d\theta$  is a maximum:

$$f(c) = \frac{27c^2}{2} - \frac{3}{4}c^4$$

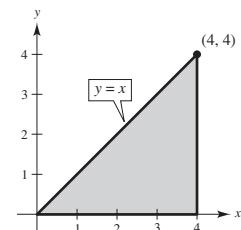
$$f'(c) = 27c - 3c^2 \Rightarrow c = 3$$

$$\text{Maximum Value: } \int_0^{2\pi} \int_0^3 (27 - 3r^2)r dr d\theta = \frac{243\pi}{2}$$

42. (a)  $\mathbf{r}(t) = \begin{cases} \mathbf{i}, & 0 \leq t \leq 4 \\ 4\mathbf{i} + (t-4)\mathbf{j}, & 4 \leq t \leq 8 \\ (12-t)\mathbf{i} + (12-t)\mathbf{j}, & 8 \leq t \leq 12 \end{cases}$

$$\begin{aligned} \int_C y^2 dx + x^2 dy &= \int_0^4 [0 dt + t^2(0)] + \int_4^8 [(t-4)^2(0) + 16 dt] + \int_8^{12} [(12-t)^2(-dt) + (12-t)^2(-dt)] \\ &= 0 + 64 - \frac{128}{3} = \frac{64}{3} \end{aligned}$$

By Green's Theorem,  $\int_R \int \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dA = \int_0^4 \int_0^x (2x - 2y) dy dx = \int_0^4 x^2 dx = \frac{64}{3}$ .

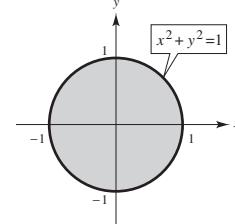


(b)  $\mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j}, 0 \leq t \leq 2\pi$

$$\begin{aligned}\int_C y^2 dx + x^2 dy &= \int_0^{2\pi} [\sin^2 t(-\sin t dt) + \cos^2 t(\cos t dt)] = \int_0^{2\pi} (\cos^3 t - \sin^3 t) dt \\ &= \int_0^{2\pi} [\cos t(1 - \sin^2 t) - \sin t(1 - \cos^2 t)] dt = \left[ \sin t - \frac{\sin^3 t}{3} + \cos t - \frac{\cos^3 t}{3} \right]_0^{2\pi} = 0\end{aligned}$$

By Green's Theorem,

$$\begin{aligned}\int_R \int \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dA &= \int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (2x - 2y) dy dx \\ &= \int_0^{2\pi} \int_0^1 (2r \cos \theta - 2r \sin \theta) r dr d\theta \\ &= \frac{2}{3} \int_0^{2\pi} (\cos \theta - \sin \theta) d\theta = \frac{2}{3}(0) = 0.\end{aligned}$$



43.  $I = \int_C \frac{y dx - x dy}{x^2 + y^2}$

(a) Let  $\mathbf{F} = \frac{y}{x^2 + y^2} \mathbf{i} - \frac{x}{x^2 + y^2} \mathbf{j}$ .

$\mathbf{F}$  is conservative because  $\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y} = \frac{x^2 - y^2}{(x^2 + y^2)^2}$ .

$\mathbf{F}$  is defined and has continuous first partials everywhere except at the origin. If  $C$  is a circle (a closed path) that does not contain the origin, then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C M dx + N dy = \int_R \int \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dA = 0.$$

- (b) Let  $\mathbf{r} = a \cos t \mathbf{i} - a \sin t \mathbf{j}, 0 \leq t \leq 2\pi$  be a circle  $C_1$  oriented clockwise inside  $C$  (see figure). Introduce line segments  $C_2$  and  $C_3$  as illustrated in Example 6 of this section in the text. For the region inside  $C$  and outside  $C_1$ , Green's Theorem applies. Note that since  $C_2$  and  $C_3$  have opposite orientations, the line integrals over them cancel. So,

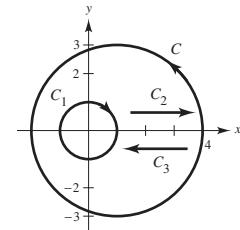
$C_4 = C_1 + C_2 + C + C_3$  and

$$\int_{C_4} \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} \mathbf{F} \cdot d\mathbf{r} + \int_C \mathbf{F} \cdot d\mathbf{r} = 0.$$

But,

$$\begin{aligned}\int_{C_1} \mathbf{F} \cdot d\mathbf{r} &= \int_0^{2\pi} \left[ \frac{(-a \sin t)(-a \sin t)}{a^2 \cos^2 t + a^2 \sin^2 t} + \frac{(-a \cos t)(-a \cos t)}{a^2 \cos^2 t + a^2 \sin^2 t} \right] dt \\ &= \int_0^{2\pi} (\sin^2 t + \cos^2 t) dt = [t]_0^{2\pi} = 2\pi.\end{aligned}$$

Finally,  $\int_C \mathbf{F} \cdot d\mathbf{r} = -\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = -2\pi$ .



**Note:** If  $C$  were oriented clockwise, then the answer would have been  $2\pi$ .

44. (a) Let  $C$  be the line segment joining  $(x_1, y_1)$  and  $(x_2, y_2)$ .

$$y = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1) + y_1$$

$$dy = \frac{y_2 - y_1}{x_2 - x_1} dx$$

$$\begin{aligned}\int_C -y dx + x dy &= \int_{x_1}^{x_2} \left[ -\frac{y_2 - y_1}{x_2 - x_1}(x - x_1) - y_1 + x \left( \frac{y_2 - y_1}{x_2 - x_1} \right) \right] dx = \int_{x_1}^{x_2} \left[ x_1 \left( \frac{y_2 - y_1}{x_2 - x_1} \right) - y_1 \right] dx \\ &= \left[ x_1 \left( \frac{y_2 - y_1}{x_2 - x_1} \right) - y_1 \right] x \Big|_{x_1}^{x_2} = \left[ x_1 \left( \frac{y_2 - y_1}{x_2 - x_1} \right) - y_1 \right] (x_2 - x_1) = x_1(y_2 - y_1) - y_1(x_2 - x_1) = x_1 y_2 - x_2 y_1\end{aligned}$$

(b) Let  $C$  be the boundary of the region  $A = \frac{1}{2} \int_C -y \, dx + x \, dy = \frac{1}{2} \int_R \int (1 - (-1)) \, dA = \int_R \int dA$ .

So,

$$\int_R \int dA = \frac{1}{2} \left[ \int_{C_1} -y \, dx + x \, dy + \int_{C_2} -y \, dx + x \, dy + \cdots + \int_{C_n} -y \, dx + x \, dy \right]$$

where  $C_1$  is the line segment joining  $(x_1, y_1)$  and  $(x_2, y_2)$ ,  $C_2$  is the line segment joining  $(x_2, y_2)$  and  $(x_3, y_3)$ , ..., and  $C_n$  is the line segment joining  $(x_n, y_n)$  and  $(x_1, y_1)$ . So,

$$\int_R \int dA = \frac{1}{2} [(x_1 y_2 - x_2 y_1) + (x_2 y_3 - x_3 y_2) + \cdots + (x_{n-1} y_n - x_n y_{n-1}) + (x_n y_1 - x_1 y_n)].$$

45. Pentagon:  $(0, 0), (2, 0), (3, 2), (1, 4), (-1, 1)$

$$A = \frac{1}{2} [(0 - 0) + (4 - 0) + (12 - 2) + (1 + 4) + (0 - 0)] = \frac{19}{2}$$

46. Hexagon:  $(0, 0), (2, 0), (3, 2), (2, 4), (0, 3), (-1, 1)$

$$A = \frac{1}{2} [(0 - 0) + (4 - 0) + (12 - 4) + (6 - 0) + (0 + 3) + (0 - 0)] = \frac{21}{2}$$

47. Because  $\int_C \mathbf{F} \cdot \mathbf{N} \, ds = \int_R \int \operatorname{div} \mathbf{F} \, dA$ , then

$$\int_C f D_N g \, ds = \int_C f \nabla g \cdot \mathbf{N} \, ds = \int_R \int \operatorname{div}(f \nabla g) \, dA = \int_R \int (f \operatorname{div}(\nabla g) + \nabla f \cdot \nabla g) \, dA = \int_R \int (f \nabla^2 g + \nabla f \cdot \nabla g) \, dA.$$

$$\begin{aligned} 48. \int_C (f D_N g - g D_N f) \, ds &= \int_C f D_N g \, ds - \int_C g D_N f \, ds \\ &= \int_R \int (f \nabla^2 g + \nabla f \cdot \nabla g) \, dA - \int_R \int (g \nabla^2 f + \nabla g \cdot \nabla f) \, dA = \int_R \int (f \nabla^2 g - g \nabla^2 f) \, dA \end{aligned}$$

$$49. \int_C f(x) \, dx + g(y) \, dy = \int_R \int \left[ \frac{\partial}{\partial x} g(y) - \frac{\partial}{\partial y} f(x) \right] \, dA = \int_R \int (0 - 0) \, dA = 0$$

50.  $\mathbf{F} = M\mathbf{i} + N\mathbf{j}$

$$\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y} \Rightarrow \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 0$$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C M \, dx + N \, dy = \int_R \int \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) \, dA = \int_R \int (0) \, dA = 0$$

## Section 15.5 Parametric Surfaces

1.  $\mathbf{r}(u, v) = u\mathbf{i} + v\mathbf{j} + uv\mathbf{k}$

$$z = xy$$

Matches (e)

3.  $\mathbf{r}(u, v) = u\mathbf{i} + \frac{1}{2}(u + v)\mathbf{j} + v\mathbf{k}$

$$2y = x + z, \text{ plane}$$

Matches (b)

2.  $\mathbf{r}(u, v) = u \cos v \mathbf{i} + u \sin v \mathbf{j} + u\mathbf{k}$

$$x^2 + y^2 = z^2, \text{ cone}$$

Matches (f)

4.  $\mathbf{r}(u, v) = u\mathbf{i} + \frac{1}{4}v^3\mathbf{j} + v\mathbf{k}$

$$4y = z^3, \text{ cylinder}$$

Matches (a)

5.  $\mathbf{r}(u, v) = 2 \cos v \cos u \mathbf{i} + 2 \cos v \sin u \mathbf{j} + 2 \sin v \mathbf{k}$

$$x^2 + y^2 + z^2 = 4 \cos^2 v \cos^2 u + 4 \cos^2 v \sin^2 u + 4 \sin^2 v = 4 \cos^2 v + 4 \sin^2 v = 4, \text{ sphere}$$

Matches (d)

6.  $\mathbf{r}(u, v) = 4 \cos u \mathbf{i} + 4 \sin u \mathbf{j} + v\mathbf{k}$

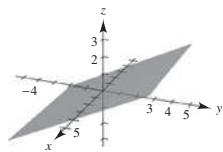
$$x^2 + y^2 = 4, \text{ circular cylinder}$$

Matches (c)

7.  $\mathbf{r}(u, v) = u\mathbf{i} + v\mathbf{j} + \frac{v}{2}\mathbf{k}$

$$y - 2z = 0$$

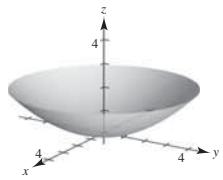
Plane



8.  $\mathbf{r}(u, v) = 2u \cos v\mathbf{i} + 2u \sin v\mathbf{j} + \frac{1}{2}u^2\mathbf{k}$

$$z = \frac{1}{2}u^2, x^2 + y^2 = 4u^2 \Rightarrow z = \frac{1}{8}(x^2 + y^2)$$

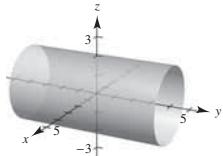
Paraboloid



9.  $\mathbf{r}(u, v) = 2 \cos u\mathbf{i} + v\mathbf{j} + 2 \sin u\mathbf{k}$

$$x^2 + z^2 = 4$$

Cylinder



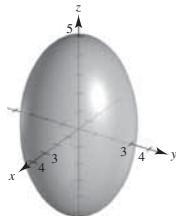
10.  $\mathbf{r}(u, v) = 3 \cos v \cos u\mathbf{i} + 3 \cos v \sin u\mathbf{j} + 5 \sin v\mathbf{k}$

$$x^2 + y^2 = 9 \cos^2 v \cos^2 u + 9 \cos^2 v \sin^2 u = 9 \cos^2 v$$

$$\frac{x^2 + y^2}{9} + \frac{z^2}{25} = \cos^2 v + \sin^2 v = 1$$

$$\frac{x^2}{9} + \frac{y^2}{9} + \frac{z^2}{25} = 1$$

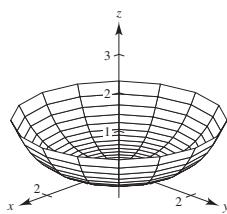
Ellipsoid



11.  $\mathbf{r}(u, v) = 2u \cos v\mathbf{i} + 2u \sin v\mathbf{j} + u^4\mathbf{k},$

$$0 \leq u \leq 1, 0 \leq v \leq 2\pi$$

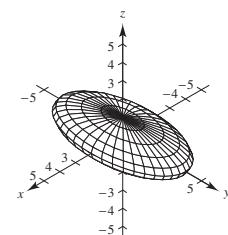
$$z = \frac{(x^2 + y^2)^2}{16}$$



12.  $\mathbf{r}(u, v) = 2 \cos v \cos u\mathbf{i} + 4 \cos v \sin u\mathbf{j} + \sin v\mathbf{k},$

$$0 \leq u \leq 2\pi, 0 \leq v \leq 2\pi$$

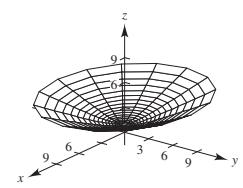
$$\frac{x^2}{4} + \frac{y^2}{16} + \frac{z^2}{1} = 1$$



13.  $\mathbf{r}(u, v) = 2 \sinh u \cos v\mathbf{i} + \sinh u \sin v\mathbf{j} + \cosh u\mathbf{k},$

$$0 \leq u \leq 2, 0 \leq v \leq 2\pi$$

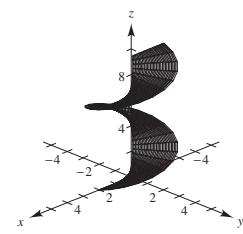
$$\frac{z^2}{1} - \frac{x^2}{4} - \frac{y^2}{1} = 1$$



14.  $\mathbf{r}(u, v) = 2u \cos v\mathbf{i} + 2u \sin v\mathbf{j} + v\mathbf{k},$

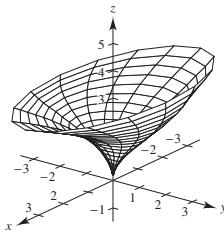
$$0 \leq u \leq 1, 0 \leq v \leq 3\pi$$

$$\tan z = \frac{y}{x}$$



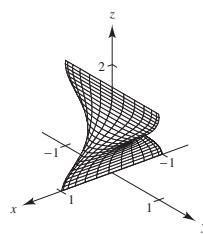
15.  $\mathbf{r}(u, v) = (u - \sin u)\cos v\mathbf{i} + (1 - \cos u)\sin v\mathbf{j} + u\mathbf{k},$

$$0 \leq u \leq \pi, 0 \leq v \leq 2\pi$$



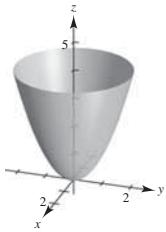
16.  $\mathbf{r}(u, v) = \cos^3 u \cos v\mathbf{i} + \sin^3 u \sin v\mathbf{j} + u\mathbf{k},$

$$0 \leq u \leq \frac{\pi}{2}, 0 \leq v \leq 2\pi$$



For Exercises 17–20,  $\mathbf{r}(u, v) = u \cos v\mathbf{i} + u \sin v\mathbf{j} + u^2\mathbf{k}$ ,  $0 \leq u \leq 2$ ,  $0 \leq v \leq 2\pi$ .

Eliminating the parameter yields  $z = x^2 + y^2$ ,  $0 \leq z \leq 4$ .



17.  $\mathbf{s}(u, v) = u \cos v\mathbf{i} + u \sin v\mathbf{j} - u^2\mathbf{k}$ ,  $0 \leq u \leq 2$ ,  $0 \leq v \leq 2\pi$

$$z = -(x^2 + y^2)$$

The paraboloid is reflected (inverted) through the  $xy$ -plane.

18.  $\mathbf{s}(u, v) = u \cos v\mathbf{i} + u^2\mathbf{j} + u \sin v\mathbf{k}$ ,  $0 \leq u \leq 2$ ,  $0 \leq v \leq 2\pi$

$$y = x^2 + z^2$$

The paraboloid opens along the  $y$ -axis instead of the  $z$ -axis.

19.  $\mathbf{s}(u, v) = u \cos v\mathbf{i} + u \sin v\mathbf{j} + u^2\mathbf{k}$ ,  $0 \leq u \leq 3$ ,  $0 \leq v \leq 2\pi$

The height of the paraboloid is increased from 4 to 9.

20.  $\mathbf{s}(u, v) = 4u \cos v\mathbf{i} + 4u \sin v\mathbf{j} + u^2\mathbf{k}$ ,  $0 \leq u \leq 2$ ,  $0 \leq v \leq 2\pi$

$$z = \frac{x^2 + y^2}{16}$$

The paraboloid is "wider." The top is now the circle  $x^2 + y^2 = 64$ . It was  $x^2 + y^2 = 4$ .

21.  $z = y$

$$\mathbf{r}(u, v) = u\mathbf{i} + v\mathbf{j} + v\mathbf{k}$$

26.  $4x^2 + y^2 = 16$

$$\mathbf{r}(u, v) = 2 \cos u\mathbf{i} + 4 \sin u\mathbf{j} + v\mathbf{k}$$

22.  $z = 6 - x - y$

$$\mathbf{r}(u, v) = u\mathbf{i} + v\mathbf{j} + (6 - u - v)\mathbf{k}$$

27.  $z = x^2$

$$\mathbf{r}(u, v) = u\mathbf{i} + v\mathbf{j} + u^2\mathbf{k}$$

23.  $y = \sqrt{4x^2 + 9z^2}$

$$\mathbf{r}(x, y) = x\mathbf{i} + \sqrt{4x^2 + 9z^2}\mathbf{j} + z\mathbf{k}$$

or,

$$\mathbf{r}(u, v) = \frac{1}{2}u \cos v\mathbf{i} + u\mathbf{j} + \frac{1}{3}u \sin v\mathbf{k},$$

$$u \geq 0, \quad 0 \leq v \leq 2\pi$$

28.  $\frac{x^2}{9} + \frac{y^2}{4} + \frac{z^2}{1} = 1$

$$\mathbf{r}(u, v) = 3 \cos v \cos u\mathbf{i} + 2 \cos v \sin u\mathbf{j} + \sin v\mathbf{k}$$

24.  $x = \sqrt{16y^2 + z^2}$

$$\mathbf{r}(y, z) = \sqrt{16y^2 + z^2}\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$

or,

$$\mathbf{r}(u, v) = u\mathbf{i} + \frac{1}{4}u \cos v\mathbf{j} + u \sin v\mathbf{k},$$

$$u \geq 0, \quad 0 \leq v \leq 2\pi$$

29.  $z = 4$  inside  $x^2 + y^2 = 9$ .

$$\mathbf{r}(u, v) = v \cos u\mathbf{i} + v \sin u\mathbf{j} + 4\mathbf{k}, \quad 0 \leq v \leq 3$$

25.  $x^2 + y^2 = 25$

$$\mathbf{r}(u, v) = 5 \cos u\mathbf{i} + 5 \sin u\mathbf{j} + v\mathbf{k}$$

30.  $z = x^2 + y^2$  inside  $x^2 + y^2 = 9$ .

$$\mathbf{r}(u, v) = v \cos u\mathbf{i} + v \sin u\mathbf{j} + v^2\mathbf{k}, \quad 0 \leq v \leq 3$$

31. Function:  $y = \frac{x}{2}$ ,  $0 \leq x \leq 6$

Axis of revolution:  $x$ -axis

$$x = u, y = \frac{u}{2} \cos v, z = \frac{u}{2} \sin v$$

$$0 \leq u \leq 6, \quad 0 \leq v \leq 2\pi$$

32. Function:  $y = \sqrt{x}$ ,  $0 \leq x \leq 4$

Axis of revolution:  $x$ -axis

$$x = u, y = \sqrt{u} \cos v, z = \sqrt{u} \sin v$$

$$0 \leq u \leq 4, \quad 0 \leq v \leq 2\pi$$

33. Function:  $x = \sin z$ ,  $0 \leq z \leq \pi$

Axis of revolution:  $z$ -axis

$$x = \sin u \cos v, y = \sin u \sin v, z = u$$

$$0 \leq u \leq \pi, \quad 0 \leq v \leq 2\pi$$

34. Function:  $z = y^2 + 1$ ,  $0 \leq y \leq 2$

Axis of revolution:  $y$ -axis

$$x = (u^2 + 1) \cos v, y = u, z = (u^2 + 1) \sin v$$

$$0 \leq u \leq 2, \quad 0 \leq v \leq 2\pi$$

35.  $\mathbf{r}(u, v) = (u + v)\mathbf{i} + (u - v)\mathbf{j} + v\mathbf{k}, (1, -1, 1)$

$$\mathbf{r}_u(u, v) = \mathbf{i} + \mathbf{j}, \mathbf{r}_v(u, v) = \mathbf{i} - \mathbf{j} + \mathbf{k}$$

$$\text{At } (1, -1, 1), u = 0 \text{ and } v = 1.$$

$$\mathbf{r}_u(0, 1) = \mathbf{i} + \mathbf{j}, \mathbf{r}_v(0, 1) = \mathbf{i} - \mathbf{j} + \mathbf{k}$$

$$\mathbf{N} = \mathbf{r}_u(0, 1) \times \mathbf{r}_v(0, 1) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 0 \\ 1 & -1 & 1 \end{vmatrix} = \mathbf{i} - \mathbf{j} - 2\mathbf{k}$$

$$\text{Tangent plane: } (x - 1) - (y + 1) - 2(z - 1) = 0$$

$$x - y - 2z = 0$$

(The original plane!)

36.  $\mathbf{r}(u, v) = u\mathbf{i} + v\mathbf{j} + \sqrt{uv}\mathbf{k}, (1, 1, 1)$

$$\mathbf{r}_u(u, v) = \mathbf{i} + \frac{v}{2\sqrt{uv}}\mathbf{k}, \mathbf{r}_v(u, v) = \mathbf{j} + \frac{u}{2\sqrt{uv}}\mathbf{k}$$

$$\text{At } (1, 1, 1), u = 1 \text{ and } v = 1.$$

$$\mathbf{r}_u(1, 1) = \mathbf{i} + \frac{1}{2}\mathbf{k}, \mathbf{r}_v(1, 1) = \mathbf{j} + \frac{1}{2}\mathbf{k}$$

$$\mathbf{N} = \mathbf{r}_u(1, 1) \times \mathbf{r}_v(1, 1) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & \frac{1}{2} \\ 0 & 1 & \frac{1}{2} \end{vmatrix} = -\frac{1}{2}\mathbf{i} - \frac{1}{2}\mathbf{j} + \mathbf{k}$$

$$\text{Direction numbers: } 1, 1, -2$$

$$\text{Tangent plane: } (x - 1) + (y - 1) - 2(z - 1) = 0$$

$$x + y - 2z = 0$$

37.  $\mathbf{r}(u, v) = 2u \cos v\mathbf{i} + 3u \sin v\mathbf{j} + u^2\mathbf{k}, (0, 6, 4)$

$$\mathbf{r}_u(u, v) = 2 \cos v\mathbf{i} + 3 \sin v\mathbf{j} + 2u\mathbf{k}$$

$$\mathbf{r}_v(u, v) = -2u \sin v\mathbf{i} + 3u \cos v\mathbf{j}$$

$$\text{At } (0, 6, 4), u = 2 \text{ and } v = \pi/2.$$

$$\mathbf{r}_u\left(2, \frac{\pi}{2}\right) = 3\mathbf{j} + 4\mathbf{k}, \mathbf{r}_v\left(2, \frac{\pi}{2}\right) = -4\mathbf{i}$$

$$\mathbf{N} = \mathbf{r}_u\left(2, \frac{\pi}{2}\right) \times \mathbf{r}_v\left(2, \frac{\pi}{2}\right) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 3 & 4 \\ -4 & 0 & 0 \end{vmatrix} = -16\mathbf{j} + 12\mathbf{k}$$

$$\text{Direction numbers: } 0, 4, -3$$

$$\text{Tangent plane: } 4(y - 6) - 3(z - 4) = 0$$

$$4y - 3z = 12$$

38.  $\mathbf{r}(u, v) = 2u \cosh v\mathbf{i} + 2u \sinh v\mathbf{j} + \frac{1}{2}u^2\mathbf{k},$

$$\mathbf{r}_u(u, v) = 2 \cosh v\mathbf{i} + 2 \sinh v\mathbf{j} + u\mathbf{k}$$

$$\mathbf{r}_v(u, v) = 2u \sinh v\mathbf{i} + 2u \cosh v\mathbf{j}$$

$$\text{At } (-4, 0, 2), u = -2 \text{ and } v = 0.$$

$$\mathbf{r}_u(-2, 0) = 2\mathbf{i} - 2\mathbf{k}, \mathbf{r}_v(-2, 0) = -4\mathbf{j}$$

$$\mathbf{N} = \mathbf{r}_u \times \mathbf{r}_v = -8\mathbf{i} - 8\mathbf{k}$$

$$\text{Direction numbers: } 1, 0, 1$$

$$\text{Tangent plane: } (x + 4) + (z - 2) = 0$$

$$x + z = -2$$

39.  $\mathbf{r}(u, v) = 4u\mathbf{i} - v\mathbf{j} + v\mathbf{k}, \quad 0 \leq u \leq 2, 0 \leq v \leq 1$

$$\mathbf{r}_u(u, v) = 4\mathbf{i}, \mathbf{r}_v(u, v) = -\mathbf{j} + \mathbf{k}$$

$$\mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & 0 & 0 \\ 0 & -1 & 1 \end{vmatrix} = -4\mathbf{j} - 4\mathbf{k}$$

$$\|\mathbf{r}_u \times \mathbf{r}_v\| = \sqrt{16 + 16} = 4\sqrt{2}$$

$$A = \int_0^1 \int_0^2 4\sqrt{2} \, du \, dv = 4\sqrt{2}(2)(1) = 8\sqrt{2}$$

40.  $\mathbf{r}(u, v) = 2u \cos v \mathbf{i} + 2u \sin v \mathbf{j} + u^2 \mathbf{k}, \quad 0 \leq u \leq 2, 0 \leq v \leq 2\pi$

$$\mathbf{r}_u(u, v) = 2 \cos v \mathbf{i} + 2 \sin v \mathbf{j} + 2u \mathbf{k}$$

$$\mathbf{r}_v(u, v) = -2u \sin v \mathbf{i} + 2u \cos v \mathbf{j}$$

$$\mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 \cos v & 2 \sin v & 2u \\ -2u \sin v & 2u \cos v & 0 \end{vmatrix} = -4u^2 \cos v \mathbf{i} - 4u^2 \sin v \mathbf{j} + 8u \mathbf{k}$$

$$\|\mathbf{r}_u \times \mathbf{r}_v\| = \sqrt{16u^4 \cos^2 v + 16u^4 \sin^2 v + 64u^2} = 4u\sqrt{u^2 + 4}$$

$$A = \int_0^{2\pi} \int_0^2 4u\sqrt{u^2 + 4} \, du \, dv = \int_0^{2\pi} \left[ \frac{4}{3}(u^2 + 4)^{3/2} \right]_0^2 \, dv = \int_0^{2\pi} \frac{4}{3}(8\sqrt{8} - 8) \, dv = \frac{4}{3}(16\sqrt{2} - 8)2\pi = \frac{64\pi}{3}(2\sqrt{2} - 1)$$

41.  $\mathbf{r}(u, v) = a \cos u \mathbf{i} + a \sin u \mathbf{j} + v \mathbf{k}, \quad 0 \leq u \leq 2\pi, 0 \leq v \leq b$

$$\mathbf{r}_u(u, v) = -a \sin u \mathbf{i} + a \cos u \mathbf{j}$$

$$\mathbf{r}_v(u, v) = \mathbf{k}$$

$$\mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -a \sin u & a \cos u & 0 \\ 0 & 0 & 1 \end{vmatrix} = a \cos u \mathbf{i} + a \sin u \mathbf{j}$$

$$\|\mathbf{r}_u \times \mathbf{r}_v\| = a$$

$$A = \int_0^b \int_0^{2\pi} a \, du \, dv = 2\pi ab$$

42.  $\mathbf{r}(u, v) = a \sin u \cos v \mathbf{i} + a \sin u \sin v \mathbf{j} + a \cos u \mathbf{k}, \quad 0 \leq u \leq \pi, \quad 0 \leq v \leq 2\pi$

$$\mathbf{r}_u(u, v) = a \cos u \cos v \mathbf{i} + a \cos u \sin v \mathbf{j} - a \sin u \mathbf{k}$$

$$\mathbf{r}_v(u, v) = -a \sin u \sin v \mathbf{i} + a \sin u \cos v \mathbf{j}$$

$$\mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a \cos u \cos v & a \cos u \sin v & -a \sin u \\ -a \sin u \sin v & a \sin u \cos v & 0 \end{vmatrix} = a^2 \sin^2 u \cos v \mathbf{i} + a^2 \sin^2 u \sin v \mathbf{j} + a^2 \sin u \cos u \mathbf{k}$$

$$\|\mathbf{r}_u \times \mathbf{r}_v\| = a^2 \sin u$$

$$A = \int_0^{2\pi} \int_0^\pi a^2 \sin u \, du \, dv = 4\pi a^2$$

43.  $\mathbf{r}(u, v) = au \cos v \mathbf{i} + au \sin v \mathbf{j} + u \mathbf{k}, \quad 0 \leq u \leq b, \quad 0 \leq v \leq 2\pi$

$$\mathbf{r}_u(u, v) = a \cos v \mathbf{i} + a \sin v \mathbf{j} + \mathbf{k}$$

$$\mathbf{r}_v(u, v) = -au \sin v \mathbf{i} + au \cos v \mathbf{j}$$

$$\mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a \cos v & a \sin v & 1 \\ -au \sin v & au \cos v & 0 \end{vmatrix} = -au \cos v \mathbf{i} - au \sin v \mathbf{j} + a^2 u \mathbf{k}$$

$$\|\mathbf{r}_u \times \mathbf{r}_v\| = au\sqrt{1 + a^2}$$

$$A = \int_0^{2\pi} \int_0^b a\sqrt{1 + a^2} u \, du \, dv = \pi ab^2\sqrt{1 + a^2}$$

44.  $\mathbf{r}(u, v) = (a + b \cos v)\cos u\mathbf{i} + (a + b \cos v)\sin u\mathbf{j} + b \sin v\mathbf{k}, a > b, 0 \leq u \leq 2\pi, 0 \leq v \leq 2\pi$

$$\mathbf{r}_u(u, v) = -(a + b \cos v)\sin u\mathbf{i} + (a + b \cos v)\cos u\mathbf{j}$$

$$\mathbf{r}_v(u, v) = -b \sin v \cos u\mathbf{i} - b \sin v \sin u\mathbf{j} + b \cos v\mathbf{k}$$

$$\begin{aligned}\mathbf{r}_u \times \mathbf{r}_v &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -(a + b \cos v)\sin u & (a + b \cos v)\cos u & 0 \\ -b \sin v \cos u & -b \sin v \sin u & b \cos v \end{vmatrix} \\ &= b \cos u \cos v(a + b \cos v)\mathbf{i} + b \sin u \cos v(a + b \cos v)\mathbf{j} + b \sin v(a + b \cos v)\mathbf{k}\end{aligned}$$

$$\|\mathbf{r}_u \times \mathbf{r}_v\| = b(a + b \cos v)$$

$$A = \int_0^{2\pi} \int_0^{2\pi} b(a + b \cos v) du dv = 4\pi^2 ab$$

45.  $\mathbf{r}(u, v) = \sqrt{u} \cos v\mathbf{i} + \sqrt{u} \sin v\mathbf{j} + u\mathbf{k}, 0 \leq u \leq 4, 0 \leq v \leq 2\pi$

$$\mathbf{r}_u(u, v) = \frac{\cos v}{2\sqrt{u}}\mathbf{i} + \frac{\sin v}{2\sqrt{u}}\mathbf{j} + \mathbf{k}$$

$$\mathbf{r}_v(u, v) = -\sqrt{u} \sin v\mathbf{i} + \sqrt{u} \cos v\mathbf{j}$$

$$\begin{aligned}\mathbf{r}_u \times \mathbf{r}_v &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\cos v}{2\sqrt{u}} & \frac{\sin v}{2\sqrt{u}} & 1 \\ -\sqrt{u} \sin v & \sqrt{u} \cos v & 0 \end{vmatrix} = -\sqrt{u} \cos v\mathbf{i} - \sqrt{u} \sin v\mathbf{j} + \frac{1}{2}\mathbf{k}\end{aligned}$$

$$\|\mathbf{r}_u \times \mathbf{r}_v\| = \sqrt{u + \frac{1}{4}}$$

$$A = \int_0^{2\pi} \int_0^4 \sqrt{u + \frac{1}{4}} du dv = \frac{\pi}{6}(17\sqrt{17} - 1) \approx 36.177$$

46.  $\mathbf{r}(u, v) = \sin u \cos v\mathbf{i} + u\mathbf{j} + \sin u \sin v\mathbf{k}, 0 \leq u \leq \pi, 0 \leq v \leq 2\pi$

$$\mathbf{r}_u(u, v) = \cos u \cos v\mathbf{i} + \mathbf{j} + \cos u \sin v\mathbf{k}$$

$$\mathbf{r}_v(u, v) = -\sin u \sin v\mathbf{i} + \sin u \cos v\mathbf{k}$$

$$\mathbf{r}_u \times \mathbf{r}_v = \sin u \cos v\mathbf{i} - \cos u \sin u\mathbf{j} + \sin u \sin v\mathbf{k}$$

$$\|\mathbf{r}_u \times \mathbf{r}_v\| = \sin u \sqrt{1 + \cos^2 u}$$

$$A = \int_0^{2\pi} \int_0^\pi \sin u \sqrt{1 + \cos^2 u} du dv = \pi \left[ 2\sqrt{2} + \ln \left| \frac{\sqrt{2} + 1}{\sqrt{2} - 1} \right| \right]$$

47. See the definition, page 1102.

50. (a) From  $(-10, 10, 0)$

48. See the definition, page 1106.

(b) From  $(10, 10, 10)$

49. Function:  $z = x$

(c) From  $(0, 10, 0)$

Axis of revolution:  $z$ -axis

(d) From  $(10, 0, 0)$

$$x = u \cos v, y = u \sin v, z = u$$

$$\mathbf{r}(u, v) = u \cos v\mathbf{i} + u \sin v\mathbf{j} + u\mathbf{k}$$

$$u \leq 0, 0 \leq v \leq 2\pi$$

51.  $\mathbf{r}(u, v) = a \sin^3 u \cos^3 v\mathbf{i} + a \sin^3 u \sin^3 v\mathbf{j} + a \cos^3 u\mathbf{k}$

$$0 \leq u \leq \pi, 0 \leq v \leq 2\pi$$

$$x = a \sin^3 u \cos^3 v \Rightarrow x^{2/3} = a^{2/3} \sin^2 u \cos^2 v$$

$$y = a \sin^3 u \sin^3 v \Rightarrow y^{2/3} = a^{2/3} \sin^2 u \sin^2 v$$

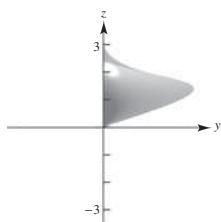
$$z = a \cos^3 u \Rightarrow z^{2/3} = a^{2/3} \cos^2 u$$

$$x^{2/3} + y^{2/3} + z^{2/3} = a^{2/3} [\sin^2 u \cos^2 v + \sin^2 u \sin^2 v + \cos^2 u] = a^{2/3} [\sin^2 u + \cos^2 u] = a^{2/3}$$

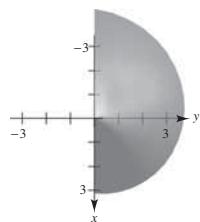
52. Graph of  $\mathbf{r}(u, v) = u \cos v\mathbf{i} + u \sin v\mathbf{j} + v\mathbf{k}$

$$0 \leq u \leq \pi, 0 \leq v \leq \pi \text{ from}$$

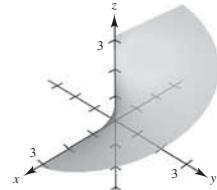
(a)  $(10, 0, 0)$



(b)  $(0, 0, 10)$



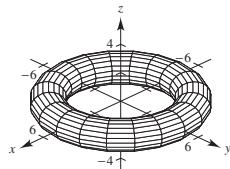
(c)  $(10, 10, 10)$



53. (a)  $\mathbf{r}(u, v) = (4 + \cos v)\cos u\mathbf{i} +$

$$(4 + \cos v)\sin u\mathbf{j} + \sin v\mathbf{k},$$

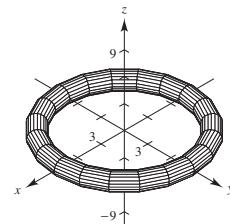
$$0 \leq u \leq 2\pi, 0 \leq v \leq 2\pi$$



(c)  $\mathbf{r}(u, v) = (8 + \cos v)\cos u\mathbf{i} +$

$$(8 + \cos v)\sin u\mathbf{j} + \sin v\mathbf{k},$$

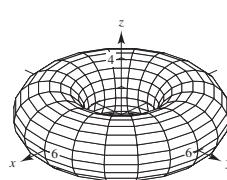
$$0 \leq u \leq 2\pi, 0 \leq v \leq 2\pi$$



(b)  $\mathbf{r}(u, v) = (4 + 2 \cos v)\cos u\mathbf{i} +$

$$(4 + 2 \cos v)\sin u\mathbf{j} + 2 \sin v\mathbf{k},$$

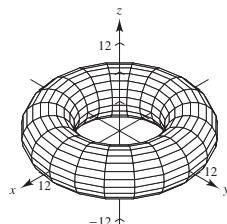
$$0 \leq u \leq 2\pi, 0 \leq v \leq 2\pi$$



(d)  $\mathbf{r}(u, v) = (8 + 3 \cos v)\cos u\mathbf{i} +$

$$(8 + 3 \cos v)\sin u\mathbf{j} + 3 \sin v\mathbf{k},$$

$$0 \leq u \leq 2\pi, 0 \leq v \leq 2\pi$$



The radius of the generating circle that is revolved about the z-axis is  $b$ , and its center is  $a$  units from the axis of revolution.

54.  $\mathbf{r}(u, v) = 2u \cos v\mathbf{i} + 2u \sin v\mathbf{j} + v\mathbf{k}, 0 \leq u \leq 1, 0 \leq v \leq 3\pi$

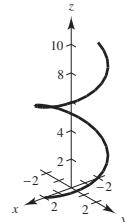
(a) If  $u = 1$ :

$$\mathbf{r}(1, v) = 2 \cos v\mathbf{i} + 2 \sin v\mathbf{j} + v\mathbf{k}$$

$$x^2 + y^2 = 4$$

$$0 \leq z \leq 3\pi$$

Helix



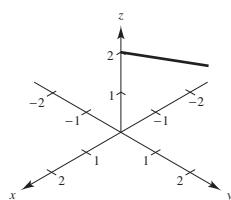
(b) If  $v = \frac{2\pi}{3}$ :

$$\mathbf{r}\left(u, \frac{2\pi}{3}\right) = -u\mathbf{i} + \sqrt{3}u\mathbf{j} + \frac{2\pi}{3}\mathbf{k}$$

$$y = -\sqrt{3}x$$

$$z = \frac{2\pi}{3}$$

Line



(c) If one parameter is held constant, the result is a **curve** in 3-space.

55.  $\mathbf{r}(u, v) = 20 \sin u \cos v \mathbf{i} + 20 \sin u \sin v \mathbf{j} + 20 \cos u \mathbf{k}, 0 \leq u \leq \pi/3, 0 \leq v \leq 2\pi$

$$\mathbf{r}_u = 20 \cos u \cos v \mathbf{i} + 20 \cos u \sin v \mathbf{j} - 20 \sin u \mathbf{k}$$

$$\mathbf{r}_v = -20 \sin u \sin v \mathbf{i} + 20 \sin u \cos v \mathbf{j}$$

$$\begin{aligned}\mathbf{r}_u \times \mathbf{r}_v &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 20 \cos u \cos v & 20 \cos u \sin v & -20 \sin u \\ -20 \sin u \sin v & 20 \sin u \cos v & 0 \end{vmatrix} \\ &= 400 \sin^2 u \cos v \mathbf{i} + 400 \sin^2 u \sin v \mathbf{j} + 400(\cos u \sin u \cos^2 v + \cos u \sin u \sin^2 v) \mathbf{k} \\ &= 400[\sin^2 u \cos v \mathbf{i} + \sin^2 u \sin v \mathbf{j} + \cos u \sin u \mathbf{k}] \\ \|\mathbf{r}_u \times \mathbf{r}_v\| &= 400 \sqrt{\sin^4 u \cos^2 v + \sin^4 u \sin^2 v + \cos^2 u \sin^2 u} = 400 \sqrt{\sin^4 u + \cos^2 u \sin^2 u} = 400 \sqrt{\sin^2 u} = 400 \sin u \\ S &= \int_S \int dS = \int_0^{2\pi} \int_0^{\pi/3} 400 \sin u \, du \, dv = \int_0^{2\pi} [-400 \cos u]_0^{\pi/3} \, dv = \int_0^{2\pi} 200 \, dv = 400\pi \text{ m}^2\end{aligned}$$

56.  $x^2 + y^2 - z^2 = 1$

Let  $x = u \cos v, y = u \sin v$ , and  $z = \sqrt{u^2 - 1}$ . Then,

$$\mathbf{r}_u(u, v) = \cos v \mathbf{i} + \sin v \mathbf{j} + \frac{u}{\sqrt{u^2 - 1}} \mathbf{k}$$

$$\mathbf{r}_v(u, v) = -u \sin v \mathbf{i} + u \cos v \mathbf{j}$$

At  $(1, 0, 0), u = 1$  and  $v = 0$ .  $\mathbf{r}_u(1, 0)$  is undefined and  $\mathbf{r}_v(1, 0) = \mathbf{j}$ . The tangent plane at  $(1, 0, 0)$  is  $x = 1$ .

57.  $\mathbf{r}(u, v) = u \cos v \mathbf{i} + u \sin v \mathbf{j} + 2v \mathbf{k}, 0 \leq u \leq 3, 0 \leq v \leq 2\pi$

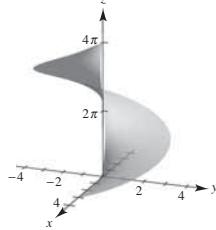
$$\mathbf{r}_u(u, v) = \cos v \mathbf{i} + \sin v \mathbf{j}$$

$$\mathbf{r}_v(u, v) = -u \sin v \mathbf{i} + u \cos v \mathbf{j} + 2 \mathbf{k}$$

$$\mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos v & \sin v & 0 \\ -u \sin v & u \cos v & 2 \end{vmatrix} = 2 \sin v \mathbf{i} - 2 \cos v \mathbf{j} + u \mathbf{k}$$

$$\|\mathbf{r}_u \times \mathbf{r}_v\| = \sqrt{4 + u^2}$$

$$A = \int_0^{2\pi} \int_0^3 \sqrt{4 + u^2} \, du \, dv = \pi \left[ 3\sqrt{13} + 4 \ln \left( \frac{3 + \sqrt{13}}{2} \right) \right]$$



58.  $\mathbf{r}(u, v) = u \mathbf{i} + f(u) \cos v \mathbf{j} + f(u) \sin v \mathbf{k}, a \leq u \leq b, 0 \leq v \leq 2\pi$

$$\mathbf{r}_u(u, v) = \mathbf{i} + f'(u) \cos v \mathbf{j} + f'(u) \sin v \mathbf{k}$$

$$\mathbf{r}_v(u, v) = -f(u) \sin v \mathbf{j} + f(u) \cos v \mathbf{k}$$

$$\mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & f'(u) \cos v & f'(u) \sin v \\ 0 & -f(u) \sin v & f(u) \cos v \end{vmatrix} = f(u) f'(u) \mathbf{i} - f(u) \cos v \mathbf{j} - f(u) \sin v \mathbf{k}$$

$$\|\mathbf{r}_u \times \mathbf{r}_v\| = f(u) \sqrt{1 + [f'(u)]^2}$$

$$A = \int_0^{2\pi} \int_a^b f(u) \sqrt{1 + [f'(u)]^2} \, du \, dv = 2\pi \int_a^b f(x) \sqrt{1 + [f'(x)]^2} \, dx \quad (\text{because } u = x)$$

59. Answers will vary.

60. Answers will vary.

## Section 15.6 Surface Integrals

1.  $S: z = 4 - x, \quad 0 \leq x \leq 4, \quad 0 \leq y \leq 3, \quad \frac{\partial z}{\partial x} = -1, \quad \frac{\partial z}{\partial y} = 0$

$$\int_S \int (x - 2y + z) dS = \int_0^4 \int_0^3 (x - 2y + 4 - x) \sqrt{1 + (-1)^2 + 0^2} dy dx = \sqrt{2} \int_0^4 \int_0^3 (4 - 2y) dy dx = \sqrt{2} \int_0^4 3 dx = 12\sqrt{2}$$

2.  $S: z = 15 - 2x + 3y, \quad 0 \leq x \leq 2, \quad 0 \leq y \leq 4, \quad \frac{\partial z}{\partial x} = -2, \quad \frac{\partial z}{\partial y} = 3, \quad dS = \sqrt{1 + 4 + 9} dy dx = \sqrt{14} dy dx$

$$\int_S \int (x - 2y + z) dS = \int_0^2 \int_0^4 (x - 2y + 15 - 2x + 3y) \sqrt{14} dy dx = \sqrt{14} \int_0^2 \int_0^4 (15 - x + y) dy dx = 128\sqrt{14}$$

3.  $S: z = 2, \quad x^2 + y^2 \leq 1, \quad \frac{\partial z}{\partial x} = \frac{\partial z}{\partial y} = 0$

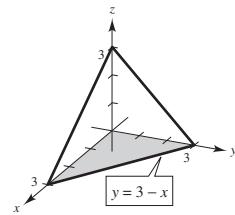
$$\begin{aligned} \int_S \int (x - 2y + z) dS &= \int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (x - 2y + 2) \sqrt{1 + 0^2 + 0^2} dy dx = \int_0^{2\pi} \int_0^1 (r \cos \theta - 2r \sin \theta + 2) r dr d\theta \\ &= \int_0^{2\pi} \left[ \frac{1}{3} \cos \theta - \frac{2}{3} \sin \theta + 1 \right] d\theta = \left[ \frac{1}{3} \sin \theta + \frac{2}{3} \cos \theta + \theta \right]_0^{2\pi} = \frac{2}{3} + 2\pi - \frac{2}{3} = 2\pi \end{aligned}$$

4.  $S: z = \frac{2}{3}x^{3/2}, \quad 0 \leq x \leq 1, \quad 0 \leq y \leq x, \quad \frac{\partial z}{\partial x} = x^{1/2}, \quad \frac{\partial z}{\partial y} = 0$

$$\begin{aligned} \int_S \int (x - 2y + z) dS &= \int_0^1 \int_0^x \left( x - 2y + \frac{2}{3}x^{3/2} \right) \sqrt{1 + (x^{1/2})^2 + (0)^2} dy dx \\ &= \int_0^1 \int_0^x \left( x - 2y + \frac{2}{3}x^{3/2} \right) \sqrt{1+x} dy dx \\ &= \frac{2}{3} \int_0^1 x^{5/2} \sqrt{x+1} dx \\ &= \frac{2}{3} \left[ \frac{1}{4} x^{5/2} (1+x)^{3/2} \right]_0^1 - \frac{5}{12} \int_0^1 x^{3/2} \sqrt{1+x} dx \\ &= \left[ \frac{1}{6} x^{5/2} (1+x)^{3/2} \right]_0^1 - \frac{5}{12} \left( \frac{1}{3} \right) \left[ x^{3/2} (1+x)^{3/2} \right]_0^1 + \frac{5}{24} \int_0^1 x^{1/2} \sqrt{1+x} dx \\ &= \frac{\sqrt{2}}{3} - \frac{5\sqrt{2}}{18} + \frac{5}{24} \int_0^1 \sqrt{x+x^2} dx \\ &= \frac{\sqrt{2}}{18} + \frac{5}{24} \int_0^1 \sqrt{\left(x+\frac{1}{2}\right)^2 - \frac{1}{4}} dx \\ &= \frac{\sqrt{2}}{18} + \frac{5}{24} \left( \frac{1}{2} \right) \left[ \left(x+\frac{1}{2}\right) \sqrt{x^2+x} - \frac{1}{4} \ln \left| \left(x+\frac{1}{2}\right) + \sqrt{x^2+x} \right| \right]_0^1 \\ &= \frac{\sqrt{2}}{18} + \frac{5}{48} \left[ \frac{3}{2} \sqrt{2} - \frac{1}{4} \ln \left| \frac{3}{2} + \sqrt{2} \right| + \frac{1}{4} \ln \left| \frac{1}{2} \right| \right] \\ &= \frac{\sqrt{2}}{18} + \frac{15\sqrt{2}}{96} + \frac{5}{192} \ln \left| \frac{1}{3+2\sqrt{2}} \right| = \frac{61\sqrt{2}}{288} - \frac{5}{192} \ln |3+2\sqrt{2}| \approx 0.2536 \end{aligned}$$

5.  $S: z = 3 - x - y \quad (\text{first octant}), \quad \frac{\partial z}{\partial x} = -1, \quad \frac{\partial z}{\partial y} = -1$

$$\begin{aligned} \int_S \int xy dS &= \int_0^3 \int_0^{3-x} xy \sqrt{1 + (-1)^2 + (-1)^2} dy dx = \sqrt{3} \int_0^3 \left[ x \frac{y^2}{2} \right]_0^{3-x} \\ &= \frac{\sqrt{3}}{2} \int_0^3 x(3-x)^2 dx = \frac{\sqrt{3}}{2} \left[ \frac{x^4}{4} - 2x^3 + \frac{9x^2}{2} \right]_0^3 = \frac{\sqrt{3}}{2} \left[ \frac{27}{4} \right] = \frac{27\sqrt{3}}{8} \end{aligned}$$



6.  $S: z = h, 0 \leq x \leq 2, 0 \leq y \leq \sqrt{4 - x^2}, \frac{\partial z}{\partial x} = \frac{\partial z}{\partial y} = 0$

$$\int_S \int xy \, dS = \int_0^2 \int_0^{\sqrt{4-x^2}} xy \, dy \, dx = \frac{1}{2} \int_0^2 x(4 - x^2) \, dx = \frac{1}{2} \left[ 2x^2 - \frac{x^4}{4} \right]_0^2 = 2$$

7.  $S: z = 9 - x^2, 0 \leq x \leq 2, 0 \leq y \leq x,$

$$\frac{\partial z}{\partial x} = -2x, \frac{\partial z}{\partial y} = 0$$

$$\int_S \int xy \, dS = \int_0^2 \int_y^2 xy \sqrt{1 + 4x^2} \, dx \, dy = \frac{391\sqrt{17} + 1}{240}$$

8.  $S: z = \frac{1}{2}xy, 0 \leq x \leq 4, 0 \leq y \leq 4, \frac{\partial z}{\partial x} = \frac{1}{2}y, \frac{\partial z}{\partial y} = \frac{1}{2}x$

$$\int_S \int xy \, dS = \int_0^4 \int_0^4 xy \sqrt{1 + \frac{y^2}{4} + \frac{x^2}{4}} \, dy \, dx = \frac{3904}{15} - \frac{160\sqrt{5}}{3}$$

9.  $S: z = 10 - x^2 - y^2, 0 \leq x \leq 2, 0 \leq y \leq 2$

$$\int_S \int (x^2 - 2xy) \, dS = \int_0^2 \int_0^2 (x^2 - 2xy) \sqrt{1 + 4x^2 + 4y^2} \, dy \, dx \approx -11.47$$

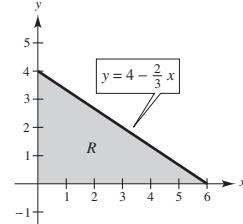
10.  $S: z = \cos x, 0 \leq x \leq \frac{\pi}{2}, 0 \leq y \leq \frac{x}{2}$

$$\int_S \int (x^2 - 2xy) \, dS = \int_0^{\pi/2} \int_0^{x/2} (x^2 - 2xy) \sqrt{1 + \sin^2 x} \, dy \, dx = \int_0^{\pi/2} \frac{x^3}{4} \sqrt{1 + \sin^2 x} \, dx \approx 0.52$$

11.  $S: 2x + 3y + 6z = 12 \quad (\text{first octant}) \Rightarrow z = 2 - \frac{1}{3}x - \frac{1}{2}y$

$$\rho(x, y, z) = x^2 + y^2$$

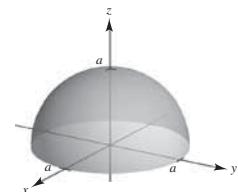
$$\begin{aligned} m &= \int_R \int (x^2 + y^2) \sqrt{1 + \left(-\frac{1}{3}\right)^2 + \left(-\frac{1}{2}\right)^2} \, dA = \frac{7}{6} \int_0^6 \int_0^{4-(2x/3)} (x^2 + y^2) \, dy \, dx \\ &= \frac{7}{6} \int_0^6 \left[ x^2 \left(4 - \frac{2}{3}x\right) + \frac{1}{3}(4 - \frac{2}{3}x)^3 \right] dx = \frac{7}{6} \left[ \frac{4}{3}x^3 - \frac{1}{6}x^4 - \frac{1}{8}(4 - \frac{2}{3}x)^4 \right]_0^6 = \frac{364}{3} \end{aligned}$$



12.  $S: z = \sqrt{a^2 - x^2 - y^2}$

$$\rho(x, y, z) = kz$$

$$\begin{aligned} m &= \int_S \int k \, dS = \int_R \int k \sqrt{a^2 - x^2 - y^2} \sqrt{1 + \left(\frac{-x}{\sqrt{a^2 - x^2 - y^2}}\right)^2 + \left(\frac{-y}{\sqrt{a^2 - x^2 - y^2}}\right)^2} \, dA \\ &= \int_R \int k \sqrt{a^2 - x^2 - y^2} \left( \frac{a}{\sqrt{a^2 - x^2 - y^2}} \right) \, dA = \int_R \int ka \, dA = ka \int_R \int dA = ka(2\pi a^2) = 2ka^3\pi \end{aligned}$$



13.  $S: \mathbf{r}(u, v) = u\mathbf{i} + v\mathbf{j} + 2v\mathbf{k}, 0 \leq u \leq 1, 0 \leq v \leq 2$

$$\mathbf{r}_u = \mathbf{i}, \quad \mathbf{r}_v = \mathbf{j} + 2\mathbf{k}$$

$$\mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 0 \\ 0 & 1 & 2 \end{vmatrix} = -2\mathbf{j} + \mathbf{k}$$

$$\|\mathbf{r}_u \times \mathbf{r}_v\| = \sqrt{5}$$

$$\int_S \int (y + 5) \, dS = \int_0^2 \int_0^1 (v + 5) \sqrt{5} \, du \, dv = \int_0^2 (v + 5) \sqrt{5} \, dv = \sqrt{5} \left[ \frac{v^2}{2} + 5v \right]_0^2 = 12\sqrt{5}$$

14.  $\mathbf{r}(u, v) = 2 \cos u \mathbf{i} + 2 \sin u \mathbf{j} + v \mathbf{k}, \quad 0 \leq u \leq \pi/2, \quad 0 \leq v \leq 1$

$$\mathbf{r}_u = -2 \sin u \mathbf{i} + 2 \cos u \mathbf{j}, \quad \mathbf{r}_v = \mathbf{k}$$

$$\mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 \sin u & 2 \cos u & 0 \\ 0 & 0 & 1 \end{vmatrix} = 2 \cos u \mathbf{i} + 2 \sin u \mathbf{j}$$

$$\|\mathbf{r}_u \times \mathbf{r}_v\| = \sqrt{4 \cos^2 u + 4 \sin^2 u} = 2$$

$$\int_S \int xy \, dS = \int_0^1 \int_0^{\pi/2} (2 \cos u)(2 \sin u) 2 \, du \, dv = 8 \int_0^1 \left[ \frac{\sin^2 u}{2} \right]_0^{\pi/2} \, dv = 4$$

15.  $S: \mathbf{r}(u, v) = 2 \cos u \mathbf{i} + 2 \sin u \mathbf{j} + v \mathbf{k}, \quad 0 \leq u \leq \pi/2, \quad 0 \leq v \leq 1$

$$\mathbf{r}_u = -2 \sin u \mathbf{i} + 2 \cos u \mathbf{j}, \quad \mathbf{r}_v = \mathbf{k}$$

$$\mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 \sin u & 2 \cos u & 0 \\ 0 & 0 & 1 \end{vmatrix} = 2 \cos u \mathbf{i} + 2 \sin u \mathbf{j}$$

$$\|\mathbf{r}_u \times \mathbf{r}_v\| = \sqrt{4 \cos^2 u + 4 \sin^2 u} = 2$$

$$\int_S \int (x + y) \, dS = \int_0^1 \int_0^{\pi/2} (2 \cos u + 2 \sin u) 2 \, du \, dv = 4 \int_0^1 [\sin u - \cos u]_0^{\pi/2} \, dv = 4 \int_0^1 2 \, dv = 8$$

16.  $S: \mathbf{r}(u, v) = 4u \cos v \mathbf{i} + 4u \sin v \mathbf{j} + 3u \mathbf{k}, \quad 0 \leq u \leq 4, \quad 0 \leq v \leq \pi$

$$\|\mathbf{r}_u \times \mathbf{r}_v\| = \| -12u \cos v \mathbf{i} - 12u \sin v \mathbf{j} + 16u \mathbf{k} \| = 20u$$

$$\int_S \int (x + y) \, dS = \int_0^\pi \int_0^4 (4u \cos v + 4u \sin v) 20u \, du \, dv = \frac{10,240}{3}$$

17.  $f(x, y, z) = x^2 + y^2 + z^2$

$$S: z = x + y, \quad x^2 + y^2 \leq 1, \quad \frac{\partial z}{\partial x} = \frac{\partial z}{\partial y} = 1$$

$$\begin{aligned} \int_S \int f(x, y, z) \, dS &= \int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} [x^2 + y^2 + (x + y)^2] \sqrt{1 + 1^2 + 1^2} \, dy \, dx \\ &= \sqrt{3} \int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} [2x^2 + 2y^2 + 2xy] \, dy \, dx = \sqrt{3} \int_0^{2\pi} \int_0^1 (2r^2 + 2r \cos \theta r \sin \theta) r \, dr \, d\theta \\ &= 2\sqrt{3} \int_0^{2\pi} \left[ \frac{r^4}{4} + \frac{r^4}{4} \cos \theta \sin \theta \right]_0^1 \, d\theta = \frac{\sqrt{3}}{2} \int_0^{2\pi} (1 + \cos \theta \sin \theta) \, d\theta = \frac{\sqrt{3}}{2} \left[ \theta + \frac{\sin^2 \theta}{2} \right]_0^{2\pi} = \sqrt{3}\pi \end{aligned}$$

18.  $f(x, y, z) = \frac{xy}{z}$

$$S: z = x^2 + y^2, \quad 4 \leq x^2 + y^2 \leq 16$$

$$\begin{aligned} \int_S \int f(x, y, z) \, dS &= \int_S \int \frac{xy}{x^2 + y^2} \sqrt{1 + 4x^2 + 4y^2} \, dy \, dx = \int_0^{2\pi} \int_2^4 \frac{r^2 \sin \theta \cos \theta}{r^2} \sqrt{1 + 4r^2} r \, dr \, d\theta \\ &= \int_0^{2\pi} \int_2^4 r \sqrt{1 + 4r^2} \sin \theta \cos \theta \, dr \, d\theta = \int_0^{2\pi} \left[ \frac{1}{12} (1 + 4r^2)^{3/2} \right]_2^4 \sin \theta \cos \theta \, d\theta \\ &= \left[ \frac{65\sqrt{65} - 17\sqrt{17}}{12} \left( \frac{\sin^2 \theta}{2} \right) \right]_0^{2\pi} = 0 \end{aligned}$$

19.  $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$

$S: z = \sqrt{x^2 + y^2}, x^2 + y^2 \leq 4$

$$\begin{aligned} \int_S \int f(x, y, z) dS &= \int_{-2}^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \sqrt{x^2 + y^2 + (\sqrt{x^2 + y^2})^2} \sqrt{1 + \left(\frac{x}{\sqrt{x^2 + y^2}}\right)^2 + \left(\frac{y}{\sqrt{x^2 + y^2}}\right)^2} dy dx \\ &= \sqrt{2} \int_{-2}^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \sqrt{x^2 + y^2} \sqrt{\frac{x^2 + y^2 + x^2 + y^2}{x^2 + y^2}} dy dx \\ &= 2 \int_{-2}^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \sqrt{x^2 + y^2} dy dx = 2 \int_0^{2\pi} \int_0^2 r^2 dr d\theta = 2 \int_0^{2\pi} \left[ \frac{r^3}{3} \right]_0^2 d\theta = \left[ \frac{16}{3} \theta \right]_0^{2\pi} = \frac{32\pi}{3} \end{aligned}$$

20.  $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$

$S: z = \sqrt{x^2 + y^2}, (x - 1)^2 + y^2 \leq 1$

$$\begin{aligned} \int_S \int f(x, y, z) dS &= \int_S \int \sqrt{x^2 + y^2 + (\sqrt{x^2 + y^2})^2} \sqrt{1 + \left(\frac{x}{\sqrt{x^2 + y^2}}\right)^2 + \left(\frac{y}{\sqrt{x^2 + y^2}}\right)^2} dy dx \\ &= \int_S \int \sqrt{2(x^2 + y^2)} \sqrt{\frac{2(x^2 + y^2)}{x^2 + y^2}} dy dx = 2 \int_S \int \sqrt{x^2 + y^2} dy dx = 2 \int_0^\pi \int_0^{2\cos\theta} r^2 dr d\theta \\ &= \frac{16}{3} \int_0^\pi \cos^3 \theta d\theta = \frac{16}{3} \int_0^\pi (1 - \sin^2 \theta) \cos \theta d\theta = \left[ \frac{16}{3} \left( \sin \theta - \frac{\sin^3 \theta}{3} \right) \right]_0^\pi = 0 \end{aligned}$$

21.  $f(x, y, z) = x^2 + y^2 + z^2$

$S: x^2 + y^2 = 9, 0 \leq x \leq 3, 0 \leq y \leq 3, 0 \leq z \leq 9$

Project the solid onto the  $yz$ -plane;  $x = \sqrt{9 - y^2}, 0 \leq y \leq 3, 0 \leq z \leq 9$ .

$$\begin{aligned} \int_S \int f(x, y, z) dS &= \int_0^3 \int_0^9 [(9 - y^2) + y^2 + z^2] \sqrt{1 + \left(\frac{-y}{\sqrt{9 - y^2}}\right)^2 + (0)^2} dz dy \\ &= \int_0^3 \int_0^9 (9 + z^2) \frac{3}{\sqrt{9 - y^2}} dz dy = \int_0^3 \left[ \frac{3}{\sqrt{9 - y^2}} \left( 9z + \frac{z^3}{3} \right) \right]_0^9 dy \\ &= 324 \int_0^3 \frac{3}{\sqrt{9 - y^2}} dy = \left[ 972 \arcsin\left(\frac{y}{3}\right) \right]_0^3 = 972 \left( \frac{\pi}{2} - 0 \right) = 486\pi \end{aligned}$$

22.  $f(x, y, z) = x^2 + y^2 + z^2$

$S: x^2 + y^2 = 9, 0 \leq x \leq 3, 0 \leq z \leq x$

Project the solid onto the  $xz$ -plane;  $y = \sqrt{9 - x^2}$ .

$$\begin{aligned} \int_S \int f(x, y, z) dS &= \int_0^3 \int_0^x [x^2 + (9 - x^2) + z^2] \sqrt{1 + \left(\frac{-x}{\sqrt{9 - x^2}}\right)^2 + (0)^2} dz dx \\ &= \int_0^3 \int_0^x (9 + z^2) \frac{3}{\sqrt{9 - x^2}} dz dx = \int_0^3 \left[ \frac{3}{\sqrt{9 - x^2}} \left( 9z + \frac{z^3}{3} \right) \right]_0^x dx \\ &= \int_0^3 \frac{3}{\sqrt{9 - x^2}} \left( 9x + \frac{x^3}{3} \right) dx = \int_0^3 27x(9 - x^2)^{-1/2} dx + \int_0^3 x^3(9 - x^2)^{-1/2} dx \end{aligned}$$

Let  $u = x^2, dv = x(9 - x^2)^{-1/2} dx$ , then  $du = 2x dx, v = -\sqrt{9 - x^2}$ .

$$= \left[ -27\sqrt{9 - x^2} \right]_0^3 + \left[ \left[ -x^2\sqrt{9 - x^2} \right]_0^3 + \int_0^3 2x\sqrt{9 - x^2} dx \right] = \left[ 81 - \frac{2}{3}(9 - x^2)^{3/2} \right]_0^3 = 81 + 18 = 99$$

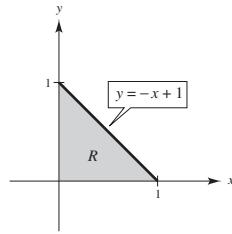
23.  $\mathbf{F}(x, y, z) = 3z\mathbf{i} - 4\mathbf{j} + y\mathbf{k}$

$S$ :  $z = 1 - x - y$  (first octant)

$$G(x, y, z) = x + y + z - 1$$

$$\nabla G(x, y, z) = \mathbf{i} + \mathbf{j} + \mathbf{k}$$

$$\begin{aligned}\int_S \int \mathbf{F} \cdot \mathbf{N} dS &= \int_R \int \mathbf{F} \cdot \nabla G dA = \int_0^1 \int_0^{1-x} (3z - 4 + y) dy dx \\ &= \int_0^1 \int_0^{1-x} [3(1-x-y) - 4 + y] dy dx \\ &= \int_0^1 \int_0^{1-x} (-1 - 3x - 2y) dy dx = \int_0^1 [-y - 3xy - y^2]_0^{1-x} dx \\ &= -\int_0^1 [(1-x) + 3x(1-x) + (1-x)^2] dx = -\int_0^1 (2 - 2x^2) dx = -\frac{4}{3}\end{aligned}$$



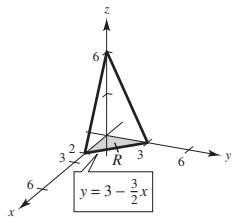
24.  $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j}$

$S$ :  $z = 6 - 3x - 2y$ , first octant

$$G(x, y, z) = 3x + 2y + z - 6$$

$$\nabla G(x, y, z) = 3\mathbf{i} + 2\mathbf{j} + \mathbf{k}$$

$$\begin{aligned}\int_S \int \mathbf{F} \cdot \mathbf{N} dS &= \int_R \int \mathbf{F} \cdot \nabla G dA \\ &= \int_0^2 \int_0^{\frac{3-\frac{3}{2}x}{2}} (3x + 2y) dy dx \\ &= \int_0^2 [3xy + y^2]_0^{\frac{3-\frac{3}{2}x}{2}} dx \\ &= \int_0^2 \left[ 3x\left(3 - \frac{3}{2}x\right) + \left(3 - \frac{3}{2}x\right)^2 \right] dx \\ &= \int_0^2 \frac{-9}{4}(x^2 - 4) dx \\ &= \frac{-9}{4} \left[ \frac{x^3}{3} - 4x \right]_0^2 = \left( \frac{-9}{4} \right) \left( \frac{-16}{3} \right) = 12\end{aligned}$$



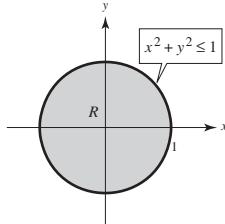
25.  $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$

$S$ :  $z = 1 - x^2 - y^2$ ,  $z \geq 0$

$$G(x, y, z) = x^2 + y^2 + z - 1$$

$$\nabla G(x, y, z) = 2x\mathbf{i} + 2y\mathbf{j} + \mathbf{k}$$

$$\begin{aligned}\int_S \int \mathbf{F} \cdot \mathbf{N} dS &= \int_R \int \mathbf{F} \cdot \nabla G dA \\ &= \int_R \int (2x^2 + 2y^2 + z) dA \\ &= \int_R \int (2x^2 + 2y^2 + (1 - x^2 - y^2)) dA \\ &= \int_R \int (1 + x^2 + y^2) dA \\ &= \int_0^{2\pi} \int_0^1 (r^2 + 1) r dr d\theta \\ &= \int_0^{2\pi} \left[ \frac{r^4}{4} + \frac{r^2}{2} \right]_0^1 d\theta = \int_0^{2\pi} \frac{3}{4} d\theta = \frac{3\pi}{2}\end{aligned}$$



26.  $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$

$S$ :  $x^2 + y^2 + z^2 = 36$  (first octant)

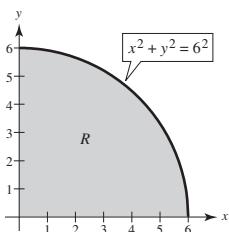
$$z = \sqrt{36 - x^2 - y^2}$$

$$G(x, y, z) = z - \sqrt{36 - x^2 - y^2}$$

$$\nabla G(x, y, z) = \frac{x}{\sqrt{36 - x^2 - y^2}}\mathbf{i} + \frac{y}{\sqrt{36 - x^2 - y^2}}\mathbf{j} + \mathbf{k}$$

$$\mathbf{F} \cdot \nabla G = \frac{x^2}{\sqrt{36 - x^2 - y^2}} + \frac{y^2}{\sqrt{36 - x^2 - y^2}} + z = \frac{36}{\sqrt{36 - x^2 - y^2}}$$

$$\int_S \int \mathbf{F} \cdot \mathbf{N} dS = \int_R \int \frac{36}{\sqrt{36 - x^2 - y^2}} dA = \int_0^{\pi/2} \int_0^6 \frac{36}{\sqrt{36 - r^2}} r dr d\theta \quad (\text{improper}) = 108\pi$$



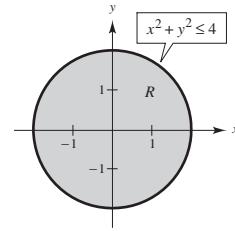
27.  $\mathbf{F}(x, y, z) = 4\mathbf{i} - 3\mathbf{j} + 5\mathbf{k}$

$$S: z = x^2 + y^2, x^2 + y^2 \leq 4$$

$$G(x, y, z) = -x^2 - y^2 + z$$

$$\nabla G(x, y, z) = -2x\mathbf{i} - 2y\mathbf{j} + \mathbf{k}$$

$$\begin{aligned}\int_S \int \mathbf{F} \cdot \mathbf{N} dS &= \int_R \int \mathbf{F} \cdot \nabla G dA = \int_R \int (-8x + 6y + 5) dA \\ &= \int_0^{2\pi} \int_0^2 [-8r \cos \theta + 6r \sin \theta + 5] r dr d\theta \\ &= \int_0^{2\pi} \left[ -\frac{8}{3}r^3 \cos \theta + 2r^3 \sin \theta + \frac{5}{2}r^2 \right]_0^2 d\theta \\ &= \int_0^{2\pi} \left[ -\frac{64}{3} \cos \theta + 16 \sin \theta + 10 \right] d\theta \\ &= \left[ -\frac{64}{3} \sin \theta - 16 \cos \theta + 10\theta \right]_0^{2\pi} = 20\pi\end{aligned}$$



28.  $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} - 2z\mathbf{k}$

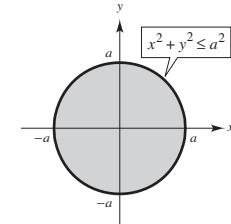
$$S: z = \sqrt{a^2 - x^2 - y^2}$$

$$G(x, y, z) = z - \sqrt{a^2 - x^2 - y^2}$$

$$\nabla G(x, y, z) = \frac{x}{\sqrt{a^2 - x^2 - y^2}}\mathbf{i} + \frac{y}{\sqrt{a^2 - x^2 - y^2}}\mathbf{j} + \mathbf{k}$$

$$\mathbf{F} \cdot \nabla G = \frac{x^2}{\sqrt{a^2 - x^2 - y^2}} + \frac{y^2}{\sqrt{a^2 - x^2 - y^2}} - 2\sqrt{a^2 - x^2 - y^2} = \frac{3x^2 + 3y^2 - 2a^2}{\sqrt{a^2 - x^2 - y^2}}$$

$$\begin{aligned}\int_S \int \mathbf{F} \cdot \mathbf{N} dS &= \int_R \int \mathbf{F} \cdot \nabla G dA = \int_R \int \frac{3x^2 + 3y^2 - 2a^2}{\sqrt{a^2 - x^2 - y^2}} dA = \int_0^{2\pi} \int_0^a \frac{3r^2 - 2a^2}{\sqrt{a^2 - r^2}} r dr d\theta \\ &= 3 \int_0^{2\pi} \int_0^a \frac{r^3}{\sqrt{a^2 - r^2}} dr d\theta - 2a^2 \int_0^{2\pi} \int_0^a \frac{r}{\sqrt{a^2 - r^2}} dr d\theta \\ &= 3 \left[ \int_0^{2\pi} \left[ -r^2 \sqrt{a^2 - r^2} - \frac{2}{3} (a^2 - r^2)^{3/2} \right]_0^a d\theta \right] - 2a^2 \int_0^{2\pi} \left[ -\sqrt{a^2 - r^2} \right]_0^a d\theta \\ &= 3 \int_0^{2\pi} \frac{2}{3} a^3 d\theta - 2a^2 \int_0^{2\pi} a d\theta = 0\end{aligned}$$



29.  $\mathbf{F}(x, y, z) = (x + y)\mathbf{i} + y\mathbf{j} + z\mathbf{k}$

$$S: z = 16 - x^2 - y^2, z = 0$$

$$G(x, y, z) = z + x^2 + y^2 - 16$$

$$\nabla G(x, y, z) = 2x\mathbf{i} + 2y\mathbf{j} + \mathbf{k}$$

$$\mathbf{F} \cdot \nabla G = 2x(x + y) + 2y^2 + z = 2x^2 + 2xy + 2y^2 + 16 - x^2 - y^2 = x^2 + y^2 + 2xy + 16$$

$$\begin{aligned}\int_S \int \mathbf{F} \cdot \mathbf{N} dS &= \int_R \int \mathbf{F} \cdot \nabla G dA \\ &= \int_0^{2\pi} \int_0^4 (r^2 + 2r^2 \cos \theta \sin \theta + 16) r dr d\theta \\ &= \int_0^{2\pi} \left[ \frac{r^4}{4} + \frac{r^4}{2} \cos \theta \sin \theta + 8r^2 \right]_0^4 d\theta = \int_0^{2\pi} [192 + 128 \cos \theta \sin \theta] d\theta = [192 + 64 \sin^2 \theta]_0^{2\pi} = 384\pi\end{aligned}$$

(The flux across the bottom  $z = 0$  is 0)

30.  $\mathbf{F}(x, y, z) = 4xy\mathbf{i} + z^2\mathbf{j} + yz\mathbf{k}$

$S$ : unit cube bounded by

$$x = 0, x = 1, y = 0, y = 1, z = 0, z = 1$$

$S_1$ : The top of the cube

$$\mathbf{N} = \mathbf{k}, z = 1$$

$$\int_{S_1} \int \mathbf{F} \cdot \mathbf{N} dS = \int_0^1 \int_0^1 y(1) dy dx = \frac{1}{2}$$

$S_2$ : The bottom of the cube

$$\mathbf{N} = -\mathbf{k}, z = 0$$

$$\int_{S_2} \int \mathbf{F} \cdot \mathbf{N} dS = \int_0^1 \int_0^1 -y(0) dy dx = 0$$

$S_3$ : The front of the cube

$$\mathbf{N} = \mathbf{i}, x = 1$$

$$\int_{S_3} \int \mathbf{F} \cdot \mathbf{N} dS = \int_0^1 \int_0^1 4(1) y dy dz = 2$$

$S_4$ : The back of the cube

$$\mathbf{N} = -\mathbf{i}, x = 0$$

$$\int_{S_4} \int \mathbf{F} \cdot \mathbf{N} dS = \int_0^1 \int_0^1 -4(0) y dy dx = 0$$

$S_5$ : The right side of the cube

$$\mathbf{N} = \mathbf{j}, y = 1$$

$$\int_{S_5} \int \mathbf{F} \cdot \mathbf{N} dS = \int_0^1 \int_0^1 z^2 dz dx = \frac{1}{3}$$

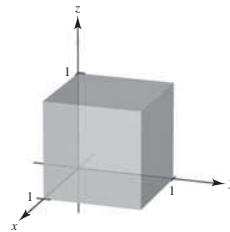
$S_6$ : The left side of the cube

$$\mathbf{N} = -\mathbf{j}, y = 0$$

$$\int_{S_6} \int \mathbf{F} \cdot \mathbf{N} dS = \int_0^1 \int_0^1 -z^2 dz dx = -\frac{1}{3}$$

So,

$$\int_S \int \mathbf{F} \cdot \mathbf{N} dS = \frac{1}{2} + 0 + 2 + 0 + \frac{1}{3} - \frac{1}{3} = \frac{5}{2}.$$



31.  $\mathbf{E} = yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}$

$S$ :  $z = \sqrt{1 - x^2 - y^2}$

$$\int_S \int \mathbf{E} \cdot \mathbf{N} dS = \int_R \int \mathbf{E} \cdot (-g_x(x, y)\mathbf{i} - g_y(x, y)\mathbf{j} + \mathbf{k}) dA$$

$$= \int_R \int (yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}) \cdot \left( \frac{x}{\sqrt{1-x^2-y^2}}\mathbf{i} + \frac{y}{\sqrt{1-x^2-y^2}}\mathbf{j} + \mathbf{k} \right) dA$$

$$= \int_R \int \left( \frac{2xyz}{\sqrt{1-x^2-y^2}} + xy \right) dA = \int_R \int 3xy dA = \int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} 3xy dy dx = 0$$

32.  $\mathbf{E} = x\mathbf{i} + y\mathbf{j} + 2z\mathbf{k}$

$$S: z = \sqrt{1 - x^2 - y^2} = g(x, y)$$

$$\int_S \int \mathbf{E} \cdot \mathbf{N} dS = \int_R \int \mathbf{E} \cdot (-g_x(x, y)\mathbf{i} - g_y(x, y)\mathbf{j} + \mathbf{k}) dA$$

$$\begin{aligned} &= \int_R \int (x\mathbf{i} + y\mathbf{j} + 2z\mathbf{k}) \cdot \left( \frac{x}{\sqrt{1-x^2-y^2}}\mathbf{i} + \frac{y}{\sqrt{1-x^2-y^2}}\mathbf{j} + \mathbf{k} \right) dA \\ &= \int_R \int \left( \frac{x^2}{\sqrt{1-x^2-y^2}} + \frac{y^2}{\sqrt{1-x^2-y^2}} + 2z \right) dA \\ &= \int_R \int \frac{x^2 + y^2 + 2(1-x^2-y^2)}{\sqrt{1-x^2-y^2}} dA = \int_R \int \frac{2-x^2-y^2}{\sqrt{1-x^2-y^2}} dA = \int_0^{2\pi} \int_0^1 \frac{2-r^2}{\sqrt{1-r^2}} r dr d\theta = \frac{8\pi}{3} \end{aligned}$$

33.  $z = \sqrt{x^2 + y^2}, 0 \leq z \leq a$

$$m = \int_S \int k dS = k \int_R \int \sqrt{1 + \left( \frac{x}{\sqrt{x^2+y^2}} \right)^2 + \left( \frac{y}{\sqrt{x^2+y^2}} \right)^2} dA = k \int_R \int \sqrt{2} dA = \sqrt{2} k \pi a^2$$

$$I_z = \int_S \int k(x^2 + y^2) dS = \int_R \int k(x^2 + y^2) \sqrt{2} dA = \sqrt{2} k \int_0^{2\pi} \int_0^a r^3 dr d\theta = \frac{\sqrt{2} k a^4}{4} (2\pi) = \frac{\sqrt{2} k \pi a^4}{2} = \frac{a^2}{2} (\sqrt{2} k \pi a^2) = \frac{a^2 m}{2}$$

34.  $x^2 + y^2 + z^2 = a^2$

$$z = \pm \sqrt{a^2 - x^2 - y^2}$$

$$\begin{aligned} m &= 2 \int_S \int k dS = 2k \int_R \int \sqrt{1 + \left( \frac{-x}{\sqrt{a^2-x^2-y^2}} \right)^2 + \left( \frac{-y}{\sqrt{a^2-x^2-y^2}} \right)^2} dA \\ &= 2k \int_R \int \frac{a}{\sqrt{a^2-x^2-y^2}} dA = 2ka \int_0^{2\pi} \int_0^a \frac{r}{\sqrt{a^2-r^2}} dr d\theta = 2ka \left[ -\sqrt{a^2-r^2} \right]_0^a (2\pi) = 4\pi ka^2 \end{aligned}$$

$$I_z = 2 \int_S \int k(x^2 + y^2) dS = 2k \int_R \int (x^2 + y^2) \frac{a}{\sqrt{a^2-x^2-y^2}} dA = 2ka \int_0^{2\pi} \int_0^a \frac{r^3}{\sqrt{a^2-r^2}} dr d\theta \text{ (use integration by parts)}$$

$$\text{Let } u = r^2, dv = r(a^2 - r^2)^{-1/2} dr, du = 2r dr, v = -\sqrt{a^2 - r^2}.$$

$$= 2ka \left[ -r^2 \sqrt{a^2 - r^2} - \frac{2}{3}(a^2 - r^2)^{3/2} \right]_0^a (2\pi) = 2ka \left( \frac{2}{3}a^3 \right) (2\pi) = \frac{2}{3}a^2 (4\pi ka^2) = \frac{2}{3}a^2 m$$

35.  $x^2 + y^2 = a^2, 0 \leq z \leq h$

$$\rho(x, y, z) = 1$$

$$y = \pm \sqrt{a^2 - x^2}$$

Project the solid onto the  $xz$ -plane.

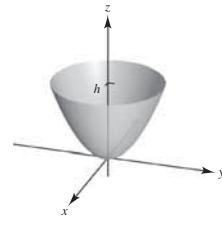
$$\begin{aligned} I_z &= 4 \int_S \int (x^2 + y^2)(1) dS = 4 \int_0^h \int_0^a \left[ x^2 + (a^2 - x^2) \right] \sqrt{1 + \left( \frac{-x}{\sqrt{a^2-x^2}} \right)^2 + (0)^2} dx dz \\ &= 4a^3 \int_0^h \int_0^a \frac{1}{\sqrt{a^2-x^2}} dx dz = 4a^3 \int_0^h \left[ \arcsin \frac{x}{a} \right]_0^a dz = 4a^3 \left( \frac{\pi}{2} \right) (h) = 2\pi a^3 h \end{aligned}$$



36.  $z = x^2 + y^2, 0 \leq z \leq h$

Project the solid onto the  $xy$ -plane.

$$\begin{aligned} I_z &= \int_S \int (x^2 + y^2) dS = \int_{-\sqrt{h}}^{\sqrt{h}} \int_{-\sqrt{h-x^2}}^{\sqrt{h-x^2}} (x^2 + y^2) \sqrt{1 + 4x^2 + 4y^2} dy dx \\ &= \int_0^{2\pi} \int_0^{\sqrt{h}} r^2 \sqrt{1 + 4r^2} r dr d\theta = 2\pi \left[ \frac{h}{12}(1+4h)^{3/2} - \frac{1}{120}(1+4h)^{5/2} \right] + \frac{2\pi}{120} \\ &= \frac{(1+4h)^{3/2}\pi}{60} [10h - (1+4h)] + \frac{\pi}{60} = \frac{\pi}{60} [(1+4h)^{3/2}(6h-1) + 1] \end{aligned}$$



37.  $S: z = 16 - x^2 - y^2, z \geq 0$

$$\mathbf{F}(x, y, z) = 0.5z\mathbf{k}$$

$$\begin{aligned} \int_S \int \rho \mathbf{F} \cdot \mathbf{N} dS &= \int_R \int \rho \mathbf{F} \cdot (-g_x(x, y)\mathbf{i} - g_y(x, y)\mathbf{j} + \mathbf{k}) dA = \int_R \int 0.5\rho z\mathbf{k} \cdot (2x\mathbf{i} + 2y\mathbf{j} + \mathbf{k}) dA \\ &= \int_R \int 0.5\rho z dA = \int_R \int 0.5\rho(16 - x^2 - y^2) dA \\ &= 0.5\rho \int_0^{2\pi} \int_0^4 (16 - r^2) r dr d\theta = 0.5\rho \int_0^{2\pi} 64 d\theta = 64\pi\rho \end{aligned}$$

38.  $S: z = \sqrt{16 - x^2 - y^2}$

$$\mathbf{F}(x, y, z) = 0.5z\mathbf{k}$$

$$\begin{aligned} \int_S \int \rho \mathbf{F} \cdot \mathbf{N} dS &= \int_R \int \rho \mathbf{F} \cdot (-g_x(x, y)\mathbf{i} - g_y(x, y)\mathbf{j} + \mathbf{k}) dA \\ &= \int_R \int 0.5\rho z\mathbf{k} \cdot \left[ \frac{x}{\sqrt{16 - x^2 - y^2}}\mathbf{i} + \frac{y}{\sqrt{16 - x^2 - y^2}}\mathbf{j} + \mathbf{k} \right] dA \\ &= \int_R \int 0.5\rho z dA = \int_R \int 0.5\rho \sqrt{16 - x^2 - y^2} dA \\ &= 0.5\rho \int_0^{2\pi} \int_0^4 \sqrt{16 - r^2} r dr d\theta = 0.5\rho \int_0^{2\pi} \frac{64}{3} d\theta = \frac{64\pi\rho}{3} \end{aligned}$$

39. The surface integral of  $f$  over a surface  $S$ , where  $S$  is given by  $z = g(x, y)$ , is defined as

$$\int_S \int f(x, y, z) dS = \lim_{\|A\| \rightarrow 0} \sum_{i=1}^n f(x_i, y_i, z_i) \Delta S_i. \text{ (page 1112)}$$

See Theorem 15.10, page 1112.

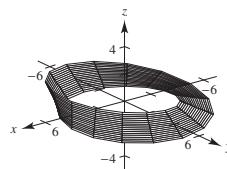
40. A surface is orientable if a unit normal vector  $N$  can be defined at every nonboundary point of  $S$  in such a way that the normal vectors vary continuously over the surface  $S$ .

41. See the definition, page 1118.

See Theorem 15.11, page 1118.

42. Orientable

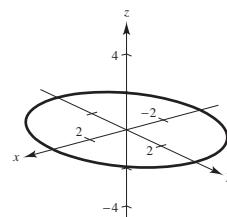
43. (a)



- (b) If a normal vector at a point  $P$  on the surface is moved around the Möbius strip once, it will point in the opposite direction.

(c)  $\mathbf{r}(u, 0) = 4 \cos(2u)\mathbf{i} + 4 \sin(2u)\mathbf{j}$

This is circle.



- (d) (construction)

- (e) You obtain a strip with a double twist and twice as long as the original Möbius strip.

44. (a)  $\mathbf{r}_u = \mathbf{i} + \mathbf{j} + 2u\mathbf{k}$

$$\mathbf{r}_v = 2v\mathbf{i} - \mathbf{j}$$

$$\mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 2u \\ 2v & -1 & 0 \end{vmatrix} = 2u\mathbf{i} + 4uv\mathbf{j} - (1+2v)\mathbf{k}$$

$\mathbf{r}_u \times \mathbf{r}_v$  is a normal vector to the surface.

(b)  $\mathbf{F}(u, v) = u^2\mathbf{i} + (u + v^2)\mathbf{j} + (u - v)\mathbf{k}$

$$\mathbf{F} \cdot (\mathbf{r}_u \times \mathbf{r}_v) = 2u^3 + 4uv(u + v^2) - (u - v)(1 + 2v) = 2u^3 + 4u^2v + 4uv^3 + v - u + 2v^2 - 2uv$$

(c)  $x = 3 = u + v^2$   
 $y = 1 = u - v$   
 $z = 4 = u^2$

$$\left. \begin{array}{l} u = 2 (u = -2 \text{ not in domain}) \\ v = 1 \end{array} \right\}$$

(d) Calculate  $\mathbf{F} \cdot \frac{\mathbf{r}_u \times \mathbf{r}_v}{\|\mathbf{r}_u \times \mathbf{r}_v\|}$  at  $P$ .

$$\mathbf{F}(3, 1, 4) = 4\mathbf{i} + 3\mathbf{j} + \mathbf{k}$$

$$(\mathbf{r}_u \times \mathbf{r}_v)(2, 1) = 4\mathbf{i} + 8\mathbf{j} - 3\mathbf{k}$$

$$\|\mathbf{r}_u \times \mathbf{r}_v\| = \sqrt{89}$$

$$\mathbf{F} \cdot \frac{\mathbf{r}_u \times \mathbf{r}_v}{\|\mathbf{r}_u \times \mathbf{r}_v\|} = \frac{1}{\sqrt{89}}(16 + 24 - 3) = \frac{37}{\sqrt{89}} = \frac{37\sqrt{89}}{89}$$

(e)  $\int_S \int \mathbf{F} \cdot \mathbf{N} dS = \int_R \int \mathbf{F} \cdot (\mathbf{r}_u \times \mathbf{r}_v) dA$   
 $= \int_{-1}^1 \int_0^2 (2u^3 + 4u^2v + 4uv^3 + v - u + 2v^2 - 2uv) du dv = \int_{-1}^1 \left( 8v^3 + 4v^2 + \frac{26v}{3} + 6 \right) dv = \frac{44}{3}$

## Section 15.7 Divergence Theorem

1. **Surface Integral:** There are six surfaces to the cube, each with  $dS = \sqrt{1} dA$ .

$$z = 0, \quad \mathbf{N} = -\mathbf{k}, \quad \mathbf{F} \cdot \mathbf{N} = -z^2, \quad \int_{S_1} \int 0 dA = 0$$

$$z = a, \quad \mathbf{N} = \mathbf{k}, \quad \mathbf{F} \cdot \mathbf{N} = z^2, \quad \int_{S_2} \int a^2 dA = \int_0^a \int_0^a a^2 dx dy = a^4$$

$$x = 0, \quad \mathbf{N} = -\mathbf{i}, \quad \mathbf{F} \cdot \mathbf{N} = -2x, \quad \int_{S_3} \int 0 dA = 0$$

$$x = a, \quad \mathbf{N} = \mathbf{i}, \quad \mathbf{F} \cdot \mathbf{N} = 2x, \quad \int_{S_4} \int 2a dy dz = \int_0^a \int_0^a 2a dy dz = 2a^3$$

$$y = 0, \quad \mathbf{N} = -\mathbf{j}, \quad \mathbf{F} \cdot \mathbf{N} = 2y, \quad \int_{S_5} \int 0 dA = 0$$

$$y = a, \quad \mathbf{N} = \mathbf{j}, \quad \mathbf{F} \cdot \mathbf{N} = -2y, \quad \int_{S_6} \int -2a dA = \int_0^a \int_0^a -2a dz dx = -2a^3$$

So,  $\int_S \int \mathbf{F} \cdot \mathbf{N} dS = a^4 + 2a^3 - 2a^3 = a^4$ .

**Divergence Theorem:** Because  $\operatorname{div} \mathbf{F} = 2z$ , the Divergence Theorem yields

$$\iiint_Q \operatorname{div} \mathbf{F} dV = \int_0^a \int_0^a \int_0^a 2z dz dy dx = \int_0^a \int_0^a a^2 dy dx = a^4.$$

**2. Surface Integral:** There are three surfaces to the cylinder.

Bottom:  $z = 0$ ,  $\mathbf{N} = -\mathbf{k}$ ,  $\mathbf{F} \cdot \mathbf{N} = -z^2$

$$\int_{S_1} \int 0 \, dS = 0$$

Top:  $z = h$ ,  $\mathbf{N} = \mathbf{k}$ ,  $\mathbf{F} \cdot \mathbf{N} = z^2$

$$\int_{S_2} \int h^2 \, dS = h^2 (\text{Area of circle}) = 4\pi h^2$$

Side:  $\mathbf{r}(u,v) = 2 \cos u \mathbf{i} + 2 \sin u \mathbf{j} + v \mathbf{k}$ ,  $0 \leq u \leq 2\pi$ ,  $0 \leq v \leq h$

$$\mathbf{r}_u = -2 \sin u \mathbf{i} + 2 \cos u \mathbf{j}, \mathbf{r}_v = \mathbf{k}$$

$$\mathbf{r}_u \times \mathbf{r}_v = 2 \cos u \mathbf{i} + 2 \sin u \mathbf{j}$$

$$\mathbf{F} \cdot (\mathbf{r}_u \times \mathbf{r}_v) = 8 \cos^2 u - 8 \sin^2 u$$

$$\int_{S_3} \int \mathbf{F} \cdot \mathbf{N} \, dS = \int_0^h \int_0^{2\pi} (8 \cos^2 u - 8 \sin^2 u) \, du \, dv = 0$$

$$\text{So, } \int_s \int \mathbf{F} \cdot \mathbf{N} \, dS = 0 + 4\pi h^2 + 0 = 4\pi h^2.$$

**Divergence Theorem:**  $\text{div } \mathbf{F} = 2 - 2 + 2z = 2z$

$$\iiint_Q 2z \, dV = \int_0^{2\pi} \int_0^2 \int_0^h 2zr \, dz \, dr \, d\theta = 4\pi h^2.$$



**3. Surface Integral:** There are four surfaces to this solid.

$z = 0$ ,  $\mathbf{N} = -\mathbf{k}$ ,  $\mathbf{F} \cdot \mathbf{N} = -z$

$$\int_{S_1} \int 0 \, dS = 0$$

$y = 0$ ,  $\mathbf{N} = -\mathbf{j}$ ,  $\mathbf{F} \cdot \mathbf{N} = 2y - z$ ,  $dS = dA = dx \, dz$

$$\int_{S_2} \int -z \, dS = \int_0^6 \int_0^{6-z} -z \, dx \, dz = \int_0^6 (z^2 - 6z) \, dz = -36$$

$x = 0$ ,  $\mathbf{N} = -\mathbf{i}$ ,  $\mathbf{F} \cdot \mathbf{N} = y - 2x$ ,  $dS = dA = dz \, dy$

$$\int_{S_3} \int y \, dS = \int_0^3 \int_0^{6-2y} y \, dz \, dy = \int_0^3 (6y - 2y^2) \, dy = 9$$

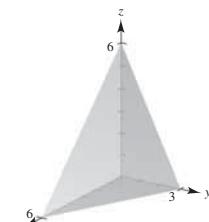
$$x + 2y + z = 6, \mathbf{N} = \frac{\mathbf{i} + 2\mathbf{j} + \mathbf{k}}{\sqrt{6}}, \mathbf{F} \cdot \mathbf{N} = \frac{2x - 5y + 3z}{\sqrt{6}}, dS = \sqrt{6} \, dA$$

$$\int_{S_4} \int (2x - 5y + 3z) \, dz \, dy = \int_0^3 \int_0^{6-2y} (18 - x - 11y) \, dx \, dy = \int_0^3 (90 - 90y + 20y^2) \, dy = 45$$

$$\text{So, } \int_s \int \mathbf{F} \cdot \mathbf{N} \, dS = 0 - 36 + 9 + 45 = 18.$$

**Divergence Theorem:** Because  $\text{div } \mathbf{F} = 1$ , you have

$$\iiint_Q dV = (\text{Volume of solid}) = \frac{1}{3}(\text{Area of base}) \times (\text{Height}) = \frac{1}{3}(9)(6) = 18.$$



4.  $\mathbf{F}(x,y,z) = xy\mathbf{i} + z\mathbf{j} + (x+y)\mathbf{k}$

$S$ : surface bounded by the planes  $y = 4$ ,  $z = 4 - x$  and the coordinate planes

**Surface Integral:** There are five surfaces to this solid.

$$z = 0, \mathbf{N} = -\mathbf{k}, \mathbf{F} \cdot \mathbf{N} = -(x+y)$$

$$\int_{S_1} \int -(x+y) dS = \int_0^4 \int_0^4 -(x+y) dy dx = -\int_0^4 (4x+8) dx = -64$$

$$y = 0, \mathbf{N} = -\mathbf{j}, \mathbf{F} \cdot \mathbf{N} = -z$$

$$\int_{S_2} \int -z dS = \int_0^4 \int_0^{4-x} -z dz dx = -\int_0^4 \frac{(4-x)^2}{2} dx = -\frac{32}{3}$$

$$y = 4, \mathbf{N} = \mathbf{j}, \mathbf{F} \cdot \mathbf{N} = z$$

$$\int_{S_3} \int z dS = \int_0^4 \int_0^{4-x} z dz dx = \int_0^4 \frac{(4-x)^2}{2} dx = \frac{32}{3}$$

$$x = 0, \mathbf{N} = -\mathbf{i}, \mathbf{F} \cdot \mathbf{N} = -xy$$

$$\int_{S_4} \int -xy dS = \int_0^4 \int_0^4 0 dS = 0$$

$$x+z=4, \mathbf{N} = \frac{\mathbf{i} + \mathbf{k}}{\sqrt{2}}, \mathbf{F} \cdot \mathbf{N} = \frac{1}{\sqrt{2}}[xy+x+y], dS = \sqrt{2} dA$$

$$\int_{S_5} \int \frac{1}{\sqrt{2}}[xy+x+y]\sqrt{2} dA = \int_0^4 \int_0^4 (xy+x+y) dy dx = 128$$

$$\text{So, } \int_S \int \mathbf{F} \cdot \mathbf{N} dS = -64 - \frac{32}{3} + \frac{32}{3} + 0 + 128 = 64.$$

**Divergence Theorem:** Because  $\operatorname{div} \mathbf{F} = y$ , you have

$$\iiint_Q \operatorname{div} \mathbf{F} dV = \int_0^4 \int_0^4 \int_0^{4-x} y dz dy dx = 64.$$

5.  $\mathbf{F}(x,y,z) = xz\mathbf{i} + yz\mathbf{j} + 2z^2\mathbf{k}$

**Surface Integral:** There are two surfaces.

Bottom:  $z = 0, \mathbf{N} = -\mathbf{k}, \mathbf{F} \cdot \mathbf{N} = -2z^2$

$$\int_{S_1} \int \mathbf{F} \cdot \mathbf{N} dS = \int_R \int -2z^2 dA = \iint 0 dA = 0$$

Side: Outward unit normal is

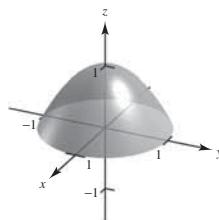
$$\mathbf{N} = \frac{2x\mathbf{i} + 2y\mathbf{j} + \mathbf{k}}{\sqrt{4x^2 + 4y^2 + 1}}$$

$$\mathbf{F} \cdot \mathbf{N} = \frac{1}{\sqrt{4x^2 + 4y^2 + 1}} [2x^2z + 2y^2z + 2z^2]$$

$$\begin{aligned} \int_{S_2} \int \mathbf{F} \cdot \mathbf{N} dS &= \int_{S_2} \int [2(x^2 + y^2)z + 2z^2] dA \\ &= \int_0^{2\pi} \int_0^1 [2r^2(1 - r^2) + 2(1 - r^2)^2] r dr d\theta = \int_0^{2\pi} \int_0^1 (2r - 2r^3) dr d\theta = \int_0^{2\pi} \frac{1}{2} d\theta = \pi \end{aligned}$$

**Divergence Theorem:**  $\operatorname{div} \mathbf{F} = z + z + 4z = 6z$

$$\begin{aligned} \iiint_Q \operatorname{div} \mathbf{F} dV &= \int_0^{2\pi} \int_0^1 \int_0^{1-r^2} 6z r dz dr d\theta \\ &= \int_0^{2\pi} \int_0^1 3(1 - r^2)^2 r dr d\theta = \int_0^{2\pi} \int_0^1 (3 - 6r^2 + 3r^4) r dr d\theta = \int_0^{2\pi} \left[ \frac{3}{2} - \frac{3}{2}r^2 + \frac{1}{2}r^4 \right] d\theta = \pi \end{aligned}$$



6.  $\mathbf{F}(x, y, z) = xy^2\mathbf{i} + yx^2\mathbf{j} + e\mathbf{k}$

S: Surface bounded by  $z = \sqrt{x^2 + y^2}$  and  $z = 4$

**Surface Integral:** There are two surfaces.

Top:  $z = 4$ ,  $\mathbf{N} = \mathbf{k}$ ,  $\mathbf{F} \cdot \mathbf{N} = e$

$$\int_{S_1} \int \mathbf{F} \cdot \mathbf{N} dS = (\text{area circle}) e = 16\pi e$$

Side:  $z = g(x, y) = \sqrt{x^2 + y^2}$ ,  $g_x = \frac{x}{\sqrt{x^2 + y^2}}$ ,  $g_y = \frac{y}{\sqrt{x^2 + y^2}}$

$$\begin{aligned} \int_{S_2} \int \mathbf{F} \cdot \mathbf{N} dS &= \int_{S_2} \int \left[ \frac{x^2 y^2}{\sqrt{x^2 + y^2}} + \frac{x^2 y^2}{\sqrt{x^2 + y^2}} - e \right] dA = \int_0^{2\pi} \int_0^4 \left( \frac{2r^2 \cos^2 \theta r^2 \sin^2 \theta}{r} - e \right) r dr d\theta \\ &= \int_0^{2\pi} \left[ \frac{2048 \sin^2 \theta \cos^2 \theta}{5} - 8e \right] d\theta = \left( \frac{512}{5} - 16e \right) \pi \end{aligned}$$

So,  $\int_S \int \mathbf{F} \cdot \mathbf{N} dS = \frac{512}{5}\pi$ .

**Divergence Theorem:**  $\text{div } \mathbf{F} = y^2 + x^2$

$$\iiint_Q \text{div } \mathbf{F} dV = \int_0^{2\pi} \int_0^4 \int_r^4 (r^2) r dz dr d\theta = \int_0^{2\pi} \int_0^4 (4r^3 - r^4) dr dz = \int_0^{2\pi} \left[ r^4 - \frac{r^5}{5} \right]_0^4 dz = \int_0^{2\pi} \frac{256}{5} dz = \frac{512\pi}{5}$$

7. Because  $\text{div } \mathbf{F} = 2x + 2y + 2z$ , you have

$$\begin{aligned} \iiint_Q \text{div } \mathbf{F} dV &= \int_0^a \int_0^a \int_0^a (2x + 2y + 2z) dz dy dx \\ &= \int_0^a \int_0^a (2ax + 2ay + a^2) dy dx = \int_0^a (2a^2x + 2a^3) dx = [a^2x^2 + 2a^3x]_0^a = 3a^4. \end{aligned}$$

8. Because  $\text{div } \mathbf{F} = 2xz^2 - 2 + 3xy$ , you have

$$\begin{aligned} \iiint_Q \text{div } \mathbf{F} dV &= \int_0^a \int_0^a \int_0^a (2xz^2 - 2 + 3xy) dz dy dx = \int_0^a \int_0^a \left( \frac{2}{3}xa^3 - 2a + 3xya \right) dy dx \\ &= \int_0^a \left( \frac{2}{3}xa^4 - 2a^2 + \frac{3}{2}xa^3 \right) dx = \frac{1}{3}a^6 - 2a^3 + \frac{3}{4}a^5. \end{aligned}$$

9. Because  $\text{div } \mathbf{F} = 2x - 2x + 2xyz = 2xyz$ ,

$$\begin{aligned} \iiint_Q \text{div } \mathbf{F} dV &= \iiint_Q 2xyz dV = \int_0^a \int_0^{2\pi} \int_0^{\pi/2} 2(\rho \sin \phi \cos \theta)(\rho \sin \phi \sin \theta)(\rho \cos \phi) \rho^2 \sin \phi d\phi d\theta d\rho \\ &= \int_0^a \int_0^{2\pi} \int_0^{\pi/2} 2\rho^5 (\sin \theta \cos \theta)(\sin^3 \phi \cos \phi) d\phi d\theta d\rho \\ &= \int_0^a \int_0^{2\pi} \frac{1}{2} \rho^5 \sin \theta \cos \theta d\theta d\rho = \int_0^a \left[ \left( \frac{\rho^5}{2} \right) \frac{\sin^2 \theta}{2} \right]_0^{2\pi} d\rho = 0. \end{aligned}$$

10. Because  $\text{div } \mathbf{F} = y + z - y = z$ , you have

$$\begin{aligned} \iiint_Q \text{div } \mathbf{F} dV &= \int_{-a}^a \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \int_0^{\sqrt{a^2-y^2}} z dz dy dx = \int_0^{2\pi} \int_0^a \int_0^{\sqrt{a^2-r^2}} zr dz dr d\theta \\ &= \int_0^{2\pi} \int_0^a \left[ \frac{a^2 r}{2} - \frac{r^3}{2} \right] dr d\theta = \int_0^{2\pi} \left[ \frac{a^2 r^2}{4} - \frac{r^4}{8} \right]_0^a d\theta = \int_0^{2\pi} \frac{a^4}{8} d\theta = \frac{\pi a^4}{4}. \end{aligned}$$

11. Because  $\text{div } \mathbf{F} = 3$ , you have

$$\iiint_Q 3 dV = 3 (\text{Volume of Sphere}) = 3 \left[ \frac{4}{3} \pi (3^3) \right] = 108\pi.$$

12. Because  $\operatorname{div} \mathbf{F} = xz$ , you have

$$\iiint_Q xz \, dV = \int_0^5 \int_{-2}^2 \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} xz \, dx \, dy \, dz = \int_0^5 \int_{-2}^2 \left[ \frac{2x^2}{2} \right]_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} dy \, dz = \int_0^5 \int_{-2}^2 0 \, dy \, dz = 0.$$

13. Because  $\operatorname{div} \mathbf{F} = 1 + 2y - 1 = 2y$ , you have

$$\iiint_Q 2y \, dV = \int_0^7 \int_{-5}^5 \int_{-\sqrt{25-y^2}}^{\sqrt{25-y^2}} 2y \, dx \, dy \, dz = \int_0^7 \int_{-5}^5 4y \sqrt{25-y^2} \, dy \, dz = \int_0^7 \left[ \frac{-4}{3} (25-y^2)^{3/2} \right]_{-5}^5 \, dz = 0.$$

14. Because  $\operatorname{div} \mathbf{F} = y^2 + x^2 + e^z$ , you have

$$\begin{aligned} \iiint_Q (x^2 + y^2 + e^z) \, dV &= \int_0^{16} \int_{-\sqrt{256-x^2}}^{\sqrt{256-x^2}} \int_{(1/2)\sqrt{x^2+y^2}}^8 (x^2 + y^2 + e^z) \, dz \, dy \, dx \\ &= \int_0^{2\pi} \int_0^{16} \int_{r/2}^8 (r^2 + e^z) r \, dz \, dr \, d\theta = \int_0^{2\pi} \int_0^{16} (8r^3 + re^8 - \frac{1}{2}r^4 - re^{r/2}) \, dr \, d\theta \\ &= \int_0^{2\pi} \left( \frac{131,052}{5} + 100e^8 \right) d\theta = \frac{262,104}{5}\pi + 200e^8\pi. \end{aligned}$$

15. Because  $\operatorname{div} \mathbf{F} = 3x^2 + x^2 + 0 = 4x^2$ , you have

$$\iiint_Q 4x^2 \, dV = \int_0^6 \int_0^4 \int_0^{4-y} 4x^2 \, dz \, dy \, dx = \int_0^6 \int_0^4 4x^2(4-y) \, dy \, dx = \int_0^6 32x^2 \, dx = 2304.$$

16. Because  $\operatorname{div} \mathbf{F} = e^z + e^z + e^z = 3e^z$ , you have

$$\iiint_Q 3e^z \, dV = \int_0^6 \int_0^4 \int_0^{4-y} 3e^z \, dz \, dy \, dx = \int_0^6 \int_0^4 3[e^{4-y} - 1] \, dy \, dx = \int_0^6 3(e^4 - 5) \, dx = 18(e^4 - 5).$$

17.  $\operatorname{div} \mathbf{F} = y + 4 + x$ . Use spherical coordinates.

$$\begin{aligned} \iiint_Q (y + 4 + x) \, dV &= \int_0^4 \int_0^\pi \int_0^{2\pi} (\rho \sin \phi \sin \theta + \rho \sin \phi \cos \theta + 4) \rho^2 \sin \phi \, d\theta \, d\phi \, d\rho \\ &= \int_0^4 \int_0^\pi \int_0^{2\pi} (\rho^3 \sin^2 \phi \sin \theta + \rho^3 \sin^2 \phi \cos \theta + 4\rho^2 \sin \phi) \, d\theta \, d\phi \, d\rho \\ &= \int_0^4 \int_0^\pi 8\pi\rho^2 \sin \phi \, d\phi \, d\rho = \int_0^4 16\pi\rho^2 \, d\rho = \frac{1024\pi}{3} \end{aligned}$$

18.  $\operatorname{div} \mathbf{F} = 2$

$$\int_S \int \mathbf{F} \cdot \mathbf{N} \, dS = \iiint_Q \operatorname{div} \mathbf{F} \, dV = \iiint_Q 2 \, dV.$$

The surface  $S$  is the upper half of a hemisphere of radius 2. Because the volume is  $\frac{1}{2} \left( \frac{4}{3}\pi(2^3) \right) = 16\pi/3$ , you have

$$\int_S \int \mathbf{F} \cdot \mathbf{N} \, dS = 2(\text{Volume}) = \frac{32\pi}{3}.$$

19. Using the Divergence Theorem, you have

$$\begin{aligned} \int_S \int \operatorname{curl} \mathbf{F} \cdot \mathbf{N} \, dS &= \iiint_Q \operatorname{div} (\operatorname{curl} \mathbf{F}) \, dV \\ \operatorname{curl} \mathbf{F}(x,y,z) &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 4xy + z^2 & 2x^2 + 6yz & 2xz \end{vmatrix} = -6y\mathbf{i} - (2z - 2z)\mathbf{j} + (4x - 4x)\mathbf{k} = -6y\mathbf{i} \\ \operatorname{div} (\operatorname{curl} \mathbf{F}) &= 0. \end{aligned}$$

$$\text{So, } \iiint_Q \operatorname{div} (\operatorname{curl} \mathbf{F}) \, dV = 0.$$

20. Using the Divergence Theorem, you have

$$\int_S \int \text{curl } \mathbf{F} \cdot \mathbf{N} dS = \iiint_Q \text{div}(\text{curl } \mathbf{F}) dV$$

$$\text{curl } \mathbf{F}(x, y, z) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy \cos z & yz \sin x & xyz \end{vmatrix} = (xz - y \sin x)\mathbf{i} - (yz + xy \sin z)\mathbf{j} + (yz \cos x - x \cos z)\mathbf{k}.$$

Now,  $\text{div curl } \mathbf{F}(x, y, z) = (z - y \cos x) - (z + x \sin z) + (y \cos x + x \sin z) = 0$ . So,

$$\int_S \int \text{curl } \mathbf{F} \cdot \mathbf{N} dS = \iiint_Q \text{div}(\text{curl } \mathbf{F}) dV = 0.$$

21. See Theorem 15.12.

22. If  $\text{div } \mathbf{F}(x, y, z) > 0$ , then source.

If  $\text{div } \mathbf{F}(x, y, z) < 0$ , then sink.

If  $\text{div } \mathbf{F}(x, y, z) = 0$ , then incompressible.

23. (a) Using the triple integral to find volume, you need  $\mathbf{F}$  so that

$$\text{div } \mathbf{F} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z} = 1.$$

So, you could have  $\mathbf{F} = x\mathbf{i}$ ,  $\mathbf{F} = y\mathbf{j}$ , or  $\mathbf{F} = z\mathbf{k}$ .

For  $dA = dy dz$  consider  $\mathbf{F} = x\mathbf{i}$ ,  $x = f(y, z)$ , then  $\mathbf{N} = \frac{\mathbf{i} + f_y\mathbf{j} + f_z\mathbf{k}}{\sqrt{1 + f_y^2 + f_z^2}}$  and  $dS = \sqrt{1 + f_y^2 + f_z^2} dy dz$ .

For  $dA = dz dx$  consider  $\mathbf{F} = y\mathbf{j}$ ,  $y = f(x, z)$ , then  $\mathbf{N} = \frac{x\mathbf{i} + \mathbf{j} + f_z\mathbf{k}}{\sqrt{1 + f_x^2 + f_z^2}}$  and  $dS = \sqrt{1 + f_x^2 + f_z^2} dz dx$ .

For  $dA = dx dy$  consider  $\mathbf{F} = z\mathbf{k}$ ,  $z = f(x, y)$ , then  $\mathbf{N} = \frac{f_x\mathbf{i} + f_y\mathbf{j} + \mathbf{k}}{\sqrt{1 + f_x^2 + f_y^2}}$  and  $dS = \sqrt{1 + f_x^2 + f_y^2} dx dy$ .

Correspondingly, you then have  $\mathbf{V} = \int_S \int \mathbf{F} \cdot \mathbf{N} dS = \int_S \int x dy dz = \int_S \int y dz dx = \int_S \int z dx dy$ .

$$(b) v = \int_0^a \int_0^a x dy dz = \int_0^a \int_0^a a dy dz = \int_0^a a^2 dz = a^3$$

$$\text{Similarly, } \int_0^a \int_0^a y dz dx = \int_0^a \int_0^a z dx dy = a^3.$$

24.  $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$

**Divergence Theorem:**  $\text{div } \mathbf{F} = 1 + 1 + 1 = 3$

$$\iiint_Q 3 dV = 3 (\text{Volume of cube}) = 3$$

**Surface Integral:** There are six surfaces.

$$x = 0: \mathbf{N} = -\mathbf{i}, \quad \mathbf{F} \cdot \mathbf{N} = -x, \quad \int_{S_1} \int 0 dS = 0$$

$$x = 1: \mathbf{N} = \mathbf{i}, \quad \mathbf{F} \cdot \mathbf{N} = x, \quad \int_{S_2} \int 1 dS = 1$$

$$y = 0: \mathbf{N} = -\mathbf{j}, \quad \mathbf{F} \cdot \mathbf{N} = -y, \quad \int_{S_3} \int 0 dS = 0$$

$$y = 1: \mathbf{N} = \mathbf{j}, \quad \mathbf{F} \cdot \mathbf{N} = y, \quad \int_{S_4} \int 1 dS = 1$$

$$z = 0: \mathbf{N} = -\mathbf{k}, \quad \mathbf{F} \cdot \mathbf{N} = -z, \quad \int_{S_5} \int 0 dS = 0$$

$$z = 1: \mathbf{N} = \mathbf{k}, \quad \mathbf{F} \cdot \mathbf{N} = z, \quad \int_{S_6} \int 1 dS = 1$$

$$\text{So, } \int_S \int \mathbf{F} \cdot \mathbf{N} dS = 1 + 1 + 1 = 3.$$

25. Using the Divergence Theorem, you have  $\int_S \int \mathbf{curl} \mathbf{F} \cdot \mathbf{N} dS = \iiint_Q \operatorname{div}(\mathbf{curl} \mathbf{F}) dV$ . Let

$$\mathbf{F}(x, y, z) = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$$

$$\mathbf{curl} \mathbf{F} = \left( \frac{\partial P}{\partial y} - \frac{\partial N}{\partial z} \right) \mathbf{i} - \left( \frac{\partial P}{\partial x} - \frac{\partial M}{\partial z} \right) \mathbf{j} + \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) \mathbf{k}$$

$$\operatorname{div}(\mathbf{curl} \mathbf{F}) = \frac{\partial^2 P}{\partial x \partial y} - \frac{\partial^2 N}{\partial x \partial z} - \frac{\partial^2 P}{\partial y \partial x} + \frac{\partial^2 M}{\partial y \partial z} + \frac{\partial^2 N}{\partial z \partial x} - \frac{\partial^2 M}{\partial z \partial y} = 0.$$

$$\text{So, } \int_S \int \mathbf{curl} \mathbf{F} \cdot \mathbf{N} dS = \iiint_Q 0 dV = 0.$$

26. If  $\mathbf{F}(x, y, z) = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ , then  $\operatorname{div} \mathbf{F} = 0$ .

So,

$$\int_S \int \mathbf{F} \cdot \mathbf{N} dS = \iiint_Q \operatorname{div} \mathbf{F} dV = \iiint_Q 0 dV = 0.$$

27. If  $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ , then  $\operatorname{div} \mathbf{F} = 3$ .

$$\int_S \int \mathbf{F} \cdot \mathbf{N} dS = \iiint_Q \operatorname{div} \mathbf{F} dV = \iiint_Q 3 dV = 3V.$$

28. If  $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ , then  $\operatorname{div} \mathbf{F} = 3$ .

$$\frac{1}{\|\mathbf{F}\|} \int_S \int \mathbf{F} \cdot \mathbf{N} dS = \frac{1}{\|\mathbf{F}\|} \iiint_Q \operatorname{div} \mathbf{F} dV = \frac{1}{\|\mathbf{F}\|} \iiint_Q 3 dV = \frac{3}{\|\mathbf{F}\|} \iiint_Q dV$$

$$29. \int_S \int f D_N g dS = \int_S \int f \nabla g \cdot \mathbf{N} dS = \iiint_Q \operatorname{div}(f \nabla g) dV = \iiint_Q (f \operatorname{div} \nabla g + \nabla f \cdot \nabla g) dV = \iiint_Q (f \nabla^2 g + \nabla f \cdot \nabla g) dV$$

$$30. \int_S \int (f D_N g - g D_N f) dS = \int_S \int f D_N g dS - \int_S \int g D_N f dS \\ = \iiint_Q (f \nabla^2 g + \nabla f \cdot \nabla g) dV - \iiint_Q (g \nabla^2 f + \nabla g \cdot \nabla f) dV = \iiint_Q (f \nabla^2 g - g \nabla^2 f) dV$$

## Section 15.8 Stokes's Theorem

$$1. \mathbf{F}(x, y, z) = (2y - z)\mathbf{i} + e^z\mathbf{j} + xyz\mathbf{k}$$

$$\mathbf{curl} \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2y - z & e^z & xyz \end{vmatrix} \\ = (xz - e^z)\mathbf{i} - (yz + 1)\mathbf{j} - 2\mathbf{k}$$

$$2. \mathbf{F}(x, y, z) = x^2\mathbf{i} + y^2\mathbf{j} + x^2\mathbf{k}$$

$$\mathbf{curl} \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 & y^2 & x^2 \end{vmatrix} = -2x\mathbf{j}$$

$$3. \mathbf{F}(x, y, z) = 2z\mathbf{i} - 4x^2\mathbf{j} + \arctan x\mathbf{k}$$

$$\mathbf{curl} \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2z & -4x^2 & \arctan x \end{vmatrix} = \left( 2 - \frac{1}{1+x^2} \right) \mathbf{j} - 8x\mathbf{k}$$

$$4. \mathbf{F}(x, y, z) = x \sin y\mathbf{i} - y \cos x\mathbf{j} + yz^2\mathbf{k}$$

$$\mathbf{curl} \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x \sin y & -y \cos x & yz^2 \end{vmatrix} \\ = z^2\mathbf{i} + (y \sin x - x \cos y)\mathbf{k}$$

5.  $\mathbf{F}(x, y, z) = e^{x^2+y^2} \mathbf{i} + e^{y^2+z^2} \mathbf{j} + xyz \mathbf{k}$

$$\begin{aligned}\operatorname{curl} \mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ e^{x^2+y^2} & e^{y^2+z^2} & xyz \end{vmatrix} \\ &= (xz - 2ze^{y^2+z^2}) \mathbf{i} - yz \mathbf{j} - 2ye^{x^2+y^2} \mathbf{k} \\ &= z(x - 2e^{y^2+z^2}) \mathbf{i} - yz \mathbf{j} - 2ye^{x^2+y^2} \mathbf{k}\end{aligned}$$

6.  $\mathbf{F}(x, y, z) = \arcsin y \mathbf{i} + \sqrt{1-x^2} \mathbf{j} + y^2 \mathbf{k}$

$$\begin{aligned}\operatorname{curl} \mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \arcsin y & \sqrt{1-x^2} & y^2 \end{vmatrix} \\ &= 2y \mathbf{i} + \left[ \frac{-x}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1-y^2}} \right] \mathbf{k} \\ &= 2y \mathbf{i} - \left[ \frac{x}{\sqrt{1-x^2}} + \frac{1}{\sqrt{1-y^2}} \right] \mathbf{k}\end{aligned}$$

7.  $C: x^2 + y^2 = 9, z = 0, dz = 0$

**Line Integral:**

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C -y \, dx + x \, dy$$

$$x = 3 \cos t, dx = -3 \sin t \, dt, y = 3 \sin t, dy = 3 \cos t \, dt$$

$$\begin{aligned}\int_C \mathbf{F} \cdot d\mathbf{r} &= \int_0^{2\pi} [(-3 \sin t)(-3 \sin t) + (3 \cos t)(3 \cos t)] \, dt \\ &= \int_0^{2\pi} 9 \, dt = 18\pi\end{aligned}$$

**Double Integral:**

$$g(x, y) = 9 - x^2 - y^2, g_x = -2x, g_y = -2y$$

$$\operatorname{curl} \mathbf{F} = 2\mathbf{k}$$

$$\int_S \int (\operatorname{curl} \mathbf{F}) \cdot \mathbf{N} \, dS = \int_R 2 \, dA = 2(\text{area circle}) = 18\pi$$



8. In this case,  $M = -y + z, N = x - z, P = x - y$  and  $C$  is the circle  $x^2 + y^2 = 1, z = 0, dz = 0$ .

**Line Integral:**  $\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C (-y + z) \, dx + (x - z) \, dy + (x - y) \, dz = \int_C -y \, dx + x \, dy$

Letting  $x = \cos t, y = \sin t$ , you have  $dx = -\sin t \, dt, dy = \cos t \, dt$  and  $\int_C -y \, dx + x \, dy = \int_0^{2\pi} (\sin^2 t + \cos^2 t) \, dt = 2\pi$ .

**Double Integral:** Consider  $F(x, y, z) = x^2 + y^2 + z^2 - 1$ .

Then  $\mathbf{N} = \frac{\nabla F}{\|\nabla F\|} = \frac{2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}}{2\sqrt{x^2 + y^2 + z^2}} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ .

Because  $z^2 = 1 - x^2 - y^2, z_x = \frac{-2x}{2z} = \frac{-x}{z}$ , and  $z_y = \frac{-y}{z}, dS = \sqrt{1 + \frac{x^2}{z^2} + \frac{y^2}{z^2}} \, dA = \frac{1}{z} \, dA$ .

Now, because  $\operatorname{curl} \mathbf{F} = 2\mathbf{k}$ , you have  $\int_S \int (\operatorname{curl} \mathbf{F}) \cdot \mathbf{N} \, dS = \int_R 2z \left( \frac{1}{z} \right) dA = \int_R 2 \, dA = 2(\text{Area of circle of radius 1}) = 2\pi$ .

## 9. Line Integral:

From the figure you see that

$C_1: z = 0, dz = 0$

$C_2: x = 0, dx = 0$

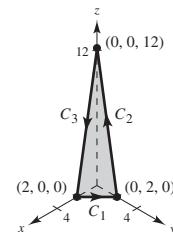
$C_3: y = 0, dy = 0$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} xyz \, dx + y \, dy + z \, dz = \int_{C_1} y \, dy + \int_{C_2} y \, dy + z \, dz + \int_{C_3} z \, dz = \int_0^2 y \, dy + \int_0^0 y \, dy + \int_{12}^{12} z \, dz + \int_{12}^0 z \, dz = 0$$

**Double Integral:**  $\operatorname{curl} \mathbf{F} = xy\mathbf{j} - xz\mathbf{k}$

Letting  $z = 12 - 6x - 6y = g(x, y), g_x = -6 = g_y$ .

$$\begin{aligned}\int_S \int (\operatorname{curl} \mathbf{F}) \cdot \mathbf{N} \, dS &= \int_R \int (\operatorname{curl} \mathbf{F}) \cdot [6\mathbf{i} + 6\mathbf{j} + \mathbf{k}] \, dA = \int_R \int (6xy - xz) \, dA \\ &= \int_0^2 \int_0^{2-x} [6xy - x(12 - 6x - 6y)] \, dy \, dx = \int_0^2 \int_0^{2-x} (12xy - 12x + 6x^2) \, dy \, dx \\ &= \int_0^2 [6xy^2 - 12xy + 6x^2y]_0^{2-x} \, dx = 0\end{aligned}$$



- 10. Line Integral:** From the figure you see that

$$C_1: y = 0, z = 0, dy = dz = 0$$

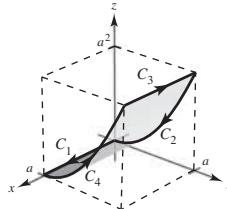
$$C_2: z = y^2, x = 0, dx = 0, dz = 2y dy$$

$$C_3: y = a, z = a^2, dy = dz = 0$$

$$C_4: z = y^2, x = a, dx = 0, dz = 2y dy.$$

So,

$$\begin{aligned} \int_C \mathbf{F} \cdot d\mathbf{r} &= \int_C z^2 dx + x^2 dy + y^2 dz = \int_{C_1} 0 dx + \int_{C_2} 2y^3 dy + \int_{C_3} a^4 dx + \int_{C_4} [a^2 dy + 2y^3 dy] \\ &= \int_a^0 2y^3 dy + \int_a^0 a^4 dx + \int_0^a (a^2 2y^3) dy = [a^4 x]_a^0 + [a^2 y]_0^a = -a^5 + a^3 = a^3(1 - a). \end{aligned}$$



**Double Integral:** Because  $S$  is given by  $-y^2 + z = 0$ , you have

$$\mathbf{N} = \frac{2y\mathbf{j} - \mathbf{k}}{\sqrt{1+4y^2}} \text{ and } dS = \sqrt{1+4y^2} dA.$$

Furthermore,  $\operatorname{curl} \mathbf{F} = 2y\mathbf{i} + 2z\mathbf{j} + 2x\mathbf{k}$ . So,

$$\begin{aligned} \int_S \int (\operatorname{curl} \mathbf{F}) \cdot \mathbf{N} dS &= \int_R \int (4yz - 2x) dA = \int_0^a \int_0^a (-4yz + 2x) dA = \int_0^a \int_0^a (-4y^3 + 2x) dydx \\ &= \int_0^a (-a^4 + 2ax) dx = [-a^4 x + ax^2]_0^a = -a^5 + a^3 = a^3(1 - a^2). \end{aligned}$$

- 11.** These three points have equation:

$$x + y + z = 2.$$

Normal vector:  $\mathbf{N} = \mathbf{i} + \mathbf{j} + \mathbf{k}$

$$\operatorname{curl} \mathbf{F} = -3\mathbf{i} - \mathbf{j} - 2\mathbf{k}$$

$$\begin{aligned} \int_S \int \operatorname{curl} \mathbf{F} \cdot \mathbf{N} dS &= \int_R \int (-6) dA = -6(\text{area of triangle in } xy\text{-plane}) \\ &= -6(2) = -12 \end{aligned}$$

- 12.** Let  $A = (0, 0, 0)$ ,  $B = (1, 1, 1)$ , and  $C = (0, 0, 2)$ . Then  $\mathbf{U} = \overrightarrow{AB} = \mathbf{i} + \mathbf{j} + \mathbf{k}$ , and  $\mathbf{V} = \overrightarrow{AC} = 2\mathbf{k}$ , and

$$\mathbf{N} = \frac{\mathbf{U} \times \mathbf{V}}{\|\mathbf{U} \times \mathbf{V}\|} = \frac{2\mathbf{i} - 2\mathbf{j}}{2\sqrt{2}} = \frac{\mathbf{i} - \mathbf{j}}{\sqrt{2}}.$$

So,  $F(x, y, z) = x - y$  and  $dS = \sqrt{2} dA$ . Because  $\operatorname{curl} \mathbf{F} = \frac{2x}{x^2 + y^2} \mathbf{k}$ , you have  $\int_S \int (\operatorname{curl} \mathbf{F}) \cdot \mathbf{N} dS = \int_R \int 0 dS = 0$ .

$$13. \operatorname{curl} \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z^2 & 2x & y^2 \end{vmatrix} = 2y\mathbf{i} + 2z\mathbf{j} + 2\mathbf{k}$$

$$z = G(x, y) = 1 - x^2 - y^2, G_x = -2x, G_y = -2y$$

$$\begin{aligned} \int_S \int \operatorname{curl} \mathbf{F} \cdot \mathbf{N} dS &= \int_R \int (2y\mathbf{i} + 2z\mathbf{j} + 2\mathbf{k}) \cdot (2x\mathbf{i} + 2y\mathbf{j} + \mathbf{k}) dA = \int_R \int [4xy + 4y(1 - x^2 - y^2) + 2] dA \\ &= \int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} [4xy + 4y - 4x^2y - 4y^3 + 2] dy dx \\ &= \int_{-1}^1 4\sqrt{1-x^2} dx = 2 \left[ \arcsin x + x\sqrt{1-x^2} \right]_{-1}^1 = 2\pi \end{aligned}$$

14.  $\mathbf{F}(x, y, z) = 4xz\mathbf{i} + y\mathbf{j} + 4xy\mathbf{k}$ ,  $S: 9 - x^2 - y^2, z \leq 0$

$$\operatorname{curl} \mathbf{F} = 4x\mathbf{i} + (4x - 4y)\mathbf{j}$$

$$G(x, y, z) = x^2 + y^2 + z - 9$$

$$\nabla G(x, y, z) = 2x\mathbf{i} + 2y\mathbf{j} + \mathbf{k}$$

$$\begin{aligned} \int_S \int (\operatorname{curl} \mathbf{F}) \cdot \mathbf{N} dS &= \int_R \int [8x^2 + 2y(4x - 4y)] dA = \int_{-3}^3 \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} [8x^2 + 8xy - 8y^2] dy dx \\ &= \int_{-3}^3 \left( 16x^2 \sqrt{9 - x^2} - \frac{16}{3}(9 - x^2)^{3/2} \right) dx = 0 \end{aligned}$$

15.  $\operatorname{curl} \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z^2 & y & z \end{vmatrix} = 2z\mathbf{j}$

$$z = G(x, y) = \sqrt{4 - x^2 - y^2}, G_x = \frac{-x}{\sqrt{4 - x^2 - y^2}}, G_y = \frac{-y}{\sqrt{4 - x^2 - y^2}}$$

$$\begin{aligned} \int_S \int \operatorname{curl} \mathbf{F} \cdot \mathbf{N} dS &= \int_R \int (2z\mathbf{j}) \cdot \left( \frac{x}{\sqrt{4 - x^2 - y^2}}\mathbf{i} + \frac{y}{\sqrt{4 - x^2 - y^2}}\mathbf{j} + \mathbf{k} \right) dA \\ &= \int_R \int \frac{2yz}{\sqrt{4 - x^2 - y^2}} dA = \int_R \int \frac{2y\sqrt{4 - x^2 - y^2}}{\sqrt{4 - x^2 - y^2}} dA = \int_{-2}^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} 2y dy dx = 0 \end{aligned}$$

16.  $\mathbf{F}(x, y, z) = x^2\mathbf{i} + z^2\mathbf{j} - xyz\mathbf{k}$ ,  $S: z = \sqrt{4 - x^2 - y^2}$

$$\operatorname{curl} \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 & z^2 & -xyz \end{vmatrix} = (-xz - 2z)\mathbf{i} + yz\mathbf{j}$$

$$G(x, y, z) = z - \sqrt{4 - x^2 - y^2}$$

$$\nabla G(x, y, z) = \frac{x}{\sqrt{4 - x^2 - y^2}}\mathbf{i} + \frac{y}{\sqrt{4 - x^2 - y^2}}\mathbf{j} + \mathbf{k}$$

$$\begin{aligned} \int_S \int (\operatorname{curl} \mathbf{F}) \cdot \mathbf{N} dS &= \int_R \int \left[ \frac{-z(x+2)x}{\sqrt{4 - x^2 - y^2}} + \frac{y^2 z}{\sqrt{4 - x^2 - y^2}} \right] dA \\ &= \int_R \int [-x(x+2) + y^2] dA = \int_{-2}^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (-x^2 - 2x + y^2) dy dx \\ &= \int_{-2}^2 \left[ -x^2 y - 2xy + \frac{y^3}{3} \right]_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} dx \\ &= \int_{-2}^2 \left[ -2x^2 \sqrt{4 - x^2} - 4x \sqrt{4 - x^2} + \frac{2}{3}(4 - x^2) \sqrt{4 - x^2} \right] dx \\ &= \int_{-2}^2 \left[ -\frac{8}{3}x^2 \sqrt{4 - x^2} - 4x \sqrt{4 - x^2} + \frac{8}{3}\sqrt{4 - x^2} \right] dx \\ &= \left[ -\frac{8}{3}\left(\frac{1}{8}\right)x(2x^2 - 4)\sqrt{4 - x^2} + 16 \arcsin \frac{x}{2} + \frac{4}{3}(4 - x^2)^{3/2} + \frac{8}{3}\left(\frac{1}{2}\right)x\sqrt{4 - x^2} + 4 \arcsin \frac{x}{2} \right]_{-2}^2 \\ &= \left[ \left(-\frac{1}{3}\right)(8\pi) + \frac{4}{3}(2\pi) + \frac{1}{3}(-8\pi) - \frac{4}{3}(-2\pi) \right] = 0 \end{aligned}$$

17.  $\mathbf{F}(x, y, z) = -\ln\sqrt{x^2 + y^2}\mathbf{i} + \arctan\frac{x}{y}\mathbf{j} + \mathbf{k}$

$$\operatorname{curl} \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -1/2 \ln(x^2 + y^2) & \arctan x/y & 1 \end{vmatrix} = \left[ \frac{(1/y)}{1 + (x^2/y^2)} + \frac{y}{x^2 + y^2} \right] \mathbf{k} = \left[ \frac{2y}{x^2 + y^2} \right] \mathbf{k}$$

$S$ :  $z = 9 - 2x - 3y$  over one petal of  $r = 2 \sin 2\theta$  in the first octant.

$$G(x, y, z) = 2x + 3y + z - 9$$

$$\nabla G(x, y, z) = 2\mathbf{i} + 3\mathbf{j} + \mathbf{k}$$

$$\begin{aligned} \int_S \int (\operatorname{curl} \mathbf{F}) \cdot \mathbf{N} dS &= \int_R \int \frac{2y}{x^2 + y^2} dA = \int_0^{\pi/2} \int_0^{2 \sin 2\theta} \frac{2r \sin \theta}{r^2} r dr d\theta \\ &= \int_0^{\pi/2} \int_0^{4 \sin \theta \cos \theta} 2 \sin \theta dr d\theta = \int_0^{\pi/2} 8 \sin^2 \theta \cos \theta d\theta = \left[ \frac{8 \sin^3 \theta}{3} \right]_0^{\pi/2} = \frac{8}{3} \end{aligned}$$

18.  $\mathbf{F}(x, y, z) = yz\mathbf{i} + (2 - 3y)\mathbf{j} + (x^2 + y^2)\mathbf{k}$

$$\operatorname{curl} \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & 2 - 3y & x^2 + y^2 \end{vmatrix} = 2y\mathbf{i} + (y - 2x)\mathbf{j} - z\mathbf{k}$$

$S$ : the first octant portion of  $x^2 + z^2 = 16$  over  $x^2 + y^2 = 16$

$$G(x, y, z) = z - \sqrt{16 - x^2}$$

$$\nabla G(x, y, z) = \frac{x}{\sqrt{16 - x^2}}\mathbf{i} + \mathbf{k}$$

$$\begin{aligned} \int_S \int (\operatorname{curl} \mathbf{F}) \cdot \mathbf{N} dS &= \int_R \int \left[ \frac{2xy}{\sqrt{16 - x^2}} - z \right] dA = \int_R \int \left[ \frac{2xy}{\sqrt{16 - x^2}} - \sqrt{16 - x^2} \right] dA \\ &= \int_0^4 \int_0^{\sqrt{16-x^2}} \left[ \frac{2xy}{\sqrt{16 - x^2}} - \sqrt{16 - x^2} \right] dy dx = \int_0^4 \left[ \frac{x}{\sqrt{16 - x^2}} y^2 - \sqrt{16 - x^2} y \right]_0^{\sqrt{16-x^2}} dx \\ &= \int_0^4 \left[ x\sqrt{16 - x^2} - (16 - x^2) \right] dx = \left[ -\frac{1}{3}(16 - x^2)^{3/2} - 16x + \frac{x^3}{3} \right]_0^4 = \left( -64 + \frac{64}{3} \right) - \left( -\frac{64}{3} \right) = -\frac{64}{3} \end{aligned}$$

19.  $\operatorname{curl} \mathbf{F} = xy\mathbf{j} - xz\mathbf{k}$

$$z = G(x, y) = x^2, G_x = 2x, G_y = 0$$

$$\int_S \int \operatorname{curl} \mathbf{F} \cdot \mathbf{N} dS = \int_R \int (xy\mathbf{j} - xz\mathbf{k}) \cdot (2x\mathbf{i} - \mathbf{k}) dA = \int_R \int xz dA = \int_0^a \int_0^a x(x^2) dy dx = \frac{a^5}{4}$$

20.  $\mathbf{F}(x, y, z) = xyz\mathbf{i} + y\mathbf{j} + z\mathbf{k}$

$$\operatorname{curl} \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xyz & y & z \end{vmatrix} = xy\mathbf{j} - xz\mathbf{k}$$

$S$ : the first octant portion of  $z = x^2$  over  $x^2 + y^2 = a^2$ . You have  $\mathbf{N} = \frac{2x\mathbf{i} - \mathbf{k}}{\sqrt{1 + 4x^2}}$  and  $dS = \sqrt{1 + 4x^2} dA$ .

$$\begin{aligned} \int_S \int (\operatorname{curl} \mathbf{F}) \cdot \mathbf{N} dS &= \int_R \int xz dA = \int_R \int x^3 dA = \int_0^a \int_0^{\sqrt{a^2 - x^2}} x^3 dy dx \\ &= \int_0^a x^3 \sqrt{a^2 - x^2} dx = \left[ -\frac{1}{3}x^2(a^2 - x^2)^{3/2} - \frac{2}{15}(a^2 - x^2)^{5/2} \right]_0^a = \frac{2}{15}a^5 \end{aligned}$$

21.  $\mathbf{F}(x, y, z) = \mathbf{i} + \mathbf{j} - 2\mathbf{k}$

$$\operatorname{curl} \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 1 & 1 & -2 \end{vmatrix} = \mathbf{0}$$

Letting  $\mathbf{N} = \mathbf{k}$ , you have  $\int_S \int (\operatorname{curl} \mathbf{F}) \cdot \mathbf{N} dS = 0$ .

22.  $\mathbf{F}(x, y, z) = -z\mathbf{i} + y\mathbf{k}$

$$S: x^2 + y^2 = 1$$

$$\operatorname{curl} \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -z & 0 & y \end{vmatrix} = \mathbf{i} - \mathbf{j}$$

Letting

$$\mathbf{N} = \mathbf{k}, \operatorname{curl} \mathbf{F} \cdot \mathbf{N} = 0 \text{ and } \int_S \int (\operatorname{curl} \mathbf{F}) \cdot \mathbf{N} dS = 0.$$

23. See Theorem 15.13.

24.  $\operatorname{curl} \mathbf{F}$  measures the rotational tendency. See page 1135.

25. (a)  $\int_C f \nabla g \cdot d\mathbf{r} = \int_S \int \operatorname{curl}[f \nabla g] \cdot \mathbf{N} dS$  (Stokes's Theorem)

$$\begin{aligned} f \nabla g &= f \frac{\partial g}{\partial x} \mathbf{i} + f \frac{\partial g}{\partial y} \mathbf{j} + f \frac{\partial g}{\partial z} \mathbf{k} \\ \operatorname{curl}(f \nabla g) &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f(\partial g / \partial x) & f(\partial g / \partial y) & f(\partial g / \partial z) \end{vmatrix} \\ &= \left[ \left[ f \left( \frac{\partial^2 g}{\partial y \partial z} \right) + \left( \frac{\partial f}{\partial y} \right) \left( \frac{\partial g}{\partial z} \right) \right] - \left[ f \left( \frac{\partial^2 g}{\partial z \partial y} \right) + \left( \frac{\partial f}{\partial z} \right) \left( \frac{\partial g}{\partial y} \right) \right] \right] \mathbf{i} \\ &\quad - \left[ \left[ f \left( \frac{\partial^2 g}{\partial x \partial z} \right) + \left( \frac{\partial f}{\partial x} \right) \left( \frac{\partial g}{\partial z} \right) \right] - \left[ f \left( \frac{\partial^2 g}{\partial z \partial x} \right) + \left( \frac{\partial f}{\partial z} \right) \left( \frac{\partial g}{\partial x} \right) \right] \right] \mathbf{j} \\ &\quad + \left[ \left[ f \left( \frac{\partial^2 g}{\partial x \partial y} \right) + \left( \frac{\partial f}{\partial x} \right) \left( \frac{\partial g}{\partial y} \right) \right] - \left[ f \left( \frac{\partial^2 g}{\partial y \partial x} \right) + \left( \frac{\partial f}{\partial y} \right) \left( \frac{\partial g}{\partial x} \right) \right] \right] \mathbf{k} \\ &= \left[ \left( \frac{\partial f}{\partial y} \right) \left( \frac{\partial g}{\partial z} \right) - \left( \frac{\partial f}{\partial z} \right) \left( \frac{\partial g}{\partial y} \right) \right] \mathbf{i} - \left[ \left( \frac{\partial f}{\partial x} \right) \left( \frac{\partial g}{\partial z} \right) - \left( \frac{\partial f}{\partial z} \right) \left( \frac{\partial g}{\partial x} \right) \right] \mathbf{j} + \left[ \left( \frac{\partial f}{\partial y} \right) \left( \frac{\partial g}{\partial x} \right) - \left( \frac{\partial f}{\partial x} \right) \left( \frac{\partial g}{\partial y} \right) \right] \mathbf{k} \\ &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} & \frac{\partial g}{\partial z} \end{vmatrix} = \nabla f \times \nabla g \end{aligned}$$

So,  $\int_C f \nabla g \cdot d\mathbf{r} = \int_S \int \operatorname{curl}[f \nabla g] \cdot \mathbf{N} dS = \int_S \int [\nabla f \times \nabla g] \cdot \mathbf{N} dS$ .

(b)  $\int_C (f \nabla f) \cdot d\mathbf{r} = \int_S \int (\nabla f \times \nabla f) \cdot \mathbf{N} dS$  (using part a)  
 $= 0$  because  $\nabla f \times \nabla f = \mathbf{0}$ .

(c)  $\int_C (f \nabla g + g \nabla f) \cdot d\mathbf{r} = \int_C (f \nabla g) \cdot d\mathbf{r} + \int_C (g \nabla f) \cdot d\mathbf{r}$   
 $= \int_S \int (\nabla f \times \nabla g) \cdot \mathbf{N} dS + \int_S \int (\nabla g \times \nabla f) \cdot \mathbf{N} dS$  (using part a)  
 $= \int_S \int (\nabla f \times \nabla g) \cdot \mathbf{N} dS + \int_S \int -(\nabla f \times \nabla g) \cdot \mathbf{N} dS = 0$

26.  $f(x, y, z) = xyz$ ,  $g(x, y, z) = z$ ,  $S: z = \sqrt{4 - x^2 - y^2}$

(a)  $\nabla g(x, y, z) = \mathbf{k}$

$$f(x, y, z)\nabla g(x, y, z) = xyz\mathbf{k}$$

$$\mathbf{r}(t) = 2 \cos t \mathbf{i} + 2 \sin t \mathbf{j} + 0\mathbf{k}, \quad 0 \leq t \leq 2\pi$$

$$\int_C [f(x, y, z)\nabla g(x, y, z)] \cdot d\mathbf{r} = 0$$

(b)  $\nabla f(x, y, z) = yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}$

$$\nabla g(x, y, z) = \mathbf{k}$$

$$\nabla f \times \nabla g = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ yz & xz & xy \\ 0 & 0 & 1 \end{vmatrix} = xz\mathbf{i} - yz\mathbf{j}$$

$$\mathbf{N} = \frac{x}{\sqrt{4 - x^2 - y^2}}\mathbf{i} + \frac{y}{\sqrt{4 - x^2 - y^2}}\mathbf{j} + \mathbf{k}$$

$$dS = \sqrt{1 + \left( \frac{-x}{\sqrt{4 - x^2 - y^2}} \right)^2 + \left( \frac{-y}{\sqrt{4 - x^2 - y^2}} \right)^2} dA = \frac{2}{\sqrt{4 - x^2 - y^2}} dA$$

$$\int_S \int [\nabla f(x, y, z) \times \nabla g(x, y, z)] \cdot \mathbf{N} dS = \int_S \int \left[ \frac{x^2 z}{\sqrt{4 - x^2 - y^2}} - \frac{y^2 z}{\sqrt{4 - x^2 - y^2}} \right] \frac{2}{\sqrt{4 - x^2 - y^2}} dA$$

$$= \int_S \int \frac{2(x^2 - y^2)}{\sqrt{4 - x^2 - y^2}} dA$$

$$= \int_0^2 \int_0^{2\pi} \frac{2r^2(\cos^2 \theta - \sin^2 \theta)}{\sqrt{4 - r^2}} r d\theta dr = \int_0^2 \left[ \frac{2r^3}{\sqrt{4 - r^2}} \left( \frac{1}{2} \sin 2\theta \right) \right]_0^{2\pi} dr = 0$$

27. Let  $\mathbf{C} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ , then  $\frac{1}{2} \int_C (\mathbf{C} \times \mathbf{r}) \cdot d\mathbf{r} = \frac{1}{2} \int_S \int \text{curl}(\mathbf{C} \times \mathbf{r}) \cdot \mathbf{N} dS = \frac{1}{2} \int_S \int 2\mathbf{C} \cdot \mathbf{N} dS = \int_S \int \mathbf{C} \cdot \mathbf{N} dS$

because  $\mathbf{C} \times \mathbf{r} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a & b & c \\ x & y & z \end{vmatrix} = (bz - cy)\mathbf{i} - (az - cx)\mathbf{j} + (ay - bx)\mathbf{k}$

and  $\text{curl}(\mathbf{C} \times \mathbf{r}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ bz - cy & cx - az & ay - bx \end{vmatrix} = 2(a\mathbf{i} + b\mathbf{j} + c\mathbf{k}) = 2\mathbf{C}$ .

28. (a)  $\mathbf{F}(x, y, z) = e^{y+z}\mathbf{i}$

From the figure you have

$$C_1: \mathbf{r}_1(t) = t\mathbf{i}, \quad 0 \leq t \leq 1, \quad \mathbf{r}'_1 = \mathbf{i}$$

$$C_2: \mathbf{r}_2(t) = \mathbf{i} + t\mathbf{j}, \quad 0 \leq t \leq 1, \quad \mathbf{r}'_2 = \mathbf{j}$$

$$C_3: \mathbf{r}_3(t) = (1-t)\mathbf{i} + \mathbf{j}, \quad 0 \leq t \leq 1, \quad \mathbf{r}'_3 = -\mathbf{i}$$

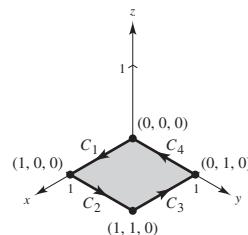
$$C_4: \mathbf{r}_4(t) = (1-t)\mathbf{j}, \quad 0 \leq t \leq 1, \quad \mathbf{r}'_4 = -\mathbf{j}$$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} \mathbf{F} \cdot \mathbf{r}'_1 dt + \int_{C_2} \mathbf{F} \cdot \mathbf{r}'_2 dt + \int_{C_3} \mathbf{F} \cdot \mathbf{r}'_3 dt + \int_{C_4} \mathbf{F} \cdot \mathbf{r}'_4 dt = \int_0^1 e^0 dt + 0 + \int_0^1 -e^{1+0} dt + 0 = 1 - e$$

**Double Integral:**  $\text{curl } \mathbf{F} = e^{y+z}\mathbf{j} - e^{y+z}\mathbf{k}$

$$G(x, y) = z = 0, \quad \mathbf{N} = \mathbf{k}$$

$$\int_S \int \text{curl } \mathbf{F} \cdot \mathbf{N} dS = \int_R \int -e^y dA = \int_0^1 \int_0^1 -e^y dx dy = \int_0^1 -e^y dy = [-e^y]_0^1 = 1 - e$$



(b)  $\mathbf{F}(x, y, z) = z^2\mathbf{i} + x^2\mathbf{j} + y^2\mathbf{k}$

From the figure you have

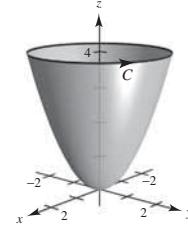
$$C: \mathbf{r}(t) = 2 \cos t \mathbf{i} + 2 \sin t \mathbf{j} + 4\mathbf{k}, \quad 0 \leq t \leq 2\pi.$$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C M dx + N dy + P dz = \int_0^{2\pi} [16(-2 \sin t) + (4 \cos^2 t)(2 \cos t) + 0] dt = \int_0^{2\pi} [-32 \sin t + 8 \cos^3 t] dt = 0$$

**Double Integral: curl F = 2y $\mathbf{i}$  + 2z $\mathbf{j}$  + 2x $\mathbf{k}$**

$$z = g(x, y) = x^2 + y^2, \quad g_x = 2x, \quad g_y = 2y$$

$$\begin{aligned} \int_S \int \text{curl } \mathbf{F} \cdot \mathbf{N} dS &= \int_R \int (2y\mathbf{i} + 2z\mathbf{j} + 2x\mathbf{k}) \cdot (-2x\mathbf{i} - 2y\mathbf{j} + \mathbf{k}) dA \\ &= \int_R \int [-4xy - 4y(x^2 + y^2) + 2x] dA \\ &= \int_0^{2\pi} \int_0^2 [-4r \sin \theta r \cos \theta - 4r \sin \theta (r^2) - 2r \cos \theta] r dr d\theta \\ &= \int_0^{2\pi} \left[ -16 \sin \theta - \frac{16}{3} \right] \cos \theta - \frac{128}{5} \sin \theta d\theta = 0 \end{aligned}$$



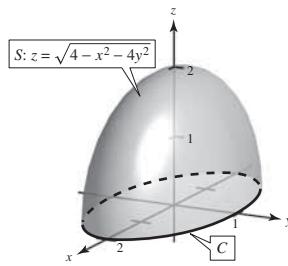
29. Let S be the upper portion of the ellipsoid

$$x^2 + 4y^2 + z^2 = 4, z \geq 0$$

Let C:  $\mathbf{r}(t) = \langle 2 \cos t, \sin t, 0 \rangle$ ,  $0 \leq t \leq 2\pi$ , be the boundary of S.

If  $\mathbf{F} = \langle M, N, P \rangle$  exists, then

$$\begin{aligned} 0 &= \int_S \int (\text{curl } \mathbf{F}) \cdot \mathbf{N} dS \quad (\text{by (i)}) \\ &= \int_C \mathbf{F} \cdot d\mathbf{r} \quad (\text{Stokes's Theorem}) \\ &= \int_C \mathbf{G} \cdot d\mathbf{r} \quad (\text{by (iii)}) \\ &= \int_0^{2\pi} \left\langle \frac{-\sin t}{4}, \frac{2 \cos t}{4}, 0 \right\rangle \cdot \langle -2 \sin t, \cos t, 0 \rangle dt = \frac{1}{4} \int_0^{2\pi} (2 \sin^2 t + 2 \cos^2 t) dt = \pi \end{aligned}$$

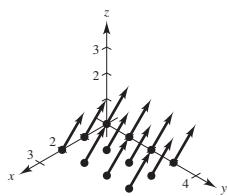


So, there is no such  $\mathbf{F}$ .

## Review Exercises for Chapter 15

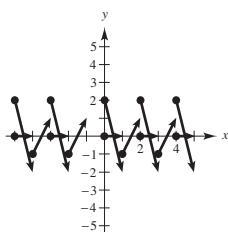
1.  $\mathbf{F}(x, y, z) = x\mathbf{i} + \mathbf{j} + 2\mathbf{k}$

$$\|\mathbf{F}\| = \sqrt{x^2 + 1^2 + 2^2} = \sqrt{x^2 + 5}$$



2.  $\mathbf{F}(x, y) = \mathbf{i} - 2y\mathbf{j}$

$$\|\mathbf{F}\| = \sqrt{1 + 4y^2}$$



3.  $f(x, y, z) = 2x^2 + xy + z^2$

$$\mathbf{F}(x, y, z) = \nabla f = (4x + y)\mathbf{i} + x\mathbf{j} + 2z\mathbf{k}$$

4.  $f(x, y, z) = x^2 e^{yz}$

$$\begin{aligned} \mathbf{F}(x, y, z) &= 2xe^{yz}\mathbf{i} + x^2ze^{yz}\mathbf{j} + x^2ye^{yz}\mathbf{k} \\ &= xe^{yz}(2\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}) \end{aligned}$$

5. Because  $\partial M/\partial y = -1/x^2 = \partial N/\partial x$ ,  $\mathbf{F}$  is conservative.

From  $M = \partial U/\partial x = -y/x^2$  and

$N = \partial U/\partial y = 1/x$ , partial integration yields

$U = (y/x) + h(y)$  and  $U = (y/x) + g(x)$  which suggests that  $U(x, y) = (y/x) + C$ .

6. Because  $\partial M/\partial y = -1/y^2 \neq \partial N/\partial x$ ,  $\mathbf{F}$  is not conservative.

7. Because  $\frac{\partial M}{\partial y} = 2xy$  and  $\frac{\partial N}{\partial x} = 2xy$ ,  $\mathbf{F}$  is conservative.

From  $M = \frac{\partial U}{\partial x} = xy^2 - x^2$  and

$N = \frac{\partial U}{\partial y} = x^2y + y^2$ , partial integration yields

$$U = \frac{1}{2}x^2y^2 - \frac{x^3}{3} + h(y)$$

and

$$U = \frac{1}{2}x^2y^2 + \frac{y^3}{3} + g(x).$$

So,  $h(y) = y^3/3$  and  $g(x) = -x^3/3$ . So,

$$U(x, y) = \frac{1}{2}x^2y^2 - \frac{x^3}{3} + \frac{y^3}{3} + C.$$

8. Because  $\partial M/\partial y = -6y^2 \sin 2x = \partial N/\partial x$ ,  $\mathbf{F}$  is conservative. From  $M = \partial U/\partial x = -2y^3 \sin 2x$  and  $N = \partial U/\partial y = 3y^2(1 + \cos 2x)$ , you obtain

$U = y^3 \cos 2x + h(y)$  and

$U = y^3(1 + \cos 2x) + g(x)$  which suggests that

$h(y) = y^3$ ,  $g(x) = C$ , and

$$U(x, y) = y^3(1 + \cos 2x) + C.$$

9. Because  $\frac{\partial M}{\partial y} = 8xy$  and  $\frac{\partial N}{\partial x} = 4x$ ,  $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ , so  $\mathbf{F}$  is not conservative.

10. Because

$$\frac{\partial M}{\partial y} = 4x = \frac{\partial N}{\partial x},$$

$$\frac{\partial M}{\partial z} = 2z = \frac{\partial P}{\partial x},$$

$$\frac{\partial N}{\partial z} = 6y \neq \frac{\partial P}{\partial y},$$

$\mathbf{F}$  is not conservative.

15. Because  $\mathbf{F} = (\cos y + y \cos x)\mathbf{i} + (\sin x - x \sin y)\mathbf{j} + xyz\mathbf{k}$ :

$$(a) \text{ div } \mathbf{F} = -y \sin x - x \cos y + xy$$

$$(b) \text{ curl } \mathbf{F} = xz\mathbf{i} - yz\mathbf{j} + (\cos x - \sin y + \sin y - \cos x)\mathbf{k} = xz\mathbf{i} - yz\mathbf{j}$$

16. Because  $\mathbf{F} = (3x - y)\mathbf{i} + (y - 2z)\mathbf{j} + (z - 3x)\mathbf{k}$ :

$$(a) \text{ div } \mathbf{F} = 3 + 1 + 1 = 5$$

$$(b) \text{ curl } \mathbf{F} = 2\mathbf{i} + 3\mathbf{j} + \mathbf{k}$$

11. Because

$$\frac{\partial M}{\partial y} = \frac{-1}{y^2z} = \frac{\partial N}{\partial x}, \frac{\partial M}{\partial z} = \frac{-1}{yz^2} = \frac{\partial P}{\partial x},$$

$$\frac{\partial N}{\partial z} = \frac{x}{y^2z^2} = \frac{\partial P}{\partial y},$$

$\mathbf{F}$  is conservative. From

$$M = \frac{\partial U}{\partial x} = \frac{1}{yz}, \quad N = \frac{\partial U}{\partial y} = \frac{-x}{y^2z}, \quad P = \frac{\partial U}{\partial z} = \frac{-x}{yz^2}$$

you obtain

$$U = \frac{x}{yz} + f(y, z), \quad U = \frac{x}{yz} + g(x, z),$$

$$U = \frac{x}{yz} + h(x, y) \Rightarrow f(x, y, z) = \frac{x}{yz} + K.$$

12. Because

$$\frac{\partial M}{\partial y} = \sin z = \frac{\partial N}{\partial x}, \quad \frac{\partial M}{\partial z} = y \cos z \neq \frac{\partial P}{\partial x},$$

$\mathbf{F}$  is not conservative.

13. Because  $\mathbf{F}(x, y, z) = x^2\mathbf{i} + xy^2\mathbf{j} + x^2z\mathbf{k}$ :

$$(a) \text{ div } \mathbf{F} = 2x + 2xy + x^2$$

$$(b) \text{ curl } \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 & xy^2 & x^2z \end{vmatrix} = -(2xz)\mathbf{j} + y^2\mathbf{k}$$

14. Because  $\mathbf{F}(x, y, z) = y^2\mathbf{j} - z^2\mathbf{k}$ :

$$(a) \text{ div } \mathbf{F} = 2y - 2z$$

$$(b) \text{ curl } \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & y^2 & -z^2 \end{vmatrix} = \mathbf{0}$$

17. Because  $\mathbf{F} = \arcsin xi + xy^2\mathbf{j} + yz^2\mathbf{k}$ :

$$(a) \text{ div } \mathbf{F} = \frac{1}{\sqrt{1-x^2}} + 2xy + 2yz$$

$$(b) \text{ curl } \mathbf{F} = z^2\mathbf{i} + y^2\mathbf{k}$$

18. Because  $\mathbf{F} = (x^2 - y)\mathbf{i} - (x + \sin^2 y)\mathbf{j}$ :

(a)  $\operatorname{div} \mathbf{F} = 2x - 2 \sin y \cos y$

(b)  $\operatorname{curl} \mathbf{F} = \mathbf{0}$

19. Because  $\mathbf{F} = \ln(x^2 + y^2)\mathbf{i} + \ln(x^2 + y^2)\mathbf{j} + z\mathbf{k}$ :

(a)  $\operatorname{div} \mathbf{F} = \frac{2x}{x^2 + y^2} + \frac{2y}{x^2 + y^2} + 1 = \frac{2x + 2y}{x^2 + y^2} + 1$

(b)  $\operatorname{curl} \mathbf{F} = \frac{2x - 2y}{x^2 + y^2}\mathbf{k}$

20. Because  $\mathbf{F} = \frac{z}{x}\mathbf{i} + \frac{z}{y}\mathbf{j} + z^2\mathbf{k}$ :

(a)  $\operatorname{div} \mathbf{F} = -\frac{z}{x^2} - \frac{z}{y^2} + 2z = z\left(2 - \frac{1}{x^2} - \frac{1}{y^2}\right)$

(b)  $\operatorname{curl} \mathbf{F} = -\frac{1}{y}\mathbf{i} + \frac{1}{x}\mathbf{j}$

22. (a) Let  $x = 5t$ ,  $y = 4t$ ,  $0 \leq t \leq 1$ , then  $ds = \sqrt{41} dt$ .

$$\int_C xy \, ds = \int_0^1 20t^2 \sqrt{41} \, dt = \frac{20\sqrt{41}}{3}$$

(b)  $C_1: x = t$ ,  $y = 0$ ,  $0 \leq t \leq 4$ ,  $ds = dt$

$C_2: x = 4 - 4t$ ,  $y = 2t$ ,  $0 \leq t \leq 1$ ,  $ds = 2\sqrt{5} dt$

$C_3: x = 0$ ,  $y = 2 - t$ ,  $0 \leq t \leq 2$ ,  $ds = dt$

So,  $\int_C xy \, ds = \int_0^4 0 \, dt + \int_0^1 (8t - 8t^2)2\sqrt{5} \, dt + \int_0^2 0 \, dt = 16\sqrt{5}\left[\frac{t^2}{2} - \frac{t^3}{3}\right]_0^1 = \frac{8\sqrt{5}}{3}$ .

23.  $x = 1 - \sin t$ ,  $y = 1 - \cos t$ ,  $0 \leq t \leq 2\pi$

$$\frac{dx}{dt} = -\cos t, \frac{dy}{dt} = \sin t, ds = \sqrt{(-\cos t)^2 + (\sin t)^2} \, dt = dt$$

$$\begin{aligned} \int_C (x^2 + y^2) \, ds &= \int_0^{2\pi} [(1 - \sin t)^2 + (1 - \cos t)^2] \, dt = \int_0^{2\pi} [1 - 2 \sin t + \sin^2 t + 1 - 2 \cos t + \cos^2 t] \, dt \\ &= \int_0^{2\pi} [3 - 2 \sin t - 2 \cos t] \, dt = [3t + 2 \cos t - 2 \sin t]_0^{2\pi} = 6\pi \end{aligned}$$

24.  $x = \cos t + t \sin t$ ,  $y = \sin t - t \cos t$ ,  $0 \leq t \leq 2\pi$ ,  $\frac{dx}{dt} = t \cos t$ ,  $\frac{dy}{dt} = t \sin t$

$$\int_C (x^2 + y^2) \, ds = \int_0^{2\pi} [(\cos t + t \sin t)^2 + (\sin t - t \cos t)^2] \sqrt{t^2 \cos^2 t + t^2 \sin^2 t} \, dt = \int_0^{2\pi} [t^3 + t] \, dt = 2\pi^2(1 + 2\pi^2)$$

25. (a) Let  $x = 3t$ ,  $y = -3t$ ,  $0 \leq t \leq 1$ .

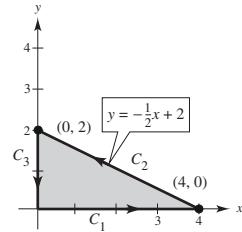
$$\int_C (2x - y) \, dx + (x + 2y) \, dy = \int_0^1 [(6t + 3t)3 + (3t - 6t)(-3)] \, dt = \int_0^1 (27t + 9t) \, dt = 18t^2 \Big|_0^1 = 18$$

(b) Let  $x = 3 \cos t$ ,  $y = 3 \sin t$ ,  $dx = -3 \sin t \, dt$ ,  $dy = 3 \cos t \, dt$ ,  $0 \leq t \leq 2\pi$ .

$$\int_C (2x - y) \, dx + (x + 2y) \, dy = \int_0^{2\pi} [(6 \cos t - 3 \sin t)(-3 \sin t) + (3 \cos t + 6 \sin t)(3 \cos t)] \, dt = \int_0^{2\pi} 9 \, dt = 18\pi$$

26.  $x = \cos t + t \sin t$ ,  $y = \sin t - t \sin t$ ,  $0 \leq t \leq \frac{\pi}{2}$ ,  $dx = t \cos t \, dt$ ,  $dy = (\cos t - t \cos t - \sin t) \, dt$

$$\int_C (2x - y) \, dx + (x + 3y) \, dy = \int_0^{\pi/2} [\sin t \cos t (5t^2 - 6t + 2) + \cos^2 t(t+1) + \sin^2 t(2t-3)] \, dt \approx 1.01$$



27.  $\int_C (2x + y) ds, \mathbf{r}(t) = a \cos^3 t \mathbf{i} + a \sin^3 t \mathbf{j}, 0 \leq t \leq \frac{\pi}{2}$

$$x'(t) = -3a \cdot \cos^2 t \sin t$$

$$y'(t) = 3a \cdot \sin^2 t \cos t$$

$$\int_C (2x + y) ds = \int_0^{\pi/2} (2(a \cos^3 t) + a \sin^3 t) \sqrt{x'(t)^2 + y'(t)^2} dt = \frac{9a^2}{5}$$

28.  $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + t^{3/2}\mathbf{k}, 0 \leq t \leq 4$

$$x'(t) = 1, y'(t) = 2t, z'(t) = \frac{3}{2}t^{1/2}$$

$$\int_C (x^2 + y^2 + z^2) ds = \int_0^4 (t^2 + t^4 + t^3) \sqrt{1 + 4t^2 + \frac{9}{4}t} dt \approx 2080.59$$

29.  $f(x, y) = 3 + \sin(x + y)$

C:  $y = 2x$  from  $(0, 0)$  to  $(2, 4)$

$$\mathbf{r}(t) = t\mathbf{i} + 2t\mathbf{j}, 0 \leq t \leq 2$$

$$\mathbf{r}'(t) = \mathbf{i} + 2\mathbf{j}$$

$$\|\mathbf{r}'(t)\| = \sqrt{5}$$

Lateral surface area:

$$\int_C f(x, y) ds = \int_0^2 [3 + \sin(t + 2t)] \sqrt{5} dt$$

$$= \sqrt{5} \int_0^2 [3 + \sin 3t] dt$$

$$= \sqrt{5} \left[ 3t - \frac{1}{3} \cos 3t \right]_0^2$$

$$= \sqrt{5} \left[ 6 - \frac{1}{3} \cos 6 + \frac{1}{3} \right]$$

$$= \frac{\sqrt{5}}{3} (19 - \cos 6) \approx 13.446$$

30.  $f(x, y) = 12 - x - y$

C:  $y = x^2$  from  $(0, 0)$  to  $(2, 4)$

$$\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j}, 0 \leq t \leq 2$$

$$\mathbf{r}'(t) = \mathbf{i} + 2t\mathbf{j}$$

$$\|\mathbf{r}'(t)\| = \sqrt{1 + 4t^2}$$

Lateral surface area:

$$\int_C f(x, y) ds = \int_0^2 (12 - t - t^2) \sqrt{1 + 4t^2} dt \approx 41.532$$

31.  $\mathbf{F}(x, y) = xy\mathbf{i} + 2xy\mathbf{j}$

$$\mathbf{r}(t) = t^2\mathbf{i} + t^2\mathbf{j}, 0 \leq t \leq 1$$

$$\mathbf{r}'(t) = 2t\mathbf{i} + 2t\mathbf{j}$$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 [t^2(t^2)(2t) + 2(t^2)(t^2)(2t)] dt$$

$$= \int_0^1 6t^5 dt = t^6 \Big|_0^1 = 1$$

32.  $d\mathbf{r} = [(-4 \sin t)\mathbf{i} + 3 \cos t\mathbf{j}] dt$

$$\mathbf{F} = (4 \cos t - 3 \sin t)\mathbf{i} + (4 \cos t + 3 \sin t)\mathbf{j}, 0 \leq t \leq 2\pi$$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} (12 - 7 \sin t \cos t) dt = \left[ 12t - \frac{7 \sin^2 t}{2} \right]_0^{2\pi} = 24\pi$$

33.  $d\mathbf{r} = [(-2 \sin t)\mathbf{i} + (2 \cos t)\mathbf{j} + \mathbf{k}] dt$

$$\mathbf{F} = (2 \cos t)\mathbf{i} + (2 \sin t)\mathbf{j} + t\mathbf{k}, 0 \leq t \leq 2\pi$$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} t dt = 2\pi^2$$

34.  $x = 2 - t, y = 2 - t, z = \sqrt{4t - t^2}, 0 \leq t \leq 2$

$$d\mathbf{r} = \left[ -\mathbf{i} - \mathbf{j} + \frac{2-t}{\sqrt{4t-t^2}} \mathbf{k} \right] dt$$

$$\mathbf{F} = (4 - 2t - \sqrt{4t - t^2})\mathbf{i} + (\sqrt{4t - t^2} - 2 + t)\mathbf{j} + 0\mathbf{k}$$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^2 (t - 2) dt = \left[ \frac{t^2}{2} - 2t \right]_0^2 = -2$$

35.  $\mathbf{F}(x, y, z) = (y + z)\mathbf{i} + (x + z)\mathbf{j} + (x + y)\mathbf{k}$

Curve of intersection:  $x = t, y = t, z = t^2 + t^2 = 2t^2$

$$\mathbf{r}(t) = t\mathbf{i} + t\mathbf{j} + 2t^2\mathbf{k}, \quad 0 \leq t \leq 2$$

$$\mathbf{r}'(t) = \mathbf{i} + \mathbf{j} + 4t\mathbf{k}$$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^2 [(t + 2t^2) + (t + 2t^2) + (2t)(4t)] dt = \int_0^2 [12t^2 + 2t] dt = [4t^3 + t^2]_0^2 = 36$$

36. Let  $x = 2 \sin t, y = -2 \cos t, z = 4 \sin^2 t, 0 \leq t \leq \pi$ .

$$d\mathbf{r} = [(2 \cos t)\mathbf{i} + (2 \sin t)\mathbf{j} + (8 \sin t \cos t)\mathbf{k}] dt$$

$$\mathbf{F} = 0\mathbf{i} + 4\mathbf{j} + (2 \sin t)\mathbf{k}$$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^\pi (8 \sin t + 16 \sin^2 t \cos t) dt = [-8 \cos t + \frac{16}{3} \sin^3 t]_0^\pi = 16$$

37. For  $y = x^2$ ,  $\mathbf{r}_1(t) = t\mathbf{i} + t^2\mathbf{j}, 0 \leq t \leq 2$

For  $y = 2x$ ,  $\mathbf{r}_2(t) = (2 - t)\mathbf{i} + (4 - 2t)\mathbf{j}, 0 \leq t \leq 2$

$$\begin{aligned} \int_C xy \, dx + (x^2 + y^2) \, dy &= \int_{C_1} xy \, dx + (x^2 + y^2) \, dy + \int_{C_2} xy \, dx + (x^2 + y^2) \, dy \\ &= \frac{100}{3} + (-32) = \frac{4}{3} \end{aligned}$$

38.  $\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C (2x - y) \, dx + (2y - x) \, dy$

$$\mathbf{r}(t) = (2 \cos t + 2t \sin t)\mathbf{i} + (2 \sin t - 2t \cos t)\mathbf{j}, 0 \leq t \leq \pi$$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = 4\pi^2 + 4\pi$$

39.  $\mathbf{F} = x\mathbf{i} - \sqrt{y}\mathbf{j}$  is conservative.

$$\text{Work} = \left[ \frac{1}{2}x^2 - \frac{2}{3}y^{3/2} \right]_{(0,0)}^{(4,8)} = \frac{1}{2}(16) - \left(\frac{2}{3}\right)8^{3/2} = \frac{8}{3}(3 - 4\sqrt{2})$$

40.  $\mathbf{r}(t) = 10 \sin t\mathbf{i} + 10 \cos t\mathbf{j} + \frac{2000/5280}{\pi/2}t\mathbf{k} = 10 \sin t\mathbf{i} + 10 \cos t\mathbf{j} + \frac{25}{33\pi}t\mathbf{k}, 0 \leq t \leq \frac{\pi}{2}$

$$\mathbf{F} = 20\mathbf{k}$$

$$d\mathbf{r} = \left( 10 \cos t\mathbf{i} - 10 \sin t\mathbf{j} + \frac{25}{33\pi}\mathbf{k} \right)$$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{\pi/2} \frac{500}{33\pi} dt = \frac{250}{33} \text{ mi} \cdot \text{ton}$$

41.  $\int_C 2xyz \, dx + x^2z \, dy + x^2y \, dz = [x^2yz]_{(0,0,0)}^{(1,3,2)} = 6$

42.  $\int_C y \, dx + x \, dy + \frac{1}{z} \, dz = [xy + \ln|z|]_{(0,0,1)}^{(4,4,4)} = 16 + \ln 4$

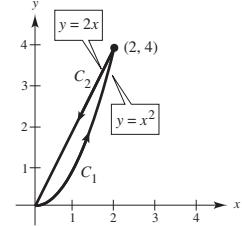
43. (a)  $\int_C y^2 \, dx + 2xy \, dy = \int_0^1 [(1+t)^2(3) + 2(1+3t)(1+t)] dt$

$$= \int_0^1 [3(t^2 + 2t + 1) + 2(3t^2 + 4t + 1)] dt = \int_0^1 (9t^2 + 14t + 5) dt = [3t^3 + 7t^2 + 5t]_0^1 = 15$$

(b)  $\int_C y^2 \, dx + 2xy \, dy = \int_1^4 [t(1) + 2(t)(\sqrt{t}) \frac{1}{2\sqrt{t}}] dt = \int_1^4 (t + t) dt = [t^2]_1^4 = 15$

(c)  $\mathbf{F}(x, y) = y^2\mathbf{i} + 2xy\mathbf{j} = \nabla f$  where  $f(x, y) = xy^2$ .

So,  $\int_C \mathbf{F} \cdot d\mathbf{r} = 4(2)^2 - 1(1)^2 = 15$ .



44.  $x = a(\theta - \sin \theta)$ ,  $y = a(1 - \cos \theta)$ ,  $0 \leq \theta \leq 2\pi$

(a)  $A = \frac{1}{2} \int_C x \, dy - y \, dx$ .

Because these equations orient the curve backwards, you will use

$$\begin{aligned} A &= \frac{1}{2} \int (y \, dx - x \, dy) \\ &= \frac{1}{2} \int_0^{2\pi} [a^2(1 - \cos \theta)(1 - \cos \theta) - a^2(\theta - \sin \theta)(\sin \theta)] d\theta + \frac{1}{2} \int_0^{2\pi a} (0 - 0) d\theta \\ &= \frac{a^2}{2} \int_0^{2\pi} [1 - 2\cos \theta + \cos^2 \theta - \theta \sin \theta + \sin^2 \theta] d\theta \\ &= \frac{a^2}{2} \int_0^{2\pi} (2 - 2\cos \theta - \theta \sin \theta) d\theta \\ &= \frac{a^2}{2}(6\pi) = 3\pi a^2. \end{aligned}$$

(b) By symmetry,  $\bar{x} = \pi a$ . From Section 15.4,

$$\bar{y} = -\frac{1}{2A} \int_C y^2 \, dx = \frac{1}{2A} \int_0^{2\pi} a^3(1 - \cos \theta)^2(1 - \cos \theta) d\theta = \frac{1}{2(3\pi a^2)} a^3(5\pi) = \frac{5}{6}a.$$

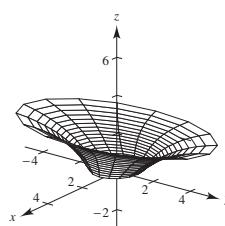
45.  $\int_C y \, dx + 2x \, dy = \int_0^1 \int_0^1 \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dy \, dx$   
 $= \int_0^1 \int_0^1 (2 - 1) dy \, dx = 1$

46.  $\int_C xy \, dx + (x^2 + y^2) \, dy = \int_0^2 \int_0^2 (2x - x) dy \, dx$   
 $= \int_0^2 2x \, dx = 4$

47.  $\int_C xy^2 \, dx + x^2y \, dy = \int_R \int \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dA$   
 $= \int_R \int (2xy - 2xy) dA = 0$

50.  $\int_C y^2 \, dx + x^{4/3} \, dy = \int_{-1}^1 \int_{-(1-x^{2/3})^{3/2}}^{(1-x^{2/3})^{3/2}} \left( \frac{4}{3}x^{1/3} - 2y \right) dy \, dx = \int_{-1}^1 \left[ \frac{4}{3}x^{1/3}y - y^2 \right]_{-(1-x^{2/3})^{3/2}}^{(1-x^{2/3})^{3/2}} dx$   
 $= \int_{-1}^1 \frac{8}{3}x^{1/3}(1 - x^{2/3})^{3/2} dx = \left[ -\frac{8}{7}x^{2/3}(1 - x^{2/3})^{5/2} - \frac{16}{35}(1 - x^{2/3})^{5/2} \right]_{-1}^1 = 0$

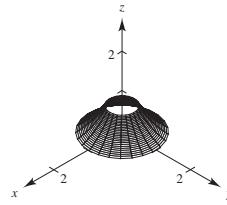
51.  $\mathbf{r}(u, v) = \sec u \cos v \mathbf{i} + (1 + 2 \tan u) \sin v \mathbf{j} + 2u \mathbf{k}$   
 $0 \leq u \leq \frac{\pi}{3}$ ,  $0 \leq v \leq 2\pi$



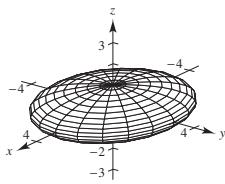
48.  $\int_C (x^2 - y^2) \, dx + 2xy \, dy = \int_{-a}^a \int_{-\sqrt{a^2 - x^2}}^{\sqrt{a^2 - x^2}} 4y \, dy \, dx$   
 $= \int_{-a}^a 0 \, dx = 0$

49.  $\int_C xy \, dx + x^2 \, dy = \int_R \int \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dA$   
 $= \int_{-1}^1 \int_{x^2}^1 (2x - x) dy \, dx$   
 $= \int_{-1}^1 [xy]_{x^2}^1 dx = \int_{-1}^1 (x - x^3) dx$   
 $= \left[ \frac{x^2}{2} - \frac{x^4}{4} \right]_{-1}^1 = 0$

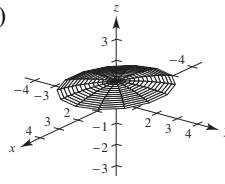
52.  $\mathbf{r}(u, v) = e^{-u/4} \cos v \mathbf{i} + e^{-u/4} \sin v \mathbf{j} + \frac{u}{6} \mathbf{k}$   
 $0 \leq u \leq 4$ ,  $0 \leq v \leq 2\pi$



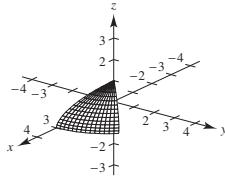
53. (a)



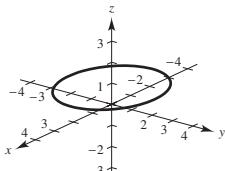
(b)



(c)



(d)



$$\text{The space curve is a circle: } \mathbf{r}\left(u, \frac{\pi}{4}\right) = \frac{3\sqrt{2}}{2} \cos u \mathbf{i} + \frac{3\sqrt{2}}{2} \sin u \mathbf{j} + \frac{\sqrt{2}}{2} \mathbf{k}$$

$$(e) \quad \mathbf{r}_u = -3 \cos v \sin u \mathbf{i} + 3 \cos v \cos u \mathbf{j}$$

$$\mathbf{r}_v = -3 \sin v \cos u \mathbf{i} - 3 \sin v \sin u \mathbf{j} + \cos v \mathbf{k}$$

$$\begin{aligned} \mathbf{r}_u \times \mathbf{r}_v &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -3 \cos v \sin u & 3 \cos v \cos u & 0 \\ -3 \sin v \cos u & -3 \sin v \sin u & \cos v \end{vmatrix} \\ &= (3 \cos^2 v \cos u) \mathbf{i} + (3 \cos^2 v \sin u) \mathbf{j} + (9 \cos v \sin v \sin^2 u + 9 \cos v \sin v \cos^2 u) \mathbf{k} \\ &= (3 \cos^2 v \cos u) \mathbf{i} + (3 \cos^2 v \sin u) \mathbf{j} + (9 \cos v \sin v) \mathbf{k} \end{aligned}$$

$$\|\mathbf{r}_u \times \mathbf{r}_v\| = \sqrt{9 \cos^4 v \cos^2 u + 9 \cos^4 v \sin^2 u + 81 \cos^2 v \sin^2 v} = \sqrt{9 \cos^4 v + 81 \cos^2 v \sin^2 v}$$

$$\text{Using a Symbolic integration utility, } \int_{\pi/4}^{\pi/2} \int_0^{2\pi} \|\mathbf{r}_u \times \mathbf{r}_v\| du dv \approx 14.44.$$

$$(f) \text{ Similarly, } \int_0^{\pi/4} \int_0^{\pi/2} \|\mathbf{r}_u \times \mathbf{r}_v\| dv du \approx 4.27.$$

54.  $S: \mathbf{r}(u, v) = (u + v)\mathbf{i} + (u - v)\mathbf{j} + \sin v \mathbf{k}, \quad 0 \leq u \leq 2, 0 \leq v \leq \pi$ 

$$\mathbf{r}_u(u, v) = \mathbf{i} + \mathbf{j}$$

$$\mathbf{r}_v(u, v) = \mathbf{i} - \mathbf{j} + \cos v \mathbf{k}$$

$$\mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 0 \\ 1 & -1 & \cos v \end{vmatrix} = \cos v \mathbf{i} - \cos v \mathbf{j} - 2 \mathbf{k}$$

$$\|\mathbf{r}_u \times \mathbf{r}_v\| = \sqrt{2 \cos^2 v + 4}$$

$$\int_S \int z dS = \int_0^\pi \int_0^2 \sin v \sqrt{2 \cos^2 v + 4} du dv = 2 \left[ \sqrt{6} + \sqrt{2} \ln \left( \frac{\sqrt{6} + \sqrt{2}}{\sqrt{6} - \sqrt{2}} \right) \right]$$

55.  $S: \mathbf{r}(u, v) = u \cos v \mathbf{i} + u \sin v \mathbf{j} + (u - 1)(2 - u) \mathbf{k}, \quad 0 \leq u \leq 2, 0 \leq v \leq 2\pi$ 

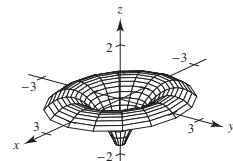
$$\mathbf{r}_u(u, v) = \cos v \mathbf{i} + \sin v \mathbf{j} + (3 - 2u) \mathbf{k}$$

$$\mathbf{r}_v(u, v) = -u \sin v \mathbf{i} + u \cos v \mathbf{j}$$

$$\mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos v & \sin v & 3 - 2u \\ -u \sin v & u \cos v & 0 \end{vmatrix} = (2u - 3)u \cos v \mathbf{i} + (2u - 3)u \sin v \mathbf{j} + u \mathbf{k}$$

$$\|\mathbf{r}_u \times \mathbf{r}_v\| = u \sqrt{(2u - 3)^2 + 1}$$

$$\int_S \int (x + y) dS = \int_0^{2\pi} \int_0^2 (u \cos v + u \sin v) u \sqrt{(2u - 3)^2 + 1} du dv = \int_0^2 \int_0^{2\pi} (\cos v + \sin v) u^2 \sqrt{(2u - 3)^2 + 1} dv du = 0$$



56. (a)  $z = a(a - \sqrt{x^2 + y^2})$ ,  $0 \leq z \leq a^2$

$$z = 0 \Rightarrow x^2 + y^2 = a^2$$

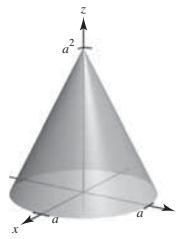
(b)  $S: g(x, y) = z = a^2 - a\sqrt{x^2 + y^2}$

$$\rho(x, y) = k\sqrt{x^2 + y^2}$$

$$m = \int_S e(x, y, z) dS = \int_R \int k\sqrt{x^2 + y^2} \sqrt{1 + g_x^2 + g_y^2} dA$$

$$= k \int_R \int \sqrt{x^2 + y^2} \sqrt{1 + \frac{a^2 x^2}{x^2 + y^2} + \frac{a^2 y^2}{x^2 + y^2}} dA = k \int_R \int \sqrt{a^2 + 1} (\sqrt{x^2 + y^2}) dA$$

$$= k\sqrt{a^2 + 1} \int_0^{2\pi} \int_0^a r^2 dr d\theta = k\sqrt{a^2 + 1} \int_0^{2\pi} \frac{a^3}{3} d\theta = \frac{2}{3} k\sqrt{a^2 + 1} a^3 \pi$$



57.  $\mathbf{F}(x, y, z) = x^2 \mathbf{i} + xy \mathbf{j} + z \mathbf{k}$

$Q$ : solid region bounded by the coordinate planes and the plane  $2x + 3y + 4z = 12$

**Surface Integral:** There are four surfaces for this solid.

$$z = 0, \quad \mathbf{N} = -\mathbf{k}, \quad \mathbf{F} \cdot \mathbf{N} = -z, \quad \int_{S_1} \int 0 dS = 0$$

$$y = 0, \quad \mathbf{N} = -\mathbf{j}, \quad \mathbf{F} \cdot \mathbf{N} = -xy, \quad \int_{S_2} \int 0 dS = 0$$

$$x = 0, \quad \mathbf{N} = -\mathbf{i}, \quad \mathbf{F} \cdot \mathbf{N} = -x^2, \quad \int_{S_3} \int 0 dS = 0$$

$$2x + 3y + 4z = 12, \quad \mathbf{N} = \frac{2\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}}{\sqrt{29}}, \quad dS = \sqrt{1 + \left(\frac{1}{4}\right)^2 + \left(\frac{9}{16}\right)} dA = \frac{\sqrt{29}}{4} dA$$

$$\begin{aligned} \int_{S_4} \int \mathbf{F} \cdot \mathbf{N} dS &= \frac{1}{4} \int_R \int (2x^2 + 3xy + 4z) dA \\ &= \frac{1}{4} \int_0^6 \int_0^{4-(2x/3)} (2x^2 + 3xy + 12 - 2x - 3y) dy dx \\ &= \frac{1}{4} \int_0^6 \left[ 2x^2 \left( \frac{12-2x}{3} \right) + \frac{3x}{2} \left( \frac{12-2x}{3} \right)^2 + 12 \left( \frac{12-2x}{3} \right) - 2x \left( \frac{12-2x}{3} \right) - \frac{3}{2} \left( \frac{12-2x}{3} \right)^2 \right] dx \\ &= \frac{1}{6} \int_0^6 (-x^3 + x^2 + 24x + 36) dx \\ &= \frac{1}{6} \left[ -\frac{x^4}{4} + \frac{x^3}{3} + 12x^2 + 36x \right]_0^6 = 66 \end{aligned}$$

**Divergence Theorem:** Because  $\operatorname{div} \mathbf{F} = 2x + x + 1 = 3x + 1$ , Divergence Theorem yields

$$\begin{aligned} \iiint_Q \operatorname{div} \mathbf{F} dV &= \int_0^6 \int_0^{(12-2x)/3} \int_0^{(12-2x-3y)/4} (3x + 1) dz dy dx \\ &= \int_0^6 \int_0^{(12-2x)/3} (3x + 1) \left( \frac{12-2x-3y}{4} \right) dy dx \\ &= \frac{1}{4} \int_0^6 (3x + 1) \left[ 12y - 2xy - \frac{3}{2}y^2 \right]_0^{(12-2x)/3} dx \\ &= \frac{1}{4} \int_0^6 (3x + 1) \left[ 4(12-2x) - 2x \left( \frac{12-2x}{3} \right) - \frac{3}{2} \left( \frac{12-2x}{3} \right)^2 \right] dx \\ &= \frac{1}{4} \int_0^6 \frac{2}{3} (3x^3 - 35x^2 + 96x + 36) dx \\ &= \frac{1}{6} \left[ \frac{3x^4}{4} - \frac{35x^3}{3} + 48x^2 + 36x \right]_0^6 = 66. \end{aligned}$$

58.  $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$

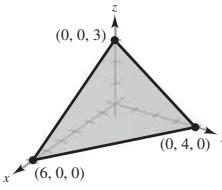
$Q$ : solid region bounded by the coordinate planes and the plane  $2x + 3y + 4z = 12$

**Surface Integral:** There are four surfaces for this solid.

$$z = 0, \quad \mathbf{N} = -\mathbf{k}, \quad \mathbf{F} \cdot \mathbf{N} = -z, \quad \int_{S_1} 0 \, dS = 0$$

$$y = 0, \quad \mathbf{N} = -\mathbf{j}, \quad \mathbf{F} \cdot \mathbf{N} = -y, \quad \int_{S_2} 0 \, dS = 0$$

$$x = 0, \quad \mathbf{N} = -\mathbf{i}, \quad \mathbf{F} \cdot \mathbf{N} = -x, \quad \int_{S_3} 0 \, dS = 0$$



$$2x + 3y + 4z = 12, \quad \mathbf{N} = \frac{2\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}}{\sqrt{29}}, \quad dS = \sqrt{1 + \left(\frac{1}{4}\right)^2 + \left(\frac{9}{16}\right)} dA = \frac{\sqrt{29}}{4} dA$$

$$\int_{S_4} \int \mathbf{N} \cdot \mathbf{F} \, dS = \frac{1}{4} \int_R \int (2x + 3y + 4z) \, dy \, dx = \frac{1}{4} \int_0^6 \int_0^{(12-2x)/3} 12 \, dy \, dx = 3 \int_0^6 \left(4 - \frac{2x}{3}\right) dx = 3 \left[4x - \frac{x^2}{3}\right]_0^6 = 36$$

**Triple Integral:** Because  $\operatorname{div} \mathbf{F} = 3$ , the Divergence Theorem yields

$$\iiint_Q \operatorname{div} \mathbf{F} \, dV = \iiint_Q 3 \, dV = 3(\text{Volume of solid}) = 3 \left[ \frac{1}{3} (\text{Area of base})(\text{Height}) \right] = \frac{1}{2} (6)(4)(3) = 36.$$

59.  $\mathbf{F}(x, y, z) = (\cos y + y \cos x)\mathbf{i} + (\sin x - x \sin y)\mathbf{j} + xyz\mathbf{k}$

$S$ : portion of  $z = y^2$  over the square in the  $xy$ -plane with vertices  $(0, 0), (a, 0), (a, a), (0, a)$

**Line Integral:** Using the line integral you have:

$$C_1: y = 0, \quad dy = 0$$

$$C_2: x = 0, \quad dx = 0, \quad z = y^2, \quad dz = 2y \, dy$$

$$C_3: y = a, \quad dy = 0, \quad z = a^2, \quad dz = 0$$

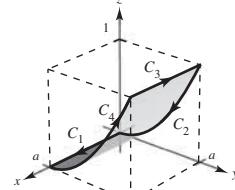
$$C_4: x = a, \quad dx = 0, \quad z = y^2, \quad dz = 2y \, dy$$

$$\begin{aligned} \int_C \mathbf{F} \cdot d\mathbf{r} &= \int_C (\cos y + y \cos x) \, dx + (\sin x - x \sin y) \, dy + xyz \, dz \\ &= \int_{C_1} dx + \int_{C_2} 0 + \int_{C_3} (\cos a + a \cos x) \, dx + \int_{C_4} (\sin a - a \sin y) \, dy + ay^3(2y \, dy) \\ &= \int_0^a dx + \int_a^0 (\cos a + a \cos x) \, dx + \int_0^a (\sin a - a \sin y) \, dy + \int_0^a 2ay^4 \, dy \\ &= a + [x \cos a + a \sin x]_a^0 + [y \sin a + a \cos y]_0^a + \left[ 2a \frac{y^5}{5} \right]_0^a \\ &= a - a \cos a - a \sin a + a \sin a + a \cos a - a + \frac{2a^6}{5} = \frac{2a^6}{5} \end{aligned}$$

**Double Integral:** Considering  $f(x, y, z) = z - y^2$ , you have:

$$\mathbf{N} = \frac{\nabla f}{\|\nabla f\|} = \frac{-2y\mathbf{j} + \mathbf{k}}{\sqrt{1 + 4y^2}}, \quad dS = \sqrt{1 + 4y^2} \, dA, \text{ and } \operatorname{curl} \mathbf{F} = xz\mathbf{i} - yz\mathbf{j}.$$

$$\text{So, } \int_S \int (\operatorname{curl} \mathbf{F}) \cdot \mathbf{N} \, dS = \int_0^a \int_0^a 2y^2 z \, dy \, dx = \int_0^a \int_0^a 2y^4 \, dy \, dx = \int_0^a \frac{2a^5}{5} \, dx = \frac{2a^6}{5}.$$



60.  $\mathbf{F}(x, y, z) = (x - z)\mathbf{i} + (y - z)\mathbf{j} + x^2\mathbf{k}$

$S$ : first octant portion of the plane  $3x + y + 2z = 12$

**Line Integral:**

$$C_1: y = 0, \quad dy = 0, \quad z = \frac{12 - 3x}{2}, \quad dz = -\frac{3}{2}dx$$

$$C_2: x = 0, \quad dx = 0, \quad z = \frac{12 - y}{2}, \quad dz = -\frac{1}{2}dy$$

$$C_3: z = 0, \quad dz = 0, \quad y = 12 - 3x, \quad dy = -3dx$$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C (x - z)dx + (y - z)dy + x^2dz$$

$$= \int_{C_1} \left[ x - \frac{12 - 3x}{2} + x^2 \left( -\frac{3}{2} \right) \right] dx + \int_{C_2} \left[ y - \frac{12 - y}{2} \right] dy + \int_{C_3} [x + (12 - 3x)(-3)] dx$$

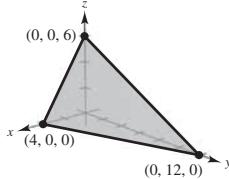
$$= \int_4^0 \left( -\frac{3}{2}x^2 + \frac{5}{2}x - 6 \right) dx + \int_0^{12} \left( \frac{3}{2}y - 6 \right) dy + \int_0^4 (10x - 36) dx = 8$$

**Double Integral:**  $G(x, y, z) = \frac{12 - 3x - y}{2} - z$

$$\nabla G(x, y, z) = -\frac{3}{2}\mathbf{i} - \frac{1}{2}\mathbf{j} - \mathbf{k}$$

$$\operatorname{curl} \mathbf{F} = \mathbf{i} - (2x + 1)\mathbf{j}$$

$$\int_S \int (\operatorname{curl} \mathbf{F}) \cdot \mathbf{N} dS = \int_0^4 \int_0^{12-3x} (x - 1) dy dx = \int_0^4 (-3x^2 + 15x - 12) dx = 8$$



61. If  $\operatorname{curl}(\mathbf{F}) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ , then  $\operatorname{div}(\operatorname{curl} \mathbf{F}) = 1 + 1 + 1 = 3$ , contradicting Theorem 15.3.

## Problem Solving for Chapter 15

1. (a)  $\nabla T = \frac{-25}{(x^2 + y^2 + z^2)^{3/2}}[x\mathbf{i} + y\mathbf{j} + z\mathbf{k}]$

$$\mathbf{N} = x\mathbf{i} + \sqrt{1 - x^2}\mathbf{k}$$

$$dS = \frac{1}{\sqrt{1 - x^2}} dA$$

$$\begin{aligned} \text{Flux} &= \int_S \int -k \nabla T \cdot \mathbf{N} dS = 25k \int_R \int \left[ \frac{x^2}{(x^2 + y^2 + z^2)^{3/2}(1 - x^2)^{1/2}} + \frac{z}{(x^2 + y^2 + z^2)^{3/2}} \right] dA \\ &= 25k \int_{-1/2}^{1/2} \int_0^1 \left[ \frac{x^2}{(x^2 + y^2 + z^2)^{3/2}(1 - x^2)^{1/2}} + \frac{1 - x^2}{(x^2 + y^2 + z^2)^{3/2}(1 - x^2)^{1/2}} \right] dy dx \\ &= 25k \int_{-1/2}^{1/2} \int_0^1 \frac{1}{(1 + y^2)^{3/2}(1 - x^2)^{1/2}} dy dx = 25k \int_0^1 \frac{1}{(1 + y^2)^{3/2}} dy \int_{-1/2}^{1/2} \frac{1}{(1 - x^2)^{1/2}} dx = 25k \left( \frac{\sqrt{2}}{2} \right) \left( \frac{\pi}{3} \right) = 25k \frac{\sqrt{2}\pi}{6} \end{aligned}$$

(b)  $\mathbf{r}(u, v) = \langle \cos u, v, \sin u \rangle$

$$\mathbf{r}_u = \langle -\sin u, 0, \cos u \rangle, \mathbf{r}_v = \langle 0, 1, 0 \rangle$$

$$\mathbf{r}_u \times \mathbf{r}_v = \langle -\cos u, 0, -\sin u \rangle$$

$$\nabla T = \frac{-25}{(x^2 + y^2 + z^2)^{3/2}}[x\mathbf{i} + y\mathbf{j} + z\mathbf{k}] = \frac{-25}{(v^2 + 1)^{3/2}}[\cos u\mathbf{i} + v\mathbf{j} + \sin u\mathbf{k}]$$

$$\nabla T \cdot (\mathbf{r}_u \times \mathbf{r}_v) = \frac{-25}{(v^2 + 1)^{3/2}}(-\cos^2 u - \sin^2 u) = \frac{25}{(v^2 + 1)^{3/2}}$$

$$\text{Flux} = \int_0^1 \int_{\pi/3}^{2\pi/3} \frac{25k}{(v^2 + 1)^{3/2}} du dv = 25k \frac{\sqrt{2}\pi}{6}$$

2. (a)  $z = \sqrt{1 - x^2 - y^2}$ ,  $\frac{\partial z}{\partial x} = \frac{-x}{\sqrt{1 - x^2 - y^2}}$ ,  $\frac{\partial z}{\partial y} = \frac{-y}{\sqrt{1 - x^2 - y^2}}$

$$dS = \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1} dA = \frac{1}{\sqrt{1 - x^2 - y^2}}$$

$$\nabla T = \frac{-25}{(x^2 + y^2 + z^2)^{3/2}}(xi + yj + zk) = -25(xi + yj + zk)$$

$$\mathbf{N} = \frac{-\frac{\partial z}{\partial x}\mathbf{i} - \frac{\partial z}{\partial y}\mathbf{j} + \mathbf{k}}{\sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}} = \left( \frac{x}{\sqrt{1 - x^2 - y^2}}\mathbf{i} + \frac{y}{\sqrt{1 - x^2 - y^2}}\mathbf{j} + \mathbf{k} \right) \sqrt{1 - x^2 - y^2}$$

$$= xi + yj + \sqrt{1 - x^2 - y^2}k = xi + yj + zk$$

$$\text{Flux} = \int_S \int -k \nabla T \cdot \mathbf{N} dS = k \int_R \int 25(xi + yj + zk) \cdot (xi + yj + zk) \frac{1}{\sqrt{1 - x^2 - y^2}} dA$$

$$= k \int_R \int \frac{25}{\sqrt{1 - x^2 - y^2}} dA = 25k \int_0^{2\pi} \int_0^1 \frac{1}{\sqrt{1 - r^2}} r dr d\theta = 50\pi k$$

(b)  $\mathbf{r}(u, v) = \langle \sin u \cos v, \sin u \sin v, \cos u \rangle$

$$\mathbf{r}_u = \langle \cos u \cos v, \cos u \sin v, -\sin u \rangle$$

$$\mathbf{r}_v = \langle -\sin u \sin v, \sin u \cos v, 0 \rangle$$

$$\mathbf{r}_u \times \mathbf{r}_v = \langle \sin^2 u \cos v, \sin^2 u \sin v, \sin u \cos u \sin^2 v + \sin u \cos u \cos^2 v \rangle$$

$$\|\mathbf{r}_u \times \mathbf{r}_v\| = \sin u$$

$$\text{Flux} = 25k \int_0^{2\pi} \int_0^{\pi/2} \sin u du dv = 50\pi k$$

3.  $\mathbf{r}(t) = \langle 3 \cos t, 3 \sin t, 2t \rangle$

$$\mathbf{r}'(t) = \langle -3 \sin t, 3 \cos t, 2 \rangle, \|\mathbf{r}'(t)\| = \sqrt{13}$$

$$I_x = \int_C (y^2 + z^2) \rho ds = \int_0^{2\pi} (9 \sin^2 t + 4t^2) \sqrt{13} dt = \frac{1}{3} \sqrt{13} \pi (32\pi^2 + 27)$$

$$I_y = \int_C (x^2 + z^2) \rho ds = \int_0^{2\pi} (9 \cos^2 t + 4t^2) \sqrt{13} dt = \frac{1}{3} \sqrt{13} \pi (32\pi^2 + 27)$$

$$I_z = \int_C (x^2 + y^2) \rho ds = \int_0^{2\pi} (9 \cos^2 t + 9 \sin^2 t) \sqrt{13} dt = 18\pi \sqrt{13}$$

4.  $\mathbf{r}(t) = \left\langle \frac{t^2}{2}, t, \frac{2\sqrt{2}t^{3/2}}{3} \right\rangle$

$$\mathbf{r}'(t) = \langle t, 1, \sqrt{2}t^{1/2} \rangle, \|\mathbf{r}'(t)\| = t + 1$$

$$\rho ds = \frac{1}{1+t}(t+1) dt = 1$$

$$I_y = \int_C (x^2 + z^2) \rho ds = \int_0^1 \left( \frac{t^4}{4} + \frac{8}{9}t^3 \right) dt = \frac{49}{180}$$

$$I_x = \int_C (y^2 + z^2) \rho ds = \int_0^1 \left( t^2 + \frac{8}{9}t^3 \right) dt = \frac{5}{9}$$

$$I_z = \int_C (x^2 + y^2) \rho ds = \int_0^1 \left( \frac{t^4}{4} + t^2 \right) dt = \frac{23}{60}$$

$$\begin{aligned}
 5. \quad w &= \frac{1}{f} = \frac{1}{\sqrt{x^2 + y^2 + z^2}} \quad \frac{\partial^2 w}{\partial x^2} = \frac{2x^2 - y^2 - z^2}{(x^2 + y^2 + z^2)^{5/2}} \\
 \frac{\partial w}{\partial x} &= -\frac{x}{(x^2 + y^2 + z^2)^{3/2}} \quad \frac{\partial^2 w}{\partial y^2} = \frac{2y^2 - x^2 - z^2}{(x^2 + y^2 + z^2)^{5/2}} \\
 \frac{\partial w}{\partial y} &= -\frac{y}{(x^2 + y^2 + z^2)^{3/2}} \quad \frac{\partial^2 w}{\partial z^2} = \frac{2z^2 - x^2 - y^2}{(x^2 + y^2 + z^2)^{5/2}} \\
 \frac{\partial w}{\partial z} &= -\frac{z}{(x^2 + y^2 + z^2)^{3/2}} \quad \nabla^2 w = \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} = 0
 \end{aligned}$$

Therefore  $w = \frac{1}{f}$  is harmonic.

$$6. \int_C y^n dx + x^n dy = \int_R \int \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dA$$

For the line integral, use the two paths

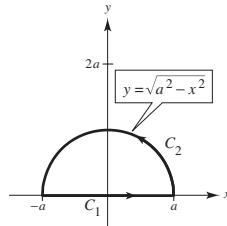
$$C_1: \mathbf{r}_1(x) = x\mathbf{i}, -a \leq x \leq a$$

$$C_2: \mathbf{r}_2(x) = x\mathbf{i} + \sqrt{a^2 - x^2}\mathbf{j}, x = a \text{ to } x = -a$$

$$\int_{C_1} y^n dx + x^n dy = 0$$

$$\int_{C_2} y^n dx + x^n dy = \int_a^{-a} \left[ (a^2 - x^2)^{n/2} + x^n \frac{-x}{\sqrt{a^2 - x^2}} \right] dx$$

$$\int_R \int \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dA = \int_{-a}^a \int_0^{\sqrt{a^2 - x^2}} [nx^{n-1} - ny^{n-1}] dy dx$$



(a) For  $n = 1, 3, 5, 7$ , both integrals give 0.

(b) For  $n$  even, you obtain

$$n = 2: -\frac{4}{3}a^3 \quad n = 4: -\frac{16}{15}a^5 \quad n = 6: -\frac{32}{35}a^7 \quad n = 8: -\frac{256}{315}a^9$$

(c) If  $n$  is odd then the integral equals 0.

$$\begin{aligned}
 7. \quad \frac{1}{2} \int_C x dy - y dx &= \frac{1}{2} \int_0^{2\pi} [a(\theta - \sin \theta)(a \sin \theta) d\theta - a(1 - \cos \theta)(a(1 - \cos \theta)) d\theta] \\
 &= \frac{1}{2}a^2 \int_0^{2\pi} [\theta \sin \theta - \sin^2 \theta - 1 + 2 \cos \theta - \cos^2 \theta] d\theta = \frac{1}{2}a^2 \int_0^{2\pi} (\theta \sin \theta + 2 \cos \theta - 2) d\theta = -3\pi a^2
 \end{aligned}$$

So, the area is  $3\pi a^2$ .

$$8. \quad \frac{1}{2} \int_C x dy - y dx = 2 \int_0^{\pi/2} \left[ \frac{1}{2} \sin 2t \cos t - \sin t \cos 2t \right] dt = 2\left(\frac{2}{3}\right)$$

So, the area is  $\frac{4}{3}$ .

$$9. (a) \quad \mathbf{r}(t) = t\mathbf{j}, 0 \leq t \leq 1$$

$$\mathbf{r}'(t) = \mathbf{j}$$

$$W = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 (\mathbf{i} + \mathbf{j}) \cdot \mathbf{j} dt = \int_0^1 dt = 1$$

$$(b) \quad \mathbf{r}(t) = (t - t^2)\mathbf{i} + t\mathbf{j}, 0 \leq t \leq 1$$

$$\mathbf{r}'(t) = (1 - 2t)\mathbf{i} + \mathbf{j}$$

$$\begin{aligned}
 W &= \mathbf{F} \cdot d\mathbf{r} = \int_0^1 \left( (2t - t^2)\mathbf{i} + [(t - t^2)^2 + 1]\mathbf{j} \right) \cdot ((1 - 2t)\mathbf{i} + \mathbf{j}) dt \\
 &= \int_0^1 [(1 - 2t)(2t - t^2) + (t^4 - 2t^3 + t^2 + 1)] dt = \int_0^1 (t^4 - 4t^2 + 2t + 1) dt = \frac{13}{15}
 \end{aligned}$$

$$(c) \quad \mathbf{r}(t) = c(t - t^2)\mathbf{i} + t\mathbf{j}, 0 \leq t \leq 1$$

$$\mathbf{r}'(t) = c(1 - 2t)\mathbf{i} + \mathbf{j}$$

$$\begin{aligned}\mathbf{F} \cdot d\mathbf{r} &= (c(t - t^2) + t)(c(1 - 2t)) + (c^2(t - t^2)^2 + 1)(1) \\ &= c^2t^4 - 2c^2t^2 + c^2t - 2ct^2 + ct + 1\end{aligned}$$

$$W = \int_C \mathbf{F} \cdot d\mathbf{r} = \frac{1}{30}c^2 - \frac{1}{6}c + 1$$

$$\frac{dW}{dc} = \frac{1}{15}c - \frac{1}{6} = 0 \Rightarrow c = \frac{5}{2}$$

$$\frac{d^2W}{dc^2} = \frac{1}{15} > 0 \text{ } c = \frac{5}{2} \text{ minimum.}$$

10.  $F(x, y) = 3x^2y^2\mathbf{i} + 2x^3y\mathbf{j}$  is conservative.

$$f(x, y) = x^3y^2 \text{ potential function.}$$

$$\text{Work} = f(2, 4) - f(1, 1) = 8(16) - 1 = 127$$

$$11. \mathbf{v} \times \mathbf{r} = \langle a_1, a_2, a_3 \rangle \times \langle x, y, z \rangle$$

$$= \langle a_2z - a_3y, -a_1z + a_3x, a_1y - a_2x \rangle$$

$$\mathbf{curl}(\mathbf{v} \times \mathbf{r}) = \langle 2a_1, 2a_2, 2a_3 \rangle = 2\mathbf{v}$$

By Stokes's Theorem,

$$\int_C (\mathbf{v} \times \mathbf{r}) \cdot d\mathbf{r} = \int_S \int \mathbf{curl}(\mathbf{v} \times \mathbf{r}) \cdot \mathbf{N} \, dS = \int_S \int 2\mathbf{v} \cdot \mathbf{N} \, dS.$$

$$13. \mathbf{F}(x, y) = M(x, y)\mathbf{i} + N(x, y)\mathbf{j} = \frac{m}{(x^2 + y^2)^{5/2}} [3xy\mathbf{i} + (2y^2 - x^2)\mathbf{j}]$$

$$M = \frac{3mxy}{(x^2 + y^2)^{5/2}} = 3mxy(x^2 + y^2)^{-5/2}$$

$$\frac{\partial M}{\partial y} = 3mxy \left[ -\frac{5}{2}(x^2 + y^2)^{-7/2}(2y) \right] + (x^2 + y^2)^{-5/2}(3mx)$$

$$= 3mx(x^2 + y^2)^{-7/2} [-5y^2 + (x^2 + y^2)] = \frac{3mx(x^2 - 4y^2)}{(x^2 + y^2)^{7/2}}$$

$$N = \frac{m(2y^2 - x^2)}{(x^2 + y^2)^{5/2}} = m(2y^2 - x^2)(x^2 + y^2)^{-5/2}$$

$$\frac{\partial N}{\partial x} = m(2y^2 - x^2) \left[ -\frac{5}{2}(x^2 + y^2)^{-7/2}(2x) \right] + (x^2 + y^2)^{-5/2}(-2mx)$$

$$= mx(x^2 + y^2)^{-7/2} [(2y^2 - x^2)(-5) + (x^2 + y^2)(-2)]$$

$$= mx(x^2 + y^2)^{-7/2} (3x^2 - 12y^2) = \frac{3mx(x^2 - 4y^2)}{(x^2 + y^2)^{7/2}}$$

So,  $\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}$  and  $\mathbf{F}$  is conservative.

12. Area =  $\pi ab$

$$\mathbf{r}(t) = a \cos t\mathbf{i} + b \sin t\mathbf{j}, 0 \leq t \leq 2\pi$$

$$\mathbf{r}'(t) = -a \sin t\mathbf{i} + b \cos t\mathbf{j}$$

$$\mathbf{F} = -\frac{1}{2}b \sin t\mathbf{i} + \frac{1}{2}a \cos t\mathbf{j}$$

$$\mathbf{F} \cdot d\mathbf{r} = \left[ \frac{1}{2}ab \sin^2 t + \frac{1}{2}ab \cos^2 t \right] dt = \frac{1}{2}ab$$

$$W = \int_0^{2\pi} \mathbf{F} \cdot d\mathbf{r} = \frac{1}{2}ab(2\pi) = \pi ab$$

Same as area.

# C H A P T E R   1 6

## Additional Topics in Differential Equations

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# C H A P T E R   1 6

## Additional Topics in Differential Equations

### Section 16.1 Exact First-Order Equations

1.  $(2x + xy^2)dx + (3 + x^2y)dy = 0$

$$\frac{\partial M}{\partial y} = 2xy$$

$$\frac{\partial N}{\partial x} = 2xy$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \text{ Exact}$$

5.  $(2x - 3y)dx + (2y - 3x)dy = 0$

$$\frac{\partial M}{\partial y} = -3 = \frac{\partial N}{\partial x} \text{ Exact}$$

$$\begin{aligned} f(x, y) &= \int M(x, y) dx \\ &= \int (2x - 3y) dx \\ &= x^2 - 3xy + g(y) \end{aligned}$$

2.  $(1 - xy)dx + (y - xy)dy = 0$

$$\frac{\partial M}{\partial y} = -x$$

$$\frac{\partial N}{\partial x} = -y$$

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x} \text{ Not exact}$$

$$\begin{aligned} f_y(x, y) &= -3x + g'(y) \\ &= 2y - 3x \Rightarrow g'(y) = 2y \end{aligned}$$

$$\Rightarrow g(y) = y^2 + C_1$$

$$\begin{aligned} f(x, y) &= x^2 - 3xy + y^2 + C_1 \\ x^2 - 3xy + y^2 &= C \end{aligned}$$

6.  $ye^x dx + e^x dy = 0$

$$\frac{\partial M}{\partial y} = e^x = \frac{\partial N}{\partial x} \text{ Exact}$$

$$\begin{aligned} f(x, y) &= \int N(x, y) dy = \int e^x dy = ye^x + g(x) \\ f_x(x, y) &= ye^x + g'(x) = ye^x \Rightarrow g'(x) = 0 \end{aligned}$$

$$\Rightarrow g(x) = C_1$$

$$f(x, y) = ye^x + C_1$$

$$ye^x = C$$

4.  $ye^{xy} dx + xe^{xy} dy = 0$

$$\frac{\partial M}{\partial y} = e^{xy} + xye^{xy}$$

$$\frac{\partial N}{\partial x} = e^{xy} + xye^{xy}$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \text{ Exact}$$

7.  $(3y^2 + 10xy^2)dx + (6xy - 2 + 10x^2y)dy = 0$

$$\frac{\partial M}{\partial y} = 6y + 20xy = \frac{\partial N}{\partial x} \text{ Exact}$$

$$\begin{aligned} f(x, y) &= \int M(x, y) dx = \int (3y^2 + 10xy^2) dx \\ &= 3xy^2 + 5x^2y^2 + g(y) \end{aligned}$$

$$\begin{aligned} f_y(x, y) &= 6xy + 10x^2y + g'(y) = 6xy - 2 + 10x^2y \\ &\Rightarrow g'(y) = -2 \Rightarrow g(y) = -2y + C_1 \end{aligned}$$

$$f(x, y) = 3xy^2 + 5x^2y^2 - 2y + C_1$$

$$3xy^2 + 5x^2y^2 - 2y = C$$

8.  $2 \cos(2x - y) dx - \cos(2x - y) dy = 0$

$$\begin{aligned}\frac{\partial M}{\partial y} &= 2 \sin(2x - y) = \frac{\partial N}{\partial x} \\ &= 2 \sin(2x - y) \text{ Exact}\end{aligned}$$

$$\begin{aligned}f(x, y) &= \int M(x, y) dx \\ &= \int 2 \cos(2x - y) dx = \sin(2x - y) + g(y)\end{aligned}$$

$$\begin{aligned}f_y(x, y) &= -\cos(2x - y) + g'(y) \\ &= -\cos(2x - y) \Rightarrow g'(y) = 0 \Rightarrow g(y) = C_1\end{aligned}$$

$$\begin{aligned}f(x, y) &= \sin(2x - y) + C_1 \\ \sin(2x - y) &= C\end{aligned}$$

9.  $(4x^3 - 6xy^2) dx + (4y^3 - 6xy) dy = 0$

$$\frac{\partial M}{\partial y} = -12xy$$

$$\frac{\partial N}{\partial x} = -6y$$

Not exact

10.  $2y^2 e^{xy^2} dx + 2xye^{xy^2} dy = 0$

$$\frac{\partial M}{\partial y} = 4(xy^3 + y)e^{xy^2}$$

$$\frac{\partial N}{\partial x} = 2(xy^3 + y)e^{xy^2}$$

Not exact

11.  $\frac{-y}{x^2 + y^2} dx + \frac{x}{x^2 + y^2} dy = 0$

$$\frac{\partial M}{\partial y} = \frac{y^2 - x^2}{(x^2 + y^2)^2} = \frac{\partial N}{\partial x} \text{ Exact}$$

$$f(x, y) = \int M(x, y) dx = -\arctan\left(\frac{x}{y}\right) + g(y)$$

$$\begin{aligned}f_y(x, y) &= \frac{x}{x^2 + y^2} + g'(y) \\ &= \frac{x}{x^2 + y^2} \Rightarrow g'(y) = 0 \Rightarrow g(y) = C_1\end{aligned}$$

$$f(x, y) = -\arctan\left(\frac{x}{y}\right) + C_1$$

$$\arctan\left(\frac{x}{y}\right) = C$$

12.  $xe^{-(x^2+y^2)} dx + ye^{-(x^2+y^2)} dy = 0$

$$\frac{\partial M}{\partial y} = -2xye^{-(x^2+y^2)} = \frac{\partial N}{\partial x} \text{ Exact}$$

$$\begin{aligned}f(x, y) &= \int M(x, y) dx \\ &= \int xe^{-(x^2+y^2)} dx = -\frac{1}{2}e^{-(x^2+y^2)} + g(y)\end{aligned}$$

$$\begin{aligned}f_y(x, y) &= ye^{-(x^2+y^2)} + g'(y) = ye^{-(x^2+y^2)} \Rightarrow g'(y) = 0 \\ &\Rightarrow g(y) = C_1\end{aligned}$$

$$\begin{aligned}f(x, y) &= -\frac{1}{2}e^{-(x^2+y^2)} + C_1 \\ e^{-(x^2+y^2)} &= C\end{aligned}$$

13.  $\left(\frac{y}{x-y}\right)^2 dx + \left(\frac{x}{x-y}\right)^2 dy = 0$

$$\frac{\partial M}{\partial y} = \frac{2xy}{(x-y)^3}$$

$$\frac{\partial N}{\partial x} = \frac{-2xy}{(x-y)^3}$$

Not exact

14.  $ye^y \cos xy dx + e^y(x \cos xy + \sin xy) dy = 0$

$$\frac{\partial M}{\partial y} = e^y \cos xy + ye^y \cos xy - xye^y \sin xy$$

$$\frac{\partial N}{\partial x} = e^y[\cos xy - xy \sin xy + y \cos xy]$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \text{ Exact}$$

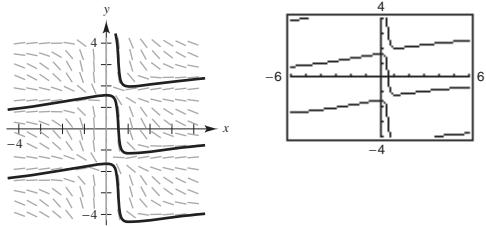
$$\begin{aligned}f(x, y) &= \int M(x, y) dx \\ &= \int ye^y \cos xy dx = e^y \sin xy + g(y)\end{aligned}$$

$$\begin{aligned}f_y(x, y) &= e^y \sin xy + xe^y \cos xy + g'(y) \\ &\Rightarrow g'(y) = 0 \Rightarrow g(y) = C_1\end{aligned}$$

$$f(x, y) = e^y \sin xy + C_1$$

$$e^y \sin xy = C$$

15. (a) and (c)



$$(b) (2x \tan y + 5) dx + (x^2 \sec^2 y) dy = 0, y\left(\frac{1}{2}\right) = \frac{\pi}{4}$$

$$\frac{\partial M}{\partial y} = 2x \sec^2 y = \frac{\partial N}{\partial x} \Rightarrow \text{Exact}$$

$$f(x, y) = \int M(x, y) dx = \int (2x \tan y + 5) dx \\ = x^2 \tan y + 5x + g(y)$$

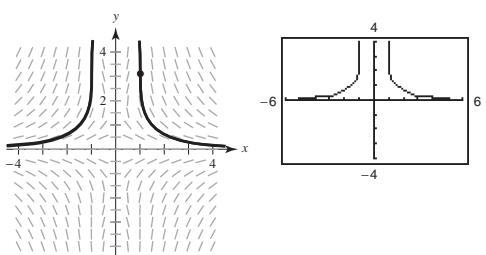
$$f_y(x, y) = x^2 \sec^2 y + g'(y) \\ = x^2 \sec^2 y \\ \Rightarrow g'(y) = 0 \Rightarrow g(y) = C$$

$$f(x, y) = x^2 \tan y + 5x = C$$

$$f\left(\frac{1}{2}, \frac{\pi}{4}\right) = \frac{1}{4} + \frac{5}{2} = \frac{11}{4} = C$$

$$\text{Answer: } x^2 \tan y + 5x = \frac{11}{4}$$

16. (a) and (c)



$$(b) 2xy dx + (x^2 + \cos y) dy = 0, y(1) = \pi$$

$$\frac{\partial M}{\partial y} = 2x = \frac{\partial N}{\partial x} \Rightarrow \text{Exact}$$

$$f(x, y) = \int M(x, y) dx = \int 2xy dx = x^2 y + g(y)$$

$$f_y(x, y) = x^2 + g'(y) = x^2 + \cos y \\ \Rightarrow g'(y) = \cos y \\ \Rightarrow g(y) = \sin y + C_1$$

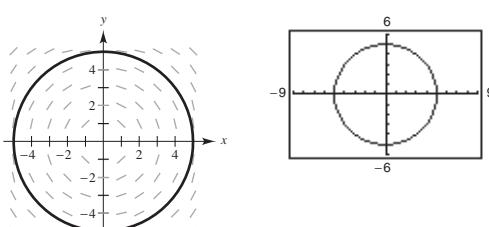
$$f(x, y) = x^2 y + \sin y + C_1$$

$$x^2 y + \sin y = C$$

$$f(1, \pi) = \pi = C$$

$$\text{Answer: } x^2 y + \sin y = \pi$$

17. (a) and (c)



$$(b) \frac{x}{\sqrt{x^2 + y^2}} dx + \frac{y}{\sqrt{x^2 + y^2}} dy = 0, y(4) = 3$$

$$\frac{\partial M}{\partial y} = -\frac{xy}{(x^2 + y^2)^{3/2}} = \frac{\partial N}{\partial x} \Rightarrow \text{Exact}$$

$$f(x, y) = \int M(x, y) dx = \int \frac{x}{\sqrt{x^2 + y^2}} dx \\ = \frac{1}{2} \int (x^2 + y^2)^{-1/2} 2x dx \\ = \sqrt{x^2 + y^2} + g(y)$$

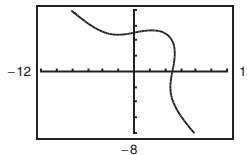
$$f_y(x, y) = \frac{y}{\sqrt{x^2 + y^2}} + g'(y) \\ = \frac{y}{\sqrt{x^2 + y^2}} \Rightarrow g'(y) = 0 \Rightarrow g(y) = C$$

$$f(x, y) = \sqrt{x^2 + y^2} = C$$

$$f(3, 4) = \sqrt{3^2 + 4^2} = \sqrt{25} = 5 = C$$

$$\text{Solution: } \sqrt{x^2 + y^2} = 5 \text{ or } x^2 + y^2 = 25$$

18. (a) and (c)



$$(b) (x^2 - y) dx + (y^2 - x) dy = 0, y(4) = 5$$

$$\frac{\partial M}{\partial y} = -1 = \frac{\partial N}{\partial x} \Rightarrow \text{Exact}$$

$$f(x, y) = \int M(x, y) dx = \int (x^2 - y) dx = \frac{x^3}{3} - xy + g(y) \\ = \frac{x^3}{3} - xy + g(y)$$

$$f_y(x, y) = -x + g'(y) = y^2 - x$$

$$\Rightarrow g'(y) = y^2 \Rightarrow g(y) = \frac{y^3}{3} + C_1$$

$$f(x, y) = \frac{x^3}{3} + \frac{y^3}{3} - xy + C_1$$

$$\frac{x^3}{3} + \frac{y^3}{3} - xy = C$$

$$f(4, 5) = 43 = C$$

$$\text{Solution: } x^3 + y^3 - 3xy = 129$$

19.  $\frac{y}{x-1} dx + [\ln(x-1) + 2y] dy = 0$

$$\frac{\partial M}{\partial y} = \frac{1}{x-1} = \frac{\partial N}{\partial x} \text{ Exact}$$

$$f(x, y) = \int M(x, y) dx = y \ln(x-1) + g(y)$$

$$f_y(x, y) = \ln(x-1) + g'(y)$$

$$\Rightarrow g'(y) = 2y \Rightarrow g(y) = y^2 + C_1$$

$$f(x, y) = y \ln(x-1) + y^2 + C_1$$

$$y \ln(x-1) + y^2 = C$$

$$y(2) = 4: 4 \ln(2-1) + 16 = C \Rightarrow C = 16$$

$$\text{Solution: } y \ln(x-1) + y^2 = 16$$

20.  $\frac{x}{x^2 + y^2} dx + \frac{y}{x^2 + y^2} dy = 0$

$$\frac{\partial M}{\partial y} = \frac{-2xy}{(x^2 + y^2)^2} = \frac{\partial N}{\partial x} \text{ Exact}$$

$$f(x, y) = \int M(x, y) dx$$

$$= \int \frac{x}{x^2 + y^2} dx = \frac{1}{2} \ln(x^2 + y^2) + g(y)$$

$$f_y(x, y) = \frac{y}{x^2 + y^2} + g'(y)$$

$$\Rightarrow g'(y) = 0 \Rightarrow g(y) = C_1$$

$$f(x, y) = \frac{1}{2} \ln(x^2 + y^2) + C_1$$

$$\ln(x^2 + y^2) = C$$

$$y(0) = 4: \ln(16) = C$$

$$\ln(x^2 + y^2) = \ln 16$$

$$\text{Solution: } x^2 + y^2 = 16$$

21.  $(e^{3x} \sin 3y) dx + (e^{3x} \cos 3y) dy = 0$

$$\frac{\partial M}{\partial y} = 3e^{3x} \cos 3y = \frac{\partial N}{\partial x} \text{ Exact}$$

$$f(x, y) = \int M(x, y) dx$$

$$= \int e^{3x} \sin 3y dx = \frac{1}{3} e^{3x} \sin 3y + g(y)$$

$$f_y(x, y) = e^{3x} \cos 3y + g'(y)$$

$$\Rightarrow g'(y) = 0 \Rightarrow g(y) = C_1$$

$$f(x, y) = \frac{1}{3} e^{3x} \sin 3y + C_1$$

$$e^{3x} \sin 3y = C$$

$$y(0) = \pi: C = 0$$

$$\text{Solution: } e^{3x} \sin 3y = 0$$

22.  $(x^2 + y^2) dx + 2xy dy = 0$

$$\frac{\partial M}{\partial y} = 2y = \frac{\partial N}{\partial x} \text{ Exact}$$

$$f(x, y) = \int M(x, y) dx$$

$$= \int (x^2 + y^2) dx = \frac{x^3}{3} + xy^2 + g(y)$$

$$f_y(x, y) = 2xy + g'(y) \Rightarrow g'(y) = 0 \Rightarrow g(y) = C_1$$

$$f(x, y) = \frac{x^3}{3} + xy^2 + C_1$$

$$\frac{x^3}{3} + xy^2 = C$$

$$y(3) = 1: 9 + 3 = 12 = C$$

$$\frac{x^3}{3} + xy^2 = 12$$

$$\text{Solution: } x^3 + 3xy^2 = 36$$

23.  $(2xy - 9x^2) dx + (2y + x^2 + 1) dy = 0$

$$\frac{\partial M}{\partial y} = 2x = \frac{\partial N}{\partial x} \text{ Exact}$$

$$f(x, y) = \int M(x, y) dx = \int (2xy - 9x^2) dx$$

$$= x^2y - 3x^3 + g(y)$$

$$f_y(x, y) = x^2 + g'(y) = 2y + x^2 + 1$$

$$\Rightarrow g'(y) = 2y + 1$$

$$\Rightarrow g(y) = y^2 + y + C_1$$

$$f(x, y) = x^2y - 3x^3 + y^2 + y + C_1$$

$$x^2y - 3x^3 + y^2 + y = C$$

$$y(0) = -3: 9 - 3 = 6 = C$$

$$\text{Solution: } x^2y - 3x^3 + y^2 + y = 6$$

24.  $(2xy^2 + 4) dx + (2x^2y - 6) dy = 0$

$$\frac{\partial M}{\partial y} = 4xy = \frac{\partial N}{\partial x} \text{ Exact}$$

$$f(x, y) = \int M(x, y) dx$$

$$= \int (2xy^2 + 4) dx = x^2y^2 + 4x + g(y)$$

$$f_y(x, y) = 2x^2y + g'(y) = 2x^2y - 6 \Rightarrow g'(y) = -6$$

$$\Rightarrow g(y) = -6y + C_1$$

$$f(x, y) = x^2y^2 + 4x - 6y + C_1$$

$$x^2 + y^2 + 4x - 6y = C$$

$$y(-1) = 8: 1 + 64 - 4 - 48 = 13 = C$$

Solution:

$$x^2 + y^2 + 4x - 6y = 13$$

**25.**  $y \, dx - (x + 6y^2) \, dy = 0$

$$\frac{(\partial N/\partial x) - (\partial M/\partial y)}{M} = -\frac{2}{y} = k(y)$$

Integrating factor:  $e^{\int k(y) \, dy} = e^{\ln y^{-2}} = \frac{1}{y^2}$

Exact equation:  $\frac{1}{y} \, dx - \left( \frac{x}{y^2} + 6 \right) \, dy = 0$

$$f(x, y) = \frac{x}{y} + g(y)$$

$$g'(y) = -6$$

$$g(y) = -6y + C_1$$

$$\frac{x}{y} - 6y = C$$

**26.**  $(2x^3 + y) \, dx - x \, dy = 0$

$$\frac{(\partial M/\partial y) - (\partial N/\partial x)}{N} = -\frac{2}{x} = h(x)$$

Integrating factor:  $e^{\int h(x) \, dx} = e^{\ln x^{-2}} = \frac{1}{x^2}$

Exact equation:  $\left(2x + \frac{y}{x^2}\right) \, dx - \frac{1}{x} \, dy = 0$

$$f(x, y) = x^2 - \frac{y}{x} + g(y)$$

$$g'(y) = 0$$

$$g(y) = C_1$$

$$x^2 - \frac{y}{x} = C$$

**27.**  $(5x^2 - y) \, dx + x \, dy = 0$

$$\frac{(\partial M/\partial y) - (\partial N/\partial x)}{N} = -\frac{2}{x} = h(x)$$

Integrating factor:  $e^{\int h(x) \, dx} = e^{\ln x^{-2}} = \frac{1}{x^2}$

Exact equation:  $\left(5 - \frac{y}{x^2}\right) \, dx + \frac{1}{x} \, dy = 0$

$$f(x, y) = 5x + \frac{y}{x} + g(y)$$

$$g'(y) = 0$$

$$g(y) = C_1$$

$$5x + \frac{y}{x} = C$$

**28.**  $(5x^2 - y^2) \, dx + 2y \, dy = 0$

$$\frac{(\partial M/\partial y) - (\partial N/\partial x)}{N} = -1 = h(x)$$

Integrating factor:  $e^{\int h(x) \, dx} = e^{-x}$

Exact equation:  $(5x^2 - y^2)e^{-x} \, dx + 2ye^{-x} \, dy = 0$

$$f(x, y) = -5x^2e^{-x} - 10xe^{-x} - 10e^{-x} + y^2e^{-x} + g(y)$$

$$g'(y) = 0$$

$$g(y) = C_1$$

$$y^2e^{-x} - 5x^2e^{-x} - 10xe^{-x} - 10e^{-x} = C$$

**29.**  $(x + y) \, dx + (\tan x) \, dy = 0$

$$\frac{(\partial M/\partial y) - (\partial N/\partial x)}{N} = -\tan x = h(x)$$

Integrating factor:  $e^{\int h(x) \, dx} = e^{\ln \cos x} = \cos x$

Exact equation:  $(x + y) \cos x \, dx + \sin x \, dy = 0$

$$f(x, y) = x \sin x + \cos x + y \sin x + g(y)$$

$$g'(y) = 0$$

$$g(y) = C_1$$

$$x \sin x + \cos x + y \sin x = C$$

**30.**  $(2x^2y - 1) \, dx + x^3 \, dy = 0$

$$\frac{(\partial M/\partial y) - (\partial N/\partial x)}{N} = -\frac{1}{x} = h(x)$$

Integrating factor:  $e^{\int h(x) \, dx} = e^{\ln(1/x)} = \frac{1}{x}$

Exact equation:  $\left(2xy - \frac{1}{x}\right) \, dx + x^2 \, dy = 0$

$$f(x, y) = x^2y - \ln|x| + g(y)$$

$$g'(y) = 0$$

$$g(y) = C_1$$

$$x^2y - \ln|x| = C$$

**31.**  $y^2 \, dx + (xy - 1) \, dy = 0$

$$\frac{(\partial N/\partial x) - (\partial M/\partial y)}{M} = -\frac{1}{y} = k(y)$$

Integrating factor:  $e^{\int k(y) \, dy} = e^{\ln(1/y)} = \frac{1}{y}$

Exact equation:  $y \, dx + \left(x - \frac{1}{y}\right) \, dy = 0$

$$f(x, y) = xy + g(y)$$

$$g'(y) = -\frac{1}{y}$$

$$g(y) = -\ln|y| + C_1$$

$$xy - \ln|y| = C$$

32.  $(x^2 + 2x + y)dx + 2dy = 0$

$$\frac{(\partial M/\partial y) - (\partial N/\partial x)}{N} = \frac{1}{2} = h(x)$$

Integrating factor:  $e^{\int h(x)dx} = e^{x/2}$

Exact equation:  $(x^2 + 2x + y)e^{x/2}dx + 2e^{x/2}dy = 0$

$$f(x, y) = 2(x^2 - 2x + 4 + y)e^{x/2} + g(y)$$

$$g'(y) = 0$$

$$g(y) = C_1$$

$$(x^2 - 2x + 4 + y)e^{x/2} = C$$

33.  $2ydx + (x - \sin\sqrt{y})dy = 0$

$$\frac{(\partial N/\partial x) - (\partial M/\partial y)}{M} = \frac{-1}{2y} = k(y)$$

Integrating factor:  $e^{\int k(y)dy} = e^{\ln(1/\sqrt{y})} = \frac{1}{\sqrt{y}}$

Exact equation:  $2\sqrt{y}dy + \left(\frac{x}{\sqrt{y}} - \frac{\sin\sqrt{y}}{\sqrt{y}}\right)dy = 0$

$$f(x, y) = 2\sqrt{y}x + g(y)$$

$$g'(y) = -\frac{\sin\sqrt{y}}{\sqrt{y}}$$

$$g(y) = 2\cos\sqrt{y} + C_1$$

$$\sqrt{y}x + \cos\sqrt{y} = C$$

34.  $(-2y^3 + 1)dx + (3xy^2 + x^3)dy = 0$

$$\frac{(\partial M/\partial y) - (\partial N/\partial x)}{N} = \frac{-3}{x} = h(x)$$

Integrating factor:  $e^{\int h(x)dx} = e^{\ln(1/x^3)} = \frac{1}{x^3}$

Exact equation:  $\left(\frac{-2y^3}{x^3} + \frac{1}{x^3}\right)dx + \left(\frac{3y^2}{x^2} + 1\right)dy = 0$

$$f(x, y) = \frac{y^3}{x^2} - \frac{1}{2x^2} + g(y)$$

$$g'(y) = 1$$

$$g(y) = y + C_1$$

$$\frac{y^3}{x^2} - \frac{1}{2x^2} + y = C$$

35.  $(4x^2y + 2y^2)dx + (3x^3 + 4xy)dy = 0$

Integrating factor:  $xy^2$

Exact equation:

$$(4x^3y^3 + 2xy^4)dy + (3x^4y^2 + 4x^2y^3)dy = 0$$

$$f(x, y) = x^4y^3 + x^2y^4 + g(y)$$

$$g'(y) = 0$$

$$g(y) = C_1$$

$$x^4y^3 + x^2y^4 = C$$

36.  $(3y^2 + 5x^2y)dx + (3xy + 2x^3)dy = 0$

Integrating factor:  $x^2y$

Exact equation:

$$(3x^2y^3 + 5x^4y^2)dx + (3x^3y^2 + 2x^5y)dy = 0$$

$$f(x, y) = x^3y^3 + x^5y^2 + g(y)$$

$$g'(y) = 0$$

$$g(y) = C_1$$

$$x^3y^3 + x^5y^2 = C$$

37.  $(-y^5 + x^2y)dx + (2xy^4 - 2x^3)dy = 0$

Integrating factor:  $x^{-2}y^{-3}$

Exact equation:

$$\left(-\frac{y^2}{x^2} + \frac{1}{y^2}\right)dx + \left(2\frac{y}{x} - 2\frac{x}{y^3}\right)dy = 0$$

$$f(x, y) = \frac{y^2}{x} + \frac{x}{y^2} + g(y)$$

$$g'(y) = 0$$

$$g(y) = C_1$$

$$\frac{y^2}{x} + \frac{x}{y^2} = C$$

38.  $-y^3dx + (xy^2 - x^2)dy = 0$

Integrating factor:  $x^{-2}y^{-2}$

$$\text{Exact equation: } \frac{-y}{x^2}dx + \left(\frac{1}{x} - \frac{1}{y^2}\right)dy = 0$$

$$f(x, y) = \frac{y}{x} + g(y)$$

$$g'(y) = -\frac{1}{y^2}$$

$$g(y) = \frac{1}{y} + C_1$$

$$\frac{y}{x} + \frac{1}{y} = C$$

39.  $y \, dx - x \, dy = 0$

(a)  $\frac{1}{x^2}, \frac{y}{x^2} \, dx - \frac{1}{x} \, dy = 0, \frac{\partial M}{\partial y} = \frac{1}{x^2} = \frac{\partial N}{\partial x}$

(b)  $\frac{1}{y^2}, \frac{1}{y} \, dx - \frac{x}{y^2} \, dy = 0, \frac{\partial M}{\partial y} = \frac{-1}{y^2} = \frac{\partial N}{\partial x}$

(c)  $\frac{1}{xy}, \frac{1}{x} \, dx - \frac{1}{y} \, dy = 0, \frac{\partial M}{\partial y} = 0 = \frac{\partial N}{\partial x}$

(d)  $\frac{1}{x^2 + y^2}, \frac{y}{x^2 + y^2} \, dx - \frac{x}{x^2 + y^2} \, dy = 0,$

$$\frac{\partial M}{\partial y} = \frac{x^2 - y^2}{(x^2 + y^2)^2} = \frac{\partial N}{\partial x}$$

40.  $(axy^2 + by) \, dx + (bx^2y + ax) \, dy = 0$

Exact equation:  $\frac{\partial M}{\partial y} = 2axy + b, \frac{\partial N}{\partial x} = 2bxy + a, \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$  only if  $a = b$ .

Integrating factor:  $x^m y^n$

$$(ax^{m+1}y^{n+2} + bx^m y^{n+1}) \, dx + (bx^{m+2}y^{n+1} + ax^{m+1}y^n) \, dy = 0$$

$$\begin{aligned} \frac{\partial M}{\partial y} &= a(n+2)x^{m+1}y^{n+1} + b(n+1)x^m y^n \\ \frac{\partial N}{\partial x} &= b(m+2)x^{m+1}y^{n+1} + a(m+1)x^m y^n \end{aligned} \quad \left. \begin{aligned} a(n+2) &= b(m+2) \\ b(m+2) &= a(m+1) \end{aligned} \right.$$

$$an - bm = 2(b-a)$$

$$bn - am = a - b$$

$$(a^2 - b^2)m = -(2b + a)(a - b)$$

$$m = -\frac{2b + a}{a + b}$$

$$bn - a\left(-\frac{2b + a}{a + b}\right) = a - b$$

$$bn = \frac{-2ab - a^2 + a^2 - b^2}{a + b} = \frac{-b(2a + b)}{a + b}$$

$$n = -\frac{2a + b}{a + b}$$

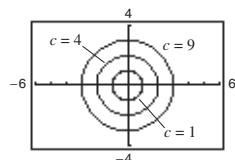
41.  $\mathbf{F}(x, y) = \frac{y}{\sqrt{x^2 + y^2}}\mathbf{i} - \frac{x}{\sqrt{x^2 + y^2}}\mathbf{j}$

$$\frac{dy}{dx} = -\frac{x}{y}$$

$$y \, dy + x \, dx = 0$$

$$y^2 + x^2 = C$$

Family of circles



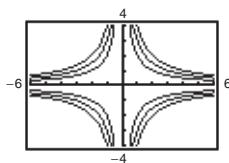
42.  $\mathbf{F}(x, y) = \frac{x}{\sqrt{x^2 + y^2}}\mathbf{i} - \frac{y}{\sqrt{x^2 + y^2}}\mathbf{j}$

$$\frac{dy}{dx} = -\frac{y}{x}$$

$$x \, dy + y \, dx = 0$$

$$xy = C$$

Family of hyperbolas



43.  $\mathbf{F}(x, y) = 4x^2y\mathbf{i} - \left(2xy^2 + \frac{x}{y^2}\right)\mathbf{j}$

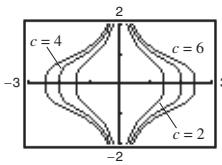
$$\frac{dy}{dx} = \frac{-y}{2x} - \frac{1}{4xy^3}$$

$$\frac{8y^3}{2y^4 + 1} dy = -\frac{2}{x} dx$$

$$\ln(2y^4 + 1) = \ln\left(\frac{1}{x^2}\right) + \ln C$$

$$2y^4 + 1 = \frac{C}{x^2}$$

$$2x^2y^4 + x^2 = C$$



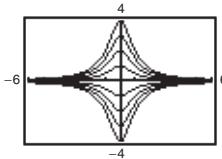
44.  $\mathbf{F}(x, y) = (1 + x^2)\mathbf{i} - 2xy\mathbf{j}$

$$\frac{dy}{dx} = \frac{-2xy}{1 + x^2}$$

$$\frac{1}{y} dy = -\frac{2x}{1 + x^2} dx$$

$$\ln y = \ln\left(\frac{1}{1 + x^2}\right) + \ln C$$

$$y = \frac{C}{1 + x^2}$$



47.  $E(x) = \frac{20x - y}{2y - 10x} = \frac{x}{y} \frac{dy}{dx}$

$$(20xy - y^2) dx + (10x^2 - 2xy) dy = 0$$

$$\frac{\partial M}{\partial y} = 20x - 2y = \frac{\partial N}{\partial x}$$

$$f(x, y) = 10x^2y - xy^2 + g(y)$$

$$g'(y) = 0$$

$$g(y) = C_1$$

$$10x^2y - xy^2 = K$$

Initial condition:  $C(100) = 500, 100 \leq x, K = 25,000,000$

$$10x^2y - xy^2 = 25,000,000$$

$xy^2 - 10x^2y + 25,000,000 = 0$  Quadratic Formula

$$y = \frac{10x^2 + \sqrt{100x^4 - 4x(25,000,000)}}{2x} = \frac{5(x^2 + \sqrt{x^4 - 1,000,000x})}{x}$$

45.  $\frac{dy}{dx} = \frac{y - x}{3y - x}$

$$(x - y) dx + (3y - x) dy = 0$$

$$\frac{\partial M}{\partial y} = -1 = \frac{\partial N}{\partial x}$$

$$f(x, y) = \frac{x^2}{2} - xy + g(y)$$

$$g'(y) = 3y$$

$$g(y) = \frac{3y^2}{2} + C_1$$

$$x^2 - 2xy + 3y^2 = C$$

Initial condition:  $y(2) = 1, 4 - 4 + 3 = C, C = 3$

Particular solution:  $x^2 - 2xy + 3y^2 = 3$

46.  $\frac{dy}{dx} = \frac{-2xy}{x^2 + y^2}$

$$2xy dx + (x^2 + y^2) dy = 0$$

$$\frac{\partial M}{\partial y} = 2x = \frac{\partial N}{\partial x}$$

$$f(x, y) = x^2y + g(y)$$

$$g'(y) = y^2$$

$$g(y) = \frac{y^3}{3} + C_1$$

$$3x^2y + y^3 = C$$

Initial condition:  $y(0) = 2, 8 = C$

Particular solution:  $3x^2y + y^3 = 8$

48.  $\frac{dy}{dx} + P(x)y = Q(x)$

$$dy + P(x)y \, dx = Q(x) \, dx$$

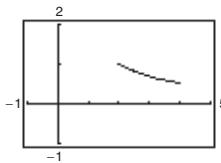
$$[P(x)y - Q(x)] \, dx + dy = 0$$

$$M(x, y) = P(x)y - Q(x) \text{ and } N(x, y) = 1$$

$$\begin{aligned} \frac{M_y(x, y) - N_x(x, y)}{N(x, y)} &= \frac{\frac{d}{dy}[P(x)y - Q(x)] - \frac{d}{dx}[1]}{1} \\ &= P(x) \end{aligned}$$

By Theorem 16.2,  $u(x) = e^{\int P(x) \, dx}$  is an integrating factor.

49. (a)  $y(4) \approx 0.5231$



(b)  $\frac{dy}{dx} = \frac{-xy}{x^2 + y^2}$

$$xy \, dx + (x^2 + y^2) \, dy = 0$$

$$\frac{1}{M} [N_x - M_y] = \frac{1}{xy} [2x - x] = \frac{1}{y} \text{ function of } y$$

alone.

Integrating factor:  $e^{\int (1/y) \, dy} = e^{\ln y} = y$

$$xy^2 \, dx + (x^2 y + y^3) \, dy = 0$$

$$f(x, y) = \int xy^2 \, dx = \frac{x^2 y^2}{2} + g(y)$$

$$f_y(x, y) = x^2 y + g'(y) \Rightarrow g(y) = \frac{y^4}{4} + C_1$$

$$f(x, y) = \frac{x^2 y^2}{2} + \frac{y^4}{4} = C$$

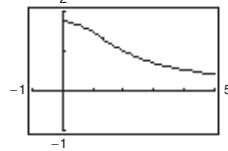
Initial condition:  $y(2) = 1, \frac{4}{2} + \frac{1}{4} = \frac{9}{4} = C$

Particular solution:  $\frac{x^2 y^2}{2} + \frac{y^4}{4} = \frac{9}{4}$  or

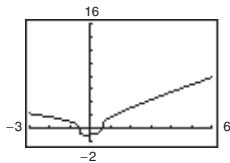
$$2x^2 y^2 + y^4 = 9.$$

For  $x = 4, 32y^2 + y^4 = 9 \Rightarrow y(4) = 0.528$

(c)



50. (a)  $y(5) \approx 6.6980$



(b)  $\frac{dy}{dx} = \frac{6x + y^2}{y(3y - 2x)}$

$$(6x + y^2) \, dx + (2xy - 3y^2) \, dy = 0$$

$$\frac{\partial M}{\partial y} = 2y = \frac{\partial N}{\partial x} \text{ Exact}$$

$$f(x, y) = \int (6x + y^2) \, dx = 3x^2 + xy^2 + g(y)$$

$$f_y = 2xy + g'(y) \Rightarrow g(y) = -y^3 + C_1$$

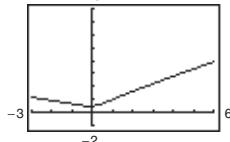
$$f(x, y) = 3x^2 + xy^2 - y^3 = C$$

Initial condition:  $y(0) = 1 \Rightarrow -1 = C$

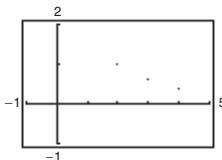
Particular solution:  $3x^2 + xy^2 - y^3 = -1$

For  $x = 5, 75 + 5y^2 - y^3 + 1 = 0 \Rightarrow y = 6.695$ .

(c)



51. (a)  $y(4) \approx 0.408$



(b)  $\frac{dy}{dx} = \frac{-xy}{x^2 + y^2}$

$$xy \, dx + (x^2 + y^2) \, dy = 0$$

$$\frac{1}{M} [N_x - M_y] = \frac{1}{xy} [2x - x] = \frac{1}{y}$$

=  $\frac{1}{y}$  function of  $y$  alone.

Integrating factor:  $e^{\int \frac{1}{y} dy} = e^{\ln y} = y$

$$xy^2 \, dx + (x^2 y + y^3) \, dy = 0$$

$$f(x, y) = \int xy^2 \, dx = \frac{x^2 y^2}{2} + g(y)$$

$$f_y(x, y) = x^2 y + g'(y) \Rightarrow g(y) = \frac{y^4}{4} + C_1$$

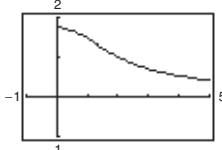
$$f(x, y) = \frac{x^2 y^2}{2} + \frac{y^4}{4} = C$$

$$\text{Initial condition: } y(2) = 1, \frac{4}{2} + \frac{1}{4} = \frac{9}{4} = C$$

$$\text{Particular solution: } \frac{x^2 y^2}{2} + \frac{y^4}{4} = \frac{9}{4} \text{ or } 2x^2 y^2 + y^4 = 9.$$

$$\text{For } x = 4, 32y^2 + y^4 = 9 \Rightarrow y(4) = 0.528$$

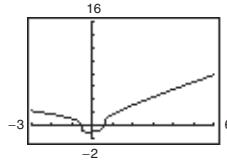
(c)



The solution is less accurate. For #49, Euler's Method gives  $y(4) \approx 0.523$ , whereas in #51, you obtain  $y(4) \approx 0.408$ . The errors are

$$0.528 - 0.523 = 0.005 \text{ and } 0.528 - 0.408 = 0.120.$$

52. (a)  $y(s) \approx 6.708$



(b)  $\frac{dy}{dx} = \frac{6x + y^2}{y(3y - 2x)}$

$$(6x + y^2) \, dx + (2xy - 3y^2) \, dy = 0$$

$$\frac{\partial M}{\partial y} = 2y = \frac{\partial N}{\partial x} \text{ Exact}$$

$$f(x, y) = \int (6x + y^2) \, dx = 3x^2 + xy^2 + g(y)$$

$$f_y = 2xy + g'(y) \Rightarrow g(y) = -y^3 + C_1$$

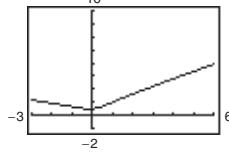
$$f(x, y) = 3x^2 + xy^2 - y^3 = C$$

$$\text{Initial condition: } y(0) = 1 \Rightarrow -1 = C$$

$$\text{Particular solution: } 3x^2 + xy^2 - y^3 = -1$$

$$\text{For } x = 5, 75 + 5y^2 - y^3 + 1 = 0 \Rightarrow y = 6.695.$$

(c)



The solution is less accurate. For #50, Euler's Method gives  $y(5) \approx 6.698$ , whereas in #52, you obtain  $y(5) \approx 6.708$ . The errors are

$$6.695 - 6.698 = -0.003 \text{ and}$$

$$6.695 - 6.708 = -0.013.$$

53. If  $M$  and  $N$  have continuous partial derivatives on an open disc  $R$ , then  $M(x, y) \, dx + N(x, y) \, dy = 0$  is exact

$$\text{if and only if } \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

54. See Theorem 16.2.

55. False

$$\frac{\partial M}{\partial y} = 2x \text{ and } \frac{\partial N}{\partial x} = -2x$$

56. False

$y \, dx + x \, dy = 0$  is exact, but  $xy \, dx + x^2 \, dy = 0$  is not exact.

57. True

$$\frac{\partial}{\partial y} [f(x) + M] = \frac{\partial M}{\partial y} \text{ and } \frac{\partial}{\partial x} [g(y) + N] = \frac{\partial N}{\partial x}$$

58. True

$$\frac{\partial}{\partial y} [f(x)] = 0 \text{ and } \frac{\partial}{\partial x} [g(y)] = 0$$

59.  $M = xy^2 + kx^2y + x^3$ ,  $N = x^3 + x^2y + y^2$

$$\frac{\partial M}{\partial y} = 2xy + kx^2, \frac{\partial N}{\partial x} = 3x^2 + 2xy$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \Rightarrow k = 3$$

60.  $M = ye^{2xy} + 2x$ ,  $N = kxe^{2xy} - 2y$

$$\frac{\partial M}{\partial y} = e^{2xy} + 2xye^{2xy}, \frac{\partial N}{\partial x} = ke^{2xy} + 2kxye^{2xy}$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \Rightarrow k = 1$$

61.  $M = g(y) \sin x$ ,  $N = y^2 f(x)$

$$\frac{\partial M}{\partial y} = g'(y) \sin x, \frac{\partial N}{\partial x} = y^2 f'(x)$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \Rightarrow g'(y) \sin x = f'(x)y^2$$

$$g'(y) = y^2 \Rightarrow g(y) = \frac{y^3}{3} + C_1$$

$$f'(x) = \sin x \Rightarrow f(x) = -\cos x + C_2$$

62.  $M = g(y)e^y$ ,  $N = xy$

$$\frac{\partial M}{\partial y} = g'(y)e^y + g(y)e^y, \frac{\partial N}{\partial x} = y$$

$$g'(y)e^y + g(y)e^y = y$$

$$[g(y)e^y]' = y$$

$$g(y)e^y = \frac{y^2}{2} + C$$

$$g(y) = e^{-y} \left[ \frac{y^2}{2} + C \right]$$

## Section 16.2 Second-Order Homogeneous Linear Equations

1.

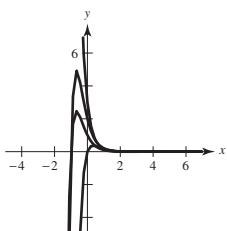
$$y = C_1 e^{-3x} + C_2 x e^{-3x}$$

$$y' = -3C_1 e^{-3x} + C_2 e^{-3x} - 3C_2 x e^{-3x}$$

$$y'' = 9C_1 e^{-3x} - 6C_2 e^{-3x} + 9C_2 x e^{-3x}$$

$$y'' + 6y' + 9y = (9C_1 e^{-3x} - 6C_2 e^{-3x} + 9C_2 x e^{-3x}) + (-18C_1 e^{-3x} + 6C_2 e^{-3x} - 18C_2 x e^{-3x}) + (9C_1 e^{-3x} + 9C_2 x e^{-3x}) = 0$$

$y$  approaches zero as  $x \rightarrow \infty$ .



2.

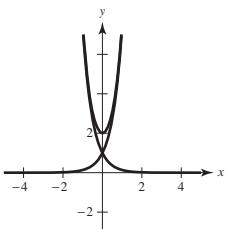
$$y = C_1 e^{2x} + C_2 e^{-2x}$$

$$y' = 2C_1 e^{2x} - 2C_2 e^{-2x}$$

$$y'' = 4C_1 e^{2x} + 4C_2 e^{-2x} = 4y$$

$$y'' - 4y = 4y - 4y = 0$$

The graphs are different combinations of the graphs of  $e^{2x}$  and  $e^{-2x}$ .



3.

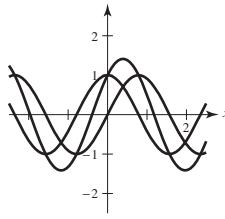
$$y = C_1 \cos 2x + C_2 \sin 2x$$

$$y' = -2C_1 \sin 2x + 2C_2 \cos 2x$$

$$y'' = -4C_1 \cos 2x - 4C_2 \sin 2x = -4y$$

$$y'' + 4y = -4y + 4y = 0$$

The graphs are basically the same shape, with left and right shifts and varying ranges.



4.  $y = C_1 e^{-x} \cos 3x + C_2 e^{-x} \sin 3x = e^{-x}(C_1 \cos 3x + C_2 \sin 3x)$

$$y' = -e^{-x}(C_1 \cos 3x + C_2 \sin 3x) + e^{-x}(-3C_1 \sin 3x + 3C_2 \cos 3x)$$

$$= e^{-x}[-C_1 \cos 3x - C_2 \sin 3x - 3C_1 \sin 3x + 3C_2 \cos 3x]$$

$$y'' = -e^{-x}[-C_1 \cos 3x - C_2 \sin 3x - 3C_1 \sin 3x + 3C_2 \cos 3x]$$

$$+ e^{-x}[3C_1 \sin 3x - 3C_2 \cos 3x - 9C_1 \cos 3x - 9C_2 \sin 3x]$$

$$y'' + 2y' + 10y = e^{-x}[C_1 \cos 3x + C_2 \sin 3x + 3C_1 \sin 3x - 3C_2 \cos 3x]$$

$$+ 3C_1 \sin 3x - 3C_2 \cos 3x - 9C_1 \cos 3x - 9C_2 \sin 3x]$$

$$+ 2e^{-x}[-C_1 \cos 3x - C_2 \sin 3x - 3C_1 \sin 3x + 3C_2 \cos 3x]$$

$$+ 10e^{-x}[C_1 \cos 3x + C_2 \sin 3x] = 0$$

$y$  approaches zero as  $x \rightarrow \infty$ . The graphs are the same only reflected.

5.  $y'' - y' = 0$

Characteristic equation:  $m^2 - m = 0$

Roots:  $m = 0, 1$

$$y = C_1 + C_2 e^x$$

6.  $y'' + 2y' = 0$

Characteristic equation:  $m^2 + 2m = 0$

Roots:  $m = 0, -2$

$$y = C_1 + C_2 e^{-2x}$$

7.  $y'' - y' - 6y = 0$

Characteristic equation:  $m^2 - m - 6 = 0$

Roots:  $m = 3, -2$

$$y = C_1 e^{3x} + C_2 e^{-2x}$$

8.  $y'' + 6y' + 5y = 0$

Characteristic equation:  $m^2 + 6m + 5 = 0$

Roots:  $m = -1, -5$

$$y = C_1 e^{-x} + C_2 e^{-5x}$$

9.  $2y'' + 3y' - 2y = 0$

Characteristic equation:  $2m^2 + 3m - 2 = 0$

Roots:  $m = \frac{1}{2}, -2$

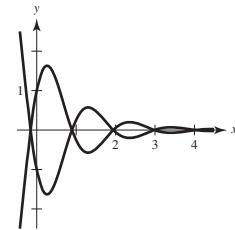
$$y = C_1 e^{(1/2)x} + C_2 e^{-2x}$$

10.  $16y'' - 16y' + 3y = 0$

Characteristic equation:  $16m^2 - 16m + 3 = 0$

Roots:  $m = \frac{1}{4}, \frac{3}{4}$

$$y = C_1 e^{(1/4)x} + C_2 e^{(3/4)x}$$



11.  $y'' + 6y' + 9y = 0$

Characteristic equation:  $m^2 + 6m + 9 = 0$

Roots:  $m = -3, -3$

$$y = C_1 e^{-3x} + C_2 x e^{-3x}$$

12.  $y'' - 10y' + 25y = 0$

Characteristic equation:  $m^2 - 10m + 25 = 0$

Roots:  $m = 5, 5$

$$y = C_1 e^{5x} + C_2 x e^{5x}$$

13.  $16y'' - 8y' + y = 0$

Characteristic equation:  $16m^2 - 8m + 1 = 0$

Roots:  $m = \frac{1}{4}, \frac{1}{4}$

$$y = C_1 e^{(1/4)x} + C_2 x e^{(1/4)x}$$

14.  $9y'' - 12y' + 4y = 0$

Characteristic equation:  $9m^2 - 12m + 4 = 0$

Roots:  $m = \frac{2}{3}, \frac{2}{3}$

$$y = C_1 e^{(2/3)x} + C_2 x e^{(2/3)x}$$

15.  $y'' + y = 0$

Characteristic equation:  $m^2 + 1 = 0$

Roots:  $m = -i, i$

$$y = C_1 \cos x + C_2 \sin x$$

16.  $y'' + 4y = 0$

Characteristic equation:  $m^2 + 4 = 0$

Roots:  $m = -2i, 2i$

$$y = C_1 \cos 2x + C_2 \sin 2x$$

17.  $y'' - 9y = 0$

Characteristic equation:  $m^2 - 9 = 0$ Roots:  $m = -3, 3$ 

$$y = C_1 e^{3x} + C_2 e^{-3x}$$

18.  $y'' - 2y = 0$

Characteristic equation:  $m^2 - 2 = 0$ Roots:  $m = -\sqrt{2}, \sqrt{2}$ 

$$y = C_1 e^{\sqrt{2}x} + C_2 e^{-\sqrt{2}x}$$

19.  $y'' - 2y' + 4y = 0$

Characteristic equation:  $m^2 - 2m + 4 = 0$ Roots:  $m = 1 - \sqrt{3}i, 1 + \sqrt{3}i$ 

$$y = e^x (C_1 \cos \sqrt{3}x + C_2 \sin \sqrt{3}x)$$

20.  $y'' - 4y' + 21y = 0$

Characteristic equation:  $m^2 - 4m + 21 = 0$ Roots:  $m = 2 - \sqrt{17}i, 2 + \sqrt{17}i$ 

$$y = e^{2x} (C_1 \cos \sqrt{17}x + C_2 \sin \sqrt{17}x)$$

21.  $y'' - 3y' + y = 0$

Characteristic equation:  $m^2 - 3m + 1 = 0$ Roots:  $m = \frac{3 - \sqrt{5}}{2}, \frac{3 + \sqrt{5}}{2}$ 

$$y = C_1 e^{[(3+\sqrt{5})/2]x} + C_2 e^{[(3-\sqrt{5})/2]x}$$

22.  $3y'' + 4y' - y = 0$

Characteristic equation:  $3m^2 + 4m - 1 = 0$ Roots:  $m = \frac{-2 - \sqrt{7}}{3}, \frac{-2 + \sqrt{7}}{3}$ 

$$y = C_1 e^{[(-2+\sqrt{7})/3]x} + C_2 e^{[(-2-\sqrt{7})/3]x}$$

23.  $9y'' - 12y' + 11y = 0$

Characteristic equation:  $9m^2 - 12m + 11 = 0$ Roots:  $m = \frac{2 + \sqrt{7}i}{3}, \frac{2 - \sqrt{7}i}{3}$ 

$$y = e^{(2/3)x} \left[ C_1 \cos \left( \frac{\sqrt{7}}{3}x \right) + C_2 \sin \left( \frac{\sqrt{7}}{3}x \right) \right]$$

24.  $2y'' - 6y' + 7y = 0$

Characteristic equation:  $2m^2 - 6m + 7 = 0$ Roots:  $m = \frac{3 + \sqrt{5}i}{2}, \frac{3 - \sqrt{5}i}{2}$ 

$$y = e^{(3/2)x} \left[ C_1 \cos \left( \frac{\sqrt{5}}{2}x \right) + C_2 \sin \left( \frac{\sqrt{5}}{2}x \right) \right]$$

25.  $y^{(4)} - y = 0$

Characteristic equation:  $m^4 - 1 = 0$ Roots:  $m = -1, 1, -i, i$ 

$$y = C_1 e^x + C_2 e^{-x} + C_3 \cos x + C_4 \sin x$$

26.  $y^{(4)} - y'' = 0$

Characteristic equation:  $m^4 - m^2 = 0$ Roots:  $m = 0, 0, -1, 1$ 

$$y = C_1 + C_2 x + C_3 e^x + C_4 e^{-x}$$

27.  $y''' - 6y'' + 11y' - 6y = 0$

Characteristic equation:  $m^3 - 6m^2 + 11m - 6 = 0$ Roots:  $m = 1, 2, 3$ 

$$y = C_1 e^x + C_2 e^{2x} + C_3 e^{3x}$$

28.  $y''' - y'' - y' + y = 0$

Characteristic equation:  $m^3 - m^2 - m + 1 = 0$ Roots:  $m = -1, 1, 1$ 

$$y = C_1 e^x + C_2 x e^x + C_3 e^{-x}$$

29.  $y''' - 3y'' + 7y' - 5y = 0$

Characteristic equation:  $m^3 - 3m^2 + 7m - 5 = 0$ Roots:  $m = 1, 1 - 2i, 1 + 2i$ 

$$y = C_1 e^x + e^x (C_2 \cos 2x + C_3 \sin 2x)$$

30.  $y''' - 3y'' + 3y' - y = 0$

Characteristic equation:  $m^3 - 3m^2 + 3m - 1 = 0$ Roots:  $m = 1, 1, 1$ 

$$y = C_1 e^x + C_2 x e^x + C_3 x^2 e^x$$

31.  $y'' + 100y = 0$

$$y = C_1 \cos 10x + C_2 \sin 10x$$

$$y' = -10C_1 \sin 10x + 10C_2 \cos 10x$$

(a)  $y(0) = 2: 2 = C_1$

$$y'(0) = 0: 0 = 10C_2 \Rightarrow C_2 = 0$$

Particular solution:  $y = 2 \cos 10x$ 

(b)  $y(0) = 0: 0 = C_1$

$$y'(0) = 2: 2 = 10C_2 \Rightarrow C_2 = \frac{1}{5}$$

Particular solution:  $y = \frac{1}{5} \sin 10x$ 

(c)  $y(0) = -1: -1 = C_1$

$$y'(0) = 3: 3 = 10C_2 \Rightarrow C_2 = \frac{3}{10}$$

Particular solution:  $y = -\cos 10x + \frac{3}{10} \sin 10x$

32.  $y = C \sin \sqrt{3}t$

$$y' = \sqrt{3}C \cos \sqrt{3}t$$

$$y'' = -3C \sin \sqrt{3}t$$

$$y'' + \omega y = -3C \sin \sqrt{3}t + \omega \sin \sqrt{3}t$$

$$= 0 \Rightarrow \omega = 3C$$

$$y'(0) = -5: -5 = \sqrt{3}C \Rightarrow C = \frac{5\sqrt{3}}{3} \text{ and}$$

$$\omega = -5\sqrt{3}$$

33.  $y'' - y' - 30y = 0, y(0) = 1, y'(0) = -4$

Characteristic equation:  $m^2 - m - 30 = 0$

Roots:  $m = 6, -5$

$$y = C_1 e^{6x} + C_2 e^{-5x}, y' = 6C_1 e^{6x} - 5C_2 e^{-5x}$$

Initial conditions:

$$y(0) = 1, y'(0) = -4, 1 = C_1 + C_2, -4 = 6C_1 - 5C_2$$

Solving simultaneously:  $C_1 = \frac{1}{11}, C_2 = \frac{10}{11}$

Particular solution:  $y = \frac{1}{11}(e^{6x} + 10e^{-5x})$

36.  $y'' + 2y' + 3y = 0, y(0) = 2, y'(0) = 1$

Characteristic equation:  $m^2 + 2m + 3 = 0$

Roots:  $m = \frac{-2 \pm \sqrt{4 - 12}}{2} = -1 \pm \sqrt{2}i$

$$y = e^{-x}(C_1 \cos \sqrt{2}x + C_2 \sin \sqrt{2}x)$$

$$y' = e^{-x}(-C_1 \sqrt{2} \sin \sqrt{2}x + C_2 \sqrt{2} \cos \sqrt{2}x) - e^{-x}(C_1 \cos \sqrt{2}x + C_2 \sin \sqrt{2}x)$$

Initial conditions:  $y(0) = 2 = C_1$

$$y'(0) = 1 = C_2 \sqrt{2} - C_1 = C_2 \sqrt{2} - 2 \Rightarrow C_2 = \frac{3}{\sqrt{2}}$$

Particular solution:  $y = e^{-x}\left(2 \cos \sqrt{2}x + \frac{3}{\sqrt{2}} \sin \sqrt{2}x\right)$

37.  $9y'' - 6y' + y = 0, y(0) = 2, y'(0) = 1$

Characteristic equation:  $9m^2 - 6m + 1 = 0$

Roots:  $m = \frac{1}{3}, \frac{1}{3}$

$$y = C_1 e^{(1/3)x} + C_2 x e^{(1/3)x}$$

$$y' = \frac{1}{3}C_1 e^{(1/3)x} + \frac{1}{3}C_2 x e^{(1/3)x} + C_2 e^{(1/3)x}$$

Initial conditions:  $y(0) = 2, y'(0) = 1$

$$\left. \begin{array}{l} C_1 = 2 \\ \frac{1}{3}C_1 + C_2 = 1 \end{array} \right\} \Rightarrow C_1 = 2, C_2 = \frac{1}{3}$$

Particular solution:  $y = 2e^{x/3} + \frac{1}{3}xe^{x/3}$

34.  $y'' - 7y' + 12y = 0, y(0) = 3, y'(0) = 3$

Characteristic equation:  $m^2 - 7m + 12 = 0$

Roots:  $m = 3, 4$

$$y = C_1 e^{3x} + C_2 e^{4x}, y' = 3C_1 e^{3x} + 4C_2 e^{4x}$$

Initial conditions:

$$y(0) = 3, y'(0) = 3, C_1 + C_2 = 3, 3C_1 + 4C_2 = 3$$

Solving simultaneously:  $C_1 = 9, C_2 = -6$

Particular solution:  $y = 9e^{3x} - 6e^{4x}$

35.  $y'' + 16y = 0, y(0) = 0, y'(0) = 2$

Characteristic equation:  $m^2 + 16 = 0$

Roots:  $m = \pm 4i$

$$y = C_1 \cos 4x + C_2 \sin 4x$$

$$y' = -4C_1 \sin 4x + 4C_2 \cos 4x$$

Initial conditions:  $y(0) = 0 = C_1$

$$y'(0) = 2 = 4C_2 \Rightarrow C_2 = \frac{1}{2}$$

Particular solution:  $y = \frac{1}{2} \sin 4x$

38.  $4y'' + 4y' + y = 0, y(0) = 3, y'(0) = 1$

Characteristic equation:  $4m^2 + 4m + 1 = 0$

$$(2m + 1)^2 = 0$$

Roots:  $m = -\frac{1}{2}, -\frac{1}{2}$

$$y = C_1 e^{(-1/2)x} + C_2 x e^{(-1/2)x}$$

$$y' = -\frac{1}{2}C_1 e^{(-1/2)x} - \frac{1}{2}C_2 x e^{(-1/2)x} + C_2 e^{(-1/2)x}$$

Initial conditions:

$$y(0) = 3 = C_1$$

$$y'(0) = 1 = -\frac{1}{2}C_1 + C_2 \Rightarrow C_2 = \frac{5}{2}$$

Particular solution:  $y = 3e^{x/2} + \frac{5}{2}xe^{-x/2}$

**39.**  $y'' - 4y' + 3y = 0$ ,  $y(0) = 1$ ,  $y(1) = 3$

Characteristic equation:  $m^2 - 4m + 3 = 0$

Roots:  $m = 1, 3$

$$y = C_1 e^x + C_2 e^{3x}$$

$$y(0) = 1: C_1 + C_2 = 1$$

$$y(1) = 3: C_1 e + C_2 e^3 = 3$$

$$\text{Solving simultaneously, } C_1 = \frac{e^3 - 3}{e^3 - e}, C_2 = \frac{3 - e}{e^3 - e}$$

$$\text{Solution: } y = \frac{e^3 - 3}{e^3 - e} e^x + \frac{3 - e}{e^3 - e} e^{3x}$$

**40.**  $4y'' + y = 0$ ,  $y(0) = 2$ ,  $y(\pi) = -5$

Characteristic equation:  $4m^2 + 1 = 0$

Roots:  $m = \pm \frac{1}{2}i$

$$y = C_1 \cos \frac{1}{2}x + C_2 \sin \frac{1}{2}x$$

$$y(0) = 2: C_1 = 2$$

$$y(\pi) = -5: C_2 = -5$$

$$\text{Solution: } y = 2 \cos \frac{1}{2}x - 5 \sin \frac{1}{2}x$$

**41.**  $y'' + 9y = 0$ ,  $y(0) = 3$ ,  $y(\pi) = 5$

Characteristic equation:  $m^2 + 9m = 0$

Roots:  $m = \pm 3i$

$$y = C_1 \cos 3x + C_2 \sin 3x$$

$$y(0) = 3: C_1 = 3$$

$$y(\pi) = 5: -C_1 = 5$$

No solution

**42.**  $4y'' + 20y' + 21y = 0$ ,  $y(0) = 3$ ,  $y(2) = 0$

Characteristic equation:  $4m^2 + 20m + 21 = 0$

Roots:  $m = -\frac{3}{2}, -\frac{7}{2}$

$$y = C_1 e^{(-3/2)x} + C_2 e^{(-7/2)x}$$

$$y(0) = 3: C_1 + C_2 = 3$$

$$y(2) = 0: C_1 e^{-3} + C_2 e^{-7} = 0 \Rightarrow C_1 + C_2 e^{-4} = 0$$

$$\text{Solving simultaneously, } C_1 = \frac{-3}{e^4 - 1}, C_2 = \frac{3e^4}{e^4 - 1}$$

$$\text{Solution: } y = \frac{-3}{e^4 - 1} e^{(-3/2)x} + \frac{3e^4}{e^4 - 1} e^{(-7/2)x}$$

**43.**  $4y'' - 28y' + 49y = 0$ ,  $y(0) = 2$ ,  $y(1) = -1$

Characteristic equation:  $4m^2 - 28m + 49 = 0$

Roots:  $m = \frac{7}{2}, \frac{7}{2}$

$$y = C_1 e^{(7/2)x} + C_2 x e^{(7/2)x}$$

$$y(0) = 2: C_1 = 2$$

$$y(1) = -1: C_1 e^{7/2} + C_2 e^{7/2} = -1 \Rightarrow C_2 = \frac{-1 - 2e^{7/2}}{e^{7/2}}$$

$$\text{Solution: } y = 2e^{(7/2)x} + \left( \frac{-1 - 2e^{7/2}}{e^{7/2}} \right) x e^{(7/2)x}$$

**44.**  $y'' + 6y' + 45y = 0$ ,  $y(0) = 4$ ,  $y(\pi) = 8$

Characteristic equation:  $m^2 + 6m + 45 = 0$

$$\text{Roots: } m = \frac{-6 \pm \sqrt{36 - 180}}{2} = -3 \pm 6i$$

$$y = C_1 e^{-3x} \cos 6x + C_2 e^{-3x} \sin 6x$$

$$y(0) = 4: C_1 = 4$$

$$y(\pi) = 8: -C_1 e^{-3\pi} = 8$$

No solution

**45.** No, it is not homogeneous because of the nonzero term  $\sin x$ .

**46.** Answers will vary. See Theorem 16.4.

**47.** Two functions  $y_1$  and  $y_2$  are linearly independent if the only solution to the equation  $C_1 y_1 + C_2 y_2 = 0$  is the trivial solution  $C_1 = C_2 = 0$ .

**48.**  $y'' + 2ky' + ky = 0$

Characteristic equation:  $m^2 + 2km + k = 0$

$$m = \frac{-2k \pm \sqrt{4k^2 - 4k}}{2} = -k \pm \sqrt{k^2 - 1}$$

(a) For  $k < -1$  and  $k > 1$ ,  $k^2 - 1 > 0$  and there are 2 distinct real roots.

(b) For  $k = \pm 1$ ,  $k^2 - 1 = 0$  and the roots are repeated.

(c) For  $-1 < k < 1$ , the roots are complex.

**49.** By Hooke's Law,  $F = kx$

$$k = \frac{F}{x} = \frac{32}{2/3} = 48.$$

$$\text{Also, } F = ma, \text{ and } m = \frac{F}{a} = \frac{32}{32} = 1.$$

$$\text{So, } y = \frac{1}{2} \cos(4\sqrt{3}t)$$

50. By Hooke's Law,  $F = kx$

$$k = \frac{F}{x} = \frac{32}{2/3} = 48.$$

Also,  $F = ma$ , and  $m = \frac{F}{a} = \frac{32}{32} = 1$ .

So,  $y = -\frac{2}{3} \cos(4\sqrt{3}t)$ .

51.  $y = C_1 \cos(\sqrt{k/m}t) + C_2 \sin(\sqrt{k/m}t)$ ,

$$\sqrt{k/m} = \sqrt{48} = 4\sqrt{3}$$

Initial conditions:  $y(0) = -\frac{2}{3}$ ,  $y'(0) = \frac{1}{2}$

$$y = C_1 \cos(4\sqrt{3}t) + C_2 \sin(4\sqrt{3}t)$$

$$y(0) = C_1 = -\frac{2}{3}$$

$$y'(t) = -4\sqrt{3}C_1 \sin(4\sqrt{3}t) + 4\sqrt{3}C_2 \cos(4\sqrt{3}t)$$

$$y'(0) = 4\sqrt{3}C_2 = \frac{1}{2} \Rightarrow C_2 = \frac{1}{8\sqrt{3}} = \frac{\sqrt{3}}{24}$$

$$y(t) = -\frac{2}{3} \cos(4\sqrt{3}t) + \frac{\sqrt{3}}{24} \sin(4\sqrt{3}t)$$

53. By Hooke's Law,  $32 = k(2/3)$ , so  $k = 48$ . Moreover, because the weight  $w$  is given by  $mg$ , it follows that

$m = w/g = 32/32 = 1$ . Also, the damping force is given by  $(-1/8)(dy/dt)$ . So, the differential equation for the oscillations of the weight is

$$m\left(\frac{d^2y}{dt^2}\right) = -\frac{1}{8}\left(\frac{dy}{dt}\right) - 48y$$

$$m\left(\frac{d^2y}{dt^2}\right) + \frac{1}{8}\left(\frac{dy}{dt}\right) + 48y = 0.$$

In this case the characteristic equation is  $8m^2 + m + 384 = 0$  with complex roots  $m = (-1/16) \pm (\sqrt{12,287}/16)i$ .

So, the general solution is  $y(t) = e^{-t/16} \left( C_1 \cos \frac{\sqrt{12,287}t}{16} + C_2 \sin \frac{\sqrt{12,287}t}{16} \right)$ .

Using the initial conditions, you have  $y(0) = C_1 = \frac{1}{2}$

$$y'(t) = e^{-t/16} \left[ \left( -\frac{\sqrt{12,287}}{16} C_1 - \frac{C_2}{16} \right) \sin \frac{\sqrt{12,287}t}{16} + \left( \frac{\sqrt{12,287}}{16} C_2 - \frac{C_1}{16} \right) \cos \frac{\sqrt{12,287}t}{16} \right]$$

$$y'(0) = \frac{\sqrt{12,287}}{16} C_2 - \frac{C_1}{16} = 0 \Rightarrow C_2 = \frac{\sqrt{12,287}}{24,574}$$

and the particular solution is

$$y(t) = \frac{e^{-t/16}}{2} \left( \cos \frac{\sqrt{12,287}t}{16} + \frac{\sqrt{12,287}}{12,287} \sin \frac{\sqrt{12,287}t}{16} \right).$$

52.  $y = C_1 \cos(4\sqrt{3}t) + C_2 \sin(4\sqrt{3}t)$

Initial conditions:  $y(0) = \frac{1}{2}$ ,  $y'(0) = -\frac{1}{2}$

$$y(0) = C_1 = \frac{1}{2}$$

$$y'(t) = -4\sqrt{3}C_1 \sin(4\sqrt{3}t) + 4\sqrt{3}C_2 \cos(4\sqrt{3}t)$$

$$y'(0) = 4\sqrt{3}C_2 = -\frac{1}{2} \Rightarrow C_2 = -\frac{1}{8\sqrt{3}}$$

$$y(t) = \frac{1}{2} \cos(4\sqrt{3}t) - \frac{1}{8\sqrt{3}} \sin(4\sqrt{3}t)$$

**54.** By Hooke's Law,  $32 = k(2/3)$ , so  $k = 48$ . Also,  $m = w/g = 32/32 = 1$ . The damping force is given by  $(-1/4)(dy/dt)$ . So,

$$m\left(\frac{d^2y}{dt^2}\right) = -\frac{1}{4}\left(\frac{dy}{dt}\right) - 48y$$

$$m\left(\frac{d^2y}{dt^2}\right) + \frac{1}{4}\left(\frac{dy}{dt}\right) + 48y = 0.$$

The characteristic equation is  $4m^2 + m + 192 = 0$  with complex roots  $m = (-1/8) \pm (\sqrt{3071}/8)i$ . So, the general solution is

$$y(t) = e^{-t/8} \left( C_1 \cos \frac{\sqrt{3071}t}{8} + C_2 \sin \frac{\sqrt{3071}t}{8} \right).$$

Using the initial conditions, you have

$$y(0) = C_1 = \frac{1}{2}$$

$$y'(t) = e^{-t/8} \left[ \left( -\frac{\sqrt{3071}}{8}C_1 - \frac{C_2}{8} \right) \sin \frac{\sqrt{3071}t}{8} + \left( \frac{\sqrt{3071}C_2}{8} - \frac{C_1}{8} \right) \cos \frac{\sqrt{3071}t}{8} \right]$$

$$y'(0) = \frac{\sqrt{3071}}{8}C_2 - \frac{C_1}{8} = 0 \Rightarrow C_2 = \frac{\sqrt{3071}}{6142}$$

and the particular solution is

$$y(t) = \frac{e^{-t/8}}{2} \left[ \cos \frac{\sqrt{3071}t}{8} + \frac{\sqrt{3071}}{3071} \sin \frac{\sqrt{3071}t}{8} \right].$$

**55.**  $y'' + 9y = 0$

Undamped vibration

Period:  $\frac{2\pi}{3}$

Matches (b)

**56.**  $y'' + 25y = 0$

Undamped vibration

Period:  $\frac{2\pi}{5}$

Matches (d)

**59.** Because  $m = -a/2$  is a double root of the characteristic equation, you have

$$\left(m + \frac{a}{2}\right)^2 = m^2 + am + \frac{a^2}{4} = 0$$

and the differential equation is  $y'' + ay' + (a^2/4)y = 0$ . The solution is

$$\begin{aligned} y &= (C_1 + C_2x)e^{-(a/2)x} \\ y' &= \left( -\frac{C_1a}{2} + C_2 - \frac{C_2a}{2}x \right) e^{-(a/2)x} \\ y'' &= \left( \frac{C_1a^2}{4} - aC_2 + \frac{C_2a^2}{4}x \right) e^{-(a/2)x} \end{aligned}$$

$$y'' + ay' + \frac{a^2}{4}y = e^{-(a/2)x} \left[ \left( \frac{C_1a^2}{4} - C_2a + \frac{C_2a^2}{4}x \right) + \left( -\frac{C_1a^2}{2} + C_2a - \frac{C_2a^2}{2}x \right) + \left( \frac{C_1a^2}{4} + \frac{C_2a^2}{4}x \right) \right] = 0.$$

**57.**  $y'' + 2y' + 10y = 0$

Damped vibration

Matches (c)

**58.**  $y'' + y' + \frac{37}{4}y = 0$

Damped vibration

Matches (a)

60. Because  $m = \alpha \pm \beta i$  are roots to the characteristic equation, you have

$$[m - (\alpha + \beta i)][m - (\alpha - \beta i)] = m^2 - 2\alpha m + (\alpha^2 + \beta^2) = 0$$

and the differential equation is  $y'' - 2\alpha y' + (\alpha^2 + \beta^2)y = 0$ . (Note:  $i^2 = -1$ .) The solution is

$$\begin{aligned}y &= e^{\alpha x}(C_1 \cos \beta x + C_2 \sin \beta x) \\y' &= e^{\alpha x}[(C_1 \alpha + C_2 \beta) \cos \beta x + (C_2 \alpha - C_1 \beta) \sin \beta x] \\y'' &= e^{\alpha x}[(C_1 \alpha^2 - C_1 \beta^2 + 2C_2 \alpha \beta) \cos \beta x + (C_2 \alpha^2 - C_2 \beta^2 - 2C_1 \alpha \beta) \sin \beta x] \\-2\alpha y' &= e^{\alpha x}[-2C_1 \alpha^2 - 2C_2 \alpha \beta] \cos \beta x + (-2C_2 \alpha^2 + 2C_1 \alpha \beta) \sin \beta x \\(\alpha^2 + \beta^2)y &= e^{\alpha x}[(C_1 \alpha^2 + C_1 \beta^2) \cos \beta x + (C_2 \alpha^2 + C_2 \beta^2) \sin \beta x]\end{aligned}$$

So,  $y'' - 2\alpha y' + (\alpha^2 + \beta^2)y = 0$ .

61. False. The general solution is  $y = C_1 e^{3x} + C_2 x e^{3x}$ .

62. True

63. True

64. False. The solution  $y = x^2 e^x$  requires that  $m = 1$  is a triple root of the characteristic equation. Because the characteristic equation is quadratic,  $m = 1$  can be at most a double root.

65.  $y_1 = e^{ax}$ ,  $y_2 = e^{bx}$ ,  $a \neq b$

$$W(y_1, y_2) = \begin{vmatrix} e^{ax} & e^{bx} \\ ae^{ax} & be^{bx} \end{vmatrix} = (b - a)e^{ax+bx} \neq 0 \text{ for any value of } x.$$

66.  $y_1 = e^{ax}$ ,  $y_2 = xe^{ax}$

$$W(y_1, y_2) = \begin{vmatrix} e^{ax} & xe^{ax} \\ ae^{ax} & e^{ax} + axe^{ax} \end{vmatrix} = e^{2ax} \neq 0 \text{ for any value of } x.$$

67.  $y_1 = e^{ax} \sin bx$ ,  $y_2 = e^{ax} \cos bx$ ,  $b \neq 0$

$$\begin{aligned}W(y_1, y_2) &= \begin{vmatrix} e^{ax} \sin bx & e^{ax} \cos bx \\ ae^{ax} \sin bx + be^{ax} \cos bx & ae^{ax} \cos bx - be^{ax} \sin bx \end{vmatrix} \\&= -be^{2ax} \sin^2 bx - be^{2ax} \cos^2 bx = -be^{2ax} \neq 0 \text{ for any value of } x.\end{aligned}$$

68.  $y_1 = x$ ,  $y_2 = x^2$

$$W(y_1, y_2) = \begin{vmatrix} x & x^2 \\ 1 & 2x \end{vmatrix} = x^2 \neq 0 \text{ for } x \neq 0.$$

69.  $x^2 y'' + ax y' + by = 0$ ,  $x > 0$

Let  $x = e^t$ .

$$(a) \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = e^{-t} \frac{dy}{dt}$$

$$\frac{d^2y}{dx^2} = \frac{d/dt [e^{-t}(dy/dt)]}{e^t} = e^{-t} \left[ e^{-t} \frac{d^2y}{dt^2} - e^{-t} \frac{dy}{dt} \right] = e^{-2t} \left[ \frac{d^2y}{dt^2} - \frac{dy}{dt} \right]$$

$$x^2 y'' + ax y' + by = 0$$

$$e^{2t} \left[ e^{-2t} \left( \frac{d^2y}{dt^2} - \frac{dy}{dt} \right) \right] + ae^t \left( e^{-t} \frac{dy}{dt} \right) + by = 0$$

$$\frac{d^2y}{dt^2} + (a - 1) \frac{dy}{dt} + by = 0$$

(b)  $x^2y'' + 6xy' + 6y = 0$

Let  $x = e^t$ . From part (a), you have:

$$\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + 6y = 0$$

$$m^2 + 5m + 6 = 0$$

$$(m+3)(m+2) = 0$$

$$m_1 = -3, m_2 = -2$$

$$y = C_1e^{-3t} + C_2e^{-2t} = C_1e^{-3\ln x} + C_2e^{-2\ln x} = C_1e^{\ln(1/x^3)} + C_2e^{\ln(1/x^2)} = \frac{C_1}{x^3} + \frac{C_2}{x^2}.$$

70.  $y'' + Ay = 0$

If  $A > 0$ , then  $y = C_1 \cos \sqrt{A}x + C_2 \sin \sqrt{A}x$ .

$$y(0) = 0 \Rightarrow C_1 = 0 \text{ and } y = C_2 \sin \sqrt{A}x$$

$$y(\pi) = 0 \Rightarrow 0 = C_2 \sin \sqrt{A}\pi \Rightarrow \sqrt{A} = \pm 1, \pm 2, \pm 3, \dots \Rightarrow A = 1, 4, 9, 16, \dots$$

So,  $A$  is a perfect square integer.

If  $A < 0$ , then  $y = C_1e^{\sqrt{-A}x} + C_2e^{-\sqrt{-A}x}$ .

$$\left. \begin{array}{l} y(0) = 0 \Rightarrow C_1 + C_2 = 0 \\ y(\pi) = 0 \Rightarrow C_1e^{\sqrt{-A}\pi} + C_2e^{-\sqrt{-A}\pi} = 0 \end{array} \right\} C_1 = C_2 = 0 \Rightarrow y = 0$$

If  $A = 0$ , then  $y'' = 0$  and the initial condition gives  $y = 0$ .

### Section 16.3 Second-Order Nonhomogeneous Linear Equations

1.  $y = 2e^{2x} - 2 \cos x$

$$y' = 4e^{2x} + 2 \sin x$$

$$y'' = 8e^{2x} + 2 \cos x$$

$$y'' + y = (8e^{2x} + 2 \cos x) + (2e^{2x} - \cos x) = 10e^{2x}$$

2.  $y = 2 \sin x + \frac{1}{2}x \sin x$

$$y' = 2 \cos x + \frac{1}{2}x \cos x + \frac{1}{2} \sin x$$

$$y'' = -2 \sin x - \frac{1}{2}x \sin x + \cos x$$

$$y'' + y = (-2 \sin x - \frac{1}{2}x \sin x + \cos x) + (2 \sin x + \frac{1}{2} \sin x) = \cos x$$

3.  $y = 3 \sin x - \cos x \ln |\sec x + \tan x|$

$$y' = 3 \cos x - 1 + \sin x \ln |\sec x + \tan x|$$

$$y'' = -3 \sin x + \tan x + \cos x \ln |\sec x + \tan x|$$

$$y'' + y = (-3 \sin x + \tan x + \cos x \ln |\sec x + \tan x|) + (3 \sin x - \cos x \ln |\sec x + \tan x|) = \tan x$$

4.  $y = (5 - \ln |\sin x|) \cos x - x \sin x$

$$y' = -(5 - \ln |\sin x|) \sin x - \cos x \cot x - \sin x - x \cos x = -6 \sin x + \sin x \ln |\sin x| - \cos x (\cot x + x)$$

$$y'' = -6 \cos x + \cos x + \cos x \ln |\sin x| - \cos x (-\csc^2 x + 1) + \sin x (\cot x + x)$$

$$= -5 \cos x + \cos x \ln |\sin x| + \csc x \cot x + x \sin x$$

$$y'' + y = \cos x (-5 + \ln |\sin x|) + \csc x \cot x + x \sin x + (5 - \ln |\sin x|) \cos x - x \sin x = \csc x \cot x$$

5.  $y'' + 7y' + 12y = 3x + 1$

$$y'' + 7y' + 12y = 0$$

$$m^2 - 7m + 12 = (m - 3)(m - 4) = 0 \text{ when } m = 3, 4$$

$$y_h = C_1 e^{3x} + C_2 e^{4x}$$

$$y_p = A_0 + A_1 x$$

$$y'_p = A_1, y''_p = 0$$

$$y''_p + 7y'_p + 12y_p = 7A_1 + 12(A_0 + A_1 x) = 3x + 1$$

$$\left. \begin{array}{l} 12A_1 = 3 \\ 7A_1 + 12A_0 = 1 \end{array} \right\} \Rightarrow A_1 = \frac{1}{4}, A_0 = -\frac{1}{16}$$

$$\text{Solution: } y_p = -\frac{1}{16} + \frac{1}{4}x$$

6.  $y'' - y' - 6y = 4$

$$y'' - y' - 6y = 0$$

$$m^2 - m - 6 = (m - 3)(m + 2) = 0 \text{ when } m = 3, -2$$

$$y_h = C_1 e^{3x} + C_2 e^{-2x}$$

$$y_p = A, y'_p = y''_p = 0$$

$$y''_p - y'_p - 6y_p = -6A = 4 \Rightarrow A = -\frac{2}{3}$$

$$\text{Solution: } y_p = -\frac{2}{3}$$

9.  $y'' - 2y' - 15y = \sin x$

$$y'' - 2y' - 15y = 0$$

$$m^2 - 2m - 15 = (m - 5)(m + 3) = 0 \Rightarrow m = 5, -3$$

$$y_p = A \sin x + B \cos x$$

$$y'_p = A \cos x - B \sin x$$

$$y''_p = -A \sin x - B \cos x$$

$$y''_p - 2y'_p - 15y_p = (-A \sin x - B \cos x) - 2(A \cos x - B \sin x) - 15(A \sin x + B \cos x) = \sin x$$

$$(-A + 2B - 15A) \sin x + (-B - 2A - 15B) \cos x = \sin x$$

$$\left. \begin{array}{l} -16A + 2B = 1 \\ -2A - 16B = 0 \end{array} \right\} \Rightarrow A = -\frac{4}{65}, B = \frac{1}{130}$$

$$\text{Solution: } y_p = -\frac{4}{65} \sin x + \frac{1}{130} \cos x$$

10.  $y'' + 4y' + 5y = e^x \cos x$

$$y'' + 4y' + 5y = 0$$

$$m^2 + 4m + 5 = 0 \Rightarrow m = \frac{-4 \pm \sqrt{-4}}{2} = -2 \pm i$$

$$y_p = Ae^x \cos x + Be^x \sin x = e^x(A \cos x + B \sin x)$$

$$y'_p = e^x(A \cos x + B \sin x) + e^x(-A \sin x + B \cos x) = e^x((B - A) \sin x + (A + B) \cos x)$$

$$y''_p = e^x((B - A) \sin x + (A + B) \cos x) + e^x((B - A) \cos x - (A + B) \sin x) = e^x(-2A \sin x + 2B \cos x)$$

$$y''_p + 4y'_p + 5y_p = e^x(-2A \sin x + 2B \cos x) + 4(e^x((B - A) \sin x + (A + B) \cos x)) + 5(Ae^x \cos x + Be^x \sin x) = e^x \cos x$$

$$(-2A + 4(B - A) + 5B) \sin x + (2B + 4(A + B) + 5A) \cos x = \cos x$$

$$\left. \begin{array}{l} -6A + 9B = 0 \\ 9A + 6B = 1 \end{array} \right\} \Rightarrow A = \frac{3}{39}, B = \frac{2}{39}$$

$$\text{Solution: } y_p = \frac{3}{39}e^x \cos x + \frac{2}{39}e^x \sin x$$

7.  $y'' - 8y' + 16y = e^{3x}$

$$y'' - 8y' + 16y = 0$$

$$m^2 - 8m + 16 = (m - 4)^2 = 0 \text{ when } m = 4$$

$$y_h = C_1 e^{4x} + C_2 x e^{4x}$$

$$y_p = Ae^{3x}, y'_p = 3Ae^{3x}, y''_p = 9Ae^{3x}$$

$$y''_p - 8y'_p + 16y_p = 9Ae^{3x} - 8(3Ae^{3x}) + 16(Ae^{3x}) \\ = e^{3x}$$

$$9A - 24A + 16A = 1 \Rightarrow A = 1$$

$$\text{Solution: } y_p = e^{3x}$$

8.  $y'' + y' + 3y = e^{2x}$

$$y'' + y' + 3y = 0$$

$$m^2 + m + 3 = 0 \Rightarrow m = \frac{-1 \pm \sqrt{-11}}{2}$$

$$y_p = Ae^{2x}, y'_p = 2Ae^{2x}, y''_p = 4Ae^{2x}$$

$$y''_p + y'_p + 3y = 4Ae^{2x} + 2Ae^{2x} + 3(Ae^{2x}) = e^{2x}$$

$$4A + 2A + 3A = 1 \Rightarrow A = \frac{1}{9}$$

$$\text{Solution: } y_p = \frac{1}{9}e^{2x}$$

11.  $y'' - 3y' + 2y = 2x$

$$y'' - 3y' + 2y = 0$$

$$m^2 - 3m + 2 = 0 \text{ when } m = 1, 2.$$

$$y_h = C_1 e^x + C_2 e^{2x}$$

$$y_p = A_0 + A_1 x$$

$$y'_p = A_1$$

$$y''_p = 0$$

$$y''_p - 3y'_p + 2y_p = (2A_0 - 3A_1) + 2A_1 x = 2x$$

$$\begin{cases} 2A_0 - 3A_1 = 0 \\ 2A_1 = 2 \end{cases} \quad \left\{ \begin{array}{l} A_1 = 1, A_0 = \frac{3}{2} \end{array} \right.$$

$$y = C_1 e^x + C_2 e^{2x} + x + \frac{3}{2}$$

12.  $y'' - 2y' - 3y = x^2 - 1$

$$y'' - 2y' - 3y = 0$$

$$m^2 - 2m - 3 = 0 \text{ when } m = -1, 3.$$

$$y_h = C_1 e^{-x} + C_2 e^{3x}$$

$$y_p = A_0 + A_1 x + A_2 x^2$$

$$y'_p = A_1 + 2A_2 x$$

$$y''_p = 2A_2$$

$$y''_p - 2y'_p - 3y_p = (-3A_2)x^2 + (-3A_1 - 4A_2)x + (-3A_0 - 2A_1 + 2A_2) = x^2 - 1$$

$$\begin{cases} -3A_2 = 1 \\ -3A_1 - 4A_2 = 0 \\ -3A_0 - 2A_1 + 2A_2 = -1 \end{cases} \quad \left\{ \begin{array}{l} A_0 = -\frac{5}{27}, A_1 = \frac{4}{9}, A_2 = -\frac{1}{3} \end{array} \right.$$

$$y = C_1 e^{-x} + C_2 e^{3x} - \frac{1}{3}x^2 + \frac{4}{9}x - \frac{5}{27}$$

13.  $y'' + 2y' = 2e^x$

$$y'' + 2y' = 0$$

$$m^2 + 2m = 0 \text{ when } m = 0, -2.$$

$$y_h = C_1 + C_2 e^{-2x}$$

$$y_p = Ae^x = y'_p = y''_p$$

$$y''_p + 2y'_p = 3Ae^x = 2e^x \text{ or } A = \frac{2}{3}$$

$$y = C_1 + C_2 e^{-2x} + \frac{2}{3}e^x$$

14.  $y'' - 9y = 5e^{3x}$

$$y'' - 9y = 0$$

$$m^2 - 9 = 0 \text{ when } m = -3, 3.$$

$$y_h = C_1 e^{-3x} + C_2 e^{3x}$$

$$y_p = Axe^{3x}$$

$$y'_p = Ae^{3x}(3x + 1)$$

$$y''_p = Ae^{3x}(9x + 6)$$

$$y''_p - 9y_p = 6Ae^{3x} = 5e^{3x} \text{ or } A = \frac{5}{6}$$

$$y = C_1 e^{-3x} + \left(C_2 + \frac{5}{6}x\right)e^{3x}$$

15.  $y'' - 10y' + 25y = 5 + 6e^x$

$$y'' - 10y' + 25y = 0$$

$$m^2 - 10m + 25 = 0 \text{ when } m = 5, 5.$$

$$y_h = C_1 e^{5x} + C_2 xe^{5x}$$

$$y_p = A_0 + A_1 e^x$$

$$y'_p = y''_p = A_1 e^x$$

$$y''_p - 10y'_p + 25y_p = 25A_0 + 16A_1 e^x = 5 + 6e^x$$

$$\text{or } A_0 = \frac{1}{5}, A_1 = \frac{3}{8}$$

$$y = (C_1 + C_2 x)e^{5x} + \frac{3}{8}e^x + \frac{1}{5}$$

16.  $16y'' - 8y' + y = 4(x + e^x)$

$$16y'' - 8y' + y = 0$$

$$16m^2 - 8m + 1 = 0 \text{ when } m = \frac{1}{4}, \frac{1}{4}$$

$$y_h = (C_1 + C_2 x)e^{(1/4)x}$$

$$y_p = A_0 + A_1 x + A_2 e^x$$

$$y'_p = A_1 + A_2 e^x$$

$$y''_p = A_2 e^x$$

$$16y''_p - 8y'_p + y_p = (A_0 - 8A_1) + A_1 x + 9A_2 e^x = 4x + 4e^x$$

$$\text{or } A_2 = \frac{4}{9}, A_1 = 4, A_0 = 32$$

$$y = (C_1 + C_2 x)e^{(1/4)x} + 32 + 4x + \frac{4}{9}e^x$$

17.  $y'' + 9y = \sin 3x$

$$y'' + 9y = 0$$

$$m^2 + 9 = 0 \text{ when } m = -3i, 3i.$$

$$y_h = C_1 \cos 3x + C_2 \sin 3x$$

$$y_p = A_0 \sin 3x + A_1 x \sin 3x + A_2 \cos 3x + A_3 x \cos 3x$$

$$y''_p = (-9A_0 - 6A_3) \sin 3x - 9A_1 x \sin 3x$$

$$+ (6A_1 - 9A_2) \cos 3x - 9A_3 x \cos 3x$$

$$y''_p + 9y_p = -6A_3 \sin 3x + 6A_1 \cos 3x = \sin 3x,$$

$$A_1 = 0, A_3 = -\frac{1}{6}$$

$$y = (C_1 - \frac{1}{6}x) \cos 3x + C_2 \sin 3x$$

18.  $y''' - 3y' + 2y = 2e^{-2x}$

$$y''' - 3y' + 2y = 0$$

$m^3 - 3m + 2 = 0$  when  $m = 1, 1, -2$ .

$$y_h = C_1 e^x + C_2 x e^x + C_3 e^{-2x}$$

$$y_p = A_0 e^{-2x} + A_1 x e^{-2x}$$

$$y'_p = (-2A_0 + A_1)e^{-2x} - 2A_1 x e^{-2x}$$

$$y''_p = (4A_0 - 4A_1)e^{-2x} + 4A_1 x e^{-2x}$$

$$y'''_p = (-8A_0 + 12A_1)e^{-2x} - 8A_1 x e^{-2x}$$

$$y''' - 3y'_p + 2y_p = 9A_1 e^{-2x} = 2e^{-2x} \text{ or } A_1 = \frac{2}{9}$$

$$y = C_1 e^x + C_2 x e^x + \left(C_3 + \frac{2}{9}x\right) e^{-2x}$$

19.  $y'' + y = x^3$ ,  $y(0) = 1$ ,  $y'(0) = 0$

$$y'' + y = 0$$

$m^2 + 1 = 0$  when  $m = i, -i$ .

$$y_h = C_1 \cos x + C_2 \sin x$$

$$y_p = A_0 + A_1 x + A_2 x^2 + A_3 x^3$$

$$y'_p = A_1 + 2A_2 x + 3A_3 x^2$$

$$y''_p = 2A_2 + 6A_3 x$$

$$\begin{aligned} y''_p + y_p &= A_3 x^3 + A_2 x^2 + (A_1 + 6A_3)x + (A_0 + 2A_2) \\ &= x^3 \end{aligned}$$

or  $A_3 = 1$ ,  $A_2 = 0$ ,  $A_1 = -6$ ,  $A_0 = 0$

$$y = C_1 \cos x + C_2 \sin x + x^3 - 6x$$

$$y' = -C_1 \sin x + C_2 \cos x + 3x^2 - 6$$

Initial conditions:

$$y(0) = 1, y'(0) = 0, 1 = C_1, 0 = C_2 - 6, C_2 = 6$$

Particular solution:  $y = \cos x + 6 \sin x + x^3 - 6x$

20.  $y'' + 4y = 4$ ,  $y(0) = 1$ ,  $y'(0) = 6$

$$y'' + 4y = 0$$

$m^2 + 4 = 0$  when  $m = 2i, -2i$ .

$$y_h = C_1 \cos 2x + C_2 \sin 2x$$

$$y_p = A_0$$

$$y''_p = 0$$

$$y''_p + 4y_p = 4A_0 = 4 \text{ or } A_0 = 1$$

$$y = C_1 \cos 2x + C_2 \sin 2x + 1$$

$$y' = -2C_1 \sin 2x + 2C_2 \cos 2x$$

Initial conditions:  $y(0) = 1$ ,  $y'(0) = 6$ ,  $1 = C_1 + 1$ ,

$$C_1 = 0, 6 = 2C_2, C_2 = 3$$

Particular solution:  $y = 3 \sin 2x + 1$

21.  $y'' + y' = 2 \sin x$ ,  $y(0) = 0$ ,  $y'(0) = -3$

$$y'' + y' = 0$$

$m^2 + m = 0$  when  $m = 0, -1$ .

$$y_h = C_1 + C_2 e^{-x}$$

$$y_p = A \cos x + B \sin x$$

$$y'_p = -A \sin x + B \cos x$$

$$y''_p = -A \cos x - B \sin x$$

$$y''_p + y'_p = (-A + B) \cos x + (-A - B) \sin x = 2 \sin x$$

$$\begin{cases} -A + B = 0 \\ -A - B = 2 \end{cases} \quad \begin{cases} A = -1, B = -1 \\ A = 2, B = 2 \end{cases}$$

$$y = C_1 + C_2 e^{-x} - (\cos x + \sin x)$$

$$y' = -C_2 e^{-x} - (-\sin x + \cos x)$$

Initial conditions:  $y(0) = 0$ ,  $y'(0) = -3$ ,

$$0 = C_1 + C_2 - 1, -3 = -C_2 - 1,$$

$$C_2 = 2, C_1 = -1$$

Particular solution:  $y = -1 + 2e^{-x} - (\cos x + \sin x)$

22.  $y'' + y' - 2y = 3 \cos 2x$ ,  $y(0) = -1$ ,  $y'(0) = 2$

$$y'' + y' - 2y = 0$$

$m^2 + m - 2 = 0$  when  $m = 1, -2$ .

$$y_h = C_1 e^x + C_2 e^{-2x}$$

$$y_p = A \cos 2x + B \sin 2x$$

$$y'_p = -2A \sin 2x + 2B \cos 2x$$

$$y''_p = -4A \cos 2x - 4B \sin 2x$$

$$\begin{aligned} y''_p + y'_p - 2y_p &= (-6A + 2B) \cos 2x + (-2A - 6B) \sin 2x \\ &= 3 \cos 2x \end{aligned}$$

$$\begin{cases} -6A + 2B = 3 \\ -2A - 6B = 0 \end{cases} \quad \begin{cases} A = -\frac{9}{20}, B = \frac{3}{20} \\ C_1 = -2C_2 + \frac{3}{10} \end{cases}$$

$$y = C_1 e^x + C_2 e^{-2x} - \frac{9}{20} \cos 2x + \frac{3}{20} \sin 2x$$

$$y' = C_1 e^x - 2C_2 e^{-2x} + \frac{9}{10} \sin 2x + \frac{3}{10} \cos 2x$$

Initial conditions:

$$y(0) = -1, y'(0) = 2, -1 = C_1 + C_2 - \frac{9}{20},$$

$$2 = C_1 - 2C_2 + \frac{3}{10}$$

$$\begin{cases} C_1 + C_2 = -\frac{11}{20} \\ C_1 - 2C_2 = \frac{17}{10} \end{cases} \quad \begin{cases} C_1 = \frac{1}{5}, C_2 = -\frac{3}{4} \\ C_1 = -2C_2 + \frac{3}{10} \end{cases}$$

Particular solution:

$$y = \frac{1}{20}(4e^x - 15e^{-2x} - 9 \cos 2x + 3 \sin 2x)$$

23.  $y' - 4y = xe^x - xe^{4x}$ ,  $y(0) = \frac{1}{3}$

$$y' - 4y = 0$$

$m - 4 = 0$  when  $m = 4$ .

$$y_h = Ce^{4x}$$

$$y_p = (A_0 + A_1x)e^x + (A_2x + A_3x^2)e^{4x}$$

$$y'_p = (A_0 + A_1x)e^x + A_1e^x$$

$$+ 4(A_2x + A_3x^2)e^{4x} + (A_2 + 2A_3x)e^{4x}$$

$$\begin{aligned} y'_p - 4y_p &= (-3A_0 - 3A_1x)e^x + A_1e^x + A_2e^{4x} \\ &\quad + 2A_3xe^{4x} = xe^x - xe^{4x} \end{aligned}$$

$$A_0 = -\frac{1}{9}, A_1 = -\frac{1}{3}, A_2 = 0, A_3 = -\frac{1}{2}$$

$$y = (C - \frac{1}{2}x^2)e^{4x} - \frac{1}{9}(1 + 3x)e^x$$

Initial conditions:  $y(0) = \frac{1}{3}, \frac{1}{3} = C - \frac{1}{9}, C = \frac{4}{9}$

Particular solution:  $y = (\frac{4}{9} - \frac{1}{2}x^2)e^{4x} - \frac{1}{9}(1 + 3x)e^x$

24.  $y' + 2y = \sin x$ ,  $y\left(\frac{\pi}{2}\right) = \frac{2}{5}$

$$y' + 2y = 0$$

$m + 2 = 0$  when  $m = -2$ .

$$y_h = Ce^{-2x}$$

$$y_p = A \cos x + B \sin x$$

$$y'_p = -A \sin x + B \cos x$$

$$\begin{aligned} y'_p + 2y_p &= (-A \sin x + B \cos x) + 2(A \cos x + B \sin x) \\ &= (2B - A) \sin x + (2A + B) \cos x = \sin x \end{aligned}$$

$$2B - A = 1, 2A + B = 0 \Rightarrow B = \frac{2}{5}, A = -\frac{1}{5}$$

$$y = y_h + y_p = Ce^{-2x} - \frac{1}{5} \cos x + \frac{2}{5} \sin x$$

Initial conditions:  $y\left(\frac{\pi}{2}\right) = \frac{2}{5}, \frac{2}{5} = Ce^{-\pi} + \frac{2}{5}, C = 0$

Particular solution:  $y = \frac{2}{5} \sin x - \frac{1}{5} \cos x$

25.  $y'' + y = \sec x$

$$y'' + y = 0$$

$m^2 + 1 = 0$  when  $m = -i, i$ .

$$y_h = C_1 \cos x + C_2 \sin x$$

$$y_p = v_1 \cos x + v_2 \sin x$$

$$v_1' \cos x + v_2' \sin x = 0$$

$$v_1'(-\sin x) + v_2'(\cos x) = \sec x$$

$$v_1' = \frac{\begin{vmatrix} 0 & \sin x \\ \sec x & \cos x \end{vmatrix}}{\begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix}} = -\tan x$$

$$v_1 = \int -\tan x \, dx = \ln |\cos x|$$

$$v_2' = \frac{\begin{vmatrix} \cos x & 0 \\ -\sin x & \sec x \end{vmatrix}}{\begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix}} = 1$$

$$v_2 = \int dx = x$$

$$y = (C_1 + \ln |\cos x|) \cos x + (C_2 + x) \sin x$$

26.  $y'' + y = \sec x \tan x$

$$y'' + y = 0$$

$m^2 + 1 = 0$  when  $m = -i, i$ .

$$y_h = C_1 \cos x + C_2 \sin x$$

$$y_p = v_1 \cos x + v_2 \sin x$$

$$v_1' \cos x + v_2' \sin x = 0$$

$$v_1'(-\sin x) + v_2' \cos x = \sec x \tan x$$

$$v_1' = \frac{\begin{vmatrix} 0 & \sin x \\ \sec x \tan x & \cos x \end{vmatrix}}{\begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix}} = -\tan^2 x$$

$$v_1 = \int -\tan^2 x \, dx = -\int (\sec^2 x - 1) \, dx$$

$$= -\tan x + x$$

$$v_2' = \frac{\begin{vmatrix} \cos x & 0 \\ -\sin x \sec x & \tan x \end{vmatrix}}{\begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix}} = \tan x$$

$$v_2 = \int \tan x \, dx = -\ln |\cos x| = \ln |\sec x|$$

$$y = y_h + y_p$$

$$= C_1 \cos x + C_2 \sin x + (x - \tan x) \cos x$$

$$+ \ln |\sec x| \sin x$$

$$= (C_1 + x - \tan x) \cos x + (C_2 + \ln |\sec x|) \sin x$$

27.  $y'' + 4y = \csc 2x$

$$y'' + 4y = 0$$

$m^2 + 4 = 0$  when  $m = -2i, 2i$ .

$$y_h = C_1 \cos 2x + C_2 \sin 2x$$

$$y_p = v_1 \cos 2x + v_2 \sin 2x = 0$$

$$v'_1 \cos 2x + v'_2 \sin 2x = 0$$

$$v'_1(-2 \sin 2x) + v'_2(2 \cos 2x) = \csc 2x$$

$$v'_1 = \frac{\begin{vmatrix} 0 & \sin 2x \\ \csc 2x & 2 \cos 2x \end{vmatrix}}{\begin{vmatrix} \cos 2x & \sin 2x \\ -2 \sin 2x & 2 \cos 2x \end{vmatrix}} = -\frac{1}{2}$$

$$v_1 = \int -\frac{1}{2} dx = -\frac{1}{2}x$$

$$v'_2 = \frac{\begin{vmatrix} \cos 2x & 0 \\ -2 \sin 2x & \csc 2x \end{vmatrix}}{\begin{vmatrix} \cos 2x & \sin 2x \\ -2 \sin 2x & 2 \cos 2x \end{vmatrix}} = \frac{1}{2} \cot 2x$$

$$v_2 = \int \frac{1}{2} \cot 2x dx = \frac{1}{4} \ln |\sin 2x|$$

$$y = \left( C_1 - \frac{1}{2}x \right) \cos 2x + \left( C_2 + \frac{1}{4} \ln |\sin 2x| \right) \sin 2x$$

28.  $y'' - 4y' + 4y = x^2 e^{2x}$

$$y'' - 4y' + 4y = 0$$

$m^2 - 4m + 4 = 0$  when  $m = 2, 2$ .

$$y_h = (C_1 + C_2x)e^{2x}$$

$$y_p = (v_1 + v_2x)e^{2x}$$

$$v'_1 e^{2x} + v'_2 x e^{2x} = 0$$

$$v'_1(2e^{2x}) + v'_2(2x+1)e^{2x} = x^2 e^{2x}$$

$$v'_1 = \frac{\begin{vmatrix} 0 & x e^{2x} \\ x^2 e^{2x} & (2x+1)e^{2x} \end{vmatrix}}{\begin{vmatrix} e^{2x} & x e^{2x} \\ 2e^{2x} & (2x+1)e^{2x} \end{vmatrix}} = \frac{-x^3 e^{4x}}{e^{4x}} = -x^3$$

$$v_1 = \int -x^3 dx = -\frac{1}{4}x^4$$

$$v'_2 = \frac{\begin{vmatrix} e^{2x} & 0 \\ 2e^{2x} & x^2 e^{2x} \end{vmatrix}}{\begin{vmatrix} e^{4x} & x^2 e^{2x} \end{vmatrix}} = \frac{x^2 e^{4x}}{e^{4x}} = x^2$$

$$v_2 = \int x^2 dx = \frac{1}{3}x^3$$

$$y = \left( C_1 + C_2x + \frac{1}{12}x^4 \right) e^{2x}$$

29.  $y'' - 2y' + y = e^x \ln x$

$$y'' - 2y' + y = 0$$

$m^2 - 2m + 1 = 0$  when  $m = 1, 1$ .

$$y_h = (C_1 + C_2x)e^x$$

$$y_p = (v_1 + v_2x)e^x$$

$$v'_1 e^x + v'_2 x e^x = 0$$

$$v'_1 e^x + v'_2(x+1)e^x = e^x \ln x$$

$$v'_1 = -x \ln x$$

$$v_1 = \int -x \ln x dx = -\frac{x^2}{2} \ln x + \frac{x^2}{4}$$

$$v'_2 = \ln x$$

$$v_2 = \int \ln x dx = x \ln x - x$$

$$y = (C_1 + C_2x)e^x + \frac{x^2 e^x}{4} (\ln x^2 - 3)$$

30.  $y'' - 4y' + 4y = \frac{e^{2x}}{x}$

$$y'' - 4y' + 4y = 0$$

$m^2 - 4m + 4 = 0$  when  $m = 2, 2$ .

$$y_h = (C_1 + C_2x)e^{2x}$$

$$y_p = (v_1 + v_2x)e^{2x}$$

$$v'_1 e^{2x} + v'_2 x e^{2x} = 0$$

$$v_1 e^{2x}(2) + v'_2(2x+1)e^{2x} = \frac{e^{2x}}{x}$$

$$v'_1 = -1$$

$$v_1 = \int -1 dx = -x$$

$$v'_2 = \frac{1}{x}$$

$$v_2 = \int \frac{1}{x} dx = \ln |x|$$

$$y = (C_1 + C_2x - x + x \ln |x|)e^{2x}$$

31.  $y'' - y' - 12y = 0$

$$m^2 - m - 12 = (m-4)(m+3) = 0 \Rightarrow m = 4, -3$$

Let  $y_p = Ax^2 + Bx + C$ . This is a generalized form of

$$F(x) = x^2.$$

32.  $y'' - y' - 12y = 0$

$$m^2 - m - 12 = (m-4)(m+3) = 0 \Rightarrow m = 4, -3$$

Because  $y_h = C_1 e^{4x} + C_2 e^{-3x}$ , let

$$y_p = Axe^{4x}.$$

33. Answers will vary. See the “Variation of Parameters” box on page 1163.

34. (a) Because  $y_p'' = 0$  and  $3(y_p) = 3(4) = 12$ .

(b)  $y_p = 2$

(c)  $y_p = 4$

35.  $q'' + 10q' + 25q = 6 \sin 5t, q(0) = 0, q'(0) = 0$

$m^2 + 10m + 25 = 0$  when  $m = -5, -5$ .

$$q_h = (C_1 + C_2 t)e^{-5t}$$

$$q_p = A \cos 5t + B \sin 5t$$

$$q'_p = -5A \sin 5t + 5B \cos 5t$$

$$q''_p = -25A \cos 5t - 25B \sin 5t$$

$$\begin{aligned} q''_p + 10q'_p + 25q_p &= 50B \cos 5t - 50A \sin 5t \\ &= 6 \sin 5t, A = -\frac{3}{25}, B = 0 \end{aligned}$$

$$q = (C_1 + C_2 t)e^{-5t} - \frac{3}{25} \cos 5t$$

Initial conditions:

$$q(0) = 0, q'(0) = 0, C_1 - \frac{3}{25} = 0, -5C_1 + C_2 = 0,$$

$$C_1 = \frac{3}{25}, C_2 = \frac{3}{5}$$

$$\text{Particular solution: } q = \frac{3}{25}(e^{-5t} + 5te^{-5t} - \cos 5t)$$

36.  $q'' + 20q' + 50q = 10 \sin 5t$

$m^2 + 20m + 50 = 0$  when  $m = -10 \pm 5\sqrt{2}$ .

$$q_h = C_1 e^{(-10+5\sqrt{2})t} + C_2 e^{(-10-5\sqrt{2})t}$$

$$q_p = A \cos 5t + B \sin 5t$$

$$q'_p = 5B \cos 5t - 5A \sin 5t$$

$$q''_p = -25A \cos 5t - 25B \sin 5t$$

$$\begin{aligned} q''_p + 20q'_p + 50q_p &= (25A + 100B) \cos 5t \\ &\quad + (25B - 100A) \sin 5t = 10 \sin 5t \end{aligned}$$

$$\left. \begin{array}{l} 25A + 100B = 0 \\ 25B - 100A = 10 \end{array} \right\} \begin{array}{l} B = \frac{2}{85}, A = -\frac{8}{85} \\ \end{array}$$

$$q = C_1 e^{(-10+5\sqrt{2})t} + C_2 e^{(-10-5\sqrt{2})t} - \frac{8}{85} \cos 5t + \frac{2}{85} \sin 5t$$

Initial conditions:

$$q(0) = 0, q'(0) = 0, C_1 + C_2 - \frac{8}{85} = 0,$$

$$(-10 + 5\sqrt{2})C_1 + (-10 - 5\sqrt{2})C_2 + \frac{2}{17} = 0,$$

$$C_1 = \frac{8 + 7\sqrt{2}}{170}, C_2 = \frac{8 - 7\sqrt{2}}{170}$$

Particular solution:

$$\begin{aligned} q &= \frac{8 + 7\sqrt{2}}{170} e^{(-10+5\sqrt{2})t} + \frac{8 - 7\sqrt{2}}{170} e^{(-10-5\sqrt{2})t} \\ &\quad - \frac{8}{85} \cos 5t + \frac{2}{85} \sin 5t \end{aligned}$$

37.  $\frac{24}{32}y'' + 48y = \frac{24}{32}(48 \sin 4t), y(0) = \frac{1}{4}, y'(0) = 0$

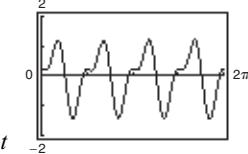
$$\frac{24}{32}m^2 + 48 = 0 \text{ when } m = \pm 8i.$$

$$y_h = C_1 \cos 8t + C_2 \sin 8t$$

$$y_p = A \sin 4t + B \cos 4t$$

$$y'_p = 4A \cos 4t - 4B \sin 4t$$

$$y''_p = -16A \sin 4t - 16B \cos 4t$$



$$\frac{24}{32}y''_p + 48y_p = 36A \sin 4t + 36B \cos 4t$$

$$= \frac{24}{32}(48 \sin 4t), B = 0, A = 1$$

$$y = y_h + y_p = C_1 \cos 8t + C_2 \sin 8t + \sin 4t$$

$$\text{Initial conditions: } y(0) = \frac{1}{4}, y'(0) = 0, \frac{1}{4} = C_1,$$

$$0 = 8C_2 + 4 \Rightarrow C_2 = -\frac{1}{2}$$

$$\text{Particular solution: } y = \frac{1}{4} \cos 8t - \frac{1}{2} \sin 8t + \sin 4t$$

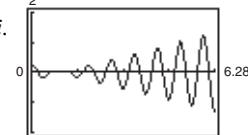
38.  $\frac{2}{32}y'' + 4y = \frac{2}{32}(4 \sin 8t), y(0) = \frac{1}{4}, y'(0) = 0$

$$\frac{2}{32}m^2 + 4 = 0 \text{ when } m = \pm 8i.$$

$$y_h = C_1 \cos 8t + C_2 \sin 8t$$

$$y_p = At \sin 8t + Bt \cos 8t$$

$$y''_p = (-64At - 16B) \sin 8t + (16A - 64Bt) \cos 8t$$



$$\frac{2}{32}y''_p + 4y_p = -B \sin 8t + A \cos 8t$$

$$= \frac{2}{32}(4 \sin 8t), A = 0, B = -\frac{1}{4}$$

$$y = C_1 \cos 8t + C_2 \sin 8t - \frac{1}{4}t \cos 8t$$

Initial conditions:

$$y(0) = \frac{1}{4}, y'(0) = 0, \frac{1}{4} = C_1, 0 = 8C_2 - \frac{1}{4} \Rightarrow C_2 = \frac{1}{32}$$

$$\text{Particular solution: } y = \frac{1}{4} \cos 8t + \frac{1}{32} \sin 8t - \frac{1}{4}t \cos 8t$$

39.  $\frac{2}{16}y'' + y' + 4y = \frac{2}{16}(4 \sin 8t), y(0) = \frac{1}{4}, y'(0) = -3$

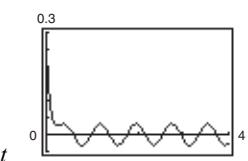
$$\frac{1}{16}m^2 + m + 4 = 0 \text{ when } m = -8, -8.$$

$$y_h = (C_1 + C_2 t)e^{-8t}$$

$$y_p = A \sin 8t + B \cos 8t$$

$$y'_p = 8A \cos 8t - 8B \sin 8t$$

$$y''_p = -64A \sin 8t - 64B \cos 8t$$



$$\frac{2}{16}y''_p + y'_p + 4y_p = -8B \sin 8t + 8A \cos 8t$$

$$= \frac{2}{16}(4 \sin 8t) - 8B$$

$$= \frac{1}{4} \Rightarrow B = -\frac{1}{32}, 8A = 0 \Rightarrow A = 0$$

Initial conditions:

$$y(0) = \frac{1}{4}, y'(0) = -3, \frac{1}{4} = C_1 - \frac{1}{32} \Rightarrow C_1 = \frac{9}{32},$$

$$-3 = -8C_1 + C_2 \Rightarrow C_2 = -\frac{3}{4}$$

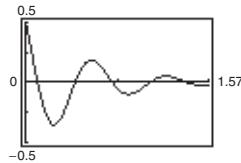
$$\text{Particular solution: } y = \left(\frac{9}{32} - \frac{3}{4}t\right)e^{-8t} - \frac{1}{32} \cos 8t$$

40.  $\frac{4}{32}y'' + \frac{1}{2}y' + \frac{25}{2}y = 0, y(0) = \frac{1}{2}, y'(0) = -4$

$$\frac{1}{8}m^2 + \frac{1}{2}m + \frac{25}{2} = 0$$

$$m^2 + 4m + 100 = 0 \text{ when } m = -2 \pm 4\sqrt{6}i$$

$$y = C_1 e^{-2t} \cos(4\sqrt{6}t) + C_2 e^{-2t} \sin(4\sqrt{6}t)$$



Initial conditions:

$$y(0) = \frac{1}{2}, y'(0) = -4, \frac{1}{2} = C_1, -4 = -2C_1 + 4\sqrt{6}C_2,$$

$$C_2 = -\frac{3}{4\sqrt{6}} = -\frac{\sqrt{6}}{8}$$

$$\text{Particular solution: } y = \frac{1}{2}e^{-2t} \cos(4\sqrt{6}t) - \frac{\sqrt{6}}{8}e^{-2t} \sin(4\sqrt{6}t)$$

41. In Exercise 37,

$$y_h = \frac{1}{4} \cos 8t - \frac{1}{2} \sin 8t - \frac{\sqrt{5}}{4} \sin \left[ 8t + \pi + \arctan \left( -\frac{1}{2} \right) \right] = \frac{\sqrt{5}}{4} \sin \left( 8t + \pi - \arctan \frac{1}{2} \right) \approx \frac{\sqrt{5}}{4} \sin(8t + 2.6779).$$

42. (a)  $\frac{4}{32}y'' + \frac{25}{2}y = 0$

$$y = C_1 \cos 10x + C_2 \sin 10x$$

$$y(0) = \frac{1}{2} : \frac{1}{2} = C_1$$

$$y'(0) = -4 : -4 = 10C_2 \Rightarrow C_2 = -\frac{2}{5}$$

$$y = \frac{1}{2} \cos 10x - \frac{2}{5} \sin 10x$$

The motion is undamped.

(b) If  $b > 0$ , the motion is damped.

(c) If  $b > \frac{5}{2}$ , the solution to the differential equation is

of the form  $y = C_1 e^{m_1 x} + C_2 e^{m_2 x}$ . There would be no oscillations in this case.

43.  $x^2 y'' - xy' + y = 4x \ln x$

$$y_1 = x \text{ and } y_2 = x \ln x$$

$$u'_1 x + u'_2 x \ln x = 0 \Rightarrow u'_1 = -u'_2 \ln x$$

$$u'_1 + u'_2 (1 + \ln x) = \frac{4}{x} \ln x \Rightarrow u'_2 = \frac{4}{x} \ln x$$

$$\text{and } u'_1 = -\frac{4}{x} (\ln x)^2$$

$$u_1 = -\frac{4}{3} (\ln x)^3 \text{ and } u_2 = 2(\ln x)^2$$

$$y_p = -\frac{4}{3} x (\ln x)^3 + 2x (\ln x)^2 = \frac{2}{3} x (\ln x)^3$$

$$y = y_h + y_p = C_1 x + C_2 x \ln x + \frac{2}{3} x (\ln x)^3$$

44. Let  $y_p = A \sin(\ln x) + B \cos(\ln x)$ .

$$y'_p = A \cos(\ln x) \frac{1}{x} - B \sin(\ln x) \frac{1}{x} = \frac{1}{x} (A \cos(\ln x) - B \sin(\ln x))$$

$$y''_p = \frac{-1}{x^2} (A \cos(\ln x) - B \sin(\ln x)) + \frac{1}{x} \left( -A \sin(\ln x) \frac{1}{x} - B \cos(\ln x) \frac{1}{x} \right) = \frac{1}{x^2} (B - A) \sin(\ln x) + \frac{1}{x^2} (-A - B) \cos(\ln x)$$

$$x^2 y''_p + xy' + 4y = (B - A) \sin(\ln x) - (A + B) \cos(\ln x) + (A \cos(\ln x) - B \sin(\ln x))$$

$$+ 4(A \sin(\ln x) + B \cos(\ln x)) = \sin(\ln x)$$

$$(B - A - B + 4A) \sin(\ln x) + (-A - B + A + 4B) \cos(\ln x) = \sin(\ln x)$$

$$3A = 1, 3B = 0 \Rightarrow A = \frac{1}{3}$$

$$\text{So, } y_p = \frac{1}{3} \sin(\ln x) \text{ and } y = y_h + y_p = C_1 \sin(\ln x^2) + C_2 \cos(\ln x^2) + \frac{1}{3} \sin(\ln x).$$

**45.** True.  $y_p = -e^{2x} \cos e^{-x}$

$$\begin{aligned}y'_p &= e^{2x} \sin e^{-x}(-e^{-x}) - 2e^{2x} \cos e^{-x} = -e^x \sin e^{-x} - 2e^{2x} \cos e^{-x} \\y''_p &= \left[ -e^x \cos e^{-x}(-e^{-x}) - e^x \sin e^{-x} \right] + \left[ 2e^{2x} \sin e^{-x}(-e^{-x}) - 4e^{2x} \cos e^{-x} \right]\end{aligned}$$

So,

$$\begin{aligned}y''_p - 3y'_p + 2y_p &= \left[ \cos e^{-x} - e^x \sin e^{-x} - 2e^x \sin e^{-x} - 4e^{2x} \cos e^{-x} \right] - 3\left[ -e^x \sin e^{-x} - 2e^{2x} \cos e^{-x} \right] - 2e^{2x} \cos e^{-x} \\&= \left[ -e^x - 2e^x + 3e^x \right] \sin e^{-x} + \left[ 1 - 4e^{2x} + 6e^{2x} - 2e^{2x} \right] \cos e^{-x} = \cos e^{-x}.\end{aligned}$$

**46.** True.

$$\begin{aligned}y_p &= -\frac{1}{8}e^{2x}, y'_p = -\frac{1}{4}e^{2x}, y''_p = -\frac{1}{2}e^{2x} \\y''_p - 6y'_p &= -\frac{1}{2}e^{2x} - 6\left(-\frac{1}{4}e^{2x}\right) = e^{2x}\end{aligned}$$

**47.**  $y'' - 2y' + y = 2e^x$

$$m^2 - 2m + 1 = 0 \Rightarrow m = 1, 1$$

$$y_h = C_1 e^x + C_2 x e^x, y_p = x^2 e^x, \text{ particular solution}$$

$$\text{General solution: } f(x) = (C_1 + C_2 x)e^x + x^2 e^x = (C_1 + C_2 x + x^2)e^x$$

$$f'(x) = (C_2 + 2x + C_1 + C_2 x + x^2)e^x = (x^2 + (C_2 + 2)x + (C_1 + C_2))e^x$$

(a) No. If  $f(x) > 0$  for all  $x$ , then  $x^2 + C_2 x + C_1 > 0 \Leftrightarrow C_2^2 - 4C_1 < 0$  for all  $x$ .

So, let  $C_1 = C_2 = 1$ . Then  $f'(x) = (x^2 + 3x + 2)e^x$  and  $f'\left(-\frac{3}{2}\right) = -\frac{1}{4} < 0$ .

(b) Yes. If  $f'(x) > 0$  for all  $x$ , then

$$(C_2 + 2)^2 - 4(C_1 + C_2) < 0$$

$$\Rightarrow C_2^2 - 4C_1 + 4 < 0$$

$$C_2^2 - 4C_1 < -4$$

$$C_2^2 - 4C_1 < 0$$

$$\Rightarrow f(x) > 0 \text{ for all } x.$$

## Section 16.4 Series Solutions of Differential Equations

**1.**  $y' - y = 0$ . Letting  $y = \sum_{n=0}^{\infty} a_n x^n$ :

$$y' - y = \sum_{n=1}^{\infty} n a_n x^{n-1} - \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n - \sum_{n=0}^{\infty} a_n x^n = 0$$

$$(n+1)a_{n+1} = a_n$$

$$a_{n+1} = \frac{a_n}{n+1}$$

$$a_1 = a_0, a_2 = \frac{a_1}{2} = \frac{a_0}{2}, a_3 = \frac{a_2}{3} = \frac{a_0}{1 \cdot 2 \cdot 3}, \dots, a_n = \frac{a_0}{n!}$$

$$y = \sum_{n=0}^{\infty} \frac{a_0}{n!} x^n = a_0 e^x$$

Check: By separation of variables, you have:

$$\int \frac{dy}{y} = \int dx$$

$$\ln y = x + C_1$$

$$y = Ce^x$$

2.  $y' - ky = 0$ . Letting  $y = \sum_{n=0}^{\infty} a_n x^n$ :

$$y' - ky = \sum_{n=1}^{\infty} n a_n x^{n-1} - k \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n - \sum_{n=0}^{\infty} k a_n x^n = 0$$

$$(n+1)a_{n+1} = ka_n$$

$$a_{n+1} = \frac{ka_n}{n+1}$$

$$a_1 = ka_0, a_2 = \frac{ka_1}{2} = \frac{k^2 a_0}{2}, a_3 = \frac{ka_2}{3} = \frac{k^3 a_0}{1 \cdot 2 \cdot 3}, \dots, a_n = \frac{k^n}{n!} a_0$$

$$y = \sum_{n=0}^{\infty} \frac{k^n}{n!} a_0 x^n = a_0 \sum_{n=0}^{\infty} \frac{(kx)^n}{n!} = a_0 e^{kx}$$

Check: By separation of variables, you have:

$$\int \frac{dy}{y} = \int k dx$$

$$\ln y = kx + C_1$$

$$y = Ce^{kx}$$

3.  $y'' - 9y = 0$ . Letting  $y = \sum_{n=0}^{\infty} a_n x^n$ :

$$y'' - 9y = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - 9 \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=0}^{\infty} 9 a_n x^n = 0$$

$$(n+2)(n+1)a_{n+2} = 9a_n$$

$$a_{n+2} = \frac{9a_n}{(n+2)(n+1)}$$

$$a_0 = a_0 \quad a_1 = a_1$$

$$a_2 = \frac{9a_0}{2} \quad a_3 = \frac{9a_1}{3 \cdot 2}$$

$$a_4 = \frac{9a_2}{4 \cdot 3} = \frac{9^2 a_0}{4 \cdot 3 \cdot 2 \cdot 1} \quad a_5 = \frac{9a_3}{5 \cdot 4} = \frac{9^2 a_1}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}$$

$$\vdots$$

$$a_{2n} = \frac{9^n a_0}{(2n)!} \quad a_{2n+1} = \frac{9^n a_1}{(2n+1)!}$$

$$y = \sum_{n=0}^{\infty} \frac{9^n a_0}{(2n)!} x^{2n} + \sum_{n=0}^{\infty} \frac{9^n a_1}{(2n+1)!} x^{2n+1} = a_0 \sum_{n=0}^{\infty} \frac{(3x)^{2n}}{(2n)!} + \frac{a_1}{3} \sum_{n=0}^{\infty} \frac{(3x)^{2n+1}}{(2n+1)!} = C_0 \sum_{n=0}^{\infty} \frac{(3x)^n}{n!} + C_1 \sum_{n=0}^{\infty} \frac{(-3x)^n}{n!}$$

$$= C_0 e^{3x} + C_1 e^{-3x} \text{ where } C_0 + C_1 = a_0 \text{ and } C_0 - C_1 = \frac{a_1}{3}.$$

Check:  $y'' - 9y = 0$  is a second-order homogeneous linear equation.

$$m^2 - 9 = 0 \Rightarrow m_1 = 3 \text{ and } m_2 = -3$$

$$y = C_1 e^{3x} + C_2 e^{-3x}$$

4.  $y = C_0 e^{kx} + C_1 e^{-kx}$ . Follow the solution to Exercise 3 with 9 replaced by  $k^2$ .

5.  $y'' + 4y = 0$ . Letting  $y = \sum_{n=0}^{\infty} a_n x^n$ :

$$\begin{aligned} y'' + 4y &= \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + 4\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} x^n + \sum_{n=0}^{\infty} 4a_n x^n = 0 \\ (n+2)(n+1)a_{n+2} &= -4a_n \\ a_{n+2} &= \frac{-4a_n}{(n+2)(n+1)} \\ a_0 &= a_0 & a_1 &= a_1 \\ a_2 &= \frac{-4a_0}{2} & a_3 &= \frac{-4a_1}{3 \cdot 2} \\ a_4 &= \frac{-4a_2}{4 \cdot 3} = \frac{(-4)^2 a_0}{4!} & a_5 &= \frac{-4a_3}{5 \cdot 4} = \frac{(-4)^2 a_1}{5!} \\ \vdots & & \vdots & \\ a_{2n} &= \frac{(-1)^n 4^n}{(2n)!} a_0 & a_{2n+1} &= \frac{(-1)^n 4^n}{(2n+1)!} a_1 \\ y &= \sum_{n=0}^{\infty} \frac{(-1)^n 4^n a_0}{(2n)!} x^{2n} + \sum_{n=0}^{\infty} \frac{(-1)^n 4^n a_1}{(2n+1)!} x^{2n+1} = a_0 \sum_{n=0}^{\infty} \frac{(-1)^n (2x)^{2n}}{(2n)!} + \frac{a_1}{2} \sum_{n=0}^{\infty} \frac{(-1)^n (2x)^{2n+1}}{(2n+1)!} = C_0 \cos 2x + C_1 \sin 2x \end{aligned}$$

Check:  $y'' + 4y = 0$  is a second-order homogeneous linear equation.

$$m^2 + 4 = 0 \Rightarrow m = \pm 2i$$

$$y = C_1 \cos 2x + C_2 \sin 2x$$

6.  $y = C_0 \cos kx + C_1 \sin kx$ . Follow the solution to Exercise 5 with 4 replaced by  $k^2$ .

7.  $y' + 3xy = 0$ . Letting  $y = \sum_{n=0}^{\infty} a_n x^n$ :

$$\begin{aligned} y' + 3xy &= \sum_{n=1}^{\infty} n a_n x^{n-1} + \sum_{n=0}^{\infty} 3 a_n x^{n+1} = 0 \\ \sum_{n=-1}^{\infty} (n+2) a_{n+2} x^{n+1} &= \sum_{n=0}^{\infty} -3 a_n x^{n+1} \Rightarrow a_1 = 0 \text{ and } a_{n+2} = \frac{-3 a_n}{n+2} \\ a_0 &= a_0 & a_1 &= 0 \\ a_2 &= -\frac{3a_0}{2} & a_3 &= -\frac{3a_1}{3} = 0 \\ a_4 &= -\frac{3}{4} \left( -\frac{3a_0}{2} \right) = \frac{3^2}{2^3} a_0 & a_5 &= -\frac{3}{5} \left( -\frac{3a_1}{3} \right) = 0 \\ a_6 &= -\frac{3}{6} \left( -\frac{3^2}{2^3} a_0 \right) = -\frac{3^3 a_0}{2^3 (3 \cdot 2)} & a_7 &= -\frac{3}{7} \left( \frac{3^2 a_1}{3 \cdot 5} \right) = 0 \\ a_8 &= -\frac{3}{8} \left( -\frac{3^3 a_0}{2^3 (3 \cdot 2)} \right) = \frac{3^4 a_0}{2^4 (4 \cdot 3 \cdot 2)} & a_9 &= -\frac{3}{9} \left( -\frac{3^3 a_1}{3 \cdot 5 \cdot 7} \right) = 0 \\ y &= a_0 \sum_{n=0}^{\infty} \frac{(-3)^n x^{2n}}{2^n n!} \end{aligned}$$

$$\lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(-3)^{n+1} x^{2n+2}}{2^{n+1} (n+1)!} \cdot \frac{2^n n!}{(-3)^n x^{2n}} \right| = \lim_{n \rightarrow \infty} \frac{3x^2}{2(n+1)} = 0$$

The interval of convergence for the solution is  $(-\infty, \infty)$ .

8.  $y' - 2xy = 0$ . Letting  $y = \sum_{n=0}^{\infty} a_n x^n$ :

$$y' - 2xy = \sum_{n=1}^{\infty} n a_n x^{n-1} - \sum_{n=0}^{\infty} 2 a_n x^{n+1} = 0$$

$$\sum_{n=-1}^{\infty} (n+2) a_{n+2} x^{n+1} = \sum_{n=0}^{\infty} 2 a_n x^{n+1} \Rightarrow a_1 = 0 \text{ and}$$

$$a_{n+2} = \frac{2a_n}{n+2}$$

$$a_0 = 0 \quad a_1 = 0$$

$$a_2 = \frac{2a_0}{2} = a_0 \quad a_3 = \frac{2a_1}{3} = 0$$

$$a_4 = \frac{2}{4} \left( \frac{2a_0}{2} \right) = \frac{2^2 a_0}{2^2 \cdot 2} = \frac{a_0}{2} \quad a_5 = \frac{2}{5} \left( \frac{2a_1}{3} \right) = 0$$

$$a_6 = \frac{2}{6} \left( \frac{2^2 a_0}{2^2 \cdot 2} \right) = \frac{2^3 a_0}{2^3 \cdot 3 \cdot 2} = \frac{a_0}{3!} \quad a_7 = \frac{2}{7} \left( \frac{2^2 a_1}{3 \cdot 5} \right) = 0$$

$$a_8 = \frac{2}{8} \left( \frac{a_0}{3!} \right) = \frac{a_0}{4!} \quad a_9 = \frac{2}{9} \left( \frac{2^3 a_1}{3 \cdot 5 \cdot 7} \right) = 0$$

$$y = a_0 \sum_{n=0}^{\infty} \frac{x^{2n}}{n!}$$

$$\lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{x^{2n+2}}{(n+1)!} \cdot \frac{n!}{2^n} \right| = \lim_{n \rightarrow \infty} \frac{x^2}{n+1} = 0$$

The interval of convergence for the solution is  $(-\infty, \infty)$ .

9.  $y'' - xy' = 0$ . Letting  $y = \sum_{n=0}^{\infty} a_n x^n$ :

$$y'' - xy' = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - x \sum_{n=1}^{\infty} n a_n x^{n-1} = 0$$

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = \sum_{n=0}^{\infty} n a_n x^n$$

$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} x^n = \sum_{n=0}^{\infty} n a_n x^n$$

$$a_{n+2} = \frac{n a_n}{(n+2)(n+1)}$$

$$a_0 = a_0 \quad a_1 = a_1$$

$$a_2 = 0 \quad a_3 = \frac{a_1}{3 \cdot 2}$$

$$\text{There are no even powered terms. } a_5 = \frac{3a_3}{5 \cdot 4} = \frac{3a_1}{5!}$$

$$a_7 = \frac{5a_5}{7 \cdot 6} = \frac{5 \cdot 3a_1}{7!}$$

$$y = a_0 + a_1 \sum_{n=0}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)x^{2n+1}}{(2n+1)!} = a_0 + a_1 \sum_{n=0}^{\infty} \frac{(2n)!x^{2n+1}}{2^n n!(2n+1)!} = a_0 + a_1 \sum_{n=0}^{\infty} \frac{x^{2n+1}}{2^n n!(2n+1)}$$

$$\lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{x^{2n+3}}{2^{n+1}(n+1)!(2n+3)} \cdot \frac{2^n n!(2n+1)}{x^{2n+1}} \right| = \lim_{n \rightarrow \infty} \frac{(2n+1)x^2}{2(n+1)(2n+3)} = 0$$

Interval of convergence:  $(-\infty, \infty)$

**10.**  $y'' - xy' - y = 0$ . Letting  $y = \sum_{n=0}^{\infty} a_n x^n$ :

$$\begin{aligned} y'' - xy' - y &= \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - x \sum_{n=1}^{\infty} na_n x^{n-1} - \sum_{n=0}^{\infty} a_n x^n = 0 \\ \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n &= \sum_{n=0}^{\infty} (n+1)a_n x^n \end{aligned}$$

$$a_{n+2} = \frac{a_n}{n+2}$$

$$a_0 = a_0 \quad a_1 = a_1$$

$$a_2 = \frac{a_0}{2} \quad a_3 = \frac{a_1}{3}$$

$$a_4 = \frac{a_2}{4} = \frac{a_0}{8} = \frac{a_0}{2^2 2!} \quad a_5 = \frac{a_3}{5} = \frac{a_1}{3 \cdot 5}$$

$$a_6 = \frac{a_4}{6} = \frac{a_0}{2^3 3!} \quad a_7 = \frac{a_5}{7} = \frac{a_1}{3 \cdot 5 \cdot 7}$$

$$a_8 = \frac{a_6}{8} = \frac{a_0}{2^4 4!} \quad a_9 = \frac{a_7}{9} = \frac{a_1}{3 \cdot 5 \cdot 7 \cdot 9}$$

$$y = a_0 \sum_{n=0}^{\infty} \frac{x^{2n}}{2^n n!} + a_1 \sum_{n=0}^{\infty} \frac{x^{2n+1}}{1 \cdot 3 \cdot 5 \cdot 7 \cdots (2n+1)}$$

$$\lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{x^{2n+2}}{2^{n+1}(n+1)!} \cdot \frac{2^n n!}{x^{2n}} \right| = \lim_{n \rightarrow \infty} \frac{x^2}{2(n+1)} = 0$$

$$\lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{x^{2n+3}}{1 \cdot 3 \cdot 5 \cdot 7 \cdots (2n+3)} \cdot \frac{1 \cdot 3 \cdot 5 \cdot 7 \cdots (2n+1)}{x^{2n+1}} \right| = \lim_{n \rightarrow \infty} \frac{x^2}{2n+3} = 0$$

Because the interval of convergence for each series is  $(-\infty, \infty)$ , the interval of convergence for the solution is  $(-\infty, \infty)$ .

**11.**  $(x^2 + 4)y'' + y = 0$ . Letting  $y = \sum_{n=0}^{\infty} a_n x^n$ :

$$\begin{aligned} (x^2 + 4)y'' + y &= \sum_{n=2}^{\infty} n(n-1)a_n x^n + 4 \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=0}^{\infty} a_n x^n \\ &= \sum_{n=0}^{\infty} (n^2 - n + 1)a_n x^n + \sum_{n=0}^{\infty} 4(n+2)(n+1)a_{n+2} x^n = 0 \\ a_{n+2} &= \frac{-(n^2 - n + 1)a_n}{4(n+2)(n+1)} \end{aligned}$$

$$a_0 = a_0 \quad a_1 = a_1$$

$$a_2 = \frac{-a_0}{4(2)(1)} = \frac{-a_0}{8} \quad a_3 = \frac{-a_1}{4(3)(2)} = \frac{-a_1}{24}$$

$$a_4 = \frac{-3a_2}{4(4)(3)} = \frac{a_0}{128} \quad a_5 = \frac{-7a_3}{4(5)(4)} = \frac{7a_1}{1920}$$

$$y = a_0 \left( 1 - \frac{x^2}{8} + \frac{x^4}{128} - \dots \right) + a_1 \left( x - \frac{x^3}{24} + \frac{7x^5}{1920} - \dots \right)$$

12.  $y'' + x^2y = 0$ . Letting  $y = \sum_{n=0}^{\infty} a_n x^n$ :

$$\begin{aligned} y'' + x^2y &= \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=0}^{\infty} a_n x^{n+2} = 0 \\ \sum_{n=-2}^{\infty} (n+4)(n+3)a_{n+4}x^{n+2} &= -\sum_{n=0}^{\infty} a_n x^{n+2} \\ a_{n+4} &= \frac{-a_n}{(n+4)(n+3)} \end{aligned}$$

Also:

$$\begin{aligned} y &= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_n x^n + \cdots \\ y'' &= 2a_2 + 3 \cdot 2a_3 x + \cdots + n(n-1)a_n x^{n-2} + \cdots \\ y'' + x^2y &= 2a_2 + 3 \cdot 2a_3 x + (a_0 + 4 \cdot 3a_4)x^2 + (a_1 + 5 \cdot 4a_5)x^3 + \cdots = 0 \\ 2a_2 &= 0, 6a_3 = 0, 12a_4 + a_0 = 0, 20a_5 + a_1 = 0 \end{aligned}$$

So,  $a_2 = 0$  and  $a_3 = 0 \Rightarrow a_6 = 0, a_7 = 0, a_{10} = 0$ , and  $a_{11} = 0$ . Therefore,  $a_{4n+2} = 0$  and  $a_{4n+3} = 0$ .

$$\begin{aligned} a_0 &= a_0 & a_1 &= a_1 \\ a_4 &= -\frac{a_0}{4 \cdot 3} & a_5 &= -\frac{a_1}{5 \cdot 4} \\ a_8 &= -\frac{a_4}{8 \cdot 7} = \frac{a_0}{8 \cdot 7 \cdot 4 \cdot 3} & a_9 &= -\frac{a_5}{9 \cdot 8} = \frac{a_1}{9 \cdot 8 \cdot 5 \cdot 4} \\ a_{12} &= -\frac{a_8}{12 \cdot 11} = -\frac{a_0}{12 \cdot 11 \cdot 8 \cdot 7 \cdot 4 \cdot 3} & a_{13} &= -\frac{a_9}{13 \cdot 12} = -\frac{a_1}{13 \cdot 12 \cdot 9 \cdot 8 \cdot 5 \cdot 4} \\ y'' + x^2y &= a_0 \left( 1 - \frac{x^4}{4 \cdot 3} + \frac{x^8}{8 \cdot 7 \cdot 4 \cdot 3} - \frac{x^{12}}{12 \cdot 11 \cdot 8 \cdot 7 \cdot 4 \cdot 3} + \cdots \right) \\ &\quad + a_1 \left( x - \frac{x^5}{5 \cdot 4} + \frac{x^7}{9 \cdot 8 \cdot 5 \cdot 4} - \frac{x^9}{13 \cdot 12 \cdot 9 \cdot 8 \cdot 5 \cdot 4} + \cdots \right) \end{aligned}$$

13.  $y' + (2x - 1)y = 0, y(0) = 2$

$$\begin{aligned} y' &= (1 - 2x)y & y'(0) &= 0 \\ y'' &= (1 - 2x)y' - 2y & y''(0) &= -2 \\ y''' &= (1 - 2x)y'' - 4y' & y'''(0) &= -10 \\ y^4 &= (1 - 2x)y''' - 6y'' & y^{(4)}(0) &= 2 \\ &\vdots & &\vdots \\ y(x) &= 2 + \frac{2}{1!}x - \frac{2}{2!}x^2 - \frac{10}{3!}x^3 + \frac{2}{4!}x^4 + \cdots \end{aligned}$$

Using the first five terms of the series,  $y\left(\frac{1}{2}\right) = \frac{163}{64} \approx 2.547$ .

Using Euler's Method with  $\Delta x = 0.1$  you have  $y' = (1 - 2x)y$ .

$i$	$x_i$	$y_i$
0	0	2
1	0.1	2.2
2	0.2	2.376
3	0.3	2.51856
4	0.4	2.61930
5	0.5	2.67169

14.  $y' - 2xy = 0, y(0) = 1$

$$\begin{aligned}y' &= 2xy & y'(0) &= 0 \\y'' &= 2(xy' + y) & y''(0) &= 2 \\y''' &= 2(xy'' + 2y') & y'''(0) &= 0 \\y^{(4)} &= 2(xy''' + 3y'') & y^{(4)}(0) &= 12 \\y^{(5)} &= 2(xy^{(4)} + 4y''') & y^{(5)}(0) &= 0 \\y^{(6)} &= 2(xy^{(5)} + 5y^{(4)}) & y^{(6)}(0) &= 120 \\\vdots & & \vdots & \\y(x) &= 1 + \frac{2}{2!}x^2 + \frac{12}{4!}x^4 + \frac{120}{6!}x^6 + \dots \\&= 1 + x^2 + \frac{1}{2}x^4 + \frac{1}{6}x^6 + \dots\end{aligned}$$

Using the first four terms of the series,

$$y(1) = \frac{8}{3} \approx 2.667.$$

Using Euler's Method with  $\Delta x = 0.1$  you have  $y' = 2xy$ .

$i$	$x_i$	$y_i$
0	0	1
1	0.1	1
2	0.2	1.02
3	0.3	1.0608
4	0.4	1.1244
5	0.5	1.2144
6	0.6	1.3358
7	0.7	1.4961
8	0.8	1.7056
9	0.9	1.9785
10	1.0	2.3346

So,  $y(1) \approx 2.335$ .

18. (a)  $m^2 + 9 = 0 \Rightarrow m = \pm 3i$

$$y = C_1 \cos 3x + C_2 \sin 3x$$

$$y(0) = 2 = C_1$$

$$y' = -3C_1 \sin 3x + 3C_2 \cos 3x$$

$$y'(0) = 6 = 3C_2 \Rightarrow C_2 = 2$$

$$y_p = 2 \cos 3x + 2 \sin 3x$$

15. Given a differential equation, assume that the solution

is of the form  $y = \sum_{n=0}^{\infty} a_n x^n$ . Then substitute  $y$  and its

derivatives into the differential equation. You should then be able to determine the coefficients  $a_0, a_1, \dots$

16. A recursion formula is a formula for determining the next term of a sequence from one or more of the preceding terms. See Example 1.

17. (a) From Exercise 9, the general solution is

$$y = a_0 + a_1 \sum_{n=0}^{\infty} \frac{x^{2n+1}}{2^n n!(2n+1)}.$$

$$y(0) = 0 \Rightarrow a_0 = 0$$

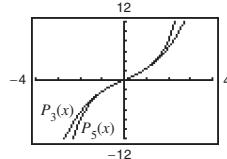
$$y' = a_1 \sum_{n=0}^{\infty} \frac{(2n+1)x^{2n}}{2^n n!(2n+1)} = a_1 \sum_{n=0}^{\infty} \frac{x^{2n}}{2^n n!}$$

$$y'(0) = 2 = a_1$$

$$y = 2 \sum_{n=0}^{\infty} \frac{x^{2n+1}}{2^n n!(2n+1)}$$

$$(b) P_3(x) = 2 \left[ x + \frac{x^3}{2 \cdot 3} \right] = 2x + \frac{x^3}{3}$$

$$P_5(x) = 2x + \frac{x^3}{3} + 2 \frac{x^5}{4 \cdot 2 \cdot 5} = 2x + \frac{x^3}{3} + \frac{x^5}{20}$$



(c) The solution is symmetric about the origin.

(b) Let  $y = \sum_{n=0}^{\infty} a_n x^n$ ,  $y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$ ,  $y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$

$$y'' + 9y = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + 9 \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + 9 \sum_{n=0}^{\infty} a_n x^n = 0$$

$$(n+2)(n+1) a_{n+2} = -9 a_n$$

$$a_{n+2} = \frac{-9}{(n+2)(n+1)} a_n$$

For  $n$  even,

$$a_2 = \frac{-9}{2} a_0$$

$$a_4 = \frac{-9}{4 \cdot 3} a_2 = \frac{(-9)^2}{4!} a_0$$

$$a_6 = \frac{(-9)^3}{6!} a_0$$

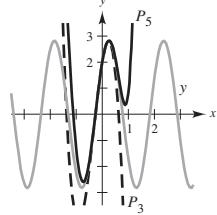
and in general,  $a_{2n} = \frac{(-9)^n}{(2n)!} a_0$

So,

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_{2n} x^{2n} + \sum_{n=0}^{\infty} a_{2n+1} x^{2n+1} \\ &= \sum_{n=0}^{\infty} \frac{(-9)^n}{(2n)!} a_0 x^{2n} + \sum_{n=0}^{\infty} \frac{(-9)^n}{(2n+1)!} a_1 x^{2n+1} = a_0 \sum_{n=0}^{\infty} \frac{(-1)^n (3x)^{2n}}{(2n)!} + a_1 \sum_{n=0}^{\infty} \frac{(-1)^n (3x)^{2n+1}}{(2n+1)!} \end{aligned}$$

Applying the initial conditions,  $a_0 = a_1 = 2$ , and  $y = 2 \left[ \sum_{n=0}^{\infty} \frac{(-1)^n (3x)^{2n}}{(2n)!} + \sum_{n=0}^{\infty} \frac{(-1)^n (3x)^{2n+1}}{(2n+1)!} \right]$ .

(c)



19.  $y'' - 2xy = 0$ ,  $y(0) = 1$ ,  $y'(0) = -3$

$$y'' = 2xy \quad y''(0) = 0$$

$$y''' = 2(xy' + y) \quad y'''(0) = 2$$

$$y^{(4)} = 2(xy'' + 2y') \quad y^{(4)}(0) = -12$$

$$y^{(5)} = 2(xy''' + 3y'') \quad y^{(5)}(0) = 0$$

$$y^{(6)} = 2(xy^{(4)} + 4y'') \quad y^{(6)}(0) = 16$$

$$y^{(7)} = 2(xy^{(5)} + 5y^{(4)}) \quad y^{(7)}(0) = -120$$

$$\vdots \quad \vdots$$

$$y = 1 - \frac{3}{1!}x + \frac{2}{3!}x^3 - \frac{12}{4!}x^4 + \frac{16}{6!}x^6 - \frac{120}{7!}x^7$$

Using the first six terms of the series,  $y\left(\frac{1}{4}\right) \approx 0.253$ .

20.  $y'' - 2xy' + y = 0$ ,  $y(0) = 1$ ,  $y'(0) = 2$

$$y'' = 2xy' - y \quad y''(0) = -1$$

$$y''' = 2xy'' + y' \quad y'''(0) = 2$$

$$y^{(4)} = 2xy''' + 3y'' \quad y^{(4)}(0) = 3$$

$$y^{(5)} = 2xy^{(4)} + 5y''' \quad y^{(5)}(0) = 10$$

$$y^{(6)} = 2xy^{(5)} + 7y^{(4)} \quad y^{(6)}(0) = -21$$

$$y^{(7)} = 2xy^{(6)} + 9y^{(5)} \quad y^{(7)}(0) = 90$$

$$\vdots \quad \vdots$$

$$y = 1 + \frac{2}{1!}x - \frac{1}{2!}x^2 + \frac{2}{3!}x^3 - \frac{3}{4!}x^4 + \frac{10}{5!}x^5 - \frac{21}{6!}x^6 + \frac{90}{7!}x^7$$

Using the first eight terms of the series,  $y\left(\frac{1}{2}\right) \approx 1.911$ .

21.  $y'' + x^2y' - (\cos x)y = 0, y(0) = 3, y'(0) = 2$

$$y'' = -x^2y' + (\cos x)y \quad y''(0) = 3$$

$$y''' = -2x^2y' - x^2y'' - (\sin x)y + (\cos x)y' \quad y'''(0) = 2$$

$$y = 3 + \frac{2}{1!}x + \frac{3}{2!}x^2 + \frac{2}{3!}x^3$$

Using the first four terms of the series,  $y\left(\frac{1}{3}\right) \approx 3.846$ .

22.  $y'' + e^xy' - (\sin x)y = 0, y(0) = -2, y'(0) = 1$

$$y'' = -e^xy' + (\sin x)y, \quad y''(0) = -1$$

$$\begin{aligned} y''' &= -e^xy' - e^xy'' + (\cos x)y + (\sin x)y' \\ &= -e^x(y' + y'') + (\cos x)y + (\sin x)y' \end{aligned} \quad y'''(0) = -(1 - 1) + (-2) = -2$$

$$y = -2 + \frac{1}{1!}x - \frac{1}{2!}x^2 - \frac{2}{3!}x^3$$

Using the first four terms of the series,  $y\left(\frac{1}{5}\right) \approx -1.823$ .

23.  $f(x) = e^x, f'(x) = e^x, y' - y = 0$ .

Assume  $y = \sum_{n=0}^{\infty} a_n x^n$ , then:

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

$$\sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} a_n x^n$$

$$\sum_{n=0}^{\infty} (n+1) a_{n+1} x^n = \sum_{n=0}^{\infty} a_n x^n$$

$$a_{n+1} = \frac{a_n}{n+1}, n \geq 0$$

$$n = 0, \quad a_0 = a_0$$

$$n = 1, \quad a_2 = \frac{a_1}{2} = \frac{a_0}{2}$$

$$n = 2, \quad a_3 = \frac{a_2}{3} = \frac{a_0}{2(3)}$$

$$n = 3, \quad a_4 = \frac{a_3}{4} = \frac{a_0}{2(3)(4)}$$

$$n = 4, \quad a_5 = \frac{a_4}{5} = \frac{a_0}{2(3)(4)(5)} \\ \vdots$$

$$a_{n+1} = \frac{a_0}{(n+1)!} \Rightarrow a_n = \frac{a_0}{n!}$$

$$y = a_0 \sum_{n=0}^{\infty} \frac{x^n}{n!} \text{ which converges on } (-\infty, \infty). \text{ When}$$

$a_0 = 1$ , you have the Maclaurin Series for  $f(x) = e^x$ .

24.  $f(x) = \cos x, f'(x) = -\sin x, f''(x) = -\cos x$ ,

$$y'' + y = 0.$$

Assume  $y = \sum_{n=0}^{\infty} a_n x^n$ , then:

$$y'' = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}$$

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=0}^{\infty} a_n x^{n-2} = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} x^n = -\sum_{n=0}^{\infty} a_n x^n$$

$$a_{n+2} = -\frac{a_n}{(n+1)(n+2)}, n \geq 0$$

$$a_0 = a_0$$

$$a_1 = a_1$$

$$a_2 = -\frac{a_0}{(1)(2)}$$

$$a_3 = -\frac{a_1}{(2)(3)}$$

$$a_4 = -\frac{a_2}{(3)(4)} = \frac{a_0}{4!}$$

$$a_5 = -\frac{a_3}{(4)(5)} = \frac{a_1}{5!}$$

$\vdots$

$$a_{2n} = \frac{(-1)^n a_0}{(2n)!}$$

$$a_{2n+1} = \frac{(-1)^n a_1}{(2n+1)!}$$

$$y = a_0 \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} + a_1 \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \text{ which}$$

converges on  $(-\infty, \infty)$

When  $a_0 = 1$  and  $a_1 = 0$ , you have the Maclaurin Series for  $f(x) = \cos x$ .

25.  $f(x) = \arctan x$

$$f'(x) = \frac{1}{1+x^2}$$

$$f''(x) = \frac{-2x}{(1+x^2)^2}$$

$$y'' = \frac{-2x}{1+x^2} y'$$

$$(1+x^2)y'' + 2xy' = 0$$

Assume  $y = \sum_{n=0}^{\infty} a_n x^n$ , then:

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

$$(1+x^2)y'' + 2xy' = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=0}^{\infty} n(n-1)a_n x^n + \sum_{n=0}^{\infty} 2na_n x^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = -\sum_{n=0}^{\infty} n(n-1)a_n x^n - \sum_{n=0}^{\infty} 2na_n x^n$$

$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} x^n = -\sum_{n=0}^{\infty} n(n+1)a_n x^n$$

$$(n+2)(n+1)a_{n+2} = -n(n+1)a_n$$

$$a_{n+2} = -\frac{n}{n+2} a_n, n \geq 0$$

$n = 0 \Rightarrow a_2 = 0 \Rightarrow$  all the even-powered terms have a coefficient of 0.

$$n = 1, \quad a_3 = -\frac{1}{3} a_1$$

$$n = 3, \quad a_5 = -\frac{3}{5} a_3 = \frac{1}{5} a_1$$

$$n = 5, \quad a_7 = -\frac{5}{7} a_5 = -\frac{1}{7} a_1$$

$$n = 7, \quad a_9 = -\frac{7}{9} a_7 = \frac{1}{9} a_1$$

⋮

$$a_{2n+1} = \frac{(-1)^n a_1}{2n+1}$$

$y = a_1 \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$  which converges on  $(-1, 1)$ . When  $a_1 = 1$ , you have the Maclaurin Series for  $f(x) = \arctan x$ .

26.  $f(x) = \arcsin x$

$$f'(x) = \frac{1}{\sqrt{1-x^2}}$$

$$f''(x) = \frac{x}{(1-x^2)^{3/2}}$$

$$y'' = \frac{1}{\sqrt{1-x^2}} \cdot \frac{x}{1-x^2} = \frac{x}{1-x^2} y'$$

$$(1-x^2)y'' - xy' = 0$$

Assume  $y = \sum_{n=0}^{\infty} a_n x^n$ , then:  $\sum_{n=2}^{\infty} a_n n(n-1)x^{n-2} - \sum_{n=0}^{\infty} a_n n(n-1)x^n - \sum_{n=0}^{\infty} a_n nx^n = 0$

$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n = \sum_{n=0}^{\infty} n^2 a_n x^n$$

$$a_{n+2} = \frac{n^2}{(n+1)(n+2)} a_n, n \geq 0$$

$n = 0 \Rightarrow a_2 = 0 \Rightarrow$  all the even-powered terms have a coefficient of 0.

$$a_1 = a_1$$

$$n = 1, \quad a_3 = \frac{1}{(2)(3)} a_1$$

$$n = 3, \quad a_5 = \frac{9}{(4)(5)} a_3 = \frac{9}{(2)(3)(4)(5)} a_1 = \frac{3}{(2)(4)(5)} a_1$$

$$n = 5, \quad a_7 = \frac{25}{(6)(7)} a_5 = \frac{(9)(25)}{(2)(3)(4)(5)(6)(7)} a_1 = \frac{(3)(5)}{(2)(4)(6)(7)} a_1$$

$$n = 7, \quad a_9 = \frac{49}{(8)(9)} a_7 = \frac{(9)(25)(49)}{(2)(3)(4)(5)(6)(7)(8)(9)} a_1 = \frac{(3)(5)(7)}{(2)(4)(6)(8)} a_1$$

$$n = 9, \quad a_{11} = \frac{81}{(10)(11)} a_9 = \frac{(9)(25)(49)(81)}{(2)(3)(4)(5)(6)(7)(8)(9)(10)(11)} a_1 = \frac{(3)(5)(7)(9)}{(2)(4)(6)(8)(10)(11)} a_1$$

⋮

$$a_{2n+1} = \frac{(2n)!}{(2^n n!)^2 (2n+1)} a_1$$

$y = a_1 \sum_{n=0}^{\infty} \frac{(2n)!}{(2^n n!)^2 (2n+1)} x^{2n+1}$  which converges on  $(-1, 1)$ . When  $a_1 = 1$ , you have the Maclaurin Series for  $f(x) = \arcsin x$ .

27.  $y'' - xy = 0$ . Let  $y = \sum_{n=0}^{\infty} a_n x^n$ .

$$y'' - xy = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - x \sum_{n=0}^{\infty} a_n x^n = \sum_{n=-1}^{\infty} (n+3)(n+2)a_{n+3} x^{n+1} - \sum_{n=0}^{\infty} a_n x^{n+1} = 0$$

$$2a_2 + \sum_{n=0}^{\infty} [(n+3)(n+2)a_{n+3} - a_n] x^{n+1} = 0$$

So,  $a_2 = 0$  and  $a_{n+3} = \frac{a_n}{(n+3)(n+2)}$  for  $n = 0, 1, 2, \dots$

The constants  $a_0$  and  $a_1$  are arbitrary.

$$a_0 = a_0$$

$$a_1 = a_1$$

$$a_3 = \frac{a_0}{3 \cdot 2}$$

$$a_4 = \frac{a_1}{4 \cdot 3}$$

$$a_6 = \frac{a_3}{6 \cdot 5} = \frac{a_0}{6 \cdot 5 \cdot 3 \cdot 2}$$

$$a_7 = \frac{a_4}{7 \cdot 6} = \frac{a_1}{7 \cdot 6 \cdot 4 \cdot 3}$$

So,  $y = a_0 + a_1 x + \frac{a_0}{6} x^3 + \frac{a_1}{12} x^4 + \frac{a_0}{180} x^6 + \frac{a_1}{504} x^7$ .

## Review Exercises for Chapter 16

1.  $(y + x^3 + xy^2)dx - x dy = 0$

$$\frac{\partial M}{\partial y} = 1 + 2xy \neq \frac{\partial N}{\partial x} = -1$$

Not exact

2.  $(5x - y)dx + (5y - x)dy = 0$

$$\frac{\partial M}{\partial y} = -1 = \frac{\partial N}{\partial x}$$

Exact

3.  $(10x + 8y + 2)dx + (8x + 5y + 2)dy = 0$

$$\text{Exact: } \frac{\partial M}{\partial y} = 8 = \frac{\partial N}{\partial x}$$

$$f(x, y) = \int (10x + 8y + 2)dx = 5x^2 + 8xy + 2x + g(y)$$

$$f_y(x, y) = 8x + g'(y) = 8x + 5y + 2$$

$$g'(y) = 5y + 2$$

$$g(y) = \frac{5}{2}y^2 + 2y + C_1$$

$$f(x, y) = 5x^2 + 8xy + 2x + \frac{5}{2}y^2 + 2y + C_1$$

$$5x^2 + 8xy + 2x + \frac{5}{2}y^2 + 2y = C$$

4.  $(2x - 2y^3 + y)dx + (x - 6xy^2)dy = 0$

$$\text{Exact: } \frac{\partial M}{\partial y} = -6y^2 + 1 = \frac{\partial N}{\partial x}$$

$$\begin{aligned} f(x, y) &= \int (2x - 2y^3 + y)dx \\ &= x^2 - 2xy^3 + xy + g(y) \end{aligned}$$

$$f_y(x, y) = -6xy^2 + x + g'(y) = x - 6xy^2$$

$$g'(y) = 0$$

$$g(y) = C_1$$

$$f(x, y) = x^2 - 2xy^3 + xy + C_1$$

$$x^2 - 2xy^3 + xy = C$$

5.  $(x - y - 5)dx - (x + 3y - z)dy = 0$

$$\frac{\partial M}{\partial y} = -1 = \frac{\partial N}{\partial x} \text{ Exact}$$

$$f(x, y) = \int (x - y - 5)dx = \frac{x^2}{2} - xy - 5x + g(y)$$

$$f_y(x, y) = -x + g'(y) = -x - 3y + 2$$

$$g'(y) = -3y + 2$$

$$g(y) = \frac{-3}{2}y^2 + 2y + C_1$$

$$\frac{x^2}{2} - xy - 5x - \frac{3}{2}y^2 + 2y + C_1 = 0$$

$$x^2 - 2xy - 10x - 3y^2 + 4y = C$$

6.  $(3x^2 - 5xy^2)dx + (2y^3 - 5xy^2)dy = 0$

$$\frac{\partial M}{\partial y} = -10xy \neq \frac{\partial N}{\partial x} = -5y^2$$

Not exact

7.  $\frac{x}{y}dx - \frac{x}{y^2}dy = 0$

$$\frac{\partial M}{\partial y} = \frac{-x}{y^2} \neq \frac{\partial N}{\partial x} = \frac{-1}{y^2}$$

Not exact

8.  $y \sin xy dx + (x \sin xy + y)dy = 0$

$$\frac{\partial M}{\partial y} = xy \cos xy + \sin xy = \frac{\partial N}{\partial x} \text{ Exact}$$

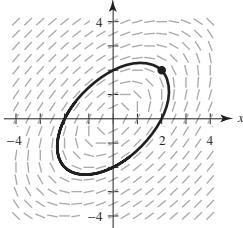
$$f(x, y) = \int y \sin xy dx = -\cos xy + g(y)$$

$$f_y(x, y) = x \sin xy + g'(y) = x \sin xy + y$$

$$g'(y) = y \Rightarrow g(y) = \frac{y^2}{2} + C_1$$

$$-\cos xy + \frac{y^2}{2} = C$$

9. (a)



(b)  $(2x - y)dx + (2y - x)dy = 0$

$$\frac{\partial M}{\partial y} = -1 = \frac{\partial N}{\partial x} \text{ Exact}$$

$$f(x, y) = \int (2x - y)dx = x^2 - xy + g(y)$$

$$f_y(x, y) = -x + g'(y) = 2y - x$$

$$g'(y) = 2y$$

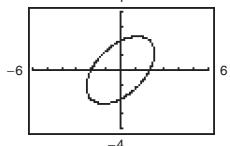
$$g(y) = y^2 + C_1$$

$$x^2 - xy + y^2 = C$$

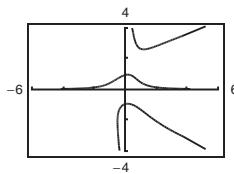
$$y(2) = 2 : 4 - 4 + 4 = 4 = C$$

$$\text{Particular solution: } x^2 - xy + y^2 = 4$$

(c)



10. (a) and (c)



(b)  $(6xy - y^3)dx + (4y + 3x^2 - 3xy^2)dy = 0$

$$\frac{\partial M}{\partial y} = 6x - 3y^2 = \frac{\partial N}{\partial x} \text{ Exact}$$

$$f(x, y) = \int (6xy - y^3)dx = 3x^2y - xy^3 + g(y)$$

$$f_y(x, y) = 3x^2 - 3xy^2 + g'(y) = 4y + 3x^2 - 3xy^2$$

$$g'(y) = 4y \Rightarrow g(y) = 2y^2 + C_1$$

$$3x^2y - xy^3 + 2y^2 = C$$

$$y(0) = 1 : 2 = C$$

$$\text{Particular solution: } 3x^2y - xy^3 + 2y^2 = 2$$

11.  $(2x + y - 3)dx + (x - 3y + 1)dy = 0$

$$\text{Exact: } \frac{\partial M}{\partial y} = 1 = \frac{\partial N}{\partial x}$$

$$f(x, y) = \int (2x + y - 3)dx = x^2 + xy - 3x + g(y)$$

$$f_y(x, y) = x + g'(y) = x - 3y + 1$$

$$g'(y) = -3y + 1$$

$$g(y) = -\frac{3}{2}y^2 + y + C_1$$

$$f(x, y) = x^2 + xy - 3x - \frac{3}{2}y^2 + y + C_1$$

$$2x^2 + 2xy - 6x - 3y^2 + 2y = C$$

Initial condition:

$$y(2) = 0$$

$$8 + 0 - 12 - 0 + 0 = C \Rightarrow C = -4$$

Particular solution:

$$2x^2 + 2xy - 6x - 3y^2 + 2y = -4$$

12.  $3x^2y^2 dx + (2x^3y - 3y^2)dy = 0, y(1) = 2$

$$\frac{\partial M}{\partial y} = 6x^2y = \frac{\partial N}{\partial x} \text{ Exact}$$

$$f(x, y) = \int 3x^2y^2 dx = x^3y^2 + g(y)$$

$$f_y(x, y) = 2x^3y + g'(y) = 2x^3y - 3y^2$$

$$g'(y) = -3y^2$$

$$g(y) = -y^3 + C_1$$

$$x^3y^2 - y^3 = C$$

$$\text{Initial condition: } y(1) = 2: 4 - 8 = C$$

$$\text{Particular solution: } x^3y^2 - y^3 = -4$$

13.  $(3x^2 - y^2)dx + 2xy dy = 0$

$$\frac{(\partial M/\partial y) - (\partial N/\partial x)}{N} = \frac{-2y - 2y}{2xy} = -\frac{2}{x} = h(x)$$

$$\text{Integrating factor: } e^{\int h(x)dx} = e^{\ln x^{-2}} = \frac{1}{x^2}$$

$$\text{Exact equation: } \left(3 - \frac{y^2}{x^2}\right)dx + \frac{2y}{x}dy = 0$$

$$f(x, y) = \int \left(3 - \frac{y^2}{x^2}\right)dx = 3x + \frac{y^2}{x} + g(y)$$

$$f_y(x, y) = \frac{2y}{x} + g'(y) = \frac{2y}{x}$$

$$g'(y) = 0 \Rightarrow g(y) = C_1$$

$$3x + \frac{y^2}{x} = C$$

14.  $2xy dx + (y^2 - x^2)dy = 0$

$$\frac{(\partial N/\partial x) - (\partial M/\partial y)}{M} = \frac{-2x - 2x}{2xy} = -\frac{2}{y} = k(y)$$

$$\text{Integrating factor: } e^{\int k(y)dy} = e^{\ln y^{-2}} = \frac{1}{y^2}$$

$$\text{Exact equation: } \frac{2x}{y}dx + \left(1 - \frac{x^2}{y^2}\right)dy = 0$$

$$f(x, y) = \int \frac{2x}{y}dx = \frac{x^2}{y} + g(y)$$

$$f_y(x, y) = -\frac{x^2}{y^2} + g'(y) = 1 - \frac{x^2}{y^2}$$

$$g'(y) = 1 \Rightarrow g(y) = y + C_1$$

$$\frac{x^2}{y} + y = C$$

15.  $dx + (3x - e^{-2y})dy = 0$

$$\frac{(\partial N/\partial x) - (\partial M/\partial y)}{M} = \frac{3 - 0}{1} = 3 = k(y)$$

Integrating factor:  $e^{\int k(y)dy} = e^{3y}$

Exact equation:  $e^{3y}dx + (3xe^{3y} - e^y)dy = 0$

$$f(x, y) = \int e^{3y} dx = xe^{3y} + g(y)$$

$$f_y(x, y) = 3xe^{3y} + g'(y) = 3xe^{3y} - e^y$$

$$g'(y) = -e^y$$

$$g(y) = -e^y + C_1$$

$$xe^{3y} - e^y = C$$

16.  $\cos y dx - [2(x - y)\sin y + \cos y]dy = 0$

$$\frac{(\partial N/\partial x) - (\partial M/\partial y)}{M} = \frac{-2\sin y + \sin y}{\cos y} = -\tan y = k(y)$$

Integrating factor:  $e^{\int k(y)dy} = \cos y$

Exact equation:

$$\cos^2 y dx - [2(x - y)\sin y \cos y + \cos^2 y]dy = 0$$

$$f(x, y) = \int \cos^2 y dx = x \cos^2 y + g(y)$$

$$f_y(x, y) = -2x \cos y \sin y + g'(y)$$

$$= -2x \sin y \cos y + 2y \sin y \cos y - \cos^2 y$$

$$g'(y) = 2y \sin y \cos y - \cos^2 y$$

$$\Rightarrow g(y) = -y \cos^2 y + C_1$$

$$x \cos^2 y - y \cos^2 y = C$$

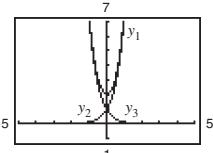
17.  $y = C_1 e^{2x} + C_2 e^{-2x}$

$$y' = 2C_1 e^{2x} - 2C_2 e^{-2x}$$

$$y'' = 4C_1 e^{2x} + 4C_2 e^{-2x}$$

$$y'' - 4y = 4C_1 e^{2x} + 4C_2 e^{-2x}$$

$$- 4(C_1 e^{2x} + C_2 e^{-2x}) = 0$$



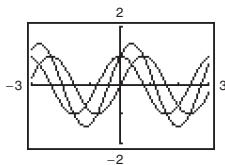
18.  $y = C_1 \cos 2x + C_2 \sin 2x$

$$y' = -2C_1 \sin 2x + 2C_2 \cos 2x$$

$$y'' = -4C_1 \cos 2x - 4C_2 \sin 2x$$

$$y'' + 4y = -4C_1 \cos 2x - 4C_2 \sin 2x$$

$$+ 4(C_1 \cos 2x + C_2 \sin 2x) = 0$$



19.  $y'' - y' - 2y = 0$

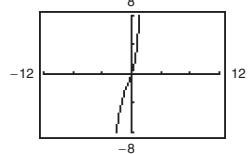
$$m^2 - m - 2 = (m - 2)(m + 1) = 0, m = 2, -1$$

$$y = C_1 e^{2x} + C_2 e^{-x}$$

$$y' = 2C_1 e^{2x} - C_2 e^{-x}$$

$$y(0) = 0 = C_1 + C_2$$

$$y'(0) = 3 = 2C_1 - C_2$$



Adding these equations,  $3 = 3C_1 \Rightarrow C_1 = 1$  and  $C_2 = -1$ .

$$y = e^{2x} - e^{-x}$$

20.  $y'' + 4y' + 5y = 0$

$$m^2 + 4m + 5 = 0 \quad m = \frac{-4 \pm \sqrt{16 - 20}}{2} = -2 \pm i$$

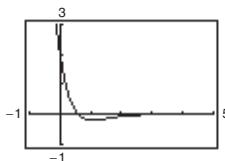
$$y = C_1 e^{-2x} \cos x + C_2 e^{-2x} \sin x$$

$$y(0) = 2 = C_1 \Rightarrow y = e^{-2x} [2 \cos x + C_2 \sin x]$$

$$y' = e^{-2x} [-2 \sin x + C_2 \cos x] - 2e^{-2x} [2 \cos x + C_2 \sin x]$$

$$y'(0) = -7 = C_2 - 2(2) \Rightarrow C_2 = -3$$

$$y = 2e^{-2x} \cos x - 3e^{-2x} \sin x$$



21.  $y'' + 2y' - 3y = 0$

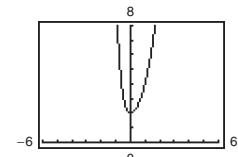
$$m^2 + 2m - 3 = (m + 3)(m - 1) = 0 \Rightarrow m = -3, 1$$

$$y = C_1 e^{-3x} + C_2 e^x$$

$$y' = -3C_1 e^{-3x} + C_2 e^x$$

$$y(0) = 2 = C_1 + C_2$$

$$y'(0) = 0 = -3C_1 + C_2$$



Subtracting these equations,  $2 = 4C_1 \Rightarrow C_1 = \frac{1}{2}$  and  $C_2 = \frac{3}{2}$ .

$$y = \frac{3}{2}e^x + \frac{1}{2}e^{-3x}$$

22.  $y'' + 12y' + 36y = 0$

$$m^2 + 12m + 36 = (m + 6)^2 = 0, m = -6, -6$$

$$y = C_1 e^{-6x} + C_2 x e^{-6x}$$

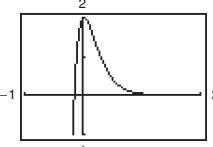
$$y' = -6C_1 e^{-6x} + C_2 e^{-6x} - 6C_2 x e^{-6x}$$

$$y(0) = 2 = C_1$$

$$y'(0) = 1$$

$$= -6(2) + C_2 \Rightarrow C_2 = 13$$

$$y = 2e^{-6x} + 13xe^{-6x}$$



23.  $y'' + 2y' + 5y = 0$

$$m^2 + 2m + 5 = 0 \Rightarrow m = \frac{-2 \pm \sqrt{4 - 20}}{2} = -1 \pm 2i$$

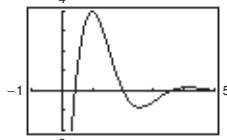
$$y = e^{-x}(C_1 \cos 2x + C_2 \sin 2x)$$

$$y(1) = 4 = e^{-1}(C_1 \cos 2 + C_2 \sin 2)$$

$$y(2) = 0 = e^{-2}(C_1 \cos 4 + C_2 \sin 4)$$

Solving this system, you obtain  $C_1 = -9.0496$ ,  $C_2 = 7.8161$ .

$$y = e^{-x}(-9.0496 \cos 2x + 7.8161 \sin 2x)$$



24.  $y'' + y = 0$

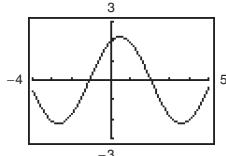
$$m^2 + 1 = 0 \Rightarrow m = \pm i$$

$$y = C_1 \cos x + C_2 \sin x$$

$$y(0) = 2 = C_1$$

$$y\left(\frac{\pi}{2}\right) = 1 = C_2$$

$$y = 2 \cos x + \sin x$$



25.  $y''$  is always positive according to the graph (concave upwards), but  $y'$  is negative when  $x < 0$  (decreasing), so  $y'' \neq y'$ .

26.  $y''$  is positive for  $x > 0$  (concave upwards), but  $-\frac{1}{2}y' < 0$  for  $x > 0$  (increasing). So,  $y'' \neq -\frac{1}{2}y'$ .

29.  $y'' + y = 2 \cos x$

$$m^2 + 1 = 0 \text{ when } m = -i, i.$$

$$y_h = C_1 \cos x + C_2 \sin x$$

$$y_p = Ax \cos x + Bx \sin x$$

$$y_p' = (Bx + A) \cos x + (B - Ax) \sin x$$

$$y_p'' = (2B - Ax) \cos x + (-Bx - 2A) \sin x$$

$$y_p'' + y_p = 2B \cos x - 2A \sin x = 2 \cos x$$

$$A = 0, B = 1$$

$$y = C_1 \cos x + (C_2 + x) \sin x$$

30.  $y'' + 5y' + 4y = x^2 + \sin 2x$

$$m^2 + 5m + 4 = 0 \text{ when } m = -1, -4.$$

$$y_h = C_1 e^{-x} + C_2 e^{-4x}$$

$$y_p = A_0 + A_1 x + A_2 x^2 + B_0 \sin 2x + B_1 \cos 2x$$

$$y_p' = A_1 + 2A_2 x + 2B_0 \cos 2x - 2B_1 \sin 2x$$

$$y_p'' = 2A_2 - 4B_0 \sin 2x - 4B_1 \cos 2x$$

$$y_p'' + 5y_p' + 4y_p = (4A_0 + 5A_1 + 2A_2) + (4A_1 + 10A_2)x + 4A_2 x^2 - 10B_1 \sin 2x + 10B_0 \cos 2x = x^2 + \sin 2x$$

$$A_0 = \frac{21}{32}, A_1 = -\frac{5}{8}, A_2 = \frac{1}{4}, B_0 = 0, B_1 = -\frac{1}{10}$$

$$y = C_1 e^{-x} + C_2 e^{-4x} + \frac{21}{32} - \frac{5}{8}x + \frac{1}{4}x^2 - \frac{1}{10} \cos 2x$$

27.  $y'' + y = x^3 + x$

$$m^2 + 1 = 0 \text{ when } m = -i, i.$$

$$y_h = C_1 \cos x + C_2 \sin x$$

$$y_p = A_0 + A_1 x + A_2 x^2 + A_3 x^3$$

$$y_p' = A_1 + 2A_2 x + 3A_3 x^2$$

$$y_p'' = 2A_2 + 6A_3 x$$

$$y_p'' + y_p = (A_0 + 2A_2) + (A_1 + 6A_3)x + A_2 x^2 + A_3 x^3 = x^3 + x$$

$$A_0 = 0, A_1 = -5, A_2 = 0, A_3 = 1$$

$$y = C_1 \cos x + C_2 \sin x - 5x + x^3$$

28.  $y'' + 2y = e^{2x} + x$

$$m^2 + 2 = 0 \text{ when } m = -\sqrt{2}i, \sqrt{2}i.$$

$$y_h = C_1 \cos \sqrt{2}x + C_2 \sin \sqrt{2}x$$

$$y_p = Ae^{2x} + B_0 + B_1 x$$

$$y_p' = 2Ae^{2x} + B_1$$

$$y_p'' = 4Ae^{2x}$$

$$y_p'' + 2y_p = 6Ae^{2x} + 2B_0 + 2B_1 x = e^{2x} + x$$

$$A = \frac{1}{6}, B_0 = 0, B_1 = \frac{1}{2}$$

$$y = C_1 \cos \sqrt{2}x + C_2 \sin \sqrt{2}x + \frac{1}{6}e^{2x} + \frac{1}{2}x$$

31.  $y'' - 2y' + y = 2xe^x$

$m^2 - 2m + 1 = 0$  when  $m = 1, 1$ .

$$y_h = (C_1 + C_2x)e^x$$

$$y_p = (v_1 + v_2x)e^x$$

$$v_1'e^x + v_2'xe^x = 0$$

$$v_1'e^x + v_2'(x+1)e^x = 2xe^x$$

$$v_1' = -2x^2$$

$$v_1 = \int -2x^2 dx = -\frac{2}{3}x^3$$

$$v_2' = 2x$$

$$v_2 = \int 2x dx = x^2$$

$$y = (C_1 + C_2x + \frac{1}{3}x^3)e^x$$

32.  $y'' + 2y' + y = \frac{1}{x^2 e^x}$

$m^2 + 2m + 1 = 0$  when  $m = -1, -1$ .

$$y_h = (C_1 + C_2x)e^{-x}$$

$$y_p = (v_1 + v_2x)e^{-x}$$

$$v_1'e^{-x} + v_2'(xe^{-x}) = 0$$

$$v_1'(-e^{-x}) + v_2'(-x+1)e^{-x} = \frac{1}{e^x x^2}$$

$$v_1' = -\frac{1}{x}$$

$$v_1 = \int -\frac{1}{x} dx = -\ln|x|$$

$$v_2' = \frac{1}{x^2}$$

$$v_2 = \int \frac{1}{x^2} dx = -\frac{1}{x}$$

$$y = (C_1 + C_2x - \ln|x| - 1)e^{-x}$$

35.  $y'' + 4y = \cos x$

$$m^2 + 4 = 0 \Rightarrow m = \pm 2i$$

$$y_h = C_1 \cos 2x + C_2 \sin 2x$$

$$y_p = A \cos x + B \sin x$$

$$y_p' = -A \sin x + B \cos x$$

$$y_p'' = -A \cos x - B \sin x$$

$$y_p'' + 4y_p = (-A \cos x - B \sin x) + 4(A \cos x + B \sin x) = \cos x$$

$$3A \cos x + 3B \sin x = \cos x \Rightarrow A = \frac{1}{3} \text{ and } B = 0$$

$$y_p = \frac{1}{3} \cos x$$

$$y = y_h + y_p = C_1 \cos 2x + C_2 \sin 2x + \frac{1}{3} \cos x$$

Initial conditions:

$$y(0) = 6: 6 = C_1 + \frac{1}{3} \Rightarrow C_1 = \frac{17}{3}$$

$$y'(0) = -6: -6 = 2C_2 \Rightarrow C_2 = -3$$

Particular solution:

$$y = \frac{17}{3} \cos 2x - 3 \sin 2x + \frac{1}{3} \cos x$$

33.  $y'' + y' - 6y = 54, y(0) = 2, y'(0) = 0$

$$m^2 - m - 6 = 0$$

$$(m-3)(m+2) = 0$$

$$m_1 = 3, m_2 = -2$$

$$y_h = C_1 e^{3x} + C_2 e^{-2x}$$

$y_p = -9$  by inspection

$$y = y_h + y_p = C_1 e^{3x} + C_2 e^{-2x} - 9$$

Initial conditions:

$$y(0) = 2: 2 = C_1 + C_2 - 9 \Rightarrow C_1 + C_2 = 11$$

$$y'(0) = 0: 0 = 3C_1 - 2C_2 \Rightarrow C_1 = \frac{22}{5}, C_2 = \frac{33}{5}$$

$$y = \frac{11}{5}(2e^{3x} + 3e^{-2x}) - 9$$

34.  $y'' + 25y = e^x, y(0) = 0, y'(0) = 0$

$$y_h = C_1 \cos 5x + C_2 \sin 5x$$

$$y_p = Ae^x, y_p' = y_p'' = Ae^x$$

$$Ae^x + 25Ae^x = e^x \Rightarrow 26A = 1 \Rightarrow A = \frac{1}{26}$$

$$y = y_h + y_p = C_1 \cos 5x + C_2 \sin 5x + \frac{1}{26}e^x$$

$$y(0) = 0: 0 = C_1 + \frac{1}{26} \Rightarrow C_1 = -\frac{1}{26}$$

$$y'(0) = 0: 0 = 5C_2 + \frac{1}{26} \Rightarrow C_2 = -\frac{1}{130}$$

$$y = -\frac{1}{26} \cos 5x - \frac{1}{130} \sin 5x + \frac{1}{26}e^x$$

36.  $y'' + 3y' = 6x$

$$m^2 + 3m = 0 \Rightarrow m_1 = 0 \text{ and } m_2 = -3$$

$$y_h = C_1 + C_2 e^{-3x}$$

$$y_p = Ax^3 + Bx^2 + Cx + D$$

$$y_p' = 3Ax^2 + 2Bx + C$$

$$y_p'' = 6Ax + 2B$$

$$y_p'' + 3y_p' = (6Ax + 2B) + 3(3Ax^2 + 2Bx + C) = 9Ax^2 + (6A + 6B)x + (2B + 3C) = 6x, A = 0, B = 1, \text{ and } C = -\frac{2}{3}$$

$$y_p = x^2 - \frac{2}{3}x + D$$

$$y = y_h + y_p = C_1 + C_2 e^{-3x} + x^2 - \frac{2}{3}x + D = C_3 + C_2 e^{-3x} + x^2 - \frac{2}{3}x$$

Initial conditions:  $y(0) = 2: 2 = C_3 + C_2$

$$y'(0) = \frac{10}{3}: \frac{10}{3} = -3C_2 - \frac{2}{3} \Rightarrow C_2 = -\frac{4}{3} \text{ and } C_3 = \frac{10}{3}$$

Particular solution:  $y = \frac{10}{3} - \frac{4}{3}e^{-3x} + x^2 - \frac{2}{3}x = \frac{1}{3}(10 - 4e^{-3x} + 3x^2 - 2x)$

37.  $y'' - y' - 2y = 1 + xe^{-x}, y(0) = 1, y'(0) = 3$

$$m^2 - m - 2 = (m - 2)(m + 1) = 0 \Rightarrow m = 2, -1$$

$$y_h = C_1 e^{2x} + C_2 e^{-x}$$

$$y_p = A + (Bx + Cx^2)e^{-x}$$

$$y_p' = -(Bx + Cx^2)e^{-x} + (B + 2Cx)e^{-x} = (B + (2C - B)x - Cx^2)e^{-x}$$

$$y_p'' = -(B + (2C - B)x - Cx^2)e^{-x} + (2C - B - 2Cx)e^{-x} = (Cx^2 + (B - 4C)x + 2C - 2B)e^{-x}$$

$$y_p'' - y_p' - 2y_p = (2C - 2B + (-4C + B)x + Cx^2)e^{-x} - (B + (2C - B)x - Cx^2)e^{-x} - 2(A + (Bx + Cx^2)e^{-x})$$

$$= -2A + (-6Cx + 2C - 3B)e^{-x} = 1 + xe^{-x} \Rightarrow A = -\frac{1}{2}, -6C = 1 \text{ and } 2C - 3B = 0.$$

So,  $C = -\frac{1}{6}$  and  $B = -\frac{1}{9}$ .

$$y = y_h + y_p = C_1 e^{2x} + C_2 e^{-x} - \frac{1}{2} + \left(-\frac{1}{9}x - \frac{1}{6}x^2\right)e^{-x}$$

Initial conditions:  $y(0) = 1 = C_1 + C_2 - \frac{1}{2} \Rightarrow C_1 + C_2 = \frac{3}{2}$

$$y'(0) = 3 = 2C_1 - C_2 - \frac{1}{9} \Rightarrow 2C_1 - C_2 = \frac{28}{9}$$

Adding,  $3C_1 = \frac{83}{18} \Rightarrow C_1 = \frac{83}{54}$ .

So,  $C_2 = -\frac{1}{27}$ .

Particular solution:  $y = \frac{83}{54}e^{2x} - \frac{1}{27}e^{-x} - \frac{1}{2} - \left(\frac{1}{9} + \frac{1}{6}x\right)xe^{-x}$

**38.**  $y''' - y'' = 4x^2$ ,  $y(0) = 1$ ,  $y'(0) = 1$ ,  $y''(0) = 1$

$$y''' - y'' = 0$$

$m^3 - m^2 = 0$  when  $m = 0, 0, 1$ .

$$y_h = C_1 + C_2x + C_3e^x$$

$$y_p = A_0x^2 + A_1x^3 + A_2x^4$$

$$y_p' = 2A_0x + 3A_1x^2 + 4A_2x^3$$

$$y_p'' = 2A_0 + 6A_1x + 12A_2x^2$$

$$y_p''' = 6A_1 + 24A_2x$$

$$y_p''' - y_p'' = (-2A_0 + 6A_1) + (-6A_1 + 24A_2)x - 12A_2x^2 = 4x^2 \text{ or } A_0 = -4, A_1 = -\frac{4}{3}, A_2 = -\frac{1}{3}$$

$$y = C_1 + C_2x + C_3e^x - 4x^2 - \frac{4}{3}x^3 - \frac{1}{3}x^4$$

$$y' = C_2 + C_3e^x - 8x - 4x^2 - \frac{4}{3}x^3$$

$$y'' = C_3e^x - 8 - 8x - 4x^2$$

Initial conditions:  $y(0) = 1$ ,  $y'(0) = 1$ ,  $y''(0) = 1$ ,  $1 = C_1 + C_3$ ,  $1 = C_2 + C_3$ ,  $1 = C_3 - 8$ ,  $C_1 = -8$ ,  $C_2 = -8$ ,  $C_3 = 9$

Particular solution:  $y = -8 - 8x - 4x^2 - \frac{4}{3}x^3 - \frac{1}{3}x^4 + 9e^x$

**39.** By Hooke's Law,  $F = kx$ ,  $k = F/x = 64/(4/3) = 48$ . Also,  $F = ma$  and  $m = F/a = 64/32 = 2$ . So,

$$\frac{d^2y}{dt^2} + \left(\frac{48}{2}\right)y = 0$$

$$y = C_1 \cos(2\sqrt{6}t) + C_2 \sin(2\sqrt{6}t).$$

Because  $y(0) = \frac{1}{2}$  you have  $C_1 = \frac{1}{2}$  and  $y'(0) = 0$  yields  $C_2 = 0$ . So,  $y = \frac{1}{2} \cos(2\sqrt{6}t)$ .

**40.** From Exercise 39 you have  $k = 48$  and  $m = 2$ . Also, the damping force is given by  $(1/8)(dy/dt)$ .

$$2\left(\frac{d^2y}{dt^2}\right) = -\frac{1}{8} \frac{dy}{dt} - 48y$$

$$y'' + \frac{1}{16}y' + 24y = 0$$

$$16y'' + y' + 384y = 0$$

The characteristic equation  $16m^2 + m = 384 = 0$  has complex roots

$$m = -\frac{1}{32} \pm \frac{\sqrt{24,575}i}{32} = -\frac{1}{32} \pm \frac{5\sqrt{983}}{32}i.$$

$$\text{So, } y(t) = e^{-t/32} \left[ C_1 \cos\left(\frac{5\sqrt{983}}{32}t\right) + C_2 \sin\left(\frac{5\sqrt{983}}{32}t\right) \right].$$

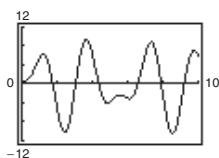
Initial conditions:

$$y(0) = \frac{1}{2} \Rightarrow C_1 = \frac{1}{2}$$

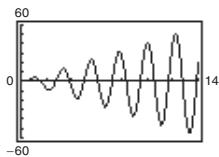
$$y'(0) = 0 \Rightarrow \frac{5\sqrt{983}}{32}C_2 - \frac{C_1}{32} = 0 \Rightarrow C_2 = \frac{\sqrt{983}}{9830}$$

$$\text{Particular solution: } y(t) = e^{-t/32} \left[ \frac{1}{2} \cos\left(\frac{5\sqrt{983}}{32}t\right) + \frac{\sqrt{983}}{9830} \sin\left(\frac{5\sqrt{983}}{32}t\right) \right]$$

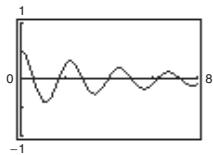
41. (a) (i)  $y = \frac{1}{2} \cos 2t + \frac{12\pi}{\pi^2 - 4} \sin 2t + \frac{24}{4 - \pi^2} \sin \pi t$



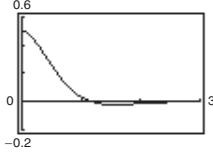
(ii)  $y = \frac{1}{2} [(1 - 6\sqrt{2}t) \cos(2\sqrt{2}t) + 3 \sin(2\sqrt{2}t)]$



(iii)  $y = \frac{e^{-t/5}}{398} \left[ 199 \cos \frac{\sqrt{199}t}{5} + \sqrt{199} \sin \frac{\sqrt{199}t}{5} \right]$



(iv)  $y = \frac{1}{2} e^{-2t} (\cos 2t + \sin 2t)$



- (b) The object comes to rest more quickly. It may not even oscillate, as in part (iv).
- (c) It would oscillate more rapidly.
- (d) Part (ii). The amplitude becomes increasingly large.

45.  $(x - 4)y' + y = 0$ . Letting  $y = \sum_{n=0}^{\infty} a_n x^n$ :

$$\begin{aligned} xy' - 4y' + y &= \sum_{n=0}^{\infty} n a_n x^n - 4 \sum_{n=1}^{\infty} n a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n \\ &= \sum_{n=0}^{\infty} (n+1)a_n x^n - \sum_{n=1}^{\infty} 4na_n x^{n-1} = \sum_{n=0}^{\infty} (n+1)a_n x^n - \sum_{n=-1}^{\infty} 4(n+1)a_{n+1} x^n = 0 \end{aligned}$$

$$(n+1)a_n = 4(n+1)a_{n+1}$$

$$a_{n+1} = \frac{1}{4}a_n$$

$$a_0 = a_0, a_1 = \frac{1}{4}a_0, a_2 = \frac{1}{4}a_1 = \frac{1}{4^2}a_0, \dots, a_n = \frac{1}{4^n}a_0$$

$$y = a_0 \sum_{n=0}^{\infty} \frac{x^n}{4^n}$$

42.  $y_p = \frac{1}{4} \cos x, y_p' = -\frac{1}{4} \sin x, y_p'' = -\frac{1}{4} \cos x$   
 $y_p'' + 4y_p' + 5y_p = -\frac{1}{4} \cos x + 4(-\frac{1}{4} \sin x) + 5(\frac{1}{4} \cos x)$   
 $= \cos x - \sin x$

False.

43. (a)  $y_p'' = -A \sin x$  and  $3y_p = 3A \sin x$ .

$$\begin{aligned} \text{So, } y_p'' + 3y_p &= -A \sin x + 3A \sin x \\ &= 2A \sin x = 12 \sin x \end{aligned}$$

(b)  $y_p = \frac{5}{2} \cos x$

(c) If  $y_p = A \cos x + B \sin x$ , then

$y_p'' = -A \cos x - B \sin x$ , and solving for  $A$  and  $B$  would be more difficult.

44.  $y = 5$ , because  $y' = y'' = 0$  and  $6(5) = 30$

**46.**  $y'' + 3xy' - 3y = 0$ . Letting  $y = \sum_{n=0}^{\infty} a_n x^n$ :

$$\begin{aligned} y'' + 3xy' - 3y &= \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + 3x \sum_{n=1}^{\infty} na_n x^{n-1} - 3 \sum_{n=0}^{\infty} a_n x^n = 0 \\ \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} x^n &= \sum_{n=0}^{\infty} (3-3n)a_n x^n \\ a_{n+2} &= \frac{3(1-n)a_n}{(n+2)(n+1)} \end{aligned}$$

$$a_0 = a_0$$

$$a_1 = a_1$$

$$a_2 = \frac{3}{2 \cdot 1} a_0$$

$$a_3 = 0$$

There are no odd-powered terms for  $n > 1$ .

$$\begin{aligned} a_4 &= -\frac{3}{4 \cdot 3} \left( \frac{3}{2 \cdot 1} a_0 \right) = -\frac{3(3)a_0}{4!} \\ a_6 &= -\frac{3(3)}{6 \cdot 5} \left( -\frac{3(3)a_0}{4!} \right) = \frac{3^3(3)a_0}{6!} \\ a_8 &= -\frac{3(5)}{8 \cdot 7} \left( \frac{3^3(3)a_0}{6!} \right) = -\frac{3^4(5 \cdot 3)a_0}{8!} \\ a_{10} &= -\frac{3(7)}{10 \cdot 9} \left( -\frac{3^4(5 \cdot 3)a_0}{8!} \right) = \frac{3^5(7 \cdot 5 \cdot 3)a_0}{10!} \end{aligned}$$

$$y = a_0 + \frac{3}{2} a_0 x^2 + a_0 \sum_{n=2}^{\infty} \frac{(-1)^{n+1} 3^n [3 \cdot 5 \cdot 7 \cdots (2n-3)]}{(2n)!} x^{2n}$$

**47.**  $y'' + y' - e^x y = 0$ ,  $y(0) = 2$ ,  $y'(0) = 0$

$$\begin{aligned} y'' &= -y' + e^x y & y''(0) &= 2 \\ y''' &= -y'' + e^x(y + y') & y'''(0) &= -2 + 2 = 0 \\ y^{(4)} &= -y''' + e^x(y + 2y' + y'') & y^{(4)}(0) &= 4 \\ y^{(5)} &= -y^{(4)} + e^x(y + 3y' + 3y'' + y''') & y^{(5)}(0) &= -4 + 8 = 4 \end{aligned}$$

$$y \approx y(0) + y'(0)x + \frac{y''(0)}{2!}x^2 + \frac{y'''(0)}{3!}x^3 + \frac{y^{(4)}(0)}{4!}x^4 + \frac{y^{(5)}(0)}{5!}x^5 = 2 + x^2 + \frac{1}{6}x^4 + \frac{1}{30}x^5$$

Using the first four terms of the series,  $y\left(\frac{1}{4}\right) \approx 2.063$ .

**48.**  $y'' + xy = 0$ ,  $y(0) = 1$ ,  $y'(0) = 1$

$$\begin{aligned} y'' &= -xy & y''(0) &= 0 \\ y''' &= -xy' - y & y'''(0) &= -1 \\ y^{(4)} &= -xy'' - y' - y' = -xy'' - 2y' & y^{(4)}(0) &= -2 \\ y^{(5)} &= -xy''' - y'' - 2y'' = -xy''' - 3y'' & y^{(5)}(0) &= 0 \\ y^{(6)} &= -xy^{(4)} - y''' - 3y''' = -xy^{(4)} - 4y''' & y^{(6)}(0) &= 4 \\ y^{(7)} &= -xy^{(5)} - y^{(4)} - 4y^{(4)} = -xy^{(5)} - 5y^{(4)} & y^{(7)}(0) &= 10 \end{aligned}$$

$$y \approx y(0) + y'(0)x + \frac{y''(0)}{2!}x^2 + \cdots + \frac{y^{(7)}(0)}{7!}x^7 = 1 + x - \frac{x^3}{6} - \frac{x^4}{12} + \frac{x^6}{180} + \frac{x^7}{504}$$

$$y\left(\frac{1}{2}\right) \approx 1.474$$

## Problem Solving for Chapter 16

1.  $(3x^2 + kxy^2)dx - (5x^2y + ky^2)dy = 0$

$$\frac{\partial M}{\partial y} = 2kxy$$

$$\frac{\partial N}{\partial x} = -10xy$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \Rightarrow k = -5$$

$$(3x^2 - 5xy^2)dx - (5x^2y - 5y^2)dy = 0 \text{ Exact}$$

$$f(x, y) = \int (3x^2 - 5xy^2)dx = x^3 - \frac{5}{2}x^2y^2 + g(y)$$

$$f_y(x, y) = -5x^2y + g'(y) = -5x^2y + 5y^2$$

$$g'(y) = 5y^2 \Rightarrow g(y) = \frac{5}{3}y^3 + C_1$$

$$x^3 - \frac{5}{2}x^2y^2 + \frac{5}{3}y^3 = C_2$$

$$6x^3 - 15x^2y^2 + 10y^3 = C$$

2.  $(kx^2 + y^2)dx - kxy dy = 0$

(a)  $\frac{1}{x^2}(kx^2 + y^2)dx - \frac{1}{x^2}kxy dy = 0$

$$\left(k + \frac{y^2}{x^2}\right)dx - \frac{ky}{x}dy = 0$$

$$\frac{\partial M}{\partial y} = \frac{2y}{x^2} = \frac{\partial N}{\partial x} = \frac{ky}{x^2} \Rightarrow k = 2$$

(b)  $\left(2 + \frac{y^2}{x^2}\right)dx - \left(\frac{2y}{x}\right)dy = 0 \text{ Exact}$

$$f(x, y) = \int \left(2 + \frac{y^2}{x^2}\right)dx = 2x - \frac{y^2}{x} + g(y)$$

$$f_y(x, y) = \frac{-2y}{x} + g'(y) = \frac{-2y}{x} \Rightarrow g(y) = C_1$$

$$2x - \frac{y^2}{x} = C$$

6. The roots of the characteristic equation  $m^2 + am + b = 0$  ( $a, b > 0$ ) are  $m = \frac{-a \pm \sqrt{a^2 - 4b}}{2}$ . You consider three cases:

(i) If the roots are equal, then  $\sqrt{a^2 - 4b} = 0$  and  $y = (C_1 + C_2x)e^{\frac{-a}{2}x} \rightarrow 0$  as  $x \rightarrow \infty$ .

(ii) If the roots are complex,  $m = \frac{-a}{2} \pm \beta i$ , then  $y = C_1e^{\frac{-a}{2}x} \cos \beta x + C_2e^{\frac{-a}{2}x} \sin \beta x \rightarrow 0$  as  $x \rightarrow \infty$

(because  $\cos \beta x$  and  $\sin \beta x$  are bounded).

(iii) If the roots are real and distinct, then  $y = C_1e^{\frac{-a+\sqrt{a^2-4b}}{2}x} + C_2e^{\frac{-a-\sqrt{a^2-4b}}{2}x}$ . The second term clearly tends to 0 as  $x \rightarrow \infty$ .

For the first term, note that  $\sqrt{a^2 - 4b} = a\sqrt{1 - \frac{4b}{a^2}} < a$ . So  $y = C_1e^{\left(\frac{-a+a\sqrt{1-\frac{4b}{a^2}}}{2}\right)x} \rightarrow 0$  as  $x \rightarrow \infty$ .

3.  $y'' - a^2y = 0, y > 0$

$$m^2 - a^2 = (m + a)(m - a) = 0 \Rightarrow m = \pm a$$

$$\begin{aligned} y &= B_1e^{ax} + B_2e^{-ax} = \frac{C_1 + C_2}{2}e^{ax} + \frac{C_1 - C_2}{2}e^{-ax} \\ &= C_1\left(\frac{e^{ax} + e^{-ax}}{2}\right) + C_2\left(\frac{e^{ax} - e^{-ax}}{2}\right) \\ &= C_1 \cosh ax + C_2 \sinh ax \end{aligned}$$

4.  $y'' + \beta^2y = 0$

$$m^2 + \beta^2 = 0 \Rightarrow m = \pm \beta i$$

$$y = C_1 \cos \beta x + C_2 \sin \beta x$$

Let  $\phi$  be given by  $\cot \phi = \frac{C_2}{C_1}, 0 \leq \phi < 2\pi$ .

Then  $C_1 \cos \phi = C_2 \sin \phi$ .

Let  $C = \frac{C_1}{\sin \phi} = \frac{C_2}{\cos \phi}$ . Then

$$y = C_1 \cos \beta x + C_2 \sin \beta x$$

$$= C \sin \phi \cos \beta x + C \cos \phi \sin \beta x = C \sin(\beta x + \phi)$$

Note that if  $C_1 = 0$ , then  $\phi = 0$  and

$$y = C \sin(\beta x). \text{ And if } C_2 = 0, \text{ then}$$

$$y = C \sin\left(\beta x + \frac{\pi}{2}\right) = C \cos(\beta x).$$

5. The general solution to  $y'' + ay' + by = 0$  is

$$y = B_1e^{(r+s)x} + B_2e^{(r-s)x}.$$

Let  $C_1 = B_1 + B_2$  and  $C_2 = B_1 - B_2$ .

Then  $B_1 = \frac{C_1 + C_2}{2}$  and  $B_2 = \frac{C_1 - C_2}{2}$ .

$$\begin{aligned} \text{So } y &= \left(\frac{C_1 + C_2}{2}\right)e^{(r+s)x} + \left(\frac{C_1 - C_2}{2}\right)e^{(r-s)x} \\ &= e^{rx} \left[ C_1 \left( \frac{e^{sx} + e^{-sx}}{2} \right) + C_2 \left( \frac{e^{sx} - e^{-sx}}{2} \right) \right] \\ &= e^{rx} [C_1 \cosh sx + C_2 \sinh sx]. \end{aligned}$$

7.  $y'' + ay = 0, y(0) = y(L) = 0$

- (a) If  $a = 0, y'' = 0 \Rightarrow y = cx + d. y(0) = 0 = d$   
and  $y(L) = 0 = cL \Rightarrow c = 0$ . So  $y = 0$  is the solution.
- (b) If  $a < 0, y'' + ay = 0$  has characteristic equation

$$m^2 + a = 0 \Rightarrow m = \pm\sqrt{-a}.$$

$$y = C_1 e^{\sqrt{-a}x} + C_2 e^{-\sqrt{-a}x}$$

$$y(0) = 0 = C_1 + C_2 \Rightarrow -C_1 = C_2$$

$$y(L) = 0 = C_1 e^{\sqrt{-a}L} + C_2 e^{-\sqrt{-a}L}$$

$$= C_1 e^{\sqrt{-a}L} - C_1 e^{-\sqrt{-a}L}$$

$$= 2C_1 \left( \frac{e^{\sqrt{-a}L} - e^{-\sqrt{-a}L}}{2} \right)$$

$$= 2C_1 \sinh(\sqrt{-a}L) \Rightarrow C_1 = 0 = C_2$$

So,  $y = 0$  is the only solution.

9.  $\frac{d^2\theta}{dt^2} + \frac{g}{L}\theta = 0, \frac{g}{L} > 0$

(a)  $\theta(t) = C_1 \sin\left(\sqrt{\frac{g}{L}}t\right) + C_2 \cos\left(\sqrt{\frac{g}{L}}t\right)$

Let  $\phi$  be given by  $\tan\left(\sqrt{\frac{g}{L}}\phi\right) = -\frac{C_1}{C_2}, -\frac{\pi}{2} < \phi < \frac{\pi}{2}$ .

Then  $C_2 \sin\left(\sqrt{\frac{g}{L}}\phi\right) = -C_1 \cos\left(\sqrt{\frac{g}{L}}\phi\right)$ .

Let  $A = \frac{C_2}{\cos\left(\sqrt{\frac{g}{L}}\phi\right)} = -\frac{C_1}{\sin\left(\sqrt{\frac{g}{L}}\phi\right)}$

$$\theta(t) = C_1 \sin\left(\sqrt{\frac{g}{L}}t\right) + C_2 \cos\left(\sqrt{\frac{g}{L}}t\right) = -A \sin\left(\sqrt{\frac{g}{L}}\phi\right) \sin\left(\sqrt{\frac{g}{L}}t\right) + A \cos\left(\sqrt{\frac{g}{L}}\phi\right) \cos\left(\sqrt{\frac{g}{L}}t\right) = A \cos\left[\sqrt{\frac{g}{L}}(t + \phi)\right]$$

(b)  $\theta(t) = A \cos\left[\sqrt{\frac{g}{L}}(t + \phi)\right], g = 9.8, L = 0.25$

$$\theta(0) = A \cos\left[\sqrt{39.2}\phi\right] = 0.1$$

$$\theta'(t) = -A\sqrt{\frac{g}{L}} \sin\left[\sqrt{\frac{g}{4}}(t + \phi)\right]$$

$$\theta'(0) = -A\sqrt{39.2} \sin\left[\sqrt{39.2}\phi\right] = 0.5$$

Dividing,  $\tan\left[\sqrt{39.2}\phi\right] = \frac{-5}{\sqrt{39.2}} \Rightarrow \phi \approx -0.1076 \Rightarrow A \approx 0.128$ .

$$\theta(t) = 0.128 \cos\left[\sqrt{39.2}(t - 0.108)\right]$$

(c) Period =  $\frac{2\pi}{\sqrt{39.2}} \approx 1 \text{ sec}$

(d) Maximum is 0.128.

(e)  $\theta(t) = 0$  at  $t \approx 0.359 \text{ sec}$ , and at  $t \approx 0.860 \text{ sec}$ .

(f)  $\theta'(0.359) \approx -0.801, \theta'(0.860) \approx 0.801$

8.  $y'' + ay = 0, a > 0, y(0) = y(L) = 0$

$$m^2 + a = 0 \Rightarrow m = \pm\sqrt{a}i$$

$$y = C_1 \cos(\sqrt{a}x) + C_2 \sin(\sqrt{a}x).$$

$$y(0) = 0 = C_1$$

$$y = C_2 \sin(\sqrt{a}x)$$

$$y(L) = 0 = C_2 \sin(\sqrt{a}L)$$

$$\text{So } \sqrt{a}L = n\pi$$

$$a = \left(\frac{n\pi}{L}\right)^2, n \text{ an integer.}$$

**10. (a)**  $Ay'' = 2Wx - \frac{1}{2}Wx^2, A > 0$

$$y'' = \frac{2W}{A}x - \frac{W}{2A}x^2 = \frac{W}{2A}(4x - x^2)$$

$$y' = \frac{W}{2A}\left(2x^2 - \frac{x^3}{3}\right) + C_1$$

$$y = \frac{W}{2A}\left(\frac{2x^3}{3} - \frac{x^4}{12}\right) + C_1x + C_2$$

$$y(0) = 0 \Rightarrow C_2 = 0$$

$$y(2) = 0 \Rightarrow \frac{W}{2A}\left(\frac{16}{3} - \frac{16}{12}\right) + 2C_1 = 0$$

$$\frac{2W}{A} = -2C_1 \Rightarrow C_1 = \frac{-W}{A}$$

$$y = \frac{W}{2A}\left(\frac{2x^3}{3} - \frac{x^4}{12} - 2x\right)$$

(b) Using a graphing utility, the maximum deflection is at  $x \approx 1.1074$ , and the deflection is

$$\frac{W}{2A}(1.43476) \approx 0.7174 \frac{W}{A}.$$

**11.**  $y'' + 8y' + 16y = 0, y(0) = 1, y'(0) = 1$

(a)  $\lambda = 4, \omega = 4, \lambda^2 - \omega^2 = 0$ , critically damped

(b)  $m_1 = m_2 = -4$

$$y = (C_1 + C_2 t)e^{-4t},$$

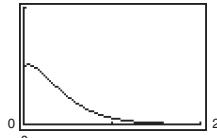
$$y' = -4(C_1 + C_2 t)e^{-4t} + C_2 e^{-4t}$$

$$y(0) = 1 = C_1$$

$$y'(0) = 1 = -4 + C_2 \Rightarrow C_2 = 5$$

$$y = (1 + 5t)e^{-4t}$$

(c)



The solution tends to zero quickly.

**12.**  $y'' + 2y' + 26y = 0, y(0) = 1, y'(0) = 4$

(a)  $\lambda = 1, \omega = \sqrt{26}$ ,

$\lambda^2 - \omega^2 = -25 < 0$ , underdamped

(b)  $m_1 = -1 + 5i, m_2 = -1 - 5i$

$$y = C_1 e^{-t} \cos(5t) + C_2 e^{-t} \sin(5t)$$

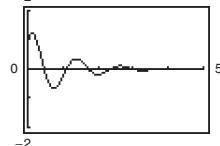
$$y(0) = 1 = C_1$$

$$y'(t) = -e^{-t}(C_1 \cos 5t + C_2 \sin 5t) + e^{-t}(-5C_1 \sin 5t + 5C_2 \cos 5t)$$

$$y'(0) = 4 = -C_1 + 5C_2 \Rightarrow C_2 = 1$$

$$y = e^{-t}(\cos 5t + \sin 5t)$$

(c)



The solution oscillates.

**13.**  $y'' + 20y' + 64y = 0, y(0) = 2, y'(0) = -20$

(a)  $\lambda = 10, \omega = 8, \lambda^2 - \omega^2 = 36 > 0$ , overdamped

(b)  $m_1 = -10 + 6 = -4, m_2 = -10 - 6 = -16$

$$y = C_1 e^{-4t} + C_2 e^{-16t}$$

$$y(0) = 2 = C_1 + C_2$$

$$y'(t) = -4C_1 e^{-4t} - 16C_2 e^{-16t}$$

$$y'(0) = -20 = -4C_1 - 16C_2$$

$$\begin{aligned} C_1 + C_2 &= 2 \\ -C_1 - 4C_2 &= -5 \end{aligned} \quad \left. \begin{aligned} C_1 &= 1, C_2 = 1 \\ y &= e^{-4t} + e^{-16t} \end{aligned} \right.$$

(c)



The solution tends to zero quickly.

**14.**  $y'' + 2y' + y = 0, y(0) = 2, y'(0) = -1$

(a)  $\lambda = 1, \omega = 1, \lambda^2 - \omega^2 = 0$ , critically damped

(b)  $m_1 = m_2 = -1$

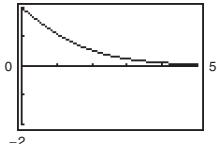
$$y = (C_1 + C_2 t)e^{-t}, y' = -(C_1 + C_2 t)e^{-t} + C_2 e^{-t}$$

$$y(0) = 2 = C_1$$

$$y'(0) = -1 = -2 + C_2 \Rightarrow C_2 = 1$$

$$y = (2 + t)e^{-t}$$

(c)



The solution tends to zero quickly.

**15.** Airy's Equation:  $y'' - xy = 0$

$$y'' - xy + y - y = y'' - (x-1)y - y = 0$$

$$\text{Let } y = \sum_{n=0}^{\infty} a_n(x-1)^n, y' = \sum_{n=1}^{\infty} na_n(x-1)^{n-1}, y'' = \sum_{n=2}^{\infty} n(n-1)a_n(x-1)^{n-2}.$$

$$y'' - (x-1)y - y = 0$$

$$\sum_{n=2}^{\infty} n(n-1)a_n(x-1)^{n-2} - (x-1)\sum_{n=0}^{\infty} a_n(x-1)^n - \sum_{n=0}^{\infty} a_n(x-1)^n = 0$$

$$\sum_{n=-1}^{\infty} (n+3)(n+2)a_{n+3}(x-1)^{n+1} - \sum_{n=0}^{\infty} a_n(x-1)^{n+1} - \sum_{n=-1}^{\infty} a_{n+1}(x-1)^{n+1} = 0$$

$$(2a_2 - a_0) + \sum_{n=0}^{\infty} [(n+3)(n+2)a_{n+3} - a_n - a_{n+1}](x-1)^{n+1} = 0$$

$$2a_2 - a_0 = 0 \Rightarrow a_2 = \frac{1}{2}a_0; a_0, a_1 \text{ arbitrary}$$

$$\text{In general, } a_{n+3} = \frac{a_n + a_{n+1}}{(n+3)(n+2)}.$$

$$a_3 = \frac{a_0 + a_1}{6}$$

$$a_4 = \frac{a_1 + a_2}{12} = \frac{a_1 + \left(\frac{1}{2}a_0\right)}{12} = \frac{2a_1 + a_0}{24}$$

$$a_5 = \frac{a_2 + a_3}{20} = \frac{\frac{1}{2}a_0 + \frac{a_0 + a_1}{6}}{20} = \frac{4a_0 + a_1}{120}$$

$$a_6 = \frac{a_3 + a_4}{30} = \frac{\left(\frac{a_0 + a_1}{6}\right) + \left(\frac{2a_1 + a_0}{24}\right)}{30} = \frac{5a_0 + 6a_1}{720}$$

$$a_7 = \frac{a_4 + a_5}{42} = \frac{\left(\frac{2a_1 + a_0}{24}\right) + \left(\frac{4a_0 + a_1}{120}\right)}{42} = \frac{9a_0 + 11a_1}{5040}$$

So, the first eight terms are

$$\begin{aligned} y &= a_0 + a_1(x-1) + \frac{a_0}{2}(x-1)^2 + \frac{a_0 + a_1}{6}(x-1)^3 + \frac{2a_1 + a_0}{24}(x-1)^4 + \frac{4a_0 + a_1}{120}(x-1)^5 \\ &\quad + \frac{5a_0 + 6a_1}{720}(x-1)^6 + \frac{9a_0 + 11a_1}{5040}(x-1)^7. \end{aligned}$$

**16.** (a)  $T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x)$

$$T_0 = 1, T_1 = x$$

$$T_2 = 2x(x) - 1 = 2x^2 - 1$$

$$T_3 = 2x(2x^2 - 1) - x = 4x^3 - 3x$$

$$T_4 = 2x(4x^3 - 3x) - (2x^2 - 1) = 8x^4 - 8x^2 + 1$$

(b)  $(1-x^2)y'' - xy' + k^2y = 0$

Substituting  $T_0, \dots, T_4$  into this equation shows that the polynomials

satisfy Chebyshev's equation. For example, for  $T_4$ ,

$$(1-x^2)[96x^2 - 16] - x[32x^3 - 16x] + 16[8x^4 - 8x^2 + 1] = 0$$

$$(c) T_5 = 2x(8x^4 - 8x^2 + 1) - (4x^3 - 3x) = 16x^5 - 20x^3 + 5x$$

$$T_6 = 2x(16x^5 - 20x^3 + 5x) - (8x^4 - 8x^2 + 1) = 32x^6 - 48x^4 + 18x^2 - 1$$

$$T_7 = 2x(32x^6 - 48x^4 + 18x^2 - 1) - (16x^5 - 20x^3 + 5x) = 64x^7 - 112x^5 + 56x^3 - 7x$$

17.  $x^2y'' + xy' + x^2y = 0$  Bessel equation of order zero

$$(a) \text{ Let } y = \sum_{n=0}^{\infty} a_n x^n, y' = \sum_{n=1}^{\infty} n a_n x^{n-1}, y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}.$$

$$x^2 y'' + xy' + x^2 y = 0$$

$$x^2 \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + x \sum_{n=1}^{\infty} n a_n x^{n-1} + x^2 \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1) a_n x^n + \sum_{n=1}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} a_n x^{n+2} = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^{n+2} + \sum_{n=-1}^{\infty} (n+2) a_{n+2} x^{n+2} + \sum_{n=0}^{\infty} a_n x^{n+2} = 0$$

$$a_1 x + \sum_{n=0}^{\infty} [(n+2)(n+1) a_{n+2} + (n+2) a_{n+2} + a_n] x^{n+2} = 0$$

$$a_1 = 0 \text{ and } a_{n+2} = \frac{-a_n}{(n+2)^2}.$$

All odd terms  $a_i$  are 0.

$$a_2 = \frac{-a_0}{2^2}$$

$$a_4 = \frac{-a_2}{4^2} = a_0 \frac{1}{2^2 \cdot 4^2} = \frac{a_0}{2^4 (1 \cdot 2)^2}$$

$$a_6 = \frac{-a_4}{6^2} = -a_0 \frac{1}{2^2 \cdot 4^2 \cdot 6^2} = \frac{-a_0}{2^6 (3!)^2}$$

$$y = a_0 \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n} (n!)^2}$$

(b) This is the same function (assuming  $a_0 = 1$ ).

$$18. (a) \text{ Let } y = \sum_{n=0}^{\infty} a_n x^n, y' = \sum_{n=1}^{\infty} n a_n x^{n-1}, y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

$$x^2 y'' + xy' + (x^2 - 1)y = 0$$

$$x^2 \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + x \sum_{n=1}^{\infty} n a_n x^{n-1} + (x^2 - 1) \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1) a_n x^n + \sum_{n=1}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} a_n x^{n+2} - \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^{n+2} + \sum_{n=-1}^{\infty} (n+2) a_{n+2} x^{n+2} + \sum_{n=0}^{\infty} a_n x^{n+2} - \sum_{n=-2}^{\infty} a_{n+2} x^{n+2} = 0$$

$$-a_0 + (a_1 - a_1)x + \sum_{n=0}^{\infty} [(n+2)(n+1) a_{n+2} + (n+2) a_{n+2} + a_n - a_{n+2}] x^{n+2} = 0$$

$$a_0 = 0 \text{ and } [(n+2)(n+1) + (n+2) - 1] a_{n+2} = -a_n \Rightarrow [n^2 + 4n + 3] a_{n+2} = -a_n \Rightarrow a_{n+2} = \frac{-a_n}{(n+1)(n+3)}$$

All even terms  $a_i$  are 0.

$$a_3 = \frac{-a_1}{2 \cdot 4} = \frac{-a_1}{2^3}$$

$$a_5 = \frac{-a_3}{4 \cdot 6} = \frac{-a_1}{2^5 3!} = \frac{-2a_1}{2^5 \cdot 2! 3!}$$

$$a_7 = \frac{-a_5}{6 \cdot 8} = \frac{-2a_1}{2^7 3! 4!}$$

$$y = 2a_1 \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2^{2n+1} n!(n+1)!}$$

(b) This is the same function (assuming  $2a_1 = 1$ ).

19. (a) Let  $y = \sum_{n=0}^{\infty} a_n x^n$ ,  $y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$ ,  $y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$ .

$$y'' - 2xy' + 8y = 0$$

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - 2x \sum_{n=1}^{\infty} n a_n x^{n-1} + 8 \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=0}^{\infty} 2na_n x^n + \sum_{n=0}^{\infty} 8a_n x^n = 0$$

$$\sum_{n=0}^{\infty} [(n+2)(n+1)a_{n+2} - 2na_n + 8a_n] x^n = 0$$

$$a_{n+2} = \frac{2(n-4)}{(n+2)(n+1)} a_n$$

$$a_4 = 16 = \frac{2(-2)}{4(3)} a_2 = -\frac{1}{3} a_2 \Rightarrow a_2 = -48$$

$$a_2 = -48 = \frac{2(-4)}{2} a_0 = -4a_0 \Rightarrow a_0 = 12$$

$$H_4(x) = 16x^4 - 48x^2 + 12$$

(b)  $H_0(x) = \frac{(2x)^0}{0!} = 1$

$$H_1(x) = \frac{(2x)^1}{1!} = 2x$$

$$H_2(x) = \sum_{n=0}^1 \frac{(-1)^n 2!(2x)^{2-2n}}{n!(2-2n)!} = \frac{2(2x)^2}{2!} - \frac{2}{1} = 4x^2 - 2$$

$$H_3(x) = \sum_{n=0}^1 \frac{(-1)^n 3!(2x)^{3-2n}}{n!(3-2n)!} = \frac{3!(2x)^3}{3!} - \frac{3!(2x)^1}{1} = 8x^3 - 12x$$

$$H_4(x) = \sum_{n=0}^2 \frac{(-1)^n 4!(2x)^{4-2n}}{n!(4-2n)!} = \frac{4!(2x)^4}{4!} - \frac{4!(2x)^2}{2!} + \frac{4!}{2!} = 16x^4 - 48x^2 + 12$$

20. (a)  $xy'' + (1-x)y' + ky = 0$

Let  $y = \sum_{n=0}^{\infty} a_n x^n$ ,  $y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$ ,  $y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$ .

$$\begin{aligned} & x \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + (1-x) \sum_{n=1}^{\infty} n a_n x^{n-1} + k \sum_{n=0}^{\infty} a_n x^n = 0 \\ & \sum_{n=2}^{\infty} n(n-1) a_n x^{n-1} + \sum_{n=1}^{\infty} n a_n x^{n-1} - \sum_{n=1}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} k a_n x^n = 0 \\ & \sum_{n=1}^{\infty} (n+1) n a_{n+1} x^n + \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n - \sum_{n=1}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} k a_n x^n = 0 \\ & (a_1 + k a_0) + \sum_{n=1}^{\infty} [(n+1) n a_{n+1} + (n+1) a_{n+1} - n a_n + k a_n] x^n = 0 \end{aligned}$$

$$a_1 + k a_0 = 0 \quad \Rightarrow \quad a_1 = -k a_0$$

$$(n+1)^2 a_{n+1} + (k-n) a_n = 0 \Rightarrow a_{n+1} = \frac{n-k}{(n+1)^2} a_n$$

Let  $a_0 = 1$ .

For  $k = 0$ ,  $a_1 = a_2 = \dots = 0 \Rightarrow L_0(x) = 1$ .

For  $k = 1$ ,  $a_1 = -1, a_2 = a_3 = \dots = 0 \Rightarrow L_1(x) = 1 - x$ .

For  $k = 2$ ,  $a_1 = -2, a_2 = \frac{-1}{2} a_1 = \frac{1}{2} \Rightarrow L_2(x) = 1 - 2x + \frac{1}{2} x^2$ .

In general, for a given integer  $k \geq 0$ ,  $a_{k+1} = a_{k+2} = \dots = 0$ . Furthermore, in the given formula for

$L_k(x)$ , you can verify that  $a_{n+1} = \frac{n-k}{(n+1)^2} a_n$ . Finally, you can see that for  $k \geq n$ ,

$$\begin{aligned} a_n &= \frac{(n-1)-k}{n^2} a_{n-1} = \frac{(-1)(k-(n-1))}{n^2} a_{n-1} = \frac{(-1)(k+1-n)}{n^2} \cdot \frac{n-2-k}{(n-1)^2} a_{n-2} \\ &= \frac{(-1)^2(k-(n-1))(k-(n-2))}{n^2(n-1)^2} a_{n-2} = \dots = \frac{(-1)^n(k-(n-1))(k-(n-2))\cdots(k-0)}{n^2(n-1)^2\cdots 2^2 \cdot 1^2} a_0 \\ &= \frac{(-1)^2(k-(n-1))(k-(n-2))\cdots k(k-n)!}{(n!)^2(k-n)!} a_0 = \frac{(-1)^n k!}{(k-n)!(n!)^2} \end{aligned}$$

(b)  $L_0(x) = \sum_{n=0}^0 \frac{(-1)^n 0! x^n}{(0-n)!(n!)^2} = 1$

$$L_1(x) = \sum_{n=0}^1 \frac{(-1)^n 1! x^n}{(1-n)!(n!)^2} = 1 - x$$

$$L_2(x) = \sum_{n=0}^2 \frac{(-1)^n 2! x^n}{(2-n)!(n!)^2} = 1 - 2x + \frac{x^2}{2}$$

$$L_3(x) = \sum_{n=0}^3 \frac{(-1)^n 3! x^n}{(3-n)!(n!)^2} = 1 - 3x + \frac{3}{2}x^2 - \frac{x^3}{6}$$

$$L_4(x) = \sum_{n=0}^4 \frac{(-1)^n 4! x^n}{(4-n)!(n!)^2} = 1 - 4x + 3x^2 - \frac{2}{3}x^3 + \frac{1}{24}x^4$$