Resolver los siguientes problemas

1. Un peso de dos libras provoca una extensión de 6 pulgadas en un resorte de dos pies de longitud. Si el peso se hala 3 pulgadas adicionales y se suelta, determinar el movimiento consiguiente despreciado la resistencia del aire. ¿Cuáles son: la amplitud, frecuencia circular y periodo del movimiento?

Rs. X $=\frac{1}{4}\cos 8t\ pies, \frac{1}{4} pie, 8\frac{rad}{s}, \frac{\pi}{4}$ s

$w = 2\,lb$

$l = 2$ pies

$\Delta y = 6\,pulg \cdot \frac{1\ pies}{12\,pulg}$

$\Delta y = \frac{1}{2}$ pies

$y(0) = 3\,pulg \cdot \frac{1\,ft}{12\,pulg} = \frac{1}{4}\,ft$

$y'(0) = 0$

$w = mg$

$m = \frac{2\,lb}{32\,ft/s^2}$

$k = \frac{2\,lb}{1/2} = 4$

$\frac{1}{16}y'' + 4y = 0$

$y'' + 64y = 0$

$r^2 + 64 = 0$

$y = C_1 \cos 8t + C_2 \operatorname{sen} 8t \rightarrow \frac{1}{4} = C_1$

$y' = -8C_1 \operatorname{sen} 8t + 8C_2 \cos 8t \rightarrow 0 = C_2$

$y(t) = \frac{1}{4}\cos(8t)$

$\omega_0 = 8\frac{rad}{s}$

$R = \frac{1}{4}$

$T = \frac{2\pi}{\omega_0} = \frac{2\pi}{8} = \frac{\pi}{4}$ seg.

2. Una masa de 100 gr. Se adhiere a un resorte de acero de longitud de original 50 cm. El resorte se extiende 5 cm por efecto de esta masa. Si se hace mover la masa hacia abajo con una velocidad de 10 cm/seg, determinar el movimiento provocado. Despreciar la resistencia del aire.

Rs. X $=\frac{5}{7} sen 14t\ cm.$

$m = 100\,gr$

$l = 50\,cm$

$\Delta y = 5\,cm$

$y'(0) = -10\,cm/seg$

$c = 0$

$y(0) = 0$

$y'(0) = 10\,cm/seg$

$k = \frac{mg}{\Delta y} = \frac{(100)(9.8)m/s}{0.05\,m}$

$k = 196\,cm \cdot s$

$my'' + cy' + ky = 0$

$100y'' + 19600y = 0$

$100r^2 + 19600 = 0$

$r_{1,2} = 14i$

$y(t) = C_1 \cos 14t + C_2 \operatorname{sen} 14t \rightarrow y(t) = \frac{5}{7} \operatorname{sen} 14t$

$\boxed{0 = C_1}$

$y'(t) = -14\,C_1 \operatorname{sen} 14t + 14\,C_2 \cos 14t$

$10 = 14\,C_2$

$C_2 = \frac{5}{7}$

→ negativo

3. un peso de 3 libras extiende 3 pulgadas un resorte espiral. Si el peso es empujado hacia arriba contrayendo el resorte una distancia de una pulgada y entonces se suelta con una velocidad hacia abajo de 2 pies/s, determinar el movimiento provocado. ¿Cuáles son: la amplitud, frecuencia circular y periodo del movimiento? Despreciar la resistencia del aire.

Rs. $X = \frac{1}{4\sqrt{2}} sen 8\sqrt{2}\,t - \frac{1}{12} cos 8\sqrt{2}\,t\ pies, \frac{\sqrt{22}}{24}\ pies, 8\sqrt{2}\frac{rad}{seg}, \frac{\pi\sqrt{2}}{8} seg.$

Datos:

$w = 3\text{ lb}$

$\Delta y = 3\text{ pulg} \cdot \frac{1\text{ pie}}{12\text{ pulg}} = \frac{1}{4}\text{ pie}$

La ecuación que govierna el sistema es:

$m y'' + c y' + k y = F(t)$

Calcular parametros de información

$w = mg$

$m = \frac{w}{g} = \frac{3\text{ lb}}{32\text{ ft}/s^2} = \frac{3}{32}\frac{\text{lb}\cdot s^2}{\text{ft}} = \frac{3}{32}\text{ slugs}$

constante de amortiguamiento: $c = 0$

$k = \frac{w}{\Delta y} = \frac{3\text{ lb}}{1/4\text{ ft}} = 12\frac{\text{lb}}{\text{ft}}$

Fuerza externa: $F(t) = 0$

Condiciones iniciales:

$y(0) = -1\text{ in} = -12\text{ ft}$

$y'(0) = 2\frac{\text{ft}}{s}$

$\frac{3}{32}y'' + 12y = 0$

$3y'' + 384y = 0$

$y'' + 128y = 0$

$r^2 + 128 = 0$

$r_{1,2} = \pm 8\sqrt{2}\,i$ → $y(t) = C_1 \cos(8\sqrt{2}\,t) + C_2 \operatorname{sen}(8\sqrt{2}\,t)$

$y'(t) = -8\sqrt{2}\,C_1 \operatorname{sen}(8\sqrt{2}\,t) + 8\sqrt{2}\,C_2 \cos(8\sqrt{2}\,t)$

$y(0) = -12 \Rightarrow -1/12 = C_1$

$y'(0) = 2 \Rightarrow 2 = 8\sqrt{2}\,C_2 \Rightarrow C_2 = \frac{1}{4\sqrt{2}}$

$y(t) = -\frac{1}{12}\cos(8\sqrt{2}\,t) + \frac{1}{4\sqrt{2}}\operatorname{sen}(8\sqrt{2}\,t)$

$A = \sqrt{\left(-\frac{1}{12}\right)^2 + \left(\frac{1}{4\sqrt{2}}\right)^2}$

$A = \sqrt{\frac{1}{144} + \frac{1}{32}}$

$A = \frac{\sqrt{22}}{24}$ pies

$y(t) = A\cos(\omega_0 t) + B \operatorname{sen}(\omega_0 t)$

$y(t) = R \cos(\omega_0 t + \delta)$

$\tan\delta = \frac{B}{A}$

$R = \sqrt{A^2 + B^2}$

$T = \frac{2\pi}{\omega_0} = \frac{2\pi}{8\sqrt{2}} = \frac{\pi}{4\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}}$

$T = \frac{\sqrt{2}\,\pi}{8}$ seg

$\omega_0 = 8\sqrt{2}$ rad/s

4. Una masa de 20 gr. Extiende 5 cm. un resorte espiral, si el resorte está conectado a un amortiguador de aceite con un coeficiente de amortiguamiento de 400 dinas/cm., determinar el movimiento provocado si la masa es halada 2 cm. adicionales y luego soltada?

Rs. X $= e^{-10t}\left(2cos4\sqrt{6}t + \frac{5}{\sqrt{6}}sen4\sqrt{6}\,t\right) cm.$

Example An object with mass 20 g stretches a spring 5 cm. Suppose that the mass is also attached to a viscous damper with a damping constant of 400 dyn-s/cm. If the mass is pulled down an additional 2 cm and then released, what is its position u at any time t? In this case, u is a solution of the initial value problem

$$mu'' + \gamma u' + ku = 0, \quad u(0) = 2, \quad u'(0) = 0,$$

where u is measured in cm and

$$m = 20\,\text{g}, \quad \gamma = 400\,\text{dyn-s/cm}, \quad k = \frac{mg}{L} = \frac{(20\,\text{g})(9.8\,\text{m/s}^2)}{0.05\,\text{m}} = 3920\frac{\text{g}}{\text{s}^2}.$$

The characteristic equation is

$$20\lambda^2 + 400\lambda + 3920 = 0,$$

which has roots

$$\lambda_{1,2} = \frac{-\gamma \pm \sqrt{\gamma^2 - 4km}}{2m} = \frac{-400 \pm \sqrt{400^2 - 4(20)(3920)}}{2(20)} = -10 \pm 4\sqrt{6}i.$$

It follows that the general solution is

$$u(t) = c_1 e^{-10t}\cos 4\sqrt{6}t + c_2 e^{-10t}\sin 4\sqrt{6}t.$$

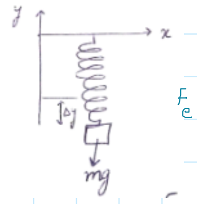
From the initial conditions, we obtain

$$u(0) = c_1 = 2, \quad u'(0) = -10c_1 + 4\sqrt{6}c_2 = 0,$$

which yields the solution

$$u(t) = \underset{c_1}{2}e^{-10t}\cos 4\sqrt{6}t + \underset{c_2}{\frac{5}{\sqrt{6}}}e^{-10t}\sin 4\sqrt{6}t.$$

The quasi frequency is $\mu = 4\sqrt{6}$ rad/s, and the quasi period is $2\pi/\mu = \pi/(2\sqrt{6})$ s. The solution oscillates between the curves $y = \pm Re^{-10t}$, where $R = \sqrt{c_1^2 + c_2^2} = 7/\sqrt{6}$. □



Elongación

$F_e = -kx = -k\Delta y$

$mg - k\Delta y = 0$ (equilibrium)

$k = \frac{mg}{\Delta y}$

$m = 20\text{ gr} \rightarrow 0.02\text{ kg}$

$\Delta y = 5\text{ cm} \rightarrow 0.05\text{ m}$

$c = 400\ \frac{\text{dinas}}{\text{cm}}$

$g = 9.8\ \frac{m}{s^2} = 980\ \frac{cm}{s^2}$

C I

$\begin{cases} y(0) = 2\text{ cm} \\ y'(0) = 0 \end{cases}$ $\quad y = 2 \quad t = 0$ $\quad y' = 0 \quad t = 0$

$k = \frac{mg}{\Delta y} = \frac{(20\text{ g})(980\text{ cm/s}^2)}{5\text{ cm}}$

$k = 3920\ \frac{g}{s^2}$

$my'' + cy' + ky = \cancel{f(t)}^{\,0}$

$20y'' + 400y' + 3920y = 0$

1 newton = 100.000 dinas

1dina = 0.00001 newton

$1\text{ newton (N)} = \frac{1\text{ Kg . m}}{\text{S}^2}$

$1\text{ dina} = \frac{1\text{ g . cm}}{\text{S}^2}$

Ec carac. . $\quad 20r^2 + 400r + 3920 = 0$

$r_{1,2} = -10 \pm 4\sqrt{6}i$

$y(t) = C_1 e^{-10t}\cos(4\sqrt{6}\,t) + C_2 e^{-10t}\operatorname{sen}(4\sqrt{6}\,t)$

$y(t) = e^{-10t}\left(C_1\cos(4\sqrt{6}t) + C_2\operatorname{sen}(4\sqrt{6}t)\right)$

$\boxed{2 = C_1}$

$y'(t) = e^{-10t}\left[-10C_1\cos(4\sqrt{6}t) - 10C_2\operatorname{sen}\cancel{(4\sqrt{6}t)}^{\,0} - 4\sqrt{6}C_1\cancel{\operatorname{sen}(4\sqrt{6}t)} + 4\sqrt{6}C_2\cos(4\sqrt{6}t)\right]$

$0 = -10C_1 + 4\sqrt{6}C_2$

$0 = -10(2) + 4\sqrt{6}C_2$

$\boxed{\frac{5}{\sqrt{6}} = C_2}$

Ec. mov : $\quad y(t) = e^{-10t}\left[2\cos(4\sqrt{6}t) + \frac{5}{\sqrt{6}}\operatorname{sen}(4\sqrt{6}t)\right]$ cm

5. Si un sistema resorte-masa sin amortiguamiento, con un peso de 6libras y una constante de resorte de 1 lb/pulg es puesto repentinamente en movimiento al tiempo t=0, por una fuerza externa de 4cos 7t libras, determinar el movimiento provocado y hacer una gráfica del desplazamiento en función del tiempo.

Rs. $X = \frac{64}{45}(\cos 7t - \cos 8t)) = \frac{128}{45}\sin\frac{1}{2}t\ sen\frac{15}{2}t\ pies.$

Datos

$\omega = 6lb$

$k = 1\frac{lb}{\cancel{pulg}} * \frac{12\cancel{pulg}}{pies}$

$f_e = 4\cos(7t)$

$w = mg$

$m = \frac{6lb}{32\ pies/s^2}$

$m = \frac{3}{16}\ \frac{lb \cdot s^2}{pies}$

C I:

$x(0) = 0$ pies

$x'(0) = 0$ pies/s

$m x'' + c x' + k x = f(t)$

$\frac{3}{16}x'' + 12x = 4\cos(7t)$

$3x'' + 192x = 64\cos(7t)$

i) Hallar x_h

$3r^2 + 192 = 0$

$r^2 = -\frac{192}{3}$

$r_{1,2} = \pm 8i$

$X_h = C_1\cos 8t + C_2\ sen\ 8t$

ii) Hallar x_p.

$x_p = A\cos(7t) + B\ sen(7t)$

$x'_p = -7A\ sen(7t) + 7B\cos(7t)$

$x''_p = -49A\cos(7t) - 49B\ sen(7t)$

$3(-49A\cos(7t) - 49B\ sen(7t)) + 192(A\cos(7t) + B\ sen(7t)) = 64\cos(7t)$

$-147A\cos(7t) - 147B\ sen(7t) + 192A\cos(7t) + 192B\ sen(7t) = 64\cos(7t)$

$-147A + 192A = 64$

$A = \frac{64}{45}$

$-147B + 192B = 0$

$B = 0$

$x(t) = C_1\cos 8t + C_2\ sen\ 8t + \frac{64}{45}\cos(7t) \rightarrow 0 = C_1 + \frac{64}{45} \Rightarrow C_1 = -\frac{64}{45}$

$x'(t) = -8C_1\ sen\ 8t + 8C_2\cos 8t - \frac{64}{45}\cdot 7\ sen(7t) \rightarrow 0 = 8C_2 \Rightarrow C_2 = 0$

Posición : $x(t) = \frac{64}{45}(\cos 7t - \cos 8t)$

$$\cos A - \cos B = -2sen\left(\frac{A+B}{2}\right).sen\left(\frac{A-B}{2}\right)$$

$\cos 7t - \cos 8t = -2\ sen\ \frac{15}{2}t \cdot sen\left(-\frac{t}{2}\right)$

$= 2\ sen\left(\frac{15}{2}t\right) sen\left(\frac{t}{2}\right)$

$x(t) = \frac{128}{45}\ sen\left(\frac{15}{2}t\right) sen\left(\frac{t}{2}\right)$ //

Resolver las siguientes ecuaciones:

$$t^2\frac{d^2z}{dt^2}+5t\frac{dz}{dt}+4z=0$$

$$\boxed{x=e^t}$$

$$\ln|x|=\ln e^t$$

$$\boxed{t=\ln|x|}$$

$$\boxed{\begin{aligned}\frac{dy}{dx}&=e^{-t}\frac{dy}{dt}\\ \frac{d^2y}{dx^2}&=e^{-2t}\left(\frac{d^2y}{dt^2}-\frac{dy}{dt}\right)\\ \frac{d^3y}{dx}&=e^{-3t}\left(\frac{d^3y}{dt^3}-3\frac{d^2y}{dt^2}+2\frac{dy}{dt}\right)\end{aligned}}$$

$$y=\frac{c_1}{x^2}+\frac{c_2\ln(x)}{x^2}$$

$$\frac{d^2w}{dt^2}+\frac{6}{t}\frac{dw}{dt}+\frac{4}{t^2}w=0 \quad * \; t^2$$

$$t^2\frac{d^2w}{dt^2}+6t\frac{dw}{dt}+4w=0$$

$$y=\frac{c_1}{x}+\frac{c_2}{x^4}$$

$$\frac{d^2w}{dt^2}+\frac{6}{t}\frac{dw}{dt}+\frac{4}{t^2}w=0 \quad t^2$$

$$t^2\frac{d^2w}{dt^2}+6t\frac{dw}{dt}+4w=0$$

$$\cancel{t^2}\cdot\frac{1}{\cancel{t^2}}\left(\frac{d^2w}{du}-\frac{dw}{du}\right)+6\cancel{t}\cdot\frac{1}{\cancel{t}}\frac{dw}{du}+4w=0$$

$$\frac{d^2w}{du}+5\frac{dw}{du}+4w=0$$

$$r^2+5r+4=0$$

$$(r+4)(r+1)=0$$

$$w(u)=C_1e^{-u}+C_2e^{-4u}$$

$$w(t)=C_1t^{-1}+C_2t^{-4} \quad //$$

$$\boxed{\begin{aligned}t&=e^u\\ u&=\ln|t|\end{aligned}} \rightarrow \frac{du}{dt}=\frac{1}{t}$$

$$w'=\frac{dw}{dt}\frac{du}{du}=\frac{dw}{du}\cdot\boxed{\frac{du}{dt}}$$

$$\boxed{\frac{dw}{dt}=\frac{1}{t}\frac{dw}{du}}$$

$$\boxed{\frac{d^2w}{dt^2}=\frac{1}{t^2}\left(\frac{d^2w}{du^2}-\frac{dw}{du}\right)}$$

$$(t-2)^2y''(t)-7(t-2)y'(t)+7y(t)=0$$

$$t-2=e^z$$

$$z=\ln|t-2|$$

$$\frac{dz}{dt}=\frac{1}{t-2}$$

$$\frac{d^2z}{dt}=-\frac{1}{(t-2)^2}$$

$$(t-2)^2y''(t)-7(t-2)y'(t)+7y(t)=0$$

$$\boxed{\begin{aligned}(t-2)&=e^u\\ u&=\ln(t-2)\end{aligned}}$$

$$\cancel{(t-2)^2}\cdot\frac{1}{\cancel{(t-2)^2}}\left(\frac{d^2y}{du^2}-\frac{dy}{du}\right)-7\cancel{(t-2)}\cdot\frac{1}{\cancel{(t-2)}}\frac{dy}{du}+7y=0$$

$$\frac{du}{dt}=\frac{1}{t-2}$$

$$\frac{d^2y}{du^2} - 8\frac{dy}{du} + 7y = 0$$

$$y' = \frac{dy}{dt}\cdot\frac{du}{dU} = \frac{dy}{dU}\cdot\boxed{\frac{du}{dt}}$$

$$r^2 - 8r + 7 = 0$$

$$(r-7)(r-1) = 0$$

$$\boxed{y' = \frac{1}{t-2}\frac{dy}{dU}}$$

$$y(u) = C_1 e^{u} + C_2 e^{7u}$$

$$\boxed{y'' = \frac{1}{(t-2)^2}\left(\frac{d^2y}{du^2} - \frac{dy}{du}\right)}$$

$$y(t) = C_1(t-2) + C_2(t-2)^7 \quad //$$

$$t^2y'' + 3ty' + y = t + t^{-1}$$

$$t = e^z$$

$$z = \ln|t|$$

$$\frac{dz}{dt} = \frac{1}{t}$$

$$t^2\cdot\frac{1}{t^2}\left(\frac{d^2y}{dz^2} - \frac{dy}{dz}\right) + 3t\cdot\frac{1}{t}\frac{dy}{dz} + y = e^z + e^{-z}$$

$$\frac{d^2y}{dz^2} - \frac{dy}{dz} + 3\frac{dy}{dz} + y = e^z + e^{-z}$$

$$y' = \frac{dy}{dt}\cdot\frac{dz}{dz} = \frac{dy}{dz}\cdot\frac{dz}{dt}$$

$$\frac{d^2y}{dz^2} + 2\frac{dy}{dz} + y = e^z + e^{-z}$$

$$y' = \frac{1}{t}\frac{dy}{dz}$$

$$y = c_1e^{-t} + c_2te^{-t} + \frac{1}{4}e^{t} + \frac{e^{-t}t^2}{2}$$

$$y'' = \frac{1}{t^2}\left(\frac{d^2y}{dz^2} - \frac{dy}{dz}\right)$$

$$r^2 + 2r + 1 = 0$$

$$(r+1)^2 = 0$$

$$y(z) = C_1 e^{-z} + C_2 z e^{-z}$$

$$y(t) = \frac{C_1}{t} + \frac{C_2\ln|t|}{t} \quad //$$

$$y_p = Ae^{z} + Bz^2e^{-z}$$

$$t^2y'' + 3ty' + y = t + t^{-1}$$

$$t = e^{U}$$

$$u = \ln|t|$$

$$\cancel{t^2}\cdot\frac{1}{\cancel{t^2}}\left(\frac{d^2y}{du^2} - \frac{dy}{du}\right) + 3\cancel{t}\cdot\frac{1}{\cancel{t}}\frac{dy}{dU} + y = e^{U} + e^{-U}$$

$$y' = \frac{1}{t}\frac{dy}{dU}$$

$$\frac{d^2y}{du^2} + 2\frac{dy}{dU} + y = e^{U} + e^{-U}$$

$$y'' = \frac{1}{t^2}\left(\frac{d^2y}{du^2} - \frac{dy}{dU}\right)$$

i). $r^2 + 2r + 1 = 0$

$(r+1)^2 = 0$

$$y(u)_h = C_1 e^{-U} + C_2 U e^{-U}$$

ii) $y(u)_p \rightarrow g(u) = e^{U} + e^{-U}$

$$y_p = Ae^{U} + BU^2e^{-U}$$

$$y'_p = Ae^{U} + 2BUe^{-U} - BU^2e^{-U}$$

$$y''_p = Ae^{U} + 2Be^{-U} - 2BUe^{-U} - 2BUe^{-U} + BU^2e^{-U}$$

$$y''_p = Ae^{U} + 2Be^{-U} - 4BUe^{-U} + BU^2e^{-U}$$

$$y'' : \quad Ae^{u} + 2Be^{-u} - 4Bue^{-u} + Bu^2 e^{-u}$$

$$2y' : \quad 2Ae^{u} + \quad + 4Bue^{-u} - 2Bu^2 e^{-u}$$

$$y : \quad Ae^{u} \quad + Bu^2 e^{-u}$$

$$e^{u} + e^{-u} = 4Ae^{u} + 2Be^{-u}$$

$$4A = 1 \Rightarrow A = 1/4 \qquad y(u)_p = \frac{1}{4}e^{u} + \frac{1}{2}u^2 e^{-u}$$

$$2B = 1 \Rightarrow B = 1/2$$

$$y(u) = C_1 e^{-u} + C_2 u e^{-u} + \frac{1}{4}e^{u} + \frac{1}{2}u^2 e^{-u}$$

$$y(t) = C_1 t^{-1} + C_2 \ln(t) e^{-1} + \frac{1}{4}t + \frac{1}{2}\ln^2(t)\, t^{-1}$$