Estimation of Laplace Series Coefficients via 2-D Numerical Quadrature

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1 Abstract

This paper provides an alternative approach to the computation of Laplace Series coefficients C_l^m and S_l^m with algorithmic complexity $O(Nl^2)$ in time and $O(Nl^2)$ in space. We make use of the fact that the spherical harmonics form a complete and orthogonal set to directly compute C_l^m as a 2-dimensional integral. We estimate this integral using numerical quadrature techniques and the resulting model is blazing fast compared to a least squares model running in $O(Nl^4)$. Harnessing the structure of the data sets, we perform many

2 Preliminaries

Any sufficiently regular function defined on a sphere can be written

$$f(\theta,\phi) = \sum_{l=0}^{+\infty}\sum_{m=0}^{l} \bar{P}_l^m(\cos\theta)[C_l^m \cos m\phi + S_l^m \sin m\phi]. \tag{1}$$

Where C_l^m, S_l^m are *spherical harmonic* coefficients (also referred to as Laplace series coefficients) of degree l and order m; $\bar{P}_l^m$ are the fully normalized Associated Legendre Functions of degree l and order m; θ is the polar angle and ϕ is the azimuthal angle, as per ISO convention[1].

The project proposal suggests computing the coefficients C_l^m, S_l^m by setting up an overdetermined linear system of equations and finding the least squares solution. Unfortunately, this problem runs in $O(Nl_{max}^4)$ time and requires about $8N(l_{max}+1)^2$ bytes of memory. That is absurdly prohibitive. Furthermore, there are no *self-evident* parallelization techniques to speed up computation time. In this report we propose an alternative approach to computing the Laplace series coefficients C_l^m and S_l^m that runs in $O(Nl_{max}^2)$ time complexity with trivial parallelization models.

[1] We have chosen the ISO standard coordinate system to ensure the integrity of future computations. For a discussion on transforming the (ϕ, λ) coordinates provided by NASA, see Appendix A

3 Data Sets

NASA data sets contain the altitude information for a spherical coordinate pair. The spherical coordinates can be represented as the discretization of a 2-d rectangle with width 2π and height π. NASA's data sets contain observations over a discretization of $[-\frac{\pi}{2}, \frac{\pi}{2}] \times [-\pi, \pi]$, represented as a $n_\theta \times n_\phi$ grid with equidistant points.

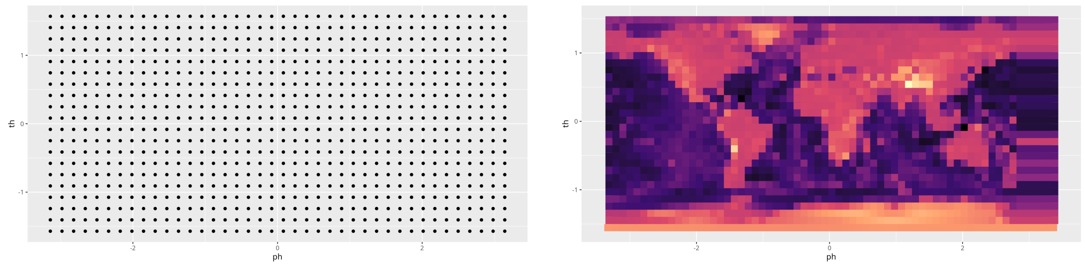


Figure 1: Discretization points (left) of a reduced data set with $n_\theta = 20$, $n_\phi = 40$. On the vertical axis we have $\theta \in [-\pi/2, \pi/2]$, or the latitude; On the horizontal axis we have $\phi \in [-\pi, \pi]$, the longitude.

Due to the linear spacing, we only need to know the number of points n_θ and n_ϕ to recreate the spherical coordinate pairs provided by the data sets, allowing us to decrease the running memory footprint of our program. In order study informative analytics about our model, we first enumerate the characteristics of each data set.

	n_θ	n_ϕ	N
small	180	360	64800
med	540	1080	583200
hi			
ultra			

$$\theta \in [0, \pi], \quad \phi \in [0, \pi 2]$$

We'll use these values later when we study the time it takes to compute our model.

3.1 Visualization

TODO: [ ] add 3-d maps

4 Mathematical Model

4.1 Coefficients

The coefficients C_l^m and S_l^m are defined as the orthogonal projection of $f(\theta, \phi)$ onto the basis functions Y_l^m, the spherical harmonics of degree l and order m. This definition is perfectly analagous to the computation of fourier coefficients for 2π periodic functions. Written in terms of the associated legendre polynomials, these integrals are defined as

$$C_l^m = \int_0^{2\pi} \int_0^{\pi} f(\theta, \phi) P_l^m(\cos\theta) \cos(m\phi) \sin\theta \, d\theta \, d\phi$$
$$S_l^m = \int_0^{2\pi} \int_0^{\pi} f(\theta, \phi) P_l^m(\cos\theta) \sin(m\phi) \sin\theta \, d\theta \, d\phi$$

Taking advantage of the structure of the data sets, we approximate these coefficients via numerical quadrature.

4.1.1 Computation via Numerical Integration

We approximate the Laplace Series coefficients via a 2 dimensional analogue of the rectangle rule.

$$\begin{aligned} C_l^m &\approx \sum_{i=1}^{n_\phi}\sum_{j=1}^{n_\theta} f(\theta_j,\phi_i)P_l^m(\cos\theta_j)\cos(m\phi_i)\sin\theta_j\Delta\theta\Delta\phi \\ &\approx \Delta\theta\Delta\phi\sum_{i=1}^{n_\phi}\cos(m\phi_i)\sum_{j=1}^{n_\theta} f(\theta_j,\phi_i)P_l^m(\cos\theta_j)\sin\theta_j \end{aligned}$$

$$\begin{aligned} S_l^m &\approx \sum_{i=1}^{n_\phi}\sum_{j=1}^{n_\theta} f(\theta_j,\phi_i)P_l^m(\cos\theta_j)\sin(m\phi_i)\sin\theta_j\Delta\theta\Delta\phi \\ &\approx \Delta\theta\Delta\phi\sum_{i=1}^{n_\phi}\sin(m\phi_i)\sum_{j=1}^{n_\theta} f(\theta_j,\phi_i)P_l^m(\cos\theta_j)\sin\theta_j \end{aligned}$$

This method computes *individual* Laplace series coefficients in $O(N)$ time! Thus, computing the coefficients for a Discrete Laplace Series (DLS) of order l_max runs in $O(Nl_{max}^2)$. More interestingly, each C_l^m or S_l^m can be computed **independently** of any other coefficient. Whereas solving the linear least squares problem requires computing every C_l^m and C_l^m in a single operation, this method allows us the ability to compute any arbitrary C_l^m directly. Why do we care? If I alraedy have the first 500 C_l^m coefficients computed and stored in a text file, I can *directly* compute the 501st in $O(N)$ time. Furthermore, this method gives rise to obvious and easy-to-implement parallelization schemes.

4.2 Comparison with Linear Squares

4.2.1 Timing

Let's start by exploring the small data set and gathering data for how long it takes to compute a model.

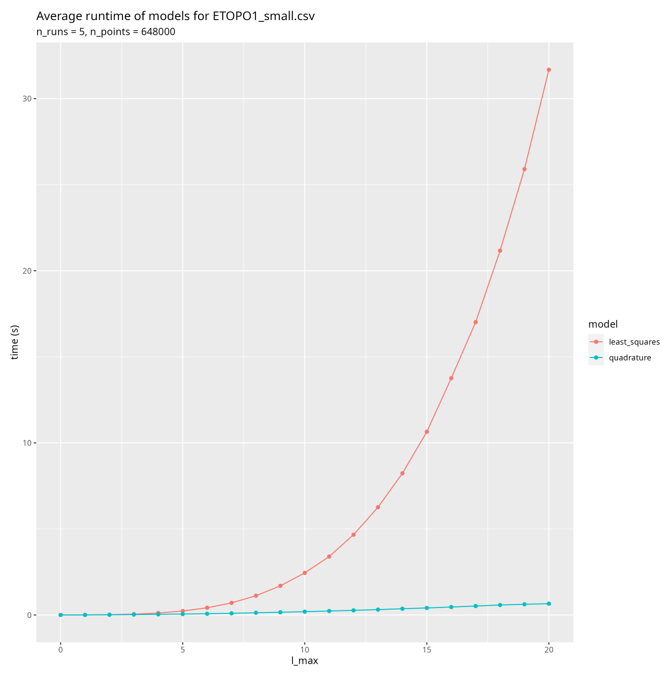


Figure 2: A timing comparison between least squares $\in O(Nl^4)$ and numerical quadrature $\in O(Nl^2)$ to compute the coefficients C_l^m and S_l^m for the small ETOPO1 data set.

Tested on the `ETOPO1_small.csv` data set with $l_{max} = 20$, the model computed via numerical integration had an average runtime of 0.656s across 5 trials, compared to the old model which ran in 31.678 seconds. And that's just for a small l. Running the algorithm in sequential for higher values of l would quickly become intractable for the least squares algorithm due to its steep algorithmic complexity. So, for an $l = 20$, numerical quadrature ran on average 50x faster than its least squares counterpart. As the degree of the model increases, so does the disparity between the two runtimes; at $l = 20$ our algorithm runs 50 times faster but at $l = 100$ our algorithm could be running 1250 times faster.

Clearly the proposed numerical integration model is faster. But how do the estimated coefficients perform relative to the coefficients that minimize the MSE? **TODO**

- Provide more detailed numeric comparison
- Provide timing analysis

section

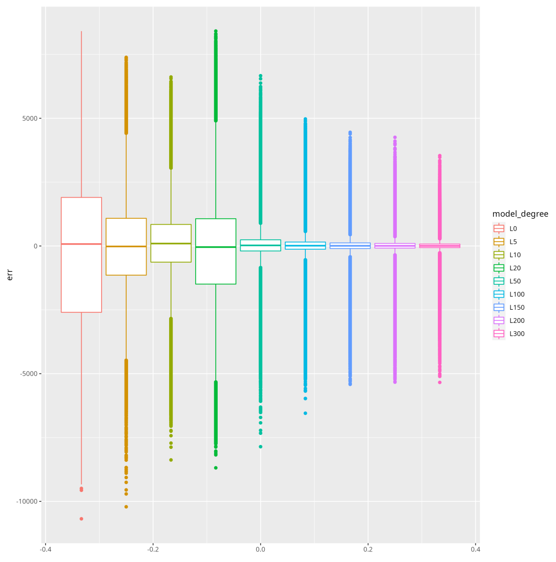
5 Improvements to the model

The least squares method provides a more accurate model, but runs in $O(Nl_{max}^4)$ time. In this section we propose improvements to our model computed via quadrature.

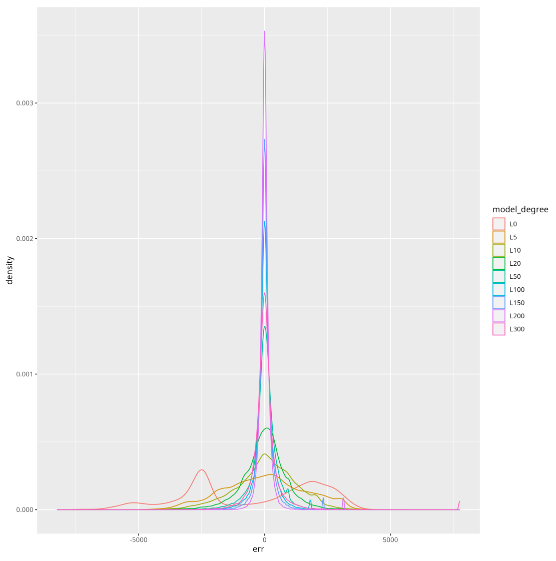
5.1 Gradient Descent

We can apply methods of optimization to minizmize the mean squared error (MSE) of our model. To begin we explicitze the MSE then compute its gradient

TODO I have already derived the gradient of the MSE with respect to all C_l^m and S_l^m on paper



err
5000
0
-5000
-10000
-0.4
-0.2
0.0
0.2
0.4
model_degree
L0
L5
L10
L20
L50
L100
L150
L200
L300



density
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0.002
0.001
0.000
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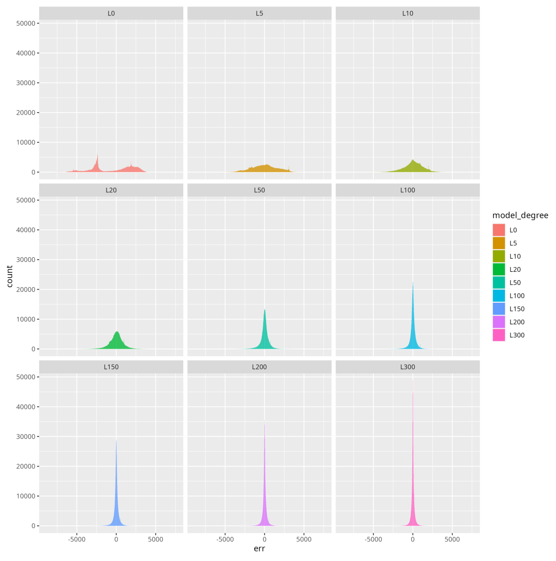
0
5000
err
model_degree
L0
L5
L10
L20
L50
L100
L150
L200
L300



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40000
30000
20000
10000
0
L0
L5
L10
L20
L50
L100
L150
L200
L300
-5000
0
5000
err
model_degree

- Write up derivation in latex

Computing the gradient vector runs in $O(Nl_{max}^2)$ time

5.1.1 Stochastic Gradient Descent

Computing the gradient with respect to a single C_l^m requires summing up certain values for all N points of the data set. For larger data sets (e.g. $N_{high} = 1800 * 3600 = 6480000$), this computation can become intractable. We can speed up our gradient computation by computing as estimate of the gradient, $\hat{\nabla}$MSE, by randomly sampling n points from our entire data set. This reduces the computation of a gradient vector from $O(Nl_{max}^2)$ to $O(nl_{max}^2)$. This is an **insane** improvement to the run time of our algorithm. Instead of summing all 6480000 points of the large data set, we could fix n at say $n = 1000$ instead, sacrificing accuracy for a speedeup of 6,480.

5.2 MCMC Algorithms

In initial tests my gradient descent learning is *sometimes* reducing the MSE but other times it's increasing it... If I don't figure out exactly what's going wrong then I plan on implementing MCMC algorithms like Gibb's sampling to explore the sample space and only accept proposal vectors that decrease the MSE.

6 Parallelization

TODO (trivial)

7 Conclusion

Instead of calculating the model with a direct method that runs in $O(Nl_{max}^4)$ time, We propose a hybrid model that first directly computes the coefficients in $O(Nl_{max}^2)$ time then iteratively improves the MSE via stochastic methods running in $O(n_{training} n_{sample} l_{max}^2)$ time.

Results are to follow :)

A Spherical Coordinates Transformation

The data sets `ETOP01_*.csv` provided by NASA represent spherical coordinates in an unconventional format:

$$(\phi, \lambda) \in [-\frac{\pi}{2}, \frac{\pi}{2}] \times [-\pi, \pi]$$

where ϕ refers to the latitude and λ refers to the longitude. In this section we outline the transformation from NASA's coordinate system to the ISO standard representation and prove equivalence between (1) and the Project presentation's $f(\phi, \lambda)$.

To avoid confusion, we denote the spherical coordinate pair in ISO coordinates as $(\theta_{iso}, \phi_{iso})$ and NASA's coordinate scheme as $(\phi_{nasa}, \lambda_{nasa})$. We remark that the polar angle θ_{iso} is simply equal to $\frac{\pi}{2} - \phi_{nasa}$. We corroborate this fact by remarking that a latitude of $\phi_{nasa} = 0$ corresponds to a polar angle of $\phi_{iso} = \frac{\pi}{2}$. Conveniently, the longitude λ_{nasa} is equivalent to the azimuthal angle ϕ_{iso}.

Proposition 1. Let the spherial coordinate pairs $(\theta_{iso}, \phi_{iso})$, $(\phi_{nasa}, \lambda_{nasa})$ be defined as above.

$$f(\theta_{iso}, \phi_{iso}) \equiv f(\phi_{nasa}, \lambda_{nasa})$$

Proof. $f(\phi_{nasa}, \lambda_{nasa})$ is defined in the Project proposal as

$$
\begin{aligned}
f(\phi_{nasa}, \lambda_{nasa}) &= \sum_{l=0}^{+\infty} \sum_{m=0}^{l} \bar{P}_l^m(\sin\phi_{nasa})[C_l^m \cos m\lambda_{nasa} + S_l^m \sin m\lambda_{nasa}] && (2) \\
&= \sum_{l=0}^{+\infty} \sum_{m=0}^{l} \bar{P}_l^m(\sin(\pi/2 - \theta_{iso}))[C_l^m \cos m\phi_{iso} + S_l^m \sin m\phi_{iso}] && (3) \\
&= \sum_{l=0}^{+\infty} \sum_{m=0}^{l} \bar{P}_l^m(\cos\theta_{iso}))[C_l^m \cos m\phi_{iso} + S_l^m \sin m\phi_{iso}] && (4) \\
&\equiv f(\theta_{iso}, \phi_{iso}) && (5)
\end{aligned}
$$

□

References

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