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Nonlinear gyrokinetics: a powerful tool for the description of microturbulence in magnetized plasmas

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Abstract
Gyrokinetics is the description of low-frequency dynamics in magnetized plasmas. In magnetic-confinement fusion, it provides the most fundamental basis for numerical simulations of microturbulence; there are astrophysical applications as well. In this tutorial, a sketch of the derivation of the novel dynamical system comprising the nonlinear gyrokinetic (GK) equation (GKE) and the coupled electrostatic GK Poisson equation will be given by using modern Lagrangian and Lie perturbation methods. No background in plasma physics is required in order to appreciate the logical development. The GKE describes the evolution of an ensemble of gyrocenters moving in a weakly inhomogeneous background magnetic field and in the presence of electromagnetic perturbations with wavelength of the order of the ion gyroradius. Gyrocenters move with effective drifts, which may be obtained by an averaging procedure that systematically, order by order, removes gyrophase dependence. To that end, the use of the Lagrangian differential one-form as well as the content and advantages of Lie perturbation theory will be explained. The electromagnetic fields follow via Maxwell's equations from the charge and current density of the particles. Particle and gyrocenter densities differ by an important polarization effect. That is calculated formally by a 'pull-back' (a concept from differential geometry) of the gyrocenter distribution to the laboratory coordinate system. A natural truncation then leads to the closed GK dynamical system. Important properties such as GK energy conservation and fluctuation noise will be mentioned briefly, as will the possibility (and difficulties) of deriving nonlinear gyrofluid equations suitable for rapid numerical solution—although it is probably best to directly simulate the GKE. By the end of the tutorial, students should appreciate the GKE as an extremely powerful tool and will be prepared for later lectures describing its applications to physical problems.

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1. Introduction: heuristic gyrokinetics

This article[1] is an introductory tutorial on the plasma-physics formalism known as *nonlinear gyrokinetics*, which is used for the description of low-frequency fluctuations in magnetized plasmas. No prior knowledge of plasma physics is assumed.

From the perspective of a conference on 'Turbulent Mixing and Beyond,' the relevance of this article is that it provides an entry point to a vast literature on research directed at the magnetic confinement of fusion plasmas. Observed particle, momentum and heat losses in tokamak devices are generally much larger than what can be attributed to classical Coulomb collisions, and are believed to result from low-frequency microturbulence. Both analytically and numerically, nonlinear gyrokinetics provides the appropriate description of that turbulence.

A comprehensive review of the nonlinear gyrokinetic (GK) formalism has been given by Brizard and Hahm (2007); that sophisticated article contains the details of many issues

[1] This article is the written version of a tutorial talk presented at the Second International Conference and Advanced School on 'Turbulent Mixing and Beyond,' 27 July–7 August, 2009, International Centre for Theoretical Physics, Trieste, Italy.

not covered here. A closely related review article by Cary and Brizard (2009) on the Hamiltonian theory of guiding-center motion is also of interest. This article merely attempts to focus on the core ideas and technical tools; if those are firmly in hand, the details will take care of themselves.

For neutral fluids, the canonical partial differential equation (PDE) is the Navier–Stokes equation, displayed here in the incompressible limit:

$$\partial_t \boldsymbol{u}(\boldsymbol{x}, t) + \boldsymbol{u} \cdot \nabla \boldsymbol{u} = -\nabla P + \mu \nabla^2 \boldsymbol{u} \qquad (\nabla \cdot \boldsymbol{u} = 0). \quad (1)$$

Of course, this dynamical equation lives in a three-dimensional (3D) configuration space.

Plasmas[2] (collections of charged particles) present several kinds of additional complexity. Because of wave–particle interactions (responsible for linear growth rates and kinetic dissipation, i.e. Landau damping) and Coulomb collisions, it is frequently necessary to study the distribution of particles in a *6D phase space* comprising both position $\boldsymbol{x}$ and velocity $\boldsymbol{v}$. Furthermore, one must calculate from Maxwell's equations a self-consistent electric field $\boldsymbol{E}$ and magnetic field $\boldsymbol{B}$. One is then led to the plasma kinetic equation for the probability density function (PDF) of the particles of species s:

$$\frac{\partial f_s(\boldsymbol{x}, \boldsymbol{v}, t)}{\partial t} + \boldsymbol{v} \cdot \nabla f + \left(\frac{q}{m}\right)_s (\boldsymbol{E} + c^{-1}\boldsymbol{v} \times \boldsymbol{B}) \cdot \frac{\partial f}{\partial \boldsymbol{v}} = -C_s[f]. \quad (2)$$

Here $C_s[f]$ is the plasma collision operator, frequently approximated by the Landau form[3]. Because it supports physics on greatly disparate length and time scales and evolves a 6D dynamics, this equation is extremely complicated; it is virtually impossible to solve it numerically in its raw form for macroscopic time and length scales characteristic of current and future fusion devices such as the ITER machine currently under construction in France (ITE 2009).

Fortunately, for the microturbulence characteristic of fusion devices, the collective modes of oscillation (generically, 'drift waves') have frequencies much lower than the ion gyrofrequency $\omega_{\mathrm{ci}} \doteq q_{\mathrm{i}} B / m_{\mathrm{i}} c$ (I use $\doteq$ for definitions):

$$\frac{\omega}{\omega_{\mathrm{ci}}} \lesssim \frac{\rho_{\mathrm{i}}}{L_n} \sim \frac{0.2\,\mathrm{cm}}{2 \times 10^2\,\mathrm{cm}} \approx 10^{-3} \ll 1. \quad (3)$$

Here $\rho_{\mathrm{i}} \doteq v_{t\mathrm{i}}/\omega_{\mathrm{ci}}$ is the ion gyroradius, $v_{t\mathrm{i}} \doteq (T_{\mathrm{i}}/m_{\mathrm{i}})^{1/2}$ is the ion thermal velocity and L_n is the characteristic scale length of the background density profile. Estimate (3) implies that one can make some analytical reductions by averaging the motion over the rapid gyration (described by gyrophase ζ). The final result of that program is the *nonlinear GK equation (GKE)*, the physics of which I will describe heuristically in the remainder of this section. In sections 2 and 3 I will sketch a more systematic derivation.

[2] An introductory textbook on plasma physics is by Chen (1983). A tutorial set of lectures that introduces many of the plasma-physics topics mentioned in this article is by Krommes (2006a).

[3] The Landau collision operator is derived in many plasma-physics textbooks and monographs, such as the one by Helander and Sigmar (2002).

1.1. The drift equations

For electromagnetic fluctuations that satisfy $\omega/\omega_{\mathrm{ci}} \ll 1$ and $k_\perp \rho_{\mathrm{i}} \ll 1$ ($k_\perp$ is a characteristic wave number perpendicular to $\boldsymbol{B}$), it is well known that the gyro-averaged particle motion obeys the *drift equations* (I now drop the species label)

$$\frac{\mathrm{d}\boldsymbol{X}}{\mathrm{d}t} = v_\parallel \widehat{\boldsymbol{b}} + \boldsymbol{V}_E + \boldsymbol{V}_d, \quad (4a)$$

$$\frac{\mathrm{d}v_\parallel}{\mathrm{d}t} = \frac{q}{m} E_\parallel + \mu \omega_c \nabla_\parallel \ln B, \quad (4b)$$

$$\frac{\mathrm{d}\mu}{\mathrm{d}t} = 0. \quad (4c)$$

Here $B \doteq |\boldsymbol{B}|$, $\widehat{\boldsymbol{b}} \doteq \boldsymbol{B}/B$, the *guiding-center drifts* are

$$\boldsymbol{V}_E \doteq c\boldsymbol{E} \times \widehat{\boldsymbol{b}}/B \quad (\boldsymbol{E} \times \boldsymbol{B}\ \text{drift}), \quad (5a)$$

$$\boldsymbol{V}_d \doteq \frac{v_\perp^2/2}{\omega_c} \widehat{\boldsymbol{b}} \times \nabla \ln B + \frac{v_\parallel^2}{\omega_c} \widehat{\boldsymbol{b}} \times (\widehat{\boldsymbol{b}} \cdot \nabla \widehat{\boldsymbol{b}}) \quad \text{(magnetic drifts)}, \quad (5b)$$

and the *magnetic moment* is approximately

$$\mu \approx \tfrac{1}{2} m v_\perp^2 / \omega_c(\boldsymbol{x}). \quad (6)$$

One may use the drifts as characteristics in the evolution equation for the guiding-center PDF:

$$\frac{\partial F(\boldsymbol{X}, v_\parallel, \mu, t)}{\partial t} + \underbrace{v_\parallel \nabla_\parallel F}_{\substack{\text{parallel}\\ \text{streaming}}} + \underbrace{(\boldsymbol{V}_E + \boldsymbol{V}_d) \cdot \nabla F}_{\substack{\text{guiding-center}\\ \text{drifts}}} + \underbrace{\left(\frac{q}{m} E_\parallel + \mu \omega_c \nabla_\parallel \ln B\right) \frac{\partial F}{\partial v_\parallel}}_{\text{parallel acceleration}} = 0. \quad (7)$$

The key features of this equation are that (i) it contains no $\partial/\partial\zeta$ (thus one has reduced the dimensionality by one and removed high-frequency motion) and (ii) it contains no $\partial/\partial\mu$ because the magnetic moment is (adiabatically) conserved. Thus, μ enters only as a parameter.

1.2. The gyrokinetic equation

For $k_\perp \rho_{\mathrm{i}} \ll 1$, the particle position and the guiding-center position are essentially coincident. But it turns out that $k_\perp \rho_{\mathrm{i}} = O(1)$ for the microturbulence of interest[4]. In this situation, one must introduce the notion of the *gyrocenter*, which is the *gyro-averaged* position of the particle. (For circular motion, the gyrocenter is precisely defined by $\boldsymbol{X} \doteq \boldsymbol{x} - \boldsymbol{\rho}$, where $\boldsymbol{\rho}$ is the vector gyroradius $\boldsymbol{\rho} = \rho \widehat{\boldsymbol{a}}$; see figure 1). Figure 2 makes it clear that gyrocenters feel *effective* potentials. In order to quantify this notion, let $\boldsymbol{E} = -\nabla\varphi$ (electrostatic approximation). Upon introducing the Fourier representation

[4] From the estimate in (3), one sees that ρ_{i} is much less than the characteristic macroscopic scale, which is why one refers to *microturbulence*.

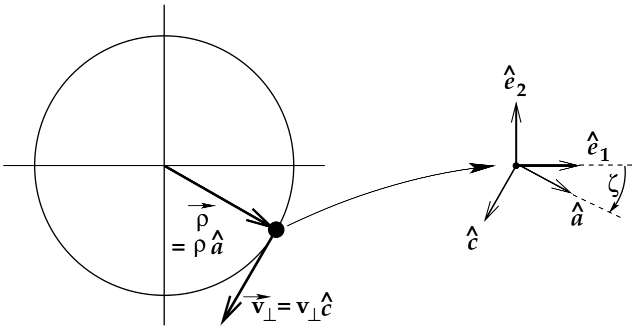

Figure 1. Illustration of the lowest-order GK variables. $\widehat{\boldsymbol{a}}$ and $\widehat{\boldsymbol{c}}$ rotate with the particle. Vectors can be decomposed with respect to either of the orthonormal sets $\{\widehat{\boldsymbol{a}}, \widehat{\boldsymbol{b}}, \widehat{\boldsymbol{c}}\}$ or $\{\widehat{\boldsymbol{e}}_1, \widehat{\boldsymbol{e}}_2, \widehat{\boldsymbol{b}}\}$, where $\widehat{\boldsymbol{e}}_1$ is chosen arbitrarily in the plane locally perpendicular to B. The gyrophase ζ is measured clockwise from $\widehat{\boldsymbol{e}}_1$.

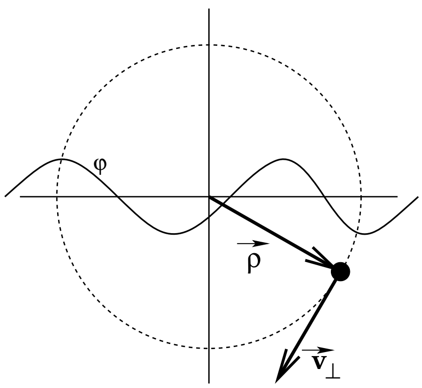

Figure 2. When $k_\perp\rho = O(1)$, the gyrocenter feels an effective potential that is averaged over the phases of the wave.

$\varphi(\boldsymbol{x}) = (2\pi)^{-3}\int \mathrm{d}\boldsymbol{k}\,\varphi_{\boldsymbol{k}}\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}$ and assuming circular motion, one has

$$\langle\varphi(\boldsymbol{x})\rangle_\zeta \equiv \overline{\varphi}(\boldsymbol{X}) = \frac{1}{2\pi}\int_0^{2\pi}\mathrm{d}\zeta\int\frac{\mathrm{d}\boldsymbol{k}}{(2\pi)^3}\varphi_{\boldsymbol{k}}\,\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot[\boldsymbol{X}+\boldsymbol{\rho}(\zeta)]} \tag{8a}$$

$$= \int\frac{\mathrm{d}\boldsymbol{k}}{(2\pi)^3}\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{X}}J_0(k_\perp\rho)\varphi_{\boldsymbol{k}}, \tag{8b}$$

where J_0 is a Bessel function. We see that the effective potential felt by the spiraling particle is reduced by the wave-number-dependent factor $J_0(k_\perp\rho)$. This is a kind of renormalization. The corrections embodied in the J_0 factor are called *finite-Larmor-radius* (FLR) *effects*.

Now, by appropriately introducing the effective fields, one can write the GKE:

$$\frac{\partial F(\boldsymbol{X}, v_\parallel, \mu, t)}{\partial t} + \underbrace{v_\parallel\nabla_\parallel F}_{\text{parallel streaming}} + \underbrace{(\overline{\boldsymbol{V}}_E + \boldsymbol{V}_d)\cdot\nabla F}_{\text{gyrocenter drifts}}$$
$$+ \underbrace{\left(\frac{q}{m}\overline{E}_\parallel + \mu\omega_c\nabla_\parallel\ln B\right)\frac{\partial F}{\partial v_\parallel}}_{\text{parallel acceleration}} = -C[F]. \tag{9}$$

This GKE is the workhorse of research on modern fusion microturbulence. It has the same structure as the guiding-center kinetic equation (7): it contains neither a $\dot{\zeta}$ term (gyration has been averaged away) nor a $\dot{\mu}$ term (the magnetic moment is conserved and enters merely as a parameter); the only new wrinkle is the presence of the gyro-averaged fields (indicated by the overlines).

This 'derivation' of the GKE is incomplete, because I have been cavalier about the definitions of the GK variables. In realistic, inhomogeneous magnetic fields, particles do not execute perfect circles around the field lines; therefore, it is unclear how to precisely define a gyrocenter. The same issue afflicts the definition of the magnetic moment. In section 2, we will see how these difficulties can be overcome by a more systematic approach.

1.3. The GK Poisson equation

To close the GKE, one needs the electromagnetic fields, so we must consider the GK Maxwell equations. For simplicity, I will consider only electrostatics in the remainder of the article[5]. Then one has $\boldsymbol{E} = -\nabla\varphi$ and one must solve Poisson's equation

$$-\nabla^2\varphi(\boldsymbol{x}, t) = 4\pi\rho(\boldsymbol{x}, t), \tag{10}$$

where ρ is the charge density[6]. Although this equation appears innocent, subtlety arises because in gyrokinetics the particle position $\boldsymbol{x}$ and the gyrocenter position $\boldsymbol{X}$ are distinct. One needs the particle charge density $\rho(\boldsymbol{x})$ but has available from solution of the GKE the *gyrocenter* (phase-space) density $F(\boldsymbol{Z})$. The route between these involves a coordinate transformation denoted by the operator T (to be defined precisely in section 2.3). Thus,

$$F(\boldsymbol{Z}) \overset{T}{\Rightarrow} \rho(\boldsymbol{x}) \overset{\text{Poisson}}{\Rightarrow} \varphi(\boldsymbol{x}) \overset{\nabla}{\Rightarrow} \boldsymbol{E}(\boldsymbol{x}) \overset{T^{-1}}{\Rightarrow} \boldsymbol{E}(\boldsymbol{X}) \overset{\text{GKE}}{\Rightarrow} F(\boldsymbol{Z}). \tag{11}$$

Now although T is quite complicated in general, the most important physical distinction between particles and gyrocenters is easy to understand; it lies in the *polarization drift* (of the ions[7]). Iterative solution of the ion fluid equation

$$m\frac{\mathrm{d}\boldsymbol{u}}{\mathrm{d}t} = q(\boldsymbol{E} + c^{-1}\boldsymbol{u}\times\boldsymbol{B}) + \cdots \tag{12}$$

for constant B and slow variations leads to $\boldsymbol{u} = \boldsymbol{u}_{\boldsymbol{E}} + \boldsymbol{u}^{\mathrm{pol}} + \cdots$, where (the lowest-order approximation to) the *polarization drift velocity* is (correct for $T_i = 0$)

$$\boldsymbol{u}^{\mathrm{pol}} \doteq \frac{1}{\omega_c}\frac{\partial}{\partial t}\left(\frac{c\boldsymbol{E}_\perp}{B}\right). \tag{13}$$

The polarization drift leads to a *polarization charge density* ρ^{pol} that heuristically obeys a simple continuity equation:

$$\partial_t\rho^{\mathrm{pol}} = -\nabla\cdot\boldsymbol{j}^{\mathrm{pol}} = -\nabla\cdot(nq\boldsymbol{u}^{\mathrm{pol}}). \tag{14}$$

Because $\boldsymbol{u}^{\mathrm{pol}}$ contains a time derivative, one can integrate immediately to find

$$\rho_{\mathrm{i}}^{\mathrm{pol}} = (nq)_{\mathrm{i}}\rho_{\mathrm{s}}^2\nabla_\perp^2\Phi, \tag{15}$$

[5] Electromagnetic corrections can be important for modern, high-pressure tokamaks, but the electrostatic analysis captures the essence of the difficulties.
[6] The distinction between the gyroradius and the charge density, both denoted by the symbol ρ, should be clear from the context.
[7] Electron polarization is negligible, as can be seen from (13).

where $\Phi \doteq e\varphi/T_\mathrm{e}$, $\rho_\mathrm{s} \doteq c_\mathrm{s}/\omega_\mathrm{ci}$ is the so-called *sound radius* and $c_\mathrm{s} \doteq (ZT_\mathrm{e}/m_\mathrm{i})^{1/2}$ is the *sound speed* (Z is the atomic number[8]). Note that $\rho_\mathrm{s} = \rho_\mathrm{i}$ for $T_\mathrm{i} = T_\mathrm{e}$, but $\rho_\mathrm{i} \to 0$ as $T_\mathrm{i} \to 0$ while ρ_s remains nonzero in that limit.

One can now process Poisson's equation by separating the total charge into a gyrocenter charge and a polarization charge:

$$-\nabla^2\varphi = 4\pi\rho = 4\pi[(\rho_\mathrm{i}^\mathrm{G} + \rho_\mathrm{i}^\mathrm{pol}) + (\rho_\mathrm{e}^\mathrm{G} + \rho_\mathrm{e}^\mathrm{pol})]. \tag{16}$$

Electron polarization is negligible, so (16) becomes

$$-\nabla^2\Phi = k_\mathrm{De}^2\left(\frac{n_\mathrm{i}^\mathrm{G}}{\overline{n}_\mathrm{i}} + \rho_\mathrm{s}^2\nabla_\perp^2\Phi - \frac{n_\mathrm{e}^\mathrm{G}}{\overline{n}_\mathrm{e}}\right), \tag{17}$$

where $k_\mathrm{De} \doteq (4\pi n_\mathrm{e}e^2/T_\mathrm{e})^{1/2}$ is the inverse of the electron Debye length λ_De. Finally, it is natural to place the ion polarization term on the left-hand side, where it renormalizes the vacuum dielectric permittivity $(= 1)$ of the original Poisson equation:

$$-\Bigg[\underbrace{\nabla^2}_{\substack{\text{original}\\ \text{Poisson}}} + \underbrace{\left(\frac{\rho_\mathrm{s}^2}{\lambda_\mathrm{De}^2}\right)\nabla_\perp^2}_{\substack{\text{ion}\\ \text{polarization}}}\Bigg]\Phi = k_\mathrm{De}^2\Bigg(\underbrace{\frac{n_\mathrm{i}^\mathrm{G}}{\overline{n}_\mathrm{i}} - \frac{n_\mathrm{e}^\mathrm{G}}{\overline{n}_\mathrm{e}}}_{\substack{\text{gyrocenter}\\ \text{charge density}}}\Bigg). \tag{18}$$

Note that the left-hand side of (18) is the only place where the polarization effect enters; in particular, *the polarization drift does not appear in the GKE.*

Equation (18) is similar to the familiar equation $\varepsilon\nabla\cdot\boldsymbol{E} = 4\pi\rho$ used for the description of dielectric media. One is therefore motivated to define the *dielectric constant of the GK vacuum* (this phrase is explained below) as

$$\varepsilon^\mathrm{G} \doteq \frac{\rho_\mathrm{s}^2}{\lambda_\mathrm{De}^2} = \frac{\omega_{pi}^2}{\omega_\mathrm{ci}^2}. \tag{19}$$

Then Poisson's equation can be written as

$$-\lambda_\mathrm{De}^2\Bigg(\overbrace{\underbrace{\nabla^2}_{\substack{\text{original}\\ \text{Poisson}}}}^{\text{small}} + \overbrace{\underbrace{\varepsilon^\mathrm{G}\nabla_\perp^2}_{\substack{\text{ion}\\ \text{polarization}}}\Bigg)\Phi = \underbrace{\frac{n_\mathrm{i}^\mathrm{G}}{\overline{n}_\mathrm{i}} - \frac{n_\mathrm{e}^\mathrm{G}}{\overline{n}_\mathrm{e}}}_{\substack{\text{gyrocenter}\\ \text{(charge) density}}}}^{\text{quasineutrality}}. \tag{20}$$

Now for fusion applications it turns out that $\varepsilon^\mathrm{G} \gg 1$. In that case, one can neglect the ∇^2 of the original Poisson equation and deal with the *quasineutrality condition*

$$-\rho_\mathrm{s}^2\nabla_\perp^2\Phi = \frac{n_\mathrm{i}^\mathrm{G}}{\overline{n}_\mathrm{i}} - \frac{n_\mathrm{e}^\mathrm{G}}{\overline{n}_\mathrm{e}}. \tag{21}$$

The left-hand side describes the ion polarization effect while the right-hand side gives the net gyrocenter (charge) density. (Equation (21) is correct only for $T_\mathrm{i} = 0$, but it can be generalized; see section 3.)

We now have a self-consistent and closed set of equations, the *GK Poisson system*. Given the potential, the GKE evolves the gyrocenter PDF F. That can be used to calculate the gyrocenter charge; then the GK Poisson equation can be solved for the potential.

Although the operations I have described are rather simple, the final result is conceptually profound. We have obtained a new nonlinear dynamical system that is not continuously deformable into the original kinetic equation. The clean separation between the $\boldsymbol{E}\times\boldsymbol{B}$ and magnetic drifts (appearing only in the gyrocenter kinetic equation) and the polarization drift (whose effect appears only in the GK Poisson equation) motivates an interpretation built upon a *GK vacuum* (Krommes 1993a). The GK vacuum is an ether-like medium with dielectric permittivity ε^G. Into that vacuum state, one now places gyrocenters (each characterized by its own conserved $\overline{\mu}$). Those move cross-field with the (effective) $\boldsymbol{E}\times\boldsymbol{B}$ and magnetic drifts, and are accelerated parallel to the lines by the effective parallel electric field and the $\mu\nabla_\parallel B$ mirroring force. The electromagnetic potentials are then determined from the GK Maxwell equations. (Here I have only discussed the electrostatic limit and the GK Poisson equation.)

This interpretation involving the GK vacuum is physically compelling, and it argues against attempts to incorporate the polarization drift into the kinetic equation. That is, in fact, done in some formulations (notably those based on the methodology of Frieman and Chen (1982)), which if carried through correctly will ultimately also give the correct low-frequency physics. But the present approach provides *the* fundamental description of gyrocenters, which is the natural entity that emerges in the low-frequency regime.

The ion polarization response is intimately related to the plasma vorticity. Indeed, if one takes the curl of the $\boldsymbol{E}\times\boldsymbol{B}$ velocity for constant $\boldsymbol{B}$, one finds that the ($\widehat{\boldsymbol{b}}$-directed) vorticity of the $\boldsymbol{E}\times\boldsymbol{B}$ motion is

$$\varpi \doteq \widehat{\boldsymbol{b}}\cdot\nabla\times(c\boldsymbol{E}\times\widehat{\boldsymbol{b}}/B) = \omega_\mathrm{ci}(\rho_\mathrm{s}^2\nabla_\perp^2\Phi), \tag{22}$$

and so the vorticity and the ion polarization charge are identical except for a scale factor. An illustration of this result is given as figure 2 of Krommes (2006b). A consequence is that fluid equations for low-frequency plasma motion invariably include a vorticity equation that is recognizably similar to the 2D Navier–Stokes equation. That implies that the plasma physics shares some properties with canonical 2D turbulence, such as the existence of dual cascades. Unfortunately, I cannot pursue that important topic here; see the review article by Krommes (2002) for discussion and references.

The elimination of high-frequency physics from the GK dynamical system has consequences in addition to the reduction in dimensionality and the adiabatic conservation of the magnetic moment. If the GK distribution is realized numerically as a collection of *discrete* gyrocenters, one can ask questions about the discreteness-induced noise, which is relevant to the fidelity of numerical schemes. The simplest case is thermal equilibrium, for which a fluctuation–dissipation theorem (FDT) is available. For *exact* dynamics based on the Klimontovich formalism, it is well known [3] that the FDT (Ecker 1972) predicts (for weak coupling)

$$\frac{\langle\delta E^2\rangle(\boldsymbol{k})}{8\pi} = \frac{T/2}{1 + k^2\lambda_D^2}; \tag{23}$$

[8] Z enters through the condition for overall charge neutrality $\overline{n}_\mathrm{e} = Z\overline{n}_\mathrm{i}$, where $\overline{n}$ denotes the mean density.

this result arises from an integration over all frequencies of the FDT-specified formula for the spectral function $\langle \delta E^2 \rangle(\boldsymbol{k}, \omega)$. If one attempts a similar calculation for the GK system, one must be very careful because the 'high'-frequency limit of gyrokinetics is nontrivial and differs from the infinite-frequency limit of the true plasma. Krommes *et al* (1986) carried out the analysis and demonstrated that the equilibrium fluctuation energy of the electrostatic GK plasma model is much smaller than that of the full plasma—a consequence of the shielding due to ion polarization. (Later, Krommes (1993a, 1993b) extended the analysis to nonzero-pressure plasmas with fluctuating magnetic fields.) The low-noise properties of GK plasma are a boon for numerical simulation, since it is easier (and less expensive) to separate signal from noise. Issues related to GK noise were reviewed by Krommes (2007).

1.4. The drift wave

To gain insight into the GK formalism, let us see how to recover the famous *drift wave*. Drift waves are driven by a gradient in the background (mean) density profile; such profiles are inevitable in any situation in which plasma is confined. A collective normal mode arises because the physics of (light) electrons and (heavy) ions are very different. Suppose a slowly varying potential fluctuation $\delta\varphi$ arises in the plasma. Because electrons have little inertia, they can respond to the potential by moving rapidly along the field lines and establishing a Maxwell–Boltzmann distribution:

$$\delta n_{\rm e}/\overline{n}_{\rm e} \to {\rm e}^{e\delta\varphi/T_{\rm e}} - 1 \approx \delta\Phi. \tag{24}$$

This is called *adiabatic response*, adiabatic being used here in the sense of slow variations, not in its thermodynamic connotation. On the other hand, the dominant motion of the ions is across the magnetic lines. The cleanest model is obtained by setting $T_{\rm i} = 0$, which removes the FLR effects (but not the polarization drift); I also assume a constant magnetic field for simplicity. Then a straightforward moment of the GKE leads to the continuity equation for ion gyrocenters:

$$\partial_t n_{\rm i}^{\rm G} + \boldsymbol{\nabla}_\perp \cdot (\boldsymbol{u_E} n_{\rm i}^{\rm G}) + \underbrace{\nabla_\parallel (u_{\parallel \rm i} n_{\rm i}^{\rm G})}_{\text{neglect}} = 0. \tag{25}$$

The parallel motion of the ions is small and is ignored in the simplest approximation. For the perpendicular motion, observe that only the $\boldsymbol{E}\times\boldsymbol{B}$ drift enters. (Remember that we are studying gyrocenters, which do not polarize.) Now linearize (25) with the ansatz $n_{\rm i}^{\rm G} = \overline{n} + n_{\rm i}^{\rm G}$; one obtains

$$\partial_t (n_{\rm i}^{\rm G}/\overline{n}) + \delta\boldsymbol{u_E} \cdot \nabla \ \ln \overline{n} = 0. \tag{26}$$

If one now defines the density scale length L_n by $L_n^{-1} \doteq -\partial_x \ \ln \overline{n}(x)$, assuming that the profile is inhomogeneous in only the x-direction, one obtains

$$\partial_t (\delta n_{\rm i}^{\rm G}/\overline{n}) + V_* \partial_y \delta\Phi = 0, \tag{27}$$

where the *diamagnetic velocity*[9] is

$$V_* \doteq \rho_{\rm s} c_{\rm s}/L_n. \tag{28}$$

Finally, the quasineutrality condition (21) closes the system. One may solve that equation for the ion response in terms of the potential. Upon substituting that result into the continuity equation (27), one obtains

$$\partial_t [(1 - \rho_{\rm s}^2 \nabla_\perp^2)\delta\Phi] + V_* \partial_y \delta\Phi = 0. \tag{29}$$

The form of this equation is invariant under a time-reversal transformation, which shows that there is no dissipation or instability at this level of approximation; it describes a wave propagating in the positive y-direction. Fourier analysis reveals that the real frequency of the eigenmode is

$$\Omega_{\boldsymbol{k}} = \frac{\omega_*(\boldsymbol{k})}{1 + k_\perp^2 \rho_{\rm s}^2}, \tag{30}$$

where $\omega_* \doteq k_y V_*$ is the *diamagnetic frequency*. Equation (30) is the dispersion relation of the drift wave. The $(k_\perp \rho_s)^2$ term in the denominator arises from ion polarization. Its presence introduces dispersion, which is an important physical property of the modes. Nondispersive wave frequencies can be transformed away; the resulting fluctuations are strongly turbulent. Dispersive waves can be weakly turbulent in at least some regimes; that is a simplifying approximation in the analytical statistical theories of turbulence. See Ottaviani and Krommes (1992) for some discussion on weakly and strongly turbulent drift waves.

1.5. The Hasegawa–Mima equation

If the $\boldsymbol{E}\times\boldsymbol{B}$ nonlinearity is retained in the continuity equation for ion gyrocenters, the calculation proceeds almost identically. One finally obtains the *Hasegawa–Mima equation* (Hasegawa and Mima 1978)[10]:

$$\Big(\underbrace{1}_{\substack{\text{adiab.}\\ \text{elect.}}} - \underbrace{\rho_{\rm s}^2 \nabla_\perp^2}_{\substack{\text{ion}\\ \text{polarization}}}\Big)\frac{\partial \delta\Phi}{\partial t} + \underbrace{V_* \frac{\partial \delta\Phi}{\partial y}}_{\substack{\text{linear}\\ \text{drift wave}}}$$

$$+ \underbrace{\boldsymbol{u_E} \cdot \nabla(-\rho_{\rm s}^2 \nabla_\perp^2 \delta\Phi)}_{\substack{\text{nonlinear } \boldsymbol{E}\times\boldsymbol{B}\\ \text{advection of vorticity}}} = 0. \tag{31}$$

We see that the $\rho_{\rm s}^2 \nabla_\perp^2$ term now appears both linearly and nonlinearly. The nonlinear term, which can be seen to describe $\boldsymbol{E}\times\boldsymbol{B}$ advection of vorticity, is frequently called the *polarization-drift nonlinearity*. That term does not change the time-reversal properties of the equation, so the Hasegawa–Mima equation is nondissipative. One can put in realistic growth and damping by hand or derive a more sophisticated equation from a more complete

[9] The diamagnetic velocity also emerges in discussions of the diamagnetic current that arises from the gyrations of particles in an inhomogeneous background, a well-known effect that is illustrated, for example, in the article by Braginskii (1965). It is important to note that diamagnetic current has nothing to do with the simple model of the drift discussed here; indeed, ion diamagnetic flow vanishes for $T_{\rm i} \to 0$ because $\rho_{\rm i} \to 0$ in that limit. Rather, the physics of the drift wave involves the $\boldsymbol{E}\times\boldsymbol{B}$ advection of parcels of background density.

[10] The original derivation of this equation by Hasegawa and Mima (1978) proceeded from particle fluid equations (the Braginskii equations), not gyrofluid equations. The GK derivation is much cleaner. In particular, the polarization drift does not appear in the continuity equation for gyrocenters, whereas it must be included in the continuity equation for particles.

GK approximation. For studies of such equations, see Horton (1986).

The Hasegawa–Mima equation is similar to the 2D Navier–Stokes equation in the vorticity representation. If the adiabatic electron response and the linear drift-wave term are dropped, the equation reduces to precisely 2D Navier–Stokes:

$$\partial_t \varpi + \boldsymbol{u_E} \cdot \nabla \varpi = 0. \tag{32}$$

The Hasegawa–Mima equation inherits many of the properties of 2D Navier–Stokes, including the dual cascade.

The Hasegawa–Mima equation is not suitable for the treatment of *zonal flows*. Zonal flows are sheared poloidal flows; their shear can regulate the nonlinear saturation of drift waves by destroying drift-wave eddies. It is crucial to note that because zonal flows have a vanishing parallel wave number $k_\parallel$, they do not respond adiabatically. Adiabatic response requires that $\omega / k_\parallel v_{te} \ll 1$, and that is violated for modes with $k_\parallel = 0$. Instead, zonal modes are in the fluid regime. That can be taken into account with various degrees of rigor. The simplest leads to the *modified* (or *generalized*) Hasegawa–Mima equation (Krommes and Kim 2000):

$$\Big(\underbrace{\widehat{\alpha}}_{\substack{\text{electron}\\ \text{response}}} - \underbrace{\rho_s^2 \nabla_\perp^2}_{\substack{\text{ion}\\ \text{polarization}}}\Big)\frac{\partial \delta\Phi}{\partial t} + \underbrace{V_* \frac{\partial \delta\Phi}{\partial y}}_{\substack{\text{linear}\\ \text{drift wave}}}$$

$$+ \underbrace{\boldsymbol{u_E} \cdot \nabla [(\widehat{\alpha} - \rho_s^2 \nabla_\perp^2)\delta\Phi]}_{\substack{\text{nonlinear } \boldsymbol{E}\times\boldsymbol{B}\\ \text{advection of ion density}}} = 0. \tag{33}$$

Here

$$\widehat{\alpha} \doteq \begin{cases} 1 & (k_\parallel \neq 0) \\ 0 & (k_\parallel = 0) \end{cases} \tag{34}$$

is a projection operator that projects onto the nonzonal modes (the drift waves). Its presence in the nonlinear term leads to an interesting coupling between the drift waves and the zonal flows. Specifically, the zonal flows are driven by drift-wave-induced *Reynolds stresses*. This observation (Diamond *et al* 1998) is the starting point for a good deal of interesting analysis (Krommes and Kim 2000, Krommes 2004a, 2004b, Krommes and Kolesnikov 2004). For example, statistical closure theory (Krommes 2002) can be used to calculate the nonlinear growth rate of the zonal modes (Krommes and Kim 2000), and a close connection to Kraichnan's theory of 2D eddy viscosity (Kraichnan 1976) can be demonstrated. However, I shall not pursue those topics here because that would divert us from our main theme, namely the GK formalism.

Let me summarize what we have learned from our heuristic introduction to gyrokinetics:

- Gyrocenters move cross-field (only) with effective $\boldsymbol{E} \times \boldsymbol{B}$ drifts and magnetic drifts (not the polarization drift).
- Knowing the gyrocenter drifts, one can write the GKE.
- In time-varying fields, particles polarize with respect to the gyrocenter motion.
- Polarization and vorticity are intimately related.
- Ion polarization endows the 'GK vacuum' with a large dielectric permittivity ε^{G}.
- For large ε^{G}, the GK Poisson equation reduces to quasineutrality:

$$(\text{net gyrocenter charge}) + (\text{ion polarization charge}) \approx 0. \tag{35}$$

- The GK Poisson system is a new nonlinear dynamical system with unique properties.
- A system of discrete gyrocenters exhibits equilibrium fluctuation noise that is much suppressed from that of the true plasma.
- Gyrokinetics predicts a turbulent 'soup' of interacting drift waves and zonal flows.
- The GK Poisson system can be simulated and used to calculate turbulent transport fluxes.

We have already accomplished a lot without really working very hard. Nevertheless, various things have been swept under the rug. Most fundamentally, the formalism assumes that μ is conserved. But in reality, what most people call μ, namely $\mu \doteq \frac{1}{2} m v_\perp^2 / \omega_c(\boldsymbol{x})$, is not exactly conserved. (It is not even Galilean invariant!) One must ask just what quantity $\overline{\mu}$ is really conserved.

Also, how does one generalize the notion of GK polarization for $T_{\mathrm{i}} \neq 0$ and $k_\perp \rho_{\mathrm{i}} = O(1)$?

When appropriately formulated, gyrokinetics is a *Hamiltonian system* (a fact that is very useful to know both analytically and numerically). However, that is not apparent from the 'derivation' given so far.

Finally, are we sure that the GK system conserves the appropriate things (e.g. energy or momentum)?

To answer questions such as these, one must engage a more systematic formalism. I will turn to that in the next several sections.

2. Systematic gyrokinetics, part I: the gyrokinetic equations of motion

2.1. A brief history of gyrokinetics

Gyrokinetics has an interesting and instructive intellectual history. (The following brief sketch does not pretend to be complete; for more details and references, see the review article by Brizard and Hahm (2007).) Several logical threads are intertwined. Catto (1978) discussed linearized gyrokinetics by implementing a straightforward transformation to guiding-center variables, then gyro-averaging. Frieman and Chen (1982) derived the first nonlinear GKE; their formalism inspired many subsequent workers. However, they wrote separate equations for the background and the fluctuating distribution. This frustrated W W Lee, who was interested in applying particle-simulation techniques (Birdsall and Langdon 1985) to gyrokinetics. That method fundamentally relies on characteristic equations of motion (here for the gyrocenters), but the Frieman–Chen equations were not in characteristic form. Lee (1983) used a recursive transformation method to reformulate gyrokinetics in terms of a single equation for the total gyrocenter distribution function that was in characteristic form, and he obtained initial numerical results; his work fathered a massive industry in GK simulation that continues to the present.

None of the works mentioned so far made explicit use of the Hamiltonian nature of the problem. But soon after Catto's (1978) work, Littlejohn (1979) described 'A guiding center Hamiltonian: a new approach.' And a few years later he discussed 'Hamiltonian perturbation theory in noncanonical coordinates' (Littlejohn 1982), which provided the key insight that paved the way for all future systematic developments.

Motivated by Littlejohn's work, Dubin *et al* (1983) reconsidered Lee's (1983) attempt and provided a Hamiltonian formulation of the nonlinear GK Poisson system. In order to focus on the most fundamental conceptual points, they worked with a constant magnetic field, thereby ignoring the magnetic drifts. But although that paper contained a number of significant advances, it still did not fully exploit modern techniques. In particular, Cary and Littlejohn (1983) discussed noncanonical Lagrangian mechanics, which provided the formalism followed by essentially all subsequent workers. It took about 5 years for those techniques to be appreciated, but finally Hahm (1988) developed nonlinear gyrokinetics in an inhomogeneous magnetic field by using the Lagrangian one-form. Brizard (1989) made further contributions, as did Qin (1989). All this and more is summarized in the review article by Brizard and Hahm (2007).

The resulting edifice of nonlinear gyrokinetics is beautiful, elegant and efficient. It is also subtle and involves nontrivial approximations; therefore, it is not surprising that questions remain. Some of those were voiced in two independent papers by Sugiyama (2008) and Parra and Catto (2008), published three decades after Catto's 1978 work on linearized gyrokinetics. The status of the ensuing debates will be described in section 4.

2.2. Modern gyrokinetics via the Lagrangian one-form

Our goal will be a systematic derivation of nonlinear gyrokinetics from first principles (meaning Newton's laws and Maxwell's equations). In the course of this, we will be led to an asymptotic construction of the 'true' adiabatic invariant $\overline{\mu}$ in complicated geometry with inhomogeneous and slowly time-varying electromagnetic fields. The methodology will follow the seminal contributions of Littlejohn (1979, 1981, 1982, 1983), including the use of Lagrange's variational principle, differential forms and Lie perturbation theory.

Much of analytical physics consists of *finding the best variables.* Contrast the particle position $\boldsymbol{x}$, the lowest-order gyrocenter position $\boldsymbol{X}$ (well defined for circular motion) and the 'true' gyrocenter position $\overline{\boldsymbol{X}}$. The procedure will be to first transform $\{\boldsymbol{x}, \boldsymbol{v}\}$ to the set of lowest-order gyrocenter variables $\boldsymbol{Z} \doteq \{\boldsymbol{X}, U, \mu, \zeta\}$, where $\boldsymbol{X} \doteq \boldsymbol{x} - \boldsymbol{\rho}$, $U \equiv v_\parallel \doteq \boldsymbol{v}\cdot\widehat{\boldsymbol{b}}$, μ is defined by (6) and ζ is the lowest-order gyrophase, measured with respect to the arbitrary unit vector $\widehat{\boldsymbol{e}}_1$ (see figure 1). Then, to deal with the consequences of fluctuations and inhomogeneous geometry, we will perturbatively transform from $\boldsymbol{Z}$ to $\overline{\boldsymbol{Z}} \doteq \{\overline{\boldsymbol{X}}, \overline{U}, \overline{\mu}, \overline{\zeta}\}$. The latter variables will be chosen specifically so that $\overline{\mu}$ is conserved.

It is well known that both Hamilton's and Lagrange's equations of motion follow from the action variational principle (Lanczos 1949)

$$0 = \delta \int_{t_0}^{t_1} L\,\mathrm{d}t, \tag{36}$$

where no variations are permitted at the endpoints. Here $L \doteq \boldsymbol{p}\cdot\boldsymbol{q} - H$ is the Lagrangian, $H \doteq |\boldsymbol{p} - q\boldsymbol{A}/c|^2/2m + q\varphi$ is the Hamiltonian (Goldstein 1951), and $\boldsymbol{p}$ is the canonical momentum: $\boldsymbol{p} \doteq m\boldsymbol{v} + q\boldsymbol{A}/c$, where $\boldsymbol{B} = \nabla\times\boldsymbol{A}$. In the language of differential forms (Flanders 1963, Fecko 2006),

$$0 = \delta \int_{t_0}^{t_1} \gamma, \tag{37}$$

where the *fundamental differential one-form* is

$$\gamma \doteq \boldsymbol{p}\cdot\mathrm{d}\boldsymbol{q} - H\,\mathrm{d}t. \tag{38}$$

A problem with standard formulations in terms of either $\{\boldsymbol{x}, \boldsymbol{v}\}$ (Lagrange) or $\{\boldsymbol{x}, \boldsymbol{p}\}$ (Hamilton) is the nonintuitive presence of $\boldsymbol{A}$ in the canonical momentum. A quick glance at the one-form γ probably does not reveal the simple physics that a charged particle spirals around a magnetic field line. Technically, it is difficult to implement a gyrophase average by working with γ as expressed in the original variables.

Fortunately, the variational principle is indifferent to the variables used to express γ. One is free to use 'better' variables (chosen to satisfy some criterion); then the variational principle will provide the equations of motion for those variables. In our case, we will choose variables such that the magnetic moment is conserved.

Technically, one can cast γ into a representation-free form by defining

$$z^\nu \doteq \{t, \boldsymbol{x}, \boldsymbol{p}\}, \tag{39a}$$

$$\gamma_\nu \doteq \{-H, \boldsymbol{p}, \boldsymbol{0}\}. \tag{39b}$$

Then γ can be written covariantly as

$$\gamma = \gamma_\nu\,\mathrm{d}z^\nu, \tag{40}$$

where the summation convention is understood. This can now be transformed to any set of variables one pleases:

$$\gamma(z) = \overline{\gamma}(\overline{z}) = \overline{\gamma}_\nu\,\mathrm{d}\overline{z}^\nu. \tag{41}$$

For any z^ν's, the Euler–Lagrange equations of motion are

$$\omega_{\sigma\nu}\frac{\mathrm{d}z^\nu}{\mathrm{d}t} = 0, \tag{42}$$

where the *symplectic two-form* is[11]

$$\omega_{\sigma\nu} \doteq \frac{\partial\gamma_\nu}{\partial z^\sigma} - \frac{\partial\gamma_\sigma}{\partial z^\nu}. \tag{43}$$

For motivation relating to the choice of good variables, recall the famous and crucially important *Noether's theorem* (Noether 1918), which shows how symmetries of the Lagrangian are intimately related to conservation laws. In the present context, Noether's theorem states that if all of the

[11] The two-form ω is the exterior derivative of the one-form γ. However, the form (43) also follows directly by carrying out the variation in (38).

γ_ν's are independent of a particular variable z^α, then γ_α is conserved. That is,

$$\gamma = \gamma_1(z^1, \ldots, \cancel{z^\alpha}, z^\alpha, \ldots, z^n)\mathrm{d}z^1 + \cdots + \underbrace{\gamma_\alpha(z^1, \ldots, \cancel{z^\alpha}, \ldots, z^n)}_{\text{conserved}}\,\mathrm{d}z^\alpha + \cdots + \gamma_n(z^1, \ldots, \cancel{z^\alpha}, \ldots, z^n)\mathrm{d}z^n. \tag{44}$$

The proof is simple and is left to the reader; it is given by Cary and Littlejohn (1983).

Since our goal is to develop a dynamical system that is independent of gyrophase, it is natural to choose $\overline{z}^\alpha = \overline{\zeta}$ (the 'true' gyrophase). Let us write

$$\overline{\gamma} = \overline{\gamma}_{(\nu)}\,\mathrm{d}\overline{z}^{(\nu)} + \overline{\mu}\,\mathrm{d}\overline{\zeta}, \tag{45}$$

where (ν) denotes all variables except $\overline{\zeta}$, and adopt $\overline{\mu}$ as one of our variables. Then, if one chooses the $\overline{\gamma}_{(\nu)}$'s so that[12]

$$\frac{\partial\overline{\gamma}_\nu}{\partial\overline{\zeta}} = 0 \qquad (\forall\nu), \tag{46}$$

$\overline{\mu}$ will be conserved according to Noether's theorem.

Thus, in summary of the general procedure, gyrokinetics is derived by removing gyrophase dependence from the one-form $\overline{\gamma}$. As a byproduct, one obtains formulas for the conserved $\overline{\mu}$ and the other gyrocenter variables.

For realistic fields, it is in practice only possible to remove gyrophase dependence perturbatively. Several small parameters are available: $\epsilon_B \doteq \rho/L_B$, the size of the magnetic inhomogeneity, and ϵ_δ, the size of the fluctuating fields. (For electrostatics, one may take $\epsilon_\delta \doteq e\delta\phi/T_e \equiv \delta\Phi$.) For the simplest asymptotic expansion in terms of a single parameter ϵ, I will order $\epsilon_B \sim \epsilon_\delta \sim \epsilon$. To explicitly display gyrophase dependence, one can express γ in terms of the lowest-order gyrocenter variables $\boldsymbol{Z} \doteq \{\boldsymbol{x}, U, \mu, \zeta\}$. Then

$$\gamma = \underbrace{\frac{q}{c}\boldsymbol{A}\cdot\mathrm{d}\boldsymbol{x}}_{O(\epsilon^{-1})} + \underbrace{m[U\widehat{\boldsymbol{b}}(\boldsymbol{x}) + v_\perp\widehat{\boldsymbol{c}}(\zeta, \boldsymbol{x})]\cdot\mathrm{d}\boldsymbol{x}}_{O(1)} - \underbrace{(\tfrac{1}{2}mU^2 + \mu\omega_c(\boldsymbol{x}))\mathrm{d}t}_{O(1)} + \underbrace{q\varphi(\boldsymbol{x}, t)\mathrm{d}t}_{O(\epsilon)}. \tag{47}$$

In determining the order of the various terms, I used

$$\frac{mv_\perp}{qA/c} = \frac{v_\perp}{L_B[q(A/L_B)/mc]} = \frac{v_\perp/\omega_c}{L_B} = \frac{\rho}{L_B} = \epsilon_B. \tag{48}$$

Let us represent the transformation from the old to the new coordinates by the operator T: $\overline{z} = Tz$. We must determine T so that $\overline{\gamma}$ is independent of $\overline{\zeta}$. But how is $\overline{\gamma}$ related to γ? Recall that the *value* of the scalar γ is independent of the coordinate system: $\overline{\gamma}(\overline{z}) = \gamma(z)$. Then

$$\underbrace{T}_{\text{'pull-back'}}\overline{\gamma}(\boldsymbol{z}) = \gamma(\boldsymbol{z}), \tag{49a}$$

[12] When $\overline{\mu}$ is chosen to be one of the independent variables, $\partial\overline{\mu}/\partial\overline{\zeta}$ vanishes automatically.

$$\overline{\gamma}(z) = \underbrace{T^{-1}}_{\text{'push-forward'}}\gamma(z). \tag{49b}$$

Because T operating on $\overline{\gamma}$ recovers the original γ, it is called a *pull-back* transformation. Similarly, T^{-1} is called a *push-forward* transformation. Equation (49*b*) is (almost) the result we need. But also note that the differential of a scalar function S does not contribute to the equations of motion because $\delta\int_{t_0}^{t_1}\mathrm{d}S = 0$ (no endpoint variations are permitted). The *gauge scalar* S provides extra freedom that helps in determining an appropriate T. Therefore, the basic formula is

$$\overline{\gamma} = \underbrace{T^{-1}(\zeta)}_{\substack{\text{push-forward}\\ \text{transformation}}}\ \gamma(\zeta) + \underbrace{\mathrm{d}S(\zeta)}_{\mathrm{d}\,(\text{gauge scalar})}. \tag{50}$$

Into this formula we must substitute (47). Then we can expand both sides of (50) order by order in ϵ, finally determining T and S such that gyrophase dependence is absent from $\overline{\gamma}$.

2.3. Lie perturbation theory

Traditionally, perturbative (canonical) transformations were implemented with the aid of mixed-variable generating functions (Goldstein 1951). But untangling the nonlinear mixture of old and new variables is quite cumbersome beyond first order. Lie methods, which untangle the variables from the start, provide a dramatic improvement. They go back to the seminal work of S Lie on groups of continuous transformations in the latter part of the 1800s, and they figure as core material in modern differential geometry (Fecko 2006). The basic idea can be motivated from the theory of ordinary differential equations. Consider the autonomous flow

$$\partial_t\overline{z} = V(\overline{z}) \qquad [\overline{z}(0) = z]. \tag{51}$$

The solution can be represented in terms of a time evolution operator $U(t)$, namely $\overline{z}(t; z) = U(t)z$. $U(t)$ is a Lie transformation in time. One has

$$U(t) = \mathrm{e}^{tL_V}, \tag{52}$$

where

$$L_V \doteq V(z)\frac{\partial}{\partial z}. \tag{53}$$

Now consider the transformation

$$\overline{z}(\epsilon; z) = T(\epsilon)z, \tag{54}$$

where ϵ is a perturbation parameter. By analogy to the temporal evolution problem, this can be represented as the solution to the equation

$$\partial_\epsilon\overline{z} = g(\overline{z}) \qquad [\overline{z}(0) = z], \tag{55}$$

and one has

$$T(\epsilon) = \mathrm{e}^{\epsilon L_g}. \tag{56}$$

Here $g(z)$ is called the *generating function* of the transformation.

For use in a perturbation expansion, one might simply allow g to contain terms of each order in ϵ. But a more efficient procedure is the following. Consider the set of generating functions $\{g_n | n = 1, 2, \ldots\}$, where $g_n = O(\epsilon^n)$, and define $L_n \equiv L_{g_n}$. Then one can construct the full transformation by compounding elemental ones (Dragt and Finn 1976*b*)

$$T = e^{L_1} e^{L_2} e^{L_3} e^{L_4} \cdots, \tag{57}$$

$$T^{-1} = \cdots e^{-L_4} e^{-L_3} e^{-L_2} e^{-L_1} \tag{58a}$$
$$= 1 - L_1 + (-L_2 + \tfrac{1}{2} L_1^2) + (-L_3 + L_2 L_1 - \tfrac{1}{6} L_1^3)$$
$$+ (-L_4 + L_3 L_1 + \tfrac{1}{2} L_2^2 - \tfrac{1}{2} L_2 L_1^2 + \tfrac{1}{24} L_1^4) + \cdots. \tag{58b}$$

The history, theory and uses of Lie perturbation theory have been reviewed by Kaufman (1978) and Cary (1981).

The algebra for determining the GK one-form is now straightforward, although tedious. At nth order one needs to determine g_n^σ and $S^{(n)}$. The goal is to remove ζ dependence from $\overline{\gamma}_\sigma^{(n)}$, and there is more than enough freedom to do this. Here is an example drawn from the midst of the algebra. At second order, one finds

$$\overline{\gamma}^{(2)}(\boldsymbol{Z}) = (\cdots) \cdot d\boldsymbol{x} + (\cdots) dt + (\cdots) dU + (\cdots) d\mu$$
$$+ \left(f_\mu + \Delta g_1^\zeta + \frac{\partial S^{(2)}}{\partial \mu} \right) d\mu + \left(f_\zeta - g_1^\mu + \frac{\partial S^{(2)}}{\partial \zeta} \right) d\zeta, \tag{59}$$

where f_ζ is a known function. Write $S = \langle S \rangle + \delta S$, such that $\langle \delta S \rangle = 0$. ($S$ must be periodic in ζ in order to avoid secularities.) δg_1^μ was determined at $O(\epsilon)$. From $\langle \overline{\gamma}_\zeta^{(2)} \rangle = \langle f_\zeta \rangle - \langle g_1^\mu \rangle$, choose $\langle g_1^\mu \rangle$ to eliminate $\langle \overline{\gamma}_\zeta^{(2)} \rangle$. Now the condition $\delta \overline{\gamma}_\zeta^{(2)} = 0$ determines $\delta S^{(2)} = \int d\zeta (-\delta f_\zeta + \delta g_1^\mu)$.

Avoidance of secularities is crucial in this kind of scheme. In plasma physics, there are many important historical precedents. See in particular the seminal work of Dewar (1973), who formalized Kaufman's treatment of quasilinear theory (Kaufman 1972) by introducing the concept of the *oscillation center*[13].

Ultimately, the procedure gives us the new $\overline{\zeta}$-independent one-form correct through some chosen order in ϵ. Through second (relative) order, one finds

$$\overline{\gamma} \approx [(q/c)\boldsymbol{A} + m\overline{U}\widehat{\boldsymbol{b}} - \overline{\mu}\boldsymbol{K}^*] \cdot d\overline{\boldsymbol{x}} + \overline{\mu}\, d\overline{\zeta}$$
$$- \underbrace{\left(\tfrac{1}{2} m\overline{U}^2 + \overline{\mu}\omega_c + q\langle \varphi \rangle \right)}_{\text{gyroaveraged Hamiltonian}} dt, \tag{60}$$

where

$$\boldsymbol{K}^* \doteq \boldsymbol{K} + \tfrac{1}{2} \underbrace{(\widehat{\boldsymbol{b}} \cdot \nabla \times \widehat{\boldsymbol{b}})}_{\text{torsion}} \widehat{\boldsymbol{b}}, \qquad \underbrace{\boldsymbol{K} \doteq (\nabla \widehat{\boldsymbol{e}}_1) \cdot \widehat{\boldsymbol{e}}_2}_{\text{gyrogauge vector}}. \tag{61}$$

From $\overline{\gamma}$, one can calculate the symplectic two-form $\overline{\omega}$; thus, one obtains the equations of motion for the gyrocenter, schematically

$$\overline{\omega}_{\sigma\nu} \frac{d\overline{z}^\nu}{dt} = 0. \tag{62}$$

[13] Forthcoming proceedings of a recent symposium held (31 October 2009, Atlanta, GA) in honor of Dewar, to be published in *Plasma Phys. Control. Fusion*, will contain various articles summarizing his research contributions.

These essentially reduce to the GK characteristic equations of motion that were introduced heuristically in section 1. One important improvement is the appearance of a velocity-dependent quantity $B^* = B + O(v_\parallel)$, which is required in order to preserve the Hamiltonian properties of the equations.

Finally, the g's give us $T(\epsilon)$, i.e. the asymptotic expansion of the proper ($\overline{\mu}$-conserving) gyrocenter variables: $\overline{\boldsymbol{Z}} = T(\epsilon)\boldsymbol{Z}$. We will see in the next section how this result is used to systematically obtain the GK Poisson equation.

The appearance of the vector $\boldsymbol{K}$ requires substantial discussion (more than space permits here). At first glance, it might appear that it introduces an unwelcome ambiguity into the formalism, since it involves the gradient of the unit vector $\widehat{\boldsymbol{e}}_1$, which was chosen arbitrarily. In fact, however, the appearance of that gradient is just what is required to make the theory independent of the choice of $\widehat{\boldsymbol{e}}_1$. As Littlejohn (1984, 1988) has shown, redefinition of the gyrophase by an arbitrary field $\chi(\boldsymbol{x})$ leaves $\overline{\gamma}$ invariant; the extra term that arises from the gradient appearing in $\boldsymbol{K}$ is canceled by a contribution from $d\overline{\zeta}$. Thus, the formalism is properly *gyrogauge invariant*.

Some residual confusion arising from the presence of $\boldsymbol{K}$ is discussed in section 4.

3. Systematic gyrokinetics, part II: the gyrokinetic Poisson equation

I now turn to a discussion of the systematic derivation of the GK Poisson equation. This is possibly the most subtle aspect of the formalism, since it is here that the fundamental closure is made that disconnects the physics of the gyrocenters from that contained in the full kinetic equation.

Poisson's equation (10) requires the charge density $\rho(\boldsymbol{x}, t)$ at the position of the particles. The goal is to express that in terms of the gyrocenter distribution, the quantity that is evolved by the GKE. Following Dubin *et al* (1983), one must consider various distribution functions: $f(\boldsymbol{z}, t)$, the particle PDF; $\widetilde{F}(\boldsymbol{Z}, t)$, the particle PDF written in lowest-order gyrocenter coordinates; $\widetilde{\widetilde{F}}(\overline{\boldsymbol{Z}}, t)$, the particle PDF written in the 'true' gyrocenter coordinates; and $\overline{F}(\overline{\boldsymbol{Z}}, t)$, the gyroaveraged PDF $\doteq \langle \widetilde{\widetilde{F}} \rangle_{\overline{\zeta}}$. Poisson's equation is explicitly

$$-\nabla^2 \varphi(\boldsymbol{x}, t) = 4\pi \sum_s (\overline{n}q)_s \int d\boldsymbol{v}'\, f_s(\boldsymbol{x}, \boldsymbol{v}', t), \tag{63}$$

where f_x must be obtained from $\overline{F}$. One proceeds as follows:

$$I \doteq \int d\boldsymbol{v}'\, f(\boldsymbol{x}, \boldsymbol{v}', t) = \int d\boldsymbol{z}'\, \delta(\boldsymbol{x} - \boldsymbol{x}') f(\boldsymbol{z}', t) \tag{64a}$$
$$= \int J(\boldsymbol{Z}) d\boldsymbol{Z}\, \delta(\boldsymbol{x} - \boldsymbol{x}'(\boldsymbol{Z})) \widetilde{F}(\boldsymbol{Z}, t), \tag{64b}$$

where the Jacobian is $J \doteq \partial(\boldsymbol{z}')/\partial(\boldsymbol{Z})$. Now recall the pull-back transformation $\widetilde{F} = T\widetilde{\widetilde{F}}$. Therefore,

$$I = \int \overline{J} d\overline{\boldsymbol{Z}} \delta(\boldsymbol{x} - \boldsymbol{x}'(\overline{\boldsymbol{Z}})) T \widetilde{\widetilde{F}}(\overline{\boldsymbol{Z}}, t). \tag{65}$$

This expression is still formally exact (assuming that $\overline{\mu}$ is conserved).

At this point, one has the following coupled system:

$$\frac{\partial \widetilde{\overline{F}}}{\partial t} + \dot{\overline{\boldsymbol{X}}} \cdot \nabla \widetilde{\overline{F}} + \dot{\overline{v}}_{\parallel} \frac{\partial \widetilde{\overline{F}}}{\partial \overline{v}_{\parallel}} + \dot{\overline{\zeta}} \frac{\partial \widetilde{\overline{F}}}{\partial \overline{\zeta}} = -C[\widetilde{\overline{F}}], \tag{66a}$$

$$-\nabla^2 \varphi = 4\pi \sum_s (\overline{n}q)_s \int \overline{J} \mathrm{d}\overline{\boldsymbol{Z}} \delta(\boldsymbol{x} - \boldsymbol{x}'(\overline{\boldsymbol{Z}})) \times \underbrace{T}_{\text{pull-back}} \widetilde{\overline{F}}(\overline{\boldsymbol{Z}}, t). \tag{66b}$$

This system is still exact; it is merely a transcription of the Vlasov–Poisson system to the barred variables. In particular, (66*a*) still involves a derivative with respect to $\overline{\zeta}$. But $\dot{\overline{\boldsymbol{X}}}$, $\dot{\overline{v}}_{\parallel}$ and $\dot{\overline{\zeta}}$ have each been constructed to be independent of $\overline{\zeta}$. Following Dubin *et al* (1983), one may therefore argue as follows. $\overline{\zeta}$ dependence can enter the kinetic equation for $\widetilde{\overline{F}}$ in just one of two ways: from (i) initial conditions or (ii) collisional effects. For *collisionless* theory, the kinetic equation does not couple the evolution of $\overline{F} \doteq \langle \widetilde{\overline{F}} \rangle_{\overline{\zeta}}$ and $\delta \overline{F} \doteq \widetilde{\overline{F}} - \langle \overline{F} \rangle$, and so (66*a*) can be trivially averaged over $\overline{\zeta}$ to obtain an equation for $\overline{F}$. One can then make in Poisson's equation the *GK closure*

$$\widetilde{\overline{F}}(\overline{\boldsymbol{Z}}, t) \approx \overline{F}(\overline{\boldsymbol{Z}}, t), \tag{67}$$

where $\overline{F}$ obeys the GKE. Furthermore, if one writes $T = 1 + \delta T$, then one has

$$I \approx \int \overline{J} \mathrm{d}\overline{\boldsymbol{Z}}\, \delta(\boldsymbol{x} - \boldsymbol{x}'(\overline{\boldsymbol{Z}}))(\underbrace{1}_{\substack{\text{gyrocenter} \\ \text{density}}} + \underbrace{\delta T}_{\substack{\text{polarization} \\ \text{density}}})\overline{F}(\overline{\boldsymbol{Z}}, t). \tag{68}$$

This gives a precise (nonperturbative, in principle) meaning to the polarization charge density (which represents the difference between the particle motion and the gyrocenter motion).

The correct generalization to collisional theory is nontrivial. It is clear that collisions drive a gyrophase-dependent part of the distribution function (Brizard 2004). Although that is small, it may be necessary to retain it, along with other small terms, when approximating the polarization charge. Failure to do this correctly may mean that some neoclassical effects[14] cannot be recovered; for more discussion, see section 4.

In the above brief description, I followed the traditional route, advocated by Dubin *et al* (1983), of writing the particle PDF in terms of the GK variables. But note that in order to achieve a tractable theory, truncations must be done at two places: in the GKE ($\dot{\overline{\boldsymbol{X}}}$ and $\dot{\overline{v}}_{\parallel}$) and in the GK Poisson equation (in the form of the polarization density). If that is not done consistently, trouble may arise, for example, in the violation of conservation laws for energy and/or momentum. With a reasonable truncation, Dubin *et al* (1983) were able to find a particular energy constant; however, their method does not really enforce consistency in general. A breakthrough occurred when Sugama (2000) showed that it is possible to derive both the GKE equation and the GK Poisson equation from a single field-theoretic variational principle. That is, one can postulate an action functional $S[\overline{F}, \varphi]$ such that a variational derivative with respect to $\overline{F}$ generates the GKE while a derivative with respect to φ gives the GK Poisson equation. The crucial property of this approach is that a form of Noether's theorem guarantees that *any* approximation to S leads to an energetically consistent theory. (This of course does not guarantee that the resulting theory is physically reasonable, but one has considerable experience in the selection of a reasonable action.) Brizard (2000) has given an alternate form of the variational principle that is easier to use than the original one of Sugama, and Scott (2009) has used that methodology to derive energetically consistent equations suitable for computer simulation. This topic is worthy of much more space than is available here.

In summary of the systematology of the GK Poisson equation:

- The particle PDF can be transformed to various variables.
- In particular, $\widetilde{F} = T\widetilde{\overline{F}}$ (use of the pull-back transformation).
- For collisionless theory, make the GK closure $\widetilde{\overline{F}} \approx \overline{F}$, where $\overline{F} \doteq \langle \widetilde{\overline{F}} \rangle_{\overline{\zeta}}$.
- Truncate T to some order in ϵ (usually $O(\epsilon)$) (being sure to not lose energy conservation).
- Alternatively, derive both the GKE and the GK Poisson equation from a single variational principle employing an approximate Lagrangian (thereby *guaranteeing* energetic consistency).
- Arrive at a *new, nonlinear dynamical system* appropriate for studies of low-frequency ($\omega \ll \omega_{\mathrm{ci}}$) collisionless plasma microturbulence.

4. The current status of gyrokinetics

Gyrokinetics is used extensively for studies of low-frequency microturbulence in both fusion and astrophysical contexts. Regarding the analytical theory of drift waves and related modes, the linear theory in general magnetic geometry is well developed; wave (gyro)kinetic equations for weak turbulence have also been studied.

Because the GKE in the presence of velocity variables is complicated, one sometimes derives nonlinear gyrofluid equations (e.g. the Hasegawa–Mima equation) by taking moments of the GKE, then implementing the important *Landau-fluid closure* (Hammett and Perkins 1990) to capture the effects of wave–particle resonances. Gyrofluid closures were pioneered by G Hammett and his collaborators (Dorland and Hammett 1993, Hammett *et al* 1993, Beer 1995). The method works well linearly, but has problems with nonlinear wave–wave–particle interactions (Mattor 1992). Therefore, as computing power has increased, the tendency has been to revert to simulations of the full GKE. There are two principal methods: the particle-in-cell (PIC) technique, pioneered by Lee (1983) (this amounts to a Monte Carlo sampling of phase space (Aydemir 1994, Hu and Krommes 1994, Krommes 2007) and the 'continuum' or 'Vlasov' approach,

[14] Neoclassical theory (Rosenbluth *et al* 1972) describes the physics of classical Coulomb collisions occurring in toroidal magnetic fields, which are necessarily inhomogeneous.

which refers to direct solution of the GK PDE (see, for example, the `GYRO` code of Candy and Waltz (2003), described by R Waltz at this conference).

This article is not a review of the many applications of nonlinear gyrokinetics, either analytically or numerically; for that, see the forthcoming article of Hammett (2010). Suffice it to say that it has had many successes. In particular, large-scale GK simulations appear to be in quantitative agreement with nontrivial experimental data on turbulent transport; see the above-mentioned article by Waltz.

However, there remain some serious concerns. I will enumerate some of them, then discuss one in more detail.

(a) In the PIC approach to solution of the GKE, Monte Carlo sampling noise may sometimes be an issue, as it can obscure measurement of the desired drift-wave signal and lead to invalid predictions for saturated fluctuation levels and turbulent transport coefficients. For further information and references, see the review article of Krommes (2007).

(b) Although conventional gyrokinetics is predicated on the conservation of $\overline{\mu}$, that conservation cannot always be assumed. Because of the Hamiltonian nature of gyrokinetics, there is an analog to Kolmogorov–Arnold–Moser (KAM) theory: conservation of $\overline{\mu}$ corresponds to good KAM surfaces. But nonlinear interaction between gyration and slower degrees of freedom can produce stochastic regions (Dubin and Krommes 1982). That is why it is said that $\overline{\mu}$ is merely an *adiabatic* invariant. Even if $\overline{\mu}$ is formally conserved through all orders in ϵ, it need not be truly conserved (Dragt and Finn 1976a). The familiar analog is to the function $\exp(-1/\epsilon)$, which has an asymptotic expansion that is zero to all orders but is not, in fact, zero. In the fusion applications, it is generally assumed that the stochastic regions are negligible. That is probably reasonable, but one must not lose sight of the possibility that certain physical phenomena can break the invariance of $\overline{\mu}$.

(c) Sugiyama (2008) has recently asserted that gyrokinetics is ill-posed for general 3D magnetic fields (torsional or stochastic). Fundamentally, her concern arises from the appearance at second relative order of the gyrogauge vector $\boldsymbol{K}$, which, as I have discussed, involves the gradient of the arbitrary unit vector $\widehat{e}_1$. Sugiyama worries that for the indicated magnetic fields it may be impossible to construct a global coordinate system with all the constraints that are invoked in the asymptotic development of gyrokinetics. In a formal comment, Krommes (2009) has argued in favor of the conventional asymptotics[15], but the response by Sugiyama (2009) shows that the issue is not yet settled.

(d) Parra and Catto (2008, PC) have argued in a series of papers that the conventional GK closure is inadequate for the study of long-wavelength phenomena such as macroscopic electric fields, plasma rotation and zonal flows. This is an important concern. Traditionally gyrokinetics has been used for studies of low-frequency microturbulence on the nonlinear saturation time scale (which is much less than the plasma confinement time). However, there is increasing demand for *simulations on the transport time scale* in order to determine profile relaxation, the development of plasma rotation, etc. One could easily worry that there might be interchanges of limits ($\epsilon \to 0$, $t \to \infty$); inconsistent truncations between the GKE and the GK Poisson equation; confusion in ordering when one takes the limit $k_\perp \rho_{\rm i} \to 0$ although the formalism was originally derived for $k_\perp \rho_{\rm i} = O(1)$; and other issues such as small errors in the numerical algorithm.

[15] The point is that the triad $(\widehat{e}_1, \widehat{e}_2, \widehat{b})$ forms an *anholonomic frame field*. This is perfectly fine for local analysis; the theory makes no requirement that the frame field be globally integrable.

One point raised by PC is that conventional truncations of the GK Poisson system may not be compatible with neoclassical theory; I have already pointed out that the gyrophase-dependent part of the PDF may be crucial in that regard. But collisionless microturbulence is also of concern. Here is brief further detail on the arguments of PC for that situation. Following Dubin *et al* (1983), consider a slab geometry with constant $\boldsymbol{B}$, in which x is the 'radial' direction of the profile gradients, y is the 'poloidal' direction and z is the direction of $\boldsymbol{B}$. Parra and Catto (2009) argue that the standard GK truncations may introduce a *spurious momentum source*. Let $\overline{Q}(x) \doteq (L_y L_z)^{-1} \int_0^{L_y} \mathrm{d}y \int_0^{L_z} \mathrm{d}z\, Q(x, y, z)$ denote the 'flux-surface average.' Then an *exact* moment of the Vlasov equation leads to

$$\partial_t (n_{\rm i} m_{\rm i} \overline{u}_{{\rm i},y}) = -\partial_x \overline{\pi}_{xy}. \tag{69}$$

Here π is the stress tensor and the relevant off-diagonal component describes *Reynolds stresses* due to the $\boldsymbol{E} \times \boldsymbol{B}$ motion:

$$\overline{\pi}_{xy} \sim n_{\rm i} m_{\rm i} \overline{V_{E,x} V_{E,y}}. \tag{70}$$

The result of PC (2009) is that when slab gyrokinetics is truncated to $O(\epsilon^2)$,

$$\partial_t (n_i m_i \overline{u}_{i,y}) = -\partial_x \overline{\pi}_{xy} + (\text{spurious momentum source}). \tag{71}$$

The spurious term is nonconservative. PC have suggested that this term can be as large as the Reynolds stresses. However, it is important to understand just what it means to 'truncate gyrokinetics to $O(\epsilon^n)$.' To PC, that means keep terms through $O(\epsilon^n)$ in both the GKE and the GK Poisson equation. However, the variational formulation of gyrokinetics (Brizard 2000) shows that, if terms through $O(\epsilon^n)$ are kept in the GKE, then only terms through $O(\epsilon^{n-1})$ should be kept in the GK Poisson equation because that is obtained by differentiating the action functional with respect to the potential, which reduces the order of a fluctuation term by one. Magnetic-field inhomogeneities do add extra complications, and final resolutions of the important points raised by PC are still pending (see note added in proof).

5. Summary and conclusions

Gyrokinetics is the appropriate description of low-frequency fluctuations in magnetized plasmas such as fusion confinement devices. It contains the physics of the drift wave (and several important related modes also driven by profile gradients). Here I have sketched the heuristic physical content and the systematic derivation of the novel dynamical system known as the coupled GK Poisson system; the formalism can be readily generalized to include electromagnetic effects. The clean separation between gyrocenter drifts (in the GKE) and polarization effects (in the GK Maxwell equations) fosters both concise analytical manipulations and efficient numerical methods. The roots of gyrokinetics in Hamiltonian and Lagrangian physics allow modern techniques drawn from disciplines such as differential geometry and Lie groups to be brought to bear. And without gyrokinetics, one's understanding of turbulent transport in fusion research devices and, ultimately, reactors would be in a much more primitive and inadequate state.

Gyrokinetics is one of the central analytical formalisms of modern plasma physics. It is therefore surprising that after a quarter of a century gyrokinetics is hardly treated at all in the textbooks. In particular, the compelling physical interpretation in terms of the GK vacuum appears to be described only in research papers (and tutorial summaries such as this). This situation should be corrected. It is clear that when the final building block of the discipline of plasma physics is in place, gyrokinetics will be seen to be an invaluable cornerstone.

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Note added in proof. A recent review paper that focuses on the numerical aspects of gyrokinetics is by Garbet X, Idomura Y, Villard L and Watanabe T H (2010 *Nucl. Fusion* **50** 043002). Many of the concerns raised by Parra and Catto have been addressed by the recent derivation of the gyrokinetic conservation law for toroidal angular momentum by Scott and Smirnov (2010 arXiv:1008.1244, submitted to *Phys. Plasmas*). However, some subtle issues remain.

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