Math 565A Final Project: Pólya Urn Model

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Due May 2, 2025 at 11:59pm (via D2L)

Prompt: Pólya Urn Model

Pólya Urn Model: We start with an urn containing W_0 white and B_0 black balls. We pull a random ball, then put it back and add another ball of the same color into the urn. What is the behavior of the total fraction of black balls, $\varphi_n = B_n/(B_n + W_n)$, in the urn, as $n \to \infty$? Does it tend to some value, or does it randomly drift in $(0, 1)$? Is there some limiting distribution of φ_n?

1 Introduction

Pólya's urn model, introduced by George Pólya in the 1930s, is a simple model for the so-called "rich–get–richer" dynamics, yet its long–run behavior reveals a deep connection to martingale theory. In this project, we first show that the sequence of black-ball proportions

$$\varphi_n \;=\; \frac{B_n}{B_n + W_n}$$

is a bounded martingale with respect to the natural filtration generated by the draws. By Doob's Martingale Convergence Theorem, φ_n converges almost surely to a limit $\varphi_\infty \in [0, 1]$. Next, we derive the exact finite–n distribution of the number of black draws—the beta-binomial law—and, via Stirling's approximation and a Riemann–sum argument, identify the limiting law

$$\varphi_\infty \;\sim\; \mathrm{Beta}(B_0, W_0).$$

The rest of the paper is organized as follows. Section 2 gives the necessary definitions and theorems needed to prove our result. Section 3 verifies the martingale property and almost-sure convergence. Subsection 5.1 derives the finite–sample beta–binomial distribution. Subsection 5.2 carries out the asymptotic analysis leading to the $\mathrm{Beta}(B_0, W_0)$ limit.

2 Definitions and Theorems

In this section, we state key definitions and theorems that will aid us in solving the Pólya Urn Model problem. We assume a basic understanding of measure theory and stochastic processes. We draw primarily from [1, 2, 3, 4, 5].

Definition 1. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Let $\{X_n\}$ and X be real-valued random variables with distributions $F_n(x) = \mathbb{P}(X_n \le x)$ and $F(x) = \mathbb{P}(X \le x)$. We say X_n converges almost surely (a.s.) to X*

$$X_n \overset{a.s.}{\to} X,$$

if

$$\mathbb{P}\left(\left\{\omega \in \Omega : \lim_{n\to\infty} X_n(\omega) = X(\omega)\right\}\right) = 1. \tag{2.1}$$

Definition 2. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Let $\{X_n\}$ and X be real-valued random variables with distributions $F_n(x) = \mathbb{P}(X_n \le x)$ and $F(x) = \mathbb{P}(X \le x)$. We say X_n converges in distribution to X*

$$X_n \xrightarrow{d} X,$$

if, for every continuity point x of F,

$$\lim_{n\to\infty} F_n(x) = F(x).$$

Definition 3. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. A collection of sub- σ-algebras $\mathcal{F}_n \subset \mathcal{F}$ for each n and $\mathcal{F}_m \subset \mathcal{F}_n$ if $m \le n$ is a filtration.*

Definition 4. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $\{\mathcal{F}_n\}_{n\ge 0}$ a filtration. A process $\{M_n\}_{n\ge 0}$ is a martingale with respect to $\{\mathcal{F}_n\}_{n\ge 0}$ if for each $n = 0, 1, \dots$*

(i) $\mathbb{E}[|M_n|] < \infty$,

(ii) M_n *is* $\mathcal{F}_n$*-measurable,*

(iii) $\mathbb{E}[M_{n+1}|\mathcal{F}_n] = M_n$ *a.s.*

Finally, we come to our most important theorem which is the Martingale Convergence Theorem.

Theorem 1. *Let a martingale $M_1, M_2, M_3, \dots$ with respect to independent random variables $X_1, X_2, X_3, \dots$ be given. Then if there exists $C < \infty$ such that for all n,*

$$\mathbb{E}(|M_n|) \le C,$$

there exists a random variable M_∞ such that, with probability 1,

$$\lim_{n\to\infty} M_n = M_\infty.$$

3 Martingale Property

Assume that at draw $n = 0$, the urn contains B_0 black balls and W_0 white balls where $B_0, W_0 \in \mathbb{N}$ are fixed. Let B_n and W_n denote the number of black and white balls, respectively, after n draws. Then let $T_n = B_n + W_n$ be the total number of balls after n draws. Let $\mathcal{F}_n = \sigma\left(B_0, W_0; X_1, \dots, X_n\right)$ where X_i is the color on the i-th draw

$$X_i = \begin{cases} 1, & \text{if draw } i \text{ is black,} \\ 0, & \text{if draw } i \text{ is white.} \end{cases} \tag{3.1}$$

Let

$$\varphi_n = \frac{B_n}{T_n}, \tag{3.2}$$

be the fraction of black balls after n draws, which is also the probability of drawing a black ball at the n-th draw. Similarly, the probability of drawing a white ball at the n-th draw is $1-\varphi_n$. To find the next draw, observe that if a black ball is drawn at $n+1$, the update is

$$B_{n+1} = B_n + 1, \quad W_{n+1} = W_n, \quad T_{n+1} = B_{n+1} + W_{n+1} = T_n + 1, \tag{3.3}$$

which leads to

$$\varphi_{n+1} = \frac{B_n + 1}{T_n + 1}. \tag{3.4}$$

If a white ball is drawn at $n+1$, the update is

$$B_{n+1} = B_n, \quad W_{n+1} = W_n + 1, \quad T_{n+1} = B_{n+1} + W_{n+1} = T_n + 1, \tag{3.5}$$

which leads to

$$\varphi_{n+1} = \frac{B_n}{T_n + 1}. \tag{3.6}$$

Then the conditional expectation given $\mathcal{F}_n$ is

$$\begin{aligned}
\mathbb{E}[\varphi_{n+1}|\mathcal{F}_n] &= \varphi_n \frac{B_n + 1}{T_n + 1} + (1-\varphi_n)\frac{B_n}{T_n + 1} \\
&= \frac{\varphi_n(B_n + 1) + (1-\varphi_n)B_n}{T_n + 1} \\
&= \frac{\frac{B_n}{T_n}(B_n + 1) + \left(1 - \frac{B_n}{T_n}\right) B_n}{T_n + 1} \\
&= \frac{B_n\,(T_n + 1)}{T_n(T_n + 1)} \\
&= \frac{B_n}{T_n} \\
&= \varphi_n.
\end{aligned} \tag{3.7}$$

Hence

$$\mathbb{E}[\varphi_{n+1}|\mathcal{F}_n] = \varphi_n, \tag{3.8}$$

so $\{\varphi_n\}$ is a martingale. Since $0 \le \varphi_n \le 1$, all φ_n are integrable, so the martingale property holds without any additional moment assumptions.

4 Existence of Limiting Distribution

Since $\varphi_n \in [0,1]$ for all $n \in \mathbb{N}$, we have that the martingale $\{\varphi_n\}$ is bounded. Then by the Martingale Convergence theorem 1, there exists a random variable φ_∞ such that

$$\varphi_n \overset{a.s.}{\to} \varphi_\infty. \tag{4.1}$$

Thus we have proven the *existence* of some random variable φ_∞ in the limit $n \to \infty$. This automatically eliminates the possibility that φ_n will randomly drift in $(0,1)$: φ_n must settle to a well-defined random value φ_∞. In the next subsection, we will identify the distribution of φ_∞.

5 The Identity of the Limiting Distribution

5.1 Exact Finite-Sample Law (Beta-Binomial)

We now find the probability of drawing exactly k black balls after n draws. Label the draws by $1, 2, \dots, n$. Fix any arbitrary pattern in which exactly k of those draws are black and the other $n-k$ are white. Then by the reinforcement rule, at draw $i-1$ we have

$$B_{i-1} = B_0 + \sum_{j=1}^{i-1} X_j, \tag{5.1}$$

$$W_{i-1} = W_0 + \sum_{j=1}^{i-1} (1 - X_j), \tag{5.2}$$

$$T_{i-1} = B_{i-1} + W_{i-1} = B_0 + W_0 + (i-1). \tag{5.3}$$

Hence

$$\mathbb{P}(X_i = 1 | X_1, \dots, X_{i-1}) = \mathbb{P}(X_i = 1 | \mathcal{F}_{i-1}) = \varphi_{i-1} = \frac{B_0 + \sum_{j=1}^{i-1} X_j}{B_0 + W_0 + (i-1)}, \tag{5.4}$$

$$\mathbb{P}(X_i = 0 | X_1, \dots, X_{i-1}) = \mathbb{P}(X_i = 0 | \mathcal{F}_{i-1}) = 1 - \varphi_{i-1} = \frac{W_0 + \sum_{j=1}^{i-1} (1 - X_j)}{B_0 + W_0 + (i-1)}. \tag{5.5}$$

So for a fixed sequence with exactly k ones and $n-k$ zeros, its probability is

$$\begin{aligned} \mathbb{P}(X_1 = 1, \dots, X_k = 1, X_{k+1} = 0, \dots, X_n = 0) &= \prod_{j=0}^{k-1} \frac{B_0 + j}{B_0 + W_0 + j} \prod_{j=0}^{n-k-1} \frac{W_0 + j}{B_0 + W_0 + k + j} \\ &= \prod_{j=0}^{k-1} \frac{B_0 + j}{B_0 + W_0 + j} \prod_{i=k}^{n-1} \frac{W_0 + i - k}{B_0 + W_0 + i}. \end{aligned} \tag{5.6}$$

where we reindexed the second product using $i = k + j$. Observe that

$$\prod_{j=0}^{k-1} (B_0 + j) = (B_0)_k, \tag{5.7}$$

$$\prod_{i=k}^{n-1} (W_0 + (i - k) = (W_0)_{n-k}, \tag{5.8}$$

$$\prod_{j=0}^{k-1} (B_0 + W_0 + j) \prod_{i=k}^{n-1} (B_0 + W_0 + i) = \prod_{i=0}^{n-1} (B_0 + W_0 + i) = (B_0 + W_0)_n, \tag{5.9}$$

where $(a)_k = a(a+1) \cdots (a+k-1)$. Then

$$\begin{aligned} \mathbb{P}(X_1 = 1, \dots, X_k = 1, X_{k+1} = 0, \dots, X_n = 0) &= \prod_{j=0}^{k-1} \frac{B_0 + j}{B_0 + W_0 + j} \prod_{j=0}^{n-k-1} \frac{W_0 + j}{B_0 + W_0 + k + j} \\ &= \frac{(B_0)_k (W_0)_{n-k}}{(B_0 + W_0)_n}. \end{aligned} \tag{5.10}$$

Based on the problem formulation, there are $\binom{n}{k}$ different ways of choosing which k of the n draws are black. Since each such pattern has the same probability, the total probability of drawing exactly k blacks in n draws is

$$\mathbb{P}(B_n = B_0 + k) = \binom{n}{k} \frac{(B_0)_k (W_0)_{n-k}}{(B_0 + W_0)_n}. \tag{5.11}$$

5.2 Asymptotic Convergence to Beta

Recall that the probability of drawing exactly k black balls after n draws is

$$\mathbb{P}(B_n = B_0 + k) = \binom{n}{k} \frac{(B_0)_k (W_0)_{n-k}}{(B_0 + W_0)_n}, \tag{5.12}$$

where $(a)_k = a(a+1)\cdots(a+k-1)$. If we use the Gamma function $\Gamma(x)$ and its property

$$\Gamma(n) = (n-1)!, \quad n \in \mathbb{N}, \tag{5.13}$$

we can write

$$(B_0)_k = \frac{\Gamma(B_0 + k)}{\Gamma(B_0)}, \quad (W_0)_{n-k} = \frac{\Gamma(W_0 + n - k)}{\Gamma(W_0)}, \quad (B_0 + W_0)_n = \frac{\Gamma(B_0 + W_0 + n)}{\Gamma(B_0 + W_0)}. \tag{5.14}$$

Additionally, we can write the binomial coefficient as

$$\binom{n}{k} = \frac{\Gamma(n+1)}{\Gamma(k+1)\Gamma(n-k+1)}. \tag{5.15}$$

Then we get

$$\begin{aligned} \mathbb{P}(B_n = B_0 + k) &= \frac{\Gamma(n+1)}{\Gamma(k+1)\Gamma(n-k+1)} \frac{\Gamma(B_0 + k)}{\Gamma(B_0)} \frac{\Gamma(W_0 + n - k)}{\Gamma(W_0)} \frac{\Gamma(B_0 + W_0)}{\Gamma(B_0 + W_0 + n)} \\ &= \frac{\Gamma(n+1)}{\Gamma(k+1)\Gamma(n-k+1)} \frac{\mathrm{B}(B_0 + k, W_0 + n - k)}{\mathrm{B}(B_0, W_0)}, \end{aligned} \tag{5.16}$$

where

$$\mathrm{B}(a, b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}, \tag{5.17}$$

is the Beta function.

To aid us in finding the asymptotic limit of (5.16), we first find the asymptotic limits of the binomial coefficient, the Gamma function, and the Beta function for large n. Recall that

$$\varphi_n = \frac{B_0 + k}{B_0 + W_0 + n} = \frac{B_0 + k}{T_0 + n}. \tag{5.18}$$

Since B_0 and W_0 are just fixed constants, $T_0 + n$ grows linearly in n. We have that

$$\varphi_n \approx \frac{k}{n} + o(1/n), \tag{5.19}$$

thus $\varphi_n \approx k/n$ for large n. Now fix any arbitrary $p \in (0,1)$ and let $k = \lfloor np \rfloor$, where $\lfloor \cdot \rfloor$ is the floor function. Then

$$k = \lfloor np \rfloor \sim np \quad \textit{and} \quad n - k = n - \lfloor np \rfloor \sim n(1-p). \tag{5.20}$$

Since we are considering large arguments in the binomial coefficient, the Gamma function, and the Beta function, we utilize Stirling's approximation

$$n! \sim \sqrt{2\pi n} n^n e^{-n}. \tag{5.21}$$

For the binomial coefficient, the Stirling approximated form is

$$\binom{n}{k} = \frac{\Gamma(n+1)}{\Gamma(k+1)\Gamma(n-k+1)} \sim \frac{n^n}{k^k (n-k)^{n-k}} \left(\frac{n}{2\pi k(n-k)} \right)^{1/2}. \tag{5.22}$$

Then we use (5.20) in the Stirling approximated binomial coefficient (5.22) to get

$$\binom{n}{k} \sim \binom{n}{\lfloor np \rfloor} \sim \frac{1}{\sqrt{2\pi np(1-p)}} \frac{n^n}{(np)^{np}(n(1-p))^{n(1-p)}}. \tag{5.23}$$

Observe that

$$\begin{aligned} \frac{n^n}{(np)^{np}(n(1-p))^{n(1-p)}} &= \exp\left[n\ln(n) - np\ln(np) - n(1-p)\ln(n(1-p))\right] \\ &= \exp[nH(p)], \end{aligned} \tag{5.24}$$

where

$$H(p) = -\left[p\ln(p) + (1-p)\ln(1-p)\right], \tag{5.25}$$

is the binary entropy. Then we can compactly write (5.23) as

$$\binom{n}{k} \sim \binom{n}{\lfloor np \rfloor} \sim \frac{1}{\sqrt{2\pi np(1-p)}} \exp[nH(p)]. \tag{5.26}$$

Now, for any fixed $a \in \mathbb{R}$, the Stirling approximated Gamma function $\Gamma(z+a)$ as $z \to \infty$ is

$$\Gamma(z+a) \sim \sqrt{2\pi} z^{z+a-\frac{1}{2}} e^{-z}. \tag{5.27}$$

We use (5.27) to get

$$\frac{\Gamma(B_0+k)}{\Gamma(k+1)} \sim \frac{k^{k+B_0-\frac{1}{2}} e^{-k}}{k^{k+1-\frac{1}{2}} e^{-k}} = k^{B_0-1}. \tag{5.28}$$

Now observe that

$$\frac{\Gamma(B_0+W_0)}{\Gamma(B_0+W_0+n)} \sim \frac{\Gamma(B_0+W_0)}{\sqrt{2\pi} n^{n+B_0+W_0-\frac{1}{2}} e^{-n}} = \frac{\Gamma(B_0+W_0)e^n}{\sqrt{2\pi} n^{n+B_0+W_0-\frac{1}{2}}}. \tag{5.29}$$

Note that in (5.28) and (5.29), B_0 and W_0 are fixed constants. Then the Beta function can be asymptotically approximated for large n as

$$
\begin{aligned}
\frac{\mathrm{B}(B_0+k, W_0+n-k)}{\mathrm{B}(B_0, W_0)} &= \frac{\Gamma(B_0+k)\Gamma(W_0+n-k)\Gamma(B_0+W_0)}{\Gamma(B_0)\Gamma(W_0)\Gamma(B_0+W_0+n)} \\
&\sim \sqrt{2\pi}\frac{\Gamma(B_0+W_0)}{\Gamma(B_0)\Gamma(W_0)}\frac{k^k(n-k)^{n-k}}{n^n}\frac{k^{B_0-\frac{1}{2}}(n-k)^{W_0-\frac{1}{2}}}{n^{B_0+W_0-\frac{1}{2}}} \\
&= C\frac{k^k(n-k)^{n-k}}{n^n}\frac{k^{B_0-\frac{1}{2}}(n-k)^{W_0-\frac{1}{2}}}{n^{B_0+W_0-\frac{1}{2}}},
\end{aligned} \tag{5.30}
$$

where

$$
C = \sqrt{2\pi}\frac{\Gamma(B_0+W_0)}{\Gamma(B_0)\Gamma(W_0)}. \tag{5.31}
$$

Since $k \sim np$ and $n-k \sim n(1-p)$, we have

$$
\frac{k^k(n-k)^{n-k}}{n^n} = \exp\left[k\ln(k) + (n-k)\ln(n-k) - n\ln(n)\right] = \exp[nH(p)], \tag{5.32}
$$

and

$$
\frac{k^{B_0-\frac{1}{2}}(n-k)^{W_0-\frac{1}{2}}}{n^{B_0+W_0-\frac{1}{2}}} \sim p^{B_0-\frac{1}{2}}(1-p)^{W_0-\frac{1}{2}}n^{-\frac{1}{2}}. \tag{5.33}
$$

Thus, we have

$$
\frac{\mathrm{B}(B_0+k, W_0+n-k)}{\mathrm{B}(B_0, W_0)} \sim C\exp[-nH(p)]p^{B_0-\frac{1}{2}}(1-p)^{W_0-\frac{1}{2}}n^{-\frac{1}{2}}. \tag{5.34}
$$

Finally, with our Stirling approximated values, we can approximate the asymptotic limit of (5.16) for large n. Let

$$
p_{n,k} = \frac{B_0+k}{B_0+W_0+n}, \quad k = 0, 1, \ldots, n. \tag{5.35}
$$

Now define the set

$$
\mathcal{P} = \{p_{n,k} | k = 0, 1, \ldots, n\}, \tag{5.36}
$$

which is the support of φ_n and we see that $\varphi_n \in \mathcal{P}$. The mesh is

$$
\Delta_n = \min_k (p_{n,k+1} - p_{n,k}) = \frac{1}{B_0+W_0+n} \sim \frac{1}{n} \underset{n\to\infty}{\to} = 0. \tag{5.37}
$$

Fix an arbitrary $p \in (0,1)$ and set $k = \lfloor np \rfloor$. Then we have by Stirling's formula

$$
\mathbb{P}(B_n = B_0+k) = \mathbb{P}(\varphi_n = p_{n,k}) = \frac{1}{n}f(p_{n,k}) + o(1/n) = \frac{1}{n}f(p) + o(1/n), \tag{5.38}
$$

where

$$
f(p) = \frac{1}{\mathrm{B}(B_0, W_0)}p^{B_0-1}(1-p)^{W_0-1}, \tag{5.39}
$$

$$
p_{n,k} = \frac{B_0+k}{B_0+W_0+n}, \quad k = 0, 1, \ldots, n. \tag{5.40}
$$

Fix any $0 \le a < b \le 1$. Then

$$\mathbb{P}(a \le \varphi_n \le b) = \sum_{\substack{k: \\ p_{n,k} \in [a,b]}} \mathbb{P}(\varphi_n = p_{n,k}) = \sum_{\substack{k: \\ p_{n,k} \in [a,b]}} \left[\frac{1}{n} f(p_{n,k}) + o(1/n) \right]. \tag{5.41}$$

Since

$$\frac{1}{n} = (1 + o(1))\, \Delta_n, \tag{5.42}$$

we have that

$$\mathbb{P}(a \le \varphi_n \le b) = \sum_{\substack{k: \\ p_{n,k} \in [a,b]}} f(p_{n,k}) \Delta_n + o(1), \tag{5.43}$$

which is a Riemann sum. As $n \to \infty$, the mesh $\Delta_n \to 0$ and the Riemann sum converges to the integral

$$\sum_{\substack{k: \\ p_{n,k} \in [a,b]}} f(p_{n,k}) \Delta_n + o(1) \to \int_a^b f(p)\, dp. \tag{5.44}$$

Since a, b, and p were arbitrary, we have that

$$\lim_{n \to \infty} \mathbb{P}(a \le \varphi_n \le b) = \int_a^b f(p)\, dp. \tag{5.45}$$

We see that $f(p)$ is the density function of a $\mathrm{Beta}(B_0, W_0)$ random variable. Thus

$$\boxed{\varphi_n \xrightarrow{d} \varphi_\infty \sim \mathrm{Beta}(B_0, W_0),} \tag{5.46}$$

namely, φ_n converges in distribution to $\varphi_\infty \sim \mathrm{Beta}(B_0, W_0)$.

6 Empirical Evidence: Numerical Simulations

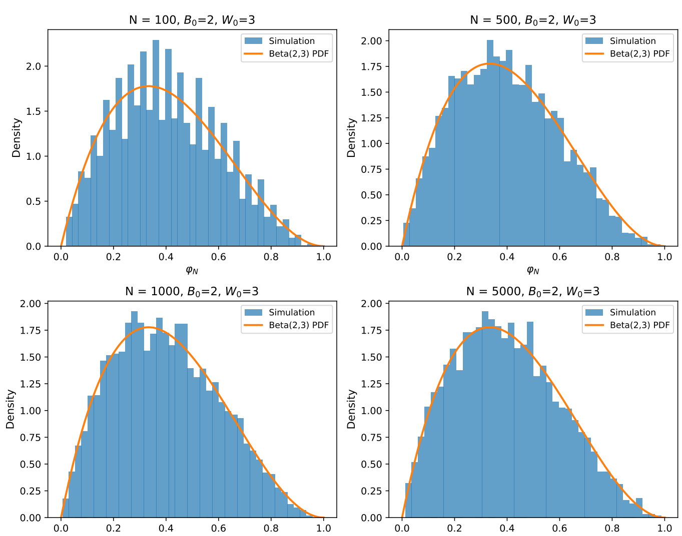

Figure 1: Empirical histogram of φ_n after N draws (blue bars) overlaid with the $\mathrm{Beta}(2,3)$ density (red curve). As N increases, the empirical distribution converges to the theoretical $\mathrm{Beta}(B_0, W_0)$ law.

To illustrate and empirically validate the theoretical Beta(B_0, W_0) limit, we ran a Monte Carlo simulation of the Pólya urn with initial counts $B_0 = 2$ black and $W_0 = 3$ white balls. We performed $M = 10000$ independent trials. In each trial, the urn was evolved for N reinforcement steps and the final proportion

$$\varphi_N = \frac{B_N}{B_N + W_N},$$

was recorded. Figure 1 shows the empirical histogram of φ_N (blue bars) for four different values of N, overlaid with the probability density of the Beta$(2, 3)$ law (red curve).

Even for moderate N, the histograms already approximate the theoretical curve quite well. At $N = 100$, the distribution of φ_N is somewhat dispersed but clearly skewed in the same manner as Beta$(2, 3)$. By $N = 500$ and $N = 1000$, the match is markedly tighter, and by $N = 5000$, the empirical frequencies fall almost exactly along the Beta density. This convergence in distribution, coupled with the almost sure stabilization of each individual trajectory, offers a compelling numerical confirmation of the martingale convergence result.

Furthermore, the simulation highlights two key facets of the Pólya urn. First, each individual sample path φ_n quickly "locks in" to a neighborhood of its random limit, which means that fluctuations of order $O(1/n)$ become negligible once n is large. Second, although every path converges, the limiting value varies from trial to trial, and the ensemble of these limits follows exactly the Beta$(2, 3)$ law predicted by theory.

Taken together, these simulation results provide both visual intuition and quantitative evidence for the almost sure convergence of φ_n to a Beta(B_0, W_0) distributed limit.

7 Conclusion

In this paper, we investigated the classical Pólya urn model, proving that the sequence of color–proportion random variables constitutes a bounded martingale. By invoking the Martingale Convergence Theorem, we established almost sure convergence to a Beta(B_0, W_0) distributed limit.

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