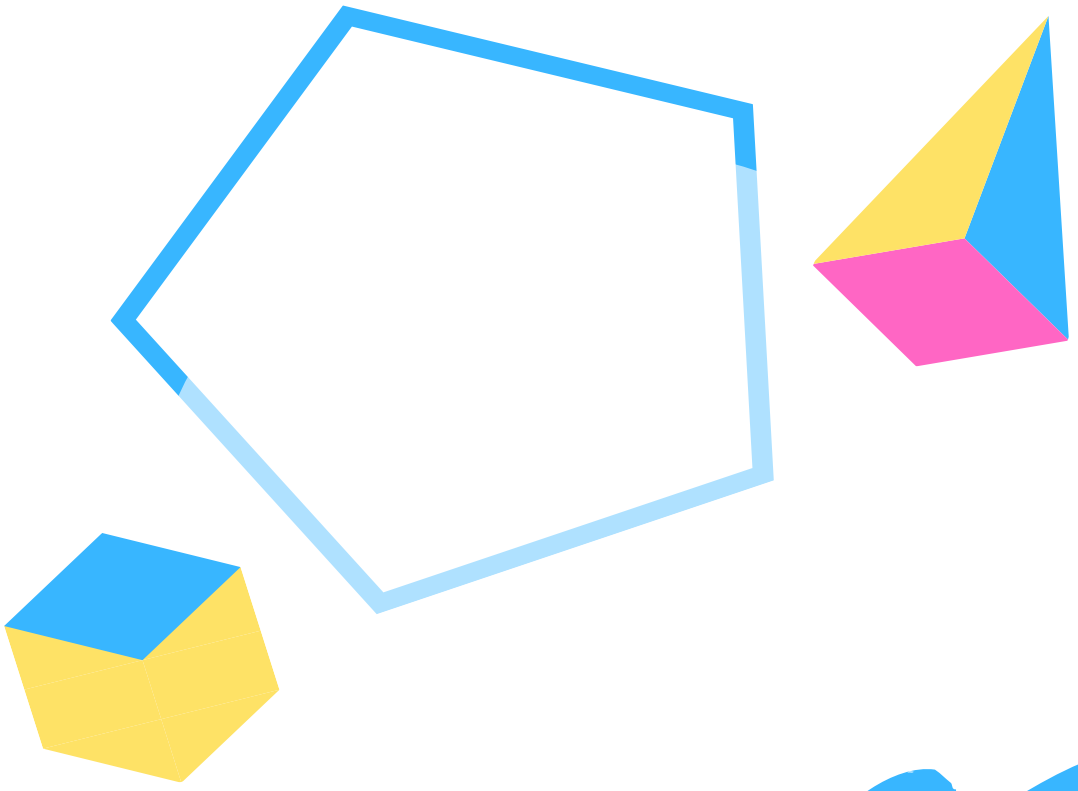
ALGORITHM DESGIN & ANALYSIS

PROBLEM BASED LEARNING



Sleeping in lectures has never been easier.

COMPUTER SCIENCE STUDENT

A collection of solved problems

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PUBLISHED BY THE INTERNET

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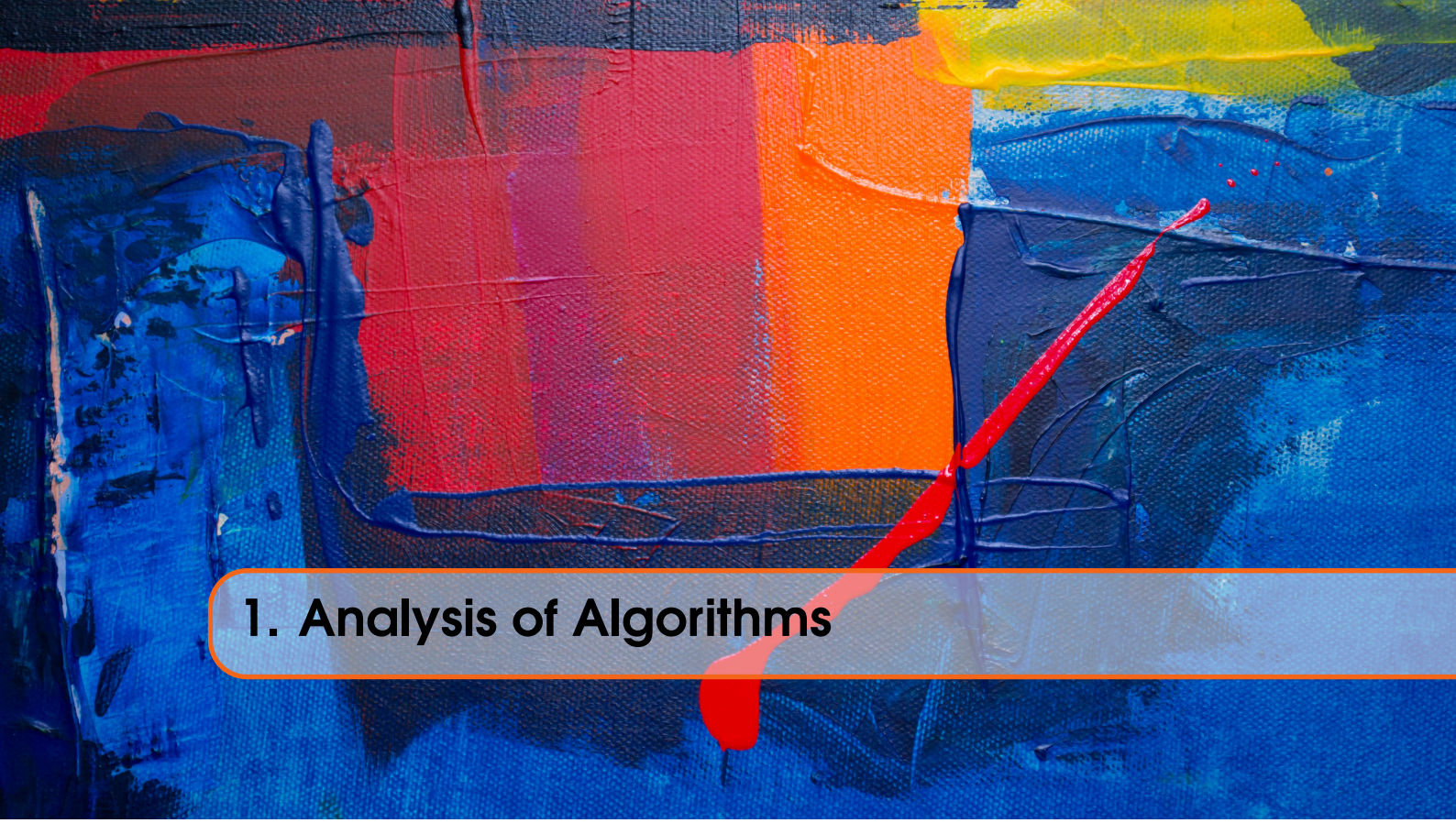
First printing, March 2019

Contents

Part One



1. Analysis of Algorithms

1.1 Basic

1.1.1 Insertion Sort

■ **Example 1.1** Given an input array $A = \langle 5,8,4,2,3,1 \rangle$ use insertion sort algorithm to show how the array looks before each iteration of the for loop in line 1. ■

Listing 1.1: Insertion-Sort(A)

```
1  for j=2 to A.length
2      key = A[j]
3      //Insert A[j] into the sorted sequence A[1..j - 1]
4      i = j-1
5      while i>0 and A[i]> key
6          A[i+1]=A[i]
7          i=i-1
8      A[i+1]=key
```

Definition 1.1.1 Iteration: A process wherein a set of instructions or structures are repeated in a sequence a specified number of times or until a condition is met.

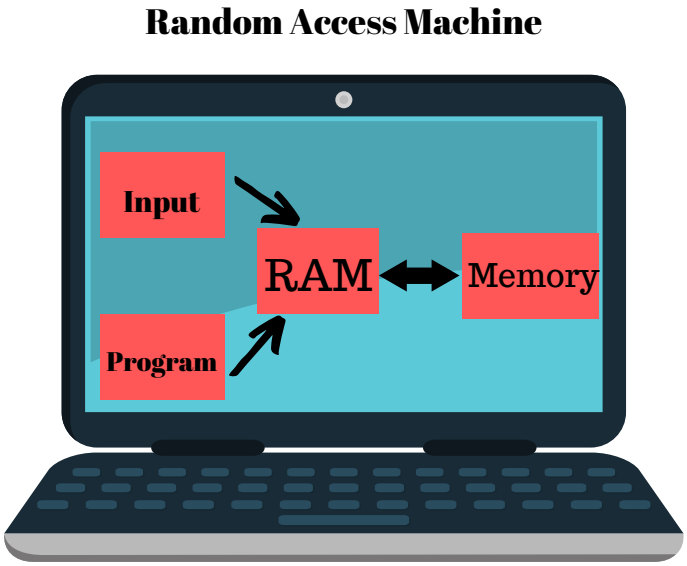
Definition 1.1.2 Insertion Sort An efficient sorting algorithm for small number of elements that builds a final sorted list one item at a time, with the movement of higher-ranked elements.

Solution:

1	j = 2	$A =< 5,8,4,2,3,1 >$
2	j = 3	$A =< 4,5,8,2,3,1 >$
3	j = 4	$A =< 2,4,5,8,3,1 >$
4	j = 5	$A =< 2,3,4,5,8,1 >$
5	j = 6	$A =< 1,2,3,4,5,8 >$

1.1.2 RAM - Random Machine Model

■ **Example 1.2** Briefly explain the RAM (Random Access Machine) computational model ■



Definition 1.1.3 Basic operations such as addition, multiplication, load, store, copy, control, initialization are assumed to take a constant amount of time each (if numbers are big, this may not be true, but we assume that this is the case unless stated otherwise).

R Not to be confused with **Random Access Memory**

Solution:

- The RAM (Random Access Machine) model is used for analyzing algorithms where we assume that instructions are executed one after the other. It is used to model how real world programs would run.
- Machine independent and theoretical.
- Simple operations such as $(+, \times, -, =)$ take $O(1)$ time.
- Loops and subroutines are not considered simple operations. Instead, they are the composition of many single-step operations

1.1.3 Merge Sort

■ **Example 1.3** Show the operation of Merge sort on the array $A =< 7,4,2,8,3,1,5,6,9 >$ ■

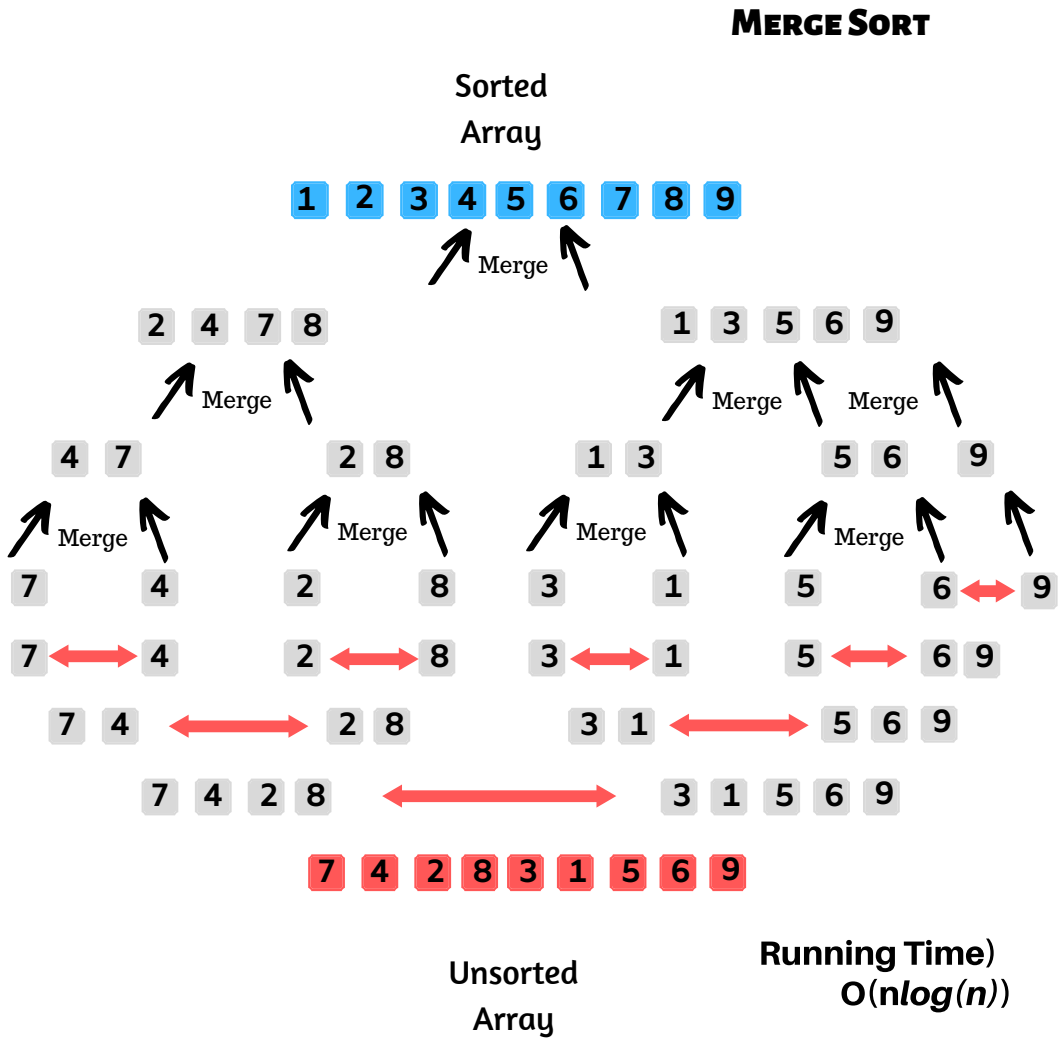
Definition 1.1.4 Merge Sort A divide and conquer algorithm. Takes an input array and divides it into two halves, calls itself for the two halves, and merges the two sorted halves.

- **merge()** function merges two halves.
- **MergeSort()** function recursively calls itself to divide the array until its size becomes one.

Listing 1.2: Merge-Sort(A,p,r)

```
1 if p < r
2     q = ⌊(p+r)/2⌋
3     Merge-Sort(A,p,q)
4     Merge-Sort(A,q+1,r)
5     Merge(A,p,q,r)
```

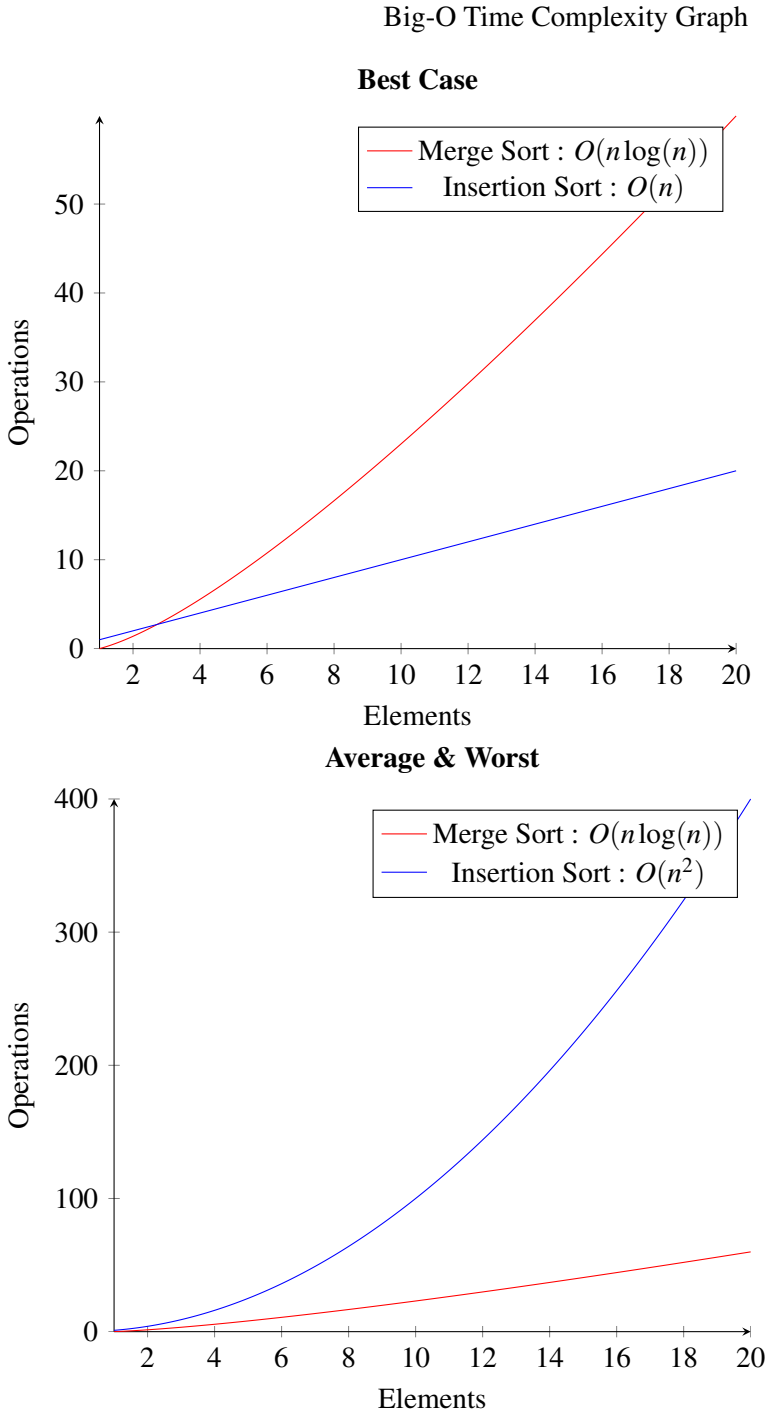
Solution:



■ **Example 1.4** *Merge Sort* is always faster than *Insertion Sort*. True or false?. Justify your answer ■

Definition 1.1.5 **Time Complexity** is the computational complexity that describes the amount of time it takes to run an algorithm. Time complexity is commonly estimated by counting the number of elementary operations performed by the algorithm, supposing that each elementary operation takes a fixed amount of time to perform.

Definition 1.1.6 Best, worst, and expected cases describe the big O (or big theta) time for particular inputs or scenarios.

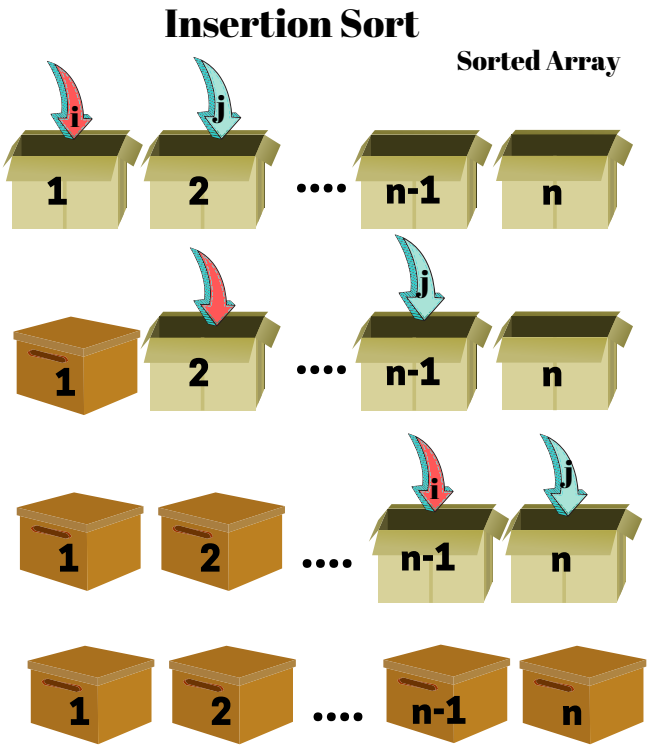


R When asked for running time of an algorithm this refers to $O(\cdot)$. Runtime time means worst case running time.

Solution: False: Merge sort is not always faster than insertion sort. In the **Best Case** Merge Sort's running time is $O(n\log(n)$ and Insertion Sort is linear $O(n)$. For **Average Case** and **Worst Case** Merge Sort is faster.

1.1.4 Insertion Sort vs Merge Sort

■ **Example 1.5** Suppose the input sequence is already sorted. For this specific input, what is the running time of Insertion-sort and Merge-sort?. Consider *listing 1.1 and 1.2* ■



Solution: For insertion sort the running time is $\Theta(n)$, for merge sort it is $\Theta(n\log(n))$.

- *Insertion Sort* must iterate through the entire array.
- The while loop (**Lines 6-8**) of Insertion Sort (*Listing 1.1*) does not execute.

1.1.5 Running Time

■ **Example 1.6** Suppose there are 2^n inputs to a certain problem. The running time of an algorithm A is exactly 2^n for exactly one of the inputs, and 1 for any other input. Then the running time is O(1) since $2^n \cdot \frac{1}{2^n} + 1 \cdot (1 - \frac{1}{2^n}) \leq 2$. Is this statement correct? ■

Solution: False: The statement is in correct. O notation refers to worst case running time, this means the running time for the algorithm is $O(2^n)$.

1.2 Intermediate

1.2.1 Selection-Sort

■ **Example 1.7** The following is a pseudo-code of Selection-Sort. Describe Selection-Sort in plain English. ■

Solution:

Listing 1.3: Selection-Sort(A)

```
1  n = A.length
2  for j = 1 to n-1
3      smallest = j
4      for i = j+1 to n
5          if A[i] < A[smallest]
6              smallest = i
7      exchange A[j] with A[smallest]
```

Definition 1.2.1 **Selection sort** finds the smallest element in the array and swaps it with the first element, then the second smallest with the second element, third smallest swaps with third element and so on.

■ **Example 1.8** State the loop invariant and prove that the algorithm is correct. What is the running time? ■

1. We want to show that the loop invariant is true prior to the first iteration of the loop, that each iteration maintains the loop invariant, and that the loop invariant gives us a useful property at loop termination.

   Loop Invariant: $A[\text{smallest}] \leq A[1..j-1]$
   - **Initialization** Prior to the first iteration A[1] is sorted. When j = 1 there is only one element and is therefore sorted. In the second for loop, i = j + 1, $A[1..j-1]$, it is only $A[i]$, one element only.
   - **Maintenance** It remains true as j increases and i is the element after j. The outer loop from j = 1 to n -1 and i = j + 1 to n, the inner loop finds smallest element from j + 1 and the outer loop exchanges the jth element with the smallest element. The sub-array A[1..j] is sorted.
   - **Termination** The sub-array A[1..n] is sorted, the outer loop iterates from j = 1 to n -1
2. The worst case running time is $O(n^2)$.

1.2.2 Merge

■ **Example 1.9** Consider the pseudo-code of Merge(A, p, q, r). Given that $A[p \ldots q]$ and $A[q+1 \ldots r]$ are both sorted, the function call merges the two sorted sub arrays into a sorted subarray $A[p \ldots r]$. Prove the correctness of Merge. For simplicity, you can assume that $L[1...n_1] = A[p \ldots q]$ and $R[1...n_2] = A[q+1...r]$., and $L[n_1+1] = R[n_2+1] = \infty$. (So you only need to consider from lines 10). You can assume that all elements stored in the array have distinct values. ■

Listing 1.4: MERGE(A,p,q,r)

```
1  n1 = q - p + 1
2  n2 = r - q
3  let L[1...n1+1] and R[1...n2+1] be new arrays
4  for i = 1 to n1
5      L[i] = A[p+i-1]
6  for j = 1 to n2
7      R[j] = A[q+j]$
8  L[n1+1] = ∞
9  R[n2+1] = ∞
10 i = 1
11 j = 1
```

```
12  for  k = p  to  r
13       if  L[i] ≤ R[j]
14            A[k] = L[i]
15            i = i + 1
16       else  A[k] = R[j]
17            j = j + 1
```

Loop Invariant: $k \in [p,r]$ elements in index k are sorted, this elements come from Left and Right subarrays, also $i <= n_1 + 1, j \leq n_2 + 1$ where $i \in [p,q], j \in [q+1,r]$ this means there are no more elements to be copied to A[p..r].

1. Initialization : Our merged array $A[p..r]$ before line 10 contains no elements from our sorted sub arrays $A[i]$,$A[j]$ recall that elements were stored in new sub arrays $L[1..n_1]$ or $R[1..n_2]$ and therefore $A[k]$ is "empty", therefore sorted.
2. Maintenance : The for loop iterates from $k \in ptor$, and elements are added to A[k], we must show this loop invariant is in sorted order. Check by Iterating through each element in L[i] and R[j] and finding the smallest element after the first iteration A[p] holds one single element, this element is sorted since it belongs to the smallest element of subarray L[1..n_1] or subarray R[1..n_2].
3. Termination : The loop iterate through (r - p + 1) element's and compares each element in the subarrays $L[1..n_1$ and $R[1..n_2$ to find the smallest, then it is stored in the array A[p..r], at the end of

1.3 Advanced

1.3.1 Searching Problem

■ **Example 1.10** Consider the searching problem The input is an array A[1..n] of n numbers and a value v. You're asked to find an index i such that A[i] = v. If there's no such index, return NIL. The following is a pseudocode of linear search, which scans through the sequence, looking for v. Using a loop invariant, prove that your algorithm is correct. Make sure that your loop invariant fulfills the there necessary properties. What is the asymptotic worst case running time? ■

Solution:

```
1    For  i  =  1  to  n
2         If  A[i]  ==  v  then  return  i
3    return  NIL
```

1. Loop Invariant: At the start of each *ith* iteration of the for loop $j \in [1, i-1]$ and A[j] is not v.
   (a) Initialization : Before the loop iteration the **variable answer** contains NIL,that is since $j = 0 \rightarrow A[0]$ does not exist. A[0] is therefore not v.
   (b) Maintenance : Assume the loop invariant holds then $A[j]$is not v. If $A[j+1]$ is v then the termination would have followed. Since termination has not followed (loop is still running) this implies that $A[j+1]$ is not v.
   (c) Termination : When the for loop terminates the variable answer contains v or NIL.It returns NIL if at index j + 1 the element is v, or returns NIL if at index j + 2 which is larger than the size of array and therefore A[j+2] is not v.
2. The asymptotic worst case is $O(n)$, that is the element is at the index n, it must examine every element.

1.4 Exam Problems

1.4.1 Insertion Sort

■ **Example 1.11** The following is a pseudocode of Insertion-Sort for instance A[1..8] = $\langle 7,4,2,9,4,3,1,6\rangle$ what is $A[1..8]$ just before the for loop starts for j = 5?

INSERTION-SORT(A)

```
1  for j = 2 to A.length
2      key = A[j]
3      //insert A[j] into the sorted
4          //sequence A[1..j−1]
5      i = j − 1
6      while i > 0 and A[i] > key
7          A[i+1] = A[i]
8          i = i − 1
9      A[i+1] = key
```

■

Solution:

	j	After
	j = 2	$\langle 4,7,2,9,4,3,1,6\rangle$
	j = 3	$\langle 2,4,7,9,4,3,1,6\rangle$
	j = 4	$\langle 2,4,7,9,4,3,1,6\rangle$
	j = 5	$\langle 2,4,4,7,9,3,1,6\rangle$

Before j = 5 the array $A[1..8]$ is $\langle 2,4,7,9,4,3,1,6\rangle$

■ **Example 1.12** Give an instance of size n for which the insertion-sort terminates in $\Omega(n^2)$ ■

Solution: The input should be n numbers in decreasing order. The while loop executes. Example: $\langle 6,5,4,3,2,1\rangle$

■ **Example 1.13** Give an instance of size n for which the Insertion-sort terminates in $O(n)$ time. ■

Solution: The input should be n number in increasing order. The while loop does execute. Example: $\langle 1,2,3,4,5,6\rangle$

■ **Example 1.14** The following is a pseudocode of Insertion-sort. Prove its correctness via loop invariant. In other words, state the loop invariant and prove it using *Initialization, Maintenance, and Termination*

INSERTION-SORT(A)

```
1  for j = 2 to A.length
2      key = A[j]
3      //Insert A[j] into the sorted
4          sequence A[1..j−1]
5      i = j − 1
6      while i > 0 and A[i] > key
7          A[i + 1] = A[i]
8          i = i − 1
9      A[i + 1] = key
```

■

Solution: Loop Invariant: At the start of each of the **for** loop of lines 1 -8, the subarray A[1..j-1] consists of the elements originally in A[1..j-1], but in sorted order.

1. **Initialization** Just at the start of the first iteration (j=2), A[1] is ordered and the number was originally in A[1].
2. **Maintenance** Say the invariant is true for iteration $j \geq 2$. At the end of the iteration. The algorithm has $A[1..i]$, A[j], and A[i+1..j-1] in this order in the prefix of A. We know that $A[1..i]$ and $A[i+1..j-1]$ are sorted by the invariant. Further, $A[i] \leq A[j] \leq A[i+1]$ by Algo's definition, meaning A[1..j] is sorted at the end of iteration. Clearly, all elements in $A[1..j]$ originate form the same subarray. Thus, the loop invariant holds before the next iteration j + 1
3. **Termination** The for loop ends when j = n + 1, and the loop invariant implies that the array is sorted as desired.

1.4.2 RAM - Random Machine Model

■ **Example 1.15** Briefly explain the Random Access Model (RAM) ■

Solution: The RAM model is used for analyzing algorithms where we assume that instructions are executed one after the other. This model is theoretical.It is used to model how real world programs would run.
Basic operations such as addition, multiplication, load, store, copy, control, initialization are assumed to take a constant amount of time each (if numbers are big, this may not be true, but we assume that this is the case unless stated otherwise).

1.4.3 Merge Sort

■ **Example 1.16** Let T(n) denote the running time of Merge-sort on input size n. In the following you can omit floor or ceiling. MERGE-SORT(A,p,r)

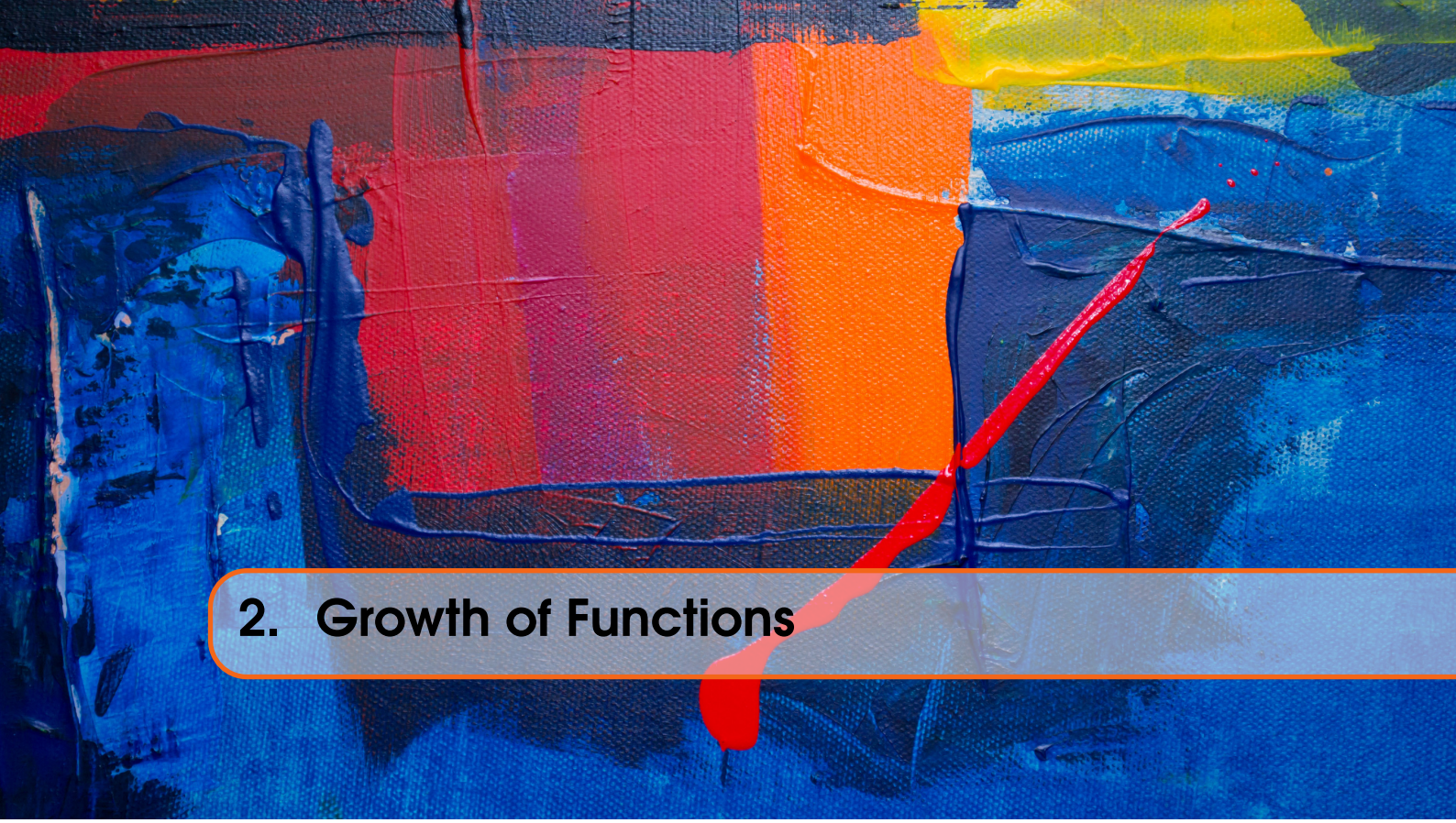
```
1 if p < r
2     q = floor((p+r)/2)
3     MERGE–SORT(A, p, q)
4     MERGE–SORT(A, q+1, r)
5     MERGE(A, p, q, r)
```

■

Running time of Merge Sort

$$f(a,b) = \begin{cases} O(1) & \text{if } n = 1 \\ 2T(n/2) + \Theta(n) & \text{if } n \geq 2 \end{cases}$$

1. What is the running time of Line 3 (of Merge-Sort)?
   Solution: A call to merge sort is $T(n/2)$
2. What is the running time of Line 4 (of Merge-Sort)?
   Solution: A call to merge sort is $T(n/2)$
3. What is the running time of LIne 5 (of Merge-Sort)?
   Solution: A call to merge is $\Theta(n)$



2. Growth of Functions

2.1 Basic

2.1.1 Ranking Functions

■ **Example 2.1** Rank the following functions by order of growth.Find an ordering $g_1, g_2, \ldots, g_k$ (Where k is the number of functions given) such that $g_1 = \mathcal{O}(g_2), g_2 = \mathcal{O}(g_3), \ldots, g_{k-1} = \mathcal{O}(g_k)$

$n^2 + 2^n$	$n\log(n)$	$n^2(\log(n))^2$	$n(\log(n))^2$	n^2	$log^{100}(n)$
$\log(\log(n))$	n^3	1	$\log(n)$	$\frac{\log(n)}{\log\log(n)}$	$\frac{n^2}{log(n)}$
$n^{10}3^n$	4^n				

■

Definition 2.1.1 Asymptotic upper bound. $\mathcal{O} - notation$

$\mathcal{O}(g(n)) = \{f(n)$: there exist positive constants c and n_o such that $0 \leq f(n) \leq c(g(n)$for all $n \geq n_o\}$

Definition 2.1.2 Dominance Pecking Order

$$1 < \alpha(n) < \log(\log(n)) < \frac{\log(n)}{\log(\log(n))} < \log(n) < \log^2(n) < \sqrt{n} < n < n\log(n) < n^{1+\varepsilon} < n^2 < 2^n < n!$$

Solution:
Follow the **Dominance Pecking Order**

g_1	g_2	g_3	g_4	g_5	g_6	g_7
1	$\log(\log(n))$	$\frac{\log(n)}{\log(\log(n))}$	$\log(n)$	$(\log(n))^{100}$	$n\log(n)$	$n(\log(n))^2$
g_8	g_9	g_{10}	g_{11}	g_{12}	g_{13}	g_{14}
$\frac{n^2}{\log(n)}$	n^2	$n^2(\log(n))^2$	n^3	$n^2 + 2^n$	$n^{10}3^n$	4^n

2.1.2 Asymptotically no smaller

■ **Example 2.2** What is $\lim_{n\to\infty} \frac{n^{10}\cdot 3^n}{4^n}$? Which one is asymptotically no smaller between the two functions? ■

Definition 2.1.3 Limit Method Process A limit quotient between f and g

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = \begin{cases} 0 & \text{then } f(n) \in \mathcal{O}(g(n)) \\ c > 0 & \text{then } f(n) \in \Theta(g(n)) \\ 0 & \text{then } f(n) \in \Omega(g(n)) \end{cases}$$

Solution:

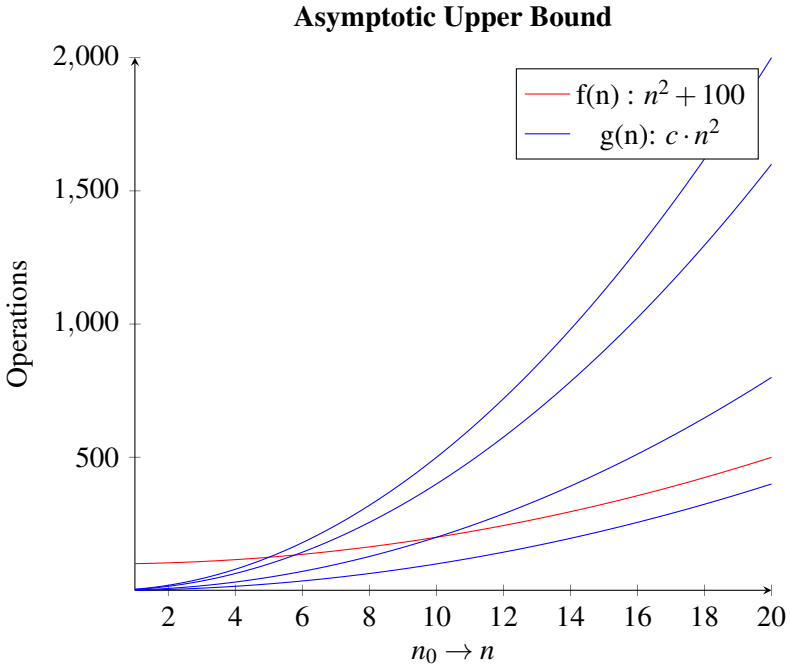
1. Since $4^n > n^{10}3^n$
2. $g = \Omega(n^{10}3^n)$
3. $f = O(4^n)$

The limit is 0, and the function that is asymptotically no smaller between the two functions is $f = O(4^n)$ since 4^n is the upper bound and $n^{10}3^n$ is the lower bound.

2.1.3 Using asymptotic definitions

■ **Example 2.3** Formally prove that $n^2 + 100 = \mathcal{O}(n^2)$ using the definition of $\mathcal{O}(\cdot)$ ■

Definition 2.1.4 $f(n) = \mathcal{O}(g(n))$ $\mathcal{O}$ - notation
$\mathcal{O}(g(n)) = \{f(n)$: there exist positive constants c and n_o such that
$0 \leq f(n) \leq c(g(n))$ for all $n \geq n_o\}$



Solution:

1. We want to show that there exist a constant c such that $n^2 + 100 \leq c(n^2)$ for some $n \geq n_0$
2. Find c:

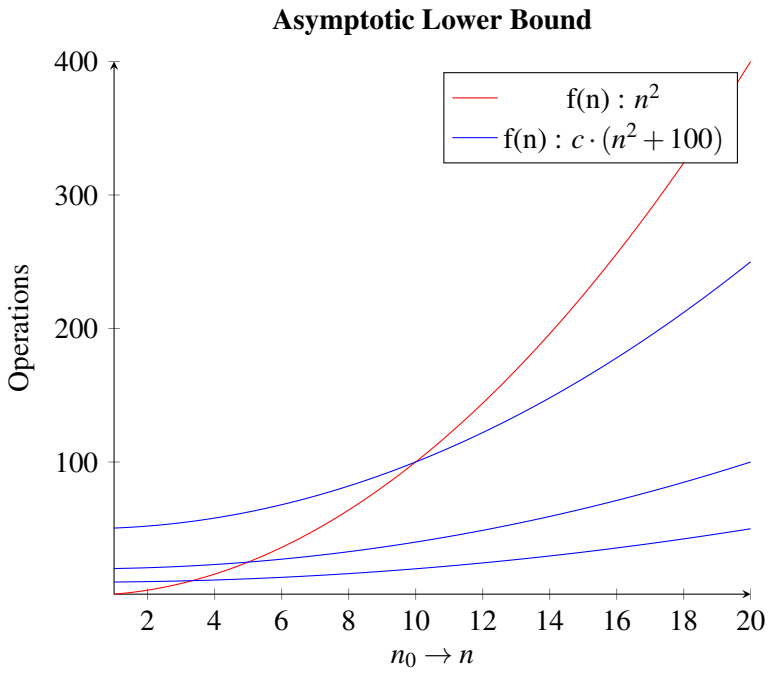
$$\frac{n^2 + 100}{n^2} \leq c \to 1 + \frac{100}{n^2} \leq c$$

3. By inspection if $n_0 = 10$ then $c \geq 2$.

R The chosen n_0 is not unique, when $n_0 >$, c converges to 1.

■ **Example 2.4** Formally prove that $n^2 = \Omega(n^2 + 100)$ Using the definition of $\Omega(\cdot)$. ■

Definition 2.1.5 Asymptotic lower bound. Ω - notation
$\Omega(g(n) = \{f(n)$: there exist positive constants c and n_0 such that
$0 \leq cg(n) \leq f(n)$ for all $n \geq n_0$



Solution:

■ **Example 2.5** Formally prove that $n^2 = \Omega(n \log_2 n + 100)$ using the definition of $\Omega(\cdot)$ ■

Definition 2.1.6 Asymptotic lower bound. Ω - notation
$\Omega(g(n) = \{f(n)$: there exist positive constants c and n_0 such that
$0 \leq cg(n) \leq f(n)$ for all $n \geq n_0$

■ **Example 2.6** Formally prove that $n + 10 = \Theta(50n + 1)$ using the definition of $\Theta(\cdot)$ ■

Definition 2.1.7 Assymptotic average bound. Θ - notation
$\Theta(g(n)) = \{f(n)$: there exist positive constants c_1, c_2 and n_0 such that
$c_1 g(n) \leq f(n) \leq c_2 g(n)$ for all $n \geq n_0$

Solution:

2.2 Intermediate

2.2.1 Using big-O notation

■ **Example 2.7** Prove that $f = O(g)$ implies $g = \Omega(f)$ ■

1. $f(n) = O(g(n)), g(n) = \Omega(f(n))$
   Meaning that if g(n) is the upper bound then this implies $f(n)$ and $f(n)$ is the lower bound of $g(n)$
2. Solution : $f(n) \leq c_1 g(n)$ Definition of O $\rightarrow \frac{1}{c_1} f(n) \leq g(n) \rightarrow c_2 f(n) \leq g(n)$ Definition of Ω (note $c_2 = \frac{1}{c_1}$)

■ **Example 2.8** Prove that $f = \Omega(g)$ and $g = \Omega(h)$ implies $f = \Omega(h)$ ■

1. $g = \Omega(h)$ means $0 \leq ch(n) \leq g(n)$ for $n_1 > n_0$

2. $f = \Omega(g)$ means $0 \leq cg(n) \leq f(n)$ for $n_2 > n_0$
3. $f = \Omega(h)$ means $0 \leq ch(n) \leq f(n)$ for $n_2 > n_1$

2.3 Exam Problems

2.3.1 True or False

For each of the following claims, decide if it is true or false. No explanation is needed.
To solve consider the following

1. **Dominance Pecking Order** *Page 56, Algo Design Manual*

$$1 < \alpha(n) < \log(\log(n)) < \frac{\log(n)}{\log(\log(n))} < \log(n) < \log^2(n)$$

$$< \sqrt{n} < n < n\log(n) < n^{1+\varepsilon} < n^2 < n^3 < c^n < n!$$

2. **Asymptotic Upper Bound**

   Definition 2.3.1 $f(n) = O(g(n))$means $c \cdot g(n)$ is an *upper bound* on f(n). Thus there exists some constant c such that f(n) is always $\leq c \cdot (n)$, for large enough n (i.e.., $n \geq n_o$ for some constant n_0).

3. **Asymptotic Lower Bound**

   Definition 2.3.2 $\Omega(g(n)) = \{f(n)$: there exist positive constant c and n_0 such that $0 \leq cg(n) \leq f(n)$ for all $n \geq n_0$

4. **Asymptotically tight bound**

   Definition 2.3.3 $\Theta(g(n)) = \{f(n)$: there exist positive constants c_1, c_2 and n_0 such that $c_1 g(n) \leq f(n) \leq c_2 g(n)$ for all $n \geq n_0$

■ **Example 2.9** $n\log(n) = O(n^2)$ ■

Solution: True, since $n\log(n) < c \cdot n^2$

■ **Example 2.10** $\log(\log(n)) = O(\log(n))$ ■

Solution: True, since $\log(\log(n)) < c \cdot \log(n)$

■ **Example 2.11** $\log^{50} n = O(n^{0.1})$ ■

Solution: True, since $\log^{50} n \leq c(n^{0.1}))$

■ **Example 2.12** $4^n = O(2^n)$ ■

Solution: False, since $4^n \not\leq c \cdot (2^n)$

■ **Example 2.13** $100^{100} = \Theta(1)$ ■

Solution: True, since $100^{100} \leq c \cdot 1$

■ **Example 2.14** If $f = O(g)$, then $g = \Omega(f)$ ■

Solution: True. If $f \leq cg(n)$ then $g(n)$ is the upper bound, this implies that $f(n)$ is the lower bound, therefore $g \geq cf(n)$

■ **Example 2.15** If $n^3 = \Omega(n^2)$ ■

Solution: True, since $n^3 \geq c \cdot (n^2)$

■ **Example 2.16** $100 + 200n + 300n^2 = \Theta(n^2)$ ■

Solution: True, since $100 + 200n + 300n^2 < c \cdot (n^2)$ and $c \cdot (100 + 200n + 300n^2) > (n^2)$

■ **Example 2.17** $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 5$ then $f(n) = O(g(n))$ ■

Solution: True.

Definition 2.3.4 If there exists a constant $c \geq 0$ such that $\lim_{n\to\infty} \frac{f(n)}{f(n)} \leq c$, then $f = O(g)$

■ **Example 2.18** $\sum_{i=1}^{n} \Theta(i) = \Omega(n^2)$ ■

Solution: True. $\Theta(1) + \Theta(2) + \cdots + \Theta(n) > c \cdot n^2$

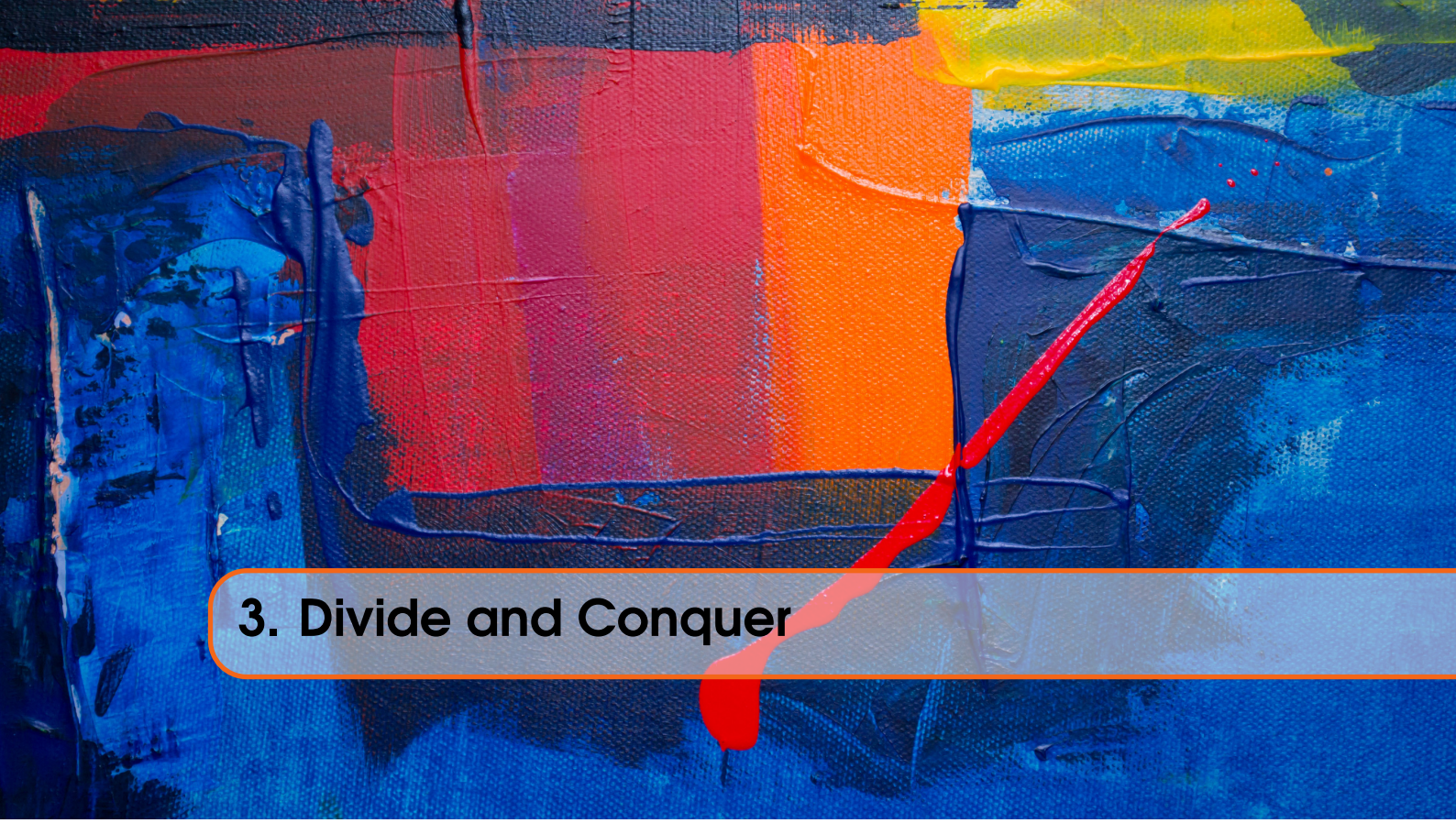
2.3.2 Proving using definition of O

■ **Example 2.19** Formally prove that $50n + 15 = O(n^2)$ using the definition of $O(\cdot)$ ■

Definition 2.3.5 $\mathcal{O}(g(n))$ = $\{f(n)$: there exist positive constants c and n_o such that $0 \leq f(n) \leq c(g(n))$for all$n \geq n_o\}$

Solution:

$$0 \leq 50n + 15 \leq c \cdot n^2$$
$$\frac{50n + 15}{n^2} \leq c$$



3. Divide and Conquer

3.1 Basic

3.1.1 Merge-Sort Pseudocode

■ **Example 3.1** What is wrong with Merge-Sort when $p < r$ condition is removed.

Listing 3.1: MERGE-SORT(A,p,r)

```
1  // if  p < r  \*Removed*\
2        q = ⌊(p+r)/2⌋
3        Merge-Sort(A,p,q)
4        Merge-Sort(A,q+1,r)
5        Merge(A,p,q,r)
```

■

Definition 3.1.1 Segmentation faults: Caused by a program trying to read or write an illegal memory location.

Solution

1. Suppose there is an array with only one element, that is **A**[] = <2>, then p = 1, q = 1, therefore the array **A** is already sorted and there is no need to call MERGE-SORT or MERGE functions.
2. If *line 1* is missing then q = 0 and MERGE-SORT(A,p = 1, q = 0) which does not make sense since element at at index 0 does not exist (Assuming we start at index 1).
3. This is a logic error, as the program reads outside of the array there will be a segmentation fault.

■ **Example 3.2** Let's slightly tweak MergeSort. Instead of partitioning the array into two subarrays, we will do into three subarrays of an (almost) equal size. Then, each recursive call will sort each

subarray, and we will merge the three sorted subarrays. What is the running time of this new merge? Write a recurrence for the running time of this new version of Merge Sort, and solve it. ■

3.1.2 Recursion Tree Method

■ **Example 3.3** Solve $T(n) = 4T(\frac{n}{2}) + \theta(n)$ using the recursion tree method. Clearly state the tree depth, each subproblem size at depth d, the number of subproblems/nodes at depth d, workload per subproblem/node at dept d, (total) workload at depth d. ■

■ **Example 3.4** Solve $T(n) = 4T(\frac{n}{2}) + \Theta(n^2)$ using the recursion tree method. ■

■ **Example 3.5** Solve $T(n) = 4T(\frac{n}{2}) + \Theta(n^3)$ using the recursion tree method. ■

3.1.3 Master Theorem

■ **Example 3.6** Solve the above recursions using the Master Theorem. You must specify which case and how you set the variables such as a,b, ε. ■

3.2 Intermediate

3.3 Advanced

3.4 Exam Problems

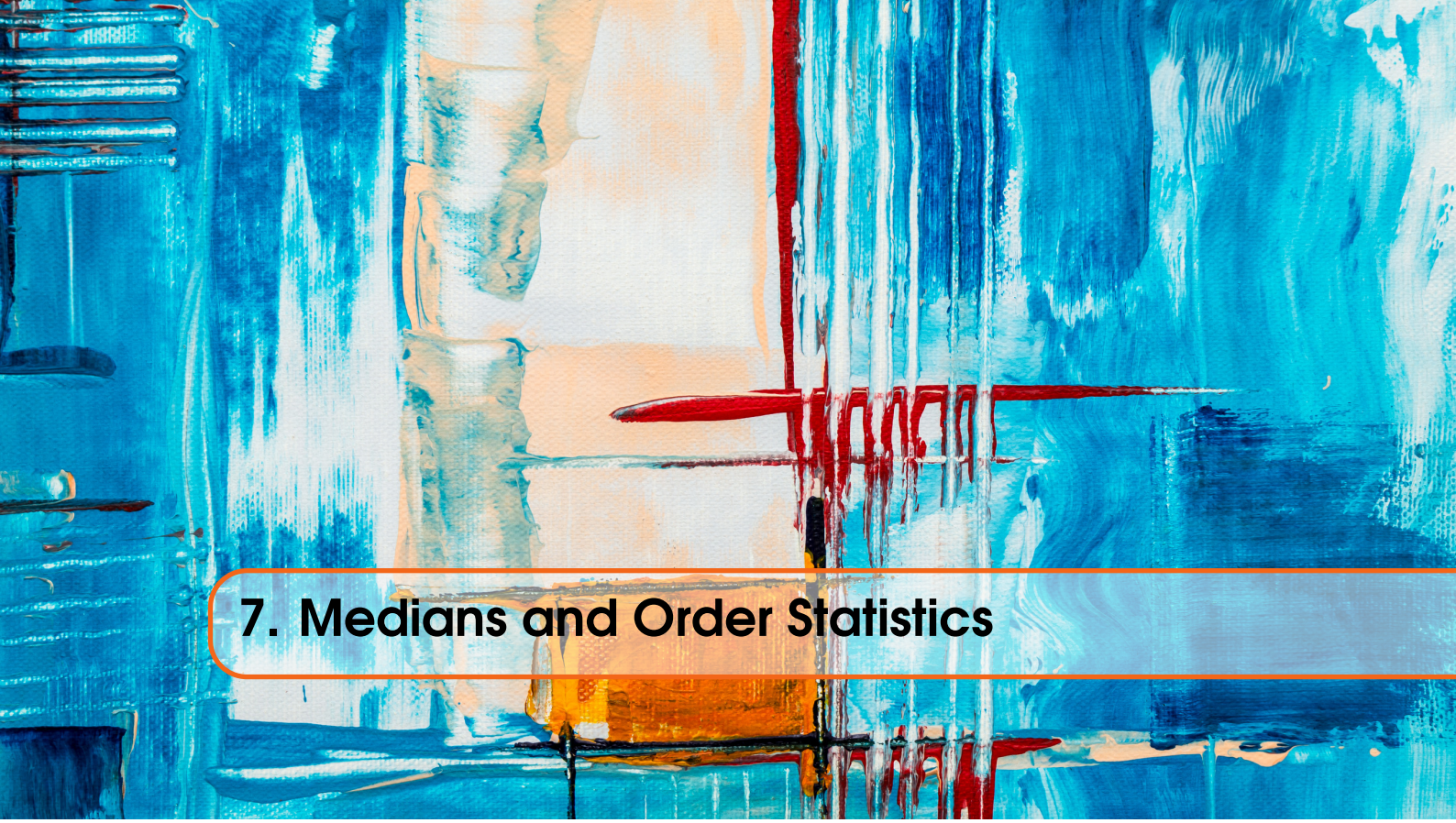
II Part Two

4. Heapsort

5. Quicksort

6. Sorting In Linear Time

sectionBasic



7. Medians and Order Statistics

8. Hash Tables

8.1 Basic

8.1.1 Properties of Hash Tables

■ **Example 8.1** What is the main advantage of hash tables over direct-address tables? ■

Definition 8.1.1 Direct-Address Tables: Uses an array and assumes that keys are integers in the universe set $U = 0, 1, \ldots, m-1$, m is small, and no two elements have the same key. Operations such as *Search, Insert*, and *Delete* run in $\mathcal{O}(1)$.

1. Stores element with key k in slot k.
2. Number of possible keys should be small.
3. Denoted as $T[0 \ldots m-1]$
4. Each slot corresponds to a key in universe U.

Definition 8.1.2 Hash Tables

Solution:

Disadvantages of Direct Addressing

1. If the universe $U = 0, 1, \ldots, m-1$ is large, storing a table T of size $|U|$ is very impractical.
2. Most space allocated for T would be wasted because the set K of keys actually stored may be small relative to U.

Advantages of Hash Tables

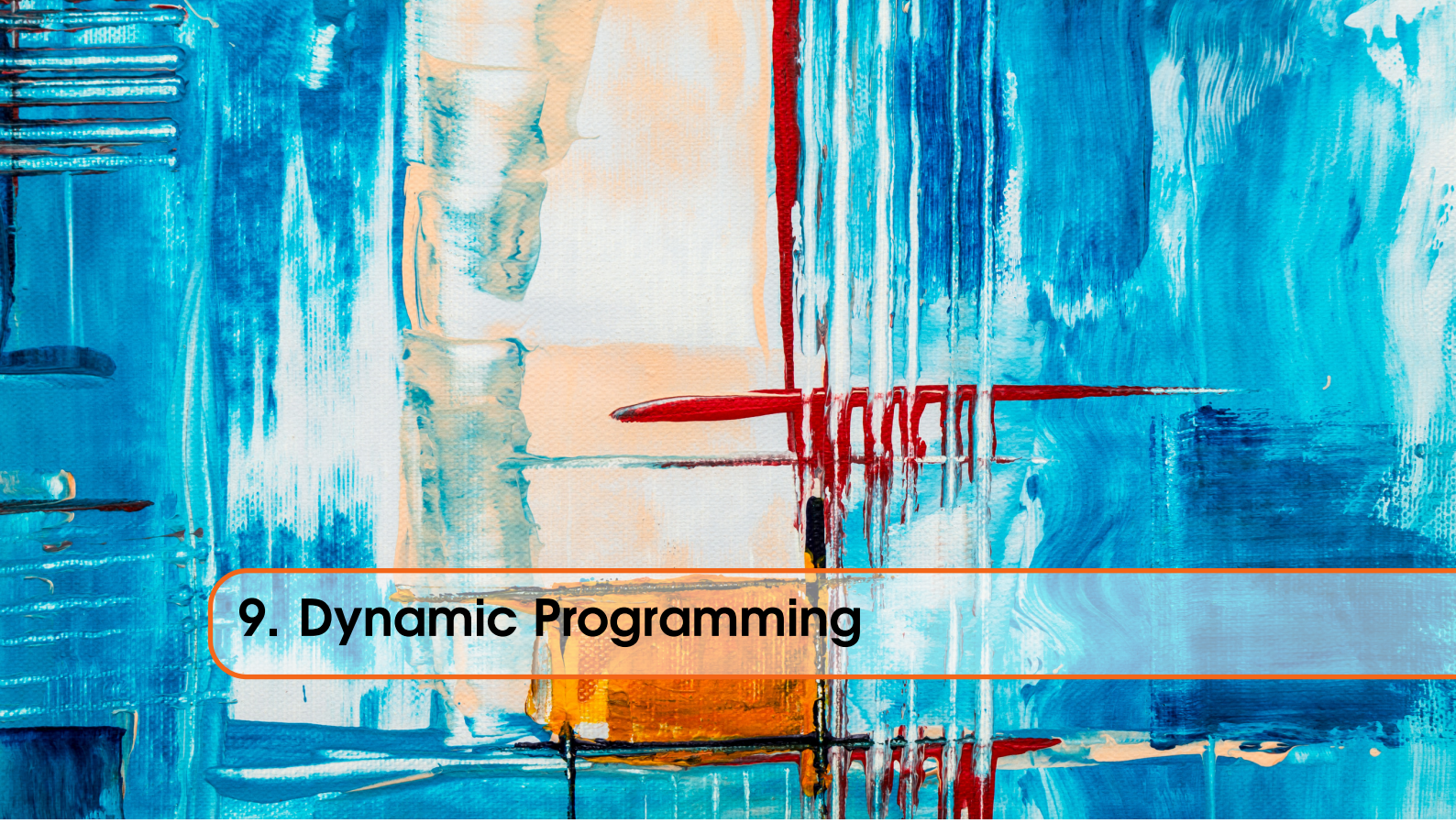
1. Requires much less storage when the set K of keys is much smaller than the universe U.
2. Hash function reduces the range o array indices and hence the size of the array.

8.2 Intermediate

8.3 Advanced

8.4 Exam Problems

Part Three



9. Dynamic Programming

sectionBasic

10. Greedy Algorithms

sectionBasic

IV Part Four

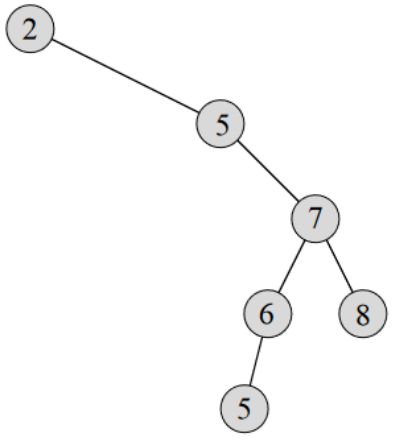
11. Binary Search Trees (BST)

In this chapter inorder, preorder, and postorder **traversal** techniques will be introduced.

11.1 Basic

11.1.1 Inorder, Pre- order, Post-order

■ **Example 11.1** Consider the BST in Fig 12.1.(b). Print out all the keys in the BST in preorder,then postorder ■



(b)

Figure 11.1: Fig 12.1(b)

R Nodes with no children are called leaves. Nodes which are not leaves are called internal nodes. Nodes with the same parent are called siblings.

Solution: Pre-order $\langle 2,5,5,6,7,8 \rangle$

Definition 11.1.1 Preorder-Tree-Walk(x) x (root), x's left subtree, and x's right subtree. (Visit the root, traverse the left subtree, traverse the right subtree)

```
1 if x ≠ NIL
2     print(x.key)
3     Preorder-Tree-Walk(x.left) //Left subtree
4     Preorder-Tree-Walk(x.right)
```

Post-order: $\langle 5,6,8,7,5,2 \rangle$

Definition 11.1.2 Postorder-Tree-Walk: x's left subtree, x's right subtree, and x (root). (Traverse the left subtree, traverse the right subtree, visit the root)

```
1 if x ≠ NIL
2     Postorder-Tree-Walk(x.left) //Left subtree
3     Postorder-Tree-Walk(x.right) //Right subtree
4     print(x.key)
```

R Keys in a BST may or may not be distinct. Figure 4.1 has duplicate values.

■ **Example 11.2** A full binary tree of 7 nodes, find Inorder, Pre-order, Post-order ■

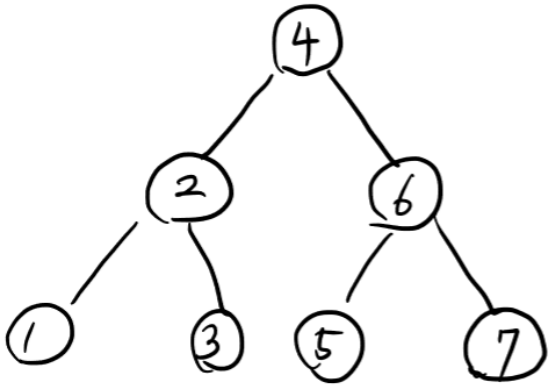


Figure 11.2: Full Binary Tree

R A full binary tree is a binary tree in which each node has exactly zero or two children.
A complete binary tree is $\Theta(\log(n))$ in worst case.

Solution:

- **Inorder** $\langle 1,2,3,4,5,6,7 \rangle$

Definition 11.1.3 Inorder-Tree-Walk: x's left subtree, x (root), and x's right subtree. (Traverse the left subtree, visit the root, traverse the right subtree)

```
1   if x ≠ NIL
2       Inorder-Tree-Walk(x.left)
3       Inorder-Tree-Walk(x.right)
4       print(x.key)
```

- **Pre-order** $\langle 4,2,1,3,6,5,7 \rangle$
- **Post-order** $\langle 1,3,2,5,7,6,4 \rangle$

The worst-case running time for most search-tree operations is proportional to the height of the tree.

11.1.2 Sequence of nodes

■ **Example 11.3** Suppose that we have numbers between 1 and 1000 in a binary search tree, and we want to search for the number 363. Which of the following sequences could not be a sequence of nodes examined? ■

Consider the following
Binary-Search-Tree Property

Definition 11.1.4 For any node x in BST.
- If y is in left subtree of x, then y.key $\leq$ x.key
- If y is in the right subtree of x, then y.key $\geq$ x.key

(a) 2,252,401,398,330,344,397,363
(b) 924,220,911,244,898,258,362,363
(c) 925,202,911,240,912,245,363
Does not meet BST property Since 911 $<$ 912. BST in Figure 4.3
(d) 2,399,387,219,266,382,381,278,363
(e) 935,278,347,621,299,392,358,363

Construct Binary Search Tree
1. Make the first element the root (x)
2. For the next element (y)
   - If value (y.key) is $\leq$ node.value (x.key)
     Place left
   - If value (y.key) is $>$ node.value (x.key)
     Place right
   - If the place is empty
     Place the node

The sequence of nodes are valid if you can easily traverse through them to find a value. If the BST has a single path, then it is valid.

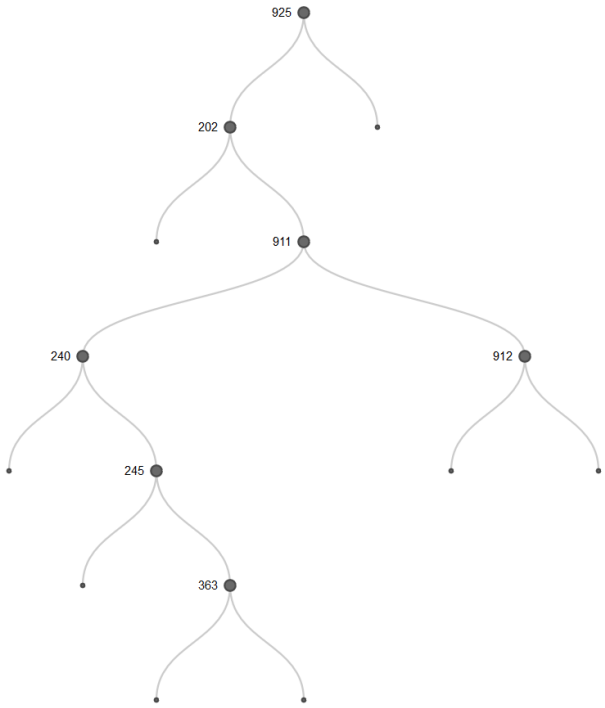


Figure 11.3: Part (c)

11.1.3 Binary Search Tree Problem

■ **Example 11.4** Consider the following binary search tree where nodes are labeled by alphabets. Here the keys are not shown in the picture; *a,b,c,...* are node labels, not keys. Assume that all keys have distinct values.

■

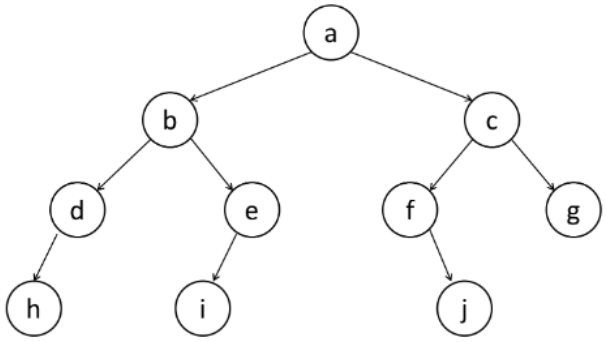


Figure 11.4: Example 3.4

(A) What is the node with the min key value?

Definition 11.1.5 TREE-MINIMUM(x): The min key is at the leftmost node.

```
1       while x.left != NIL
```

```
2           x = x.left
3       return x
```

Solution : Node h has the min key value.

(B) What is the node with the max key value?

Definition 11.1.6 TREE-MAXIMUM(x): The max key is at rightmost node.

```
1       while x.right != NIL
2           x = x.right
3       return x
```

Solution : Node g has the max key.

R Running time for TREE-MINIMUM and Tree-MAXIMUM is O(h) where h is the subtree's height.

(C) What is the node with the upper median key value?

Definition 11.1.7

(D) What is the node with the lower median key value?

(E) What is e's successor?

Definition 11.1.8 Successor: The successor of a node x is the node y such that y.key is the smallest key greater than x.key

```
1       if x.right != NIL
2           return TREE–MINIMUM(x.right)
3       y = x.p //(Parent of x)
4       while y != NIL and x == y.right
5           x = y
6           y = y.p //(Parent of y)
7       return(y)
```

Solution : The successor is a.

(a) **Lines 1-2:** Start at node label e (x), the right subtree of e is NIL (x.right) therefore **line 2** does not execute.

(b) **Line 3:** y is set to the parent of x (x.p) which is node label b

(c) **Lines 4: While Loop** Since y is b it is not NIL and (y.right) is x which is e.

(d) **Lines 5-6: Inside While** Set x equal to y which is b, and y equal to parent of y which is a.

(e) **Lines 4: While Loop** y is b and not NIL and (y.right) is x which is a

(f) **Lines 5-6: Inside While** Set x equal to y which is b, and y equal to the parent of y which is NIL (y.p).

(g) **Lines 7** While loop terminates. Returns y which node label **a**

(F) What is i's successor?

Solution: The successor is e

(a) **Lines 1-2:** Start at node label i, the right subtree of i is NIL (x.right) therefore **line 2** does not execute.

(b) **Line 3:** y is set to the parent of x (x.p) which is node label e

(c) **Line 4: While Loop** Since y is *e* it is not NIL and (y.right) is NIL not x, the while loop does not execute **Lines 5-6**.
(d) **Line 7:** Returns y which is node label **e**.

(G) What is *g*'s successor?
Solution: The successor is *a*
(a) **Line 1-2:** Start at node label *g* (x), the right subtree is NIL (x.right) therefore **line 2** does not execute.
(b) **Line 3:** y is set to the parent x (x.p) which is node label *c*
(c) **Line 4: While Loop** Since y is *c* it is not NIL and (y.right) is x which is *g*
(d) **Line 5-6: Inside While** Set x equal to y which is **c**, and y equal to the parent of y which is *a*.
(e) **Line 4: While Loop** Since y is *a* and (y.right) is x which is *c*.
(f) **Line 5-6: Inside While** Set x equal to y which is **a** and y equal to the parent of y which is **NIL**
(g) **Line 4: While Loop** Since y is NIL **Lines 5-6** do not execute.
(h) **Line 7:** Returns y which is node label **a**.

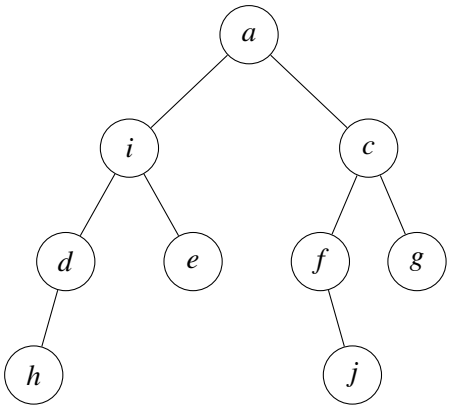
(H) What is *j*'s predecessor?
Solution: The predecessor is *c*

R The running time for **successor** and **predecessor** is $O(h)$

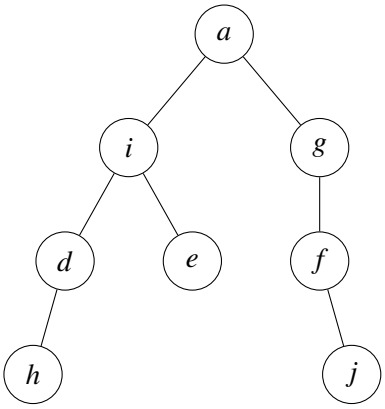
■ **Example 11.5** Consider the BST in the above problem. ■

Definition 11.1.9 Find the minimum of right subtree, copy that value to the node that will be deleted, and delete the duplicate from the right subtree.
Find maximum in left subtree, copy that value to the node that will be deleted, and then delete the duplicate from the right subtree.

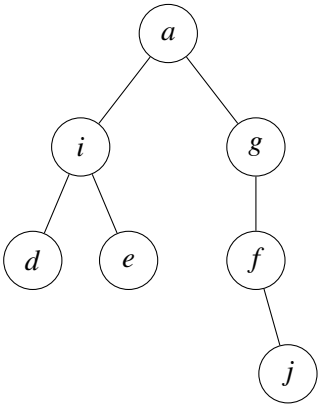
(A) Delete node b and show the resulting BST
Solution: Remove *b* and replace with minimum in rights subtree. We find *e* to be the minimum.



(B) Continue to delete another node c and show the resulting BST.



(C) Continue to delete another node h and show the resulting BST.

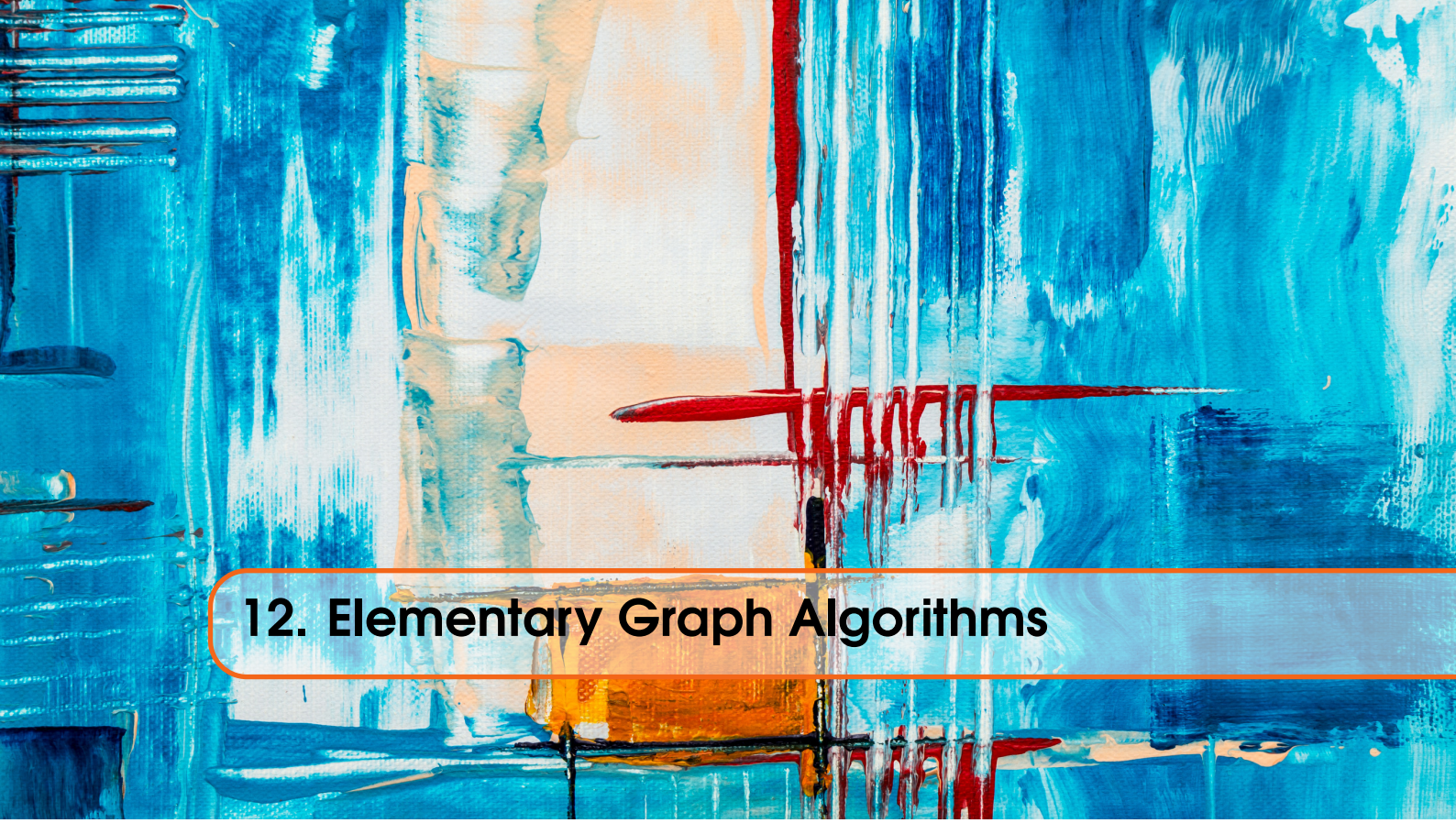


R When leaf nodes are deleted the BST property is maintained. Leaf nodes do not have right or left children. When deleting an internal node that has only one child remove internal node and link its parent to the child.

11.2 Advanced

11.3 Sources and Resources

In order to fully understand the concepts here. Consider the following items.
https://www.cs.rochester.edu/ gildea/csc282/slides/C12-bst.pdf



12. Elementary Graph Algorithms

12.1 Basic

12.1.1 Properties of BFS and DFS

■ **Example 12.1** To implement BFS, what data structure do you use? What about DFS (if we use no recursion)? ■

Definition 12.1.1 Breadth First Search (BFS):

1. **INPUT:** Graph $G = (V, E)$ and a source s.
2. **OUTPUT:** A tree (Breadth First Tree) consisting of vertices reachable from s encoding distance from source vertex s
   - The vertices that are reachable from s
   - The shortest distance from s to each reachable vertex
   - A shortest paths tree that allows reporting the shortest path from s to a reachable vertex.
3. Works on Direct and Undirected graphs
4. Time Complexity is $\Theta(V+E)$
5. Uses a queue (FIFO) data structure.

Definition 12.1.2 Depth First Search (DFS):

1. **INPUT:** Graph $G = (V, E)$ and no source.
2. **OUTPUT:**
   - π to record predecessors (to encode the resulting DFF)
   - two timestamps on each vertex v
3. Time complexity is $\Theta(V+E)$ if g is represented using **adjency lists**
4. Information about structure of graph.

Solution:

1. To implement Breadth First Search (BFS) we use a queue data structure. First In First Out

(FIFO)

2. For Depth First Search (DFS) we use a stack. Last In First Out (LILO)

12.1.2 Undirected and Directed Graph

■ **Example 12.2** We're given a directed graph G, along with a pair of vertices, u and v. We would like to know if there is a path from u to v. What algorithm would you like to use? ■

Definition 12.1.3 Graph: $G = (V, E)$. V is set of vertices. E is set of edges.

Definition 12.1.4 Undirected Graph: Edges are unordered pairs.

- Edges (u, v) and (v, u) are the same.
- Edges go from one vertex to another.
- No self loop. A *self-loop* is an edge that connects a vertex to itself. (v,v) $\notin E$

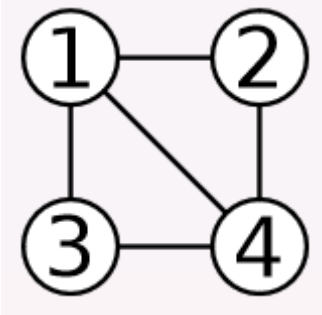


Figure 12.1: Undirected Graph

Definition 12.1.5 Directed Graph: Edges are ordered pairs.

- Edges (u, v) and (v, u) are different.
- u is called tail, and v is the head.

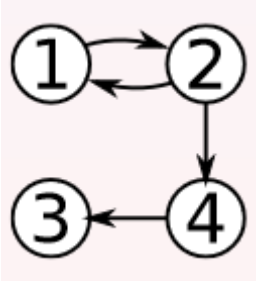


Figure 12.2: Directed Graph

Solution: The algorithm Breadth First Search (BFS).

12.1.3 Definition of Breadth First Tree (BFT)

■ **Example 12.3** What is the definition of BFT (breadth first tree)?
Suppose the graph consideration is undirected. Could there be an edge between two vertices whose depths differ by more than one? ■

Definition 12.1.6 Breadth First Tree

12.1.4 BFS and DFS problem

■ **Example 12.4** Consider the following directed graph.

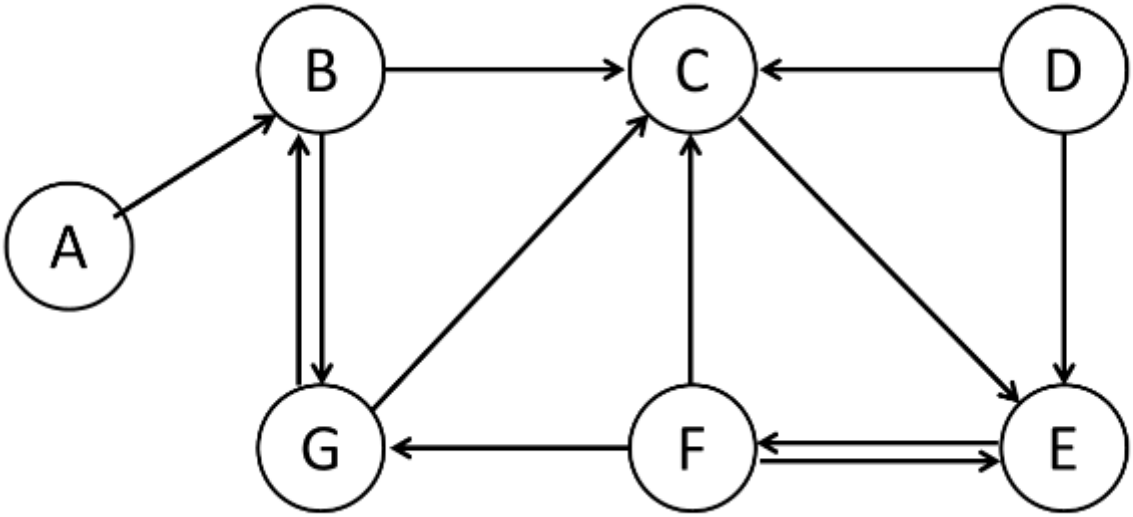


Figure 12.3: (a) through (h)

■

(a) Draw the adjacency-list representation of G, with each list sorted in increasing alphabetical order.

Definition 12.1.7 Adjacency-list Represents the graph $G = (V, E)$ as an array of linked list. The index of the array represents a vertex and each element in its linked list represents the other vertices that form an edge with the vertex.

(1) Each vertex in the graph is placed in a box in the left hand column
(2) Each adjacent vertex is shown as a node in the list to its right.
(3) The arrows indicate the next node in the list.

R Generally the order of the vertices in an adjacency list do not matter unless specified.

Solution:

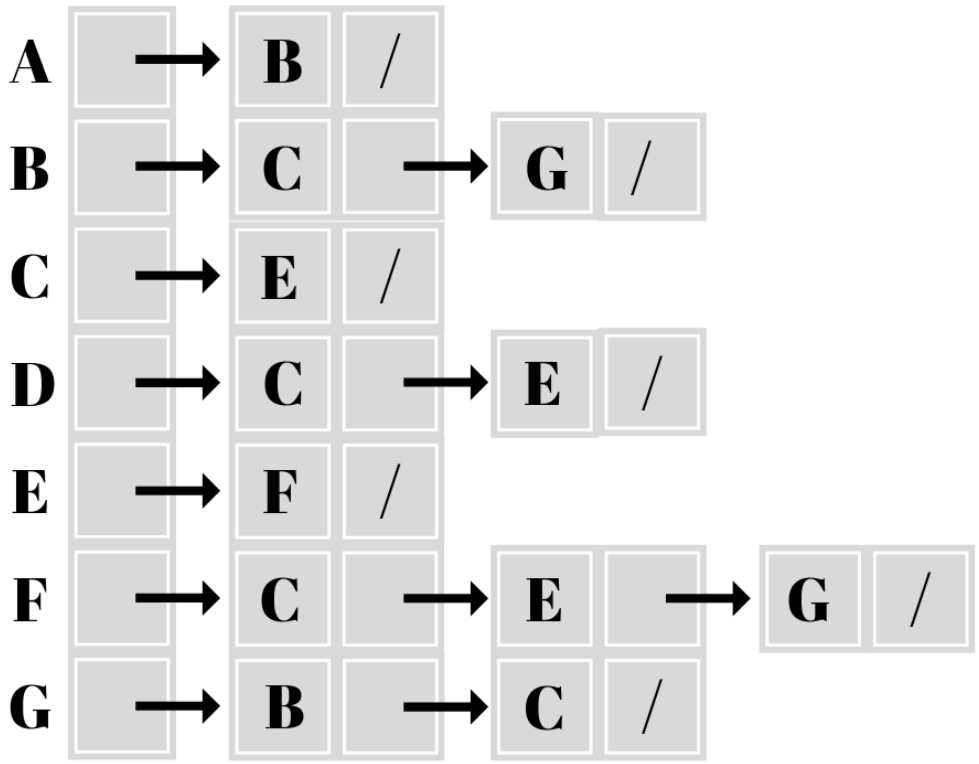


Figure 12.4: (a)

(b) Give the adjacency matrix of G.

Definition 12.1.8 Adjacency Matrix of a graph $G = (V, E)$ is represented by a $|V| \times |V|$ matrix, $A = (a_{ij})$ where $a_{i,j} = 1$ if $(i, j) \in E$ and 0 otherwise.

(1) There are 7 vertices (|V|). The matrix should be 7×7
(2) If there exists and edge between two vertices then a_{ij} is *True (1)* otherwise *False (0)*

$\mathbf{A} =$

	A	B	C	D	E	F	G
A	0	1	0	0	0	0	0
B	0	0	1	0	0	0	1
C	0	0	0	0	1	0	0
D	0	0	1	0	1	0	0
E	0	0	0	0	0	1	0
F	0	0	1	0	1	0	1
G	0	1	1	0	0	0	0

(c) Draw the graph, the adjacency-list representation (with each list sorted in increasing alphabetical order), and the adjacency matrix for the transpose graph G^T.

Definition 12.1.9 Transpose of Graph: Reverse the order of edges. $(u, v) \to (v, u)$ and $(v, u) \to (u, v)$ where $(u, v) \in E$.

- Find the transpose G^T

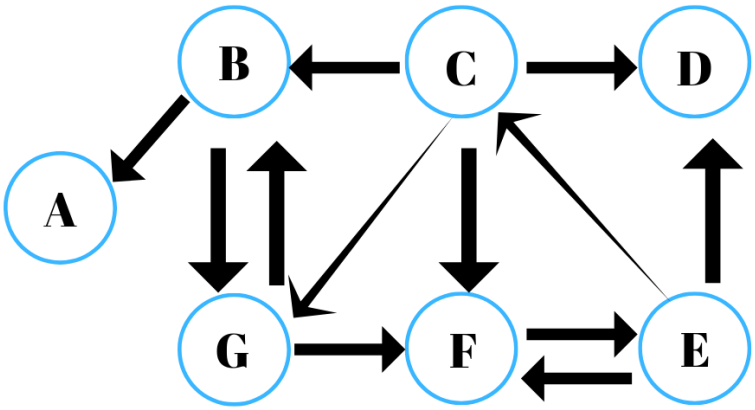


Figure 12.5: (c) G^T

- Draw adjacency-list representation.

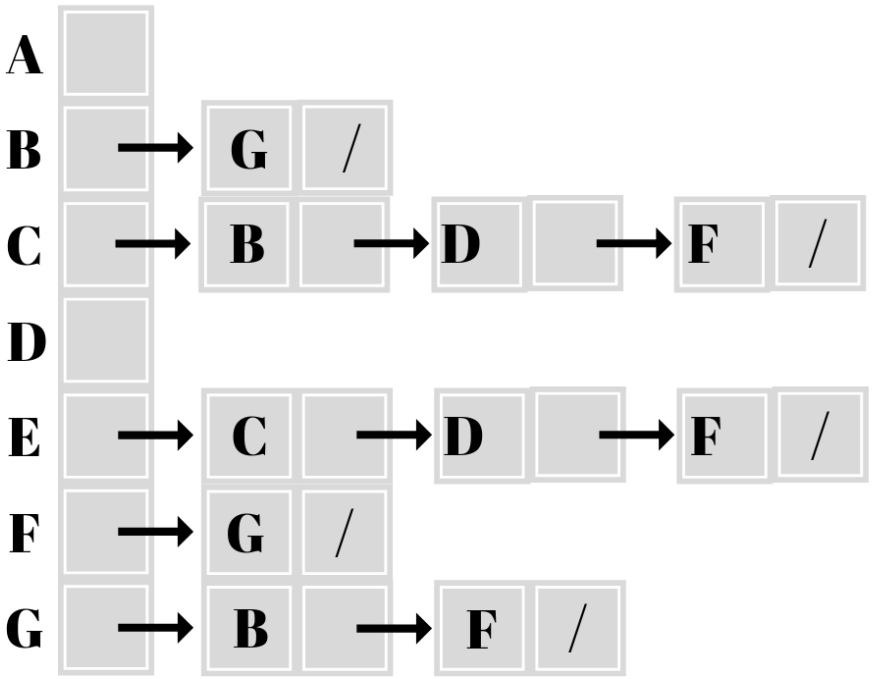


Figure 12.6: (c) Adjacency-list

- Give adjacency matrix.

(d) Do depth-first search in G, considering vertices in increasing alphabetical order. Show the final result, with vertices labeled with their starting and finishing times, and edges labeled with their type (T/B/F/C).
(e) Based on your results, proceed to to run the algorithm to find the strongly connected components of G (show the result of the DFS in G^T, with vertices labeled with their starting and finishing times.
(f) Draw the component graph G^{SCC} of G
(g) Find a topological sort of G^{SCC} using the following algorithms. (label each vertex with its DFS finishing time)
(h) Run BFS with B as a starting vertex. Show the tree edges produced by the BFS along with v.d of each vertex v. You must draw the current tree edges in each itration together with the queue status. More precisely, run BFS with B starting point assuming that each adjacency list is sorted in increasing alphabetical order.

12.1.5 DFF

■ **Example 12.5** If u is reachable from v, u muse be a descendant of v in any DFF. ■

12.2 Intermediate

12.3 Advanced

13. Minimum Spanning Trees

13.1 Basic

13.1.1 Terminology

■ **Example 13.1** What is the definition of a Tree. ■

Solution: A connected graph with no cycles.

■ **Example 13.2** If a tree **T** has n vertices, then **T** must have $n-1$ edges? ■

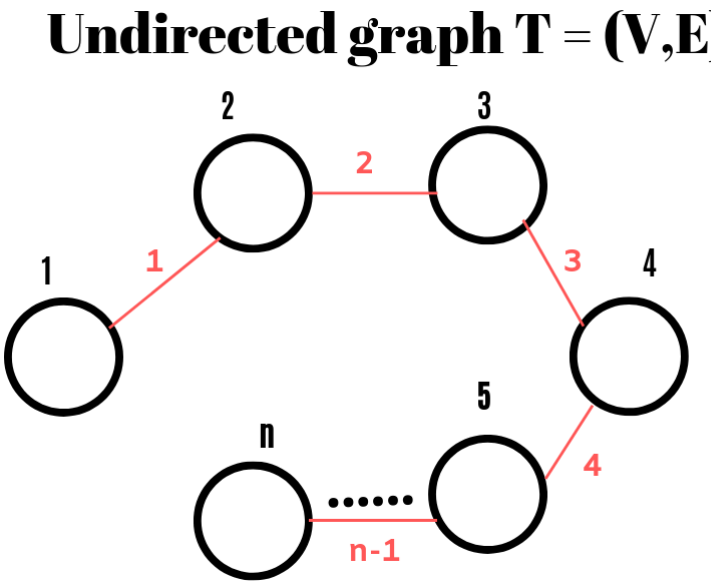


Figure 13.1: Conneced **T**

Solution: *True*. A tree does not have cycles. A tree is an undirected graph in which any two vertices are connected by exactly one path.

TRUE: If a graph **G** = (V,E) has |V| edges or more, then it must have a cycle.

■ **Example 13.3** Consider the pseudocode of **GENERIC-MST(G,w)**. Where the **Input:** Undirected graph $G = (V,E)$ and weight/cost $w(u,v)$ for every edge $(u,v) \in E$. What is the output? ■

GENERIC-MST(G,w)

```
1 A = ∅
2 While A does not form a spanning tree
3       find and edge (u,v) that is safe for A
4       A = A ∪ (u,v)
5 return (A)
```

Definition 13.1.1 Spanning Tree: A tree that connects all vertices of $G = (V,E)$

Definition 13.1.2 Minimum Spanning Tree: A spanning tree denoted **T** whose total edge weight is minimized.

Solution: A minimum spanning tree $T \subseteq E$

To find the **Spanning Tree** of a graph $G = (V,E)$ the vertices must be connected.

■ **Example 13.4** What is the definition of safe edges (assuming that edges have all distinct weights)? The lecture slides use a definition that is different from the textbook. Use the definition in the lecture slides, which is simpler. ■

Solution:

14. Single-Source Shortest Paths

Basic

14.0.1 Optimality of subpaths

■ **Example 14.1** Explain why the following lemma is true.

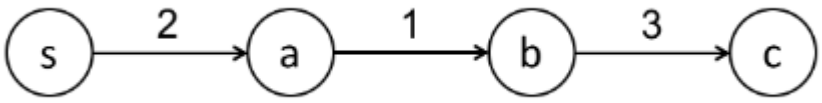
Lemma 14.0.1 Optimality of subpaths: If $P = \langle v_0, v_1, \ldots, v_k \rangle$ is a shortest path from v_0 to v_k, then for any $0 \leq i \leq j \leq k$, $\langle v_i, v_{i+1}, \ldots, v_j$ is a shortest path from v_i to v_j. In other words, subpaths of shortest subproblems are also shortest paths.

■

Solution: One could get a "better" shortest path from v_0 to v_k by replacing $\langle v_i, v_{i+1}, \ldots, v_j$ with a better path from v_i to v_j

14.0.2 Bellman-Ford Algorithm

■ **Example 14.2** The Bellman Ford algorithm repeats relaxing the entire set of edges $|V| - 1$ times. Each iteration edges can be relaxed in an arbitrary order. Consider the following chain graph. The source vertex is s.

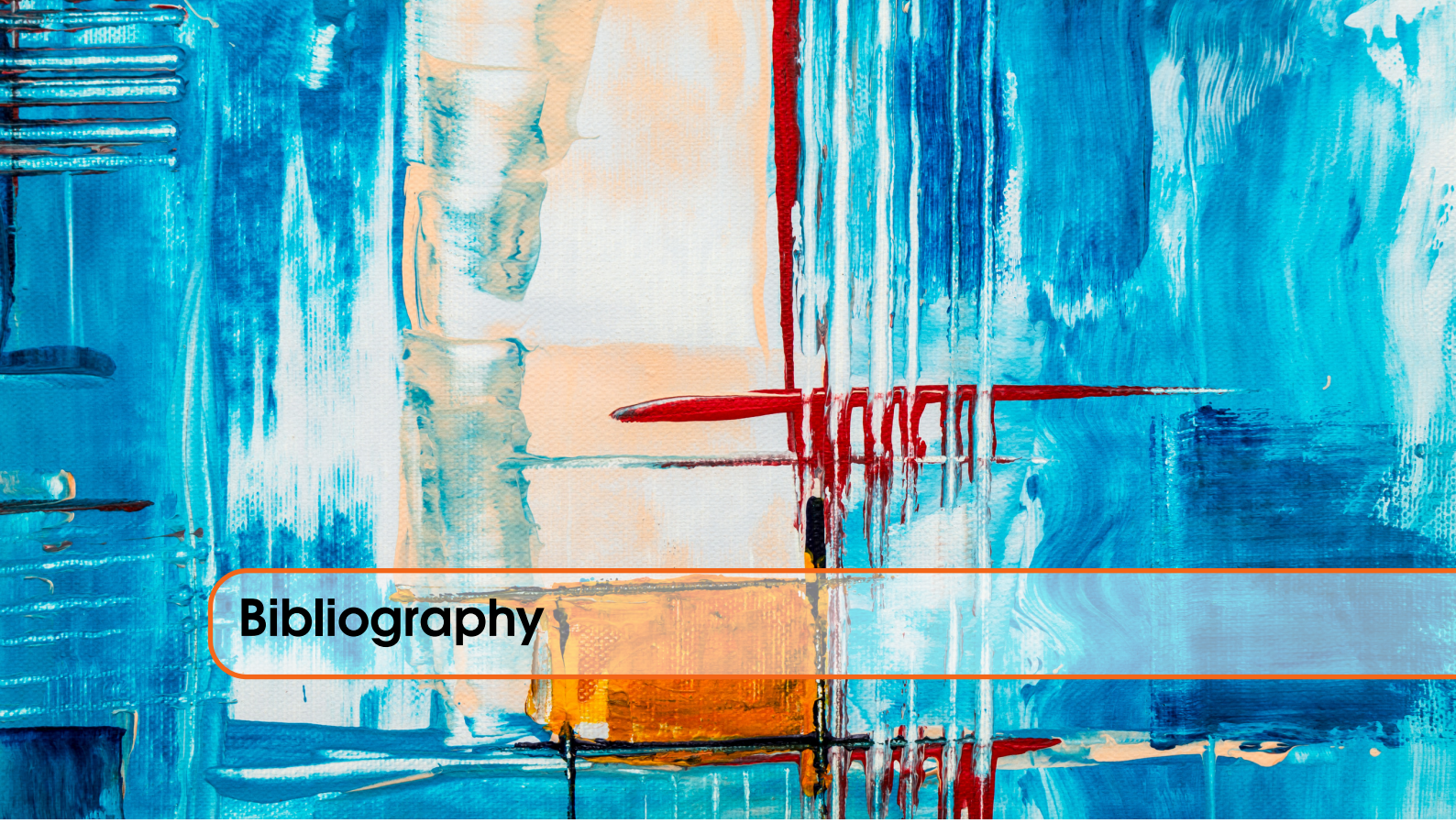


Find the values of $s.d, a.d, b.d$ and $c.d$ after exactly one iteration, i.e. just after you relax all edges for $i = 1$, when edges are considered in each of the following orders?

(a) (s,a), (a,b), (b,c).

(b) (b,c), (a,b), (s,a).

■



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