Lecture 2

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Qualifying Prep Course – Numerical

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Outline

Let $\Omega = (0,1)^2$ and u be the solution of the second order elliptic problem:

$$-\Delta u : -u_{x_1x_1} - u_{x_2x_2} = f(x), \quad \text{for} x \in \Omega \tag{1.1}$$

$$\frac{\partial u}{\partial n} + u = g(x), \quad \text{for} x \in \partial\Omega \tag{1.2}$$

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Question 1: What kind of boundary condition do we have?

Problem 2.

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Remark 1: $\frac{\partial u}{\partial n} = \nabla u \cdot \boldsymbol{n}$

(a) Derive the weak formulation of this problem in the form $a(u, v) = F(v)$, where $a(u, v)$ and $F(v)$ are the appropriate bilinear and linear forms defined on the Sobolev space $H^1(\Omega)$.

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(b) Let S_h be a finite element space of continuous piece-wise polynomial functions defined over a regular partitioning of Ω into triangles and let $a_h(u, v)$ **and** $F_h(v)$**(!!!)** be the bilinear forms where all integrals are computed approximately. Derive Strang's lemma for the error of the FEM: find $u_h \in S_h$ such that $a_h(u_h, v) = F_h(v)$, $\forall v \in S_h$.

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(c) Let S_h be the finite element space of piece-wise linear functions. Let all integrals in $a(u,v)$ and $F(v)$ be computed using quadratures. Namely, for τ and e being triangle and edge defined by the vertexes P_1, P_2, P_3 and P_1, P_2 respectively,

$$\int_\tau w(x)dx \approx \frac{|\tau|}{3}\Big(w(P_1)+w(P_2)+w(P_3)\Big), \quad \int_e w(x)ds \approx \frac{|e|}{2}\Big(w(\alpha)+w \tag{1.3}$$

where $|\tau|$ is the area of τ and $|e|$ is the length of e, and α and β are the Gaussian quadrature nodes. Explain why $a(w,v) = a_h(w,v)$ for all $w, v \in S_h$.

Solution to (a)

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\begin{aligned}
-\int_{\Omega} (\Delta u) v dx &\overset{\text{IBP}}{=} -\int_{\partial\Omega} v(\boldsymbol{n} \cdot \nabla u) ds + \int_{\Omega} \nabla u \cdot \nabla v dx \\
&\overset{(1.2)}{=} -\int_{\partial\Omega} v(g - u) ds + \int_{\Omega} \nabla u \cdot \nabla v dx
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RHS:

$$\int_{\Omega} f(x) v dx = \int_{\Omega} f(x) v dx$$

Q: How do we group these terms?

The weak formulation is given as follows:

Find $u \in V := H^1(\Omega)$ such that for any $v \in V$

$$a(u, v) = F(v)$$

where

$$a(u, v) := \int_{\Omega} \nabla u \cdot \nabla v dx + \int_{\partial\Omega} uvds \tag{1.4}$$

$$F(v) := \int_{\Omega} f(x) v dx + \int_{\partial\Omega} gvds \tag{1.5}$$

(First) Solution to (b)

Let S_h be a finite element space of continuous piece-wise polynomial functions defined over a regular partitioning of Ω into triangles and let $a_h(u, v)$ **and** $F_h(v)$**(!!!)** be the bilinear forms where all **integrals are computed approximately**. Derive Strang's lemma for the error of the FEM: find $u_h \in S_h$ such that $a_h(u_h, v) = F_h(v)$, $\forall v \in S_h$.

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Strang's First Lemma

Let $V_h \subset V$ and let the bilinear form $a_h(\cdot,\cdot)$ be uniformly $V_h - \textit{elliptic}$. Then, there exists a constant $c > 0$ such that

$$\|u - u_h\| \leq c\left[\inf_{z_h \in V_h}\{\|u - z_h\| + \|a(z_h,\cdot) - a_h(z_h,\cdot)\|_{*,h}\} + \|f - f_h\|_{*,h}\right].$$

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Proof.

(i) First recall the Galerkin orthogonality property:

$$a_h(u - u_h, v_h) = 0, \quad \text{for all } v_h \in S_h.$$

This holds since $S_h \subset H^1(\Omega)$.

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(ii) Let $z_h, v_h \in S_h$. Then we see that

$$a_h(u_h - z_h, v_h) \;=\; a_h(u_h, v_h) - a_h(z_h, v_h)$$

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$$\begin{aligned} a_h(u_h - z_h, v_h) &= a_h(u_h, v_h) - a_h(z_h, v_h) \\ &= F_h(v_h) - a_h(z_h, v_h) \end{aligned}$$

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&= F_h(v_h) - a_h(z_h, v_h) \\
&= F_h(v_h) - a_h(z_h, v_h) + \underbrace{a(u, v_h) - F(v_h)}_{=0} \\
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&= a(u - z_h, v_h) + a(z_h, v_h) - a_h(z_h, v_h) + \\
&\qquad F_h(v_h) - F(v_h)
\end{aligned}$$

(iii) Now let $v_h = u_h - z_h$. Then we see that

$$a_h(u_h - z_h, u_h - z_h) \quad = \quad a(u - z_h, u_h - z_h) + a(z_h, u_h - z_h)$$

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where we invoked V-ellipticity and continuity of $a(\cdot, \cdot)$.

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Recall: $\|w\|_{*,h} := \sup_{v_h \in V_h} \frac{|w(v_h)|}{\|v_h\|_h}$ **for** $w \in V_h^*$ **is the norm in the dual space** V_h^***.**

(iv) Now, we divide by $\alpha\|u_h - z_h\|$ and take the supremum over $v_h \in V_h$:

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(v) Recall by the triangle inequality

$$\|u - u_h\| = \|u - z_h + z_h - u_h\| \leq \|u - z_h\| + \|u_h - z_h\|$$

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(vi) Thus, we have that

$$\begin{aligned}\|u - u_h\| &\leq \|u - z_h\| + \underbrace{\|u_h - z_h\|} \\ &\leq \left(1 + \frac{C}{\alpha}\right)\|u - z_h\| + \\ &\quad \frac{1}{\alpha}\Big(\|a(z_h, v_h) - a_h(z_h, v_h)\|_{*,h} + \|F_h(v_h) - F(v_h)\|_{*,h}\Big)\end{aligned}$$

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(vii) Then take infimum over z_h to get final result.

Alternative solution to (b)

An alternative way to approach this problem is to assume that it is asking to derive a generic error estimate.

Alternative solution to (b) (technical) version

We say that the problem[2]

Find $u_h \in v_h \in S_h$ such that

$$a_h(u_h, v_h) = F_h(v_h), \quad \text{for all } v_h \in S_h$$

is **stable** (or well-posed) whenever

$$\inf_{v_h \in S_h} \sup_{w_h \in S_h} \frac{|a_h(v_h, w_h)|}{\|v_h\|_{H^1(\Omega)} \|w_h\|_{H^1(\Omega)}} =: \alpha_h > 0 \tag{1.6}$$

[2]See discrete inf-sup condition on Page 91
https://www.math.tamu.edu/~guermond/M610_SPRING_2005/lect2.pdf

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(i) Assume the problem is stable. Then, an immediate consequence of (1.6) is: For every $v_h \in S_h$

$$\sup_{w_h \in S_h} \frac{|a_h(v_h, w_h)|}{\|v_h\|_{H^1(\Omega)} \|w_h\|_{H^1(\Omega)}} \geq \alpha_h \implies \sup_{w_h \in S_h} \frac{|a_h(v_h, w_h)|}{\|w_h\|_{H^1(\Omega)}} \geq \alpha_h \|v_h\|_{H^1(\Omega)}.$$

[2]See discrete inf-sup condition on Page 91 `https://www.math.tamu.edu/~guermond/M610_SPRING_2005/lect2.pdf`

(ii) Since $u_h - v_h \in S_h$, we have that:

$$\alpha_h \|u_h - v_h\|_{H^1(\Omega)} \leq \sup_{w_h \in S_h} \frac{|a_h(u_h - v_h, w_h)|}{\|w_h\|_{H^1(\Omega)}} \tag{1.7}$$

Note that we call this a **stability** condition.

(ii) Since $u_h - v_h \in S_h$, we have that:

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Note that we call this a **stability** condition.

(iii) Using that $a_h(u_h - v_h, w_h) = F_h(w_h) - a_h(v_h, w_h)$, we get the following **consistency** term:

$$\alpha_h \|u_h - v_h\|_{H^1(\Omega)} \leq \sup_{w_h \in S_h} \frac{|F_h(w_h) - a_h(v_h, w_h)|}{\|w_h\|_{H^1(\Omega)}} \tag{1.8}$$

(iv) **Triangle inequality.** The triangle inequality implies $\|u - u_h\|_{H^1(\Omega)} \leq \|u - v_h\|_{H^1(\Omega)} + \|u_h - v_h\|_{H^1(\Omega)}$. Thus using this and combining the above, we have that:

$$\begin{aligned} \|u - u_h\|_{H^1(\Omega)} &\leq \|u - v_h\|_{H^1(\Omega)} + \frac{1}{\alpha_h} \sup_{w_h \in S_h} \frac{|F_h(w_h) - a_h(v_h, w_h)|}{\|w_h\|_{H^1(\Omega)}} \\ &\leq \inf_{v_h \in S_h} \Bigg(\|u - v_h\|_{H^1(\Omega)} \\ &\qquad + \frac{1}{\alpha_h} \sup_{w_h \in S_h} \frac{|F_h(w_h) - a_h(v_h, w_h)|}{\|w_h\|_{H^1(\Omega)}} \Bigg) \end{aligned}$$

Solution to (c)

Let S_h be the finite element space of piece-wise **linear** functions. Let all integrals in $a(u, v)$ and $F(v)$ be computed using quadratures. Namely, for τ and e being triangle and edge defined by the vertexes P_1, P_2, P_3 and P_1, P_2 respectively,

$$\int_\tau w(x)dx \approx \frac{|\tau|}{3}\Big(w(P_1)+w(P_2)+w(P_3)\Big), \quad \int_e w(x)ds \approx \frac{|e|}{2}\Big(w(\alpha)+w(\beta)\Big) \tag{1.9}$$

where $|\tau|$ is the area of τ and $|e|$ is the length of e, and α and β are the Gaussian quadrature nodes. Explain why $a(w, v) = a_h(w, v)$ for all $w, v \in S_h$.

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Recall that

$$a(u, v) = \int_{\Omega} \nabla u \cdot \nabla v dx + \int_{\partial\Omega} uvds.$$

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$$\begin{aligned} \int_{\Omega} \nabla w \cdot \nabla v \, dx &= \sum_{\tau \in \mathcal{T}_h} \int_{\tau} \nabla w \cdot \nabla v dx \\ &= \sum_{\tau \in \mathcal{T}_h} (\nabla w \cdot \nabla v) \int_{\tau} dx \end{aligned}$$

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&= \sum_{\tau \in \mathcal{T}_h} |\tau| (\nabla w \cdot \nabla v)
\end{aligned}$$

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&= \sum_{\tau \in \mathcal{T}_h} \frac{|\tau|}{3} \Big((\nabla w \cdot \nabla v)(P_1) + (\nabla w \cdot \nabla v)(P_2) \\
&\quad + (\nabla w \cdot \nabla v)(P_3) \Big)
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Also recall that the Gaussian quadrature is defined on the interval $[-1, 1]$. Let $e = [a, b]$ be an arbitrary edge.

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Also recall that the Gaussian quadrature is defined on the interval $[-1,1]$. Let $e = [a,b]$ be an arbitrary edge. Then, we need the transformation

$$T(t) = \frac{1}{2}(1-t)a + \frac{1}{2}(t+a)b$$

so that $T(-1) = a$ and $T(1) = b$. Note that $T^{'}(t) = \frac{1}{2}(b-a) = \frac{1}{2}\,|e|$.

Now we can compute the boundary integral:

$$\begin{aligned}\int_{\partial\Omega} wv ds &= \sum_{e\in\partial\Omega}\int_e wv dx \\ &= \sum_{e\in\partial\Omega}\int_{-1}^{1} wv \underbrace{\frac{1}{2}|e|\,dt} \\ &= \sum_{e\in\partial\Omega}\frac{|e|}{2}\Big((wv)(\alpha)+(wv)(\beta)\Big)\end{aligned}$$

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Thus, combining everything, we see that $a_h(w,v)=a(w,v)$ for all $w,v\in S_h$.

Solution to (d)

Using the estimate of Part (b) estimate the error $\|u - u_h\|_{H^1}$.

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This is justified since a_h is V_h-elliptic since a is V-elliptic and $a_h = a$ on V_h. Since we want an error estimate, we need to do the following: **1. Define a projection; 2. Transform to the reference element; 3. Use Bramble-Hilbert to get error on reference element; 4. Transform back to physical element;**

Review of affine transformations

For reference, see 4.4.1 of Grossman book pg 219.

- We first define the reference element as the triangle:

$$\widehat{\tau} = \left\{ \begin{pmatrix} \widehat{x} \\ \widehat{y} \end{pmatrix} : \widehat{x} \geq 0, \widehat{y} \geq 0, \widehat{x} + \widehat{y} \leq 1 \right\}. \tag{1.9}$$

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- Consider the mapping $F_\tau : \widehat{\tau} \to \tau$ that goes from the reference element to the triangle τ (physical element) with the vertices $(x_1, y_1)^\mathsf{T}, (x_2, y_2)^\mathsf{T}, (x_3, y_3)^\mathsf{T}$:

$$F_\tau(\widehat{p}) = \underbrace{\begin{pmatrix} x_2 - x_1 & x_3 - x_1 \\ y_2 - y_1 & y_3 - y_1 \end{pmatrix}}_{B} \begin{pmatrix} \widehat{x} \\ \widehat{y} \end{pmatrix} + \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \tag{1.10}$$

Note that $|\det B| = \left|\det(F'_\tau(\widehat{p}))\right| = \frac{|\tau|}{|\widehat{\tau}|} \sim ch^2$. ($h$ is defined on the next slide)

- Setting $v(\widehat{p}) = u(F(\widehat{p}))$ for $\widehat{p} \in \widehat{\tau}$, we see that each function $u(\boldsymbol{x})$ for $\boldsymbol{x} \in \tau$ is mapped to a function $v(\widehat{p})$ defined on the reference element. **Notation: We will interchange $v(\widehat{p}) = u(F(\widehat{p}))$ with $\widehat{u}$.**

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- So, if F is differentiable, then by the chain rule, we have that:

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- Let h be the maximum length of the edges of the triangles in the triangulation of Ω. Then there exists $c > 0$ such that

$$\|F^{'}(\widehat{p})\| = \|B\| \leq ch \qquad \text{for all } \widehat{p} \in \widehat{\tau} \tag{1.12}$$

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- Using the above we see that

$$\int_{\tau} u^2(\boldsymbol{x}) dx = \int_{\widehat{\tau}} v^2(\widehat{p}) \left| \det F^{'}(\widehat{p}) \right| d\widehat{p} \tag{1.13}$$

- According to Lemma 4.23 in Grossman book:

$$\int_{\tau} |(\nabla u)|^2 \, d\boldsymbol{x} \leq c\|B^{-1}\|^2 \int_{\widehat{\tau}} \left|\nabla_{\widehat{p}} \widehat{u}\right|^2 |\det B| \, d\widehat{p} \tag{1.14}$$

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- Assuming that the mesh is shape-regular[3] (which we can always do for these error estimates), we have the following bound:

$$\|B^{-1}\| \leq ch^{-1}.$$

[3] see pg 222 of Grossman book

A note on Bramble-Hilbert

For applying Bramble-Hilbert (and any other major result), it is recommended that students first state the result in their solution and then apply it appropriately (by making sure the assumptions are satisfied). You should not just say “by Bramble-Hilbert” in a sub-step and move on.

A note on Bramble-Hilbert

Bramble-Hilbert

Let $B \subset \mathbb{R}^n$ be a domain with a Lipschitz boundary and let q be a bounded sub-linear functional on $H^{k+1}(B)$. Assume that

$$q(w) = 0, \qquad \text{for all } w \in P^k.$$

Then there exists a constant $c = c(B) > 0$, which depends on B, such that

$$|q(v)| \leq c\,|v|_{k+1,B}\,, \qquad \text{for all } v \in H^{k+1}(B).$$

Back to solution...

We first focus on the term $\inf_{z_h \in S_h} \left(1 + \frac{C}{\alpha}\right) \|u - z_h\|$.

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$$\left(\inf_{z_h \in S_h} \left(1 + \frac{C}{\alpha}\right) \|u - z_h\| \right)^2 \leq c \|u - \Pi_h u\|^2_{H^1(\Omega)} \tag{1.15}$$

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We want to compute this term on the reference element:

$$\|u - \Pi_h u\|^2_{H^1(\Omega)} \quad = \quad \sum_{\tau \in \mathcal{T}_h} \|u - \Pi_h u\|^2_{H^1(\tau)}$$

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\|u - \Pi_h u\|^2_{H^1(\Omega)} &= \sum_{\tau \in \mathcal{T}_h} \|u - \Pi_h u\|^2_{H^1(\tau)} \\
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&\leq c \sum_{\tau \in \mathcal{T}_h} \int_{\widehat{\tau}} \left(\left| \widehat{u} - \widehat{(\Pi_h u)} \right|^2 + \right.
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In this next step, we bound $\|B^{-1}\| \leq ch^{-1}$ and replace $\frac{|\tau|}{|\widehat{\tau}|}$ by ch^2:

$$\|u - \Pi_h u\|^2_{H^1(\Omega)} \leq ch^2 \sum_{\tau \in \mathcal{T}_h} \int_{\widehat{\tau}} \left| \widehat{u} - \widehat{(\Pi_h u)} \right|^2 + c \sum_{\tau \in \mathcal{T}_h} \int_{\widehat{\tau}} \left| \nabla(\widehat{u} - \widehat{(\Pi_h u)}) \right|^2 d\widehat{p}$$

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&= ch^2 \sum_{\tau \in \mathcal{T}_h} \|(\mathrm{Id} - \widehat{\Pi}_h)\widehat{u}\|^2 + \\
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\end{aligned}
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Notice that $||(\mathrm{Id} - \widehat{\Pi}_h)(\cdot)||_{L^2(\tau)}$ and $|(\mathrm{Id} - \widehat{\Pi}_h)(\cdot)|_{H^1(\tau)}$ are both sublinear functionals defined on $H^2(\widehat{\tau})$ and are zero exactly zero for linear polynomials on τ, therefore the Bramble-Hilbert lemma can be applied.

Recall: $\|B\| \leq ch$, $\frac{|\widehat{\tau}|}{|\tau|} \to ch^{-2}$

$$\sum_{\tau \in \mathcal{T}_h} ||u - \Pi_h u||^2_{H^1(\tau)} \quad \leq \quad ch^2 \sum_{\tau \in \mathcal{T}_h} |\widehat{u}|^2_{H^2(\widehat{\tau})} + c \sum_{\tau \in \mathcal{T}_h} |\widehat{u}|^2_{H^2(\widehat{\tau})}$$

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\text{(Lemma 4.23(ii))} \quad &\le c(h^2+1) \sum_{\tau \in \mathcal{T}_h} \int_{\tau} \sum_{|\alpha|=2} \underbrace{\|B\|^4}\, \left|D^\alpha u\right|^2 \underbrace{\frac{|\widehat{\tau}|}{|\tau|}} \, d\boldsymbol{x}
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\sum_{\tau \in \mathcal{T}_h} ||u - \Pi_h u||^2_{H^1(\tau)} &\leq ch^2 \sum_{\tau \in \mathcal{T}_h} |\widehat{u}|^2_{H^2(\widehat{\tau})} + c \sum_{\tau \in \mathcal{T}_h} |\widehat{u}|^2_{H^2(\widehat{\tau})} \\
&\leq c(h^2+1) \sum_{\tau \in \mathcal{T}_h} \int_{\widehat{\tau}} \sum_{|\alpha|=2} |\widehat{D}^\alpha \widehat{u}|^2 \, d\widehat{p} \\
\text{(Lemma 4.23(ii))} \quad &\leq c(h^2+1) \sum_{\tau \in \mathcal{T}_h} \int_{\tau} \sum_{|\alpha|=2} \underbrace{\|B\|^4} \left|D^\alpha u\right|^2 \underbrace{\frac{|\widehat{\tau}|}{|\tau|}} \, d\boldsymbol{x} \\
&\leq c(h^2+1) \sum_{\tau \in \mathcal{T}_h} \int_{\tau} h^2 \sum_{|\alpha|=2} |D^\alpha u|^2 \, d\boldsymbol{x} \\
&= c(h^4+h^2) \sum_{\tau \in \mathcal{T}_h} \int_{\tau} \sum_{|\alpha|=2} |D^\alpha u|^2 \, d\boldsymbol{x}
\end{aligned}
$$

Recall: $\|B\| \leq ch$, $\frac{|\widehat{\tau}|}{|\tau|} \to ch^{-2}$

$$
\begin{aligned}
\sum_{\tau \in \mathcal{T}_h} ||u - \Pi_h u||^2_{H^1(\tau)} &\leq ch^2 \sum_{\tau \in \mathcal{T}_h} |\widehat{u}|^2_{H^2(\widehat{\tau})} + c \sum_{\tau \in \mathcal{T}_h} |\widehat{u}|^2_{H^2(\widehat{\tau})} \\
&\leq c(h^2+1) \sum_{\tau \in \mathcal{T}_h} \int_{\widehat{\tau}} \sum_{|\alpha|=2} |\widehat{D}^\alpha \widehat{u}|^2 \, d\widehat{p} \\
\text{(Lemma 4.23(ii))} &\leq c(h^2+1) \sum_{\tau \in \mathcal{T}_h} \int_{\tau} \sum_{|\alpha|=2} \underbrace{\|B\|^4} \left|D^\alpha u\right|^2 \underbrace{\frac{|\widehat{\tau}|}{|\tau|}} \, d\boldsymbol{x} \\
&\leq c(h^2+1) \sum_{\tau \in \mathcal{T}_h} \int_{\tau} h^2 \sum_{|\alpha|=2} |D^\alpha u|^2 \, d\boldsymbol{x} \\
&= c(h^4+h^2) \sum_{\tau \in \mathcal{T}_h} \int_{\tau} \sum_{|\alpha|=2} |D^\alpha u|^2 \, d\boldsymbol{x} \\
&\leq ch^2 |u|^2_{H^2(\Omega)}.
\end{aligned}
$$