07-30-2020

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Qualifying Prep Course – Numerical

07-30-2020

Outline

A note on problem 2 boundary conditions

(open PDF)

Problem 3

Let $u(x,t)$ be a smooth solution satisfying

$$\partial_t u + \beta \partial_x u = 0, \quad x \in \Omega := (0,1), \quad t > 0 \quad \text{and} \quad u(0,x) = \phi(x), x \in \Omega,$$

where $\beta \in \mathbb{R}$ and ϕ is a given smooth function. In addition, we assume that $u(x,t)$ satisfies the periodic boundary condition $u(0,t) = u(1,t), t > 0$. Let $\mathbb{V} = \{v \in H^1(\Omega) : v(0) = v(1)\}$.

(b) Show that the Finite Element approximation $u_h(t)$ satisfies

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(c) Show that

$$c^{-1}\int_{\Omega} u_h^2(t,x)dx \leq h\sum_{i=0}^{N} u_h(t,x_i)^2 \leq c\int_{\Omega} u_h^2(t,x)dx$$

and deduce the estimate

$$\int_{\Omega} u_h^2(t,x)dx \leq C\int_{\Omega} \phi_h^2(0,x)dx.$$

Here c and C are constants independent of h.

Solution to (b)

First recall that for a smooth solution u, we have $u\partial_t u = \frac{1}{2}\partial_t(u^2)$ and $u\partial_x au = \frac{1}{2}\partial_x(u^2)$.

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Let us take $v_h(x) := u_h(t,x)$ for some fixed t. Substituting this into the discrete yields the following

$$\frac{h}{2}\sum_{i=0}^{N}\left(\tfrac{1}{2}\partial_t(u_h(t,x_{i+1})^2) + \tfrac{1}{2}\partial_t(u_h(t,x_i))^2\right) + \beta\int_\Omega \tfrac{1}{2}\partial_x(u_h(t,x)^2)\,dx = 0.$$

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Note that the integral term will vanish because we have periodic boundary conditions. Then, the remaining summation term can be written as follows

$$\frac{h}{2}\frac{d}{dt}\Big[\sum_{i=1}^{N} u_h(t,x_i)^2 + \tfrac{1}{2}u_h(t,x_0) + \tfrac{1}{2}u_h(t,x_{N+1})\Big] = 0$$

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But, because we have periodic boundary conditions, we know that $u_h(t,x_{N+1}) = u_h(t,x_0)$. Thus we can combine these terms with the summation:

$$\frac{d}{dt}\sum_{i=0}^{N} u_h(t,x_i)^2 = 0.$$

Solution to (c)

Definition A.5 (Equivalent norms). *Two norms* $\|\cdot\|_{V,1}$ *and* $\|\cdot\|_{V,2}$ *are said to be* equivalent *on* V *if there exists a positive real number* c *such that*

$$c\,\|v\|_{V,2} \leq \|v\|_{V,1} \leq c^{-1}\,\|v\|_{V,2}, \qquad \forall v \in V. \tag{A.1}$$

Whenever (A.1) *holds true,* V *is a Banach space for the norm* $\|\cdot\|_{V,1}$ *if and only if it is a Banach space for the norm* $\|\cdot\|_{V,2}$.

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&= \sum_{T\in\mathcal{T}_h} \frac{|T|}{|\widehat{T}|} \|\widehat{u}_h\|^2_{L^2(\widehat{T})},
\end{aligned}
$$

where $\widehat{u}_h = u_h \circ F_T$ where $F_T : \widehat{T} \to T$ is the affine map from the reference element to the physical element T.

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for positive constants c, c' where we used the norm $\|u_h\|_E := \sqrt{u_h(t,x_i)^2 + u_h(t,x_{i+1})^2}$ (Verify that this is indeed a norm).

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for positive constants c, c' where we used the norm $\|u_h\|_E := \sqrt{u_h(t, x_i)^2 + u_h(t, x_{i+1})^2}$ (Verify that this is indeed a norm). Note that we have used the fact that we can write $\widehat{u}_h(t, 0) = u_h(x_i)$ and $\widehat{u}_h(t, 1) = u_h(x_{i+1})$ via the mapping F_T.

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The other direction can be shown by using the other side of the inequalities.

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Outline

Problem 2

Let T be the unit triangle in $\mathbb{R}^2$, with vertices $v_1 = (0,0)$, $v_2 = (1,0)$, and $v_3 = (0,1)$ and edges $e_1 = v_1 v_2$, $e_2 = v_2 v_3$, and $e_3 = v_3 v_1$. Let $RT_0 = \{(a + cx, b + cy) \ : \ a, b, c \in \mathbb{R}\}$ (so that members of RT_0 are vector functions over T, and $[\mathbb{P}_0]^2 \subsetneq RT_0 \subsetneq [\mathbb{P}_1]^2$). Finally, let $\sigma_i(\vec{u}) = \int_{e_i} \vec{u} \cdot \vec{n}_i$ where $\vec{n}_i$ is the outward pointing unit normal vector to T on e_i, and let $\Sigma = \{\sigma_1, \sigma_2, \sigma_3\}$.

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(a) Show that (T, RT_0, Σ) is unisolvent.

(b) Find a basis $\{\vec{\varphi}_1, \vec{\varphi}_2, \vec{\varphi}_3\}$ for RT_0 that is dual to Σ, i.e. $\sigma_i(\vec{\varphi}_j) = \delta_{ij}$ with $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ otherwise.

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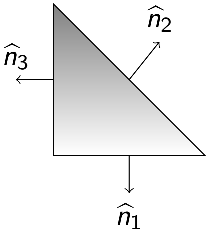

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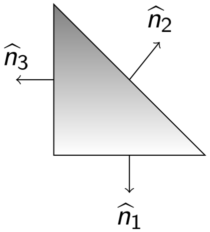

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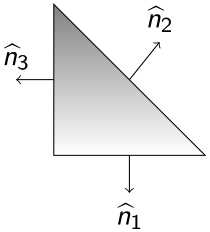

First note that $\mathrm{card}(\Sigma) = 3 = \dim RT_0$. We now want to verify that $\sigma(\vec{p}) = 0$ for all $\sigma \in \Sigma$ implies that $\vec{p} = 0$ for any $\vec{p} \in RT_0$.

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$$\sigma_1(\vec{p}) = \int_{e_1} (a + cx, b + cy) \cdot (0, -1)\, ds$$

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$$\begin{aligned}\sigma_2(\vec{p}) &= \int_{e_2} (cx, cy) \cdot (1/\sqrt{2}, 1/\sqrt{2})\, ds \\ &= \int_0^1 c(1-t, t) \cdot (1/\sqrt{2}, 1/\sqrt{2}) \underbrace{\sqrt{2}\, dt} \\ &= \int_0^1 c\, dt\end{aligned}$$

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where we have parameterized the line segment e_2 by $\gamma(t) := (1-t, t)$ for $t \in [0, 1]$.

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where we have parameterized the line segment e_2 by $\gamma(t) := (1-t, t)$ for $t \in [0,1]$.Thus, we have $c = 0$ and therefore $\vec{p} = \vec{0}$. We have unisolvence on the referenc eelement and thus (T, RT_0, Σ) is unisolvent.

Solution to (b)

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$$\begin{aligned}
\sigma_3(\vec{\varphi}_1) &= \int_{e_3} (cx, -1 + cy) \cdot (1/\sqrt{2}, 1/\sqrt{2}) ds \\
&= \int_0^1 (c(1-t), -1 + ct) \cdot (1, 1) dt \\
&= \int_0^1 c - 1 \, dt \\
&= 0.
\end{aligned}$$

This implies that $c = 1$.

Solution to (b)

We now want to find three elements $\vec{\varphi}_i$ of RT_0 such that $\sigma_i(\vec{\varphi}_j) = \delta_{i,j}$.Following the work from part (a), we see that if $\sigma_1(\vec{\varphi}_1) = 1$ and $\sigma_2(\vec{\varphi}_1) = \sigma_3(\vec{\varphi}_1) = 0$, we will have that $b = -1$ and $a = 0$. Finding c requires a little bit more work:

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This implies that $c = 1$. Thus $\vec{\varphi}_1 = (x, y - 1)$.

We can repeat the same process for the other basis functions. Doing so, we find,

$$\begin{aligned}\vec{\varphi}_1(x,y) &= (x, y-1)\\ \vec{\varphi}_2(x,y) &= (x, y)\\ \vec{\varphi}_3(x,y) &= (x-1, y).\end{aligned}$$

Outline

Hermite polynomials

Consider the finite elements $(K_i, \mathbb{P}^3, \Sigma_i)$ where $K_i = [x_{i-1}, x_i]$ for $i = 1, \ldots, N$ and $x_i - x_{i-1} = h$ and $\mathbb{P}^3$ is the space of cubic polynomials. The degrees of freedom are defined to be $\Sigma_i = \{\sigma_{i-1}, \sigma'_{i-1}, \sigma_i, \sigma'_i\}$, for $i = 1, \ldots, N$ where

$$\begin{aligned} \sigma_{i-1}(f) &= f(x_{i-1}) & \qquad \sigma_i(f) &= f(x_i) \\ \sigma'_{i-1}(f) &= f'(x_{i-1}) & \qquad \sigma'_i(f) &= f'(x_i). \end{aligned}$$

Note that one can show that the finite element is unisolvent. For some arbitrary finite element approximation, consider the finite dimensional space

$$V_h = \{v \in C^1(\Omega) \;:\; v|_{K_i} \in \mathbb{P}^3,\ i = 1, \ldots, N,\ v(0) = v(1) = 0\}.$$

A basis for this space is given by $\cup_{i=1}^{N-1}\{\phi_i\}\bigcup\cup_{i=0}^{N}\{\psi_i\}$, where ϕ_i and ψ_i are the cubic Hermite polynomials (note that we remove ϕ_0 and ϕ_N because of the boundary conditions). Specifically, ϕ_i and ψ_i are defined by the following conditions,

$$\begin{aligned}
&\sigma_k(\psi_j) = 0,\ \sigma_k'(\psi_j) = \delta_{kj}, && \text{for } j,k = 0,\dots,N\\
&\sigma_k(\phi_j) = \delta_{kj},\ \sigma_k'(\phi_j) = 0, && \text{for } j,k = 1,\dots,N-1
\end{aligned}$$

Precisely speaking, ψ_i and ϕ_i are defined as

$$\psi_i(x) = \begin{cases} \frac{1}{h^2}(x-x_i)(x-x_{i-1})^2 & \text{for } x \in [x_{i-1}, x_i],\\ \frac{1}{h^2}(x-x_{i+1})^2(x-x_i) & \text{for } x \in [x_i, x_{i+1}],\\ 0 & \text{otherwise,} \end{cases}$$

$$\phi_i(x) = \begin{cases} \frac{1}{h^2}(x-x_{i-1})^2\left(\frac{2}{h}(x_i-x)+1\right) & \text{for } x \in [x_{i-1}, x_i],\\ \frac{1}{h^2}(x_{i+1}-x)^2\left(\frac{2}{h}(x-x_i)+1\right) & \text{for } x \in [x_i, x_{i+1}],\\ 0 & \text{otherwise.} \end{cases}$$

Recall that Hermite cubic polynomials are useful when solving fourth-order boundary value problems. See: **January 2009, Problem 1**, **January 2017, Problem 2**, **January 2020, Problem 2**

Exercise 1: Let $K := [0,1], P := \mathbb{P}_3$, and $\Sigma := \{\sigma_1, \sigma_2, \sigma_3, \sigma_4\}$ be the linear forms on P such that
$\sigma_1(p) := p(0), \sigma_2 := p'(0), \sigma_3(p) := p(1), \sigma_4(p) := p'(1)$ for all $p \in P$.

(a) Show that (K, P, Σ) is a finite element.

(b) Compute the shape functions for this finite element.

(c) Indicate possible choices for the domain $V(K)$ of the canonical interpolation operator.

Exercise 2: Assume we have a conforming finite dimensional space $V_h \subset V$ and everything is "well-defined". Here V_h is the space defined two slides above. Let $\pi_h : V \to V_h$ be the canonical interpolation operator. What is:

$$\begin{aligned}
\|u - \pi_h u\|_{L^2(\Omega)} &\leq \quad ? \\
\|u - \pi_h u\|_{H^1(\Omega)} &\leq \quad ? \\
\|u - \pi_h u\|_{H^2(\Omega)} &\leq \quad ? \\
\|u - \pi_h u\|_{H^3(\Omega)} &\leq \quad ?
\end{aligned}$$

Outline

Elliptic projection