NUMERICAL QUALIFIER SOLUTION: JANUARY 2011

Problem 1. Consider the following two-points boundary value second order problem in 1-D: Find a function u define a.e. in (0,1) such that

$$-(xK(x)u'(x))' + xq(x)u(x) = xf(x) \text{ a.e. in } (0,1), \tag{0.1}$$

$$\lim_{x\to 0}(xu'(x)) = 0 \text{ and } K(1)u'(1) + u(1) = 0, \tag{0.2}$$

where $K \in \mathcal{C}^1([0,1])$ and $f \in L^2(0,1)$ are given functions. Assume that there exists a constant $\kappa_0 > 0$ such that $K(x) \geq \kappa_0$ and $q(x) \geq 0$ for all $x \in [0,1]$. Let

$$V = \{v \in L^2_{\text{loc}}(0,1) : \sqrt{x}v \in L^2(0,1), \sqrt{x}v' \in L^2(0,1)\}$$

Accept as a fact that V is a Hilbert space for the norm

$$||v||_V = \left(||\sqrt{x}v||^2_{L^2(0,1)} + ||\sqrt{x}v'||^2_{L^2(0,1)}\right)^{1/2}$$

and $\mathcal{C}^1([0,1])$ is dense in V for this norm.

(a) Derive the variational formulation (also called weak formulation) of problem 1 in the space V.

(b) Prove that the corresponding bilinear form of this variational formulation is elliptic (or coercive) in V. **Hint.** First show that all functions v of $C^1([0,1])$ satisfy

$$\int_0^1 v(x)^2\,dx = v^2(1) - 2\int_0^1 xv(x)v'(x)\,dx \tag{0.3}$$

and then establish the following variant Poincaré's inequality

$$\forall v \in V, \quad ||\sqrt{x}v||_{L^2(0,1)} \leq \alpha\left(v^2(1) + ||\sqrt{x}v'||^2_{L^2(0,1)}\right)^{1/2} \tag{0.4}$$

for some constant $\alpha > 0$. Based on this equality deduct the ellipticity.

(c) Choose an integer $N \geq 2$, set $h = 1/N$, let $x_i = ih$, $0 \leq i \leq N$ and define the finite element space,

$$V_h = \{v_h \in C^0([0,1]);\ v_h|_{(x_i,x_{i+1})} \in \mathbb{P}_1, 0 \leq i \leq N-1\}. \tag{0.5}$$

Show that V_h is a subspace of V. Discretize the variational problem in this space. Prove existence and uniqueness of the discrete solution and establish an error estimate without estimating the norms of the interpolation errors.

Solution. **(a)** Let $v \in V$. We multiply the PDE in the two-point boundary value problem by v and integrate over the domain $\Omega = (0,1)$. We see from the left hand side:

$$\begin{aligned}
\int_0^1 \Big(-(xKu')'v + xquv\Big)dx &\overset{\text{IBP}}{=} \int_0^1 \Big(xKu'v' + xquv\Big)dx - \lim_{t\to 0}\,[xKu'v]_t^1\,,\\
&= \int_0^1 \Big(xKu'v' + xquv\Big)dx - K(1)u'(1)v(1) + \lim_{t\to 0} tK(t)u'(t)v(t),\\
&\overset{(0.2)}{=} \int_0^1 \Big(xKu'v' + xquv\Big)dx + u(1)v(1) + (\lim_{t\to 0} tu'(t))(K(0)v(0)),\\
&= \int_0^1 \Big(xKu'v' + xquv\Big)dx + u(1)v(1).
\end{aligned}$$

$\blacksquare$

Thus, we have the following weak formulation: Find $u \in V$ such that $a(u,v) = F(v)$ for all $v \in V$ where

$$
\begin{aligned}
a(u,v) &= \int_0^1 \Big(xKu'v' + xquv\Big)dx + u(1)v(1),\\
F(v) &= \int_0^1 (xfv)dx.
\end{aligned}
$$

(b) Let us first prove the hint. Let $v(x) \in C^1([0,1])$ be arbitrary. Then we have that

$$
\begin{aligned}
\int_0^1 v(x)^2 dx &= \int_0^1 (1 \cdot v(x)^2)dx,\\
&\overset{\text{IBP}}{=} [xv(x)^2]_0^1 - 2\int_0^1 (xv(x)v'(x))dx,\\
&= v(1)^2 - 2\int_0^1 (xv(x)v'(x))dx.
\end{aligned}
$$

Note that $\sqrt{x}v \in C^1((0,1])$ (Not sure if $[0,1]$ is necessary). Then substituting this into the above relation yields:

$$
\begin{aligned}
||\sqrt{x}v||^2_{L^2(0,1)} &= \int_0^1 xv^2 \, dx,\\
&= \frac{1}{2}v^2(1) - \int_0^1 (x^2vv') \, dx.
\end{aligned}
$$

This can also be shown by direct computation using integration by parts.

We now want to establish the variant Poincare inequality. Since $x^2 \leq x$ on $[0,1]$, we have that

$$
\begin{aligned}
||\sqrt{x}v||^2_{L^2(0,1)} &\leq \frac{1}{2}v^2(1) + \int_0^1 \sqrt{x}^2|v||v'| \, dx,\\
&\overset{\text{C.S}}{\leq} \frac{1}{2}v^2(1) + \Big(\int_0^1 xv^2 \, dx\Big)^{1/2}\Big(\int_0^1 x(v')^2 \, dx\Big)^{1/2},\\
&= \frac{1}{2}v^2(1) + ||\sqrt{x}v||_{L^2(0,1)}||\sqrt{x}v'||_{L^2(0,1)}\\
&\leq \frac{1}{2}v^2(1) + \frac{1}{2}||\sqrt{x}v||^2_{L^2(0,1)} + \frac{1}{2}||\sqrt{x}v'||^2_{L^2(0,1)}
\end{aligned}
$$

Then, subtracting both sides by $\frac{1}{2}||\sqrt{x}v||^2_{L^2(0,1)}$ and multiplying by 2, we see that

$$
||\sqrt{x}v||^2_{L^2(0,1)} \leq v^2(1) + ||\sqrt{x}v'||^2_{L^2(0,1)}.
$$

The Poincare inequality follows by taking the square root of both sides and setting $\alpha = 1$.

We now want to show coercivity of $a(\cdot,\cdot)$. Recall that $a(\cdot,\cdot)$ is coercive if there exists a $\gamma > 0$ such that for $a(u,u) \geq \gamma\|u\|_V^2$ for all $v \in V$. Since we want to use the above results, let us first consider $v \in C^1([0,1])$. Then we see that

$$
\begin{aligned}
a(v,v) &= \int_0^1 \Big(xK(x)(v'(x))^2 + xq(x)v^2(x) \Big)\, dx + v^2(1), \\
&\geq \int_0^1 \Big(\kappa_0(\sqrt{x}v'(x))^2 + xq(x)v^2(x) \Big)\, dx + v^2(1), \\
&\geq \kappa_0 ||\sqrt{x}v'||^2_{L^2(0,1)} + v^2(1), \\
&\geq \min\{\kappa_0, 1\} \Big(||\sqrt{x}v'||^2_{L^2(0,1)} + v^2(1) \Big).
\end{aligned}
$$

Recall from our Poincare inequality we have that

$$
||\sqrt{x}v'||^2_{L^2(0,1)} \geq ||\sqrt{x}v||^2_{L^2(0,1)} - v^2(1). \tag{0.6}
$$

Using this, we see that

$$
\begin{aligned}
a(v,v) &\geq \min\{\kappa_0, 1\}(\frac{1}{2}||\sqrt{x}v'||^2_{L^2(0,1)} + \frac{1}{2}||\sqrt{x}v'||^2_{L^2(0,1)} - v^2(1)), \\
&\geq \min\{\kappa_0, 1\}(\frac{1}{2}||\sqrt{x}v'||^2_{L^2(0,1)} + \frac{1}{2}||\sqrt{x}v||^2_{L^2(0,1)} + \frac{1}{2}v^2(1)), \\
&\geq \frac{1}{2}\min\{\kappa_0, 1\}||v||^2_V.
\end{aligned}
$$

Thus $a(v,v) \geq \frac{1}{2}\min\{\kappa_0, 1\}||v||^2_V$ for $v \in C^1([0,1])$.

We now want to invoke the fact that $C^1([0,1])$ is dense in V to show the Poincare inequality also holds for $v \in V$. Let $v \in V$. First note that $||\sqrt{x}v'||^2_{L^2(0,1)} \leq ||v||^2_V$, $||\sqrt{x}v||^2_{L^2(0,1)} \leq ||v||^2_V$ and by a trace inequality, there exists a $c > 0$ such that $v^2(1) \leq c||v||^2_V$. That is, we can bound all the terms in the Poincare inequality by $c||v||^2_V$. By density of $C^1([0,1])$ in V, there exists a sequence $v_n \in C^1([0,1])$ such that $v_n \to v \in V$ as $n \to \infty$. Let us consider the Poincare inequality for $v_n \in C^1([0,1])$:

$$
||\sqrt{x}v_n||^2_{L^2(0,1)} \leq v_n^2(1) + ||\sqrt{x}v_n'||^2_{L^2(0,1)}.
$$

Since all these terms are bounded by $c||v||^2_V$, we can take the limit as $n \to \infty$. That is to say, as $n \to \infty$: $||\sqrt{x}v_n||^2_{L^2(0,1)} \to ||\sqrt{x}v||^2_{L^2(0,1)}$, $||\sqrt{x}v_n'||^2_{L^2(0,1)} \to ||\sqrt{x}v'||^2_{L^2(0,1)}$ and since the quantity $v(1)$ "makes sense" we have that $v_n^2(1) \to v^2(1)$. Thus, the Poincare inequality holds for $v \in V$:

$$
||\sqrt{x}v||^2_{L^2(0,1)} \leq v^2(1) + ||\sqrt{x}v'||^2_{L^2(0,1)}.
$$

Then, we can repeat the above argument for $v \in V$ to show that $a(v,v) \geq \frac{1}{2}\min\{\kappa_0, 1\}\|v\|^2_V$. $\blacksquare$

(c) We want to show that the space V_h is a subspace of V. Recall that

$$
V = \{v \in L^2_{\text{loc}}(0,1) : \sqrt{x}v \in L^2(0,1), \sqrt{x}v' \in L^2(0,1)\},
$$

with norm

$$
||v||_V = \Big(||\sqrt{x}v||^2_{L^2(0,1)} + ||\sqrt{x}v'||^2_{L^2(0,1)} \Big)^{1/2}.
$$

Let $v_h \in V_h$. Recall that $L^2_{\text{loc}}(\Omega) := \{f : \Omega \to \mathbb{R}\ \textit{measurable} \,\big|\, f|_K \in L^2(K) \text{ for all compact } K \subset \Omega\}$. Note that v_h is a continuous, piecewise linear function. Then $\sqrt{x}v_h$ is a bounded, continuous function on $(0,1)$ so $\sqrt{x}v_h \in L^2(0,1)$ and consequently $v_h \in L^2_{\text{loc}}$. Note that v_h' is piecewise constant function and $\sqrt{x}v_h'$ is a discontinuous, but bounded function. Since the discontinuities occur on sets of measure 0, then $\sqrt{x}v_h'$ is also in $L^2(0,1)$.

Our discrete variational problem is: Find $u_h \in V_h$ such that $a(u_h, v_h) = F(v_h)$ for all $v_h \in V_h$. Note that V_h is a finite dimensional space, and therefore V_h is a closed subspace of V. Since the bilinear form $a(\cdot,\cdot)$ is continuous (how to show?) and coercive, the Lax-Milgram Lemma applies to the variational problem on V_h which guarantees existence and uniqueness of a solution $v_h \in V_h$.

Let us now derive an error estimate. Since $V_h \subset V$, we have that for $u \in V$, $a(u, v_h) = F(v_h)$ for all $v_h \in V_h$. Then, by linearity of the bilinear form and using the fact that $u_h \in V_h$ also solves $a(u_h, v_h) = F(v_h)$ for all $v_h \in V_h$, we have the Galerkin orthogonality property:

$$a(u - u_h, v_h) = 0,$$

for all $v_h \in V_h$. Galerkin orthogonality and linearity of the bilinear form imply

$$a(u - u_h, u - u_h) = a(u - u_h, u - v_h), \quad \text{for all } v_h \in V_h.$$

Then, by coercivity and continuity of $a(\cdot,\cdot)$ we have that

$$\gamma \|u - u_h\|_V^2 \leq \|a\| \cdot \|u - u_h\|_V \|u - v_h\|_V, \quad \text{for all } v_h \in V_h,$$

where $\|a\|$ is the boundedness constant. Then, dividing by $\|u - u_h\|_V$ and applying the triangle inequality yields

$$\gamma \|u - u_h\|_V \leq \|a\| \cdot \|u - v_h\|_V, \quad \text{for all } v_h \in V_h.$$

Since v_h was arbitrary, we have that

$$\|u - u_h\|_V \leq \frac{\|a\|}{\gamma} \inf_{v_h \in V_h} \cdot \|u - v_h\|_V.$$

Then, letting Π_h denote the canonical interpolation operator onto V_h, we have that

$$\|u - u_h\|_V \leq \frac{\|a\|}{\gamma} \inf_{v_h \in V_h} \|u - v_h\|_V \leq \|u - \Pi_h u\|_V.$$

Then, since $v_h|_{K_i} \in \mathbb{P}_1$ where $K_i := (x_i, x_{i+1})$, we can bound the interpolant error using standard results as follows:

$$\|u - \Pi_h u\|_V \leq ch \, |u|_{H^2(\Omega)}.$$

Problem 2. Let Ω be a bounded domain in $\mathbb{R}^2$ with polygonal boundary $\partial\Omega$. Let

$$H_0^1(\Omega) = \{v \in H^1(\Omega) : v(x) = 0 \, \forall x \in \partial\Omega\} \tag{0.7}$$

be the standard Sobolev space of functions defined on Ω that vanish on the boundary.

In all that follows $T > 0$ is a given final time, $c > 0$ is a constant and $u_0 \in C^0(\Omega)$ are given functions. Consider the parabolic equation: Find a function u defined a.e. in $\Omega \times (0,T)$ solution of

$$\begin{aligned} \frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x_1^2} - \frac{\partial^2 u}{\partial x_2^2} + cu = 0 &\quad \text{a. e. in } \Omega \times (0,T) \\ u(x,t) = 0 &\quad \text{a.e. in } \partial\Omega \times (0,T) \\ u(x,0) = u_0(x) &\quad \text{a.e. in } \Omega. \end{aligned} \tag{0.8}$$

Accept as a fact that problem (0.8) has one and only one solution u in $C^0([0,T]; L^2(\Omega))$.

Let $\mathcal{T}_h$ be a finite element partition of Ω into triangles τ of diameter $h_\tau \le h$. Further, let

$$W_h = \{v_h \in C^0(\overline{\Omega}) \, : \, \forall \tau \in \mathcal{T}_h, \; v_h|_\tau \in \mathcal{P}_1, \; v_h|_{\partial\Omega} = 0\}, \tag{0.9}$$

be a finite element space of continuous piecewise linear functions over $\mathcal{T}_h$.

Consider the fully discrete backward Euler implicit approximation of (0.8): for K a positive integer, set $k = T/K$, define $t_n = nk$, $0 \le n \le K$, and for each $0 \le n \le K-1$, knowing $u_h^n \in W_h$ find $u_h^{n+1} \in W_h$ such that for all $v_h \in W_h$,

$$\frac{1}{k}(u_h^{n+1} - u_h^n, v_h) + a(u_h^{n+1}, v_h) = 0, \tag{0.10}$$

for $n = 0, 1, \ldots, K$ and $u_h^0 = I_h(u_0)$. Here $(\cdot,\cdot)$ is the inner product in $L^2(\Omega)$, the bilinear form $a(u_h^{n+1}, v_h)$ comes from the variational formulation of problem (0.8), and I_h is the lagrange interpolation operator in W_h. Write the expression of $a(u_h^{n+1}, v_h)$.

(a) Show that (0.10) defines a unique function u_h^{n+1} in W_h.

(b) Prove the following stability estimate,

$$\sup_{1 \le n \le K} ||u_h^n||_{L^2(\Omega)}^2 + k \sum_{n=1}^{K} |u_h^n|_{H^1(\Omega)}^2 \le ||u_h^0||_{L^2(\Omega)}^2 \tag{0.11}$$

(c) Also prove the estimate

$$\sup_{1 \le n \le K} |u_h^n|_{H^1(\Omega)} \le |u_h^0|_{H^1(\Omega)}. \tag{0.12}$$

An alternative inequality one can prove is as follows:

$$a(u_h^{n+1}, u_h^{n+1}) \le a(u_h^0, u_h^0).$$

Solution. **(a)** Let us first define the expression for $a(u_h^{n+1}, v_h)$. For notation purposes, let $x := x_1$ and $y := x_2$. Let $v_h \in W_h$. Then doing the following: (i) multiplying the PDE in the IVBP by v_h; (ii) integrating over the domain Ω; (iii) applying the boundary conditions; (iv) using the backward Euler implicit approximation; yields

$$a(u_h^{n+1}, v_h) = \int\limits_{\Omega} (\nabla u_h^{n+1} \cdot \nabla v_h) d\boldsymbol{x} + c \int\limits_{\Omega} u_h^{n+1} v_h d\boldsymbol{x}.$$

We now want to show we have a unique solution. Assume there exists two solutions u_h^{n+1} and $\tilde{u}_h^{n+1}$ such that

$$\begin{aligned}
&\frac{1}{k}(u_h^{n+1} - u_h^n, v_h) + a(u_h^{n+1}, v_h) = 0, \\
&\frac{1}{k}(\tilde{u}_h^{n+1} - u_h^n, v_h) + a(\tilde{u}_h^{n+1}, v_h) = 0.
\end{aligned}$$

Subtracting the two equations, we find

$$\frac{1}{k}(u_h^{n+1} - \tilde{u}_h^{n+1}, v_h) + a(u_h^{n+1} - \tilde{u}_h^{n+1}, v_h) = 0.$$

Note that this equation holds for any $v_h \in W_h$. Taking $v_h = u_h^{n+1} - \tilde{u}_h^{n+1}$, we see that

$$\frac{1}{k}||u_h^{n+1} - \tilde{u}_h^{n+1}||^2_{L^2(\Omega)} + |u_h^{n+1} - \tilde{u}_h^{n+1}|^2_{H^1(\Omega)} + c||u_h^{n+1} - \tilde{u}_h^{n+1}||^2_{L^2(\Omega)} = 0.$$

However, all of these terms are positive and hence each of the norms must be identically zero. Therefore, $||u_h^{n+1} - \tilde{u}_h^{n+1}||_{H^1(\Omega)} = 0$ and so $u_h^{n+1} = \tilde{u}_h^{n+1}$.

We just showed uniqueness of our problem and now want to show existence. Substituting $u_h = \sum_{i=1}^N U_i \phi_i$ and $v_h = \phi_j$ into the discrete equation where $U_i \in \mathbb{R}$ are the unknown coefficients and ϕ_i are the basis functions, we get the following $N \times N$ linear system of equations:

$$(\boldsymbol{M} + k\boldsymbol{A})\boldsymbol{U}^{n+1} = \boldsymbol{U}^n,$$

where $\boldsymbol{U} = [U_i]_{i=1}^N$, $\boldsymbol{M} = [(\phi_i, \phi_j)]_{i,j=1}^N$, and $\boldsymbol{A} = [a(\phi_i, \phi_j)]_{i,j=1}^N$. Since we showed we have a unique solution to this problem, we know that $\text{nullity}(\boldsymbol{M} + k\boldsymbol{A}) = 0$. Then, by the Rank-Nullity Theorem we have that $\text{rank}(\boldsymbol{M} + k\boldsymbol{A}) = N$ and so $\boldsymbol{U}^n \in \text{colspace}(\boldsymbol{M} + k\boldsymbol{A})$, and thus a solution must exist.

Alternative solution. Since the discrete equation no longer contains time derivatives, we can reformulate the problem as an elliptic boundary value problem. We then show boundedness and coercivity of $a(\cdot, \cdot)$ in $H_0^1(\Omega)$ and then invoke the Lax-Milgram Lemma.

(b) We now want to prove the stability estimate in the given problem. Let $0 \leq n \leq K$ be arbitrary. Let $u_h^0 = \mathcal{I}_h(u_0)$ be the Lagrange interpolant of the initial condition u_0.

Letting $v_h = u_h^{n+1}$ and substituting into our discrete scheme yields

$$\frac{1}{k}(u_h^{n+1} - u_h^n, u_h^{n+1}) + a(u_h^{n+1}, u_h^{n+1}) = 0.$$

Rearranging the terms and expanding the inner products yields

$$\begin{aligned}
(u_h^{n+1}, u_h^{n+1}) &= (u_h^n, u_h^{n+1}) - ka(u_h^{n+1}, u_h^{n+1}), \\
&= (u_h^n, u_h^{n+1}) - k\Big(\int_\Omega (\nabla u_h^{n+1} \cdot \nabla u_h^{n+1}) d\boldsymbol{x} + c \int_\Omega u_h^{n+1} u_h^{n+1} d\boldsymbol{x} \Big), \\
&\overset{\text{C.S}}{\leq} ||u_h^n||_{L^2(\Omega)} ||u_h^{n+1}||_{L^2(\Omega)} - k \left| u_h^{n+1} \right|^2_{H^1(\Omega)} - ck ||u_h^{n+1}||^2_{L^2(\Omega)}, \\
&\leq \frac{1}{2} ||u_h^n||^2_{L^2(\Omega)} + \frac{1}{2} ||u_h^{n+1}||^2_{L^2(\Omega)} - k \left| u_h^{n+1} \right|^2_{H^1(\Omega)}.
\end{aligned}$$

Subtracting both sides by $\frac{1}{2}||u_h^{n+1}||^2_{L^2(\Omega)}$, we have the following inequality,

$$\begin{aligned}
\frac{1}{2}||u_h^{n+1}||^2_{L^2(\Omega)} &\leq \frac{1}{2}||u_h^n||^2_{L^2(\Omega)} - k|u_h^{n+1}|^2_{H^1(\Omega)}, \\
&\leq \frac{1}{2}||u_h^n||^2_{L^2(\Omega)} - \frac{1}{2}k|u_h^{n+1}|^2_{H^1(\Omega)}.
\end{aligned}$$

Then an application of this inequality for each $n = 0, 1, \ldots, N-1$ yields the following the relation

$$\frac{1}{2}||u_h^{n+1}||^2_{L^2(\Omega)} \leq \frac{1}{2}||u_h^0||^2_{L^2(\Omega)} - \frac{1}{2}k \sum_{j=1}^{n+1} |u_h^j|^2_{H^1(\Omega)}.$$

Rearranging the equation we have,

$$||u_h^{n+1}||^2_{L^2(\Omega)} + k \sum_{j=1}^{n+1} |u_h^j|^2_{H^1(\Omega)} \leq ||u_h^0||^2_{L^2(\Omega)}.$$

(c) We now want to prove the estimate in part (c). We want to make a "guess" for the test function v_h in the discrete equation to arrive at our estimate.

Let us define the discrete Laplacian operator $A_h : W_h \to W_h$ to be the action

$$(A_h v_h, w_h) = \int_\Omega \nabla v_h \cdot \nabla w_h d\boldsymbol{x}.$$

Let us recall our discrete equation:

$$\frac{1}{k}(u_h^{n+1} - u_h^n, v_h) + a(u_h^{n+1}, v_h) = 0.$$

Expanding $a(\cdot, \cdot)$ using the inner product notation yields:

$$\frac{1}{k}(u_h^{n+1} - u_h^n, v_h) + (\nabla u_h^{n+1}, \nabla v_h) + c(u_h^{n+1}, v_h) = 0$$

Note that the second term can written as follows: $(\nabla u_h^{n+1}, \nabla v_h) = (A_h u_h^{n+1}, v_h)$.

Now let us choose $v_h = A_h u_h^{n+1}$ and substitute into the above equation:

$$
\begin{aligned}
\frac{1}{k}(u_h^{n+1} - u_h^n, A_h u_h^{n+1}) + (A_h u_h^{n+1}, A_h u_h^{n+1}) + c(u_h^{n+1}, A_h u_h^{n+1}) &= 0, \\
&\Longrightarrow \\
\frac{1}{k}(u_h^{n+1} - u_h^n, A_h u_h^{n+1}) + \underbrace{(A_h u_h^{n+1}, A_h u_h^{n+1})}_{\geq 0} + c(A_h u_h^{n+1}, u_h^{n+1}) &= 0, \\
\frac{1}{k}(u_h^{n+1} - u_h^n, A_h u_h^{n+1}) + c\underbrace{\int_\Omega \nabla u_h^{n+1} \cdot \nabla u_h^{n+1} d\boldsymbol{x}}_{\geq 0} &\leq 0, \\
\frac{1}{k}(u_h^{n+1} - u_h^n, A_h u_h^{n+1}) &\leq 0, \\
\frac{1}{k}(A_h u_h^{n+1}, u_h^{n+1} - u_h^n) &\leq 0, \\
&\Longrightarrow \\
\int_\Omega \nabla u_h^{n+1} \cdot \nabla(u_h^{n+1} - u_h^n) d\boldsymbol{x} &\leq 0, \\
&\Longrightarrow \\
\int_\Omega \nabla u_h^{n+1} \cdot \nabla u_h^{n+1} d\boldsymbol{x} &\leq \int_\Omega \nabla u_h^{n+1} \cdot \nabla u_h^n d\boldsymbol{x}, \\
\|\nabla u_h^{n+1}\|_{L^2(\Omega)}^2 &\leq \|\nabla u_h^{n+1}\|_{L^2(\Omega)} \|\nabla u_h^n\|_{L^2(\Omega)}, \\
&\Longrightarrow \\
\|\nabla u_h^{n+1}\|_{L^2(\Omega)} &\leq \|\nabla u_h^n\|_{L^2(\Omega)}
\end{aligned}
$$

Applying this inequality for each time step n and then taking the supremum yields the result.

Alternate solution We now want to prove the alternative estimate in part (c). Let us show first show coercivity of $a(\cdot,\cdot)$ in $H_0^1(\Omega)$. Let $u \in V$, then note that

$$
\begin{aligned}
a(u,u) &= \int_\Omega |\nabla u|^2 \, d\boldsymbol{x} + c\int_\Omega |u|^2 \, d\boldsymbol{x}, \\
&= |u|_{H^1(\Omega)}^1 + c\|u\|_{L^2(\Omega)}^2, \\
&\geq \min\{1, c\}\|u\|_{H^1(\Omega)}^2
\end{aligned}
$$

Let us now show boundendness of $a(\cdot,\cdot)$. Let $u, v \in H^1(\Omega)$, then we have that:

$$
\begin{aligned}
|a(u,v)| &= \left| \int_\Omega \nabla u \cdot \nabla v d\boldsymbol{x} + c\int_\Omega uv d\boldsymbol{x} \right|, \\
&\leq |u|_{L^2(\Omega)} |v|_{L^2(\Omega)} + c\|u\|_{L^2(\Omega)}\|v\|_{L^2(\Omega)}, \\
&\leq \|u\|_{H^1(\Omega)}\|v\|_{H^1(\Omega)} + c\|u\|_{H^1(\Omega)}\|v\|_{H^1(\Omega)}, \\
&\leq \max\{1, c\}\|u\|_{H^1(\Omega)}\|v\|_{H^1(\Omega)}
\end{aligned}
$$

Now note that the $V_h \subset V := H_0^1(\Omega)$. Then, substituting $v_h = u_h^{n+1} - u_h^n$ into the discrete equation we have

$$
\begin{array}{rcl}
\frac{1}{k}(u_h^{n+1}-u_h^n, u_h^{n+1}-u_h^n) + a(u_h^{n+1}, u_h^{n+1}-u_h^n) & = & 0, \\
& \Longrightarrow & \\
a(u_h^{n+1}, u_h^{n+1}-u_h^n) & \leq & 0, \\
& \Longrightarrow & \\
a(u_h^{n+1}, u_h^{n+1}) & \leq & a(u_h^{n+1}, u_h^n), \\
& = & \int\limits_\Omega \nabla u_h^{n+1} \cdot \nabla u_h^n d\boldsymbol{x} + c \int\limits_\Omega u_h^{n+1} u_h^n d\boldsymbol{x}, \\
& \leq & |u_h^{n+1}|_{H^1(\Omega)} |u_h^n|_{H^1(\Omega)} + c\|u_h^{n+1}\|_{L^2(\Omega)} \|u^n\|_{L^2(\Omega)}, \\
& \leq & \sqrt{|u_h^{n+1}|^2_{H^1(\Omega)} + c\|u_h^{n+1}\|^2_{L^2(\Omega)}} \sqrt{|u_h^n|^2_{H^1(\Omega)} + c\|u_h^n\|^2_{L^2(\Omega)}}, \\
& = & \sqrt{a(u_h^{n+1}, u_h^{n+1}} \sqrt{a(u_h^n, u_h^n)}, \\
& \Longrightarrow & \\
a(u_h^{n+1}, u_h^{n+1}) & \leq & a(u_h^n, u_h^n), \\
& \Longrightarrow & \\
a(u_h^{n+1}, u_h^{n+1}) & \leq & a(u_h^0, u_h^0)
\end{array}
$$

Problem 3. Consider the interval $(0,1)$ and the set of continuous functions $\hat{v}$ defined on $[0,1]$. Let $\hat{a}_1 = 0$, $\hat{a}_2 = \frac{1}{2}$, $\hat{a}_3 = 1$.

(a) Consider the following two sets of degrees of freedom,

$$\Sigma_1 = \{\hat{v}(\hat{a}_j),\ j = 1,2,3\} \quad \text{and} \quad \Sigma_2 = \{\hat{v}(\hat{a}_1), \hat{v}(\hat{a}_3), \int_0^1 \hat{v}(s)\, ds\}. \tag{0.13}$$

Write down the basis functions of $\mathcal{P}_2$ (for both sets of degrees of freedom) such that

1. $p_i \in \mathcal{P}_2$, $1 \le i \le 3$, satisfying: $p_i(\hat{a}_j) = \delta_{i,j}$, $1 \le i,j \le 3$ for the set Σ_1;
2. $q_i \in \mathcal{P}_2$, $1 \le i \le 3$, satisfying:

$$q_i(\hat{a}_j) = \delta_{i,j} \quad \text{and} \quad \int_0^1 q_i(s)\, ds = 0, \tag{0.14}$$

for $i = 1,3$, and $j = 1,3$ and

$$\int_0^1 q_2(s)\, ds = 1, \quad \text{and } q_2(\hat{a}_1) = q_2(\hat{a}_3) = 0. \tag{0.15}$$

In both cases, write down the FE interpolant $\hat{\Pi}(\hat{w})$ of a given function $\hat{w} \in C^0([0,1])$.

(b) Consider the interval $[a,b]$, let F map $[0,1]$ onto $[a,b]$, and let v be given in $H^3(a,b)$. Define $\Pi(v)$ by $(\Pi(v)) \circ F = \hat{\Pi}(v \circ F)$. Give the Bramble-Hilbert argument to get an estimate in terms of $h = b - a$ for the error

$$||v' - \Pi(v)'||_{L^2(a,b)}. \tag{0.16}$$

Explain how to modify the proof when v is less regular, e.g. $v \in H^2(a,b)$.

Solution. (a)
<u>**Case Σ_1:**</u> We want to find the basis functions for the set of degrees of freedom Σ_1. That is, we want to find polynomials $p_i, 1 \le i \le 3$ such that $p_i(\hat{a}_j) = \delta_{ij}$ for $1 \le i,j \le 3$ where δ_{ij} is the Kronecker delta function. Let us begin with p_1 $(i = 1)$. By definition of the DOFs, $p_1(\hat{a}_1) = 1$ and $p_1(\hat{a}_2) = p_1(\hat{a}_3) = 0$. Since $p_1 \in \mathcal{P}_2$, it must be in them form of $p_1 = c(x - \frac{1}{2})(x-1)$. To find c, we use the fact that $p_1(\hat{a}_1) = 1 = c(-\frac{1}{2})(-1) = 1$ which implies that $c = 2$. A similar process can be followed to find p_2 and p_3. Thus, our basis functions are given by:

$$\begin{aligned} p_1(x) &= 2(x - \frac{1}{2})(x-1), \\ p_2(x) &= -4x(x-1), \\ p_3(x) &= 2x(x - \frac{1}{2}). \end{aligned}$$

<u>**Case Σ_2:**</u> We want to find the basis functions for the set of degrees of freedom Σ_2. Let $i = 1$. Then note that $q_1(\hat{a}_1) = q_1(0) = 1$, $q_1(\hat{a}_3) = q_1(1) = 0$ and $\int_0^1 q_1(s)ds = 0$. Let $q(x)$ be a general quadratic polynomial: $q(x) = ax^2 + bx + c$. Then, we see what $q_1(0) = 1 \implies c = 1$. The other two conditions give:

$$\begin{aligned} q_1(1) =& a + b + 1 = 0, \\ \int_0^1 q_1(s)dx = \Big[\frac{1}{3}ax^3 + \frac{1}{2}bx^2 + 1\Big]_0^1 =& \frac{1}{3}a + \frac{1}{2}b + 1 = 0. \end{aligned}$$

Solving this linear system of equations yields $a = 3$ and $b = -4$. We can repeat this process to find q_2 and q_3. Thus, our basis functions for Σ_2 are given by:

$$\begin{aligned} q_1(x) &= 3x^2 - 4x + 1, \\ q_2(x) &= -6x(x-1), \\ q_3(x) &= 3x^2 - 2x. \end{aligned}$$

The finite element interpolants for both sets Σ_1 and Σ_2 are given by:

$$\begin{aligned} \hat{\Pi}_1(\hat{w}) &= \hat{w}(\hat{a}_1)p_1(x) + \hat{w}(\hat{a}_2)p_2(x) + \hat{w}(\hat{a}_3)p_3(x) \\ &= 2\hat{w}(0)(x - 1/2)(x-1) - 4\hat{w}(1/2)x(x-1) + 2\hat{w}(1)x(x-1/2), \\ \hat{\Pi}_2(\hat{w}) &= \hat{w}(0)(3x^2 - 4x + 1) - 6\int_0^1 \hat{w}(s)\,ds\,x(x-1) + 2\hat{w}(1)x(x-1/2). \end{aligned}$$

(b) Let F map $[0,1]$ onto $[a,b]$ be explicitly defined by $F(\xi) = h\xi + a$ with $\det(F') = h$ where $h = b - a$. Let $v \in H^3(a,b)$. Define $\Pi(v)$ by $(\Pi(v)) \circ F = \hat{\Pi}(v \circ F)$. Let $\hat{\Pi}$ be either $\hat{\Pi}_1$ or $\hat{\Pi}_2$ from part (a). This relationship of Π and $\hat{\Pi}$ can be seen with the following diagram:

$$\begin{tikzcd} H^3(a,b) \arrow[r, "\Pi"] \arrow[d, "\psi"'] & \mathcal{P}_2 \arrow[d, "\psi"] \\ H^3(0,1) \arrow[r, "\widehat{\Pi}"] & P \end{tikzcd}$$

where $P = \mathcal{P}_2 \circ F^{-1}$ and $\psi(v) = v \circ F$ is the *pullback*. Note that $\psi \circ \Pi = \widehat{\Pi} \circ \psi$. We want to apply a Bramble-Hilbert argument to get an estimate on the error in terms of the mesh size h.

We want to compute the norm $\|v' - \Pi(v)'\|_{L^2(a,b)}$ on the interval $[0,1]$. First note that by the chain rule

$$\frac{d}{dx} = \frac{d\xi}{dx}\frac{d}{d\xi} = \frac{1}{h}\frac{d}{d\xi}.$$

Then we see that

$$\begin{aligned} ||v' - \Pi(v)'||^2_{L^2(a,b)} &= \int_a^b \Big|\frac{d}{dx}(v(x) - \Pi(v)(x))\Big|^2 dx, \\ &= \int_0^1 \Big|\frac{1}{h}\frac{d}{d\xi}\big((v\circ F)(\xi) - \hat{\Pi}(v \circ F)(\xi)\big)\Big|^2 h\,d\xi. \end{aligned}$$

Defining $\hat{v} = v \circ F$, we can rewrite the the above by equality as

$$\begin{aligned} ||v' - \Pi(v)'||^2_{L^2(a,b)} &= \frac{1}{h}\int_0^1 \Big|\frac{d}{d\xi}\big(\hat{v} - \hat{\Pi}(\hat{v})\big)\Big|^2 d\xi, \\ &= \frac{1}{h}\int_0^1 \Big|\frac{d}{d\xi}\big((\mathrm{Id} - \hat{\Pi})(\hat{v})\big)\Big|^2 d\xi, \\ &= \frac{1}{h}|(\mathrm{Id} - \hat{\Pi})(\hat{v})|^2_{H^1(0,1)}. \end{aligned}$$

Then notice that $|(\mathrm{Id}-\hat{\Pi})(\cdot)|_{H^1(0,1)}$ is a sub-linear functional which is exactly zero for all $\hat{v} \in \mathcal{P}_2$. (Maybe we should recall definition of sub-linear functional). Let us recall the Bramble-Hilbert Lemma and apply it to our setting:

LEMMA 0.1. *Let $B \subset \mathbb{R}^n$ be a domain with a Lipschitz boundary and let q be a bounded sub-linear functional on $H^{k+1}(B)$. Assume that*

$$q(w) = 0, \qquad \text{for all } w \in P^k.$$

Then there exists a constant $c = c(B) > 0$, which depends on B, such that

$$|q(v)| \leq c\,|v|_{k+1,B}\,, \qquad \text{for all } v \in H^{k+1}(B).$$

Note that $(0,1) \subset \mathbb{R}$ is a domain with Lipschitz boundary. Let q be the sub-linear functional $q(w) = |(\mathrm{Id}-\hat{\Pi})(w)|_{H^1(0,1)}$. Since $q(w) = 0$ for all $w \in \mathcal{P}_2$, we can apply the lemma:

$$\begin{aligned}
||v' - \Pi(v)'||^2_{L^2(a,b)} &\leq \frac{1}{h} c \int_0^1 \left|\frac{d^3}{d\xi^3}\hat{v}\right|^2 d\xi,\\
&= \frac{c}{h}\int_a^b \left|h^3 \frac{d^3}{dx^3} v\right|^2 \frac{1}{h}\, dx,\\
&= ch^4 \int_a^b \left|\frac{d^3}{dx^3} v\right|^2 dx,\\
&= ch^4 |v|^2_{H^3(a,b)}.
\end{aligned}$$

Thus,

$$||v' - \Pi(v)'||_{L^2(a,b)} \leq ch^2 |v|_{H^3(a,b)}.$$

Let us now consider the case where we have lower regularity on v, that is, now $v \in H^2(a,b)$. Let us assume that $\hat{\Pi} = \hat{\Pi}_1$. (Note that this is okay since $\hat{\Pi}$ is not given). The goal is to redo the previous proof and modify it appropriately for when we have lower regularity.

Note that for us to make sense of the interpolant operator $\hat{\Pi}(\widehat{w}) = \hat{\Pi}_1(\widehat{w})$, we need the values of $\widehat{w}$ to make sense at the nodes $\widehat{a}_i$ for $i = 1,2,3$. That is, we need continuity of $\widehat{w}$. Since we want to work in a Sobolev space setting, we recall the result of the Sobolev Embedding theorem to note that functions in $H^1(\Omega)$ where $\Omega \subset \mathbb{R}$ are continuous functions. Note that functions in $H^2(\Omega)$ are also in $H^1(\Omega)$. We now repeat the above arguments for functions in $H^2(\Omega)$. We have that

$$\begin{aligned}
||v' - \Pi(v)'||^2_{L^2(a,b)} &= \frac{1}{h}\int_0^1 \left|\frac{d}{d\xi}\big(\hat{v} - \hat{\Pi}(\hat{v})\big)\right|^2 d\xi,\\
&= \frac{1}{h}\int_0^1 \left|\frac{d}{d\xi}\big((\mathrm{Id}-\hat{\Pi})(\hat{v})\big)\right|^2 d\xi,\\
&= \frac{1}{h}|(\mathrm{Id}-\hat{\Pi})(\hat{v})|^2_{H^1(0,1)}.
\end{aligned}$$

Then, applying the Bramble-Hilbert lemma yields

$$
\begin{aligned}
||v' - \Pi(v)'||^2_{L^2(a,b)} &\leq \frac{1}{h} c \int_0^1 \left| \frac{d^2}{d\xi^2} \hat{v} \right|^2 d\xi, \\
&= \frac{c}{h} \int_a^b \left| h^2 \frac{d^2}{dx^2} v \right|^2 \frac{1}{h} \, dx, \\
&= ch^2 \int_a^b \left| \frac{d^2}{dx^2} v \right|^2 dx, \\
&= ch^2 |v|^2_{H^2(a,b)}.
\end{aligned}
$$

Thus our new estimate is given as follows:

$$
||v' - \Pi(v)'||_{L^2(a,b)} \leq ch|v|_{H^2(a,b)}.
$$