Lecture 3

Eric Tovar

Qualifying Prep Course – Numerical

07-03-2020

Outline

Problem 2

Problem 2.

Let $\Omega = (0,1)^2$ and u be the solution of the second order elliptic problem:

$$-\Delta u : -u_{x_1x_1} - u_{x_2x_2} = f(x), \quad \text{for} x \in \Omega \tag{1.1}$$

$$\frac{\partial u}{\partial n} + u = g(x), \quad \text{for} x \in \partial\Omega \tag{1.2}$$

where n is the outward normal unit vector to the boundary $\partial\Omega$ and $f(x)$ and $g(x)$ are given functions.

Let $\Omega = (0,1)^2$ and u be the solution of the second order elliptic problem:

$$-\Delta u : -u_{x_1x_1} - u_{x_2x_2} = f(x), \quad \text{for} x \in \Omega \tag{1.1}$$

$$\frac{\partial u}{\partial n} + u = g(x), \quad \text{for} x \in \partial\Omega \tag{1.2}$$

where n is the outward normal unit vector to the boundary $\partial\Omega$ and $f(x)$ and $g(x)$ are given functions.

Question 1: What kind of boundary condition do we have?

Let $\Omega = (0,1)^2$ and u be the solution of the second order elliptic problem:

$$-\Delta u : -u_{x_1x_1} - u_{x_2x_2} = f(x), \quad \text{for} x \in \Omega \tag{1.1}$$

$$\frac{\partial u}{\partial n} + u = g(x), \quad \text{for} x \in \partial\Omega \tag{1.2}$$

where n is the outward normal unit vector to the boundary $\partial\Omega$ and $f(x)$ and $g(x)$ are given functions.

Question 1: What kind of boundary condition do we have?
Remark 1: $\frac{\partial u}{\partial n} = \nabla u \cdot \boldsymbol{n}$

(a) Derive the weak formulation of this problem in the form $a(u, v) = F(v)$, where $a(u, v)$ and $F(v)$ are the appropriate bilinear and linear forms defined on the Sobolev space $H^1(\Omega)$.

(a) Derive the weak formulation of this problem in the form $a(u, v) = F(v)$, where $a(u, v)$ and $F(v)$ are the appropriate bilinear and linear forms defined on the Sobolev space $H^1(\Omega)$.

(b) Let S_h be a finite element space of continuous piece-wise polynomial functions defined over a regular partitioning of Ω into triangles and let $a_h(u, v)$ **and** $F_h(v)$**(!!!)** be the bilinear forms where all integrals are computed approximately. Derive Strang's lemma for the error of the FEM: find $u_h \in S_h$ such that $a_h(u_h, v) = F_h(v)$, $\forall v \in S_h$.

(a) Derive the weak formulation of this problem in the form $a(u, v) = F(v)$, where $a(u, v)$ and $F(v)$ are the appropriate bilinear and linear forms defined on the Sobolev space $H^1(\Omega)$.

(b) Let S_h be a finite element space of continuous piece-wise polynomial functions defined over a regular partitioning of Ω into triangles and let $a_h(u, v)$ **and** $F_h(v)$**(!!!)** be the bilinear forms where all integrals are computed approximately. Derive Strang's lemma for the error of the FEM: find $u_h \in S_h$ such that $a_h(u_h, v) = F_h(v)$, $\forall v \in S_h$.

(c) Let S_h be the finite element space of piece-wise linear functions. Let all integrals in $a(u, v)$ and $F(v)$ be computed using quadratures. Namely, for τ and e being triangle and edge defined by the vertexes P_1, P_2, P_3 and P_1, P_2 respectively,

$$\int_\tau w(x)dx \approx \frac{|\tau|}{3}\Big(w(P_1)+w(P_2)+w(P_3)\Big), \quad \int_e w(x)ds \approx \frac{|e|}{2}\Big(w(\alpha)+w \tag{1.3}$$

where $|\tau|$ is the area of τ and $|e|$ is the length of e, and α and β are the Gaussian quadrature nodes. Explain why $a(w, v) = a_h(w, v)$ for all $w, v \in S_h$.

Solution to (c)

Let S_h be the finite element space of piece-wise **linear** functions. Let all integrals in $a(u, v)$ and $F(v)$ be computed using quadratures. Namely, for τ and e being triangle and edge defined by the vertexes P_1, P_2, P_3 and P_1, P_2 respectively,

$$\int_\tau w(x)dx \approx \frac{|\tau|}{3}\Big(w(P_1)+w(P_2)+w(P_3)\Big), \quad \int_e w(x)ds \approx \frac{|e|}{2}\Big(w(\alpha)+w(\beta)\Big) \tag{1.4}$$

where $|\tau|$ is the area of τ and $|e|$ is the length of e, and α and β are the Gaussian quadrature nodes. Explain why $a(w, v) = a_h(w, v)$ for all $w, v \in S_h$.

Solution to (c)

Recall that

$$a(u,v) = \int_{\Omega} \nabla u \cdot \nabla v dx + \int_{\partial\Omega} uvds.$$

Let $w, v \in S_h$.

Solution to (c)

Recall that

$$a(u,v) = \int_{\Omega} \nabla u \cdot \nabla v dx + \int_{\partial\Omega} uvds.$$

Let $w, v \in S_h$. **Q: What kind of function is $\nabla w \cdot \nabla v$?**

Solution to (c)

Recall that

$$a(u, v) = \int_{\Omega} \nabla u \cdot \nabla v dx + \int_{\partial\Omega} uvds.$$

Let $w, v \in S_h$. **Q: What kind of function is $\nabla w \cdot \nabla v$?** $\nabla w \cdot \nabla v$ is a piecewise constant function. Thus, we see that

Solution to (c)

Recall that

$$a(u,v) = \int_{\Omega} \nabla u \cdot \nabla v dx + \int_{\partial\Omega} uvds.$$

Let $w, v \in S_h$. **Q: What kind of function is $\nabla w \cdot \nabla v$?** $\nabla w \cdot \nabla v$ is a piecewise constant function. Thus, we see that

$$\begin{aligned} \int_{\Omega} \nabla w \cdot \nabla v \, dx &= \sum_{\tau \in \mathcal{T}_h} \int_{\tau} \nabla w \cdot \nabla v dx \\ &= \sum_{\tau \in \mathcal{T}_h} (\nabla w \cdot \nabla v) \int_{\tau} dx \end{aligned}$$

Solution to (c)

Recall that

$$a(u, v) = \int\limits_{\Omega} \nabla u \cdot \nabla v dx + \int\limits_{\partial\Omega} uvds.$$

Let $w, v \in S_h$. **Q: What kind of function is $\nabla w \cdot \nabla v$?** $\nabla w \cdot \nabla v$ is a piecewise constant function. Thus, we see that

$$\begin{aligned}
\int_{\Omega} \nabla w \cdot \nabla v \, dx &= \sum_{\tau \in \mathcal{T}_h} \int_{\tau} \nabla w \cdot \nabla v dx \\
&= \sum_{\tau \in \mathcal{T}_h} (\nabla w \cdot \nabla v) \int_{\tau} dx \\
&= \sum_{\tau \in \mathcal{T}_h} |\tau| (\nabla w \cdot \nabla v)
\end{aligned}$$

Solution to (c)

Recall that

$$a(u, v) = \int_{\Omega} \nabla u \cdot \nabla v dx + \int_{\partial\Omega} uvds.$$

Let $w, v \in S_h$. **Q: What kind of function is $\nabla w \cdot \nabla v$?** $\nabla w \cdot \nabla v$ is a piecewise constant function. Thus, we see that

$$\begin{aligned}
\int_{\Omega} \nabla w \cdot \nabla v \, dx &= \sum_{\tau \in \mathcal{T}_h} \int_{\tau} \nabla w \cdot \nabla v dx \\
&= \sum_{\tau \in \mathcal{T}_h} (\nabla w \cdot \nabla v) \int_{\tau} dx \\
&= \sum_{\tau \in \mathcal{T}_h} |\tau| (\nabla w \cdot \nabla v) \\
&= \sum_{\tau \in \mathcal{T}_h} \frac{|\tau|}{3} \Big((\nabla w \cdot \nabla v)(P_1) + (\nabla w \cdot \nabla v)(P_2) \\
&\qquad + (\nabla w \cdot \nabla v)(P_3) \Big)
\end{aligned}$$

Now we want to compute the boundary term $\int_{\partial\Omega} wvds$ for $w, v \in S_h$.

Now we want to compute the boundary term $\int_{\partial\Omega} wvds$ for $w, v \in S_h$. **Q: What kind of function is the product wv on a triangle edge?**

Now we want to compute the boundary term $\int_{\partial\Omega} wvds$ for $w, v \in S_h$. **Q: What kind of function is the product wv on a triangle edge?** wv is a quadratic one-dimensional polynomial.

Now we want to compute the boundary term $\int_{\partial\Omega} wvds$ for $w, v \in S_h$. **Q: What kind of function is the product wv on a triangle edge?** wv is a quadratic one-dimensional polynomial.

Recall that we are dealing with Gaussian quadrature which are exact for polynomials of degree $2n-1$ where n is the number of points used (2 in our case).

Now we want to compute the boundary term $\int_{\partial\Omega} wvds$ for $w, v \in S_h$. **Q: What kind of function is the product wv on a triangle edge?** wv is a quadratic one-dimensional polynomial.

Recall that we are dealing with Gaussian quadrature which are exact for polynomials of degree $2n - 1$ where n is the number of points used (2 in our case).

Also recall that the Gaussian quadrature is defined on the interval $[-1, 1]$. Let $e = [a, b]$ be an arbitrary edge.

Now we want to compute the boundary term $\int_{\partial\Omega} wvds$ for $w, v \in S_h$. **Q: What kind of function is the product wv on a triangle edge?** wv is a quadratic one-dimensional polynomial.

Recall that we are dealing with Gaussian quadrature which are exact for polynomials of degree $2n - 1$ where n is the number of points used (2 in our case).

Also recall that the Gaussian quadrature is defined on the interval $[-1, 1]$. Let $e = [a, b]$ be an arbitrary edge. Then, we need the transformation

$$T(t) = \frac{1}{2}(1 - t)a + \frac{1}{2}(t + a)b$$

so that $T(-1) = a$ and $T(1) = b$. Note that $T^{'}(t) = \frac{1}{2}(b - a) = \frac{1}{2}|e|$.

Now we can compute the boundary integral. Recall that the Gaussian quadrature here is exact up to polynomials of degree 3.

$$
\begin{aligned}
\int_{\partial\Omega} wv\,ds &= \sum_{e\in\partial\Omega} \int_e wv\,dx \\
&= \sum_{e\in\partial\Omega} \int_{-1}^{1} wv \underbrace{\frac{1}{2}\,|e|\,dt} \\
&= \sum_{e\in\partial\Omega} \frac{|e|}{2}\Big((wv)(\alpha) + (wv)(\beta)\Big)
\end{aligned}
$$

Now we can compute the boundary integral. Recall that the Gaussian quadrature here is exact up to polynomials of degree 3.

$$
\begin{aligned}
\int_{\partial\Omega} wv ds &= \sum_{e\in\partial\Omega} \int_e wv dx \\
&= \sum_{e\in\partial\Omega} \int_{-1}^{1} wv \underbrace{\frac{1}{2}|e|\, dt} \\
&= \sum_{e\in\partial\Omega} \frac{|e|}{2}\Big((wv)(\alpha) + (wv)(\beta)\Big)
\end{aligned}
$$

Thus, combining everything, we see that $a_h(w,v) = a(w,v)$ for all $w, v \in S_h$.

Solution to (d)

Using the estimate of Part (b) estimate the error $\|u - u_h\|_{H^1}$.

Solution to (d)

We just showed that $a_h(w, v) = a(w, v)$ for all $w, v \in S_h$.

Solution to (d)

We just showed that $a_h(w, v) = a(w, v)$ for all $w, v \in S_h$. Recall by Strang's first Lemma above, we know that

$$\begin{aligned} \|u - u_h\|_{H^1(\Omega)} \quad &\leq \quad \inf_{z_h \in S_h} \left(1 + \frac{C}{\alpha}\right) \|u - z_h\| + \\ &\quad \frac{1}{\alpha}\Big(\underbrace{\|a(z_h, v_h) - a_h(z_h, v_h)\|_{*,h}}_{=0} + \|F_h(v_h) - F(v_h)\|_{*,h} \Big) \end{aligned}$$

Solution to (d)

We just showed that $a_h(w,v) = a(w,v)$ for all $w, v \in S_h$. Recall by Strang's first Lemma above, we know that

$$\begin{aligned}
\|u - u_h\|_{H^1(\Omega)} &\leq \inf_{z_h \in S_h} \left(1 + \frac{C}{\alpha}\right)\|u - z_h\| + \\
&\qquad \frac{1}{\alpha}\Big(\underbrace{\|a(z_h, v_h) - a_h(z_h, v_h)\|_{*,h}}_{=0} + \|F_h(v_h) - F(v_h)\|_{*,h}\Big) \\
&= \inf_{z_h \in S_h} \left(1 + \frac{C}{\alpha}\right)\|u - z_h\| + \frac{1}{\alpha}\|F_h(v_h) - F(v_h)\|_{*,h}
\end{aligned}$$

Solution to (d)

We just showed that $a_h(w, v) = a(w, v)$ for all $w, v \in S_h$. Recall by Strang's first Lemma above, we know that

$$\begin{aligned}
\|u - u_h\|_{H^1(\Omega)} &\leq \inf_{z_h \in S_h} \left(1 + \frac{C}{\alpha}\right)\|u - z_h\| + \\
&\qquad \frac{1}{\alpha}\Big(\underbrace{\|a(z_h, v_h) - a_h(z_h, v_h)\|_{*,h}}_{=0} + \|F_h(v_h) - F(v_h)\|_{*,h}\Big) \\
&= \inf_{z_h \in S_h} \left(1 + \frac{C}{\alpha}\right)\|u - z_h\| + \frac{1}{\alpha}\|F_h(v_h) - F(v_h)\|_{*,h}
\end{aligned}$$

This is justified since a_h is V_h-elliptic since a is V-elliptic and $a_h = a$ on V_h.

Solution to (d)

We just showed that $a_h(w,v) = a(w,v)$ for all $w, v \in S_h$. Recall by Strang's first Lemma above, we know that

$$\begin{aligned}
\|u - u_h\|_{H^1(\Omega)} &\leq \inf_{z_h \in S_h} \left(1 + \frac{C}{\alpha}\right)\|u - z_h\| + \\
&\qquad \frac{1}{\alpha}\Big(\underbrace{\|a(z_h, v_h) - a_h(z_h, v_h)\|_{*,h}}_{=0} + \|F_h(v_h) - F(v_h)\|_{*,h}\Big) \\
&= \inf_{z_h \in S_h} \left(1 + \frac{C}{\alpha}\right)\|u - z_h\| + \frac{1}{\alpha}\|F_h(v_h) - F(v_h)\|_{*,h}
\end{aligned}$$

This is justified since a_h is V_h-elliptic since a is V-elliptic and $a_h = a$ on V_h. Since we want an error estimate, we need to do the following:

Solution to (d)

We just showed that $a_h(w, v) = a(w, v)$ for all $w, v \in S_h$. Recall by Strang's first Lemma above, we know that

$$
\begin{aligned}
\|u - u_h\|_{H^1(\Omega)} &\leq \inf_{z_h \in S_h} \left(1 + \frac{C}{\alpha}\right) \|u - z_h\| + \\
&\qquad \frac{1}{\alpha}\Big(\underbrace{\|a(z_h, v_h) - a_h(z_h, v_h)\|_{*,h}}_{=0} + \|F_h(v_h) - F(v_h)\|_{*,h} \Big) \\
&= \inf_{z_h \in S_h} \left(1 + \frac{C}{\alpha}\right) \|u - z_h\| + \frac{1}{\alpha} \|F_h(v_h) - F(v_h)\|_{*,h}
\end{aligned}
$$

This is justified since a_h is V_h-elliptic since a is V-elliptic and $a_h = a$ on V_h. Since we want an error estimate, we need to do the following: **1. Define a projection;**

Solution to (d)

We just showed that $a_h(w, v) = a(w, v)$ for all $w, v \in S_h$. Recall by Strang's first Lemma above, we know that

$$
\begin{aligned}
\|u - u_h\|_{H^1(\Omega)} &\leq \inf_{z_h \in S_h} \left(1 + \frac{C}{\alpha}\right) \|u - z_h\| + \\
&\qquad \frac{1}{\alpha}\Big(\underbrace{\|a(z_h, v_h) - a_h(z_h, v_h)\|_{*,h}}_{=0} + \|F_h(v_h) - F(v_h)\|_{*,h} \Big) \\
&= \inf_{z_h \in S_h} \left(1 + \frac{C}{\alpha}\right) \|u - z_h\| + \frac{1}{\alpha} \|F_h(v_h) - F(v_h)\|_{*,h}
\end{aligned}
$$

This is justified since a_h is V_h-elliptic since a is V-elliptic and $a_h = a$ on V_h. Since we want an error estimate, we need to do the following: **1. Define a projection; 2. Transform to the reference element;**

Solution to (d)

We just showed that $a_h(w, v) = a(w, v)$ for all $w, v \in S_h$. Recall by Strang's first Lemma above, we know that

$$
\begin{aligned}
\|u - u_h\|_{H^1(\Omega)} &\leq \inf_{z_h \in S_h} \left(1 + \frac{C}{\alpha}\right) \|u - z_h\| + \\
&\qquad \frac{1}{\alpha} \Big(\underbrace{\|a(z_h, v_h) - a_h(z_h, v_h)\|_{*,h}}_{=0} + \|F_h(v_h) - F(v_h)\|_{*,h} \Big) \\
&= \inf_{z_h \in S_h} \left(1 + \frac{C}{\alpha}\right) \|u - z_h\| + \frac{1}{\alpha} \|F_h(v_h) - F(v_h)\|_{*,h}
\end{aligned}
$$

This is justified since a_h is V_h-elliptic since a is V-elliptic and $a_h = a$ on V_h. Since we want an error estimate, we need to do the following: **1. Define a projection; 2. Transform to the reference element; 3. Use Bramble-Hilbert to get error on reference element;**

Solution to (d)

We just showed that $a_h(w, v) = a(w, v)$ for all $w, v \in S_h$. Recall by Strang's first Lemma above, we know that

$$\begin{aligned}
\|u - u_h\|_{H^1(\Omega)} &\leq \inf_{z_h \in S_h} \left(1 + \frac{C}{\alpha}\right) \|u - z_h\| + \\
&\qquad \frac{1}{\alpha}\left(\underbrace{\|a(z_h, v_h) - a_h(z_h, v_h)\|_{*,h}}_{=0} + \|F_h(v_h) - F(v_h)\|_{*,h} \right) \\
&= \inf_{z_h \in S_h} \left(1 + \frac{C}{\alpha}\right) \|u - z_h\| + \frac{1}{\alpha} \|F_h(v_h) - F(v_h)\|_{*,h}
\end{aligned}$$

This is justified since a_h is V_h-elliptic since a is V-elliptic and $a_h = a$ on V_h. Since we want an error estimate, we need to do the following: **1. Define a projection; 2. Transform to the reference element; 3. Use Bramble-Hilbert to get error on reference element; 4. Transform back to physical element;**

Review of affine transformations

For reference, see 4.4.1 of Grossman book pg 219.

- We first define the reference element as the triangle:

$$\widehat{\tau} = \left\{ \begin{pmatrix} \widehat{x} \\ \widehat{y} \end{pmatrix} : \widehat{x} \geq 0, \widehat{y} \geq 0, \widehat{x} + \widehat{y} \leq 1 \right\}. \tag{1.4}$$

For reference, see 4.4.1 of Grossman book pg 219.

- We first define the reference element as the triangle:

$$\widehat{\tau} = \left\{ \begin{pmatrix} \widehat{x} \\ \widehat{y} \end{pmatrix} : \widehat{x} \geq 0, \widehat{y} \geq 0, \widehat{x} + \widehat{y} \leq 1 \right\}. \tag{1.4}$$

- Consider the mapping $F_\tau : \widehat{\tau} \to \tau$ that goes from the reference element to the triangle τ (physical element) with the vertices $(x_1, y_1)^\mathsf{T}, (x_2, y_2)^\mathsf{T}, (x_3, y_3)^\mathsf{T}$:

$$F_\tau(\widehat{p}) = \underbrace{\begin{pmatrix} x_2 - x_1 & x_3 - x_1 \\ y_2 - y_1 & y_3 - y_1 \end{pmatrix}}_{B} \begin{pmatrix} \widehat{x} \\ \widehat{y} \end{pmatrix} + \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \tag{1.5}$$

Note that $|\det B| = \left|\det(F'_\tau(\widehat{p}))\right| = \frac{|\tau|}{|\widehat{\tau}|} \sim ch^2$. ($h$ is defined on the next slide)

- Setting $v(\widehat{p}) = u(F(\widehat{p}))$ for $\widehat{p} \in \widehat{\tau}$, we see that each function $u(\boldsymbol{x})$ for $\boldsymbol{x} \in \tau$ is mapped to a function $v(\widehat{p})$ defined on the reference element. **Notation: We will interchange $v(\widehat{p}) = u(F(\widehat{p}))$ with $\widehat{u}$.**

- Setting $v(\widehat{p}) = u(F(\widehat{p}))$ for $\widehat{p} \in \widehat{\tau}$, we see that each function $u(\boldsymbol{x})$ for $\boldsymbol{x} \in \tau$ is mapped to a function $v(\widehat{p})$ defined on the reference element. **Notation: We will interchange $v(\widehat{p}) = u(F(\widehat{p}))$ with $\widehat{u}$.**
- So, if F is differentiable, then by the chain rule, we have that:

$$\nabla_{\widehat{p}} v(\widehat{p}) = F^{'}(\widehat{p}) \nabla u(F(\widehat{p})). \tag{1.6}$$

- Setting $v(\widehat{p}) = u(F(\widehat{p}))$ for $\widehat{p} \in \widehat{\tau}$, we see that each function $u(\boldsymbol{x})$ for $\boldsymbol{x} \in \tau$ is mapped to a function $v(\widehat{p})$ defined on the reference element. **Notation: We will interchange $v(\widehat{p}) = u(F(\widehat{p}))$ with $\widehat{u}$.**
- So, if F is differentiable, then by the chain rule, we have that:

$$\nabla_{\widehat{p}} v(\widehat{p}) = F^{'}(\widehat{p}) \nabla u(F(\widehat{p})). \tag{1.6}$$

- Let h be the maximum length of the edges of the triangles in the triangulation of Ω. Then there exists $c > 0$ such that

$$\|F^{'}(\widehat{p})\| = \|B\| \leq ch \qquad \text{for all } \widehat{p} \in \widehat{\tau} \tag{1.7}$$

- Setting $v(\widehat{p}) = u(F(\widehat{p}))$ for $\widehat{p} \in \widehat{\tau}$, we see that each function $u(\boldsymbol{x})$ for $\boldsymbol{x} \in \tau$ is mapped to a function $v(\widehat{p})$ defined on the reference element. **Notation: We will interchange $v(\widehat{p}) = u(F(\widehat{p}))$ with $\widehat{u}$.**
- So, if F is differentiable, then by the chain rule, we have that:

$$\nabla_{\widehat{p}} v(\widehat{p}) = F^{'}(\widehat{p}) \nabla u(F(\widehat{p})). \tag{1.6}$$

- Let h be the maximum length of the edges of the triangles in the triangulation of Ω. Then there exists $c > 0$ such that

$$\|F^{'}(\widehat{p})\| = \|B\| \leq ch \qquad \text{for all } \widehat{p} \in \widehat{\tau} \tag{1.7}$$

- Using the above we see that

$$\int_{\tau} u^2(\boldsymbol{x}) dx = \int_{\widehat{\tau}} v^2(\widehat{p}) \left| \det F^{'}(\widehat{p}) \right| d\widehat{p} \tag{1.8}$$

- According to Lemma 4.23 in Grossman book:

$$\int_{\tau} |(\nabla u)|^2 \, d\boldsymbol{x} \leq c \|B^{-1}\|^2 \int_{\widehat{\tau}} \left|\nabla_{\widehat{p}} \widehat{u}\right|^2 |\det B| \, d\widehat{p} \tag{1.9}$$

[1] see pg 222 of Grossman book

- According to Lemma 4.23 in Grossman book:

$$\int_{\tau} |(\nabla u)|^2 \, d\boldsymbol{x} \leq c\|B^{-1}\|^2 \int_{\widehat{\tau}} \left|\nabla_{\widehat{p}} \widehat{u}\right|^2 |\det B| \, d\widehat{p} \tag{1.9}$$

- Assuming that the mesh is shape-regular[1] (which we can always do for these error estimates), we have the following bound:

$$\|B^{-1}\| \leq ch^{-1}.$$

[1] see pg 222 of Grossman book

A note on Bramble-Hilbert

For applying Bramble-Hilbert (and any other major result), it is recommended that students first state the result in their solution and then apply it appropriately (by making sure the assumptions are satisfied). You should not just say “by Bramble-Hilbert” in a sub-step and move on.

A note on Bramble-Hilbert

Bramble-Hilbert

Let $B \subset \mathbb{R}^n$ be a domain with a Lipschitz boundary and let q be a bounded sub-linear functional on $H^{k+1}(B)$. Assume that

$$q(w) = 0, \qquad \text{for all } w \in P^k.$$

Then there exists a constant $c = c(B) > 0$, which depends on B, such that

$$|q(v)| \leq c\,|v|_{k+1,B}\,, \qquad \text{for all } v \in H^{k+1}(B).$$

Back to solution...

We first focus on the term $\inf_{z_h \in S_h} \left(1 + \frac{C}{\alpha}\right) \|u - z_h\|$.

Back to solution...

We first focus on the term $\inf_{z_h \in S_h} \left(1 + \frac{C}{\alpha}\right) \|u - z_h\|$. Let $\Pi_h : H^1(\Omega) \to S_h$ be the projection onto the finite element subspace.

Back to solution...

We first focus on the term $\inf_{z_h \in S_h} \left(1 + \frac{C}{\alpha}\right) \|u - z_h\|$. Let $\Pi_h : H^1(\Omega) \to S_h$ be the projection onto the finite element subspace. Recall that the projection error bounds this term from above:

$$\left(\inf_{z_h \in S_h} \left(1 + \frac{C}{\alpha}\right) \|u - z_h\| \right)^2 \leq c \|u - \Pi_h u\|^2_{H^1(\Omega)} \tag{1.10}$$

Back to solution...

We first focus on the term $\inf_{z_h \in S_h} \left(1 + \frac{C}{\alpha}\right)\|u - z_h\|$. Let $\Pi_h : H^1(\Omega) \to S_h$ be the projection onto the finite element subspace. Recall that the projection error bounds this term from above:

$$\left(\inf_{z_h \in S_h} \left(1 + \frac{C}{\alpha}\right)\|u - z_h\|\right)^2 \leq c\|u - \Pi_h u\|^2_{H^1(\Omega)} \tag{1.10}$$

We want to compute this term on the reference element:

$$\|u - \Pi_h u\|^2_{H^1(\Omega)} = \sum_{\tau \in \mathcal{T}_h} \|u - \Pi_h u\|^2_{H^1(\tau)}$$

Back to solution...

We first focus on the term $\inf_{z_h \in S_h} \left(1 + \frac{C}{\alpha}\right) \|u - z_h\|$. Let $\Pi_h : H^1(\Omega) \to S_h$ be the projection onto the finite element subspace. Recall that the projection error bounds this term from above:

$$\left(\inf_{z_h \in S_h} \left(1 + \frac{C}{\alpha}\right) \|u - z_h\| \right)^2 \leq c \|u - \Pi_h u\|^2_{H^1(\Omega)} \tag{1.10}$$

We want to compute this term on the reference element:

$$\begin{aligned} \|u - \Pi_h u\|^2_{H^1(\Omega)} &= \sum_{\tau \in \mathcal{T}_h} \|u - \Pi_h u\|^2_{H^1(\tau)} \\ &= \sum_{\tau \in \mathcal{T}_h} \int_\tau \left(|u - \Pi_h u|^2 + |\nabla(u - \Pi_h u)|^2 \right) d\boldsymbol{x} \\ &\leq c \sum_{\tau \in \mathcal{T}_h} \int_{\widehat{\tau}} \left(\left| \widehat{u} - \widehat{(\Pi_h u)} \right|^2 + \right. \end{aligned}$$

Back to solution...

We first focus on the term $\inf_{z_h \in S_h} \left(1 + \frac{C}{\alpha}\right) \|u - z_h\|$. Let $\Pi_h : H^1(\Omega) \to S_h$ be the projection onto the finite element subspace. Recall that the projection error bounds this term from above:

$$\left(\inf_{z_h \in S_h} \left(1 + \frac{C}{\alpha}\right) \|u - z_h\| \right)^2 \leq c \|u - \Pi_h u\|^2_{H^1(\Omega)} \tag{1.10}$$

We want to compute this term on the reference element:

$$\begin{aligned}
\|u - \Pi_h u\|^2_{H^1(\Omega)} &= \sum_{\tau \in \mathcal{T}_h} \|u - \Pi_h u\|^2_{H^1(\tau)} \\
&= \sum_{\tau \in \mathcal{T}_h} \int_{\tau} \left(|u - \Pi_h u|^2 + |\nabla (u - \Pi_h u)|^2 \right) d\boldsymbol{x} \\
&\leq c \sum_{\tau \in \mathcal{T}_h} \int_{\widehat{\tau}} \left(\left| \widehat{u} - \widehat{(\Pi_h u)} \right|^2 + \right. \\
&\qquad \left. \|B^{-1}\|^2 \left| \nabla_{\widehat{p}} (\widehat{u} - \widehat{(\Pi_h u)}) \right|^2 \right) \underbrace{\frac{|\tau|}{|\widehat{\tau}|}} d\widehat{p}
\end{aligned}$$

In this next step, we bound $\|B^{-1}\| \leq ch^{-1}$ and replace $\frac{|\tau|}{|\widehat{\tau}|}$ by ch^2:

$$
\begin{aligned}
\|u - \Pi_h u\|^2_{H^1(\Omega)} \quad \leq \quad & ch^2 \sum_{\tau \in \mathcal{T}_h} \int_{\widehat{\tau}} \left| \widehat{u} - \widehat{(\Pi_h u)} \right|^2 + \\
& c \sum_{\tau \in \mathcal{T}_h} \int_{\widehat{\tau}} \left| \nabla(\widehat{u} - \widehat{(\Pi_h u)}) \right|^2 d\widehat{p}
\end{aligned}
$$

In this next step, we bound $\|B^{-1}\| \leq ch^{-1}$ and replace $\frac{|\tau|}{|\widehat{\tau}|}$ by ch^2:

$$
\begin{aligned}
\|u - \Pi_h u\|^2_{H^1(\Omega)} &\leq ch^2 \sum_{\tau \in \mathcal{T}_h} \int_{\widehat{\tau}} \left|\widehat{u} - \widehat{(\Pi_h u)}\right|^2 + \\
&\qquad c \sum_{\tau \in \mathcal{T}_h} \int_{\widehat{\tau}} \left|\nabla(\widehat{u} - \widehat{(\Pi_h u)})\right|^2 d\widehat{p} \\
&= ch^2 \sum_{\tau \in \mathcal{T}_h} \|(\mathrm{Id} - \widehat{\Pi}_h)\widehat{u}\|^2 + \\
&\qquad c \sum_{\tau \in \mathcal{T}_h} \left|(\mathrm{Id} - \widehat{\Pi}_h)\widehat{u})\right|^2_{H^1(\Omega)}
\end{aligned}
$$

In this next step, we bound $\|B^{-1}\| \leq ch^{-1}$ and replace $\frac{|\tau|}{|\widehat{\tau}|}$ by ch^2:

$$
\begin{aligned}
\|u - \Pi_h u\|^2_{H^1(\Omega)} &\leq ch^2 \sum_{\tau \in \mathcal{T}_h} \int_{\widehat{\tau}} \left| \widehat{u} - \widehat{(\Pi_h u)} \right|^2 + \\
&\qquad c \sum_{\tau \in \mathcal{T}_h} \int_{\widehat{\tau}} \left| \nabla(\widehat{u} - \widehat{(\Pi_h u)}) \right|^2 d\widehat{p} \\
&= ch^2 \sum_{\tau \in \mathcal{T}_h} \|(\mathrm{Id} - \widehat{\Pi}_h)\widehat{u}\|^2 + \\
&\qquad c \sum_{\tau \in \mathcal{T}_h} \left| (\mathrm{Id} - \widehat{\Pi}_h)\widehat{u}) \right|^2_{H^1(\Omega)}
\end{aligned}
$$

Notice that $||(\mathrm{Id} - \widehat{\Pi}_h)(\cdot)||_{L^2(\tau)}$ and $|(\mathrm{Id} - \widehat{\Pi}_h)(\cdot)|_{H^1(\tau)}$ are both sublinear functionals defined on $H^2(\widehat{\tau})$ and are zero exactly zero for linear polynomials on τ, therefore the Bramble-Hilbert lemma can be applied.

Recall: $\|B\| \leq ch$, $\frac{|\widehat{\tau}|}{|\tau|} \to ch^{-2}$

$$\sum_{\tau\in\mathcal{T}_h} ||u - \Pi_h u||^2_{H^1(\tau)} \quad \leq \quad ch^2 \sum_{\tau\in\mathcal{T}_h} |\widehat{u}|^2_{H^2(\widehat{\tau})} + c \sum_{\tau\in\mathcal{T}_h} |\widehat{u}|^2_{H^2(\widehat{\tau})}$$

Recall: $\|B\| \leq ch$, $\frac{|\widehat{\tau}|}{|\tau|} \rightarrow ch^{-2}$

$$\begin{aligned}
\sum_{\tau\in\mathcal{T}_h} ||u-\Pi_h u||^2_{H^1(\tau)} &\leq ch^2 \sum_{\tau\in\mathcal{T}_h} |\widehat{u}|^2_{H^2(\widehat{\tau})} + c \sum_{\tau\in\mathcal{T}_h} |\widehat{u}|^2_{H^2(\widehat{\tau})} \\
&\leq c(h^2+1) \sum_{\tau\in\mathcal{T}_h} \int_{\widehat{\tau}} \sum_{|\alpha|=2} |\widehat{D}^\alpha \widehat{u}|^2 \, d\widehat{p}
\end{aligned}$$

Recall: $\|B\| \leq ch$, $\frac{|\widehat{\tau}|}{|\tau|} \to ch^{-2}$

$$
\begin{aligned}
\sum_{\tau \in \mathcal{T}_h} ||u - \Pi_h u||^2_{H^1(\tau)} &\leq ch^2 \sum_{\tau \in \mathcal{T}_h} |\widehat{u}|^2_{H^2(\widehat{\tau})} + c \sum_{\tau \in \mathcal{T}_h} |\widehat{u}|^2_{H^2(\widehat{\tau})} \\
&\leq c(h^2 + 1) \sum_{\tau \in \mathcal{T}_h} \int_{\widehat{\tau}} \sum_{|\alpha|=2} |\widehat{D}^\alpha \widehat{u}|^2 \, d\widehat{p} \\
\text{(Lemma 4.23(ii))} \quad &\leq c(h^2 + 1) \sum_{\tau \in \mathcal{T}_h} \int_{\tau} \sum_{|\alpha|=2} \underbrace{\|B\|^4}\left|D^\alpha u\right|^2 \underbrace{\frac{|\widehat{\tau}|}{|\tau|}} \, d\boldsymbol{x}
\end{aligned}
$$

Recall: $\|B\| \leq ch$, $\frac{|\widehat{\tau}|}{|\tau|} \to ch^{-2}$

$$
\begin{aligned}
\sum_{\tau\in\mathcal{T}_h} ||u-\Pi_h u||^2_{H^1(\tau)} &\leq ch^2 \sum_{\tau\in\mathcal{T}_h} |\widehat{u}|^2_{H^2(\widehat{\tau})} + c \sum_{\tau\in\mathcal{T}_h} |\widehat{u}|^2_{H^2(\widehat{\tau})} \\
&\leq c(h^2+1) \sum_{\tau\in\mathcal{T}_h} \int_{\widehat{\tau}} \sum_{|\alpha|=2} |\widehat{D}^\alpha \widehat{u}|^2 \, d\widehat{p} \\
\text{(Lemma 4.23(ii))} \quad &\leq c(h^2+1) \sum_{\tau\in\mathcal{T}_h} \int_{\tau} \sum_{|\alpha|=2} \underbrace{\|B\|^4}\, |D^\alpha u|^2 \underbrace{\frac{|\widehat{\tau}|}{|\tau|}} \, d\mathbf{x} \\
&\leq c(h^2+1) \sum_{\tau\in\mathcal{T}_h} \int_{\tau} h^2 \sum_{|\alpha|=2} |D^\alpha u|^2 \, d\mathbf{x} \\
&= c(h^4+h^2) \sum_{\tau\in\mathcal{T}_h} \int_{\tau} \sum_{|\alpha|=2} |D^\alpha u|^2 \, d\mathbf{x}
\end{aligned}
$$

Recall: $\|B\| \le ch$, $\frac{|\widehat{\tau}|}{|\tau|} \to ch^{-2}$

$$
\begin{aligned}
\sum_{\tau\in\mathcal{T}_h} ||u-\Pi_h u||^2_{H^1(\tau)} &\le ch^2 \sum_{\tau\in\mathcal{T}_h} |\widehat{u}|^2_{H^2(\widehat{\tau})} + c\sum_{\tau\in\mathcal{T}_h} |\widehat{u}|^2_{H^2(\widehat{\tau})} \\
&\le c(h^2+1)\sum_{\tau\in\mathcal{T}_h}\int_{\widehat{\tau}}\sum_{|\alpha|=2} |\widehat{D}^\alpha \widehat{u}|^2\, d\widehat{p} \\
\text{(Lemma 4.23(ii))} \quad &\le c(h^2+1)\sum_{\tau\in\mathcal{T}_h}\int_{\tau}\sum_{|\alpha|=2} \underbrace{\|B\|^4}\, |D^\alpha u|^2 \underbrace{\frac{|\widehat{\tau}|}{|\tau|}}\, d\boldsymbol{x} \\
&\le c(h^2+1)\sum_{\tau\in\mathcal{T}_h}\int_{\tau} h^2 \sum_{|\alpha|=2} |D^\alpha u|^2\, d\boldsymbol{x} \\
&= c(h^4+h^2)\sum_{\tau\in\mathcal{T}_h}\int_{\tau}\sum_{|\alpha|=2} |D^\alpha u|^2\, d\boldsymbol{x} \\
&\le ch^2 |u|^2_{H^2(\Omega)}.
\end{aligned}
$$

This implies

$$
\|u-\Pi_h u\|_{H^1(\Omega)} \le ch\,|u|_{H^2(\Omega)}\,.
$$

We now want to bound the **consistency** term: $\|F(\cdot) - F_h(\cdot)\|_{*,h}$. (**see pg 247 in Grossman book for similar process**)

We now want to bound the **consistency** term: $\|F(\cdot) - F_h(\cdot)\|_{*,h}$. (**see pg 247 in Grossman book for similar process**) Recall that $F(v) = \int_\Omega fv\, dx + \int_{\partial\Omega} gv\, ds$ for all $v \in H^1(\Omega)$. Also recall that $S_h \subset H^1(\Omega)$.

We now want to bound the **consistency** term: $\|F(\cdot) - F_h(\cdot)\|_{*,h}$. (**see pg 247 in Grossman book for similar process**) Recall that $F(v) = \int_\Omega fv\, dx + \int_{\partial\Omega} gv\, ds$ for all $v \in H^1(\Omega)$. Also recall that $S_h \subset H^1(\Omega)$. Let us define the following **error functionals**:

We now want to bound the **consistency** term: $\|F(\cdot) - F_h(\cdot)\|_{*,h}$. (**see pg 247 in Grossman book for similar process**) Recall that $F(v) = \int_\Omega fv\, dx + \int_{\partial\Omega} gv\, ds$ for all $v \in H^1(\Omega)$. Also recall that $S_h \subset H^1(\Omega)$. Let us define the following **error functionals**:

$$E_h(z) := \int_\Omega z(\boldsymbol{x})\, d\boldsymbol{x} - \sum_{\tau \in \mathcal{T}_h} \frac{|\tau|}{3} \sum_{i=1}^{3} z(P_i) \tag{1.11}$$

$$G_h(z) := \int_{\partial\Omega} z\, ds - \sum_{e \in \partial\Omega} \frac{|e|}{2}(z(\alpha) + z(\beta)) \tag{1.12}$$

We now want to bound the **consistency** term: $\|F(\cdot) - F_h(\cdot)\|_{*,h}$. (**see pg 247 in Grossman book for similar process**) Recall that $F(v) = \int_\Omega fv\,dx + \int_{\partial\Omega} gv\,ds$ for all $v \in H^1(\Omega)$. Also recall that $S_h \subset H^1(\Omega)$. Let us define the following **error functionals**:

$$E_h(z) := \int_\Omega z(\boldsymbol{x})\,d\boldsymbol{x} - \sum_{\tau \in \mathcal{T}_h} \frac{|\tau|}{3} \sum_{i=1}^{3} z(P_i) \tag{1.11}$$

$$G_h(z) := \int_{\partial\Omega} z\,ds - \sum_{e \in \partial\Omega} \frac{|e|}{2}(z(\alpha) + z(\beta)) \tag{1.12}$$

Note that $E_h : H^1(\Omega) \to \mathbb{R}$ and is 0 for constant functions. That is, $E_h(p) = 0$ for $p \in P_0$.

We now want to bound the **consistency** term: $\|F(\cdot) - F_h(\cdot)\|_{*,h}$. (**see pg 247 in Grossman book for similar process**) Recall that $F(v) = \int_\Omega fv \, dx + \int_{\partial\Omega} gv \, ds$ for all $v \in H^1(\Omega)$. Also recall that $S_h \subset H^1(\Omega)$. Let us define the following **error functionals**:

$$E_h(z) := \int_\Omega z(\boldsymbol{x}) \, d\boldsymbol{x} - \sum_{\tau \in \mathcal{T}_h} \frac{|\tau|}{3} \sum_{i=1}^{3} z(P_i) \tag{1.11}$$

$$G_h(z) := \int_{\partial\Omega} z \, ds - \sum_{e \in \partial\Omega} \frac{|e|}{2} (z(\alpha) + z(\beta)) \tag{1.12}$$

Note that $E_h : H^1(\Omega) \to \mathbb{R}$ and is 0 for constant functions. That is, $E_h(p) = 0$ for $p \in P_0$. Let us first measure the error $|E_h(fv_h)|$.

We now want to bound the **consistency** term: $\|F(\cdot) - F_h(\cdot)\|_{*,h}$. (**see pg 247 in Grossman book for similar process**) Recall that $F(v) = \int_\Omega fv\,dx + \int_{\partial\Omega} gv\,ds$ for all $v \in H^1(\Omega)$. Also recall that $S_h \subset H^1(\Omega)$. Let us define the following **error functionals**:

$$E_h(z) := \int_\Omega z(\boldsymbol{x})\,d\boldsymbol{x} - \sum_{\tau\in\mathcal{T}_h} \frac{|\tau|}{3} \sum_{i=1}^{3} z(P_i) \tag{1.11}$$

$$G_h(z) := \int_{\partial\Omega} z\,ds - \sum_{e\in\partial\Omega} \frac{|e|}{2}(z(\alpha) + z(\beta)) \tag{1.12}$$

Note that $E_h : H^1(\Omega) \to \mathbb{R}$ and is 0 for constant functions. That is, $E_h(p) = 0$ for $p \in P_0$. Let us first measure the error $|E_h(fv_h)|$. **The strategy is the same: we transform to the reference element and apply Bramble-Hilbert.**

We have that:

$$|E_h(fv_h)| = \left| \sum_{\tau \in \mathcal{T}_h} \int_\tau f v_h \, d\boldsymbol{x} - \frac{|\tau|}{3} \sum_{i=1}^{3} f(P_i) v_h(P_i) \right| \tag{1.13}$$

We have that:

$$
\begin{aligned}
|E_h(fv_h)| &= \left| \sum_{\tau\in\mathcal{T}_h} \int_\tau fv_h \, d\boldsymbol{x} - \frac{|\tau|}{3} \sum_{i=1}^{3} f(P_i)v_h(P_i) \right| && (1.13) \\
&\leq \sum_{\tau\in\mathcal{T}_h} \left| \int_\tau fv_h \, d\boldsymbol{x} - \frac{|\tau|}{3} \sum_{i=1}^{3} f(P_i)v_h(P_i) \right| && (1.14)
\end{aligned}
$$

We have that:

$$
\begin{aligned}
|E_h(fv_h)| &= \left|\sum_{\tau\in\mathcal{T}_h}\int_\tau fv_h\,d\boldsymbol{x} - \frac{|\tau|}{3}\sum_{i=1}^{3} f(P_i)v_h(P_i)\right| && (1.13)\\
&\le \sum_{\tau\in\mathcal{T}_h}\left|\int_\tau fv_h\,d\boldsymbol{x} - \frac{|\tau|}{3}\sum_{i=1}^{3} f(P_i)v_h(P_i)\right| && (1.14)\\
&= \sum_{\tau\in\mathcal{T}_h}\underbrace{\frac{|\tau|}{|\widehat{\tau}|}}\underbrace{\left|\int_{\widehat{\tau}} \widehat{f}\,\widehat{v_h}\,d\widehat{p} - \underbrace{\frac{|\widehat{\tau}|}{3}}\sum_{i=1}^{3} \widehat{f}(\widehat{P}_i)\widehat{v}_h(\widehat{P}_i)\right|}_{\widehat{E_h}(\widehat{f}\,\widehat{v_h})} && (1.15)
\end{aligned}
$$

We have that:

$$
\begin{aligned}
|E_h(fv_h)| &= \left| \sum_{\tau\in\mathcal{T}_h} \int_\tau fv_h \, d\boldsymbol{x} - \frac{|\tau|}{3}\sum_{i=1}^{3} f(P_i)v_h(P_i) \right| && (1.13)\\
&\le \sum_{\tau\in\mathcal{T}_h} \left| \int_\tau fv_h \, d\boldsymbol{x} - \frac{|\tau|}{3}\sum_{i=1}^{3} f(P_i)v_h(P_i) \right| && (1.14)\\
&= \sum_{\tau\in\mathcal{T}_h} \underbrace{\frac{|\tau|}{|\widehat{\tau}|}} \underbrace{\left| \int_{\widehat{\tau}} \widehat{f}\,\widehat{v_h} \, d\widehat{p} - \underbrace{\frac{|\widehat{\tau}|}{3}} \sum_{i=1}^{3} \widehat{f}(\widehat{P}_i)\widehat{v}_h(\widehat{P}_i) \right|}_{\widehat{E_h}(\widehat{f}\,\widehat{v_h})} && (1.15)\\
(B.H) \quad &\le ch^2 \sum_{\tau\in\mathcal{T}_h} \left| \widehat{f}\,\widehat{v_h} \right|_{H^1(\widehat{\tau})} && (1.16)
\end{aligned}
$$

We have that:

$$
\begin{aligned}
|E_h(fv_h)| &= \left| \sum_{\tau\in\mathcal{T}_h} \int_\tau fv_h \, d\boldsymbol{x} - \frac{|\tau|}{3} \sum_{i=1}^{3} f(P_i) v_h(P_i) \right| && (1.13) \\
&\leq \sum_{\tau\in\mathcal{T}_h} \left| \int_\tau fv_h \, d\boldsymbol{x} - \frac{|\tau|}{3} \sum_{i=1}^{3} f(P_i) v_h(P_i) \right| && (1.14) \\
&= \sum_{\tau\in\mathcal{T}_h} \underbrace{\frac{|\tau|}{|\widehat{\tau}|}} \underbrace{\left| \int_{\widehat{\tau}} \widehat{f}\,\widehat{v_h} \, d\widehat{p} - \underbrace{\frac{|\widehat{\tau}|}{3}} \sum_{i=1}^{3} \widehat{f}(\widehat{P}_i) \widehat{v}_h(\widehat{P}_i) \right|}_{\widehat{E_h}(\widehat{f}\,\widehat{v_h})} && (1.15) \\
(B.H) &\leq ch^2 \sum_{\tau\in\mathcal{T}_h} \left| \widehat{f}\,\widehat{v_h} \right|_{H^1(\widehat{\tau})} && (1.16) \\
(Lemma\,4.23) &\leq ch^2 \sum_{\tau\in\mathcal{T}_h} \left(\int_\tau h^2 \left| \nabla (fv_h) \right|^2 \frac{|\widehat{\tau}|}{|\tau|} d\,\boldsymbol{x} \right)^{1/2} && (1.17)
\end{aligned}
$$

We have that:

$$
\begin{aligned}
|E_h(fv_h)| &= \left| \sum_{\tau\in\mathcal{T}_h} \int_\tau fv_h \, d\boldsymbol{x} - \frac{|\tau|}{3} \sum_{i=1}^{3} f(P_i)v_h(P_i) \right| && (1.13)\\
&\le \sum_{\tau\in\mathcal{T}_h} \left| \int_\tau fv_h \, d\boldsymbol{x} - \frac{|\tau|}{3} \sum_{i=1}^{3} f(P_i)v_h(P_i) \right| && (1.14)\\
&= \sum_{\tau\in\mathcal{T}_h} \underbrace{\frac{|\tau|}{|\widehat{\tau}|}} \underbrace{\left| \int_{\widehat{\tau}} \widehat{f}\widehat{v_h} \, d\widehat{p} - \underbrace{\frac{|\widehat{\tau}|}{3}} \sum_{i=1}^{3} \widehat{f}(\widehat{P}_i)\widehat{v}_h(\widehat{P}_i) \right|}_{\widehat{E_h}(\widehat{f}\widehat{v_h})} && (1.15)\\
(B.H) \quad &\le ch^2 \sum_{\tau\in\mathcal{T}_h} \left|\widehat{f}\widehat{v_h}\right|_{H^1(\widehat{\tau})} && (1.16)\\
(Lemma\,4.23) \quad &\le ch^2 \sum_{\tau\in\mathcal{T}_h} \left(\int_\tau h^2 \left|\nabla(fv_h)\right|^2 \frac{|\widehat{\tau}|}{|\tau|} d\,\boldsymbol{x} \right)^{1/2} && (1.17)\\
&\le ch^2 \sum_{\tau\in\mathcal{T}_h} |fv_h|_{H^1(\tau)} && (1.18)
\end{aligned}
$$

where we assumed the mesh was **shape-regular** so that $\|B\| \leq h^2$ in (1.17).

Since f is a given function, we can bound $|fv_h|_{H^1(\tau)}$ by $c\|f\|_{W^{1,\infty}(\tau)}\|v_h\|_{H^1(\tau)}$. **HW. Verify the following:** $|fv_h|^2_{H^1(\tau)} \leq c\|f\|^2_{W^{1,\infty}(\tau)}\|v_h\|^2_{H^1(\tau)}$. (*hint: use product rule to expand then use Cauchy-Schwarz*)

Since f is a given function, we can bound $|fv_h|_{H^1(\tau)}$ by $c\|f\|_{W^{1,\infty}(\tau)}\|v_h\|_{H^1(\tau)}$. **HW. Verify the following:** $|fv_h|^2_{H^1(\tau)} \leq c\|f\|^2_{W^{1,\infty}(\tau)}\|v_h\|^2_{H^1(\tau)}$. (*hint: use product rule to expand then use Cauchy-Schwarz*) Thus, we have that

$$|E_h(fv_h)| \leq ch^2 \sum_{\tau\in\mathcal{T}_h} \|f\|_{W^{1,\infty}(\tau)}\|v_h\|_{H^1(\tau)}$$

Since f is a given function, we can bound $|fv_h|_{H^1(\tau)}$ by $c\|f\|_{W^{1,\infty}(\tau)}\|v_h\|_{H^1(\tau)}$. **HW. Verify the following:** $|fv_h|^2_{H^1(\tau)} \le c\|f\|^2_{W^{1,\infty}(\tau)}\|v_h\|^2_{H^1(\tau)}$. (*hint: use product rule to expand then use Cauchy-Schwarz*) Thus, we have that

$$
\begin{aligned}
|E_h(fv_h)| &\le ch^2 \sum_{\tau\in\mathcal{T}_h} \|f\|_{W^{1,\infty}(\tau)}\|v_h\|_{H^1(\tau)} \\
&\le ch^2 \|f\|_{W^{1,\infty}(\Omega)} \sum_{\tau\in\mathcal{T}_h} \|v_h\|_{H^1(\tau)}
\end{aligned}
$$

Since f is a given function, we can bound $|fv_h|_{H^1(\tau)}$ by $c\|f\|_{W^{1,\infty}(\tau)}\|v_h\|_{H^1(\tau)}$. **HW. Verify the following:** $|fv_h|^2_{H^1(\tau)} \leq c\|f\|^2_{W^{1,\infty}(\tau)}\|v_h\|^2_{H^1(\tau)}$. (*hint: use product rule to expand then use Cauchy-Schwarz*) Thus, we have that

$$
\begin{aligned}
|E_h(fv_h)| &\leq ch^2 \sum_{\tau\in\mathcal{T}_h} \|f\|_{W^{1,\infty}(\tau)}\|v_h\|_{H^1(\tau)} \\
&\leq ch^2 \|f\|_{W^{1,\infty}(\Omega)} \sum_{\tau\in\mathcal{T}_h} \|v_h\|_{H^1(\tau)} \\
\text{Use C.S for sum} &\leq ch^2 \|f\|_{W^{1,\infty}(\Omega)} \Big(\sum_{\tau\in\mathcal{T}_h} 1^2\Big)^{1/2} \Big(\sum_{\tau\in\mathcal{T}_h} \|v_h\|^2_{H^1(\tau)}\Big)^{1/2}
\end{aligned}
$$

Since f is a given function, we can bound $|fv_h|_{H^1(\tau)}$ by $c\|f\|_{W^{1,\infty}(\tau)}\|v_h\|_{H^1(\tau)}$. **HW. Verify the following:** $|fv_h|^2_{H^1(\tau)} \leq c\|f\|^2_{W^{1,\infty}(\tau)}\|v_h\|^2_{H^1(\tau)}$. (*hint: use product rule to expand then use Cauchy-Schwarz*) Thus, we have that

$$
\begin{aligned}
|E_h(fv_h)| &\leq ch^2 \sum_{\tau\in\mathcal{T}_h} \|f\|_{W^{1,\infty}(\tau)}\|v_h\|_{H^1(\tau)} \\
&\leq ch^2\|f\|_{W^{1,\infty}(\Omega)} \sum_{\tau\in\mathcal{T}_h} \|v_h\|_{H^1(\tau)} \\
\text{Use C.S for sum} &\leq ch^2\|f\|_{W^{1,\infty}(\Omega)} \Big(\sum_{\tau\in\mathcal{T}_h} 1^2\Big)^{1/2} \Big(\sum_{\tau\in\mathcal{T}_h} \|v_h\|^2_{H^1(\tau)}\Big)^{1/2} \\
&= ch^2\|f\|_{W^{1,\infty}(\Omega)} \Big(|\mathcal{T}_h|\Big)^{1/2} \|v_h\|_{H^1(\Omega)}
\end{aligned}
$$

where $|\mathcal{T}_h|$ denotes the number of elements in $\mathcal{T}_h$.

Since f is a given function, we can bound $|fv_h|_{H^1(\tau)}$ by $c\|f\|_{W^{1,\infty}(\tau)}\|v_h\|_{H^1(\tau)}$. **HW. Verify the following:** $|fv_h|^2_{H^1(\tau)} \leq c\|f\|^2_{W^{1,\infty}(\tau)}\|v_h\|^2_{H^1(\tau)}$. (*hint: use product rule to expand then use Cauchy-Schwarz*) Thus, we have that

$$
\begin{aligned}
|E_h(fv_h)| &\leq ch^2 \sum_{\tau\in\mathcal{T}_h} \|f\|_{W^{1,\infty}(\tau)}\|v_h\|_{H^1(\tau)} \\
&\leq ch^2\|f\|_{W^{1,\infty}(\Omega)} \sum_{\tau\in\mathcal{T}_h} \|v_h\|_{H^1(\tau)} \\
\text{Use C.S for sum} &\leq ch^2\|f\|_{W^{1,\infty}(\Omega)}\Big(\sum_{\tau\in\mathcal{T}_h} 1^2\Big)^{1/2}\Big(\sum_{\tau\in\mathcal{T}_h} \|v_h\|^2_{H^1(\tau)}\Big)^{1/2} \\
&= ch^2\|f\|_{W^{1,\infty}(\Omega)}\Big(|\mathcal{T}_h|\Big)^{1/2}\|v_h\|_{H^1(\Omega)}
\end{aligned}
$$

where $|\mathcal{T}_h|$ denotes the number of elements in $\mathcal{T}_h$. Since we assumed we have a shape-regular mesh $|\mathcal{T}_h| \sim h^{-2}$ (pg 248 of Grossman book)

Thus, we have that

$$|E_h(fv_h)| \leq ch\|f\|_{W^{1,\infty}(\Omega)}\|v_h\|_{H^1(\Omega)} \tag{1.19}$$

Thus, we have that

$$|E_h(fv_h)| \leq ch\|f\|_{W^{1,\infty}(\Omega)}\|v_h\|_{H^1(\Omega)} \tag{1.19}$$

HW. Show that :

$$|G_h(gv_h)| \leq ch\|g\|_{W^{1,\infty}(\partial\Omega)}\|v_h\|_{H^1(\Omega)}.$$

Thus, we have that

$$|E_h(fv_h)| \leq ch\|f\|_{W^{1,\infty}(\Omega)}\|v_h\|_{H^1(\Omega)} \tag{1.19}$$

HW. Show that :

$$|G_h(gv_h)| \leq ch\|g\|_{W^{1,\infty}(\partial\Omega)}\|v_h\|_{H^1(\Omega)}.$$

The process is VERY similar, but requires the trace inequality $\|v_h\|^2_{H^1(e)} \leq c\|v_h\|^2_{H^1(\tau_e)}$ where τ_e is the triangle corresponding to the boundary edge e.

Thus, we have that

$$|E_h(fv_h)| \leq ch\|f\|_{W^{1,\infty}(\Omega)}\|v_h\|_{H^1(\Omega)} \tag{1.19}$$

HW. Show that :

$$|G_h(gv_h)| \leq ch\|g\|_{W^{1,\infty}(\partial\Omega)}\|v_h\|_{H^1(\Omega)}.$$

The process is VERY similar, but requires the trace inequality $\|v_h\|^2_{H^1(e)} \leq c\|v_h\|^2_{H^1(\tau_e)}$ where τ_e is the triangle corresponding to the boundary edge e.

The final estimate is given by:

$$\|u - u_h\|_{H^1(\Omega)} \leq ch\Big(|u|_{H^1(\Omega)} + \|f\|_{W^{1,\infty}(\Omega)} + \|g\|_{W^{1,\infty}(\partial\Omega)}\Big) \tag{1.20}$$

Problem 3

Let $\Omega = (0,1)^2$ and u be the solution of the elliptic problem:

$$-\Delta u + u = f(\boldsymbol{x}) \text{ for } \boldsymbol{x} \in \Omega, \quad u = g(\boldsymbol{x}) \text{ for } \boldsymbol{x} \in \partial\Omega \tag{1.21}$$

[2] Section 2.5 of http://citeseerx.ist.psu.edu/viewdoc/download?doi=10.1.1.

Problem 3

Let $\Omega = (0,1)^2$ and u be the solution of the elliptic problem:

$$-\Delta u + u = f(\boldsymbol{x}) \text{ for } \boldsymbol{x} \in \Omega, \quad u = g(\boldsymbol{x}) \text{ for } \boldsymbol{x} \in \partial\Omega \tag{1.21}$$

(a) Let
$\omega_h = \{\boldsymbol{x} = (x_{1,i}, x_{2,j}) : x_{1,i} = ih,\ x_{2,j} = jh,\ i,j = 0,1,...,N,\ h = 1/N\}$ be a square mesh in Ω.[2]

[2] Section 2.5 of `http://citeseerx.ist.psu.edu/viewdoc/download?doi=10.1.1.`

Problem 3

Let $\Omega = (0,1)^2$ and u be the solution of the elliptic problem:

$$-\Delta u + u = f(\boldsymbol{x}) \text{ for } \boldsymbol{x} \in \Omega, \quad u = g(\boldsymbol{x}) \text{ for } \boldsymbol{x} \in \partial\Omega \tag{1.21}$$

(a) Let
$\omega_h = \{\boldsymbol{x} = (x_{1,i}, x_{2,j}) : x_{1,i} = ih,\ x_{2,j} = jh,\ i,j = 0,1,...,N,\ h = 1/N\}$ be a square mesh in Ω.[2]

(b) Show that

$$\max_{\boldsymbol{x}\in\omega_h} |U(\boldsymbol{x})| \leq \max_{\boldsymbol{x}\in\omega_h\cap\partial\Omega} |g(\boldsymbol{x})| + \max_{\boldsymbol{x}\in\omega_h} |f(\boldsymbol{x})|.$$

[2] Section 2.5 of `http://citeseerx.ist.psu.edu/viewdoc/download?doi=10.1.1.`

Problem 3

Let $\Omega = (0,1)^2$ and u be the solution of the elliptic problem:

$$-\Delta u + u = f(\boldsymbol{x}) \text{ for } \boldsymbol{x} \in \Omega, \quad u = g(\boldsymbol{x}) \text{ for } \boldsymbol{x} \in \partial\Omega \tag{1.21}$$

(a) Let
$\omega_h = \{\boldsymbol{x} = (x_{1,i}, x_{2,j}) : x_{1,i} = ih,\ x_{2,j} = jh,\ i,j = 0,1,...,N,\ h = 1/N\}$ be a square mesh in Ω.[2]

(b) Show that

$$\max_{\boldsymbol{x}\in\omega_h} |U(\boldsymbol{x})| \leq \max_{\boldsymbol{x}\in\omega_h\cap\partial\Omega} |g(\boldsymbol{x})| + \max_{\boldsymbol{x}\in\omega_h} |f(\boldsymbol{x})|.$$

(c) Using this a priori estimate and the estimation of the local truncation

error in a. conclude that for sufficiently smooth solution $u(\boldsymbol{x})$ the following error estimate (with a constant independent of h):

$$\max_{x\in\omega_h} |U(\boldsymbol{x}) - u(\boldsymbol{x})| \leq Ch^2 \tag{1.22}$$

[2] Section 2.5 of `http://citeseerx.ist.psu.edu/viewdoc/download?doi=10.1.1.`

Solution to part (a)

For notation purposes, let $x := x_1$ and $y = x_2$.

Solution to part (a)

For notation purposes, let $x := x_1$ and $y = x_2$. We use the forward and backward finite difference approximation for the second derivative operators:

$$\frac{\partial^2 u}{\partial x^2} \approx \frac{U_{i+1,j} - 2U_{i,j} + U_{i-1,j}}{h^2}, \quad \frac{\partial^2 u}{\partial y^2} \approx \frac{U_{i,j+1} - 2U_{i,j} + U_{i,j-1}}{h^2}$$

Solution to part (a)

For notation purposes, let $x := x_1$ and $y = x_2$. We use the forward and backward finite difference approximation for the second derivative operators:

$$\frac{\partial^2 u}{\partial x^2} \approx \frac{U_{i+1,j} - 2U_{i,j} + U_{i-1,j}}{h^2}, \quad \frac{\partial^2 u}{\partial y^2} \approx \frac{U_{i,j+1} - 2U_{i,j} + U_{i,j-1}}{h^2}$$

We can substitute this into (1.21) to obtain the 5-point stencil finite difference scheme:

$$-\Big(\frac{U_{i+1,j} - 2U_{i,j} + U_{i-1,j}}{h^2} + \frac{U_{i,j+1} - 2U_{i,j} + U_{i,j-1}}{h^2}\Big) + U_{i,j} = f(x_i, y_j)$$

The previous equation can be simplified as follows:

$$-\frac{1}{h^2}(U_{i+1,j} + U_{i-1,j} + U_{i,j+1} + U_{i,j-1} - 4U_{i,j}) + U_{i,j} = f(x_i, y_j)$$

The previous equation can be simplified as follows:

$$-\frac{1}{h^2}(U_{i+1,j} + U_{i-1,j} + U_{i,j+1} + U_{i,j-1} - 4U_{i,j}) + U_{i,j} = f(x_i, y_j)$$

We now want to derive the local truncation error (LTE).

The previous equation can be simplified as follows:

$$-\frac{1}{h^2}(U_{i+1,j} + U_{i-1,j} + U_{i,j+1} + U_{i,j-1} - 4U_{i,j}) + U_{i,j} = f(x_i, y_j)$$

We now want to derive the local truncation error (LTE). THe LTE is defined as the residual after replacing the numerical solution U by the true solution u in the numerical scheme evaluated at (x_i, y_j).

We first expand $u(x, y)$ in a Taylor series about (x_i, y_j) assuming that u is sufficiently smooth:

We first expand $u(x, y)$ in a Taylor series about (x_i, y_j) assuming that u is sufficiently smooth:

$$
\begin{aligned}
u(x,y) \approx\ & u(x_i,y_j) + \frac{\partial u}{\partial x}(x_i,y_j)(x-x_i) + \frac{\partial u}{\partial y}(x_i,y_j)(y-y_j) \\
& + \frac{1}{2}\Big[\frac{\partial^2 u}{\partial x^2}(x_i,y_j)(x-x_i)^2 + 2\frac{\partial^2 u}{\partial x \partial y}(x_i,y_j)(x-x_i)(y-y_j) + \frac{\partial^2 u}{\partial y^2}(x_i,y_j)(y- \\
& + \frac{1}{6}\Big[\frac{\partial^3 u}{\partial x^3}(x_i,y_j)(x-x_i)^3 + 3\frac{\partial^3 u}{\partial x^2 \partial y}(x_i,y_j)(x-x_i)^2(y-y_j) \\
& + 3\frac{\partial^3 u}{\partial x \partial y^2}(x_i,y_j)(x-x_i)(y-y_j)^2 + \frac{\partial^3 u}{\partial y^3}(x_i,y_j)(y-y_j)^3\Big] + \text{H.O.T.}
\end{aligned}
$$

where "H.O.T." represents "higher order terms".

We first expand $u(x, y)$ in a Taylor series about (x_i, y_j) assuming that u is sufficiently smooth:

$$
\begin{aligned}
u(x,y) \approx u(x_i, y_j) &+ \frac{\partial u}{\partial x}(x_i, y_j)(x - x_i) + \frac{\partial u}{\partial y}(x_i, y_j)(y - y_j) \\
&+ \frac{1}{2}\Big[\frac{\partial^2 u}{\partial x^2}(x_i, y_j)(x - x_i)^2 + 2\frac{\partial^2 u}{\partial x \partial y}(x_i, y_j)(x - x_i)(y - y_j) + \frac{\partial^2 u}{\partial y^2}(x_i, y_j)(y - \\
&+ \frac{1}{6}\Big[\frac{\partial^3 u}{\partial x^3}(x_i, y_j)(x - x_i)^3 + 3\frac{\partial^3 u}{\partial x^2 \partial y}(x_i, y_j)(x - x_i)^2(y - y_j) \\
&+ 3\frac{\partial^3 u}{\partial x \partial y^2}(x_i, y_j)(x - x_i)(y - y_j)^2 + \frac{\partial^3 u}{\partial y^3}(x_i, y_j)(y - y_j)^3\Big] + \text{H.O.T.}
\end{aligned}
$$

where "H.O.T." represents "higher order terms". Now we need to compute the terms in the 5-point stencil using the expansion above.

We first expand $u(x,y)$ in a Taylor series about (x_i, y_j) assuming that u is sufficiently smooth:

$$
\begin{aligned}
u(x,y) \approx{}& u(x_i,y_j) + \frac{\partial u}{\partial x}(x_i,y_j)(x-x_i) + \frac{\partial u}{\partial y}(x_i,y_j)(y-y_j) \\
&+ \frac{1}{2}\Big[\frac{\partial^2 u}{\partial x^2}(x_i,y_j)(x-x_i)^2 + 2\frac{\partial^2 u}{\partial x \partial y}(x_i,y_j)(x-x_i)(y-y_j) + \frac{\partial^2 u}{\partial y^2}(x_i,y_j)(y - \\
&+ \frac{1}{6}\Big[\frac{\partial^3 u}{\partial x^3}(x_i,y_j)(x-x_i)^3 + 3\frac{\partial^3 u}{\partial x^2 \partial y}(x_i,y_j)(x-x_i)^2(y-y_j) \\
&+ 3\frac{\partial^3 u}{\partial x \partial y^2}(x_i,y_j)(x-x_i)(y-y_j)^2 + \frac{\partial^3 u}{\partial y^3}(x_i,y_j)(y-y_j)^3\Big] + \text{H.O.T.}
\end{aligned}
$$

where "H.O.T." represents "higher order terms". Now we need to compute the terms in the 5-point stencil using the expansion above. For example, to compute $u(x_{i+1}, y_j)$ substitute (x_{i+1}, y_j). Notice that all the $y_j - y_j$ terms will drop out and the terms $x_{i+1} - x_i$ will produce powers of h.

We see that:

We see that:

$$u(x_{i+1}, y_j) = u(x_i, y_j) + h\frac{\partial u}{\partial x}(x_i, y_j) + \frac{1}{2}h^2\frac{\partial^2 u}{\partial x^2}(x_i, y_j) + \frac{1}{6}h^3\frac{\partial^3 u}{\partial x^3}(x_i, y_j) + \mathcal{O}(h^4)$$

We see that:

$$u(x_{i+1}, y_j) = u(x_i, y_j) + h\frac{\partial u}{\partial x}(x_i, y_j) + \frac{1}{2}h^2\frac{\partial^2 u}{\partial x^2}(x_i, y_j) + \frac{1}{6}h^3\frac{\partial^3 u}{\partial x^3}(x_i, y_j) + \mathcal{O}(h^4)$$
$$u(x_{i-1}, y_j) = u(x_i, y_j) - h\frac{\partial u}{\partial x}(x_i, y_j) + \frac{1}{2}h^2\frac{\partial^2 u}{\partial x^2}(x_i, y_j) - \frac{1}{6}h^3\frac{\partial^3 u}{\partial x^3}(x_i, y_j) + \mathcal{O}(h^4)$$

We see that:

$$u(x_{i+1}, y_j) = u(x_i, y_j) + h\frac{\partial u}{\partial x}(x_i, y_j) + \frac{1}{2}h^2\frac{\partial^2 u}{\partial x^2}(x_i, y_j) + \frac{1}{6}h^3\frac{\partial^3 u}{\partial x^3}(x_i, y_j) + \mathcal{O}(h^4)$$
$$u(x_{i-1}, y_j) = u(x_i, y_j) - h\frac{\partial u}{\partial x}(x_i, y_j) + \frac{1}{2}h^2\frac{\partial^2 u}{\partial x^2}(x_i, y_j) - \frac{1}{6}h^3\frac{\partial^3 u}{\partial x^3}(x_i, y_j) + \mathcal{O}(h^4)$$
$$u(x_i, y_{j+1}) = u(x_i, y_j) + h\frac{\partial u}{\partial y}(x_i, y_j) + \frac{1}{2}h^2\frac{\partial^2 u}{\partial y^2}(x_i, y_j) + \frac{1}{6}h^3\frac{\partial^3 u}{\partial y^3}(x_i, y_j) + \mathcal{O}(h^4)$$

We see that:

$$u(x_{i+1}, y_j) = u(x_i, y_j) + h\frac{\partial u}{\partial x}(x_i, y_j) + \frac{1}{2}h^2\frac{\partial^2 u}{\partial x^2}(x_i, y_j) + \frac{1}{6}h^3\frac{\partial^3 u}{\partial x^3}(x_i, y_j) + \mathcal{O}(h^4)$$
$$u(x_{i-1}, y_j) = u(x_i, y_j) - h\frac{\partial u}{\partial x}(x_i, y_j) + \frac{1}{2}h^2\frac{\partial^2 u}{\partial x^2}(x_i, y_j) - \frac{1}{6}h^3\frac{\partial^3 u}{\partial x^3}(x_i, y_j) + \mathcal{O}(h^4)$$
$$u(x_i, y_{j+1}) = u(x_i, y_j) + h\frac{\partial u}{\partial y}(x_i, y_j) + \frac{1}{2}h^2\frac{\partial^2 u}{\partial y^2}(x_i, y_j) + \frac{1}{6}h^3\frac{\partial^3 u}{\partial y^3}(x_i, y_j) + \mathcal{O}(h^4)$$
$$u(x_i, y_{j-1}) = u(x_i, y_j) - h\frac{\partial u}{\partial y}(x_i, y_j) + \frac{1}{2}h^2\frac{\partial^2 u}{\partial y^2}(x_i, y_j) - \frac{1}{6}h^3\frac{\partial^3 u}{\partial y^3}(x_i, y_j) + \mathcal{O}(h^4)$$

A direct substitution (with a lot of messy algebra) yields

$$-\Delta u(x_i, y_j) + u(x_i, y_j) + \mathcal{O}(h^2) = f(x_i, y_j) \tag{1.23}$$

A direct substitution (with a lot of messy algebra) yields

$$-\Delta u(x_i, y_j) + u(x_i, y_j) + \mathcal{O}(h^2) = f(x_i, y_j) \tag{1.23}$$

Thus, our LTE $\tau_{i,j} = f(x_i, y_j) - (-\Delta u(x_i, y_j) + u(x_i, y_j)) = \mathcal{O}(h^2)$ as $h \to 0$.

Solution to (b)

Show that

$$\max_{x\in\omega_h} |U(x)| \leq \max_{x\in\omega_h\cap\partial\Omega} |g(x)| + \max_{x\in\omega_h} |f(x)|.$$

Solution to (b)

There are two cases that can occur: (i) the maximum occurs on the boundary; (ii) the maximum occurs on the interior $\omega_h \setminus \partial\Omega$.

Solution to (b)

There are two cases that can occur: (i) the maximum occurs on the boundary; (ii) the maximum occurs on the interior $\omega_h \setminus \partial\Omega$. Let us focus on the first case.

Solution to (b)

There are two cases that can occur: (i) the maximum occurs on the boundary; (ii) the maximum occurs on the interior $\omega_h \setminus \partial\Omega$. Let us focus on the first case. Recall that our boundary condition gives that $u = g(\boldsymbol{x})$ on $\boldsymbol{x} \in \partial\Omega$.

Solution to (b)

There are two cases that can occur: (i) the maximum occurs on the boundary; (ii) the maximum occurs on the interior $\omega_h \setminus \partial\Omega$. Let us focus on the first case. Recall that our boundary condition gives that $u = g(\boldsymbol{x})$ on $\boldsymbol{x} \in \partial\Omega$. Thus, we see that

$$\max_{x \in \omega_h} |U(\boldsymbol{x})| = \max_{x \in \omega_h \cap \partial\Omega} |U(\boldsymbol{x})|$$

Solution to (b)

There are two cases that can occur: (i) the maximum occurs on the boundary; (ii) the maximum occurs on the interior $\omega_h \setminus \partial\Omega$. Let us focus on the first case. Recall that our boundary condition gives that $u = g(\boldsymbol{x})$ on $\boldsymbol{x} \in \partial\Omega$. Thus, we see that

$$\max_{x\in\omega_h} |U(\boldsymbol{x})| = \max_{x\in\omega_h\cap\partial\Omega} |U(\boldsymbol{x})| = \max_{x\in\omega_h\cap\partial\Omega} |g(\boldsymbol{x})|$$

Solution to (b)

There are two cases that can occur: (i) the maximum occurs on the boundary; (ii) the maximum occurs on the interior $\omega_h \setminus \partial\Omega$. Let us focus on the first case. Recall that our boundary condition gives that $u = g(\boldsymbol{x})$ on $\boldsymbol{x} \in \partial\Omega$. Thus, we see that

$$\max_{x\in\omega_h} |U(\boldsymbol{x})| = \max_{x\in\omega_h\cap\partial\Omega} |U(\boldsymbol{x})| = \max_{x\in\omega_h\cap\partial\Omega} |g(\boldsymbol{x})| \leq \max_{x\in\omega_h\cap\partial\Omega} |g(\boldsymbol{x})| + \underbrace{\max_{x\in\omega_h} |f(\boldsymbol{x})}_{\geq 0}$$

Now, let us assume that the maximum occurs in the interior $\omega_h \setminus \partial\Omega$.

Now, let us assume that the maximum occurs in the interior $\omega_h \setminus \partial\Omega$. Recall our 5-point stencil finite difference scheme:

$$-\frac{1}{h^2}(U_{i+1,j} + U_{i-1,j} + U_{i,j+1} + U_{i,j-1} - 4U_{i,j}) + U_{i,j} = f(x_i, y_j).$$

Now, let us assume that the maximum occurs in the interior $\omega_h \setminus \partial\Omega$.
Recall our 5-point stencil finite difference scheme:

$$-\frac{1}{h^2}(U_{i+1,j} + U_{i-1,j} + U_{i,j+1} + U_{i,j-1} - 4U_{i,j}) + U_{i,j} = f(x_i, y_j).$$

This can be rewritten as

$$\begin{aligned}
(4 + h^2)U_{i,j} \quad &= \quad h^2 f_{i,j} + U_{i+1,j} + U_{i-1,j} + U_{i,j+1} + U_{i,j-1} \\
&\Longrightarrow \\
(4 + h^2)\left|U_{i,j}\right| \quad &\leq \quad \left|h^2 f_{i,j} + U_{i+1,j} + U_{i-1,j} + U_{i,j+1} + U_{i,j-1}\right| \\
&\leq \quad h^2\left|f_{i,j}\right| + \left|U_{i+1,j}\right| + \left|U_{i-1,j}\right| + \left|U_{i,j+1}\right| + \left|U_{i,j-1}\right|
\end{aligned}$$

Now, let us assume that the maximum occurs in the interior $\omega_h \setminus \partial\Omega$. Recall our 5-point stencil finite difference scheme:

$$-\frac{1}{h^2}(U_{i+1,j} + U_{i-1,j} + U_{i,j+1} + U_{i,j-1} - 4U_{i,j}) + U_{i,j} = f(x_i, y_j).$$

This can be rewritten as

$$\begin{aligned}
(4+h^2)U_{i,j} &= h^2 f_{i,j} + U_{i+1,j} + U_{i-1,j} + U_{i,j+1} + U_{i,j-1} \\
&\Longrightarrow \\
(4+h^2)\left|U_{i,j}\right| &\leq \left|h^2 f_{i,j} + U_{i+1,j} + U_{i-1,j} + U_{i,j+1} + U_{i,j-1}\right| \\
&\leq h^2\left|f_{i,j}\right| + \left|U_{i+1,j}\right| + \left|U_{i-1,j}\right| + \left|U_{i,j+1}\right| + \left|U_{i,j-1}\right|
\end{aligned}$$

Then, we take the maximum over $0 \leq i, j \leq N$:

$$(4+h^2)\max_{0\leq i,j\leq N}\left|U_{i,j}\right| \leq h^2 \max_{x\in\omega_h}\left|f(x)\right| + 4\max_{0\leq i,j\leq N}\left|U_{i,j}\right|$$

Now, let us assume that the maximum occurs in the interior $\omega_h \setminus \partial\Omega$. Recall our 5-point stencil finite difference scheme:

$$-\frac{1}{h^2}(U_{i+1,j} + U_{i-1,j} + U_{i,j+1} + U_{i,j-1} - 4U_{i,j}) + U_{i,j} = f(x_i, y_j).$$

This can be rewritten as

$$\begin{aligned}
(4+h^2)U_{i,j} &= h^2 f_{i,j} + U_{i+1,j} + U_{i-1,j} + U_{i,j+1} + U_{i,j-1} \\
&\implies \\
(4+h^2)\left|U_{i,j}\right| &\leq \left|h^2 f_{i,j} + U_{i+1,j} + U_{i-1,j} + U_{i,j+1} + U_{i,j-1}\right| \\
&\leq h^2\left|f_{i,j}\right| + \left|U_{i+1,j}\right| + \left|U_{i-1,j}\right| + \left|U_{i,j+1}\right| + \left|U_{i,j-1}\right|
\end{aligned}$$

Then, we take the maximum over $0 \leq i, j \leq N$:

$$(4+h^2)\max_{0\leq i,j\leq N}\left|U_{i,j}\right| \leq h^2 \max_{x\in\omega_h}\left|f(x)\right| + 4\max_{0\leq i,j\leq N}\left|U_{i,j}\right|$$

This implies that

$$\max_{0\leq i,j\leq N}\left|U_{i,j}\right| \leq \max_{x\in\omega_h}\left|f(x)\right| \leq \max_{x\in\omega_h}\left|f(x)\right| + \max_{x\in\omega_h\cap\partial\Omega}\left|g(x)\right|$$

Solution to (c)

Using this a priori estimate and the estimation of the local truncation error in a. conclude that for sufficiently smooth solution $u(\boldsymbol{x})$ the following error estimate (with a constant independent of h):

$$\max_{x \in \omega_h} |U(\boldsymbol{x}) - u(\boldsymbol{x})| \leq Ch^2 \tag{1.24}$$

Assume that $u(\boldsymbol{x})$ is sufficiently smooth. **The goal is to formulate a BVP for the difference $U - u$, so we can use the previous results.**

Solution to (c)

Assume that $u(\boldsymbol{x})$ is sufficiently smooth. **The goal is to formulate a BVP for the difference $U - u$, so we can use the previous results.** Let us define the following continuous and discrete operators:

$$
\begin{aligned}
Lu &:= -\Delta u + u \\
L_h U &:= -\frac{1}{h^2}(U_{i+1,j} + U_{i-1,j} + U_{i,j+1} + U_{i,j-1} - 4U_{i,j}) + U_{i,j}.
\end{aligned}
$$

Solution to (c)

Assume that $u(\boldsymbol{x})$ is sufficiently smooth. **The goal is to formulate a BVP for the difference $U - u$, so we can use the previous results.** Let us define the following continuous and discrete operators:

$$
\begin{aligned}
Lu &:= -\Delta u + u \\
L_h U &:= -\frac{1}{h^2}(U_{i+1,j} + U_{i-1,j} + U_{i,j+1} + U_{i,j-1} - 4U_{i,j}) + U_{i,j}.
\end{aligned}
$$

Note that for $\boldsymbol{x} \in \omega_h$:

$$
L_h U - L_h u = L_h U - Lu + Lu - L_h u
$$

Solution to (c)

Assume that $u(\boldsymbol{x})$ is sufficiently smooth. **The goal is to formulate a BVP for the difference $U - u$, so we can use the previous results.** Let us define the following continuous and discrete operators:

$$
\begin{aligned}
Lu &:= -\Delta u + u \\
L_h U &:= -\frac{1}{h^2}(U_{i+1,j} + U_{i-1,j} + U_{i,j+1} + U_{i,j-1} - 4U_{i,j}) + U_{i,j}.
\end{aligned}
$$

Note that for $\boldsymbol{x} \in \omega_h$:

$$
\begin{aligned}
L_h U - L_h u &= L_h U - Lu + Lu - L_h u \\
&= \underbrace{f - f}_{=0} + Lu - L_h u
\end{aligned}
$$

Solution to (c)

Assume that $u(\boldsymbol{x})$ is sufficiently smooth. **The goal is to formulate a BVP for the difference $U - u$, so we can use the previous results.** Let us define the following continuous and discrete operators:

$$\begin{aligned} Lu &:= -\Delta u + u \\ L_h U &:= -\frac{1}{h^2}(U_{i+1,j} + U_{i-1,j} + U_{i,j+1} + U_{i,j-1} - 4U_{i,j}) + U_{i,j}. \end{aligned}$$

Note that for $\boldsymbol{x} \in \omega_h$:

$$\begin{aligned} L_h U - L_h u &= L_h U - Lu + Lu - L_h u \\ &= \underbrace{f - f}_{=0} + Lu - L_h u \\ &= Lu - L_h u \end{aligned}$$

Note that u does not necessarily solve the equation $L_h u = f$.

Note that u does not necessarily solve the equation $L_h u = f$. Also note as a consequence of the BC, we have $U|_{\omega_h \cap \partial\Omega} = u|_{\omega_h \cap \partial\Omega} = g$. So, now we can consider this BVP for the operator L_h:

Note that u does not necessarily solve the equation $L_h u = f$. Also note as a consequence of the BC, we have $U|_{\omega_h \cap \partial\Omega} = u|_{\omega_h \cap \partial\Omega} = g$. So, now we can consider this BVP for the operator L_h:

$$\begin{cases} L_h(U - u) = Lu - L_h u & \text{on } \omega_h \setminus \partial\Omega \\ U - u = 0 & \text{on } \omega_h \cap \partial\Omega. \end{cases}$$

Note that u does not necessarily solve the equation $L_h u = f$. Also note as a consequence of the BC, we have $U|_{\omega_h \cap \partial\Omega} = u|_{\omega_h \cap \partial\Omega} = g$. So, now we can consider this BVP for the operator L_h:

$$\begin{cases} L_h(U - u) = Lu - L_h u & \text{on } \omega_h \setminus \partial\Omega \\ U - u = 0 & \text{on } \omega_h \cap \partial\Omega. \end{cases}$$

We can now apply the bound from part (b) for this discrete problem:

Note that u does not necessarily solve the equation $L_h u = f$. Also note as a consequence of the BC, we have $U|_{\omega_h \cap \partial\Omega} = u|_{\omega_h \cap \partial\Omega} = g$. So, now we can consider this BVP for the operator L_h:

$$\begin{cases} L_h(U - u) = Lu - L_h u & \text{on } \omega_h \setminus \partial\Omega \\ U - u = 0 & \text{on } \omega_h \cap \partial\Omega. \end{cases}$$

We can now apply the bound from part (b) for this discrete problem:

$$\max_{x \in \omega_h} |U(x) - u(x)| \leq + \max_{x \in \omega_h \cap \partial\Omega} \underbrace{|0|} + \max_{x \in \omega_h} |Lu - L_h u| \leq Ch^2. \qquad (1.24)$$

Recall that $Lu - L_h u$ is the residual which is bounded by the LTE.