Lecture 6

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Qualifying Prep Course – Numerical

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Outline

Problem 3

Consider the following initial boundary value problem: find $u(x,t)$ such that

$$u_t - u_{xx} + u = 0, \qquad 0 < x < 1, \quad t > 0 \tag{1.1}$$

$$u_x(0,t) = u_x(1,t) = 0, \qquad t > 0 \tag{1.2}$$

$$u(x,0) = g(x), \quad 0 < x < 1. \tag{1.3}$$

(a) Derive the semi-discrete approximation of this problem using linear finite elements over a uniform partition of $(0,1)$. Write it as a system of linear ordinary differential equations for the coefficient vector.

(b) Further, derive discretizations in time using backward Euler and Crank-Nicolson methods, respectively.

(c) Show that both fully discrete schemes are unconditionally stable with respect to the initial data in the spatial $L^2(0,1)$-norm.

Solution to (a)

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Let us proceed "informally". We first multiply the PDE in our IBVP by a sufficiently smooth test function $v(x)$ in some real vector space V, and integrate over the domain $(0,1)$:

$$0 = \int_0^1 \Big(u_t(x,t)v(x) - u_{xx}(x,t)v(x) + u(x,t)v(x) \Big)\, dx,$$

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$$\overset{\text{IBP}}{=} (u_t, v) + \int_0^1 \Big(u_x(x,t)v_x(x) + u(x,t)v(x) \Big)\, dx - \cancel{\big[u_x(x,t)v(x)\big]_0^1}$$

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\begin{aligned}
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&= (u_t, v) + a(u,v),
\end{aligned}
$$

where we define the inner product $(\cdot,\cdot)$ and bilinear form $a(\cdot,\cdot)$ to be:

$$
\begin{aligned}
(u_t, v) &= \int_0^1 u_t v \, dx \\
a(u,v) &= \int_0^1 (u_x v_x + uv)\, dx.
\end{aligned}
$$

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Thus, our weak formulation in the finite element discretization becomes: Find $u_h(t) \in V_h$ such that $(u_{h,t}, v_h) + a(u_h, v_h) = 0$ for all $v_h \in V_h$. Note, $u_h(t) \in V_h$ for each $t > 0$.

Now, let us consider the standard tent functions as our basis for V_h: $\text{span}\{\phi_i\}_{i=0}^N$. We set $v = \phi_j(x)$ and express u_h in terms of the basis functions

$$u_h(x,t) = \sum_{i=0}^{N} u_i(t)\phi_i(x),$$

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where $\{u_i\}_{i=0}^N$ are the unknown coefficients. Substituting these into our weak formulation yields:

$$\int_0^1 \sum_{i=0}^{N} u_i'(t)\phi_i(x)\phi_j(x)\,dx + \int_0^1 \Big(\sum_{i=0}^{N} u_i(t)\phi_i'(x)\phi_j'(x) + \sum_{i=0}^{N} u_i(t)\phi_i(x)\phi_j(x) \Big)\,dx,$$

for $j = 0, \ldots, N$.

Using the vector/matrix notation, $\boldsymbol{U}(t) = [u_i(t)]_{i=1}^N$, $\boldsymbol{M} = [(\phi_i, \phi_j)]_{i,j=1}^N$, and $\boldsymbol{A} = [a(\phi_i, \phi_j)]_{i,j=1}^N$, Using the vector/matrix notation, $\boldsymbol{U}(t) = [u_i(t)]_{i=1}^N$, $\boldsymbol{M} = [(\phi_i, \phi_j)]_{i,j=1}^N$, and $\boldsymbol{A} = [a(\phi_i, \phi_j)]_{i,j=1}^N$, we can rewrite our semi-discrete variational equation as,

$$\begin{cases} \boldsymbol{M}\boldsymbol{U}'(t) + \boldsymbol{A}\boldsymbol{U}(t) = \mathbf{0}, \\ \boldsymbol{U}(0) = \boldsymbol{G}, \end{cases}$$

where $\boldsymbol{G} = [g(x_i)]_{i=1}^N$.

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where $\boldsymbol{G} = [g(x_i)]_{i=1}^N$.This is our system of linear ordinary differential equations.

Solution to (b)

Further, derive discretizations in time using backward Euler and Crank-Nicolson methods, respectively.
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$$\boldsymbol{M}\frac{\boldsymbol{U}^{n+1} - \boldsymbol{U}^n}{\Delta t} + \boldsymbol{A}\Big(\theta \boldsymbol{U}^{n+1} + (1-\theta)\boldsymbol{U}^n\Big) = \mathbf{0},$$

where $\theta \in [0, 1]$.

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where $\theta \in [0, 1]$. The backward Euler scheme is obtained by using $\theta = 1$ and the Crank-Nicolson scheme by using $\theta = \frac{1}{2}$.

Solution to (c)

Show that both fully discrete schemes are unconditionally stable with respect to the initial data in the spatial $L^2(0,1)$-norm. Let us consider the following variational problem:

$$\left(\frac{u^{n+1}-u^n}{\Delta t}, v\right) + a\Big(\theta u^{n+1} + (1-\theta)u^n, v\Big) = 0, \tag{1.4}$$

where $u^n := u(x, n\Delta t)$.

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Let us consider the following variational problem:

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Note that $a(u^{n+1}, u^{n+1}) \geq 0$, Dropping this term gives us that $LHS \leq 0$. Expanding and rearranging yields the following inequality:

$$||u^{n+1}||^2_{L^2(0,1)} = (u^{n+1}, u^{n+1}) \leq (u^n, u^{n+1}) \leq ||u^n||_{L^2(0,1)}||u^{n+1}||_{L^2(0,1)}.$$

Then, dividing by $||u^{n+1}||_{L^2(0,1)}$, we have that:

$$||u^{n+1}||_{L^2(0,1)} \leq ||u^n||_{L^2(0,1)} \leq \cdots \leq ||u^0||_{L^2(0,1)} = ||g||_{L^2(0,1)}.$$

Thus the backward Euler is unconditionally stable.

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$$(\frac{u^{n+1} - u^n}{\Delta t}, v) + a(\frac{u^{n+1} + u^n}{2}, v) = 0.$$

We make the *ansatz* $v = \frac{u^{n+1}+u^n}{2}$ and proceed similarly as in the backward Euler case:

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Expanding the inner product yields:

$$||u^{n+1}||^2_{L^2(0,1)} \leq ||u^n||^2_{L^2(0,1)}.$$

Thus, we have that

$$||u^{n+1}||_{L^2(0,1)} \leq ||g||_{L^2(0,1)}.$$

and the Crank-Nicolson scheme is unconditionally stable.

Outline

Problem 3

Let $\Omega_e = \{x \in \mathbb{R}^2 : \|x\| > 1\}$. Show that the Poincare inequality does not hold in $H_0^1(\Omega)$,i.e.,, there does not exist a constant $c > 0$ satisfying

$$c\|u\|^2_{L^2(\Omega_e)} \leq \int\limits_{\Omega_e} \|\nabla u\|^2 dx, \quad \text{for all } u \in H_0^1(\Omega_e).$$

The space $H_0^1(\Omega_e)$ is the completion of $C_0^\infty(\Omega_e)$ in the norm:

$$\|u\|_{H^1(\Omega)} = \left(\|v\|^2_{L^2(\Omega_e)} + \|\nabla v\|^2_{(L^2(\Omega_e))^2}\right)^{1/2}$$

(Hint: Consider dilating a fixed function.)

Solution

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Loosely speaking, the Poincare inequality defined above allows us to obtain bounds on our function u using bounds on its gradient. For the Poincare inequality to hold, both the norms $\|u\|^2_{L^2(\Omega_e)}$ and $\|\nabla v\|^2_{(L^2(\Omega_e))^2}$ need to make sense. We will show that this is not the case.

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$$\phi(r,\theta) := \phi(r) := \begin{cases} \exp(\frac{-1}{r(1-r)}) & \text{for } 0 < r < 1 \\ 0 & \text{for } r \geq 1, \end{cases}$$

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which is only a function of r. Let us now consider the sequence of bump functions defined by,

$$\phi_n(r) := \phi(\frac{r-1}{n}).$$

Note that ϕ_n has the support $\text{supp}(\phi_n) \subset [1, n+1]$.

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&= 2\pi \int_0^1 \phi^2(s) n(ns+1) \, ds \\
&= 2\pi n \int_0^1 \phi^2(s) \, ds + 2\pi n^2 \int_0^1 s\phi^2(s) \, ds \\
&\geq 2\pi n^2 \int_0^1 s\phi^2(s) \, ds \\
&= C_0 n^2.
\end{aligned}
$$

Thus $\lim_{n\to\infty} ||\phi_n||_{L^2(\Omega_e)} = \infty$.

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&= 2\pi \int_0^{1} \Big|\frac{1}{n}\frac{\partial}{\partial s}\phi(s)\Big|^2 n(ns+1)\, ds \\
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&= 2\pi \int_0^{1} \left|\frac{\partial}{\partial s}\phi(s)\right|^2 \frac{n(ns+1)}{n^2}\, ds \\
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&\leq 2\pi \frac{n(n+1)}{n^2}\int_0^1 |\phi'(s)|^2\, ds \\
&\leq C_1
\end{aligned}$$

Thus since the H^1-semi norm is bounded for all n and the L^2 norm blows up to infinity, this implies that we cannot have a Poincare inequality on this domain.

Problem 2

Consider the Neumann Problem:

$$-\Delta u = f \quad \text{in } \Omega \tag{2.1}$$

$$\frac{\partial u}{\partial n} = g \quad \text{on } \partial\Omega. \tag{2.2}$$

Here Ω is a bounded domain in $\mathbb{R}^2$ and f and g are suitably smooth.

(a) Derive a weak form of the above problem using a test function in $H^1(\Omega)$.

(b) Discuss when the weak form of Part a. has a solution and if it is unique.

(c) Describe a variational formulation of (2.1) in terms of an appropriate Hilbert space V. Be sure to explicitly define V.

(d) Prove coercivity of the form of Part a. on the V of Part c. when $\Omega = (0,1)^2$.

Solution to (a)

We want to derive a weak formulation of the BVP.

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We want to derive a weak formulation of the BVP. We begin by multipying the PDE by a test function $v \in H^1(\Omega)$ and integrating over Ω:

$$
\begin{aligned}
-\int_\Omega \Delta u v \, dx &= \int_\Omega \nabla u \cdot \nabla v \, dx - \int_{\partial\Omega} \frac{\partial u}{\partial n} v \, ds \\
&= \int_\Omega \nabla u \cdot \nabla v \, dx - \int_{\partial\Omega} g v \, ds.
\end{aligned}
$$

Solution to (a)

We want to derive a weak formulation of the BVP. We begin by multipying the PDE by a test function $v \in H^1(\Omega)$ and integrating over Ω:

$$
\begin{aligned}
-\int_\Omega \Delta u v \, dx &= \int_\Omega \nabla u \cdot \nabla v \, dx - \int_{\partial\Omega} \frac{\partial u}{\partial n} v \, ds \\
&= \int_\Omega \nabla u \cdot \nabla v \, dx - \int_{\partial\Omega} g v \, ds.
\end{aligned}
$$

So our weak formulation becomes: Find $u \in H^1(\Omega)$ such that $a(u, v) = L(v)$ for all $v \in H^1(\Omega)$ where

$$
a(u, v) := \int_\Omega \nabla u \cdot \nabla v \, dx = \int_\Omega f v \, dx + \int_{\partial\Omega} g v \, ds =: L(v).
$$

Solution to (b)

We first note (**HW: show this**) that a and F can both be shown to be continuous (for F you need to invoke the Trace Lemma 3.3 in Grossmann book).

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This is called the **compatibility condition** or **solvability condition** and is necessary for existence of the solution. (**See Remark 3.27 on Neumann boundary conditions in Grossmann book**)

Now, note that if u is a solution then $u + c$ must also be a solution for $c \in \mathbb{R}$.

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In order to guarantee a *unique* solution, we can impose the following condition on u:

$$\int_\Omega u \, dx = 0.$$

This assumption will allows us to derive a Poincare inequality, which will then allows us apply the Lax-Milgram lemma.

Solution to (c)

Describe a variational formulation of (2.1) in terms of an appropriate Hilbert space V. Be sure to explicitly define V. The variational formulation will be as follows: Find $u \in V := \{v \in H^1(\Omega) \; : \; \int_\Omega v \, ds = 0\}$ such that $a(u, v) = L(v)$ for all $v \in V$.

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Solution to (d)

We want to show coercivity of the bilinear form $a(\cdot, \cdot)$. Recall that $a(\cdot, \cdot)$ is coercive when there exists a $\gamma > 0$ such that $\gamma \|u\|_V^2 \leq a(u, u)$ for all $u \in V$.

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Note that our domain $\Omega = [0,1]^2$ is a Lipschitz domain. Let us recall the Poincare-Steklov Lemma:

Lemma 3.24 (Poincaré–Steklov). *Let D be a Lipschitz domain in $\mathbb{R}^d$. Let $\ell_D := \mathrm{diam}(D)$. Let $p \in [1,\infty]$. There is $C_{\mathrm{PS},p}$ (the subscript p is omitted when $p = 2$) s.t.*

$$C_{\mathrm{PS},p}\|v - \underline{v}_D\|_{L^p(D)} \leq \ell_D |v|_{W^{1,p}(D)}, \qquad \forall v \in W^{1,p}(D), \tag{3.8}$$

where $\underline{v}_D := \frac{1}{|D|}\int_D v\,\mathrm{d}x$. The following holds true when D is convex:

$$C_{\mathrm{PS},1} = 2, \quad C_{\mathrm{PS}} := C_{\mathrm{PS},2} = \pi, \quad C_{\mathrm{PS},p} \geq \frac{1}{2}\left(\frac{2}{p}\right)^{\frac{1}{p}},\ p > 1. \tag{3.9}$$

Since we have that $\int_\Omega v \, ds = 0$, coercivity follows in the usual way with an application of the Poincare-Steklov Lemma.