Lecture 1

Eric Tovar

Qualifying Prep Course – Numerical

06-30-2020

Outline

Course details

- Meeting Tuesday, Thursday Fridays 9am - 11am via Zoom
- Goal is to cover a handful of exams in detail and try to cover all topics
- Interaction between all of us is very important
- Suggestions?

Generic exam details

- Official syllabi and previous exams can be found at: `https://www.math.tamu.edu/graduate/phd/quals.html`
- The applied/numerical qualifying exam is 4 hours long with no particular time allotted to each section
- Each section has roughly 4 to 5 problems

Syllabus for numerical part of exam

https://www.math.tamu.edu/graduate/phd/quals.html

Outline

Reference Books

- I think "Numerical Treatment of Partial Differential Equations" by Grossman et al is the most appropriate book for this prep course.

Online resources

- I will put up the lectures in
  https://www.math.tamu.edu/~ejtovar/teaching.html
- Shared document?
  https://www.overleaf.com/6598876616fybpqcprytmw
- https://courses.maths.ox.ac.uk/node/view_material/3407
- https://arxiv.org/pdf/1709.08618.pdf

Outline

Let $\Omega = (0,1)$ and u be the solution of the boundary value problem

$$u^{(4)} - (k(x)u')' + q(x)u = f(x) \tag{3.1}$$

$$u(0) = u''(0) = 0 \tag{3.2}$$

$$u(1) = 0, \quad u''(1) + \beta u'(1) = \gamma, \tag{3.3}$$

for $x \in \Omega$ where $k(x) \geq 0$, $q(x) \geq 0$, $f(x)$, γ, and $\beta > 0$ are given data.

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Question 1: What are some things that you notice?
Question 2: What might the 4th order derivatives imply?
Question 3: What kind of boundary conditions are we dealing with?

(a) Derive the weak formulation of this problem. Specify the appropriate Sobolev spaces and show that the corresponding bilinear form is coercive.

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(b) Suggest a finite element approximation to this problem using piecewise polynomial functions over a uniform partition of Ω into subintervals with length $h = 1/N$.

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(b) Suggest a finite element approximation to this problem using piecewise polynomial functions over a uniform partition of Ω into subintervals with length $h = 1/N$.

(c) Derive an error estimate for the finite element solution.

Solution to (a)

Let $v \in V$ such that v is sufficiently smooth (we will be more precise about V later). We proceed "formally".

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$$\int_0^1 \left(u^{(4)} v\right) dx \overset{\text{IBP}}{=} \underbrace{[u^{'''} v]_0^1}_{=0} - \int_0^1 (u^{'''} v) dx$$

$$= -\underbrace{[u^{''} v^{'}]_0^1} + \int_0^1 (u^{''} v^{''}) dx$$

$$\overset{(3.3)}{=} -\left(\gamma - \beta u^{'}(1)\right) v^{'}(1) + \int_0^1 (u^{''} v^{''}) dx$$

Q: Why is first boundary term 0?

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$$-\int_0^1 \Big((k(x)u^{'})^{'}v\Big)dx \overset{\text{IBP}}{=} \underbrace{[(k(x)u^{'})v]_0^1}+\int_0^1 (k(x)u^{'}v^{'})dx$$

$$= \int_0^1 (k(x)u^{'}v^{'})dx$$

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$$\int_0^1 \Big(q(x)uv\Big)dx \;=\; \int_0^1 \Big(q(x)uv\Big)dx$$

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Let $v \in V$ such that v is sufficiently smooth (we will be more precise about V later). We proceed "formally".

(i) We first multiply (3.1) by v and integrate over $\Omega := (0, 1)$: (we focus on each term separately)

(ii) We now have to combine everything together. **Q: Where do we "put" the boundary terms?**

Solution to (a)

Let $v \in V$ such that v is sufficiently smooth (we will be more precise about V later). We proceed "formally".

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(ii) We now have to combine everything together. **Q: Where do we "put" the boundary terms?**

(iii) **Q1: What kind of "smoothness" is needed for v? Q2: What did we assume about v on the boundary?**

Recall that we assumed $v(0) = v(1) = 0$ and we need up to v'' to make sense. Thus, the weak formulation for the problem is given as follows:

Weak formulation

Find $u \in V := H^2(\Omega) \cap H_0^1(\Omega)$ such that for all $v \in V$:

$$a(u, v) = F(v)$$

where

$$a(u, v) := \int_0^1 \Big(u'' v'' + k(x) u' v' + q(x) uv \Big) dx + \beta u'(1) v'(1)$$

$$F(v) := \int_0^1 f(x) v dx + \gamma v'(1)$$

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$$a(u,u) \geq c\|u\|_V^2$$

for all $u \in V$.
We see that

$$a(u,u) = \int_0^1 \left((u^{''})^2 + k(x)(u^{'})^2 + q(x)u^2\right)dx + \beta(u^{'}(1))^2$$

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Q1: What is our goal?
Q2: What can we do with the $k(x)$ and $q(x)$ term?

Since $k(x) \geq 0$ and $q(x) \geq 0$,

$$\begin{aligned} a(u,u) &= \int_0^1 \left((u^{''})^2 + k(x)(u^{'})^2 + q(x)u^2\right)dx + \beta(u^{'}(1))^2 \\ &\geq \|u^{''}\|^2_{L^2(\Omega)} + \beta(u^{'}(1))^2 \end{aligned}$$

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Now we need some kind of "information" on $(u^{'}(1))^2$.

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$$\begin{aligned} \|u'\|^2_{L^2(\Omega)} &= \int_0^1 \left(u'\right)^2 dx = \int_0^1 \left(u'(1) - \int_x^1 u''(s)ds \right)^2 dx \\ (why?) &\leq \int_0^1 \left(2(u'(1))^2 + 2\Big(\int_x^1 u''(s)ds \Big)^2 \right) dx \\ &\leq \int_0^1 \left(2(u'(1))^2 + 2\Big(\int_x^1 (u''(s))^2 ds \Big) \right) dx \\ &\leq 2(u'(1))^2 + 2\|u''\|^2_{L^2(\Omega)} \end{aligned}$$

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These type of inequalities come from practice!

We now have that $(u'(1))^2 + \|u''\|^2_{L^2(\Omega)} \geq \frac{1}{2}\|u'\|^2_{L^2(\Omega)}$. So

$$\begin{aligned}
a(u,u) &= \int_0^1 \Big((u'')^2 + k(x)(u')^2 + q(x)u^2\Big)dx + \beta(u'(1))^2 \\
&\geq \|u''\|^2_{L^2(\Omega)} + \beta(u'(1))^2 \\
&= \frac{1}{2}\|u''\|^2_{L^2(\Omega)} + \underbrace{\frac{1}{2}\|u''\|^2_{L^2(\Omega)} + \beta(u'(1))^2} \\
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Q: What do we do with $\|u^{'}\|^2_{L^2(\Omega)}$?

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Q: What do we do with $\|u'\|^2_{L^2(\Omega)}$? A: We need a Poincare inequality

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Q: What do we do with $\|u'\|^2_{L^2(\Omega)}$? A: We need a Poincare inequality

Show for hw: $\|u\|^2_{L^2(\Omega)} \leq \|u'\|^2_{L^2(\Omega)}$

We now have that

$$
\begin{aligned}
a(u,u) &\geq \|u''\|^2_{L^2(\Omega)} + \beta(u'(1))^2 \\
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&\geq \frac{1}{2}\|u''\|^2_{L^2(\Omega)} + \frac{1}{4}\min(\frac{1}{2},\beta)\|u'\|^2_{L^2(\Omega)} + \underbrace{\frac{1}{4}\min(\frac{1}{2},\beta)\|u\|^2_{L^2(\Omega)}}
\end{aligned}
$$

Note that now we have all the proper terms needed. Combining everything above:

$$
\begin{aligned}
a(u,u) &\geq \|u''\|^2_{L^2(\Omega)} + \beta(u'(1))^2 \\
&\geq \frac{1}{2}\|u''\|^2_{L^2(\Omega)} + \frac{1}{4}\min(\frac{1}{2},\beta)\|u'\|^2_{L^2(\Omega)} + \underbrace{\frac{1}{4}\min(\frac{1}{2},\beta)\|u\|^2_{L^2(\Omega)}} \\
&\geq \frac{1}{4}\min(\frac{1}{2},\beta)\left(\|u''\|^2_{L^2(\Omega)} + \|u'\|^2_{L^2(\Omega)} + \|u\|^2_{L^2(\Omega)}\right) \\
&\geq \frac{1}{4}\min(\frac{1}{2},\beta)\|u\|^2_{H^2(\Omega)}
\end{aligned}
$$

Solution to (b)

Suggest a finite element approximation to this problem using piecewise polynomial functions over a uniform partition of Ω into subintervals with length $h = 1/N$.

[1] "In numerical analysis cubic splines that are globally C2 are commonly used, but the application of the finite element method to fourth-order differential equations requires only the global C1 property." Grossman pg 185

Solution to (b)

Solution. We can use the finite elements $(K_i, \mathbb{P}^3, \Sigma_i)$[1] where

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Solution to (b)

Solution. We can use the finite elements $(K_i, \mathbb{P}^3, \Sigma_i)$[1] where

(i) $K_i = [x_{i-1}, x_i]$ for $i = 1, \ldots, N$ and $x_i - x_{i-1} = h$

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(ii) $\mathbb{P}^3$ is the space of cubic polynomials

(iii) the unisolvent linear functionals are defined to be $\Sigma_i = \{\sigma_{i-1}, \sigma'_{i-1}, \sigma_i, \sigma'_i\}$, where

$$\sigma_{i-1}(f) = f(x_{i-1}) \qquad \sigma_i(f) = f(x_i)$$
$$\sigma'_{i-1}(f) = f'(x_{i-1}) \qquad \sigma'_i(f) = f'(x_i).$$

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For the finite element approximation, we consider the subspace

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A basis for this space is given by $\bigcup_{i=0}^{N}\{\phi_i, \psi_i\} - \phi_0, \phi_N$, where ϕ_i and ψ_i are the cubic Hermite polynomials (note we remove ϕ_0 and ϕ_N because of the boundary conditions).

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Specifically, ϕ_i and ψ_i are defined by the following conditions,

$$\sigma_k(\psi_j) = 0 \quad \text{and} \quad \sigma_k'(\psi_j) = \delta_{kj}$$
$$\sigma_k(\phi_j) = \delta_{kj} \quad \text{and} \quad \sigma_k'(\phi_j) = 0.$$

Precisely speaking, ψ_i and ϕ_i are defined as

$$\psi_i(x) = \begin{cases} \frac{1}{h^2}(x - x_i)(x - x_{i-1})^2 & \text{for } x \in [x_{i-1}, x_i], \\ \frac{1}{h^2}(x - x_{i+1})^2(x - x_i) & \text{for } x \in [x_i, x_{i+1}], \\ 0 & \text{otherwise} \end{cases}$$

$$\phi_i(x) = \begin{cases} \frac{1}{h^2}(x - x_{i-1})^2\left(\frac{2}{h}(x_i - x) + 1\right) & \text{for } x \in [x_{i-1}, x_i], \\ \frac{1}{h^2}(x_{i+1} - x)^2\left(\frac{2}{h}(x - x_i) + 1\right) & \text{for } x \in [x_i, x_{i+1}], \\ 0 & \text{otherwise .} \end{cases}$$

Solution to (c)

Derive an error estimate for the finite element solution.

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(iv) Thus, we can apply Cea's Lemma (**see: 4.4.1 of Grossman book**):

$$\|u - u_h\|_{H^2(\Omega)} \leq c \inf_{v_h \in V_h} \|u - v_h\|_{H^2(\Omega)} \leq c\|u - \Pi_h u\|_{H^2(\Omega)}$$

Solution to (c)

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We now want to estimate the projection error. This is done in three steps: (i) transformation to the reference element; (ii) estimation on reference element (using Bramble-Hilbert); (iii) inverse transformation to the finite (or physical) element

Let $\xi = \frac{x - x_{i-1}}{h}$ be the coordinate on the reference element and let $\overline{u}$ denote the function evaluated on the reference element.

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$$
\begin{aligned}
\|u - \Pi_h u\|^2_{H^2(\Omega)} &= \sum_{i=1}^{N} \|u - \Pi_h u\|^2_{H^2([x_{i-1},x_i])} \\
&= \sum_{i=1}^{N} \int_{x_{i-1}}^{x_i} \left(|u - \Pi_h u|^2 + \left|(u - \Pi_h u)'\right|^2 + \left|(u - \Pi_h u)''\right|^2 \right) dx \\
&= \sum_{i=1}^{N} \int_0^1 \left(\left|\overline{u} - \overline{\Pi_h u}\right|^2 + \frac{1}{h^2} \left|(\overline{u} - \overline{\Pi_h u})'\right|^2 + \frac{1}{h^4} \left|(\overline{u} - \overline{\Pi_h u})''\right|^2 \right) \underbrace{h d\xi} \\
&= \sum_{i=1}^{N} \left(h\|\overline{u} - \overline{\Pi_h u}\|^2_{L^2([0,1])} + \frac{h}{h^2} \|(\overline{u} - \overline{\Pi_h u})'\|^2_{L^2([0,1])} \right. \\
&\qquad \left. + \frac{h}{h^4} \|(\overline{u} - \overline{\Pi_h u})''\|^2_{L^2([0,1])} \right)
\end{aligned}
$$

Let us first recall the Bramble-Hilbert Lemma as stated in Lemma 4.25 in `Numerical Treatment of Partial Differential Equations`:

Bramble-Hilbert Lemma

Let $B \subset \mathbb{R}^n$ be a domain with a Lipschitz boundary and let q be a bounded sub-linear functional on $H^{k+1}(B)$. Assume that

$$q(w) = 0, \qquad \text{for all } w \in P^k.$$

Then there exists a constant $c = c(B) > 0$, which depends on B, such that

$$|q(v)| \leq c\, |v|_{k+1,B}\,, \qquad \text{for all } v \in H^{k+1}(B).$$

See also: Lemma 4.27 and Theorem 4.28

Note that $||(\mathrm{Id} - \overline{\Pi}_h)(\cdot)||_{L^2([0,1])}$, $||\frac{d}{d\overline{x}} \circ (\mathrm{Id} - \overline{\Pi}_h)(\cdot)||_{L^2([0,1])}$, and $||\frac{d^2}{d\overline{x}^2} \circ (\mathrm{Id} - \overline{\Pi}_h)(\cdot)||_{L^2([0,1])}$ are all sub-linear functionals defined on $H^4([0,1])$ which are zero for $p \in \mathbb{P}_3$. (**HW: verify this statement**)

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$$||u-\Pi_h u||^2_{H^2(\Omega)} \leq C\sum_{i=1}^{N}\left(h|\overline{u}|^2_{H^4([0,1])}+\frac{h}{h^2}|\overline{u}|^2_{H^4([0,1])}+\frac{h}{h^4}|\overline{u}|^2_{H^4([0,1])}\right)$$

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$$\begin{aligned} ||u - \Pi_h u||^2_{H^2(\Omega)} &\leq C \sum_{i=1}^{N} \left(h|\overline{u}|^2_{H^4([0,1])} + \frac{h}{h^2}|\overline{u}|^2_{H^4([0,1])} + \frac{h}{h^4}|\overline{u}|^2_{H^4([0,1])} \right) \\ &= C \sum_{i=1}^{N} \int_0^1 \left(\left|\frac{d^4}{d\xi^4}\overline{u}\right|^2 + \frac{1}{h^2}\left|\frac{d^4}{d\xi^4}\overline{u}\right|^2 + \frac{1}{h^4}\left|\frac{d^4}{d\xi^4}\overline{u}\right|^2 \right) h \, d\xi \end{aligned}$$

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So taking the square root of both sides, we arrive out our error estimate,

$$||u - u_h||_{H^2(\Omega)} \leq Ch^2|u|_{H^4(\Omega)}.$$

Let $\Omega = (0,1)^2$ and u be the solution of the second order elliptic problem:

$$-\Delta u : -u_{x_1 x_1} - u_{x_2 x_2} = f(x), \quad \text{for} x \in \Omega \tag{3.4}$$

$$\frac{\partial u}{\partial n} + u = g(x), \quad \text{for} x \in \partial\Omega \tag{3.5}$$

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Question 1: What kind of boundary condition do we have?

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Question 1: What kind of boundary condition do we have?
Remark 1: $\frac{\partial u}{\partial n} = \nabla u \cdot \boldsymbol{n}$

(a) Derive the weak formulation of this problem in the form $a(u, v) = F(v)$, where $a(u, v)$ and $F(v)$ are the appropriate bilinear and linear forms defined on the Sobolev space $H^1(\Omega)$.

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(b) Let S_h be a finite element space of continuous piece-wise polynomial functions defined over a regular partitioning of Ω into triangles and let $a_h(u,v)$ **and** $F_h(v)$**(!!!)** be the bilinear forms where all integrals are computed approximately. Derive Strang's lemma for the error of the FEM: find $u_h \in S_h$ such that $a_h(u_h, v) = F_h(v)$, $\forall v \in S_h$.

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(c) Let S_h be the finite element space of piece-wise linear functions. Let all integrals in $a(u, v)$ and $F(v)$ be computed using quadratures. Namely, for τ and e being triangle and edge defined by the vertexes P_1, P_2, P_3 and P_1, P_2 respectively,

$$\int_\tau w(x)dx \approx \frac{|\tau|}{3}\Big(w(P_1)+w(P_2)+w(P_3)\Big), \quad \int_e w(x)ds \approx \frac{|e|}{2}\Big(w(\alpha)+w \tag{3.6}$$

where $|\tau|$ is the area of τ and $|e|$ is the length of e, and α and β are the Gaussian quadrature nodes. Explain why $a(w, v) = a_h(w, v)$ for all $w, v \in S_h$.

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Q: How do we group these terms?

The weak formulation is given as follows:

Find $u \in V := H^1(\Omega)$ such that for any $v \in V$

$$a(u, v) = F(v)$$

where

$$a(u, v) := \int_{\Omega} \nabla u \cdot \nabla v dx + \int_{\partial\Omega} uvds \tag{3.7}$$

$$F(v) := \int_{\Omega} f(x) v dx + \int_{\partial\Omega} gvds \tag{3.8}$$

(First) Solution to (b)

Let S_h be a finite element space of continuous piece-wise polynomial functions defined over a regular partitioning of Ω into triangles and let $a_h(u, v)$ **and** $F_h(v)$**(!!!)** be the bilinear forms where all **integrals are computed approximately**. Derive Strang's lemma for the error of the FEM: find $u_h \in S_h$ such that $a_h(u_h, v) = F_h(v)$, $\forall v \in S_h$.

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(First) Solution to (b)

Note that the "statement" of the problem is ambiguous. That is, what does it mean to *derive* a known result? So let us consider the first solution. Recall the statement to Strang's First Lemma[2];

Strang's First Lemma

Let $V_h \subset V$ and let the bilinear form $a_h(\cdot,\cdot)$ be uniformly $V_h - \textit{elliptic}$. Then, there exists a constant $c > 0$ such that

$$\|u - u_h\| \leq c\left[\inf_{z_h \in V_h}\{\|u - z_h\| + \|a(z_h,\cdot) - a_h(z_h,\cdot)\|_{*,h}\} + \|f - f_h\|_{*,h}\right].$$

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