NUMERICAL QUALIFIER SOLUTION: AUGUST 2013

Problem 1. Let V be a closed subspace of $H^1(\Omega), V_h \subset V$ be a finite element approximation space and Ω a domain in $\mathbb{R}^d$. Given $W^0 \in V_h$, we consider the forwward Euler approximation: $W^{n+1} \in V_h$, $n = 0, 1, \ldots$ satisfying

$$((W^{n+1} - W^n)/k, \theta) + A(W^n, \theta) = (f^n, \theta), \qquad \text{for all } \theta \in V_h.$$

Here $k > 0$ is the time step size, $t^n = nk$, $f^n \in V_h$ $(\cdot,\cdot)$ is the inner product in $L^2(\Omega)$, $\|\cdot\|$ is the corresponding norm and $A(\cdot,\cdot)$ is a symmetric, coercive, and bounded bilinear form on V.

Let $\{\psi_i\}$, $i = 1, \ldots, M$ be an orthonormal basis with respect to $(\cdot,\cdot)$ for V_h of eigenfunctions satisfying

$$A(\psi_i, \theta) = \lambda_i(\psi_i, \theta), \qquad \text{for all } \theta \in V_h.$$

(a) Expand

$$W^n = \sum_{i=1}^{M} c_i^m \psi_i, \quad \text{and} \quad f^n = \sum_{i=1}^{M} d_i^n \psi_i,$$

and set $\delta_i = 1 - k\lambda_i$. Derive a recurrance relation for c_i^{n+1} in terms of δ_i, c_i^n, k and d_i^n.

(b) Assume that the CFL condition, $k\lambda_i \leq 2$, holds for all eigenvalues λ_i. Show that

$$|c_i^n| \leq \begin{cases} \left|c_i^0\right|, & if\, f^n = 0\, \forall n, \\ (t^n)^{1/2}\Big(k\sum_{j=0}^{n-1} \left|d_i^j\right|^2\Big)^{1/2}, & if\, W^0 = 0. \end{cases}$$

(c) Use Part (b) above and superposition principle to derive the stability estimate

$$\|W^n\| \leq \|W^0\| + (t^n)^{1/2}\Big(k\sum_{j=0}^{n-1} \|f^j\|^2\Big)^{1/2}$$

Solution. **(a)** We want to derive a recurrence relation for c_i^{n+1}. Let $W^n = \sum_{i=1}^{M} c_i^n \psi_i$ and $f^n = \sum_{i=1}^{M} d_i^n \psi$ and substitute into the discrete equation in the problem statement:

$$\frac{1}{k}\Big(\sum_{i=1}^{M} c_i^{n+1}\psi_i - \sum_{i=1}^{M} c_i^n \psi_i, \theta\Big) + A(\sum_{i=1}^{M} c_i^n \psi_i, \theta) = \Big(\sum_{i=1}^{M} d_i^n \psi_i, \theta\Big).$$

Note that this holds for any $\theta \in V_h$. Then, by linearity of integration and linearity of the bilinear form, we have that

$$\sum_{i=1}^{M}(c_i^{n+1} - c_i^n)(\psi_i, \theta) + k\sum_{i=1}^{M} c_i^n A(\psi_i, \theta) = k\sum_{i=1}^{M} d_i^n (\psi_i, \theta).$$

where we multiplied both sides by k. Then, by the eigenvalue problem above, we can rewrite the above equation as

$$\sum_{i=1}^{M}(c_i^{n+1} - c_i^n)(\psi_i, \theta) + k\sum_{i=1}^{M} c_i^n \lambda_i(\psi_i, \theta) = k\sum_{i=1}^{M} d_i^n(\psi_i, \theta).$$

Since $\{\psi_i\}$, $i = 1, \ldots, M$ is an **orthonormal** basis, let us set $\theta = \psi_i$. This yields

$$\begin{aligned}
(c_i^{n+1} - c_i^n) + kc_i^n\lambda_i &= kd_i^n, \\
&\Longrightarrow \\
c_i^{n+1} &= (1 - k\lambda_i)c_i^n + kd_i^n, \\
&= \delta_i c_i^n + kd_i^n.
\end{aligned}$$

This our recurrence relation for c_i^{n+1}.

(b) Let us assume the CFL condition holds $k\lambda_i \leq 2$ for all λ_i.
<u>**Case** $f^n = 0$</u>. Let $f^n = 0$ for all n. Then, by part (a) we know that

$$\begin{aligned}
c_i^{n+1} &= \delta_i c_i^n, \\
&= (1 - k\lambda_i)c_i^n.
\end{aligned}$$

Since A is **coercive**, the eigenvalue problem above gives us that $\lambda_i \geq 0$. Then note that $0 \leq k\lambda_i \leq 2 \implies 0 \geq -k\lambda_i \geq -2 \implies 1 \geq 1 - k\lambda_i \geq -1 \implies |\delta_i| \leq 1$.

Then, since $d_i^n = 0$, we see that

$$\begin{aligned}
\left|c_i^{n+1}\right| &= |\delta_i c_i^n| \\
&\leq |c_i^n|.
\end{aligned}$$

Since this holds for each time step, we have that $|c_i^n| \leq \left|c_i^0\right|$ for all n.

<u>**Case** $W^0 = 0$</u>. Let $W^0 = 0$. From the recurrence relation, we see that for $n = 0$, $c_i^1 = \cancel{\delta_i^0 c_i^0} + kd_i^0 = kd_i^0 \implies \left|c_i^1\right| \leq k\left|d_i^0\right| = k^{1/2}\left|k^{1/2}d_i^0\right| = (1 \cdot k)^{1/2}\left(k\left|d_i^0\right|^2\right)^{1/2}$.
Here, we show the "1" to represent $n = 1$.

Let us proceed by induction. Assume the following holds

$$|c_i^n| \leq (t^n)^{\frac{1}{2}}\left(k\sum_{j=0}^{n-1}\left|d_i^j\right|^2\right)^{\frac{1}{2}}$$

for some $n \geq 1$. Then, we have that

$$\begin{aligned}
\left|c_i^{n+1}\right| &= \left|\delta_i c_i^n + k d_i^n\right|, \\
&\leq \left|\delta_i\right| \left|c_i^n\right| + \left|k d_i^n\right|, \\
&\leq \left|c_i^n\right| + \left|k d_i^n\right|, \\
&\leq (t^n)^{\frac{1}{2}} \left(k \sum_{j=0}^{n-1} \left|d_i^j\right|^2\right)^{\frac{1}{2}} + k \left|d_i^n\right|, \\
&= (t^n)^{\frac{1}{2}} \left(k \sum_{j=0}^{n-1} \left|d_i^j\right|^2\right)^{\frac{1}{2}} + k^{\frac{1}{2}} \left(k \left|d_i^n\right|^2\right)^{\frac{1}{2}}, \\
\left(ab + cd \leq \sqrt{a^2 + c^2}\sqrt{b^2 + d^2}\right) &\leq (t^n + k)^{\frac{1}{2}} \left(k \sum_{j=0}^{n-1} \left|d_i^j\right|^2 + k \left|d_i^n\right|^2\right)^{\frac{1}{2}}, \\
&= (t^{n+1})^{\frac{1}{2}} \left(k \sum_{j=0}^{n} \left|d_i^j\right|^2\right)^{\frac{1}{2}}
\end{aligned}$$

(c) We want to show the stability inequality in the problem statement. Let us use the results from the previous sub-problems. Let $W^n \in V_h$ be defined as $W^n = W_1^n + W_2^n$ where $W_1^n, W_2^n \in V_h$ solve the respective problems:

$$\begin{cases} (W_1^{n+1} - W_1^n, \theta) + A(W_1^n, \theta) = 0 \\ W_1^0 = W^0 & \forall \theta \in V_h, \end{cases}$$

$$\begin{cases} (W_2^{n+1} - W_2^n, \theta) + A(W_2^n, \theta) = (f^n, \theta) \\ W_2^0 = 0 & \forall \theta \in V_h, \end{cases}$$

(Here we used the superposition principle). Then, by part (b) we know that for each W_1^n and W_2^n:

$$\begin{aligned}
\left|c_{1,i}^n\right| &\leq \left|c_i^0\right|, \\
\left|c_{2,i}^n\right| &\leq (t^n)^{\frac{1}{2}} \left(k \sum_{j=0}^{n-1} \left|d_i^j\right|^2\right)^{\frac{1}{2}}.
\end{aligned}$$

Note that $(W_1^n, W_1^n) = \|W_1^n\|^2 = \sum_{i=1}^M \left|c_{1,i}\right|^2 \leq \sum_{i=1}^M \left|c_i^0\right|^2 = \|W^0\|^2$. Similarly, $\|W_2^n\|^2 \leq t^n k \sum_{j=0}^{n-1} \|f^j\|^2$. Then, by the triangle inequality we get the desired estimate:

$$\|W^n\| \leq \|W_1^n\| + \|W_2^n\| \leq \|W^0\| + (t^n)^{1/2} \left(k \sum_{j=0}^{n-1} \|f^j\|^2\right)^{\frac{1}{2}}$$

Problem 2. In this problem, C (with or without subscript) denotes generic positive constants which are independent of the triangle diameters h_τ and $\mathbb{P}^j$ denotes the space of polynomials $\mathbb{R}^2$ of degree at most j.

Let Ω be a polynomial domain in $\mathbb{R}^2$ and u be the solution in $H_0^1(\Omega)$ of

$$\int_\Omega \nabla u \cdot \nabla v dx = \int_\Omega f v dx, \qquad \text{for all } v \in H_0^1(\Omega), \tag{0.1}$$

with $f \in L^2(\Omega)$.

Let $\mathcal{T}_h$ for $0 < h < 1$ be shape regular triangulations of Ω. Set

$$X_h = \{v_h \in H^1(\Omega) : v_h|_\tau \in \mathbb{P}^2\} \quad \text{and} \quad V_h = X_h \cap H_0^1(\Omega).$$

(a) For $\tau \in \mathcal{T}_h$, let $b_\tau \in \mathbb{P}^3$ be the "bubble" function defined by the conditions b_τ equal one on the barycenter of τ and b_τ vanishes on $\partial\tau$. Show that for any function $w_h \in X_h$

$$C_2\|w_h\|_{L^2(\tau)} \geq \|b_\tau^{1/2} w_h\|_{L^2(\tau)} \geq C_1\|\|_{L^2(\tau)}.$$

(b) For $f_h \in X_h$ and $v_h \in V_h$, let z_h denote the function given by

$$z_h|_\tau = b_\tau(f_h + \Delta v_h)|_\tau, \quad \tau \in \mathcal{T}_h.$$

Explain why $z_h \in H_0^1(\Omega)$.

(c) Show that

$$\|b_\tau^{1/2}(f_h + \Delta v_h)\|^2_{L^2(\tau)} = \int_\tau (f_h - f) z_h dx + \int_\tau (\nabla(u - v_h) \cdot \nabla z_h dx.$$

Solution. **(a)** Let $\lambda_1, \lambda_2, \lambda_3$ be the usual barycentric coordinates on τ. Recall that the "bubble" function defined by the conditions b_τ equal one on the barycenter of τ and b_τ vanishes on $\partial\tau$ is given by:

$$b_\tau = 27\lambda_1\lambda_2\lambda_3.$$

Let $b_\tau(x, y)$ be the representation of the bubble function in Cartesian coordinates. Let $F_\tau : \hat{\tau} \to \tau$ be the affine transformation from the reference element $\hat{\tau}$ to the physical element τ. Let $\hat{w}_h = w_h \circ F_\tau$ and $\hat{b}_{\hat{\tau}} = b_\tau \circ F_\tau$. Then we have the following equalities:

$$\|b_\tau^{1/2} w_h\|_{L^2(\tau)} = \left|\det F_\tau^{'}\right|^{1/2} \|\hat{b}_{\hat{\tau}}^{1/2} \hat{w}_h\|_{L^2(\hat{\tau})},$$

and

$$\|w_h\|_{L^2(\tau)} = \left|\det F_\tau^{'}\right|^{1/2} \|\hat{w}_h\|_{L^2(\hat{\tau})}.$$

Then, by norm equivalence in polynomial spaces (since they are finite dimensional) we have that:

$$C_2\|\hat{w}_h\|_{L^2(\hat{\tau})} \geq \|\hat{b}_{\hat{\tau}}^{1/2} \hat{w}_h\|_{L^2(\hat{\tau})} \geq C_1\|\hat{w}_h\|_{L^2(\hat{\tau})}.$$

Then transforming back to the physical element τ by the affine mapping F_τ yields the inequalities on τ.

(b) First note that z_h vanishes on the boundary of τ, $\partial\tau$, since b_τ vanishes on the boundary for all $\tau \in \mathcal{T}_h$. Thus, z_h must vanish on $\partial\Omega$. Note that $(\Delta v_h)|_\tau$ is a constant function, thus $\big(f_h + (\Delta v_h)\big)|_\tau \in \mathbb{P}^2 \implies b_\tau\big(f_h + (\Delta v_h)\big)|_\tau \in \mathbb{P}^5$. So $z_h|_\tau \in H_0^1(\tau)$. Since z_h vanishes on all $\partial\tau$, we will have continuity over all Ω. Thus, $z_h \in H_0^1(\Omega)$.

(c) We want to show the property in part (c). Let us begin with the left hand side and use integration by parts:

$$\begin{aligned}
\|b_\tau^{1/2}(f_h + \Delta v_h)\|^2_{L^2(\tau)} &= \int_\tau b_\tau^{1/2}(f_h + \Delta v_h) b_\tau^{1/2}(f_h + \Delta v_h)dx, \\
&= \int_\tau z_h(f_h + \Delta v_h)dx, \\
&= \int_\tau z_h f_h + \int_\tau z_h \Delta v_h dx, \\
&= \int_\tau z_h f_h dx + \cancel{\int_{\partial\tau} (\nabla v_h \cdot \boldsymbol{n}) z_h ds} - \int_\tau \nabla v_h \cdot \nabla z_h dx, \\
&= \int_\tau z_h f_h dx - \int_\tau \nabla v_h \cdot \nabla z_h dx, \\
&= \int_\tau z_h f_h dx - \int_\tau \nabla v_h \cdot \nabla z_h dx \underbrace{- \int_\tau f z_h dx + \int_\tau \nabla z_h \cdot \nabla u dx}_{=0}, \\
&= \int_\tau (f_h - f) z_h dx + \int_\tau (\nabla(u - v_h)) \cdot \nabla z_h dx.
\end{aligned}$$

Problem 3. Let K be a nondegenerate triangle in $\mathbb{R}^2$. Let a_1, a_2, a_3 be the three vertices of K. Let $a_{ij} = a_{ji}$ denote the midpoint of the segment $(a_i, a_j), i, j \in \{1, 2, 3\}$. Let $\mathbb{P}^2$ be the set of the polynomial functions over K of total degree at most 2. Let $\Sigma = \{\sigma_1, \sigma_2, \sigma_3, \sigma_{12}, \sigma_{23}, \sigma_{31}\}$ be the functions (or degrees of freedom) on $\mathbb{P}^2$ defined as

$$\sigma_i(p) = p(a_i), i \in \{1, 2, 3\}, \qquad \sigma_{ij}(p) = p(a_i) + p(a_j) - 2p(a_{ij}),\ i, j = 1, 2, 3,\ i \neq j.$$

(a) Show that Σ is a unisolvent set for $\mathbb{P}^2$.
(b) Compute the "nodal" bases of $\mathbb{P}^2$ which corresponds to $\{\sigma_1, \ldots, \sigma_{31}\}$.
(c) Evaluate the entry m_{11} of the element mass matrix.

Solution. **(a)** We want to show that the set Σ is a unisolvent set for $\mathbb{P}^2$. First note that $\text{card}(\Sigma) = 6 = \dim \mathbb{P}^2$. We now want to verify that $\sigma(p) = 0$ for all $\sigma \in \Sigma$ implies that $p = 0$ for any $p \in \mathbb{P}^2$. Before we continue, we want to invoke affine equivalence of the physical and reference element to show unisolvence on the reference element.

Let $\hat{K} := \{(x, y) \big| x \geq 0, y \geq 0, x + y \leq 1\}$ be the reference triangular element. Let $p(x, y) = a + bx + cy + dxy + ex^2 + fy^2$ be an arbitrary polynomial in $\mathbb{P}^2$ defined on the reference element. Let $a_1 = (0, 0), a_2 = (1, 0), a_3 = (0, 1), a_{12} = (\frac{1}{2}, 0), a_{13} = (0, \frac{1}{2}), a_{23} = (\frac{1}{2}, \frac{1}{2})$. Then for each degree of freedom, we have that

$$\begin{aligned} \sigma_1(p) = a \quad & \sigma_2(p) = a + b + e, \\ \sigma_3(p) = a + c + f, \quad & \sigma_{12}(p) = e/2, \\ \sigma_{13}(p) = f/2, \quad & \sigma_{23}(p) = \frac{1}{2}(-d + e + f). \end{aligned}$$

We immediately see that $\sigma_1(p) = \sigma_4(p) = \sigma_5(p) = 0$ imply that $a = e = f = 0$. Then from $\sigma_2(p) = \sigma_3(p) = 0 = \sigma_6(p)$ implies that $b = c = d = 0$. Thus, our polynomial is exactly 0 and have a unisolvent set.

(b) We now want to compute the "nodal" basis functions which correspond to the set Σ on the reference element. That is, we want to find the 6 polynomials w_j such that $\tilde{\sigma}_i(w_j) = \delta_{ij}$, $i, j = 1, \ldots, 6$ where δ_{ij} is the Kronecker delta and $\tilde{\sigma}$ represent the 6 DOFS in our set Σ. Solving this system of equations, yields the following basis functions:

$$w_1 = 1 - x - y, w_2 = x, w_3 = y,$$

$$w_4 = -2x(1 - x - y), w_5 = -2y(1 - x - y), w_6 = -2xy$$

(Try to find these basis functions using barycentric coordinates: $w_1 = \lambda_1, w_2 = \lambda_2, w_3 = \lambda_3, w_4 = -2\lambda_1\lambda_2, w_5 = -2\lambda_1\lambda_3, w_6 = -2\lambda_2\lambda_3$.)

(c) We now want to evaluate the entry m_{11} of the **element** mass matrix. Recall that the entry m_{11} is defined by:

$$m_{11} = \int_K w_1 w_1 dx,$$

where K is the physical triangular element. Since we defined the basis functions on the reference element, let $\tilde{w}_1 = w_1 \circ F_K^{-1}$. Then, we have that:

$$\begin{aligned}
\int_K \tilde{w}_1 \tilde{w}_1 dx &= \frac{|K|}{\left|\hat{K}\right|} \int_{\hat{K}} w_1 w_1 d\hat{x}, \\
&= \frac{|K|}{\left|\hat{K}\right|} \int_{\hat{K}} (1 - \hat{x} - \hat{y})^2 dx, \\
&= \frac{|K|}{\left|\hat{K}\right|} \frac{1}{12} = \frac{1}{6} |K|
\end{aligned}$$

This integration can also be done can with the following integration formula for barycentric coordinates on a simplex $K \subset \mathbb{R}^d$:

$$\int_K \prod_{i=1}^{d+1} \lambda_k^{m_i} = \frac{\prod_{i=1}^{d+1} m_i!}{\left(d + \sum_{i=1}^{d+1} m_i\right)!} d! |K|$$

Here, $d = 2, m_1 = 2, m_2 = 0, m_3 = 0$. Thus, we have that

$$\begin{aligned}
\int_K \lambda_1 \lambda_1 &= \frac{2!0!0!}{(2+2+0+0)!} 2! |K|, \\
&= \frac{1}{6} |K|.
\end{aligned}$$