Lecture 4

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Qualifying Prep Course – Numerical

07-07-2020

- Comments/review about last week's exam
- Note that January 2010 exam and August 2010 exam are the same
- Solution exercise

Outline

Problem 1

Consider the system

$$\begin{aligned} -\Delta u - \phi &= f \\ u - \Delta \phi &= g \end{aligned} \tag{1.1}$$

in the bounded smooth domain Ω, with boundary conditions $u = \phi = 0$ on $\partial\Omega$.

(a) Derive a weak formulation of the system (1.1), using suitable test functions for each equation. Define a bilinear form $a((u, \phi), (v, \psi))$ such that this weak formulation amounts to,

$$a((u, \phi), (v, \psi)) = (f, v) + (g, \psi). \tag{1.2}$$

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(c) Show that the weak formulation (1.2) has a unique solution. Hint: Lax-Milgram.

(d) For a domain $\Omega_d = (-d, d)^2$, show that

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(e) Now change the second "-" in the first equation of (1.1) to a "$+$". Use (1.3) to show stability for the modified equation on Ω_d, provided that d is sufficiently small.

Solution to (a)

Let X and Y be real vector spaces[1] and let $v \in X$ and $\psi \in Y$ be sufficiently smooth to make sense of the following computations. We proceed "informally" and make more precise statements later.

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$$\int_\Omega \Big(- \Delta u v - \phi v \Big) \, dx \overset{\text{IBP}}{=} - \int_{\partial\Omega} \frac{\partial u}{\partial n} v \, ds + \int_\Omega \Big(\nabla u \cdot \nabla v - \phi v \Big) \, dx$$

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$$\int_\Omega \Big(u\psi - \Delta \phi \psi \Big)\, dx \overset{\text{IBP}}{=} - \int_{\partial\Omega} \frac{\partial \phi}{\partial n} \psi \, ds + \int_\Omega \Big(u\psi + \nabla \phi \cdot \nabla \psi \Big)\, dx.$$

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$$a((u,\phi),(v,\psi)) := \int_\Omega \Big(\nabla u \cdot \nabla v + \nabla\phi \cdot \nabla\psi + u\psi - \phi v\Big)\, dx = (f,v) + (g,\psi), \tag{1.4}$$

where (f, v) denotes the inner product of f and v: $(f, v) = \int_\Omega fv\, dx$

Solution to (b)

Since u and ϕ vanish on the boundary and only up to their gradients show up in the bilinear form, it is sufficient to take
$(u, \phi) \in V := H_0^1(\Omega) \times H_0^1(\Omega)$.

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We must first define a norm for the space V. For $v \in V$ in the form (u, ϕ), we define the norm as follows:

$$|||(u,\phi)|||_V = \sqrt{||u||^2_{H^1(\Omega)} + ||\phi||^2_{H^1(\Omega)}} \tag{1.5}$$

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For notation purposes, we will drop the subscript V.

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&\le (||u||_{H^1(\Omega)} + ||\phi||_{H^1(\Omega)})(||v||_{H^1(\Omega)} + ||\psi||_{H^1(\Omega)})
\end{aligned}
$$

Note that $\|u\|_{H^1(\Omega)} = \sqrt{\|u\|^2_{L^2(\Omega)} + \|\nabla u\|^2_{L^2(\Omega)}} \le \sqrt{\|u\|^2_{L^2(\Omega)} + \|\nabla u\|^2_{L^2(\Omega)} + \|\phi\|^2_{L^2(\Omega)} + \|\nabla\phi\|^2_{L^2(\Omega)}} = |||(u,\phi)|||.$

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\end{aligned}$$

Thus,

$$a((u,\phi),(v,\psi)) \leq 4|||(u,\phi)||| \cdot |||(v,\psi)|||$$

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$$\begin{aligned} a((u,\phi),(u,\phi)) &\geq \frac{1}{2}|u|^2_{H^1(\Omega)} + \frac{1}{2}|\phi|^2_{H^1(\Omega)} + \frac{c}{2}||u||^2_{L^2(\Omega)} + \frac{c}{2}||\phi||^2_{L^2(\Omega)} \\ &\geq \frac{1}{2}\min\{1,c\}\left(||u||^2_{H^1} + ||\phi||^2_{H^1}\right) \\ &= \gamma|||(u,\phi)|||^2. \end{aligned}$$

We now want to use a Poincare inequality to get the $\|\cdot\|_{L^2(\Omega)}$ terms out. Since u and ϕ are both in $H_0^1(\Omega)$, it is easy (**HW**) to show that $c\|\nabla u\|_{L^2(\Omega)} \geq \|u\|_{L^2(\Omega)}$ and $c\|\nabla \phi\|_{L^2(\Omega)} \geq \|\phi\|_{L^2(\Omega)}$.

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Since we showed $a(\cdot,\cdot)$ is continuous and coercive, we can apply the Lax-Milgram Lemma to our setting:

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Let $a(\cdot,\cdot) : V \times V \to \mathbb{R}$ be a continuous, coercive bilinear form. Then for each $f \in V^*$ (here V^* is the space V equipped with the dual norm $\|\cdot\|_*$), the variational problem

$$a(u,v) = f(v) \qquad \textit{for all } v \in V,$$

has a unique solution $u \in V$. Furthermore, we have the a priori estimate

$$\|u\|_V \leq \frac{1}{\gamma}\|f\|_*.$$

Solution to (d)

We want to show that for $\Omega_d = (-d, d)^2$

$$\|u\|_{L^2(\Omega_d)} \leq cd^2 \|\nabla u\|_{L^2(\Omega_d)},$$

holds for any function $u \in H_0^1(\Omega_d)$.

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holds for any function $u \in H_0^1(\Omega_d)$. We proceed with a straight forward computation.

$$||u||^2_{L^2(\Omega_d)} = \int_{-d}^{d}\int_{-d}^{d} u(x,y)^2 \, dx \, dy$$

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&= \int_{[-d,d]^2} \left(\frac{1}{2}\Big(\int_{-d}^{x} \frac{\partial}{\partial \xi}u(\xi,y)\, d\xi\Big)^2 + \frac{1}{2}\Big(\int_{-d}^{y} \frac{\partial}{\partial \eta}u(x,\eta)\, d\eta\Big)^2\right) dx\, dy
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(why?) &\leq \frac{1}{2}\int_{[-d,d]^2} \Bigg((x+d)\int_{-d}^{x}\Big(\frac{\partial}{\partial \xi}u(\xi,y)\Big)^2 d\xi \\
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&\leq d^2 \int_{-d}^{d}\int_{-d}^{d}\left(\frac{\partial}{\partial \xi}u(\xi,y)\right)^2 d\xi\, dy + \int_{-d}^{d}\int_{-d}^{d}\left(\frac{\partial}{\partial \eta}u(x,\eta)\right)^2 d\eta\, dx
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&= \int_{[-d,d]^2} \left(\frac{1}{2}\left(\int_{-d}^{x} \frac{\partial}{\partial\xi}u(\xi,y)\, d\xi\right)^2 + \frac{1}{2}\left(\int_{-d}^{y} \frac{\partial}{\partial\eta}u(x,\eta)\, d\eta\right)^2\right) dx\, dy \\
(why?) &\leq \frac{1}{2}\int_{[-d,d]^2} \Bigg((x+d)\int_{-d}^{x}\left(\frac{\partial}{\partial\xi}u(\xi,y)\right)^2 d\xi \\
&\qquad\qquad + (y+d)\int_{-d}^{y}\left(\frac{\partial}{\partial\eta}u(x,\eta)\right)^2 d\eta\Bigg)\, dx\, dy \\
&\leq d^2\int_{-d}^{d}\int_{-d}^{d}\left(\frac{\partial}{\partial\xi}u(\xi,y)\right)^2 d\xi\, dy + \int_{-d}^{d}\int_{-d}^{d}\left(\frac{\partial}{\partial\eta}u(x,\eta)\right)^2 d\eta\, dx \\
&= d^2\int_{\Omega_d} |\nabla u|^2\, dx\, dy \\
&= cd^2||\nabla u||^2_{L^2(\Omega_d)}.
\end{aligned}
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Solution to (e)

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Repeating the same process for this new set of equations (**HW: show details**), we arrive at the following bilinear form

$$a((u,\phi),(v,\psi)) = \int_{\Omega_d} \Big(\nabla u \cdot \nabla v + \nabla \phi \cdot \nabla \psi + u\psi + \phi v \Big)\, dx$$

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Let us assume for now that this is equivalent to showing that the bilinear form $a(\cdot, \cdot)$ is coercive. Need to work out the details

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We now want to properly treat the terms $\frac{1}{2cd^2}(|u|^2 + |\phi|^2) + 2u\phi$.

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&= \tilde{\alpha}|||(u,\phi)|||^2
\end{aligned}$$

Thus, since $\tilde{\alpha} > 0$ and this holds for any $w \in V$, $a(\cdot, \cdot)$ is coercive for the modified problem.