NUMERICAL QUALIFIER SOLUTION: AUGUST 2011

Problem 1. Let $\mathbb{P}_2$ be the space of polynomials in two variables spanned by

$$\{1, x_1, x_2, x_1^2, x_1x_2, x_2^2\}, \tag{0.1}$$

let $\widehat{T}$ be the reference unit triangle, $\hat{\gamma}$ one side of $\widehat{T}$, and $\hat{\pi}$ the standard Lagrange interpolant in $\widehat{T}$ with values in $\mathbb{P}_2$.

Recall that there exists a constant C only depending on the geometry of $\widehat{T}$ such that $\forall v \in H^3(\widehat{T})$,

$$\inf_{p\in\mathbb{P}_2} ||v + p||_{H^3(\widehat{T})} \leq C|v|_{H^3(\widehat{T})} \tag{0.2}$$

(a) State the trace theorem relating $L^2(\hat{\gamma})$ and $H^1(\widehat{T})$.

(b) Prove that there exists a constant $\hat{C}$ only depending on the geometry of $\widehat{T}$ and $\hat{\gamma}$ such that $\forall \hat{u} \in H^3(\widehat{T})$,

$$||\hat{u} - \hat{\pi}(\hat{u})||_{L^2(\hat{\gamma})} \leq \hat{C}|\hat{u}|_{H^3(\widehat{T})} \tag{0.3}$$

(c) Let

$$X_h = \{v_h \in C^0(\overline{\Omega}) \,:\, \forall T \in \mathcal{T}_h,\ v_h|_T \in \mathbb{P}_2\}. \tag{0.4}$$

Let T be a triangle of $\mathcal{T}_h$ with diameter h_T and diameter of inscribed disc ϱ_T, and let γ be one side of T. Let F_T be the affine mapping from $\widehat{T}$ onto T and let $\pi_{2,h}$ denote the standard Lagrange interpolant on X_h. Prove that there exists a constant C only depending on the geometry of $\widehat{T}$ and $\hat{\gamma}$ such that $\forall u \in H^3(T)$,

$$||u - \pi_{2,h}(u)||_{L^2(\hat{\gamma})} \leq C\sigma_T h_T^{2+1/2}|u|_{H^3(T)}, \tag{0.5}$$

where $\sigma = h_T/\varrho_T$.

Solution. **(a)** Let us recall the Trace theorem as stated in **??**:

Theorem 0.1 (Trace). *Let D be a Lipschitz domain in $\mathbb{R}^2$. Let $p = 2$ and $s = 1$. There is a bounded, linear map $\varphi : W^{1,2}(D) \to L^2(\partial D)$ such that*

(i) $\varphi(v) = v\big|_{\partial D}$ where v is smooth, i.e., $v \in C^0(\overline{D})$.

(ii) The kernel of φ is $W_0^{1,2}(D)$.

(iii) $\varphi : W^{1,2}(D) \to W^{\frac{1}{2},2}(\partial D)$ is bounded and surjective, that is, there exists a C_φ such that for every $g \in W^{\frac{1}{2},2}(\partial D)$ one can find a function $u_g \in W^{1,2}(D)$ called a lifting of g such that

$$\varphi(u_g) = g, \quad \textit{and } \|u_g\|_{W^{1,2}(D)} \leq C_\varphi \ell^{\frac{1}{2}} \|g\|_{W^{\frac{1}{2},2}(\partial D)}$$

where ℓ_D is a characteristic length of D, e.g., $\ell_D := diam(D)$.

Note that the map φ is usually denoted by γ, but to avoid confusion with the triangle side $\hat{\gamma}$ we use the variable φ. For notation purposes, let us write $W^{1,2}(D) := H^1(D)$. Also recall that

$$W^{\frac{1}{2},2}(\partial D) =: H^{\frac{1}{2}}(\partial D) := \{w \in L^2(\partial D) : \text{there exists } v \in H^1(\Omega) \text{ with } w = \varphi(v).\}$$

(b) We want to show there exists a constant $\hat{C}$ only depending on the geometry of $\widehat{T}$ and $\hat{\gamma}$ such that $\forall \hat{u} \in H^3(\widehat{T})$,

$$||\hat{u} - \hat{\pi}(\hat{u})||_{L^2(\hat{\gamma})} \leq \hat{C}|\hat{u}|_{H^3(\widehat{T})}.$$

Let $\hat{u} \in H^3(\widehat{T})$. Note that $\hat{\Pi}(\widehat{w}) \in \mathbb{P}^2 \subset H^3(\widehat{T})$. Let $\hat{\varphi}$ be the bounded, linear map $\hat{\varphi} : H^1(\widehat{T}) \to L^2(\partial\widehat{T})$ that satisfies the properties in the Trace Theorem above.

Observe that $||\hat{u} - \hat{\pi}(\hat{u})||_{L^2(\hat{\gamma})} \leq ||\hat{u} - \hat{\pi}(\hat{u})||_{L^2(\partial\widehat{T})}$. Also note that $\hat{u}, \hat{\Pi}(\hat{u}) \in H^3(\widehat{T}) \subset H^1(\widehat{T})$. Then, the boundedness of the trace map $\hat{\varphi}$ implies there exists a constant $C_{\widehat{T},p}$ such that

$$\begin{aligned} ||\hat{u} - \hat{\pi}(\hat{u})||_{L^2(\hat{\gamma})} &\leq ||\hat{u} - \hat{\pi}(\hat{u})||_{L^2(\partial\widehat{T})}, \\ &\leq C_{\widehat{T},2}||\hat{u} - \hat{\pi}(\hat{u})||_{H^1(\widehat{T})}. \end{aligned}$$

Note that for any $p \in \mathbb{P}_2$, we have $\hat{\pi}(p) = p$. In addition, notice that $||(\mathrm{Id} - \hat{\pi})(\hat{u})||_{H^1(\widehat{T})}$ is a bounded, sublinear functional on $H^3(\widehat{T})$ that is zero for all polynomials in $\mathbb{P}^2$. Thus, applying the Bramble-Hilbert lemma yields

$$C_{\widehat{T},2}||(\mathrm{Id} - \hat{\pi})(\hat{u})||_{H^1(\widehat{T})} \leq \hat{C}|\hat{u}|_{H^3(\widehat{T})}. \tag{0.6}$$

Let us state the result that was used for the above problem:

LEMMA 0.2 (Bramble-Hilbert). *Let $B \subset \mathbb{R}^n$ be a domain with a Lipschitz boundary and let q be a bounded sub-linear functional on $H^{k+1}(B)$. Assume that*

$$q(w) = 0, \qquad \textit{for all } w \in P^k.$$

Then there exists a constant $c = c(B) > 0$, which depends on B, such that

$$|q(v)| \leq c\,|v|_{k+1,B}\,, \qquad \textit{for all } v \in H^{k+1}(B).$$

(c) Let $u \in H^3(T)$. Let us define $\hat{u} = u \circ F_T$. From the following diagram

$$\begin{tikzcd} H^3(T) \arrow[r, "\pi_{2,h}"] \arrow[d, "\psi"'] & \mathbb{P}_2 \arrow[d, "\psi"] \\ H^3(\widehat{T}) \arrow[r, "\hat{\pi}"] & P \end{tikzcd}$$

where $P = \mathbb{P}_2 \circ F_T^{-1}$ and $\psi(u) = u \circ F_T$ is the *pullback* map, we see that $(\pi_{2,h}(u)) \circ F_T =$

$\hat{\pi}(u \circ F_T) = \hat{\pi}(\hat{u})$. Then note that

$$\begin{aligned}
||u - \pi_{2,h}(u)||^2_{L^2(\gamma)} &= \int_\gamma |u - \pi_{2,h}(u)|^2 \, ds, \\
&= \int_{\hat{\gamma}} |\hat{u} - \hat{\pi}(\hat{u})|^2 \frac{|\gamma|}{|\hat{\gamma}|} \, d\hat{s}, \\
&\leq h_T \int_{\hat{\gamma}} |\hat{u} - \hat{\pi}(\hat{u})|^2 \, d\hat{s}, \\
&= h_T ||\hat{u} - \hat{\pi}(\hat{u})||^2_{L^2(\hat{\gamma})} \\
&\leq \hat{C}_{\widehat{T},2} h_T |\hat{u}|^2_{H^3(\hat{T})},
\end{aligned}$$

where we used the estimate from part (b) in the last step. Now we want to transform back to the physical element. However, let us recall some relationships between the triangle T and the diameter ρ_T. Let $F_T^{'} = B$ where B is the Jacobian matrix of the transformation. Recall that $|\det(F_T')| = |\det(B)| = |T|/|\hat{T}| = 2|T|$. Let us recall the formula for the area of a triangle in terms of ϱ_T:

$$|T| = \frac{1}{2}(a + b + c)\frac{\varrho_T}{2}, \tag{0.7}$$

where a, b, and c are the side lengths of the triangle T. From this formula, we can get the following inequalities:

$$\frac{1}{2} h_T \varrho_T = \frac{1}{2}(2h_T)\frac{\varrho_T}{2} \leq |T| \leq \frac{1}{2}(3h_T)\frac{\varrho_T}{2} = \frac{3}{4} h_T \varrho_T,$$

where we have used the fact that the sum of any two sides of the triangle is greater than h_T. Thus we have that

$$\begin{aligned}
|\det(B)| &\leq \frac{3}{2} h_T \varrho_T, \\
|\det(B^{-1})| = |(\det(B))^{-1}| &\leq (h_T \varrho_T)^{-1}.
\end{aligned}$$

Then, we have that

$$\begin{aligned}
||u - \pi_{2,h}(u)||^2_{L^2(\gamma)} &\leq \hat{C}_{\widehat{T},2} h_T |\hat{u}|^2_{H^3(\hat{T})} \\
&\leq C h_T ||B||^6 \cdot |\det(B)|^{-1} \cdot |u|^2_{H^3(T)}, \\
&\leq C h_T \int_T h_T^6 \sum_{|\beta|=3} |(D^\beta u)(x)|^2 h_T^{-1} \varrho_T^{-1} \, dx \\
&= C \frac{h_T}{\varrho_T} h_T^5 |u|^2_{H^3(T)}.
\end{aligned}$$

Taking the square root we have,

$$||u - \pi_{2,h}(u)||_{L^2(\gamma)} \leq C \sigma_T^{1/2} h_T^{2+1/2} |u|_{H^3(T)}.$$

Problem 2. (From Ben's solution) Let $\delta > 0$ be a given constant parameter and $u \in H_0^1(\Omega)$ a given function. Consider the problem: Find $\varphi^\delta \in H_0^1(\Omega)$ such that

$$\begin{aligned} -\delta^2 \Delta \varphi^\delta(x) + \varphi^\delta(x) &= u(x) \text{ a.e. in } \Omega, \\ \varphi^\delta(x) &= 0 \text{ a.e. on } \partial\Omega. \end{aligned} \tag{0.8}$$

a. Define the bilinear form

$$a^\delta(w, v) = \delta^2 \int_\Omega \nabla w(x) \cdot \nabla v(x)\, dx + \int_\Omega w(x) v(x)\, dx. \tag{0.9}$$

Write the variational formulation of Problem (0.8) and prove that it has one and only one solution $\varphi^\delta \in H_0^1(\Omega)$.

Proof: Define $f_u(v) := \int_\Omega u(x)v(x)\, dx$. Then the variational problem is: Find $\varphi^\delta \in H_0^1(\Omega)$ such that

$$a^\delta(\varphi^\delta, v) = f_u(v)$$

for all $v \in H_0^1(\Omega)$. To show that there is one and only one solution, we can use Lax-Milgram theorem. We first need to show that a^δ and f_u are both continuous and a^δ is coercive. So consider,

$$\begin{aligned} a^\delta(w, v) &= \delta^2 \int_\Omega \nabla w(x) \cdot \nabla v(x)\, dx + \int_\Omega w(x) v(x)\, dx \\ &\le \delta^2 |w|_{H^1(\Omega)} |v|_{H^1(\Omega)} + ||w||_{L^2(\Omega)} ||v||_{L^2(\Omega)} \\ &\le \max\{\delta^2, 1\} \big(|w|_{H^1(\Omega)} |v|_{H^1(\Omega)} + ||w||_{L^2(\Omega)} ||v||_{L^2(\Omega)} \big) \\ &\le \max\{\delta^2, 1\} ||w||_{H^1(\Omega)} \big(|v|_{H^1(\Omega)} + ||v||_{L^2(\Omega)} \big) \\ &\le 2 \max\{\delta^2, 1\} ||w||_{H^1(\Omega)} ||v||_{H^1(\Omega)}. \end{aligned}$$

Thus a^δ is continuous. Cauchy-Schwarz shows that f_u is continuous, so we only need to check that a^δ is coercive. So consider,

$$\begin{aligned} a^\delta(v, v) &= \delta^2 \int_\Omega |\nabla v|^2\, dx + \int_\Omega v^2\, dx \\ &\ge \min\{\delta^2, 1\} ||v||^2_{H^1(\Omega)}. \end{aligned}$$

Thus by Lax-Milgram, there exists a unique solution to the variational formulation. ■

b. Prove that

$$||\varphi^\delta||_{L^2(\Omega)} \le ||u||_{L^2(\Omega)}. \tag{0.10}$$

Proof: From the variational equation, we have

$$a(\varphi^\delta, \varphi^\delta) = L(\varphi^\delta).$$

Hence we can write,

$$||\varphi^\delta||^2_{L^2(\Omega)} \leq \int_\Omega \delta^2 |\nabla\varphi^\delta|^2 + (\varphi^\delta)^2 \, dx = \int_\Omega u\varphi^\delta \, dx \leq ||u||_{L^2(\Omega)} ||\varphi^\delta||_{L^2(\Omega)}.$$

Dividing by $||\varphi^\delta||_{L^2(\Omega)}$ we have the result. ■

c. Prove that

$$||\nabla\varphi^\delta||_{L^2(\Omega)} \leq ||\nabla u||_{L^2(\Omega)}. \tag{0.11}$$

Hint: observe that $\Delta\varphi^\delta$ belongs to $L^2(\Omega)$, take the scalar product of (0.8) with $-\Delta\varphi^\delta$ and apply Green's formula.

Proof: To see that $\Delta\varphi^\delta$ is in $L^2(\Omega)$, consider,

$$||\delta^2\Delta\varphi^\delta||_{L^2(\Omega)} = ||u - \varphi^\delta||_{L^2(\Omega)} \leq ||u||_{L^2(\Omega)} + ||\varphi^\delta||_{L^2(\Omega)} < \infty.$$

So, following the hint, we have,

$$\begin{aligned}
\int_\Omega \delta^2(\Delta\varphi^\delta)^2 - \varphi^\delta\Delta\varphi^\delta \, dx &= \delta^2 \int_\Omega (\Delta\varphi^\delta)^2 \, dx + \int_\Omega |\nabla\varphi^\delta|^2 \, dx - \int_{\partial\Omega} \varphi^\delta \frac{\partial\varphi^\delta}{\partial n} \, ds \\
&= \delta^2 \int_\Omega (\Delta\varphi^\delta)^2 \, dx + \int_\Omega |\nabla\varphi^\delta|^2 \, dx \\
&= \delta^2 ||\Delta\varphi^\delta||^2_{L^2(\Omega)} + ||\nabla\varphi^\delta||^2_{L^2(\Omega)}
\end{aligned}$$

For the right hand side, we have,

$$-\int_\Omega u\Delta\varphi^\delta \, dx = \int_\Omega \nabla u \cdot \nabla\varphi^\delta \, dx - \int_{\partial\Omega} u \frac{\partial\varphi^\delta}{\partial n} \, ds = \int_\Omega \nabla u \cdot \nabla\varphi^\delta \, dx$$

Hence we have $\delta^2||\Delta\varphi^\delta||^2_{L^2(\Omega)} + ||\nabla\varphi^\delta||^2_{L^2(\Omega)} = \int_\Omega \nabla u \cdot \nabla\varphi^\delta \, dx$. Using the usual inequality tricks, we have,

$$||\nabla\varphi^\delta||^2_{L^2(\Omega)} \leq ||\nabla u||_{L^2(\Omega)} ||\nabla\varphi^\delta||_{L^2(\Omega)}.$$

Dividing by $||\nabla\varphi^\delta||_{L^2(\Omega)}$ we thus have the result.

d. Now let

$$X_{0,h} = \{v_h \in C^0(\Omega) \, : \, \forall T \in \mathcal{T}_h, \, v_h|_T \in \mathbb{P}_1, \, v_h|_{\partial\Omega} = 0\}. \tag{0.12}$$

Given $u_h \in X_{0,h}$, consider the discrete problem: Find $\varphi_h^\delta \in X_{0,h}$, satisfying $\forall v_h \in X_{0,h}$,

$$a^\delta(\varphi_h^\delta, v_h) = \int_\Omega u_h(x) v_h(x)\, dx. \tag{0.13}$$

(i) Prove that problem (0.13) has one and only one solution $\varphi_h^\delta \in X_{0,h}$.
(ii) Prove that

$$||\varphi_h^\delta||_{L^2(\Omega)} \le ||u_h||_{L^2(\Omega)}. \tag{0.14}$$

Proof: For part (i) note that $X_{0,h}$ is a subspace of $H_0^1(\Omega)$, then Lax-Milgram applies in this case. For part (ii) the same method we used in part b. can be applied again here. ■

e. Assume that φ^δ belongs to $H^2(\Omega)$. Let $\pi_{1,h}$ denote the standard Lagrange interpolant on $X_{0,h}$.

(i) Prove that

$$a^\delta(\varphi^\delta - \varphi_h^\delta, \varphi^\delta - \varphi_h^\delta) = a^\delta(\varphi^\delta - \varphi_h^\delta, \varphi^\delta - \pi_{1,h}(\varphi^\delta)) - \int_\Omega (u - u_h)(\varphi_h^\delta - \varphi^\delta + \varphi^\delta - \pi_{1,h}(\varphi^\delta)) dx.$$

(ii) Assuming that u is smooth enough, $u_h = \pi_{1,h}(u)$, and $\delta = h$, derive an estimate for $||\varphi^\delta - \varphi_h^\delta||_{L^2(\Omega)}$.

Proof: For part (i) consider,

$$\begin{aligned}
a^\delta(\varphi^\delta - \varphi_h^\delta, \varphi^\delta - \varphi_h^\delta) &= a^\delta(\varphi^\delta - \varphi_h^\delta, \varphi^\delta - \pi_{1,h}(\varphi^\delta) + \pi_{1,h}(\varphi^\delta) - \varphi_h^\delta) \\
&= a^\delta(\varphi^\delta - \varphi_h^\delta, \varphi^\delta - \pi_{1,h}(\varphi^\delta)) + a^\delta(\varphi^\delta, \pi_{1,h}(\varphi^\delta) - \varphi_h^\delta) - a^\delta(\varphi_h^\delta, \pi_{1,h}(\varphi^\delta) - \varphi_h^\delta) \\
&= a^\delta(\varphi^\delta - \varphi_h^\delta, \varphi^\delta - \pi_{1,h}(\varphi^\delta)) + \int_\Omega u(\pi_{1,h}(\varphi^\delta) - \varphi_h^\delta)\, dx \\
&\qquad\qquad - \int_\Omega u_h(\pi_{1,h}(\varphi^\delta) - \varphi_h^\delta)\, dx \\
&= a^\delta(\varphi^\delta - \varphi_h^\delta, \varphi^\delta - \pi_{1,h}(\varphi^\delta)) - \int_\Omega (u - u_h)(\varphi_h^\delta - \pi_{1,h}(\varphi^\delta))\, dx \\
&= a^\delta(\varphi^\delta - \varphi_h^\delta, \varphi^\delta - \pi_{1,h}(\varphi^\delta)) - \int_\Omega (u - u_h)(\varphi_h^\delta - \varphi^\delta + \varphi^\delta - \pi_{1,h}(\varphi^\delta))\, dx.
\end{aligned}$$ ■

Note we have used the fact that $\pi_{1,h}(\varphi^\delta) - \varphi_h^\delta \in X_{0,h}$, so φ^δ and φ_h^δ will solve their respective variational equations.

Now for part (ii). Using the identity we proved in part (i) and continuity and coercivity of a, we have,

$$
\begin{aligned}
h^2|\varphi^\delta - \varphi_h^\delta|^2_{H^1(\Omega)} + ||\varphi^\delta - \varphi_h^\delta||^2_{L^2(\Omega)} &= a^\delta(\varphi^\delta - \varphi_h^\delta, \varphi^\delta - \varphi_h^\delta) \\
&= a^\delta(\varphi^\delta - \varphi_h^\delta, \varphi^\delta - \pi_{1,h}(\varphi^\delta)) \\
&\quad - \int_\Omega (u - u_h)(\varphi_h^\delta - \varphi^\delta + \varphi^\delta - \pi_{1,h}(\varphi^\delta))\, dx \\
&\leq ||\varphi^\delta - \varphi_h^\delta||_{L^2(\Omega)} ||\varphi^\delta - \pi_{1,h}(\varphi^\delta)||_{L^2(\Omega)} \\
&\quad + h^2|\varphi^\delta - \varphi_h^\delta|_{H^1(\Omega)} |\varphi^\delta - \pi_{1,h}(\varphi^\delta)|_{H^1(\Omega)} \\
&\quad + ||u - u_h||_{L^2(\Omega)} (||\varphi_h^\delta - \varphi^\delta||_{L^2(\Omega)} + ||\varphi^\delta - \pi_{1,h}(\varphi^\delta)||_{L^2(\Omega)}) \\
&\leq \frac{1}{2}||\varphi^\delta - \varphi_h^\delta||^2_{L^2(\Omega)} + \frac{h^2}{2}|\varphi^\delta - \varphi_h^\delta|^2_{H^1(\Omega)} \\
&\quad + \frac{1}{2}||\varphi^\delta - \pi_{1,h}(\varphi^\delta)||^2_{L^2(\Omega)} + \frac{h^2}{2}|\varphi^\delta - \pi_{1,h}(\varphi^\delta)|^2_{H^1(\Omega)} \\
&\quad + Ch^2|u|_{H^2(\Omega)} (||\varphi_h^\delta - \varphi^\delta||_{L^2(\Omega)} + Ch^2|\varphi^\delta|_{H^2(\Omega)}) \\
&\leq \frac{1}{2}||\varphi^\delta - \varphi_h^\delta||^2_{L^2(\Omega)} + \frac{h^2}{2}|\varphi^\delta - \varphi_h^\delta|^2_{H^1(\Omega)} \\
&\quad + Ch^4|\varphi^\delta|^2_{H^2(\Omega)} + Ch^4|\varphi^\delta|^2_{H^2(\Omega)} \\
&\quad + Ch^4|u|^2_{H^2(\Omega)} + \frac{1}{4}||\varphi_h^\delta - \varphi^\delta||^2_{L^2(\Omega)} + Ch^4|u|_{H^2(\Omega)}|\varphi^\delta|_{H^2(\Omega)}. \blacksquare
\end{aligned}
$$

There are a few things to note about the inequalities above. First, we have used the standard results from the Bramble-Hilbert lemma to get the coefficients in terms of h. Second, in the last inequality, we used the inequality $ab \leq \frac{1}{2}(a^2 + b^2)$, but in the form,

$$
Ch^2|u|_{H^2(\Omega)} ||\varphi_h^\delta - \varphi^\delta||_{L^2(\Omega)} = \big(\sqrt{2}Ch^2|u|_{H^2(\Omega)}\big)\big(\frac{1}{\sqrt{2}}||\varphi_h^\delta - \varphi^\delta)||_{L^2(\Omega)}\big).
$$

Now subtract $\frac{h^2}{2}|\varphi^\delta - \varphi_h^\delta|^2_{H^1(\Omega)}$ and $\frac{3}{4}||\varphi^\delta - \varphi_h^\delta||^2_{L^2(\Omega)}$ from both sides of the inequality to get,

$$
\frac{h^2}{2}|\varphi^\delta - \varphi_h^\delta|^2_{H^1(\Omega)} + \frac{1}{4}||\varphi^\delta - \varphi_h^\delta||^2_{L^2(\Omega)} \leq Ch^4(|\varphi^\delta|^2_{H^2(\Omega)} + |u|^2_{H^2(\Omega)} + |u|_{H^2(\Omega)}|\varphi^\delta|_{H^2(\Omega)}).
$$

Dropping the H^1-semi norm on the left hand side, we are left with our estimate,

$$
||\varphi^\delta - \varphi_h^\delta|| \leq Ch^2(|\varphi^\delta|_{H^2(\Omega)} + |u|_{H^2(\Omega)}).
$$

Problem 3. Let $T > 0$ be a given final time, let $\boldsymbol{b}$ be a given vector valued function with components in $L^2(0,T;H^1(\Omega)) \cap C^0(\Omega \times [0,T])$ and let u_0 be a given real valued function in $C^0(\Omega)$. We suppose that

$$\text{div } \boldsymbol{b} = 0 \text{ a.e. in } \Omega, \ \boldsymbol{b} = \vec{0} \text{ on } \Gamma. \tag{0.15}$$

Consider the time-dependent problem: Find u such that

$$\begin{aligned}\frac{\partial u}{\partial t}(x,t) + \boldsymbol{b}(x,t)\cdot\nabla u(x,t) = 0 \text{ a.e. in } \Omega\times(0,T),\\ u(x,0) = u_0(x) \text{ a.e. in } \Omega,\end{aligned} \tag{0.16}$$

where

$$\boldsymbol{b}\cdot\nabla u = b_1\frac{\partial u}{\partial x_1} + b_2\frac{\partial u}{\partial x_2} \tag{0.17}$$

Accept as a fact that (0.16) has one and only one solution u that is sufficiently smooth. It is discretized as follows in space and time. Let

$$X_h = \{v_h \in C^0(\Omega) : \forall T \in \mathcal{T}_h,\ v_h|_T \in \mathbb{P}_1\}. \tag{0.18}$$

Choose an integer $K \geq 2$, set $k = T/K$, $t_n = nk$ and $u_h^0 = \pi_{1,h}(u_0)$. For $1 \leq n \leq K$, define $u_h^n \in X_h$ from u_h^{n-1} recursively by,

$$\frac{1}{k}\int_\Omega (u_h^n - u_h^{n-1})(x)v_h(x)\,dx + \int_\Omega (\boldsymbol{b}(x,t_n)\cdot\nabla u_h^n(x))v_h(x)\,dx = 0, \tag{0.19}$$

for all $v_h \in X_h$.

(a) Prove that

$$\int_\Omega (\boldsymbol{b}(x,t_n)\cdot\nabla v_h(x))v_h(x)\,dx = 0, \tag{0.20}$$

for all $v_h \in X_h$.

(b) Show that, given $u_h^{n-1} \in X_h$, (0.19) has one and only one solution $u_h^n \in X_h$.

(c) Prove for $1 \leq n \leq K$

$$||u_h^n||_{L^2(\Omega)} \leq ||u_h^0||_{L^2(\Omega)}. \tag{0.21}$$

(d) Is the matrix of the system (0.19) symmetric ? Justify your answer.

Solution. **(a)** Let $v_h \in X_h$. We will show the property in the problem statement by using integration by parts and using the fact that div$\boldsymbol{b} = 0$ a.e. in Ω and $\boldsymbol{b} = \mathbf{0}$ on Γ. We see that:

$$\begin{aligned}\int_\Omega (\boldsymbol{b}(x,t_n)\cdot\nabla v_h(x))v_h(x)\,dx &= \int_\Omega (v_h(x)\boldsymbol{b}(x,t_n))\cdot\nabla v_h(x)\,dx\\ &= \cancel{\int_{\partial\Omega}(v_h^2(x)\boldsymbol{b}(x,t_n))\cdot\mathbf{n}\,ds} - \int_\Omega \nabla\cdot(v_h(x)\boldsymbol{b}(x,t_n))v_h(x)\,dx\\ &= -\int_\Omega \Big(\boldsymbol{b}\cdot\nabla v_h(x) + \cancel{v_h\nabla\cdot\boldsymbol{b}}\Big)v_h(x)\,dx\\ &= -\int_\Omega \big(\boldsymbol{b}(x,t_n)\cdot\nabla v_h(x)\big)v_h(x)\,dx\\ &= -\int_\Omega (\boldsymbol{b}(x,t_n)\cdot\nabla v_h(x))v_h(x)\,dx.\end{aligned}$$

Thus, adding the right hand side to the left side, we have the result.

(b) We want to show that given $u_h^{n-1} \in X_h$, the equation (0.19) has one and only one solution. Let us first show that we have a unique solution.

Assume there exists two solutions u_h^n and $\tilde{u}_h^n$. That is, u_h^n and $\tilde{u}_h^n$ both solve (0.19):

$$\frac{1}{k}\int_\Omega (u_h^n - u_h^{n-1})(x)v_h(x)\,dx + \int_\Omega (\boldsymbol{b}(x,t_n)\cdot\nabla u_h^n(x))v_h(x)\,dx = 0,$$
$$\frac{1}{k}\int_\Omega (\tilde{u}_h^n - u_h^{n-1})(x)v_h(x)\,dx + \int_\Omega (\boldsymbol{b}(x,t_n)\cdot\nabla \tilde{u}_h^n(x))v_h(x)\,dx = 0.$$

Subtracting the two equations gives

$$\frac{1}{k}\int_\Omega (u_h^n - \tilde{u}_h^n)(x)v_h(x)\,dx + \int_\Omega \boldsymbol{b}(x,t)\cdot(\nabla u_h^n - \nabla\tilde{u}_h^n)v_h\,dx = 0.$$

Since this holds for every $v_h \in X_h$, we can take $v_h = u_h^n - \tilde{u}_h^n$. Thus by part (a), we have that

$$\frac{1}{k}||u_h^n - \tilde{u}_h^n||^2_{L^2(\Omega)} + \cancel{\int_\Omega \boldsymbol{b}(x,t)\cdot(\nabla u_h^n - \nabla\tilde{u}_h^n)(u_h^n - \tilde{u}_h^n)\,dx} = 0.$$

Thus $u_h^n = \tilde{u}_h^n$.

To show existence, one must write the discrete equation into a linear system of equations and invoke the Rank-Nullity theorem (Hint: the system $A\boldsymbol{x} = \boldsymbol{b}$ has a unique solution when $\text{nullity}(A) = 0$).

(c) We want to prove the estimate in part (c). Let $u_h^0 = \pi_{1,h}(u_0)$. Let us take $v_h = u_h^n$ and substitute into (0.19). As a consequence of the property in part (a), we see that

$$\frac{1}{k}(u_h^n - u_h^{n-1}, u_h^n) + \cancel{\int_\Omega (\boldsymbol{b}(x,t_n)\cdot\nabla u_h^n)u_h^n\,dx} = 0.$$

Then applying the Cauchy-Schwarz inequality, we have that

$$||u_h^n||^2_{L^2(\Omega)} \le ||u_h^n||_{L^2(\Omega)}||u_h^{n-1}||_{L^2(\Omega)}.$$

Dividing by $||u_h^n||_{L^2(\Omega)}$ yields

$$||u_h^n||_{L^2(\Omega)} \le ||u_h^{n-1}||_{L^2(\Omega)}.$$

Then applying this inequality for each n yields the result:

$$||u_h^n||_{L^2(\Omega)} \le ||u_h^0||_{L^2(\Omega)}.$$

(d) To show that the matrix of the system (0.19), is or is not symmetric, we consider the terms of the discrete equation.

Note that for $u_h, v_h \in X_h$:

$$a(u_h, v_h) := \int_\Omega (\boldsymbol{b}(x, t_n) \cdot \nabla u_h) v_h(x)\, dx,$$

is not a symmetric bilinear form. To show this, we apply integration by parts to get the following:

$$a(u_h, v_h) = -\int_\Omega \nabla\cdot(v_h(x)\boldsymbol{b}(x, t_n))u_h dx = -\int_\Omega \boldsymbol{b}(x, t_n)\cdot\nabla v_h(x) u_h(x) dx = -a(v_h, u_h).$$

Note that the other integral term is symmetric. Thus, we will have the sum of a symmetric and non-symmetrix matrix, so the final matrix of system will not be symmetric.