Lecture 5

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Qualifying Prep Course – Numerical

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Outline

Problem 2

Consider the two finite elements (τ, Q_1, Σ) and $(\tau, \tilde{Q}_1, \Sigma)$ where $\tau = [-1, 1]^2$ is the reference square and

$$Q_1 = \text{span}\{1, x, y, xy\} \tag{1.1}$$

$$\tilde{Q}_1 = \text{span}\{1, x, y, x^2 - y^2\}. \tag{1.2}$$

$\Sigma = \{w(1, 0), w(-1, 0), w(0, 1), w(0, -1)\}$ is the set of values of a function $w(x, y)$ at the midpoints of the edges τ.

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(a) Which of the two elements is unisolvent? Prove it!

(b) Show that the unisolvent element leads to a finite element space, which is not H^1-conforming.

Definition 5.2 (Finite element). *Let $d \geq 1$, an integer $n_{\rm sh} \geq 1$, and the set $\mathcal{N} := \{1{:}n_{\rm sh}\}$. A* finite element *consists of a triple (K, P, Σ) where:*

(i) *K is a* polyhedron *in $\mathbb{R}^d$ or the image of a polyhedron in $\mathbb{R}^d$ by some smooth diffeomorphism. More generally, K could be the closure of a Lipschitz domain in $\mathbb{R}^d$ (see §3.1). K is nontrivial, i.e.,* $\mathrm{int}(K) \neq \emptyset$.

(ii) *P is a* finite-dimensional vector space *of functions $p : K \to \mathbb{R}^q$ for some integer $q \geq 1$ (typically $q \in \{1, d\}$). P is nontrivial, i.e., $P \neq \{0\}$. The members of P are polynomial functions, possibly composed with some smooth diffeomorphism.*

(iii) *Σ is a* set of $n_{\rm sh}$ linear forms *from P to $\mathbb{R}$, say $\Sigma := \{\sigma_i\}_{i\in\mathcal{N}}$, such that the map $\Phi_\Sigma : P \to \mathbb{R}^{n_{\rm sh}}$ defined by $\Phi_\Sigma(p) := \big(\sigma_i(p)\big)_{i\in\mathcal{N}}$ is an isomorphism. The linear forms σ_i are called* degrees of freedom *(in short* dofs*), and the bijectivity of the map Φ_Σ is referred to as* unisolvence.

Remark 5.3 (Proving unisolvence). To prove unisolvence, it suffices to show that $\dim P \geq n_{\rm sh} = \mathrm{card}\,\Sigma$ and that Φ_Σ is injective, i.e.,

$$[\,\sigma_i(p) = 0,\ \forall i \in \mathcal{N}\,] \implies [\,p = 0\,], \qquad \forall p \in P. \tag{5.4}$$

Owing to the rank nullity theorem Φ_Σ is then bijective and $\dim P = n_{\rm sh}$. $\square$

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We now want to check that $\Sigma \ni \sigma_i(p) = 0 \, \forall i \in \{1, 2, 3, 4\}$ implies that $p = 0$ for any $p \in Q_1$. Let $w(x, y)$ be an arbitrary polynomial in Q_1:

$$w(x, y) = a + bx + cy + dxy$$

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Then (1.3) implies $a = -b$; (1.4) implies $b = 0 \implies a = 0$; (1.5) implies $c = 0$. However, d is a free variable and can be chosen to be any value. Thus, $w(x, y)$ is not identically zero and consequently (τ, Q_1, σ) is **not** unisolvent.

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Thus, $w(x, y)$ is identically zero and consequently $(\tau, \tilde{Q}_1, \sigma)$ **is** unisolvent.

Solution to (b)

We now want to show that $(\tau, \tilde{Q}_1, \sigma)$ leads to a finite element space which is not H^1-conforming.

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Thus we want to construct a finite element space for $(\tau, \tilde{Q}_1, \sigma)$ such that $V_h \not\subset H^1$.

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Then, the unisolvent finite element leads to the following finite element space:

$$V_h := \{v : \Omega \to \mathbb{R} \,\big|\, v|_{K_i} \in \tilde{Q}_1 \text{ for } i = 1,2 \text{ and } v|_{K_1}(1,0) = v|_{K_2}(1,0)\}. \tag{1.11}$$

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We claim that $V_h \not\subset H^1(\Omega)$.

Let $w \in V_h$ and let us define $w_1 := w\big|_{K_1}$ and $w_2 := w\big|_{K_2}$ such that

$$w_1(x, y) := x^2 - y^2,$$
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Therefore, $w \notin H^1(\Omega)$. Note that this discontinuity is not *removable*[2]. Hence, the finite element space is not H^1-conforming.

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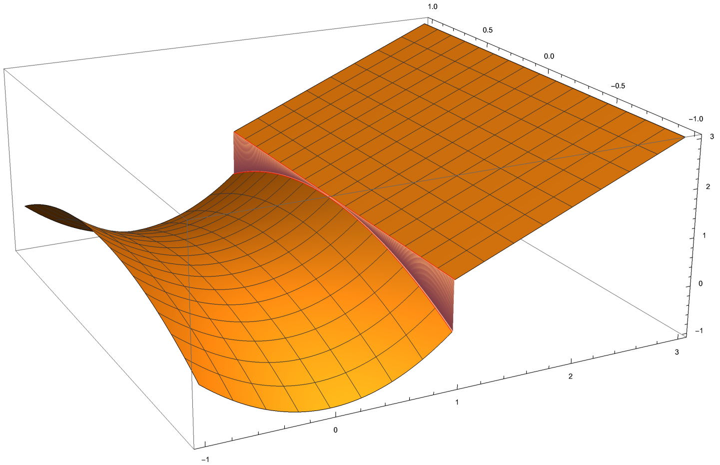
1.0
0.5
0.0
−0.5
−1.0
3
2
1
0
−1
−1
0
1
2
3

Outline

Problem 1

Consider the following finite element triple:

- K = a rectangle with vertices $\{a^i\}$, $i = 1, 2, 3, 4$.
- $P = Q^3 = \text{span}\{x_1^i x_2^j : i, j = 0, \ldots, 3\}$
- $N = \{p(a^i), p_1(a^i), p_2(a^i), p_{12}(a^i),\ i = 1, 2, 3, 4\}$. (Here p_i denotes differentiation with respect to x_i.

(a) Show that the above finite element is unisolvent.

(b) What do you need to do to check if the above element with a rectangular mesh results in a C^1 finite element space?

(c) Does the above element (with a rectangular mesh) result in a C^1 finite element space? (Explain your answer).

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To further simplify our computations, we will show unisolvence on the unit square $\hat{K} = [0, 1] \times [0, 1]$.

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Definition 3.11 Affine equivalence of finite elements

Let $\mathcal{T}_h$ be a triangulation of $\Omega \subset \mathbb{R}^d$, $K \in \mathcal{T}_h$ and P_K, Σ_K as in Definition 3.9. Let further $(\hat{K}, \hat{P}_{\hat{K}}, \hat{\Sigma}_{\hat{K}})$ be a **reference element**. Then, the finite elements (K, P_K, Σ_K), $K \in \mathcal{T}_h$, are said to be **affine equivalent** to the reference element $(\hat{K}, \hat{P}_{\hat{K}}, \hat{\Sigma}_{\hat{K}})$, if there exists an invertible affine mapping $F_K : \mathbb{R}^d \to \mathbb{R}^d$ such that for all $K \in \mathcal{T}_h$

$$K = F_K(\hat{K}) \ , \tag{3.15}$$

$$P_K = \{p : K \to \mathbb{R} \mid p = \hat{p} \circ F_K^{-1} \ , \ \hat{p} \in \hat{P}_{\hat{K}}\} \ , \tag{3.16}$$

$$\Sigma_K = \{\ell_i : P_K \to \mathbb{R} \mid \ell_i = \hat{\ell}_i \circ F_K^{-1} \ , \ \hat{\ell}_i \in \hat{\Sigma}_{\hat{K}} \ , \ 1 \le i \le n_K\} \ . \tag{3.17}$$

[2] https://www.math.uh.edu/~rohop/Fall_16/downloads/Chapter3.pdf

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We now want to consider the mixed partial derivative of f and show that this must vanish at each a_i for $i = 1, 2, 3, 4$. A simple computation yields:

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\begin{aligned}
f_{xy} = (2x-1)(2y-1)g + (2x-1)(y-1)g_y \\
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Solution to (b)

What do you need to do to check if the above element with a rectangular mesh results in a C^1 finite element space?

Solution to (b)

To check that our finite element space is C^1 globally, we need to check the continuity of our shape functions and their respective derivatives across the edges of adjacent elements.

Solution to (c)

Does the above element (with a rectangular mesh) result in a C^1 finite element space? (Explain your answer).

Solution to (c)

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Let $p \in Q^3$ and $q \in Q^3$ be defined on K_1 and K_2 respectively. Let us define:

$$f = p|_e - q|_e$$
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\begin{aligned}
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We want to show that f and g are identically zero since this will imply that the piecewise defined functions and their derivatives will be continuous across the edge e.

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The same argument can be used to the case of a horizontal line segment e. Thus, the above element will result in a C^1 finite element space.

Outline

Problem 3

Consider the following initial boundary value problem: find $u(x,t)$ such that

$$
\begin{aligned}
u_t - u_{xx} + u &= 0, \qquad 0 < x < 1, \quad t > 0 && (3.1)\\
u_x(0,t) = u_x(1,t) &= 0, \qquad t > 0 && (3.2)\\
u(x,0) &= g(x), \quad 0 < x < 1. && (3.3)
\end{aligned}
$$

(a) Derive the semi-discrete approximation of this problem using linear finite elements over a uniform partition of $(0,1)$. Write it as a system of linear ordinary differential equations for the coefficient vector.

(b) Further, derive discretizations in time using backward Euler and Crank-Nicolson methods, respectively.

(c) Show that both fully discrete schemes are unconditionally stable with respect to the initial data in the spatial $L^2(0,1)$-norm.

Solution to (a)

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Let us proceed "informally". We first multiply the PDE in our IBVP by a sufficiently smooth test function $v(x)$ in some real vector space V, and integrate over the domain $(0,1)$:

$$0 = \int_0^1 \Big(u_t(x,t)v(x) - u_{xx}(x,t)v(x) + u(x,t)v(x) \Big)\, dx,$$

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$$\overset{\text{IBP}}{=} (u_t, v) + \int_0^1 \Big(u_x(x,t)v_x(x) + u(x,t)v(x) \Big)\, dx - \cancel{\big[u_x(x,t)v(x) \big]_0^1}$$

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0 &= \int_0^1 \Big(u_t(x,t)v(x) - u_{xx}(x,t)v(x) + u(x,t)v(x) \Big)\, dx, \\
&\overset{\text{IBP}}{=} (u_t, v) + \int_0^1 \Big(u_x(x,t)v_x(x) + u(x,t)v(x) \Big)\, dx - \cancel{\big[u_x(x,t)v(x)\big]_0^1} \\
&= (u_t, v) + a(u, v),
\end{aligned}$$

where we define the inner product $(\cdot, \cdot)$ and bilinear form $a(\cdot, \cdot)$ to be:

$$(u_t, v) = \int_0^1 u_t v\, dx$$

$$a(u, v) = \int_0^1 (u_x v_x + uv)\, dx.$$

For the integrals above, it is sufficient to take our function space to be $H^1(0, 1)$.

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$$V_h := \{v \in H^1(0,1) \ : \ v|_{[x_i,x_{i+1}]} \in \mathbb{P}_1,\ i = 0,\ldots,N-1\}.$$

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Thus, our weak formulation in the finite element discretization becomes: Find $u_h(t) \in V_h$ such that $(u_{h,t}, v_h) + a(u_h, v_h) = 0$ for all $v_h \in V_h$. Note, $u_h(t) \in V_h$ for each $t > 0$.

Now, let us consider the standard tent functions as our basis for V_h: $\text{span}\{\phi_i\}_{i=0}^N$. We set $v = \phi_j(x)$ and express u_h in terms of the basis functions

$$u_h(x,t) = \sum_{i=0}^{N} u_i(t)\phi_i(x),$$

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where $\{u_i\}_{i=0}^N$ are the unknown coefficients. Substituting these into our weak formulation yields:

$$\int_0^1 \sum_{i=0}^{N} u_i'(t)\phi_i(x)\phi_j(x)\,dx + \int_0^1 \Big(\sum_{i=0}^{N} u_i(t)\phi_i'(x)\phi_j'(x) + \sum_{i=0}^{N} u_i(t)\phi_i(x)\phi_j(x) \Big)\,dx,$$

for $j = 0, \dots, N$.

Using the vector/matrix notation, $\boldsymbol{U}(t) = [u_i(t)]_{i=1}^N$, $\boldsymbol{M} = [(\phi_i, \phi_j)]_{i,j=1}^N$, and $\boldsymbol{A} = [a(\phi_i, \phi_j)]_{i,j=1}^N$, Using the vector/matrix notation, $\boldsymbol{U}(t) = [u_i(t)]_{i=1}^N$, $\boldsymbol{M} = [(\phi_i, \phi_j)]_{i,j=1}^N$, and $\boldsymbol{A} = [a(\phi_i, \phi_j)]_{i,j=1}^N$, we can rewrite our semi-discrete variational equation as,

$$\begin{cases} \boldsymbol{M}\boldsymbol{U}'(t) + \boldsymbol{A}\boldsymbol{U}(t) = \mathbf{0}, \\ \boldsymbol{U}(0) = \boldsymbol{G}, \end{cases}$$

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where $\boldsymbol{G} = [g(x_i)]_{i=1}^N$.This is our system of linear ordinary differential equations.