August 2017

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Qualifying Prep Course – Numerical

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Outline

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Outline

Problem 1

Let K be a non-degenerate triangle in $\mathbb{R}^2$. Let a_1, a_2, a_3 be the three vertices of K. Let $a_{ij} = a_{ji}$ denote the midpoint of the segment $(a_i, a_j), i, j \in \{1, 2, 3\}$. Let $\mathbb{P}^1$ be the set of of linear functions $p(x_1, x_2)$ over K and $\Sigma = \{\sigma_1, \sigma_2, \sigma_3\}$ be the linear forms (or degrees of freedom) on $\mathbb{P}^1$ defined as

$$\sigma_{ij}(p) = p(a_{ij}),\ i, j = 1, 2, 3,\ i \neq j.$$

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(b) Compute the "nodal" basis of $\mathbb{P}^1$ which corresponds to $\{\sigma_{12}, \sigma_{23}, \sigma_{31}\}$.

(c) Let $\mathcal{T}_h$ be a triangulation of the domain Ω with polygonal boundary and let the finite dimensional space $\mathbb{V}$ consist of functions whose restrictions to to each K are the functions from the FE $(K, \mathbb{P}^1, \Sigma)$. Show that in general these functions are NOT in $H^1(\Omega)$.

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(b) Compute the "nodal" basis of $\mathbb{P}^1$ which corresponds to $\{\sigma_{12}, \sigma_{23}, \sigma_{31}\}$.
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(d) If M_K is the element "mass" matrix, evaluate its entries m_{ij}.

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We want to show that the set Σ is a unisolvent set for $\mathbb{P}^1$. First note that $\operatorname{card}(\Sigma) = 3 = \dim \mathbb{P}^1$. We now want to verify that $\sigma(p) = 0$ for all $\sigma \in \Sigma$ implies that $p = 0$ for any $p \in \mathbb{P}^1$. Before we continue, we will show unisolvence on the reference element and invoke affine equivalence of the physical and reference element to show unisolvence on the physical element.

2D triangle	$\widehat{z}_1 = (0,0)$, $\widehat{z}_2 = (1,0)$, $\widehat{z}_3 = (0,1)$
3D tetrahedron	$\widehat{z}_1 = (0,0,0)$, $\widehat{z}_2 = (1,0,0)$, $\widehat{z}_3 = (0,1,0)$, $\widehat{z}_4 = (0,0,1)$
2D square	$\widehat{z}_1 = (0,0)$, $\widehat{z}_2 = (0,1)$, $\widehat{z}_3 = (1,0)$, $\widehat{z}_4 = (1,1)$
3D cube	$\widehat{z}_1 = (0,0,0)$, $\widehat{z}_2 = (1,0,0)$, $\widehat{z}_3 = (0,1,0)$, $\widehat{z}_4 = (0,0,1)$ $\widehat{z}_5 = (1,1,0)$, $\widehat{z}_6 = (1,0,1)$, $\widehat{z}_7 = (0,1,1)$, $\widehat{z}_8 = (1,1,1)$

Table 10.1 Enumeration of the vertices in the reference simplex and in the reference cuboid in dimensions two and three.

Unless specified otherwise we enumerate the vertices of the reference element $\widehat{K}$ by using the convention described in Table 10.1. Moreover $\widehat{K}$ is oriented by using the convention of the increasing vertex-index enumeration as in Figure 10.2.

Let $\hat{K} := \{(x, y) \big| x \geq 0, y \geq 0, x + y \leq 1\}$ be the reference triangular element.

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$$\sigma_{12}(p) = a + \frac{1}{2}b, \qquad \sigma_{23}(p) = a + \frac{1}{2}b + \frac{1}{2}c, \qquad \sigma_{13}(p) = a + \frac{1}{2}c.$$

We see that $\sigma_{12}(p) = 0 \implies a + \frac{1}{2}b = 0$ and from $\sigma_{23} = 0 \implies \cancel{a + \frac{1}{2}}b + c = 0$ implies that $c = 0$ which implies $a = 0$ from $\sigma_{13} = 0$.

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$$w_1 = 1 - 2y, \quad w_2 = 1 - 2(1 - x - y), \quad w_3 = 1 - 2x.$$

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We can also show this with a specific example. Take $\Omega = K_1 \cup K_2$ where K_1 is the reference triangle and K_2 its adjacent triangle. See solution to problem 2 on January 2010 exam.

Solution to (d)

Let $\widetilde{w}_i = w_i \circ F_K^{-1}$, $i = 1, 2, 3$ be basis functions on the physical element. The entries of the mass matrix M_K are defined as

$$m_{ij} = \int_K \widetilde{w}_i \widetilde{w}_j dx, \quad i, j = 1, 2, 3.$$

Note that

$$\int_K \widetilde{w}_i \widetilde{w}_j dx = \frac{|K|}{|\hat{K}|} \int_{\hat{K}} w_i w_j d\hat{x}, \quad i, j = 1, 2, 3,$$

where w_i are the basis functions on the reference element defined above. Then computing these integrals on the reference elements yields the following mass matrix:

$$M_K = \frac{|K|}{|\hat{K}|} \begin{pmatrix} \frac{1}{6} & -\frac{1}{3} & 0 \\ -\frac{1}{3} & \frac{17}{6} & -\frac{1}{3} \\ 0 & -\frac{1}{3} & \frac{1}{6} \end{pmatrix}.$$

Note that we only need to compute the upper or lower triangular matrix since the matrix is symmetric.

Outline

Problem 2

(a) Let $\Omega = (0,1)$. Assume that $u \in H^1(\Omega)$ and let $x_0 \in \overline{\Omega}$. Prove that

$$\|u\|^2_{L^2(\Omega)} \leq C_1\Big(u^2(x_0) + \|u'\|^2_{L^2(\Omega)}\Big), \tag{3.1}$$

with a constant C_1 independent of x_0.

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(b) Consider the fourth-order boundary value problem

$$u^{(iv)} = f \text{ in } \Omega, u(0) = 0,\ u^{''}(0) = 0,\ u^{''}(1) + u^{'}(1) = 1,\ u^{'''}(1) = 0.$$

Derive a weak formulation of this problem assuming that $f \in L^2(\Omega)$.

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(e) Show that the finite element method you derived has a unique solution u_h and derive an optimal order error estimate for $u - u_h$ in the $H^2(\Omega)$-norm. *Hint:* A correct proof will involve using an interpolation error bound. You may state and use such a bound without proving it.

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$$\int_0^1 u^2(x)dx \leq 2\left(u^2(x_0) + \|u^{'}\|^2_{L^2(\Omega)} \right).$$

Note that one can also prove $|u(x_0)|^2 \leq 2\|u\|^2 + 2\|u^{'}\|$ similarly.

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$$\int_0^1 u^{(iv)} v dx \;=\; \left[u^{'''} v\right]_0^1 - \int_0^1 u^{'''} v^{'} dx,$$

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$$\begin{aligned} \int_0^1 u^{(iv)} v dx &= \left[u''' v\right]_0^1 - \int_0^1 u''' v' dx, \\ &= \left[u''' v\right]_0^1 - \left[u'' v'\right]_0^1 + \int_0^1 u'' v'' dx, \end{aligned}$$

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(\textit{Using BCs}) &= -v^{'}(1) + u^{'}(1) v^{'}(1) + \int_0^1 u^{''} v^{''} dx
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\end{aligned}$$

We define the space V to be:

$$V = \{v \in H^2(0,1) : v(0) = 0\}.$$

Thus our weak formulation is as follows: Find $u \in V$ such that $a(u, v) = F(v)$ for any $v \in V$ where

$$a(u, v) := \int_0^1 u'' v'' dx + u'(1)v'(1),$$

$$F(v) := \int_0^1 fv dx + v'(1).$$

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$$a(u,v) = \int_0^1 u'' v'' \, dx + u'(1) v'(1)$$

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a(u,v) &= \int_0^1 u^{''} v^{''}\, dx + u^{'}(1) v^{'}(1) \\
&\leq \|u^{''}\| \|v^{''}\| + \left|u^{'}(1)\right| \left|v^{'}(1)\right|
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&\leq \sqrt{\|u''\|^2 + 2\|u'\|^2 + 2\|u''\|^2} \sqrt{\|v''\|^2 + 2\|v'\|^2 + 2\|v''\|^2} \\
&= \sqrt{3\|u''\|^2 + 2\|u'\|^2} \sqrt{3\|v''\|^2 + 2\|v'\|^2}
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\begin{aligned}
a(u,v) &= \int_0^1 u^{''} v^{''}\, dx + u^{'}(1) v^{'}(1) \\
&\leq \|u^{''}\| \|v^{''}\| + \left|u^{'}(1)\right| \left|v^{'}(1)\right| \\
&\leq \sqrt{\|u''\|^2 + |u'(1)|^2} \sqrt{\|v''\|^2 + |v'(1)|^2} \\
&\leq \sqrt{\|u''\|^2 + 2\|u'\|^2 + 2\|u''\|^2} \sqrt{\|v''\|^2 + 2\|v'\|^2 + 2\|v''\|^2} \\
&= \sqrt{3\|u''\|^2 + 2\|u'\|^2} \sqrt{3\|v''\|^2 + 2\|v'\|^2} \\
&\leq 3\|u\|_{H^2(0,1)} \|v\|_{H^2(0,1)}
\end{aligned}
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Similarly, for $v \in V$

$$|F(v)|^2 = \left| \int_0^1 fv dx + v^{'}(1) \right|^2$$

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\textit{(Using part 1 inequality with } u^{'}) &\leq 2\left(\|f\|^2 \|v\|^2 + 2\|v^{'}\|^2 + 2\|v^{''}\|^2 \right)
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&\leq 2 \left(\|f\|^2 \|v\|^2 + \left| v^{'}(1) \right|^2 \right) \\
\text{(Using part 1 inequality with } u^{'}\text{)} \quad &\leq 2 \left(\|f\|^2 \|v\|^2 + 2\|v^{'}\|^2 + 2\|v^{''}\|^2 \right) \\
&\leq 4 \cdot \max\{1, \|f\|^2\} \|v\|^2_{H^2(0,1)}
\end{aligned}
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Now, for coercivity of $a(\cdot, \cdot)$ we see that for any $v \in V$

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&\geq \frac{1}{2}\|v''\|^2 + \frac{1}{4}\|v'\|^2 \\
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&= \frac{1}{2}\|v''\|^2 + \frac{1}{8}\|v'\|^2 + \frac{1}{8}\|v'\|^2 \\
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&= \frac{1}{2}\|v''\|^2 + \frac{1}{8}\|v'\|^2 + \frac{1}{8}\|v'\|^2 \\
&\geq \frac{1}{2}\|v''\|^2 + \frac{1}{8}\|v'\|^2 + \frac{1}{16}\|v\|^2 \\
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&\geq \frac{1}{16}\|v\|^2_{H^2(0,1)}
\end{aligned}$$

Note that in the second to last inequality, we applied the poincare inequality for $x_0 = 0$ (by the boundary condition $v^2(0) = 0$).

Then, by the following lemma, we have uniqueness of a solution for our weak formulation

Lemma (Lax-Milgram)

Let $a(\cdot, \cdot) : V \times V \to \mathbb{R}$ be a continuous, V-coercive bilinear form. Then for each $F \in V^*$ the variational equation

$$a(u, u) = F(v) \qquad \textit{for all } v \in V$$

has a unique solution $u \in V$. Furthermore, the a priori estimate

$$\|u\| \leq \frac{1}{\frac{1}{16}} \|f\|_*,$$

is valid.

Solution to (d)

We now want to suggest a finite element method of the above problem using piecewise polynomial functions over a uniform partition of Ω into subintervals with length $h = 1/N$.

Solution to (d)

We now want to suggest a finite element method of the above problem using piecewise polynomial functions over a uniform partition of Ω into subintervals with length $h = 1/N$. We can use the finite elements $(K_i, \mathbb{P}^3, \Sigma_i)$ where $K_i = [x_{i-1}, x_i]$ for $i = 1, \dots, N$ and $x_i - x_{i-1} = h$ and $\mathbb{P}^3$ is the space of cubic polynomials. The degrees of freedom are defined to be $\Sigma_i = \{\sigma_{i-1}, \sigma'_{i-1}, \sigma_i, \sigma'_i\}$, for $i = 1, \dots, N$ where

$$\sigma_{i-1}(f) = f(x_{i-1}) \qquad \sigma_i(f) = f(x_i)$$
$$\sigma'_{i-1}(f) = f'(x_{i-1}) \qquad \sigma'_i(f) = f'(x_i).$$

For the finite element approximation, we consider the subspace

$$V_h = \{v \in C^1(\Omega) \,:\, v|_{K_i} \in \mathbb{P}^3,\ i = 1, \ldots, N,\ v(0) = 0\}.$$

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$$V_h = \{v \in C^1(\Omega) \ : \ v|_{K_i} \in \mathbb{P}^3, \ i = 1, \dots, N, \ v(0) = 0\}.$$

A basis for this space is given by $\cup_{i=1}^N \{\phi_i\} \bigcup \cup_{i=0}^N \{\psi_i\}$, where ϕ_i and ψ_i are the cubic Hermite polynomials (note that we remove ϕ_0 because of the boundary condition). Specifically, ϕ_i and ψ_i are defined by the following conditions,

$$\begin{aligned} &\sigma_k(\psi_j) = 0, \, \sigma'_k(\psi_j) = \delta_{kj}, \qquad &\text{for } j, k = 0, \dots, N \\ &\sigma_k(\phi_j) = \delta_{kj}, \, \sigma'_k(\phi_j) = 0, \qquad &\text{for } j, k = 1, \dots, N \end{aligned}$$

Then, the finite element method is as follows: Find $u_h \in V_h$ such that $a(u_h, v_h) = F(v_h)$ for all $v_h \in V_h$.

Solution to (e)

Since $V_h \subset V$, then we know by the Lax-Milgram Lemma there exists a unique $u_h \in V_h$ for the finite element method.

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Solution to (e)

Since $V_h \subset V$, then we know by the Lax-Milgram Lemma there exists a unique $u_h \in V_h$ for the finite element method. We now we want to derive an optimal order error estimate for $u - u_h$. Since we have a unique solution, we can invoke Cea's Lemma to obtain the "best approximation error". Let us recall Cea's Lemma and adapt it to our setting:

Lemma (Cea)

Let $a(\cdot,\cdot)$ be a continuous, V-coercive blinear form. Then for each $F \in V^*$ the continuous problem $a(u,v) = F(v)$ for all $v \in V$ has a unique solution $u \in V$ and the discrete problem $a(u_h, v_h) = F(v_h)$ for all $v_h \in V_h$ has a unique solution $u_h \in V_h$. The error $u - u_h$ satisfies the inequality

$$\|u - u_h\|_{H^2(\Omega)} \leq \frac{3}{\frac{1}{16}} \inf_{v_h \in V_h} \|u - v_h\|_{H^2(\Omega)}.$$

Now, let $\pi_h : C^1(\Omega) \to V_h$ be the canonical Lagrange interpolant.

Now, let $\pi_h : C^1(\Omega) \to V_h$ be the canonical Lagrange interpolant. Then, using standard results we learned in class, we have that for $u \in H^4(\Omega)$:

$$\begin{aligned}
\|u - u_h\|_{H^2(\Omega)} &\leq \frac{3}{\frac{1}{16}} \inf_{v_h \in V_h} \|u - v_h\|_{H^2(\Omega)}, \\
&\leq c\|u - \pi_h u\|_{H^2(\Omega)}, \\
&\leq ch^2 \left|u\right|_{H^4(\Omega)} .
\end{aligned}$$

Outline

Problem 3

Let $u(x,t)$ be a smooth solution satisfying

$$\partial_t u + \beta \partial_x u = 0, \quad x \in \Omega := (0,1), \quad t > 0 \quad \text{and} \quad u(0,x) = \phi(x), x \in \Omega,$$

where $\beta \in \mathbb{R}$ and ϕ is a given smooth function. In addition, we assume that $u(x,t)$ satisfies the periodic boundary condition $u(0,t) = u(1,t), t > 0$. Let $\mathbb{V} = \{v \in H^1(\Omega) : v(0) = v(1)\}$.

(a) Let $N \in \mathbb{N} \setminus \{0\}$, set $h := \frac{1}{N+1}$ and consider the uniform mesh $\mathcal{T}_h$ composed of the cells $[x_i, x_{i+1}], i = 0, \ldots, N$. Let $\mathcal{P}(\mathcal{T}_h)$ be the finite element space composed of continuous piecewise linear functions on $\mathcal{T}_h$. Given $\phi_h \in \mathbb{V} \cap \mathcal{P}(\mathcal{T}_h)$ an approximation of ϕ, consider the semi-discrete method: For $t > 0$, find $u_h(t, \cdot) \in \mathbb{V} \cap \mathcal{P}(\mathcal{T}_h)$ such that $u_h(0, x) = \phi_h(x)$ and for every $v_h \in \mathcal{P}(\mathcal{T}_H)$ with $v_h(0) = v_h(1)$ there holds

$$\frac{h}{2} \sum_{i=0}^{N} \Big(\partial_t u_h(t, x_{i+1}) v_h(x_{i+1}) + \partial_t u_h(t, x_i) v_h(x_i) \Big) + \beta \int_\Omega \partial_x u_h(t, x) v_h(x) dx = 0.$$

Show that the above problem can be reformulated as a system of ODEs and express this system in matrix-vector form. Note: we assume that as a function of t, $u_h(t) \to \mathbb{V} \cap \mathcal{P}(\mathcal{T}_h)$ is smooth.

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(b) Show that the Finite Element approximation $u_h(t)$ satisfies

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$$\frac{h}{2} \sum_{i=0}^{N} \Big(\partial_t u_h(t, x_{i+1}) v_h(x_{i+1}) + \partial_t u_h(t, x_i) v_h(x_i) \Big) + \beta \int_\Omega \partial_x u_h(t, x) v_h(x) dx = 0.$$

Show that the above problem can be reformulated as a system of ODEs and express this system in matrix-vector form. Note: we assume that as a function of t, $u_h(t) \to \mathbb{V} \cap \mathcal{P}(\mathcal{T}_h)$ is smooth.

(b) Show that the Finite Element approximation $u_h(t)$ satisfies

$$\frac{d}{dt} \sum_{i=0}^{N} u_h(t, x_i)^2 = 0.$$

(c) Show that

$$c^{-1} \int u_h^2(t, x) dx \leq h \sum^{N} u_h(t, x_i)^2 \leq c \int u_h^2(t, x) dx$$

Solution to (a)

Let $\{\varphi_i\}_{i=0}^{N+1}$ be the basis of $\mathcal{P}(\mathcal{T}_h)$ consisting of the usual "tent" functions.

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$$u_h(t,x) = \sum_{j=0}^{N+1} U_j(t)\varphi_j(x).$$

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$$u_h(t,x) = \sum_{j=0}^{N+1} U_j(t)\varphi_j(x).$$

Substituting this into the discrete equation and taking $v_h(x) = \varphi_k(x)$ for some $k \in \{0 : N+1\}$, gives the following

$$\frac{h}{2}\sum_{i=0}^{N}\left[\Big(\sum_{j=0}^{N+1} U_j'(t)\varphi_j(x_{i+1})\Big)\varphi_k(x_{i+1}) + \Big(\sum_{j=0}^{N+1} U_j'(t)\varphi_j(x_i)\Big)\varphi_k(x_i)\right] + \beta\sum_{j=0}^{N+1} U_j(t)\int_\Omega \varphi_j'(x)\varphi_k(x)\,dx = 0.$$

Then, recall that that $\varphi_j(x_i) = \delta_{j,i}$.

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$$\sum_{j=0}^{N+1} U_j'(t) \Big[\frac{h}{2} \sum_{i=0}^{N} \big(\delta_{j,i+1}\delta_{k,i+1} + \delta_{j,i}\delta_{k,i} \big) \Big] + \beta \sum_{j=0}^{N+1} U_j(t) \int_\Omega \varphi_j'(x)\varphi_k(x)\, dx = 0.$$

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Let $\boldsymbol{M}$ and $\boldsymbol{A}$ be matrices with entries defined as $m_{j,k} := \sum_{i=0}^{N}(\delta_{j,i+1}\delta_{k,i+1} + \delta_{j,i}\delta_{k,i})$ and $a_{j,k} := \int_\Omega \varphi_j'(x)\varphi_k(x)\,dx$, respectively.

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$$\tfrac{h}{2}\boldsymbol{M}\mathbf{U}'(t) + \beta\boldsymbol{A}\mathbf{U}(t) = 0,$$

where $\boldsymbol{U}(t) = [U_0(t), \ldots, U_{N+1}]^{\mathsf{T}}$.

Solution to (b)

First recall that for a smooth solution u, we have $u\partial_t u = \frac{1}{2}\partial_t(u^2)$ and $u\partial_x au = \frac{1}{2}\partial_x(u^2)$.

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First recall that for a smooth solution u, we have $u\partial_t u = \frac{1}{2}\partial_t(u^2)$ and $u\partial_x au = \frac{1}{2}\partial_x(u^2)$.
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$$\frac{h}{2}\sum_{i=0}^{N}\left(\tfrac{1}{2}\partial_t(u_h(t,x_{i+1})^2) + \tfrac{1}{2}\partial_t(u_h(t,x_i))^2\right) + \beta\int_\Omega \tfrac{1}{2}\partial_x(u_h(t,x)^2)\,dx = 0.$$

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Note that the integral term will vanish because we have periodic boundary conditions.

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Note that the integral term will vanish because we have periodic boundary conditions. Then, the remaining summation term can be written as follows

$$\frac{h}{2}\frac{d}{dt}\Big[\sum_{i=1}^{N} u_h(t,x_i)^2 + \tfrac{1}{2}u_h(t,x_0) + \tfrac{1}{2}u_h(t,x_{N+1})\Big] = 0$$

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Let us take $v_h(x) := u_h(t,x)$ for some fixed t. Substituting this into the discrete yields the following

$$\frac{h}{2}\sum_{i=0}^{N}\left(\tfrac{1}{2}\partial_t(u_h(t,x_{i+1})^2) + \tfrac{1}{2}\partial_t(u_h(t,x_i))^2\right) + \beta\int_\Omega \tfrac{1}{2}\partial_x(u_h(t,x)^2)\,dx = 0.$$

Note that the integral term will vanish because we have periodic boundary conditions. Then, the remaining summation term can be written as follows

$$\frac{h}{2}\frac{d}{dt}\Big[\sum_{i=1}^{N} u_h(t,x_i)^2 + \tfrac{1}{2}u_h(t,x_0) + \tfrac{1}{2}u_h(t,x_{N+1})\Big] = 0$$

But, because we have periodic boundary conditions, we know that $u_h(t,x_{N+1}) = u_h(t,x_0)$. Thus we can combine these terms with the summation:

$$\frac{d}{dt}\sum_{i=0}^{N} u_h(t,x_i)^2 = 0.$$

Solution to (c)

Definition A.5 (Equivalent norms). *Two norms* $\|\cdot\|_{V,1}$ *and* $\|\cdot\|_{V,2}$ *are said to be* equivalent *on* V *if there exists a positive real number* c *such that*

$$c\,\|v\|_{V,2} \leq \|v\|_{V,1} \leq c^{-1}\,\|v\|_{V,2}, \qquad \forall v \in V. \tag{A.1}$$

Whenever (A.1) *holds true,* V *is a Banach space for the norm* $\|\cdot\|_{V,1}$ *if and only if it is a Banach space for the norm* $\|\cdot\|_{V,2}$.

Not sure about this one...

We want to show the first string of inequalities in part (c). We will invoke the equivalence of norms on finite dimensional spaces. Let u_h be an element of our finite dimensional space for some arbitrary h. Let us define the following $\|u_h\|_h := \sqrt{h\sum_{i=0}^{N+1} u_h^2(t, x_i)}$. Then, since are working on a finite dimensional space, we have that for some $c > 0$:

$$\frac{1}{\sqrt{c}}\|u_h\|_{L^2(\Omega)} \leq \|u_h\|_h \leq \sqrt{c}\|u_h\|_{L^2(\Omega)}$$

Then, squaring each term yields the result.