Ideas and Hints for your HW 5

Dr. Diaz

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1 Problems

1. If H is a subgroup of G, then $\phi[H] = \{\phi(h) | h \in H\}$ (and isomorphism from $\langle G, * \rangle$ onto $\langle G', *' \rangle$) is a subgroup of G'. That is an isomorphism carries subgroups into subgroups.

   Hint: We have to prove:

   (a) The multiplication of two elements of $\phi(H)$ is an element in $\phi(H)$.
   Recall that an element $a \in \phi(H)$ has the form $a = \phi(h)$ for some $h \in H$.

   (b) The identity e is an element of $\phi(H)$.
   Under an isomorphism the identity is being mapped to the identity. You may want to prove this again by yourself.

   (c) for all $a \in \phi(H)$ it is true that $a^{-1} \in \phi(H)$
   You may to use the fact that under an homomorphism $\phi(h^{-1}) = [\phi(h)]^{-1}$. If you use this fact, you may need to prove it. Just notice that

   $$\phi(hh^{-1}) = \phi(e) = e'$$

   and then use the fact that ϕ is homomorphism.

2. If G is cyclic, then G' is cyclic under isomorphism.

   Hint: If G is cyclic is generated by and element, let us say that element is a, then I bet that the generator of G' is $\phi(a)$. You may need to use the fact that $\phi(a^n) = \phi(a)^n$

3. Show that a nonempty subset H of a group G is a subgroup if and only if $ab^{-1} \in H, \forall a, b \in H$.

   Hint: With if and only if ($\iff$) we prove this by directions, first we assume the H is a subgroup of G and we prove that $ab^{-1} \in H, \forall a, b \in H$. And second we assume that $ab^{-1} \in H, \forall a, b \in H$ and we prove that H is a subgroup.

   First, $\implies$. So, in this case we assume that H is a subgroup of G and we want to prove $ab^{-1} \in H, \forall a, b \in H$.

Hint: Just use the hypothesis that H is a subgroup which means that if $b \in H$ then $b^{-1} \in H$ and it is close.

Second, $\Longleftarrow$, So, we are asuming $ab^{-1} \in H, \forall a, b \in H$ and we have to prove that H is a subgroup.

Hint: For example, to prove that the identity e is in H, just consider that $ab^{-1} \in H, \forall a, b \in H$ in particular $aa^{-1} \in H$, but $aa^{-1} = e$, therefore, $e \in H$.

4. Prove that if G is an abelian group, written multiplicatively, with identity element e, then all elements x of G satisfying the equation $x^2 = e$ form a subgroup G with identity e.

   Hint: There is no much to prove here, but you may want to use the previous characterization of a subgroup. So, let us take $a, b \in H$ we just need to prove that $ab^{-1} \in H$, therefore we need to show that $(ab^{-1})^2 = e$, but this is

$$(ab^{-1})^2 = ab^{-1}ab^{-1}$$

   which by the group being Abelian is

$$aab^{-1}b^{-1} = a^2(b^2)^{-1} = ee = e$$

5. Let H be a subgroup of a group G. For $a, b \in G$, let $a \sim b \iff ab^{-1} \in H$. Show that $\sim$ is an equivalence relation on G.

   Hint: Observe that $a \sim a$ because $aa^{-1} = e$.

   If $a \sim b$ we need to prove that $b \sim a$, but if $a \sim b$ then $ab^{-1} \in H$ that means $ab^{-1} = h$ for some $h \in H$, this implies

$$e = hba^{-1}$$

   and this implies

$$h^{-1} = ba^{-1}$$

   Because H is a subgroup $h^{-1} \in H$ therefore, $ba^{-1} \in H$