Optimal statistical filtering of noisy data using Wiener filter

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Abstract—**This project demonstrates that the Wiener filter can be used for estimation of random noisy signals in cases where the signal is wide spread stationary. An optimal FIR filter was implemented to estimate three different input signals which were compared to the corresponding target signals to evaluate the performance of the estimation. The first part of the work was to check whether the noise corrupted signal is widespread stationary or not and then apply the statistical filtering method for estimating the signal. Good matches between the input and target signals were achieved in both time and frequency domain. However, the most interesting result appeared when an attempt was made to implement this type of filter in random voice recordings that do not satisfy the wide-spread stationary condition. The results of this implementation showed that the Wiener filter approximated efficiently our initial signal, even when it was not wide-spread stationary.**

Keywords—wiener filter, noise cancellation, correlation, stochastic processes, wide-spread stationary

Introduction

The task of noise reduction and precisely of noise cancellation, becomes more and more relevant these days, as more applications of it appear, such as air-to-ground telecommunications.

A typical example is observed inside a cockpit where audible noise is created by the engine, the wind, as well as from the driver's voice, which is the sound that we eventually want to transmit. A potential solution might be the creation of a bandpass filter which will isolate frequencies that lie within the range of human's voice, however, this wouldn't work on the grounds that both the engine's noise, as well as that of the wind, may interfere with frequencies similar to the ones of human voice. Consequently, active noise canceling seems like the most efficient choice so as to achieve our goal. Such a system would demand at least two sensors, one which will measure the corrupted signal and another one destined to measure the interference itself, isolated from the signal of interest. A process of simply subtracting the interference measured at the secondary sensors from the signal containing both the desired signal and interference would in most cases achieve little improvement. This is because the primary sensor rarely measures the interference with the exact same amplitude, phase, and distortion as the secondary sensors. [1]

We can suggest that both the noise measured in the primer sensor and the one measured in the secondary will be highly correlated due to the fact that they are generated from the same process. The optimal filter choice for signals mixed with noise that are stationary linear stochastic processes with known autocorrelation and cross correlation is the Wiener filter. [2] Hence, the Wiener filter is used in performing different active noise cancellation experiments. In the first part of our paper, we will define what a Wiener filter is, how we are going to set it up computationally using Matlab software and then we are going to perform different experiments with simple sinusoidal functions and real-time signals. Lastly, we conclude by evaluating the performance of the Wiener filter in the various set up experiments.

I. Technical Approach

A. Wiener filter

First, we must define what such a filter is and how it is used for model cancellation. A Wiener filter calculates a statistical estimation of a signal we do not know, by using a correlated input signal and then attempting to remove the noise and eventually estimate our initial signal. To perform that we assume that we have knowledge of the signal's properties, as well as of noise's.

Assume we filter a corrupted signal x[n] such that the output of the filter y[n] approximates some other signal d[n]. The output error e[n] represents the mismatch between y[n] and d[n]. [3]

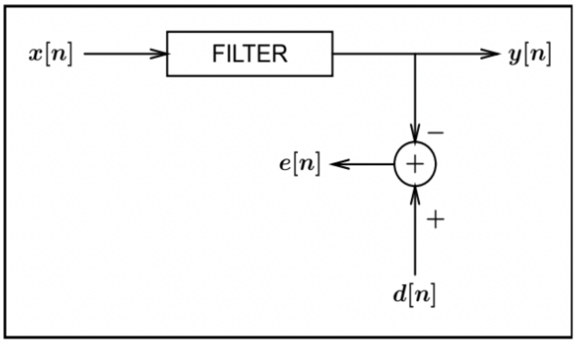

Figure 1. Wiener filtering concept [3]

In order to discuss an "optimal" filter for estimating d[n] from x[n], we must be able to quantify how well the filter performs. A "cost function" may take many different forms and is used to evaluate performance. The mean square error (MSE) is most frequently used as our cost function.

$$\xi = E[e^2[n]]$$

where E[·] represents statistical expectation, and

$$E[e^2[n]] = \int_{-\infty}^{\infty} x^2 p_e(x)\, dx$$

where p_e(x) is the probability density function of the error. The filter that is optimum in the MSE sense is called a Wiener filter.

In each of our analyses, we will assume:

1. x[n] is wide-sense stationary, i.e., it has a constant (and finite) mean and variance, as well as a correlation function that is solely a function of time shift:

$$E[x[n]]\, x[m_1 + n]] = E[x[m_2]\, x[m_2 + n]] \quad \forall m_1, m_2$$

2. All of the signals are zero-mean.

3. MSE is our error criterion.

The definition of correlation between two random processes x[n] and y[n] (when x and y are wide sense

stationary) is:

$$\varphi_{xy}[k] = E[x[n]\, y[k + n]]$$

B. Two sided (Unconstrained) Wiener filter

$$\xi = E[e^2[n]]$$

$$= E[(d[n] - y[n])]^2$$

$$= E[d^2[n]] - 2E[y[n]d[n]] + E[y^2[n]]$$

$$= \varphi_{dd}[0] - 2\varphi_{yd}[0] + \varphi_{yy}[0]$$

So, the cost function is itself a function of the correlation and cross correlation of the output of the filter and the desired output. We know that

$$y[n] = \sum_{m=-\infty}^{\infty} h[m]x[n - m]$$

Observe that, in general, h[m] represents a non-causal filter with infinite impulse response (IIR).

$$\varphi_{yd}[0] = E[y[n]\, d[n]]$$

$$= E\left[\sum_{m=-\infty}^{\infty} h[m]x[n - m]d[n]\right]$$

$$\sum_{m=-\infty}^{\infty} h_{opt}[m]\varphi_{xx}[n_i - m] = \varphi_{xd}[n_i]$$

which must be satisfied for all n_i

This equation specifies the impulse response of the optimal filter. Observe that it depends on the cross correlation between x[n] and d[n] and the autocorrelation function of x[n].

The optimal filter is known as the Wiener solution. Observing that the left hand side of the equation is equal to the convolution of h_{opt}[n] and φ_{xx}[n],

$$H_{opt}(z)\, \Phi_{xx}(z) = \Phi_{xd}(z)$$

$$H_{opt}(z) = \Phi_{xd}(z)\, \Phi_{xx}(z)^{-1}$$

[4] Note:

- The filter obtained may be non-causal!
- $\xi_{min} = \varphi_{dd}[0] - \sum_{n=-\infty}^{\infty} h_{opt}[n]\varphi_{xd}[n] \neq 0,$ in general

Example: Noise canceling

Assuming that s[n] and v[n] are uncorrelated, zero mean:

$$E[x[n]\, x[k + n]] = E[(s[n + k] + u[n + k])]$$
$$= \varphi_{ss}[k] + \varphi_{\upsilon\upsilon}[k]$$

$$\Phi_{xx}(z) = \Phi_{ss}(z) + \Phi_{\upsilon\upsilon}(z)$$

$$E[x[n]\, d[k + n]] = E[(s[n] + u[k])\, s[n + k]$$
$$= \varphi_{ss}[k]$$

$$\Phi_{xd}(z) = \Phi_{ss}(z)$$

Thus, $H_{opt}(z) = \frac{\Phi_{ss}(z)}{\Phi_{ss}(z) + \Phi_{\upsilon\upsilon}(z)}$

We notice two properties from this solution:

1. At “frequencies” where $\Phi_{\upsilon\upsilon}(z) \to 0$ (i.e. the noise is zero) there is no need to filter the signal and $H_{opt}(z) \to 1$.

2. At “frequencies” where $\Phi_{\upsilon\upsilon}(z) \to \infty$ (i.e. where the noise dominates) the filter is “turned off” and

C. Optimal FIR Wiener filter

Instead of using a given data matrix X and output vector Y, the causal finite impulse response (FIR) Wiener filter determines suitable tap weights based on the statistics of the input and output signals. It populates the input matrix X with estimates of the input signal's (T) autocorrelation and the output vector Y with estimates of the cross-correlation between the output and input signals (V).[5]

In order to derive the coefficients of the Wiener filter, consider the signal w[n] being fed to a Wiener filter of order (number of past taps) N and with coefficients {a0,a1,a2,...an}.[6] The output of the filter is denoted x[n] which is given by the expression**:**

$$x[n] = \sum_{i=0}^{N} a_i w[n - i]$$

The residual error is denoted e[n] and is defined as e[n] = x[n] − s[n]. The Wiener filter is designed so as to minimize the mean square error which can be stated concisely as follows:

$$a_i = \arg\min E\left[e^2[n]\right]$$

where {a0,a1,a2,...an} denotes the expectation operator. For simplicity, the following considers only the case where all these quantities are real. The mean square error (MSE) may be rewritten as

$$\begin{aligned} E\left[e^2[n]\right] &= E\left[(x[n]-s[n])^2\right] \\ &= E\left[x^2[n]\right]+E\left[s^2[n]\right]-2E[x[n]s[n]] \\ &= E\left[\left(\sum_{i=0}^{N} a_i w[n-i]\right)^2\right]+E\left[s^2[n]\right]-2E\left[\sum_{i=0}^{N} a_i w[n-i]s[n\right. \end{aligned}$$

To find the vector {a0,a1,a2,...an} which minimizes the expression above, calculate its derivative with respect to each term in the above vector

$$\begin{aligned} \frac{\partial}{\partial a_i} E\left[e^2[n]\right] &= \frac{\partial}{\partial a_i}\left\{E\left[\left(\sum_{i=0}^{N} a_i w[n-i]\right)^2\right]+E\left[s^2[n]\right]-2E\left[\sum_{i=0}^{N} a_i w[n-i]s[n]\right]\right\} \\ &= 2E\left[\left(\sum_{j=0}^{N} a_j w[n-j]\right)w[n-i]\right]-2E[w[n-i]s[n]] \\ &= 2\left(\sum_{j=0}^{N} E[w[n-j]w[n-i]]a_j\right)-2E[w[n-i]s[n]] \end{aligned}$$

Assuming that w[n] and s[n] are each stationary and jointly stationary, the sequences Rw[m] and Rws[m] known respectively as the autocorrelation of w[n] and the cross-correlation between w[n] and s[n] can be defined as follows:

$$\begin{aligned} R_w[m] &= E\{w[n]w[n+m]\} \\ R_{ws}[m] &= E\{w[n]s[n+m]\} \end{aligned}$$

The derivative of the MSE may therefore be rewritten as:

$$\frac{\partial}{\partial a_i} E\left[e^2[n]\right] = 2\left(\sum_{j=0}^{N} R_w[j-i]a_j\right)-2R_{ws}[i] \qquad i=0,\cdots,N.$$

Note that for real w[n], the autocorrelation is symmetric:

$$R_w[j-i] = R_w[i-j]$$

Letting the derivative be equal to zero results in:

$$\sum_{j=0}^{N} R_w[j-i]a_j = R_{ws}[i] \qquad i=0,\cdots,N.$$

which can be rewritten (using the above symmetric property) in matrix form.

$$\underbrace{\begin{bmatrix} R_w[0] & R_w[1] & \cdots & R_w[N] \\ R_w[1] & R_w[0] & \cdots & R_w[N-1] \\ \vdots & \vdots & \ddots & \vdots \\ R_w[N] & R_w[N-1] & \cdots & R_w[0] \end{bmatrix}}_{\mathbf{T}} \underbrace{\begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_N \end{bmatrix}}_{\mathbf{a}} = \underbrace{\begin{bmatrix} R_{ws}[0] \\ R_{ws}[1] \\ \vdots \\ R_{ws}[N] \end{bmatrix}}_{\mathbf{v}}$$

The name for these equations is Wiener–Hopf equations. In the equation, the matrix T is a symmetric Toeplitz matrix. Under certain circumstances on R, it is known that these matrices are positive definite and, therefore, non-singular, resulting in a unique solution for determining the Wiener filter coefficient vector, a=(T)*v. In addition, the efficient Levinson-Durbin algorithm exists for solving such Wiener–Hopf equations; therefore, an explicit inversion of T is not required.

II. EXPERIMENTS

In this problem, there are two sensors- primary and secondary. The primary sensor receives the signal x[n], corrupted with noise v1[n]. While the secondary sensor receives v2[n], a signal that is correlated with the noise v1[n]. The aim is to estimate x[n] using an optimal FIR Wiener filter. The algorithm for our experiment is to first check whether the signal is wide spread stationary or not, if yes, only then continue to estimate x[n].

We have tried the following three cases:

Case-1: A sinusoidal signal d[n] is used as the original signal and is then corrupted with noise. Then we try to get the original sinusoidal signal from the corrupted signal x[n].

we have our desired signal d[n] = sin(0.04n + θ) and we corrupt it with random noise e[n], the corrupted signal x[n] is then tested for being widespread stationary and since it qualifies we go ahead with estimating the corrupted signal. We can see from the results shown in figure 2-7. figure 6 and 7 shows that the estimation is quite good both in time and frequency domain.

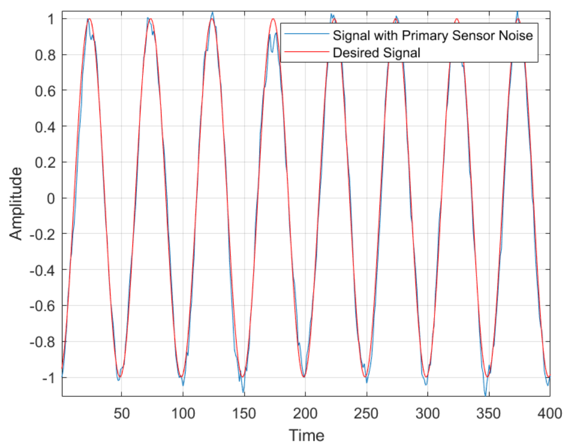

Figure 2

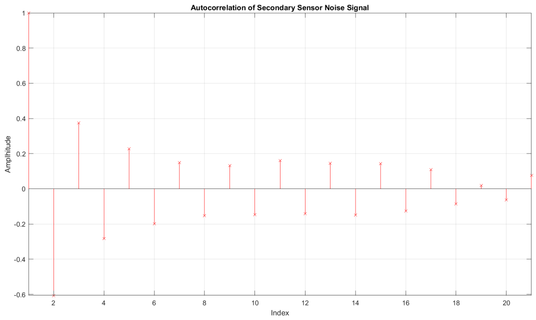

Figure 3

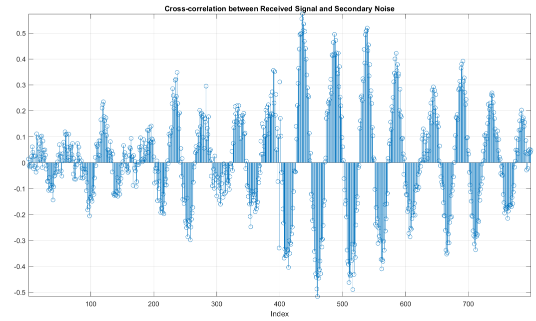

Figure 4

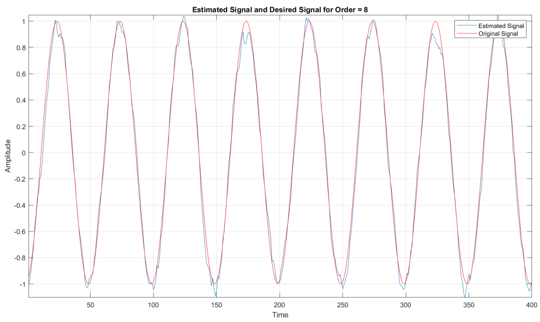

Figure 5

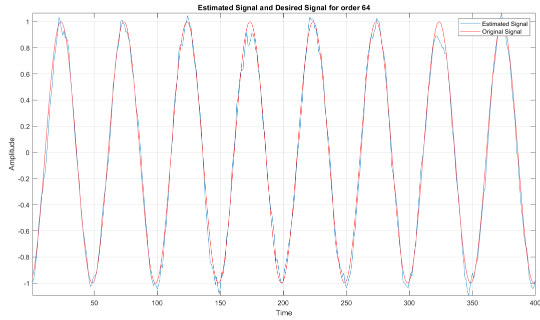

Figure 6

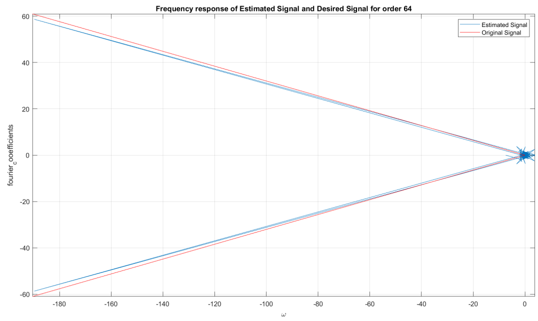

Figure 7

Case-2: An audio file is implemented as our desired signal d[n], and is afterwards corrupted with noise e[n]. The corrupted signal x[n] is to be estimated with the original desired signal. Widespread stationary conditions were not met for the corrupted signal so it does not make any sense to go ahead with estimating this signal using a Wiener filter. Nonetheless, we still proceeded with estimating our corrupted signal and realized that even in this case, the estimation of x[n] gave a really close match with d[n], as seen from figure 8-13. Additionally, we can observe from figure 12, and 13 that there is a good match between the input and desired signals in both time and frequency domain.

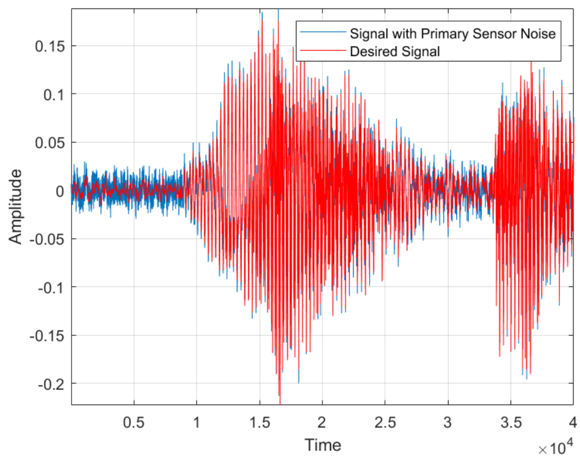

Figure 8

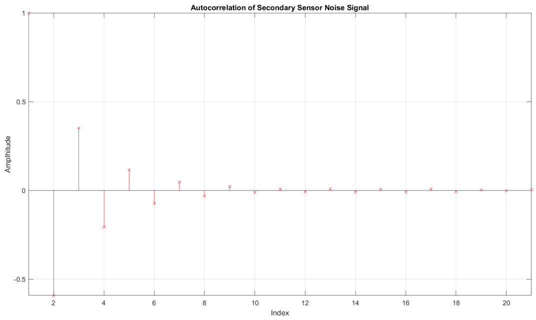

Figure 9

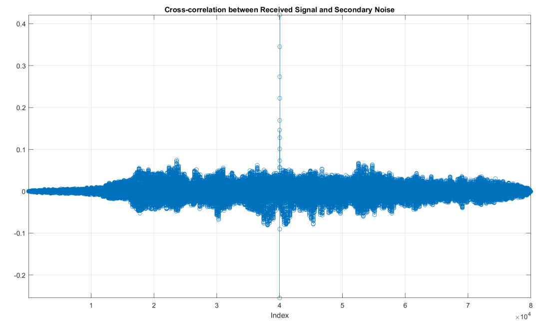

Figure 10

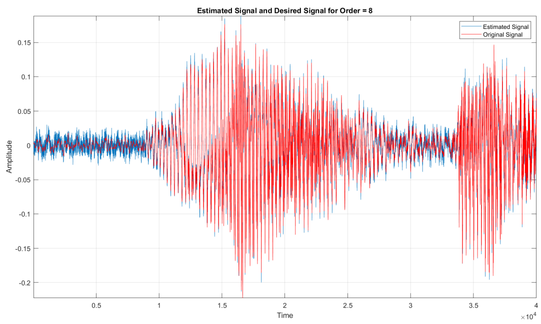

Figure 11

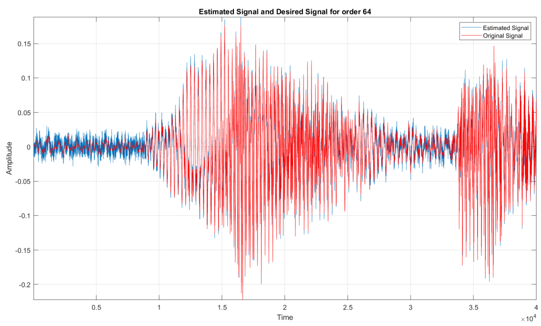

Figure 12

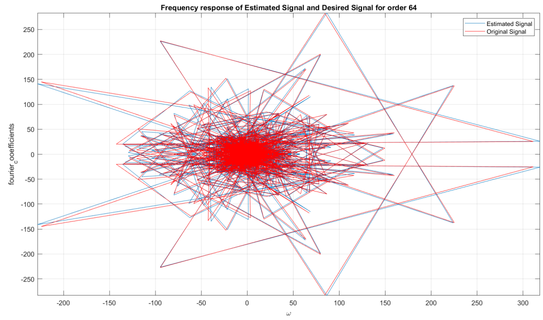

Figure 13

case-3: Here our input signal x[n] was a 10 second voice recording, combined with music playing in the background. The noise e[n] was the same 10 second part of the music being replayed and re-recorded for the same spatial configuration between the recorder and the noise source. Here, the motive was to comprehend whether the estimation of x[n] is either good or not with respect to x[n] - e[n]. Widespread stationary conditions were again not met for the corrupted signal so it does not make any sense to go ahead with estimating this signal using a Wiener filter. However, we still went ahead with estimating the and say that even in this case the estimation of x[n] gave a really close match with d[n] as seen from figure 14-19, and again we can see (from figure 18, and 19) that there is a good match between the input and desired signals in both time and frequency domain.

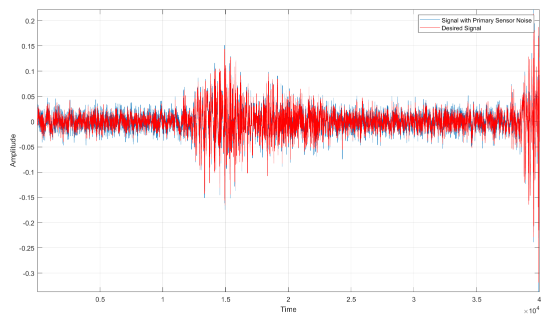

Figure 14

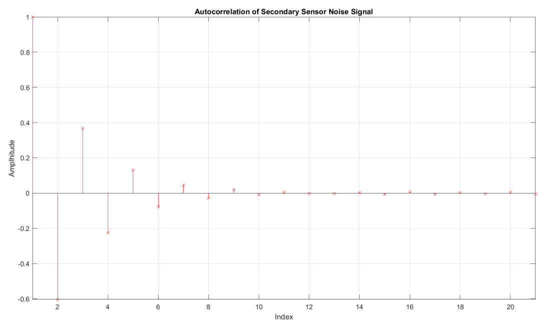

Figure 15

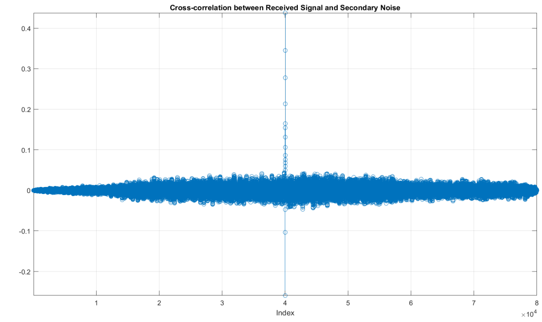

Figure 16

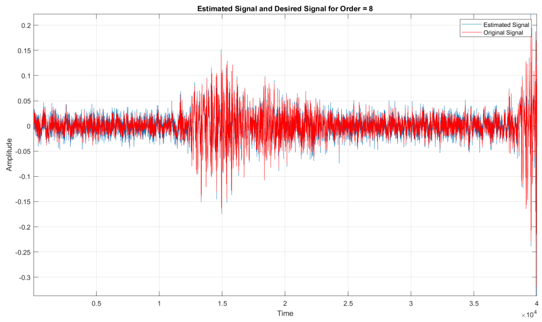

Figure 17

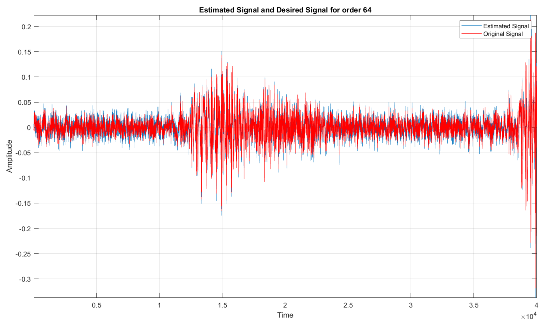

Figure 18

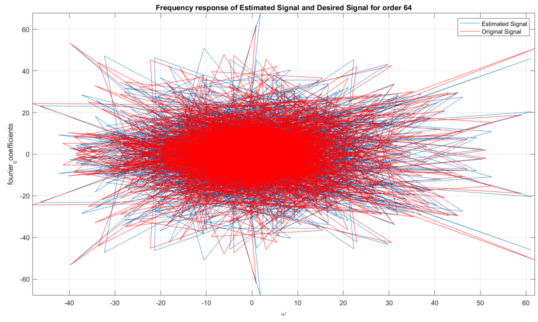

Figure 19

III. DISCUSSION

From our experiments it is clear that a Wiener filter can constitute an efficient filter for noise cancellation in the case of a wide-spread stationary noisy signal. The Wiener filter is capable of estimating the input signal which is corrupted by noise to the desired signal in a statistical manner. However, when we tried applying it to audio recordings the procedure should have ideally not worked, since it failed to satisfy the assumption that the initial signal must be wide-spread stationary. Nevertheless, we found that even when the audio recording did not satisfy the widespread stationary condition, the filter was still able to estimate the desired signal well enough in both time and frequency domain. In our future work, we will try to estimate the coefficients of the filter or weights appearing in MSE cost function by machine learning. We will also try to again test our audio signal which did not meet the widespread stationary condition in case-2 of our experiment by making it widespread stationary using a noise signal which makes the input signal widespread statistical stationary and thus making it a good fit to test again our Wiener filter.

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