Verification of the CVM algorithm with a Functional Probabilistic Invariant

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September 27, 2025

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- No hashing

Illustration of Streaming Algorithms

$\boxed{a_1}\boxed{a_1}\boxed{a_2} \quad \cdots \quad \boxed{a_l}$

The CVM Algorithm

Input: Stream elements $a_1, \ldots, a_l$, $0 < \varepsilon$, $0 < \delta < 1$.
Output: A cardinality estimate R for set $A = \{a_1, \ldots, a_l\}$

$\chi \leftarrow \{\}, p \leftarrow 1, n = \left\lceil \frac{12}{\varepsilon^2} \ln\left(\frac{6l}{\delta}\right) \right\rceil$
for $i \leftarrow 1$ to l **do**
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  if $|\chi| = n$ **then return** $\perp$ ▷ fail if χ remains full
return $\frac{|\chi|}{p}$ ▷ estimate cardinality of A

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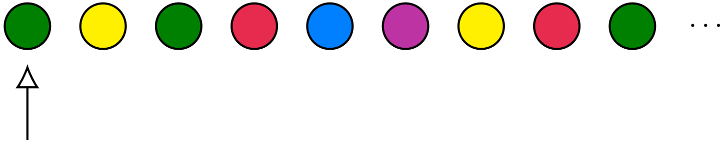

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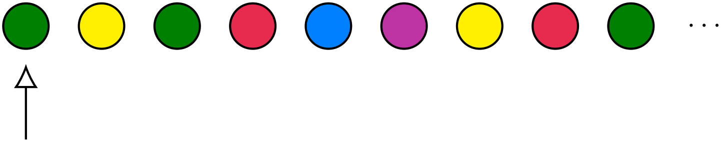

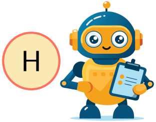

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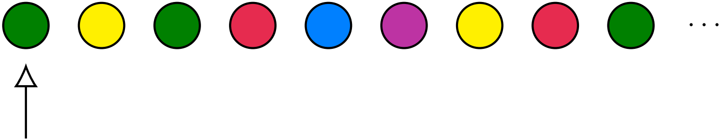
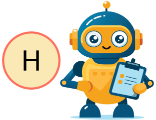
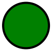




$\chi = \{$  $\}$

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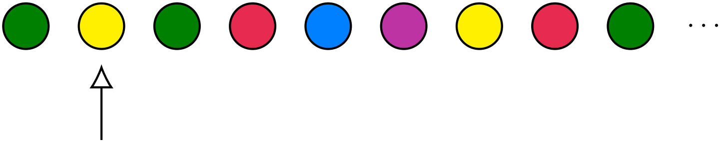

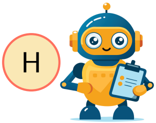

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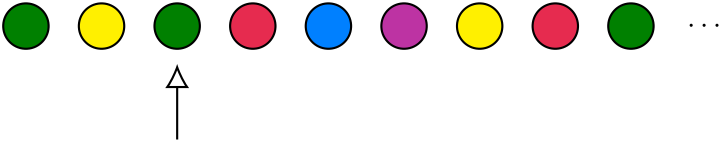
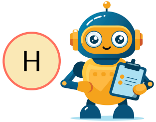
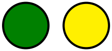




$\chi = \{$  $\}$

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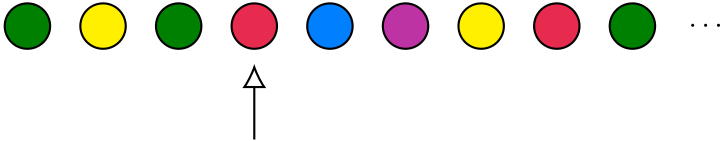
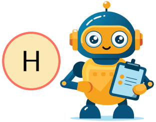
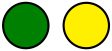




$\chi = \{$  $\}$

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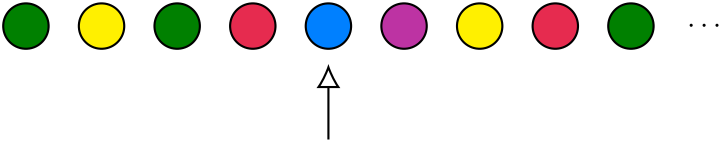
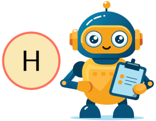
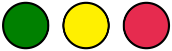




$\chi = \{$  $\}$

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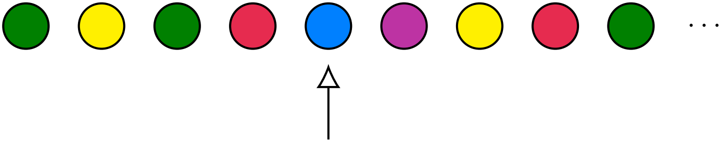
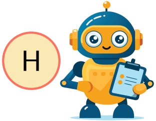
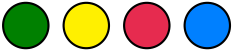




$\chi = \{$  $\}$

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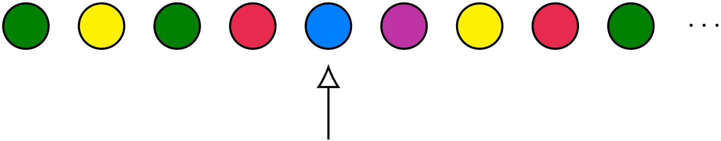
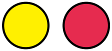


$\chi = \{$  $\}$

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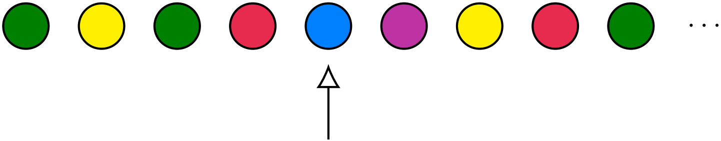
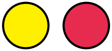


$\chi = \{$  $\}$

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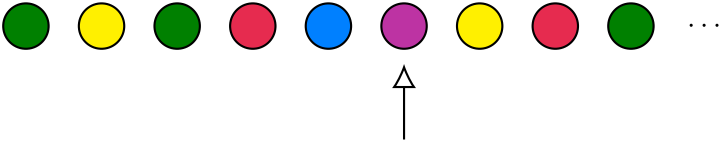
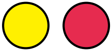


$\chi = \{$  $\}$

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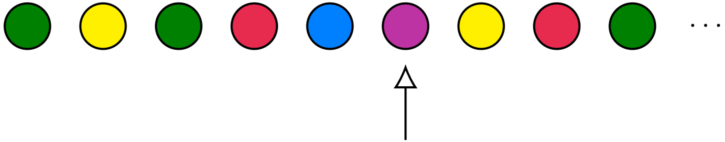
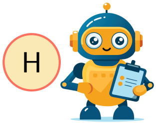
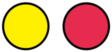




$\chi = \{$  $\}$

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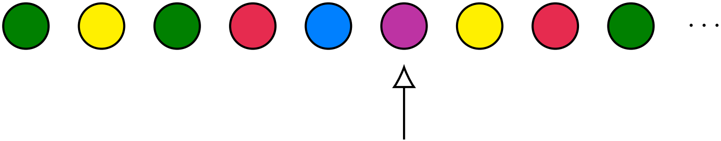
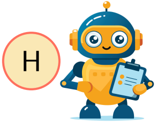
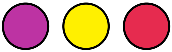




$\chi = \{$  $\}$

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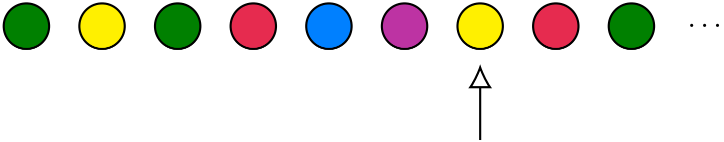
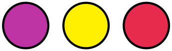


$\chi = \{$  $\}$

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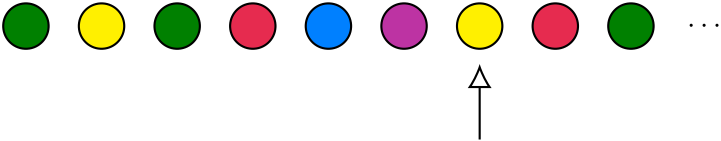
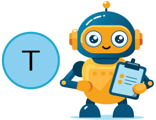
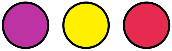




$\chi = \{$  $\}$

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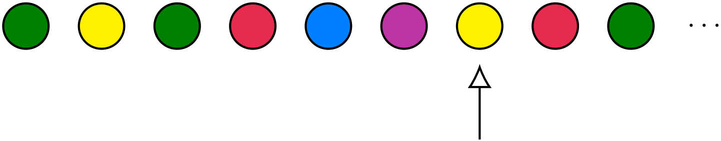

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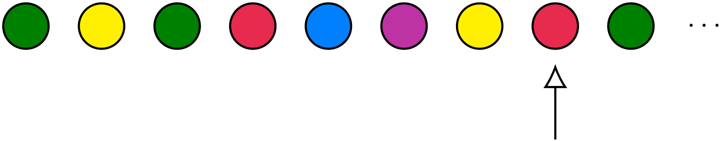

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Comparison with Previous Algorithms

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2007	HLL [4]	$O(\ln(\delta^{-1}\varepsilon^{-2} \ln b)$	O. M. H.
2010	Kane et al. [5]	$O(\ln(\delta^{-}1)(\varepsilon^{-2} + b))$	H.
2020	Błasiok [2]	$O(\varepsilon^{-2} \ln(\delta^{-1}) + b)$	H. E.
2022	Karayel [6]	$O(\varepsilon^{-2} \ln(\delta^{-1}) + b)$	H. E. M.
2023	Chakraborty et al. [3]	$O(\varepsilon^{-2} \ln(\delta^{-1} l) b)$	-

Abbreviations:

- M: Supports merging of sketches.
- H: Relies on hash functions.
- E: Relies on expander graphs.
- O: Random oracle model.

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Abbreviations:

- M: Supports merging of sketches.
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- O: Random oracle model.

Comparison with Previous Algorithms

Year	Author(s)	Space	Remarks
2001	Bar-Yossef et al. [1]	$\tilde{O}(\ln(\delta^{-1})(\varepsilon^{-2} + b))$	M. H.
2007	HLL [4]	$O(\ln(\delta^{-1}\varepsilon^{-2} \ln b)$	O. M. H.
2010	Kane et al. [5]	$O(\ln(\delta^{-}1)(\varepsilon^{-2} + b))$	H.
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- Discovery of an unbiased total variant of the algorithm. (Application of our new technique.)

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$\chi \leftarrow \{\}, p \leftarrow 1, n = \left\lceil \frac{12}{\varepsilon^2} \ln\left(\frac{6l}{\delta}\right) \right\rceil$
for $i \leftarrow 1$ to l **do**
    $b \xleftarrow{\$} \mathrm{Ber}(p)$
    if b **then**
        $\chi \leftarrow \chi \cup \{a_i\}$
    else
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    if $|\chi| = n$ **then**
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Estimate

Giry Monad

We can represent randomized algorithms using the Giry monad:

- Primitive random operations, e.g., $\mathrm{Ber}(p)$
- Return operation $\mathrm{return}\, x$
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$$\mathcal{P}_{m \ggg= f}(E) \quad = \quad \int_m P_{f(x)}(E)\, dx$$

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For an event E:

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\mathcal{P}_{m \ggg f}(E) &= \int_m P_{f(x)}(E)\, dx = \sum_x P_{f(x)}(E) P_m(x) \\
\mathrm{E}_{m \ggg f}[g] &= \int_m \mathrm{E}_{f(x)}[g]\, dx
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Our algorithm in monadic notation

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\mathrm{init} \quad &\gg\!= \quad \mathrm{step}_1\, a_1 \gg\!= \mathrm{step}_2 \gg\!= \mathrm{step}_1\, a_2 \gg\!= \mathrm{step}_2 \cdots \\
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Example invariant:

$$\mathrm{E}\left[\frac{\mathrm{I}(s \in \chi)}{p}\right] = 1$$

for all s that are present in the stream.

Verifying the invariant: Induction step for Step 2

Assumption

$\mathrm{E}_m \left[\frac{\mathrm{I}(s \in \chi)}{p} \right] = 1$

Goal

$L := \mathrm{E}_{m \gg= \mathrm{step}_2} \left[\frac{\mathrm{I}(s \in \chi)}{p} \right] = 1$

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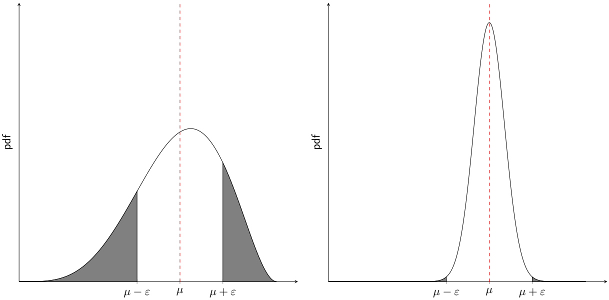

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- More details in the paper.

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- For example we can select the subsampling ratio dynamically.

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- More interestingly: It is possible to use a different subsampling operation.
- Select a random nf-subset (where $\frac{1}{2} \leq f < 1$, $nf \in \mathbb{Z}$).
- $\Rightarrow$ Unbiased algorithm

Another slide with the algorithm

$\chi \leftarrow \{\}, p \leftarrow 1, n = \left\lceil \frac{12}{\varepsilon^2} \ln\left(\frac{6l}{\delta}\right) \right\rceil$
for $i \leftarrow 1$ to l **do**

$b \xleftarrow{\$} \mathrm{Ber}(p)$
if b **then**
$\chi \leftarrow \chi \cup \{a_i\}$
else
$\chi \leftarrow \chi - \{a_i\}$

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if $|\chi| = n$ **then**
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$p \leftarrow \frac{p}{2}$

Step 2

~~**if** $|\chi| = n$ **then return** $\perp$~~

return $\frac{|\chi|}{p}$

Estimate

Another slide with the algorithm

$\chi \leftarrow \{\}, p \leftarrow 1, n = \left\lceil \frac{12}{\varepsilon^2} \ln\left(\frac{3l}{\delta}\right) \right\rceil$
for $i \leftarrow 1$ to l **do**

$b \xleftarrow{\$} \mathrm{Ber}(p)$
if b **then**
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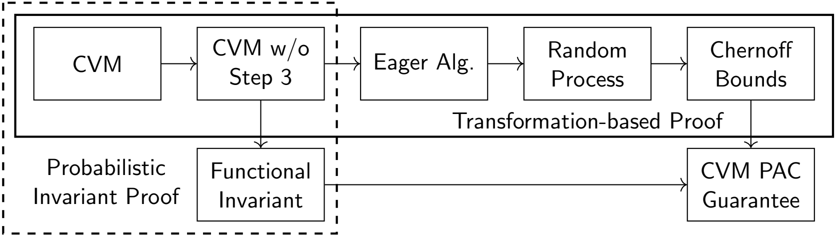

CVM

CVM → CVM w/o Step 3 → Eager Alg. → Random Process

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CVM Algorithm II

CVM → **CVM w/o Step 3** → Eager Alg. → Random Process

$\chi \leftarrow \{\}, k \leftarrow 0, n = \left\lceil \frac{12}{\varepsilon^2} \ln\left(\frac{6l}{\delta}\right) \right\rceil$

for $i \leftarrow 1$ to l **do**

 $b \xleftarrow{\$} \mathrm{Ber}(2^{-k})$

 if b **then**

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 else

  $\chi \leftarrow \chi - \{a_i\}$

 if $|\chi| = n$ **then**

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  $k \leftarrow k + 1$

return $2^k|\chi|$

Eager Algorithm

CVM → CVM w/o Step 3 → **Eager Alg.** → Random Process

$\chi \leftarrow \{\}, k \leftarrow 0, n = \left\lceil \frac{12}{\varepsilon^2} \ln\left(\frac{6l}{\delta}\right) \right\rceil$

$b[i,j] \xleftarrow{\$} \mathrm{Ber}(1/2)$ for $i, j \in \{1, \cdots, l\}$

for $i \leftarrow 1$ to l **do**

  if $b[i,1] = b[i,2] = \cdots = b[i,k] = 1$ **then**

    $\chi \leftarrow \chi \cup \{a_i\}$

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  if $|\chi| = n$ **then**

    $\chi \leftarrow \{a \in \chi \mid b[\mathrm{last\_index}(a), k+1] = 1\}$

    $k \leftarrow k + 1$

return $2^k|\chi|$

Eager Algorithm

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Random Process

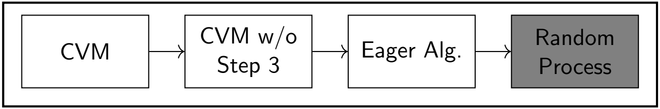
CVM $\rightarrow$ CVM w/o Step 3 $\rightarrow$ Eager Alg. $\rightarrow$ **Random Process**

Observation

The eager algorithm preserves the invariant:

$$s \in \chi \leftrightarrow \text{For all } j < k\text{:}\, b[\mathrm{last\_index}(s), j] = 1$$

Random Process



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Non-deterministic algorithm:

$b[i,j] \xleftarrow{\$} \mathrm{Ber}(1/2)$ for $i, j \in \{1, \cdots, l\}$
$k \leftarrow \{0, \ldots, k_{\max}\}$ $\triangleright$ Non-deterministc step.
$\chi \leftarrow \{s \in A \mid b[\mathrm{last\_index}(s), j] = 1 \text{ for all } j < k\}$
return $2^k|\chi|$

Conclusion/Summary

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- Thank You!

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- Artwork:  By vecteezy.com.

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