Контрольная работа

Задача 1

$X^n \sim N(\theta, 1)$ $\hat{\theta}_n = a\langle X^n \rangle$, $0 \le a \le 1$

$\mathbb{E}\hat{\theta}_n = a\mathbb{E}X_i = a\theta$

$bias = \mathbb{E}\hat{\theta}_n - \theta = (a-1)\theta$

$\mathbb{V}\hat{\theta}_n = \frac{a^2}{n^2} n \mathbb{V}X_i = \frac{a^2}{n}$

$MSE\,\hat{\theta}_n = bias^2\hat{\theta}_n + \mathbb{V}\hat{\theta}_n = (a-1)^2\theta^2 + \frac{a^2}{n}$

$$\frac{\partial MSE\,\hat{\theta}_n}{\partial a} = 2(a-1)\theta^2 + \frac{2a}{n} = 0 \Rightarrow a = \frac{n\theta^2}{n\theta^2+1}$$

$$MSE_{min} = \frac{\theta^2}{n\theta^2+1}$$

Ответ: $MSE\,\hat{\theta}_n = (a-1)^2\theta^2 + \frac{a^2}{n}$

$a = \frac{n\theta^2}{n\theta^2+1}$

$MSE_{min} = \frac{\theta^2}{n\theta^2+1}$

Задача 2

$X^n \sim \Gamma(\alpha,\beta)$, $\mathbb{E}X = \alpha\beta$, $\mathbb{V}X = \alpha\beta^2$

$$\begin{cases} \frac{1}{n}\sum_{i=1}^{n} X_i = \alpha\beta \\ \frac{1}{n}\sum_{i=1}^{n} X_i^2 = \mathcal{D}X + (\mathbb{E}X)^2 = \alpha\beta^2 + (\alpha\beta)^2 \end{cases}$$

$$\frac{1}{n}\sum_{i=1}^{n} X_i^2 = \frac{1}{n}\sum_{i=1}^{n} X_i\beta + \left(\frac{1}{n}\sum_{i=1}^{n} X_i\right)^2 \Rightarrow \beta = \frac{\langle X^2\rangle - (\langle X\rangle)^2}{\langle X\rangle}$$

$$\alpha = \frac{(\langle X\rangle)^2}{\langle X^2\rangle - (\langle X\rangle)^2}$$

Ответ: $\alpha = \frac{(\langle X\rangle)^2}{\langle X^2\rangle - (\langle X\rangle)^2}$ $\beta = \frac{\langle X^2\rangle - (\langle X\rangle)^2}{\langle X\rangle}$

Задача 3

$\theta = (\mu; \sigma),\ \mu \in \mathbb{R},\ \sigma > 0$

$p(x, \theta) = \frac{1}{2\sigma} e^{-\frac{|x-\mu|}{\sigma}},\ x \in \mathbb{R}$

$L = \prod_{i=1}^{n} p(x_i, \theta) = \left(\frac{1}{2\sigma}\right)^n e^{-\sum_{i=1}^{n} \frac{|x_i-\mu|}{\sigma}}$

$l = \ln L = -n \ln(2\sigma) - \sum_{i=1}^{n} \frac{|x_i - \mu|}{\sigma}$

$\frac{\partial l}{\partial \sigma} = -\frac{n}{\sigma} + \frac{1}{\sigma^2} \sum_{i=1}^{n} |x_i - \mu| = 0 \Rightarrow \hat{\sigma} = \frac{1}{n} \sum_{i=1}^{n} |x_i - \hat{\mu}|$

$l \to \max \Leftrightarrow -\sum_{i=1}^{n} |x_i - \mu| \to \max \Rightarrow \hat{\mu} = \text{median } x_i$

$\boxed{\text{Ответ: } \hat{\theta} = (\hat{\mu}, \hat{\sigma}) = (\text{median } x_i,\ \frac{1}{n}\sum_{i=1}^{n} |x_i - \text{median } x_i|}$

Задача 4

$X^n \sim \text{Uniform}(\theta, 2\theta),\ \theta > 0$

а) $\theta_{MLE} = ?$ $\quad MSE\ \theta_{MLE} = ?$ $\quad \theta_{MLE}$ состоят?

$l = \prod_{i=1}^{n} \frac{1}{2\theta - \theta} I[\theta \le x_i \le 2\theta] = \theta^{-n} I[\theta \le x_{(1)} \le x_{(n)} \le 2\theta] =$

$= \theta^{-n} I\left[\frac{x_{(n)}}{2} \le \theta \le x_{(1)}\right]$, θ^{-n} - убыв ф-ия $\Rightarrow \boxed{\theta_{MLE} = \frac{x_{(n)}}{2}}$

Распределение $x_{(n)}$

$F(y) = P^n(x_i < y) = \frac{(y-\theta)^n}{\theta^n},\ y \in [\theta, 2\theta]$

$p(y) = n(y-\theta)^{n-1} \cdot \frac{1}{\theta^n},\ y \in [\theta, 2\theta]$

$E \frac{x_{(n)}}{2} = \frac{n}{2\theta^n} \int_{\theta}^{2\theta} x(x-\theta)^{n-1} dx = \theta - \frac{\theta}{2(n+1)}$

$V\left(\frac{x_{(n)}}{2}\right) = \frac{1}{4} V x_{(n)} = \frac{1}{4}\left(4\theta^2 - 4\frac{\theta^2}{n+1} + \frac{2\theta^2}{(n+1)(n+2)} - \left(2\theta - \frac{\theta}{n+1}\right)^2\right)$

$\boxed{MSE = bias^2 + V = \frac{\theta^2}{4(n+1)} + V = \frac{\theta^2}{2(n+1)(n+2)}}$

$MSE \to 0 \Rightarrow$ оценка состоят

$\boxed{\text{Ответ: } \theta_{MLE} = \frac{x_{(n)}}{2},\ MSE\ \theta_{MLE} = \frac{\theta^2}{2(n+1)(n+2)},\ \text{состоят.}}$

б) Метод моментов

$$\frac{1}{n}\sum_{i=1}^{n} X_i = \frac{\theta+2\theta}{2} = \frac{3\theta}{2} \Rightarrow \boxed{\theta_M = \frac{2}{3}\cdot\frac{1}{n}\sum_{i=1}^{n} X_i}$$

По ЦПТ: $$\frac{\sqrt{n}\left(\frac{1}{n}\sum_{i=1}^{n} X_i - \frac{3\theta}{2}\right)}{\sqrt{\frac{\theta^2}{12}}} \to N(0,1)$$

$$\frac{\sqrt{n}\left(\theta_M - \frac{3}{2}\cdot\frac{2}{3}\theta\right)}{\sqrt{\frac{\theta^2}{12}\cdot\frac{4}{9}}} \to N(0,1)$$

$$\sqrt{n}(\theta_M - \theta) \to N\left(0, \frac{\theta^2}{27}\right)$$

$$\boxed{\text{Ответ: } \theta_M = \frac{2}{3}\langle X^n\rangle, \ \sqrt{n}(\theta_M - \theta) \to N\left(0, \frac{\theta^2}{27}\right)}$$

Задача 5

$P(X_i = 1) = p_1$, $P(X_i = 2) = p_2$, $P(X_i = 3) = p_3$, $p_1 + p_2 + p_3 = 1$

а) $p_{MLE} = ?$

$$L = \frac{n!}{n_1!\, n_2!\, n_3!} p_1^{n_1} p_2^{n_2} p_3^{n_3}, \text{ где } n_i = \sum_{j=1}^{n} I[X_j = i]$$

$$\begin{cases} \log L = \log n! - \sum_{i=1}^{3} \log n_i! + \sum_{i=1}^{3} n_i \log p_i \to \max\limits_{p} \\ \sum p_i = 1 \end{cases}$$

ф-ия Лагранжа:

$$L(x,\lambda) = \log n! - \sum_{i=1}^{3} \log n_i! + \sum_{i=1}^{3} n_i \log p_i + \lambda\left(\sum_{i=1}^{3} p_i - 1\right)$$

$$\frac{\partial L(x,\lambda)}{\partial p_i} = \frac{n_i}{p_i} + \lambda = 0 \Rightarrow p_i = -\frac{n_i}{\lambda}$$

$$p_1 + p_2 + p_3 = -\left(\frac{n_1}{\lambda} + \frac{n_2}{\lambda} + \frac{n_3}{\lambda}\right) = -\frac{n}{\lambda} = 1 \Rightarrow \lambda = n \Rightarrow p_i = \frac{n_i}{\lambda}$$

$$\boxed{\text{Ответ: } p_{MLE} = (\hat{p}_1, \hat{p}_2, \hat{p}_3) = \left(\frac{n_1}{n}, \frac{n_2}{n}, \frac{n_3}{n}\right)}$$

б) $p_1 + p_2 + p_3 = 1.$

Построим матрицу Фишера, зависящую от p_1 и p_2 (p_3 можно выразить через p_1 и p_2: $p_3 = 1 - p_1 - p_2$)

$$I(p_1, p_2) = \begin{pmatrix} -\mathbb{E}\frac{\partial^2 \ell}{\partial p_1^2} & -\mathbb{E}\frac{\partial^2 \ell}{\partial p_1 \partial p_2} \\ -\mathbb{E}\frac{\partial^2 \ell}{\partial p_1 \partial p_2} & -\mathbb{E}\frac{\partial^2 \ell}{\partial p_2^2} \end{pmatrix}$$

$$\ell = \log\left(\frac{n!}{n_1! n_2! n_3!} p_1^{n_1} p_2^{n_2} p_3^{n_3}\right) = \log c + n_1 \log p_1 + n_2 \log p_2 + n_3 \log \underbrace{p_3}_{1-p_1-p_2}$$

$$\frac{\partial \ell}{\partial p_1} = \frac{n_1}{p_1} - \frac{n_3}{1-p_1-p_2} \qquad \frac{\partial \ell}{\partial p_2} = \frac{n_2}{p_2} - \frac{n_3}{1-p_1-p_2}$$

$$\frac{\partial^2 \ell}{\partial p_1^2} = -\frac{n_1}{p_1^2} - \frac{n_3}{(1-p_1-p_2)^2} \qquad \frac{\partial^2 \ell}{\partial p_2^2} = -\frac{n_2}{p_2^2} - \frac{n_3}{(1-p_1-p_2)^2}$$

$$-\mathbb{E}\frac{\partial^2 \ell}{\partial p_1^2} = +\frac{np_1}{p_1^2} + \frac{np_3}{p_3^2} = +\frac{n}{p_1} + \frac{n}{1-p_1-p_2}$$

$$-\mathbb{E}\frac{\partial^2 \ell}{\partial p_2^2} = +\frac{n}{p_2} + \frac{n}{(1-p_1-p_2)}$$

$$\frac{\partial^2 \ell}{\partial p_1 \partial p_2} = -\frac{n_3}{(1-p_1-p_2)^2} \qquad -\mathbb{E}\frac{\partial^2 \ell}{\partial p_1 \partial p_2} = \frac{n(1-p_1-p_2)}{(1-p_1-p_2)^2} = \frac{n}{1-p_1-p_2}$$

$$\boxed{I(p_1, p_2) = n\begin{pmatrix} \frac{1}{p_1} + \frac{1}{1-p_1-p_2} & \frac{1}{1-p_1-p_2} \\ \frac{1}{1-p_1-p_2} & \frac{1}{p_2} + \frac{1}{1-p_1-p_2} \end{pmatrix}}$$

в) Согласно дельта методу

$$\frac{\log \hat{p}_1 - \log p_1}{\left|\frac{1}{\hat{p}_1}\right| se\, \hat{p}_1} = \frac{\log \hat{p}_1 - \log p_1}{\frac{1}{\hat{p}_1} \cdot \sqrt{\frac{n}{n^2}\hat{p}_1(1-\hat{p}_1)}} = \frac{\log \hat{p}_1 - \log p_1}{\sqrt{\frac{1}{n}\frac{1-\hat{p}_1}{\hat{p}_1}}} \to N(0,1)$$

$$\boxed{C = \left(\log \hat{p}_1 - z_{\alpha/2}\sqrt{\frac{1-\hat{p}_1}{\hat{p}_1 n}},\ \log \hat{p}_1 + z_{\alpha/2}\sqrt{\frac{1-\hat{p}_1}{\hat{p}_1 n}}\right)}$$

Задача 6

$$\hat{\mu}(\alpha) = \left(\frac{1}{n}\sum_{i=1}^{n} X_i^{\alpha}\right)^{\frac{1}{\alpha}}$$

- $\mathbb{E}\frac{1}{n}\sum_{i=1}^{n} X_i^{\alpha} = \mathbb{E}X_i^{\alpha} = \int_{\theta}^{+\infty} x^{\alpha-k-1} k\theta^k dx = \frac{x^{\alpha-k}}{\alpha-k} k\theta^k \Big|_{\theta}^{+\infty} = -\frac{\theta^{\alpha-k} k\theta^k}{\alpha-k} =$

$= \frac{k\theta^{\alpha}}{k-\alpha}$, $\alpha < k$

$$\boxed{\mathbb{E}\hat{\mu}(\alpha) = \left(\frac{k\theta^{\alpha}}{k-\alpha}\right)^{\frac{1}{\alpha}}}$$

$$\mathbb{V}\frac{1}{n}\sum_{i=1}^{n} X_i^{\alpha} = \frac{1}{n}\mathbb{V}X_i^{\alpha} = \frac{1}{n}\left(\mathbb{E}X_i^{2\alpha} - (\mathbb{E}X_i^{\alpha})^2\right) = \frac{1}{n}\left(\frac{k\theta^{2\alpha}}{k-2\alpha} - \frac{k^2\theta^{2\alpha}}{(k-\alpha)^2}\right) =$$

$$= \frac{k\theta^{2\alpha}\alpha^2}{n(k-2\alpha)(k-\alpha)^2}$$

$$\boxed{\hat{\mathbb{V}}\left(\frac{1}{n}\sum_{i=1}^{n} X_i^{\alpha}\right)^{1/\alpha} = \left(\frac{1}{\alpha}\left(\frac{1}{n}\sum_{i=1}^{n} X_i^{\alpha}\right)^{\frac{1}{\alpha}-1}\right)^2 \cdot \frac{k\theta^{2\alpha}\alpha^2}{n(k-2\alpha)(k-\alpha)^2} = \left(\frac{1}{n}\sum_{i=1}^{n} X_i^{\alpha}\right)^{\frac{2}{\alpha}-2} \cdot \frac{k\theta^{2\alpha}}{n(k-2\alpha)(k-\alpha)^2}}$$

- Чтобы $\hat{V}\left(\frac{1}{n}\sum_{i=1}^{n} X_i^{\alpha}\right)^{1/\alpha} > 0 \Rightarrow k-2\alpha > 0 \Rightarrow \boxed{\alpha < \frac{k}{2}}$

а) $\alpha \to 0$

$$\mathbb{E}\hat{\mu}(\alpha) = \left(\frac{k\theta^{\alpha}}{k-\alpha}\right)^{1/\alpha} \xrightarrow[\alpha\to 0]{} \theta e^{\frac{1}{k}}$$

б) $\alpha \to \frac{k}{2}$

$$\mathbb{E}\hat{\mu}(\alpha) \xrightarrow[\alpha\to\frac{k}{2}]{} \left(\frac{k\cdot\theta^{\frac{k}{2}}}{k-\frac{k}{2}}\right)^{\frac{2}{k}} = \frac{k^{\frac{2}{k}}\theta}{(\frac{k}{2})^{\frac{2}{k}}} = 2^{\frac{2}{k}}\theta$$

Ответ: $\mathbb{E}\hat{\mu}(\alpha) = \left(\frac{k\theta^{\alpha}}{k-\alpha}\right)^{\frac{1}{\alpha}}$

$\hat{\mathbb{V}}\left(\frac{1}{n}\sum_{i=1}^{n} X_i^{\alpha}\right)^{1/\alpha} = \hat{V}\hat{\mu}(\alpha) = \left(\frac{1}{n}\sum_{i=1}^{n} X_i^{\alpha}\right)^{\frac{2}{\alpha}-2} \cdot \frac{k\theta^{2\alpha}}{n(k-2\alpha)(k-\alpha)^2}$

$\alpha < \frac{k}{2}$

при $\alpha \to 0$: $\mathbb{E}\hat{\mu}(\alpha) \to \theta e^{\frac{1}{k}}$

при $\alpha \to \frac{k}{2}$: $\mathbb{E}\hat{\mu}(\alpha) \to 2^{\frac{2}{k}}\theta$

Задача 7

$X^n \sim Uniform(0, \theta)$

а) $H_0: \theta = 2$, если $\langle X \rangle \le 3$

$H_1: \theta = 4$

$\boxed{\text{Ошибка первого рода} = P(\text{отклон } H_0 \mid H_0)} =$

$= P_{H_0}\left(\frac{1}{n}\sum_{i=1}^{n} X_i > 3\right) = P_{H_0}\left(\frac{\langle X \rangle - \mathbb{E}\langle X \rangle}{\sqrt{\mathbb{V}\langle X \rangle}} > \frac{3 - \mathbb{E}\langle X \rangle}{\sqrt{\mathbb{V}\langle X \rangle}}\right) =$

$\boxed{= 1 - \Phi\left(\frac{(3-1)\sqrt{n}}{\sqrt{\frac{\theta^2}{12}}}\right) = 1 - \Phi(\sqrt{12n}) \xrightarrow[n\to\infty]{} 0}$

$\boxed{\text{Ошибка второго рода} = P(H_0 \mid H_1)} =$

$\boxed{= P_{H_1}(\langle X \rangle \le 3) = \Phi\left(\frac{(3-2)\sqrt{n}}{\sqrt{\frac{4^2}{12}}}\right) = \Phi\left(\sqrt{\frac{12}{16}n}\right) \xrightarrow[n\to\infty]{} 1}$

б) $H_0: \theta = 2$, $\max X_i < 3$

$H_1: \theta = 4$

$\boxed{\text{Ош. 1 рода} = P(\text{отклон } H_0 \mid H_0) = P(\max X_i \ge 3 \mid \theta = 2) = 0}$

$\boxed{\text{Ош. 2 рода} = P(H_0 \mid H_1) = P(\max X_i < 3 \mid \theta = 4) = \left(\frac{3-0}{4-0}\right)^n = \left(\frac{3}{4}\right)^n}$

Задача 8

- Тест на основе отношения правдоподобия

$H_0: \theta = 0$

$H_1: \theta \ne 0$

$X^n \sim N(\theta, 1)$

$$\lambda = 2\log \frac{\sup\limits_{\theta_i \ne 0} \prod\limits_{i=1}^{n} \cancel{\frac{1}{\sqrt{2\pi}}} e^{-\frac{(X_i - \theta_i)^2}{2}}}{\prod\limits_{i=1}^{n} \cancel{\frac{1}{\sqrt{2\pi}}} e^{-\frac{X_i^2}{2}}} = 2\log \frac{e^{-\frac{1}{2}\sum_{i=1}^{n}(X_i - \langle X \rangle)^2}}{e^{-\frac{1}{2}\sum X_i^2}} =$$

$$= \cancel{2}\left(-\frac{1}{\cancel{2}}\sum_{i=1}^{n}(X_i - \langle X \rangle)^2 + \frac{1}{\cancel{2}}\sum X_i^2\right) = -\sum_{i=1}^{n}(X_i^2 - 2X_i\langle X \rangle + (\langle X \rangle)^2) + \sum_{i=1}^{n} X_i^2 =$$

$$= 2\sum_{i=1}^{n} X_i \langle X \rangle - n(\langle X \rangle)^2 = 2n(\langle X \rangle)^2 - n(\langle X \rangle)^2 = n(\langle X \rangle)^2$$

$\boxed{\text{Ответ: } \lambda = n(\langle X \rangle)^2}$

• Тест Неймана-Пирсона

$H_0 : \theta = 0$

$H_1 : \theta = a, \ a > 0$

$$T = \frac{\prod_{i=1}^{n} f(x_i; a)}{\prod_{i=1}^{n} f(x_i; 0)} = \frac{e^{-\frac{1}{2}\sum_{i=1}^{n}(x_i - a)^2}}{e^{-\frac{1}{2}\sum_{i=1}^{n} x_i^2}} = e^{-\frac{1}{2}\left(\sum_{i=1}^{n}(x_i-a)^2 - x_i^2\right)} =$$

$$= e^{-\frac{1}{2}\sum_{i=1}^{n}(-2ax_i + a^2)} = \boxed{e^{an\langle x\rangle - \frac{n}{2}a^2} > k}$$

$$an\langle x\rangle - \frac{n}{2}a^2 > \ln k$$

$$\boxed{\langle x\rangle > \frac{\ln k}{an} + \frac{a}{2}}$$

Если выполняется, то принимается гипотеза H_1, иначе - H_0

Мощность теста $= P(\text{откл } H_0 \mid H_1) =$

$$= P\left(\langle x\rangle > \frac{\ln k}{an} + \frac{a}{2} \mid H_1\right) = P\left(\frac{\langle x\rangle - a}{\frac{1}{\sqrt{n}}} > \frac{\frac{\ln k}{an} + \frac{a}{2} - a}{\frac{1}{\sqrt{n}}}\right) =$$

$$= 1 - \Phi\Big(\underbrace{\frac{\ln k}{a\sqrt{n}}}_{\to 0} - \underbrace{\frac{a}{2}\sqrt{n}}_{\infty}\Big) \xrightarrow[n\to\infty]{} 1$$

$$P\left(\langle x\rangle > \frac{\ln k}{an} + \frac{a}{2} \mid H_0\right) = 1 - \Phi\left(\left(\frac{\ln k}{an} + \frac{a}{2}\right)\sqrt{n}\right) = \alpha$$

$$\Phi^{-1}(1-\alpha) = \left(\frac{\ln k}{an} + \frac{a}{2}\right)\sqrt{n}$$

$$\left(\frac{z_{1-\alpha}}{\sqrt{n}} - \frac{a}{2}\right)an = \ln k \implies \boxed{k = \exp\left(a\sqrt{n}\, z_{1-\alpha} - \frac{a^2 n}{2}\right)}$$ - критич. порог в тесте

Задача 9

$X^n \sim Exp(\theta)$, т.е. $p(x;\theta) = \theta e^{-x\theta}, x>0$ и $\theta>0$

- $\theta_{MLE} = ?$

$$L = \prod_{i=1}^{n} \theta e^{-X_i\theta} = \theta^n e^{-\theta\sum_{i=1}^{n} X_i}$$

$$\ell = \log L = n \log\theta - \theta\sum_{i=1}^{n} X_i$$

$$\frac{\partial \ell}{\partial\theta} = \frac{n}{\theta} - \sum_{i=1}^{n} X_i = 0 \Rightarrow \boxed{\theta_{MLE} = \frac{n}{\sum_{i=1}^{n} X_i}}$$

θ_{MLE} явл асимпт. норм. оценкой, если:

1) независимость от θ мн-ва $X = \{x : p(x|\theta) \neq 0\}$
2) $\exists \frac{\partial p}{\partial\theta}$ и $\exists \frac{\partial^2 p}{\partial\theta^2}$
3) $\exists\ \mathbb{E}\left[\frac{\partial \ln p}{\partial\theta}\right]^2$

Действительно, 1 пункт выполняется согласно определению плотности.

$$\frac{\partial p}{\partial\theta} = e^{-x\theta} - \theta x e^{-x\theta}, \quad \frac{\partial^2 p}{\partial\theta^2} = -xe^{-x\theta} - xe^{-x\theta} + \theta x^2 e^{-x\theta} \Rightarrow 2 \text{ пункт выполнен}$$

$$\mathbb{E}\left[\frac{\partial \ln p}{\partial\theta}\right]^2 = \int_0^{+\infty} \left(\frac{1}{\theta} - x\right)^2 \theta e^{-x\theta} dx = \frac{1}{\theta^2} \Rightarrow 3 \text{ пункт выполнен}$$

Таким образом, θ_{MLE} - асимпт норм. оценка

- Информация Фишера

$$I(\theta) = -\mathbb{E}\frac{\partial^2}{\partial\theta^2}(\log\theta - \theta X_i) = -\mathbb{E}\frac{\partial}{\partial\theta}\left(\frac{1}{\theta} - X_i\right) = -\mathbb{E}\left(-\frac{1}{\theta^2}\right) = \frac{1}{\theta^2}$$

$$I_n(\theta) = nI(\theta) = \frac{n}{\theta^2}$$

$$\boxed{\text{Ответ: } I_n(\theta) = \frac{n}{\theta^2}}$$

- $(1-\alpha)$ асимпт. доверит интервал для $\varphi = \log\theta$

$$\varphi(\theta_{MLE}) = \log\theta_{MLE}$$

Согласно дельта-методу:

$$\frac{\log\theta_{MLE} - \log\theta}{\left|\frac{1}{\theta_{MLE}}\right| \cdot se(\theta_{MLE})} = \frac{\log\theta_{MLE} - \log\theta}{\underbrace{\left|\frac{1}{\theta_{MLE}}\right|}_{>0} \cdot \frac{n\theta_{MLE}}{(n-1)\sqrt{n-2}}} \to N(0,1)$$

$$\frac{\log\theta_{MLE} - \log\theta}{\frac{1}{\sqrt{n}}} \to N(0,1) \qquad C = \left(\log\theta_{MLE} - z_{\alpha/2}\frac{1}{\sqrt{n}},\ \log\theta_{MLE} + z_{\alpha/2}\frac{1}{\sqrt{n}}\right)$$

Как найдено $se(\theta_{MLE})$?

Пусть $\sum_{i=1}^{n} X_i = y$, тогда $\theta_{MLE} = \frac{n}{y}$.

Известно, что $y \sim \Gamma(n, \frac{1}{\theta})$: $p(x) = \begin{cases} 0, & x < 0 \\ \frac{x^{n-1} e^{-\theta x} \theta^n}{\Gamma(n)}, & x \geq 0 \end{cases}$

$$\mathbb{E}(\theta_{MLE}) = \int_{-\infty}^{+\infty} \frac{n}{y} p(y) dy = \int_{0}^{+\infty} \frac{n y^{n-2} e^{-\theta y} \theta^n}{\Gamma(n)} dy = \frac{\theta n}{n-1} \sim \theta$$

$$\mathbb{E}(\theta_{MLE}^2) = \int_{-\infty}^{+\infty} \left(\frac{n}{y}\right)^2 p(y) dy = \frac{n^2 \theta^2}{(n-1)(n-2)}$$

$$se(\theta_{MLE}) = \sqrt{\mathbb{E}(\theta_{MLE}^2) - (\mathbb{E}\theta_{MLE})^2} = \frac{n\theta}{(n-1)\sqrt{n-2}} \sim \frac{\theta}{\sqrt{n}}$$

$$\boxed{\text{Ответ: } C = \left(\log \theta_{MLE} - z_{\alpha/2} \frac{1}{\sqrt{n}}, \log \theta_{MLE} + z_{\alpha/2} \frac{1}{\sqrt{n}}\right)}$$

- Тест Вальда

$H_0: \theta = \theta_0$

$H_1: \theta \neq \theta_0$

$$|W| \approx \left| \frac{\theta_{MLE} - \theta_0}{\frac{\theta_{MLE}}{\sqrt{n}}} \right| = \left| \frac{\sqrt{n}(\theta_{MLE} - \theta_0)}{\theta_{MLE}} \right| > z_{\alpha/2} \text{ - отклоняется гипотеза } H_0$$

- Тест на основе отношения правдоподобия

$$\lambda = 2\log \frac{\mathcal{L}(\hat{\theta})}{\mathcal{L}(\theta_0)} = 2\log \frac{\sup\limits_{\theta \neq \theta_0} \theta^n e^{-\theta \sum_{i=1}^{n} X_i}}{\theta_0^n e^{-\theta_0 \sum_{i=1}^{n} X_i}} = \begin{cases} 0, & \theta_0 = \theta_{MLE} \\ 2\log\left[\left(\frac{\theta_{MLE}}{\theta_0}\right)^n e^{(-\theta_{MLE} + \theta_0)\sum_{i=1}^{n} X_i}\right] \end{cases}$$

$$= \begin{cases} 0, & \theta_0 = \theta_{MLE} \\ -2n(\log\langle X\rangle + \log\theta_0 + 1 - \theta_0\langle X\rangle) \end{cases}$$

(10) $y_i = w_0 + w_1 x_i + \varepsilon_i,\ \varepsilon \sim N(0, \sigma^2)$

$w_{MLE} = (X^T X)^{-1} X^T y$, где $X = \begin{pmatrix} 1 & x_1 \\ \vdots & \vdots \\ 1 & x_n \end{pmatrix}$

$\mathrm{Var}\, w_{MLE} = \mathbb{E}(w_{MLE} - w)(w_{MLE} - w)^T = \mathbb{E}((X^T X)^{-1} X^T (Xw + \varepsilon) - w)(\ldots)^T =$

$= \mathbb{E}((X^T X)^{-1} X^T \varepsilon \varepsilon^T X (X^T X)^{-1}) = \sigma^2 (X^T X)^{-1} =$

$= \sigma^2 \left(\begin{pmatrix} 1 & 1 & \ldots & 1 \\ x_1 & x_2 & \ldots & x_n \end{pmatrix} \begin{pmatrix} 1 & x_1 \\ 1 & x_2 \\ \vdots & \vdots \\ 1 & x_n \end{pmatrix} \right)^{-1} = \sigma^2 \begin{pmatrix} n & \sum x_i \\ \sum x_i & \sum x_i^2 \end{pmatrix}^{-1} =$

$= \dfrac{\sigma^2}{n(\sum x_i^2 - (\sum x_i)^2)} \begin{pmatrix} \sum x_i^2 & -\sum x_i \\ -\sum x_i & n \end{pmatrix} = \dfrac{\sigma^2}{n(\langle x^2 \rangle - (\langle x \rangle)^2)} \begin{pmatrix} \langle x^2 \rangle & -\langle x \rangle \\ -\langle x \rangle & 1 \end{pmatrix}$

$\mathbb{V}(y^* | X) = \mathbb{V}(\hat{w}_0 + \hat{w}_1 x^*) = \mathbb{V}\hat{w}_0 + \mathbb{V}(\hat{w}_1 x^*) + 2\mathrm{cov}(\hat{w}_0, \hat{w}_1 x^*) =$

$= \dfrac{\sigma^2}{n(\langle x^2 \rangle - (\langle x \rangle)^2)} \left(\langle x^2 \rangle + (x^*)^2 \cdot 1 - 2 \langle x \rangle x^* \right)$

Ответ: $\mathbb{V}(y^* | X) = \dfrac{\sigma^2}{n(\langle x^2 \rangle - (\langle x \rangle)^2)} \left(\langle x^2 \rangle + (x^*)^2 - 2 \langle x \rangle x^* \right)$

(11) $y_i = x_i w + \varepsilon$

$(y - Xw)^T (y - Xw) \to \min \Rightarrow \hat{w} = (X^T X)^{-1} X^T y = \dfrac{\sum x_i y_i}{\sum x_i^2}$

$\mathbb{V}\hat{w} = \mathbb{E}(\hat{w} - w)(\hat{w} - w)^T = \sigma^2 (X^T X)^{-1} = \dfrac{\sigma^2}{\sum x_i^2}$

$\mathbb{E}\hat{w} = (X^T X)^{-1} X^T \mathbb{E}(Xw + \varepsilon) = w + \mathbb{E}\varepsilon$

Если $MSE \to 0$, то оценка состоятельна

$MSE = bias^2 + \mathbb{V}\hat{w} = (w + \mathbb{E}\varepsilon - w)^2 + \dfrac{\sigma^2}{\sum x_i^2} = (\mathbb{E}\varepsilon)^2 + \dfrac{\sigma^2}{\sum x_i^2} \to 0,$

если $\boxed{\mathbb{E}\varepsilon = 0 \text{ и } \dfrac{\sigma^2}{\sum x_i^2} \to 0}$