HIDDEN MARKOV MODELS

David Talbot, Yandex Translate

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Yandex School of Data Analysis

You see a single sequence of coin flips and are told they were generated by two different coins but it's not clear which observations are associated with which coin.

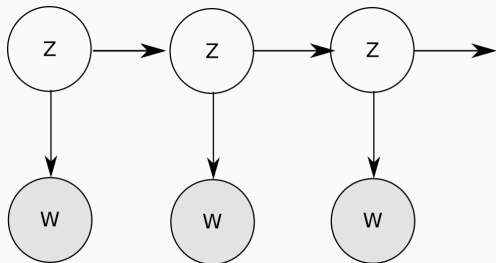H, H, H, H, H, H, T, H, T, T, T, H, T, T, H, T, T, H, H, H, H, H, H, ...

How could you model this data?

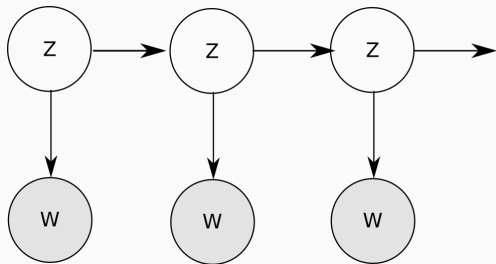What parameters would you need?

What are the sufficient statistics?

How could you estimate the parameters?

Each state emits one symbol then state transition occurs.

What independence assumptions does it make?

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What are the sufficient statistics?

In the observed case, we need the following statistics:

$$\#(Z_0 = i)$$

$$\#(Z_{t-1} = i, Z_t = j)$$

$$\#(X_t = x, Z_t = i)$$

In the hidden case, we need expectations for each sample:

$$\#(Z_0 = i) \rightarrow \Pr(Z_0 = i | X_{0:T} = x_{0:T}, \theta)$$

$$\#(Z_{t-1} = i, Z_t = j) \rightarrow \Pr(Z_{t-1} = i, Z_t = j | X_{0:T} = x_{0:T}, \theta)$$

$$\#(X_t = x, Z_t = i) \rightarrow \Pr(Z_t = i | X_{0:T} = x_{0:T}, \theta)\#(X_t = x)$$

We want to compute:

$$\Pr(Z_t = z | X_{0:T} = x_{0:T}, \theta) = \frac{\Pr(Z_t = z, X_{0:T} = x_{0:T})}{\Pr(X_{0:T} = x_{0:T})}$$

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But the computation looks exponential in the length T ...

$$\Pr(X_{0:T} = x_{0:T}) = \sum_{z_0, z_1, \ldots, z_T} \Pr(x_{0:T}, z_0, z_1, \ldots, z_T | \theta)$$

Use HMM independence assumptions to factorize

$$\Pr(x_0, \ldots, x_t, z_t, x_{t+1}, \ldots, x_T | \theta) = \Pr(x_0, \ldots, x_t, z_t | \theta)\Pr(x_{t+1}, \ldots, x_T | z_t, \theta)$$

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If we can compute this, then the denominator is easy

$$\Pr(x_0, \ldots, x_T | \theta) = \sum_{z_t} \Pr(x_0, \ldots, x_t, z_t | \theta) \Pr(x_{t+1}, \ldots, x_T | z_t, \theta).$$

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$$
\begin{aligned}
\Pr(x_0, \ldots, x_t, z_t | \theta) &= \sum_{z_{t-1}} \Pr(x_0, \ldots, x_{t-1}, x_t, z_{t-1}, z_t | \theta) \\
&= \sum_{z_{t-1}} \Pr(x_0, \ldots, x_{t-1}, z_{t-1} | \theta) \Pr(z_t | z_{t-1}) \Pr(x_t | z_t) \\
&= \sum_{z_{t-1}} \Pr(x_0, \ldots, x_{t-1}, z_{t-1} | \theta) A_{z_{t-1}}(z_t) O_{z_t}(x_t)
\end{aligned}
$$

Definition:
$$\alpha_t(z) \equiv \Pr(x_0, \ldots, x_t, z_t | \theta)$$

Initialization:
$$\alpha_0(i) = \pi_i O_i(x_0)$$

Recursion:
$$\alpha_{t+1}(i) = \sum_j \alpha_t(j) A_j(i) O_i(x_{t+1})$$

Gives us the probability of observed sequence since,

$$\Pr(x_0, \ldots, x_T | \theta) = \sum_{z_T} \Pr(x_0, \ldots, x_T, z_T | \theta) = \sum_i \alpha_T(i).$$

Definition:

$$\beta_t(z) \equiv \Pr(x_{t+1}, \ldots, x_T | z_t, \theta)$$

Initialization:

$$\beta_T(i) = 1$$

Recursion:

$$\beta_t(i) = \sum_j \beta_{t+1}(j) A_i(j) O_j(x_{t+1})$$

Posterior probabilities over single states

$$\Pr(Z_t = i | x_0, \ldots, x_T; \theta) = \frac{\Pr(Z_t = i, x_0, \ldots, x_T | \theta)}{\Pr(x_0, \ldots, x_T | \theta)}$$

$$= \frac{\alpha_t(i)\beta_t(i)}{\sum_j \alpha_t(j)\beta_t(j)}$$

Posterior probabilities over state transitions

$$\Pr(Z_t = i, Z_{t+1} = j | x_0, \ldots, x_T; \theta) = \frac{\Pr(Z_t = i, Z_{t+1} = j, x_0, \ldots, x_T | \theta)}{\Pr(x_0, \ldots, x_T | \theta)}$$

$$= \frac{\alpha_t(i) A_i(j) O_j(x_{t+1}) \beta_{t+1}(j)}{\sum_i \sum_j \alpha_t(i) A_i(j) O_j(x_{t+1}) \beta_{t+1}(j)}$$

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How could you use an HMM to solve this problem?