# Calculus II Recap: Integration

January 2026

# 7. Integral Calculus

## 7.1 Definition

The **Indefinite Integral** (antiderivative) of a function $f(x)$ is a function $F(x)$ such that $F'(x) = f(x)$. It is denoted as:

$$\int f(x)\,dx = F(x) + C$$

where $C$ is the constant of integration. The **Definite Integral** represents the signed area under the curve from $a$ to $b$:

$$\int_a^b f(x)\,dx = F(b) - F(a) \quad \text{(Fundamental Theorem of Calculus)}$$

[Image of area under a curve definite integral]

## 7.2 Basic Integration Rules

- **Power Rule:** $\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \quad (n \neq -1)$
- **Logarithmic Rule:** $\int \frac{1}{x}\,dx = \ln|x| + C$
- **Exponential Rule:** $\int e^x\,dx = e^x + C$
- **Constant Multiple Rule:**
$$\int k \cdot f(x)\,dx = k \int f(x)\,dx$$
- **Sum Rule:**
$$\int [f(x) + g(x)]\,dx = \int f(x)\,dx + \int g(x)\,dx$$
- **Difference Rule:**
$$\int [f(x) - g(x)]\,dx = \int f(x)\,dx - \int g(x)\,dx$$
- **Linearity of Integration:** Combined rule: $\int [af(x) + bg(x)]\,dx = a \int f(x)\,dx + b \int g(x)\,dx$
- **Trigonometric Rules:**
  - $\int \sin x\,dx = -\cos x + C$
  - $\int \cos x\,dx = \sin x + C$
  - $\int \sec^2 x\,dx = \tan x + C$
- **Hyperbolic Rules:**
  - $\int \sinh x\,dx = \cosh x + C$
  - $\int \cosh x\,dx = \sinh x + C$
  - $\int \text{sech}^2 x\,dx = \tanh x + C$
  - $\int \frac{1}{\sqrt{x^2+1}}\,dx = \text{arsinh}\ x + C = \ln(x + \sqrt{x^2+1}) + C$
  - $\int \frac{1}{\sqrt{x^2-1}}\,dx = \text{arcosh}\ x + C = \ln(x + \sqrt{x^2-1}) + C$

## 7.3 Partial Integration (Integration by Parts)

Derived from the product rule of differentiation, used when the integrand is a product of two functions:

$$\int u\,dv = uv - \int v\,du$$

**Strategy (LIATE):** Choose $u$ in order of: **L**ogarithmic, **I**nverse trig, **A**lgebraic, **T**rigonometric, **E**xponential.

## 7.4 Integration by Substitution ($u$-substitution)

Used to simplify an integral by changing variables, essentially the "reverse chain rule":

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du, \quad \text{where } u = g(x)$$

For definite integrals, remember to change the limits of integration:

$$\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du$$

### Trigonometric Substitution

Used when evaluating integrals of the form $\sqrt{a^2 - x^2}, \sqrt{a^2 + x^2}, \sqrt{x^2 - a^2}$ as trigonometric integrals.

**Reference Triangles:** Helpful for determining where to place the square root and what the trig functions are.

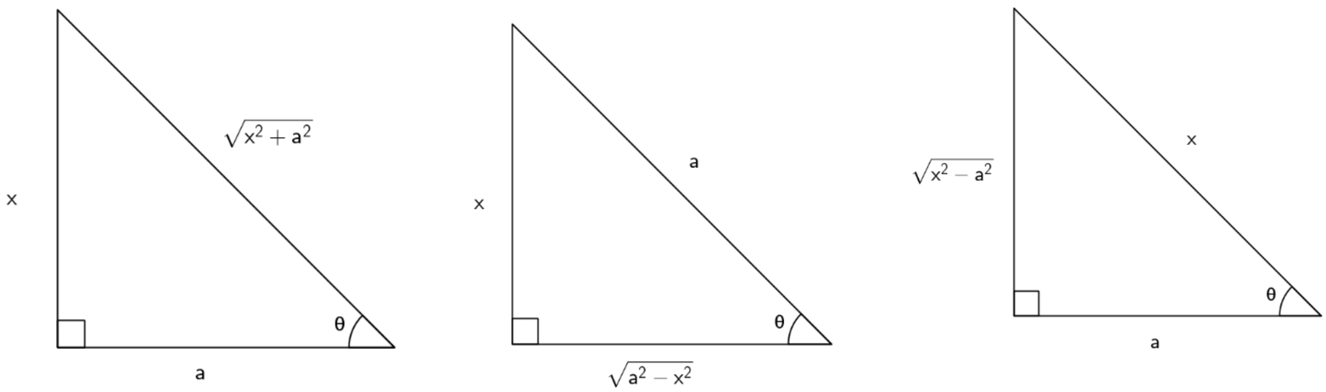


Figure 1: Reference triangles

**Hypotenuse**

$$a^2 + x^2 = (\sqrt{x^2 + a^2})^2$$

$x$ and number are positive

Use the substitution: $x = a\sin\theta$

Result: $a\cos\theta$

**Adjacent**

$$(\sqrt{a^2 - x^2})^2 + x^2 = a^2$$

$x$ is negative, number is positive

Use the substitution: $x = a\tan\theta$

Result: $a\sec$

**Opposite**

$$a^2 + (\sqrt{x^2 - a^2})^2 = x^2$$

$x$ is positive, number is negative

Use the substitution: $x = a\sec\theta = a/\cos\theta$

Result: $a\tan\theta$

**Common Hyperbolic Substitutions:**

| Integral | Circle trig substitution | Hyperbolic trig substitution |
|---|---|---|
| $\int \frac{dx}{\sqrt{a^2 - x^2}}$ | $x = a\sin(u)$ | $x = a\tanh(u)$ |
| $\int \frac{dx}{\sqrt{a^2 + x^2}}$ | $x = a\tan(u)$ | $x = a\sinh(u)$ |
| $\int \frac{dx}{\sqrt{x^2 - a^2}}$ | $x = a\sec(u)$ | $x = a\cosh(u)$ |

## 7.5 Partial Fraction Decomposition

Used to integrate rational functions $\frac{P(x)}{Q(x)}$ by breaking them into simpler fractions.

- **Linear Factors:** $\frac{1}{(x-a)(x-b)} = \frac{A}{x-a} + \frac{B}{x-b}$
- **Repeated Factors:** $\frac{1}{(x-a)^2} = \frac{A}{x-a} + \frac{B}{(x-a)^2}$
- **Irreducible Quadratic:** $\frac{1}{x^2+px+q} = \frac{Ax+B}{x^2+px+q}$

### 7.6 Mean Value Theorem for Integrals

If $f$ is continuous on $[a, b]$, then there exists a number $c$ in $[a, b]$ such that:

$$f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx$$

Here, $f(c)$ represents the **average value** of the function $f$ on the interval $[a, b]$.

# 8. Sequences and Series

## 8.1 Sequences

A sequence $\{a_n\}$ is an ordered list of numbers. It can be defined:

- **Explicitly:** $a_n = f(n)$, e.g., $a_n = \frac{1}{n}$.
- **Recursively:** $a_{n+1} = f(a_n)$, e.g., $a_{n+1} = \sqrt{a_n} + 1$.

#### Special Sequences

- **Geometric Sequence:** $a_n = a \cdot r^{n-1}$. Constant ratio $r$.
- **Harmonic Sequence:** $a_n = \frac{1}{n}$.

#### Convergence and Boundedness

A sequence **converges** if $\lim_{n\to\infty} a_n = L$.

- **Monotone Convergence Theorem:** If a sequence is **monotonic** (always increasing or decreasing) and **bounded**, then it must converge.

## 8.2 Series

A series is the sum of the terms of a sequence: $S = \sum_{n=1}^{\infty} a_n$.

- **Partial Sums ($s_n$):** $s_n = \sum_{k=1}^{n} a_k$. A series converges if the sequence of partial sums $\{s_n\}$ has a finite limit.
- **Necessary Condition:** If $\sum a_n$ converges, then $\lim_{n\to\infty} a_n = 0$. (Note: If the limit is not 0, the series diverges).

#### Convergence Tests

| **Test** | **Condition for Convergence** | **Notes** |
|---|---|---|
| Geometric Series | $\sum r^n$ converges if $\lvert r\rvert < 1$ | Sum $S = \frac{a}{1-r}$ |
| Harmonic Series | $\sum \frac{1}{n}$ always **diverges** | $p$-series with $p = 1$ |
| Ratio Test | $\lim_{n\to\infty} \left\lvert \frac{a_{n+1}}{a_n} \right\rvert = L < 1$ | Diverges if $L > 1$ |
| Root Test | $\lim_{n\to\infty} \sqrt[n]{\lvert a_n\rvert} = L < 1$ | Diverges if $L > 1$ |
| Comparison Test | $a_n \le b_n$ and $\sum b_n$ converges | $\sum a_n$ must converge |

## 8.3 Power Series

A power series is of the form $\sum_{n=0}^{\infty} c_n (x - x_0)^n$.

- **Convergence Radius ($R$):** The series converges for $|x - x_0| < R$.
- **Formula (Ratio Test):** $R = \lim_{n\to\infty} \left| \frac{c_n}{c_{n+1}} \right|$ or $R = \frac{1}{\lim_{n\to\infty} \sqrt[n]{|c_n|}}$

# 9. Taylor Expansion

## 9.1 Taylor Series Formula

The Taylor series of $f(x)$ centered at $x_0$ is:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$$

If $x_0 = 0$, it is called a **Maclaurin Series**.

## 9.2 Common Taylor Expansions (at $x_0 = 0$)

- $e^x = 1 + x + \frac{x^2}{2!} + \cdots = \sum \frac{x^n}{n!}, \quad R = \infty$
- $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots, \quad R = \infty$
- $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots, \quad R = \infty$
- $\frac{1}{1-x} = 1 + x + x^2 + \ldots, \quad R = 1$

## 9.3 Applications

1. **Approximation:** Using the first few terms (Taylor Polynomial $P_n(x)$) to estimate function values.
2. **Limits:** Solving indeterminate forms by replacing functions with their series.
3. **Integration:** Approximating integrals that have no elementary antiderivative:

$$\int e^{-x^2} dx \approx \int \left(1 - x^2 + \frac{x^4}{2}\right) dx$$