1. Elementary Functions

A function $f : A \to B$ assigns each $x \in A$ exactly one $y \in B$.

- **Injective:** $f(x_1) = f(x_2) \implies x_1 = x_2$.
- **Surjective:** Range equals Codomain.
- **Bijective:** Both injective and surjective; implies an **inverse** f^{-1} exists.

1. Inverse function

f is a one-to-one function on a domain D with range R. The inverse function f^{-1} is defined by

$$f^{-1}(b) = a \quad \text{if} \quad f(a) = b.$$

The domain of f^{-1} is R and the range of f^{-1} is D.

2. Function transformation

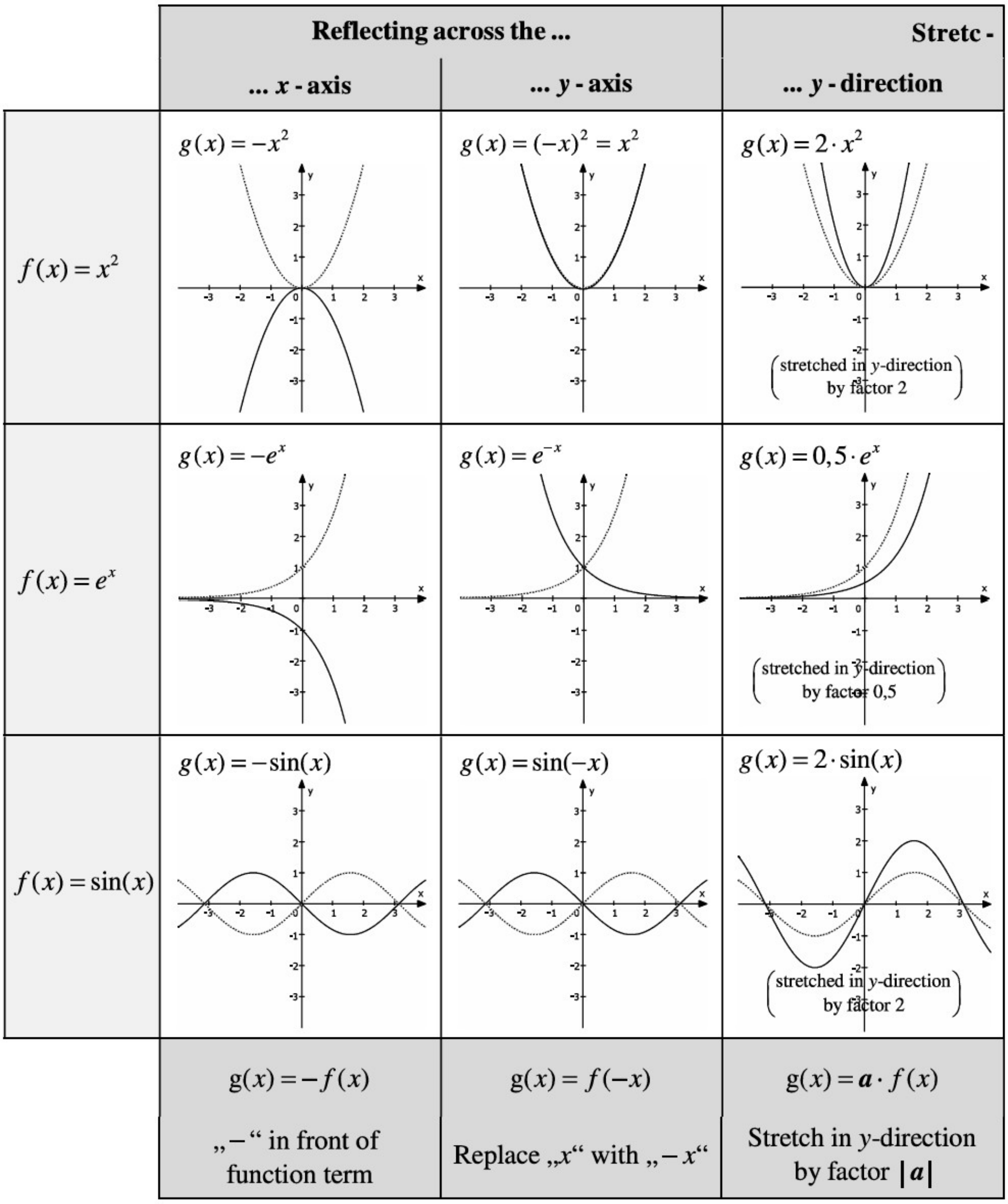

Figure 1: Function Transformation 01

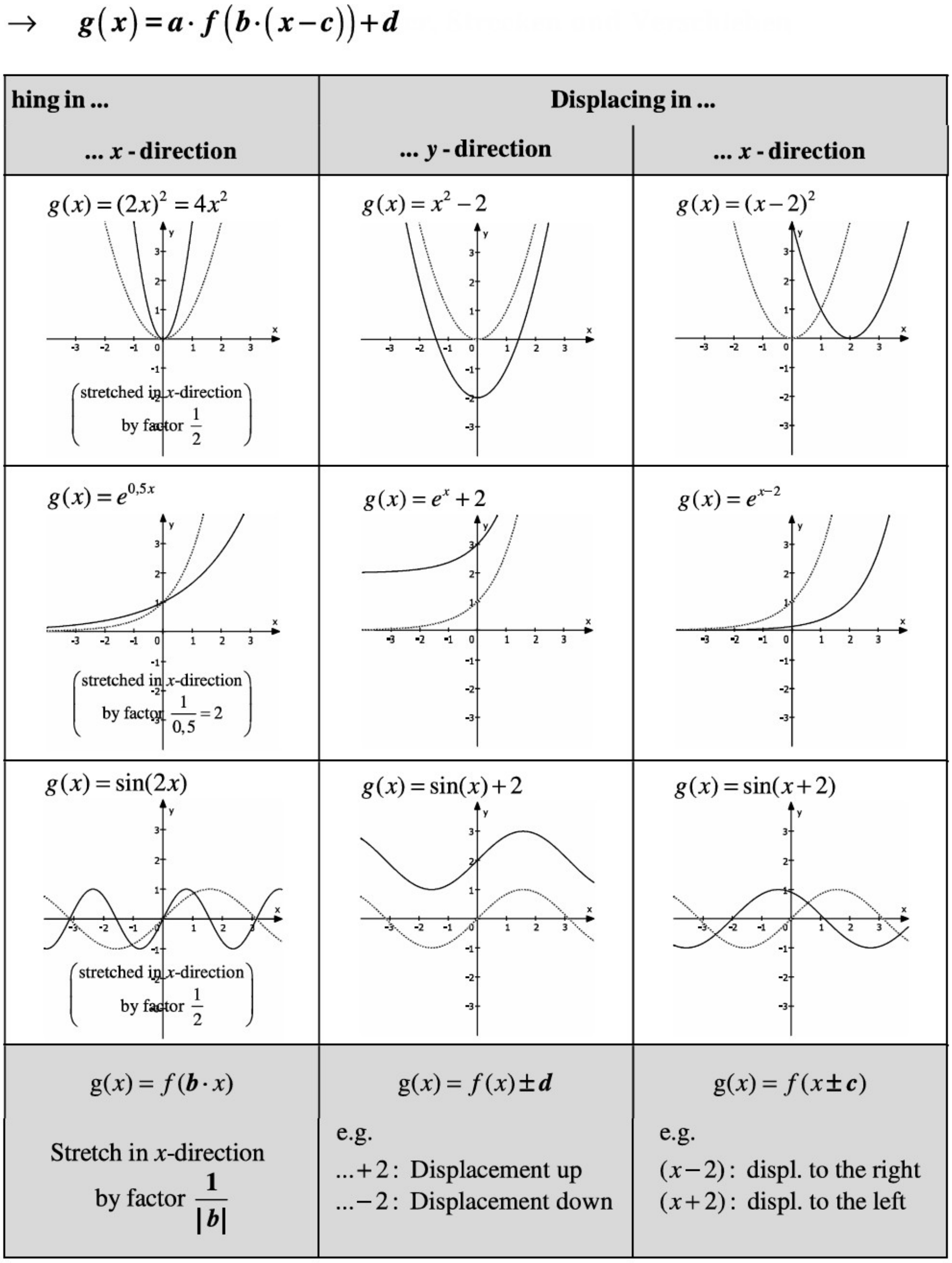

Figure 2: Function Transformation 02

2. Logarithmic and exponential functions

The logarithm function with base a, $y = \log_a x$, is the inverse of the base-a exponential function $y = a^x$ (with $a > 0$ and $a \neq 1$).

1. Logarithm Rules

- **Product Rule:** $\log_b(xy) = \log_b(x) + \log_b(y)$
- **Quotient Rule:** $\log_b\left(\frac{x}{y}\right) = \log_b(x) - \log_b(y)$
- **Power Rule:** $\log_b(x^n) = n\log_b(x)$
- **Change of Base Rule:** $\log_b(x) = \frac{\log_c(x)}{\log_c(b)}$
  or $\log_b(x) \cdot \log_c(b) = \log_c(x)$
- **Zero Rule:** $\log_b(1) = 0$
- **Identity Rule:** $\log_b(b) = 1$

- **Equality Rule:** $\log_b(x) = \log_b(y) \Rightarrow x = y$
- **Inverse Rule:** $b^{\log_b(x)} = x$ and $\log_b(b^x) = x$
- **Reciprocal Rule:** $\log_b\left(\frac{1}{x}\right) = -\log_b(x)$

2. Exponent Rules

- **Product Rule:** $a^m \cdot a^n = a^{m+n}$
- **Quotient Rule:** $\frac{a^m}{a^n} = a^{m-n}$
- **Power of a Power Rule:** $(a^m)^n = a^{mn}$
- **Power of a Product Rule:** $(ab)^n = a^n \cdot b^n$
- **Power of a Quotient Rule:** $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$
- **Zero Exponent Rule:** $a^0 = 1$ (for $a \neq 0$)
- **Negative Exponent Rule:** $a^{-n} = \frac{1}{a^n}$
- **Fractional Exponent Rule:** $a^{1/n} = \sqrt[n]{a}$ and $a^{m/n} = \sqrt[n]{a^m}$

3. Applications

- Exponential decay/growth: $y = y_0 e^{kt}$ (growth if $k > 0$, decay if $k < 0$)
- Compound interest: $A = P\left(1 + \frac{r}{n}\right)^{nt}$ (compounded n times per year)
  $A = Pe^{rt}$ (continuous compounding)

3. Trigonometric functions

1. Definition

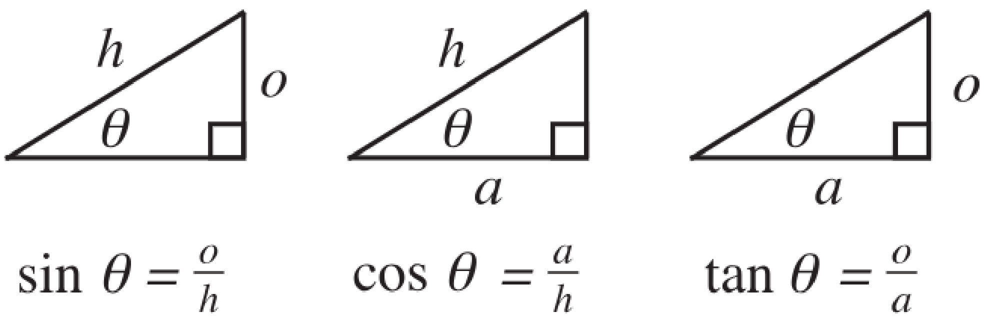

Figure 3: Trigonometric basics

2. Properties of Trigonometric Functions

Function	Periodicity	Symmetry	Continuity
$\sin x$	2π	Odd	Continuous everywhere
$\cos x$	2π	Even	Continuous everywhere
$\tan x$	π	Odd	Discontinuous at $\frac{\pi}{2} + k\pi$
$\cot x$	π	Odd	Discontinuous at $k\pi$
$\sec x$	2π	Even	Discontinuous at $\frac{\pi}{2} + k\pi$
$\csc x$	2π	Odd	Discontinuous at $k\pi$
$\arcsin x$	None	Odd	Continuous on $[-1, 1]$
$\arccos x$	None	Neither	Continuous on $[-1, 1]$
$\arctan x$	None	Odd	Continuous everywhere
$\cot^{-1} x$	None	Neither	Continuous everywhere except $x = 0$ (depending on convention)
$\sec^{-1} x$	None	Neither	Continuous on $(-\infty, -1] \cup [1, \infty)$
$\csc^{-1} x$	None	Neither	Continuous on $(-\infty, -1] \cup [1, \infty)$

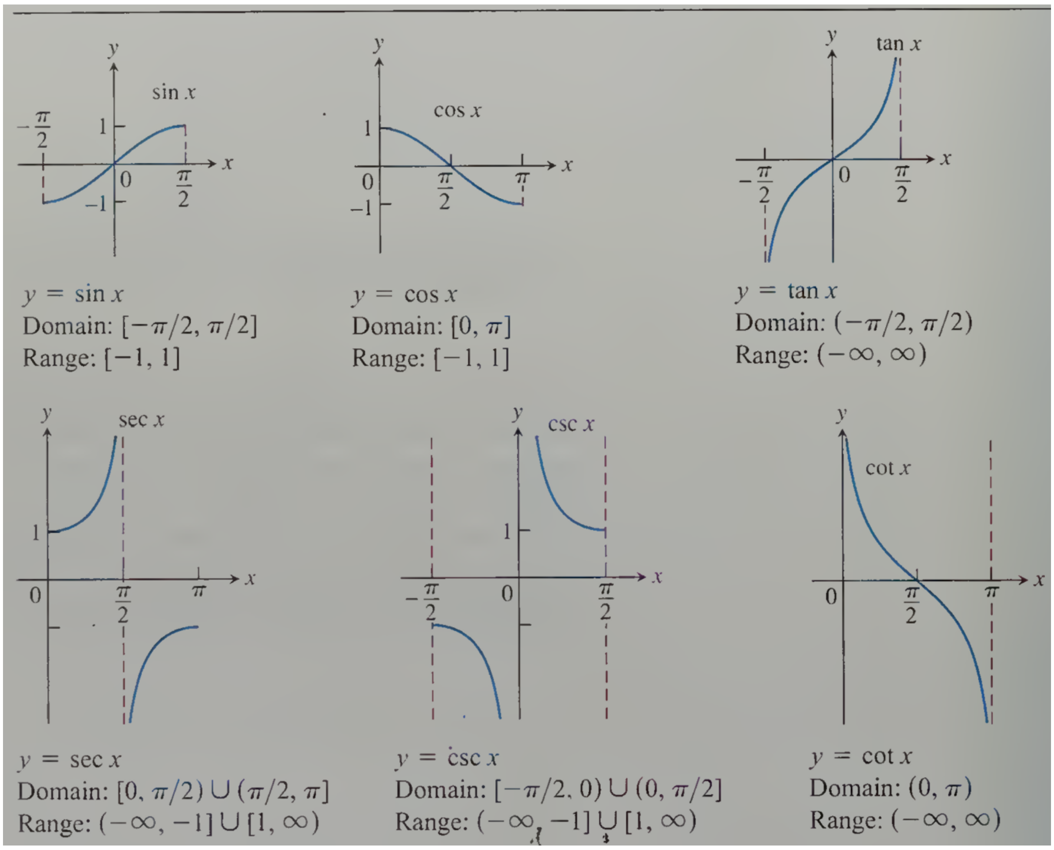

Figure 4: Trig Functions and their inverses

0.1 3. Trigonometric Identities

Pythagorean Identities

$$\sin^2\theta + \cos^2\theta = 1$$
$$\tan^2\theta + 1 = \sec^2\theta$$
$$1 + \cot^2\theta = \csc^2\theta$$

Reciprocal Identities

$$\sin\theta = \frac{1}{\csc\theta} \qquad \csc\theta = \frac{1}{\sin\theta}$$
$$\cos\theta = \frac{1}{\sec\theta} \qquad \sec\theta = \frac{1}{\cos\theta}$$
$$\tan\theta = \frac{1}{\cot\theta} \qquad \cot\theta = \frac{1}{\tan\theta}$$

Sum Identities

$$\sin(a+b) = \sin a\cos b + \cos a\sin b$$
$$\cos(a+b) = \cos a\cos b - \sin a\sin b$$
$$\tan(a+b) = \frac{\tan a + \tan b}{1 - \tan a\tan b}$$

Double Angle Identities

$$\sin(2a) = 2\sin a\cos a$$
$$\cos(2a) = \cos^2 a - \sin^2 a = 2\cos^2 a - 1 = 1 - 2\sin^2 a$$
$$\tan(2a) = \frac{2\tan a}{1 - \tan^2 a}$$

4. Polar coordinates

- **Polar Conversion:**

$$x = r\cos(\theta), \quad y = r\sin(\theta)$$

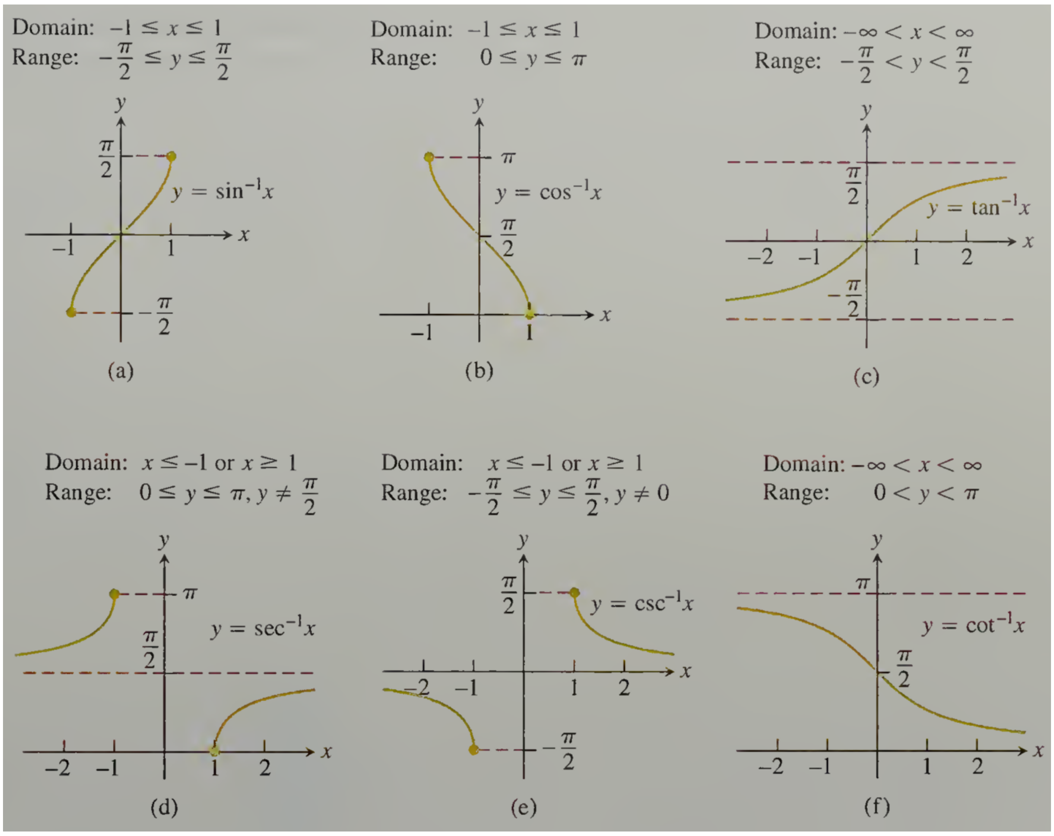

Figure 5: Arc functions

.

- **Inverse Conversion:**

$$r = \sqrt{x^2 + y^2}, \quad \theta = \tan^{-1}\left(\frac{y}{x}\right)$$

4. Hyperbolic Functions

1. Definition

Hyperbolic functions are defined using the exponential function e^x.

$$\sinh x = \frac{e^x - e^{-x}}{2} \qquad \text{csch } x = \frac{1}{\sinh x}$$

$$\cosh x = \frac{e^x + e^{-x}}{2} \qquad \text{sech } x = \frac{1}{\cosh x}$$

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}} \qquad \coth x = \frac{1}{\tanh x}$$

2. Properties of Hyperbolic Functions

Function	**Domain**	**Symmetry**	**Continuity**
$\sinh x$	$(-\infty, \infty)$	Odd	Continuous everywhere
$\cosh x$	$(-\infty, \infty)$	Even	Continuous everywhere
$\tanh x$	$(-\infty, \infty)$	Odd	Continuous everywhere
$\coth x$	$x \neq 0$	Odd	Discontinuous at $x = 0$
$\text{sech } x$	$(-\infty, \infty)$	Even	Continuous everywhere
$\text{csch } x$	$x \neq 0$	Odd	Discontinuous at $x = 0$
$\text{arsinh } x$	$(-\infty, \infty)$	Odd	Continuous everywhere
$\text{arcosh } x$	$[1, \infty)$	Neither	Continuous on $[1, \infty)$
$\text{artanh } x$	$(-1, 1)$	Odd	Continuous on $(-1, 1)$

3. Hyperbolic Identities

Fundamental Identity (Hyperbolic Pythagorean)

$$\cosh^2 x - \sinh^2 x = 1$$
$$1 - \tanh^2 x = \text{sech}^2 x$$
$$\coth^2 x - 1 = \text{csch}^2 x$$

Sum Identities

$$\sinh(a \pm b) = \sinh a \cosh b \pm \cosh a \sinh b$$
$$\cosh(a \pm b) = \cosh a \cosh b \pm \sinh a \sinh b$$
$$\tanh(a \pm b) = \frac{\tanh a \pm \tanh b}{1 \pm \tanh a \tanh b}$$

Double Angle Identities

$$\sinh(2x) = 2 \sinh x \cosh x$$
$$\cosh(2x) = \cosh^2 x + \sinh^2 x = 2\cosh^2 x - 1 = 1 + 2\sinh^2 x$$

4. Inverse Hyperbolic Functions (Logarithmic Form)

Hyperbolic inverses can be expressed directly using natural logarithms:

- $\text{arsinh } x = \ln(x + \sqrt{x^2 + 1})$
- $\text{arcosh } x = \ln(x + \sqrt{x^2 - 1}), \quad x \geq 1$
- $\text{artanh } x = \frac{1}{2} \ln\left(\frac{1+x}{1-x}\right), \quad |x| < 1$

5. Differential Calculus

5.1 Tangent

The derivative $f'(x)$ is the **slope** of the tangent line to the function at any given point.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

5.2 General Rules

- **Power Rule:** $(x^n)' = nx^{n-1}$
- **Product Rule:** $(uv)' = u'v + uv'$
- **Quotient Rule:** $\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$
- **Chain Rule:** $[f(g(x))]' = f'(g(x)) \cdot g'(x)$

5.3 Implicit Differentiation

When we cannot put an equation $F(x, y) = 0$ in the form $y = f(x)$ to differentiate it in the usual way, we may still be able to find dy/dx by implicit differentiation.

1. Differentiate both sides of the equation with respect to x, treating y as a differentiable function of x.
2. Collect all terms containing $\dfrac{dy}{dx}$ on one side of the equation and solve for $\dfrac{dy}{dx}$.

5.4 Curve Analysis

Used to find extrema, turning points, and inflection points:

1. **Necessary Criteria / First Derivative Test:** at $f'(x) = 0$ identifies critical points (maxima, minima, or plateaus).
2. **Sufficient Criteria / Second Derivative Test:**
   - $f''(x) > 0 \implies$ Local Minimum (concave up).
   - $f''(x) < 0 \implies$ Local Maximum (concave down).
   - $f''(x) = 0 \implies$ Potential Inflection Point.

Derivative Rules

Exponential Functions

$$\frac{d}{dx}\left(e^{x}\right) = e^{x}$$

$$\frac{d}{dx}\left(a^{x}\right) = a^{x}\ln a$$

$$\frac{d}{dx}\left(e^{g(x)}\right) = e^{g(x)}g'(x)$$

$$\frac{d}{dx}\left(a^{g(x)}\right) = \ln(a)\,a^{g(x)}\,g'(x)$$

Logarithmic Functions

$$\frac{d}{dx}(\ln x) = \frac{1}{x},\ x > 0$$

$$\frac{d}{dx}\ln(g(x)) = \frac{g'(x)}{g(x)}$$

$$\frac{d}{dx}\left(\log_a x\right) = \frac{1}{x\ln a},\ x > 0$$

$$\frac{d}{dx}\left(\log_a g(x)\right) = \frac{g'(x)}{g(x)\ln a}$$

Trigonometric Functions

$$\frac{d}{dx}(\sin x) = \cos x$$

$$\frac{d}{dx}(\cos x) = -\sin x$$

$$\frac{d}{dx}(\tan x) = \sec^2 x$$

$$\frac{d}{dx}(\csc x) = -\csc x\cot x$$

$$\frac{d}{dx}(\sec x) = \sec x\tan x$$

$$\frac{d}{dx}(\cot x) = -\csc^2 x$$

Inverse Trigonometric Functions

$$\frac{d}{dx}\left(\sin^{-1} x\right) = \frac{1}{\sqrt{1-x^2}},\ x \neq \pm 1$$

$$\frac{d}{dx}\left(\cos^{-1} x\right) = \frac{-1}{\sqrt{1-x^2}},\ x \neq \pm 1$$

$$\frac{d}{dx}\left(\tan^{-1} x\right) = \frac{1}{1+x^2}$$

$$\frac{d}{dx}\left(\cot^{-1} x\right) = \frac{-1}{1+x^2}$$

$$\frac{d}{dx}\left(\sec^{-1} x\right) = \frac{1}{x\sqrt{x^2-1}},\ x \neq \pm 1, 0$$

$$\frac{d}{dx}\left(\csc^{-1} x\right) = \frac{-1}{x\sqrt{x^2-1}},\ x \neq \pm 1, 0$$

Hyperbolic Functions

$$\frac{d}{dx}(\sinh x) = \cosh x$$

$$\frac{d}{dx}(\cosh x) = \sinh x$$

$$\frac{d}{dx}(\tanh x) = \operatorname{sech}^2 x$$

$$\frac{d}{dx}(\operatorname{csch} x) = -\operatorname{csch} x\coth x$$

$$\frac{d}{dx}(\operatorname{sech} x) = -\operatorname{sech} x\tanh x$$

$$\frac{d}{dx}(\coth x) = -\operatorname{csch} x$$

Inverse Hyperbolic Functions

$$\frac{d}{dx}\left(\sinh^{-1} x\right) = \frac{1}{\sqrt{1+x^2}}$$

$$\frac{d}{dx}\left(\cosh^{-1} x\right) = \frac{1}{\sqrt{x^2-1}},\ x > 1$$

$$\frac{d}{dx}\left(\tanh^{-1} x\right) = \frac{1}{1-x^2},\ |x| < 1$$

$$\frac{d}{dx}\left(\operatorname{csch}^{-1} x\right) = \frac{-1}{|x|\sqrt{1-x^2}},\ x \neq 0$$

$$\frac{d}{dx}\left(\operatorname{sech}^{-1} x\right) = \frac{-1}{x\sqrt{1-x^2}},\ 0 < x < 1$$

$$\frac{d}{dx}\left(\coth^{-1} x\right) = \frac{1}{1-x^2},\ |x| > 1$$

Figure 6: Derivation Rules

6. Limits

6.1 Definition

The limit $\lim_{x \to a} f(x) = L$ exists if and only if the left-hand and right-hand limits are equal:

$$\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = L$$

2. Limit Laws

If L, M, c, and k are real numbers, and

$$\lim_{x \to c} f(x) = L \quad \text{and} \quad \lim_{x \to c} g(x) = M,$$

then:

- **Sum Rule:** $\lim\limits_{x \to c}(f(x) + g(x)) = L + M$
- **Difference Rule:** $\lim\limits_{x \to c}(f(x) - g(x)) = L - M$
- **Constant Multiple Rule:** $\lim\limits_{x \to c}(k \cdot f(x)) = k \cdot L$
- **Product Rule:** $\lim\limits_{x \to c}(f(x) \cdot g(x)) = L \cdot M$
- **Quotient Rule:** $\lim\limits_{x \to c} \dfrac{f(x)}{g(x)} = \dfrac{L}{M}, \quad M \neq 0$
- **Power Rule:** $\lim\limits_{x \to c}[f(x)]^n = L^n, \quad n$ a positive integer
- **Root Rule:** $\lim\limits_{x \to c} \sqrt[n]{f(x)} = \sqrt[n]{L} = L^{1/n}, \quad n$ a positive integer
  (If n is even, assume $\lim_{x \to c} f(x) = L > 0$.)

6.2 Limit Types

- **Limits at Infinity:** Describes the horizontal asymptotes of a function as $x \to \pm\infty$.
- **Infinite Limits:** Occur at vertical asymptotes where the function grows without bound.
- **Indeterminate Forms:** Limits that result in $\frac{0}{0}$ or $\frac{\infty}{\infty}$ require further algebraic manipulation or specialized rules.

6.3 Specialized Techniques

- **L'Hôpital's Rule:** If a limit results in an indeterminate form ($\frac{0}{0}$ or $\frac{\infty}{\infty}$), then:

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

- **Squeeze (Sandwich) Theorem:** If $g(x) \leq f(x) \leq h(x)$ and $\lim_{x \to a} g(x) = \lim_{x \to a} h(x) = L$, then $\lim_{x \to a} f(x) = L$.
- **Eliminating Zero Denominators:**
  - Cancel common factors in the numerator and denominator to reduce the fraction.
  - Substitute the limit in the simplified fraction.
- **Limit of** $\sin\theta/\theta$

$$\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$$