1. Vectors

1. Unit vectors

2. Vector properties

- Addition
- Multiplication
- Dot product: $\vec{a} \cdot \vec{b} = |\vec{a}| * |\vec{b}| * \cos\alpha$
- Cross product: $\vec{a} \times \vec{b} = \vec{a}| * |\vec{b}| * \sin\alpha$
- Triple product: $(\vec{a} \times \vec{b}) \cdot \vec{c} = \vec{a}| * |\vec{b}| * |\vec{c}| * \sin\alpha$

3. Linear Independence

$c_1 * \vec{a_1} + c_2 * \vec{a_2} + ...c_n * \vec{a_n} = 0$ only for $c_1 = c_2 = ... = c_n = 0$

2. Coordinate Geometry (Cartesian)

1. Lines & Planes Representation

- **Parametric:** $\vec{r} = \vec{a} + t\vec{v}$ (Line) or $\vec{r} = \vec{a} + s\vec{u} + t\vec{v}$ (Plane)
- **Linear (General):** $ax + by + cz = d$ (plane) & $y = mx + b$ (line)
- **HNF (Hesse Normal Form):** $\vec{r} \cdot \vec{n}_0 = d$ (where $|\vec{n}_0| = 1$)

2. Minimal Distances

Scenario	**Formula**
Point P to Line	$d = \frac{\vert\vec{v} \times \vec{AP}\vert}{\vert\vec{v}\vert}$
Point P to Plane	$d = \frac{\vert\vec{n} \cdot \vec{AP}\vert}{\vert\vec{n}\vert}$ (or use HNF: $\vert\vec{P} \cdot \vec{n}_0 - d\vert$)
Skew Lines	$d = \frac{\vert(\vec{v}_1 \times \vec{v}_2) \cdot \vec{A_1 A_2}\vert}{\vert\vec{v}_1 \times \vec{v}_2\vert}$

3. Line Relationships

To determine how two lines interact, check the direction vectors $\vec{v}_1, \vec{v}_2$ and the system of equations:

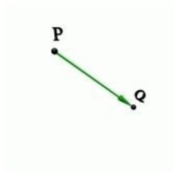
- **Parallel:** $\vec{v}_1 = k\vec{v}_2$ and no common points.
- **Skew:** Not parallel and no intersection point.
- **Intersecting:** Unique solution for t and s.
- **Perpendicular:** $\vec{v}_1 \cdot \vec{v}_2 = 0$.

Distances overview

DISTANCE POINT-POINT (3D). If P and Q are two points, then

$$d(P, Q) = |\vec{PQ}|$$

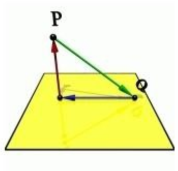
is the distance between P and Q. We use the notation $|\vec{v}|$ instaed of $||\vec{v}||$ in this handout.



DISTANCE POINT-PLANE (3D). If P is a point in space and $\Sigma : \vec{n} \cdot \vec{x} = d$ is a plane containing a point Q, then

$$d(P, \Sigma) = \frac{|\vec{PQ} \cdot \vec{n}|}{|\vec{n}|}$$

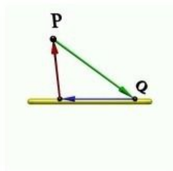
is the distance between P and the plane. Proof: use the angle formula in the denominator.



DISTANCE POINT-LINE (3D). If P is a point in space and L is the line $\vec{r}(t) = Q + t\vec{u}$, then

$$d(P, L) = \frac{|(\vec{PQ}) \times \vec{u}|}{|\vec{u}|}$$

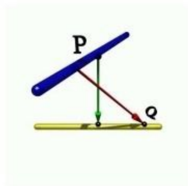
is the distance between P and the line L. Proof: the area divided by base length is height of parallelogram.



DISTANCE LINE-LINE (3D). L is the line $\vec{r}(t) = Q + t\vec{u}$ and M is the line $\vec{s}(t) = P + t\vec{v}$, then

$$d(L, M) = \frac{|(\vec{PQ}) \cdot (\vec{u} \times \vec{v})|}{|\vec{u} \times \vec{v}|}$$

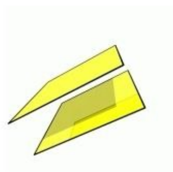
is the distance between the two lines L and M. Proof: the distance is the length of the vector projection of $\vec{PQ}$ onto $\vec{u} \times \vec{v}$ which is normal to both lines.



DISTANCE PLANE-PLANE (3D). If $\vec{n} \cdot \vec{x} = d$ and $\vec{n} \cdot \vec{x} = e$ are two parallel planes, then their distance is

$$\frac{|e - d|}{|\vec{n}|} .$$

Non-parallel planes have distance 0. Proof: use the distance formula between point and plane.



4. Angles

The angle θ between two objects is found using the dot product of their direction/normal vectors:

- **Between two lines:** $\cos\theta = \frac{|\vec{v}_1 \cdot \vec{v}_2|}{|\vec{v}_1||\vec{v}_2|}$
- **Between two planes:** $\cos\theta = \frac{|\vec{n}_1 \cdot \vec{n}_2|}{|\vec{n}_1||\vec{n}_2|}$

3. Matrices

1. Basic Properties

- **Addition:** $A + B$ (Matrices must have identical dimensions).
- **Multiplication:** $A_{m \times n} \cdot B_{n \times l} = C_{m \times l}$.

$$\begin{pmatrix} 0 & \mathbf{-2} & 3 & 1 & 0 & 1 \\ 0 & 0 & 0 & \mathbf{3/2} & -4/3 & 17 \\ 0 & 0 & 0 & 0 & \mathbf{1} & 11 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Figure 1: REF

$$\begin{pmatrix} 0 & \mathbf{1} & -3/2 & 0 & 0 & 181/18 \\ 0 & 0 & 0 & \mathbf{1} & 0 & 190/3 \\ 0 & 0 & 0 & 0 & \mathbf{1} & 11 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Figure 2: RREF

- **Identity:** $A \cdot I = I \cdot A = A$.
- **Dimensionality:** square, triangular.

2. Gaussian Elimination

Used to reach **REF** (Row Echelon Form) or **RREF** (Reduced REF) via:

- Row swaps, scalar multiplication, and row addition/subtraction.

3. Kernel (Null Space):

The set of vectors $\{\vec{v}\}$ such that $A\vec{v} = 0$.

4. The Determinant $\det(A)$

- If $\det(A) = 0$, the matrix is **singular** (non-invertible).
- **Laplace Expansion:** $\det(A) = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} \cdot \det(M_{ij})$
- **Cofactor** C_{ij}**:** $(-1)^{i+j} \cdot \det(M_{ij})$ where M_{ij} is the minor of an element.
- **Minor** M_{ij}**:**
- **Adjoint:** $adj(A) = C^T$

5. Inverse Matrix: $A \cdot A^{-1} = I$

Exists only if $\det(A) \neq 0$.

- **Laplace Expansion:** $A^{-1} = \frac{1}{\det(A)} * adj(A)$

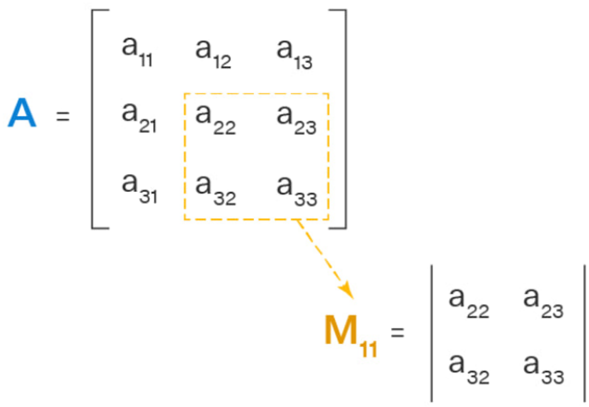


Figure 3: Minor

6. **Rank $\rho(A)$:**
   - **The number of non-zero rows in REF.**
   - **non-singular** $A_{n\times n}$ p(A) = n.
   - **singular** $A_{m\times n}$ p(A) ¡ m.
   - **REF(**$A_{m\times n}$**)** p(A) = non-zero m.

4. Systems of Linear Equations (SLE)

$a_1 * x_1 + a_2 * x_2 + ...a_n * x_n = b$

1. Matrix Notation

A system of equations can be written as $A\vec{x} = \vec{b}$.

- **Homogeneous:** $\vec{b} = 0$. Always has at least the trivial solution $\vec{x} = 0$.
- **Non-homogeneous:** $\vec{b} \neq 0$.

2. Solution Methods

- **Substitution/Elimination:** Best for small systems.
- **Augmented Matrix** $[A|\vec{b}]$**:** Use Gaussian Elimination to reach RREF.
- **Inverse Matrix:** $\vec{x} = A^{-1}\vec{b}$ (Only if A is square and $\det(A) \neq 0$).
- **Cramer's Rule:** $x_i = \frac{\det(A_i)}{\det(A)}$ where A_i is A with column i replaced by $\vec{b}$.

3. Consistency and Number of Solutions

Check the rank of the coefficient matrix $\rho(A)$ and the augmented matrix $\rho(A|\vec{b})$:

- **Unique Solution:** $\rho(A) = \rho(A|\vec{b}) = n$ (number of variables).
- **Infinitely Many:** $\rho(A) = \rho(A|\vec{b}) < n$ (Introduction of free parameters).
- **No Solution:** $\rho(A) \neq \rho(A|\vec{b})$ (The system is inconsistent).

5. Linear Transformations

1. Definition

$L : V \to W$ is a linear transformation if for all $\vec{u}, \vec{v} \in V$ and scalars c:

1. $L(\vec{u} + \vec{v}) = L(\vec{u}) + L(\vec{v})$ (Additivity)
2. $L(c\vec{u}) = cL(\vec{u})$ (Homogeneity)

2. The Transformation Matrix

Any linear transformation can be represented as a matrix multiplication:

$$\vec{v}' = A \cdot \vec{v}$$

where the columns of A are the images of the standard basis vectors.

3. Common Operations (2D/3D)

- **Scaling:** $\begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix}$
- **Rotation (2D):** $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$
- **Shearing:** $\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$

. Add all-ones row to extend to 3D.