Deferred-Acceptance Auctions and Radio Spectrum Reallocation*

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Abstract

Deferred-acceptance auctions choose allocations by iteratively rejecting the least attractive bids. For single-minded bidders with private values, DA clock auctions are obviously strategy-proof, (weakly) group strategy-proof, and are the only dominant-strategy mechanisms that preserve winners' privacy. If restricted to implement efficient allocations, and/or to use myopic bid rejection rules, DA auctions could only handle settings in which bidders are substitutes. However, DA auctions with heuristic bid reduction rules can also be useful for some environments with complementarities, particularly computationally challenging ones such as the US auction to repurchase television broadcast rights. DA auctions can balance a number of goals, including efficiency, revenue, and a budget constraint. Dominant-strategy

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incentives in these auctions need not be costly: the outcome of a strategy-proof DA auction coincides with the complete-information Nash equilibrium outcome of the paid-as-bid auction with the same allocation rule. A paid-as-bid auction with a non-bossy allocation rule is dominance solvable if and only if the rule is a deferred-acceptance rule.

1 Introduction

This paper formulates and analyzes a general class of clock auction mechanisms, which may be suitable for some computationally challenging resource allocation problems. [1] Our specific motivation arises from the US government's effort to reallocate electromagnetic spectrum from UHF television broadcasting to use for wireless broadband services. This reallocation involves purchasing television broadcast rights from some TV stations, reassigning ("repacking") the remaining over-the-air broadcasters into a smaller set of channels, using the cleared spectrum to create licenses suitable for use in wireless broadband, and selling those licenses to cover the costs of acquiring broadcast rights.[2] The repacking of about 2,000 TV stations must be done in a way that satisfies over 600,000 interference constraints, which preclude certain pairs of (geographically close) stations from being assigned to the same or adjacent channels. Even determining whether a given set of broadcasters can be feasibly repacked into a given set of channels is a com-

[1]The problem of designing computationally feasible economic mechanisms is studied in the field of "Algorithmic Mechanism Design" (Nisan and Ronen (1999)). While economists have long been concerned about the computational properties of economic allocation mechanisms (e.g., see Hayek (1945)), the formal economic literature has focused on modeling communication costs (e.g., Hurwicz 1977, Mount and Reiter 1974, Segal 2007), which are trivial in the setting of single-minded bidders considered in this paper.

[2]Middle Class Tax Relief and Job Creation Act of 2012, Pub. L. No. 112-96, §§ 6402, 6403, 125 Stat. 156 (2012). The legislation aims to make it possible to reallocate spectrum from a lower-valued use to a higher-valued one, generating billions in net revenues for the federal government.

putationally challenging (NP-hard) problem, at least as hard as the "graph coloring problem" (see Aardal et al. (2007)). The problem of selecting a set of stations to repack to maximize the total broadcast value is, in practice, even harder and cannot be solved exactly in reasonable time using today's state-of the art algorithms and hardware.

These computational difficulties pose a challenge for designing an auction that bidders can understand and trust, and in which they can easily decide what bids to make. For easy bidding, it is tempting to choose a strategy-proof auction such as the Vickrey auction, but computing Vickrey outcomes requires optimizations that are not possible in a problem of this size. One might think that an "approximate Vickrey auction" – in which winners are selected and prices set as in the Vickrey auction, but with approximate optimization replacing exact optimization – could be a suitable alternative, but viability requires that the near-optimizations used be extremely accurate. For an average bidder whose value is about 1/2000 of the total value, a 1% approximation error in one term of the Vickrey pricing formula with an exact computation of the other term would lead to a pricing error of 2000%, possibly leading to negative prices. Even if tiny computation errors could be guaranteed, bidders may fail to understand the complexities of Vickrey price computation, or may distrust the auctioneer's ability to perform those computations sufficiently precisely, again undermining their incentives for truthful bidding.

The U.S. Federal Communications Commission has avoided those difficulties by adopting a strategically simple descending-clock auction design (see Milgrom et al. 2012). The auction initially offers high prices to bidders and then reduces those prices, giving each bidder the opportunity to exit the auction and be repacked into the TV band each time its price is reduced. The auction maintains feasibility by never reducing the price offered to a bidder that cannot be feasibly added to the set of stations to be packed in the TV band. Since checking whether a station could be feasibly added is a large-scale NP-hard problem, the checker will occasionally fail to return either a positive or a negative answer in the allotted time.[3] However, by treating time-outs as negative answers and not reducing the bidder's price in such cases, the auction preserves both feasibility and strategic simplicity regardless of the frequency of time-outs.

This paper formulates the general class of clock auctions of the kind described above and examines both their strategic and algorithmic properties. For concreteness, we focus on a procurement auction like the reverse auction proposed by the FCC, which offers descending prices to sellers, but the analysis also applies with obvious sign changes to selling auctions, which offer ascending prices to buyers, as well as to double auctions, which offer prices to both buyers and sellers. A general clock auction for the procurement setting reduces prices using a rule that depends on the history of bidder exits, and may disclose to bidders some information in addition to their prices. When the auction stops, all bidders who have not exited become winning at the final prices. In analyzing such auctions, we restrict attention the case in which each bidder is "single-minded" (has a single object for sale) and knows its "value" (minimum acceptable price) for its object when the auction begins.

A clock auction with any price reduction rule and any information disclosure rule has a property that is stronger than traditional strategy-proofness: not only is it a dominant strategy for a bidder in such an auction to bid truthfully (i.e., to keep bidding while its price offer is at least its value and exit immediately afterwards), but the optimality of this strategy can be understood even without a full understanding of the computations involved and even if the bidder does not understand the auctioneer's policy for setting clock prices or does not trust the auctioneer to adhere to the stated policy. To determine that truthful bidding is optimal, all a bidder needs to know

[3]Nevertheless, using recent advances in Artificial Intelligence and certain problem-specific innovations, Frechette at al. (2015) developed a feasibility checker that solves more than 99% of the problems in auction simulations within 2 seconds, ensuring that time-outs would be rare.

is that its clock price can only descend over time. One way to formalize this strong property is with the notion of an "obviously dominant strategy" in an extensive-form game proposed by Li (2015): at any first information set at which the alternative strategy prescribes a different action, the deviation's best-case payoff (against even the most favorable profile of strategies of the other players that is consistent with this information set) is weakly lower than the worst-case payoff from truthful bidding (achieved against the least favorable such strategy profile). To verify that truthful bidding is an obviously dominant strategy in a clock auction, observe that any deviation from truthful bidding involves either exiting at a price weakly above value or continuing at a price below value, that any such deviation yields a nonpositive payoff in the continuation regardless of the behavior of other bidders, and that the truthful strategy guarantees a nonnegative payoff regardless of others' behavior. Thus, a bidder can deduce that the strategy is dominant just by comparing two numbers: a best payoff and a worst payoff. In contrast, truthful bidding is not obviously dominant for a sealed-bid Vickrey auction, because the worst payoff from truthful bidding is zero and the best payoff from almost any deviation is strictly positive. In laboratory experiments, bidders in Vickrey auctions often fail to understand the optimality of truthful bidding, even in the simplest single-object case (see Kagel et al. (1987)).

The obvious dominance of truthful bidding implies that no *coalition* of bidders could deviate from truthful bidding in a way that makes all of its members strictly better off: Indeed, the coalition's member who deviates first will not benefit from the coalitional deviation. Thus, the social choice function implemented by a clock auction is weakly group strategy-proof.[4]

Another attractive property of clock auctions is that, instead of requiring all bidders to reveal their values, it only requires winners to reveal the

[4]It is not strongly group strategyproof: since winners' price are determined by losing bids, thus a weakly Pareto-improving deviation could be achieved by a loser deviating to raise a winner's price.

minimal information about their values that is necessary to prove that they should be winning. This is a strong notion of privacy, known as "unconditional privacy" in computer science. Aside from alleviating bidders' concerns about revealing their values, this notion of privacy is useful because it makes bidding easier for bidders who find it costly to figure out their exact values. We show that clock auctions are the only dominant-strategy mechanisms that preserve winners' unconditional privacy.

The clock auctions we describe may be called "deferred-acceptance auctions" due to their similarity to the famous deferred acceptance algorithm of Gale and Shapley (1982). Indeed, as noted by Hatfield and Milgrom (2005), the Gale-Shapley algorithm modified to a setting with monetary transfers (as in Kelso and Crawford (1982)) and applied to the case of a single buyer and multiple sellers can be interpreted as a clock auction. In this auction, a price reduction to a seller amounts to irreversible rejection of the seller's offer to sell at this price, but permits the seller to accept the reduced price (submit a lower-priced offer) so as to remain active ("provisionally accepted") in the auction. The cited literature on deferred-acceptance auctions has focused on just a subset of these clock auctions – ones that decrement prices to bidders whose offers are not among the auctioneer's optimal choices from the offers that are currently available. We show that if we restrict attention to such myopic clock auctions or to auctions that always implement an optimal allocation, then clock auctions can implement only allocation rules in which bidders are substitutes. The class of clock auctions defined in this paper is much broader and can be applied in situations with complementarities, or in which exactly optimal allocations cannot be computed, or in which optimality is defined with respect to different objectives and constraints.[5]

[5]It should be noted that a number of "clock auctions" have been studied that are not deferred-acceptance auctions, and are therefore not covered by this paper's analysis. For example, "Dutch auctions" are descending-price selling auctions that are strategically equivalent to pay-as-bid auctions and so are not strategy-proof. Ausubel and Milgrom (2002) propose "cumulative-offer" clock auctions, which implement efficiency even in setting with complements, but do so by sometimes "recalling" exited bids, causing violations

For understanding the objectives that could be achieved with clock auctions, it is useful to consider the equivalent strategy-proof direct revelation mechanisms, which we call "sealed-bid deferred-acceptance auctions." In such an auction, each bidder bids its value, and the auction rejects bidders iteratively and irreversibly, while keeping the remaining bidders active. (The irreversibility distinguishes this from that of a clock auction, in which a rejected bidder may be permitted to improve his bid.) To determine which bidders should be rejected, in each round the deferred acceptance auction first calculates a "score" for each active bidder, which is an increasing function of its bid and may also depend on the rejected bids. It then rejects the bidder or bidders with the highest score. The algorithm stops when all the remaining active bidders have zero scores, and all those bidders become winning. In order for a sealed-bid auction to be strategy-proof, each winning bidder must be paid its "threshold price," which is the maximum bid it could have made that would still be winning. We show that such sealed-bid DA auctions with threshold prices are exactly the direct mechanisms for clock auctions: Namely, for each clock auction with some price decrementing rule there exist scoring functions that yield an equivalent direct DA auction, and conversely each DA auction with some scoring functions is a DA mechanism for a clock auction with some price decrementing rule.

The direct DA auctions are closely related to the "greedy heuristics" proposed for designing computationally feasible incentive-compatible mechanisms in the field of "Algorithmic Mechanism Design" (an approach pioneered by Lehmann et al. (2002)[6]). The difference is that those heuristics are typically "greedy acceptance" algorithms, which iteratively and irreversibly accept the "most attractive" bids as determined by some scoring. In contrast,

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of strategy-proofness. Lehmann et al. (2002) propose a "clock auction" to implements their heuristic algorithm, which sustains truthful bidding as a Nash equilibrium, but not as an obviously dominant strategy. This paper will use the term "clock actions" narrowly to describe only deferred-acceptance clock auctions and exclude all the other variations.

[6]For other examples of such "greedy heuristic" auctions, see also Mu'alem and Nisan (2008), Babaioff and Blumrosen (2008), and the references therein.

DA auctions iteratively reject the “least attractive” bids.[7] Both kinds of algorithms can be used as part of a strategy-proof mechanism in which each winner is paid its threshold price, and for both the threshold price is easy to compute. Yet, in other respects the two classes of algorithms have very different properties, which is due to the fact that only accepted bids lead to money changing hands. In particular, unlike DA auctions, “greedy-acceptance” auctions are not equivalent to deferred-acceptance clock auctions, do not have an obviously strategy-proof implementation, and are not group strategy-proof. DA auctions can also have an advantage in revenues (costs) which we discuss below. (See Appendix A for a simple example illustrating these points.)

This brings us to the important question of which objectives can be optimized or nearly optimized using DA auctions and how that depends on the constraints. One set of cases in which a DA auction leads to the efficient (surplus-maximizing) outcome subject to feasibility constraints arises when the sets of feasibly rejected bids form a matroid. In this case, efficiency can be obtained without using scoring, i.e., by a clock auction that offers the same price to all the bidders who could still be feasibly rejected. (This is similar to the result of de Vries, Schummer, and Vohra (2007), but the reverse auction algorithm differs from the algorithm for a selling auction, which they study. See also Ausubel (2004), Gul and Stachetti (2000), Milgrom (2000).) Second, when the feasibility constraint limits the total volume of the rejected bids (a “knapsack” problem), efficiency cannot be achieved exactly by a clock auction but can be well approximated by a DA auction that is based on the Dantzig greedy heuristic, in which each bid is scored by dividing the bid amount divided by the volume of the corresponding item. While we are unaware of any general theory of when DA auctions can achieve a good approximation of efficiency, we do describe below a set of cases involving constraints that are qualitatively similar to those faced by FCC and

[7] While “greedy acceptance” mechanisms have been mostly proposed for selling auctions, and we illustrate “deferred acceptance” mechanisms for the case of procurement auctions, either kind of mechanism can be used for either kind of auctions.

in which efficiency can be closely approximated with a DA auction. (Other examples have been considered in work of Duetting et al. (2014a) and Kim (2015) following an earlier draft of this paper). In simulations of the FCC's problem, a clock auction can yield a very good approximation of efficiency, provided that it uses appropriate scoring and a sufficiently good "feasibility checker."

In many practical settings, auction designers care not just about efficiency but also about budget balance or cost. In the FCC example, the reverse auction's cost is not allowed to exceed the revenue obtained from selling the spectrum in the forward auction. It is therefore important that DA auctions can be designed to guarantee satisfaction of a budget constraint. Examples of budget-balanced mechanisms already in the literature include cost-sharing mechanisms of Moulin (1999), Juarez (2007), and Mehta et al. (2007), double auctions of McAfee (1982), and procurement auctions by Ensthaler and Giebe (2009, 2014). All of these papers propose mechanisms in the broad class of DA auctions introduced here. Follow-up work to an earlier version of this paper includes Duetting et al. (2014b) who construct approximately optimal budget-constrained DA double auctions, and Jarman and Meisner (2015), who show that optimal budget-constrained procurement auctions can be implemented as DA auctions.

We show that DA auctions can also be used to promote expected cost minimization in two different kinds of environments. First, if bidders' values are independently drawn from known "regular" distributions, then scoring according to Myerson's "virtual valuations" leads to a DA auction that achieves minimal expected cost with matroid constraints, or otherwise approximates minimal expected costs. If the values are instead correlated (e.g., independently drawn from a distribution that is unknown to the designer), then a DA auction can implement "yardstick competition" among bidders to reduce expected costs as in Segal (2003). Follow-up work by Marx and Loertscher (2015) constructs DA auctions that achieve asymptotically optimal profits in such settings as the number of agents grows.

Another cost question concerns whether DA threshold auctions, with their dominant-strategy incentives, are more costly than an auction using the same DA allocation algorithm but with paid-as-bid pricing. It is not obvious what bidding behavior can be reasonably expected in such auctions and the setting is too complicated to allow a general Bayesian equilibrium analysis, so we adopt a theoretical competitive standard: the full-information equilibrium. We show that a paid-as-bid auction with a DA allocation algorithm has a Nash equilibrium in which winners bid and pay their threshold prices and losers bid their values, resulting in exactly the same outcome as the corresponding threshold-price DA auction. While a paid-as-bid auction can, in general, have many Nash equilibria, we show that for any non-bossy deferred acceptance heuristic, a DA paid-as-bid auction with discrete bid spaces and generic values is dominance-solvable (i.e., its payoffs are completely determined by iterated elimination of dominated strategies) and has a unique equilibrium outcome in undominated strategies, and the outcome is the same as that of the DA clock auction.[8]

This equivalence result is surprising, because no parallel result holds for optimizing paid-as-bid auctions when bidders are not substitutes. For auctions based on optimization, first-price auctions can have many different equilibrium outcomes and requiring strategy-proofness tends to increase costs: using the selection criteria of Bernheim and Whinston (1986), all selected full-information Nash equilibria of optimizing paid-as-bid auctions have weakly lower costs than the threshold (Vickrey) prices (see Ausubel and Milgrom (2006)). [9] We establish the special character of DA auctions with

[8]Since optimization is implementable via a DA algorithm when bidders are substitutes, this result can be viewed as extending the finding by Bernheim and Whinston (1986) that, when bidders are substitutes, the coalition-proof equilibrium outcome of an optimizing paid-as-bid auction coincides with the Vickrey outcome. However, our new result is limited neither to coalition-proof equilibrium – it applies to all Nash equilibria in undominated strategies – nor just to the case of optimization - it applies to all DA auctions.

[9]Indeed, this high costs/low revenue problem of the Vickrey auction is the motiva-

a converse result: any dominance-solvable paid-as-bid auction that selects winners using a monotonic, non-bossy allocation rule is a DA auction.

The paper is organized as follows: Section 2 illustrates a simple heuristic clock auction in a three-bidder example and contrasts its properties with those of the Vickrey auction. Section 3 defines general clock auctions and shows that truthful bidding strategies are obviously dominant, and also form a strong Nash equilibrium of the game. Section 4 shows that clock auctions are essentially the only auctions that preserve winners' unconditional privacy. Section 5 characterizes direct-revelation deferred-acceptance auctions and show that they are equivalent to clock auctions. Section 6 provides examples of useful DA auctions. Section 7 characterizes the domain of applicability of optimizing clock auctions, as well as of myopic clock auctions, as environments that satisfy substitutes. Section 8 analyzes paid-as-bid DA auctions, comparing them to clock auctions and deriving their properties described above. Section 9 concludes.

2 A Simple Example

In this section, we illustrate some advantages of DA auctions in a very simple setting. In our example, three TV stations (labelled 1,2,3) are bidding to relinquish their broadcast rights in a reverse auction, with only a single channel available to assign the losing bidders. Stations 1 and 2 are "peripheral" stations that could be both assigned to broadcast on the remaining channel without interfering with each other, but either of them would interfere with the "central" station 3. Thus, the auctioneer needs to choose whether to clear the desired spectrum by acquiring station 3 (assigning both stations 1 and 2 to the remaining channel) or by acquiring stations 1 and 2 (assigning

tion for the proposed "core-selecting auctions" (Day and Milgrom 2008), which sacrifice strategy-proofness in order to reduce costs or increase revenue. The present paper proposes a different solution to the problem for some applications, preserving strategy-proofness but sacrificing outcome efficiency.

station 3).[10] The three stations' values for their broadcast rights are denoted by v_1, v_2, v_3, respectively.

In this setting, the Vickrey auction implements the efficient allocation, which is to acquire stations 1 and 2 when $v_1+v_2 \leq v_3$ and station 3 otherwise. (For concreteness, we break ties in favor of the lower-numbered bidders.) If the auction acquires stations 1 and 2, the Vickrey payments to them are $v_3 - v_2$ and $v_3 - v_1$, respectively. While the Vickrey auction is efficient and strategy-proof, it suffers from a number of problems:

- It requires the auctioneer to compute the exactly optimal allocation, both with all bidders and with each winner not participating. (This is trivial in this simple setting but impossible in a computationally challenging setting.)

- It requires all bidders to submit their exact values, which they may be reluctant or unable to do.

- The optimality of truthful bidding is not obvious to a bidder who does not understand or trust the auctioneer's computations (which would be of particular concern in a computationally challenging setting).

- The auction is not (even weakly) group strategy-proof: stations 1 and 2 can raise each other's payments by reducing their bids, and in particular can each obtain the maximal payment of v_3 by both bidding zero.

- The total payment to bidders 1 and 2, which equals $2v_3 - v_1 - v_2$, exceeds bidder 3's value v_3. In contrast, in the *paid-as-bid* auction in which the auctioneer selects the cost-minimizing combination of bids (a "menu auction" of Bernheim and Whinston (1986)), there is no full-information Nash equilibrium in which the total payment to bidders

[10]It is also feasible, but clearly suboptimal, to acquire and clear station combinations {1,3}, {2,3}, and {1,2,3}.

1 and 2 exceeds bidder 3's value, since bidder 3 could have profitably deviated by undercutting $v_1 + v_2$.[11]

The DA clock auctions considered in this paper avoid all these problems. For a simple example, consider the clock auction with a single descending clock q that determines a price offer $w_i q$ to each bidder i, where the constant $w_i > 0$ is interpreted as the bidder's "volume" (for example, we may set $w_1 = w_2 = 1$ and $w_3 > 1$ to account for the symmetry of stations 1 and 2 and for the greater interference created by station 3). The opening level of q should be high enough to induce all three stations to participate. As the clock declines, an active station may choose exit the auction to be assigned to broadcast. When an active station could no longer feasibly exit, it becomes a winner and is paid the offer $w_i q$ made at the moment in which it became winning.[12] In our simple example, when station 3 exits first, stations 1 and 2 become winners, while when either station 1 or station 2 exits first, station 3 becomes a winner, and the clock runs down until the remaining station exits as well.

Unlike the Vickrey auction, this simple clock auction does not require the auctioneer to compute an optimal allocation (instead, it only needs to check the feasibility of each bidder's exit), it spares winners the need to reveal their exact values (instead, they only need to establish that their values are low enough to win); it makes the optimality of truthful bidding obvious to a bidder; and it is weakly group strategy-proof (if both bidders 1 and 2 win, neither affects the other's payment).

The described clock auction has a "direct" implementation, which assigns each bidder i with reported value v_i a score equal to v_i / w_i, and iteratively

[11] In the general setting, Vickrey payments are sometimes higher, but never lower, than those in the full-information equilibrium of the corresponding paid-as-bid ("menu") auction (Ausubel and Milgrom (2006)).

[12] If more than one active station chooses to exit at the same time, their exits are processed sequentially, e.g., in the order of their numbers, and those bidders whose exits are precluded by the previous exits become winners.

rejects the bidder with the highest score that could still be feasibly rejected. Thus, the DA algorithm acquires stations 1 and 2 if $\max\{v_1/w_1, v_2/w_2\} \leq v_3/w_3$ (so that station 3 is rejected in the first round), and acquires station 3 otherwise. To ensure strategy-proofness, when stations 1 and 2 are acquired, they are paid v_3w_1/w_3, v_3w_2/w_3 respectively (these are the stations' "threshold prices," which are their maximal bids that would have still been accepted given the bids of the others). When station 3 is acquired, it is paid its "threshold price" $w_3 \max\{v_1/w_1, v_2/w_2\}$.

Note that the DA algorithm satisfies a "revenue equivalence" property: in a paid-as bid auction with the same allocation rule, when the stations' values satisfy $v_1/w_1, v_2/w_2 \leq v_3/w_3$, the unique full-information Nash equilibrium in weakly undominated strategies is for station 3 to lose by bidding its true value v_3, and for stations 1 and 2 to win by bidding their "threshold prices" v_3w_1/w_3, v_3w_2/w_3.

Of course, all the nice features of the DA algorithm in the example are only made possible by sacrificing the guarantee of choosing the efficient allocation given the bids. In fact, we will show that this sacrifice is unavoidable due to the complementarities between bidders 1 and 2 in the example: any deferred-acceptance auction would reject the first bid based on the pairwise comparison of its score with those of the other bids, rather than on the comparison of $v_1 + v_2$ to v_3, which is required to guarantee efficiency. Yet, we believe that this sacrifice is worthwhile in some important practical settings, for any of the following reasons: (1) Exact efficiency may be unachievable anyway due to computational limitations. (2) Efficiency would be undermined by bidder manipulation in mechanisms that are not strategy-proof (such as core-selecting auctions or approximate-Vickrey auctions) or, while being strategy-proof, not obviously strategy-proof or group strategy-proof (such as Vickrey auctions). (3) The auction designer may be interested in goals other than efficiency, such as procurement costs and budget constraints. In such settings, the designer may prefer to use an auction in the class of DA clock auctions studied in this paper.

3 Clock Auctions and Truthful Bidding

We consider procurement mechanisms with N bidders, in which each bidder can either "win" (which means that his bid to supply a given good or bundle of goods is "accepted") or "lose" (which means that his bid is rejected). We restrict attention to mechanisms in which winning bidders receive payments but losing bidders do not, which we henceforth refer to as "auctions".

The preferences of each bidder i depend on whether he wins or loses, and, if he wins, on the payment p_i. We assume that these preferences are strictly increasing in the payment and that there exists some payment $v_i \in \mathbb{R}_+$ that makes him indifferent between winning and losing; we call v_i his "value".[13]

Informally, a Deferred-Acceptance ("descending") clock auction is a dynamic mechanism that presents a declining sequence of prices to each bidder, with each presentation followed by a decision period in which any bidder decides whether to exit or continue. Bidders that have not exited are called "active". Bidders who choose to continue when their prices are reduced are said, informally, to "accept" the lower price. When the auction ends, the active bidders become the winners and are paid their last (lowest) accepted prices. Different clock auctions are distinguished by their pricing functions, which determine the sequence of prices presented.[14]

To avoid technical complications, we restrict attention to auctions with discrete time periods, indexed by $t = 1, ...$ The set of active bidders in period t is denoted by $A_t \subseteq N$. A period-t history consists of the sets of active bidders in all periods up to period t: $A^t = (A_1, ..., A_t)$ such that

[13] For unmixed outcomes, such a preference can be expressed by a quasilinear utility $p_i - v_i$ when the bidder wins and zero when he loses.

[14] The described descending clock auctions for the procurement setting are the mirror image of the ascending clock auctions for the selling setting, which in turn generalize the classic English auction for selling a single item.

$A_t \subseteq ... \subseteq A_1$. Let H denote the set of all such histories for all possible $t \geq 1$. A descending clock auction is described by a price mapping $p : H \to \mathbb{R}^N$ such that $p(A^t) \leq p(A^{t-1})$ for all $t \geq 2$ and all A^t.

The clock auction describes the following extensive-form game: the set of participants is initialized as $A_1 = N$ and the opening prices are set at $p(N)$. In each period $t \geq 1$, given history A^t, the auction offers a profile of prices $p(A^t)$ to the bidders. If $t \geq 2$ and $p(A^t) = p(A^{t-1})$, the auction stops; bidder i is a winner if and only if $i \in A_t$ and in that case i is paid $p_i(A^t)$. Each bidder $i \in A_t$ may choose to refuse the new price and exit the set of active bidders.[15] Letting $E_t \subseteq A_t$ denote the set of bidders who choose to exit, the auction continues in period $t+1$ with the new set of active bidders $A_{t+1} = A_t \backslash E_t$ and the new history $A^{t+1} = (A^t, A_{t+1})$. We restrict attention to *finite* clock auctions, in which the set $\{p(A^t)\}_{h \in H}$ of possible price offers is finite (ensuring that the auction ends in a bounded number of periods).

To complete the description of the extensive-form game, we need to describe bidders' information sets, given by functions $I_i : H \to S_i$. We allow arbirtrary information sets except that we require that each bidder i observe his current price, i.e., that for any two histories $h, h' \in H$ such that $I_i(h) = I_i(h')$ we have $p_i(h) = p_i(h')$.

The *truthful strategy* of agent i with value v_i in a clock auction accepts at history h if and only if $p_i(h) \geq v_i$. (Note that this is a feasible strategy with any information disclosure since the agent always observes his own price.) Clearly, the truthful strategy is a dominant strategy. We can prove more: it is an obviously dominant strategy in the sense of Li (2015):

Definition 1 (Li 2015) *Strategy σ_i of agent i is* obviously dominant *if, for any alternative strategy σ_i', at any information set I_i of the agent at which σ_i'*

[15]In a practical variant of the auction, an active bidder $i \in A_t$ is allowed to exit in a period $t \geq 2$ only if his price is reduced in this period, i.e., $p_i(A^t) < p_i(A^{t-1})$. This variant allows to maintain feasibility constraints (as in Example 1 below) throughout the auction provided that the set A_1 of bidders who accepted the opening prices satisfies these constraints. Note that this variant preserves the possibility of truthful bidding and therefore all the results below apply to it.

and σ_i prescribe different actions, the agent's payoff achieved by σ_i and any strategy profile σ_{-i} of other players such that (σ_i, σ_{-i}) visits I_i is at least as high as his payoff achieved by σ'_i and any strategy profile σ'_{-i} of other players such that $\left(\sigma'_i, \sigma'_{-i}\right)$ also visits I_i.

Proposition 1 *The truthful strategy is an obviously dominant strategy in a clock auction.*

Proof. A deviation from the truthful strategy involves either exiting at a price weakly above value or continuing at a price below value. Either deviation yields a nonpositive payoff for any behavior of others that is consistent with the information set, while the truthful strategy yields a nonnegative payoff for any behavior of others. ■

This proposition also implies that *coalitional* deviations from truthful bidding cannot be strictly Pareto improving:

Corollary 1 *In a clock auction, for every strategy profile σ, if all members of a coalition $S \subseteq N$ switch to truthful strategies then at least one member in S will receive a weakly higher payoff.*

Proof. Consider the first history $h \in H$ at which a player $i \in N$ bids non-truthfully under strategy profile σ. If all members of S switch to truthful strategies, then history h will also be reached, and agent i's payoff will be weakly increased according to Proposition 1. ■

Thus, truthful strategies form a strong Nash equilibrium of any clock auction. Since truthful strategies do not condition on other bidders' values, it is an "ex post" strong Nash equilibrium, i.e., it is a strong Bayesian-Nash equilibrium no matter what information bidders have about each other's values.

4 Winners' Privacy

Here we formalize the notion that clock auctions alone preserve the privacy of winners. Computer scientists (e.g., Brandt and Sandholm (2005)) call a protocol (extensive-form communication game plus a profile of agents' strategies for this game) that computes a given allocation rule "unconditionally fully private" if it prevents any coalition of agents from inferring any information about the other players in the course of the protocol besides what the coalition could infer from the resulting allocation and the initial information of the coalition's members.[16] We modify this traditional definition in two ways. First, we weaken it by focusing only on each *winners'* privacy. Clock auctions do not preserve losers' privacy, since their drop-out points can reveal a lot of information about their values. Second, we strengthen the definition by focusing on each individual winner: even when $N-1$ players pool their information, if the remaining player is a winner, they should be unable to infer any information about his value beyond that needed to establish that he is a winner. Thus, we introduce the following definition:

Definition 2 *For each bidder i, let $V_i \subseteq \mathbb{R}_+$ denote the set of bidder i's possible values, and consider an allocation rule $\alpha : \Pi_{i \in N} V_i \to 2^N$, where $\alpha(v) \subseteq N$ is the set of accepted bidders in state $v \in V$. A communication protocol that computes allocation rule α satisfies Unconditional Winner Privacy (UWP) if for any player i, any values $v_{-i} \in V_{-i}$ of the other players, and any pair $v_i, v_i' \in V_i$ such that $i \in \alpha(v_i, v_{-i}) \cap \alpha(v_i', v_{-i})$, the protocol finishes in the same terminal node of the game in states (v_i, v_{-i}) and (v_i', v_{-i}).*

Proposition 2 *A clock auction with truthful bidding satisfies UWP. Furthermore, any allocation rule on a finite value space that is implementable in*

[16] This notion is also known as "non-cryptographic privacy," since it permits neither private-key cryptography, which relies on private communication channels, nor public-key cryptography, which relies on agents' computational constraints. A definition of privacy that allowed such cryptographic tricks would not much restrict what can be implemented (see Izmalkov et al. (2005)).

dominant strategies with a protocol satisfying UWP is implementable with a clock auction with truthful bidding.

Proof. For the first statement, for player i to win in a clock auction with truthful bidding for two different values $v_i, v_i' \in V_i$ given any strategy profile of the other players, player i must not exit in either case, and so the protocol must finish in the same terminal node in both cases.[17]

For the second statement, we first observe that any allocation rule α that is implementable in dominant strategies must be monotonic (see Section 5 below). Then, starting with any protocol that satisfies UWP and implements α, we construct a clock auction with truthful bidding that implements α. We do this by induction on the histories of the protocol, in such a way that the clock auction reveals at least as much information about players at each history as the original protocol does at its corresponding history. We initialize the opening price of each bidder i at $p_i(N) = \max V_i$ for each i (so that all types truthfully accept it). Then, at any history h, let $\bar{V} \subseteq V$ denote the set of states in which the history is reached in the original protocol. By the usual "communication complexity" argument (e.g., Kushilevitz and Nisan (1997), Segal (2007)), this set must be a product set: $\bar{V} = \bar{V}_1 \times ... \times \bar{V}_N$. Let the player who sends a message at history h be player i. We replace his message with several rounds of bidder i's price reduction in the clock auction. Namely, for each $v_i \in \mathbb{R}$, let $v_i^+ = \begin{cases} \min\{v \in V_i : v > v_i\} & \text{if } v_i < \max V_i, \\ \max V_i + 1 & \text{otherwise,} \end{cases}$ and $v^- = \begin{cases} \max\{v \in V_i : v < v\} & \text{if } v_i > \min V_i, \\ \min V_i - 1 & \text{otherwise.} \end{cases}$ The rounds reduce the bidder's price from p_i to p_i^- for as long as $i \notin \cup_{v_{-i} \in \bar{V}_{-i}} \alpha(p_i, v_{-i})$ – i.e., as long as bidder i with value p_i could never win at this history.

[17]This argument actually establishes a somewhat stronger property than UWP: that other players cannot learn additional information about a winner's value in a clock auction without causing him to lose, even if they deviate to non-truthful strategies (in computer science lingo, even if they are "Byzantine" rather than selfish).

At the end of the price reductions, all bidder i types exceeding the new price p_i, who could not win at history h in the original protocol, fully reveal themselves in the clock auction by exiting. Suppose now $p_i \in \bar{V}_i$, in which case there exists $v_{-i} \in \bar{V}_{-i}$ such that $i \in \alpha(p_i, v_{-i})$. Note that by monotonicity of α, for any type $v_i \leq p_i$ we must also have $i \in \alpha(v_i, v_{-i})$, and therefore by UWP of the original protocol any such type must send the same message at history h as type p_i.

Thus, when bidder i responds truthfully to the price reductions, all bidder i types with values not exceeding the new price p_i do not reveal themselves in either the original protocol or the constructed clock auction, while all types with values above p_i fully reveal themselves in the clock auction by exiting. Hence the clock auction reveals at least as much information as the original protocol at the corresponding history. In each round of the clock auction, only bidders who could never win in α exit. At any terminal history h, $\alpha\left(\bar{V}\right)$ must be a singleton, and so any bidders who could still win (i.e. who have not exited in the clock auction) must be winners. Thus, the constructed clock auction implements allocation rule α. ■

5 Direct Deferred-Acceptance Auctions

Appealing to the Revelation Principle, for every clock auction with truthful bidding strategies we can construct an equivalent direct revelation mechanism. Namely, let $V_i \subseteq \mathbb{R}_+$ denote the set of bidder i's possible values. In the direct mechanism, each bidder reveals his value, and the mechanism implements an *allocation rule* $\alpha : \Pi_{i\in N} V_i \to 2^N$ and a *payment rule* $\pi : \Pi_{i\in N} V_i \to \mathbb{R}^N$ such that $\pi_i(v) = 0$ for all $i \in N \backslash \alpha(v)$ (i.e., losing bidders are not paid). Such triples $\langle V, \alpha, \pi \rangle$ will be called direct auctions.

Definition 3 *Allocation rule α is* monotonic *if $i \in \alpha(v_i, v_{-i})$ and $v_i' < v_i$ imply $i \in \alpha(v_i', v_{-i})$.*

Definition 4 *A direct auction* $\langle V, \alpha, \pi \rangle$ *is a* threshold auction *if* α *is monotonic and the price paid to any winning bidder* $i \in \alpha(v)$ *is given by the* threshold pricing *formula:*[18]

$$\pi_i(v_{-i}) = \sup \{ v_i' \in V_i : i \in \alpha(v_i', v_{-i}) \} . \tag{1}$$

The following characterization of strategy-proof direct auctions is well known:

Proposition 3 *Any threshold auction is strategy-proof. Conversely, any strategy-proof direct auction has a monotonic allocation rule, and, if* $V = \mathbb{R}_+^N$, *it must be a threshold auction.*[19]

Now we specify direct mechanisms that are equivalent to clock auctions. A direct deferred-acceptance (DA) algorithm on value spaces V is described by a collection of *scoring functions* $\left(s_i^A\right)_{A \subseteq N, i \in A}$, where for each $A \subseteq N$ and each $i \in A$, the function $s_i^A : V_i \times V_{N \setminus A} \to \mathbb{R}_+$ is nondecreasing in its first argument. The algorithm operates as follows: Let $A_t \subseteq N$ denote the set of "active bidders" at the beginning of iteration t. We initialize the algorithm with $A_1 = N$. In each iteration $t \geq 1$, if $s_i^{A_t}\left(v_i, v_{N \setminus A_t}\right) = 0$ for all $i \in A_t$ then stop and output $\alpha(v) = A_t$; otherwise, let $A_{t+1} = A_t \setminus \arg\max_{i \in A_t} s_i^{A_t}\left(v_i, v_{N \setminus A_t}\right)$

[18]Note that $\pi_i(v_{-i}) = +\infty$ if $i \in \alpha(v_i', v_{-i})$ for all $v_i' \in V_i$ and V_i is unbounded; this case does not present any formal complications but clearly should be avoided in practice. For example, modifying α so that a bidder's bid above a given "reserve price" can never win would guarantee that the bidder's threshold price cannot exceed his reserve price.

[19]It is obvious that a threshold auction is strategy-proof. For the converse, monotonicity follows from the incentive compatibility constraints. To derive the threshold pricing in the case $V = \mathbb{R}_+^N$, apply the envelope theorem to see that the outcome function and strategy-proofness determine the bidders' payoffs up to a constant of integration. Since losers must be paid zero and have a payoff of zero, the strategy-proof payment rule is unique. If V is a (possibly finite) subset of $\mathbb{R}_+^N$, the payment rule implementing a given α is not unique, but it must be a function $\pi_i(v_{-i})$ of the others' bids v_{-i} that satisfies

$$\sup \{ v_i' \in V_i : i \in \alpha(v_i', v_{-i}) \} \leq \pi_i(v_{-i}) \leq \inf \{ v_i' \in V_i : i \in N \setminus \alpha(v_i', v_{-i}) \} .$$

and continue. In words, the algorithm iteratively rejects the least desirable (highest-scoring) bids until only zero scores remain. We say that the DA algorithm *computes* allocation rule α if for every value profile $v \in V$, when the algorithm stops the set of active bidders is exactly $\alpha(v)$.

By inspection, every DA algorithm computes a monotonic allocation rule. Thus, we can define a *DA threshold auction* as a sealed-bid auction which computes its allocation using a DA algorithm and makes the corresponding threshold payments (1) to the winners. Like any threshold auction, each DA threshold auction is strategy-proof. Furthermore, the threshold prices can be computed in the course of the DA algorithm, by initializing the prices as $p_i^0 = +\infty$ for all i, and then updating them in each round $t \geq 1$ as follows:

$$p_i^t = \min\left\{p_i^{t-1}, \sup\left\{v_i' \in V_i : s_i^{A_t}\left(v_i', v_{N\backslash A_t}\right) < s_j^{A_t}\left(v_j, v_{N\backslash A_t}\right) \text{ for } j \in A_t \backslash A_{t+1}\right\}\right\}$$

for every bidder $i \in A_{t+1}$. In the final round of the algorithm, for every winner $i \in A^T$, p_i^T is the winner's threshold price.

The next two results show that the direct mechanisms corresponding to clock auctions with truthful strategies (i.e., social choice rules implemented with such auctions) are exactly direct DA threshold auctions.

Proposition 4 *The direct mechanism for a finite clock auction with truthful bidding, with state space $\mathbb{R}_+^N$ (more generally, such that $\{p_i(h) : h \in H\} \subseteq V_i \subseteq \mathbb{R}_+$), is a DA threshold auction.*

Proof. Given a finite clock auction p, we construct the scoring rule for the DA auction in the following manner: Holding fixed a set of bidders $S \subseteq N$ and their values $v_S \subseteq \mathbb{R}^S$, let $A_t^S(v_S)$ denote the set of active bidders in round t of the clock auction if every bidder from S bids truthfully and every bidder from $N\backslash S$ never exits.[20] Now for any given $A \subseteq N$ and $i \in A$, define

[20] Formally, initialize $A_1^S(v_S) = N$ and iterate by setting $A_{t+1}^S(v_S) = A_t^S(v_S) \backslash \left\{j \in S : v_j > p_j\left(A_t^S(v_S)\right)\right\}$. The sequence $\left(A_t^S(v_S)\right)_{t=1}^{\infty}$ must start repeating eventually (when the clock auction stops).

the score of agent i as the inverse of how long he would remain active in clock auction if he bids truthfully with value v_i and all bidders from $N \backslash A$ bid truthfully with values $v_{N \backslash A}$, while bidders in $A \backslash \{i\}$ never quit:

$$s_i^A \left(v_i, v_{N \backslash A}\right) = 1/ \sup \left\{ t \geq 1 : i \in A_t^{\{i\} \cup (N \backslash A)} \left(v_i, v_{N \backslash A}\right) \right\}.$$

(Note that the score is $1/\infty = 0$ if agent i remains active for the remainder of the auction.) This score is by construction nondecreasing in v_i. Also by construction, given a set A of active bidders, the set of bidders to be rejected by the algorithm in the next round ($\arg\max_{i \in A} s_i^A \left(v_i, v_{N \backslash A}\right)$) is the set of bidders who would quit the soonest in the clock auction under truthful bidding. If no more bidders would ever exit the auction, then all active bidders have the score of zero, so the DA algorithm stops. Finally, each winner's final clock auction prices is its threshold price: it would have lost by bidding any higher value in V_i in the DA auction, since this would correspond to rejecting the final price in the clock auction. ■

Proposition 5 *Every direct DA threshold auction with a finite state space is a direct mechanism for some finite clock auction with truthful bidding.*

Proof. Given a direct DA threshold auction with a scoring rule s and a finite state space V, we construct an equivalent clock auction. This auction sets the opening prices as $p_i(N) = \max V_i$ for each i, so that all truthful bidders participate. Then, in each round, the price to every highest-scoring active bidder is reduced by the minimal decrement, while the prices for all other bidders are left unchanged:

$$p_i\left(A^t\right) = \begin{cases} p_i\left(A^{t-1}\right)^- & \text{if } i \in \arg\max_{j \in A_t} s_j^{A_t} \left(p_j\left(A^{t-1}\right), p_{N \backslash A_t}\left(A^t\right)^+\right) \\ p_i\left(A^{t-1}\right) & \text{otherwise.} \end{cases}$$

Note in particular that the auction maintains $p_i\left(A^t\right) = p_i\left(A^{t-1}\right)$ for all $i \in N \backslash A_t$ – thus memorizing the prices rejected by the bidders who exited, which are one decrement below their values.

Then equivalence is easy to see: First, for every history of the clock auction, under truthful bidding, the next set of bidders to exit consists of the bidders who have the maximum scores among the active bidders, and iterating this argument establishes that the final set of winners is the same in both auctions. Second, for each bidder who has not exited by the auction's end and thus became a winner, his final clock price is his highest value that would be still winning – i.e., his threshold price. ■

These results imply that sealed-bid DA threshold auctions are strategy-proof and weakly group strategy-proof, by Proposition 1 and Corollary 1, respectively.[21,22] Clearly, the sealed-bid auctions lose *obvious* strategy-proofness as well as winners' privacy, so they may be less attractive for some practical purposes.[23] At the same time, sealed-bid DA threshold auctions are a useful theoretical construct for understanding the objectives that may be achieved by means of clock auctions.

6 Examples of Deferred-Acceptance Auctions

In this section we describe a number of examples of practical objectives that could be achieved with DA auction. These examples expand on some observations made in the earlier literature, and in follow-up work to an earlier version of the present paper.

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[21]Note that such auctions are not generally "strongly" group strategy-proof, because a bid change by a losing bidder might increase a winner's threshold price, and so this change would be a weakly Pareto improving deviation by the coalition consisting of the two bidders.

[22]The immediate implication is for DA threshold auctions on finite value spaces, but the arguments are easily extended to infinite value spaces.

[23]The downside of clock auctions is that they might require many rounds to achieve good precision. In practice it is possible to use "hybrid" designs that achieve a compromise between the speed of sealed-bid auctions and the attractive features of clock auctions. These hybrid designs include, for example, clock auctions with sealed intra-round bids (restricted to be between "start-of-round" and "end-of round" prices), and the "survival auctions" of Fujishima et al. (1999).

Example 1 (Feasibility and Non-Wastefulness) *Suppose that it is only feasible for the auctioneer to accept bids of a subset of bidders* $A \in \mathcal{F}$*, where* $\mathcal{F} \subseteq 2^N$ *is a given family of sets, with* $N \in \mathcal{F}$ *(so that feasibility is achievable).*[24] *To ensure that the algorithm maintains feasibility we require that* $s_i^A\left(b_i, b_{N\setminus A}\right) > 0$ only if $A\setminus\{i\} \in \mathcal{F}$*, and that there are no ties, i.e.,* $s_i^A\left(b_i, b_{N\setminus A}\right) \neq s_j^A\left(b_j, b_{N\setminus A}\right)$ *for all* $i \neq j$*,* A*,* $b_i, b_j, b_{N\setminus A}$*. In the corresponding clock auction* p*, no ties implies that* $\left|\left\{i \in A_t : p_i\left(A^t\right) < p_i\left(A^{t-1}\right)\right\}\right| \leq 1$*, and maintaining feasibility implies that* $p_i\left(A^t\right) < p_i\left(A^{t-1}\right)$ *only if* $A_t\setminus\{i\} \in \mathcal{F}$*.* [25]

Say that family $\mathcal{F} \subseteq 2^N$ *is* comprehensive *if* $A \in \mathcal{F}$ *and* $A \subseteq A'$ *imply* $A' \in \mathcal{F}$ *(i.e.,* $\mathcal{F}$ *has "free disposal" of winners). Say that a DA algorithm has* perfect feasibility checking *if its scoring functions satisfy* $s_i^A\left(b_i, b_{N\setminus A}\right) > 0$ if and only if $A\setminus\{i\} \in \mathcal{F}$ *– that is, it stops only when it is infeasible to reject any more active bids. If both conditions hold, then the algorithm has the property of* non-wastefulness*, which means that it stops at a* minimal *set* $A \in \mathcal{F}$*. (Indeed, if there is some* $A' \in \mathcal{F}$ *that is a strict subset of* A*, then for each* $i \in A\setminus A'$ *we have* $A\setminus\{i\} \in \mathcal{F}$ *by comprehensiveness, and therefore* $s_i^A\left(b_i, b_{N\setminus A}\right) > 0$ *by perfect feasibility checking, hence the algorithm cannot stop at* A*.) For the corresponding clock auction this means that the auction cannot stop while* $A_t\setminus\{i\} \in \mathcal{F}$ *for some* $i \in A_t$*.*

In the FCC's reverse-auction problem, perfect feasibility checking is computationally unachievable, because it requires checking whether a given set $N\setminus A$ *of the rejected bidders could be assigned to the available channels in a way that satisfies all interference constraints, which is NP-hard problem. When a feasibility checker has a limited time to run on a computationally challenging problem, it may generate one of the following three outputs: (i) a proof that* $A\setminus\{i\} \in \mathcal{F}$ *(which consists of a feasible assignment of bidders*

[24] E.g., in the example of Section 2, $\mathcal{F} = \{\{1,2\}, \{3\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$.

[25] This maintains feasibility throughout the auction provided that the set of initial participants is feasible, which is not guaranteed if the auction imposes reserve prices (or with a clock auction, whose opening prices $p(N)$ serve as reserve prices).

$(N \backslash A) \cup \{i\}$ to channels), (ii) a proof that $A \backslash \{i\} \notin \mathcal{F}$ (which could be a long logical argument), or (iii) a time-out, which means that the algorithm had to be terminated before producing either (i) or (ii). In case (i), we set $s_i^A\left(b_i, b_{N\backslash A}\right) > 0$, while in cases (ii) and (iii), we must set $s_i^A\left(b_i, b_{N\backslash A}\right) = 0$ (and so, do not decrement bidder i's price in the corresponding clock auction) in order to guarantee that the algorithm always yields a feasible assignment. Thus, without perfect feasibility checking, we cannot ensure non-wastefulness.

Example 2 (Optimization with Matroid Constraints) *Suppose that the goal is to find an "efficient" (i.e. social cost-minimizing) set of winning bids subject to the feasibility constraint in Example 1:*

$$\alpha(v) \in \arg\min_{A \in \mathcal{F}} \textstyle\sum_{i \in A} v_i. \tag{2}$$

Consider the "standard" DA algorithm with perfect feasibility checking, which scores the bids that are feasible to reject by their bid amounts, i.e., uses scoring functions $s_i^A\left(v_i, v_{N\backslash A}\right) = \begin{cases} v_i & \text{when } A\backslash\{i\} \in \mathcal{F}, \\ 0 & \text{otherwise,} \end{cases}$ and assume there are no ties (i.e., $V_i \cap V_j = \varnothing$ for $i \neq j$). By a classical result in matroid theory (see Oxley (1992)), the standard DA algorithm computes an efficient allocation rule α in this setting if and only if the set family $\{R \subseteq N : N \backslash R \in \mathcal{F}\}$ – i.e., the feasible sets of rejected *bids – form the independent sets of a matroid on the ground set N. The standard DA algorithm corresponds to the clock auction that offers the same descending price to all the active bidders who could still be feasibly rejected, "freezing" the price to a bidder who can no longer be feasibly rejected.*[26]

[26] This setting is related to the analysis of Bikchandani et al. (2011), who consider "selling"' auctions in which the family of sets of bids that could be feasibly *accepted* is a matroid. In their setting, the DA algorithm achieving efficiency is the "greedy worst-out" heuristic, which starts with the infeasible allocation in which all bids are provisionally accepted and iteratively rejects the lowest bid that could be rejected while the set of active bids remains a spanning set of the matroid (i.e., contains a maximal set of bids that could be feasibly accepted) until a feasible allocation obtains. This DA auction is equivalent to

Intuitively, the matroid property in Example 2 involves a notion of "one-for-one substitution"' between bidders (in particular, it implies that all the maximal feasible sets of rejected bids – the "bases" of the matroid – have the same cardinality.) This notion does not hold, even approximately, in the FCC setting, which involves a trade-off between acquiring a larger number of stations with smaller coverage areas and a smaller number of stations with larger coverage areas, as illustrated in Section 2. In such cases, instead of the standard DA algorithm it is desirable to use scoring that assigns greater scores to larger stations, given the same bid values. This is illustrated in the following stylized example:

Example 3 (Knapsack Problem) *Suppose that the family of feasible sets takes the form*

$$\mathcal{F} = \left\{ A \subseteq N : \textstyle\sum_{i \in N \setminus A} w_i \leq 1 \right\}.$$

The problem of maximizing total surplus subject to this constraint is known as the "knapsack problem." Interpreting $w_i > 0$ as the "volume"' of bidder i, the problem (2) is to minimize the total value of the accepted bids subject to a constraint on the total volume of rejected bids.[27] *While this optimization problem is NP-hard, it can be approximated with the famous heuristic of Dantzig (1957), which corresponds to the direct DA algorithm with the "per-volume" scoring* $s_i^A\left(v_i, v_{N\setminus A}\right) = \begin{cases} v_i/w_i & \text{when } A\setminus\{i\} \in \mathcal{F} \\ 0 & \text{otherwise,} \end{cases}$ *provided that there are no ties. (This is equivalent to the clock auction in which a single descending clock price q determines the current price offer $w_i q$ to each*

the clock auction that offers the same ascending price to all the active bidders that could still be rejected while maintaing the spanning condition. While both the "greedy best-in" and the "greedy worst-out" algorithms can be used to compute the efficient allocation rule in both the procurement and selling matroid auctions, only one of them is a DA algorithm: in the procurement auction it is the best-in algorithm, and in the selling auction it is the worst-out algorithm.

[27]In follow-work, Duetting, Gkatzelis and Roughgarden (2014) consider instead the "selling" problem in which the *accepted* bids must fit into a knapsack, and examine the approximation power of DA algorithms for this problem.

active bidder i, "freezing"' the price to a bidder who can no longer be feasibly rejected.) It is well known that this heuristic yields a good approximation of efficiency if the values of individual items are small relative to the possible value in the knapsack. Indeed, the inefficiency cannot exceed the value of the first item that did not fit in the knapsack, since if we were allowed to put a maximal fraction of this item in the knapsack, we would have obtained a solution to the linear relaxation of the problem.

Example 4 (Approximate Matroid Constraints) *Suppose that the family $\mathcal{F}$ of feasible sets does not have the matroid structure of Example 2, but there exists a smaller family $\bar{\mathcal{F}} \subseteq \mathcal{F}$ such that $\left\{R \subseteq N : N \backslash R \in \bar{\mathcal{F}}\right\}$ is a matroid, and that $\bar{\mathcal{F}}$ is "not to far" from $\mathcal{F}$, implying that the optimal solution for $\bar{\mathcal{F}}$ is "not too suboptimal" for $\mathcal{F}$. Then we would achieve a good approximation of the optimal solution for $\mathcal{F}$ by first following the simple DA algorithm for $\bar{\mathcal{F}}$, and then, once no more active bidders could be feasibly rejected given $\bar{\mathcal{F}}$, switching to the standard DA algorithm for $\mathcal{F}$ to possibly reject additional bidders.*[28]

For example, in the FCC's reverse-auction setting, with n TV channels available, suppose that the bidders (stations) could be partitioned into geographic areas, so that no two stations in the same area could be assigned to the same channel. In addition, suppose that the number of cross-area interference constraints is not too large, so that for some given d, we would always be able to assign any $n - d$ stations in each area to channels in a way satisfying all interference constraints. (See Appendix B for a formalization of this setting.) Let us define $\bar{\mathcal{F}}$ by the constraint that no more than $n - d$ stations in each area could be assigned. This set of constraints has the matroid structure of Example 2, and therefore we could first use the standard DA algorithm for $\bar{\mathcal{F}}$ to assign the $n - d$ most valuable stations in each area, and

[28] Of course, the same approach would work if $\bar{\mathcal{F}}$ instead takes the knapsack form of Example 3; in this case instead of the simple DA algorithm for $\bar{\mathcal{F}}$ we should use the Dantzig heuristic for $\bar{\mathcal{F}}$.

then assign any additional stations that would fit (with repacking) in under the true constraints $\mathcal{F}$.

In the FCC reverse-auction setting, keeping total procurement cost low may also be an important goal. A cost-minimization objective can be easily handled using the "virtual value" approach of Myerson (1981).

Example 5 (Expected Cost Minimization with Independent Values) *Suppose that bidders' values v_i are independently drawn from distributions $F_i(v) = \Pr\{v_i \leq v\}$ on $V_i = [0, \bar{v}_i]$ for each i. Then, following the logic of Myerson (1981), the expected cost of a threshold auction that implements allocation rule α can be expressed as $\mathbb{E}\left[\sum_{i\in\alpha(v)} \gamma_i(v_i)\right]$, where $\gamma_i(v_i) = (v_i + F_i(v_i)/F_i'(v_i))$ (bidder i's "virtual cost function"). Assume that the virtual cost functions are strictly increasing. Then, if we are given a DA algorithm with scoring rule s that exactly or approximately minimizes the expected social cost subject to feasibility constraints as in the above examples, it can be modified to yield a DA threshold auction that exactly or approximately minimizes the total expected cost of procurement subject to the constraints. The modified auction uses the scoring rule $\hat{s}_i^A\left(b_i, b_{N\setminus A}\right) = s_i^A\left(\gamma_i(b_i), \left(\gamma_j(b_j)\right)_{j\in N\setminus A}\right)$.*

Example 6 (Expected Cost Minimization with Correlated Values) *Suppose that the auctioneer again cares about minimizing the expected total cost, but relax the assumption that bidders" values are independent of each other. In such cases, as noted in Segal (2003), the auctioneer would optimally condition the price offered to one bidder on the bids of the others, even those that no longer have a chance of winning. Thus, in contrast to the preceding examples, in this context it is helpful to condition scoring on the values of already-rejected bids.*

For a simple example, if the auctioneer values acquiring each bidder at π and faces no feasibility constraints, it might use scoring functions $s_i^A\left(v_i, v_{N\setminus A}\right) = \max\left\{v_i - p_A^*\left(v_{N\setminus A}\right), 0\right\}$, where $p_A^*\left(v_{N\setminus A}\right) = \arg\max_p (\pi - p)\Pr\left\{v_i \leq p | v_{N\setminus A}\right\}$ is the optimal monopsony price for the posterior distribution of values given *the rejected bids. In general, the optimal auction for this setting, characterized in Segal (2003), could not be implemented as DA auction, as it requires winning bidders to be "bossy" (able to affect the allocation while still winning) which cannot happen in a DA auction. However, in follow-up work, Marx and Loertscher (2015) show that the optimal expected profits can be approximated asymptotically with a DA clock auction for a large number of bidders whose values are drawn i.i.d. from an unknown distribution.*

Another important objective in FCC's "incentive auction" is a balanced budget: the reverse auction's cost must be at least covered by the forward auction's revenues. This objective can also be achieved with a clock auction.

Example 7 (Budget Constraint) *Suppose that the designer faces the "budget constraint" that the total payment to the winners cannot exceed $R(A)$ when the set of accepted bids is $A \subseteq N$. In order to ensure that a clock auction satisfies this constraint, it cannot stop when while is violated, i.e., $p(A^t) \neq p(A^{t-1})$ if $\sum_{i \in A_t} p_i(A^{t-1}) > R(A_t)$. (Note that the equivalent direct DA threshold auction must use scoring that is contingent on already-rejected bids, since it is those bids that determine the current threshold prices of the still-active stations.). In the mirror-image "selling" setting, budget-balanced "cost-sharing"' clock auctions have been proposed by Moulin (1999) and Mehta et al. (2009). Their clock auctions have the special features of always offering budget-balanced prices to all active bidders (in our notation, $\sum_{i \in A_{t-1}} p_i(A^{t-1}) = R(A_{t-1})$ for all histories A^{t-1}), and stopping as soon as all active bidders accept their decrements (i.e., $A_t = A_{t-1}$). A different kind of budget-balanced (sealed-bid and clock) DA auctions for procurement is proposed by Ensthaler and Giebe (2009, 2014) for the case where the target revenue $R(A)$ is a constant. In a follow-up to our work, Jarman and Meisner (2015) consider optimal budget-constrained auctions for this case, and show that they can be implemented as DA auctions.*

Budget constraints may be combined with various feasibility constraints on the set of accepted bids. For example, McAfee (1992) proposes a budget-balanced clock double auction for a homogeneous-good market with unit buyers and unit sellers, in which the feasibility constraint dictates that the number of the accepted buy bids ("demand") equal the number of the accepted sell bids ("supply").[29] *The FCC's "incentive auction" is similarly a double auction for spectrum sellers (TV broadcasters) and spectrum buyers (mobile broadband companies) that is constrained to generate a certain amount of net revenue, but subject to the added complication that buyers demand and sellers supply different kinds of differentiated goods and the feasible combinations of accepted bids are quite complicated. However, the FCC's setting also admits a DA double auction that is similar to McAfee's double auction, which sets a sequence of possible spectrum targets for the number of channels to clear, and starts by trying to clear the largest number, reducing the target whenever the revenue goals cannot be achieved with the current spectrum clearing target.*[30]

7 Optimization, Substitutes, and Myopic Clock Auctions

A substantial literature on many-to-one matching (Gale and Shapley (1982), Kelso and Crawford (1982), Hatfield and Milgrom (2005)) has examined when an allocation that is stable given both sides' preferences could be found using a "myopic" deferred-acceptance algorithm that, in every round, rejects the proposing side's offers that are not part of the responding side's optimal choices from the best offers that are currently available. A central finding

[29]McAfee's clock auction offers the same ascending "buy"' price to all buyers and the same descending "sell" price to all sellers, "freezing" the price to a "short" side of the market to keep demand within 1 of supply, and stopping as soon as both (i) the "sell"' price falls weakly below the "buy" price and (ii) demand equals supply.

[30]Duetting, Roughgarden, and Talgam-Cohen (2014) consider the approximation power of balanced-budget DA double auctions for settings in which buyers and sellers must be matched one-to-one subject to some constraints.

of this literature is that the myopic algorithm finds a stable allocation when the responding side's optimal choice rule has an appropriate "substitutes" property. In this section, we derive two stronger results for our setting.[31] The first drops the limitation to myopic clock auctions and characterizes the rules that optimize some objective – not necessarily total value – that can be implemented by some DA auction. The second drops the limitation to optimizing rules and characterizes the ones that can be implemented in a myopic way.

Optimizing allocation rules are ones that can be described as follows:

$$\alpha(v) \in \underset{A \subseteq N}{\arg\max} F(A) - \textstyle\sum_{i \in A} \gamma_i(v_i), \tag{3}$$

for some function $F : 2^N \to \mathbb{R}^N \cup \{-\infty\}$. For example, for $\gamma_i(v_i) \equiv v_i$, this is a total-surplus maximization problem, with $F(A)$ representing the auctioneer's gross benefit from accepting bid combination A (the combinations with $F(A) = -\infty$ are infeasible). This objective (3) can also describe maximization of the auctioneer's expected profit, or minimization of its expected cost (as in Example 5), when bidders' values are independently drawn from regular distributions whose virtual values are given by $\gamma_i(v_i)$.

We say that allocation rule α has *substitutes* if $i \in \alpha(v)$ and $v'_j > v_j$ for some $j \neq i$ implies $i \in \alpha(v'_j, v_{-j})$.

Proposition 6 *Let $\bar{V}_i = (\underline{v}_i, \bar{v}_i)$ and $\bar{V} = \Pi_{i \in N} \bar{V}_i$ and suppose allocation rule $\alpha : \bar{V} \to 2^N$ solves (3) for each $v \in \bar{V}$, for some nondecreasing continuous functions $\gamma_i : \bar{V}_i \to \mathbb{R}$ for each i. Then, if α is implementable with a clock auction on any finite product subset $V = \Pi_{i \in N} V_i \subseteq \bar{V}$, it must have substitutes on any such V on which the objective in (3) has no ties.*

Before proving the proposition, we prove a useful lemma. Say that allocation rule α is *non-bossy* if for any $i \in N$, $v \in V$, and $v'_i \in V_i$,

[31] Note that our setting is a special case of that of Kelso and Crawford (1982), in that it has is a single agent on the responding side.

$\alpha(v_i', v_{-i}) \cap \{i\} = \alpha(v) \cap \{i\}$ implies $\alpha(v_i', v_{-i}) = \alpha(v)$ (i.e., a bidder cannot affect the allocation without changing his own winning status.)

Lemma 1 *If allocation rule $\alpha : V \to 2^N$ is the unique solution to (3) for each $v \in V$, for some nondecreasing functions $\gamma_i : V_i \to \mathbb{R}$ for each i, then α is monotonic and non-bossy.*

Proof. For monotonicity, note that increasing bidder i's value from v_i to $v_i' > v_i$ does not increase the objective (3) on any $A \subseteq N$ such that $i \in A$ and does not change the objective (3) on any $A \subseteq N \backslash \{i\}$.

For non-bossiness, note that (3) implies that for all $i \in N$, $v_i, v_i' \in V_i$, $v_{-i} \in V_{-i}$,

$$\begin{aligned} i &\notin \alpha(v_i, v_{-i}) \cup \alpha(v_i', v_{-i}) \Longrightarrow \{\alpha(v_i, v_{-i})\} = \arg\max_{A \subseteq N : i \notin A} F(A) - \sum_{j \in A} \gamma_j(v_j) = \{\alpha(v_i', v_{-i})\}, \\ i &\in \alpha(v_i, v_{-i}) \cap \alpha(v_i', v_{-i}) \Longrightarrow \{\alpha(v_i, v_{-i})\} = \arg\max_{A \subseteq N : i \in A} F(A) - \sum_{j \in A \backslash \{i\}} \gamma_j(v_j) = \{\alpha(v_i', v_{-i})\}. \end{aligned}$$

■

Proof of Proposition 6. Let $\Lambda(v) \subseteq 2^N$ denote the set of maximizers in (3) at value profile $v \in \bar{V}$. Suppose in negation that α violates substitutes when there are no ties: for two agents $i \neq j$, for some $v_i, v_i' \in \bar{V}_i$ such that $v_i < v_i'$, and some $v_{-i} \in \bar{V}_{-i}$ we have $\Lambda(v_i, v_{-i}) = \{A\}$, $\Lambda(v_i', v_{-i}) = \{A'\}$, with $j \in A \backslash A'$. Due to non-bossiness and monotonicity established in Lemma 1, this is only possible if we also have $i \in A \backslash A'$. By monotonicity and continuity of γ_i, there exists $\hat{v}_i \in (v_i, v_i')$ such that $\Lambda(\hat{v}_i, v_{-i}) = \{A, A'\}$. Then, by continuity of γ_i, γ_j there exists $\varepsilon > 0$ small enough so that $\Lambda\left(\hat{v}_i + \delta_i, v_j + \delta_j, v_{-\{i,j\}}\right) \subseteq \{A, A'\}$ whenever $|\delta_i|, |\delta_j| \leq \varepsilon$. By monotonicity, we have $\Lambda\left(\hat{v}_i - \varepsilon, v_j - \varepsilon, v_{-\{i,j\}}\right) = \{A\}$, and using also continuity of γ_i, γ_j, there exists $\delta \in (0, \varepsilon)$ small enough such that $\Lambda\left(\hat{v}_i + \delta, v_j - \varepsilon, v_{-\{i,j\}}\right) = \Lambda\left(\hat{v}_i - \varepsilon, v_j + \delta, v_{-\{i,j\}}\right) = \{A\}$. At the same time, by monotonicity, we have $\Lambda\left(\hat{v}_i + \delta, v_j + \delta, v_{-\{i,j\}}\right) = \{A'\}$. But then α

cannot be implemented by any clock auction on value spaces $V_k = \begin{cases} \{\hat{v}_k - \varepsilon, \hat{v}_k + \delta\} & \text{for } k \in \{i,j\}, \\ \{v_k\} & \text{otherwise} \end{cases}$ (where we let $\hat{v}_j = v_j$). Indeed, the clock auction cannot stop while $(p_i(h), p_j(h)) \geq (\hat{v}_i + \delta, \hat{v}_j + \delta)$, since bidders i and j must both lose when their values are $(\hat{v}_i + \delta, \hat{v}_j + \delta)$, but reducing the price to a bidder $k \in \{i,j\}$ below $\hat{v}_k + \delta$ would prevent him from winning in state $\left(\hat{v}_k + \delta, \hat{v}_{\{i,j\}\backslash k} - \varepsilon, v_{-\{i,j\}}\right)$. ■

While the substitute property is quite restrictive, numerous examples of optimizing allocation rules with this property are known (see, e.g., Hatfield and Milgrom (2005), Ostrovsky and Leme (2015)). A simple example is given by Example 2 above, in which the total cost is minimized subject to matroid constraints on the set of rejected bids.

Now, to show a converse to Proposition 6, we describe a particular kind of clock auction that could be used to implement an allocation rule α with substitutes (on a finite value space V). This auction, which we call the *myopic clock auction*, initializes prices as $p_i(N) = \max V_i$ for each i, and then for every other history A^t, sets

$$p_i\left(A^t\right) = \begin{cases} p_i\left(A^{t-1}\right)^- & \text{if } i \in A_t \backslash \alpha\left(p_{A_t}\left(A^{t-1}\right), p_{N\backslash A_t}\left(A^{t-1}\right)^+\right), \\ p_i\left(A^{t-1}\right) & \text{otherwise.} \end{cases}$$

In words, the myopic clock auction decrements prices (by the lowest possible decrement) to those active bidders who wouldn't be accepted in α given bidders' current best offers – the active bidders' current prices and the exited bidders' last accepted prices.

Myopic clock auctions can be used to implement very general allocation rules, and not just those resulting from optimization (3).[32] Namely, in addi-

[32]A similar observation is made by Oygun and Sonmez (2013); however, they assume "irrelevance of rejected contracts," which in our case implies non-bossiness and so excludes some allocation rules that could be implemented with a myopic clock auction. On the other hand, our restriction of non-bossy winners, while without loss in our setting with single-minded bidders, would be restrictive in the more setting of Hatfield and Milgrom (2005),

tion to substitutes, we only require allocation rule α to be monotonic, and to have *non-bossy* winners: $i \in \alpha(v) \cap \alpha(v_i', v_{-i})$ should imply $\alpha(v) = \alpha(v_i', v_{-i})$. For optimizing allocation rules, these assumptions are innocuous by Lemma 1. In fact, these three properties fully characterize allocation rules that are implementable by myopic auctions:

Proposition 7 *Let $\bar{V}_i \subseteq \mathbb{R}_+$ for each i. Allocation rule $\alpha : \Pi_{i\in N}\bar{V}_i \to 2^N$ is implemented by the myopic clock auction with truthful bidding on any finite product subset $V = \Pi_{i\in N} V_i \subseteq \bar{V}$ if and only if α is monotonic, has non-bossy winners, and has substitutes.*[33]

Proof. The "if" part: First observe that an agent $i \in \alpha(v)$ could never exit the myopic auction under truthful bidding. Indeed, at any history A^t at which $i \in A_t$ and i faces price $p_i(A^{t-1}) = v_i$, while for the other (truthful) bidders $p_{A_t\setminus\{i\}}(A^{t-1}) \geq v_{A_t\setminus\{i\}}$ and $p_{N\setminus A_t}(A^{t-1})^+ = v_{N\setminus A_t}$, the substitute property and $i \in \alpha(v)$ imply that $i \in \alpha\left(p_{A_t}(A^{t-1}), p_{N\setminus A_t}(A^{t-1})^+\right)$, and so the auction sets $p_i(A^t) = p_i(A^{t-1})$, ensuring that $i \in A_{t+1}$. Thus, we have $\alpha(v) \subseteq A_t$ throughout the auction under truthful bidding. In particular, this implies that $\alpha\left(p_{A_t}(A^{t-1}), p_{N\setminus A_t}(A^{t-1})^+\right) \subseteq A_t$, and so when the myopic auction stops in period T we must have $\alpha\left(p_{A_T}\left(A^{T-1}\right), p_{N\setminus A_T}\left(A^{T-1}\right)^+\right) = A_T$. Since under truthful bidding we have $p_{N\setminus A_T}\left(A^{T-1}\right)^+ = v_{N\setminus A_T}$ and $v_{A_T} \leq p_{A_T}\left(A^{T-1}\right)$, iteratively applying monotonicity and non-bossiness of winners for members of A_T yields $\alpha(v) = A_T$.

The "only if" part: Clearly, monotonicity and non-bossiness of winners are necessary conditions for α to be implementable by *any* clock auction. To see the necessity of the substitutes condition, take any $v \in \bar{V}$, any $j \in N$, and any $v_j' \in \bar{V}_j$ such that $v_j' > v_j$. Consider value spaces $V_j = \left\{v_j, v_j'\right\}$ and $V_k = \{v_k\}$ for all $k \neq j$. For these value spaces, the myopic clock auction

in which a winner could affect the outcome by switching between different contracts.

[33]If we did not require the myopic clock auction to work on any value subspace (effectively, for any opening prices), then, as shown by Hatfield and Kojima (2009), weaker notions of substitutability may suffice.

starts with $p_k(N) = \left(v'_j, v_{-j}\right)$, and in order to implement α in state v it cannot decrement the price to any bidder from $\alpha(v) \setminus \{j\}$ in round 1, thus we must have $\alpha(v) \setminus \{j\} \subseteq \alpha(p(N)) \setminus \{j\} = \alpha\left(v'_j, v_{-j}\right) \setminus \{j\}$. ■

The assumptions of monotonicity and non-bossiness of winners are not dispensable: i.e., they are not implied by substitutes.[34] On the other hand, both assumptions are satisfied by any DA-implementable allocation rule.[35] Thus, an allocation rule that is implementable by *some* clock auction is implementable by the *myopic* clock auction if and only if it has the substitute property.

The above propositions establish that if we want an optimizing clock auction, and/or a myopic clock auction, then we must restrict attention to allocation rules that have substitutes. But such a restriction would exclude some important goals that could be attained by clock auctions with heuristic bid rejection rules, which are not guaranteed to optimize an objective of the form (3). The simplest example is given by the setting in Section 2, in which the efficient allocation rule does *not* have substitutes and therefore is not implementable with a clock auction, while the proposed heuristic clock auction implements an allocation rule that does not have substitutes and therefore is not myopic. The knapsack setting in Example 3 has the same features (for example, with $N = 3$ agents with weights $w_1 = w_2 = 1/2$ and $w_3 = 1$ the knapsack setting is isomorphic to the example in Section 2). Similar complementarities may be induced by the budget constraint in Example 7, and by correlation in expected cost minimization in Example 6 (in which one bidder's exit may make it optimal to set a higher price to other bidders). One contribution of our paper lies in extending the class

[34] For an allocation rule that has substitutes but bossy winners, let $N = 2$, $V_1 = V_2 = \{1, 2\}$,and $\alpha(v) = \arg\min_{i\in\{1,2\}} v_i$ on value spaces $V_1 = V_2 = \{1, 2\}$ (so that in case of a tie for the lowest value both tied bidders are winners). Then α satisfies substitutes, but does not satisfy non-bossiness of winners: $1 \in \alpha(1, 2) \cap \alpha(2, 2)$ but $\alpha(1, 2) = \{1\} \neq \{1, 2\} = \alpha(2, 2)$.

[35] In particular, non-bossines of winners follows from winners' privacy, discussed in Section 4.

of deferred-acceptance auctions to include heuristic auctions, which do not optimize an objective of the form (3), use heuristic price adjustment (bid rejection) rules, and are therefore suited for settings with complementarities.

8 Pay-as-Bid: Full-info equivalence

Proposition 8 *For every paid-as-bid DA auction with bid space V and allocation rule α, and every value profile $v \in V$ there is a complete-information Nash equilibrium bid profile b with $b_i = \max\{v_i, \pi_i(v_{-i})\}$ for each $i \in N$, resulting in the equilibrium allocation $\alpha(b) = \alpha(v)$.*

Proof. Let $A = \alpha(v)$, and note that the definition of threshold prices (1) implies that

$$b_i = \max\{v_i, \pi_i(v_{-i})\} = \begin{cases} \pi_i(v_{-i}) & \text{for } i \in A, \\ v_i & \text{for } i \in N \backslash A. \end{cases}$$

Since changing winning bids so that they are still winning does not alter the outcome of the DA algorithm, we have

$$\begin{aligned} \alpha(v) &= \alpha\left(\pi_A(v), v_{N\backslash A}\right) = \alpha(b), \text{ and} \\ \pi_i(v_{-i}) &= \pi_i\left(\pi_{A\backslash i}(v), v_{N\backslash A}\right) = \pi_i(b_{-i}) \text{ for each } i \in A. \end{aligned}$$

Now, we verify that the bid profile b is a Nash equilibrium. Every bidder $i \in A$ is winning by bidding $b_i = \pi_i(v_{-i}) = \pi_i(b_{-i}) \geq v_i$, and since any bid above $\pi_i(b_{-i})$ would make him lose, he has no profitable deviation. Every bidder $i \in N \backslash A$ is losing with his bid of $b_i = v_i$, and since he could only win by lowering his bid, he has no profitable deviation. ■

Proposition 8 only describes one Nash equilibrium outcome among many. In the remainder of this section, we show how the described outcome may be selected as the unique prediction by eliminating dominated strategies (either iteratively, or in a single round and then considering Nash equilibria in the remaining strategies). However, in order to ensure uniqueness, we need to eliminate examples like the following one (which could arise as a particular instance of yardstick competition described in Example 6):

Example 8 *Let $N = 2$, $V_1 = \{1, 3\}$, $V_2 = \{2, 4\}$, $\alpha(v) = \{1\}$ if $v_2 = 4$, and $\alpha(v) = \varnothing$ otherwise. With $v_1 < 3$, bidder 1's dominant strategy is to bid 3, but bidder 2 has no dominated strategies. There are two Nash equilibrium profiles in undominated strategies: they are $(3, 4)$ (in which bidder 1 wins) and $(3, 2)$ (in which there is no winner). Both strategy profiles also survive iterated deletion of dominated strategies.*

Intuitively, what leads to multiplicity of outcomes in the example is that neither iterated dominance nor undominated Nash equilibrium nails down the behavior of the losing bidder 2, and this behavior in turn affects the allocation of bidder 1. In order to rule out such situations, we restrict attention to assignment rules that are non-bossy (see Section 7). In addition, for technical reasons, we consider DA allocation rule with finite bid spaces V, but consider value profiles $v \in \mathbb{R}_+^N$ that are "generic," defined as $v_i \in \mathbb{R}_+ \backslash V_i$ and $v_i < \max V_i$ for each i. Rounded-up values are defined as $v_i^+ = \min \{b_i \in V_i : b_i > v_i\}$.

Definition 5 *An auction is* dominance-solvable *in state v if under full information, there exists a unique payoff profile that remains after iterated deletion of (weakly) dominated strategies, regardless of the order of elimination.*

For "generic" value profiles, a unique payoff profile implies a unique outcome (allocation and winning bids).

Intuitively, iterated deletion of weakly dominated strategies closely resembles a deferred-acceptance procedure, because it works by iterated rejections using a myopic criterion and finally accepting all strategies that are not rejected. We find that, for paid-as-bid auctions, iterated elimination of weakly dominated bids is closely related to the iterated deletions of always-losing bids that characterize DA auctions, and that this implies a similarly close connection between dominance-solvable auctions and DA auctions:

Proposition 9 *Consider a paid-as-bid auction with a monotonic, non-bossy allocation rule α and finite bid spaces.*

(i) The auction is pure-strategy dominance-solvable for all generic value profiles if and only if α can be implemented with a DA algorithm.

(ii) If α can be implemented with a DA algorithm, then for every generic value profile, the unique payoff profile surviving iterated deletion of dominated strategies is also the unique payoff profile associated with any (pure or mixed) Nash equilibrium in undominated strategies.

(iii) If α can be implemented with a DA algorithm, then one strategy profile that survives iterated deletion of dominated strategies and is a Nash equilibrium in undominated strategies is $b_i = \max\left\{v_i^+, \pi_i\left(v_{-i}^+\right)\right\}$ for each $i \in N$, resulting in the equilibrium allocation $\alpha(b) = \alpha(v^+)$.

The equivalence of dominance solvability and implementation using a DA algorithm – equivalently, using the results of Section 5, a clock-auction algorithm – is constructive and intuitive. Indeed, a clock auction decrements the price to a bidder when the bidder could no longer win at that price, given the prices already accepted by the other bidders. Similarly, in a round of iterated deletion of dominated strategies in a paid-as-bid auction, a bid may be dominated because it always loses against all remaining possible bid profiles of the other bidders (when a lower bid, still above the bidder's value, might sometimes be winning). The extra work in the proof arises because, in a paid-as-bid auction, bids may also be dominated in other ways: A bid is also dominated when it is below the bidder's value, or when there exists a higher bid that wins against exactly the same opposing bid profiles. In a non-bossy auction, by a result of Marx and Swinkels (1997), dominated bids can be eliminated in any order without affecting the final outcome. The proof of the proposition, given in Appendix C, distinguishes the reasons for domination to identify steps that correspond to price reductions in a clock auction.

Recall that one class of non-bossy allocation rules is given by the optimizing allocation rules defined in Section 7. Another class contains allocation rules computed by DA algorithm with fixed scoring and perfect feasibility checking:

Lemma 2 *Suppose that $\mathcal{F} \subseteq 2^N$ is a comprehensive family of feasible sets (see Example 1), and that scoring is given by* $s_i^A\left(v_i, v_{N\setminus A}\right) = \begin{cases} \sigma_i\left(v_i\right) & \text{if } A\setminus\{i\} \in \mathcal{F}, \\ 0 & \text{otherwise,} \end{cases}$ *where $\sigma_i : V_i \to \mathbb{R}_{++}$ are strictly increasing functions that have no ties (so feasibility is always maintained). The allocation rule computed by the resulting DA heuristic is non-bossy.*

Proof. As observed above, every DA procedure satisfies non-bossiness of winners. To check that condition for the losers, too, we show that if given value profile v agent i is rejected in some round t, then replacing his value with some $v_i' > v_i$ does not affect the final outcome of the algorithm. Note that it suffices to check situations in which the replacement results in bidder i is rejected *prior to* round t: Indeed, otherwise he will be rejected in round t and the replacement will not affect the behavior of the algorithm. Suppose first that the replacement results in bidder i being rejected in round $t-1$, and thus does not affect the behavior of the algorithm prior to round $t-1$. Letting A_{t-1} be the set of accepted bids in round $t-1$, and letting bidder j be the bidder rejected in round $t-1$, we must have $A_{t-1}\setminus\{i,j\} \in \mathcal{F}$, and

$$\max_{k \in A_{t-1}\setminus\{i,j\} : A_{t-1}\setminus\{j,k\} \in \mathcal{F}} \sigma_k\left(v_k\right) < \sigma_i\left(v_i\right) < \sigma_j\left(v_j\right).$$

Using this and the comprehensiveness of $\mathcal{F}$, we obtain

$$\max_{k \in A_{t-1}\setminus\{i,j\} : A_{t-1}\setminus\{i,j,k\} \in \mathcal{F}} \sigma_k\left(v_k\right) \leq \max_{k \in A_{t-1}\setminus\{i,j\} : A_{t-1}\setminus\{j,k\} \in \mathcal{F}} \sigma_k\left(v_k\right) < \sigma_j\left(v_j\right),$$

which implies that bidder j must be rejected in round t. Then after round t the algorithm will be unaffected by the replacement of v_i with v_i'. Iterating this argument, we see by induction on τ that any increase in agent i's bid, resulting in it being rejected in some round $t-\tau$, will not affect the allocation. ■

9 Conclusion

The analysis and results reported in this paper were developed in response to questions arising from an engagement to advise the FCC on the design of its "incentive auction." The possibility that some heuristic clock auction could achieve nearly optimal results is of obvious importance for that application: it inspired simulations to identify the price reduction (or scoring) algorithms that work best for the FCC's particular constraints. Clock auctions have the huge advantage for this application that, in contrast to sealed-bid auctions, they are obviously strategy-proof and, in contrast to Vickrey auctions, are weakly group strategy-proof. Obvious strategy-proofness is important because it reduces the cost of participation, especially for small local broadcasters whose participation is needed for a successful incentive auction.[36] Our outcome-equivalence results suggest that obvious strategy-proofness may not increase acquisition costs, even with the same set of bidders, compared to a paid-as-bid rule. Since there are many Nash equilibria in paid-as-bid auctions, the credibility of the outcome equivalence result is enhanced by our equilibrium outcome uniqueness result. In the incentive auction, revenues

[36] Two additional advantages help account for the popularity of clock auctions in practice and make them suitable for FCC's problem. First, clock auctions permit information feedback so as to help bidders aggregate common-value information and thereby improve efficiency and revenues (as in Migrom and Weber 1982), Second, they allow bidders to express their preferences for multiple bidding options (e.g., auction bundles) by switching among those options (as in Gul and Stachetti 2000). (Expressing such preferences in a sealed-bid auction may require a larger message space). However, these advantages are beyond the scope of this paper, which focuses on single-minded bidders with private values.

for the forward auction portion must be sufficient to pay the costs the broadcasters incur in moving to new broadcast channels, as well as meeting certain other gross and net revenue goals, so the possibility of including a cost target in the scoring rule is necessary to make the whole design feasible. Including yardstick competition could allow the FCC to set maximum prices for broadcasters in regions with little competition based on bids in other, more competitive regions.

One important limitation of our analysis is its focus on single-minded bidders. In practice, bidders are often interested in selling or buying various goods or packages and have different values for those packages. For example, in the FCC's "reverse auction," in addition to selling their UHF stations to go off-air, bidders may also be permitted to bid to switch to a less-congested lower-frequency VHF band. Also, some broadcasters own multiple stations and may interested in selling different subsets of their stations.

A number of deferred-acceptance clock auctions for multidimensional bidders have also been proposed and examined. Some DA clock auctions sustain truthful bidding as an ("ex post") Nash equilibrium in some multidimensional settings, by virtue of their implementation of the Vickrey outcome (e.g. Kelso and Crawford 1987, Ausubel 2004, Bikhchandani et al. 2011). However, these auctions are not even strategy-proof (let alone obviously strategy-proof): A bidder who expects others to use strategies that are inconsistent with any type may prefer not to be truthful himself. Other proposed DA clock auctions do not even sustain truthful bidding as a Nash equilibrium: for example, in the simultaneous multiple-item ascending auctions studied by Milgrom (2000) and Gul and Stachetti (2000), bidders who are willing to purchase more than one item at a time have an incentive to engage in monopsonistic "demand reduction." (Such incentives may be minimized in "large-market" settings in which competition limits the power of each bidder to affect prices.) Finally, some obviously strategy-proof clock auctions for multidimensional bidders have been proposed (see, e.g., Bartal et al. (2003)), but such auctions only guarantee a fraction of efficiency. Further examination of DA auctions for "multi-minded" bidders remains an important direction for future research.

Roth (2002) has observed that "Market design involves a responsibility for detail, a need to deal with all of a market's complications, not just its principal features." Over the past two decades, variants of the original DA algorithm have had remarkable success in accommodating the diverse details and complications of a wide set of matching market design problems. In this paper, we extend that success to new class of auction design problems, connect DA auctions to encompass clock auctions, and reaffirm the deferred-acceptance idea as the basis for many of the most successful mechanisms of market design.

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10 Appendix A: Greedy-Acceptance Auctions

In this appendix, we illustrate the greedy-acceptance auctions of Lehmann, O'Callaghan and Shoham (2002). To be consistent with their setting, we consider a three-bidder "forward" (selling) auction (although it is also possible to construct a corresponding reverse auction) in which it is feasible to satisfy either both bidders 1 and 2 or bidder 3. (Say, bidder 3 desires a bundle of two objects while bidders 1 and 2 each desire a single object from the bundle.) For a simple example, the bidders' scores could be defined as their bids. The auction iterates accepting the highest bid that is still feasible to accept. The winners are paid their threshold prices, i.e., the minimal bids that would have been accepted.

Consider the case in which both bidders 1 and 2 bid above bidder 3's bid. In this case they both win and pay zero, since each of them would have still won by bidding zero, letting the other bid be accepted in the first round, which makes bid 3 infeasible to accept. This implies that the auction is not weakly group strategyproof: when the true values of bidders 1,2 are below bidder 3's value but strictly positive, they would both strictly benefit from both bidding above bidder 3's value, so that they both win and pay zero. Also, the auction's revenue in this case is zero, while any full-information Nash equilibrium of the corresponding paid-as-bid auction could not sell to bidders 1 and 2 at a total price below bidder 3's value, since otherwise bidder 3 could have profitably deviated to win the auction. Finally, this allocation rule cannot be implemented with a DA clock auction, which is an ascending-price clock auction in the selling-auction setting, since the allocation is completely determined by which bidder has the highest value, while the first bidder exiting in a clock auction has the worst (lowest) value according to some scoring criterion.[37]

[37]Lehmann et al. (2002) propose a descending-price "clock" auction, in which, when bidder 3 buys first and the allocation is determined, the prices of bidders 1 and 2 must continue descending to determine the winner's threshold payment. Note that in this

11 Appendix B: A Near-Optimal DA Algorithm for TV Spectrum Repurchasing

Consider a setting in which the bidders are television stations who bid to relinquish their broadcast rights. Broadcast channels must be assigned to the stations whose bids are rejected in a way that satisfies non-interference constraints. In this simplified model, those constraints are represented by an interference graph Z: a set of two-elements subsets of 2^N ("edges"). We interpret $\{i, j\} \in Z$ to mean that stations i and j cannot both be assigned to the same channel without causing unacceptable interference. Letting $\{1, ..., n\}$ denote the set of channels left after the auction. Then, the feasible sets of accepted bids for the FCC's repacking problem can be written as

$$\mathcal{F} = \{A \subseteq N : (\exists c : N \backslash A \to \{1, ..., n\})(\forall i, j \in N \backslash A)(\{i, j\} \in Z \Rightarrow c(i) \neq c(j)\}. \tag{4}$$

Proposition 10 *Suppose there exists an ordered partition of the set N of stations into m disjoint sets $N_1, ..., N_m$ such that*

(i) for each $k = 1, ..., m$ and each $i, j \in N_k$, $\{i, j\} \in Z$ (that is, each N_k is a "clique"),

(ii) there exists some $d < n$ such that for each $k = 1, ..., m$ and each $S \subseteq N_k$ satisfying $|S| \leq n - d$, we have

$$|S| + |\cap_{i \in S} \cup_{l<k} \{j \in N_l : \{i, j\} \in Z\}| \leq n.$$

Consider a DA algorithm that iterates rejecting the most valuable bid in each partition element N_k as long as that is feasible and there are no more than $n - d$ stations rejected from each element, and then continues in any way.

"clock auction," in contrast to the deferred-acceptance clock auctions, studied in this paper, truthful bidding stragegies are not dominant strategies (although they do form a Nash equilibrium).

The total value of stations assigned by such an algorithm is at least a fraction $1 - d/n$ of the optimal value.

Proof. Let

$$
\begin{aligned}
F_{Zn} &= \{S \subseteq N : (\exists c : S \to \{1,...,n\})(\forall s, s' \in S)(\{s,s'\} \in Z \Longrightarrow c(s) \neq c(s')\}, \\
F^d_{mn} &= \{S \subseteq N : (\forall j)\, |S \cap N_j| \leq n - d\}, \text{ and } F^d_{Zmn} = F_{Zn} \cap F^d_{mn}.
\end{aligned}
$$

For each $S \subseteq N$, define $S_j = S \cap (\cup_{j' \leq j} N_{j'})$, $S'_j = S \cap N_j$, and $Z^*(S'_j) = \cap_{s \in S'_j}\{s' \in S_{j-1} : \{s,s'\} \in Z\}$ – i.e., the set of stations in S_{j-1} that interfere with *all* of the stations in S'_j. If $Z^*(S'_j) \subseteq N_{j-1}$, then to avoid interference it is necessary to assign a different channel to each station in $S'_j \cup Z^*(S'_j)$. A necessary condition for this is $\left|S'_j \cup Z^*(S'_j)\right| = \left|S'_j\right| + |Z^*(S'_j)| \leq n$, which is assumption (ii) of the proposition. Less obviously, Hall's Marriage theorem[38] implies that this condition is also sufficient, allowing us to prove the following lemma.

Lemma 3 $F^d_{Zmn} = F^d_{mn}$.

Proof. It is obvious that $F^d_{Zmn} \subseteq F^d_{mn}$ (the first set imposes all the same within-area constraints plus additional ones). For the reverse inclusion, consider any $S \in F^d_{mn}$. We will establish that $S \in F^d_{Zmn}$ by showing the possibility of constructing the required channel assignment function $c : S \to \{1,...,n\}$.

Begin the construction by assigning a different channel $c(s)$ to each station $s \in S_1$, which is possible because $|S_1| \leq n$. Then, c is feasible for S_1.

[38]Hall's Marriage Theorem concerns bipartite graphs linking two sets – "men" and "women." Given any set of women S, let $A(S)$ be the set of men who linked ("acceptable") to at least one woman in S. Hall's theorem asserts that there exists a one-to-one match in which every woman is matched to some acceptable man (some men may be unmatched) if and only if for every subset S' of the women, $|S'| \leq |A(S')|$.

In our application, Hall's theorem is used to show that channels can be assigned to the stations in each area without violating the channel constraints implied by the assignments in the lower indexed areas.

Inductively, suppose that the channel assignment c has been constructed to be feasible for stations S_{j-1}. We show how to extend c to a feasible channel assignment for $S_j = S_{j-1} \cup S'_j$.

For any $S''_j \subseteq S'_j$, define $J(S''_j) = \{1, ..., n\} - c(Z^*(S''_j))$ and notice that $|J(S''_j)| = n - |c(Z^*(S''_j))|$. According to Hall's Marriage Theorem (informally, think of the stations in S'_j as the women and the n channels as the men), there exists a one-to-one map $c : S'_j \to \{1, ..., n\}$ with the property that $(\forall s \in S'_j) c(s) \in J(\{s\})$ if and only if $(\forall S''_j \subseteq S'_j)$, $|J(S''_j)| \geq |S''_j|$. Substituting for $|J(S''_j)|$, this last inequality is equivalent to $|c(Z^*(S''_j))| + \left|S''_j\right| \leq n$.

For all $S''_j \subseteq S'_j$, since $S \in F^d_{mn}$, we have $\left|S''_j\right| \leq n - d$. Then, since $S''_j \cup S_{j-1} \in F^d_{mn}$, it follows from assumption (ii) of the Proposition that $|S''_j| + \left|Z^*(S''_j)\right| \leq n$. Combining that inequality with $\left|c(Z^*(S''_j))\right| \leq |Z^*(S''_j)|$, we obtain the condition required by Hall's Marriage theorem, implying the existence of a one-to-one function $c : S'_j \to \{1, ..., n\}$ such that $(\forall s \in S'_j) c(s) \in J(\{s\})$. This extends c to a feasible channel assignment for the expanded domain $S_j = S'_j \cup S_{j-1}$. ■

Finally, to establish the proposition, consider the following DA algorithm. At any round t, if the set of stations already assigned is T, then any station that is essential gets a score of zero. Among inessential stations at round t, the score for any station s with $m(s) = j$ is $n - |T \cap N_j| + v(s)/(1 + v(s))$. (Intuitively, this algorithm tries to keep the number of stations assigned in each area roughly equal at every round.) By the above Lemma, this algorithm will first assign the most valuable station in each area, then the second most valuable station in each area, and so on until at least the $n - d$ most valuable stations in each area are assigned. ■

Intuitively, the Proposition applies to the setting in which the stations can be partitioned into m "metropolitan areas" in such a way that (i) no two television stations in the same area can be assigned to the same channel and (ii) the cross-area constraints are limited in the sense that if we have a set S of no more than $n - d$ stations in one metropolitan area, there are no more than $n - |S|$ stations in lower-indexed areas that have interference constraints with *all* the stations in S.[39] Using an argument based on Hall's Marriage Theorem, condition (ii) ensures that it is possible to select any arbitrary $n - d$ stations in each area independently of each other and still be able to find a feasible assignment of these stations to channels. Since the optimal value is bounded above by assigning the n most valuable stations in each area (which would be feasible if there were no inter-area constraints), the worst-case fraction of efficiency loss is bounded above by $(n - d)/n$.

We make several observations.

1. It is possible to satisfy condition (ii) while having several times more inter-area constraints than within-area constraints. To illustrate, suppose that all stations are arranged from north to south on a line and that each station interferes with its $n-1$ closest neighbors to the north as well as its $n-1$ closest neighbors to the south. Suppose that each successive group of n stations is described as a metropolitan area. Then, it is possible to assign all stations ($d = 0$) to channels without creating interference just by rotating through the n channel numbers. In this example, there are just $x = n(n-1)/2$ constraints among stations within an area but $2x$ constraints between those stations and ones in the next lower indexed area and another $2x$ constraints involving stations in the next higher indexed area.

2. In general, there may be many ways to partition stations into cliques, and many ways to order any given partition. The Proposition formally applies to each partition, but the number d and therefore the approximation guarantee will vary depending on which partition is selected and how it is ordered.

[39] While we assume that the partition is totally ordered for notational simplicity, the proposition can also be extended to cases in which partition elements form an ordered acyclic graph, by interpreting $<$ and $\leq$ as referring to a precedence relation.

3. The worst-case bound applies over all possible station values and incorporates both a conservatively high estimate of the optimum and a conservatively low estimate of the DA algorithm performance. To approach the worst-case bound, the optimum must be similar to assigning the n most valuable stations in each area, and the DA algorithm must be unable to assign any other stations after the $n - d$ most valuable stations have been assigned in each area.

4. The standard DA algorithm discussed in Example 2 is not among the ones described in the Proposition, and does not achieve the same performance guarantee. For example, suppose that there is some central area – area 1 – such that the stations in any area are linked to all other stations in that area and to the stations in area 1, but not to any other stations. Suppose that there are 2 channels available. Then the DA algorithms described in the proposition assign the single most valuable station in each area, thus achieving at least half of the optimal value. On the other hand, the standard DA algorithm could in this case achieve as little as $1/(m-1)$ of the optimal value: This could happen if the two most valuable stations happen to be in area 1, in which the standard DA algorithm assigns just those two stations and no others. Thus, the example in this section demonstrates how it may be possible to design a DA algorithm to improve upon the standard DA algorithm by taking advantage of known properties of the feasible set. In applications like the FCC auction, in which interference graph is known before the auction, it may be possible to apply a variety of analytical tools and simulations to find a DA algorithm that performs much better than the standard one.

12 Appendix C: Proof of Proposition 9

For the "if" direction of (i), recall from Proposition 5 that any assignment rule α that is implementable via a DA threshold auction is also implementable with a clock auction. Furthermore, we can compute the outcome of the *paid-as-bid* auction with assignment rule α using a "two-phase clock auction" mechanism, as follows. In phase 1, the clock auction algorithm described above is run to determine the set of winners. In phase 2, the payments to the winners are determined by allowing prices to continue falling (through points in V_i) until all bidders "quit," and then paying each winning bidder the last price it has accepted. In this two-phase clock auction game, if each bidder i is restricted to use a truthful strategy corresponding to some value profile $b_i \in V_i$ (i.e., exit when his price falls below b_i but not before), that obviously leads to the same outcome as the paid-as-bid DA auction game with bid profile b.

If the assignment rule is non-bossy, then for generic values the game satisfies the TDI condition of Marx and Swinkels (1997). Hence, by their rsults, the payoff profiles surviving iterated deletion of weakly dominated strategies do not depend on the order of deletion, and on whether we delete "very weakly" dominated strategies (which include payoff-equivalent strategies to a suriving strategy, in addition to those weakly dominated by it).[40] Hence, iterated deletion of either weakly or very weakly dominated strategies in any order leads to the same set of possible outcomes.

We specify the following deletion process of very weakly dominated (henceforth, VW-dominated) strategies: Begin by deleting for each agent i all the bids $b_i < v_i^+$ (which are VW-dominated by the bid v_i^+). In the game that remains after these initial deletions, every bidder strictly prefers any outcome in which it wins to any in which it loses. We specify the next deletions

[40]Note that in a non-bossy paid-as-bid auction, two bids of bidder i are payoff-equivalent to each other against others' bids from $\hat{B}_{-i}$ if and only if they both make bidder i a sure loser, i.e., $i \notin \alpha(b_i, b_{-i}) \cup \alpha(b_i', b_{-i})$ for all $b_{-i} \in \hat{B}_{-i}$.

iteratively by referring to the sequence of prices $\{p(A^t)\}$ that would emerge during phase 1 if each bidder were to bid v_i^+. At the beginning of each step $t = 1, 2, ...$ of our iterated deletion process, the set of strategies remaining to each bidder i is $\hat{B}_i^{t-1} = B_i \cap [v_i^+, \max\left\{v_i^+, p_i\left(A^{t-1}\right)\right\}]$. With just these strategies remaining, when the clock auction offers new prices $p\left(A^t\right)$ in iteration t, for each bidder i, all the bids $b_i \in \hat{B}_i^{t-1}$ such that $b_i > \max\left\{v_i^+, p_i\left(A^t\right)\right\}$ are sure to lose and are therefore VW-dominated in the remaining subgame by bidding v_i^+. Deletion of these VW-dominated strategies yields new strategy sets $\hat{B}_i^t = B_i \cap [v_i^+, \max\left\{v_i^+, p_i\left(A^t\right)\right\}]$ for each i. These iterated deletions continue until phase 1 ends at some iteration T, at which the set of winners $\alpha(\hat{B}^T)$ is uniquely determined. For each agent i, if $\hat{B}_i^T$ is not a singleton, then its largest element, $\max \hat{B}_i^T = \max\left\{v_i^+, p_i\left(A^T\right)\right\}$, is dominant in the remaining game with strategy sets $\hat{B}^T$ (because it wins at the highest remaining price). So, we may perform one more round of deletions, taking $\hat{B}_i^{T+1} = \{\max(v_i^+, p_i\left(A^T\right))\}$. Hence, the only possible outcome of iterative elimination of VW-dominated bids in any order is the outcome corresponding to the bid profile $(\max(v_i^+, p_i\left(A^T\right)))_{i \in N}$.

For (ii), fix an undominated mixed Nash equilibrium profile. For each bidder i with a zero equilibrium payoff, all bids of v_i^+ or more must be always losing. Hence, by non-bossiness, we may replace all bids of such bidder by the pure strategy bid v_i^+ to obtain another mixed strategy profile σ with the same distribution of outcomes.

For any bidder i with strictly positive equilibrium expected payoffs, all bids in the support of σ_i have positive expected payoffs, so all must win with a positive probability against σ_{-i}. Consider the maximum bid profile in the support of σ. Referring to the clock auction process, we infer that if any positive-payoff bidder's bid is losing for that profile, then it is losing for all profiles in the support of σ, which contradicts positive expected payoffs. Since reducing a winning bid in the clock auction does not affect the allocation, for every bid profile in the support of σ, the positive-payoff players are the winners. Since the highest always-winning bid earns strictly more than any lower winning bid, this further implies that the winners' equilibrium mixtures are degenerate: winning bidders play pure strategies. Therefore, σ assigns probability one to some single bid profile b.

Next, we claim that the iterative deletions described in the proof of (i) above do not delete any of the component bids in b. Phase I of the iterative deletion procedure deletes only bids above v_i^+ for zero-payoff bidders and only always-losing bids for positive-payoff bidders, so all the component bids in b survive that phase. Phase II deletes all but the highest remaining bid of each winning bidder: the lower bids are never best replies to the highest surviving bids (they always win, but they are paid less). Hence, the full procedure never deletes any component bid in the profile b. It follows that $b = (\max\left\{v_i^+, p_i\left(A^T\right)\right\})_{i \in N}$ and that the outcome of b is the outcome of every undominated Nash equilibrium.

To prove (iii): in the surviving bid profile b, each agent $i \in A^T$ bids its threshold price, which is $p_i\left(A^T\right) \geq v_i^+$, while each $i \in N \backslash A^T$ bids v_i^+, which is by definition above its threshold price. Thus by Proposition 8 it is a Nash equilibrium and it contains only undominated strategies, and as argued above it survives iterated deletion of dominated strategies.

It remains to prove the "only if" direction of (i): we assume that the paid-as-bid auction for allocation rule α is dominance solvable and construct a clock-auction price mapping $p : H \to \mathbb{R}^N$ that implements α. For each possible clock auction history A^t of the auction and each bidder i, let $\hat{B}_i\left(A^t\right) \subseteq B_i$ denote the set of bidder i's bids that are consistent with history A^t, i.e.,

$$\hat{B}_i\left(A^t\right) = \begin{cases} \{b_i \in B_i : b_i \leq p_i\left(A^{t-1}\right)\} & \text{for } i \in A_t, \\ \{(p_i\left(A^{t-1}\right))^+\} & \text{for } i \in N \backslash A_t. \end{cases}$$

We will show by induction that, for each possible history A^t, the strategy sets $\hat{B}_i\left(A^t\right)$ have two important properties: (a) $\cup_{b \in \hat{B}(A^t)} \alpha\left(b\right) \subseteq A_t$ (only bidders who are still active could become winners), and (b) the sets $\hat{B}\left(A^t\right)$ can be obtained by an iterative process that, at each step, deletes some bids that are then always-losing bids for bidders in A_t or all bids below some cutoffs for bidders in $N \backslash A_t$.

To construct the clock auction p, we initialize the clock prices at the null history N as $p(N) = \max B$, so that $\hat{B}_i(N) = B_i$ for each i and properties (a) and (b) are trivially satisfied. For each clock round $t = 1, 2, ...$, given any history A^t at which properties (a) and (b) are satisfied, we show that either we can stop the auction at this point with the set of winners known to be A_t, or we can reduce the price to a single bidder in such a way that properties (a) and (b) are inherited by any history A^{t+1} that could succeed A^t. We do so using the following claim:

Claim 1 *For all possible histories $A^t \in H$, either (i) $\alpha(b) = A_t$ for all $b \in \hat{B}(A^t)$ (all active bidders must win), or (ii) there exists $i \in A_t \backslash \cup_{b_{-i} \in \hat{B}_{-i}(A^t)} \alpha(p_i(A^{t-1}), b_{-i})$ (there is an active bidder whose highest remaining bid $p_i(A^{t-1})$ must lose).*

Proof. We begin by noting that if, given some history A^t, the set of winners is uniquely determined to be some $\hat{A} \subseteq N$ (i.e., $\hat{A} = \alpha(b)$ for all $b \in \hat{B}(A^t)$), then by inductive property (a), either $\hat{A} = A_t$ and so we are in case (i) of the claim, or we are in case (ii) of the claim for some bidder $i \in A_t \backslash \hat{A}$. Thus, it remains only to prove the claim for the case in which $\alpha(\hat{B}(A^t))$ is not a singleton.

Call two bids $b_i, b_i' \in \hat{B}_i(A^t)$ of bidder i *allocation-equivalent* (at A^t) if $\alpha(b_i', b_{-i}) = \alpha(b_i, b_{-i})$ for all $b_{-i} \in \hat{B}_{-i}(A^t)$. For each bidder i, we construct a strategy set $\bar{B}_i \subseteq \hat{B}_i(A^t)$ by eliminating from $\hat{B}_i(A^t)$ all of i's allocation-equivalent bids except for the highest one from each equivalence class. Note that by construction $\max \bar{B}_i = \max \hat{B}_i(A^t) = p_i(A^{t-1})$ for $i \in A_t$, and $\bar{B}_i = \hat{B}_i(A^t) = \left\{(p_i(A^{t-1}))^+\right\}$ for $i \in N \backslash A_t$. Note also that the elimination of allocation-equivalent bids preserves the possible sets of winners: $\alpha\left(\bar{B}\right) = \alpha(\hat{B}(A^t))$, and that, by assumption, this is not a singleton.

Now consider a generic value profile v such that $v_i^+ = \begin{cases} \min B_i & \text{for } i \in A_t, \\ (p_i(A^{t-1}))^+ & \text{for } i \in N \backslash A_t \end{cases}$ (thus, bidders in A_t always prefer to win, and the other bidders preferred to exit at the last prices they were offered). Observe that for this value profile, inductive property (b) allows us to obtain strategy sets $\hat{B}(A^t)$ by iterated deletion of strategies that are VW-dominated by bidding v_i^+, by deleting all bids below $(p_i(A^{t-1}))^+$ for all bidders $i \in N \backslash A_t$, and iteratively deleting always-losing bids. Next, note that when a bid b_i is allocation-equivalent to a bid $b_i' > b_i$, then b_i is VW-dominated by b_i'. Iterated deletion of such VW-dominated strategies from $\hat{B}(A^t)$ yields $\bar{B}$.

Dominance solvability for value profile v implies that if the set $\alpha\left(\bar{B}\right)$ of winners is not uniquely determined, then for some bidder $i \in A_t$ some bid $b_i \in \bar{B}_i$ must be VW-dominated by some bid $b_i' \in \bar{B}_i \backslash \{b_i\}$ against $\bar{B}_{-i}$. If we had $b_i' > b_i$, then (by monotonicity) the two bids would have to win against the same set of opposing bid profiles $b_{-i} \in \bar{B}_{-i}$ and hence (by non-bossiness) they would be allocation-equivalent, which is impossible given our construction of $\bar{B}$.[41] Hence, we must have $b_i' < b_i$. Furthermore, since $\bar{B}$ was obtained from $\hat{B}(A^t)$ by deleting allocation-equivalent bids, bid b_i' must also VW-dominate bid b_i against $\hat{B}_{-i}(A^t)$. Since $v_i^+ = \min B_i \leq b_i' < b_i$, such VW-dominance is only possible if b_i never wins against $\hat{B}_{-i}(A^t)$, which, by monotonicity, implies that the bid $p_i(A^{t-1}) = \max \bar{B}_i \geq b_i$ also never wins against $\hat{B}_{-i}(A^t)$. This establishes the claim. ■

Now, if we are in case (i) of the claim, then the auction can be finished in round t: if the bid profile is $b \in \hat{B}(A^t)$, then the auction has found the desired allocation $\alpha(b) = A_t$.

If we are instead in case (ii) of the claim, then in iteration t of the clock auction, our construction decrements the price to the bidder i iden-

[41] This argument relies on the observation that deleting allocation-equivalent bids for one player does not affect the allocation-equivalence of other players' bids: hence, when two bids of bidder i are allocation-equivalent against $\bar{B}_{-i}$, they must also be allocation-equivalent against $\hat{B}_{-i}(A_t)$.

tifed in the claim and leave the other prices unchanged, that is, we set $p_j(A^t) = \begin{cases} (p_i(A^{t-1}))^- & \text{for } i = j, \\ p_j(A^{t-1}) & \text{for } j \neq i. \end{cases}$. It remains to show that the two inductive properties are inherited in iteration $t+1$ by both the history $A^{t+1} = (A^t, A_t)$ in which bidder i accepts the reduced price and the history $A^{t+1} = (A^t, A_t \backslash \{i\})$ in which bidder i quits. For property (a), using the fact that $\hat{B}(A^{t+1}) \subseteq \hat{B}(A^t)$ and the inductive hypothesis, we see that $\cup_{b \in \hat{B}(A^{t+1})} \alpha(b) \subseteq \cup_{b \in \hat{B}(A^t)} \alpha(b) \subseteq A_t$, which establishes the property for history $A^{t+1} = (A^t, A_t)$; as for history $A^{t+1} = (A^t, A_t \backslash \{i\})$, we use in addition the fact that $i \notin \cup_{b \in \hat{B}(A^{t+1})} \alpha(b)$ since we are in case (ii) of the claim. For property (b), it extends to history (A^t, A_t) since $\hat{B}_i(A^t, A_t) = \hat{B}_i(A^t) \setminus \{p_i(A^{t-1})\}$ and we are in case (ii) of the claim, and it extends to history $(A^t, A_t \backslash \{i\})$ since $\hat{B}_i(A^t, A_t \backslash \{i\}) = \left\{b_i \in \hat{B}_i(A^t) : b_i \geq p_i(A^{t-1})\right\}$.