블록행렬

김수환

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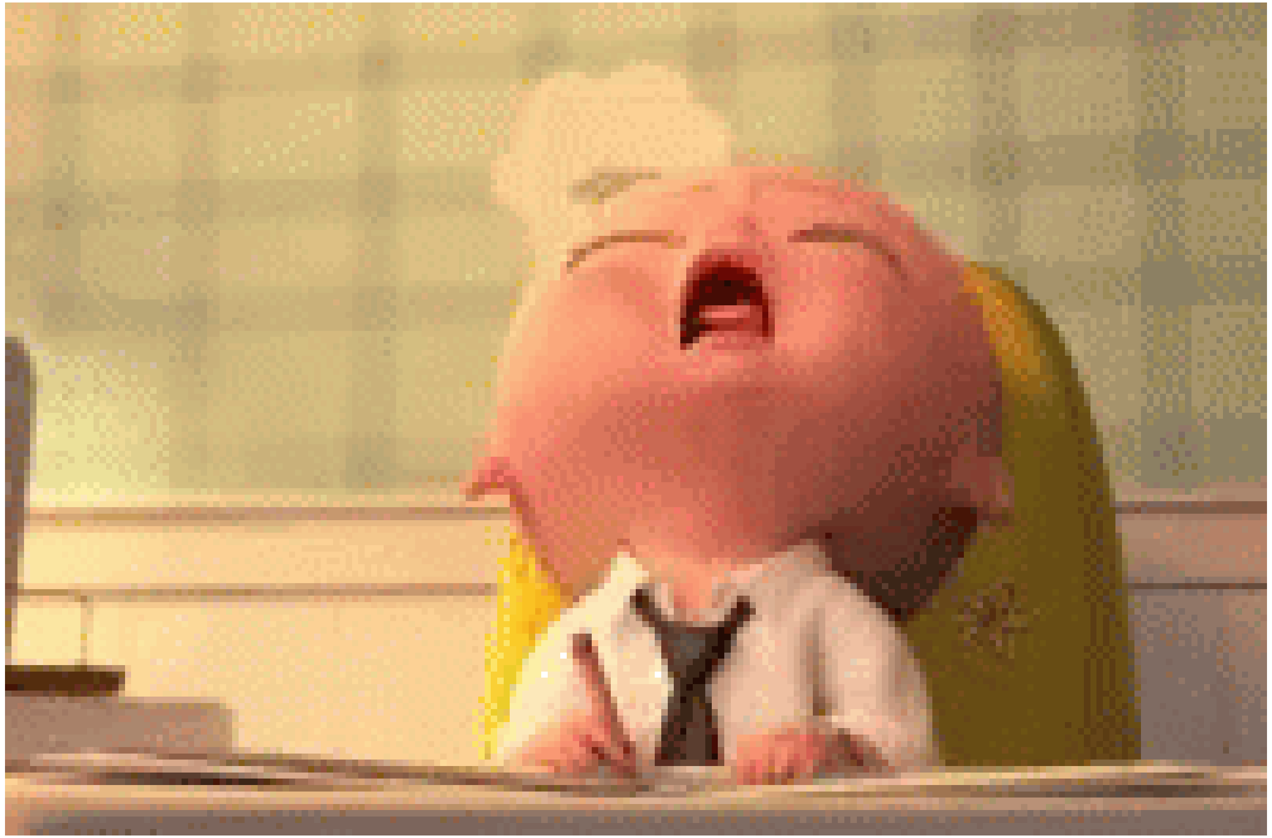
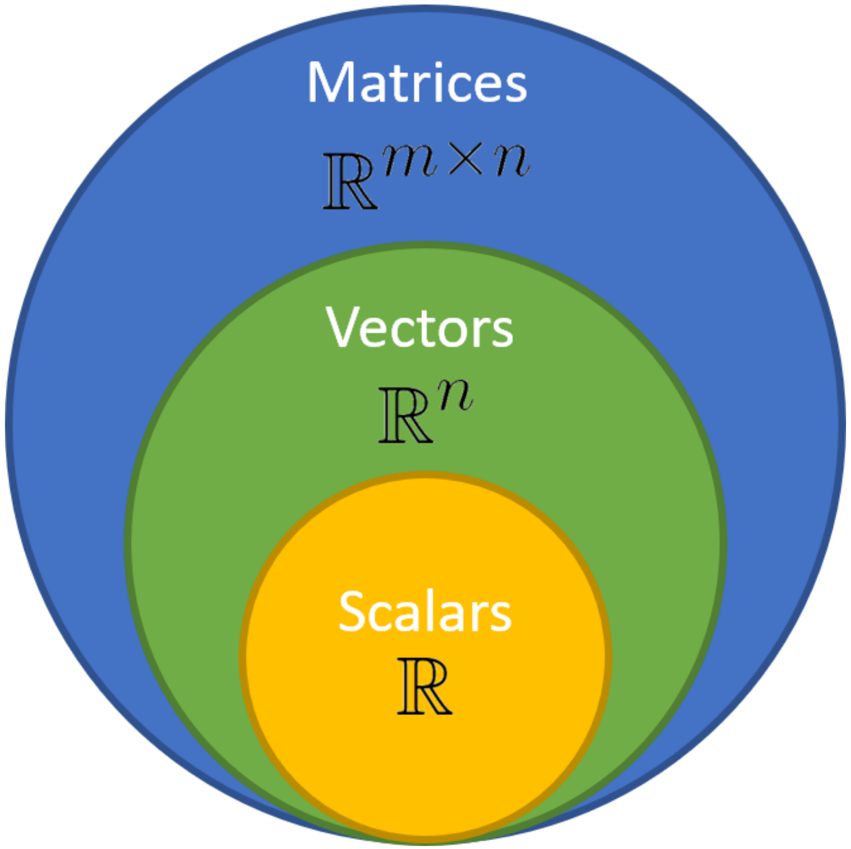
복습

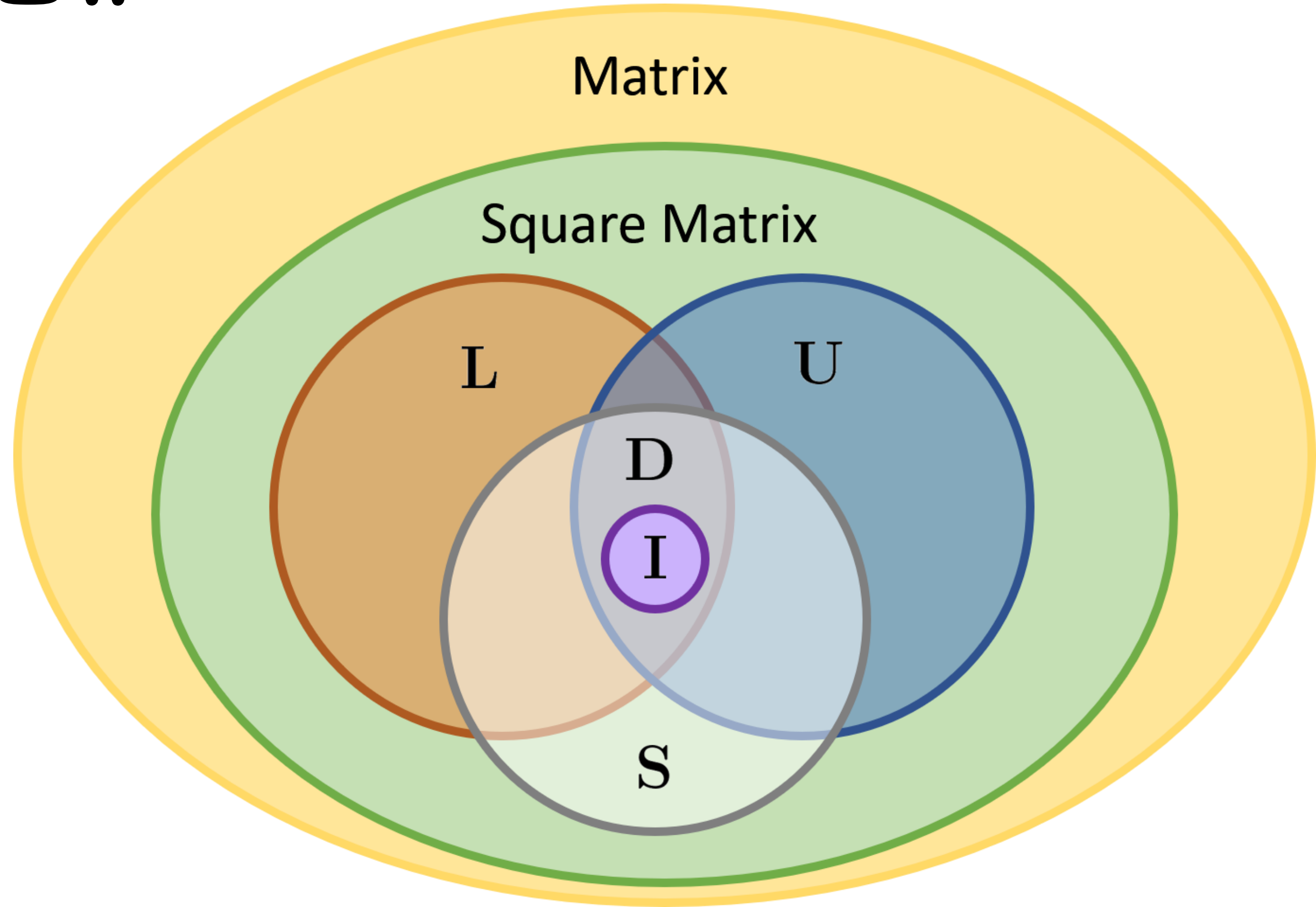


https://giphy.com/gifs/bored-sleepy-boring-LTYT5GTIiAMBa

스칼라(Scalars) vs. 벡터(Vectors) vs. 행렬(Matrices)



행렬의 종류



벡터의 연산

- 두 벡터

$$\mathbf{a},\ \mathbf{b} \in \mathbb{R}^n$$

- 행렬의 덧셈/뺄셈

$$[\mathbf{a} \pm \mathbf{b}]_i = [\mathbf{a}]_i \pm [\mathbf{b}]_i$$

- 행렬의 스칼라배

$$[c\mathbf{a}]_i = c[\mathbf{a}]_i$$

행렬의 연산

- 두 행렬

$$\mathbf{A},\ \mathbf{B} \in \mathbb{R}^{m\times n}$$

- 행렬의 덧셈

$$[\mathbf{A} \pm \mathbf{B}]_{ij} = [\mathbf{A}]_{ij} \pm [\mathbf{B}]_{ij}$$

- 행렬의 스칼라배

$$[c\mathbf{A}]_{ij} = c[\mathbf{A}]_{ij}$$

- 행렬의 전치

$$[\mathbf{A}^\top]_{ij} = [\mathbf{A}]_{ji}$$

- 두 행렬

$$\mathbf{A} \in \mathbb{R}^{m\times n},\ \mathbf{B} \in \mathbb{R}^{n\times l}$$

- 행렬의 곱셈

$$[\mathbf{A}\mathbf{B}]_{ij} = \mathbf{a}_i^\top \mathbf{b}_j$$

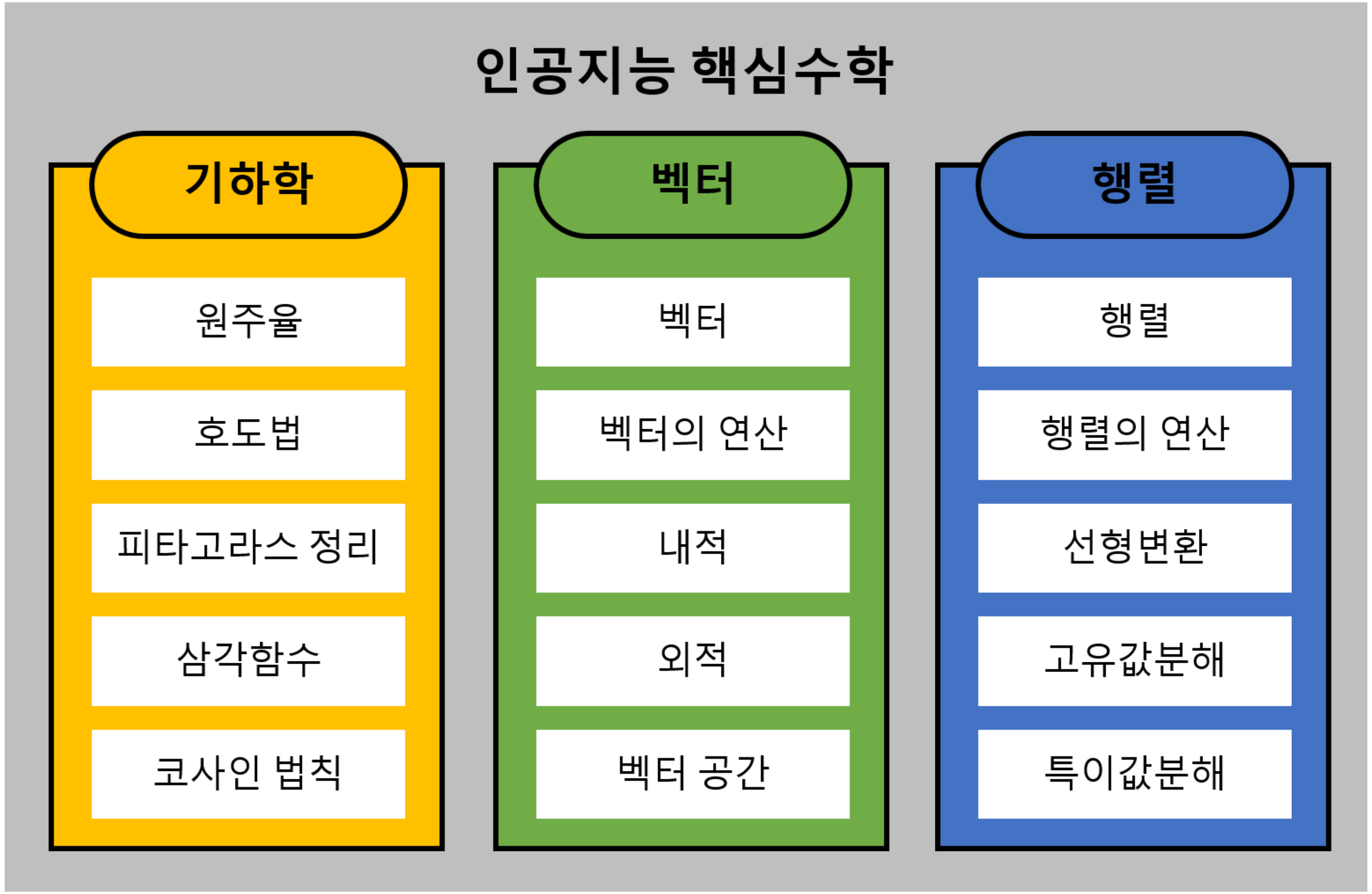
- 행렬의 곱은 교환법칙이 성립하지 않는다

$$\mathbf{A}\mathbf{B} \neq \mathbf{B}\mathbf{A}$$

- 행렬의 대각합

$$\mathrm{Trace}(\mathbf{A}) = \sum_{i=1}^{n} [\mathbf{A}]_{ii}$$

Big Picture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기하학
원주율
호도법
피타고라스 정리
삼각함수
코사인 법칙
벡터
벡터
벡터의 연산
내적
외적
벡터 공간
행렬
행렬
행렬의 연산
선형변환
고유값분해
특이값분해

🎯 학습목표

학습목표

- 행렬
    - 정의
    - 연산
        - 덧셈
        - 뺄셈
        - 스칼라배
        - 전치
        - 곱셈
    - 종류

- 블록행렬
    - 정의
    - 연산
        - 덧셈
        - 뺄셈
        - 스칼라배
        - 전치
        - 곱셈
    - 종류

🍱 블록행렬의 정의와 연산

블록행렬의 덧셈

- 덧셈

$$\begin{bmatrix} \mathbf{M}_{11} & \mathbf{M}_{12} & \cdots & \mathbf{M}_{1q} \\ \mathbf{M}_{21} & \mathbf{M}_{22} & \cdots & \mathbf{M}_{2q} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{M}_{p1} & \mathbf{M}_{p2} & \cdots & \mathbf{M}_{pq} \end{bmatrix} + \begin{bmatrix} \mathbf{N}_{11} & \mathbf{N}_{12} & \cdots & \mathbf{N}_{1q} \\ \mathbf{N}_{21} & \mathbf{N}_{22} & \cdots & \mathbf{N}_{2q} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{N}_{p1} & \mathbf{N}_{p2} & \cdots & \mathbf{N}_{pq} \end{bmatrix} = \begin{bmatrix} \mathbf{M}_{11}+\mathbf{N}_{11} & \mathbf{M}_{12}+\mathbf{N}_{12} & \cdots & \mathbf{M}_{1q}+\mathbf{N}_{1q} \\ \mathbf{M}_{21}+\mathbf{N}_{21} & \mathbf{M}_{22}+\mathbf{N}_{22} & \cdots & \mathbf{M}_{2q}+\mathbf{N}_{2q} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{M}_{p1}+\mathbf{N}_{p1} & \mathbf{M}_{p2}+\mathbf{N}_{p2} & \cdots & \mathbf{M}_{pq}+\mathbf{N}_{pq} \end{bmatrix}$$

- 예

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ \hline 9 & 10 & 11 & 12 \end{array}\right] + \left[\begin{array}{ccc|c} 2 & 0 & -1 & 1 \\ -2 & 1 & 0 & -2 \\ \hline 0 & 1 & -2 & 1 \end{array}\right] = \left[\begin{array}{ccc|c} 3 & 2 & 2 & 5 \\ 3 & 7 & 7 & 6 \\ \hline 9 & 11 & 9 & 13 \end{array}\right]$$

블록행렬의 뺄셈

- 뺄셈

$$\begin{bmatrix} \mathbf{M}_{11} & \mathbf{M}_{12} & \cdots & \mathbf{M}_{1q} \\ \mathbf{M}_{21} & \mathbf{M}_{22} & \cdots & \mathbf{M}_{2q} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{M}_{p1} & \mathbf{M}_{p2} & \cdots & \mathbf{M}_{pq} \end{bmatrix} - \begin{bmatrix} \mathbf{N}_{11} & \mathbf{N}_{12} & \cdots & \mathbf{N}_{1q} \\ \mathbf{N}_{21} & \mathbf{N}_{22} & \cdots & \mathbf{N}_{2q} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{N}_{p1} & \mathbf{N}_{p2} & \cdots & \mathbf{N}_{pq} \end{bmatrix} = \begin{bmatrix} \mathbf{M}_{11} - \mathbf{N}_{11} & \mathbf{M}_{12} - \mathbf{N}_{12} & \cdots & \mathbf{M}_{1q} - \mathbf{N}_{1q} \\ \mathbf{M}_{21} - \mathbf{N}_{21} & \mathbf{M}_{22} - \mathbf{N}_{22} & \cdots & \mathbf{M}_{2q} - \mathbf{N}_{2q} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{M}_{p1} - \mathbf{N}_{p1} & \mathbf{M}_{p2} - \mathbf{N}_{p2} & \cdots & \mathbf{M}_{pq} - \mathbf{N}_{pq} \end{bmatrix}$$

- 예

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ \hline 9 & 10 & 11 & 12 \end{array}\right] - \left[\begin{array}{rrr|r} 2 & 0 & -1 & 1 \\ -2 & 1 & 0 & -2 \\ \hline 0 & 1 & -2 & 1 \end{array}\right] = \left[\begin{array}{rrr|r} -1 & 2 & 4 & 3 \\ 7 & 5 & 7 & 10 \\ \hline 9 & 9 & 13 & 11 \end{array}\right]$$

블록행렬의 스칼라배

- 스칼라배

$$c\begin{bmatrix} \mathbf{M}_{11} & \mathbf{M}_{12} & \cdots & \mathbf{M}_{1q} \\ \mathbf{M}_{21} & \mathbf{M}_{22} & \cdots & \mathbf{M}_{2q} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{M}_{p1} & \mathbf{M}_{p2} & \cdots & \mathbf{M}_{pq} \end{bmatrix} = \begin{bmatrix} c\mathbf{M}_{11} & c\mathbf{M}_{12} & \cdots & c\mathbf{M}_{1q} \\ c\mathbf{M}_{21} & c\mathbf{M}_{22} & \cdots & c\mathbf{M}_{2q} \\ \vdots & \vdots & \ddots & \vdots \\ c\mathbf{M}_{p1} & c\mathbf{M}_{p2} & \cdots & c\mathbf{M}_{pq} \end{bmatrix}$$

- 예

$$2\left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ \hline 9 & 10 & 11 & 12 \end{array}\right] = \left[\begin{array}{ccc|c} 2 & 4 & 6 & 8 \\ 10 & 12 & 14 & 16 \\ \hline 18 & 20 & 22 & 24 \end{array}\right]$$

블록행렬의 곱셈

- 곱셈

$$
\begin{bmatrix} \mathbf{M}_{11} & \mathbf{M}_{12} & \cdots & \mathbf{M}_{1q} \\ \mathbf{M}_{21} & \mathbf{M}_{22} & \cdots & \mathbf{M}_{2q} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{M}_{p1} & \mathbf{M}_{p2} & \cdots & \mathbf{M}_{pq} \end{bmatrix}
\begin{bmatrix} \mathbf{N}_{11} & \mathbf{N}_{12} & \cdots & \mathbf{N}_{1r} \\ \mathbf{N}_{21} & \mathbf{N}_{22} & \cdots & \mathbf{N}_{2r} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{N}_{q1} & \mathbf{N}_{q2} & \cdots & \mathbf{N}_{qr} \end{bmatrix}
=
\begin{bmatrix} \sum_{i=1}^{q} \mathbf{M}_{1i}\mathbf{N}_{i1} & \sum_{i=1}^{q} \mathbf{M}_{1i}\mathbf{N}_{i2} & \cdots & \sum_{i=1}^{q} \mathbf{M}_{1i}\mathbf{N}_{ir} \\ \sum_{i=1}^{q} \mathbf{M}_{2i}\mathbf{N}_{i1} & \sum_{i=1}^{q} \mathbf{M}_{2i}\mathbf{N}_{i2} & \cdots & \sum_{i=1}^{q} \mathbf{M}_{2i}\mathbf{N}_{ir} \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{i=1}^{q} \mathbf{M}_{pi}\mathbf{N}_{i1} & \sum_{i=1}^{q} \mathbf{M}_{pi}\mathbf{N}_{i2} & \cdots & \sum_{i=1}^{q} \mathbf{M}_{pi}\mathbf{N}_{ir} \end{bmatrix}
$$

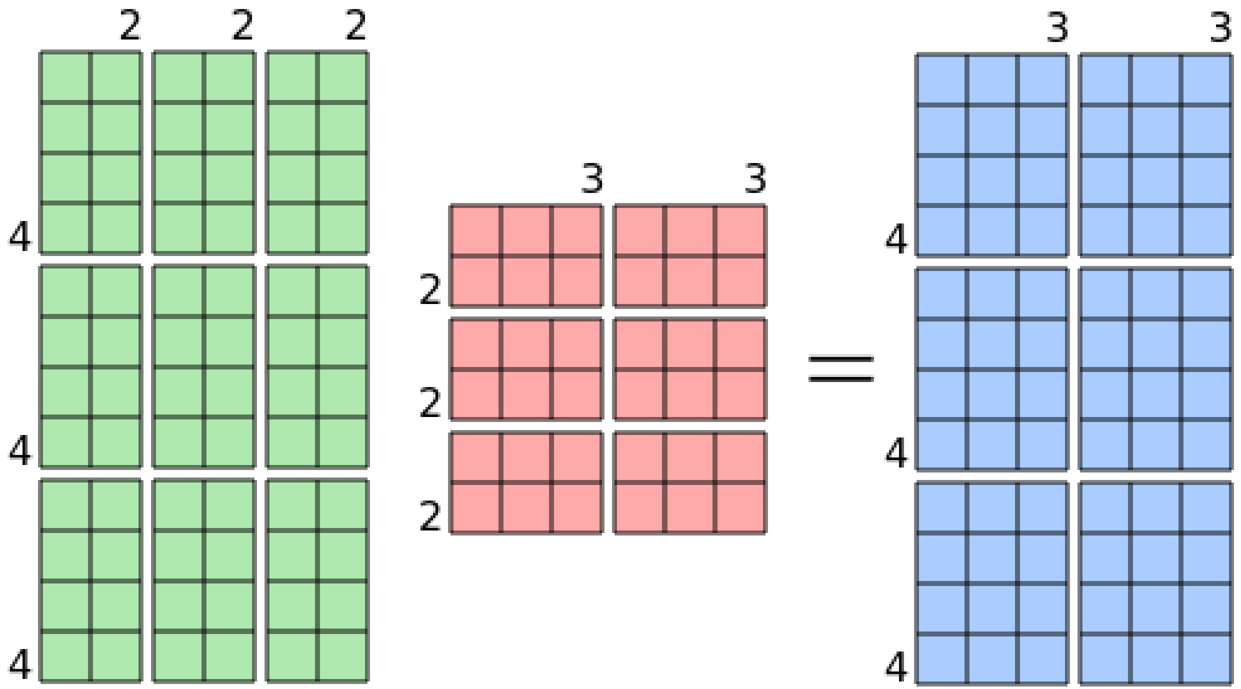
- 예

$$
\left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ \hline 9 & 10 & 11 & 12 \end{array}\right]
\left[\begin{array}{cc|c} 2 & -2 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & -2 \\ \hline -1 & -2 & 1 \end{array}\right]
=
\left[\begin{array}{c|c}
\begin{bmatrix} 1 & 2 & 3 \\ 5 & 6 & 7 \end{bmatrix}\begin{bmatrix} 2 & -2 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 4 \\ 8 \end{bmatrix}\begin{bmatrix} -1 & -2 \end{bmatrix} & \begin{bmatrix} 1 & 2 & 3 \\ 5 & 6 & 7 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 4 \\ 8 \end{bmatrix}\begin{bmatrix} 1 \end{bmatrix} \\ \hline
\begin{bmatrix} 9 & 10 & 11 \end{bmatrix}\begin{bmatrix} 2 & -2 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 12 \end{bmatrix}\begin{bmatrix} -1 & -2 \end{bmatrix} & \begin{bmatrix} 9 & 10 & 11 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 12 \end{bmatrix}\begin{bmatrix} 1 \end{bmatrix}
\end{array}\right]
$$

$$
= \left[\begin{array}{cc|c} 1 & -8 & 0 \\ 9 & -20 & 0 \\ \hline 17 & -32 & 0 \end{array}\right]
$$

블록행렬의 곱셈

- 곱셈



https://ko.wikipedia.org/wiki/블록_행렬

블록행렬의 전치

- 전치(Transpose)

$$\begin{bmatrix} \mathbf{M}_{11} & \mathbf{M}_{12} & \cdots & \mathbf{M}_{1q} \\ \mathbf{M}_{21} & \mathbf{M}_{22} & \cdots & \mathbf{M}_{2q} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{M}_{p1} & \mathbf{M}_{p2} & \cdots & \mathbf{M}_{pq} \end{bmatrix}^\top = \begin{bmatrix} \mathbf{M}_{11}^\top & \mathbf{M}_{21}^\top & \cdots & \mathbf{M}_{p1}^\top \\ \mathbf{M}_{12}^\top & \mathbf{M}_{22}^\top & \cdots & \mathbf{M}_{p2}^\top \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{M}_{1q}^\top & \mathbf{M}_{2q}^\top & \cdots & \mathbf{M}_{pq}^\top \end{bmatrix}$$

- 예

$$\left[\begin{array}{rrr|r} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ \hline 9 & 10 & 11 & 12 \end{array}\right]^\top = \left[\begin{array}{rr|r} 1 & 5 & 9 \\ 2 & 6 & 10 \\ 3 & 7 & 11 \\ \hline 4 & 8 & 12 \end{array}\right]$$

👓 행렬의 두 가지 관점

행렬의 두 가지 관점

- 행렬: 가로보기

$$\mathbf{A} = \left[\begin{array}{cccc} a_{11} & a_{12} & \cdots & a_{1n} \\ \hline a_{21} & a_{22} & \cdots & a_{2n} \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline a_{m1} & a_{m2} & \cdots & a_{mn} \end{array}\right] \in \mathbb{R}^{m\times n}$$

- 행렬: 세로보기

$$\mathbf{A} = \left[\begin{array}{c|c|c|c} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{array}\right] \in \mathbb{R}^{m\times n}$$

- 행벡터의 열벡터

$$\mathbf{A} = \begin{bmatrix} - & \mathbf{a}_1^\top & - \\ - & \mathbf{a}_2^\top & - \\ & \vdots & \\ - & \mathbf{a}_m^\top & - \end{bmatrix}, \quad \mathbf{a}_i \in \mathbb{R}^n$$

- 열벡터의 행벡터

$$\mathbf{A} = \begin{bmatrix} | & | & & | \\ \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \\ | & | & & | \end{bmatrix}, \quad \mathbf{a}_i \in \mathbb{R}^m$$

[돌아보기] 행렬의 곱셈: 정의

- 두 행렬

$$\mathbf{A} = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \in \mathbb{R}^{m\times n}, \quad \mathbf{B} = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1l} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nl} \end{bmatrix} \in \mathbb{R}^{n\times l}$$

- 두 행렬의 곱셈

$$\mathbf{AB} = \begin{bmatrix} \sum_{i=1}^{n} a_{1i}b_{i1} & \cdots & \sum_{i=1}^{n} a_{1i}b_{il} \\ \vdots & \ddots & \vdots \\ \sum_{i=1}^{n} a_{mi}b_{i1} & \cdots & \sum_{i=1}^{n} a_{mi}b_{il} \end{bmatrix} \in \mathbb{R}^{m\times l}$$

참고: 앞행렬의 열의 개수와 뒷행렬의 행의 개수가 다르면 곱셈을 할 수 없다.

행렬의 곱셈: 두 블록행렬의 곱셈

- (행벡터의) 열벡터, (열벡터의) 행백터

$$\mathbf{A} = \begin{bmatrix} - & \mathbf{a}_1^\top & - \\ - & \mathbf{a}_2^\top & - \\ & \vdots & \\ - & \mathbf{a}_m^\top & - \end{bmatrix} \in \mathbb{R}^{m\times n}, \quad \mathbf{B} = \begin{bmatrix} | & | & & | \\ \mathbf{b}_1 & \mathbf{b}_2 & \cdots & \mathbf{b}_n \\ | & | & & | \end{bmatrix} \in \mathbb{R}^{n\times l}$$

- (행벡터의) 열벡터와 (열벡터의) 행백터의 곱 (Outer Product)

$$\mathbf{AB} = \begin{bmatrix} \mathbf{a}_1^\top \mathbf{b}_1 & \cdots & \mathbf{a}_1^\top \mathbf{b}_l \\ \vdots & \ddots & \vdots \\ \mathbf{a}_m^\top \mathbf{b}_1 & \cdots & \mathbf{a}_m^\top \mathbf{b}_l \end{bmatrix} \in \mathbb{R}^{m\times l}$$

각 원소는 앞행렬의 행벡터와 뒷행렬의 열벡터의 내적

[돌아보기] 행렬의 전치: 정의

- 전치
  - 轉: 구르다, 회전하다, 예: 회전, 운전, 역전, 반전, 전화위복, 기승전결, 심기일전
  - 置: 두다, 예: 위치, 배치

- 행렬

$$\mathbf{A} = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \in \mathbb{R}^{m \times n}$$

- 정의: 전치행렬

$$\mathbf{A}^\top = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{bmatrix} \in \mathbb{R}^{n \times m}$$

전치: 열과 행을 바꾼다. $[\mathbf{A}^\top]_{ij} = \mathbf{A}_{ji}$

행렬의 전치: 블록행렬의 전치

- (행벡터의) 열벡터 ⇒ (열벡터의) 행벡터

$$\mathbf{A} = \begin{bmatrix} - & \mathbf{a}_1^\top & - \\ - & \mathbf{a}_2^\top & - \\ & \vdots & \\ - & \mathbf{a}_m^\top & - \end{bmatrix} \Rightarrow \mathbf{A}^\top = \begin{bmatrix} | & | & & | \\ \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_m \\ | & | & & | \end{bmatrix}$$

- (열벡터의) 행벡터 ⇒ (행벡터의) 열벡터

$$\mathbf{A} = \begin{bmatrix} | & | & & | \\ \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \\ | & | & & | \end{bmatrix} \Rightarrow \mathbf{A}^\top = \begin{bmatrix} - & \mathbf{a}_1^\top & - \\ - & \mathbf{a}_2^\top & - \\ & \vdots & \\ - & \mathbf{a}_n^\top & - \end{bmatrix}$$

📝 요약

[돌아보기] 행렬의 연산

- 두 행렬

$$\mathbf{A},\ \mathbf{B} \in \mathbb{R}^{m\times n}$$

- 행렬의 덧셈

$$[\mathbf{A} \pm \mathbf{B}]_{ij} = [\mathbf{A}]_{ij} \pm [\mathbf{B}]_{ij}$$

- 행렬의 스칼라배

$$[c\mathbf{A}]_{ij} = c[\mathbf{A}]_{ij}$$

- 행렬의 전치

$$[\mathbf{A}^\top]_{ij} = [\mathbf{A}]_{ji}$$

- 두 행렬

$$\mathbf{A} \in \mathbb{R}^{m\times n},\ \mathbf{B} \in \mathbb{R}^{n\times l}$$

- 행렬의 곱셈

$$[\mathbf{A}\mathbf{B}]_{ij} = \mathbf{a}_i^\top \mathbf{b}_j$$

- 행렬의 곱은 교환법칙이 성립하지 않는다

$$\mathbf{A}\mathbf{B} \neq \mathbf{B}\mathbf{A}$$

- 행렬의 대각합

$$\mathrm{Trace}(\mathbf{A}) = \sum_{i=1}^{n} [\mathbf{A}]_{ii}$$

블록행렬의 연산

- 두 행렬

$$\mathbf{A} = [\mathbf{M}_{ij}],\ \mathbf{B} = [\mathbf{N}_{ij}] \in \mathbb{R}^{m \times n}$$

- 행렬의 덧셈

$$[\mathbf{A} \pm \mathbf{B}]_{ij} = \mathbf{M}_{ij} \pm \mathbf{N}_{ij}$$

- 행렬의 스칼라배

$$[c\mathbf{A}]_{ij} = c\mathbf{M}_{ij}$$

- 행렬의 전치

$$[\mathbf{A}^\top]_{ij} = \mathbf{M}_{ji}^\top$$

- 두 행렬

$$\mathbf{A} = [\mathbf{M}_{ij}] \in \mathbb{R}^{m \times n},\ \mathbf{B} = [\mathbf{N}_{ij}] \in \mathbb{R}^{n \times l}$$

- 행렬의 곱셈

$$[\mathbf{A}\mathbf{B}]_{ij} = \sum_{k=1}^{n} \mathbf{M}_{ik}\mathbf{N}_{kj}$$

🍒 References

References

- 칸아카데미 - 행렬